Fulvia Skof (Ed.)

# Giuseppe Peano
## between Mathematics and Logic

Springer

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Giuseppe Peano (1858–1932) in 1928 – Department of Mathematics G. Peano, University of Torino

Fulvia Skof
*Editor*

# Giuseppe Peano between Mathematics and Logic

Proceeding of the International Conference in honour of Giuseppe Peano on the 150th anniversary of his birth and the centennial of the *Formulario Mathematico*, Turin (Italy), October 2–3, 2008

Springer

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# Preface

Giuseppe Peano was one of the greatest figures in modern mathematics and logic, and without a doubt the most important Italian mathematical logician, esteemed by Bertrand Russell as well as Rudolf Carnap and Kurt Gödel. Born in a small village near Cuneo, in southern Piedmont, in August 1858 – on the eve of Italian unity – he studied in Turin, where he would then rapidly advance through the successive levels of his academic career: he received his habilitation in Infinitesimal Calculus in 1884, became a professor at the Military Academy in 1889, entered the University of Torino in 1890 and was a full professor of Infinitesimal Calculus from 1895 to 1931, and was also a full professor of Complementary Mathematics until his death in April 1932. As early as 1891 he was a member of Turin's Accademia delle Scienze, and in 1905 became a member of the Accademia Nazionale dei Lincei as well.

On the occasion of the one hundred fiftieth anniversary of Peano's birth, and a century after the publication of the fifth edition of the *Formulario Mathematico*, a grandiose attempt to systematise mathematics in symbolic form, the Accademia delle Scienze of Torino and the University of Torino (in particular, the Faculty of Mathematical, Physical and Natural Sciences and the Department of Mathematics), together with the Italian Society for the History of Mathematics, created a committee for the celebration of this dual occasion, the presidency of which was entrusted to Prof. Clara Silvia Roero. Among the many initiatives organised by the committee, one of the most important with respect to science was the international conference entitled "Giuseppe Peano between Mathematics and Logic", which took place on 2–3 October 2008 under the auspices of the President of the Republic and with the sponsorship of the Accademia Nazionale dei Lincei and the Istituto Lombardo Accademia di Scienze, Lettere e Arti. The conference provided an examination of the various aspects of Peano's work, presented by the greatest scholars from Italy and abroad. This present volume contains the papers that developed out of the presentations given during the conference.

The conference was made possible by funding contributed by the Region of Piedmont, the Compagnia di San Paolo and the Cassa di Risparmio di Cuneo, as well as the Accademia delle Scienze of Torino – under the presidency of Angelo Raffaele Meo – which hosted the conference in the "Sala dei Mappamondi". We are most

grateful to these institutions. Particular thanks go to Prof. Roero, then president of the Italian Society for the History of Mathematics, for her efficient and indefatigable organisational work, and for her valuable collaboration with Prof. Fulvia Skof in collecting and editing the papers that appear here.

Turin, September 2010 Accademia delle Scienze

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# Contents

# 1

# Giuseppe Peano and Mathematical Analysis in Italy

Fulvia Skof

If you enter what was latterly the office of the Head of the former *Istituto di Analisi Matematica* of the University of Torino, you find before you a group of photographs which recall those Professors who held the Chair of *Analisi* (or *Calcolo*) in the period from 1811 to 1972; among them, side by side with Giovanni Plana (who occupied the Chair from 1811 to 1864), Angelo Genocchi (from 1864 to 1889), Enrico D'Ovidio (from 1872 to 1918), is the mild countenance of Giuseppe Peano, Professor of Calculus from 1890 to 1931; next to him is his successor Francesco G. Tricomi, Professor of Analysis from 1925 to 1972.

Peano, in fact, who left the mark of his brilliant mind and results in a variety of fields of knowledge, started from Analysis as a teacher and scientist. Graduating in 1880, he spent a year as assistant of D'Ovidio, then was assistant of Genocchi from 1881 to 1890; *Libero docente di Calcolo infinitesimale* in 1884, he became a full professor of Infinitesimal Calculus in December 1890 and remained in this role until 1931; he subsequently transferred to the Chair of *Matematiche complementari* in the last year of his life (1931/32).

## 1.1 Peano and Classical Analysis

It is his interest in Analysis that seems really to have been the basis on which were to develop the various fields of research which made clear the characteristic originality, the perspicacity, the independence of Peano's thought: fields of research that ranged from Analysis to Geometry, to numerical Calculus, to Logic, to History, and finally to Linguistics, to say nothing of his many other studies, some of which were of a practical or social origin.

In his first years as a University teacher, teaching itself had a crucial influence on his research, leading him to clarify and go more deeply into the various topics of the Calculus courses, thanks to his inborn, exceptional critical spirit as well as his vast classical culture.

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The decisive opportunity for a profound re-examination of Analysis was offered to him, in 1884, by the writing of the text "A. Genocchi, *Calcolo differenziale e principii di calcolo integrale, pubblicato con aggiunte da G. Peano*" (Peano 1884c), which originated from Genocchi's lectures. In reality these lectures were greatly enriched by the addition of *Annotazioni* (*Notes*), which are the most original part of the book and which, as Genocchi himself chose to make clear, almost as though he wanted to keep his distance from them, were entirely the work of the young Peano: here were critical observations, here, often by means of brilliant counter-examples, were stressed – and then corrected simply and rigorously – inexactitudes frequently repeated and included in most texts of the time. Peano himself specified, in the commemoration (Peano 1890a) of his Master, Genocchi:

> Ill at that time, he wished to remain extraneous to the whole undertaking. Making use of summaries made by students during his lectures, I compared them point by point with all the main treatises on calculus, and with original papers, thus taking into account the work of many. I consequently made many additions to his lectures, and some modifications.[1]

The book was also published in a German edition (Bohlman, Schepp 1899t).

Meanwhile, the characteristics of his keen mind and the scrupulousness, typical of a careful scientist, with which he enriched his knowledge of the developments of contemporary mathematics made it possible for him to perceive the need for rigour and abstract thought which were beginning to be developed and to spread in the mathematical environment, thanks to the works of a number of writers among whom we may mention – in Italy – Ulisse Dini, Salvatore Pincherle, Cesare Arzelà. The new spirit that was animating the Analysts in those years, "heroic" for the evolution of Analysis after the construction of the theory of continuous functions by Lagrange and Cauchy, is clearly perceptible in the Introduction to Dini's treatise *Fondamenti per la teorica delle funzioni di variabili reali* (Dini 1878), where the author states:

> I shall be happy if [...] it contributes to make known certain remarks and results which in recent times have shaken the foundational principles of Analysis, only to rebuild them immediately on more solid bases.[2]

Peano's brilliant results, which are still currently studied in the first two years of our courses in Analysis and which, as we have seen, sprang up on the margins of his teaching activities in the last twenty years of the 19th century, fit in with this critical spirit: they are expounded, not only in some printed scientific papers and in the

[1] G. Peano (1890a), 198–199: "Egli, allora malato, volle rimanere estraneo a tutto il lavoro. Io, servendomi di sunti fatti da allievi alle sue lezioni, li paragonai punto per punto con tutti i principali trattati di calcolo, e con Memorie originali, tenendo così conto dei lavori di molti. Feci in conseguenza alle sue lezioni molte aggiunte, e qualche modificazione."

[2] U. Dini (1878), viii: "Sarò lieto se [...] esso contribuirà a rendere familiari alcune osservazioni e alcuni risultati che in questi ultimi tempi hanno scosso per poi riedificare, immediatamente, su basi più solide i principii fondamentali dell'Analisi."

*Annotazioni*, mentioned above, to the treatise on Calculus (the "Genocchi-Peano") of 1884, but also in the treatise (in two volumes) *Lezioni di Analisi infinitesimale* (Peano 1893h) written for the students of the Scuola di Artiglieria e Genio (Artillery and Engineer Corps School) in Turin, where Peano was teaching regularly from 1886 to 1901. Both treatises were highly praised, in Italy and abroad, and are recalled by the mathematician A. Pringsheim, in the volume 2 of Enzyklopädie der Mathematischen Wissenschaften, among the twenty most important treatises for the development of the theory of functions, starting from Euler's work.

Suffice it to mention only some of these famous results, selecting those which seem best to indicate Peano's peculiarities:

- On the integrability of the functions of a real variable, with reference to the upper and lower bounds of the integral sums (Peano 1882b, 1895n).
- An explicit expression (in the form of an iterated limit) of "Dirichlet's function" (which is 0 on rationals and 1 on irrationals) (Peano 1884c: Annotazione n. 28).
- The theorem (later known as "Peano's theorem") of the three functions (Peano 1884c: Annotazione n. 45).
- A theorem on the existence and derivability of the implicit function (Peano 1893h vol. 2).
- On the convergence of numerical series, in relation to the behaviour of the $n$th term (Peano 1884c: Annotazione n. 55).
- On the inversion of mixed second derivatives (Peano 1890e).
- The famous theorem on the existence of the solution to Cauchy's problem for the first-order differential equation in the sole assumption of continuity (Peano 1885a) and its extension to the case of the system of differential equations (Peano 1890f).
- On the definition of "area" of a curved surface (Peano 1890c – *Opere scelte*, vol. 1, 107–109).
- A new form of the complementary term in Taylor's formula for real functions (Peano 1889e) and its generalisation to "complexes" of any order (Peano 1894a – *Opere scelte*, vol. 1, 226–227).
- Introduction of the concept of "asymptotic " development of a function in power series (Peano 1892a).
- On the definition of derivative (Peano 1884a,b and 1892s).
- On the definition of limit of a function (Peano 1892l and 1895c).
- The famous example of a curve filling a square (the "Peano's curve") (Peano 1890b – *Opere scelte*, vol. 1, 115–116).

And finally: on the integration by series of linear differential equations, on Jacobians, Wronskians and still more.

Separate attention is merited by the volumes *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* (Peano 1888a), *Gli elementi di calcolo geometrico* (Peano 1891b), *Die Grundzüge des Geometrischen Calculs* (transl. Schepp, 1891t), to which we will return later.

Particularly worthy of note is his recourse to *counter-examples*, which displays Peano's critical and at the same time practical spirit.

## 1.2 Symbolic Writing and New Directions of Research

While he was carrying out the already mentioned critical revision of the basic topics of classical Analysis, Peano felt increasingly the need to make mathematical propositions succinct, rigorous and free of ambiguity, to a degree that ordinary language cannot guarantee. Taking over the problem, already posed by Leibniz, of expressing any logical proposition by means of a small number of symbols having constant meaning, subjected to precise logical rules as though dealing with calculus, Peano immersed himself in this new research, which was to lead him to the well known mathematical ideography, and to the series of studies on criticism of the foundations, which was to win him a very prominent position in Mathematical Logic.

From this time, also his work on Analysis was gradually enriched with extensive parts written in ideographic form, which at first were accompanied with their translation, for didactic purposes. A good example of this is the paper on the definition of limit, published in the *American Journal of Mathematics* (Peano 1895c), where the proofs written in symbolic form and accompanied by detailed explanations in current language, are followed by a final remark aimed at convincing the reader of the usefulness of symbolic writing. To this end, he quotes Condillac:

> Tout l'art de raisonner se réduit à bien faire la langue de chaque science. Plus vous abrégerez votre discours, plus vos idées se rapprocheront ; et plus elles seront rapprochées, plus il vous sera facile de les saisir sous tous leurs rapports.[3]

This is the spirit in which, in 1891, he founded the *Rivista di Matematica* (which came out in eight volumes, finishing in 1906), of which he was himself the director.

From some of Peano's letters contained in the "collected papers" of renowned Italian Analysts of the time – for example, his correspondence with Giuseppe Vitali – it is interesting to learn that Peano was in the habit of making strong appeals to his interlocutor, who had asked his opinion of work he had carried out, appeals that the statements expressed in ordinary language be rewritten in strictly symbolic form, here and there betraying his impatience with the ambiguity inherent in the common language used by the author. For example, he wrote to Vitali in 1905: "Take steps to make your work intelligible to me ..."[4].

It should be remarked that the strong defence of symbolic writing which characterises his work as a whole contributed to win him the esteem he enjoys today in the mathematical community, and especially among logical mathematicians. But it was also an obstacle to the timely achievement of the success promised by Peano's brilliant results in Analysis in the early years and by his innovative ideas; for the presentation of the results in ideographic form, distancing the majority of readers, in the end made them little known. This is what happened, for instance, to the important theorem on the existence of a solution to Cauchy's problem for first-order differ-

[3] G. Peano (1895c), 68.

[4] G. Peano to G. Vitali, Turin 25 March 1905, in Vitali 1984, letter n. 37, 452: "Ella faccia un passo per rendermi intelligibile il suo lavoro, senza obbligarmi a ricerche."

ential equations (Peano 1885a) and the subsequent extensions to systems (Peano 1890f): Beppo Levi recalls, in his commemoration (Levi 1932) of his Master, that they were virtually unknown until the German mathematician Gustav Mie, in his *Beweis der Integrierbarkeit gewoehnlicher Differentialgleichungssysteme nach Peano* (Mie 1893), based on Peano's 1890 paper – with addition of a study of the uniqueness of the solution – supplied, as Levi says:

> [...] free re-exposition of Peano's Memoir [...], freeing it from the new hindrance to reading caused by the use of logical ideography, which Peano had introduced in his own exposition, and from that prolixity that resulted from an excessive preciseness in the statement of introductory observations. It was only after Mie's work that Peano's result and his procedure to proof could be universally appreciated at its true value, and could prompt further studies by De La Vallée-Poussin, Arzelà, and Osgood.[5]

Levi adds that: "Among the critical observations of this work, one regarding the necessity – because of mathematical rigour – of avoiding 'infinite choices' deserves to be recalled"[6]. But we shall return to this point.

The fulfillment of Peano's *Formulario mathematico* goes back to this period: it was prepared in response to the idea suggested to him by the wealth of his mathematical (and historical) knowledge on the one hand, and by ideography seen here as a tool, on the other. He had forecast it in 1892, remarking:

> It would be of the greatest use to publish the collections of all the theorems now known referring to given branches of the mathematical sciences [...]. Such a collection, extremely difficult and long in ordinary language, is notably facilitated by the use of the notations of logical mathematics; and the collection of the theorems on a given subject perhaps becomes less long than its bibliography.[7]

The *Formulario* came out, thanks in part to the collaboration of a number of Peano's students (among them G. Vailati, C. Burali-Forti, G. Vivanti, R. Bettazzi, G. Fano, A. Padoa), between 1895 and 1908 in five editions, the last of them (Peano 1908a) written in *latino sine flexione* in accordance with the linguistic studies Peano had

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[5] B. Levi (1932), 258: "[...] libera trasposizione della Memoria del Peano [...], liberandola dal nuovo ostacolo posto alla lettura dall'uso della ideografia logica che il Peano aveva introdotto nella propria esposizione, e dalle prolissità derivanti da un eccesso di precisione nella enunciazione di osservazioni introduttorie. Soltanto dopo questo lavoro del Mie il risultato e il procedimento dimostrativo del Peano poté essere universalmente stimato al suo giusto valore e provocare altri studii da parte del De La Vallée-Poussin, dell'Arzelà, dell'Osgood."

[6] B. Levi (1932), 258: "Merita di essere ricordata, fra le osservazioni critiche di questo lavoro, una relativa alla necessità – per il rigore matematico – di evitare le infinite scelte."

[7] G. Peano (1892k), 76: "Sarebbe cosa della più grande utilità il pubblicare delle raccolte di tutti i teoremi ora noti riferentisi a dati rami delle scienze matematiche [...]. Una siffatta raccolta, difficilissima e lunga con il linguaggio comune, è notevolmente facilitata servendoci delle notazioni della logica matematica; e la raccolta dei teoremi su un dato soggetto diventa forse meno lunga della sua bibliografia."

undertaken starting from 1903: this is a work which, both for the wealth of mathematical content and the painstaking historical reports, and for the critical rigour and choice of language – which, in the present writer's opinion, should make it readily understood by a broad range of scholars – is the most expressive "monument" capable of representing the outstanding, polyhedric personality of its author.

From the early years of the 20th century, Peano was devoting himself to a variety of projects which consequently dispersed his attention. The result was that he drew away from specific research on Analysis. Perhaps this dispersion is more imagined than real, for the subsequent developments of his activity, seen as a whole, reveal a continuity whose *fil rouge* is the constant search for simplicity and for the rigorous clarification of the basic elements of every problem he faced. This is also true of his remarkably deep linguistic studies, which seem to have become his main activity after 1909, made concrete in the creation of the *Academia pro Interlingua* and in the publication of the *Vocabulario commune ad linguas de Europa* (Peano 1909b).

Alongside these activities arose the need to solve new typographic problems linked to the increasing use of ideography in his writings; this drastically limited the number of journals in which he could publish his scientific papers, while the constant need to update the *Formulario* meant that new editions had to be prepared. Faced with all this, as is well known, he did not hesitate to make personal provision – once he had learned the necessary techniques – for the printing of the volumes at a small, specially equipped printing shop in his house in Cavoretto. This experience also led him to make rational simplifications to the writing of certain symbols or formulae, after painstaking historical research on the forms of the commonest mathematical symbols. This, for example, is the topic of the short – interesting and highly readable – paper *Sulla forma dei segni in algebra* (Peano 1920b) or of *L'esecuzione tipografica delle formule matematiche* (Peano 1916a).

In the early years of the 20th century Peano also wrote papers dedicated to mathematics as applied to human activities. For example, he derived a form of shorthand from the binary representation of numbers; he also studied problems of actuarial mathematics, in a group of works which he wrote in his role as a member of the Board nominated by the Turin "Commissione nominata dalla Cassa Mutua Cooperativa per le Pensioni" for the study of their social bases: these are interesting works, not only because of the perspicacious mathematical method and the exhaustive analysis of the various possible situations, but also because of the human aspect of the considerations contained in them, which allow us to discover further qualities of his personality.

## 1.3 Peano and Analysis after 1900

In such a variety of interests, what place did Analysis occupy for Peano after 1900? He continued to be Professor of Analysis at the University. He kept up a dense correspondence with well-known Analysts, both Italian and foreign. He took part in international Conferences on mathematics and logic. He was certainly up to date on the trends of contemporary research, as is also clear from his reviews of new treatises by distinguished authors. It appears that teaching Analysis no longer offered him spurs towards a better critical organisation of the subject, whereas it appears natural that Peano now planned to direct his students towards the correct use of symbolism and of the rules of logic.

In any case, his student Beppo Levi in the commemoration of his Master, which we have already mentioned, was careful to note:

> [...] one as it were practical characteristic of Peano's thinking and teaching, which may seem to be opposed to the abstractness of absolute rigour[8]

and, referring to the text used by Peano in his lectures (the 1893 treatise), states:

> Anyone who examines his lectures on infinitesimal Analysis [...] will find neither the quest for generalities, nor trifling conditions for the validity of propositions; despite some digressions towards his favourite topics, logical notation and geometric calculus, the author proceeds rapidly admitting all the conditions of continuity that hold true in practice, and that permit the greatest simplicity in statements and proofs. In compensation, Peano insists on numerical application, actual calculation, the determination of the approximations that accompany this calculation. These topics, about which he was already thinking in his youth with his research on Taylor's formula, on the interpolation formulae, on approximate integration and on the rest of these formulae, ended by having the upper hand in his more recent scientific writings and in the direction he gave to the research of his last disciples.[9]

Yet in the work on Analysis in which Peano was engaged in the first period, there are aspects of modernity which herald Functional Analysis. They can already be found in the important 1888a treatise *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* and in its subsequent short but important reworking *Gli elementi di Calcolo geometrico* (1891b), which was also published in German (transl. Schepp 1891t). These are works in which Peano achieved a *vectorial calculus*, in the sense of a system of operations to be performed directly on the geometric entities, in a modern view, which however has its roots, yet again, in the thinking of Leibniz.

[8] B. Levi (1932), 262: "Una caratteristica per così dire pratica del pensiero e dell'insegnamento del Peano, che potrebbe parere in opposizione colle astrattezze dell'assoluto rigore."

[9] B. Levi (1932), 262: "Chi esamini le Lezioni di Analisi infinitesimale [...] non trova né ricerca di generalità, né minuzia di condizioni per la validità delle proposizioni; nonostante qualche divagazione attraverso gli argomenti prediletti, notazioni logiche e calcolo geometrico, l'Autore procede rapido ammettendo tutte le condizioni di continuità che nella pratica si verificano e che consentano agli enunciate e alle dimostrazioni la massima semplicità. In compenso, insiste il Peano sull'applicazione numerica, sul calcolo effettivo, sulla determinazione delle approssimazioni che a questo calcolo si accompagnano. Questi argomenti, ai quali già si volgeva il suo pensiero giovanile colle ricerche sulla formula di Taylor, sulle formule di interpolazione, sull'integrazione approssimata, sui resti delle dette formole, finirono per prendere il sopravvento nella sua produzione scientifica degli ultimi tempi e nell'indirizzo che egli diede alle ricerche degli ultimi suoi discepoli."

Peano himself did not hesitate to use direct calculus on vectors (which he referred to as *complessi*) in Vol. 2 of his *Lezioni di Analisi infinitesimale* (1893h). Applications to problems of Mechanics or Geometry were demonstrated by Cesare Burali-Forti, who like Peano taught at the Turin School of Artillery and Military Engineering, and later by other writers. It is also worth noting that in the important work of 1890 on Cauchy's problem for the systems of differential equations, Peano conceived the system abstractly as representing a single equation.

In the Italian cultural world, Peano's positions, in advance of their time, found convinced supporters, but also, as we shall see, opponents. Among the former, outside the circle of his students who revered their Master, was the famous analyst Salvatore Pincherle who, in his comprehensive *Mémoire sur le calcul fonctionnel distributif* (Pincherle 1897), felt called upon to indicate the point of view of Peano's *Calcolo geometrico* in the following terms:

> Il nous reste enfin à citer quelques travaux qui regardent encore le calcul fonctionnel, mais qui s'en occupent à un point de vue nouveau, qui permet de rendre très claires et presque intuitives certaines généralités de ce calcul. C'est le point de vue vectoriel ou du calcul géométrique, inspiré par l'Ausdehnungslehre de Grassmann et par les écrits de Hamilton et de Tait sur les quaternions. Dans cet ordre d'idées, M. Peano a écrit quelques pages très intéressantes où, d'une façon aussi sobre que claire, il donne les propriétés les plus simples des opérations distributives appliquées à des éléments déterminés par *n* coordonnées : on peut dire que c'est une esquisse de la théorie des opérations fonctionnelles distributives exécutée sur les fonctions d'un ensemble linéaire *n* fois infini ; [...] L'auteur note encore, sans y insister, qu'on pourrait aussi considérer des systèmes linéaires à un nombre infini de dimensions.[10]

Clear evidence of the respect and fame enjoyed by Peano in Italy and abroad is found in the important honours granted him by scientific Academies and Associations in those years: Member of the Accademia dei Lincei from 1905; Member of the Istituto Lombardo di Scienze e Lettere from 1922, of the Circolo Matematico di Palermo, of the Geneva National Institute, of the Kazan Physico-mathematical Society, of the "Antonino Alzate" Scientific Society in Mexico. The Accademia delle Scienze of Torino had elected him a member as early as 1891.

But he also knew the bitterness of being an underestimated thinker, both in the philosophical and in the mathematical environment in Italy, even in his own University in Turin.

A lively picture of Peano's situation with regard to the Italian cultural environment, and specifically in Turin, emerges from the recollections of the philosopher Ludovico Geymonat, who was Peano's student for a year in the Mathematics courses. On the occasion of the *Celebrations* in memory of Peano held in Turin in 1982, he recalled:

> When, back in 1934, I went to Vienna to go more deeply into the neopositivism of Schlick, I took with me various letters of introduction; [...] to my

[10] S. Pincherle (1897), 331–332.

> surprise, what carried most weight in my favour was the fact that in 1930–31 I had been Peano's student. I have allowed myself to mention what is in itself an unimportant fact for two reasons: 1) to emphasise the very high regard which Peano enjoyed, even after his death, outside Italy; 2) to admit that I, like many other young people who had just graduated from the University of Turin, had not realised the exceptional value of the man whose lectures I had attended for a whole academic year, and with whom I had also had so many opportunities to converse outside the lecture halls. The truth is that for some years this great mathematician and logician had been in a very singular position in the Turin Science Faculty: he had been deprived of the basic teaching of infinitesimal analysis and had been, so to speak, confined to what was then regarded as secondary, the course on complementary mathematics; there were valid didactic reasons for this measure, but Peano had been greatly embittered by it, as it implied a somewhat limiting view of his capabilities. And it was precisely this limiting view on the part of his colleagues in the Faculty that hindered us students from recognising all the value of the scientific – and not only scientific – personality that stood before us. But if this was the general atmosphere at the University of Turin in the years 1929–30, 30–31, 31–32, it was in truth merely a reflection of the situation in force throughout Italy with regard to Peano's work, in the last years of his life. Of course, his fortunes were very different in the early years of the 20th century. Yet even then something hindered his thinking from exercising on Italian philosophical and scientific culture all the influence which in our view today he certainly deserved to exercise.[11]

Geymonat recalls harsh judgments expressed in Italy on Peano's research in Logic and on his school – especially on the work of his student Giovanni Vailati:

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[11] L. Geymonat (1986), 7–8: "Quando nel lontano 1934 mi recai a Vienna per approfondire il neopositivismo di Schlick, portai con me diverse lettere di presentazione; [. . . ] con mia sorpresa, ciò che pesò più di tutti a mio vantaggio fu il fatto che nel 1930–31 io ero stato allievo di Peano. Mi sono permesso di ricordare questo fatto in se stesso di nessun rilievo, a due scopi: 1) per sottolineare l'altissima stima di cui Peano godeva, anche dopo la sua morte, fuori d'Italia; 2) per confessare che purtroppo io pure, come molti altri giovani appena usciti dall'Università di Torino, non mi rendevo conto dell'eccezionale valore dell'uomo di cui tuttavia avevo seguito le lezioni per un intero anno accademico, e col quale avevo avuto tante occasioni per discorrere anche fuori dalle aule accademiche. La realtà è che da qualche anno il grande matematico e logico occupava nella Facoltà di scienze di Torino una posizione assai singolare: era stato privato dell'insegnamento fondamentale di analisi infinitesimale ed era stato, per così dire, confinato in quello (allora ritenuto secondario) di matematiche complementari; provvedimento che aveva delle valide giustificazioni di ordine didattico, ma che aveva profondamente amareggiato Peano, implicando un giudizio limitativo nei suoi riguardi. Ed è proprio questo giudizio limitativo dei suoi colleghi di Facoltà, che impediva a noi studenti di riconoscere tutto il valore della personalità scientifica, e non solo scientifica, che stava innanzi a noi. Ma se questa era l'atmosfera diffusa all'Università di Torino negli anni 1929–30, 30–31, 31–32, essa in verità non era che un riflesso della situazione vigente in tutta l'Italia riguardo all'opera di Peano, negli ultimi anni della sua vita. Ben diversa era stata, naturalmente, la sua fortuna all'inizio del secolo. Eppure anche allora vi fu qualcosa che impedì al suo pensiero di esercitare sulla cultura italiana, sia filosofica sia scientifica, tutta l'influenza che a nostro giudizio di oggi avrebbe senza dubbio meritato di esercitare."

> Croce wrote that Peano's logic is 'risible stuff', and Gentile said [...] that Vailati would never have 'any place at all in the history of philosophical thought'.[12]

Sadly, some mathematicians in Italy, while recognising the importance of the brilliant results Peano had achieved in Analysis in his early years of activity, ended by marginalising him, not only because of the difficulties involved in reading his work – of which we have already spoken – but also because the characteristic logico-critical interest which continued to inform his scientific and didactic writing finally distanced him from the technical, specialised nature towards which much of the new research in Analysis was being directed.

As for the Torinese environment, thirty years after Peano's death his profile was drawn by his former colleague Francesco G. Tricomi, in a Memoir dedicated to *Matematici italiani del primo secolo dello Stato unitario* (Italian mathematicians of the first hundred years after the unification of the State) (Tricomi 1962). Here, *inter alia*, there emerges one of the two contrasting opinions that had spread among the lecturers in the local Science Faculty regarding Peano's teaching:

> G. Peano was undoubtedly one of the greatest mathematicians of the century and his name remains linked, together with those of Cauchy, Weierstrass, Dini, etc., to the rigorous organisation of Analysis and of Mathematics in general, which previously rested on rather shaky foundations. Peano's arithmetical postulates, Peano's curve (a continuous curve filling an entire square), Peano's existential theorem for ordinary differential equations are milestones in the history of science. Yet his work was not always accepted with general consensus, a fact which can perhaps be explained by bearing in mind that Peano was a precursor of certain modern developments of mathematics ('bourbakism') which, partly because of their aggressive, iconoclastic spirit, still today meet with lively resistance. In Peano, however, there is no trace of that modern vice of making things artificially difficult and complicated; on the contrary, one of his best features was the simplifying spirit that was revealed above all in the brilliant simplicity of some of his classic examples showing the not general validity of certain basic theorems of Calculus [...] From the beginning of this century Peano gradually cut himself off from active mathematics, ending by concerning himself only with some marginal aspects of it (history, numerical approximation, etc.) and finally almost exclusively with auxiliary international languages (Latin 'sine flexione'). Correlatively, his university teaching gradually lost usefulness and effectiveness as, in the words of his student Beppo Levi (1875–1961): 'The apostle limited the work of the mathematician and sometimes hindered its complete appraisal'.[13]

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[12] L. Geymonat (1986), 9: "Croce scriveva che la logica peaniana è 'cosa risibile', e Gentile affermava [...] che Vailati non avrebbe mai avuto un qualsiasi posto nella storia del pensiero filosofico."

[13] F.G. Tricomi (1962), 85–86: "G. Peano è stato indubbiamente uno dei più grandi matematici italiani del secolo e il suo nome resta legato, assieme a quelli di Cauchy, Weierstrass, Dini, ecc. alla sistemazione rigorosa dell'Analisi e della Matematica in genere, che antecedentemente riposava su

## 1.4 Peano and Functional Analysis

At this point it is natural to wonder what, in the new century, can have driven Peano to pursue deeper studies in the direction of the universal language, rather than undertaking the construction of a "theory of operators" in the spirit of the emerging Functional Analysis, as a development of his modern anticipatory ideas. It is not easy to find an answer. Certainly, at root there will have been a spur that arose from his character and his tastes, which led him to attribute the greatest importance to the correct formulation of the bases of a given theory and then to abandon it when, weighed down by the inevitable superstructures of its subsequent development, it became external to the ambit in which Peano's brilliance could find its best expression. In contrast, the creation of a simple form of linguistic communication, suitable for overcoming the incomprehensions and barriers raised by idioms that differed too greatly from one another, was consistent with the programme of simplification which had led him to symbolic writing and to its subsequent applications; moreover it responded to his humanitarian and social tendencies and to a demand that, once again, had its roots in the work of Leibniz.

But another plausible motivation can be conjectured, one which is more scientific and hidden, for Peano's setting aside the study of a "theory of linear operators" (in the sense of Functional Analysis): this is suggested by a reading of his 1915 paper *Le grandezze coesistenti di Cauchy* (Peano 1915i) which leads us to the position of rejection taken up by Peano towards Zermelo's *axiom of infinite choices*.

At the beginning of this paper, Peano asked whether there are additive (real) functions different from proportionality, concluding that "thus far we do not know how to express a definite function for all the real values of the variable, which is distributive, and is not proportionality[14]".

This statement came about ten years after the work of Georg Hamel (Hamel 1905) who, once the existence of a suitable *basis* of the field of real numbers (now known as "Hamel's basis") had been established, had explained how the form of the

basi poco salde. I postulati aritmetici del Peano, la curva di Peano (curva continua riempiente tutto un quadrato), il teorema esistenziale di Peano per le equaz. differenziali ordinarie, restano pietre miliari nella storia della scienza. Tuttavia la sua opera non fu sempre accettata con generale consenso, ciò che forse si spiega tenendo conto che il Peano fu un precursore di certi moderni sviluppi della matematica ("bourbakismo") che, anche pel loro spirito spesso aggressivo e iconoclastico, incontrano tuttora vivaci resistenze. In Peano però non c'è traccia di certo moderno malcostume di rendere le cose artificialmente difficili e complicate, anzi uno dei suoi lineamenti migliori fu lo spirito semplificatore, che si rivelò soprattutto nella geniale semplicità di certi suoi classici esempi mostranti la non generale validità di alcuni fondamentali teoremi del Calcolo. [...] Dall'inizio di questo secolo il Peano si straniò gradualmente dalla matematica attiva, finendo con l'interessarsi soltanto di alcuni aspetti marginali di essa (storia, approssimazioni numeriche, ecc.) e finalmente quasi esclusivamente delle lingue internazionali ausiliarie (latino "sine flexione"). Correlativamente il suo insegnamento universitario andò gradualmente perdendo di utilità ed efficacia ché, come disse il suo allievo Beppo Levi (1875–1961): 'L'apostolo limitò l'opera del matematico e ne impedì talvolta la completa estimazione.'"

[14] G. Peano (1915i), 1147 – *Opere scelte*, vol. 1, 433: "finora non si sa esprimere una funzione definita per tutti i valori reali della variabile, che sia distributiva, e che non sia la proporzionalità."

more general additive function can be expressed. It is significant that the existence of such a *basis* rests on the “axiom of choice”. It is likely that Peano knew of Hamel’s result, since in 1906 Peano himself had devoted a paper (Peano 1906b, 1906e) to Zermelo’s principle, but that he deliberately ignored it since the existence of the *basis*, and hence of the more general additive function, had been established with a proof which he considered unfounded.

It should be noted, incidentally, that already in the proof of his famous theorem of existence for differential equations Peano had adopted a procedure (“successive approximations”) of a constructive nature.

But if we reflect on the role of the “axiom of choice” in Functional Analysis, where important properties and theorems of existence are ensured on the basis of Zorn’s lemma (which exists precisely thanks to this axiom), it cannot be excluded that Peano’s acumen perceived the centrality of this postulate, which he considered unacceptable, and that this crucial question provoked him to withdraw from a new theory of linear operators and to choose to return to the problems of numerical calculus, approximation and interpolation.

In any case, some years later, when the evolution of Mathematics, and especially of Analysis, made manifest the modernity of many of his ideas and the strength of his thinking, the role of Peano’s work and its exceptional value were appropriately re-evaluated by Italian culture.

Here I think we must remember the contribution of his devoted disciple Ugo Cassina who analysed the work of the Master in depth and wanted to transmit his teaching to his own students, but above all promoted the reissue of Peano’s most important papers. And all credit, of course, to the Unione Matematica Italiana and to the city of Cuneo for arranging the publication, respectively, of the three volumes of *Opere scelte* and of the *Formulario*, *editio V*, which contributed decisively to the knowledge of the ideas of our great Mathematician.

# 2

# Some Contributions of Peano to Analysis in the Light of the Work of Belgian Mathematicians

Jean Mawhin

## 2.1 Introduction

The period of the main original contributions of Giuseppe Peano (1858–1932) to analysis goes from 1884 till 1900, and his work deals mostly with a critical analysis of the foundations of differential and integral calculus and with the fundamental theory of ordinary differential equations. During this period, the main analysts in Belgium were Louis-Philippe Gilbert (1832–1892) and his successor Charles-Jean de La Vallée Poussin (1866–1962), at the *Université Catholique de Louvain*, and Paul Mansion (1844–1919) at the *Université de Gand*. At the *Université de Liège*, Eugène Catalan (1814–1894) retired in 1884, and his successor was Joseph Neuberg (1840–1926), an expert in the geometry of the triangle. Analysis at the *Université Libre de Bruxelles* was still waiting to be awakened by Théophile De Donder (1872–1957)[1].

The aim of this paper is to analyze the relations between Peano and Gilbert, Mansion, de La Vallée Poussin. In the case of Gilbert and Mansion, they are essentially connected with the foundations of differential and integral calculus. They are direct, polemical in the first case, courteous in the second, but in both cases Peano's work has been fundamental in instilling rigour in teaching differential and integral calculus in Belgian universities. We show in the Appendix that a part of Gilbert–Peano's polemic, had they pushed it further, could have come close to the definition of a powerful integral introduced in the second half of last century. The relations with de La Vallée Poussin appear to be less direct and more scientific than personal. They deal with the fundamental theory of differential equations and with the concept of generalized derivative of higher order. Their posterity, however, is important, both *Peano's derivatives* and *de La Vallée Poussin's derivatives* still playing a basic role in the study of trigonometric series and non-absolute integration.

[1] See e. g. J. Mawhin (2001), 99–115 and the references therein.

F. Skof (Ed.), *Giuseppe Peano between Mathematics and Logic* 

## 2.2 Peano, Gilbert and the Mean Value Theorem

Louis-Philippe Gilbert[2] was the successor of the Piedmontese mathematician Gaspard Pagani (1796–1855)[3] in the Chair of analysis and mechanics of the *Université Catholique de Louvain*. Although he was born in Belgium and spent all his life there, Gilbert kept the French citizenship of his father. More famous in mechanics for his *barogyroscope*, a mechanical device showing Earth rotation, Gilbert, in analysis, seems to be better known for his polemic than for his contributions. One of these opposed him to Peano, when Gilbert defended an argument, criticized by Peano, in the first edition of Jordan's *Cours d'analyse*[4], for proving the mean value theorem. The letters exchanged on this occasion in 1884 in the *Nouvelles annales de mathématiques* have been partly published in Peano's *Opere Scelte*[5] and discussed by Flett[6], Dugac[7], Guitard[8], Kennedy[9], Gispert[10], Bottazzini[11], Borgato[12]. This polemic is also reported in letters between Genocchi and Hermite, recently published by Michelacci[13]. Hermite writes to Genocchi, on this occasion:

> [...] votre adversaire a blessé d'autres encore que vous par son caractère désobligeant. [...] Pour employer une locution française, M. Gilbert est un mauvais coucheur, s'étant fait connaître comme tel. [...] M. Picard [...] m'a exprimé au sujet de la lettre de M. Gilbert au rédacteur, p. 153, une opinion que je partage complètement. [...] Il juge la communication de M. Gilbert archistupide, elle ne mérite par conséquent point de vous occuper.[14]

A recent interesting paper by Luciano[15] considers the polemic in the light of letters between Angelo Genocchi and Placido Tardy.

We just recall the main lines of the polemic. Jordan, on p. 21 of volume 1 of the first edition of his *Cours d'analyse*, gives a 'proof' of the following version of the mean value theorem:

> Soit $y = f(x)$ une fonction de $x$ dont la dérivée reste finie et déterminée lorsque $x$ varie dans un certain intervalle. Soient $a$ et $a + h$ deux valeurs de

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[2] See e. g. J. Mawhin (1989), 385–396 and the references therein.

[3] See e. g. J. Mawhin (2003), 196–215 and the references therein.

[4] C. Jordan, *Cours d'analyse*, vol. 1, Paris, Gauthier-Villars, 1882.

[5] Peano (1884a), Peano (1884b), Gilbert (1884a).

[6] T. Flett (1974), 69–70, 72; T. Flett (1980), 62–63.

[7] P. Dugac (1979), 32–33.

[8] Th. Guitard (1986), 20.

[9] H.C. Kennedy (1980), 15–16.

[10] H. Gispert (1982), 44–47; H. Gispert (1983), 55.

[11] U. Bottazzini (1991), 45.

[12] M.T. Borgato (1991), 68–76.

[13] G. Michelacci (2005), 8–270.

[14] Michelacci (2005), 168–170.

[15] E. Luciano (2007), 226–227, 255–262.

> $x$ prises dans cet intervalle. On aura
>
> $$f(a+h) - f(a) = \mu h \ , \tag{2.1}$$
>
> $\mu$ désignant une quantité intermédiaire entre la plus grande et la plus petite valeur de $f'(x)$ dans l'intervalle de $a$ à $a+h$.

His argument goes as follows:

> Donnons successivement à $x$ une série de valeurs $a_1, \ldots, a_{n-1}$ intermédiaires entre $a$ et $a+h$. Posons
>
> $$f(a_r) - f(a_{r-1}) = (a_r - a_{r-1})[f'(a_{r-1}) + \epsilon_r] \ . \tag{2.2}$$
>
> [...] Supposons maintenant les valeurs intermédiaires $a_1, \ldots, a_{n-1}$ indéfiniment multipliées. Les quantités $\epsilon_1, \epsilon_2, \ldots$ tendront toutes vers zéro car $\epsilon_1$, par exemple, est la différence entre $(f(a_1) - f(a))/(a_1 - a)$ et sa limite $f'(a)$.

Jordan then sums the inequalities (2.2) to conclude that, if $m$ and $M$ respectively denote the smallest and the largest value of $f'(x)$ for $x$ between $a$ and $a+h$, $(\min_{1\leq r\leq n} \epsilon_r)h + mh \leq f(a+h) - f(a) \leq (\max_{1\leq r\leq n} \epsilon_r)h + Mh$, and hence, as the $\epsilon_r \to 0$ when $n$ increases, $mh \leq f(a+h) - f(a) \leq Mh$, which implies (2.1).

In his first letter published in the *Nouvelles annales*[16], Peano contests the fact that all quantities $\epsilon_1, \epsilon_2, \ldots$ tend to zero, because:

> $f'(a_{r-1}) = \lim[f(a_r) - f(a_{r-1})](a_r - a_{r-1})$ quand on suppose $a_{r-1}$ fixe, et $a_r$ variable et s'approchant indéfiniment de $a_{r-1}$ ; mais on ne le peut pas affirmer quand varient en même temps $a_r$ et $a_{r-1}$, si l'on ne suppose pas que la dérivée soit continue.

And Peano gives the counter-example defined by $f(0) = 0$, $f(x) = x^2 \sin(1/x)$ for $x \neq 0$. For $a = 0$, $h > 0$ arbitrary, the choice of $a_1 = 1/(2n\pi)$, $a_2 = 1/[(2n+1)\pi]$, $a_3, a_4, \ldots$ arbitrary gives $\epsilon_2 = 1$. Peano adds that:

> [...] on démontre très facilement la formule
>
> $$f(x_0 + h) - f(x_0) = hf'(x_0 + \theta h) \tag{2.3}$$
>
> sans supposer la continuité de la dérivée.

Jordan graciously agrees with Peano's criticism in a letter[17] following Peano's in the *Nouvelles annales*:

> Je n'ai rien à répondre à la critique de M. le Dr Peano, qui est parfaitement fondée. J'ai admis implicitement dans ma démonstration que $\frac{f(x+h)-f(x)}{h}$ tendait *uniformément* vers $f'(x)$ dans l'intervalle de $a$ à $b$.

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[16] Peano (1884a) 45–47.

[17] Peano (1884a), 47.

And he adds that Peano:

> [...] ferait plaisir en me communiquant sa démonstration [of (2.3)], car je n'en connais pas qui me paraisse satisfaisante.

Peano's answer is published in Borgato's paper mentioned above, and Jordan accordingly modifies the second edition of his *Cours d'analyse*[18].

This should have been the happy end of the story if Gilbert had not decided to defend Jordan. In a letter sent to the same journal[19]:

> [...] à laquelle M. Jordan n'aurait eu aucune peine à répondre lui-même, s'il n'eût probablement aperçu derrière quelque difficulté plus subtile.

Gilbert observes (correctly as we will see later) that:

> [...] il n'est pas nécessaire que les $\epsilon$ tendent vers zéro pour *tout* mode de division de l'intervalle $h$ en parties indéfiniment décroissantes $\delta$ ; il suffit que cela ait lieu pour *un* mode de division.

Gilbert does not prove the existence of this mode of division but argues that the fact that Peano's division does not work in his counter-example does not mean that *some* division could not work. With respect to Peano's last assertion concerning the validity of formula (2.3), Gilbert makes the wrong comment:

> M. Peano croit qu'il est facile de démontrer la formule (2.3) *sans supposer la continuité de la dérivée*. M. Jordan demande, non sans malice, à voir cette démonstration, laquelle est impossible, puisque le théorème est inexact.

And he gives the 'counter-example' of the function $f(x)$ equal to $\sqrt{2px}$ between 0 and $a$ and equal to $\sqrt{2p(2a-x)}$ between $a$ and $2a$, which does not contradict Peano's statement (2.3) because this $f$ has *no* derivative at $x = a$!

In his answer subsequently published in the *Nouvelles annales*[20], Peano carefully refutes Gilbert's arguments. He gives explicitly the (now classical) proof of formula (2.3) for a not necessarily continuous derivative, based upon Rolle's theorem deduced from Weierstrass' maximum theorem, referring to various recent German and Italian treatises of analysis. But the most interesting point of Peano's answer is probably the following:

> M. Gilbert dit que le théorème sera démontré si l'on prouve que, pour *un* mode de division, les $\epsilon$ ont pour limite zéro. Si l'on entend par ces mots que, pour un mode de division, le maximum des $\epsilon$ a pour limite zéro, la proposition est juste ; mais, comme cela n'arrive pas pour *tout* mode de division, le théorème résultera démontré lorsque M. Gilbert aura trouvé ce mode particulier de division, pour lequel la condition précédente est satisfaite. Et je dis cela

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[18] C. Jordan, *Cours d'analyse de l'École polytechnique*, vol. 1, 2e éd., Paris, Gauthier-Villars, 1893.

[19] P. Gilbert (1884a), 153–155.

[20] Peano (1884b), 252–256.

sans malice, parce que ce mode existe, mais je laisserai le soin de le trouver à M. Gilbert; et pour bien fixer la question, je lui propose de démontrer ce théorème, dont il se sert :

*Si* $f(x)$ *a une dérivée déterminée et finie* $f'(x)$ *pour toutes les valeurs de* $x$ *appartenant à un intervalle fini* $(a, b)$, *étant fixée une quantité* $\epsilon$ *arbitrairement petite, on peut toujours diviser l'intervalle* $(a, b)$ *avec les points* $a_0 = a, a_1, a_2, \ldots, a_{n-1}, a_n = b$ *de façon que chacune des différences* $\frac{f(a_{r+1})-f(a_r)}{a_{r+1}-a_r} - f'(a_r)$, $(r = 0, 1, \ldots, n-1)$, *soit, en valeur absolue, moindre que* $\epsilon$.

Strangely, Gilbert's answer to Peano's challenge, published in the same volume of the *Nouvelles annales*[21], is not reproduced in Peano's *Opere Scelte*, and not mentioned in the literature quoted above about Gilbert-Peano's quarrel. This is unfair to Gilbert, who, in this letter, not only explains the reason for his inadequate 'counterexample' (for him Peano's statement 'without assuming the continuity of the derivative' included the possibilty of a jump discontinuity), but, more importantly, proposes a proof of Peano's statement, starting with the preliminary remark:

M. Peano n'ignore pas que du moment où la dérivée $f'(x)$ a une valeur *unique* en chaque point, [...] on démontre rigoureusement, *sans faire usage de la proposition qu'il énonce,* que le rapport $\frac{f(b)-f(a)}{b-a}$ est compris entre la plus petite et la plus grande valeur de $f'(x)$ dans l'intervalle $(a, b)$ (théorème de M. Jordan). [...] La proposition en question ne peut donc servir à rien pour mon but, c'est pourquoi je n'ai pas beaucoup cherché à perfectionner la démonstration que je donne ci-dessous; mais comme le théorème offre par lui-même quelque intérêt, j'espère que M. Peano voudra publier sa démonstration, qui sera sans doute meilleure.

Peano apparently never did it, and Gilbert's proof is of interest. He first considers the largest $\delta_1 > 0$ such that $\left|h^{-1}[f(a+h) - f(a)] - f'(a)\right| < \epsilon$ when $0 < h \leq \delta_1$, takes $a_1 = a + \delta_1$, and continues in the same way to find a (possibly infinite) sequence of positive numbers $\delta_2, \delta_3, \ldots$ such that, with $a_r = a_{r-1} + \delta_r$, $\left|h^{-1}[f(a_r + h) - f(a_r)] - f'(a_r)\right| < \epsilon$ when $0 < h \leq \delta_{r+1}$. The increasing sequence $(a_r)$ either reaches $b$ in a finite number of steps, in which case the proposition is proved, or remains strictly smaller than $b$, in which case it is infinite and has a limit $c \leq b$. Hence one can find $\sigma > 0$ such that

$$\left|h^{-1}[f(c+h) - f(c)] - f'(c)\right| < \epsilon/2 \tag{2.4}$$

when $-\sigma \leq h < 0$. Using Bonnet's version (2.3) of the mean value theorem, there exists some $\xi \in ]-\sigma, 0[$ such that $\frac{f(c-\sigma)-f(c)}{-\sigma} = f'(\xi)$, which, introduced in (2.4) with $h = -\sigma$, gives $|f'(\xi) - f'(c)| < \frac{\epsilon}{2}$. This last inequality and inequality (2.4) with $c + h = \xi$ imply that

$$\left|\frac{f(c) - f(\xi)}{c - \xi} - f'(\xi)\right| \leq \left|\frac{f(c) - f(\xi)}{c - \xi} - f'(c)\right| + |f'(c) - f'(\xi)| < \epsilon\ .$$

[21] P. Gilbert (1884b), 475–482.

Now, $\xi < c$ is equal to some of the $a_r$ or contained between $a_r$ and $a_{r+1}$, so that one has, for some $r$, $\left|\frac{f(\xi)-f(a_r)}{\xi-a_r} - f'(a_r)\right| < \epsilon$, and the division $a, a_1, \ldots, a_r, \xi, c$ works for $[a, c]$. The result is proved if $c = b$ and if $c < b$, Gilbert states that:

> En raisonnant sur l'intervalle $(c, b)$ comme on a raisonné sur l'intervalle $(a, b)$, on finira par établir qu'on peut toujours passer de $a$ à $b$ par un nombre fini d'intervalles $\delta$, qui satisfont à la condition formulée dans l'énoncé du théorème.

In the *Appendix I* of his paper mentioned above, Flett proves Peano's statement (which he calls 'Theorem $(J)$'), and suggests that his proof may be close to the one Peano mentions in his letter to Gilbert. Flett's proof uses essentially the same ingredients as Gilbert's unquoted one, in particular Bonnet's mean value theorem. Flett just replaces the increasing sequence argument by defining $c$ as the supremum of all $x > a$ such that for each $y \in ]a, x]$, there is a finite Peano's division of $[a, y]$, which renders the argument when $c < b$ more transparent. Flett notes that:

> In stating (J), Peano was, of course, attempting only to deal with Gilbert's observation concerning Jordan's proof rather than to rescue the proof itself. But had Peano really wished to rescue the proof, then he might easily have been led to the following modification of (J), which also implies (D) [Jordan's statement].
>
> (K) *Let* $f : [a, b] \to \mathbb{R}$ *be a function possessing a derivative at each point of the closed interval* $[a, b]$*, and let* $\epsilon > 0$*. Then there exists a positive integer* $n$*, numbers* $a_0 = a < a_1 < \ldots < a_n = b$ *and numbers* $\xi_r$ *equal to* $a_{r-1}$ *or* $a_r$ $(r = 1, \ldots, n)$ *such that for* $r = 1, \ldots, n$, $\left|\frac{f(a_r)-f(a_{r-1})}{a_r-a_{r-1}} - f'(\xi_r)\right| \leq \epsilon$.[22]

Flett then observes that Theorem (K) can be proved by an obvious modification of the argument of Theorem (J), except that the recourse to Bonnet's mean value theorem is no longer necessary, and that a more direct proof can be given using Heine–Borel theorem. We discuss a simpler version of this argument in the *Appendix*, and show how Peano–Gilbert's discussion is linked to fruitful recent concepts in analysis and how Peano, who had already simplified the definition of Riemann's integral in 1883[23], came somewhat close to, but missed, in his statement challenging Gilbert, a much more powerful concept of integration.

In his additions to Genocchi's *Calcolo differenziale*[24], Peano criticizes some existing textbooks on analysis. Remark 18 deals with the proof of the intermediate value theorem:

> The proof of this number was given by Cauchy *Analyse algébrique,* Paris 1821, note III. The geometrical proof (also due to Cauchy, id., p. 44) in which

[22] T. Flett (1974), 69.

[23] G. Peano (1882b), 439–446.

[24] A. Genocchi (1884).

> one observes that the curve of equation $y = f(x)$, which has two points located at opposite sides with respect to the axis of $x$, meets this axis in some point, is not satisfactory [...] but will be exact if continuous functions are defined as those that cannot pass from one value to another one without passing through all the intermediate values. And this definition can be found in several treatises, and in particular in the ones recently quoted by Gilbert, *Cours d'analyse infinitésimale*, Louvain 1872; but, erroneously, the author, on p. 55 tries to prove its equivalence with the one we use here. In fact, if, when $x$ tends to $a$, $f(x)$ oscillates between values including $f(a)$, without tending to any limit, $f(x)$ is discontinuous at $x = a$, according to our definition, and is continuous, according to Gilbert's definition.[25]

Indeed, Gilbert defines as follows the concept of continuity in the first edition of his *Cours d'analyse infinitésimale*[26]:

> Désignons par $f(x)$ une fonction de la variable $x$, que nous supposerons réelle pour toutes les valeurs de la variable $x$ comprises entre deux valeurs données $x = a$ et $x = b$. Si nous concevons que la variable passe successivement par toutes les valeurs depuis $x = a$ jusqu'à $x = b$, et si alors la fonction $f(x)$, *conservant toujours une valeur finie, ne peut passer d'une valeur quelconque à une autre sans passer par toutes les valeurs intermédiaires*, nous disons que cette fonction est *continue* par rapport à $x$ depuis $x = a$ jusque $x = b$.[27]

He then claims to prove the equivalence of this definition with Cauchy's:

> Toute fonction continue jouit d'une propriété remarquable, qui peut servir aussi, soit à définir, soit à constater la continuité. Concevons qu'une fonctin $f(x)$ soit continue entre deux valeurs $a$ et $b$ de la variable et soient $x, x+h$ deux valeurs comprises dans cet intervalle, $h$ étant infiniment petit. Il est clair que la différence des valeurs correspondantes de la fonction $f(x+h) - f(x)$ sera infiniment petite en même temps que $h$. En effet, si la différence entre $f(x+h)$ et $f(x)$ ne décroissait pas au-dessous de toute grandeur donnée lorsque $h$ tend vers zéro, [...] la fonction passerait brusquement d'une valeur à une autre en $x$, et ne serait pas continue.

Clearly Gilbert neglects the possibility of an oscillatory discontinuity.

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[25] Peano (1884c), x–xi: "La dimostrazione di questo numero fu data da Cauchy, *Analyse algébrique*, Paris 1821, nota III. La dimostrazione geometrica (pure data dal Cauchy, id., pag. 44) in cui si ritiene che la linea di equazione $y = f(x)$, che ha due punti giacenti da parte opposta dell'asse delle $x$, incontra questo asse in qualche punto, non è soddisfacente ... sarebbe esatta qualora si definisse per funzione continua quella che non può passare da un valore ad un altro senza passare per tutti i valori intermedii. E questa definizione trovasi appunto in alcuni trattati, e fra i recenti citeró il Gilbert, *Cours d'analyse infinitésimale*, Louvain 1872; ma erroneamente, l'A. a pag. 55 cerca di dimostrare la sua equivalenza con quella di cui noi ci serviamo. Invero, se col tendere di $x$ ad $a$, $f(x)$ oscilla entro valori che comprendono $f(a)$, senza tendere ad alcun limite, $f(x)$ è discontinua per $x = a$, secondo la nostra definizione, ed è continua, secondo la definizione del Gilbert."

[26] P. Gilbert (1872).

[27] P. Gilbert (1872), 53–55.

A positive characteristic of Gilbert, also revealed on other occasions, is to recognize his errors and sacrifice his self-respect on the altar of rigour. In the *Preface* of the third edition of his *Cours d'analyse infinitésimale*[28], Gilbert writes:

> Quoique la destination de ce livre et son plan général soient restés ce qu'ils étaient lors de la première édition (1872), sa rédaction a subi des modifications profondes. [...] Or, depuis une vingtaine d'années, des écrits nombreux ont eu pour but, surtout en Allemagne, de présenter d'une manière plus précise et plus rigoureuse les théories de l'analyse infinitésimale [...]. Je me suis particulièrement inspiré à ce point de vue des traités publiés dans les dernières années par MM. C. Jordan (*Cours d'analyse*), Lipschitz (*Lehrbuch der Analysis*), Dini (*Fondamenti per la teoria delle funzioni di variabili reali*), P. du Bois-Reymond (*Allgemeine Functionenlehre*), Stolz (*Vorlesungen über allgemeine Arithmetik*), J. Tannery (*Introduction à la théorie des fonctions d'une variable*), Peano (*Calcolo differenziale*).

This time, Gilbert adopts Cauchy's definition of a continuous function, deduces from it in a classical way, the intermediate value theorem, and refrains from proving the (false) converse. As observed by Luciano[29], in her well documented article, the rigorous treatment of this third edition is mentioned by Peano in a hand-written note in the margin of his copy of Genocchi–Peano's *Calcolo*.

## 2.3 Peano, Mansion and Foundations of Calculus

Paul Mansion[30] (1844–1919) was a professor of analysis at the *Université de Gand* and an indefatigable animator of mathematics in Belgium. With his colleague Joseph Neuberg from the *Université de Liège*, in 1880 he created the journal *Mathesis*, devoted to elementary mathematics, to which Peano will contribute a few papers[31]. When in 1884 Peano published Genocchi's *Calcolo differenziale*, Mansion was preparing a printed version of his corresponding lectures in Gand, and wrote to Genocchi to have a copy of the *Calcolo* made available to him[32]:

> Je suis bien curieux de voir le livre de M. Peano, car je publie moi même, à l'heure qu'il est, la partie la plus originale de mon cours de Calcul différentiel. Pourriez-vous m'envoyer [...] pour une huitaine de jours, le livre de M. Peano.

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[28] P. Gilbert, *Cours d'analyse infinitésimale. Partie élémentaire*, 3e éd., Paris, Gauthier-Villars, 1887, 57.

[29] Luciano (2007), 252

[30] A. Demoulin (1929), 1–71.

[31] G. Peano (1889b), (1889c), 110–112; Peano (1889e), 182–183; Peano (1890d), 73–74; Peano (1890e), 153–154; Peano (1892s), 12–14.

[32] U. Cassina (1952), 337–362.

Genocchi generously sent Mansion a complimentary copy of the book and the Belgian mathematician took benefit of it in his *Résumé du Cours d'analyse infinitésimale de l'université de Gand*[33], with as subtitle the exact French translation of the title of Genocchi–Peano's treatise. We read in the *Introduction*:

> Mais nous devons signaler spécialement, parmi les écrits qui nous ont servi de guide, [...] à partir du n° 214, le Cours de MM. Genocchi et Peano.[34]

The explicit mention 'proof of Genocchi and Peano' is attached to Rouquet's theorem (a version of L'Hospital rule, p. 91), Euler's formula for homogeneous functions (p. 100), and Cauchy's convergence criterion (p. 235).

In his *Cours d'analyse infinitésimale*[35], as a sequence of the *Résumé*, Mansion states and names *théorème de Peano*[36] his existence theorem of 1886 for Cauchy problem associated to a scalar ordinary differential equation:

> Voir la démonstration de ce théorème dans une note de Peano, publiée dans les *Atti della R. Accademie delle Scienze di Torino,* t. 21, 20 Juin 1886 à laquelle nous avons emprunté ce qui précède. Peano a étendu ce théorème aux équations simultanées, *Mathematische Annalen,* 1890, t. 37, pp. 182–228.[37]

It is remarkable to see this result of Peano[38] mentioned in undergraduate university lectures only one year after its publication.

Other mathematical contributions from Peano inspired two articles of Mansion in *Mathesis*, namely his counterexample to Serret's definition of the area of a curved surface in 1890[39], and his result on the error in quadrature formulas in 1915[40]. More surprisingly, Mansion also reported upon Peano's linguistic interests in 1904 and 1907[41].

## 2.4 Peano, de La Vallée Poussin and Ordinary Differential Equations

Charles-Jean de La Vallée Poussin[42] was a cousin of Gilbert and his successor, in 1892, in the Chair of analysis at the *Université Catholique de Louvain*. He became

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33 P. Mansion (1887).

34 Cassina (1952), 349.

35 P. Mansion (1891).

36 G. Peano (1885a), 677–685.

37 Mansion (1891), p. 57

38 G. Peano (1890f), 182–228.

39 P. Mansion (1890), 222–224.

40 P. Mansion (1914), 168–174.

41 P. Mansion (1904), 254–257; P. Mansion (1907), 213–218.

42 See e. g. P. Butzer, J. Mawhin (2000), 3–9 and the references therein.

world famous in 1896 for his proof of Gauss' prime numbers conjecture, and his *Cours d'analyse infinitésimale*[43] in two volumes is classic. His first papers were motivated by his reading of Darboux and Jordan's papers in real analysis and, in 1893, he submited to the *Académie royale de Belgique* a substantial memoir on the integration of differential equations[44]. The aim of this work was to extend to the Cauchy problem $y' = f(x, y)$, $y(x_0) = y_0$, Riemann's approach for the special case $y' = f(x)$, $y(x_0) = y_0$, when $f(x)$ need not be continuous (Riemann integral). Following the tradition of the *Académie*, the manuscript was submitted to three referees and the first one, Mansion, wrote a report, quite positive on the mathematical value of the memoir, but ending with the following sounded suggestion:

> Au point de vue historique, il serait peut-être bon que M. de la Vallée Poussin indiquât les points de contact, ou plutôt de *quasi-contact*, entre son travail et celui de Peano sur l'intégrabilité des équations différentielles où les dérivées sont des fonctions continues des variables. [...] Il suffit que l'auteur [...], *s'il le juge utile*, [...] compare sommairement, dans un paragraphe final, sa méthode à celle de Peano, pour en faire ressortir les analogies et les différences.[45]

De La Vallée Poussin scrupulously followed Mansion's suggestion and wrote in the *Appendice* to his *Mémoire*:

> Méthode de M. Peano. [...] M. Peano se place à un point de vue tout différent de celui des auteurs précédents [Cauchy, Picard]: [...] il montre qu'étant donné le système d'équations entre $n$ fonctions inconnues de $t$
>
> $$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t), \quad (i = 1, 2, \ldots, n) ,$$
>
> la continuité des fonctions $f_i$ par rapport aux variables suffit pour établir l'existence d'un système *au moins* d'intégrales, prenant des valeurs initiales données, mais il peut, en général en exister une infinité. La démonstration dont nous parlons a fait l'objet d'un article étendu publié dans les *Mathematische Annalen* (t. XXXVII, pp. 182–228)[46], mais elle ne présente, ni dans son objet propre, ni dans ses procédés, aucune analogie avec la nôtre. [...] Assez longtemps avant de publier la démonstration générale dont nous venons de parler, M. Peano avait démontré le théorème, pour le cas particulier d'une seule équation. Cette démonstration, qui a paru dans les *Atti de Turin* (1886), repose sur des principes spéciaux, qu'il est impossible d'étendre au

[43] Ch.-J. de La Vallée Poussin (1903).

[44] Ch.-J. de La Vallée Poussin (1893a), 1–82

[45] P. Mansion (1892), 227–236.

[46] Cette démonstration rédigée à l'aide des symboles de la logique algébrique, est d'une étude très pénible pour ceux qui ne sont pas familiers avec ces notations. Nous avons donné du même théorème une démonstration plus simple dans les *Annales de la Société Scientifique de Bruxelles* (t. XVII, 1$^{\text{ère}}$ partie, p. 8–12). Tout ce que nous disons ici de la démonstration de Peano s'applique aussi à celle-là, qui ne diffère pas essentiellement de celle de Peano.

> cas de plusieurs équations ; mais sa portée, que la comparaison fait mieux saisir, est exactement la même que celle de la démonstration générale dans le cas particulier dont il s'agit. Il y a dans cette démonstration particulière un point de contact avec la méthode exposée dans la première partie du présent travail. Nous allons l'indiquer.[47]

De La Vallée Poussin's other paper mentioned in the footnote of his *Appendice* is a short note published the same year[48], which starts as follows:

> Soit le système de $n$ équations différentielles
>
> $$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t), \quad (i = 1, 2, \ldots, n) \ ;$$
>
> M. Peano a montré (*Math. Annal.* t. XXXVII, pp. 182–228) que la continuité des fonctions $f_i$ dans le voisinage du point $(x_{10}, x_{20}, \ldots, x_{n0}, t_0)$ suffit pour établir l'existence d'un système *au moins* d'intégrales se réduisant à $(x_{10}, x_{20}, \ldots, x_{n0})$ pour $t = t_0$. Nous nous proposons de retrouver le même résultat par une méthode peut-être un peu plus simple que celle de Peano.

The originality of de La Vallée Poussin's approach in proving Peano's theorem of 1890[49] is to avoid an explicit use of Ascoli–Arzelá's theorem, by reproving the part of this theorem necessary for his existence proof. These two papers of de La Vallée Poussin are not mentioned in Peano's survey of 1897[50]. But Peano quotes de La Vallée Poussin's *Mémoire*[51], for the existence of a solution to Cauchy's problem when $f(x, y)$ is continuous with respect to $y$ and integrable with respect to $x$.

## 2.5 Peano, de La Vallée Poussin and Generalized Derivatives

In 1891, Peano introduced a concept of generalized $n$th derivative for a real function $f$ of a real variable[52]:

> Sia $f(x)$ una funzione reale della variable reale $x$ [...]. *Noi converremo di scrivere :* $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n +$ *etc. per indicare che*
>
> $$\lim_{x \to 0} \frac{f(x) - a_0 - a_1x - a_2x^2 - \ldots - a_{n-1}x^{n-1}}{x^n} = a_n \ . \qquad (2.5)$$

[47] Ch.-J. de La Vallée Poussin (1893a), 79.

[48] Ch.-J. de La Vallée Poussin (1893b), 8–12.

[49] G. Peano (1890f), 182–228.

[50] G. Peano (1897c), 9–18.

[51] G. Peano (1908a), 429.

[52] G. Peano (1892a), 40–46.

[...] Questa formula si può pure scrivere sotto le forme

$$\lim_{x\to 0} \frac{f(x) - a_0 - a_1x - a_2x^2 - \ldots - a_{n-1}x^{n-1} - a_nx^n}{x^n} = 0 ,$$
$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \alpha x^n, \text{ ove } \lim_{x\to 0} \alpha = 0 . \quad [\ldots]$$

L'eguaglianza (2.5) sta per indicare che la differenza fra $f(x)$ e il polinomio $a_0 + a_1x + \ldots + a_nx^n$ è infinitesima con $x$, d'ordine superiore all'$n^{\text{mo}}$.[53]
[Let $f(x)$ be a real function of the real variable $x$ [...]. *We will write* $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n +$ *etc. to indicate that*

$$\lim_{x\to 0} \frac{f(x) - a_0 - a_1x - a_2x^2 - \ldots - a_{n-1}x^{n-1}}{x^n} = a_n . \tag{2.5}$$

[...] This formula can also be written in the form

$$\lim_{x\to 0} \frac{f(x) - a_0 - a_1x - a_2x^2 - \ldots - a_{n-1}x^{n-1} - a_nx^n}{x^n} = 0 ,$$
$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \alpha x^n, \text{ where } \lim_{x\to 0} \alpha = 0 . \quad [\ldots]$$

The equality (2.5) means that the difference between $f(x)$ and the polynomial $a_0 + a_1x + \ldots + a_nx^n$ is an infinitesimal of order superior to the $n$th with respect to $x$.]

$f_{(n)}(x) := n!a_n$ is Peano's *generalized derivative of order $n$ of $f$ at $x$.*
In a memoir of 1908 on the approximation of functions[54], de La Vallée Poussin introduced generalized *symmetric* derivative of arbitrary order:

Supposons que l'on puisse écrire $[f(x+h) - f(x-h)]/2 = a_1h + a_3\frac{h^3}{3!} + \ldots + a_{2n-1}\frac{h^{2n-1}}{(2n-1)!} + (a_{2n+1} + \omega)\frac{h^{2n+1}}{(2n+1)!}$, les coefficients $a$ étant des constantes relativement à $h$ et $\omega$ une quantité qui tend vers 0 avec $h$. Nous dirons que les coefficients $a_1, a_3, \ldots, a_{2n+1}$ sont les *dérivées généralisées successives d'ordre impair de* $f(x)$. De même supposons qu'on puisse écrire $[f(x+h) + f(x-h)]/2 = a_0 + a_2\frac{h^2}{2!} + a_4\frac{h^4}{4!} + \ldots + a_{2n-2}\frac{h^{2n-2}}{(2n-2)!} + (a_{2n} + \omega)\frac{h^{2n}}{(2n)!}$, où les $a$ sont des constantes par rapport à $h$ et où $\omega$ tend vers 0 avec $h$. Nous dirons que $a_2, a_4, \ldots, a_{2n}$ sont les *dérivées généralisées successives d'ordre pair de* $f(x)$.

De La Vallée Poussin does not quote Peano's paper, but it is clear that his $(2n+1)^{\text{th}}$ (resp. $(2n)^{\text{th}}$) generalized derivative of $f(x)$ at $x$ is nothing but Peano's $(2n+1)^{\text{th}}$ (resp. $(2n)^{\text{th}}$) generalized derivative at 0 of the odd function $[f(x+\cdot) - f(x-\cdot)]/2$ (resp. even function $[f(x+\cdot) + f(x-\cdot)]/2$).

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53 Peano (1892a), 40–42.

54 Ch.-J. de La Vallée Poussin (1908), 193–254.

The lack of acknowledgement of Peano's contribution was to last for many years. In 1935, Arnaud Denjoy[55] rediscovered Peano's generalized derivative in an apparently independent way when he wrote:

> Selon une définition particulièrement commode, nous dirons que $f(x)$ possède au point $x$ une *différentielle d'ordre* $n$ si [...] $f(x+h)$ est la somme d'un polynome en $h$ et d'un infiniment petit d'ordre supérieur à $n$, $h$ étant l'infiniment petit principal. Le coefficient de $\frac{h^n}{n!}$ dans le polynome, soit $f_n(x)$, sera appelé le $n^{\mathrm{e}}$ *quotient différentiel de* $f$ *au point* $x$.

One year later, in the same journal, Jozéf Marcinkiewicz and Antoni Zygmund[56] attributed Peano's concept to de La Vallée Poussin when they wrote, with reference neither to Peano nor to de La Vallée Poussin:

> If, for a given $x$, we have an equation $f(x+t) = a_0 + a_1 t + a_2 t^2/2! + \ldots + a_k t^k/k! + o(t^k)$ where the numbers $a_j = a_j(x)$ are independent of $t$, then $a_k$ will be called the $k$-th de la Vallée-Poussin derivative of $f$ at the point $x$, and will be denoted by $f_{(k)}(x)$.

The wrong attribution was repeated one year later, in the same journal, by Marcinkiewicz[57], with a reference to de La Vallée Poussin's paper:

> On dit que *la fonction* $f(x)$ *admet dans le point* $x$ *la dérivée généralisée d'ordre* $k$ *au sens de de la Vallée-Poussin*, s'il y a $k+1$ constantes $a_0(x)$, $a_1(x), \ldots, a_k(x)$ telles que $f(x+t) = a_0(x) + a_1(x)t + a_2(x)t^2/2! + \ldots + a_k(x)t^k/k! + o(t^k)$.

It seems that the first attribution of the concept to Peano was made in 1946 by Ernesto Corominas Vignaux[58], who wrote in a note[59] presented to the *Académie des Sciences de Paris* by Denjoy:

> Nous appelons dérivées de Peano celles qu'a étudiées Denjoy dans son mémoire fondamental *Sur l'intégration des coefficients différentiels d'ordre supérieur (Fundamenta Mathematicae, 1935)*. Il semble que Peano fut le premier à donner une définition convenable des dérivées en employant le développement de Taylor.

By the way, seven years later, Corominas[60] forgot Peano when writing:

> Nous dirons avec M. Denjoy que, $f(x)$, continue et définie sur $[a,b]$ possède au point $x$ une *différentielle d'ordre* $n$ si $x+h$ appartient à $[a,b]$, $f(x+h)$ est la somme d'un polynome en $h$ et d'un infiniment petit d'ordre supérieur à $n$, $h$ étant l'infiniment petit principal

and quoting only Denjoy's paper!

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[55] A. Denjoy (1935), 273–326.

[56] J. Marcinkiewicz, A. Zygmund (1936), 1–43.

[57] J. Marcinkiewicz (1937), 38–69.

[58] E. Corominas Vigneaux (1946), 88–93.

[59] E. Corominas Vigneaux (1947), 89–91.

[60] E. Corominas Vigneaux (1953), 177–222.

Despite its title, the ambiguous paternity survives in an important paper of H. William Oliver[61]:

> A real-valued function $f(x)$, defined for $a \leq x \leq b$, is said to have a $n$th Peano derivative at $x_0$, $n = 1, 2, \ldots$ (also called an $n$th derivative of de la Vallée-Poussin), if there exists numbers $f_1(x_0), f_2(x_0), \ldots, f_n(x_0)$ such that $f(x_0 + h) = f(x_0) + hf_1(x_0) + \ldots + \frac{h^n}{n!}[f_n(x_0) + \epsilon(x_0, h)]$, where $\epsilon(x_0, h) \to 0$ as $h \to 0$.

Children are finally correctly attributed to their respective fathers in volume 2 of the second edition of Zygmund's famous treatise on trigonometric series. Zygmund, who quotes de La Vallée Poussin's paper but not Peano's, writes:

> Suppose that a function $f(x)$ is defined in the neighborhood of a point $x_0$ and that there exist constants $\alpha_0, \alpha_1, \ldots, \alpha_r$ such that for small $|t|$,
>
> $$f(x_0 + t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_{r-1}\frac{t^{n-1}}{(r-1)!} + (\alpha_r + \epsilon_t)\frac{t^r}{r!}\,, \qquad (2.6)$$
>
> where $\epsilon_t$ tends to 0 with $t$. We then say that $f$ has a *generalized r-th derivative* $f_{(r)}(x_0)$ at $x_0$ and define $f_{(r)}(x_0) = \alpha_r$. [...] The above definition is due to Peano. For applications to trigonometric series a certain modification of it, due to de la Vallée Poussin, is of importance. We define it separately for $r$ even and odd, ...

and then repeats de La Vallée Poussin's definition of generalized symmetric derivative before noticing that:

> [...] taking the semi-sum and semi-differences of (2.6) for $\pm t$, we see that the existence of the unsymmetric derivative implies the existence of the symmetric one of the same order, and equality of both.[62]

This authoritative reference was not to prevent the propagation of the confusion between Peano and de La Vallée Poussin, as examplified by Kassimatis[63], who refers to Peano's derivative as:

> [...] called, following Marcinkiewicz and Zygmund, the *nth de la Vallée Poussin derivative of $f$ at $x$, $f_{(n)}(x)$*.

And by P.S. Bullen and S.N. Mukhopadhyay who in 1973[64] repeated Oliver's ambiguous paternity statement.

It seems therefore that Peano and de La Vallée Poussin had introduced similar concepts in independent ways and for different reasons. Peano's motivation was to

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[61] H. Oliver (1954), 444–456.

[62] A. Zygmund (1959), 59–60.

[63] C. Kassimatis (1965), 1171–1172.

[64] P.S. Bullen, S.N. Mukhopadhyay (1973), 127–140.

extend his result[65] on Taylor's finite development, namely that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + o(h^n)\,,$$

when $f$ is $(n-1)$-times differentiable in a neighborhood of $x$ and $f^{(n)}(x)$ exists. The motivation of de La Vallée Poussin comes from approximation theory. Given a real function $f$ Lebesgue-integrable over [0, 1], and a positive integer $n$, he defines the approximating polynomial $P_n$ of degree $2n$ by

$$P_n(x) := \frac{1}{2}\left(\frac{3\cdot 5\cdot\ldots\cdot(2n+1)}{2\cdot 4\cdot\ldots\cdot 2n}\right)\int_0^1 f(u)[1-(u-x)^2]^n\,du\,,$$

and proves that, *given a positive integer $r$, the derivative of order $r$ of $P_n$ converges for $n \to \infty$ to the generalized (symmetric) derivative of the same order of $f$ at any point $x$ where this last generalized derivative exists,* as well as a similar result for some approximating trigonometrical polynomials for a real $2\pi$-periodic function $f$ Lebesgue-integrable over $[-\pi, \pi]$. We can then conclude, with Butzer and Nessel[66] that:

> It may be said that, together with B. Riemann (1854) and G. Peano (1891), de La Vallée Poussin belongs to the first three mathematicians to work in the important area of generalized derivatives.

## 2.6 Appendix

To simplify Flett's argument, first observe that Theorem (K) in Sect. 2.2 is easily shown to be equivalent to the following:

**Theorem (K′)** *Let $f : [a,b] \to \mathbb{R}$ be a function possessing a derivative at each point of the closed interval $[a,b]$, and let $\epsilon > 0$. Then there exists a positive integer $n$, numbers $a_0 = a < a_1 < \ldots < a_n = b$ and numbers $x_r \in [a_{r-1}, a_r]$ $(r = 1,\ldots,n)$ such that $\left|\frac{f(a_r)-f(a_{r-1})}{a_r-a_{r-1}} - f'(x_r)\right| \le \epsilon$ for $r = 1,\ldots,n$.*

*Proof.* Given $\epsilon > 0$ and $x \in [a,b]$, it follows from the derivability of $f$ at $x$ that there exists $\delta(x) > 0$ such that

$$|f(z) - f(y) - f'(x)(z-y)| \le \epsilon(z-y) \tag{2.7}$$

when $x - \delta(x) \le y \le x \le z \le x + \delta(x)$. Now, Cousin's lemma[67], which is equivalent to Heine–Borel's theorem, just states that given any positive mapping $\delta$ on $[a,b]$, there exists a positive integer $n$, numbers

$$a_0 = a < a_1 < \ldots < a_n = b \tag{2.8}$$

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[65] G. Peano (1889e), 182–183.

[66] P.L. Butzer, R.J. Nessel (1993), 72–73; P.L. Butzer, R.J. Nessel (2004), 381.

[67] See e. g. J. Mawhin (1992), 120.

and numbers $x_r \in [a_{r-1}, a_r]$ $(r = 1, \ldots, n)$ such that for $r = 1, \ldots, n$

$$x_r - \delta(x_r) \leq a_{r-1} \leq x_r \leq a_r \leq x_r + \delta(x_r)\ . \tag{2.9}$$

Consequently, taking $x = x_r$, $y = a_{r-1}$, $z = a_r$ in (2.7), we have

$$|f(a_r) - f(a_{r-1}) - f'(x_r)(a_r - a_{r-1})| \leq \epsilon(a_r - a_{r-1}) \quad (r = 1, \ldots, n)\ . \tag{2.10}$$

To deduce Jordan's version of the mean value theorem from Theorem (K′) (or Theorem (K)), one sums over $r$ both members of (2.10) to obtain $(m - \epsilon)(b - a) \leq f(b) - f(a) \leq (M + \epsilon)(b - a)$ and then let $\epsilon \to 0$.

But another consequence can be obtained from (2.10), namely that *given $\epsilon > 0$, there is a positive mapping $\delta$ on $[a, b]$ such that*

$$\left| f(b) - f(a) - \sum_{r=1}^{n} f'(x_r)(a_r - a_{r-1}) \right| \leq \epsilon(b - a)$$

*for all tagged partitions of* $[a, b]$

$$a = a_0 \leq x_1 \leq a_1 \leq x_2 \leq a_2 \leq \ldots \leq a_{n-1} \leq x_n \leq a_n = b \tag{2.11}$$

*satisfying (2.8) and (2.9).* In other words, the *Riemann sums* $\sum_{r=1}^{n} f'(x_r)(a_r - a_{r-1})$ of $f'$ approach $f(b) - f(a)$ in a way reminiscent of Riemann's integration, except that Riemann's integral corresponds, as easily verified, to the special choice of a *constant* mapping $\delta$. This observation led Jaroslav Kurzweil[68] and Ralph Henstock[69] to define independently, some fifty years ago, a new type of integration. Namely, a function $g : [a, b] \to \mathbb{R}$ is *Kurzweil–Henstock-integrable* over $[a, b]$ if there is some $J \in \mathbb{R}$ having the property that, for each $\epsilon > 0$, there exists a positive mapping $\delta$ on $[a, b]$ such that $\left| J - \sum_{r=1}^{n} g(x_r)(a_r - a_{r-1}) \right| \leq \epsilon$ for all tagged partitions (2.11) of $[a, b]$ satisfying (2.8) and (2.9). It can be shown that this integral is more general than Lebesgue's and indeed equivalent to Denjoy–Perron's integral. In particular, our argument above shows that *any derivative $f'$ is Kurzweil–Henstock integrable on $[a, b]$ with integral equal to $f(b) - f(a)$*. This approach of integration has been widely developed and generalized in the last forty years[70]. Notice finally that the absence of a similar direct proof for Peano's statement $(J)$ comes from the fact that no Cousin's lemma exists for tagged divisions of $[a, b]$ such that $x_r = a_{r-1}$ for $r = 1, \ldots, n$.

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[68] J. Kurzweil (1957), 418–446.

[69] R. Henstock (1961), 402–418.

[70] For more details see e. g. J. Mawhin (1992) or A. Fonda (2001). For comments on its history, see e. g. J. Mawhin (2007), 47–63 and the references therein.

# 3

# Peano, his School and . . . Numerical Analysis

Giampietro Allasia

*Lo scopo della Matematica è di dare il valore*
*numerico delle incognite che si presentano*
*nei problemi pratici.*
Giuseppe Peano

## 3.1 Introduction

Giuseppe Peano, an outstanding mathematician of unusual versatility, made fundamental contributions to many branches of mathematics; in numerical analysis, noteworthy results concern representation of linear functionals, quadrature formulas, ordinary differential equations, Taylor's formula, interpolation, and numerical approximations[1]. Many results are still of great interest, whereas a few others appear obsolete.

[1] Some of these works (e. g. those on differential equations and Taylor's formula) are considered by some people as pertaining mainly to mathematical analysis; this is not astonishing because numerical analysis is based on mathematical analysis. The book by J.B. Scarborough, *Numerical Mathematical Analysis*, Baltimore, Johns Hopkins Press, 6th ed. 1966 (1st ed. 1930), amongst the earliest courses in numerical analysis, must be remembered. The title is clearly motivated by the following statement in the Preface to the first edition: "Applied mathematics comes down ultimately to numerical results, and the student [. . . ] will do well to supplement his usual mathematical equipment with a definite knowledge of the numerical side of mathematical analysis." These words sound quite similar to the following considerations by Peano (1918c), 693, referring to the ordered list of Peano's publications in the CD-ROM *L'Opera omnia e i Marginalia di Giuseppe Peano (with English version)*, C.S. Roero (ed.), Torino, Dipartimento di Matematica, 2008, see the URL www.peano2008.unito.it): "Lo scopo della Matematica è di dare il valore numerico delle incognite che si presentano nei problemi pratici [. . . ] tutti i grandi matematici sviluppano le loro mirabili teorie fino al calcolo numerico delle cifre decimali necessarie. È di somma importanza che le teorie matematiche che si insegnano nelle scuole di vario grado siano coronate dal calcolo numerico" (The aim of mathematics is to give the numerical value to the unknowns arising from practical problems; all the great mathematicians develop their wonderful theories up to the numerical evaluation of the required decimal digits. It is extremely important that the mathematical theories, which are thought in schools of various levels, are crowned with numerical calculus).

Several members of Peano's school worked in numerical analysis, following the suggestions of the master, but their contributions are undoubtedly less significant, either because superseded by more general results or because too strictly connected with problems of that period. Among the followers a prominent position is held by Ugo Cassina, author of several scientific notes and two textbooks on numerical analysis[2], the latter is an extended version of the former, and a wide-ranging encyclopedia chapter[3], in which the contributions of Peano and his school are precisely pointed out[4].

In order to make clear our point of view presenting the contributions by Peano (and his school) to numerical analysis and in order to correctly place them in contemporary problems, we think it is convenient to present short considerations about the nature of numerical analysis and its development.

Numerical analysis[5] is a branch of mathematics, which has in some respects quite a long history, going back to the rising of mathematics itself, but it became a topic in a mathematical degree only in the twentieth century. Actually, numerical analysis enjoyed a very rapid growth in the last century, particularly in the second half, and underwent also a significant evolution thanks to the development of modern computers. The contents of numerical analysis and its development in the nineteenth and twentieth centuries are not adequately reflected in most textbooks on the history of mathematics and have been insufficiently studied in historical research so far.

Numerical analysis is concerned with the mathematical derivation, description and analysis of constructive methods of obtaining numerical solutions of mathematical problems. A *constructive method* describes how to determine effectively the solution of a mathematical problem and not only to prove its existence. A *numerical solution* is a set of approximate values of the solution of a mathematical problem.

The main current domains, which numerical analysis investigates from its specific point of view, are approximation theory, interpolation and extrapolation, linear algebra, optimization and nonlinear equations, quadrature and orthogonal polyno-

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[2] U. Cassina (1922) and (1928).

[3] U. Cassina (1943).

[4] "Indice, per quanto possibile completo, de auctores italiano, que [. . . ] pertine ad schola de Peano" (A list, as complete as possible, of Italian authors, who pertain to Peano's school), in general, is drawn up by U. Cassina (1932), 124 and a shorter one, restricted to numerical analysis, is given by C. Migliorero (1928), 36, who writes: "Peano reincipe studio de [. . . ] questiones [de calculo numerico], et suo labores fundamentale da origine ad serie de alio studios interessante facto per suo discipulos de [. . . ] ultimo periodo" (Peano restarts studying problems of numerical calculus, and his basic work gives rise to a series of other interesting contributions by his followers of the last period). Further information on the most significant contributions by members of Peano's school is supplied by U. Cassina in the notes to the list of Peano's works (1932), 133–148. Peano's scientific work as a whole is illustrated by Cassina (1933), 323–389.

[5] The appellation "numerical analysis" is the most frequent today, but other expressions with the same (or almost the same) meaning are: numerical mathematics, computational mathematics, numerical methods, computational methods, computing methods, numerical computation, numerical calculus, and in Italy "calcolo numerico".

mials, ordinary differential equations and integral equations, and partial differential equations.

A comprehensive "state-of-the-art" picture at the beginning of the 20th century is offered by the articles on numerical and graphical methods in the *Encyklopädie der mathematischen Wissenschaften* or in its French counterpart *Encyclopédie des sciences mathématiques*[6].

By the Second World War, there was little in the way of numerical literature and numerical analysis was hardly a mathematical topic. As an example, published in the English language, there were only half dozen books with a numerical content, namely those by D. Brunt, E.T. Whittaker and G. Robinson, J.F. Steffensen, J.B. Scarborough, L.M. Milne-Thomson, and H. Levy and E.A. Baggott[7]. In Italian we can only recall the books by Cassina (1928), already quoted, and G. Cassinis[8], which is decidedly more modern[9]. Moreover, the booklet by E. Maccaferri deserves to be recalled; it is entitled *Calcolo numerico approssimato*, and it is partially inspired by Peano's notes on numerical approximations[10].

By the beginning of the century and then for many decades, when all arithmetic was done by pencil and paper, multiplication and division at least were tedious and time-consuming operations. To ease this many mathematical tables were produced, in particular tables of logarithms[11]. Then, as applied mathematicians developed their skills, their computational problems became increasingly complex, and more advanced mathematical tables were needed and indeed produced.

Finally, mechanical calculating machines evolved into electronic computers in the 1940. But the invention of the computer also influenced the field of numerical analysis, since longer and more complicated calculations could now be made.

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[6] *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Leipzig, Teubner, 1898–1935; *Encyclopédie des sciences mathématiques pures et appliquées*, Paris, Gauthier-Villars, 1904–1916. Interesting viewpoints are expressed in G. Fano (1911), 106–126, and in F.G. Tricomi (1927), 102–108.

[7] D. Brunt (1923); E.T. Whittaker, G. Robinson, *The calculus of observations. A treatise on numerical mathematics*, London, Blackie & Son, 4th ed. 1944 (1st ed. 1924); J.F. Steffensen, *Interpolation*, New York, Chelsea, 1950 (reprint of the 1st ed., Baltimore, Williams & Wilkins, 1927); J.B. Scarborough (1930); L.M. Milne-Thomson (1933), further ed. 1951; H. Levy, E.A. Baggott (1934).

[8] G. Cassinis (1928).

[9] The former is reviewed by Peano in (1928h), and by G. Vivanti in U. Cassina (1929), 199–201; the latter by Tricomi in G. Cassinis (1928), 74–75: "Nel complesso mi sembra che questa del Cassinis sia un'opera veramente notevole, completamente nuova nella nostra letteratura scientifica" (As a whole, this book by Cassinis seems to be a really remarkable work, completely new in our scientific literature).

[10] E. Maccaferri (1919).

[11] In the Preface of the book by Whittaker and Robinson (1944), vi, we read: "The material equipment essential for a student's mathematical laboratory is very simple. Each student should have a copy of Barlow's tables of squares, etc., a copy of Crelle's *Calculating Tables*, and a seven-place table of logarithms. Further, it is necessary to provide a stock of computing paper [...], and lastly a stock of computing forms for practical Fourier analysis. [...] With this modest apparatus nearly all computations hereafter described may be performed, although time and labour may often be saved by the use of multiplying and adding machines when these are available."

## 3.2 Peano's Works on Numerical Analysis

The first result by Peano in numerical analysis appeared in 1887[12] and the last in 1919[13]. Such a long period of time shows clearly the constant interest of Peano in problems of numerical analysis[14].

For greater convenience, we list Peano's papers referable to numerical analysis partitioning them by subject.
**(A)** The notes devoted to **quadrature formulas** are five[15]:

- *Generalizzazione della formula di Simpson* (1892);
- *Formule di quadratura* (1893);
- *Resto nelle formule di quadratura, espresso con un integrale definito* (1913);
- *Residuo in formulas de quadratura* (1914);
- *Resto nella formula di Cavalieri-Simpson* (1915).

Almost all of them concern quadrature formulas which are exact for polynomials of a certain degree. The third is the most important, because the content goes beyond the title, discussing a general property of an important class of linear functionals. Here we shall be able to consider only this topic in detail for lack of space.

It must be remarked that interesting results on quadrature formulas are already contained in the books[16]:

- *Applicazioni geometriche del calcolo infinitesimale* (1887);
- *Lezioni di Analisi infinitesimale* (1893).

[12] G. Peano (1887b).

[13] G. Peano (1919b).

[14] Hence, the following comment by S. Di Sieno (S. Di Sieno, A. Guerraggio, P. Nastasi (eds.) 1998), 5, must be corrected: "Altri sono i suoi [di Peano] interessi prevalenti, sin dall'inizio del secolo. Né la situazione viene modificata dalla redazione di due brevi Note [Peano 1918d and 1919b], che rimangono le uniche del periodo in questione ascrivibili in qualche modo alla disciplina [Analisi matematica]" (Ever since the beginning of the century Peano has other prevalent interests. The situation is not even changed by the writing of two short notes, which are the only ones in the considered period somehow pertaining to mathematical analysis). Indeed, U. Cassina (1932), 122–123, states more precisely: "In anno 1903 Peano initia studios philologico et opera interlinguistico, ad que illo post vol dedica, usque ad morte, parte extra grande de suo activitate [...]. Tamen, hoc non porta (ut, in modo vero erroneo, aliquo crede) ad relinque studios mathematico, que, ad contrario, recipe novo impulso. In vero, circa anno 1913, illo initia et cultiva cum grande successu novo campo de investigationes mathematico: illo dedicato ad *Calculo numerico*, neglecto et considerato quasi cum dedignatione ab aliquo mathematico moderno (per quanto jam mathematicos illustre [...] ne habe omisso de dedica scriptos ad quaestiones de calculo numerico)" (In 1903 Peano begins some philological studies and interlinguistic works, whom he devotes the greatest part of his activity till his death. Nevertheless, this fact does not imply, as some people think quite wrongly, that he leaves mathematical studies, which on the contrary receive a new impulse. As a matter of fact, nearly in 1913, he starts to investigate with great success a new mathematical field: the one that is devoted to Numerical Calculus, which is neglected and thought almost unworthy of consideration by some modern mathematicians (even though previously famous mathematicians did not omit writing papers on topics of numerical calculus).

[15] G. Peano (1892q); (1893c); (1913g); (1914b); (1915c).

[16] G. Peano (1893h).

**(B)** Three notes are pertinent to **interpolation formulas**[17]:

- *Sulle funzioni interpolari* (1882)
  (in which the divided differences of a complex variable function are expressed by a contour integral and several consequences are discussed);
- *Sulle differenze finite* (1906);
- *Resto nelle formule di interpolazione* (1918)
  (which gives a form of the remainder term for Lagrange interpolation formula. It aroused the prompt interest of Whittaker and Robinson, who devoted a section of their classical treatise to it[18]).

**(C)** Peano's form of the remainder term in **Taylor's formula**, given in the paper[19]:

- *Une nouvelle forme du reste dans la formule de Taylor* (1889)
  (renowned and quoted everywhere as *Peano remainder*). Moreover, Peano in the note[20]
- *Sulla formula di Taylor* (1892)
  anticipates the concept of asymptotic expansion, then widely developed in 1886 by Poincaré[21].

**(D)** Peano proposed a constructive method for the solution of **ordinary differential equations** (or systems of such equations), which satisfy given initial conditions, in two papers[22]:

- *Integrazione per serie delle equazioni differenziali lineari* (1887),
  then translated with extensions into
- *Intégration par séries des équations différentielles linéaires* (1888).

To solve Cauchy's problem for a first-order differential equation, Peano was the first to introduce the "method of successive approximations", often known as the Peano–Picard method[23]. This method can be used for solving both a system of differential

[17] G. Peano (1882c); (1906a); (1918d).

[18] E.T. Whittaker, G. Robinson (1944), 32–33.

[19] G. Peano (1889e).

[20] G. Peano (1892a).

[21] H. Poincaré (1886), 295–344.

[22] G. Peano (Peano 1887a); (Peano 1888b). These papers should be considered jointly with the followings: (Peano 1885a); (Peano 1890f); (Peano 1892aa); (Peano 1897c).

[23] Tricomi, discussing in detail the theorem of existence and uniqueness for the solution of Cauchy's problem (see F. Tricomi (1948), 18–29), writes (note at p. 26): "Al metodo delle approssimazioni successive si associa generalmente il nome del grande matematico francese E. Picard (1856–1941) che ne ha fatto vedere tutta l'importanza. Tuttavia già qualche anno prima che dal Picard, esso era stato usato dal nostro Peano" (The method of successive approximations is generally associated with the name of the great French mathematician E. Picard (1856–1941) who showed the full importance of the method. Nevertheless a few years before Picard, it had been used by our Peano). A chronologically ordered list of references to the papers by Peano, Picard and Lindelöf, which are pertinent to the topic, is given by U. Cassina (1943), 190, note 354. Interesting considerations are also made by B. Levi in (1955, 14–18), or (1932), 253–262, and by T. Viola in (1985), 33–35. Note that Peano seems to prefer the term "method of successive integrations", see (Peano 1897c), 12.

equations and a differential equation of higher order, if the latter is written in the form of a system. From a numerical point of view, the method of successive approximations has its own shortcoming, which consists in computing more and more cumbersome integrals.

**(E)** In the years 1916–1918, Peano organized series of lectures and working teams to promote the correct use of the procedures of **numerical approximations**, publishing five interesting papers[24]:

- *Approssimazioni numeriche* (1916);
- *Valori decimali abbreviati e arrotondati* (1917);
- *Approssimazioni numeriche* (1917, two notes);
- *Interpolazione nelle tavole numeriche* (1918);
- *Risoluzione graduale delle equazioni numeriche* (1919).

Only Peano's papers on numerical approximations are included in the section *Calcolo numerico* of *Opere scelte*[25], whereas the other contributions, as listed above, are included in the section *Analisi matematica*. This editorial choice may find its explanation recalling the considerations developed in our Introduction. In fact, the same topic may be pertinent to different areas and, moreover, the expression *Calcolo numerico* is used in *Opere scelte* in a very restricted meaning, which is clearly obsolete.

## 3.3 Integral Representation of Remainders

Many approximation processes are linear on a space of functions and exact for a class of functions on which a certain linear functional vanishes. Examples of such processes are the following: the approximation of a definite integral by a linear combination of values of the integrand and its derivatives, the approximating formula being such as to be exact if the integrand is a polynomial of degree $n$; the approximation of a function by that polynomial of degree $n$ which minimizes a weighted integral of the square of the error; the approximation of a function by a Lagrangian interpolating polynomial of degree $n$ or by the first $n + 1$ terms of its Taylor series.

In each of the cited cases , the approximating process is exact for functions whose derivatives of order $n$ vanish. In 1913 Peano[26] pointed out that the error committed can then be expressed as a single integral of the $(n + 1)$th derivative.

Rémès (1939)[27] generalized Peano's theorem to the case in which the approximating process is exact for functions satisfying a linear homogeneous differential equation. Peano's theorem is the special case of Rémès's theorem obtained when

[24] G. Peano (1916b); (1917b); (1917c) and (1917d); (1918c); (1919b).

[25] G. Peano, *Opere scelte. Volume I: Analisi matematica – Calcolo numerico, Volume II: Logica matematica – Interlingua e algebra della grammatica, Volume III: Geometria e fondamenti – Meccanica razionale – Varie*, a cura di U. Cassina, edito dall'Unione Matematica Italiana col contributo del Consiglio Nazionale delle Ricerche, Edizioni Cremonese, Roma, 1957–59.

[26] G. Peano (1913g) and (1914b).

[27] E.J. Rémès (1939), 21–62; (1940a), 47–82; (1940b), 129–133.

the linear homogeneous differential equation on the function $f(x)$ is the equation $f^{(n+1)}(x) = 0$.

Sard (1948)[28] gave further generalizations of the results obtained by Peano and Rémès, considering the case of an approximating process in which a linear operator is exact for all functions on which another linear operator vanishes.

Today Peano's result is investigated and generalized in many ways, but we would like to recall some considerations by H. Brass and K.-J. Förster:

> For more than 80 years, Peano kernel theory has proven to be an important tool in numerical analysis. [...] It seems to us that [more] general representations [of Peano's theorem] have hardly ever been applied in a concrete way to problems in numerical analysis.[29]

In particular, Sard called attention to the importance of Peano's theorem and stated precisely the attribution of the result to Peano. Then, illustrating some applications of Peano's theorem, Sard observed that significant results for a direct treatment of the remainders in mechanical quadrature had been obtained by Birkhoff, Radon, and von Mises[30].

Before Sard's paper, Peano's theorem was not widely known and appreciated. Indeed, it is quoted by Rémès, but among several other contributions[31], and by Radon, but only as an addition to page proofs[32], whereas mention of it is significantly lacking in other authors[33].

Cassina in (1943)[34] accurately reported the results achieved by Peano and his school on Peano's theorem and its applications, even though he does not seem quite aware of the power of Peano's result. Similarly, Ghizzetti and Ossicini in their very important monograph in (1970), and then Ghizzetti in (1986) did not highlight the generality and the significance of Peano's theorem[35]. Seemingly they ignore, or do

[28] A. Sard (1948), 333–345.

[29] H. Brass, K.-J. Föster (1998), 175–202. See also H. Brass, K. Petras (2003), 195–207, and references therein.

[30] G.D. Birkhoff (1906), 107–136; J. Radon (1935), 389–396; R. von Mises (1936), 56–67.

[31] E.J. Rémès (1940b), 130: "On obtient ainsi, comme cas particuliers, diverses représentations intégrales des termes complémentaires, qui ont été indiquées, comme conséquences de différentes considérations théoriques, par Mises ['Über allgemeine Quadraturformeln', 1936], Radon ['Restausdrücke bei Interpolations- und Quadraturformeln', 1935], et par d'autres auteurs (Peano 1913g, [...], Kowalewski [G. Kowalewski, *Interpolation und genäherte Quadratur*, Leipzig, Teubner, 1932] [...], Birkhoff ['General mean value and remainder theorems', 1906])."

[32] J. Radon (1935), 396: "Zusatz bei der Korrektur [...]: Für den Fall $D_n[f(x)] = f^{(n)}(x)$ (polynomiale Annäherung) findet sich die Formel (3.2) bereits bei G. Peano [Peano 1913g]."

[33] It is odd, for example, that in 1944 Whittaker and Robinson (1924) do not quote Peano's theorem, in spite of their knowledge about the researches of Peano's school. In fact, they cite by Peano the papers *Applicazioni geometriche* (1887b), and 'Resto nelle formule di interpolazione' (1918d), and by Cassina, 'Formole sommatorie e di quadratura ad ordinate estreme' (1939a), 225–274, and 'Formole sommatorie e di quadratura con l'ordinata media' (1939b), 300–325. W.E. Milne, in (1949a) and (1949b) gives no cross-reference to Peano.

[34] U. Cassina (1943), 183–186, and also 168–177.

[35] A. Ghizzetti, A. Ossicini (1970), in particular the preface; A. Ghizzetti (1986), in particular 54.

not share, the interpretation given by Sard on the work of Peano, in particular, and on the works of Rémès, von Mises, and Radon[36].

Without striving for full generality[37], consider functions of class $C^{n+1}[a,b]$, and let linear functionals of the following type be defined over this class

$$R_n f = \int_a^b \left[a_0(x) f(x) + a_1(x) f'(x) + \cdots + a_n(x) f^{(n)}(x)\right] \mathrm{d}x - \\ - \sum_{i=1}^{j_0} b_{i0} f(x_{i0}) - \sum_{i=1}^{j_1} b_{i1} f'(x_{i1}) - \cdots - \sum_{i=1}^{j_n} b_{in} f^{(n)}(x_{in}) \,. \tag{3.1}$$

The functions $a_i(x)$ are assumed to be piecewise continuous over $[a,b]$, the points $x_{ij}$ to lie in $[a,b]$, and $b_{i0}, b_{i1}, \ldots, b_{in}$ to be real constants. Note that the functional $R_n f$ in (3.1) can be interpreted, in particular, as the *remainder term* or *error term* of a numerical integration formula.

**Theorem 3.1 (Peano)** *Let $R_n p = 0$ for all $p \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the space of all polynomials of degree less than or equal to $n$. Then, for all $f \in C^{n+1}[a,b]$,*

$$R_n f = \int_a^b K_n(t) f^{(n+1)}(t) \mathrm{d}t \,, \tag{3.2}$$

*where*

$$K_n(t) = \frac{1}{n!} R_n \left[(x-t)_+^n\right] \tag{3.3}$$

*and*

$$(x-t)_+^n = \begin{cases} (x-t)^n \,, & \text{if } x-t \geq 0 \,, \\ 0 \,, & \text{if } x-t < 0 \,. \end{cases} \tag{3.4}$$

*The functional $R_n$ in* (3.3) *is applied to $(x-t)_+^n$ considered as a function of $x$.*

[36] In fact, they observe: "It is reasonable to think that it is possible to obtain quadrature formulae, without using interpolation methods. In a paper of 1913, G. Peano first made an attempt in this way and succeeded in obtaining Cavalieri–Simpson's formula, with an integral expression of the remainder, only by means of integration by parts. This method was systematically employed by R. von Mises, who in a paper of 1936 showed how it is possible to get every quadrature formula with the sole tool of integration by parts. In 1935, J. Radon also showed how the integral expression of the remainder can be obtained by means of the Green–Lagrange identity, relative to a linear differential operator and its adjoint. [...] It is worthwhile to note that Radon's method includes that of von Mises, since the integrations by parts used by von Mises, are already performed in the Green–Lagrange identity, used by Radon."

[37] General hypotheses are considered by Sard in the quoted works. Here we follow the presentation by P.J. Davis, *Interpolation and approximation*, New York, Dover, 1975, and P.J. Davis, P. Rabinowitz, *Methods of numerical integration*, New York, Academic Press, 2nd ed., 1984. A more recent and interesting chapter on Peano's theorem was also written by G.M. Phillips (G.M. Phillips 2003, in particular 147–162).

The method followed by Peano to prove this theorem starts with the Taylor formula with the remainder in integral form[38]

$$\begin{aligned} f(x) = f(a) + (x-a)f'(a) + \cdots + (x-a)^n \frac{f^{(n)}(a)}{n!} \\ + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\mathrm{d}t\,. \end{aligned} \tag{3.5}$$

The last integral can be extended to $t = b$ if we use the truncated power function (3.4). Thus,

$$\int_a^x (x-t)^n f^{(n+1)}(t)\mathrm{d}t = \int_a^b (x-t)_+^n f^{(n+1)}(t)\mathrm{d}t\,. \tag{3.6}$$

Now applying the functional $R_n$ to both sides of (3.5), with the integral written as in (3.6), yields by the hypothesis $R_n p = 0$ for all $p \in \mathcal{P}_n$

$$R_n f = \frac{1}{n!} R_n \left\{ \int_a^b (x-t)_+^n f^{(n+1)}(t)\mathrm{d}t \right\} = \frac{1}{n!}\int_a^b R_n\left[(x-t)_+^n\right] f^{(n+1)}(t)\mathrm{d}t\,, \tag{3.7}$$

because the interchange of $R_n$ with the integral is legitimate for functionals of the type (3.1). Defining $K_n(t)$ as in (3.3), we thus have the representation (3.2) for the functional $R_n$. This is called the *Peano representation* of the remainder $R_n$, and $K_n$ the $n$th *Peano kernel* for $R_n$. Theorem 1 is known as *Peano's remainder theorem* or *Peano's kernel theorem*[39].

**Corollary 3.1 (Peano)** *If, in addition to the above hypotheses, the Peano kernel $K_n(t)$ does not change sign on $[a,b]$, then for all $f \in C^{n+1}[a,b]$*

$$R_n f = \frac{f^{(n+1)}(\xi)}{(n+1)!} R_n x^{n+1}\,, \quad a < \xi < b\,.$$

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[38] Rémès starts with the Taylor formula as well. A few possible inaccuracies in Peano's proof are pointed out by Sard ('Integral representations of remainders', 1948, 339).

[39] Precisely, Peano obtains the representation, equivalent to (3.2):

$$R_n f = \int_{-\infty}^{+\infty} \tilde{K}_n(t) f^{(n+1)}(t)\mathrm{d}t,$$

where

$$\tilde{K}_n(t) = R_n\left[\frac{(x-t)^n}{n!}\frac{1}{2}\mathrm{sgn}\,(x-t)\right],$$

the variable $x$ takes all real values, and $\mathrm{sgn}\,x$, sign of $x$, takes for $x > 0$ the value $+1$, for $x < 0$ the value $-1$, and for $x = 0$ the value 0. Another form, also suggested by Peano, is

$$\hat{K}_n(t) = R_n\left[\frac{(x-t)^n}{n!}\frac{1}{2}\big(1 + \mathrm{sgn}\,(x-t)\big)\right].$$

In fact, we can use the mean-value theorem for integrals to write (3.2) in the form

$$R_n f = f^{(n+1)}(\xi) \int_a^b K_n(t)\mathrm{d}t \,, \tag{3.8}$$

and the integral on the right (which is nonzero by assumption) is easily evaluated by putting $f(x) = x^{n+1}$ in (3.8). This gives

$$R_n x^{n+1} = (n+1)! \int_a^b K_n(t)\mathrm{d}t \,.$$

The functional $R_n$ is called *definite of order n* if its Peano kernel $K_n$ does not change sign, and we then also say that $K_n$ is definite.

For nondefinite functional $R_n$, we must estimate by

$$|R_n f| \leq \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right| \int_a^b |K_n(t)|\mathrm{d}t \,, \tag{3.9}$$

which, in view of the form (3.3) of $K_n$, can be rather laborious[40].

The constant

$$c = \int_a^b |K_n(t)|\mathrm{d}t \,,$$

plays an important role in certain theories of error. With this constant, we may write the error estimate (3.9) in the form

$$|R_n f| \leq c \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right| . \tag{3.10}$$

**Theorem 3.2** *The error estimate (3.10) is best possible in the sense that there exists a function $f(x)$ with a bounded, piecewise continuous $(n+1)$th derivative for which the inequality is an equality*[41].

## 3.4 Applications of the Remainder Functional

Several error functionals that occur in numerical analysis fall within the frame of Peano's theorem. The general point of view offered by Peano's theorem is useful in

[40] From a computational viewpoint a little more can be said; see, e. g., G. Allasia, M. Allasia (1976), 353–358; G. Allasia, P. Patrucco (1976), 263–274; G. Allasia, C. Giordano (1979), 1103–1110; G. Allasia, C. Giordano (1980), 257–269. See also P.J. Davis, P. Rabinowitz (1975).

[41] See the proof in P.J. Davis, P. Rabinowitz (1975), 290–291.

that it establishes attractive similarities in mathematical situations that might otherwise appear different. It should not be expected that the general theory will afford results which cannot be obtained directly. A direct attack on any problem is at least as powerful as any other attack, inasmuch as the direct attack can include a particularization of any general argument.

We offer a glance at some of the most interesting applications.

**Numerical quadrature.** If we set

$$R_r f = \int_{-1}^{1} w(x) f(x) \mathrm{d}x - \sum_{k=1}^{n} w_k f(x_k)$$

for $-1 \leq x_k \leq 1$, and if we assume that $R_r f = 0$ for $f \in \mathcal{P}_r$, then the Peano kernel for $R_r$ is given explicitly by

$$K_r(t) = \frac{1}{r!} R_r \left[ (x-t)_+^r \right]$$

or

$$r! K_r(t) = \int_{-1}^{1} w(x)(x-t)_+^r \mathrm{d}x - \sum_{x_k > t} w_k (x_k - t)^r .$$

In the special case $w(x) = 1$,

$$r! K_r(t) = \frac{(1-t)^{r+1}}{r+1} - \sum_{x_k > t} w_k (x_k - t)^r . \tag{3.11}$$

Peano considers in particular the application of his theorem to the remainder of Simpson's rule, but this is just an example[42]. In fact, Peano is quite conscious of the general meaning of his theorem, which must hold at least for functionals of the class (3.1) considered above[43].

Let

$$R_4 f = \int_{-1}^{+1} f(x) \mathrm{d}x - \frac{1}{3} \left[ f(-1) + 4 f(0) + f(1) \right],$$

[42] See Peano (1913g). Peano was interested in this example because in 1887 he gave first the remainder of Simpson's rule in the form (3.9) (see Peano 1887b, 204). In fact, Peano (and Rémès too) also consider the functionals relating to divided differences and numerical differentiation.

[43] Indeed, Peano observes (Peano 1914b, 7): "Non es necesse que formula de approximatione, de que nos determina residuo, contine in modo explicito uno integrale. Suffice que es lineare in functione $f$" (It is not necessary that the approximation formula, whose remainder has to be determined, contains explicitly an integral term. It is sufficient for the formula to be linear with respect to the function $f$). A precise characterization of the considered remainder functionals is given in Peano 1913g, 563.

where $R_4 p = 0$ if $p \in \mathcal{P}_3$. Applying (3.11) we find

$$K_4(t) = \begin{cases} -\dfrac{1}{72}(1-t)^3(3t+1)\,, & \text{if } 0 \le t \le 1\,, \\ K_4(-t)\,, & \text{if } -1 \le t \le 0\,. \end{cases}$$

Note that $K_4(t) \le 0$ so that Corollary 3.1 is applicable and this leads to the following error for Simpson's rule

$$R_4 f = \frac{1}{4!} f^{(4)}(\xi) R_4(x^4) = -\frac{1}{90} f^{(4)}(\xi). \tag{3.12}$$

Some members of Peano's school then applied formula (3.2) to the evaluation of the remainders of several quadrature formulas, both in the exact form with a definite integral and in the approximate form with a derivative. These works are carefully quoted by Cassina[44].

For a direct treatment of the remainder in mechanical quadrature, one can consult the pertinent papers by Birkhoff, Radon, von Mises, and the definitive contribution by Steffensen on Newton–Cotes formulas[45].

**Finite differences, differentiation formulas.** Let us consider the divided difference of order $n+1$ on the points $x_0 < x_1 < \cdots < x_{n+1}$

$$f[x_0, x_1, \ldots, x_{n+1}] = \sum_{i=0}^{n+1} \frac{f(x_i)}{\omega'(x_i)},$$

where $\omega(x) = (x - x_0)(x - x_1)\cdots(x - x_{n+1})$. Since this functional annihilates all polynomials of degree $\le n$, it follows from Peano's theorem

$$f[x_0, x_1, \ldots, x_{n+1}] = \int_{x_0}^{x_{n+1}} K_n(t) f^{(n+1)}(t)\mathrm{d}t,$$

where

$$K_n(t) = \frac{1}{n!} \sum_{i=0}^{n+1} \frac{(x_i - t)_+^n}{\omega'(x_i)}\ .$$

Now $n!K_n(t) \equiv B_n(t)$ satisfies the properties:

$B_n(t)$ is a polynomial of degree $n$ for $x_i \le t \le x_{i+1}, \quad i = 0, 1, \ldots, n$ ;
$B_n(t) \in C^{n-1}[x_0, x_{n+1}]$.

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[44] U. Cassina (1943), 186.

[45] G.D. Birkhoff (1906); J. Radon (1935); R. von Mises (1936); J.F. Steffensen (1927).

Moreover, it is $B_n(t) = 0$ for $t \le x_0$ and $t \ge x_{n+1}$. Hence, $B_n(t)$ is a spline of degree $n$ and nodes $x_0, x_1, \ldots, x_{n+1}$. This is called the *Peano form of divided differences* and $B_n$ is called the Peano kernel for divided differences.

Since for equally spaced points,

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{\Delta^k f(x_i)}{k!h^k},$$

a similar argument can also be developed for finite differences[46]. The same observation can be made for numerical differentiation formulas based on finite differences.

Peano's theorem was applied by Cassina and other members of Peano's school to a number of formulas based on finite differences[47].

**Polynomial interpolation.** Let $x, x_0, \ldots, x_n$ be fixed in $[a, b]$ and $f \in C^{n+1}[a, b]$. Let

$$R_n f = f(x) - \sum_{k=0}^{n} f(x_k) l_k(x),$$

where $l_k$ are the fundamental polynomials for pointwise interpolation. Then,

$$\begin{aligned} n!K_n(t) = R_n\Big[(x-t)_+^n\Big] &= (x-t)_+^n - \sum_{k=0}^{n}(x_k - t)_+^n \, l_k(x) \\ &= \sum_{k=0}^{n} \left[(x-t)_+^n - (x_k - t)_+^n\right] \, l_k(x). \end{aligned}$$

For fixed $k$, by (3.4) we have

$$\begin{aligned} &\int_a^b \left[(x-t)_+^n - (x_k - t)_+^n\right] f^{(n+1)}(t)\mathrm{d}t = \int_a^x (x-t)^n f^{(n+1)}(t)\mathrm{d}t - \\ &- \int_a^{x_k} (x_k - t)^n f^{(n+1)}(t)\mathrm{d}t = \int_a^x \left[(x-t)^n - (x_k - t)^n\right] f^{(n+1)}(t)\mathrm{d}t + \\ &+ \int_{x_k}^{x} (x_k - t)^n f^{(n+1)}(t)\mathrm{d}t. \end{aligned}$$

46 P.J. Davis, *Interpolation and approximation*, 1975, 72–73; L. Schumaker, *Spline functions: basic theory*, Krieger Publ. Co., Malabar, Florida, 1993.

47 U. Cassina (1943), 184–185.

Hence,

$$n!\int_a^b K_n(t)f^{(n+1)}(t)\mathrm{d}t = \int_a^x f^{(n+1)}(t)\sum_{k=0}^{n}[(x-t)^n-(x_k-t)^n]\,l_k(x)\mathrm{d}t+ \\ +\sum_{k=0}^{n} l_k(x)\int_{x_k}^{x}(x_k-t)^n f^{(n+1)}(t)\mathrm{d}t.$$

The inner sum in the second integral can be transformed as follows

$$\sum_{k=0}^{n}[(x-t)^n-(x_k-t)^n]\,l_k(x) = (x-t)^n-\sum_{k=0}^{n}(x_k-t)^n\,l_k(x)\,.$$

Since $\sum_{k=0}^{n}(x_k-t)^n\,l_k(x) = (x_k-t)^n$, the inner sum vanishes identically. Thus, finally,

$$R_n(f) = \frac{1}{n!}\sum_{k=0}^{n} l_k(x)\int_{x_k}^{x}(x_k-t)^n f^{(n+1)}(t)\mathrm{d}t, \tag{3.13}$$

which is *Kowalewski's exact remainder for polynomial interpolation*[48].

**Linear interpolation.** The case $n=1, x_0=a, x_1=b$ is particularly noteworthy[49]. From (3.13),

$$R_1 f \equiv f(x)-\left[\frac{x-b}{a-b}f(a)+\frac{x-a}{b-a}f(b)\right] = \tag{3.14}$$

$$=\frac{x-b}{b-a}\int_a^x (t-a)f''(t)\mathrm{d}t+\frac{x-a}{b-a}\int_x^b (t-b)^n f''(t)\mathrm{d}t.$$

Introduce the following function defined over the square $a\le x\le b, a\le t\le b$

$$G(x,t) = \begin{cases} \dfrac{(t-a)(x-b)}{b-a}, & t\le x\,, \\ \dfrac{(x-a)(t-b)}{b-a}, & t\ge x\,. \end{cases} \tag{3.15}$$

Then, we can write (3.14) in the form

$$R_1 f = \int_a^b G(x,t)f''(t)\mathrm{d}t. \tag{3.16}$$

The function $G(x,t)$ is, for fixed $x$, the Peano kernel for $R_1(f)$.

48 G. Kowalewski (1932), 21–24, or P.J. Davis (1975), 71–72.

49 Davis (1975), 72.

Let $h(x) \in C[a,b]$ and $H''(x) = h(x)$. Set

$$\phi(x) = \int_a^b G(x,t)h(t)\mathrm{d}t. \tag{3.17}$$

Then, by (3.16),

$$H(x) - L_1(H;x) = \phi(x)\,,$$

where $L_1(H;x)$ is the linear interpolation polynomial for $H(x)$, so that $\phi''(x) = H''(x) = h(x)$. Furthermore, $\phi(a) = (R_1 H)(a) = 0$, $\phi(b) = (R_1 H)(b) = 0$. Therefore the integral (3.17) solves the differential problem

$$\phi''(x) = h(x)\,, \quad \phi(a) = \phi(b) = 0\,. \tag{3.18}$$

The function $G(x,t)$ is known as *Green's function* for the differential problem (3.18). These remarks indicate the close relationship between Peano's kernel and Green's function, and hence between interpolation theory and the theory of linear differential equations.

**Least squares approximations by algebraic and trigonometric polynomials, approximations by sums of periodic functions.** For the discussion of these cases we refer to Sard[50].

## 3.5 Trapezoidal and Parabolic Formulas

Peano concerned himself several times, in the period from 1887 to 1915, with the theoretical and practical aspects of quadratures. The fundamental remainder theorem can be considered as a brilliant but logical consequence of Peano's previous results. In fact, Peano writes:

> The remainders of some quadrature formulas are represented by definite integrals. Taylor's formula is of this type. Also Euler's summation formula has a remainder represented by Jacoby in the form of a definite integral, from which one can deduce other forms in terms of mean values. The trapezoidal rule is a particular case. As regard to other formulas the remainders are only known in terms of derivative mean values. Examples of this situation are the so called Simpson's formula and the Gaussian quadrature formulas, whose remainder has been found by prof. Mansion in 1887[51]. No remainder repre-

[50] A. Sard (1948), 341–343.

[51] The reference is to the following paper on the error terms for the Gauss–Lagrange formulas: P. Mansion (1886), 293–307. P. Mansion, editor of the journal *Mathesis*, had a close relationship with Peano. In particular, he noticed the interest of Peano's remainder theorem and wrote a note on the topic: P. Mansion (1914), 169–174. About this point see G. Allasia (2005), 43–61. E. Picard, the editor of Hermite's Collected Works, wrote (see *Œuvres de Charles Hermite*, É. Picard (ed.), vol.

sentation is known for all the other quadrature formulas. The remainder of every quadrature formula can always be represented in integral form.[52]

Peano obtains directly the error terms for the midpoint, trapezoidal and Simpson formulas, both simple and composite. First, Peano obtains the trapezoidal rule with its error in a way which is today standard[53]. In fact, if $f \in C^2[a,b]$, integrating on $[a,b]$ both sides of the interpolation formula

$$f(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a} + (x-a)(x-b)\frac{f''(t)}{2!},$$

where $a < t < b$, he obtains

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} - \frac{1}{12}(b-a)^3 f''(u),$$

where $a < u < b$. Besides, Peano in (1893h)[54] proposes the much more interesting representation

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} + \frac{1}{2}\int_a^b (x-a)(x-b)f''(x)\mathrm{d}x,$$

which he recalls introducing his note on the remainder theorem. However, this formula is not obtained by reasoning as in the theorem proof, but it is derived by means of integration by parts.

Then, Peano considers the midpoint rule[55] and, identifying it with the first of the Gauss–Legendre formulas, writes

$$\int_a^b f(x)\mathrm{d}x = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{12}\frac{f''(u)}{2!},$$

1–4, Paris, Gauthier-Villars, 1905–1917, in particular, vol. 4, *Avertissement*): "J'ai encore le devoir de rappeler l'aide que m'a apportée l'esquisse biographique et bibliographique écrite quelques semaines après la mort d'Hermite par M. Mansion, professeur à l'Université de Gand. [...] Puisse mon souvenir atteindre le vénéré doyen de la science mathématique en Belgique dans la ville où il est retenu depuis près de trois ans."

[52] Peano (1913g), 562: "Alcune formule di quadratura hanno il resto espresso mediante un'integrale definito. Tale è la formula di Taylor. Anche la formula sommatoria di Eulero ha un resto calcolato da Jacobi sotto forma di integrale definito, e da cui si deducono le espressioni con valori medii. Caso particolare è la formula del trapezio [...] Di altre formule di quadratura si conosce solo il resto espresso mediante il valore medio di una derivata. Tale è la formula detta di Simpson, e le formule di quadratura di Gauss, il cui resto fu calcolato dal prof. Mansion nell'anno 1887. Di tutte le altre formule di quadratura, non si conosce alcuna espressione del resto. Il resto in ogni formula di quadratura si può sempre ridurre ad integrale."

[53] Peano (1887b), 202–205.

[54] Peano (1893h), vol. I, 238.

[55] Peano (1887b), 219.

where $a < u < b$. Though some authors do not explicitly consider the midpoint rule as a Gaussian formula[56], Peano's argument is obviously right and, later, Cassina recalls it, saying that the first Gauss–Legendre formula is also known as "formola del trapezio con l'ordinata media"[57].

Peano offers, though indirectly, a different and independent way to get the midpoint formula with its error[58]. Namely, it is obtained by integrating on $[a, b]$ the Taylor expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)^2 \frac{f''(u)}{2!}, \quad x_0 = \frac{a+b}{2}. \tag{3.19}$$

If instead of (3.19), one considers the same formula but with the remainder in integral form, then one gets the result of Peano's theorem for the midpoint rule.

The pioneering representations of the error terms, given in Peano's work, did not have a prompt diffusion. As an example, J. Boussinesq seems not to know Peano's representations and achieves only incomplete results[59]. To confirm this late spread, one can observe that the exact representation of the error term for Simpson's or parabolic formula, obtained first by Peano in 1887[60], seems unknown to C. de la Vallèe Poussin, who proposed an error term representation 32 times less approximated than Peano's one[61]. The lack of knowledge about the error term of the (composite) Simpson rule had strange effects, as is shown by the following comment due to H. Laurent:

> Dans la méthode de Simpson, une des plus mauvaises que l'on puisse employer, parce que rien n'indique la limite ni même le sens de l'erreur qu'elle comporte, on substitue à l'aire que l'on veut évaluer une série d'aires paraboliques.[62]

In fact, Laurent prefers Poncelet's rule, "une méthode des plus simples et des plus rapides", for which there exists an upper bound for the error.

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[56] See, e. g., E.T. Whittaker, G. Robinson, *The calculus of observations*, 1944, 160, and M. Abramowitz, I.A. Stegun (1964), 916.

[57] U. Cassina (1943), 174.

[58] Peano (1887b), 209, note.

[59] J. Boussinesq, *Cours d'analyse infinitésimale*, vol. 2, Paris, Gauthier-Villars, 1890, 70–73.

[60] This is asserted in E.T. Whittaker, G. Robinson, *The calculus of observations*, 1944, 156, and in U. Cassina (1943), 169, but see also A.A. Markoff (1896), 59.

[61] C. de la Vallée Poussin, *Cours d'analyse infinitésimale*, vol. 1, 7e éd., Louvain, 1930, 332; U. Cassina (1943), 169.

[62] H. Laurent (1885–1888), vol. 3, 498–499.

Peano with colleagues and scholars at the Basilica of Superga, near Turin – Department of Mathematics G. Peano, University of Torino, by courtesy of Chinaglia Family

# 4

# Peano and the Foundations of Arithmetic

Gabriele Lolli

At the end of the 1880s two episodes occurred in rapid succession which formed the bases of what we call the foundations of arithmetic: the publication in 1888 of *Was sind und was sollen die Zahlen* by Richard Dedekind and in 1889 of *Arithmetices Principia, nova methodo exposita* by Giuseppe Peano. This work was to give Peano lasting fame, in that he had for the first time expounded the axioms for the system of natural numbers; from that time on they were linked to his name, and from the English "Peano Arithmetic" were known by the acronym PA.

Some historians insist on using the term "Dedekind–Peano axioms". Hao Wang, for example, asserted that Peano himself had admitted to having taken his axioms from the work of Dedekind[1]; his source for this is Jourdain (1912), who however says nothing of the sort (Jourdain 1989, 187), but simply mentions that Peano acknowledges the usefulness to him of Dedekind's essay.

A comedy of misunderstandings has arisen around Peano (1889a), nourished by the failure to take seriously what Peano clearly states at the end of the *Praefatio*, namely that "This booklet of mine has to be taken as a specimen of the new method"[2].

It was the author's intention in this work to present an example of the new method, and he appears to be motivated by this, and not by specific reflections on the foundations of arithmetic. To discover what Peano thought about it we must look elsewhere. First however we must look back, in order to understand the context.

## 4.1 Prehistory

Even after 1654, when Pascal had explicitly formulated the two rules for the proof by induction, the basis and the inductive step, and had published them in Pascal (1665), little use was made of this technique; examples are found in Fermat and occasionally in Euler.

When it was mentioned as such, what we call induction was attributed rather to Bernoulli and was called "the passage from $n$ to $n + 1$" or "Bernoulli's method",

[1] "Peano borrowed his axioms from Dedekind", quoted by H. Wang 1964, Repr. 1970, 68.

[2] G. Peano (1889a), v: "Hic meus libellus ut novae methodi specimen habendum est."

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for instance by A. G. Kästner, towards the end of the 18th century, or subsequently actually "Kästner's method" in recognition of his calling attention to it.

Jacobi in 1826 makes a fleeting reference to Gauss's method of "reaching his results with a difficult induction which, by means of the so-called Kästner's method of proving that, when something is true for the number $n$, therefore is also true for $n + 1$, can then be raised to complete generality". Gauss in 1816 states in effect repeatedly: "This induction is easily converted into a rigorous proof by a well known method ... such induction leads easily to full certainty ... it is easy to give a demonstrative strength to the induction." (Hæcce inductio facile in demonstrationem rigorosam convertitur per methodum vulgo notam ... Hæcce quoque inductio facillime ad plenam certitudinem evehitur ... cui inductioni facile est demonstrationis vim conciliare")[3]; unfortunately he gives no further clarification.

Meanwhile, at least the name was defined: in 1830 George Peacock began calling it "demonstrative induction", and Augustus de Morgan named it "Mathematical Induction". The first systematic, explicit use of induction is found in a treatise by Herman Grassmann, written with his brother for schools[4]; their declared aim of rigour was attained by giving ample, systematic room to proofs by induction; this type of proof is imposed naturally because it rests on the equations which characterise the operations by means of a definition by primitive recursion.

Grassmann's importance in the history of induction is twofold: he brought out the role of recursive definitions, and their link with inductive proofs, and he actually hypothesised the existence of type of proof for arithmetic which he called *induktorisch*, and which he put side by side with the "forward", "backward" and indirect proofs of the logical tradition; I say he put them side by side because he did not attempt to reduce them to these kinds of proofs, as would seem to be suggested by Gauss's talk of "converting them into rigorous proof".

Grassmann's overall view of arithmetic must have been extremely lucid if the exercise carried out by Wang (1964) is correct: namely, he claims that from Grassmann's exposition we can obtain the laws he used, without proof, taking them to be obvious or presenting them as definitions; and from them it is possible to infer that Grassmann considered the whole numbers, in modern terminology, as an ordered domain of integrity in which each set of positive elements has a minimum.

We are fortunate in having, in the person of Gottlob Frege, a witness and implacable judge of all the attempts in this period at justifying arithmetic. He provides a varied and instructive picture of the systematizations proposed, few of which are based on a logical analysis. The prevalent formulations were those in which the basic assumptions, such as the independence of numbers from the manner of counting, were given by an internal intuition (for example, Frege quotes Lipschitz).

After a comparison with the archeological digging up of various layers of ground, when one does not know what to expect, Frege discusses the objection that in the dig the layers are there, whereas:

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[3] Quoted by H. Freudenthal (1953), 36.

[4] H. Grassmann (1861).

> Numbers are actually created, determined in their whole being, by the addition of a unit. We reply: this can only mean that it is possible to deduce all the properties of a number, e. g. of 8, from the way in which it is formed by the addition of successive units. But in so doing one concedes what we wanted, namely that the properties of numbers derive from their definitions; and this opens up the possibility of proving the general laws of numbers by the method of producing them [which is] common to them all, whereas the special properties of each of them should be derived from the special way in which each single number is originated by the addition of successive units.

There was a widespread feeling that numbers should be defined by means of the progressive addition of a unit, and that they are characterised as those which are obtained in precisely this way; but it was difficult to expound it in a non circular manner.

For example, Frege criticises Grassmann's manner of proceeding, saying that:

> [Grassmann] tries to make us reach, by means of a definition, the law $a + (b + 1) = (a + b) + 1$, and to this end he writes: 'If $a$ and $b$ are any terms of the fundamental numerical series, we will mean by the sum $a + b$ that term of it by which the formula $a + (b + e) = (a + b) + e$ is true, where $e$ denotes a positive unit.' Against such a manner of proceeding, however, two objections may be raised. In the first place, that this is supposed to explain the sum by means of itself ... In the second place one may object that the sign $a + b$ would prove empty, in the event that there was no term of the natural series with the property assigned.[5]

Schröder, though greatly appreciated by Dedekind, presents the associative property of addition in this way:

$$a + (b + c) = a + b + c = (a + b) + c$$

*Beweis*

$$\begin{aligned} 2 + (4 + 3) &= (1 + 1) + \{(1 + 1 + 1 + 1) + (1 + 1 + 1)\} \\ &= (1 + 1) + \{1 + 1 + 1 + 1 + 1 + 1 + 1\} = \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = \\ &= (1 + 1) + (1 + 1 + 1 + 1) + (1 + 1 + 1) = \\ &= 2 + 4 + 3\,. \end{aligned}$$

...

Ebenso zeigt man, dass $(2 + 4) + 3 = 2 + 4 + 3$ ist. Daraus folgt dann: $2 + (4 + 3) = 2 + 4 + 3 = (2 + 4) + 3$, q. e. d.[6]

[5] G. Frege (1884).

[6] E. Schröder (1874), 26, §41.

In the proof, he inserts an illustration "in words", where it is said that from the concept of addition there follows the possibility of eliminating the brackets where sums of units are concerned, and that the passages above show how this is extended to sums of arbitrary numbers. However, though he treats numbers as sums of units, Schröder defined them by means of the abstraction of equinumerosity.

Grassmann's treatment was the first to systematically exploit the definition of numbers as sums of "+1", although it was still inevitably inaccurate, because in the treatment of recursion the risk is indeed that of circularity. Frege saw only this, and reproached him for it, in the proof of the associative property.

Grassmann forgot to mention explicitly, among the properties of numbers, that the successors of two different numbers are different and that 1 is not the successor of any positive number. This confirms that he most certainly did not have Dedekind's outlook.

## 4.2 Dedekind

Richard Dedekind reflected at length on the concept of natural number starting from 1872 and several times going back to his work on this topic, after interruptions, until 1887[7].

In the interest of simplicity, and of maximum efficacy, we reproduce here his letter of 27 February 1890 to Hans Keferstein of Hamburg, an *Oberlehrer* who had made a number of criticisms of Dedekind (1888) in the *Mitteilungen* of the Hamburg Mathematical Society, showing that, among other things, he had not understood the concept of the chain:

> My dear Doctor,
>
> I should like to express my sincere thanks for your kind letter of the 14th of this last month, and for your willingness to publish my reply. But I would ask you not to rush anything in this matter and to come to a decision only after you have once more carefully read and thoroughly considered the most important definitions and proofs in my essay on numbers, if you have the time. For I think that most probably you will then be converted on all the points to my conception and to my treatment of the subject; and that is just what I would value most, since I am convinced that you really have a deep interest in the matter.
>
> In order to further this rapproachment wherever possible, I should like to ask you to lend your attention to the following train of thought, which constitutes the genesis of my essay. How did my essay come to be written? Certainly not in one day; rather, it is a synthesis constructed after protracted labor, based a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, to our consideration. What are the mutually inde-

[7] For the evolution of Dedekind's investigation see J. Ferreirós (1999; 2nd ed. 2007).

pendent fundamental properties of the sequence $N$, that is, those properties that are not derivable from one another but from which all others follow? And should we devest these properties of their specifically arithmetical character so that they are subsumed under more general notions and under activities of understanding *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions?

When the problem is posed in this way, one is, I believe, forced to accept the following facts:

(1) The number sequence $N$ is a *system* of individuals, or elements, called numbers. This leads to the general consideration of systems as such (§1 of my essay).
(2) The elements of the system $N$ stand in a certain relation to one another; a certain order obtains, which consists, to begin with, in the fact that to each definite number $n$ there corresponds a definite number $n'$, the succeeding, or next greater number. This leads to the consideration of the general notion of a *mapping* $\varphi$ of a system (§2), and since the image $\varphi(n)$ of every number $n$ is again a *number*, $n'$, and therefore $\varphi(N)$ is a part of $N$, we are here concerned with the mapping $\varphi$ of a system $N$ *in itself*, of which we must therefore make a general investigation (§4).
(3) Distinct numbers $a$ and $b$ are succeeded by distinct numbers $a'$ and $b'$; the mapping $\varphi$, therefore, has the property of distinctness, or *similarity* (§4).
(4) Not every number is a successor $n'$; in other words, $\varphi(N)$ is a proper part of $N$. This, together with the preceeding, is what makes the number sequence $N$ infinite (§5).
(5) And, in particular, the number 1 is the *only* number that does not lie in $\varphi(N)$. Thus we have listed the facts that you regard [...] as the complete characterisation of an ordered, simply infinite system $N$.
(6) I have shown in my reply [...] however, that these facts are still far from being adequate for completely characterizing the nature of the number sequence $N$. All these facts would hold also for every system $S$ that, besides the number sequence $N$, contained a system $T$ of arbitrary additional elements $t$, to which the mapping $\varphi$ could always be extended while remaining similar and satisfying $\varphi(T) = T$. But such a system $S$ is obviously something quite different from our number sequence $N$, and I could so choose it that scarsely a single theorem of arithmetic would be preserved in it. What then, must we add to the facts above in order to cleance our system $S$ again of such alien intruders $t$ as disturb every vestige of order and to restrict it to $N$? This was one of the most difficult points of my analysis and its mastery required lengthy reflection. If one presupposes knowledge of the sequence $N$ of natural numbers and, accordingly, allows himself the use of language of arithmetic, then of course it has an easy time of it. He need only say: an element $n$ belongs to the sequence $N$ if and only if, starting with the element 1 and on and on steadfastly, that is,

going through a finite number of iterations of the mapping $\varphi$ (see the end of article 131 in my essay), I actually reach the element $n$ at some time; by this procedure, however, I shall never reach an element $t$ outside the sequence $N$. But this way of characterizing the distinction between those elements $t$ that are to be ejected from $S$ and those elements $n$ that alone are to remain is surely quite useless for our purpose; it would, after all, contain the most pernicious and obvious kind of vicious circle. The mere words "finally get there at some time", of course, will not do either; they would be of no more use than, say, the words "karam sipo tatura", which I invent at this instant without giving them any clearly defined meaning. Thus, how can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements $n$ and the elements $t$? Merely through consideration of the *chains* (articles 37 and 44 of my essay), and yet, by means of these, completely! If I wanted to avoid my technical expression "chain" I would say: an element $n$ of $S$ belongs to the sequence $N$ if and only if $n$ is an element of *every* part $K$ of $S$ that possesses the following two properties: (i) the element 1 belongs to $K$, and (ii) the image $\varphi(K)$ is a part of $K$. In my technical language: $N$ is the intersection [*Gemeinheit*] $1_0$, or $\varphi_0(1)$, of all those chains $K$ (in $S$) to which the element 1 belongs. Only now is the sequence $N$ characterised completely. In passing, I whould like to make the following remark on this point. Frege's *Begriffsschrift* and *Grundlagen der Arithmetik* came into my possession for the first time for a brief period last summer (1889), and I noted with pleasure that his way of defining the non-immediate succession of an element upon another in a sequence agrees in *essence* with my notion of chain (articles 37 and 44); only, one must not be put off by his somewhat inconvenient terminology.

(7) After the essential nature of the simply infinite system, whose abstract type is the number sequence $N$, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system *exist* at all in the realm of our ideas? Without a logical proof of existence there would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay).

(8) After this, too, had been settled, tehre was the question: does what has been said so far also contain a *method of proof* sufficient to establish, in full generality, propositions that are supposed to hold for *all* numbers $n$? Yes! The famous method of proof by induction rests upon the secure foundation of the notion of chain (articles 59, 60 and 80 of my essay).

(9) Finally, is it possible also to set up the *definitions* of numerical operations consistently for *all* numbers $n$? Yes! This is in fact accomplished by the theorem of article 126 of my essay.

Thus the analysis was completed and the synthesis could begin: but this still caused me trouble enough! Indeed the reader of my essay does not have an easy task either; apart from sound common sense, it requires very

> strong determination is necessary to work everything through completely. I shall now turn to some parts of your paper...[8]
> [...] the meaning of these lines [of yours][9] is not quite clear to me. Do they perhaps express the desire that my definition of the number sequence $N$ and of the way in which the element $n'$ follows the element $n$ be propped up, if possible, by an *intuitive* sequence? If so, I would resist that with the utmost determination, since the danger would immediately arise that from such an intuition we might perhaps unconsciously also take as self-evident theorems that must rather be derived quite abstractely from the logical definition of $N$. If I *call* (article 73) $n'$ the element *following* $n$, that is only a new *technical expression* by means of which I merely bring some variety into my *language*; this language would sound even more monotonous and repelling if I had to deny myself this variety and were always to call $n'$ only the *map* $\varphi(n)$ of $n$. But one expression is to *mean* exactly the same as the other.
> [...] Repeating the wish I expressed at the beginning and begging you to excuse the thoroughness of my discussion, I remain with kindest regards. Yours very truly Richard Dedekind.[10]

In comparison with Peano, the logicistic approach of Dedekind is apparent: "the theorems [must] be derived in totally abstract manner from the logical definition of $N$". Dedekind also recognises elsewhere[11] that in Frege's *Grundlagen der Arithmetik* there are " points of very close contact with my paper, especially with my definition (44) [that of the chain of the successors of an element, or Frege's posterity]", and that "the positiveness with which the author speaks of the logical inference form $n$ to $n+1$ shows plainly that here he stands upon the same ground with me"[12].

Frege was to say in his *Grundgesetze* that "system" and "belonging" are not usual concepts in logic and are not reduced by Dedekind to accepted logical notions. But apart from the disagreement as to what "logic" is, Dedekind explicitly expresses his wish to strip these properties of their specifically arithmetical character in such a way that they are subsumed under "the ability of the mind to relate things to things [...] or to represent a thing by a thing, an ability without which no thinking is possible"[13].

The "basic properties" of $N$ are "those properties that are not derivable from one another, but from which all others follow"[14].

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[8] Now Dedekind answers some of Keferstein's objections and misunderstandings, which are not especially interesting here, except for the following.

[9] Keferstein claims that since Dedekind does not stress the fact that $N$ can be considered a sequence in which $\varphi(n) = n'$ immediately follows $n$, the notions of sequence and of successor in a sequence "make an *abrupt* appearance when we come to the definition of ordinal numbers".

[10] R. Dedekind, 'Letter to H. Keferstein', 27 February (1890), in *From Frege to Gödel*, J. van Heijenoort (ed.), Cambridge (MA), Harvard University Press, 1967, 98–103.

[11] R. Dedekind (1888). Quotations and page references are from the Dover edition 1963.

[12] In the second preface to Dedekind (1888), (1893), 42–3.

[13] R. Dedekind (1888), 32.

[14] From the letter to Keferstein, above.

These properties are explicitly listed by way of recapitulation in §8 of the 1888 essay, "Simply infinite systems. Series of natural numbers", in the often-quoted paragraph 71[15]:

> 71. *Definition*. A system $N$ is said to be *simply infinite*, if there is a similar representation $\varphi$ of $N$ in itself such that $N$ proves to be the chain of an element not contained in $\varphi(N)$. We call this element, which will be indicated below with the symbol 1, the *fundamental element* of $N$, and we say that the simply infinite system $N$ is *ordered* by the representation $\varphi$. Preserving the previous notations of the images and of the chains, we can say that the essence of a simply infinite system $N$ is characterised by the existence of a representation $\varphi$ of $N$, and of an element 1 which satisfies the following conditions $\alpha, \beta, \gamma, \delta$:
>
> $\alpha$. $N' \subseteq N$.
> $\beta$. $N = 1_0$.
> $\gamma$. The element 1 is not contained in $N'$.
> $\delta$. The representation is similar.
> ...[16]

These properties constitute a definition, thanks to which "now the succession $N$ is completely characterised". The structure $N$ is defined, and its defining properties at the same time constitute a system of propositions from which all the others follow.

Was this really a definition? The conviction that he had characterised $N$ completely follows from the considerations of point 6 of the letter, where the reasoning is expounded which excludes the systems having "alien intruders".

Later logical research has cleared how this proof, and hence uniqueness up to isomorphisms of the structure satisfying Dedekind's conditions depends on the logic used. The possibility of a plurality of logics occurred neither to Dedekind, nor to Frege, nor to Peano.

The question of how much theory of sets can be a part of logic for mathematics is still being debated today: see, for example, Parsons and Steiner particularly as regards the definition of the system of numbers[17].

## 4.3 Peano

### The New Method

The new method of which the *Arithmetices principia* are an example, followed by *I principii di geometria logicamente esposti*, consists in an original analysis which

[15] To simplify reading we use, as Peano was to do, the symbology later applied, diverging from what was then new and invented *ad hoc*, specifically for inclusion and implication (we save $1_0$, with which Dedekind indicates the chain of 1, i. e. the intersection of all the sets that contain 1 and are closed with respect to the function $\varphi$).

[16] Dedekind (1888), 67.

[17] Ch. Parsons (1965), 180–203, reprinted with a postscriptum, in Ch. Parsons (1983), 150–175; M. Steiner (1975), 28–41.

to him was apt to lay the base of the discipline. The solutions then current were adjudged unsatisfactory by Peano:

> Questions pertaining to the foundations of mathematics, though recently addressed by many people, still lack a satisfying solution. Here the problems arise mainly from the ambiguities of language.[18]

The path Peano followed consisted in paying close attention to the words used:

> And so it is of the highest interest to consider the very words we use. I set myself this task and in this paper I present the results of my investigations, with application to artihmetic.[19]

This examination led him to a sort of formalisation:

> I have attached signs to all the ideas that occurr in the principles of arithmetic, in such a way that any proposition can be expressed with these only signs.[20]

It cannot be said at this point that Peano was aware of the novelty of the axiomatic method, though his work greatly contributed to its success. His ideas on the axiomatic method are expounded only in the context of the discussions of the definition in mathematics, and they consistently, over and over again, always express the same position: for example, in Peano (1921d) he states: "Given an order of ideas of a science, not all of them can be defined" – not the first, for instance, and not equality.

Those which cannot be defined are primitive ideas, but:

> [. . . ] being a primitive idea is not an absolute character, but only relative to the group of ideas which are taken to be known [. . . ] The fundamental properties of primitive ideas are determined by 'primitive propositions', or propositions which are not proven and from which all the other properties of the entities being considered are deduced. Primitive propositions function in a certain way as definitions of primitive ideas. The authors cited [Burali-Forti, Padoa, Pieri, Russell and Whitehead, Korselt, Dickson, Huntington] expounded for the various parts of mathematics many complete systems of primitive ideas and of primitive propositions.[21]

The hypothetical-deductive method, which was emerging mainly from the field of geometry, was motivated by the desire to exploit a multiplicity of interpretations,

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[18] G. Peano (1889a), I: "Quaestiones, quae ad mathematicae fundamenta pertinent, etsi hisce temporibus a multis tractatae, satisfacienti solutione et adhunc carent. Hic difficultas maxime ex sermonis ambiguitate oritur."

[19] G. Peano (1889a), I: "Quare summi interest verba ipsa, quibus utimur attente perpendere. Hoc examen mihi proposui, atque mei studii resultatus, et arithmeticae applicationes in hoc scripto expono."

[20] G. Peano (1889a), I: "Ideas omnes quae in arithmeticae principiis occurrunt, signis indicavi, ita ut quaelibet propositio his tantum signis enuncietur."

[21] G. Peano (1921d), 186. Repr. in *Opere scelte*, vol. II, in particular 432–433.

as explained in Enriques[22]. Peano's inspiration was instead derived from Leibniz, and from the idea of identifying the simple ideas of which all others are compositions:

> One might imagine an alphabet of human thought, and that everything could be discovered and distinguished by means of the combination of the letters of this alphabet.[23]

The goal Peano was expecting to reach, with his method, was the application to proofs of the methods of algebraic manipulation:

> With these notations, every proposition assumes the form and the precision that equations have in algebra, and from propositions so written other propositions are deduced by a process that can be assimilated to equations resolution.[24]

The signs referred either to logic or to arithmetic proper. The use and properties of signs are explained at the beginning in ordinary language: "Arithmetical signs, where they occur, are explained"[25].

Most of the signs can be defined in terms of others:

> I have defined all signs, with the exception of four, which are listed in the explications in §1. If, as I conjecture, these cannot be further reduced, also the ideas expressed by them cannot be defined in terms of ideas known in advance.[26]

Of the four primitive ideas, one is equality, the other three are number, unit and successor. Peano must not have reflected much on his choice, since he subsequently promoted equality among logical ideas.

After the introduction, §1 of *Arithmetices principia* begins with:

*Explicationes*

| | | | |
|---|---|---|---|
| Signo | N | significatur | *numerus (integer positivus).* |
| ″ | 1 | ″ | *unitas.* |
| ″ | $a + 1$ | ″ | *sequens a*, sive *a plus* 1. |
| ″ | $=$ | ″ | *est aequalis* .... |

[22] F. Enriques (1922).

[23] G. W. Leibniz (1875–90), VII, 185.

[24] G. Peano (1889a), I: "His notationibus quaelibet propositio formam assumit atque praecisionem, qua in algebra aequationes gaudent, et a propositionibus ita scriptis aliae deducuntur, idque processis qui aequationum resolutioni assimilantur."

[25] G. Peano (1889a), I: "Aritmeticae signa, ubi occurrunt, explicantur."

[26] G. Peano (1889a), I: "Ita omnia definivi signa, si quatuor excipias, quae in explicationibus §1 continentur. Si, ut puto, haec ulterius reduci nequeunt, ideas ipsis espressas, ideis quae prius notae supponuntur, definire non licet."

The axioms[27] are:

*Axiomata*

1. $1 \in N.$
2. $a \in N. \supset . a = a.$
3. $a, b \in N. \supset : a = b. \equiv . b = a.$
4. $a, b \in N. \supset \therefore a = b. b = c : \supset . a = c.$
5. $a = b. b \in N : \supset . a \in N.$
6. $a \in N. \supset . a + 1 \in N.$
7. $a, b \in N. \supset : a = b. \equiv . a + 1 = b + 1.$
8. $a \in N. \supset . a + 1 \neq 1.$
9. $k \in K \therefore 1 \in K \therefore x \in N. x \in k : \supset_x . x + 1 \in k :: \supset . N \subseteq k.$

from which, excluding those of equality, the famous five are obtained:

1. $1 \in N.$
6. $a \in N. \supset . a + 1 \in N.$
7. $a, b \in N. \supset : a = b. \equiv . a + 1 = b + 1.$
8. $a \in N. \supset . a + 1 \neq 1.$
9. $k \in K \therefore 1 \in K \therefore x \in N. x \in k : \supset_x . x + 1 \in k :: \supset . N \subseteq k.$

At the end of the preface, Peano explains that:

> For the arithmetical proofs, I have made use of the book: H. Grassmann, *Lehrbuch der Arithmetik*, Berlin 1861. Quite useful was also the recent paper: R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig, 1888, in which questions pertaining to the foundations of numbers are keenly investigated.[28]

The debt to Grassmann is apparent in all the proofs by induction and in the recursive definitions of the operations. The reference to Dedekind's essay has thus far remained obscure; Peano was to return to this subject in order better to explain the relationships with his own[29].

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[27] We use some modern symbols for typographical reasons: dots (one or more) are for conjunction, $\supset$ for implication, $\equiv$ for bi-implication (Peano used $=$); we cross a symbol for negation, while the original symbol was a sort of ink fulled rectangle.

[28] G. Peano (1889a), v: "In arithmeticae demonstrationibus usus sum libro: H. Grassmann, *Lehrbuch der Arithmetik*, Berlin 1861. Utilius quoque mihi fuit recens scriptum: R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur."

[29] The content of *Arithmetices Principia* is not restricted to the natural numbers, with the four basic operations and exponentiation, but includes the rationals, the reals and a few theorems on the topology of the line; we are however interested for now only in the axioms.

**The Concept of Number**

In *Sul concetto di numero*, Peano presents a new system, motivated by the fact that in his opinion, from recent works it had been seen that:

> [...] introducing some brief mention of the theory of operations (functions), certain properties of numbers can be made to depend on other, more general [properties], and treated in a more concise form.[30]

Recall that Dedekind's work was based on two concepts, that of system and that of application.

The first section of Peano (1891i) is devoted to "Correspondences", and contains, for example, the notation $a \mid b$ for the class of functions from $a$ in $b$.

The text is much richer than Peano 1889a in comments, collected in "Observations" at the end of each section.

The new system consists of five "Primitive propositions", based on three arithmetical signs:

1. $1 \epsilon N$
2. $+ \epsilon N \mid N$
3. $a, b \epsilon N . a+ = b+ : \supset . a = b$
4. $1 \not\epsilon N+$
5. $s \epsilon K . 1 \epsilon s . s+ \supset s : \supset . N \supset s.$

In the "Observations" Peano discusses the possibility of "defining the unit, the number, the sum of two numbers":

> The common definition of number, which is Euclid's definition, 'number is the aggregate of several units', may serve as clarification, but is not satisfactory as definition. Indeed a child, in its earliest years uses the words *one, two, three*, etc.; subsequently he uses the word *number*; only much later does the word *aggregate* appears in his lexicon. And in the same order, as philology teaches, these words are presented in the development of the Aryan languages. Hence, from the practical viewpoint the problem seems to me to be solved; that is to say, it is pointless in teaching to give any definition of number, this being a very clear idea for the pupils, and any definition having no effect but to confuse them.
>
> On the theoretical side [...] one should first say what ideas are to be used.[31]

If it is supposed that only the ideas represented by logical signs are known, "*number cannot be defined*, because it is obvious that however these words are combined together, one can never have an expression equivalent to *number*".

Further on he repeats that number cannot be defined on the basis of simpler ideas, although various authors give different answers, "since simplicity may be differently

[30] G. Peano (1891i), 87. Repr. *Opere scelte*, vol. III, 81.

[31] G. Peano (1891i), 90–91. Repr. *Opere scelte*, vol. III, 84–85.

understood”; but in his view “the ideas of order, succession, aggregate etc. are just as complex as that of number”[32].

Anyway “if number cannot be defined, those properties can be expounded from which are derived as a consequence all the innumerable and well known properties of numbers”[33].

There follows an illustration of the primitive propositions, with the comment: “The preceding primitive propositions are due to Dedekind op. cit. n. 71”, adding that there is a slight difference in proposition n. 5 (due to its not mentioning the chains).

> These in substance are identical to those expounded by me in the *Arith. Princ.*, except that the introduction of the sign $a \mid b$ makes it possible to simplify its form.[34]
> These propositions express the conditions necessary and sufficient for the entities of a system to be made to correspond univocally to the series of $N$s.[35]

He illustrates their independence, and concludes:

> Between the above, and what Dedekind says, there is an apparent contradiction, which is best dealt with immediately. Here number is not being defined, but its fundamental properties are being expounded. Dedekind instead defines number, and to be precise calls number what satisfies the above conditions. Obviously the two things coincide.[36]

The section on addition concludes with the recognition that:

> The rigorous proofs of these properties, and which we have given, are due to H. Grassmann. They were then repeated by Hankel, Peirce, Dedekind, etc., the last of whom also expounded the principle of mathematical induction, which others made use of ‘nach einer bekannten Schlussweise’ but without explicitly expounding it.[37]

Peano returns to the discussion of the bases of numbers in the expositions for the *Formulario*[38]. Here (Peano 1898f, 1) the primitive ideas are 0 for zero, $N_0$ for whole number, positive or null, and $a+$ for the successor of $a$. The axioms:

1. $0 \in N_0$.
2. $a \in N_0 . \supset . a + \in N_0$.
3. $a, b \in N_0 . a+ = b+ . \supset . a = b$.

[32] G. Peano (1891o), 256. Repr. *Opere scelte*, vol. III, 98.

[33] Peano (1891i), 91. Repr. *Opere scelte*, vol. III, 85.

[34] Peano (1891i), 93. Repr. *Opere scelte*, vol. III, 86–87. Proposition n. 2 means that $+$ is a function from $N$ in $N$, and n. 4 that 1 does not belong to the image of $+$.

[35] Peano (1891i), 93. Repr. *Opere scelte*, vol. III, 87.

[36] Peano (1891i), 94. Repr. *Opere scelte*, vol. III, 88.

[37] Peano (1891i), 96. Repr. *Opere scelte*, vol. III, 90.

[38] G. Peano (1898f).

4. $a \in N_0 . \supset . a+ \neq 0.$
5. $s \in Cls . 0 \in s : x \in s . \supset_x . x + \in s : \supset . N_0 \subseteq s.$

This time ample extracts from Dedekind are repeated, including the whole of section 71.

In his paper *Sul §2 del Formulario, t. II*, in addition to stating that "the nearest work is Dedekind's, 1888"[39], Peano expands on the relationship of this work with his own work of 1889 (Peano 1889a): he states that his is independent, that it was being printed when he saw Dedekind's essay, and that he drew from it only moral support for his conviction of the independence of the axioms. Peano's biographer accepts these declarations at their face value[40]. Without wishing to cast doubt on Peano's honesty, it is legitimate to suspect that he "drew" something more. At the very least his reticence regarding how he arrived at carrying out the reduction to primitive ideas and propositions is disturbing. Peano's general attitude is to formalise the theories which are consolidated, with the current proofs of their theorems. Zermelo too has explained that the discovery of his axioms for the theory of sets was based on an examination of the proofs as they appeared in the literature, and not on a conceptual reflection; but in the case of arithmetic, these proofs – apart from those by induction – were practically non-existent; Grassmann himself does not bring out the properties of the successor which alone, together with induction, are sufficient to constitute the system of axioms. Peano's path from Grassmann to his own axioms seems to have been very rapid, whereas one has the impression that his reflection on the bases of number began after the publication of Peano (1889a).

## Consistency

Ludovico Geymonat (1953) wondered whether Peano had a philosophy of numbers of his own; he was almost disappointed that Peano had not answered Bertrand Russell's criticisms[41].

Russell had several times remarked – e. g. in (1903) and (1919) – that Peano's three basic ideas [natural number, zero, successor] are open to an infinite number of different interpretations [for each progression], each of which satisfies the five basic propositions, and there is nothing in Peano's system that allows us to distinguish among the different interpretations of his basic ideas. Peano presupposes that we know what he means by 'zero' and therefore presupposes that we will not attribute to this symbol the meaning of one hundred or of anything else[42]. Russell had also conceived in (1903) a sort of justification of the *aporia*, remarking that since only the serial and ordinal properties of finite numbers are generally used in mathematics, any one of the successions could be taken as the basis[43]. But since in daily life an

[39] G. Peano (1898e), 84. Repr. *Opere scelte*, vol. III, 243.

[40] H.C. Kennedy (1980), 20.

[41] L. Geymonat (1955), 51–63.

[42] B. Russell (1919), 20.

[43] Russell (1903), XIV §122.

unusual mathematics would not be applied, Russell held that it was necessary to define the single numbers.

Peano certainly knew that "zero" could be realised in many ways; he had tackled the problem on the consistency of the postulates in Peano (1906b), a work seemingly devoted to another topic[44].

The article contains an important contribution to the Cantor-Schröder-Bernstein theorem, largely ignored probably because of the inaccessible language used. In Bernstein's and Schröder's proofs of the theorem, natural numbers are used, assumed to be already defined[45]. Poincaré had asked whether the numbers could not be eliminated from the proof[46].

After a general discussion of the impossibility of eliminating all the mathematical ideas from the proofs, otherwise only the logical signs would remain – and Peano on this occasion shows clearly that he is no logicist – Peano remarks that in this case one may do so.

Natural numbers enter in the proof in the following manner: given a set $a$, with $c \subset b \subset a$, and a bijection $g$ between $a$ and $c$, it must be proved that there is a bijection between $a$ and $b$. To this end $a$ is divided into three subsets one of which is $Z(a \setminus b)$ and another $Z(b \setminus c)$ (plus a remainder), where $Z(u)$ is the union of $u = g^0(u), g''u, g^2(u) = g''(g''u), \ldots, g^n(u), \ldots$, for $n \in \mathbb{N}$:

$$Z(u) = \bigcup\{g^n(u) : n \in \mathbb{N}\} .$$

(The bijection between $a$ and $b$ is defined as $g$ on $Z(a \setminus b)$ and the identity on $Z(b \setminus c)$ and on the remainder $a \setminus (Z(a \setminus b) \cup Z(b \setminus c))$.)

Peano replaces this definition with the one whereby $Z(u)$ is the "The part common to all classes $v$ such that the function $g$ trasforms $v$ into $v$ and contain $u$"[47], in current notation:

$$Z(u) = \bigcap\{v : u \subseteq v \,.\, g''v \subseteq v\} .$$

The solution can only have been inspired by the property of the set of natural numbers of being the intersection of all the classes containing an initial element and closed as regards the successor. Peano was probably thinking of the definition of posterity given by Frege, as well as Dedekind's chains[48]. It should however be pointed out that here the equivalence between the inductive definitions from below and the inductive definitions from above is exploited in a wholly abstract, not

[44] G. Peano (1906b). Also the "Additione" (Peano 1906e) treats with another topic, namely the antinomies.

[45] The later proofs which avoid recourse to numbers are based on Tarski's fixed point lemma.

[46] Cf. H. Poincaré (1905), 815–835; (1906), 17–34, 294–317.

[47] Peano (1906b), 362: "parte comune ad classes $v$ tale que functione $g$ transforma $v$ in parte de $v$, et que contine $u$".

[48] Also Zermelo, who was familiar with Dedekind's chain theory, in 1906 communicates to Poincaré a similar solution.

a numerical context. Dedekind had already pointed out this fact in article 131 of *Was sind und was sollen die Zahlen*.

But in addition, indifferent to conventions as was his wont, Peano went on to discuss a further possibility suggested by his very definition. Suppose that only logic, and not arithmetic, is known, so that the symbols 0, $N_0$ and $+$ are meaningless. If $u$ is not empty, and one of its elements is indicated with 0, if we let $N_0 = Z(\{0\})$ and $x+ = g(x)$:

> And I read 0, $N_0, +$ as in arithmetic [...] we deduce theorems identical with the axioms of arithmetic.[49]

In other words, we obtain a model of the arithmetical axioms.

Having found an interpretation for the arithmetical symbols which satisfies the system of postulates:

> So it is proved (if a proof is necessary) that the axioms of arithmetic [...] do not involve in them a contradiction.[50]

However, after recalling that other examples of entities which satisfy the postulates have been given by Burali-Forti[51] and by Russell, Peano concludes:

> But a proof that the system of axioms for arithmetic, or for geometry, do not involve contradictions is not, to my mind, necessary. For we do not create axioms arbitrarily, we rather assume as axioms the simplest propostions we find, explicitely written or implicit, in every treatise of arithmetic or geometry. Our analysis of the principles of these sciences consists in reducing these common statements to the minimum number of necessary and sufficient conditions. The sistem of axioms for arithmetic, or geometry, is satisfied by the ideas of number, or point, shared by whoever writes of arithmetic or geometry. We have the idea of number, hence the number exists.[52]

He goes on, conceding that the proof of the compatibility of the system of axioms may be useful if they are hypothetical, i. e. not answering to a real fact.

Peano and his pupils take the interpretation of the axioms into consideration only for the proofs of independence, which serve to prove that the axioms are necessary; this is the only metamathematical activity that remains, if we exclude consistency. This check is always made; for instance, in Peano (1898f) the independence

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[49] Peano (1906b), 364–365: "et me lege 0, $N_0, +$ ut in Arithmetica... nos deduce theoremas, identico ad postulatos de Arithmetica."

[50] Peano (1906b), 365: "ita est probato (se proba es necessario), que postulatos de Arithmetica ... non involve in se contradictione."

[51] The reference is perhaps C. Burali-Forti (1896), 34–52.

[52] Peano (1906b), 365: "Sed proba que systema de postulatos de Arithmetica, aut de Geometria, non involve contradictione, non es, me puta, necessario. Nam nos non crea postulatos ad arbitrio, sed nos sume ut postulatos propositiones simplicissimo, scripto in modo explicito aut implicito, in omni tractatus de Arithmetica, aut de Geometria. Nostro analysi de principios de ce scientias es reductione de affirmationes commune ad numero minimo, necessario et sufficiente. Systema de postulatos de Arithmetica et de Geometria es satisfacto per ideas que de numero et de puncto habe omni scriptore de Arithmetica et de Geometria. Nos cogita numero, ergo numero es."

of the axiom of induction is shown adding another system to $N_0$, as it might be $N_0 \cup (i + N_0)$, where $i$ is the imaginary unit, and the operation of the successor is always "+1".

The interpretation $N_0 = Z(\{0\}), x+ = g(x)$ in any case constitutes the recognition that a wholly general simply infinite set of Dedekind is a model of the axioms.

## 4.4 The School of Peano

Among the contributions of Peano's pupils on the problem of the foundations of arithmetic we must recall those of Pieri and Padoa[53].

In (1906d) Mario Pieri proposed an important variation of the axiomatisation of natural numbers, which made it possible to distinguish the property of well ordering from that of induction, or, if you prefer, to isolate the naturals as the first segment of the ordinals, made up, under $\omega$, of the finite ordinals[54].

Pieri recalls the systemitazation reached in the immediately preceding years, quoting Dedekind, Peano and Padoa[55]; from Padoa he assumes the current axioms, which in ordinary language state:

$\alpha$ The successor of a number is a number.

$\beta$ Two numbers which have as successor the same number are equal to each other.

$\gamma$ There is at least one number which does not follow any number.

$\delta$ If a class (of numbers) contains a number which does not follow any other number, and if the successor of each number of the class belongs to the class, then every number belongs to the class.

Padoa had been the first to use neither 0 nor 1 as primitive idea; from the axioms he derived the fact that there is a sole number which is not the successor of any number, and called it 0, as in Peano (1898f).

Pieri, on the other hand, still in natural language, proposed that

I) There is at least one number.

II) The successor of a number is a number.

III) Two numbers, neither of which is the successor of a number, are equal to each other.

IV) In any non-illusory class of numbers there is at least one number which is not the successor of any number of the class.

The axioms are formalised in Burali-Forti[56] Burali-Forti comments that the substitution of the principle of the minimum for the principle of induction is important didactically too, as IV is so much "simpler, i. e. more intuitive, than the principle

[53] Burali-Forti will be mentioned when talking of Pieri (1906d).

[54] M. Pieri (1906d), 196–207.

[55] A. Padoa (1906), 45–54.

[56] C. Burali-Forti, *Logica matematica*, Milano, Hoepli, 1919$^2$, 42.

of induction" (Pieri had called it a surrogate of induction, maintaining that it was preferable "because easier and plainer"). Burali-Forti observes, however, that "we cannot do without the principle of induction in the proof of simple propositions such as the commutative property of sum and product"[57].

Burali-Forti does not perceive that in order to make this substitution Pieri had had to make an appropriate modification of the other axioms regarding the successor, in such a way as to postulate that there is a *unique* number which is not the successor of any other number (otherwise the succession of the ordinals $0, 1, \ldots, \omega, \omega + 1, \ldots$ would be a model of the axioms). The subject was one of the topics of the dispute with Enriques[58], and it was to be Enriques who pointed out this essential detail[59].

But in addition to axiomatisation, Padoa and Pieri also showed interest, in different ways, in the problem of the proof of consistency.

After waiting two years for an answer to the publication of his contribution at the Paris Congress, reworked in Padoa 1900[60], irked by the silence of Hilbert, who had not even attended the section of Padoa's communication, Alessandro Padoa took up his pen to repeat his explanation that "le problème n. 2 [de M. Hilbert] n'etait qu'une causerie"[61].

In his communication at the Congress[62] Padoa had explained very precisely how to analyse the formal structure of a deductive theory; he had fixed the canon of the axiomatic; and now he summarised it, recalling in particular:

> Les postulats doivent être compatible, c'est-à-dire qu'ils *ne doivent pas* se contradire. Pour démontrer la compatibilité d'un système de postulats, *il faut trouver* une interprétation des symbols non définis, qui vérifie simultanément tous les postulats.

An interpretation of this kind, for the system of axioms for relative wholes, is presented as follows:

> III Compatibilité de nos postulats
> Nos *postulats* sont *compatibles*. En effet, voici une *interprétation* de nos *symboles non définis* qui *vérifie simultanément* tous nos *postulats* :
>
> > **entier** signifie nombre entier relatif,
> > et, si $x$ est **entier** quelconque,
> > **suc**$x$ signifie $1 + x$,
> > **sym**$x$ signifie $-x$.[63]

[57] C. Burali-Forti, *Logica matematica*, 1919, 344.

[58] C. Burali-Forti, F. Enriques (1921), 354–365.

[59] For the interaction between induction and the various possible axioms for the successor see L. Henkin (1960), 323–338.

[60] A. Padoa (1902a), 249–56.

[61] A. Padoa (1903), 85.

[62] Padoa repeated his communication at both congresses, preceding it at the philosophy congress with a "logical introduction to any deductive theory". An English translation of this introduction can be found in the van Heijenoort anthology, *From Frege to Gödel*, 1967, 118–123.

Again, as in Peano, the interpretation is made on with the "ideas" we already have.

Padoa remarks in a note that "on constate immédiatement" that the postulates are satisfied. For the postulate of induction the explanation is as follows:

> Soit $a$ un *entier* qui appartient à la classe $u$ [...], alors [...] *tout entier plus grand* que $a$ appartient aussi à la classe $u$ ; et [...] *tout entier plus petit* que $a$ appartient aussi à la classe $u$ ; par suite, tout *entier* appartient à la classe $u$.[64]

A note at the end of Pieri (1904a) shows that Pieri was fairly obviously taking his distance:

> But for all it is vain to seek in the field of Arithmetic itself for a direct and absolute proof of the *compatibility* of the arithmetical axioms, it is not necessarily the case that such a proof cannot be found on the field of pure Logic (setting aside, of course, the grave difficulties encountered to exclude from this field the notion of whole number and the principles of Arithmetic). In this sense, the canon of D. Hilbert (Götting. Nachricht., 1900, p. 264; and Comptes rendus du deuxième congrès intern. Des mathém., Paris 1900, pp. 71–74) seems acceptable to me – as a *desideratum* of a logico-deductive order: that is, that the proof of the compatibility of the arithmetical axioms demands a direct, absolute method; which should be sought, therefore, in the domain of pure logic, without recourse to any other auxiliary system, of which the mathematical existence may be called in doubt; and I cannot share the opinion of Prof. A. Padoa (L'enseignement mathématique, Mars 1903, p. 6) that any effort to obtain "une démostration directe de la non-contradiction des axiomes de l'arithmétique, en appliqant à ce but les méthodes de raisonnement connues, dont on se sert dans la théorie des nombres irrationelles (as in fact Mr. D. Hilbert wishes) *is vain a priori*.[65]

The note, taken up and amply developed in Pieri (1906d), shows that Pieri had reflected on the demand put forward in Hilbert *Über den Zahlbegriff*[66] and in the second problem of Hilbert[67]. According to Hilbert, the proof of the consistency of the theory of numbers, represented by the "constitutive" system of axioms proposed in Hilbert , which must necessarily be carried out "directly" and not relative to other mathematical entities, required "only an appropriate modification of known methods of proof", or an exact redevelopment and appropriate modification of the "known inferential methods of the theory of irrational numbers".

Pieri interprets the inferential methods invoked by Hilbert as those of pure logic, but is aware of the "grave difficulties encountered to exclude from this field the

[63] Padoa (1903), 87.

[64] Padoa (1900) e Padoa (1903), 85–87.

[65] M. Pieri (1904a), 21.

[66] D. Hilbert (1900a).

[67] D. Hilbert (1900b).

notion of whole number", a *caveat* which was to be taken up again by Poincaré in his criticism of Hilbert's *Über die Grundlagen der Logik und der Arithmetik*[68].

In any case, in (1906d) Pieri believed in the possibility of a logical proof of compatibility with a logic that includes the concept of class:

> Je me propose précisément d'établir la *compatibilité des axiomes arithmetiques de R. Dedekind et G. Peano* [sic] *dans un domaine* $\Delta$ *de Logique pure* [...] en raisonnant dans les limites de la Logique des classes.[69]

Pieri takes up the demonstration in Burali-Forti (1896) that the finite classes constitute a model of Peano's arithmetic, making a few fundamental corrections[70]. The definition assumed for "finite" is Dedekind's "non-reflexive" one.

Burali-Forti had shown, without using induction, that if a property is possessed by the empty class, and when it is possessed by a class *u* which is not the Whole possessed by any class following *u* (for an appropriate definition), then the property is possessed by any finite class. There is no intimation that the finite classes from a class of their own. Burali-Forti's and Pieri's proof succeeds by relying on the logical axioms of the *Formulario* and the two principes suivants:

I. *Il y a au moins une classe infinie* (Le Tout est une classe infinie).
II. *Étant donnée une classe infinie, dont les élements sont à leur tour des classes, la classe formée par tous les élements de celles-ci est elle-méme infinie.*

Pieri maintains that he can include these two principles without scruple among the logical axioms:

> [...] car je n'y vois qu'une *détermination* convenable des concepts de *classe* et *représentation*.[71]

But, independently of a discussion of this conviction:

> Je crois avoir établi que le concept de nombre entier, avec ses propriétés fondamentales (y compris le *principe d'induction*) peut être construit sur la Logique des classes de M. Peano, au moyen des propositions I et II.[72]

He concludes with the correct observation that his proof is preferable to Russell's of 1903, since while the latter avoids assumption II, it introduces the principle of induction in the very definition of numbers: greater deductive simplicity is attained at the cost of "renoncer à toute *analyse* de ce principe".

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68 Cf. D. Hilbert (1905), 174–185 and H. Poincaré (1905), 815–835; (1906), 17–34 and 294–317; (1908), 152–71.

69 Pieri (1906d), 203.

70 Replacing it with II below, he has to modify an axiom on the classes used by Burali-Forti, which Whitehead had pointed out as erroneous, and Poincaré (1908), 209, had derided.

71 Pieri (1906d), 207.

72 Pieri (1906d), 207.

Giuseppe Peano at the International Mathematical Congress in Toronto, 1924 – Department of Mathematics G. Peano, University of Torino

# 5

# Geometric Calculus and Geometry Foundations in Peano

Paolo Freguglia

## 5.1 Grassmann's Legacy Background

It is well known that Hermann Günther Grassmann (1809–1877) published his *Ausdehnungslehre* in 1844, a work full of philosophical reflections, written in language that had little to do with the mathematical mentality. In Germany this work made no impression on the mathematical world; but in Italy Giusto Bellavitis read it and began an exchange of letters with Grassmann. It was also appreciated by Luigi Cremona.

When, in 1862, Grassmann published a second edition in which he devoted considerable space to geometrical interpretations and applications, it was no more successful than the first. Grassmann's fundamental idea is proposed in the following definition:

> [Def. 5]: Every expression derived by a system of units (which have not only the absolute unit, that is the real number 1) by means of [real] numbers, named *numbers of derivation of the units* [...], is called *extensive magnitude* [*extensive Grösse*]. For instance the polynomial:
>
> $$\sum_{i=1}^{n} a_i e_i$$
>
> where $a_i$ are real numbers and $e_i$ form a system of units, is an extensive magnitude. A magnitude is called *numerical* if the system is constituted only by the absolute unit.[1]

Grassmann introduces algebraic operations (addition, product) among extensive magnitudes. Actually, Grassmann proposes an abstract and general theory about

[1] H. Grassmann (1862), 415; (1896), 11–12.

the magnitudes. These are seen as the bases of the more general, and not necessarily geometrical, mathematical thought. We would like to present a historical reconstruction and analysis of the theoretical development which, in the context of Peano's works and of the school of Peano, led from H. Grassmann's legacy to the realization of vector calculus and the theory of homographies. Peano was to give to Grassmann's ideas a Euclidean interpretation which broadens the very bases of Euclidean geometry. Indeed, first he introduces oriented segment and other concepts which do not belong to Euclidean geometry. Moreover the interpretation by Peano of the product among extensive magnitudes ("geometrical formations") has projective meanings.

When from the academic year 1885–86 to that of 1888–89 Giuseppe Peano (1858–1932) held the post of lecturer in *Applicazioni geometriche del calcolo infinitesimale* (Geometrical applications of infinitesimal calculus) at the University of Torino, he was well aware of the problems regarding geometric calculus, so that, when in 1887 he published his lectures in a book entitled *Applicazioni geometriche del calcolo infinitesimale* (*Geometrical applications of infinitesimal calculus*), he had in mind Bellavitis, Möbius, Hamilton and Grassmann. In particular, in this first treatise on the subject, he gave importance to Bellavitis' manner of expression, in part because of the influence of his colleague and master Genocchi[2], who was linked to Bellavitis by friendship and respect. But it was in 1888 that Peano published the basic work on these topics: *Calcolo geometrico secondo l'Ausdehnungslehre di Hermann Grassmann, preceduto dalle operazioni della logica deduttiva* (Peano 1888a) (*Geometric Calculus according to H. Grassmann's Ausdehnungslehre preceded by the operations of deductive logic*), a work crucial also for the history of logic. Here he shows that he is decidedly convinced by Grassmann's approach, making reference to the 1844 edition of the *Ausdehnungslehre*.

The student of Peano who devoted himself above all to the studies of geometric calculus was Cesare Burali Forti (1861–1931); but Filiberto Castellano (1860–1919), Tommaso Boggio (1877 1963) and Mario Pieri (1860–1904) also took an interest in the subject.

## 5.2 Peano's *Calcolo geometrico*

In (1888a) Peano presents Grassmann's ideas in an original way: as we have already said, he gives a Euclidean interpretation to the fundamental Grassmannian notions, by limiting his considerations on the nature of the system of units, not beyond the three dimensions. At first, he introduces the notion of *geometrical formation* so:

$$\sum_{i=1}^{n} m_i \tau_i^q$$

[2] The Genocchi–Bellavitis letters are analyzed in G. Canepa, P. Freguglia (1991), 211–219.

where $m_i$ are real numbers and $\tau_i^q$ are $q$-hedrons, with $1 \leq q \leq 4$, which are Peano's interpretations of Grassmann's units. That is, we have for the system of units the following possibilities:

| | |
|---|---|
| 1-hedron | point |
| 2-hedron | segment |
| 3-hedron | triangle |
| 4-hedron | tetrahedron |

With Peano, we call the geometrical formations which have as system of units points (and only points) of the *first kind* (or degree), of the *second kind* if the units are segments, of the *third kind* if the units are triangles and finally of the *fourth kind* if the units are tetrahedrons. We will in general denote a geometrical formation by $\mathrm{F}_q$, where $q$ expresses the kind of formation considered. Even if the possibility of geometrical formations with $q \geq 5$ is not contemplated by Peano, it should be an easy and natural generalization. Veronese, who was a contemporary of Peano, should have done it. But Peano considered only a traditional vision of geometry. Between two geometrical formations we can establish the operation of algebraic addition, which complies with the rules of the algebra of polynomials. But conceptually the more important operation is the *alternated product*, which is introduced by Peano (and by Burali-Forti)[3] thus: if we have two geometrical formations in 3D, $\mathrm{F}_r$ and $\mathrm{F}_s$, the alternated product is a product which complies with the rules of the algebra of polynomials, but without changing the order of the letters which denote points. If $r+s \leq 4$ the product is called *progressive*[4] and expresses the geometrical operation of projection. If $r+s > 4$ the product is called *regressive*[5] and represents the geometrical operation of section. In the plane case, 2D, in the definition we must replace respectively $r+s \leq 3$ and $r+s > 3$. The alternated product is not commutative.[6]

For instance, a segment is represented by the product AB, if A and B are two points which determine the segment. But a segment can be represented by the formation of the first kind B–A. We also have that: $\mathrm{AB} = -\mathrm{BA}$ and $\mathrm{AA} = 0$. Some particular progressive products, equalized to 0, have interesting geometrical interpretations:

| | |
|---|---|
| $\mathrm{ABCD} = 0:$ | points A, B, C, D are coplanar |
| $\mathrm{ABC} = 0:$ | points A, B, C are in a straight line |
| $\mathrm{AB} = 0:$ | points A and B coincide. |

If $\alpha$ is a plane and A a point, the expression $\mathrm{A}\alpha = 0$ means that A lies on the plane $\alpha$.

If a, b, c are straight lines $\mathrm{abc} = 0$ means that three straight lines a, b, c have a point in common.
etc.

[3] C. Burali-Forti (1926), 5–6.

[4] Peano (1888a), 30.

[5] Peano (1888a), 107.

[6] Peano (1888a), 110–111.

By means of their geometric calculus, Peano and Burali-Forti are able to show theorems of projective geometry. Even Bellavitis[7] proposed the applications of his equipollence calculus to elementary geometry and to projective geometry. Peano does not utilize figures. He considers the figures which we find in the traditional treatises of synthetic (elementary and) projective geometry as heuristic representations. Indeed, a figure influences the actual proof of a theorem and the solution of a problem. Instead, Peano and Burali-Forti consider only chains of expressions-identities of their calculus and subsequently they interpret the last expression geometrically. Hence, according to these mathematicians, a theorem is an interpretation, a model of an identity which concludes a sequence of geometrical calculus. Now, for instance, we present (Theorem 5.1) Peano's exposition of Menelaus-Ptolemy theorem[8]. Because Peano states the respective proof in a very concise way, our philological mathematical reconstruction has been necessary. To simplify, we will explain this theorem by utilizing a figure. The reader can see that the role of this figure to prove the theorem is superfluous. But first it is necessary to introduce the following lemma:

**Lemma 5.1** *If* C *is a point which belongs to the straight line* AB*, then we can write:*

$$\mathrm{C} = x\mathrm{A} + y\mathrm{B} \tag{5.1}$$

*where* $x = \frac{\mathrm{CB}}{\mathrm{AB}}$ *and* $y = \frac{\mathrm{AC}}{\mathrm{AB}}$.

The Menelaus-Ptolemy theorem says:

**Theorem 5.1** *If* $\mathrm{AB} \cdot \mathrm{B'C'} = \mathrm{C'}$; $\mathrm{BC} \cdot \mathrm{B'C'} = \mathrm{A'B}$; $\mathrm{AC} \cdot \mathrm{B'C'} = \mathrm{B'}$ *then*

$$\mathrm{AC'} \cdot \mathrm{BA'} \cdot \mathrm{CB'} = \mathrm{BC'} \cdot \mathrm{CA'} \cdot \mathrm{AB'}$$

*that is:*

$$\frac{\mathrm{AC'} \cdot \mathrm{BA'} \cdot \mathrm{CB'}}{\mathrm{BC'} \cdot \mathrm{CA'} \cdot \mathrm{AB'}} = -1\ . \tag{5.2}$$

*Proof.* In virtue of the previous Lemma, because $\mathrm{C'} \in \mathrm{AB}$, $\mathrm{B'} \in \mathrm{AC}$ and $\mathrm{A'} \in \mathrm{BC}$ we have:

$$\begin{aligned}
\mathrm{A'} &= \left(\frac{\mathrm{A'C}}{\mathrm{BC}}\right)\mathrm{B} + \left(\frac{\mathrm{BA'}}{\mathrm{BC}}\right)\mathrm{C} \\
\mathrm{B'} &= \left(\frac{\mathrm{B'A}}{\mathrm{CA}}\right)\mathrm{C} + \left(\frac{\mathrm{CB'}}{\mathrm{CA}}\right)\mathrm{A} \\
\mathrm{C'} &= \left(\frac{\mathrm{C'B}}{\mathrm{AB}}\right)\mathrm{A} + \left(\frac{\mathrm{AC'}}{\mathrm{AB}}\right)\mathrm{B}\ .
\end{aligned}$$

[7] G. Bellavitis (1854), 13–85.

[8] Peano (1888a), 47.

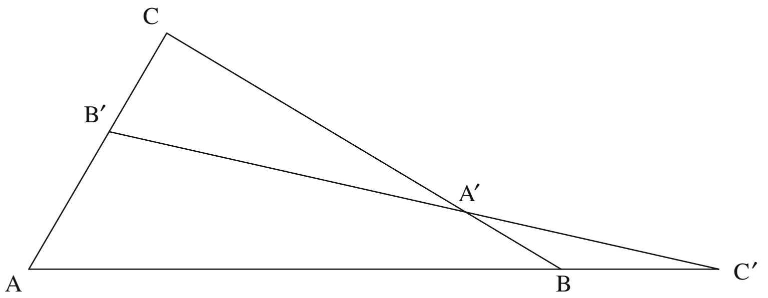


**Fig. 1**

Now if we apply the progressive product $A'B'C'$ we have:

$$A'B'C' = \left[\left(\frac{A'C}{BC}\right)\left(\frac{B'A}{CA}\right)\left(\frac{C'B}{AB}\right) + \left(\frac{BA'}{BC}\right)\left(\frac{CB'}{CA}\right)\left(\frac{AC'}{AB}\right)\right] ABC\,. \tag{5.3}$$

Hence Peano says: "if ABC is different from zero, the necessary and sufficient condition for the three points $A', B', C'$ to be on a straight line, i. e. $A'B'C' = 0$, is that the coefficient of ABC be set to zero"[9]. This proposition is equivalent to saying: if the points $A', B', C'$, which lie respectively on the sides BC, CA, AB of the triangle ABC, are in a straight line, then (5.3) is true and vice versa. In this proposition the symbol "·" denotes both the progressive product (geometrical operation of section) and the arithmetical product.

Peano also analyses Desargues' classic theorem[10] (plane case of homological triangles) and Pascal's theorem. At this point we will examine (according to our philological reconstruction) Desargues' Theorem 5.2.

**Theorem 5.2** *The points* T, U, V, *where the corresponding sides* BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ *respectively intersect, are collinear only if the straight lines* $AA'$, $BB'$, $CC'$ *intersect at the point* S.

*Proof.* Independently of the figure (Fig. 2), we obtain the following steps. We can start from the identity:

$$AB \cdot CD = (ABD) \cdot C - (ABC) \cdot D\,. \tag{5.4}$$

The (5.4) says that the product $AB \cdot CD$ (the result of this product is a fourth kind formation on the plane) between the segments AB and CD (respectively formations of second species) is regressive and hence it expresses an intersection on the plane, i. e. a point, which is represented by the right-hand expression of (5.4). We shall

[9] Peano (1888a), 47: "Supposto ABC non nullo, la condizione necessaria e sufficiente affinchè i tre punti $A'B'C'$ siano in linea retta, ossia $A'B'C' = 0$, è l'annullarsi del coefficiente di ABC. Dunque se i punti $A'B'C'$, che stanno sui lati BC, CA, AB del triangolo ABC sono in linea retta, si ha $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A}\frac{AC'}{C'B} = -1$ e vice versa."

[10] Peano (1888a), 92.

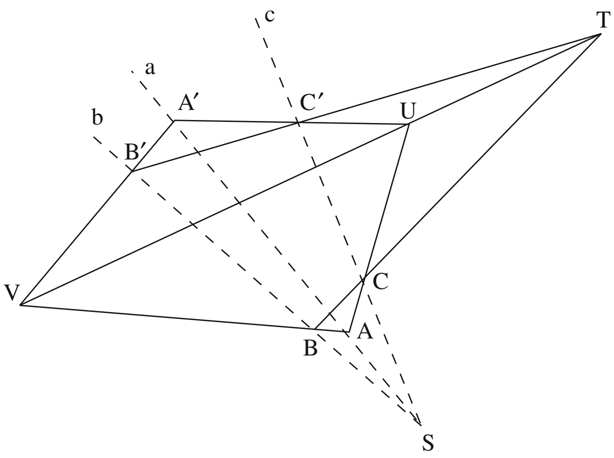


**Fig. 2**

denote the points with capital letters and the segments with small letters. Thus (5.4), putting p in the place of CD, will be written as follows:

$$\mathrm{AB} \cdot \mathrm{p} = \mathrm{Ap} \cdot \mathrm{B} - \mathrm{Bp} \cdot \mathrm{A}\ . \tag{5.5}$$

Now we will take into consideration the following expression:

$$(\mathrm{BC} \cdot \mathrm{a})\ (\mathrm{CA} \cdot \mathrm{b})\ (\mathrm{AB} \cdot \mathrm{c})\ . \tag{5.6}$$

Applying (5.5) to (5.6) we obtain consecutively:

$$\begin{aligned}
&(\mathrm{Ba} \cdot \mathrm{C} - \mathrm{Ca} \cdot \mathrm{B})\ (\mathrm{Cb} \cdot \mathrm{A} - \mathrm{Ab} \cdot \mathrm{C})\ (\mathrm{Ac} \cdot \mathrm{B} - \mathrm{Bc} \cdot \mathrm{A}) = \\
&= (\mathrm{Ba} \cdot \mathrm{Cb} \cdot \mathrm{CA} + \mathrm{Ca} \cdot \mathrm{Ab} \cdot \mathrm{BC})\ (\mathrm{Ac} \cdot \mathrm{B} - \mathrm{Bc} \cdot \mathrm{A}) = \\
&= \mathrm{Ba} \cdot \mathrm{Cb} \cdot \mathrm{Ac} \cdot \mathrm{CAB} - \mathrm{Ca} \cdot \mathrm{Ab} \cdot \mathrm{Bc} \cdot \mathrm{ABC} = \\
&= \mathrm{Ba} \cdot \mathrm{Cb} \cdot \mathrm{Ac} \cdot \mathrm{ABC} - \mathrm{Ca} \cdot \mathrm{Ab} \cdot \mathrm{Bc} \cdot \mathrm{ABC} = \\
&= (\mathrm{Ba} \cdot \mathrm{Cb} \cdot \mathrm{Ac} - \mathrm{Ca} \cdot \mathrm{Ab} \cdot \mathrm{Bc})\ \mathrm{ABC}\ .
\end{aligned}$$

Hence we have the identity:

$$(\mathrm{BC} \cdot \mathrm{a})(\mathrm{CA} \cdot \mathrm{b})(\mathrm{AB} \cdot \mathrm{c}) = (\mathrm{Ba} \cdot \mathrm{Cb} \cdot \mathrm{Ac} - \mathrm{Ca} \cdot \mathrm{Ab} \cdot \mathrm{Bc})\ \mathrm{ABC}\ . \tag{5.7}$$

Dually, i. e. putting capital letters in the place of the small letters and vice versa, we obtain the true expression:

$$(\mathrm{bc} \cdot \mathrm{A})(\mathrm{ca} \cdot \mathrm{B})(\mathrm{ab} \cdot \mathrm{C}) = (\mathrm{bA} \cdot \mathrm{cB} \cdot \mathrm{aC} - \mathrm{cA} \cdot \mathrm{aB} \cdot \mathrm{bC})\ \mathrm{abc}\ . \tag{5.8}$$

Multiplying (5.7) by abc and (5.8) by ABC and adding member to member, we obtain:

$$\mathrm{abc}(\mathrm{BC} \cdot \mathrm{a})(\mathrm{CA} \cdot \mathrm{b})(\mathrm{AB} \cdot \mathrm{c}) + \mathrm{ABC}(\mathrm{bc} \cdot \mathrm{A})(\mathrm{ca} \cdot \mathrm{B})(\mathrm{ab} \cdot \mathrm{C}) = 0\ . \tag{5.9}$$

If in (5.9) we put $\mathrm{a} = \mathrm{B'C'}$, $\mathrm{b} = \mathrm{C'A'}$, $\mathrm{c} = \mathrm{A'B'}$ we will have:

$$\mathrm{A'B'C'}(\mathrm{BC} \cdot \mathrm{B'C'})(\mathrm{CA} \cdot \mathrm{C'A'})(\mathrm{AB} \cdot \mathrm{A'B'}) + \mathrm{ABC}(\mathrm{A'A} \cdot \mathrm{B'B} \cdot \mathrm{C'C}) = 0\ . \tag{5.10}$$

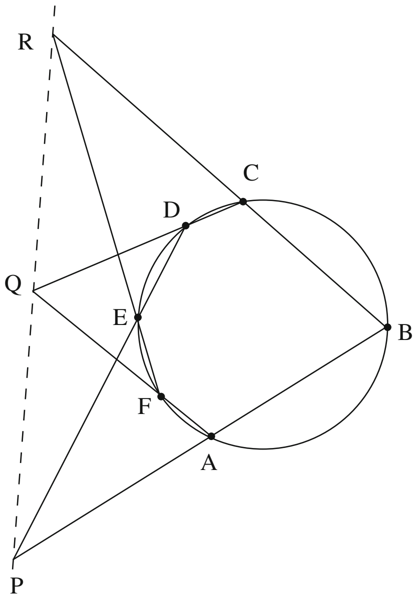


**Fig. 3**

Assuming that (the triangles) ABC and A′B′C′ are different from zero, (5.10) leads us to:

$$(\mathrm{BC}\cdot\mathrm{B'C'})(\mathrm{CA}\cdot\mathrm{C'A'})(\mathrm{AB}\cdot\mathrm{A'B'})=0$$

and

$$(\mathrm{AA'}\cdot\mathrm{BB'}\cdot\mathrm{CC'})=0\ .$$

In conclusion and interpreting this, we have (see Fig. 2):

$$(\mathrm{BC}\cdot\mathrm{B'C'})(\mathrm{CA}\cdot\mathrm{C'A'})(\mathrm{AB}\cdot\mathrm{A'B'})=\mathrm{TUV}=0\ ,$$

i. e. "*the points* T*,* U*,* V*, where the corresponding sides* BC*,* B′C′*;* CA*,* C′A′*;* AB*,* A′B′ *respectively intersect, are collinear, if (*AA′.BB′.CC′) = abc = 0*"*, i. e. "*the straight lines* AA′*,* BB′*,* CC′ *intersect at the point* S".

While Bellavitis' calculus of equipollence is more directly connected with elementary (plane) synthetic geometry, Peano and Bellavitis use their calculus in a more compact way. For instance, Pascal's hexagon theorem (Theorem 5.3)[11] is presented only through the expression (see Fig. 3):

$$(\mathrm{AB}\cdot\mathrm{DE})(\mathrm{BC}\cdot\mathrm{EX})(\mathrm{CD}\cdot\mathrm{AX})=0 \tag{5.11}$$

and in the interpretation of (5.11) the proof of the theorem also remains. The (5.11) is a second degree monomial for X (number of times X appears). Because X is an

[11] Peano (1888a), 95.

unknown variable and A, B, C, D, E, X are points, then (5.11) is a second degree equation, i. e. a conic equation which passes through the previous points. Besides, (5.4) is identically equal to zero if $X = A$, or B, or C, or D, or E. Therefore, when $X = FX$, the six different points A, B, C, D, E, F belonging to the conics determine a hexagon inscribed in these same conics. We can observe that: $(AB \cdot DE) = P$, $(BC \cdot EF) = R$, $(CD \cdot AF) = Q$. In virtue of (5.11) we have: $PQR = 0$, i. e. the points P, Q, R are aligned. Hence:

**Theorem 5.3** *If a hexagon* ABCDEF *is inscribed in a conic section then the three intersections* P, Q, R *of opposite sides belong to the same straight line, i. e.* P, Q, R *are aligned, and vice versa.*

If we write (5.4) in lower case, we obtain, by means of duality, Brianchon's theorem. Hence Peano and his student propose a very synthetic geometric analysis.

## 5.3 The Minimum System

In the context of the Grassmann–Peano trend, Burali-Forti and Roberto Marcolongo developed their studies about vector calculus. Marcolongo was not, in the strict sense, a student of Peano, but with Burali-Forti he had a systematic scientific collaboration. According to Peano and to Burali-Forti, the co-ordinates method constitutes a numerical intermediation for the study of geometrical objects and their properties, while geometric calculus proposes absoluteness and conciseness, and the approach through it is immediate and direct for the study of geometrical problems. However, this calculus does not exclude the use of co-ordinates. We must consider the following important works of these mathematicians:

- *Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla fisica-matematica* (1st edition 1909, 2nd enriched edition 1921).
- *Analyse vectorielle générale, I. Transformations linéaires* (1912), *II. Applications à la Mécanique et à la Physique* (1913).

In the 1909 book they present vector calculus as a structure of vector space with the operations of scalar product and vector product. This is the *minimum system*, while "we have the *general system* when the geometrical formations are introduced and we use the alternated product" (Burali-Forti 1909). In the case of the minimum system, they introduce the notions of gradient, rotor and divergence. The Hamilton quaternions are presented through the techniques of the minimum system in the edition of 1920. The sixth chapter of both editions is devoted to applications to electrodynamics and to Maxwell equations and the Lorentz transformations.

In Chapter IX (first pages) of his 1888 treatise, Peano presents the definition of *linear system*, that is, according to our language, he establishes the notion of *vector space* (on $\boldsymbol{R}$); and he also introduces the notion of *linear transformations*.

Now we will examine how, according to Burali-Forti and Marcolongo[12], the general system contains the minimum system, i. e. from the general system we deduce the minimum system. To this end they introduce the notions of bi-vector and of tri-vector (inherited from Peano and Grassmann). The first is seen as an alternated product of two vectors, so:

$$\hat{u} = \boldsymbol{v}\boldsymbol{w} = (\mathrm{B} - \mathrm{A})(\mathrm{C} - \mathrm{A}) \ .$$

A bi-vector is a geometrical object determined by three elements: a modulus, a plane position and an orientation. mod $\hat{u}$ is the area of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$. These two vectors also determine a plane position and the direction depends on the property $\boldsymbol{u}\boldsymbol{w} = -\boldsymbol{w}\boldsymbol{u}$. We define *index of a bi-vector* $\hat{u}$, and we denote it by $|\hat{u}$, a vector such that:

1. $\mathrm{mod}|\hat{u} = \mathrm{mod}\,\hat{u}$,
2. if $\hat{u} \neq 0$ the direction of $|\hat{u}$ is normal to the plane position of $\hat{u}$,
3. the orientation of $|\hat{u}$ is such that the number $\mathrm{O}\hat{u}(|\hat{u})$ is either positive or zero.

If $\boldsymbol{v}$ is a vector, we define *index of a vector* $\boldsymbol{v}$, and we will denote it thus: $|\boldsymbol{v}$, the bi-vector $\hat{u}$ which has $\boldsymbol{v}$ as index.

A tri-vector is, in its turn, an alternated product of three vectors. If we consider three vectors: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, we can show that the product $\mathrm{O}\boldsymbol{u}\boldsymbol{v}\boldsymbol{w}$ is a real number (which represents the affine volume of the oriented parallelepiped constructed through the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$). In particular we put $\Omega = \mathrm{O}\boldsymbol{i}\boldsymbol{j}\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ determine the usual unitary orthogonal right system. $\Omega$ is called unitary tri-vector.

Hence we can establish the following definitions:

**Definition 5.1** If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors, we will call scalar product (or inner product) the operation: $\boldsymbol{u} \times \boldsymbol{v} = \frac{\boldsymbol{u}(|\boldsymbol{v})}{\Omega}$.

We can show that: $\boldsymbol{u} \times \boldsymbol{v} = \mathrm{mod}\boldsymbol{u} \cdot \mathrm{mod}\boldsymbol{v} \cdot \cos(\boldsymbol{u}, \boldsymbol{v})$.

**Definition 5.2** If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors, we will call vector product (or outer product) the operation: $\boldsymbol{u} \wedge \boldsymbol{v} = |(\boldsymbol{u}\boldsymbol{v})$.

So, from the general system of geometrical calculus we have obtained the minimum system.

The theory of homographies is well presented and analysed in the treatise *Analyse vectorielle générale*. Burali-Forti and Marcolongo define a homography as a linear operator which transforms vectors into vectors.

Certainly, also from a foundational point of view, the Grassmann–Peano statement shows its importance. But nevertheless, as we have seen, the goal of geometrical calculus, or vector calculus or homographies goes beyond a "philosophical" justification of the bases of geometry. In fact the applications to mathematical physics and to the geometry of transformations is very important and, notwithstanding the matrix techniques, even today proposable. Of course, suitable adaptations

[12] C. Burali-Forti, R. Marcolongo (1909), 27–45.

could be made. Analogously, *mutatis mutandis*, as the reappraisal of the quaternion techniques has been made. In the Preface (to *Analyse vectorielle générale*) Burali-Forti and Marcolongo[13] again explain that their calculus is not, in itself, seen as tachygraphical in comparison with the concepts of calculus which are introduced by means of Cartesian co-ordinates. In fact, geometrical synthetic calculus is, according to these mathematicians, "intrinsic, or absolute, or autonomous". In our opinion what is important is its absoluteness, because this calculus presents concepts and procedures which disregard the particular Cartesian co-ordinate system. We may say that we have an "intrinsic invariance".

## 5.4 The Axiomatic Foundations of Geometry

Peano's works, concerning in particular the study of axioms of Geometry, are: *I principi di geometria logicamente esposti* (1889d) and 'Sui fondamenti della geometria' (1894c). Historically, these works lie between Pasch's *Vorlesungen über neuere Geometrie* (1882) and Hilbert's *Grundlagen der Geometrie* (1899). In Peano's first work, his explicit aim is to propose the fundamental propositions of position geometry as theorems, i. e. the propositions about order and belonging. Peano's goal is to establish the smallest possible set of geometrical concepts as primitive (from which we can define other concepts), i. e. not attributable to preceding concepts, and the smallest possible set of axioms (from which we can show the other propositions). Peano proposes the concepts "point" and "segment" as primitive, even if this second concept is expressed through the ternary relation[14] $c \epsilon ab$, which means "$c$ is between (the points) $a$ and $b$". In particular Mario Pieri (1860–1913) was the student of Peano who followed this field of research[15].

The set of all points is denoted by **1**. If $a$ and $b$ are points, we can define the following two sets of points which we call *radii*:

$$a'b =: \mathbf{1}.[x \epsilon] \quad (b \epsilon ax)$$
$$ab' =: \mathbf{1}.[x \epsilon] \quad (a \epsilon xb)$$

i. e. $a'b$ is the set of $x$ points, in such a way that $b$ is inside the segment $ax$. Meanwhile, $ab'$ is the set of $x$ points, and $a$ is inside the segment $xb$. The axioms introduced by Peano were taken from Pasch's *Vorlesungen*, although he presents some interesting additions and modifications.

*Peano's preliminary axioms about segments*:

**0.1**. $a, b \epsilon \mathbf{1}. \supset .ab \epsilon \mathrm{K}\mathbf{1}$

i. e. if $a$ and $b$ are points, then the segment $ab$ is a set of points.

Besides "…The symbol $=$ between two points denotes their identity", hence:

[13] C. Burali-Forti, R. Marcolongo (1912), IX (footnote 4).

[14] Typographically we find in the Peano's text the symbol $\epsilon$ instead of the modern $\in$.

[15] See e. g. M. Pieri (1900a); (1901a), 367–404; (1908a), 345–450.

**0.2**. $a, b, c, d \in \mathbf{1}.a = b.c = d :\supset .ac = bd$
i.e. if $a$, $b, c, d$, are points and $a$ is equal to $b$ and $c$ is equal to $d$, then the segment $ac$ is equal to the segment $bd$.

*Peano's straight line axioms*:

1. $\mathbf{1}- = \Lambda$ , that is $\mathbf{1} \neq \emptyset$
2. $a \in \mathbf{1}. \therefore\supset x \in \mathbf{1}. x- = a : - =_x \Lambda$ (that is: "If $a$ is a point then a point $x \neq a$ exists")
3. $a \in \mathbf{1}. \supset: aa = \Lambda$
4. $a, b \in \mathbf{1}. a- = b :\supset. ab- = \Lambda$
5. $a, b \in \mathbf{1}. \supset. ab = ba$
6. $a, b \in \mathbf{1}. \supset. a - \in ab$
7. $a, b \in \mathbf{1}. a- = b :\supset. a'b- = \Lambda$
8. $a, b, c, d \in \mathbf{1}. c \in ad. b \in ac :\supset. b \in ad$
9. $a, d \in \mathbf{1}. b, c \in ad :\supset: b = c. \cup. b \in ac. \cup. b \in cd$
10. $a, b \in \mathbf{1}. c, d \in a'b :\supset: c = d. \cup. d \in bc. \cup. c \in bd$
11. $a, b, c, d \in \mathbf{1}. b \in ac. c \in bd :\supset. c \in ad$

The *plane geometry* is obtained by adding the following axioms:

12. $r \in \mathbf{2}. \supset: x \in \mathbf{1}.x - \in r. - =_x \Lambda$ where $\mathbf{2}$ is the class of straight lines.
13. (aside from the respective symbolism): "If the points $a$, $b$, $c$ are not collinear and if $d$ belongs to the segment $ad$, and a point $f$ belonging to $ac$ exists, then $e$ belongs to $bf$. This means that the segment $ac$ and the prolongation of the segment $bc$ have a common point".
14. "The points $a$, $b$, $c$ are not collinear. If $d$ is a point of $bc$ and $f$ of $ac$, then a point $e$ which is common to the segments $ad$ and $bf$ exists".
15. Peano 3D *space geometry* is obtained by adding the following axioms:
16. $h \in \mathbf{3}. \supset: x \in \mathbf{1}. x - \in h. - =_x \Lambda$ where $\mathbf{3}$ is the set of planes.

By means of the previous axioms, Peano can prove Desargues's theorem on homological triangles (space case):

**Theorem 5.4** *(Desargues's theorem on homological triangles, space case, see Fig. 4) If among ten points e, a, b, c, d, h, m, n, f, x the first four are not coplanar and nine of their following relations:*

$$h \in ad,\ h \in bc,\ e \in am,\ n \in ed,\ n \in mh,\ f \in mb,\ n \in cf,\ a \in xb,\ c \in xd,\ e \in xf$$

*are verified, then the remaining relation will be verified.*

It is interesting to observe the different language which Peano utilizes in the Theorem 5.4 in comparison to that used in the Theorem 5.2. In fact, we have two different foundational points of view.

The last axiom of Peano's 3D space geometry is:

17. Given a plane $h$ and a point $a \notin h$, and given a point $b$ which belongs to a radium $a'h$ (from a point $a$ to some points of $h$), then every point $x$ of the space lies on $h$, or the segment $ax$ intersects $h$ or the segment $bx$ intersects $h$.

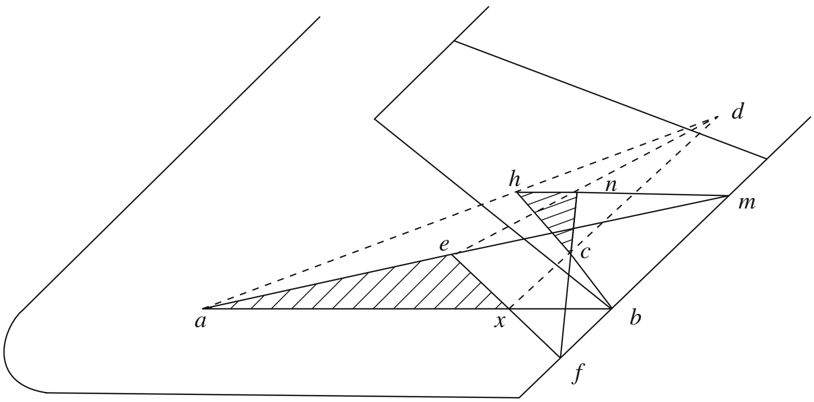


**Fig. 4**

At first Peano in 'Sui fondamenti della geometria' (1894c) drew on the axioms previously presented in *I principi di geometria logicamente esposti* (1889d). Afterwards, he presented the axioms about motion in geometry (i. e. congruences, etc.). Pasch in the *Vorlesungen* (1882) also presented axioms on the congruences. Peano considers geometrical motion as an affine transformation which he characterizes by some axioms.

Peano's geometrical-epistemological point of view is not unlike the Euclidean position, in comparison with Hilbert. But, the introduction of a symbolic and logical apparatus and the axiomatic enrichment enables him to establish a rigorous analysis and presentation of the basis of geometry. In this case, the comparison with Pasch's *Vorlesungen* is very important.

Now, let us look at the controversy between Peano and Giuseppe Veronese (1854–1917)[16]. In this period, Veronese was the Italian mathematician who presented an alternative approach to the foundations of Geometry which we can see in his treatise *Fondamenti di geometria a più dimensioni e a più specie di unità rettilinee, esposti in forma elementare. Lezioni per la Scuola di Magistero in Matematica* (1891). He begins as follows:

> Empirical remark: When we consider outside bodies, which appear to us by means of our senses, in particular touch and sight, we associate these bodies with the object that contains them. We call this *outer environment* or *intuitive space* where each body occupies a certain *place*.[17]

[16] D. Palladino, § 5 *Appendix*, in M. Borga, P. Freguglia, D. Palladino (1985), 244–250.

[17] G. Veronese (1891), Parte I, Libro I, Capitolo I, § 1, 209: "*Oss. emp.* Alla presenza dei corpi fuori di noi, che ci appariscono per mezzo dei sensi, specialmente per mezzo della vista e del tatto, è collegata l'idea di ciò che li contiene, e, si chiama *ambiente esterno* o *spazio intuitivo*, nel quale i corpi occupano ciscuno un determinato *posto* o *luogo*." See also P. Cantù, *Giuseppe Veronese e i fondamenti della geometria*, Milano, Unicopli, 1999.

To determinate a $n$D Euclidean space $S^n$ ,Veronese introduces the following (methodological) principle:

If $S^{n-1}$ is a $(n-1)$D space, then a point $x$ exists outside this. By means of Peano's symbolism we can write:

$$s \in (\boldsymbol{n-1}) . \supset : x \in \mathbf{1} . x- \in s- =_x \Lambda \ .$$

Hence:

The hyperstar of straight lines which pass by the point $x$ and intersect the *hyperspace* $S^{n-1}$ constitutes $S^n$.

This generating space principle has an intuitive validity if we use, according to Veronese, projective and descriptive geometry: from a $S^n$, through consecutive projections we arrive at $S^2$ (or $S^3$) and the geometrical properties of the plane $S^2$ or of the space $S^3$ are intuitively verifiable. This principle – as Veronese explicitly says – is based on the following logic or epistemological law:

> Given a determined thing A, if we do not establish that A is the set of all possible things by us considered, then we can think of another thing which does not belong to A (which is outside it) and independent of A.[18]

Peano observes that previous proposition is equivalent to:

> Given a set A, if A does not contain all objects, then A does not contain all objects.[19]

Peano's remark is understandable, but Veronese's epistemological point of view is different from Peano's foundational ideas. Peano's aim is to propose an optimal and rigorous system of axioms, while Veronese wants to present a large number of theorems and of properties and to study the *general space* which is a container with a very high number of dimensions. Besides, Peano's point of reference is Pasch while Veronese's point of reference is Riemann, so their respective approaches are necessarily different.

## 5.5 Conclusion

First, Peano's geometrical calculus theory is a general theory which is of intrinsic mathematical interest and which is also applied to mechanics and to physics. Peano's contributions, which come from an elaboration of Grassmann's ideas, consist in an Euclidean interpretation of relative concepts. Moreover, in this context,

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[18] G. Veronese (1891), 13–14: "Data una cosa A determinata, se non è stabilito che A è il gruppo di tutte le cose possibili che vogliamo considerare, possiamo pensarne un'altra non contenuta in A (vale a dire fuori di A) e indipendente da A."

[19] G. Peano (1892n), Review G. Veronese (1891), 144: "Data una classe A, se essa non contiene tutti gli oggetti, allora essa non contiene tutti gli oggetti". Actually in this review Peano blasts the Veronese's treatise.

Peano proves important fundamental theorems of projective geometry. For this reason, Peano's geometrical calculus has an implicit foundational interest. In our opinion, the protophysical role of Euclidean geometry in Peano's works is essential and decisive. He distinguishes *position geometry* from Euclidean geometry, and from a theoretical point of view, it is appropriate. In his 'Sui fondamenti della geometria' the congruence theory is well determined and regulated. Classical geometry constitutes the crucial model for the study of the foundations of geometry. Even Hilbert, deep down, takes Euclid into account[20]. During this period, we have many proposals of systems with different essential or primitive notions and axioms. Hence, we can observe "equivalent theories" for the foundation of elementary geometry, and in this way we have a "theoretical relativism" regarding the choice of primitive elements and fundamental axioms. This is epistemologically and historiographically[21] very important[22].

[20] Cf. U. Bottazzini (2000), 123–148.

[21] We must observe that in a letter (11 February 1894) to Frege, Pasch abandons his foundational idea of 1882 and writes "The rigorous foundation of geometry should be preceded by the foundation of arithmetic."

[22] See V. Benci, P. Freguglia, Modelli e realtà: una riflessione sulle nozioni di spazio e di tempo, Bollati Boringhieri, Torino, 2011.

# 6

# The *Formulario* between Mathematics and History

Clara Silvia Roero

*Dal libro di Lebesgue potrà risultare un rigo, o mezza pagina.*
G. Peano to G. Vitali, 3 April 1905

For almost twenty years, from 1888 to 1908, Peano devoted all his energies to formulating and realising a project, which throughout his life he was to acknowledge as one of the most important results of his mathematical research[1]. This was the *Formulaire de Mathématiques*, a huge collection of mathematical propositions expressed in symbols, especially written with his own logic, capable of concentrating in a single volume the knowledge of mathematics of his time. To this end, Peano founded a journal and invited to collaborate on it scholars, assistants, colleagues at the University and at the military Academy, teachers and other mathematicians in Italy and abroad. His total commitment to this enterprise was also accompanied by his voluntary decision to leave his post as Professor of infinitesimal Calculus at the military Academy[2], keeping only his University position, and by the purchase of

[1] Peano (1916e), 8: "Formulario Mathematico t. V, a. 1908 è un trattato più completo dei miei precedenti di Calcolo infinitesimale, incluse le parti introduttorie, Aritmetica, Algebra e Geometria." (The Formulario 5th edition of 1908 is a treatise of infinitesimal Calculus, more complete than my previous ones, including the preliminary parts of Arithmetic, Algebra and Geometry).

[2] From the documents held at the military Academy and from Peano's correspondence with his collaborators it emerges that he himself presented his resignation in order to devote himself full time to other occupations, such as the publication of the *Formulario*. Cf. Peano's letter to F. Amodeo, 22 February 1901, in F. Palladino, N. Palladino (2006), 252. Cf. also the minutes of 18 February 1901 in Torino Military Academy Archive: "Il Prof. Cav. Peano ha presentato le dimissioni da insegnante presso questa Accademia Militare ed il Ministero della Guerra con suo dispaccio del 14 corrente le ha accettate. Nel dare la partecipazione di questa Superiore disposizione esprimo il dispiacere vivissimo da me provato che venga a mancare all'Accademia l'opera efficace del

a printing press so that he could set up the text himself, in view of the difficulties that the mathematical symbols created for the publishing houses[3].

This paper will highlight the genesis and the aims of the project, the main sources of inspiration, the stages of realisation and some differences among the five editions, the difficulties and the limits remarked by Peano himself in the course of his work, the controversies and debates on the front of research and teaching, and finally some of the cultural influences and repercussions.

There are four main areas in which the enterprise of the *Formulaire* was gradually built up: mathematics, logic, history and language, i.e. the 'mathematical vocabulary' and how it could be spread. The first and second dictated the contents and made possible the organisation, while the history and language had the role of creating a context in which for the first time these concepts, definitions, theorems, methods, etc. had been conceived, and of communicating them exactly and rigorously to the widest possible public. Here we shall dwell above all on mathematics and on its history, since logic and language have already been the subject of thorough historical articles to which we make reference[4].

## 6.1 The Genesis and Aims of the *Formulario*

From the 1890s, influenced by research on the foundations of mathematics, by the discovery of logic, and by the philosophy of positivism which pervaded the sciences of the time, in his *Rivista di Matematica* Peano stressed the importance of collecting and cataloguing the theorems, with a view to the development of new research:

> It would be extremely useful to publish the collections of all the theorems now known referring to given branches of the mathematical sciences in such a way that the scholar need not consult this collection in order to know how much had been done on a given point, and whether his research was new or not. Such a collection, extremely difficult and lengthy in ordinary language, is notably facilitated by the use of the notations of mathematical logic; and the

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Prof. Peano per l'istruzione degli Allievi, ed il prestigio che ad essa procurava la spiccata personalità del Prof. Peano e la reputazione da lui acquisita nel mondo scientifico." Hence the affirmations of H.C. Kennedy on "an undesired interruption" of his teaching at the military school do not correspond to the facts. (H.C. Kennedy (1980), 101).

[3] Peano had bought the printing press from the typography of F. Faà di Bruno, who had been his professor at the University. He installed it in his villa in Cavoretto, and for three months went to a workshop in Turin to learn the art of typographical composition, and he paid three workers to help him with the printing at his own home.

[4] U. Cassina (1955), 244–265, 544–574; N.I. Styazhkin (1969), 276–282; G. Lolli, 'Quasi alphabetum: logica ed enciclopedia in G. Peano', in G. Lolli (1985), 49–83; F.A. Rodriguez-Consuegra (1991), 91–113; W.O. Quine (1986), 33–43; I. Grattan-Guinness (1986), 17–31; (ed.) Philip E.B. Jourdain (1999); E.A. Zaitsev (1994), 367–383; I. Grattan-Guinness (2000), 219–267. As far as concern the language cf. C.S. Roero (1999), 159–182 and E. Luciano, C.S. Roero (2005), LX–LXV.

> collection of the theorems on a given subject perhaps becomes less long than its bibliography.[5]

Like many of his contemporaries, he was fascinated by the *Encyclopädie der Mathematischen Wissenschaften* which was being published in Leipzig, because it provided "an excellent collection of results"[6], and he read with interest the archives and collections of formulae published by W. Laska and J.G. Hagen and subsequently, in Italy, by E. Pascal[7].

In addition Peano, who in his youth had cultivated classical studies and a passion for history, loved to read the works of mathematicians of the past and articles and books on the history of mathematics and on logic. Like others before him, at the end of one century or the beginning of the next[8], he too decided to leave to posterity an encyclopaedic work, written in condensed form by means of symbols, useful not only as a source of inspiration for new studies and research, but as a basis of comparison with other axiomatic arrangements or treatises, as a catalogue of results in mathematics and in history, and finally as a bibliographical list.

On 25 August 1894 he described the features of the project, which he was in the process of writing, to Felix Klein:

> And here I pause for a moment, in order to draw your attention to mathematical Logic, and to the *Formulario*. Mathematical logic with a very limited number of signs (7 used, and these can be reduced still further) has succeeded in expressing all the logical relations imaginable between classes and propositions; or rather the analysis of these relations has led to the use of these signs, with which everything can be expressed, even the most complicated relations, which it is difficult and laborious to express with ordinary language. But its advantage is not limited to the simplification of writing; its usefulness lies especially in the analysis of the ideas and reasonings that are carried out in mathematics. Meanwhile, to illustrate its usefulness, the *Formulario* of mathematics is being printed. [...] Each of the parts dealt with must contain all the propositions, theorems and definitions, to which reference is made. Since, once the *Formulario* is well advanced, anyone may wish to get up to date on science, on a given point already dealt with in the *Formulario*, he need only

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[5] G. Peano (1892k), 76: "Sarebbe cosa della più grande utilità il pubblicare delle raccolte di tutti i teoremi ora noti riferentisi a dati rami delle scienze matematiche, sicché lo studioso non abbia che a confrontare siffatta raccolta onde sapere quanto fu fatto sopra un dato punto, e se una sua ricerca sia nuova ovvero no. Una siffatta raccolta, difficilissima e lunga col linguaggio comune, è notevolmente facilitata servendoci delle notazioni della logica matematica; e la raccolta dei teoremi su un dato soggetto diventa forse meno lunga della sua bibliografia."

[6] Peano (1916e), 1.

[7] Cf. Peano (1892k), 77; W. Laska (1888–1894); J.G. Hagen (1891); E. Pascal, Repertorium der höheren Analysis, Leipzig, Teubner, 2 vol., 1910.

[8] This is the case of Luca Pacioli's *Summa* (1494), of the collections of classics prepared by Christophorus Clavius and by Francesco Maurolico, with comments and developments of contemporaries in the 16th century, of the *Cursus seu Mundus Mathematicus* (1690) by Claude François Milliet Descales, of Christian Wolff's *Elementa Matheseos universae* at the beginning of the 18th century, etc.

> look it up and will find all the known propositions there. [...] Quotations and historical information make it possible to compare books where the questions are discussed at greater length. This *Formulario* could not be put into effect in ordinary language. But it becomes possible, and relatively simple with the notations of mathematical logic. These not only condense the writing, but show that many propositions which, in ordinary language, seem to be distinct, are transformed into symbols in the same way, and hence are actually one and the same proposition. I could cite many so-called theories, which, translated into symbols, *vanish*; they vanish only apparently because the name of an old idea has been changed. Suffice it to say that a fair number of parts of Dedekind's theory of *fields*, *modules*, are simply logical propositions and hence are contained in part I of the *Formulario*. I am now going in for the composition of this *Formulario*; and every day a new part is translated into symbols. For the translation into symbols of a part of mathematics is no easy matter at present; one must examine all the ideas that appear in it, and reduce them to the smallest possible number. So far it is the parts of Analysis that are most easily transformed; in these, in fact, are found a smaller number of fundamental ideas; but let us hope that before very long Geometry too will be analysed, and translated into symbols. I am working on the publication of the *Formulario*, and am happy to have the collaboration of a number of colleagues, and of several recent young graduates, who have taken on the various parts with enthusiasm. But my efforts are directed at making known these methods to the scientific world.[9]

Peano asked his German colleague about the possibility of forming other systems of symbols, in order to represent precisely all mathematical ideas, systems that would be easier and better than the one he was developing, and he concluded:

> So I will not cease to work on this, until the importance of the question is sufficiently recognised.

He had already presented his project for the *Formulario* in France, at Caen, at the conference of the *Association française pour l'avancement des sciences* and on 6 November 1894, writing to Camille Jordan, he reaffirmed the great importance of the logic:

> C'est la première fois qu'on a appliqué la logique mathématique à l'analyse d'une question de mathématiques supérieures ; et cette application est, selon moi, la chose plus importante de mon travail. Mais les symboles et les opérations de la logique exigent du temps pour être appris ; et ma démonstration est peu connue. M. Mie a publié un article explicatif dans les Mathematische Annalen, Bd. 43, p. 553. Mais ensuite ont parus plusieurs travaux sur le même sujet, sans y ajouter rien de nouveau (sauf quelque inexactitude), et sans faire mention de mon travail. Je regrette cela, parce que je crois que la logique

[9] G. Peano to F. Klein, 25 August 1894, in M. Segre (1997), 119–120, repr. E. Luciano, C.S. Roero (2008), 91–92. Cf. also G. Peano to G. Frege, 10 February 1894 and 14 October 1896, in C. Mangione (1983), 146–147, 158–162.

> mathématique apportera des grands avantages dans l'analyse des questions difficiles.[10]

Peano's conception of logic and its goals emerges in numerous points of the *Formulario*. Logic is an instrument not only of expression, but above all of research, a cognitive tool, which makes it possible to "analyse" mathematics and "make it rigorous"[11]. With this tool one can examine the principles of arithmetic, of geometry and of any other theory, in order to single out the primitive ideas and those which are derived, the definitions, the axioms and the theorems:

> L'idéographie, qui résulte de la combinaison des symboles logiques avec les algébriques, a été bientôt appliquée par divers Auteurs. Dans quelques travaux elle sert seulement à énoncer sous forme plus claire des théorèmes. En général elle est l'instrument indispensable pour analyser les principes de l'Arithmétique et de la Géométrie, et pour y démêler les idées primitives, les dérivées, les définitions, les axiomes et les théorèmes.[12]

Though he understood all the importance of theoretical studies of logic, Peano insisted that his aim in the *Formulario* was the application of logic to mathematics:

> Comme vous le remarquez bien, mon but est d'appliquer la logique aux sciences mathématiques. Je comprends toute l'importance des études théoriques sur la logique ; mais, vu la vastité de ces études, je préfère de diriger mes forces du côté de l'application.[13]

In his school Pieri chose the image of the microscope to define the capabilities of the 'mathematical' logic tool, in the hands of the researcher, presenting it in Paris at the international philosophy conference in 1900:

> La Logique mathématique ressemble à un microscope propre à observer les plus petites différences d'idées, différences que les défauts du langage ordinaire rendent le plus souvent imperceptibles, en l'absence de quelque instrument qui les agrandisse. Quiconque méprise les avantages d'un tel instrument, notamment dans cet ordre d'études (où souvent l'erreur résulte d'équivoques

[10] G. Peano to C. Jordan, 6 November 1894, in M.T. Borgato (1991), 96.

[11] Cf. Lolli (1985), 49–83 and C. Cellucci (1993), 73–138.

[12] Cf. G. Peano (1901b), v; cf. also Peano (1913i), 48: "Symbolismo da alas ad mente de homo; sed suo usu exige studio et labore. Illos que, per defectu de exercitio, judica que symbolismo es ligamen, non es obligato ad adopta illo. Nos strue novo instrumento, et non destrue instrumentos existente. [...] Auctores adopta, in parte, symbolos de *Formulario mathematico*. In aliquo casu, illos varia aut forma aut extensione de symbolos; et introduce numeroso symbolo novo. Ratione de divergentia es scopo differente de symbolismo in *Formulario* et in libro de Auctores. In *Formulario*, logica-mathematica es solo instrumento pro exprime et tracta propositiones de mathematica commune; non es fine ad se; logica-mathematica es explicato in 16 pagina; uno hora de studio suffice pro cognosce quod es necessario in applicationes de isto novo scientia ad mathematica. Libro de nostro Auctores tracta logica-mathematica ut scientia in se, et suo applicationes ad theoria de numeros transfinito de vario ordine; quod exige symbolismo multo plus amplo."

[13] G. Peano to L. Couturat, 1 June 1899, in E. Luciano, C.S. Roero (2005), 19.

> et de malentendus dans des détails en apparence insignifiants) se prive à mon avis, de propos délibéré, du plus puissant auxiliaire qu'on possède aujourd'hui pour soutenir et diriger notre esprit dans les opérations intellectuelles qui réclament une grande précision.[14]

Peano was also well aware of the difficulties and the limits he was facing, and of the necessity of a collective action and revision, whose amplifier was the *Rivista*, and he had involved new young graduates and researchers:

> Naturally every new work presents hitches. Here and there gaps are still easily perceived; but the *Rivista di Matematica* always gladly welcomes all the additions and corrections that may be indicated; thus in a short time this *Formulario* will have reached the desirable perfection.[15]

## 6.2 The Influence of Leibniz

*Omnis humana ratiocinatio signis quibusdam sive characteribus perficitur.*
G.W. Leibniz

In the various editions of the *Formulario* it is clear that the source of inspiration for Peano was the ambitious idea of the *Characteristica universalis*, conceived by G.W. Leibniz[16], which gave him the basis for an *Enciclopedia generalis*:

> Leibniz a énoncé, il y a deux siècles, le projet de créer une écriture universelle [...] Il dit : "Ea si recte constituta fuerit et ingeniose, scriptura haec universalis aeque erit facilis quam communis, et quae possit sine omni lexico legi, simulque imbibetur omnium rerum fundamentalis cognitio." À la solution de ce problème a contribué d'abord le développement de l'écriture algébrique, qui s'est beaucoup perfectionnée après Leibniz. Au moyen des signes $+$, $-$, $=$, $>$, etc., des parenthèses, et des lettres de l'alphabet, elle permet d'écrire en symboles quelques propositions. Mais ce qui a le plus contribué à la solution du problème, c'est la nouvelle et importante science qu'on appelle Logique mathématique, et qui étudie les propriétés formelles des opérations et des relations de logique.[17]

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[14] M. Pieri (1901a), 382.

[15] G. Peano to Felix Klein, 25 August 1894, in M. Segre (1997), 120: "Naturalmente ogni lavoro nuovo presenta degli inconvenienti. Qua e là si scorgono ancora facilmente delle lacune; ma la *Rivista di Matematica* accoglie sempre con piacere tutte le aggiunte e correzioni che verranno indicate; sicchè fra non molto questo *Formulario* avrà raggiunta la perfezione desiderabile."

[16] Cf. G.W. Leibniz, *Scientia generalis, Characteristica, Calculus universalis*, in G.W. Leibniz (1999); E. Pasini (1995), 385–412; M. Mugnai (1996), 61–88; M. Mugnai, E. Pasini (2000); E. Luciano (2006), 525–531.

[17] Peano (1894g), 3.

The insistence with which Peano acknowledged his cultural debt to Leibniz was not simply a manner of speaking in order to promote his contemporaries' approval of the project, by pointing to the German philosopher and mathematician as his distinguished predecessor. In his personal library were found the editions of Leibniz's works and manuscripts, as well as many notes with passages taken from various of Leibniz's essays, all testifying to a constant, profound interest throughout his life[18]. Moreover it was Peano who suggested to his student Vacca that he should go to Hanover in 1899 to examine Leibniz's unpublished works, and who acted as go-between with Couturat for the continuation of the French philosopher's historical research, the fruits of which appeared in the two massive volumes *La logique de Leibniz* and the *Opuscules et fragments inédits de Leibniz*[19].

In 1896 Peano several times returned to the fulfilment of Leibniz's dream of producing "une spécieuse générale ou une manière de langue ou d'écriture universelle, où toutes les vérités de raison seraient réduites à une façon de calcul"[20]. Like Leibniz, he held that it was one of the main problems to be faced and that it had as much value in science as the discovery of the telescope and the microscope[21]. The echoes of Leibniz's remarks are almost identical to the German's original words, and are scattered throughout the writings of Peano and his collaborators. They insisted that mathematical logic is not simply a tachigrafy, but a "sort of calculus" which not only made the exposition of mathematics simpler and clearer, but made it possible to distinguish primitive ideas, derived ideas, definitions, axioms and theorems:

> Car toutes les recherches qui dépendent du raisonnement se feroient par la transposition de ces caractères, et par une espèce de calcul ; ce qui rendroit l'invention des belles choses tout a fait aisée. Car il ne faudroit pas se rompe la teste autant qu'on est obligé de faire aujourd'huy, et neantmoins on seroit asseuré de pouvoir faire tout ce qui seroit faisable, ex datis.[22]

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[18] In Peano's manuscript notes it is clear that the topics dealt with from Leibniz's works concerned logic, the international language, minimum simplified Latin, binary arithmetic, analysis, the theory of determinants, the encyclopaedia, the history of mathematics, etc.

[19] G. Vacca (1899), 113–116; (1903), 64–74; L. Couturat (1901); (1903); G. Vailati (1901a), 148–159; (1901b), 103–110.

[20] G. Peano (1896i), 169.

[21] G. Peano (1896b), 1: "Il [Leibniz] énonce ce projet dans son premier travail, ou, comme il l'appelle, dans son "essai d'écolier" intitulé "De arte combinatoria a. 1666". Il fixe le temps nécessaire à la former : "aliquot selectos homines rem intra quinquennium absolvere posse puto". Il trouve cette découverte plus importante que l'invention des télescopes et des microscopes ; elle est l'étoile polaire du raisonnement ... Dans ses dernières lettres il regrette "que si j'avois été moins distrait, ou si j'étois plus jeune, ou assisté par des jeunes gents bien disposés, j'espérerois donner une manière de cette spécieuse (p. 701)". Il dit aussi (p. 703) "J'ai parlé de ma spécieuse générale à Mr. Le Marquis de l'Hospital, et à d'autres ; mais ils n'y ont point donné plus d'attention que si je leur avois conté un songe."

[22] Cf. G.W. Leibniz, 'Linguae Philosophicae Specimen in Geometriam edendum', 1680, in G.W. Leibniz (1999), vol. 4, 155.

> But the main use of the symbols of logic is that they facilitate reasoning [...] So symbolism is clearer; make it possible to develop series of reasonings when the imagination would be quite unable to maintain itself without symbolic aid.[23]
>
> This ideography, which derives from the studies of mathematical logic, is not just a conventional abbreviated way of writing, or tachigraphy. Thus our symbols do not represent words, but ideas. So one must write the same symbol, where one finds the same idea, whatever the expression used in ordinary language to represent it : and different symbols must be used, where one finds the same word, which because of its position, represents different ideas.[24]

Proudly, Peano said he had been successful:

> Nous avons donc la solution du problème proposé par Leibniz. Je dis "la solution" et non "une solution", car elle est unique. La Logique mathématique, la nouvelle science composée de ces recherches, a pour objet les propriétés des opérations et des relations de logique. Son objet est donc un ensemble de vérités, et non de conventions [...] Ces résultats sont merveilleux, et bien dignes des éloges de Leibniz à la science qu'il avait deviné.[25]

Peano also shared the concept of the role that the history of mathematics had in research, both as a source of inspiration to increase the *ars inveniendi*, as Leibniz maintained, and to attribute the "paternity" of the results. History is useful for the instruction of the young in that it makes the study of mathematics more attractive. This was why scientific literature should be side by side with humanistic literature. The historical introduction is, in fact, of great utility to show that mathematics is not a static nor a dogmatic science, that it is not a set of rules or formulae, but the fruit of a development of human thinking. Hence in the *Formulario* great attention was given to identifying the authors of concepts, theorems, methods, symbols, to the research dome by earlier scholars, to the history of signs, etc. Peano was in the habit of introducing into his university lectures much information about the history of mathematics. His assistant Vacca, who inherited the same passion for history, recalled:

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[23] G. Peano (1915j), 170, 172: "Ma l'utilità principale dei simboli di logica si è che essi facilitano il ragionamento [...] Perciò il simbolismo è più chiaro; permette di costruire serie di ragionamenti quando l'immaginazione sarebbe interamente inabile a sostenere se stessa senza aiuto simbolico."

[24] G. Peano (1896j), 565–583: "Questa ideografia, che deriva dagli studii di logica matematica, non è solo una scrittura convenzionale abbreviata, o tachigrafia. Poiché i nostri simboli non rappresentano delle parole, ma delle idee. Si dovrà pertanto scrivere lo stesso simbolo, ove trovasi una stessa idea, qualunque sia l'espressione usata dal linguaggio ordinario per rappresentarla: e si dovranno usare simboli distinti, ove trovasi una stessa parola, che, a causa della sua posizione, rappresenta idee distinte."

[25] G. Peano (1896b), 2.

> He knew by heart, and willingly repeated, long pages of Newton's *Principia* and of the two famous letters from Newton to Leibniz. He admired (with Abel) Cauchy's clear volume, the *Cours d'Analyse* (1821) [...] His lectures, different every year, represented a constant effort to arrive at clearer expositions. I remember the first part of the 1903 course, begun following Bonaventura Cavalieri's geometric methods of the indivisibles. I remember the lectures on the theory of irrational numbers, illustrated with the Fifth Book of Euclid, the lectures on the rectification of curves, starting from the expositions of Archimedes. Finally, I remember the reading of the pages of Galileo and of Torricelli on the fall of heavy bodies, and the lectures on the calculus of variations, in which he interpreted in a new form the classic memories of Euler and of Lagrange.[26]

The editions of 1897–99, 1901 and 1903 are rich in historical Notes which accompany the various sections. Peano pointed out their importance in his *Rivista* and as editor, together with Vacca and Vailati, stressed the need for absolute precision if they were to be really useful:

> The historical indications, both as to the propositions, and to the notations, always useful, are especially so in the *Formulaire*, because they rest the reader a little, and show better the importance of the propositions, and often the advantage of the ideography. But they too require much labour in order to have some value. The indications found in the books of the past generations, and also in some modern books [...] have no precision at all [...] In consequence one has to go back to the origin of the quoted passages; and the quotations in *Formulaire* are accompanied by precise indications, so that anyone can easily compare the quoted passage; and often the cited passage is repeated. This as far as it was possible; because in the *Formulaire* too, some quotations await greater specification. Note too that the historical indications contained in the *Formulaire* do not pretend at all to go back to the first origin of the P [proposition] in question; but simply to indicate an Author where it is found. A further study will be able to substitute for them other citations relative to a more ancient period. After all, here use has been made of the historical research of M. Marie, M. Cantor, of those contained in the *Intermédiaire des Mathématiciens,* and in various other works mentioned.[27]

From 1901 the philological-linguistic aspect of mathematical terms was also taken into account, with the gradual insertion of a mathematical Dictionary[28].

The *Formulaire* was intended to be to all effects an encyclopaedia in which the reader would be able to find mathematics, history and philology. The readership to

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[26] G. Vacca (1933), 97–99.

[27] G. Peano (1898e), 83, 85–86.

[28] The *Dizionario di Matematica* begun by Peano and Vailati included the section on Logic and was presented in 1901 in Leghorn to the teachers of the Italian Associazione Mathesis. Cf. Peano (1901j), 160–172.

whom it was mainly addressed was composed of university colleagues, students and teachers. In this context of the encyclopaedia too the inspiration of Leibniz can be seen[29]:

> A *Dictionary of Mathematics*, that is, a collection of terms that are met with in the current mathematical works, together with the remarks that serve to specify the meaning or meanings of every term, such as the etymology, the history, the definition, when possible, will be a work which is useful as much from the scientific as from the didactic point of view. The multiplicity of terms used to represent a single idea, and the multiplicity of meanings in which a single term has been used are an all too widespread and well-known inconvenience. The dictionary will be capable of guiding every author individually in the choice of the most suitable terms for his work.[30]

With regard to the international language too, Peano acknowledged his cultural debt to Leibniz, whose echo can be heard in many passages:

> Leibniz went into this subject in depth and at length, but published nothing. His study has remained buried in the Hanover library to this day. Some of his manuscripts were discovered and published first by Vacca in the RdM, then by Couturat in *Opuscules et fragments inédits de Leibniz*, Paris a. 1903, p. XVI–682, which contains Leibniz's study, invaluable for the assembling of the Philosophical Dictionary. If analysis and synthesis should come together in the future, like two teams of miners working in a tunnel from its opposite ends, then Leibniz's "Rational Language" and "Universal Characteristic" will do likewise." See also: H. Diels, *Über Leibniz und das Problem der Universalsprache*, Berlin, Sitzung. d. Akademie, a. 1899 p. 579.[31]

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[29] G.W. Leibniz, 'Initia et Specimina Scientiae Generalis', 1679, n. 86, in G.W. Leibniz (1999), 360: "Consilium de Encyclopaedia condenda, velut Inventario cognitionis humanae condendo in quod referantur utiliora, certiora, universaliora et magis sufficientia pro reliquis omnibus determinandis; quacunque sive in melioribus autoribus extant sive inter homines in primis certa vivendi genera sectantibus adhuc latent, additis semper rationibus eorum quae fiunt originibusque inventionibus. Quod opus non nimis erit prolixum ... Hujus operas usus erit ut occurratur confusioni librorum eadem repetentium, paucaque interdum utilia sub magna farragine obruentium, si sit Basis aliqua ad quam omnia imposterum nova per modum supplementorum referri possint."

[30] G. Peano (1901j), 1: "Un Dizionario di Matematica, cioè una raccolta dei termini che si incontrano nelle opere matematiche attuali, insieme alle osservazioni che servono a precisare il significato o i significati d'ogni termine, quali l'etimologia, la storia, la definizione, quando è possibile, riuscirà un lavoro utile tanto sotto l'aspetto scientifico quanto sotto quello didattico. La moltiplicità dei termini usati per rappresentare una stessa idea, e la moltiplicità dei significati in cui è usato uno stesso termine costituiscono un inconveniente troppo diffuso e ben noto. Il dizionario potrà guidare individualmente ogni autore nella scelta dei termini più opportuni pel suo lavoro." G. Peano (1903f), viii: "Le *Formulaire* maintenant, par l'abondance des propositions, des indications historiques et bibliographiques joue le rôle d'une Encyclopédie. Toutes les idées du *Formulaire* sont introduites par des définitions régulières. Dans plusieurs théories les propositions sont accompagnées de la démonstration (et aussi de plusieurs démonstrations)."

[31] G. Peano (1903d), 80, 82.

## 6.3 The Stages of the Production of the *Formulario*

From correspondence with his students and collaborators and from materials recently rediscovered on Peano's personal library and in the archives of Vacca, Vailati and Cassina, it is possible to reconstruct the stages of production and the changes in editorial choices[32]. Chronological scanning of the contents and the indication of the contributions of the various authors and of the main novelties can be seen in Tables 1–5, which summarises the information obtained from the *marginalia* and from the galley proofs[33]. From the very first, the *Formulaire* was conceived as a collective work. Peano stressed this aspect in the first edition and in 27 points laid out the rules that future collaborators would have to follow in writing the chapters.[34] Payment consisted in the annual subscription to his *Rivista di Matematica*.

[32] The rediscovery of the catalogue of Peano's personal library has made it possible to trace the volumes belonged to him and to identify those with autograph marginal notes. Some of these volumes were sold between 1935 and 1938 to the Library of the Department of Mathematics of the University of Milan (BDM Milano) to finance the journal *Schola et Vita*; others are held in the *Fondo Cassina* of the Library of the Department of Mathematics of the University of Parma, and yet others in the *Lascito Peano* at the Cuneo Civic Library. Cf. E. Luciano, C.S. Roero (2008), 86–88 and website www.peano2008.unito.it. Peano's correspondence, donated in 1954 to the Cuneo Civic Library by Cassina and Gliozzi, is available in digital format on the cd-rom of C.S. Roero, N. Nervo, T. Armano (2002). In the Library of the G. Peano Department of Mathematics at the University of Torino the acquisition has recently been made of documents, books, correspondence and manuscripts which were in possession of G. Vacca, of M. Gliozzi and N. Mastropaolo, and this is currently being catalogued. Among the papers there are the proofs of the *Formulario* sent by Peano to Vacca, with his corrections and marginal notes. Among the books there are some editions of the *Formulario* with Vacca's marginal notes and comments.

[33] The volumes of the *Formulario* with Peano's autograph notes are reproduced on the dvd of C.S. Roero (2008). The texts in question are Peano (1894g)* *Notations de logique mathématique (Introduction au Formulaire)*, Torino, Guadagnini (BDM Milano: Op. I 46); (1895r)* *Formulaire de Mathématiques, tome 1 publié par la Rivista di matematica*, Torino, Bocca (BDM Milano: Op. I 46); (1895r)** *Formulaire de Mathématiques, tome 1 publié par la Rivista di matematica*, Torino, Bocca (BDM Milano: Op. A 138); (1897b)* *Formulaire de Mathématiques*, t. II, n. 1, *Logique mathématique*, Torino, Bocca (BDM Milano: Op. A 140); (1897e)* *Formulaire de Mathématiques*, t. II §1, "Logique mathématique", Turin, Bocca-Clausen (BDM Milano: Op. I 46); (1898h)* *Formulaire de mathématiques*, t. II, §2 Aritmetica, Torino, Bocca (BDM Milano: Op. I 46); (1899b)* *Formulaire de Mathématiques*, t. II, n. 3, Torino, Bocca (BDM Milano: Op. I 46); (1901a)* *Formulaire de Mathématiques*, t. III. Turin, Bocca-Clausen (BDM Milano: Op. I 46); (1903f)* *Formulaire mathématique*, tome IV de l'édition complète, Torino, Bocca (BDM Parma: Per 0831709 999653); (1906g)* *Formulario mathematico ed. V. Indice et Vocabulario*, Torino, Bocca (BDM Milano: Op. A 139); (1908a)* *Formulario Mathematico*, t. V, Torino, Bocca (BDM Milano: Op. A 141). The examination of all Peano's autograph notes and of those of his followers in copies of the various editions of the *Formulario*, together with the critical reading of the correspondence of those who were actively engaged in the undertaking, and with the analysis of the manuscript notes given by Peano to Vacca, when the latter was his assistant in Turin, with the proofs annotated by Peano and by Vacca, will cast new light on the whole story and may perhaps clarify some of the problems rightly pointed out in Grattan-Guinness (2000), 262–267.

[34] G. Peano (1895aa), 'Preface', iii–vii.

The fact that the *Formulaire* was work in progress was repeated on many occasions:

> Quelques théories sont déjà suffisamment analysée, mais dans d'autres cas il n'y a que l'énoncé de quelque propositions, pour indiquer la place où un collaborateur de bonne volonté pourra insérer une théorie complète. Ces lacunes sont inévitables dans notre publication, car le Formulaire, toujours en construction, procède par perfectionnements successifs ; d'un coté l'on ordonne et complète des théories, déjà publiées, de l'autre on introduit les esquisse de théorie nouvelles, qu'on perfectionnera dans la suite.[35]

A constant exchange with readers was activated both in the *Rivista di Matematica*, and in the Prefaces. In the preface to the 2nd edition, for example, composed on 11 August 1897, Peano inserted a note to the effect that the most important propositions were indicated with an asterisk[36]. Among the most important novelties of this edition is the presence of many original passages from historical sources (not only books, but also manuscripts and correspondence and the introduction of the concepts of ordered couple, of the symbols $F$ for the defined function and $\exists$ for the existential quantifier, the change in the axioms of arithmetic of 0 in place of 1, and the statements of his theorems on the systems of several differential equations. In the 3rd edition, published on 1 January 1901, the historical parts, edited by Vacca and by Vailati, are further amplified with passages in Greek from Aristotle, Euclid, Apollonius and Diophantus[37]. Peano here introduces the analytical functions, the transformations of vectors (1895q) and completes the parts on the derivatives, on the integrals and on the complex numbers. In the section on the primitive propositions of arithmetic is inserted the variation proposed by Padoa in Rome in 1900[38], which makes it possible to reduce their number.

Peano was well aware of the limitations and of the difficulties to be overcome, but he was also optimistic and trusted in the contribution that many collaborators would offer. He was reading, studying, making comparisons with other layouts, convinced that he was offering *one* possible structure on which mathematics could be based, not *the only one* possible. On this point he wrote to his assistants in 1902 and in 1905:

> These difficulties which are encountered in Mathematical Logic are not worrying. I remember that in '88 I introduced the sign '$x \ni$' in another form, as well as following Schröder, but I met with difficulty in the sign of deduction, still only one both for the general and for the individual Props. This difficulty was resolved in 1889 with the distinction of the two signs '$\in$' and '$\supset$'. Other difficulties, which made certain transformations impossible with fixed rules,

[35] G. Peano (1898f), ii. Cf. also Peano (1901b), vii: "Le Formulaire est toujours en construction ... on trouvera ici la place d'une proposition, déjà écrite en symboles, à peu près comme on trouve la place d'un mot dans un dictionnaire."

[36] Peano (1897b), 2.

[37] Cf. G. Vailati to G. Vacca, 16 December 1899, in G. Lanaro (1971) and Vacca to Vailati, 6 April 1905, in E. Luciano, C.S. Roero (2008), 101.

[38] A. Padoa (1901a).

> were solved with the introduction of the signs '$\iota$' and '$\overline{\iota}$', which I think was done in 1890. Another difficulty presented itself and was solved with the two signs '$f$' and '$F$'. It may be that there are others. But I have great faith that these too will find a way to become quite clear, and in consequence to be solved. In this difficulty Burali found himself, in his article in the RdM. It is intimately linked to the theory of the relations or classes of couples, studied by Russell, to which he and Whitehead attribute great importance, I do not know whether rightly or wrongly, because I have not yet formed a clear idea of this study.[39]

> I am not worried about the application that my contemporaries may make of it, and much less those in the future. However, once the first astonishment has passed, it is possible that many may think it useful to create the same tool, or a similar tool, to express similar ideas [...] In the system of symbols adopted in the Formulario, the possibility emerges to recognise the apparent letter from its position with regard to the three signs '$\supset\ \ni\ /$'. But in another ideography – and several are possible – and I would be very glad to see others arise – which would not at all mean damaging competition – it may be that the variable letter is accompanied by a single sign.[40]

In the review of Whitehead and Russell's *Principia*, which he often praised in his writings, Peano affirmed:

> Symbolism gives wings to the human mind, but its use requires study and effort. Those who for lack of exercise regard symbolism as a liability, are not obliged to adopt it. We are building a new tool and we are not destroying the existing tools.[41]

He was always fond of the product he had conceived, as were his pupils, who did not attempt to follow other authors, such as Russell and Hilbert, but in the end were still walking the path trodden by the Peano in the 1940s and '50s.

In the 4th edition (1903) the novelties regarded continuous fractions, calculus of differences, probability, elementary geometry, the applications of differential geometry to twenty or so curves, the singularities of real curves, the definition of the area of a curved surface (Peano-Schwarz) and the definition of the length of a curve. The part regarding the vocabulary of mathematics and the biographical information was greatly expanded, edited by Vacca. In 1906 a limited print run of 100 copies of the 5th edition, in the international language encouraged by Peano, *latino sine flexione*, which in June 1908 appeared on its definitive form. Here we find the bibliography of texts on mathematical logic published between 1900 and 1908[42].

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[39] G. Peano to G. Vacca, 28 December 1902, in *Fondo Peano-Vacca*, Dep. Mathematics Peano University of Torino.

[40] G. Peano to G. Vailati and G. Vacca, 29 November 1905, in *Fondo Peano-Vacca*, Dep. Mathematics Peano University of Torino.

[41] G. Peano (1913i), 48.

[42] Among the authors quoted, outside Peano's circle, are L. Couturat, E. Huntington, P.H. Jourdain, E.H. Moore, B. Russell, O. Veblen, A.N. Whitehead, A.T. Shearman.

While Peano originally aspired to provide an encyclopaedia of higher mathematics, in symbolic form, what he actually succeeded in producing was a compendium of elementary mathematics, i.e. the notions that were imparted in Italy in the first two years of university study of mathematics. Apropos the suppression of parts of advanced mathematics, which had been inserted in previous editions, such as the theory of algebraic numbers edited by Fano, the motivation Peano put forward was that ideas that are not precise and theories that are not consolidated cannot be represented, and that the main objective was rigour:

> But rigour does not proceed by degrees to the infinite. The books of one generation do not destroy, but rather complete the books of the preceding generation. The solution of some obscure point is not given by books of great bulk, but by a new combination of known ideas. [...] The Formulario, fairly complete as regards the mathematics of past centuries, is very incomplete for the modern, living authors. In fact the reduction of a theory to symbols demands the analysis of all the ideas, the enunciation of all the hypotheses, a lengthy and often difficult business. Many modern theories are not sufficiently rigorous. The Formulario does not contain all the propositions already reduced to symbols. There are many other applications of Mathematical Logic to different questions made by many authors who adopt the symbols and the methods of Mathematical Logic.[43]

Moreover Peano was constantly using his *Formulario* in his teaching, as is testified both by the syllabi of his courses (which are simply the index of the *Formulario*), and by the handouts prepared by his students Meriano and de Finis. He also held an open free course on mathematical Logic in the academic year 1906–07, during which he developed the following topics:

> Ideas of Logic that arise in mathematics. Equality, deduction. Syllogism, according to Aristotle. Commutative and associative properties of multiplication and logical addition according to Leibniz. Distributive properties according to Lambert. Algebra of Logic, according to Boole and Schröder. Characteristics of mathematical definitions. Primitive ideas and derived ideas. Characteristics of mathematical proofs. Primitive propositions and Theorems. Analysis of the principles of Arithmetic, according to Dedekind, and Russell. Analysis of the principles of Geometry, according to Pieri and Hilbert. Theory of groups of points, cardinal numbers and transfinite ordinal numbers, according to Can-

[43] Peano (1908a), xii–xiii: "Sed rigore non procede per gradu, usque ad infinito. Libros de uno generatione non destrue, sed completa libros de generatione praecedente. Solutione de aliquo puncto obscuro non es dato per magno libro, sed per aliquo novo combinatione de ideas noto [...] Formulario, satis completo pro mathematica de seculos praeterito, es multo incompleto pro auctores moderno et vivente. Nam reductione in symbolos de aliquo theoria exige analysi de omni idea, enunciatione de omni hypotesi, quod es longo et saepe difficile. Plure theoria moderno non es satis rigoroso. Formulario non contine omni propositione jam reducto in symbolos; existe numeroso alio applicatione de Logica-Mathematica ad differente quaestiones, per plure Auctore, que adopta symbolos, vel methodos de Logica-Mathematica."

> tor. Antinomies which are found, according to Russell, and others. Attempts by Borel, Hadamard, Poincaré, Lebesgue, Baire, Jourdain to solve them.[44]

Also in the courses on infinitesimal Calculus, after the usual explanation of the theorems, Peano went on to the translation of each passage into symbols. There were conflicting reactions from the student audience. At the end of the handouts, which he had prepared, Igino De Finis concluded:

> With this we have finished expounding what the syllabus demands, or more correctly has our beloved professor said we have learned to read the *Formulaire Mathématique*. I think it is my duty to beg pardon of all my colleagues if these few pages have not answered their aim. You all know in what conditions of time I have had to re-order my notes, translate them and then with my own hand write them on lithographic paper. It would be folly and vain pretension on your part if you thought that you would find here that truly original mark of the Lectures that our famous Professor gave us. Only someone who has had the honour of following them all and with the concentration necessary to understand properly such a delicate subject can appreciate how great is the subtlety and the sublime art that transpires from the wise words of Prof. G. Peano; and be rightly proud to have had such a teacher.[45]

On the other hand C. Botto, who attended as a student of Engineering, expressed great perplexity:

> The textbook which Peano followed had instead become the *Formulaire* of which, with supreme love and great patience, he taught the first pages, devoted

[44] G. Peano, *Programma di Logica Matematica, corso libero per l'anno 1906-07 presso la R. Università di Torino*, Archive University of Torino, Affari ordinati per classe, XIV B 227, Programmi di corsi liberi, Torino 20 March 1906, in E. Luciano, C.S. Roero (2008), 133–134: "Idee di Logica che si presentano in matematica. Eguaglianza, deduzione. Sillogismo, secondo Aristotele. Proprietà commutativa e associativa della moltiplicazione e dell'addizione logica secondo Leibniz. Proprietà distributiva secondo Lambert. Algebra della Logica, secondo Boole e Schröder. Caratteri delle definizioni matematiche. Idee primitive e idee derivate. Caratteri delle dimostrazioni matematiche. Proposizioni primitive e Teoremi. Analisi dei principii di Aritmetica, secondo Dedekind, e Russell. Analisi dei principii di Geometria, secondo Pieri e Hilbert. Teoria dei gruppi di punti, numeri cardinali e numeri ordinali transfiniti, secondo Cantor. Antinomie che vi si riscontrano, secondo Russell, ed altri. Tentativi di Borel, Hadamard, Poincaré, Lebesgue, Baire, Jourdain per risolverle." For the use of the *Formulario* in his teaching, cf. E. Luciano, 'Un sessantennio di ricerca e di insegnamento dell'Analisi infinitesimale a Torino: da Genocchi a Peano', Quaderni di Storia dell'Università di Torino, 9, 2008, 65–72 and 76–84.

[45] G. Peano (1904d), 219–220: "Abbiamo con questo finito di esporre quanto esige il programma, o più propriamente come ha detto il nostro amato professore abbiamo imparato a saper leggere il *Formulaire Mathématique*. Credo mio dovere il chiedere scusa a tutti i miei colleghi se queste poche pagine non hanno risposto allo scopo. Voi tutti sapete in quali condizioni di tempo io ho dovuto riordinare i miei appunti, tradurli e quindi di mio pugno scriverli su carta litografica. Sarebbe follia e vana pretesa la vostra se credeste trovare qui dentro quell'impronta veramente originale che hanno le Lezioni che il nostro illustre professore ci ha fatte. Solo chi ha avuto l'onore di seguirle tutte e con quel raccoglimento necessario per ben comprendere una materia così delicata può capire quanto grande sia la finezza e l'arte sublime che traspira dalla sapiente parola del Prof. G. Peano; ed essere con diritto orgoglioso di avere avuto un tale maestro."

> to the symbols of logic and then a few lines of some other pages, devoted to very detailed definitions of concepts, to the various operations and to developments of various parts of mathematics. Only in the last few months of the academic year did Peano reach the point of covering briefly, still with his symbols, Calculus with the system of vectors, and expounding some applications to curves, with deductions of length, area, etc. [...] But we students knew that this teaching was too lofty for us, we understood that these very subtle analyses of concepts, these very minute criticisms of the definitions used by other authors, were not suitable for beginners, and especially were of no use to engineering students. We were sorry to have to devote time and effort to "symbols" which in subsequent years we would never again have used.[46]

Firmly convinced of the importance of mathematical logic as a research tool in mathematics, Peano proposed its use to the students in their degree dissertations of Higher Analysis, and to his assistants to go more deeply into concepts and theorems. He wrote to Vacca in 1906:

> These properties were born once more in the mind of Boole, from whence, by way of Jevons, Schröder, and others, they arrived at the *Formulario*, where their importance emerges as research method, and not only their toy laboratory.[47]

> I consider it my duty, and that of all those with the responsibility of teachers, to perfect it, with relevant studies and publications That is why I am publishing the *Formulario*. [...] The questions that are important, useful for our young people immediately, or useful later on, are in heaps in the *Formulario*; and only a little attention is needed to discover some [...]. So, to be clear and to conclude something, take the proofs of the *Formulario* favourably; read them with care, wherever they are new to you. You will find many threads that will lead you to use its broad though chaotic knowledge. Others I myself will point out, and thus you can continue to work, and conclude, as you did with the preceding volumes, excel, and in essence do your duty. From this point you can take flight and do those tasks and publications, in which my help would be nil.[48]

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[46] C. Botto (1934), 19–20: "il libro di testo che il Peano seguiva era diventato invece il *Formulaire* del quale Egli insegnava, con sommo amore e grande pazienza, le prime pagine destinate ai simboli della logica e poi alcune linee di alcune altre pagine, dedicate alle accuratissime definizioni dei concetti, alle diverse operazioni e ad alcuni svolgimenti di varie parti della Matematica. Solo negli ultimi mesi dell'anno scolastico il Peano arrivava a svolgere brevemente, sempre con i suoi simboli, il Calcolo col sistema dei vettori, e ad esporre qualche applicazione alle curve, con deduzioni di lunghezze, di aree, ecc. [...] Ma noi studenti sapevamo che quell'insegnamento era troppo alto per noi, capivamo che quelle analisi così sottili dei concetti, quelle critiche così minute delle definizioni usate da altri autori, non erano adatte a dei principianti, e specialmente non servivano a degli allievi ingegneri. Ci spiaceva dover dedicare tempo e fatiche attorno a dei "simboli" che negli anni seguenti non avremmo mai più adoperato."

[47] G. Peano to G. Vacca, 15 November 1906: "Quelle proprietà nacquero un'altra volta nella mente di Boole, donde passando per Jevons, Schröder, ed altri, arrivarono al *Formulario*, ove ne risulta l'importanza come metodo di ricerca, e non solo il loro ufficio di giocattoli."

[48] G. Peano to G. Vacca, 19 February 1905: "Io reputo dovere mio, e di quanti sono incaricati di insegnamento, di perfezionarlo, con studii e pubblicazioni relative. Perciò io pubblico il *For-*

In the two academic years 1908–10, in the course on Higher Analysis, Peano made his students study the *Formulario* more deeply, with other texts with contributions of new, original research expounded with logico-mathematical symbolism. The degree dissertations by Gramegna, Mago and Peyroleri, written under his supervision, were published in the form of articles which show that the *Formulario* was the main research tool. For instance Mago, in his *Teoria degli ordini* wrote:

> The propositions can be found in my work written not only in ordinary language, but also in symbols. The ideographical signs can be used both to analyse more certainly and to expound briefly, precisely and completely the propositions of logic and of mathematics (and in this sense they are used especially in the *Rivista Matematica* and in the *Formulario* published by Peano), and as tools suitable to suggest new classes of entities and constant, I might almost say mechanical, methods with which to develop the theory. Perhaps when their usefulness is quite clear in creating and expounding new mathematical theories which are either of great elegance in themselves or more suited to the description of natural phenomena, around which our knowledge is growing daily more complex, the ideographical signs will gradually come to be universally accepted.[49]

This anomalous kind of teaching and of introducing research in analysis prompted protests from his colleagues, who at the Faculty meeting on 17 March 1910 decided not to renew Peano's appointment to the course on advanced Analysis, forcing him to confine himself to the first two years of university study. Peano confided bitterly to Vacca:

> I am giving up advanced teaching, against my will and with great regret. I have done all my lessons, succeeding in interesting the students, who in effect took an interest in it. I succeeded in agreeing with my colleagues, on whom I depend. But they want me to give up symbols, not to talk about the *Formulario* any longer and still more. I rejected any confirmation on those terms. I held that course out of pleasure and not self-interest. So it's all over. It will be difficult to bring out another volume of the *Rivista*. I have worked a good deal,

*mulario*. [. . . ] Le questioni importanti, utili per i nostri giovani immediatamente, o utili più tardi, sono nel *Formulario* a mucchi; e basta un po' di attenzione per scoprirne alcune. [. . . ] Dunque, per intenderci, e per conchiudere qualche cosa, prenda alle buone le bozze del *Formulario*; le legga con attenzione, ovunque sonvi novità. Troverà molti fili che la condurranno ad utilizzare le sue cognizioni ampie, ma caotiche. Altri ne indicherò io stesso, e così potrà continuare a lavorare, e conchiudere, come fece per tomi precedenti, farsi onore, e essenzialmente fare il proprio dovere. Di qui potrà spiccare il volo a fare quei lavori e pubblicazioni, in cui il mio aiuto sarebbe nullo."

[49] V. Mago (1912/13), 1–25, in particular note 8: "Le proposizioni si trovano scritte nel mio lavoro oltre che in linguaggio ordinario, anche in simboli. I segni ideografici si possono usare sia per analizzare con maggior sicurezza ed esporre in forma breve, precisa e completa le proposizioni di logica e di matematica (e in questo senso sono specialmente usati nella *Rivista Matematica* e nel *Formulario* editi dal Peano), sia come strumenti atti a suggerire nuove classi d'enti e metodi costanti, meccanici, direi quasi, onde svolgerne la teoria. Forse quando sarà del tutto palese la loro utilità nel creare ed esporre nuove teorie matematiche o di grande eleganza in sé o meglio atte alla descrizione dei fenomeni di natura, intorno alla quale la nostra conoscenza si fa di giorno in giorno più complessa, i segni ideografici finiranno a poco a poco per essere universalmente accettati."

> and I have a right to rest, all the more since my colleagues find my theories dangerous. Whoever cares to, can defend the *Formulario*. Anyway it is already a rather well-known book, and will no longer die. It may be that I will dedicate these last years to interlingua or to gardening. [...] I am a member of the Genoa philosophical society; I enrolled with great ideas, but I have no longer any desire to work.[50]

In the spring of 1910 his mathematical research interests changed and although he continued to follow and read works on mathematical logic he no longer took the field with new, important research and results.

In his review of Whitehead and Russell's *Principia* he stressed the differences from his enterprise with the *Formulario*:

> The Authors adopt, in part, the symbols from the *Formulario mathematico*. In some cases they vary either the form or the extent of the symbols and introduce many new symbols. The reason for this divergence is the different aim of the symbolism in the *Formulario* and in the books by these Authors. In the *Formulario* mathematical-logic is simply a tool to express and deal with propositions of ordinary mathematics; it is not an end in itself. Mathematical-logic is explained in 16 pages and one hour of study is sufficient to know what is necessary in the applications of this new science to mathematics. In contrast our Authors' book deals with mathematical-logic as science in itself, and its applications to the theory of transfinite numbers of various orders, and this demands a much broader symbolism.[51]

Polemical tones towards the authoritarian, excessively drastic positions of certain of his colleagues regarding logic and symbols can be detected in the words:

> Those who for lack of exercise regard symbolism as a liability, are not obliged to adopt it. We are building a new tool and we are not destroying the existing tools.

Peano loved freedom and democracy both in the context of research and in that of teaching. He was not an anarchic individualist, as is clear from the fact that in order to live in harmony with his colleagues he decided to turn his energies to the world of school and to the preparation of future teachers, as well as to the spread of an international language that would promote the peaceful exchange of ideas among scientists.

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[50] G. Peano to G. Vacca, 24 April 1910: "Io abbandono l'insegnamento superiore, contro la mia volontà e con dolore. Ho fatto tutte le mie lezioni, procurando di interessare gli allievi, che si sono effettivamente interessati. Ho procurato di vivere d'accordo coi colleghi, da cui dipendo. Ma questi vogliono che io abbandoni i simboli, che non parli più del Formulario e altro ancora. Rifiutai ogni conferma in tali condizioni. Facevo quel corso per piacere e non per interesse. Così è finita. Difficilmente farò ancora uscire un volume della Rivista. Ho lavorato abbastanza, ed ho diritto di riposare, tanto più che i colleghi ritengono le mie teorie pericolose. La difesa del Formulario la faccia chi vuole. Del resto esso è un libro già abbastanza noto, e non muore più. Può essere che io dedichi questi ultimi anni all'interlingua o al giardinaggio. [...] Io sono socio della società filosofica di Genova; mi sono iscritto con grandi idee, ma non ho più volontà di lavorare."

[51] Peano (1913i), 48.

## 6.4 Outcome of the *Formulario* and Cultural Spin-off

It is well known that the subsequent developments of logic[52] took another path, thanks above all to Russell, Hilbert and Gödel, and that Peano's *Formulario* was overtaken by their works. However, these authors publicly recognised their cultural debt to Peano:

> M. Peano a forgé un instrument de grande puissance pour certains ordres de recherches. Quelques-uns d'entre nous s'intéressent à ces recherches, et par suite honorent M. Peano, qui est allé, selon nous, tellement plus loin et plus haut que les mathématiciens "aptères", que ceux-ci l'ont perdu de vue et ne savent pas combien il est en avance sur eux.[53]
>
> On the one hand we have the works of analysts and geometers, in the way of formulating and systematising their axioms, and the work of Cantor and others on such matters as the theory of aggregates. On the other hand we have symbolic logic, which, after a necessary period of growth, has now, thanks to Peano and his followers, acquired the technical adaptability and the logical comprehensiveness that are essential to a mathematical instrument for dealing with what have hitherto been the beginnings of mathematics.[54]
>
> Wie Sie bemerken, ist ein wesentliches Hilfsmittel für meine Beweistheorie die Begriffsschrift, und wir verdanken dem Klassiker dieser Begriffsschrift, Peano, die sorgfältigste Pflege und weitgehendste Ausbildung derselben. Die Form, in der ich die Begriffsschrift brauche, ist wesentlich diejenige, die Russell zuerst eingeführt hat.[55]

Judgments of the *Formulaire* by its contemporaries were mixed: flattering in Britain and in America, where Peano's symbols were adopted by some mathematicians, but harsher in France and in Italy. In 1910 Eliakim Hastings Moore proposed its introduction into mathematical analysis, printing the list of logic signs in the fifth edition of the *Formulario,* and Clarence Irving Lewis of the University of Berkeley stated in 1918 that the "Peano's *Formulaire de Mathématiques* marks a new era in the history of symbolic logic"[56]. In 1971 Kurt Gödel suggested to Ralph Hwastecki to use the Peano's *Formulaire* with the students[57].

In France and in Italia the *Formulario* was involved in the controversy on intuition and rigour, which flared up between 1905 and 1907 in the pages of the *Revue de metaphysique et de morale*, with echoes in the Italian journal *Leonardo*. The debate was wide-ranging and well-expressed and involved mathematicians and

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[52] Cf. Cellucci (1993), 73–138.

[53] B. Russell (1906), 628. Cf. also B. Russell (1917). The influence of the *Formulario* on Russell is well documented in F.A. Rodriguez-Consuegra (1991), 91–165, 175–177, 181–184.

[54] A.N. Whitehead, B. Russell (1910).

[55] D. Hilbert (1929), 137.

[56] C.I. Lewis (1918), 115; also 278–281.

[57] Cf. S. Feferman, J.W. Dawson Jr., W. Goldfarb, Ch. Parsons, W. Sieg (eds.) *The Collected Works of Kurt Gödel*, vol. V, Correspondence H–Z, Oxford, University Press, 2003, 80–81.

philosophers of the calibre of H. Poincaré, B. Russell, A.N. Whitehead, L. Couturat, E. Borel, M. Winter, G. Peano, G. Vacca, G. Vailati, M. Pieri and B. Croce.

The emergence of the antinomies of the theory of sets and the doubts regarding the axiom of choice – subjects which had had great resonance after the publication of the famous *Cinq lettres sur la théorie des ensembles by* R. Baire, E. Borel, H. Lebesgue and J. Hadamard – contributed to attract attention to the relationships between logic and mathematics and on the usefulness of the former in the latter. Faced by the proliferation of paradoxes, harsh criticisms were moved against the symbolic logic of Peano, Russell and Hilbert, accused of hindering the momentum of intuition and creativity and of not safeguarding the theories of vicious circles. What in Francia proved harmful to the reception of the *Formulario* was the action of the philosopher Couturat, who presented it with excessive emphasis as a work destined to carry out the refoundation of the logic of all mathematics, misunderastanding its more modest didactic range. Thus he finally provoked the caustic irony of Poincaré who, secure in his scientific and academic prestige, announced his refusal to read the *Formulario* and challenged the experts in logic to use the wings of symbolism to take flight towards the construction of new theories:

> En ce qui concerne la fécondité, il semble que L. Couturat se fasse de naïves illusions. La logistique d'après lui, prête à l'invention « des échasses et des ailes » et à la page suivante : « il y a dix ans que M. Peano a publié la première édition de son *Formulaire*. » Comment, voilà dix ans que vous avez des ailes, et vous n'avez pas encore volé ! J'ai la plus grande estime pour M. Peano qui a fait des très jolies choses (par exemple sa courbe qui remplit toute une aire) ; mais enfin il est allé ni plus loin, ni plus haut, ni plus vite que la plupart des mathématiciens aptères, et il aurait pu faire tout aussi bien avec ses jambes.[58]

Though severe and heated, the controversy was not a sterile debate, but laid the foundations for a dialogue between mathematicians and philosophers on logico-foundational topics, rare in other European countries and very superficial in Italy. This bore fruit in original results. Among these may be citated the studies on compatibility, the independence and logical irreducibility of the axioms of arithmetic, conducted by Pieri and by Padoa between 1906 and 1912, the simplifications of Cantor–Bernstein's theorem, thanks to Peano and to Padoa, the theory of types developed by Russell to overcome the obstacle of the antinomies and the distinction between logical and semantic paradoxes introduced by Ramsey, following the brief mention by Peano in the note *Super theorema de Cantor Bernstein*[59].

In Italy the main cultural spin-off of the operation carried out by Peano in the *Formulario* can be seen in the encyclopaedic collections, edited by F. Enriques and by Berzolari, Vivanti and Gigli, and in the dozens of texts for upper schools written by members of the School of Peano. The encyclopaedias of elementary mathematics were prepared by a team of mathematical researchers, some of whom had collabo-

[58] H. Poincaré (1906), 295.

[59] M. Pieri (1906d), 196–207; (1908b), 26–30 – *Opere...*, Roma, 1980, 449–453; G. Peano (1906b), 360–366; (1906e), 143–157; B. Russell (1906), 627–650; A. Padoa (1911), (1912); F. Ramsey (1925), 338–384.

rated on the *Formulario*. A certain importance here can be attributed to the history of mathematical ideas, of concepts, of theorems, of methods and of theories. The chapter on mathematical logic for Enriques' Encyclopaedia was requested of Padoa, the one on History of Vacca, though it was not finally entrusted to him.

The manuals on arithmetic, geometry and analysis for middle schools, written by the teachers of Peano's group, demonstrate the absorption of those criteria of rigour, simplicity and essentiality in the transmission of knowledge, typical of the *Formulario.* The insertion of the content in a historical context which justified its choices, and the attempt to avoid excessive, cold symbolism, had positive effects for the spread of the theories on the foundations of arithmetic and of geometry However, for Peano and for his principal followers the *Formulaire* always remained the most meritorious work they had carried out in mathematics[60]. At the ripe age of 70, in 1929, Peano proposed a new edition to the President of the Mathematical Committee of the Research Council:

> A collective task which can be carried out is the publication of a new edition of the *Formulario matematico*, whose fifth and last edition of 1908 has sold out. This *Formulario* is a mathematical encyclopaedia, or collection of all the mathematical propositions written in symbol, with their proof and history. The use of symbols offers the primary advantage of brevity; in addition, many propositions which in ordinary language appear to be distinct, prove to be identical; and the propositions take on a precise form, much more than with ordinary language. Prof. Cipolla of Palermo writes to me: "I consider it very timely, indeed necessary, to publish a new edition of the *Formulario*." And Profs. Boggio of Turin, Cassina of Milan, Padoa of Genoa and many others are in favour of its continuation. The language used in the last edition is Latin-sine-flexione, very useful to make the work known abroad offering greater diffusion, both to express the ideas more clearly, not confused by grammatical inflexions. The history is made up of passages taken from the authors, in the original form and language. [...] I should be glad to dedicate the rest of my life to it, now I am in my seventies.[61]

[60] Cf. U. Cassina (1933), 323–389 and G. Vacca (1946), 30–44.

[61] G. Peano to G. Scorza, Torino 24 February 1929, in C.S. Roero, N. Nervo, T. Armano (2002): "Un lavoro collettivo che si può fare è la pubblicazione di una nuova edizione del *Formulario matematico*, di cui la quinta ed ultima edizione del 1908 è ora esaurita. Questo *Formulario* è una enciclopedia matematica, o raccolta di tutte le proposizioni matematiche scritte in simboli, colla dimostrazione e storia. L'uso dei simboli offre il primo vantaggio della brevità; inoltre molte proposizioni che col linguaggio comune paiono distinte, si rivelano identiche; e le proposizioni assumono una forma precisa, molto più che col linguaggio comune. Il prof. Cipolla di Palermo mi scrive: 'Ritengo opportunissima, anzi necessaria la pubblicazione di una nuova edizione del *Formulario*.' E sono in caso di continuarlo i proff. Boggio di Torino, Cassina di Milano, Padoa di Genova e molti altri. La lingua usata nell'ultima edizione è il Latino-sine-flexione, molto utile per far conoscere il lavoro all'estero dandoci maggior diffusione, sia per esprimere le idee in modo più chiaramente, non confuse dalle flessioni grammaticali. La storia è fatta riportando i passi degli autori, nella lingua e forma originale [...] io sarei lieto di dedicare ad esso il restante della mia vita, dopo gli anni settanta."

As for Leibniz, so for Peano the mathematical encyclopaedia written in symbols remained a Utopian dream, as he succeeded in completing only the part regarding elementary and classical mathematics. In this sense his enterprise stands side by side with the series of encyclopaedias of elementary mathematics, which were issued in his own time. Peano's dream of a work collecting all mathematical research, even the most advanced, was to find the worthy fulfilment of his initial ideals only in the 20th century, in the Bourbaki group[62]. Peano had scattered the first seeds for the immense undertaking which was then presented by the Bourbaki group: advanced mathematics, expounded in abstract, symbolic mode, and accompanied by the historical context of the most important stages in the various branches of research. What they wrote in 1960 is a tribute to Peano:

> Le but de Peano était à la fois plus vaste et plus terre à terre [Frege] ; il s'agissait de publier un *Formulaire de mathématiques*, écrit entièrement en langage formalisé et contenant non seulement la logique mathématique, mais tous les résultats des branches des mathématiques les plus importantes. La rapidité avec laquelle il parvint à réaliser cet ambitieux projet, aidé d'une pléiade de collaborateurs enthousiastes (Vailati, Pieri, Padoa, Vacca, Vivanti, Fano, Burali-Forti) témoigne de l'excellence du symbolisme qu'il avait adopté : suivant de près la pratique courante des mathématiciens, et introduisant de nombreux symboles abréviateurs bien choisis, son langage reste en outre assez aisément lisible, grâce notamment à un ingénieux système de remplacement des parenthèses par des points de séparation.[63]

Of importance, too, is the fact that they recognised that certain criticisms from Poincaré were exaggerated and unjust; these criticisms had contributed to hinder the spread of the *Formulaire* in France:

> Bien des notations dues à Peano sont aujourd'hui adoptées par la plupart des mathématiciens : citons $\in$, $\supset$ (mais, contrairement à l'usage actuel, au sens de est contenu ou implique), $\cup$, $\cap$, A–B (ensemble des différences $a$–$b$, où $a \in$ A et $b \in$ B). D'autre part, c'est dans le *Formulaire* qu'on trouve pour la première fois une analyse poussée de la notion générale de fonction [...]. Mais la quantification, chez Peano, est soumise à des restrictions gênantes [...]. En outre le zèle presque fanatique de certains de ses disciples prêtait aisément le flanc au ridicule ; la critique, souvent injuste, de Poincaré en particulier, porta un coup sensible à l'école de Peano et fit obstacle à la diffusion de ses doctrines dans le monde mathématique. Avec Frege et Peano sont acquis les éléments essentiels des langages formalisés utilisés aujourd'hui. Le plus répandu est sans doute celui forgé par Russell et Whitehead dans leur grand ouvrage *Principia Mathematica*, qui associe heureusement la précision de Frege et la commodité de Peano.[64]

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[62] B. Segre (1955), 31–39.

[63] N. Bourbaki (1960), 20–21; (1970), *Structures Note historique* E IV 42.

[64] N. Bourbaki (1960), 21.

In Italy certain of the protagonists of active mathematical research, such as Volterra and Enriques, were somewhat disdainful of the work carried out by Peano in the *Formulario*, stressing above all its philosophical aspect. In 1908 at the international congress of mathematicians, which was held in Rome, Volterra expressed the following judgment of the progress made in the second half of the 19th century, in particular in the field of analysis:

> Research on the functions of real variables and their singularities, which were called the studies on the deformities and monstrosities of mathematics, in which the aid of the so-called physiological laws of geometry are missing, and not only is every intuition lacking, but all the simple persuasive forecasts most of the time lead to error. [...] It was Dini who introduced and spread in Italy the passion for this research with his works, and even more, with his effective and original teaching. [...] Weierstrass and Riemann, moving from ideas which had somewhat infiltrated into analysis, had begun them, Georg Cantor had astonished everyone with his unexpected revelations, Du Bois-Reymond had penetrated many obscure problems and Darboux had discovered many fine, original propositions. Dini, coordinating this set of doctrines, enriching them with new truths, had the courage to bring them to Italy in school at the very beginning of the studies in infinitesimal analysis and as their basis. [...] Attracted by these studies, a school was formed in Italy of mathematicians who dedicated the energies of their genius to the development of these doctrines and brought about important results. And the studies themselves took on a double direction among us: one led Ascoli, Arzelà and others to concrete research on the series, the limits and the theory of functions; the other, with Peano and the School that took its inspiration from him, aimed to give an increasingly solid basis to the fundamental concepts, merged with those doctrines that were going more deeply into the criticism of the postulates and drove on from day to day into ever more abstract regions, taking on a more philosophical aspect.[65]

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[65] V. Volterra (1909), 62: "[...] ricerche sopra le funzioni di variabili reali e le più riposte singolarità loro, che efficacemente furon chiamate gli studi sulle deformità e le mostruosità della matematica, in cui l'aiuto delle leggi, per dir così, fisiologiche della geometria viene a mancare, e non solo ogni intuizione fa difetto, ma tutte le facili e seducenti previsioni inducono il più spesso in errore. [...] Fu il Dini che introdusse e diffuse in Italia l'amore per queste ricerche colle sue opere, e più ancora, con l'efficace ed originale suo insegnamento. [...] Weierstrass e Riemann, movendo da idee che si erano un poco alla volta infiltrate nell'analisi, le avevano iniziate, Giorgio Cantor aveva fatto strabiliar tutti colle sue inattese rivelazioni, il Du Bois-Reymond era penetrato addentro a molti oscuri problemi ed il Darboux aveva scoperto tante belle ed originali proposizioni. Il Dini, coordinando questo insieme di dottrine, arricchendole di nuove verità ebbe il coraggio di portarle in Italia nella scuola all'inizio stesso degli studi di analisi infinitesimale e come base di essi. [...] Attratta da questi studi, si formò in Italia una scuola di matematici che consacrarono le forze del loro ingegno allo sviluppo di queste dottrine ed apportarono loro importanti risultati. E presero gli studi stessi doppia direzione fra noi: l'una condusse l'Ascoli, l'Arzelà ed altri a ricerche concrete sopra le serie, i limiti e la teoria delle funzioni; l'altra mirò, col Peano e colla Scuola che ebbe l'impulso da lui, a dare una base sempre più solida ai concetti fondamentali, si fuse con quelle dottrine che approfondivano la critica dei postulati e si spinse di giorno in giorno in regioni sempre più astratte, acquistando un carattere vieppiù filosofico." Cf. also F. Enriques (1913), 77.

The way in which Peano did research in the *Formulario* was a long way from, almost antithetical to that of Poincaré and of Volterra, in that it looked more to the past than to the future, more directed at the codification and structuring of theories that had already been learned, rather than at the conception and development of new branches of mathematics. One of Peano's best and most enthusiastic collaborators, Mario Pieri, was able to grasp this 'static' aspect at the basis of the logical research to be found in the *Formulario* and characteristic of the style of the Piedmontese scholar, observing shrewdly that:

> The direct and immediate discovery by way of brilliant intuition, the artistic divination, will always have great status and power in the kingdom of knowledge: but opposing the fact of invention to the progress of demonstrative Logic would be like denying faith and value to counterpoint out of respect for musical inspiration. [...] Not sufficient distinction is made (I believe) between science and art, between the *static* and rational structure of a scientific discipline and its operative and *dynamic* qualities. The tendencies of logistics (it should be recognised) aim more at the static equilibrium of the various deductive disciplines and at science, as a body of established truths, than at the operative function of the scientific discovery.[66]

For his part, Peano had become further convinced that by means of the education of the young in the clear, simple, rigorous exposition of mathematics, through the use of logic, there would be an improvement in Italy of school and also research would take on new drive. At the lemma "Logica matematica" in the *Dizionario di cognizioni utili* in 1919, he insisted on precisely this point:

> With these ten or twelve symbols, together with the symbols to represent the ideas of arithmetic and of geometry, all the propositions of mathematics can be expressed, as can be seen in Peano's *Formulario mathematico*. With this tool analysis has been made of the definition encountered in mathematical texts, and it has been found that they satisfy special rules, never before expressed. Analysis has been made of the forms of reasoning used in mathematical proofs, and it has been seen that they are not reduced to the types considered in the treatises on logic. We have found what are the primitive ideas of arithmetic and of geometry, especially by the work of the late lamented Pieri; the principles of mathematics have been analysed, at the hands especially of Russell and Whitehead. This tool was useful to Moore for the integration of

[66] M. Pieri (1906–07), 60: "La scoperta diretta e immediata per intuizione geniale, la divinazione artistica, avranno sempre grande stato e potere nel regno della conoscenza: ma opporre il fatto dell'invenzione ai progressi della Logica dimostrativa sarebbe come negar fede e valore al contrappunto in ossequio all'ispirazione musicale. ... Non si distingue abbastanza (io credo) fra scienza ed arte, fra l'assetto *statico* e razionale di una disciplina scientifica e le sue qualità operative e *dinamiche*. Le tendenze logistiche (conviene riconoscerlo) mirano più all'equilibrio statico delle varie discipline deduttive e alla scienza, come corpo di verità stabilite, che alla funzione operativa della scoperta scientifica."

> differential equations. Some school books are already formed on mathematical logic, and it is in the field of teaching that this science can prove its dazzling simplicity.[67]

Simplicity, brevity and rigour were the pivotal elements of Peano's mathematics. His most famous results arose as he was preparing his university lectures and reading the works of the great mathematicians of the past. The *Formulario* was none other than the distillation of the disciplines with which he dealt in his university courses, expounded in symbolic manner, and linked to their history. The value of the exposition condensed in symbols was intended to permit dialogue among specialists in several different sectors of mathematics and the young researchers would thus be able more readily to have command of a field which was becoming ever more extensive. Hence it is not strange that Peano should have written to that "Out of a book by Lebesgue there may be one line, or half a page." Just as for Joseph Joubert, for Peano too the summit of art lies in "Concentrating a page in one line, and a line in one word."

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[67] G. Peano (1919e), 960: "Con questa decina di simboli, uniti ai simboli per rappresentare le idee di aritmetica e di geometria, si possono esprimere tutte le proposizioni di matematica, come si può vedere nel *Formulario mathematico* di Peano. Con questo strumento si sono analizzate le definizioni che si incontrano nei libri di matematica, e si è trovato che esse soddisfano a regole speciali, non enunciate prima. Si sono analizzate le forme di ragionamento usate nelle dimostrazioni matematiche, e si è visto che esse non si riducono ai tipi considerati nei trattati di logica. Si è trovato quali sono le idee primitive dell'aritmetica e della geometria, per opera specialmente del compianto Pieri; si sono analizzati i principi della matematica, per opera specialmente di Russell e Whitehead. Questo strumento servì a Moore per l'integrazione di equazioni differenziali. Già alcuni libri scolastici sono formati sulla logica matematica, ed è nel campo dell'insegnamento che questa scienza può dimostrare la sua fulgida semplicità."

**Table 1** Mathematics and History of Mathematics in Peano's *Formulaire* 1895–1908: *Formulaire de Mathématiques* 1895

| ***Formulaire* 1895** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| I. **Logique mathématique** | March 1892–1895 | Peano, Vailati | 1–7 | Peano, 115–116 *Notes*, 127–129 | Leibniz, Boole, Peirce, Aristoteles, MacColl, Segner, De Morgan, Schröder, Hauber, Jevons, Dedekind | Peano 1891g, 1891m, Vailati RdM 1, 1891, 103; RdM 1, 1891, 24–31, 182–184; 3, 1893, 4–5 |
| 1. $\supset, =, \cap$ déduction, égalité, conjunction<br>2. $-, \cup$ négation, disjonction<br>3. $\Lambda, o$ absurde, disjonction complète<br>4. $K, \varepsilon, \iota$ Classes<br>5. $f, \bar{f}$ Fonctions | | | | | | |
| II. **Opérations algébriques** | January 1893 | Peano, Castellano | 8–21 | Peano, 117–119 *Notes*, 130–131 | Pythagoras, Euclides, Archimedes, Diophantus, Nicomacus, Ariabhatas, Leonardus Pisanus, Jordanus Nemorarius, Chuquet, Napier, Bachet, Fermat, Newton, Euler, Lagrange, Cavalieri, Cauchy; M. Cantor; *Hermite, Todhunter, Segar* | Peano 1892o; 1892p, 1–8; RdM 4, 189–197 |
| III. **Arithmétique** | | Peano, Burali-Forti | 22–27 | Burali-Forti, 119–120; *Notes*, 132 | Euclides, Fermat, Leibniz, Euler, Wilson, Waring, Lagrange, Legendre; *Burkhardt, Serret, Peruchine, Legendre* | RdM 3, 1893, 75 |
| IV. **Théorie des grandeurs** | | Burali-Forti | 28–57 | Burali-Forti, 120 | | RdM 3, 1893, 76–101 |
| V. **Classe de nombres** | | Peano | 58–64 | Peano, 120–121 *Notes*, 132 | Weierstrass, Bolzano, Dini, Pincherle, Stolz, Grassmann, Cayley, G. Cantor, De Paolis, Bendixon, Jordan; *Frege, Weierstrass, Cantor* | Peano 1889a, 1890f, 1894b |

**Table 1** (continued)

| *Formulaire* **1895** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| VI. **Théorie des ensembles** | | Vivanti | 65–70 | Vivanti<br>*Notes*, 132 | G. Cantor, Dedekind, Bendixon, Thomae, Lüroth, Netto, Milesi, Peano, Hilbert, De Paolis, Scheeffer, Gutberlet, Schwarz, Phragmén, Jordan; *Frege, Cantor, Jordan* | RdM 2, 1892, 165–167; RdM 4, 1894, 135–139 |
| **Liste bibliographique jusq'à l'an 1893** | | | 71–74 | Vivanti, 122 | Pincherle | |
| VII. **Limites** | | Bettazzi | 75–82 | Bettazzi, 122–126<br>*Notes*, 133 | Peano, Cauchy, Bettazzi, du Bois-Reymond, Bolzano, Cesàro, Stolz, Giudice, Euler, Schlömilch, Genocchi-Peano, Novi, Laska, Laisant; *Capelli, Laisant, Marcolongo, Archimedes, Fermat, Baltzer, La Maestra, Jensen, Stolz* | RdM 4, 1894, 161–162 |
| VIII. **Séries** | | Giudice | 83–100 | Giudice, 126<br>*Notes*, 133 | Pringsheim, Cesàro, Euler, Nicole, Cauchy, Riemann, du Bois-Reymond, Abel, Dirichlet, Faifofer, Joh. Bernoulli, Giudice, Bonnet, Dini, Bertrand, De Morgan, Kummer, Raabe-Duhamel, Gauss, Dini, Ermakof, Capelli-Garbieri, Mertens; *Abel, Jac. Bernoulli, Dini, Cesàro, Cauchy, Bonnet, Kummer, Lerch, Ed. Weyr, MacLaurin, Hoèevar, Bertrand, Giudice, Weierstrass* | RdM 3, 1893, 185–188; RdM 4, 1894, 163–165 |
| IX. **Théorie des nombres algébriques** | Oct. 1894 | Fano | 101–114 | Fano, 126<br>*Notes*, 133 | Dedekind, Dirichlet | RdM 5, 1895, 1–8 |
| Table des Signes | | | 134–139 | | | |
| Table des Auteurs | | | 140–141 | | | |
| Table Générale | | | 142–144 | | | |

**Table 2** Mathematics and History of Mathematics in Peano's *Formulaire* 1895–1908: *Formulaire de Mathématiques* 1897–1899

| ***Formulaire* 1897** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| II. §1 **Logique Mathématique** | 1897.8.11 | Peano, Vailati | 1–63 | 1–531 | *Prop.*: Leibniz, Aristoteles, Mc Coll, Frege, Boetius, Lambert, Boole, Segner, Padoa, Schröder, Peirce, Vailati, De Morgan, Pieri, Jevons, Richeri, Hauber, Burali-Forti | Peano 1896b |
| $K, \varepsilon, \supset, \cap, =$ | | | 3–7 | *Notes*, 19–63 | *Notes*: Legendre, Euclide, Mc Coll, Segner, Lambert, Wilson, Schröder, Leibniz, Fermat, Maass, Venn, Jevons, Peirce, Frege, Richeri, Lagrange, Abel, Macfarlane, Dedekind, Pieri | Peano 1896j, 1895c, 1893h |
| $; -, \cup$ | | | 6–11 | | | |
| $-, \cup; = \Lambda$ | | | 11–13 | | | |
| $= V; \exists$ | | | 13–14 | | | |
| $\iota, \bar{\iota}, K'; \cup', \cap'$ | | | 14–16 | | | |
| $f$; Sim; rcp | | | 16–17 | | | |
| Bibliographie | | | 18 | | | |

**Table 2** (continued)

| ***Formulaire* 1898** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| II. §2 **Arithmétique** [théorie des nombres entiers, fractionaires, positifs et négatifs] | 1898.4.19–1898.8.9 | Peano | i–viii, 1–53 | 001–390.5 | | |
| Signes de Logique adoptés | | Peano | i–vii | | Aristoteles, Segner, Lambert, Leibniz. | |
| Ordre des signes d'Arithmétique<br>Sigles des collaborateurs | | Peano | viii | | Peano, Castellano, Burali-Forti, Chini, Vacca | |
| $N_0 + 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ X$ | | Peano, Padoa | 1–9 | 010–019<br>*Notes*, 1–3<br>*Note sur les chiffres*, 5<br>*Note*, 6 | *Notes* [idées primitives]: G. Cantor, Dedekind; *Note sur les chiffres*: M. Cantor, Bayley, Lindemann, Boetius, Pythagoras; *Note* [somme]: Diophantus, Pacioli, de la Roche, Stifel, Widman, De Morgan, Clifton, Eneström | Peano 1889a, 1890f, 1894b |
| $N_0 + N_1$<br>$N_0 + -n$ | | Peano, Castellano, Burali-Forti, Chini, Vacca | 9<br>10–12 | 020<br>021–034<br>*Note sur les nombres*, 11 | *Note sur les nombres positifs et négatifs*: Brahmagoupta, Rodet, Pacioli | F 1895, I & III |
| $N_0 + \times$<br>$+N_1\times$<br>$N_0 + -\times$<br>$N_0 + -n\times$ | | Peano, Vacca, Castellano, Burali-Forti | 13–14<br>14<br>14<br>15 | 041–044<br>045<br>046–047<br>048–049 | Barrow, Euclides, Oughtred, Diophantus, Pacioli | F 1895, I & III |
| $+N_1 \times /R$<br>$+N_1 - /R$<br>$+N_1 - n \times /Rr$ | | Peano, Castellano, Burali-Forti, Chini | 16–17<br>18<br>18–19 | 050–065; *Note*, 16<br>066–067<br>070–075 | *Note sur les R*: Macfarlane; Euclides, Aryabhata, Diophantus | F 1895, I & III |
| $N_0 + \times \uparrow$<br>$N_0 + N_1\times \uparrow$<br>$N_0 + -n\times \uparrow$ | 1898.4.19–1898.8.9 | Peano, Castellano, Burali-Forti, Chini, Padoa, Vacca | 20–21<br>21–23<br>23–25 | 080–083;<br>084–085;<br>090–099 | Chuquet, Tschu Schi Kih, Tartaglia; Bachet, Fermat, Lagrange, Alchodschandi, Euclides; Oltramare, Leibniz, Euclides, Legendre, Platon, Diophantus, Euler, Lagrange, Young, Graves | |

**Table 2** (continued)

| *Formulaire* 1898 | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| $N_0 + \ldots \sum$<br>$+N_1 \times \uparrow \ldots \sum$ | | Padoa, Vacca | 26–31 | 100–115<br>*sur les systèmes*, 28–30 | Pythagoras, Theon, Diophantus, Aryabhata, Alqachani; *Note sur les systèmes de numération*: M. Cantor, Lindemann, Archimedes, Rodet, Leibniz, Lucas, Cauchy; Archimedes, Aryabhata, Nicomachus, Waring, Jacobi, Oltramare, Euclides, Alqachani, Fermat, Amigues | F 1895, I & III |
| $N_0 + \times \ldots \prod$ | | Peano, Castellano, Burali-Forti | 32 | 121–130 | Fermat | F 1895, I & III |
| $N_0 + N_1 > < \geqslant \leqslant$<br>$N_0 + N_1 + -n \times >$<br>$+N_1 \times / R >$<br>$N_0 + \times \uparrow >$<br>$N_0 + - \times \uparrow >$ | | Peano, Castellano, Burali-Forti, Chini, Padoa, Vacca | 33–36 | 150–189 | Harriot, Euclides, Cauchy | |
| $N_0 + N_1 - n \times / Rr \uparrow < \ldots \sum \prod$<br>mod<br>$+ - n \times / Rr >$ mod sgn | | Peano, Castellano, Burali-Forti, Vacca | 37 | 190–196 | Cauchy, Weierstrass, Kronecker | F 1895, I & III |
| $N_0 +$ num<br>$N_0 \times \leqslant$ num<br>$N_0$ num $\infty$<br>$Nc'$ infn | | Peano, Vivanti | 38–39 | 200–210 | G. Cantor, Vivanti | F 1895, I & III |
| $N_0 >$ max min | | Peano, Castellano, Burali-Forti | 40 | 220 | | |
| Cmb $N_0 + \times !$ | | Vacca | 41–43 | 240–244<br>*Note*, 41–42 | Kramp, Gauss, Pascal, Leibniz, Tartaglia, Euler | F 1895, I & III |

**Table 2** (continued)

| ***Formulaire* 1898** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| quot rest | | Padoa, Peano, Castellano, Burali-Forti | 44 | 300 | | F 1895, I & III |
| Dvr *le plus grand commun diviseur* | | Peano, Castellano, Burali-Forti | 45–46 | 310–313 *Note*, 45 | V.A. Lebesgue, Euclides, Stieltjes | F 1895, I & III |
| mlt *le plus petit multiple commun* | | Peano, Castellano, Burali-Forti, Vacca | 47 | 320 | Euclides, Stieltjes, V.A. Lebesgue | F 1895, I & III |
| Np *nombre premier* | | Peano, Vacca | 48–49 | 330–340 | Burckhardt, Glaisher, Dase-Rosenberg, Euclides, Goldbach, Bertrand, Fermat, Leibniz, Pervouchine, S. Germain, Legendre, Wilson, Waring, Dirichlet, *G. Cantor* | F 1895, I & III |
| mp | | | 50 | 350–351 | *Wallis*, *Legendre* | F 1895, I & III |
| $N_1 \ldots$ num Dvr $\Phi$ | | Vacca | 51 | | Gauss, Euler | F 1895, I & III |
| $E$ *Entier de* | | Vacca | 52 | 380, *Note*, 52 | Legendre, Gauss, *Maupin* | |
| $\Theta$ *partie fractionnaire de* | | Vacca | 53 | 390 | Hermite, Gauss, Cesàro, *Bertrand* | |
| Table des Signes | | | 54–56 | | | |
| Bibliographie | | | 57–59 | | | |

**Table 2** (continued)

| *Formulaire* 1899 | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| II. §3 Logique mathématique – Arithmétique – Limites – Nombres complexes – Vecteurs – Dérivées – Intégrales | 1899 | Peano, Burali-Forti, Castellano, Chini, Padoa, Vacca, Vailati, Vivanti, Bettazzi, Giudice, Fano, Pieri | 3–199 | | | |
| Préface | | Peano | 3–4 | | | RdM 6, 1899, 65–74 |
| § 1 Notations Cls $\varepsilon \ni \cap, ; \supset$ ; § 2 $=$ ; § 3 $\cup$ ; § 4 – *non* ; § 5 $\exists$ ; § 6 $\iota$ ; § 7 ˋı; § 8 ⁚ ; § 9 f ⅃; § 11 Sim rcp ; § 12 idem ; § 13 ‘’ ; § 14 Variab $F$ Funct | | | 5–28 | 24 | Aristoteles, Girard, Chuquet, Leibniz, Euler, Segner, Lambert, *Gergonne*, Pell, Peirce, Schröder, Burali-Forti, Padoa, Couturat, Viète, Record, Wallis, Newton, *Hamilton*, Boole, Mac Coll, Lambert, Whitehead, De Morgan, Hauber | Peano 1898h, Vacca RdM 6, 1899, 121–125, Peano 1889a; F 1897; Padoa RdM 6, 1899, 105–121, Peano 1898a, 1899c |
| § 20 $0 N_0 +$ ; § 21 $012 \ldots 9 X$ ; § 22 $N_1$ ; § 23 – ; § 24 $\boldsymbol{n}$ ; § 25 $\times$ ; § 26 / ; § 27 $\boldsymbol{R}$ ; § 28 $\boldsymbol{r}$ ; § 29 $N_0 + \times \uparrow$ *élevé à la puissance* ; § 30 $>$ ; § 31 $\ldots$ ; § 32 Num infn ; § 33 $\sum$ | | | 29–69 | 29–30; *sur les chiffres*, 35–36; *nombres positifs et négatifs*, 37–38; *nombres rationnels* 45–46; 49; 61; 62–63; *systèmes* 65–66; 68 | Pythagoras, Euclides, Padoa; *Note sur les chiffres*: M. Cantor, Bayley, Lindemann, Boetius, Pythagoras; *Note sur les nombres positifs et négatifs*: Ahmes, Diophantus, Chuquet, Pacioli, Stifel, MacLaurin, Cauchy; Brahmagupta, Rodet, Leibniz, Euclides, L. Fibonacci, Oughtred, Pell, MacLaurin, *Méray*, *Couturat*; *Note sur les nombres rationnels*: Ahmes, Macfarlane; Aryabhata, Diophantus, Lucas, Euclides, Descartes, Pell, Girard, Chuquet, Tschu Schi Khi, Tartaglia, Bachet, Fermat, Lagrange, P. Tannery, Alchodschandi, Legendre, Euler, Pythagoras, Platon, Proclus, Cauchy, Degen, Young, Lamé, Oltramare, Leibniz, Chuquet, Girard, Oughtred, Harriot, Bertrand; G. Cantor; | Peano 1898m |

**Table 2** (continued)

| *Formulaire* **1899** | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| | | | | | Pythagoras, Theon, Aryabhata, Alqachani; *Note sur les systèmes de numération*: M. Cantor, Lindemann, Archimedes, Aryabhata, Rodet, Leibniz, Lucas, Cauchy; Archimedes, Aryabhata, Nicomachus, Ibn Albanna, Waring, Jacobi, Oltramare, Alqachani, Fermat, Jac. Bernoulli, Amigues, Lucas, Ahmes, Eisenlohr, Euclides, Regiomontanus, Viète, Stevin, Kepler, Mercator, Cauchy, Jacobi | |
| § 34 $\prod$ ; § 35 ! ; § 40 mod ; § 41 sgn ; § 42 max min ; § 43 quot rest ; § 44 Dvr ; § 45 mlt ; § 46 Cmb. | 1899 | Vacca (§ 46) | 70–81 | 71; 79 | Fermat; Kramp, Gauss, Pascal, Leibniz, Joh. Bernoulli, Cauchy, Pringsheim, *Schlömilch*; *Stolz*, Argan, Weierstrass; Kronecker; V.A. Lebesgue, Euclides, Stieltjes, Euler, Lucas, Bertrand, Barrieu; Pascal, *Hindenburg*, Raabe, Tartaglia, Herigone, Jac. Bernoulli, *Legendre*, Abel, Lagrange | |
| § 50 Np ; § 51 mp ; § 52 $\Phi$ ; § 53 nt dt ; § 60 $\vartheta$ ; § 61 Sgm | | | 82–91 | 88; 90 | Burckhardt, Glaisher, Dase, Dase-Rosenberg, Euclides, Goldbach, P. Bongi, Bertrand, Tchebychef, Girard, Fermat, Leibniz, S. Germain, Heans, Legendre, Euler, Pervouchine, Bikmore, Protii, Lucas, Dirichlet, L. Fibonacci, Wilson, Waring; Girard, Wallis, Liouville; Gauss, Euler | Burali-Forti, F 1895; Padoa RdM 6, 1898, 90–94; Peano 1899c |

**Table 2** (continued)

| *Formulaire* 1899 | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| § 62 $l' l_i \infty$ ; § 63 $Q$ ; § 64 $q$ ; § 65 Log ; § 66 $E$ ; § 67 $\beta$ (Partie fractionnaire) ; § 70 Med (nombre moyen) | | Vacca (§ 67) | 92–109 | 96; 99; 105; 106; 108 | Weierstrass, Darboux, Pringsheim, Guilmin; Chuquet, Oresme, M. Cantor, Girard, Newton, Euclides, Leibniz, D. Bernoulli, MacLaurin, Joh. Bernoulli; Euclides, L. Fibonacci Pisano, Brahmagoupta, Rodet, Euclides, Diophantus, Bachet, Tartaglia; Neperus; Legendre, Gauss, Bertrand, Cesàro; Zehfuss, Wallis, Euclides, Euler, Lucas; Cauchy | Peano 1899c |
| § 71 $\lambda \Lambda \delta$ ; § 72 cresc decr ; § 73 Lm ; § 74 lim ; § 75 Chf ; § 76 e ; § 77 log ; § 78 $C$ | | Peano, Bettazzi, Giudice, *Nassò* (§ 75), Vacca (§ 78) | 110–128 | 114, 116, 124, 125, 128 | Cauchy, G. Cantor, Vivanti; Cauchy, Bolzano, Leibniz, Brouncker, MacLaurin, Abel, Cauchy, Eisenstein, Dirichlet, Mertens, Euler, Joh. Bernoulli, D. Bernoulli, Encke, Newton, Tchebychef, Markoff, Cesàro, Euler, Dirichlet; Cotes, Euler, Vega, Shanks, Boorman, Tichánek, Cauchy, Liouville, Newton, Leibniz, Lambert; § 77 Mercator, Gregorius, Lambert, Jensen; § 78 Euler, Glaisher, Mascheroni, Gauss, Nicolai, Adams | F 1895; RdM |
| § 80 $q_n$ *nombre complexe*; § 81 *intervalles*; § 82 cont *fonction continue*; § 83 perm *permutation*; § 84 Dtrm *déterminant*; § 85 lin *fonction linéaire* Subst Sb *Substitution ou transformation linéaire*; § 86 i$q'$ *nombre imaginaire*; § 87 $\pi$ ; § 88 sin cos tng ; § 89 $\sin^{-1} \cos^{-1} \mathrm{tng}^{-1}$ ; § 90 $B$ *nombres de Jac. Bernoulli* | | | 129–150 | 129, 133, 134, 139, 140, 143, 148 | § 80 Eisenstein; § 82 Thomae, Cauchy, Heine, Lüroth; § 84 Binet, Cauchy, *Leibniz*, Mansion, Smith, Koch; § 86 Gauss, Abel; § 87 Jones, Euler, Ahmes, Archimedes, Ptolemaeus, Aryabhata, P. Metius, A. Metius, Lambert, Legendre, Viète, Adrianus Romanus, Ludolphus, Snell, De Haan, Grienberger, Sharp, Sherwin, Machin, Lagny, Vega, Thibaut, Dahse, Clausen, Richter, Rutherford, Shanks, Wallis, Leibniz, Joh. Bernoulli, Euler, | Peano 1892l; 1895c; 1893h; 1899t, 371; 1888a |

**Table 2** (continued)

| *Formulaire* **1899** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| | | | | | Stirling, Gauss, Cotes; § 88 Euler, Ptolemaeus, Newton, Abel, Hessel, Viète, Wendt, Albategnius, Virgilius, Delambre, Bernoulli, Legendre; § 89 Newton, Leibniz, J. Gregory, Euler, Störmer, Clausen, *Vega*, Machin, Dahse, Jac. Bernoulli, Euler, Raabe | |
| § 91 pnt vct *point vecteur* | | Peano, Vacca | 151–164 | 151, 152, 154, 155, 156, 158–159, 161, 163, 164 | Euclides, Wessel, Grassmann, Hamilton, Möbius, Resal, Somoff, Simpson, Legendre, Pythagoras, Plutarchos, Apollonius, Euler, Thales, Diogenes, Regiomontanus | Peano 1894c, 1898c; Pieri 1897, 1899 |
| § 100 *D dérivée* | | | 165–173 | | Leibniz, Newton, Taylor, MacLaurin, Lagrange, Cauchy, *Arbogast*, Joh.Bernoulli, Schwarz, Genocchi | Peano 1893h, 1899t, 1890f |
| § 101 *S Intégrale* | | | 174–180 | | Cavalieri, Mercator, Leibniz, Fourier, Darboux, Fermat, *Stirling*, MacLaurin, Joh. Bernoulli, Euler, Stolz, Plana, Bierens de Haan | Peano 1899t |
| § 102 rectaTang planOscul Arc | | | 181 | | | |
| Table des Signes | | | 182–188 | | | |
| Publications périodiques citées | | | 189 | | | |
| Bibliographie | | | 190–198 | | | |
| Table des Matières | | | 199 | | | |

**Table 3** Mathematics and History of Mathematics in Peano's *Formulaire* 1895–1908: *Formulaire de Mathématiques* 1901

| *Formulaire* **1901** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| | 1901 | Peano, Nassò, Castellano, Vacca, Vailati, Chini, Boggio; Eneström, Vivanti, Ciamberlini, Padoa, Ramorino, *Buhl* | | | | |
| Préface | 1901.1.1 | Peano | iii–viii | | Möbius, Grassmann, Hamilton, Aristoteles, Leibniz, Erdmann, Gerhardt, Vacca, Lambert, Boole, De Morgan, Schröder, McColl, Tait, Plarr | RdM 6, 1898, 65–74; Peano 1900a; Nassò RdM 7, 42–55; Castellano RdM 7, 58; Vacca RdM 7, 59–66, Chini RdM 7, 66; Boggio RdM 7, 70–72 |
| I. **Logique mathématique** | 1901 | Peano, Vailati, Vacca | 1–38 | 2–6; 33–34 | | F 1897; |
| § 1 Cls $\varepsilon$ $\ni$; $\supset$ $\cap$ $=$ Notations | | Peano, Vailati, Vacca, Zignago | 1–18 | 2–6 | § 1 *Notes*: Chuquet, Leibniz, Girad, Aristoteles, Euler, Segner, Lambert, Gergonne, Pell, Abel, Padoa, Viète, Recorde, Henry, Wallis, Newton; Padoa (Paris 1900); Pieri (Acc. To 1898); Leibniz, Aristoteles, McColl, Boole, Peirce | Vacca RdM 6, 121–125, 183–186; Padoa RdM 6, 105–121; (Paris 1900); Peano 1891i, 1894g; Burali-Forti RdM 3, 1893, 79; 6, 1899, 141; RdM 5, 1895, 185 |
| § 2 $\breve{}$ (ou) | | | 19–21 | | Leibniz, De Morgan, Schröder, Lambert, Peirce, Padoa, Pieri, McColl | F 1897, Peano 1889a, |
| § 3 $\Lambda$ (classe nulle) | | | 22–23 | | Boole, Aristoteles, De Morgan | Peano 1888a, 1889a, F 1897, F 1895 |
| § 4 – (non) | | | 24–27 | | Leibniz, Vailati, Peirce, Boole, Whitehead, De Morgan, Schröder, Lambert | Vailati RdM 1, 1891, 103; Peano 1891g |
| § 5 $\exists$ (existe) | | | 28–29 | | Padoa | Peano 1889a, F 1895, F 1897, F 1899 |
| § 6 $\iota$ (égal à) | | | 30 | | Padoa | |

**Table 3** (continued)

| *Formulaire* **1901** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| § 7 ̇ɿ (le) | | | 31 | | Padoa | |
| § 8 ⁚ (avec) | | | 32 | | De Morgan, Schröder, Hauber, Jevons, Dedekind; *Notes*, 127–129 | |
| § 10 ⅃ (fonction) | | | 33–34 | 33–34 | *Note sur les fonctions*: Euler, Legendre | |
| § 11 \| (inverse) | | | 35 | | | |
| § 12 ‘’ (quelque) | | | 35–36 | | Padoa | |
| § 13 sim rcp idem | | | 37 | | | F 1895, F 1899 |
| § 14 Variab $F$ Funct | | | 37–38 | | | F 1899, F 1897 |
| II. **Arithmétique** | 1901 | | | | | |
| § 20 0 $N_0$+ ; § 21 $\geqslant$> ; § 22 $-$ ; § 23 × ; § 24 / $R$ $r$ (rationnel, rationnel relatif); § 25 (puissance); § 31 … ; § 32 Num infn ; § 33 $\sum$ (somme) ; § 34 $\prod$ (produit) ; § 35 ! (factorielle) $C$ (combinaisons) ; § 36 mod sgn ; § 40 max min ; § 41 quot rest ; § 42 $E$ (Entier de) $\beta$ (partie fractionnaire) ; § 43 Chf (chiffre) ; § 44 Dvr ; § 45 mlt ; § 46 nt dt ; § 51 Np (nombre premier) ; § 52 mp ; § 53 $\Phi$ ; § 54 Nprf (nombre parfait) ; § 60 $\vartheta$ (fraction propre) ; | | Peano, Padoa, Vacca | 39–120 | Notes, 39; Note sur les chiffres, 40, Note sur les nombres positifs et négatifs, 48–49; Note sur les systèmes de numération, 75–78 | § 20 Pythagoras, *Boetius*, *Dedekind*, *Wallis*, Pascal, Babbage; § 21 Girard, Oughtred, Harriot; § 22 *Note sur les nombres positifs et négatifs:* Ahmes, Diophantus, Chuquet, Pacioli, Grammateus, M. Cantor, Widmann, MacLaurin, Cauchy; *Tannery*, *Méray*, *Couturat*, Brahmagupta, Rodet; § 23 Oughtred, Euclides, Diophantus; § 24 L. Fibonacci, Oughtred, Pell, MacLaurin, Leibniz, Euclides, Macfarlane, Ahmes, Pappus, Aryabhata, Diophantus, Prior; § 25 Euclides, Chuquet, Girard, Descartes, Pell, Wallis, Diophantus, Tschu Schi Khi, Stifel, Tartaglia, Cauchy, Fermat, Legendre, Bachet, Lagrange, P. Tannery, Frenicle, Euler, Landry, Pervouchine, Seelhoff, Alchodschandi, *Legendre*, *Dirichlet*, *Lamé*, *Kummer*, *Gambioli*, | Peano 1889a, F 1898, F 1899, F 1895, Peano 1898m, Padoa RdM 6, 1898, 90–94, Nassò RdM 7, 1900, 52 |

**Table 3** (continued)

| *Formulaire* 1901 | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| § 61 $l'$ (limite sup.) $\infty$ (l'infini) $l_i$ (limite inf.) § 62 $Q$ (quantité positive) ; § 63 Log ; § 64 Med (moyen) ; § 65 $\lambda$ (classe limite) $\Lambda$ (limite généralisée) ; § 66 $\delta$ (dérivé) ; § 67 Int (intérieur) | | | | | Harriot, Betrand, Euler, Pythagoras, Platon, Proclus, Friedlein, Cauchy, Degen, Lamé, Oltramare, Leibniz, Chuquet; § 32 G. Cantor, Vivanti; § 33 Lagrange, Cauchy, Pythagoras, Theon, Aryabhata, Alqachani, Nicomachus, Fermat, Wallis, Jac. Bernoulli, Jacobi, Amigues, Lucas, Ibn Albanna, Waring, Oltramare, Ahmes, Eisenlohr, Euclides, Catalan; *Note sur les systèmes de numération*: M. Cantor, Lindemann, Archimedes, Aryabhata, Rodet, Leibniz, Lucas, Cauchy; Archimedes, Aryabhata, Rodet, Cauchy, Leibniz, Lucas, Regiomontanus, Viète, Stevin, Bürgi, Kepler, Mercator, Cauchy, Jacobi; § 34 Fermat, Nicole; § 35 Kramp, Gauss, Pascal, Euler, Cauchy, Raabe, Frenicle, Tartaglia, Herigone, Jac. Bernoulli, Abel, Lagrange, Lucas, Dixon, Vivanti, Joh. Bernoulli, Leibniz, Gergonne, Pringsheim; § 36 Leibniz, Cauchy, Weierstrass, Kronecker; § 42 Legendre, Gauss, Bertrand, Cesàro, Zehfuss, Wallis; § 43 *Planudes*, Euler, Pascal, Wallis, Sibt-el Maridini; § 44 V.A. Lebesgue, Euclides, Stieltjes, Leibniz, Euler, Lucas, Gauss, Bertrand, Barrieu; § 45 Euclides, Stieltjes, V.A. Lebesgue, Bertrand, Barrieu; § 46 Murer, Barrieu, Padoa; § 51 Burckhardt, Glaisher, Dase, Dase-Rosenberg, Euclides, Goldbach, G. Cantor, Aubry, Bungus, Bertrand, Tchebychef, L. Pisano, Girard, | |

**Table 3** (continued)

| *Formulaire* **1901** | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| | | | | | Fermat, Legendre, Euler, Pervouchine, Seelhoff, Gergonne, Goldbach, S. Germain, Heans, Leibniz, Euler, Bikmore, Proth, Lucas, Eisenstein, Matrot, Pappit, Leibniz, Wilson, Waring, Lagrange, Tchebychef, Legendre, Dirichlet, Osborn; § 52 Girard, Legendre, Wallis, Liouville, Barrrieu; § 53 Gauss, Euler, Cauchy; § 54 Nassò, Ciamberlini, Euclides, Descartes; § 61 Weierstrass, Darboux, Pringsheim, Guilmin, Mittag-Leffler, Stifel, Wallis; § 62 Cauchy, Chuquet, Oresme, M. Cantor, Girard, Newton, Euclides, Darboux, L. Pisano, Brahmagoupta, Rodet, Euclides, Diophantus, Bachet, Tartaglia, G. Cantor, Euler; § 63 Napier, Euclides; § 64 Cauchy; § 66 G. Cantor, Vivanti | |
| III. **Fonctions analytiques** | 1901 | | 121–159 | | | |
| § 70 cresc decr<br>§ 71 Lm<br>§ 72 lim<br>§ 73 cont<br>§ 74 ***D*** (dérivée)<br>§ 75 ∫ ***S*** (intégrale)<br>§ 76 e<br>§ 77 log<br>§ 78 ***C*** | | Peano | 121<br>122–124<br>125–135<br>136–137<br>138–146<br>147–153<br>154–156<br>157–158<br>159 | § 74, 138–139 | § 71 Cauchy; § 72 Wallis, Cauchy, Duhamel, Bolzano, Du Bois-Reymond, Catalan, Mansion, Jac. Bernoulli, Eisenstein, Joh. Bernoulli, MacLaurin, Leibniz, Abel, Brouncker, Stirling, D. Bernoulli, Lambert, Euler, Dirichlet, Riemann, Mansion, Cesàro, Mertens, Weierstrass, Dini, Arzelà, Stern, Cauchy, Newton, Encke, Tchebychef, Markoff, Cesàro, Euler, Dirichlet; § 73 Abel, Heine, Lüroth, Cauchy, Weierstrass, G. Cantor, | RdM 2, 1892, 76–77; Peano 1895c; 1893h; 1884c; 1899t, 321; 1889c; 1891a |

**Table 3** (continued)

| *Formulaire* **1901** | Chronology | Authors | Pages | Notes | Historical Sources | Peano's works and RdM |
|---|---|---|---|---|---|---|
| | | | | | Euler, *Pincherle*; § 74 Leibniz, Newton, Lagrange, Arbogast, Cauchy, Jacobi, Rolle, Cavalieri, L'Hospital, Joh. Bernoulli, Taylor, MacLaurin, Schlömilch, Schwarz, Stieltjes; § 75 Cavalieri, Leibniz, Jac. Bernoulli, Euler, Fourier, Darboux, Dirichlet, *Pringsheim*, Bonnet, Weierstrass, Du Bois-Reymond, Cavalieri, Fermat, Mersenne, Stirling, *Wallis*, Legendre, Binet, Cauchy, Riemann, MacLaurin, Weierstrass, Thomé, Darboux, Joh. Bernoulli, Euler, Stolz, Pringsheim, Lagrange, Cotes, *Torricelli, Simpson*; § 76 Cotes, Euler, Vega, Shanks, Boorman, Lambert, Liouville, Newton, Leibniz, Fourier, Stainville, Cauchy, Hermite, Gordan; § 77 Mercator, Gregorius, Adams, Joh. Bernoulli, Euler, Eisenstein, Jensen, Tchebychef; § 78 Euler, Mascheroni, Gauss, Nicolai, Glaisher, Adams | |
| IV. **Nombres complexes** | 1901 | | 160–191 | | | |
| § 80 $q_n$ (nombre complexe)<br>§ 81 Dtrm (déterminant)<br>§ 82 lin (fonct. linéaire) Subst Sb (Substitution)<br>§ 83 i$q'$ (nombre imaginaire)<br>§ 84 $\pi$<br>§ 85 sin cos tng $\sin^{-1}$ $\cos^{-1}$ $\text{tng}^{-1}$<br>§ 86 $B$ (nombres de Jac. Bernoulli) | | Peano, Vacca | 160–163<br>164–166<br>167–170<br>171–174<br>175–180<br>181–189<br>190–191 | 164, § 85, 182–183 | § 80 Weierstrass, Grassmann, G. Cantor, Eisenstein, *Cauchy*, Hilbert, Moore, Cauchy, Lipschitz; § 81 Leibniz, Cramer, Laplace, Cauchy, Binet, Vandermonde, Mansion, Smith, Koch; § 82 Weierstrass, Cayley, Laguerre, Frobenius, Lagrange; § 83 *Note* Bombelli, Euler, Gauss, Weierstrass, Hamilton, Cauchy, Girard, Loria, Abel, Glaisher; § 84 Jones, Euler, Ahmes, Archimedes, Ptolemaeus, Aryabhata, Anthonisz, A. Metius, | Peano 1890f, 1887a, 1888a, 1897c, 1895q, Loria RdM 1, 1891, 185–248 |

**Table 3** (continued)

| ***Formulaire* 1901** | **Chronology** | **Authors** | **Pages** | **Notes** | **Historical Sources** | **Peano's works and RdM** |
|---|---|---|---|---|---|---|
| | | | | | Lambert, Legendre, Mascheroni, Viète, Kochanski, Specht, Gergonne, Lindemann, Gordan, A. Romanus, Ludolphus, Grienberger, Sharp, Machin, Lagny, Vega, Thibaut, Dahse, Clausen, Schuhmacher, Richter, Rutherford, Shanks, Leibniz, Jac. Bernoulli, Gauss, Cotes, Wallis, Joh. Bernoulli, Stirling, Cesàro, Bertrand, Parseval; § 85 Euler, Cotes, Ptolemaeus, Hessel, Lambert, Wendt, Albategnius, Virgilius, Delambre, Bernoulli, Legendre, Gudermann, Abu l Wefa, Viète, Gauss, Bertrand, Machin, Dahse, Vega, Störmer, Newton, Leibniz, J. Gregory, Fourier, Abel, Lobatto, Fresnel, Kepler, Gauss, Cotes; § 86 Jac. Bernoulli, Euler, Rothe, Adams, Serret, Genocchi, Staudt, Clausen, Stirling | |
| V. **Vecteurs** | | | | | | |
| § 91 vct | 1901 | Burali-Forti | 192–209 | | Pieri, Perry, Euclides, Wessel, Grassmann, Hamilton, Möbius, Bellavitis, Archimedes, Carnot, Lagny, Legendre, Pythagoras, Plutarchos, Apollonius, Euler, Thales, Diogenes, Stewart, Heron, Al Battani, Regiomontanus, Abu l Wefa, Delambre, Napier, Macfarlane, *Heaviside, Gibbs*, Hamilton, Lamé, Leibniz | Peano 1889d, 1894c, 1898c, |
| Table des signes | 1901 | | 210–212 | | | |
| Vocabulaire mathématique | 1901 | | 213–216 | | | |
| Publications périodiques citées | 1901 | | 217–218 | | | |
| Bibliographie | 1901 | | 219–230 | | | |
| Table des Matières | 1901 | | 231 | | | |

**Table 4** Mathematics and History of Mathematics in Peano's *Formulaire* 1895–1908: *Formulaire de Mathématiques* 1902–1903

| ***Formulaire* 1902–1903** | **Chronology** | **Pages** | **Authors** | **Historical Sources, Notes & Additions** | **Peano's works and RdM** |
|---|---|---|---|---|---|
| | 1902.2.17 | | Arbicone, Boggio, Cantoni, Castellano, Ciamberlini, Eneström, Padoa, Peano, Ramorino, Stolz, Vacca, Beman, Burali-Forti, Couturat, D'Arcais, Ferrari, Giudice, Invrea, Korselt, Morera, Nassò, Rius y Casas, Severi, Zignago | | |
| Préface | | v–ix | Peano | Aristoteles, Leibniz, Vacca, Couturat, Lambert, Boole, Russell, Whitehead, Euclides, Heiberg | RdM 7, 85–110 |
| Exercices de Logique Mathématique | | x–xvi | | | |
| I. **Logique mathématique**<br>§ 1 $=$ ( ) [ ] { } $\supset$ ˆ<br>§ 2 $\varepsilon$ Cls ,<br>§ 3 Df (Définitions)<br>§ 4 Dm (Démonstration)<br>§ 5 ;<br>§ 6 $\ni$ (qui)<br>§ 7 $-$ (non)<br>§ 8 ˘ (ou)<br>§ 9 $\exists$ | 1902.2.17 | 1–28 | | § 1 *Notes – Lettres – Points et parenthèses – Variables réelles et apparentes*, 3–6; Viète, Leibniz, Recorde, Henry, Wallis, Newton, Aristoteles, Segner, Gergonne, Pell, Abel, (Peano: Vailati, De Morgan, Heiberg), (Invrea: Scolastiques), (Couturat: Leibniz); § 2 *Notes*: Aristoteles, Leibniz, Euler, Boole, McColl, (Korselt: Voigt, Husserl); § 3 (Peano: Burali-Forti); § 4 *Idées et propositions primitives*: Padoa, Pieri, *Substitutions*: Eisenstein § 7 *Notes* Leibniz, Segner, Boole, *Théorie* (Invrea: Scolastiques, Vacca: Aristoteles, Diogenes L., A. Magnus, Leibniz; Peano); § 8 *Notes – Indications historiques – Théories* Leibniz, Couturat, Lambert, De Morgan, Peirce; § 9 *Notes – Théories* Boole, Leibniz, De Morgan, Schröder | F 1901, Peano 1901a, 1891i, 1894c, Burali RdM 3, 1893, 79; Padoa RdM 5, 1895, 185; *Add.* Peano 1902b |

**Table 4** (continued)

| *Formulaire* 1902–1903 | Chronology | Pages | Authors | Historical Sources, Notes & Additions | Peano's works and RdM |
|---|---|---|---|---|---|
| II. **Arithmétique** | 1902.2.17 | 29–52 | Peano, Vacca | § + *Notes – Prop. Primitives – Indépendance – Notes sur les chiffres – Note historique* Maurolicus, Dedekind, Gazzaniga, Burali-Ramorino, Nassò, Stolz-Gmeiner, Mannoury (Peano: Helmolz, Huntington, Dickson, Mannoury); § × *Notes historiques* Euclides, Legendre, Dirichlet, Baltzer, Humbert, Hankel, Oughtred (Rius y Casas; Ferrari; Vacca: Cauchy); § ↑ (Vacca: De Morgan; Korselt; Ferrari); § 15 *Note sur les systèmes de numération*: base 2: Leibniz, Legendre; *Règles pour les opérations mathém.*: Boetius, Crelle, Zimmermann, Ernst, Colson, Fourier, Cauchy, Napier, Genaille-Lucas, Thomas, D'Ocagne, Mehmke | Peano 1889a, 1898m |
| III. **Théorie des nombres** | 1902.2.17 | 53–73 | | § / Rahn, Beman; § Dvr (Vacca; Arbicone; Peano) § Np Kulik, Davis, Haussner, Plana, Mertens (Vacca: Cunningham, Euler; Arbicone; Korselt) | |
| IV. **Algèbre** | 1902.2.17–1902.2.18 | 75–104 | | § 31 *Note sur les fonctions*: Babbage, Servois; § n Padoa, Blater, Arnaudeau (Ferrari; Boggio; Peano: Lagrange; Vacca: Euler, Young, Cayley, Hurwitz, Genocchi; Borio; Korselt: Lindemann); § mod Leibniz, Argand, Cauchy, Gauss, Weierstrass; § R. Stolz, J. Tannery, Couturat, Méray, Pappus, Euclides, Macfarlane, Hamilton | RdM 7, 1901, 73–84 |
| V. **Nombres réels** | 1902.2.19 | 105–121 | | § Q (Peano: Viète; Boggio; Vacca); § Log Briggs; § $\Lambda$ (Peano); § $\delta$ G. Cantor | Peano1889a |
| VI. **Fonctions définies** | 1902.2.19–1902.2.20 | 123–144 | | § F Burali-Forti, Schröder, Russell; § 56 Num infn G. Cantor, Bernstein, Borel, Vivanti; § $\sum$ Cardano (Vacca: Fermat, Cauchy, Legendre, Maurolicus, Hypsicles, Diophantus; Korselt: Muir; Boggio: Hatzidakis; Peano: Sylvester, Lagrange, Carlini); § Nprf Frenicle, Huygens, Sylvester | RdM 6, 1899, 142; F 1895, RdM 7, 52 |
| VII. **Calcul infinitésimal** | 1902.2.20–1902.2.22 | 145–200 | Peano | § lim Hadamard, Legendre, Torricelli, Jacobi, Bonnet (Peano; Giudice; D'Arcais: J. Tannery; Vacca:Goldbach, Euler, Weierstrass, Peano 1893h; Peano: Pringsheim, Ames); § $D$ *Note* [*formule Taylor*] Cauchy, Genocchi, Poincaré (Peano: Genocchi, Goursat; Borio; Boggio: Morera; Vacca); | D'Arcais RdM 5, 1895, 186–189; Peano 1902c; Peano 1884c, 1899t, 1895n, 1887b, 1892q, 1893h |

**Table 4** (continued)

| *Formulaire* **1902–1903** | **Chronology** | **Pages** | **Authors** | **Historical Sources, Notes & Additions** | **Peano's works and RdM** |
|---|---|---|---|---|---|
| | | | | § $\int$ Torricelli, Perelli, Grandi, Simpson (Peano: Darboux; Boggio; Vacca: Cauchy, Cavalieri, Newton, Euler, Lagrange; Morera); § e Cotes, Euler, Laguerre, Soldner, Mascheroni, Caluso, Jac. Joh. Bernoulli; § log Lambert, Koralek, Cauchy, Seidel, Alfonso de Sarasa (Peano: Bradshaw, Euler, Cayley); § 70 Fc [fraction continue] Baltzer, Cataldi, Müller, Schwenter, Favaro, Cayley, Lagrange, Serret (Severi), Euler, Oppermann; § prob Moivre, Andrade (Peano) | |
| VIII. **Nombres complexes** | 1902.2.22–1902.5.2 | 201–223 | Peano | § Dtrm *Note* Günther, Kronecker, Zeipel, Stern (Peano: Cayley), Siacci; § Subst (Vacca: Gauss, Boole, Cayley, Laguerre; Peano: Grassmann, Peirce; Boggio); § i$q'$ *Note* Cauchy | Peano 1890b, 1890f, 1885a, 1892bb, 1887a, 1888b, 1897c, 1895q |
| IX. **Fonctions circulaires** | 1902.5.2 | 225–250 | Peano, Vacca | § sin *Note* Oughtred, Jones, Gudermann, Euler, Stolz, Werner, M. Cantor, Hessel, Aristarchus (Vacca: Le Verrier, Legendre, Jacobi, Gauss, Archimedes, Euler; Peano: Lagrange; Boggio: Meyer; Ramorino); § $\pi$ Peirce, V. Riccati, Plana, Wallis, Kepler, Pappus, Bertrand, Fresnel, Hartmann | |
| X. **Calcul géométrique** | 1902.4.30 | 251–285 | Peano, Pieri, Castellano, Padoa, Vacca | § pnt vct Pieri, Schur, Moore, Peano, Poinsot, Cauchy, Siacci, Burali-Forti, Leibniz, Cagnoli § [motor] Halphen, Stephanos, Mozzi, Chasles (Vacca; Zignago; Peano: Euler, Möbius, Chasles, Carnot, Staudt, Mackay, Stewart, Simpson, Lebon; Cantoni: Desargues, Ceva, Carnot, Steiner, Euclides, Euler, Lexell; Boggio; Pieri: Euler; Peano); § 81 [produit alterné] Grassmann, Hamilton, Saint Venant, Cauchy, Poinsot, Chelini (Peano: Grassmann, Descartes, Möbius, Poinsot, Cayley, Plücker, Hamilton, Appell, Königs, Carvallo, Burali-Forti; Castellano: Resal); § 82 [rotor quaternio] Wessel, Boué, Argand, Cauchy, Hamilton, Maxwell, Heaviside, Macfarlane (Peano: Hamilton, Bellavitis) | Peano 1889d, 1894c, 1898c, 1903a; 1888a, |

**Table 4** (continued)

| *Formulaire* **1902–1903** | **Chronology** | **Pages** | **Authors** | **Historical Sources, Notes & Additions** | **Peano's works and RdM** |
|---|---|---|---|---|---|
| XI. **Géométrie différentielle** | 1902.5.1–1902.5.3 | 287–311 | Peano (§ 83–90), Burali-Forti (§ 91–95) | § 83–84 (Peano: Euclides, Descartes, Monge); § 85 [rectaT] Saint Venant, van Heuraet, Descartes, Archimedes, Apollonius, Leibniz, Mersenne, Pascal (Peano: Archimedes, Grégoire St Vinc., Descartes, Mersenne, van Heuraet, Huygens, Torricelli, Loria, Leibniz, Wallis, Roberval, Joh. Bernoulli, Jac. Bernoulli, Mansion, Wren, Wallis, Galilei, Pascal); § 87 [Tang] Descartes; § 89 [Long Area Volum] Euclides, Bricard, Sforza, Dehn, Cavalieri, Schwarz, Borchardt, Minkovski, Archimedes, Kepler, Harriot, Girard, Viviani, Euler; § 90 [paramètre différentiel] Hamilton, Lamé, Leibniz; § 91 [curvatura] Burali-Forti, Frenet, Serret; § 92 [flex cusp] Burali-Forti | Peano 1887b, 1890c, 1890g |
| Additions | 1903. 3.12 | 313–366 | | | |
| Abbréviations | 1903. 3.12 | 368 | Vacca | | |
| Notices biographiques et bibliographiques | 1903. 3.12 | 369–385 | Vacca | | |
| Table des noms d'auteurs | 1903. 3.12 | 386–390 | | | |
| Publications périodiques citées | 1903. 3.12 | 390–392 | | | |
| Vocabulaire mathématique | 1903. 3.13 | 393–406 | | | |
| Table des Matières | | 407 | | | |

**Table 5** Mathematics and History of Mathematics in Peano's *Formulaire* 1895–1908: *Formulaire de Mathématiques* 1906–1908

| ***Formulario* 1906** *Proba de 100 exemplare* | **Chronology** | **Pages** | **Authors** | **Historical Sources & Marginal Notes** in Peano 1906g* |
|---|---|---|---|---|
| Indice | 1906 | i–v | Peano | |
| Vocabulario | 1906 | vii–xlvii | Peano | |
| I. **Logica-Mathematica**<br>§ 1 aequale,tunc, et<br>§ 2 Classe, $\varepsilon$<br>§ 3 (que)<br>§ 4 – (non)<br>§ 5 ˘ (aut)<br>§ 6 $\Lambda$ (classe nullo), $\exists$ (existe)<br>§ 7 $\iota$ (aequale ad), $\dot{\rceil}$ (illo)<br>§ 8 Df (Définitione), Dfp (Définitione possibile), Vocabulario I | 1906 | 1–16<br>17–24 | Peano | *Vailati,* Viète, Leibniz, Recorde, Newton, Chuquet, Bernoulli, Euler, *Aristotele, Trendelenburg, Kant,* Gergonne, Abel, De Morgan, Aristotele, Möbius, J.S. Mill, Vailati; Historia (p. 16–17): Leibniz, Couturat, Lambert, De Morgan, Boole, Schröder, Burali-Forti, Russell, Wilson, Whitehead, Huntington |
| II. **Arithmetica**<br>§1–18, Vocabulario II | 1906 | 25–64,<br>65–70 | Peano, Castellano | *Pieri*, Lindemann, *Carra de Vaux*, Huntington, Dickson, Burali-Forti, Fine, Mannoury, *Leibniz*, Descartes, De Morgan, *Todhunter*, Bungus |
| III. **Algebra**<br>§1–27, Vocabulario III | 1906 | 71–154,<br>154–162 | Peano, Burali-Forti | *Hamilton, Lagrange*, *Bonatelli*, *Maccaferri*, *Peano, Burali-Forti*, *Whitehead-Russell*, *Marcolongo*, *Catania*, *L. Pisano, Ta-yen*, *Amodeo*, Ahamesu, *Calvitti, Quarra, Darboux*, *Cardano, Jac. Bernoulli, Hermite, Bertolani, Pascal, Dedekind,Dini, Hadamard, Descartes* |
| IV. **Geometria**<br>§ 1–4, Vocabulario IV | 1906 | 163–201,<br>202–208 | Peano, Pieri, Padoa, Burali-Forti | *Föppl*, Indep. Prop. primitivo (p. 166–167), Historia (p. 167) |

**Table 5** (continued)

| ***Formulario* 1906** *Proba de 100 exemplare* | **Chronology** | **Pages** | **Authors** | **Historical Sources & Marginal Notes** in Peano 1906g* |
|---|---|---|---|---|
| V. **Limites**<br>§ 1–8 | 1906 | 209–272 | Peano | *Sannia, Weierstrass, W.H. Young, Arzelà, Zermelo, Cesàro, Frobenius,* Peano 1890b, Hilbert, Cesàro, Moore, H. Lebesgue, *Broglio, Bessel, Oughtred,Ptolemaeo, Legendre* |
| VI. **Calculo differentiale**<br>§ 1 D (derivata) Theor. de max, min; Applicationes ad Geometria, Theor. de Rolle, de valore medio, de De L'Hospital, de Bernoulli-Taylor, de Lagrange, rectaT, planN, planO, curvatura, torsio, Serie de Lagrange, Derivatas partiale, Derivata de functione de numero complexo, Tang (Figura tangente), Derivata de potentiale, Applicationes | 1906 | 273–312 | Peano | Nota (p. 276–277), Morera RdM 2, 1892, 36, Tinseau, Leibniz, Poinsot, Steiner, Peano 1887b, Hurwitz, Wetzig, Baker, Sturm, Frenet |
| VII. **Calculo integrale**<br>§ 1 S (integrale), integrale supero, integrale infero, integrale de $f$, theor. de valore medio, integratione per serie, integrales improprio, relatione inter derivata et integrale, formulas de quadratura, Serie de Fourier, Integrale multiplo, Variatione de integrale, Arc, Long Area Volum, centro de gravitate, momento de inertia, variatione de arcu | 1906 | 313–370 | Peano | Historia (p. 321): Cauchy, Peano 1895n, Dirichlet, Pringsheim, Thomae, Joh. Bernoulli, Stolz, G. Cantor, Cavalieri, Gregory, Simpson, Rimondini, Moigno, Coqué, Fuchs, Peano 1887b, 1888b, Picard, Soldner, Mascheroni, Caluso, Grégoire S. Vinc., A. de Sarasa, Fresnel, Boggio, H. Lebesgue, Sibirani, Fréquet, *Legendre,* Dainelli, Viviani, Stokes, Gibbs, Marcolongo, *Müller, Lagrange, Gauss* |
| VIII. **Theoria de curvas**<br>§§1–27: Parabola, Ellipsi, Hyperbola, Parabola de vario ordine, Linea exponentiale, Catenaria Tractoria, Sinusoide, Tangentoide, Curva de luce, Spira mirabile, Spirale de ordine $m$,Spirale de Archimede, Spirale de ordine –1, Cochleoide, Sinus-spirale, Cycloide, Evolvente de circulo, Asteroide, Epicycloide, Limace de Pascal, Cardioide, Cissoide de Diocle, Podaria, Conchoide, Conchoide de Nicomede, Helice, Inversione | 1906 | 371–391 | Pagliero | De la Goupillière, Castiglioni |

**Table 5** (continued)

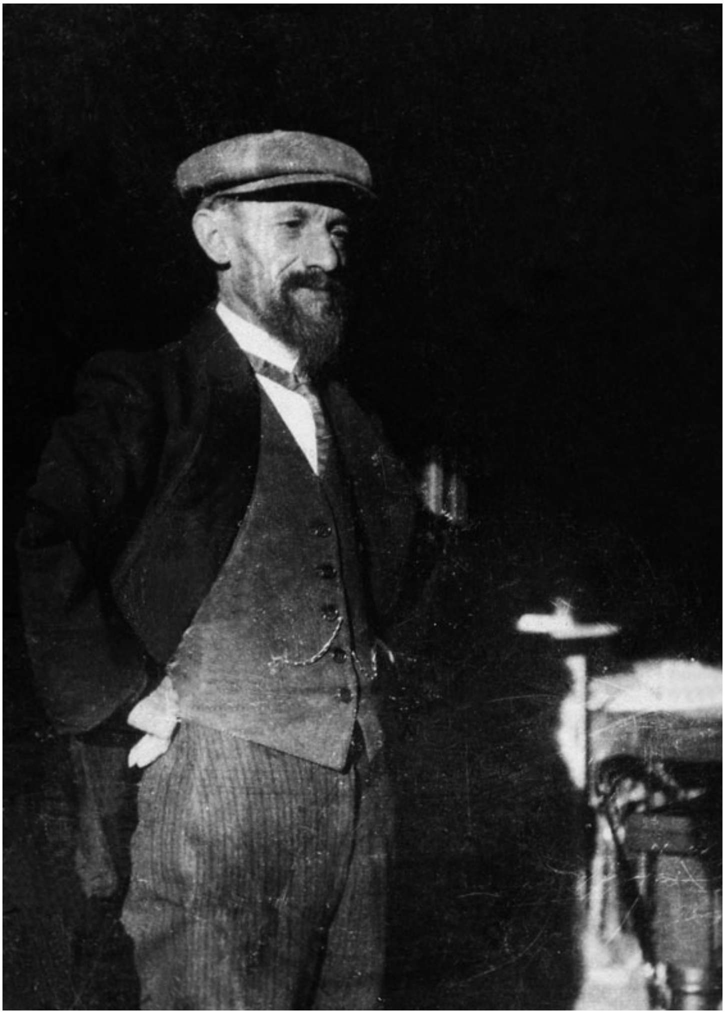
| *Formulario* **1908** | **Chronology** | **Pages** | **Authors** | **Historical Sources & Marginal Notes** in Peano 1908a* |
|---|---|---|---|---|
| Praefatione | | i–xiii | Peano | |
| Bibliographia de Logica-Mathematica post anno 1900 | Junio 1908 | xiv–xvi | Peano | *Huntington*, *Bôcher*, *Pierpont*, *Whitehead-Russell*, *Catania*, *Moore*, *Burali-Forti*, *Maccaferri*, *Mago*, *Pastore*, *Peano*, *Della Casa*, *Quarra*, *Shearman*, *Vacca*, *Wilson* |
| Tabula de symbolos | 1908 | xvii–xix | Peano | |
| Indice alphabetico et abbreviationes | 1908 | xix–xxii | Peano | |
| Publicationes periodico ... | 1908 | xxii–xxiii | Peano | |
| Bibliographia | 1908 | xxiv–xxxv | Vacca, Pagliero | *Peet*, *van Roomen*, *Chu Shi-ki* |
| Correctiones | 1908 | xxxv–xxxvi | Chionio, Korselt, Pagliero, Pensa, Sannia | |
| I. **Logica-Mathematica** §1–8, Vocabulario I | 1908 | 1–17, 17–24 | Peano | Historia, 16–17 |
| II. **Arithmetica** §1–18, Vocabulario II | 1908 | 25–64, 65–70 | Peano, Castellano | |
| III. **Algebra** §1–27, Vocabulario III | 1908 | 71–154, 154–162 | Peano, Burali-Forti | |
| IV. **Geometria** § 1–4, Vocabulario IV | 1908 | 163–201, 202–208 | Peano, Pieri, Castellano, Padoa, Burali-Forti, Vacca | |
| V. **Limites** § 1–8 | 1908 | 209–272 | Peano | |

**Table 5** (continued)

| ***Formulario* 1908** | **Chronology** | **Pages** | **Authors** | **Historical Sources & Marginal Notes** in Peano 1908a* |
|---|---|---|---|---|
| VI. **Calculo differentiale** § 1 D (derivata) Nota, differentiale, 5 D de summa, 6 D de producto, 7 D de quotiente, 8 D de potestate, 9 D de functione de functione, 10 Functione inverso, 11 D de radice, 12 D de exponentiale, 13 D de logarithmo, 14 Exercitio, 15 D de numero complexo, de vectore et de puncto, functione de variabile reale, 16 D de productointerno et alterno, 17 Functione imaginario de variabile imaginario, 18 D de functiones trigonometrico, 20 Theor. de maximo et minimo; Applicationes … ad Geometria, 21 Theor. de Rolle, 22 Theor. de valore medio, 24 Approximationes, 25 Integrale de polynomio, 26 Altero theor. de valore medio, Exemplo, 27 Theor. de De L'Hospital, 28 D de serie, 30 D de ordine sup; 31 interpolatione de primo gradu; interpolatione in tabula de log; 32 Serie asymptotico de potestates; 33 Max et min de functione; 34 Theor. de Lagrange, 35 Serie de potestates, Serie de Taylor et de MacLaurin, 36 Serie duplo, 37 Rationes incrementale successivo; 38 functione interpolante, 39 ratione incrementale de ordine ***n***, 40 functione integro; 42 Serie de Lagrange; 43 Functione complexo; 44 Valore medio pro functione complexo; 45 Recta Tangente, 46 Plano Normale, 47 48 Plano Osculatore, Axi, Centro de curvatura, Radio de curvatura; 49 curvatura, 50 torsio, 54 Coordinatas, 57 Aeq. diff. lineare; 60 Motu de puncto grave; 61 Motu centrale; 62 Puncto grave in medio resistente; 63 Aeq. lineare de ordine duo; 64 Motu harmonico; 65 Systema de Aeq. diff. lineare; 66 D partiale, 67 D de functione de numero complexo, 68 Tang (Figura tangente), 70 Plano tangente ad superficie; 71 D de potentiale; 73 Relatione inter potentiale et energia | 1908 | 273–336 | Peano | *Theor. de valore medio*: Grassmann, Weierstrass, G. Cantor, Ossian-Bonnet, Serret; Perry, *Altero theor. de valore medio*: Mercator, Cauchy; *Theor. de Lagrange*: Dem. 1, Dem. 2: Bernoulli, Dem. 3, exemplo; *Serie de Taylor et de Mac Laurin*: Stirling, D'Arcais, Cauchy; *Historia* (p. 303–304): Joh. Bernoulli, Taylor, Pringsheim, MacLaurin, Arbogast, Lagrange, Cauchy; *Rationes incrementale successivo*: Ampère, Newton, Waring, Lagrange, Cauchy, Schwarz, Stieltjes, *Serie de Lagrange*: Rouché, Lagrange, Laplace, Levi-Civita; *Motu de puncto grave*: Galilei; *Motu centrale*: Newton; *Puncto grave in medio resistente*: Newton; *Motu harmonico*: Newton; *D de functione de numero complexo*: Jacobi, Grassmann, Hedrick; *Tang*: Descartes, *D de potentiale*: Lamé, Hamilton, Laplace, Green |

**Table 5** (continued)

| *Formulario* **1908** | **Chronology** | **Pages** | **Authors** | **Historical Sources & Marginal Notes** in Peano 1908a* |
|---|---|---|---|---|
| VII. **Calculo integrale** § 1 S (integrale) 1 Polygono circum., 2 Polygono irscr., 3 Integrale supero, Integrale infero, 4 Integrale de $f$, 6 Decompositione de intervallo basi, 7 Integrale de summa, 8 cres, 9 cont, 10 Theor. de valore medio, 11 Integrale inter limites, 12 Relatione inter derivata et integrale, 14 Integratione per partes, 15 Substitutione in Integrale, 16 Integrale de potestate, 17 Integrale de functione integro, 19 Integrale de functione rationale, 20 Integrales improprio, 24 Functiones irrationale, 25 Integrale Euleriano, 26 e S, 28 sin cos S, 40 Integratione per serie, 44 Formulas de quadratura, 46 Arcu, 48 Long, 51 Novo conditione de Integrabilitate, 53 Area, 56 Volum, 59 Volumen de cylindro et de cono, 60 Volumen de sphaera, 62 Volumen in coordinatas curvilineo, 63 Area de superficienon in plano | 1908 | 337–386 | Peano | *Historia* (p. 342–344): Euclide, Archimede, Kepler, Cavalieri, Wallis, Leibniz, Jac.Bernoulli, Ascoli, Thomae, Encyclopädie, Darboux; 13 *Lüroth*; 16 *Integrale de potestate Historia* (p. 351–352): Euclide, Archimede, Cavalieri, Fermat; 22 *Integrales improprio*: Plana, Euler, 24 *Functiones irrationale*: Wallis, 45 *Newton, MacLaurin, Steiner*, *Quarra*, *Peano* 1887b, *Tchebysceff, Lampe*, 47 Jordan, Scheffer, H. Lebesgue, *de la Vallée Poussin, Tonelli, Fubini,* 48 Du Bois Reymond, 52 *H. Lebesgue*, Vitali, 57 Kepler, Cavalieri, Peano 1887b, 60 Archimede, 63 Borchardt, Minkowski, Lebesgue, Fréquet, Sibirani |
| VIII. **Applicationes ad geometria et complemento** § 28 $\gamma$ (constante de Eulero), § 29 Complemento super numeros complexo, Dg (derivata generale), § 30 Aequationes Differentiale, § 31 Integrale elliptico, § 32 Producto de duo serie, § 33 Functione de variabile imaginario, § 34 Serie Fourier, § 35 Limite de integrale, § 36 Derivata de integrale, § 37 Commutatione de integratione, § 38 Integrale multiplo, Centro de gravitate, Momento de inertia, Variatione de integrale, Variatione de arcu, § 39 Substitutiones de vectores, determinante, scalare, homographia, velocitate, derivata, Integrale de linea et de superficie | 1908 | 387–407, 408–459 | Pagliero (§§1–27) Peano (§§28–38) | *Archimede*, Torricelli, L'Hôpital, Huygens, Cotes; 29 *Nota* (p. 412) Bolzano,Cauchy, Weierstrass, *Dini*, *Du Bois Reymond, Hedrick*, 30 Euler, Lacroix, Cauchy, Jourdain, Zermelo, *Historia* (p. 429): Cauchy, Lipschitz, Peano 1890f, Peano 1892bb, Mie, Encyclopädie, de la Vallée Poussin, Arzelà, Osgood, Bliss, Bolza, *Brouwer*, Methodo de approximationes: Cauchy, Coqué, Fuchs, Peano 1887a, Peano 1888b, Peano 1897c, Encyclopädie, Bôcher, Picard, *Lindelöf,* 34 *Feyer*, *Euler, d'Alembert, Clairaut,* 35 Osgood, Richardson, 36 *Leibniz, Markoff,* 39 Hamilton, Grassmann, Maxwell, Fick, Heaviside, Burali-Forti, Marcolongo, *Wilson*, *Green,* Stokes, *Hankel, Gibbs* |
| Indice | 1908 | 461–463 | Peano | |



Giuseppe Peano and the printing-press in his villa in Cavoretto (Turin) – Department of Mathematics G. Peano, University of Torino

# 7

# Giuseppe Peano: a Revolutionary in Symbolic Logic?

Ivor Grattan-Guinness

## 7.1 Three Early Mathematical Interests

In this paper I consider Peano's main mathematical concerns in the 1880s, and the relations between them. I shall propose that he had a sort of magical moment that led him to create his mathematical logic, but also that he was obscure, or at least unclear, about one of the major attendant changes in thought. The material covered is summarised historically in Grattan-Guinness (2000, especially chs. 2, 4 and 5), and treated in more detail in various works cited there.

One concern was mathematical analysis, where, in common with many Continental mathematicians, Peano was aware of the stress currently being laid upon rigour and wished to convey at least some of its features in his university teaching. The chief source of ideas was the lecture courses given quite frequently at Berlin University by Karl Weierstrass (1815–1897): his views were propagated, at both research and teaching levels, both by Weierstrass's students and by older mathematicians such as Peano who were sympathetic to the approach. Staple features included the subtleties required in handling multiplelimits in contexts such as the convergence of functions rather than series of constants, *definitions* of irrational numbers rather than just remarks about them, exploration of the range of mathematical functions that could be given analytic expressions, and the clarification and even extension of the definition of the integral as the limiting value (if it exists) of a sequence of partition sums (Grattan-Guinness 1970, ch. 6). The first major outcome for Peano was his edition of the lecture course of his teacher Angelo Genocchi, which he rather upstaged with his own notes (Peano 1884c). He presented an excellent selection of points in these and related areas, some of the examples and contributions being his own.

The main successor to this edition was Peano's textbook on 'the application of the calculus to geometry' (1887b). Presumably for educational reasons he did not impose all the Weierstrassian refinements; on the other hand he made an attempt to extend the integral to the notion of content[1].

A rather unusual feature of the book was Peano's deployment of new algebraic techniques, especially vectors and determinants. One of his sources for the former was the calculus of extension of Hermann Grassmann (1809–1877); Peano made limited use of it, but he recognised its significance to the extent of quickly producing a short textbook (Peano 1888a) on it. Grassmann had developed his theory in the 1840s, especially the book *Ausdehnungslehre* (1844), to little early response; but it had gained attention from the 1860s onwards after a second edition of his book. Most of the interest was taken in German-speaking countries[2], but there was some also in Italy[3], and Peano saw it worthy of reaching a wider audience.

One unusual feature of Peano's presentation was the emphasis that he laid upon the pertaining logic, which he outlined in the opening chapter. His main inspiration seems not to have come from the Grassmann literature itself (a point to which we return in § 4) but to a short book (1877) by the German mathematician Ernst Schröder (1841–1902). As a rule, in all ages mathematicians take logic for granted, but just invoke results when convenient (for example, the modus ponens rule of inference, and the proof method by contradiction). However, in recent decades a modest tradition of symbolic logic had been developing, especially in Britain (Liard 1878), in which logical principles and properties were cast into algebraic forms of various kinds. The principal initiatives had started in the 1840s and 1850s: Augustus De Morgan (1806–1871) on the symbolisation of various aspects of syllogistic logic (such as the forms of the constituent propositions in a syllogism), and also the introduction of a logic of relations; and George Boole (1815–1864) with a version of the logical laws and methods that are still known after him, applied to terms such as 'man' associated with collections and building propositions stating relationships between collections. The next main stages had been taken elsewhere, in the 1870s, chiefly by the American C.S. Peirce (1839–1914) and then by Schröder, who ran together both the logic of relations and a modified form of Boole's algebra. This whole tradition has become known as 'algebraic logic', and in their respective short books Schröder and Peano gave good basic expositions of the calculus of collections, although excluding the logic of relations.

Peano's next short book is the most important of the sequence, and the one that inspired the word 'revolutionary' of my title: his *New method of expounding the principles of arithmetic*, which contains among other things his axioms for arithmetic (Peano 1889a). However, he also handled symbolic logic in a way that has changed its character in a fundamental way, as we shall see in the next two sections.

The last in this sequence of Peano's short books treated 'the principles of geometry expounded logically' (Peano 1889d). He presented his new logic, and also used parts of Grassmann's algebra to symbolise the basic notions of geometry; however, in a note he credited only Boole as a predecessor in symbolic logic!

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1 T.W. Hawkins (1970), 87–91.

2 G. Schubring (1996).

3 U. Bottazzini (1985).

## 7.2 Peano on 'Classes'

In addition to its use of algebraic principles, algebraic logic has some other noteworthy characteristics. In particular, it made use of the traditional theory of collections often known as 'part-whole theory', where a collection of objects may be divided into parts, such as the collection of Italians having Italian women as a part (and Italian non-women as its complement). These collections were parts of some all-embracing universe, sometimes given names such as 'all'; its empty complement was 'nothing' or 'null'. There was no distinction between membership of an object J to a collection and inclusion of J within it as a part. All the algebraic logicians named above, and also Grassmann, used this theory.

In his book on Grassmann Peano followed the main symbols and properties of Grassmann's algebra. Thus when he spoke of 'classes' he was surely using the traditional part-whole theory of collections, as Grassmann himself had done[4]. Now in his book on arithmetic Peano also wrote 'classes'; but here he certainly intended Cantor's set theory, for early on he introduced the symbols '$\epsilon$' for membership (associating it with 'is') and 'C' for (improper) inclusion, along with '[$x\epsilon$]' (later, '$\ni$') for the converse notion 'such that' to membership (1889a, 27–28). It is two great pities that he used the same word 'classes' for this different theory of collections, and that he did not explicitly indicate the change of sense[5]. At one place he muddled up an individual with its unit set, so falsely claiming a theorem (no. 56); but this was just a slip made by someone working in new territory, not a sign of him reverting to the traditional theory. In the succeeding short book on geometry he also deployed set theory to handle *class, or category of entity*, again without mentioning the change (1889d, 59).

The question arises of Peano's knowledge of Cantor's set theory at this time. He was aware of its origins in Cantor's work on Fourier series in the early 1870s, for he cited the paper involved (and also the short book of 1872 on irrational numbers by Richard Dedekind (1831–1916)) in the first page of his notes 1884c of Genocchi's textbook. But the main development of Cantor's theory was much more recent, effected in around a dozen papers published between 1877 and 1885, and it is not clear how familiar was this material to Peano. The use made of it in his book 1887d on geometry suggests that he knew at least the basic properties of manipulating sets and of point set topology, especially in the chapter on 'geometrical magnitudes' where he attempted to extend the notion of integral, as indeed Cantor had tried to do before him. He also knew the short book of 1888 on the foundations of arithmetic by Richard Dedekind (1831–1916), for he cited it in his own account of arithmetic (Peano 1889a, 22); and he could have picked up some elements of Cantor's theory there also.

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[4] The English translation of the book on Grassmann renders at one point Peano's 'nullo' (G. Peano 1888a, 18) as 'the empty set' (L.C. Kannenberg 2000, 14), which is surely far too Cantorian. Note also 'nulla' as 'empty' at G. Peano (1888a), L.C. Kannenberg (2000), 2.

[5] Peano's own copy of the arithmetic book contains many annotations, but none relates to this point (C.S. Roero 2002, Peano's file 1889a+). Borga notes the change in theory of collections, but he does not bring out its significance (M. Borga 1985, 26).

Unfortunately Peano remained unclear on the distinction between the two theories of collections on later occasions. For example, he did not mention it in a partly autobiographical paper on mathematical logic 1896j presented to the Turin Academy. Worse was to come in 1912 over an article on his logical work written by the English logician and historian Philip Jourdain (1879–1919), who sent him the manuscript for comments to include on publication. To the passage on the early period, in which Jourdain himself was not clear on the change of theories of collections, Peano added this remark (Jourdain 1912, 273), in which it was explicitly ignored:

> In 1888, I followed the calculus of classes after Schröder; it is very clear. In 1889, I followed the calculus of propositions; it is more extensive. At the present day, I give the preference to the calculus of classes, as it is more precise and rigorous.[6]

## 7.3 Peano's Unsung Quantification Theory

Peano's note to Jourdain was opaque also on his version of symbolic logic, to which we now turn. An important feature of algebraic logic is quantification theory, which was introduced in the early 1880s by Peirce and his student O.H. Mitchell (1851–1889)[7]. In the algebraic spirit they construed universal quantification as an extension of conjunction and existential quantification similarly of disjunction, and gave them the appropriate respective symbols '$\prod$' and '$\sum$'. Peano seems to have known of this origin, for he cited two of Peirce's papers in his book by Grassmann (Peano 1888a, x); but his own version of it was quite different, for reasons linked to his change in theory of collections.

As we have seen, Peano was drawn into algebraic logic especially by the book by Schröder, with its attendant part-whole theory. The magic moment that I conjecture came to him was the insight, presumably some time during the winter of 1888–1889, that *Weierstrassian mathematical analysis, including its underlying arithmetic, was also susceptible to treatment by symbolic logic*. But what kind of logic? Part of it was the propositional calculus (1888a, 24–27), which had been introduced by the logician Hugh MacColl (1837–1909) in 1877–78; Peano cited this paper (1889a, x), and presumably took over from it the need for such a calculus in mathematics in general. But he also needed to be able to express logically the properties studied in mathematical analysis, which required some kind of dissection of propositions. The appropriate form to adopt in many cases was conditional propositions involving universal quantification *over a Cantorian set of values* of some pertinent argument variable $x$: for all $x$, if $f(x)$ then $g(x)$. Note that this is not quite the predicate calculus as we know it, where the predicate is itself notated by, say, '$f(x)$', but

[6] Note by Peano in P.E.B. Jourdain (1912), 273.

[7] R. Dipert (1994).

a proposition in which $x$ was associated with one or more predicates and available for universal quantification, maybe along with other predicates and variables. For propositions of this kind he proposed the notation '$\supset_x$' to express implication with universal quantification over $x$, along with '$\supset_{x.y}$' for quantification over $x$ and $y$, and '$=_x$' for equivalence over $x$. He did not introduce a symbol for existential quantification, but specified it in the form 'the set of $x$'s such that.... is not empty' (for example, Peano 1889a, 16 on rational numbers). Astonishingly, he omitted both notations from his lists of symbols (Peano 1889a, vi–viii).

As regards collections, the difference from Peirce lies in the fact that he was quantifying over Cantorian sets, not over part-whole collections; his 'mathematical logic', as he was to call it later, differed *fundamentally* from its algebraic parent in this respect. Algebraic and mathematical logic differed over the propositional calculus merely on points of presentation (for example, only the algebraists emphasised duality). However, the two traditions were poles apart concerning their handing and use of the predicate calculi, including quantification and also relations, especially because of their different theories of collections and also the much closer links of mathematical logic to mathematical theories. Thus, in launching mathematical logic in his book on arithmetic and continuing it in his succeeding book on geometry, he started a significant change – a revolution – in the domain of symbolic logic[8]; indeed, his mathematical version would come to *eclipse* the older algebraic tradition. It is incomprehensible that he did not mention these points in any of his remarks to Jourdain, especially in the one quoted above.

## 7.4 Two Further Figures

I finish this survey of the early Peano by noting the place in it of the work of two other logicians. The first one is Grassmann – that is, Robert of that ilk (1815–1901), the younger brother of Hermann, and even more obscure at that time. His potential relevance is that he adopted Hermann's theory to develop an algebraic logic; for example, the early 1870s he formed a Boolean algebra, seemingly unaware of his predecessor[9]. Peano cited both brothers in the preface of his book on Hermann; but his source on Robert may only have been the citation in Schröder's book (1877, iii), not personal reading. Indeed, while in general Peano's knowledge of the literature was impressive, Robert Grassmann appears to have escaped his net; for example, he did not include him in the bibliography given in the last edition of Peano's *Formulario* (Peano 1908, xxiv–xxxv), nor, as far as I can see, in earlier editions.

The same judgment and source appear to obtain concerning the second figure, Gottlob Frege (1848–1925). He had pioneered parts of mathematical logic (including propositional and predicate calculi, quantification, and some bits of the logic of relations and of set theory) in his *Begriffsschrift* (1879) in order to elucidate his

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[8] I. Grattan-Guinness (2004).

[9] G. Schubring (1996), 211–227.

claim that this logic could ground arithmetic. Unfortunately the reception of this and his later writings was as modest as had been the initial reactions to the Grassmanns: in particular, Peano's awareness of it in the late 1880s seems to have been confined to a review (Schröder 1880) of Frege's book. Peano annotated extensively his own copy of his book on Grassmann, and in the notes on Frege he cited the review as well as Frege's book; and he wrote out exactly, and only, the examples of the propositional calculus that Schröder had transcribed[10].

## 7.5 On the Aftermath

We have seen Peano negotiate his way through some pretty fundamental changes: from collections to sets, and from algebraic to mathematical logic, including the calculus of propositions. Now a characteristic of the algebraic logicians, especially Schröder, was enthusiasm over analogies between notions in arithmetic and in the calculi of classes and propositions; for example among many, the numbers 0 and 1, the null and universal classes, and contradictions and tautologies; and the pairs of operations addition and multiplication, union and intersection, and disjunction and conjunction. In these early writings Peano was amenable to such analogies, but gradually they decreased in significance, and became largely absent from the mathematical logicians (and indeed already so in Frege).

But from another point of view there is a attitude *common* to both kinds of logician, which was quite marked in the mathematics of the late 19th century, especially in algebra and geometry[11]: a profusion of *new kinds of theory*. We see algebras both common and uncommon (for example among many of the latter, Grassmann's and Boole's), and also geometries Euclidian and non-Euclidian (and also projective, line and descriptive). It was essential to specify the basic assumptions upon which such a theory rests, and this need led to a rise in the status of axiomatics. Peano's work is a typical and significant example of this approach, not only during this early period but equally so in his later developments. One consequence was the need to distinguish axioms (or laws) from definitions from theorems: he was conscious of the importance of these distinctions, although he was not always successful in effecting them[12].

Unusual among mathematicians was Peano's extensive study of symbolic logic, where axioms, definitions and theorems also had to be sorted out. One later occasion when he discussed that issue was a talk (Peano 1901a) on definitions in mathematics, given at the International Congress of Philosophy in August 1900; it led to a dispute in the discussion period with fellow mathematician-logician Schröder over the correct manner of defining sets. This event was a magic moment for a young English member of the audience then deeply involved in the foundations of mathematics, Bertrand Russell (1872–1970); he was sure that Peano had won, and importantly so.

[10] U. Bottazzini (1985), 48–49.

[11] I. Grattan-Guinness (1997), ch. 13.

[12] F. Rodriguez-Consuegra (1991), ch. 3.

Peano always distinguished mathematics from logic, giving separate lists of symbols for them in his presentation of a mathematical theory. However, from the 1890s onwards the difference had become somewhat porous, in that notions of set theory could appear in both lists. This blemish may have been part of his influence upon his new English supporter in the new century. After his magic moment in August 1900 Russell quickly adopted Peano's mathematical logic and extended it with a logic of relations, and merged Peano's two lists by conceiving a philosophical position that has become known as 'logicism', which claims that mathematical logic already contains enough stuff in it to encompass also all the needs of mathematics[13]. Here we see Russell as a revolutionary, for in adopting this stance he not only rejected Peano's position but in applying his post-Peanist logic to mathematics he also reversed the relationship between the two subjects adopted by the algebraists.

[13] I. Grattan-Guinness (2000), ch. 6.

# 8

# At the Origins of Metalogic

Ettore Casari

## 1.

Among the many aspects of the logical and foundationalistic work of Peano, there is one which, though not ignored, seems not yet to have attracted all the attention it deserves: that which deals with *metalogic*. Yet the relationship between Peano and this basic theme is decisive both for full appreciation of his work, and for a genuine understanding of its destiny. While he was, indeed, – as we will try to show – one of the genuine founders also of this logical discipline, it was precisely its development that played a decisive role in the decline and even the disappearance from the international scene of his logical tradition.

## 2.

Our considerations must necessarily start from the organisation of the theory of natural numbers. The difference between Frege's problem on the one hand and Dedekind's and Peano's on the other – the former essentially interested in the nature of numbers as such, whereas for Dedekind and Peano interest focussed on the system constituted by numbers as a whole – is well known even to those who have never really approached the problem. What is less generally known is that there are important differences between the position of Dedekind and that of Peano.

We may set aside the question – which is of essentially biographical interest – whether, in accordance with the *opinio communis*, Peano arrived at his axioms by reading Dedekind, or, as he himself had occasion to affirm in a passage recently recalled by Grattan-Guinness[1], he did so quite independently[2] – a declaration in whose

[1] I. Grattan-Guinness (2000), 228.

[2] G. Peano (1898e), 85: "La composizione del mio lavoro a. 1889 fu ancora indipendente dallo scritto menzionato del Dedekind; prima della stampa, ebbi la prova morale dell'indipendenza delle proposizioni primitive da cui io partivo, nella loro coincidenza sostanziale colle definizioni del Dedekind. In seguito riuscii a dimostrarne l'indipendenza" (The composition of my work in 1889 was still independent of the Dedekind's mentioned writing; before it was printed, I had the moral proof of the independence of the primitive propositions from which I started, in their substantially coinciding with Dedekind's definitions. I was subsequently able to prove this independence).

favour, by the by, there is really only his honesty and his almost manic 'scrupulousness in attribution'. What concerns us here is instead the existence of two important differences, of which only the first seems to have been given due attention.

## 3.

It is a matter, in the first case, of that difference that has its roots in the profound difference in the philosophical attitudes of the two scholars. Whereas Dedekind, who saw in mathematics the unfolding of the creative activity of the human spirit, pursues the ideal of a logico-philosophical *justification* of the system of numbers – his genuine question was, precisely, *Was sind und was sollen die Zahlen?* – Peano, who was not interested in such questions, as will be clearly shown subsequently, (consider, for example, his reviews of Frege's *Grundgesetze der Arithmetik* or Russell and Whitehead's *Principia Mathematica*[3]), was not in the least affected by the idea that this system needed justification; in his view all it needed was a *rigorous exposition* of its governing principles[4].

Peano himself was well aware of this difference: in the note *Sul concetto di numero* (1891i), he wrote:

> Between what is said above, and what Dedekind says, there is a seeming contradiction, which should at once be pointed out. Here the number is not defined, but its fundamental properties are enunciated. But Dedekind defines the number, and precisely calls 'number' what satisfies the predicted conditions.[5]

And, clearly looking towards the immediate mathematical value of the two constructions, added: "Evidently the two things coincide"[6].

It is here worth recalling that this diversity, made explicit, as has just been stated, by Peano only in (1891i), was immediately grasped by Gino Loria who had already written, in 1889, presenting the *Arithmetices principia*[7]:

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[3] G. Peano (1895l), 122–128; Peano (1913i), 47–53, 75–81.

[4] Peano's lack of interest in the attempts at a logical foundation for the concept of natural number was not perhaps so much the fruit of a certain philosophical insensitivity on his part, sometimes to the point of almost boasting with a touch of coquettishness, as of the idea that such a purpose could not be achieved. In fact he wrote in Peano (1891o), 256: "Per mio conto [. . . ] il numero (intero positivo) non si può definire (poiché le idee di ordine, successione, aggregato, ecc., sono altrettanto complesse come quella di numero)." (In my opinion [. . . ] the (positive integer) number cannot be defined (since the ideas of order, succession, aggregate, etc., are just as complex as that of number)).

[5] G. Peano (1891i), 94: "Fra quanto precede, e quanto dice il Dedekind, vi ha una contraddizione apparente, che conviene subito rilevare. Qui non si definisce il numero, ma se ne enunciano le proprietà fondamentali. Invece il Dedekind definisce il numero, e precisamente chiama numero ciò che soddisfa alle condizioni predette."

[6] G. Peano (1891i), 94: "Evidentemente le due cose coincidono."

[7] G. Peano (1889a).

> He is not concerned, as Dedekind was, to arrive at the idea of number by way of pure reasoning; but he admits the existence of entities, which he calls numbers, defined by certain characteristic properties, which are sufficient both to generate the whole group starting from one of its elements (*the unit*) and to establish all the properties of the group itself.[8]

## 4.

The second difference on which we feel it is worth pausing, and which is much more important for our subject, has to do with the *conceptual tools* involved in the two proposals – a difference which, note, Peano himself did not grasp, as clearly shows his comment: "Evidently the two things coincide". Moreover, this difference offers us a further illuminating example of a phenomenon which is not unusual in the history of mathematics: the fact that different stipulations put forward for one and the same object and at first sight proving equivalent and being recognised as such by their proposers, though involving different concepts, have subsequently come to be developed along completely different lines. Let me recollect that what is perhaps the best known, certainly the most evident example of this phenomenon consists in the definitions of the real continuum as completion of the rational line, provided in 1872 simultaneously by Cantor and by Dedekind. The two definitions were certainly equivalent in the sense that they generated isomorphic systems of reals, but the concepts involved were profoundly different, and in the subsequent developments had totally different destinies: the two procedures of completion advanced ended by being the supporting pillars of the general theory of metric spaces, and of the general theory of orders, respectively. As far as our case is concerned, setting aside merely terminological questions, we can see how Dedekind's proposal involves only *set-theoretical* concepts: a numerical system is constituted by a set and an application of it which have certain properties. In contrast, Peano's, in addition to *set-theoretical* concepts, involves – and this is actually the first time it has happened – two fundamental *metalogical* concepts: that of *formal language* and that of the *model* of a formal system, or, in other words, of a set-theoretical structure which *satisfies* the propositions of a formal system.

If today we want to rediscover the ancestors, the origins, of all that by now very wide-ranging area of research centred on the so-called 'models of arithmetic', we must look, not to Dedekind, but to Peano; it is he who, making the concept of numerical system *depend* on the concepts of *formal language* and of the *satisfaction of formal conditions*, created, albeit quite unaware, the premises for the possible *variation* of the very concept of numerical system; this, in fact, is transformed when the language adopted and the accepted relationship of satisfaction, vary. From the point of view of the subsequent history of logic, then, Peano's proposal seems much

[8] G. Loria (1889), 154–156*: "Egli non si occupa, come fece Dedekind, di pervenire alla nozione di numero col puro ragionamento; ma ammette l'esistenza di enti, che chiama numeri, definiti da certe proprietà caratteristiche, le quali bastano e per generare tutto il gruppo partendo da un suo elemento (*l'unità*) e per stabilire tutte le proprietà del gruppo stesso."

the most significant, and this remains true despite the fact that from other 'logical' standpoints, Peano's work was less advanced than Dedekind's. In contrast with what happens with the latter, in fact, there is no trace in Peano of the idea that the recursive definitions of the arithmetical functions demand a 'theorem of recursion', a theorem, that is, which assures their existence and uniqueness, just as there is no attempt to show that his arrangement is, as was later said, 'categorical' – that structurally speaking it has a single model; on this point he limits himself to declaring that his axioms "express the necessary and sufficient conditions so that the entities of a system can be made to correspond univocally to the series of *N*s".

Further – as is obvious from the perspective within which Peano moved – there is not the slightest attempt to show that there is at least a model; Dedekind, in contrast, had of course to make just such an attempt and, as we know, he carried it out along a road already travelled, though for other aims, by Bernard Bolzano.

## 5.

It is also important to stress that the clear view of the distinction between a formal language and the set-theoretical structure which satisfies its propositions emerges clearly above all in Peano's discussion, in his work (1891i), of the independence of his axioms. In his famous *Introduction to mathematical logic* (1956), Alonzo Church, after presenting the method for the proof of independence of a certain postulate by showing an interpretation which falsifies it while making true all that remains, adds in a note:

> This method of establishing independence of postulates was used by Peano, *Rivista di Matematica*, vol. I (1891), pp. 93–94 and by Hilbert in his *Grundlagen der Geometrie*, first edition (1899). However, the origin of the method is to be seen still earlier in connection with the non-Euclidean geometry of Bolyai and Lobachevsky – models of the postulates of this geometry, found by Eugenio Beltrami (1968) and Felix Klein (1871), being in effect independence examples for Euclid's parallel postulate.[9]

Distinguishing clearly between the *use* of this method by Peano and subsequently by Hilbert in order to prove an independence and the *origins* of this method in work on non-Euclidean geometries by Beltrami and by Klein, Church very precisely shows the merits of each. That – *in effect*, as Church says – the interpretation of Lobacewsky's planimetry offered in (1868) by Eugenio Beltrami in his *Essay of Interpretation of Non-Euclidean Geometry*[10], constitutes a proof of independence of the postulate of parallels and similarly, more generally, the first presentation of a method designed to provide proofs of independence, is obvious; but this should be clearly distinguished from the authentic aim pursued by Beltrami and from the consequent approach to the use of this method. Otherwise it is difficult to grasp fully

[9] A. Church (1956), n. 539.

[10] E. Beltrami (1868), 284–312.

the real contribution made by Peano to the development of the two fundamental metalogical concepts of 'model' and 'independence'.

## 6.

Beltrami's explicitly declared intention is, in fact, to "find a real substratum" to the "doctrine of Lobatschewsky". It is significant that Felix Klein, in his first work dedicated to non-Euclidean geometries (a conference at the Academy of Sciences of Göttingen on August 30th 1871),[11] spoke of Beltrami's interpretation as a "materialisation [*Versinnlichung*] of hyperbolic geometry"[12] and, indeed, with respect to certain limits of this interpretation – which he glimpsed, although rigorously demonstrated only thirty or so years later by Hilbert – observed that "this unfortunately never carries the whole plane to the intuition [*zur Anschauung*]"[13]. He himself presented his work as an attempt to respond to the "need to materialise [*versinnlichen*] the very abstract speculations which have led to the constitution of the three types of geometry"[14]. Beltrami's starting point is that although the postulates are always advanced in order to account for the properties of a well-defined "category of entities" which one has in mind, their consequences hold in general for all those systems of entities which prove to satisfy the conditions actually employed in the various proofs, which can thus be found to possess "a greater generality than was being sought" and to be valid for "categories of entities", in which properties are also given that are very far from those present in the originally intended category. Thus the problem will be to overcome the unease produced by the "apparent incongruencies" which may thus arise, and the exhibition of an interpretation will make it possible to give plasticity and legitimacy to these "apparent incongruencies". In a word: the presentation of a model serves to legitimate realities very different from those one has in mind and which present themselves when certain conditions are not actually being employed.

## 7.

Peanos's perspective is in a sense dual: the method does not serve to show how many nice sensible things there are when a certain condition is not made use of, but to show the indispensability of a certain assumption when the intention is to show that, in Beltrami's words, "those determinations that identify the category itself as

[11] F. Klein (1871), 419–433.

[12] F. Klein (1871), 424: "Beltrami, dem man die betreffende Versinnlichung der hyperbolischen Geometrie verdankt."

[13] F. Klein (1871), 425: "Diese letztere Interpretation bringt leider, wie es scheint, nie das gesammte Gebiet der Ebene zur Anschauung."

[14] F. Klein (1871), 424: "Das Bedürfniss, die sehr abstracten Speculationen, welche zur Aufstellung der dreierlei Geometrieen geführt haben, zu versinnlichen, hat dahingeführt, Massbestimmungen aufzusuchen, die als Bilder der gennanten Geometrien aufgefasst werden könnten."

compared with a broader category have effectively been introduced[15]". Of course, it was quite clear to Peano that Beltrami's construction was in a certain way the starting point of the whole of the subsequent developments; for example, in the note presented in support of the candidacy of Pieri for the Lobacevski prize, Peano said:

> The irreducible nature, or independence, of a proposition, a provisional postulate, is shown giving the example of an interpretation of the geometric symbols, in such a way that all the preceding postulates are proved, with the exception of the one under consideration. This method received a classic application in Pangeometry. To show that the postulate of parallels was irreducible, the example was given of pseudospherical surfaces, which also prove all the postulates of Geometry, with the exception of that of parallels.[16]

## 8.

And here we come to what for many reasons may be considered the *summa* of Peano's experience as regards the general metalogical problem, the *Logical introduction to every deductive theory*, which Alessandro Padoa prefaced to his *Essay of an algebraic theory of integers*, presented in Paris at the Conference of philosophers in 1900[17]. In this authentic gem of scientific literature is found the exposition, with exemplary clarity, of the ideas which in the preceding decade had been developed and adjusted within the school on the *general concept of deductive theory* and on the *properties of independence in this theory of concepts and axioms*.

## 9.

As regards the presentation of the general concept of deductive theory, it is developed by means of listing and illustrating a whole series of concepts and of distinctions which are to be found today in any *Introduction to logic*, but which, in point of fact, are here collected and briefly expounded for the first time. In particular, it proceeds by distinguishing clearly, and terminologically, the three levels: *linguistic*, *ontologic* and *semantic*. The concepts of the first are those of *symbol* and *proposition*; of the second, those of *idea* and of *fact*; and of the third, those of *representing*, as relationship between symbols and ideas, and of *enunciating*, as relationship between propositions and facts. The first subdivision introduced, as regards the linguistic level, is

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[15] E. Beltrami (1868), 286: "effettivamente introdotte quelle determinazioni che individuano la categoria stessa in confronto di una categoria più estesa."

[16] G. Peano (1904b), 96: "On prouve l'irreductibilité, ou indépendance d'une proposition, postulat provisoire, en donnant l'exemple d'une interprétation des symboles géométriques, de façon que tous les postulats précédents soient vérifiés, à l'exception de celui qu'on considère. Cette méthode a reçu une application classique en Pangéometrie. Pour prouver que le postulat des parallèles était irréductible, on a donné l'exemple des surfaces pseudosphériques, qui verifient tous les postulats de la Géométrie, à l'exeption de celui des parallèles."

[17] A. Padoa (1901b), (1900). There is an English [Van Heijenoort 1967, 118–123] and an Italian [Mugnai 1982, 382–394] translation of the Introduction.

between *the symbols and the propositions of general logic*, whose meaning is presupposed, and *particular symbols and propositions*. Specifying then that *defining* a symbol means "expressing it by means of others already considered" and that *proving* a proposition means "deducing it from others already enunciated" the symbols are subdivided into *indefinite* and *definite* and the propositions into *unproved* and *proved*, specifying that the assertions of *definability* of a symbol or of *provability* of a proposition must always be accompanied by the indication of the symbols respectively of the propositions starting from which the definition, respectively the proof is possible. Under the name of 'symbolic' translation, is also illustrated the procedure of elimination of the symbols defined by means of replacing them with their respective *definientia*. Stress is also laid on the partially arbitrary nature of these allocations, observing that the identification of the *undefined* symbols with those which represent the *simplest* ideas and of the *unproved* propositions with those which enunciate the *most obvious* facts, despite its undeniable psychological valence and a certain importance in the *development* phase of the theory, is quite unfounded from the logical viewpoint, since no rule can be imagined by which one might "choose with absolute certainty" of two ideas the simpler and, of two facts, the more obvious. Hence in the phase of *formulation* (we would say of '*formalisation*') of the theory:

> We may imagine that the undefined symbols are *completely without meaning* and that the unproved propositions (instead of enunciating *facts*, i. e. *relationships* between the *ideas* represented by the undefined symbols) are simply *conditions* to which the undefined symbols are subjected.[18]

In this way, the text goes on:

> The system of the ideas which we initially chose is simply an *interpretation* of the *system* of the *undefined symbols*; but, from the deductive point of view, this interpretation can be ignored by the reader, who can freely replace it, in his thinking, with another interpretation which verifies the conditions enunciated by the unproved propositions. And since these, from the deductive point of view, enunciate not *facts*, but *conditions*, they cannot be considered true *postulates*. Thus the *logical* questions take on complete independence as compared with *empirical* or *psychological* questions (and, particularly, the *problem of knowledge*); and every question relative to the *simplicity* of the *ideas* and the *evidence* of the *facts* disappears.[19]

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[18] A. Padoa (1901b), 318: "nous pouvons imaginer que les symboles non-definis soient *complètement dépourvus de signification* et que les P non-démontrées (au lieu d'énoncer des *faits*, c'est-à-dire des *relations* entre les *idées* représentées par les symboles non-définis) ne soient que des *conditions* auxquelles les symboles non-définis sont assujettis."

[19] A. Padoa (1901b), 318–319: "Alors, le *système* des *idées* que nous avons choisi d'abord n'est qu'*une interprétation* du *système* des *symboles non-définis* ; mais au point de vue déductif, cette interprétation peut être ignorée par le lecteur, qui peut librément la remplacer, dans sa pensée, par *une autre interprétation* qui vérifie les conditions énoncées par les P *non-démontrées*. Et, comme celles-ci, au point de vue déductif, n'énoncent pas des *faits*, mais des *conditions*, on ne peut les considérer comme de vrais *postulats*. Ainsi les questions *logiques* acquièrent une complète indé-

## 10.

These ideas are further developed and an idea also emerges which today might be called ‘substitutional semantics’:

> It may happen that there are several (even an infinite number of) interpretations of the system of *undefined* symbols, which prove the system of the *unproved* propositions and hence, all the propositions of a theory. In this case the system of the *undefined* symbol can be considered the *abstraction* from all these interpretations, and the *generic theory* can be considered the *abstraction* from the *specialised theories* which are obtained by separately replacing the system of undefined symbols with each of its interpretations.[20]

And it is with reference to this concept of generic theory that the observation – which, as is well known, goes back to Moritz Pasch – on the practical value of formalism derives:

> By means of a single reasoning which proves a proposition of the generic theory, a proposition is implicitly proved in each of its specialised theories.[21]

A distinctive feature of the whole essay which also merits emphasis is the unusually clear, conscious view of the strict parallelism between the conceptualisations regarding the couple (*idea*, *definition*) and those regarding the couple (*proposition*, *proof*), to the point that some considerations are actually developed only with reference to ideas and definitions, while, as to the respective assertions on propositions and proofs, these are obtainned by means of the explicit suggestion that the words ‘symbol’, ‘defined’, ‘idea’ and ‘simple’ should everywhere be replaced by the words ‘proposition’, ‘proved, ‘fact’ and ‘evident’. In conclusion it must be noted that, regarding the elucidation of the basic ideas of the new axiomatic, Padoa’s essay certainly wins the contest with the almost contemporary *Grundlagen der Geometrie* by Hilbert.

## 11.

As regards the second question, namely the *independence*, or, as Padoa prefers to call it, the *irreducibility* of primitive propositions and symbols, this is first and foremost specified as, respectively: 1) impossibility for every *primitive proposition* to be

pendance à l’égard des questions *empiriques* ou *psychologiques* (et, en particulier, du *problème de la connaissance*) : et toute question relative à la *simplicité* des idées et à l’*évidence* des *faits* disparait.”

[20] A. Padoa (1901b), 319–320: “Il peut se faire qu’il y ait plusieurs (et même un nombre infini d’) *interprétations* du système des symboles *non-définis*, qui *vérifient* le système des P *non-démontrées* et, par consequent, toutes les P d’une théorie. Alors le système des symboles non-definis peut être considerée comme l’*abstraction* de toutes ces interprétations, et la *théorie générique* peut être considérée comme l’*abstraction* des *théories specialisées* qu’on obtient en y remplaçant séparément le système des symboles non-définis par chacune de ses interprétations.”

[21] A. Padoa (1901b), 320: “Par un seul raisonnement, qui démontre une P de la théorie générique, on démontre alors implicitement une P dans chacune de ses théories spécialisées.”

deduced from one or more of the other primitive propositions; 2) impossibility, for every *primitive symbol*, to deduce from the primitive propositions an explicit definition of it in terms of other primitive symbols. Going on to the problem of how to *prove* these irreducibilities, Padoa recalls, in the case of the *primitive propositions*, the method, which he claims has been "known for some time", of exhibiting an *interpretation* of the system of primitive symbols which satisfies the system of primitive propositions, with the exception of *one and only one of them*. Regarding the proof of the irreducibility of *primitive symbols*, rightly stressing its novelty, he puts forward the method which, recovered only after more than thirty years by Tarski[22], became standard in the literature of logic under the name of '*Padoa's method*' and which consists in exhibiting, for every primitive symbol, a *couple of interpretations* which differ from each other only as regards the meaning attributed to that symbol and yet make all the primitive propositions true.

## 12.

As to the efficacy of such procedures, this is taken for granted in the already well-known case of propositions, whereas, in the case of symbols, it is just its presentation that makes it plausible. Thus it remains established that the existence of an interpretation which makes all but one of the primitive propositions true is *sufficient* to ensure the independence of this last from the rest, just as the existence of two interpretations which differ only as regards the meaning attributed to a symbol and make all the primitive propositions true is *sufficient* to ensure the indefinability of that symbol. In fact later too, when the conceptual tools involved underwent the suitable specifications, the justification of both assertions did not create any real problem.

But Padoa does not content himself with this and, in both cases, explicitly states that these conditions are not only sufficient but also *necessary*. In the very brief *Avant-propos* to the communication on *A new system of irreducible postulates for algebra* which Padoa presented only few days after, at the second – the mathematicians'[23] – Paris Congress, he adds, to the *irreducibility* of systems of propositions and of symbols as "the main conditions of logical perfection for any deductive theory" the *compatibility* of a system of propositions, defining it as reciprocal consistency and noting that irreducibility itself may be reformulated in terms of compatibility: a system of propositions is irreducible if "separately replacing each postulate with its negation we obtain a system of compatible propositions". The most significant thing, however, is that to the characterisations of these properties he adds "the practical rules for recognising whether such conditions are proved in a given theory" and these are given, in all three cases, according to the pattern: "In order to prove ..., *it is necessary* to find ... an interpretation ..."[24]. So, sufficiency of the conditions – of course, for compatibility, not mentioned in the first memoir, it is

[22] A. Tarski, *Some Investigations on the Definability of Concepts*, in A. Tarski 1956, 296–319.

[23] A. Padoa (1902a), 249–256.

[24] A. Padoa (1902a), ∗: "Pour *démontrer* [...] il faut trouver [...] une interprétation [...]."

a matter of the existence of an interpretation which verifies all the propositions – is here not even mentioned, but only their *necessity* is stated.

**13.**

Now, as every logician knows, on the basis of the successive specifications of the concepts involved it has been ascertained that the *necessity*, for the indeducibility of one proposition from others, of the existence of a realisation which falsifies the first while it makes true all the others, is valid for that particular type of propositions called 'of the first order', not, however, in general; it is certainly false, for instance, when the propositions are of the so-called 'second order'. The proofs of these facts are, however, anything but trivial. The first is none other than an equivalent reformulation of the famous *Theorem of completeness* proved by Gödel only in 1930; the second is a corollary of the still more famous *Theorem of incompleteness* of Gödel which goes back to 1931. Moreover, that, at least in the case of the first order, the existence of two interpretations that differ only in the meaning attributed to a symbol and make all the primitive propositions true is a *necessary* condition for the indefinability of that symbol, is the content of the famous *Theorem of Beth* which was proved only in 1953; apart from this case, the situation is more complex and in any case is not true in general. It is, however, not inappropriate to remark, on this point, that the very position of the *problem* of a possible *necessity* of these conditions became possible only after the clarification of the concept *formal deducibility* which took laboriously form only in the Twenties and the Thirties within Hilbert' school.

**14.**

By way of integration of this last remark, it is worth recollecting a controversial aftermath on the part of Padoa towards Hilbert, of which, however, we have not been able to find any echo in the works of the latter.

Here are the facts. On 3 August at the Congress of philosophers Padoa presented the *Summa* of which we have spoken. On 8 August, at the Congress of mathematicians, Hilbert presented his famous memoir on *Mathematical Problems* which was to characterise the century which was just beginning, and in which he assigned the second place to the problem of the *consistency of arithmetic*. At the end of this communication, Peano spoke up, declaring that the previously mentioned communication from Padoa, which had been programmed for the following day, answered the second problem[25]. Hilbert said nothing and did not even attend Padoa's communication the next day. A couple of years later the *Compte rendu* of the congress was published, but Hilbert still showed not the least interest in Padoa's work. At this point Padoa became extremely irritated and in 1903, in *L'enseignement mathématique*, published a note on *The problem no. 2 of M. David Hilbert*, in which having stated

[25] A. Padoa (1902a), 21.

that if Hilbert had even read only the *l'Avant-propos* of his memoir, he could have "understood that his problem no. 2 was just gobbledygook [*une causerie*] and could be suppressed", he returned to Hilbert's memoir and commented on it in detail[26].

## 15.

To do justice to Padoa's point of view which, for us who come after almost a hundred years of Proof theory, may at first sight appear to be only the fruit of simple blindness, it should be said that in Hilbert's proposal there are at least three distinct things. The first is the basic identification of the *consistency* of a system of axioms with the impossibility of deducing from them a contradiction "with a finite number of logical steps" – and this expression, which was to become standard, is repeated no fewer than *four* times in the two pages containing the problem. The second is that the *mathematical existence* of a concept is peremptorily identified with the consistency of the axioms which define it. The third is the affirmation that, in order to complete the proofs of *relative* consistency thus far obtained for geometric theories, it is necessary to find a *direct* proof of this property for arithmetic.

As regards the identification of consistency and existence, which will certainly have disturbed a Peano disciple for whom the axiomatic method had only a *descriptive*, and certainly not a *creative* function, it must be admitted that in the last analysis this proves to be arbitrary and substantially unsuitable. In fact, we have learned from Gödel, among many other things, that there are systems of axioms from which it is impossible to derive a contradiction 'with a finite number of logical steps', but to which no convincingly structured world corresponds. But this apart, which might risk sliding into a merely terminological dispute, apropos the 'direct' proof of the consistency of arithmetic, two points must be emphasised. In the first place, the consistency Hilbert is talking about is not, as it was later to become, the consistency of *natural* arithmetic, but that of *real* arithmetic.

Secondly, in hoping for a 'direct' proof of such consistency, Hilbert was not deploying – as would be done later – the proofs considered in themselves as specific mathematical objects – indeed, at this time he was still a long way from an authentic interest in logic – but seems rather directed towards some mathematical construction; it was his conviction and these are his own words – that:

> We must succeed in finding a direct proof for the consistency of arithmetic axioms by precisely redeveloping and appropriately modifying to this end the inferential methods known in the theory of irrational numbers.[27]

From this it would seem that Hilbert was at this time thinking of a sort of redevelopment of the methods of arithmetization of analysis, somehow taking for granted

[26] A. Padoa (1903), 85–91.

[27] D. Hilbert (1900a), 180: "Ich bin nun überzeugt, dass es gelingen muss, einen direkten Beweis für die Wiederspruchslosigkeit der arithmetischen Axiome zu finden, wenn man die bekannten Schlussmethoden in der Theorie der Irrationalzahlen im Hinblick auf das bezeichnete Ziel genau durcharbeitet und in geeigneter Weise modifiziert."

natural arithmetic (or at least, as he was to specify further on, a part of it). Of course, the undeniable historical merit of this 'second problem' remains that of having laid down the *requirement* of a 'direct' proof – a proof, that is, which would bring to an end the regress of relative proofs. But it must not be forgotten that in order to make sense of this vague aspiration, Hilbert would have to modify his perspective radically and develop a new mathematics which would have as its object precisely those things of which, for the moment, he knew only that they were "a finite number of logical inferences".

This being the case, we may say that, in a sense, Padoa's reckless observation, according to which the very idea of a direct proof of consistency seemed to him to show that Hilbert:

> [...] has not understood that to demonstrate the independence or consistency of a system of propositions we may choose the interpretations of the indefinite symbols in any appropriate domain, as long as the knowledge of that domain is admitted in advance[28]

is at least comprehensible, because from Hilbert's words it really seems that what he is trying to do is merely a model of real arithmetic.

## 16.

In conclusion, I should like to say that when – in the light of what later occurred – we look at Peano's and his followers' metalogical achievements, a crucial question arises: how could it be that these skills, these competences, not many years after what Hans Freudenthal liked to call "the Parisian triumph of the Italian phalanx"[29], should have ceased not only to be a reference point for worldwide research, but even to appear on the Italian cultural scene?

In the creation of the conceptual premises of what was to be the subsequent evolution of logic and of the investigation into its foundations, in fact, Peano and his disciples were certainly further on than anyone else at the time, including, specifically, Hilbert, who, if I may crudely oversimplify, learned logic from Russell who had learned it from Peano. They had a rather articulate idea of what a formal language should be and had fully mastered the idea of what an axiomatic system was. They were also able to present the proofs of entire far from trivial chapters of mathematics as chains of symbolic expressions, because, as Peano had said as early as 1889, in his essay on *I principi di geometria logicamente esposti*:

[28] A. Padoa (1903), 88: "« Quant à la démonstration de la non-contradiction des axiomes de l'Arithmétique, elle demande à être effectuée par voie directe » nous prouve que M. Hilbert n'a pas compris que, pour démontrer l'indépendance ou la non-contradiction d'un système de propositions, l'on peut choisir les interprétations des symboles non définis dans un domaine convenable quelconque, pourvu seulement que la connaissance de ce domaine soit préalablement admise."

[29] Cf., for example, H. Freudenthal (1962), 613–621.

> [...] when once reduced [...] the propositions in formulae analogous to algebraic equations, then, examining the common proofs, one sees that they consist in transformations of propositions and groups of propositions, having the greatest analogy with the transformations of simultaneous algebraic equations. These transformations, or identities logical, of which we make continual use in our reasonings, can be stated and studied.[30]

And apropos these logical identities he had dallied, evidently still unaware of Frege's – in any case 'illeggible' *Begriffsschrift*:

> It would make an interesting study, which so far is lacking, to distinguish the fundamentals [logical identities], which must certainly be admitted from those which remain, contained in the fundamentals. This research would lead to a study, on Logic, analogous to what has been done here for Geometry, and in the previous leaflet for Arithmetic.[31]

Peano's followers knew, on the basis of a clear view of semantic relationships, how to manage very skilfully, as has been said, the models of axiomatic systems. Hence they had, so to speak, in their hands the basic concepts on which were to be constituted those new chapters on logic and mathematics that were to be the *Proof theory* and the *Model theory*, and yet it appears that they had never been so much as grazed by the idea that this chain of symbolic expressions which they knew how to construct so skilfully might constitute in itself a new mathematical object, nor by the thought that the models, the structures satisfying certain formal conditions, might for their part become in themselves the object of mathematical investigation.

It is my opinion that beyond the various circumstances, both internal and external, frequently cited, which certainly played a significant part, at the base of this decline there is a wholly internal reason which I would call the 'systematory obsession': namely, that mathematics needed only to be reorganised and clearly expounded; they declined to accept that mathematics might need – as others wished – a foundation.

## 17.

Today we know that these foundationalistic dreams have not come true. Neither the 'logicist' plan to transform all mathematics into a branch of logic, nor the 'formalist' plan to ensure the certainty of mathematics by means of a somehow 'absolute'

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[30] G. Peano (1889d), 28–29: "ridotte [...] le proposizioni in formule analoghe alle equazioni algebriche, allora, esaminando le comuni dimostrazioni, si scorge che esse consistono in trasformazioni di proposizioni e gruppi di proposizioni, aventi massima analogia colle trasformazioni delle equazioni algebriche simultanee. Queste trasformazioni, o identità logiche, di cui facciamo continuamente uso nei nostri ragionamenti, si possono enunciare e studiare."

[31] G. Peano (1889d), 29: "sarebbe uno studio interessante, e che finora manca, il distinguere le fondamentali [*scl.* identità logiche], che si devono ammettere senz'altro, dalle rimanenti, contenute nelle fondamentali. Questa ricerca porterebbe ad uno studio, sulla Logica, analogo a quello qui fatto per la Geometria, e nel precedente opuscolo per l'Aritmetica."

proof of consistency succeeded in reaching the goal they had set themselves. Paradoxically, then, it might also be said that the rejection of these dreams by Peano and his followers was justified. But that would be rather like saying that since Columbus, travelling westwards, could not reach the Indies, those who held that the earth was flat were right. Pursuing these logicist and formalist objectives, exploiting many tools prepared by Peano and his followers, they discovered many Americas, confirming too yet once again that often, in pursuit of a great objective, what is most important may not be reaching that goal, which you may even miss, but rather all that you collect along the way.

Peano's greatness, seen in his own country as a sort of achievement of the 'fullness of logical times', actually, and unfortunately, ended by being a hindrance to the attempts to resume discussion with those who had headed West in search of the Indies.

Not least because of the small-mindedness and nationalistic arrogance that connote it, what happened to Geymonat is emblematic: in 1942, when he asked the Ministry of Popular Culture to authorise publication of his collection of Frege's writings, he was first turned down on the basis of the "view of the Italian Royal Academy", that this work was "by now long out of date". Subsequently, the explanation from the two members of the Academy involved, to whom Geymonat had addressed his request, was that: "it was not possible to give value to the work of a foreigner on the foundations of arithmetic, without at the same time highlighting the definitive work of Giuseppe Peano"[32].

Even the last exponent of this school, Ugo Cassina, though on the one hand he deserves to be recognised here today for his tireless, invaluable work to safeguard and hand on the memory of his master, on the other was not without responsibility, given the authoritative voice he had in view of the power of his academic position, for hindering the diffusion in the Italian mathematical world of the new logical knowledge. As late as the end of 1953, when Löwenheim's and Skolem's theorems were already known, as well as those on completeness, incompleteness and compactness by Gödel, and those of Tarski, Church, Kleene, Turing, Beth, etc. he was fiercely critical of the "experts on modern symbolic logic" for whom Peano's work, in the field of logic, today has only historical value, and they barricaded themselves behind a prolix, imprecise, incomplete symbolic language, which is to Peano's as a cubist or surrealist painting by Picasso ... is to Titian's woman lying down ... or Correggio's Danae![33]

[32] L. Geymonat, *Prefazione* to G. Frege (1948), 12–13: "non si poteva mettere in valore l'opera di uno straniero sui fondamenti dell'aritmetica, senza contemporaneamente lumeggiare quella definitiva di Giuseppe Peano."

[33] Quoted in L. Geymonat (1959), 109–118*: "si trincerano dietro un linguaggio simbolico prolisso, impreciso e incompleto, che colle debite proporzioni sta a quello di Peano come un quadro cubista o surrealista di Picasso [...] sta alla donna sdraiata di Tiziano [...] od alla Danae del Correggio!"

# 9

# Foundations of Geometry in the School of Peano

Elena Anne Marchisotto

## 9.1 Introduction

Giuseppe Peano is well known in the history of mathematics not only for the important contributions he made to mathematical logic and foundations, arithmetic, geometry, and analysis, but also for his impact on the many scholars who gathered around him at the University of Turin. One of the more notable mathematicians of the famous Peano school was Mario Pieri (1860–1913).

Pieri was Peano's junior colleague in Turin, both at the Royal Military Academy and at the University of Turin. Pieri's friendship with Peano and allegiance to the goals of the Peano school extended well beyond the nearly fourteen years he spent in Turin. A review of the publications of these two scholars clearly demonstrates the impact that each had on the other's research.

Largely because of the influence of Peano, Pieri changed his original research direction on algebraic and differential geometry to investigations into mathematical logic and axiomatics. Between 1895 and 1912, Pieri published seventeen papers[1] in foundations of geometry, presenting geometry as an abstract formal system rather than as a study of space. These papers each reflected Pieri's allegiance to goals of the Peano school, and explicitly or implicitly used Peano's logical calculus.

The cross-fertilization of ideas between the scholars is so deep that there are many topics from which I could have chosen. The idea for this paper arose from a statement Pieri made in his first axiomatization of projective geometry that is a series of three notes written and published between 1894 and 1896. Pieri indicated that a transformation central to his development is related to one adopted by Peano in his paper on the foundations of geometry (Peano 1894c). Pieri compared what he called *seg-*

[1] The citations in this paper of page numbers from Pieri's publications refer to *Opere sui fondamenti della matematica*, a cura dell'Unione Matematica Italiana, Roma, Cremonese, 1980. Wherever possible paragraph references (§) are also given. The reader is invited to consult the list of Pieri's publications in the larger context of his entire *opus* as specified in E.A. Marchisotto, J.T. Smith (2007), 373–399.

F. Skof (Ed.), *Giuseppe Peano between Mathematics and Logic*

*mental transformations* to what Peano called *affinities*[2]. I decided to explore the relationship between these two transformations within the context of the papers in which they appeared: Pieri (1895a), Pieri (1896a), Pieri (1896b)[3], and Peano (1894c). The scholars wrote these papers at a time when interest in projective geometry and the role of geometric transformations was high. In 1893, F. Klein republished (with improvements), in both the *Mathematische Annalen* and the *Bulletin of the New York Mathematical Society*, his 1872 article outlining the *Erlangen Program*. In (1872), Klein had observed that among the advances of the last fifty years in the field of geometry, the development of *projective geometry* occupied the first place (1893b, 215). An Italian translation of Klein's article was produced by G. Fano in (1890)[4].

## 9.2 Projective Geometry. Two Different Views

Projective geometry can be defined as the study of properties of figures that remain invariant under the process of projection and section. Properties of incidence are fixed under this process and so are called projectively invarant. Distance, angle magnitude, linear order, and parallelism are not *projectively invariant*, and so are not considered purely projective concepts under the given definition. However, from its roots in antiquity, projective geometry had been envisioned as an extension of elementary geometry. So, many of its central concepts were defined metrically. Even those who attempted to provide synthetic treatments of the subject, admitted non-projective concepts such as linear order (e. g., Fano 1892, Enriques 1894) and distance (e. g., De Paolis 1880–81).

Peano and Pieri both referred to projective geometry as the *geometry of position*[5]. They had similar goals for developing it axiomatically, but their views of the subject itself differed significantly. Peano envisioned real projective geometry as part of *general* geometry (Euclidean or non-Euclidean) that could be derived from elementary geometry (ordinary geometry of the plane and space). In (1894c), Peano sought a metric-free construction of real projective geometry. He introduced a set of postulates[6] "to treat the geometry of position", noting that they made no reference to rigid motions or geometric magnitudes. He made it clear however that his postulates for the geometry of position could not contradict the axioms of elementary geome-

[2] Pieri also compared segmental transformations to ordered correspondences in F. Enriques (1894).

[3] This collection of three notes will from now on be denoted as the 1895–1896 Notes.

[4] A French translation by H.E. Padé appeared in 1891. Russian and Hungarian translations appeared in 1896 and 1897 respectively. For an analysis of Klein's paper which compares research in different areas of geometry and its historical impact see J. Gray (2005).

[5] The term first appeared in L. Euler (1736), a paper on the famous problem of the bridges of Königsberg. Euler credited G. Leibniz for the idea of a geometry concerned only with the properties of position.

[6] F. Schur gave a simplified version of Peano's system due to G. Ingrami in F. Schur (1902), §1, 267 ff. Schur referred to axioms of connection and order given by D. Hilbert as "projective axioms", and to the simplified version of Peano's axioms as "another grouping" of Hilbert's projective axioms.

try, indeed they *started* from them[7]. Peano's postulates established the relationships among the ordinary entities of general geometry. He then could derive the projective entities from them (e. g., a projective point can be described as an ordinary point or an ideal point) with the introduction of "opportune definitions" (e. g., an ideal point can be represented as the center of a star of lines). Peano also emphasized the view that although it is possible to arbitrarily develop the logical consequences of the postulates, for a work to merit the name "geometry" it is necessary that these postulates express the result of observations of physical figures (Peano 1894c, 54–55, 75).

In all his axiomatizations of projective geometry, Pieri also sought metric-free constructions. But unlike Peano, he endeavored to sever the ties between projective and elementary geometry by establishing the geometry of position as an autonomous subject. Unlike Peano who started from ordinary geometry and proceeded to the place where ideal entities could be introduced, Pieri would assume a projective environment *a priori*, postulating the relationships between its fundamental entities. Unlike Peano who viewed geometry as a categorical system with necessary postulates derived from experience, Pieri believed that geometry should be developed as a purely speculative and abstract formal system. The path to Pieri's first axiomatization of projective geometry begins with C. Segre.

## 9.3 Geometry of Position, according to Pieri

Pieri has been called a *true bridge* between the schools of Segre and Peano at the University of Turin[8]. While this observation applies, from a broad perspective, to Pieri's double research interests in algebraic geometry and in foundations, it also applies in a more narrow sense to Pieri's efforts in establishing projective geometry as a fully independent and rigorous mathematical theory with its own foundations[9]. Notwithstanding the fact that Pieri's focus on foundations was nurtured by his association with Peano, his interest in projective geometry was sparked when, in 1887, Segre invited him to edit and translate into Italian the seminal work by G.K.C. von Staudt, *Geometrie der Lage* (Staudt 1847)[10].

In (1847) Staudt had endeavored to produce a metric-free projective geometry, but he was not completely successful in doing so. In his 1889 translation, Pieri addressed the gaps in Staudt's reasoning[11]. Of particular significance was his demonstration of a fundamental theorem n. 106 about projective transformations

[7] U. Cassina called Peano's postulates "graphic postulates", observing that in their presence, two hypotheses are possible: Euclidean and non-Euclidean geometry (Cassina 1961b, §12.4, 319–320).

[8] A. Brigaglia, G. Masotto (1982), 137.

[9] See M. Avellone, A. Brigaglia, C. Zappulla (2002), §§3, 7. Segre also invited Pieri to edit a foundational paper of R. De Paolis (E.A. Marchisotto, J.T. Smith 2007, §2, 123–124).

[10] Letter of C. Segre to M. Pieri, Torino 11 October 1887, in Arrighi 1997, Nr. 114, 113. Cf. M. Pieri (1889).

[11] M. Pieri (1889), XXIV–XXV. Pieri was one of many, including, for example, Peano in 1891, who attempted to make Staudt's reasoning rigorous. See E.A. Marchisotto (2006), §§6, 7.

that Staudt had not proved rigorously[12]. Pieri's (1889) proof of this theorem was constructed in the context of Staudt's environment that admits parallel lines. But a few years later Pieri would publish a purely projective proof. It appeared in his first axiomatization of projective geometry that he entitled *Sui principii che reggono la geometria di posizione*, likely in tribute to Staudt's *Geometrie der Lage*[13]. In this paper (1895–1896 Notes), Pieri established three-dimensional real projective geometry up to Staudt's fundamental theorem on the basis of three primitives, and nineteen postulates. It was written in Peano's symbolic language, interspersed with commentary in ordinary language.

Pieri explained in the first note (1895a) that his goal was to provide a series of postulates which construct projective geometry as a deductive science independent from other mathematical or physical doctrines and which are governed by such fundamental projective laws as the principles of projection and duality. He began by establishing the incidence properties of points, lines and planes with a set of ten postulates based on two primitives (projective[14] point and line) that satisfy the principle of duality. Pieri then introduced his third primitive, projective segment, observing that this undefined term is needed so that the notion of order could be established with no appeal to ideas extraneous to projective geometry. On the basis of seven more postulates, he developed the segmental incidence properties, and defined projective order on the basis of projective segment. Pieri's last two postulates were concerned respectively with Dedekind continuity of the line adapted from Fano (1892, §17), and the projective character of the segment, credited to Enriques (1894, §9).

In his second note (1896a), Pieri focused on projective figures (quadrangles, triangles, and their duals), and the roles of these figures in separation and connection properties of the plane. For example, via the complete quadrangle, Pieri established the separation of pairs of points on a line by harmonic sets of four points[15]. He defined projective triangle and used it to partition the plane. Here, and throughout his development, Pieri replaced the affine approach of Staudt (1847) with a projective one, establishing the properties of figures with no appeal to principles external to projective geometry. For example, he gave a purely projective proof that the line

[12] M. Pieri (1889), §106, 43–44, footnote (*). Pieri built on the results of F. Klein, G. Darboux and T. Reye to construct his proof. For the details, see E.A. Marchisotto (2006), §7, 295–298. For a discussion of the evolution of proofs, see J.D. Voelke (2008).

[13] L. Carnot had written a pioneering volume on projective geometry entitled *Géométrie de position* (L. Carnot 1803). Reye produced a study of projective geometry emanating from Staudt's ideas entitled *Die Geometrie der Lage* (Reye 1886–92). Pieri cited Reye's book in his translation of Staudt as well as in his own axiomatizations of projective geometry (Pieri 1889, XXV). Reye 1886–92 appeared in 5 editions up to 1923.

[14] Pieri noted that he used the terms *projective point*, *projective line*, etc. to distinguish these primitives from the common physical understanding of point and line (Pieri 1895a, §2, 5–8 – *Opere* 1980, 15–18).

[15] G. Vailati has been cited for characterizing the fundamental properties of the quaternary relation of separation of points of a closed line on the basis of incidence. See Veblen, Young 1908, §4, 362, Borga, Palladino 1992, 32. In G. Vailati (1895a) (*Scritti* 1911, 26) Vailati indicated that his "repeated discussions" with Pieri led him to such considerations. In G. Vailati (1895b) (*Scritti* 1911, 30), Vailati again acknowledged M. Pieri (1895a), §7.

joining two arbitrary points on two sides (projective segments opposite a vertex) of a projective triangle never intersects the third side, observing that Staudt had derived this result using principles of *analysis situs* (topology).

In his third note (1896b), Pieri introduced a special class of transformations that he called *segmental.* He defined them as injective functions between lines that preserve segments. He then described *harmonic transformations* as bijective mappings between lines that preserve harmonic sets, and proved they are segmental. Pieri's harmonic transformations are what Staudt called *projectivities* (Staudt 1847, n. 103). Prior to Staudt, a projectivity had been described as a correspondence that is the composition of a finite number of correspondences called perspectivities.[16] Staudt instead chose invariance of harmonicity as the defining property of projective transformations between lines. But his failure to reconcile his definition with the classical one was what had compromised his own proof of the fundamental theorem n. 106. Pieri succeeded in correcting this defect, and gave purely synthetic projective proof of Staudt's fundamental theorem that a projectivity having more than two fixed points is the identity. He would reprise this proof in an axiomatization of $n$-dimensional projective geometry (Pieri 1898c)[17] that fully realized Staudt's dream of a metric-free projective geometry. This axiomatization has been characterized as a "highlight of the Italian geometrical enterprise in rigorising geometry undertaken by Peano and his school"[18]. Clearly Pieri has sowed the seeds for this achievement in his first axiomatizaton (1895–1896 Notes). A salient question is: To what extent was Pieri's first effort at axiomatizing projective geometry influenced by the ideas of Peano?

## 9.4 Comparing Peano 1894c to Pieri 1895–1896 Notes

Peano's (1894c) axiomatization of geometry built upon a previous paper (1889d) where he had constructed that part of elementary geometry involving incidence and betweenness on the basis of two primitives (point and segment) and seventeen postulates[19]. In (1894c) Peano reproduced these postulates, acknowledging that the first eleven were inspired by the work of M. Pasch (Pasch 1882).

Peano's seventeen postulates establish three-dimensional projective geometry. They are sufficient to prove the postulate of Archimedes (projective version) and Desargues' theorem[20], but not Staudt's fundamental theorem. A proof of the fundamen-

[16] E.A. Marchisotto (2006), §3, 281–283.

[17] E.A. Marchisotto (2006) traces the development to M. Pieri (1898c), in the contributions of Pieri and others that led to it. Marchisotto, Rodriguez-Consuegra, Smith 2011 (to appear) will provide an English translation and analysis of Pieri 1898c.

[18] J. Barrow-Green, J. Gray (2006), 275.

[19] The publication of G. Peano (1889d) coincided with the publication of Pieri's translation of G.K.C. Staudt (1847). Cf. M. Pieri (1889).

[20] Calling two triangles *perspective* from a point when the joins of three pairs of corresponding vertices are on the point, Desargues' Theorem states that two triangles that are perspective from a point are perspective from a line. In other words their three pairs of corresponding sides meet in collinear points.

tal theorem in the context of Peano's development could be obtained with postulates of motion. Peano did add direct motion as a third primitive and gave eight postulates characterizing it. He used these postulates to develop the notion of congruence. He did not further analyze the geometry of position, noting simply that one *could* continue studying it, introducing ideal points, etc. as Pasch had in his work. Peano's disciple, Cassina, eventually demonstrated how to introduce ideal entities on the basis of Peano's (1894c) postulates of position and motion[21]. Cassina also noted that by appealing to the full complement of Peano's postulates of (1894c), the fundamental theorem of Staudt[22] could be proved. Pieri actually proved Staudt's theorem on the basis of his postulate set in 1895–1896 Notes. So both scholars provided axiomatic systems to develop the geometry of position, albeit to different extents and in different ways. Any comparision of these two axiomatizations needs to keep these differences in mind. That being said, several analogies between the works bear mention.

Pieri and Peano shared a common goal to achieve rigor and simplicity for geometry. Peano expressed this objective in his introductions to (1889d) and (1894c). Pieri (1895a, §1 – *Opere* 1980, 13) indicated how he would achieve it, for example, with his use of Peano's algebraic logic. Pieri's endorsement of symbolic language coincided only in part with Peano's intention for its use. Like Peano, he appealed to the logical calculus for clarity of exposition and ease of analysis[23]. But unlike Peano, he also saw it as basis for the construction of geometry as an abstract, hypothetical-deductive system. For this and other reasons, Pieri observed that while his first axiomatization of projective geometry owed a great debt to Peano 1889d, it agreed with Peano's treatment only in form (Pieri 1895a, §1 – *Opere* 1980, 15). This observation applies to an even greater degree to Peano (1894c), which, with respect content, bears more resemblance to Pasch (1882) than to Pieri's construction.

Peano's (1889d) and (1894c) treatments of geometry express and expand the modern axiomatic view initiated by Pasch. Peano defined geometric terms as those that appear in geometry books but do not belong to Logic. He distinguished between primitive ideas and those derived from them. He made it clear that the primitives should be simple ideas "known to all men" (Peano 1894c, 52). Pieri fully endorsed Peano's modern axiomatic view and strictly adhered to his practice of providing precise definitions of all concepts in terms of the primitive ones. But Pieri's primitives had no connections with experience.

The scholars were in agreement that primitives in an axiom system should be reduced to the smallest possible number. In (1894c), Peano reduced the four taken by Pasch (point, segment, coplanar set of points, congruence of figures) to only three (point, segment, motion). Pieri also relied on three primitives (point, segment, and line) in Pieri (1895–1896) Notes.

[21] See U. Cassina (1940), §2.9, U. Cassina (1961a), 414–416 and U. Cassina (1961b), 320–325.

[22] Cassina said that by appealing to Peano's graphic postulates and postulates of motion, in the presence of the postulate of Archimedes, the fundamental theorem of Staudt could be proved (Cassina 1940, §3.11).

[23] The language proposed by Peano had a goal of obtaining "a highly synthetic and rigorous exposition" of mathematical theory, achieved through the expression of mathematical propositions "without the prolixities and ambiguities found in ordinary language" (Borga, Palladino 1992, 28).

Both Peano and Pieri took point as an undefined term. But their other choices for primitives differed. Although they both included segment as undefined, Pieri intended the projective segment (*segmento individuato per via di tre punti*), while Peano following Pasch (1882) meant a segment defined by its extremities (*retta limitata*). Peano, unlike Pieri, did not choose line as primitive. By assuming rectilinear segment as primitive (so that the class of points lying between any two points is undefined), Peano was able to define line as the class of points collinear with two points and lying beyond them. Peano's reasons for eschewing line may have gone beyond his desire to reduce the number of primitives. Since he envisioned geometry as the science of the space in which we live, taking rectilinear segment as primitive may have made better sense for a reason E.H. Moore identified: limited segment a "more fundamental notion" than line for that part of geometry which establishes a body of postulates based on spatial intuition (Moore 1902, 144). Pieri rejected the use of the rectilinear segment as primitive, not only because its interpretation is based on sensory experience; but because it is not invariant under the operations of projection and duality, and is therefore not an object of *pure* projective geometry. But why did Pieri not simply define line in terms of projective segment? The reason is that Pieri believed that incidence is more basic than separation. It is important to further note that Pieri only included segment as undefined in his first two axiomatizations of projective geometry. By his third construction (1897c), he was able to eliminate segment as primitive, and define it entirely in terms of incidence. Peano would embrace Pieri's decision, calling this reduction of primitives "truly remarkable" (Peano 1904b, 93).

Because their approaches to projective geometry differed, Peano and Pieri naturally chose different sets of postulates. Peano's postulates were rooted in experience. Pieri's were not. Peano did envision postulates as implicit defininitions of the primitives. Pieri endorsed this view, as well as Peano's method of gradually introducing postulates only as the need for them arose[24]. The scholars also agreed that the postulates should be simple statements. Pieri noted that he created his postulates in such a way that they could not easily be broken down into smaller components (Pieri 1895a, §1, 607 – *Opere* 1980, 13). Peano reduced Pasch's composite postulates IV–VIII of the line to simple statements VIII, IX, VII, X and XI (1894c, 56–60).

Both scholars were concerned with completeness of their sets of postulates (in the sense that they provide the raw materials for the rigorous proofs of all the theorems), and both believed that demonstrating the independence of the postulates is an important goal. Although he could not provide the absolute independence of all the postulates of his geometry of position, Peano did prove relative independence of several by exhibiting models. Pieri did not attempt to prove the independence of his postulates in Pieri (1895–1896) Notes, but would do so in subsequent axiomatizations. In Pieri (1896c) (§1 – *Opere* 1980, 84), Pieri called the independence as well as the irreducibility of postulates "ideal goals".

[24] In (1894c), Peano demonstrated this process, using his first eleven postulates to develop the ideas of segments and rays, and to define lines and their properties. His next six postulates, developed the geometry of position of the plane and space.

## 9.5 Affinities and Segmental Transformations

Peano discussed affinities in several of his works. Today most mathematicians would interpret the word *affinity* as an affine transformation of a vector space over a field of scalars satisfying certain conditions. In the late nineteenth century, affine transformations were not only described analytically, by means of linear equations, but also synthetically, using the language of projections and sections. The motivation for the word affinity has been ascribed to Euler who called the plane curves related by them affine curves. A.F. Möbius had adopted the word for affine transformations that he described analytically via barycentric coordinates. He distinguished an affinity from a *collineation*, the word he reserved for projective transformations. In his geometric calculus (1888a), Peano followed Möbius in viewing affinities as particular cases of projective transformations of linear spaces that he called *homographies*[25].

Compare now Peano's discussion of affinities in his axiomatizations of (1889d) and (1894c). In (1889d), Peano only included a brief note in the appendix indicating that a homography can be defined in terms of the logical concept of correspondence or function between points of space. In (1894c), he didn't mention homographies, and simply defined an affinity synthetically as a correspondence $m$ between the points of space such that if the point $c$ lies between $a$ and $b$, then the same relation holds between its correspondents $mc$, $ma$, and $mb$. So by definition, Peano's (1894c) affinities preserve the betweenness relation of points on a line. But they do not necessarily preserve segments. Peano explicitly indicated that the image of a segment $ab$ under an affinity *m is contained in* the segment $(ma)(mb)$, but that he did not know if the image would be $(ma)(mb)$ (1894c, 78).

To understand why Peano made preservation of betweenness a part of his definition of affinities and why he did not prove that they preserve segments, it is useful to compare his remarks in (1889d) and (1894c). In (1889d), Peano indicated that if $a$ and $b$ are points, by $ab$ he intended the class formed by points *internal to* (emphasis added) the segment $ab$, hence the formula "$c \epsilon ab$" signifies that $c$ is a point internal to (or a point of)[26] the segment $ab$. In (1894c) he noted that instead of saying that "$c$ is a point of segment $ab$", it is more convenient to say that "*c lies between* (emphasis added) $a$ and $b$"; so that all of geometry could be based on the concepts of points and a relation between three points $a, b, c$ expressed by the phrase "$c$ lies between $a$ and $b$". Peano indicated that Vailati had demonstrated in (1892) that the binary relation, "the point $b$ follows the point $a$" in a certain direction, can be used on a fixed line to express the ternary relation, "$c$ lies between $a$ and $b$"[27]. Peano observed, however, that the idea of "between" or "internal to" as a consequence of the relation "following" cannot apply to points of space because the fundamental

[25] Peano followed M. Chasles who had used the word homography instead of collineation (Chasles 1837, II, 695). In general, usage of the terms homographies and collineation varied among geometers, and many, like Peano, used them interchangeably.

[26] Formula $a \epsilon b$ indicates that "$a$ è un $b$" or $a$ is an individual of the class $b$ (Peano 1889d, *Notazioni*, 6 – *Opere scelte*, vol. 2, 1958, 59).

[27] Vailati was able to replace the ternary relation with the binary one for seven of the eleven axioms for the line that Peano had adopted in (1889d).

relation between three points cannot be expressed by means of a relation between only two.

Staudt (1847, n. 121–122) had proved, in the presence of his fundamental theorem, it is possible to define homographies in the plane or in space by the simple condition that points of a straight line correspond to points of a straight line. In (1880), Darboux provided that homographies between lines are ordered correspondences. Peano knew this result[28]. Why then in (1894c) did Peano choose to define an affinity as a point transformation of space that preserves the order of points on a line? With respect to Darboux's results, the short answer is that Peano's assumptions and definitions were different from those of Darboux. But the larger answer reveals considerations that also provide an opportunity to examine why Pieri in (1896b) compared his segmental transformations to Peano's affinities, despite the fact that affinities are point transformations of space, while segmental transformations are point transformations between lines.

Peano's definition of affinity in (1894c) is a type of *nominal definition* in which the defined entity is an entity or combinations of previously introduced entities that can be described in terms of the primitives of the system. Both he and Pieri advocated the use of nominal definitions (*definizione nominale*)[29], as opposed, for example, to real definitions (*definizione reale o di cosa*) that list properties sufficient to characterize the concept for some intended purpose. Pieri noted, in particular, that nominal definitions have an advantage over other types of definitions in that they imply *ipso facto* the existence of the defined concept. Peano defined an affinity in terms of his primitive, betweenness of points on a line. Pieri similarly defined segmental transformations in terms of his primitive, projective segment.

Order of points on a line is by definition explicitly preserved by Peano's affinities, and implicitly by Pieri's segmental transformations (since he defined order in terms of projective segments). Pieri would not have made preservation of order part of his definition of segment transformations. He believed that the derived properties of transformations should be separated from their primitive properties. His nominal definition of segmental transformation would therefore exclude any reference to order.

Underlying the idea of order is an intuitive concept of motion, understood in the Euclidean sense of a distance-preserving transformation, or, appealing to the idea of Staudt, as the projective order of points on a continuous line. Both Peano and Pieri sought to interpret motion exclusively using logical terms. Peano's congruence motions, particular types of affinities, were treated as logical correspondences. Pieri's definition of order in terms of incidence enabled him to provide a logical basis for Staudt's interpretation. Pieri indicated that all segmental properties of the line de-

[28] Peano had cited Darboux's paper in Peano (1884c) for the purpose of showing the hypothesis of a proposition of A. Cauchy that the continuous solution of the functional equation $f(x+y) = f(x) + f(y)$ is $ax$, for some constant $a$, could be weakened.

[29] Pieri only used nominal definitions in his system and Peano also attempted to do so. See, for example, M. Pieri (1900a), Prefazione, 173 – *Opere* 1980, 183–184, G. Peano (1889d), 25–28 – *Opere scelte*, vol. 2, 1958, 77–80. F. Rodriguez-Consuegra (1991), §3.4.2, 121, reports that Peano only resorted to other types when he could not offer nominal ones.

duced from his postulates can be interpreted as statements of order in the sense of that generated by the motion of a point[30].

Peano's affinities and Pieri's segmental transformations are both injective, albeit in different contexts[31]. These injective transformations were used by both scholars as a context to discuss a bijective subset of them. In (1894c) Peano introduced motions as a particular case of affinities[32]. In (1896b), Pieri introduced harmonic transformations as a particular case of segmental transformations. However, it is important to note that although he defined harmonic transformations in Pieri (1896b) as bijective, by (1906b), Pieri would prove this condition too strong. The evolution of Pieri's thought on harmonic transformations reveals a deliberate use of language in his treatment of them, which I pause to reflect upon now. The story begins with Staudt.

Staudt intended plane or space collineations as one-to-one correspondences that preserves points on lines. Collineations on lines that preserve harmonic sets are projectivities. Unless referring to Staudt, Pieri eschewed both of the terms collineation and projectivity in all his axiomatizations of real projective geometry. He used the phrase *harmonic transformations*[33] for Staudt projectivities. It is reasonable to assume that Pieri chose that phrase precisely because he wanted the reader to know he was emulating Staudt's reformulation of the classical definition of projectivity in terms of the preservation of harmonic sets.

In (1896b), Pieri introduced the notion of harmonic transformation as a particular case of a *segmental transformation*[34]. He assumed segment as primitive and defined a harmonic transformation as a bijective segmental transformation that preserves harmonic sets. In (1898c) he made the same definition of harmonic transformation, but without assuming segment as primitive[35]. It is here that Pieri first explicitly ob-

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[30] De Paolis, for example, adopted Staudt's interpretation, indicating for example that a segment $E_1E_2$ is decribed by fixing the direction of the motion if a point E leaves from an initial position $E_1$ and arrives at a final position $E_2$ (De Paolis 1880–81, Parte 1 §1, 489). Darboux had used such a characterization: "Supposons qu'un point M se meuve de P′ en Q′. . . " (Darboux 1880, 58). But Peano (1894c, 76) indicated that such statements as "a moveable point describes a line" should be excluded from geometry books.

[31] In introducing segmental transformations, Pieri referred the reader to (G. Peano 1894g, §26, 31–32) where Peano had discussed different categories of functions, and where he had distinguished *Sim* from *sim*, the term used for bijective. In (M. Pieri 1896b, §13, 457 – *Opere* 1980, 69 footnote), Pieri explicitly observed that his segmental transformations are not required to be bijective are therefore more general than *similes*.

[32] In (1889d), Peano had introduced motion as a special case of homographies (without mentioning the word affinity). But in both (1888a) and (1894c), Peano introduced motions as special cases of affinities.

[33] Reye had used the term *harmonic projectivity* (harmonische Projectivitaeten) in lecture 12 of the third edition of the second volume of his *Die Geometrie der Lage* (Reye 1886–1892).

[34] Pieri treated harmonic transformations in terms of segmental transformations in (1896b) and (1898c). In (1898b), (1904a) and (1905c), he discussed them in the context of homographies.

[35] G. Darboux (1880, 58) had already proved that in the presence of continuity separation can be defined from harmonicity. Pieri went one step further, defining separation from harmonicity without appealing to continuity. See E.A. Marchisotto (2006), §7.

served that there are certain conditions (without saying which) in his definition of harmonic transformations that are superfluous. But it wasn't until 1906 that Pieri finally proved what he considered superfluous conditions on the definition of harmonic transformations. Without appealing to continuity, he showed that since the postulates of projective geometry ensure these properties, it is not necessary to: (1) define harmonic transformations as *bijective* (it being sufficient that they are simply injective), nor (2) stipulate that they take *any* harmonic set into a harmonic set (it being sufficient to say that there exist in $r$ two distinct elements such that each harmonic set that contains one or the other of them is mapped to a harmonic set)[36].

## 9.6 Conclusion

The question that motivated this paper – why Pieri made an analogy to Peano's affinities when he introduced segmental transformations – revealed several plausible answers. But perhaps more importantly, seeking to answer the question provided an opportunity to explore the commonalities and differences about the scholars' views and treatments of projective geometry and its transformations. In this regard, there is one more avenue to explore. What is not evident from his axiomatizations, but is clear from his lectures[37] to students, is the evolution of Pieri's thoughts about projective geometry. In his (1891) notes for a course in projective geometry at the Military Academy – prior to writing his first axiomatization, but after he had translated Staudt (1847) – Pieri took the same approach to projective geometry as had Peano[38]. But in his notes for the University of Parma (1909–10), after he had written all his foundational papers in projective geometry, Pieri alerted students to the more "desirable" direction of Staudt as opposed to that pursued by J. Poncelet, Möbius, J. Steiner and Chasles, who studied projective geometry as an extension of elementary geometry.

Pieri had learned well from Peano, but was not reluctant to forge his own path. For example, Pieri would adopt Peano's ideas on point transformations, but took their use to new levels. In (1898b), he demonstrated the possibility of constructing real projective geometry entirely on the basis of point and a projective point transformation that preserves lines. In (1900), he constructed absolute geometry, the theory common to Euclidean and Bolyai–Lobatchevskian geometry, solely on the undefined notions of point and motion[39]. In that paper, Pieri observed that although the distinction between the synthetic concept of a congruence transformation (motions)

[36] M. Pieri (1906f), 1–5.

[37] Pieri's lecture notes will be analyzed in E.A. Marchisotto, F. Rodriguez-Consuegra, J.T. Smith 2010 (to appear).

[38] Pieri taught projective and descriptive geometry at Royal Military Academy from 1886 to 1900. He translated G.K.C. Staudt (1847) in 1889.

[39] Pieri was the first to establish absolute geometry (1900) on the basis of only two primitives. In E.A. Marchisotto, F. Rodriguez-Consuegra, J.T. Smith 2011 (to appear), an English translation and analysis of this work will be given.

from points to points rather than from figure to figure is not, from a logical perspective, significant, the first idea is more "manageable" to the deductive process. Pieri acknowledged Peano (1889d) and Peano (1894c), noting that Peano's primitives and postulates could be derived from his Pieri (1900a, *Prefazione*, 174 – *Opere* 1980, 184).

And the path would come full circle. Peano would be inspired by Pieri's fertile ideas. For example in 1903 Peano proposed a construction of geometry based on the ideas of point and distance[40]. His proposal combined Pieri's plan (announced in Pieri 1901b) to establish elementary geometry on the basis of point and two points equidistant from a third (that would be realized in Pieri 1908), with his own Peano (1898c) construction on point and vector. Using Pieri's idea of equidistance, Peano was able to define the equivalence of vectors instead of taking it as primitive, as he had previously, and reformulate definitions (include that of vector) on solely on the basis of it. He produced a systemization of geometry founded on three primitives (point, the relation of equidifference between pairs of points, and inner product of two vectors) and nineteen postulates (reducible to seventeen).

It is impossible to exclude the influence of Peano on Pieri's immersion into the world of foundations. After his (1895–1896) Notes, Pieri would continue to refine his ideas on projective geometry and ultimately produce what Russell (1903)[41] called "the best work" on the subject. As I have observed, Peano was involved in a substantial way in propelling Pieri on the path to that achievement. And he shared Russell's evaluation of it. Peano wrote: "The results reached by Pieri constitute an epoch in the study of foundations of geometry, and all those who later treated the foundations of geometry have made ample use of Pieri's work and have echoed Russell's evaluation". Pieri had made his mentor proud![42]

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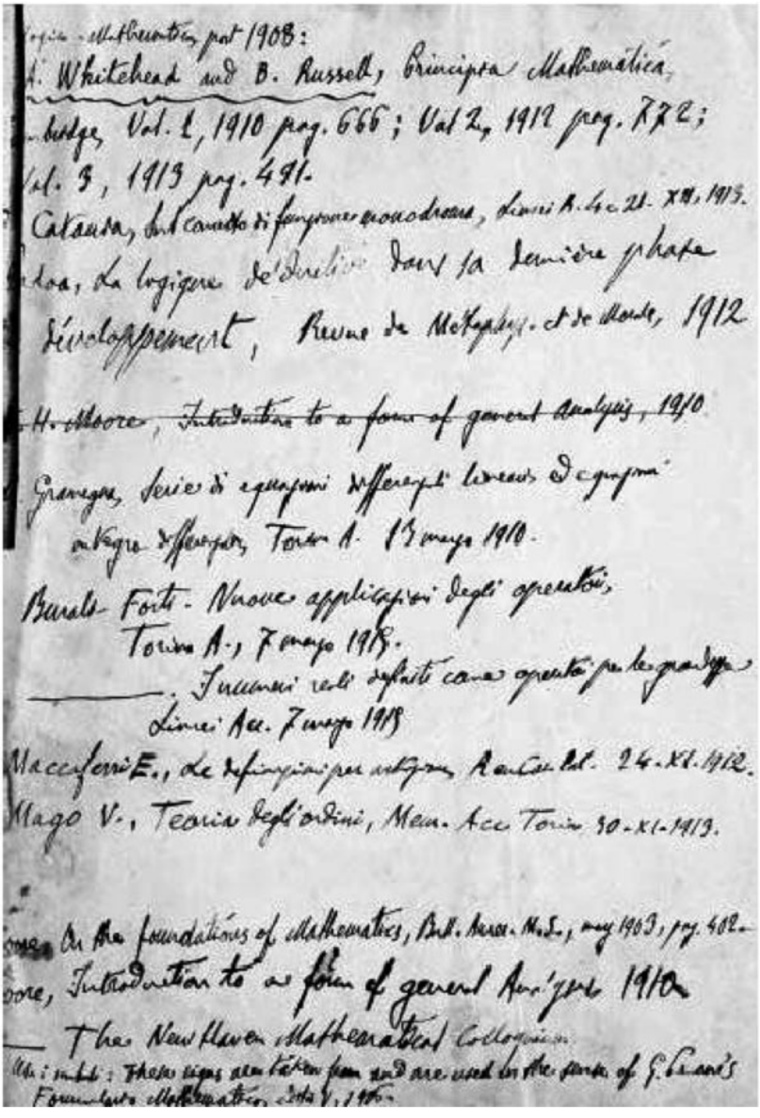
[40] G. Peano (1903a).

[41] Russell believed that Pieri was the one who demonstrated the true nature of the purely projective method (Gandon 2004, 189).

[42] G. Peano (1915j), 171.



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