you failed your math test, Comrade Einstein

EDITED BY M. SHIFMAN

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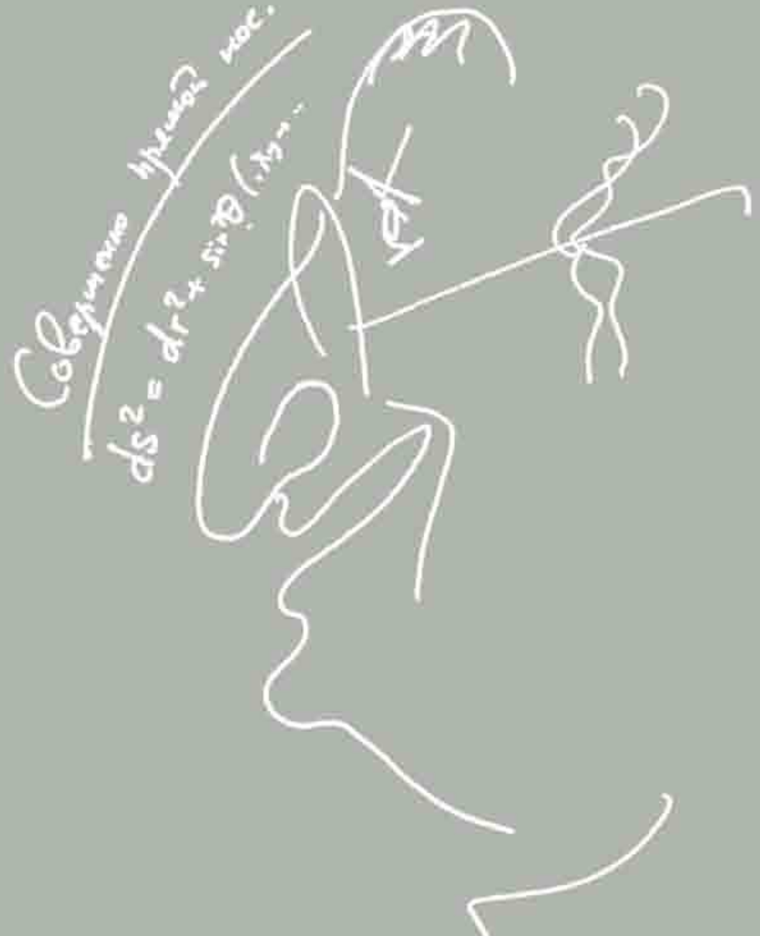
Adventures and Misadventures of Young Mathematicians

Edited by M. Shifman



Никакого отношения
к Мисс Shifton'у

FROM THE EDITOR

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Everybody knows that the Soviet Union had a great culture of chess. Many outstanding chess players of the 20$^{\text{th}}$ century were from the USSR. Much less known, however, is another remarkable cultural tradition, which I will refer to as the "Math Movement," with capital M's. Quite different from recreational mathematics in the West, Math Movement mathematics was a unique phenomenon in the social life of the country, if the term "social life" is at all applicable to communist regimes. The tradition was upheld and promoted by a great variety of enthusiasts – from 13-year-old schoolboys and girls, to seasoned mathematics professors. The phenomenon hit every large city of the country that spanned eleven time zones. These enthusiasts were engaged in creating contrived, complex and intellectually challenging math problems which could be solved, in principle, on the basis of elementary mathematics (i.e. "mathematics before calculus"), as it was taught in Soviet schools. They strived to get nonstandard solutions to these problems, and to disseminate knowledge about such problems and their solutions in every school and every class.[a] This Math Movement became widespread in the 1930s; and it attracted the best and the brightest. Its basis was formed from so-called *mathematical circles* – groups of school students, math teachers and mathematicians that existed virtually in every school, university, and in many other places. I remember that I myself belonged, at different times, to several such circles: one at my school, another at the Moscow Institute for Physics and Technology, and a third one associated with the Moscow Palace of Pioneers. They held regular meetings, once or twice a week, where advanced problems were discussed in classes and offered for personal analysis on one's own time. I looked forward to

[a]Some idea of the character of these problems can be inferred from a selection *Mathematical Circles* published by the American Mathematical Society.[1]

and enjoyed every meeting – they provided me with brain gymnastics, which were otherwise so scarce and so discouraged, to put it mildly, in every other aspect of Soviet life.

The Math Movement had its Grandmasters, who were highly esteemed. Most of them were research mathematicians and university professors who had drawn experience from years spent within the same mathematical circles. Their books, which contained selections of problems with commentaries, or thorough analyses of selected topics from "elementary mathematics," were in high demand. Many of these books were superb and unparalleled in their quality and depth. Remarkably, they were swept from the bookstore shelves, immediately upon arrival. These books were a captivating read, and I hunted for some of them in secondhand bookstores for weeks and even months.

Mathematical circles were just one element of the Math Movement. *Olympiads* (or math competitions) presented another pillar. They were organized on a regular basis and at every level, beginning in school districts, through city competitions, and finally at the national level. The highest achievers at every level were admired. National prize winners were praised by the media just as winners of the national spelling-bee competitions are praised in the US.

Finally, the third pillar of the Math Movement was a network of special "mathematical schools." Every city had at least one, and large cities, such as Moscow and Leningrad, had, perhaps, a dozen. Even small towns tended to arrange a "mathematical class" in a school. At age 13 or so, mathematically and scientifically gifted students were selected for such schools through a competitive process – usually a skillfully tailored entrance examination. What made these schools really special was a unique academic and social environment. They were run by enthusiastic teachers who worked not for money – salaries were meager, as they were everywhere else in the USSR – but for the excitement and joy that naturally emerged in the creative atmosphere produced by enthusiastic students united by a common appreciation for the beauty of math and science. Classes often continued far into the night. Students and teachers often went on weekend trips, summer retreats and hiking expeditions. My daughter was a student at one of the best Moscow math schools, # 57, and I remember many events of this type, and so does she – fifteen years after graduation.

The Math Movement was an element of culture scarcely mentioned in the literature accessible to the western reader. One of my goals in this book is to familiarize the western reader with elementary math problems, of various levels of complexity, which constituted a fertile ground, the very basis of the Math Movement. The core of this book, two excellent essays written by Dr. Ilan Vardi, serve this purpose.

There is another goal, however, which is as important to me as the first one. This second objective is explained below.

Mathematics at the service of ideology of "real socialism"

> "All students are treated equally,
> but some more equally than others..."
> *Achievements of Real Socialism*
> (Moscow University Press, 1982)
> Vol. 1982, p. 1982.

> "Don't worry, we will flunk them all..."
> *From an overheard conversation of*
> *a mathematics professor with*
> *the Chairman of an Admission Committee.*

An important part of this book is devoted to a bizarre and, I would say, unique page in the history of science. It tells a story of how high-school mathematics was used as a weapon of racism in the USSR – a country which gave to the world many brilliant mathematicians whose role in shaping 20$^{\text{th}}$-century mathematics was absolutely instrumental.[2] This topic deserves the attention of professional historians, and I am admittedly an amateur in this field. Since professional historians are in no hurry, and time is rapidly erasing the recollections of live witnesses, I would like to, at least, set the stage. My role is more than modest. I collected, at a rather fragmentary level, relatively accessible notes and recollections of live participants in these events. Some were published in the Russian media, and thus the only effort needed to make them accessible to the western reader was translation; others were not published.

The place of action is the Soviet Union, the time is the 1970s and 80s, a time of a general decline of the regime that accelerated after the 1968 Soviet invasion of Czechoslovakia. The social and economic climate was rapidly deteriorating. Stagnation, moral degradation and decay became permanent components of the everyday life of Soviet citizens. Among other ugly phenomena of the so-called *real socialism* that flourished under Brezhnev was state–sponsored anti-Semitism. Vehemently denied in public, it was in fact orchestrated by the highest echelons of the Communist Party and, behind the scenes, encouraged and promoted by the state–party machine.

As a particular manifestation, discrimination against Jews in the admission policies of Soviet universities reached its peak. Of course, it was not the first peak, and not even the strongest, but it was strong enough to virtually close all reputable departments of mathematics in the Soviet Union, as well as some physics departments, to Jewish applicants.[b] I do not know why, but it is a well-known fact that the Russian mathematical establishment was pathologically anti-Semitic. Such outstanding mathematicians as Pontryagin, Shafarevich and Vinogradov, who had enormous administrative power in their hands, were ferocious anti-Semites. The tactics used for cutting off Jewish students were very simple. At the entrance examination, special groups of "undesirable applicants" were organized.[c] They were then offered killer problems which were among the hardest from the set circulated in mathematical circles, quite frequently at the level of international mathematical competitions. Sometimes they were deliberately flawed. Even if an exceptionally bright Jewish student occasionally overcame this barrier in the written examination, zealous professors would adjust the oral exam appropriately, to make sure that this student flunked the oral exam.

What else is there to say on this issue? Everybody knew that "pu-

[b]Statistical data illustrating this fact in the most clear-cut manner were presented in the *samizdat* essays [3,4], see also the book [5].

[c]The working definition of "Jewishness" was close to that of Nazis; having at least one Jewish parent of even one grandparent would almost certainly warrant one's placement in the category of undesirables.

rification" of the student body, Nazi style, had taken place for years. It was a part of Soviet state ideology. Needless to say, the Soviet state did not want it to become public knowledge, especially in the West. The silence was first broken by dissidents and Jewish *refuseniks*[d] in the 1980s in a series of *samizdat*[e] essays, one of the first and the most famous of which, *Intellectual Genocide*, was written by Boris Kanevsky and Valery Senderov in 1980. This book presents the first publication of this essay. As you will see, it is very factual and is based on a study of 87 Moscow high school graduates from six special math schools, many of whom had won prizes in national mathematics Olympiads. The bulk of the essay is an unemotional comparative analysis of various math problems given to "desirable" and "undesirable" applicants, with statistically motivated conclusions at the end. The essay was deemed a political provocation, and heavy consequences ensued shortly. One of the authors, Valery Senderov, was sentenced to seven years in prison and 5 years in exile on charges of anti-Soviet agitation and propaganda. Boris Kanevsky was also arrested and spent three years in prison. It is hard to believe it now, but this is a true story. This is how it was ... and, unfortunately, this is not just "the past, long gone." Although anti-Semitism is no longer encouraged by authorities in the new Russia, some of the zealous professors who were part of the "intellectual

[d] A group of people treated as political enemies in the USSR in the 1970s and 80s. The only "crime" committed by these people was that they had applied for and got denied exit visas to Israel. And yet, they were treated essentially as criminals: fired from jobs and blacklisted, with no access to work (with the exception of low-paid manual labor), constantly intimidated by the KGB, at the verge of arrest. In fact, the most active of them, those who tried to organize and fight back for their rights, were imprisoned.

[e] A strict censorship existed in the USSR. Nothing could be published without preapproval from *Glavlit*, an omnipotent State Agency implementing censorship. The class of suppressed books and other printed materials included not only those with political connotations, but, in general, everything that was not considered helpful for Soviet ideology. Forbidden publications circulated in typewritten form. People retyped them, using mechanical type-writers and carbon paper, or photographed them, page by page, using amateur cameras, and then printed them at home on photopaper, producing huge piles. The process was called *samizdat*, which can be loosely translated from Russian as self-publishing. *Samizdat* was forbidden by the Soviet law.

genocide" in the 1980s continue to occupy high positions and flourish at Moscow University and some other institutions. Alas, there is no full stop in this story yet. Apology or remorse is not in sight.

To put a personal touch on this picture and to give a clearer idea of the atmosphere in which we lived, I would like to tell of an episode which happened to me in 1985 or '86. A friend of mine gave me a wonderful gift: a photocopy of Feynman's book *Surely You're Joking, Mr. Feynman.* I swallowed it overnight. It was so fascinating that I could not keep it to myself. I badly wanted to share my fascination with others. Upon reflection, I decided that the only way for me to do so was to translate it into Russian and try to publish the translation.

I called a person – let us call him RA – who was in charge of one of the departments of the popular magazine *Nauka i Zhizn* (*Science and Life*). From time to time, he would provide me with small writing jobs, so I could make an extra 20 or 30 roubles to make ends meet. In those days, this magazine had a circulation of three million plus. Now it is almost extinct; a meager 30 thousand is all that the new Russia can support.

RA met the idea with enthusiasm and was very supportive. He told me that I could go ahead and translate from a quarter to a third of Feynman's book, at my choice. He would push it through the board and take care of the copyright issues.

"Just make sure you stay away from chapters with political connotations, and passages where he might mention our spy at Los Alamos, Klaus Fuchs," he added. "Focus on science."

I worked for a month or so, and came up with 120 typewritten pages which I brought to *Nauka i Zhizn*'s office and left with RA.

In a few days he called me and said: "Are you mad?"

"What happened?"

"In your translation I found at least three paragraphs where Feynman mentioned he was Jewish. The board will never authorize this material for publication. Cut them out!"

"I do not understand, RA ... You said yourself, just steer clear from Klaus Fuchs and political issues, and so I did ... this is not political..."

"This *is* political. Just do what I am telling you, or say farewell to the project."

What could I do but comply? My crippled translation was serialized and published.[f]

On this book

A few words on the structure of this book. As I have already said, mathematics – the purest and the most beautiful of all sciences – is not responsible for the abuses associated with it. In the mid-1990s Alexander Shen, professor at the Independent University of Moscow, published[6] in *The Mathematical Intelligencer* a selection of problems which were offered to "undesirable" applicants at the entrance examinations at the Department of Mechanics and Mathematics (Mekh-mat) of Moscow University. Many of these problems are captivating. Their solution does not require knowledge of a higher level of mathematics; what you learned in high school will do. The solution does require, however, ingenuity, creativity and unorthodox attitudes. Solutions to these problems were thoroughly analyzed by Dr. Ilan Vardi. He wrote two excellent essays which are being published in Part 1: *Mekh-mat entrance examinations problems* and *Solutions to the year 2000 International Mathematical Olympiad*. The second essay is meant to complement the first one by providing a natural frame of reference for evaluating the relative complexity of various problems.

Part 2 provides the reader with necessary historical background. English translation of Kanevsky and Senderov's essay *Intellectual Genocide* opens this chapter. Among other things in Part 2 the reader will find an essay *Science and Totalitarianism* written by A. Vershik, which has

[f] *Nauka i Zhizn*, # 10, 12 (1986); 2, 8 (1987) and 8 (1988). There is a funny continuation to this story. *Surely You're Joking, Mr. Feynman* was published in Russian in full only in 2001.[7] My 1/3 of translation was incorporated; the remaining 2/3 of the book were translated by Natasha Zubchenko. Apparently, she was educated in classical British English. Many nuances of Feynman's English puzzled her and she found them incomprehensible. We had exchanged innumerable messages, and I had a few phone conversations with Natasha, trying to help her out. And yet, quite a few hilarious misinterpretations slipped unnoticed. For instance, orthodox rabbi was translated as "православный раввин."

never been published in English previously.

Part 3 describes a little-known page of 1970s-80s Soviet history, one rather rare example of the oppressed organizing to defend their dignity and trying to fight back. This is the story of the so-called *Jewish People's University*, the inception of which is associated with the names of Kanevsky, Senderov and Bella Abramovna Subbotovskaya.

It opens with the article *Free Education at the Highest Price* written by K. Tylevich, a young friend of mine who became interested in the story, made an extensive literature search, and summarized what little was known in the literature, by the beginning of 2004. Then follow personal recollections of D. Fuchs, A. Zelevinsky (both taught at the Jewish People's University) and I. Muchnik (Subbotovskaya's ex-husband), live witnesses of the events. They are emotionally charged and immerse us in the depths of this dramatic story.

Bella Subbotovskaya's idea was to launch something along the lines of unofficial extension classes, where students unfairly barred from the official universities could get food for their hungry minds from the hands of first-class mathematicians and physicists. It was supposed that the classes would take place on a regular basis through the entire school year, that they would be open to everybody (no registration or anything of the like was required) and that the spectrum of courses offered would be broad and deep enough to provide a serious educational background in the exact sciences.

Andrei Zelevinsky (now at the Northeastern University, Boston) recollects:[8] "... I was truly impressed with her [Subbotovskaya's] courage and quiet determination to run the whole thing. All the organizational work, from finding the places for our regular meetings to preparing sandwiches for participants, was done by her and two other activists: Valery Senderov and [...] Boris Kanevsky. Both, as I understand, were active dissidents at the time. I think they made a deliberate effort to separate mathematics from politics, in order to protect us, professional research mathematicians."

Another professor of this "university," Dmitry Fuchs (now at UC, Davis) writes: "We taught there major mathematical disciplines corresponding to the first two years of the Mekh-Mat curriculum: mathematical analysis, linear algebra and geometry, abstract algebra and so on. I taught there since 1980 through 1982, until Subbotovskaya's sudden death in September or October of 1982 put an end to this enterprise. Bella Abramovna was tragically killed in a hit-and-run accident which was universally believed to be the act of KGB. This has never been officially confirmed, though. At this time we did not use any particular name for our courses [...]. The number of students varied between 60 and 20. The place of our meetings was not permanent: we met at an elementary school where Bella Abramovna worked as a teacher, at the Gubkin Institute for Oil and Gas, at the Chemistry and Humanities Buildings of the Moscow State University, and so on. Anywhere, where we could get permission to occupy a large enough room... All instructors prepared notes which were photocopied and distributed among the students. An article on our "school" was published by a Russian-Israeli newspaper some time ago.[9] It was interesting but excessively emotional. Among other things, the authors had a tendency to exaggerate the Jewish nature of our "university." It is certainly true that a substantial number of both students and teachers were ethnic Jews. This was the result of the well-known policy of the Mekh-Mat Admission Committee, and the Soviet State at large, rather than the deliberate aim of the organizers. After all, we taught only exact sciences. No plans were made to teach Jewish culture, history, or language."

Needless to say, all teachers of the Jewish People's University received no reward other than the wonderful feeling that what they were doing was a good deed. Although the goals of the university were purely educational, the very fact of its existence was considered a political act of resistance. The end of Jewish People's University, which gave excellent mathematical education to over 350 young people, was tragic. The actual circumstances of the death of Bella Subbotovskaya remain uninvestigated so far. Perhaps the materials I present in this book will attract the attention of a historian or a novelist, who knows?

Thank you

This book was in the making for four years. The work on it became my weekly hobby, sucking me in like a good detective story. In fact, it was a detective story which spanned two decades and three continents. As the story unfolded, increasing numbers of people became involved, and it is their generous assistance that made this publication possible. I am deeply grateful to all participants.

It was Gregory Korchemsky, a theoretical physicist from Orsay and an old friend of mine, who came to Minneapolis in the spring of 2000 and brought, as a present, Ilan Vardi's Mekh-Mat essay. From this essay I learned of Shen's article in *The Mathematical Intelligencer*, which, in turn, quotes *Intellectual Genocide* of Kanevsky and Senderov. When I tried to google Kanevsky and Senderov, I got a few dozen hits, none of them being particularly useful since they contained only marginal mentions of Kanevsky and Senderov's essay. Archive searches became inevitable. I wrote to many archives in an attempt to get a copy of the essay and/or locate the authors. In the autumn of 2000, three positive responses came — from G. Superfin (*Forschungsstelle Osteuropa*, Universität Bremen, Germany), N. Zanegina (Open Society Archives at Central European University, Budapest, Hungary), and G. Kuzovkin (Memorial Archive, Moscow). Along with other useful materials, they sent me photocopies of *Intellectual Genocide*, none of them complete, all taken from typewritten n-th carbon copy of distinct originals where I estimate $n > 3$. The tedious work of comparing three distinct copies, restoring the full original, checking all mathematical expressions and translating the original into English was done by Nodira Dadabayeva and Alexei Kobrinsky, my student helpers who were also responsible for typesetting all of the materials.

In November 2003, I managed to get in touch with Valery Senderov through the Editorial Office of *The Herald of Europe*. In December of the same year I interviewed him in Moscow. Not only did he share with me a treasure trove of personal recollections of the 1980's, but he also connected me to Boris Kanevsky in Jerusalem who owns a large private archive of *Samizdat* documents (and shared some of them with me), and to M. Vyalyi, A. Belov-Kanel and A. Reznikov, the Editors of the *Matematicheskoye Prosveshcheniye*.[g] This Russian magazine is intended for the general public and devoted to the advancement of mathematics. The 2005 Almanac of *Matematicheskoye Prosveshcheniye* will include a collection of articles on the Jewish People's University and B. Subbotovskaya. Two of these articles are translated into English and published in Part 3, with the kind permission of the Editors.

It is my pleasure to say thank you to Sally Menefee who handled all financial aspects of this project and to Roxanne Keen for proofreading the articles translated from Russian. I am grateful to Roman Kovalev who carried the main burden of translation. As usual, my World Scientific contact, Lakshmi Narayanan — "my Editor" — was instrumental in the speedy completion of this project. Special thanks go to Poline Tylevich, the graphic designer, who did a great job making this book visually appealing. And, above all, I want to say *thank you* to the contributors and advisors who are listed below.

Minneapolis, December 31, 2004

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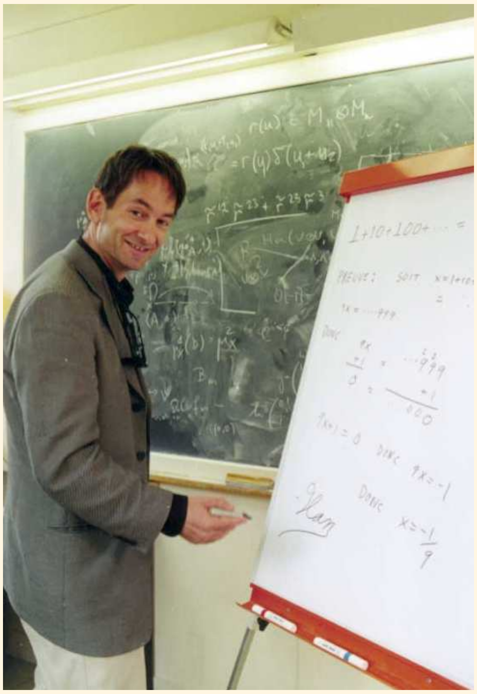


Ilan Vardi

Ilan Vardi is a mathematician with broad interests including number theory, analysis, combinatorics, computer science, history of mathematics, and recreational mathematics. He is currently living in France.

MEKH-MAT ENTRANCE EXAMINATIONS PROBLEMS

ILAN VARDI

The recent articles [a] of Anatoly Vershik and Alexander Shen [1,2] describe discrimination against Jews in Soviet universities during the 1970's and 1980's. The articles contain a report by Alexander Shen on the specific role of examinations in discrimination against Jewish applicants to the Mekh-mat at Moscow State University during the 1970's and 1980's. The article goes on to list "killer problems" that were given to Jewish candidates. However, solutions to the problems were not given in the article, so in order to judge their difficulty, one must try the problems for oneself. The aim of this note is to relieve readers of this time consuming task by providing a full set of solutions to the problems. Hopefully, this will help readers gain some insight into the ethical questions involved.

Section 2 consists of a personal evaluation of the problems in the style of a referee's report. It was written to provide a template for readers to make a similar evaluation of the problems. This evaluation also reflects the author's own mathematical strengths and weaknesses as well as his approach to problem solving. Readers are therefore encouraged to make up their own minds.

The problems are given exactly as in Ref. 2 with the names of the examiners and the year (A. Shen has explained that in his article, the name of the examiners and year is given by a set of problems ending with the name). Some inaccuracies of Ref. 2 both in the statement of the problems and attribution of examiners have have been corrected, see Section 4. Some of the statements are nevertheless incorrect. These errors are a reflection of either the examinations themselves, the reports given by the students, or the article of Ref. 2. In any case, this is further evidence for the need of a complete solution set.

[a] These articles are reprinted in Part 2 of the present Collection. – Editor's note

These solutions were worked out during a six week period in July and August 1999. In order to retain some aspect of an examination, no sources were consulted. As a result, the solutions reflect gaps in the author's background. However, this might offer some insight into how one can deal with a wide range of elementary problems without the help of outside references. An effort was therefore made to explain how the solutions were found. The solutions are the most direct that the author could come up with, so some unobvious tricks may have been overlooked.

After completing these problems, the author discussed them with other mathematicians who, in some cases, found much better solutions. These solutions are therefore given along with the author's solutions in Section 3. Section 4 provides notes on the problems such as outside references and historical remarks.

The most egregious aspect of these problems is the fact that they are, to the author's knowledge, the only example in which mathematics itself has been used a political tool. It is important to note that there is absolutely no controversy about whether this discrimination actually took place – it appears that antisemitism at the Mekh-mat was accepted as a fact of life. It is the author's conviction that the best course of action now is to provide as much information as possible about what took place. A more detailed account of the political practices described by Vershik and Shen should follow.

Acknowledgment

I would like to thank Marcel Berger, Jean-Pierre Bourgignon, Ofer Gabber, Mikhail Grinberg, Leonid Polterovich, Igor Rivin, and Anatoly Vershik for helpful remarks. A. Shen was kind enough to supply supplementary information and explanations about the problems. I would like to thank the following persons for communicating their solutions: Marcel Berger (Pr. 1), David Fried, Ofer Gabber (Pr. 22), Georg Illies (Pr. 2), Igor Rivin (Prs. 2 and 22), David Ruelle (Pr. 2), Pavol Severa (Prs. 1 and 2), and Michel Waldschmidt (Pr. 4).

Finally, I would like to thank Victor Kač for bringing these problems to my attention.

Problems

1. K is the midpoint of a chord AB. MN and ST are chords that pass through K. MT intersects AK at a point P and NS intersects KB at a point Q. Show that $KP = KQ$. /Lavrentiev, Gnedenko, Vinogradov, 1973; Maksimov, Falunin, 1974/

2. A quadrangle in space is tangent to a sphere. Show that the points of tangency are coplanar. /Maksimov, Falunin, 1974/

3. The faces of a triangular pyramid have the same area. Show that they are congruent. /Nesterenko, 1974/

4. The prime decompositions of different integers m and n involve the same primes. The integers $m+1$ and $n+1$ also have this property. Is the number of such pairs (m,n) finite or infinite? /Nesterenko, 1974/

5. Draw a straight line that halves the area and perimeter of a triangle. /Podkolzin, 1978/

6. Show that, for $0 < x < \pi/2$,

$$(1/\sin^2 x) \le (1/x^2) + 1 - 4/\pi^2 .$$

/Podkolzin, 1978/

7. Choose a point on each edge of a tetrahedron. Show that the volume of at least one of the resulting tetrahedrons is $\le 1/8$ of the volume of the initial tetrahedron. /Podkolzin, 1978/

8. We are told that $a^2 + 4b^2 = 4$, $cd = 4$. Show that

$$(a-d)^2 + (b-c)^2 \ge 1.6 .$$

/Sokolov, Gashkov, 1978/

9. We are given a point K on the side AB of a trapezoid $ABCD$. Find a point M on the side CD that maximizes the area of the quadrangle which is the intersection of the triangles AMB and CDK. /Fedorchuk, 1979; Filimonov, Proshkin, 1980/

10. Can one cut a three-faced angle by a plane so that the intersection is an equilateral triangle? /Pobedrya, Proshkin, 1980/

11. Let H_1, H_2, H_3, H_4, be the altitudes of a triangular pyramid. Let O be an interior point of the pyramid and let h_1, h_2, h_3, h_4 be the perpendiculars from O to the faces. Show that

$$H_1^4 + H_2^4 + H_3^4 + H_4^4 \geq 1024\, h_1 \cdot h_2 \cdot h_3 \cdot h_4\,.$$

/Vavilov, Ugol'nikov, 1981/

12. Solve the system of equations

$$\begin{cases} y(x+y)^2 = 9 \\ y(x^3 - y^3) = 7\,. \end{cases}$$

/Vavilov, Ugol'nikov, 1981/

13. Show that if a, b, c are the sides of a triangle and A, B, C are its angles, then

$$\frac{a+b-2c}{\sin(C/2)} + \frac{b+c-2a}{\sin(A/2)} + \frac{a+c-2b}{\sin(B/2)} \geq 0\,.$$

/Dranishnikov, Savchenko, 1984/

14. In how many ways can one represent a quadrangle as the union of two triangles? /Dranishnikov, Savchenko, 1984/

15. Show that

$$\sum_{n=1}^{1000} \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{4}\,.$$

/Bogatyi, 1984/

16. Solve the equation

$$x^4 - 14x^3 + 66x^2 - 115x + 66.25 = 0\,.$$

/Evtushik, Lyubishkin, 1984/

17. Can a cube be inscribed in a cone so that 7 vertices of the cube lie on the surface of the cone? /Evtushik, Lyubishkin, 1984/

18. The angle bisectors of the exterior angles A and C of a triangle ABC intersect at a point of its circumscribed circle. Given the sides

AB and BC, find the radius of the circle.[b] /Evtushik, Lyubishkin, 1986/

19. A regular tetrahedron $ABCD$ with edge a is inscribed in a cone with a vertex angle of $90°$ in such a way that AB is on a generator of the cone. Find the distance from the vertex of the cone to the straight line CD. /Evtushik, Lyubishkin, 1986/

20. Compare

$$\log_3 4 \cdot \log_3 6 \cdot \ldots \cdot \log_3 80 \qquad \text{and} \qquad 2\log_3 3 \cdot \log_3 5 \cdot \ldots \cdot \log_3 79\,.$$

/Smurov, Balsanov, 1986/

21. A circle is inscribed in a face of a cube of side a. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles. /Smurov, Balsanov, 1986/

22. Given k segments in a plane, show that the number of triangles all of whose sides belong to the given set of segments is less than $C\,k^{3/2}$, for some positive constant C which is independent of k. /Andreev, 1987/

23. Use ruler and compasses to construct the coordinate axes from the parabola $y = x^2$. /Kiselev, Ocheretyanskii, 1988/

24. Find all a such that for all $x < 0$ we have the inequality

$$ax^2 - 2x > 3a - 1\,.$$

/Tatarinov, 1988/

25. Let A, B, C be the angles and a, b, c the sides of a triangle. Show that

$$60° \le \frac{aA + bB + cC}{a + b + c} \le 90°\,.$$

/Podol'skii, Aliseichik, 1989/

[b] From Ref. 2: "The condition is incorrect: this doesn't happen."

1. Evaluation of the Problems

I have classified the problems according to difficulty, inherent interest, and correctness. The first two criteria are subjective, however, the fact that the problems listed in categories *VI* and *VII* have incorrect statements is proved in Section 3. This evaluation reflects the solutions presented in Section 3.

Since these problems appear to be at a level similar to Olympiad problems [2], it seems that Olympiad problems are an appropriate standard for comparison [3–8].

However, it must be stressed that these problems were given in *oral* examinations. This makes the comparison to Olympiad problems valid only in the sense that given similar conditions, the problems have the same level of difficulty. Note that the International Mathematical Olympiad consists of a written examination given over two days, with a total of hours to solve 6 problems.

It should be noted that these problems also differ from Olympiad problems by being, in many cases, either false or poorly stated. Such defects have the side effect of making the problems more interesting in some cases, as they are less artificial than Olympiad problems in which a certain type of solution is often expected.

I. Easy: 15, 24

By this, I mean problems which, once one has understood the statement, offer no conceptual or technical difficulty – there is no idea or difficult computation to challenge the solver. I also include problems which require ideas which are completely standard and should be known to students wishing to pursue a college mathematics education.

II. Tricky: 4, 7, 8, 13, 14, 18, 23

By this, I mean problems which can be quite challenging until one has found a simple but not well motivated idea after which the result is immediate. This applies to the proof that the statement of Problem 18 is false. Note that Problem 7 is much more difficult than the others, see Remark 7.2. Problem 14 has a “trap” which caught some students [9], but the examiners themselves overlooked a trap, see Remark 14.1.

III. Challenging and interesting: **3, 9, 11, 20, 22, 25**

These are problems whose solutions require interesting ideas and whose statements are also of interest. In other words, these would make good Olympiad problems.

IV. Straightforward and difficult: **1, 2, 5, 6, 17, 21**

These are problems which can be solved by a direct computation which does not require any clever idea, though the computation may be quite involved. The problems have alternate solutions with interesting conceptual content and thus put them in category *III* (this applies to Problems 1, 2, and 21).

V. Difficult and uninteresting: **10, 12, 16, 19**

These are problems with an uninteresting statement and whose solution is a long and unmotivated computation.

VI. Inaccurate statement: **5, 7, 9, 14, 19, 22**

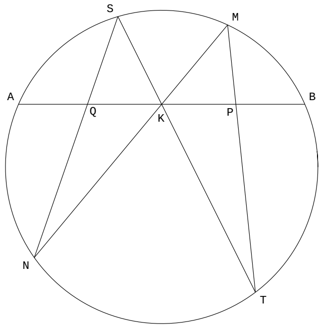
These problems have statements with alternative interpretations most likely not intended by the authors.

VII. Completely wrong: **18**

This problem asks for conclusions about situations which cannot occur. (So A. Shen's [2] comment about it is correct.)

2. Solutions

Problem 1. *K is the midpoint of a chord AB. MN and ST are chords that pass through K. MT intersects AK at a point P and NS intersects KB at a point Q. Show that $KP = KQ$.*



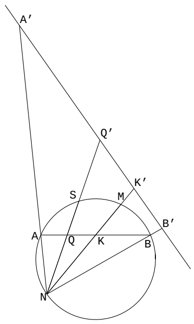
Solution S. The following solution is due to Pavol Severa.

The claim can be made obvious using Lobačevskij geometry. In the Klein disk, i.e. projective, model, the Lobačevskij plane is a disk and straight lines are chords. Let the notation be as in the statement of the problem. If K is an arbitrary point on the Lobačevskij line AB then QK and PK are congruent, since a 180° rotation about K preserves the picture, except that it exchanges P and Q. It follows that P and Q are equidistant to K in the Lobačevskij metric. Now let CD be the diameter of the circle passing through K, then the previous remark shows that the Lobačevskij reflection about CD takes P to Q. But since we chose K to be the Euclidean center of AB, CD is perpendicular to AB, so that the Lobačevskij reflection about CD equals the Euclidean reflection about CD. It follows that $|QK| = |PK|$ in Euclidean sense as well.

Solution R. The following solution is due to David Ruelle. The idea is to use the cross ratio of four points A, B, C, D which can be defined by

$$[A, B, C, D] = \frac{|AC| \cdot |BD|}{|AD| \cdot |BC|}. \tag{1}$$

One simply notes that A, Q, K, B are the stereographic projections of A, S, M, B with pole N and that A, K, P, B are the stereographic projections of A, S, M, B with pole T. Since stereographic projection preserves cross ratios, it follows that $[A, Q, K, B] = [A, K, P, B]$. The result follows from a simple computation using the above algebraic definition of cross ratio.



It should be noted that the above stereographic projection is not the standard one but still preserves cross ratios. To see this, let L be a line perpendicular to a diameter through N and let A', Q', K', B' be the projections of A, Q, K, B onto L. This projection preserves the cross ratio. The stereographic projection of A, S, N, B onto A', Q', K', B' is now the standard one, i.e. is an inversion, and thus preserves cross ratios [10].

Algebraic solution. The following argument uses an algebraic approach which seems to be the most direct, i.e. requires the least amount of ingenuity or knowledge.

Let the circle be $\{(x, y) : x^2 + y^2 = 1\}$ and let $K = (0, \beta)$, so that

$$A = (-\sqrt{1-\beta^2}, \beta)\,, \quad B = (\sqrt{1-\beta^2}, \beta)\,.$$

One first excludes the trivial case: $M = S$ and $N = T$, when $P = Q = K$ and the result holds. Otherwise, one considers lines L_1 and L_2 passing through K which determine the chords. Assuming for the time being that neither L_1 nor L_2 is parallel to the y-axis, the lines L_1 and L_2 can be defined by the equations $y = m_1 x + \beta$ and $y = m_2 x + \beta$, respectively.

Without loss of generality, one can assume that $m_1 > 0$. First, one considers the case when $m_2 < 0$. Let L_1 intersect the circle at

$$M = (x_1\,, y_1) \text{ and } N = (x_3\,, y_3)\,, \quad \text{where } x_1 > x_3\,,\ y_1 > y_3\,,$$

and L_2 intersect the circle at

$$T = (x_2\,, y_2) \text{ and } S = (x_4\,, y_4)\,, \quad \text{where } x_2 > x_4\,,\ y_2 < y_4\,.$$

One now computes P, i.e. one finds the x-coordinate of the point on the line segment MT which has y-coordinate equal to β. The line segment is represented by

$$\lambda M + (1-\lambda)T\,, \quad 0 \le \lambda \le 1\,,$$

so

$$\lambda y_1 + (1-\lambda)y_2 = \beta\,,$$

and one gets

$$\lambda = \frac{\beta - y_2}{y_1 - y_2}\,.$$

Letting $P = (\alpha, \beta)$, one has

$$\alpha = \frac{\beta\,(x_1 - x_2) + y_1 x_2 - y_2 x_1}{y_1 - y_2}\,. \tag{2}$$

One observes that

$$y_1 x_2 - y_2 x_1 = x_1 x_2 \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right) = x_1 x_2 \left(m_1 + \frac{\beta}{x_1} - m_2 - \frac{\beta_2}{x_2}\right)$$

$$= x_1 x_2 (m_1 - m_2) + \beta x_2 - \beta x_1\,.$$

Substituting this into (2) gives

$$\alpha = \frac{m_1 - m_2}{\frac{m_1}{x_2} - \frac{m_2}{x_1}}\,. \tag{3}$$

Since M and T lie on the unit circle, one has

$$x_1^2 + (m_1 x_1^2 + \beta)^2 = 1 \quad \text{and} \quad x_2^2 + (m_2 x_2^2 + \beta)^2 = 1\,,$$

so with the above assumptions,

$$x_1 = \frac{-m_1\beta + \sqrt{1+m_1^2-\beta^2}}{1+m_1^2}\,, \quad x_2 = \frac{-m_2\beta + \sqrt{1+m_2^2-\beta^2}}{1+m_2^2}\,.$$

This implies that

$$\begin{aligned}\frac{m_1}{x_2} - \frac{m_2}{x_1} &= \frac{m_1\big(1+m_2^2\big)\big(-m_2\beta - \sqrt{1+m_2^2-\beta^2}\,\big)}{m_2^2\beta^2 - \big(1+m_2^2-\beta^2\big)} \\ &\quad - \frac{m_2\big(1+m_1^2\big)\big(-m_1\beta - \sqrt{1+m_1^2-\beta^2}\,\big)}{m_1^2\beta^2 - \big(1+m_1^2-\beta^2\big)} \\ &= \frac{-m_1\sqrt{1+m_2^2-\beta^2} + m_2\sqrt{1+m_1^2-\beta^2}}{\beta^2-1}\,,\end{aligned}$$

and gives

$$\alpha = \frac{(m_1-m_2)(1-\beta^2)}{m_1\sqrt{1+m_2^2-\beta^2} - m_2\sqrt{1+m_1^2-\beta^2}}\,.$$

One now observes that the value of α is invariant under $\beta \mapsto -\beta$. This in fact proves the result in this case. To see this, one notes that $\beta \mapsto -\beta$ corresponds to a 180° rotation which interchanges M and N and interchanges S and T, and therefore interchanges P and Q. Moreover, since $\angle BKM = \angle AKN$ and $\angle BKT = \angle AKS$, this preserves the slopes of L_1 and L_2. Since the value of α does not change, this shows that $|KP| = |KQ|$.

Next, one considers the case in which $m_2 > 0$. Without loss of generality, assume that $m_2 > m_1$. One then lets $M = (x_1, y_1)$ be the intersection of L_1 with the circle and $T = (x_2, y_2)$ be the intersection of the circle with L_2, where $x_1, y_1 > 0$ and $x_2, y_2 < 0$. Arguing exactly as above one lets $P = (\alpha, \beta)$ and once again (3) holds. Solving for T, M on the unit circle, one now obtains

$$x_1 = \frac{-m_1\beta + \sqrt{1+m_1^2-\beta^2}}{1+m_1^2}\,, \quad x_2 = \frac{-m_2\beta - \sqrt{1+m_2^2-\beta^2}}{1+m_2^2}\,.$$

Substituting this into (3) yields

$$\alpha = \frac{(m_2 - m_1)(1-\beta^2)}{m_1\sqrt{1+m_2^2-\beta^2} + m_2\sqrt{1+m_1^2-\beta^2}}\,.$$

Once again, α is invariant under $\beta \mapsto -\beta$ and the result for this case follows as above.

Finally, there remain the cases when L_1 or L_2 are parallel to the x-axis or to the y-axis. Since $|PK|$ and $|QK|$ are obviously continuous functions of M and S, the result follows by continuity from the previous cases.

Elementary geometry solution. After much effort the following elementary "geometric" argument was found. However, this proof seems more difficult, as some of the intermediate results appear to be at least as deep as the main result. On the other hand, this argument does not require knowledge of hyperbolic geometry.

A trivial case occurs if one of the chords equals AB, so it will be assumed that this is not the case. The first observation is that the result follows from

$$\frac{\operatorname{area} KMT}{\operatorname{area} KMBT} = \frac{\operatorname{area} KSN}{\operatorname{area} KSAN}\,. \tag{4}$$

To see why this is the case, define h_1 to be the distance between M and KB, i.e. the altitude of KMB, and similarly let h_2 the distance between T and KB, h_3 the distance between S and AK, and h_4 the distance between N and AK. It follows that

$$\begin{aligned} &\operatorname{area} KMT = |KP|\,(h_1+h_2)\,, \quad \operatorname{area} KMBT = |KB|\,(h_1+h_2)\,, \\ &\operatorname{area} KSN = |KQ|\,(h_3+h_4)\,, \quad \operatorname{area} KSAN = |KA|\,(h_3+h_4)\,. \end{aligned} \tag{5}$$

Since $|KA| = |KB|$, equation (4) implies that $|KQ| = |KP|$, which is the statement of the result.

The proof of (4) begins by recalling that if two chords XY and ZW of a circle intersect at T, then $|XT|\cdot|TY| = |ZT|\cdot|TW|$. Since the proof of this result is much easier than what is to follow, it is left as a preparatory exercise for the reader.

The intersection of chords implies that triangles KMT and KSN are similar and it will be convenient to let $\rho = |KM|/|KS|$ be the common ratio between the corresponding sides. Then the area of triangle KMT equals $\rho^2 \operatorname{area} NKS$ (this follows from the same argument as Lemma 1.1 below). Equation (4) is therefore equivalent to $\operatorname{area} KMBT = \rho^2 \operatorname{area} KSAN$, and (5) shows that this is in turn equivalent to

$$h_1 + h_2 = \rho^2 (h_3 + h_4)\,. \tag{6}$$

The proof of this follows from the following lemma.

Lemma 1.1. *Let the notation be as above. Then*

(a) $h_1 h_2 = \rho^2 h_3 h_4\,,$

(b) $\dfrac{1}{h_1} + \dfrac{1}{h_2} = \dfrac{1}{h_3} + \dfrac{1}{h_4}\,.$

Assuming this for the moment, one has

$$\frac{h_3 + h_4}{h_3 h_4} = \frac{1}{h_3} + \frac{1}{h_4} = \frac{1}{h_1} + \frac{1}{h_2} = \frac{h_1 + h_2}{h_1 h_2} = \frac{h_1 + h_2}{\rho^2 h_3 h_4}\,,$$

which implies (6) and the main result.

Proof of Lemma 1.1. In order to prove part (a) one uses the above result about chords which shows that triangle KMB is similar to triangle KAN and triangle KTB is similar to triangle KAS. As before, one has

$$\operatorname{area} KMB = \left(\frac{|KM|}{|AK|}\right)^2 \cdot \operatorname{area} KAN\,,$$

$$\operatorname{area} KTB = \left(\frac{|KB|}{|KS|}\right)^2 \cdot \operatorname{area} KAS\,.$$

Since $|KB| = |KA|$ it follows that

$$(\operatorname{area} KMB)\,(\operatorname{area} KTB) = \rho^2\,(\operatorname{area} KAN)\,(\operatorname{area} KAS)\,. \tag{7}$$

But one also has

$$\operatorname{area} KMB = h_1\,|KB|\,, \qquad \operatorname{area} KTB = h_2\,|KB|\,,$$

$$\operatorname{area} KAN = h_4\,|AK|\,, \qquad \operatorname{area} KAS = h_3\,|AK|\,,$$

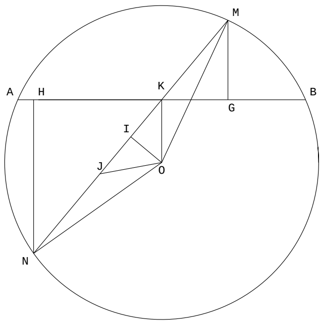
which, upon using $|AK| = |KB|$, gives part (a) of the lemma. Part (b) is a consequence of the following surprising result.

Lemma 1.2. *Given a chord AB of a circle, let MN be any other chord bisecting AB, x the distance from N to the line AB and y the distance from M to the line AB. Then $\left(\frac{1}{x} - \frac{1}{y}\right)$ is independent of MN except for its sign which only depends on the side of AB that M lies on.*

Assuming this, one notes that M and N lie on different sides of AB, so applying Lemma 1.2 gives

$$\frac{1}{h_1} - \frac{1}{h_4} = -\frac{1}{h_2} + \frac{1}{h_3}\,,$$

which is equivalent to part (b) of Lemma 1.1.



Proof of Lemma 1.2. Let O be the center of the circle and assume that N and O lie on the same side of AB. Draw a perpendicular from O to KN meeting KN at I. Since N and M lie on a circle with center O, one has $|OM| = |ON|$ and so the triangle OIM is congruent to the triangle OIN. It follows that I bisects NM. Now let J lie on the ray IN and be such that $|JI| = |IK|$. By the previous argument $|IK| < |IM|$, so J lies strictly between N and I. Since I

also bisects JK, it follows that $|NJ| = |KM|$ and thus

$$|KN| = 2\,|KI| + |KM|\,. \tag{8}$$

Now drop a perpendicular from N to AK meeting the line AK at H, so that the signed length of NH is η. Since $\angle KHN = \angle HKO = 90°$ and $\angle HKN + \angle IKO = 90°$, it follows that triangle KHN is similar to triangle OIK. From similar triangles, one gets

$$\frac{|KI|}{x} = \frac{|OK|}{|KN|}\,.$$

Substituting this into (8) gives

$$\frac{|KN|}{|KM|} = 1 + \frac{2x\,|OK|}{|NK|\cdot|KM|} = 1 + \frac{2x\,|OK|}{R^2 - |OK|^2}\,,$$

where R is the radius of the circle. The last equality follows from the fact that NM intersects the diameter DE containing K. By intersection of chords,

$$|NK|\cdot|KM| = |DK|\cdot|KE| = \big(R + |OK|\big)\big(R - |OK|\big)\,.$$

One therefore gets

$$\frac{1}{x} - \frac{|KN|}{x\,|KM|} = \frac{-2\,|OK|}{R^2 - |OK|^2}\,. \tag{9}$$

One now drops a perpendicular from M to KB meeting KB at G, so that the length of MG is y, since G and O lie on opposite sides of AB. The triangle KGM is similar to KHN so that

$$y = \frac{x|KM|}{|KN|}\,.$$

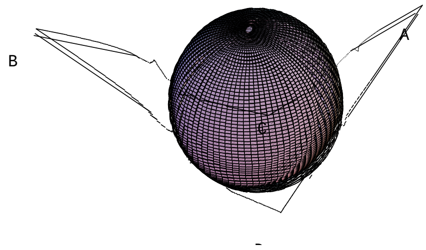
Plugging this into (9) shows that

$$\frac{1}{x} - \frac{1}{y} = \frac{-2\,|OK|}{R^2 - |OK|^2}$$

and has a constant value, which proves the result in this case.

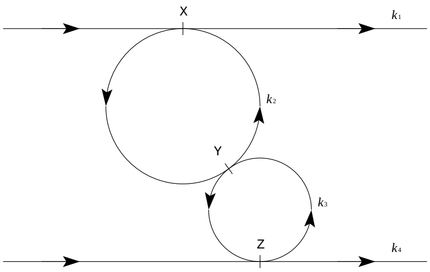
Similarly, if it is M and O which lie on the same side of AB, then one replaces N with M in the above argument, i.e. one lets I lie on KM, etc., and the result carries through in the same way and one arrives at (9) with x and y interchanged, which proves that value of $\left(\frac{1}{x} - \frac{1}{y}\right)$ changes sign if N lies on the opposite side of AB. $\Box$

Problem 2. *A quadrangle in space is tangent to a sphere. Show that the points of tangency are coplanar.*



Solution. This solution was independently found by Pavol Severa and Igor Rivin.

Let $A_1, \ldots, A_4$ be the vertices of the quadrangle given in cyclic order e.g., A, C, B, D in the diagram. For each point A_i let k_i be the circle on the sphere where the tangents passing through A_i touch the sphere. Notice that the cyclic order of the vertices induces a cyclic order on the k_i's, in particular, these can be oriented so that their orientations at the four points of tangency agree. Now we make a stereographic projection from one of these points of tangency so the picture looks like the following, with X, Y, Z being the images of three of the four vertices of the quadrangle under the stereographic projection.



We have to prove that X, Y and Z lie on a line. This is visually obvious, but just to help: the homothety with center at Y that maps k_2 to k_4, maps k_1 to a tangent of k_3 parallel to k_4. Actually, since the orientations agree, it has to map k_1 to k_4 and therefore X to Z.

Alternate solution. This solution was found by Georg Illies.

Excluding trivial special cases we assume that the points T_1, T_3, T_4 and T_2, at which the sides of the quadrangle ABCD touch the sphere (with center M), lie in the interior of the sides, i.e. $T_1 \in AB$, $T_1 \neq A, B$ and so on. We also assume that A, B, C, D are not coplanar.

Consider the plane $\mathcal{E}$ determined by T_1, T_3 and T_4. If one edge of $ABCD$ were in $\mathcal{E}$ the others would also, by our assumptions. So by the above, A and C lie on different sides of $\mathcal{E}$ as do C and B as well as B and D. Thus, A and D also lie on different sides of $\mathcal{E}$.

Let points $A', B', C', D' \in \mathcal{E}$ be such that $AA' \perp \mathcal{E}$, $BB' \perp \mathcal{E}$, etc. Let Q be the intersection point of $\mathcal{E}$ and AD, it is thus the point in which AD and $A'D'$ intersect. (Observe that $AA' \| DD'$, so the points A, A', D, D' are coplanar; the same argument shows that T_1 is the point in which AC and $A'C'$ intersect and so on.) We have to show that $T_2 = Q$.

Now we have $|AT_2| = |AT_1|$, $|CT_1| = |CT_3|$ and so on (as the right triangles AMT_2 and AMT_1 are congruent etc.). Thus

$$\frac{|AT_2|}{|DT_2|} = \frac{|AT_1|}{|CT_1|} \cdot \frac{|CT_3|}{|BT_3|} \cdot \frac{|BT_4|}{|DT_4|} = \frac{|AA'|}{|CC'|} \cdot \frac{|CC'|}{|BB'|} \cdot \frac{|BB'|}{|DD'|} = \frac{|AA'|}{|DD'|} = \frac{|AQ|}{|DQ|},$$

where the second and fourth equality follow by considering the similar right triangles $AA'T_1$ and $CC'T_1$. One therefore gets $T_2 = Q$, as claimed.

Algebraic solution. The following approach puts the problem into purely algebraic form and minimizes geometric intuition.

Let the quadrangle be $ABCD$ and the sphere S. Assume, without loss of generality, that the points of tangency lie on AC, AD, BC, BD. If the quadrangle lies in the plane, then the result is trivial, so it will be assumed that this is not the case.

Let K be the center of S, then A, B, and K lie on a plane. Without loss of generality, one can assume to be the xy-plane, that A and B lie on the x-axis and that K lies on the y-axis, so that

$A=(a,0,0)$, $B=(b,0,0)$, and $K=(0,k,0)$. One can also assume that $b>a$, $k\geq 0$, and that the sphere has radius 1. Let the points of tangency be at $T_i=(x_i,y_i,z_i)$, $i=1,\ldots,4$, where T_1 lies on AC, T_2 on AD, T_3 on BC, and T_4 on BD.

The approach begins by noticing that A, B, C, T_1, T_3 lie in a plane and the same holds for A, B, D, T_2, T_4. In fact, choose a plane P containing the x-axis and of slope m with respect to the xy-plane, and two points $T,T'\in S\cap P$ so that AT and BT' are tangent to the sphere. Then, generically, there is an interval of slopes m such that AT and BT' meet at a point. One concludes that characterizing T and T' in terms of m will lead to all possible quadrangles with 4 points of tangency on S.

Thus, consider a plane P of slope m with respect to the x-axis, so that (x,y,z) lies on P if and only if $z=my$. Now let $T=(x,y,z)$ be a point on S such that TA is tangent to S. Thus $(T-K)\cdot(T-A)=0$, so that

$$x(x-a)+y(y-k)+z^2=0\,.$$

Moreover, since S has radius 1 and center K, one gets

$$x^2+(y-k)^2+z^2=1\,. \tag{10}$$

Subtracting these equations yields

$$x=\frac{ky+1-k^2}{a}\,, \tag{11}$$

where one assumes for the time being that $a\neq 0$. It follows that

$$T=\left(\frac{ky+1-k^2}{a}\ ,\ y\ ,\ my\right)\,, \tag{12}$$

for some y. One can now use (10) to solve for y and this leads to

$$y^2\left(\frac{k^2+a^2m^2+a^2}{k^2+a^2-1}\right)-2ky+k^2-1=0\,. \tag{13}$$

Note that this equation is well defined since the conditions of the problem imply that the distance from A to the center of the circle K is greater than the radius of the circle, i.e. $a^2+k^2>1$.

Instead of solving directly for the y_i's, it seems more efficient to continue by using (12) to gain information about $T_1, \dots, T_4$. Thus, assume that A, B, C lie on a plane of slope m with respect to the xy-plane, and that A, B, D lie on a plane of slope n with respect to the xy-plane. Applying (12) yields

$$\begin{aligned} T_1 &= \left(\tfrac{ky_1+1-k^2}{a}, y_1, my_1\right), \quad & T_2 &= \left(\tfrac{ky_2+1-k^2}{a}, y_2, ny_2\right), \\ T_3 &= \left(\tfrac{ky_3+1-k^2}{b}, y_3, my_3\right), \quad & T_4 &= \left(\tfrac{ky_4+1-k^2}{b}, y_4, ny_4\right). \end{aligned}$$

In order to tell whether $T_1, \dots, T_4$ lie in a plane, one checks to see if $T_2 - T_1$, $T_3 - T_1$, and $T_4 - T_1$ form a linearly independent set. Since $T_4 - T_1 = (T_4 - T_1) - (T_3 - T_1)$, this is equivalent to verifying whether the following determinant vanishes.

$$D = \begin{vmatrix} \frac{k}{a}(y_2 - y_1) & y_2 - y_1 & ny_2 - my_1 \\ k\left(\frac{y_3}{b} - \frac{y_1}{a}\right) + (1-k^2)\left(\frac{1}{b} - \frac{1}{a}\right) & y_3 - y_1 & m(y_3 - y_1) \\ \frac{k}{b}(y_4 - y_3) & y_4 - y_3 & ny_4 - my_3 \end{vmatrix}.$$

Assuming for the moment that $y_1 \neq y_2, y_3$ and $y_4 \neq y_3$, one gets

$$\begin{aligned} D = {} & (y_2 - y_1)(y_3 - y_1)(y_4 - y_3) \\ & \times \begin{vmatrix} \frac{k}{a} & 1 & \frac{ny_2 - my_1}{y_2 - y_1} \\ \frac{k}{y_3 - y_1}\left(\frac{y_3}{b} - \frac{y_1}{a}\right) + \frac{1-k^2}{y_3 - y_1}\left(\frac{1}{b} - \frac{1}{a}\right) & 1 & m \\ \frac{k}{b} & 1 & \frac{ny_4 - my_3}{y_4 - y_3} \end{vmatrix}. \end{aligned}$$

Subtracting the first row from the other two rows yields

$$D = (y_2 - y_1)(y_3 - y_1)(y_4 - y_3)$$
$$\times \begin{vmatrix} \frac{k}{a} & 1 & \frac{ny_2 - my_1}{y_2 - y_1} \\ \left(\frac{1}{b} - \frac{1}{a}\right) \frac{ky_3 + 1 - k^2}{y_3 - y_1} & 0 & \frac{(m-n)y_2}{y_2 - y_1} \\ k\left(\frac{1}{b} - \frac{1}{a}\right) & 0 & (m-n)\,\frac{y_1y_4 - y_2y_3}{(y_2 - y_1)(y_4 - y_3)} \end{vmatrix} .$$

Expanding from the top row and factoring out common terms in rows and columns yields

$$D = -\,(y_3 - y_1)\,(y_4 - y_3)\,(m-n)\left(\frac{1}{b} - \frac{1}{a}\right) \begin{vmatrix} \frac{ky_3 + 1 - k^2}{y_3 - y_1} & y_2 \\ k & \frac{y_1y_4 - y_2y_3}{y_4 - y_3} \end{vmatrix} .$$

This last determinant equals

$$\frac{(ky_3 + 1 - k^2)\,(y_1y_4 - y_2y_3)}{(y_3 - y_1)\,(y_4 - y_3)} - ky_2$$
$$= \frac{k\,(-y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3) + (1 - k^2)\,(y_1y_4 - y_2y_3)}{(y_3 - y_1)\,(y_4 - y_3)} .$$

It follows that

$$D = (m-n)\left(\frac{1}{a} - \frac{1}{b}\right)\Big[k\,(y_2y_3y_4 - y_1y_3y_4 - y_1y_2y_4 + y_1y_2y_3) + (k^2 - 1)\,(y_1y_4 - y_2y_3)\Big] . \qquad (14)$$

It is easily seen that this formula holds also if any of $y_1 = y_2$, $y_1 = y_3$, or $y_3 = y_4$ holds, since both sides of (14) are analytic in $y_1, \ldots, y_4$.

Now let $w_i = 1/y_i$, $i = 1, \ldots, 4$. Since A and B both lie on the x-axis, it is clear that to lie on a quadrangle, none of the points of tangency can satisfy $y_i = 0$, so the w_i's are well defined. Substituting this in (14) results in

$$D = \frac{m-n}{w_1 w_2 w_3 w_4}\left(\frac{1}{a} - \frac{1}{b}\right) \times \left[k(w_1 - w_2 - w_3 + w_4) + (k^2-1)(w_2 w_3 - w_1 w_4)\right]. \tag{15}$$

One now solves for the w_i's by applying (13) which gives

$$(k^2-1)w^2 - 2kw + \left(\frac{k^2 + a^2 m^2 + a^2}{k^2 + a^2 - 1}\right) = 0\,,$$

so that, assuming that $k \neq 1$,

$$\begin{aligned} w &= \frac{k \pm \sqrt{k^2 - \frac{(k^2-1)\,(k^2+a^2m^2+a^2)}{k^2+a^2-1}}}{k^2-1} \\ &= \frac{k \pm \sqrt{\frac{(m^2-k^2m^2+1)\,a^2}{k^2+a^2-1}}}{k^2-1} \\ &= \frac{k \pm f(m)g(a)}{k^2-1}\,, \end{aligned} \tag{16}$$

where

$$f(m) = \sqrt{m^2 - k^2 m^2 + 1}\,, \quad g(a) = \sqrt{\frac{a^2}{k^2+a^2-1}}\,.$$

One thus gets

$$w_1 = \frac{k \pm f(m)g(a)}{k^2-1}\,, \qquad w_2 = \frac{k \pm f(n)g(a)}{k^2-1}\,,$$

$$w_3 = \frac{k \pm f(m)g(b)}{k^2-1}\,, \qquad w_4 = \frac{k \pm f(n)g(b)}{k^2-1}\,.$$

Once again, one can make a further simplification before using this formula. By writing $w_i = (k+s_i)/(k^2-1)$, one gets

$$k(w_1 - w_2 - w_3 + w_4) + (k^2-1)(w_2w_3 - w_1w_4)$$

$$= \frac{k}{k^2-1}\,(s_1 - s_2 - s_3 + s_4)$$

$$+ \frac{1}{k^2-1}\big[(k+s_2)(k+s_3) - (k+s_1)(k+s_4)\big] = \frac{s_2s_3 - s_1s_4}{k^2-1}\,.$$

In other words,

$$D = \frac{m-n}{w_1w_2w_3w_4}\,\frac{s_2s_3 - s_1s_4}{k^2-1}\left(\frac{1}{a} - \frac{1}{b}\right).$$

One now observes that

$$|s_2s_3| = f(n)g(a)f(m)g(b) = |s_1s_4|\,.$$

It follows that $D = 0$ depends only on the signs of s_i, $i = 1, \ldots, 4$. In other words, if σ_i is the sign of s_i, then $T_1, \ldots, T_4$ are coplanar if and only if $\sigma_1\sigma_4 = \sigma_2\sigma_3$, that is *if s_1, s_4 and s_2, s_3 either both have the same signs or both have unequal signs.*

In order to characterize this last condition, one examines the geometrical significance of the sign of σ_i. Note first that one has the explicit computation

$$x_1 = \frac{ky_1 + 1 - k^2}{a} = \frac{\frac{k}{w_1} + 1 - k^2}{a} = -\,\frac{(k^2-1)s_1}{a(k+s_1)}\,.$$

Assuming for the time being that $k > 1$, one has

$$s_1^2 = \frac{a^2}{a^2 + k^2 - 1}\,\big(1 - m^2(k^2-1)\big) \le 1\,,$$

so that $k + s_1 > 0$. It follows that σ_1 equals the sign of x_1 if $a > 0$, and is minus the sign of x_1 is $a < 0$. In other words, if the sphere S is divided into two sides S_1 and S_{-1} according to whether sign$\,(a) > 0$ or < 0, then σ_1 determines the side of the sphere that T_1 lies in, namely, T_1 lies in $S_{\sigma_1\,\mathrm{sign}(a)}$. The same holds true for $T_2, \ldots, T_4$ and one gets

$$T_1 \in S_{\sigma_1 \,\mathrm{sign}(a)}, \; T_2 \in S_{\sigma_2 \,\mathrm{sign}(a)}, \; T_3 \in S_{\sigma_3 \,\mathrm{sign}(b)}, \; T_4 \in S_{\sigma_4 \,\mathrm{sign}(b)} \,. \quad (17)$$

Now consider the two points of tangency T_1, T_3 lying in the plane of A, B. One will say that these are of *Type I* with respect to A, B if T_1 and T_3 lie on different sides of the sphere, as defined above, and of *Type II* with respect to A, B if they lie on the same side of the sphere. Thus, in the diagram, the two figures on the left represent Type I and the figure on the right Type II.

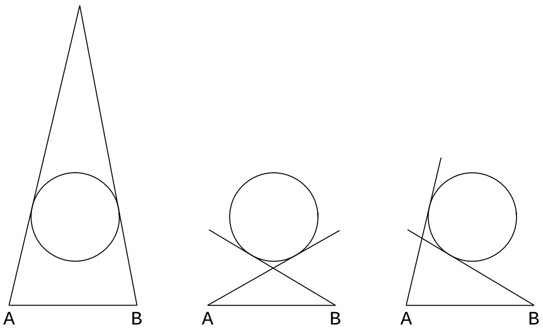
One now translates the condition that $\sigma_1\sigma_4 = \sigma_2\sigma_3$ into this notation.

(i) If $\sigma_1 = \sigma_4$ and $\sigma_2 = \sigma_3$, and $\mathrm{sign}\,(a) = \mathrm{sign}\,(b)$, then $T_1, \ldots, T_4$ all lie on S_1 so T_1, T_4 and T_2, T_3 are both of Type II with respect to A, B.

(ii) If $\sigma_1 = \sigma_4$ and $\sigma_2 = \sigma_3$, and $\mathrm{sign}\,(a) = -\mathrm{sign}\,(b)$, then T_1, T_4 and T_2, T_3 are both of Type I with respect to A, B.

(iii) If $\sigma_1 = -\sigma_4$ and $\sigma_2 = -\sigma_3$, and $\mathrm{sign}\,(a) = \mathrm{sign}\,(b)$, then T_1, T_4 and T_2, T_3 are both of Type I with respect to A, B.

(iv) If $\sigma_1 = -\sigma_4$ and $\sigma_2 = -\sigma_3$, and $\mathrm{sign}\,(a) = -\mathrm{sign}\,(b)$, then T_1, T_4 and T_2, T_3 are both of Type II with respect to A, B.



One concludes that $\sigma_1\sigma_4 = \sigma_2\sigma_3$ implies that T_1, T_4 and T_2, T_3 are of the same type with respect to A, B. If $\sigma_1\sigma_4 = -\sigma_2\sigma_3$, then the exact same argument shows that T_1, T_4 and T_2, T_3 cannot be of the same type with respect to A, B. It follows that the condition $\sigma_1\sigma_4 = \sigma_2\sigma_3$

is equivalent to T_1, T_4 and T_2, T_3 being of the same type with respect to A, B. In other words, *the points of tangency are coplanar if and only if T_1, T_4 and T_2, T_3 are of the same type with respect to A, B.*

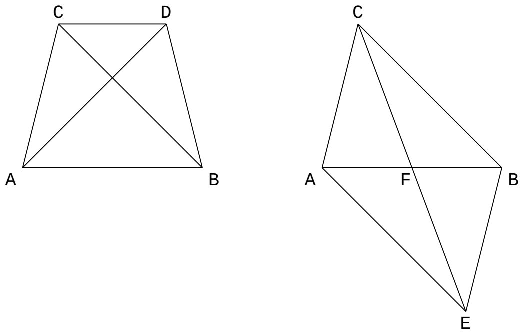
This will be shown in the case at hand. In fact, if the points of tangency lie on the edges of a quadrangle, then they must all be of Type I. In fact, it is easily seen that AT_1 and BT_3 cannot lie on the same side of the sphere when T_1 and T_3 lie on line segments AC and BC. This is obvious from the above diagram and a rigorous proof is left as an exercise.

Finally, as can be easily checked, the case $a = 0$ can be proved by continuity from the above argument.

Remark 2.1. The above argument proves the slightly more general result. *Given a quadrangle and a sphere such that the lines extending the edges of the quadrangle are tangent to the sphere, then the points of tangency are coplanar if and only if there are two vertices such that the pairs of points of tangency are of the same type with respect to these vertices.*

Problem 3. *The faces of a triangular pyramid have the same area. Show that they are congruent.*

Solution. A good way to approach this problem is to first characterize the consequence of the following statement. *There exists a tetrahedron all of whose sides are congruent to a given triangle if and only if all the angles of the triangle are acute.*



Proof. Let the triangle be ABC, and assume, without loss of generality, that $\angle BAC$ is greater or equal the other angles of the triangle. One places a triangle ABD on the line AB such that C and D lie on the same side of AB and such that ABC is congruent to BAD, in other words, ABD is a mirror image of ABC. As in the above, one rotates ABD about AB to form a triangle ABD'. Clearly, any tetrahedron with all sides congruent to ABC will be formed in this way, so if a solution exists, then it is unique up to rotational symmetry.

(a) First, assume that all angles of ABC are acute. It follows that $|CD| < |AB|$. Now let ABE be the triangle ABD rotated 180° about AB, and let F be the intersection of AB and CE. By construction, it follows that triangles ACF and BEF are congruent, so CE bisects AB. By assumption, $\angle BAC \geq \angle ACB$, so $\angle FAC > \angle ACF$. But in the triangle FAC one has, by the law of sines, that

$$\frac{|CF|}{\sin \angle FAC} = \frac{|AF|}{\sin \angle ACF},$$

and one concludes that $|CF| > |AF|$, since $\angle FAC < 90°$ and $\sin z$ is increasing for $0 < z < 90°$.

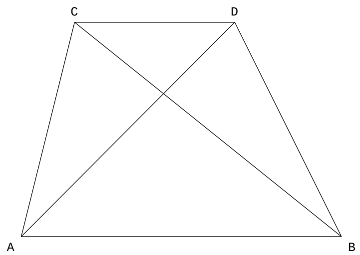
It follows that $|CE| > |AB|$. This implies that there must be a rotation with angle strictly between zero and 180° such that $|CD'| = |AB|$. This value of D' then gives the required tetrahedron.

(b) Let us assume now that at least one angle of ABC is not acute, i.e. $\angle BAC \geq 90°$. Then $|CD| \geq |AB|$, and rotating ABD about AB will only increase the value of $|CD'|$ so that it is strictly greater than $|AB|$. It follows that there can be no solution in this case. □

The solution of the problem uses similar ideas but will require the following technical point.

Lemma 3.1. *Let a, b, c, d be positive real numbers such that a, b are not equal to c, d in some order. Then there is at most one value of x such that there are two triangles with side lengths a, b, x and c, d, x, and with equal areas.*

Assuming this holds, one proves the result by contradiction. One begins as above by trying to construct a tetrahedron all of whose sides have equal areas. Thus, let ABC and ABD be non-congruent triangles with equal areas.



One now forms a tetrahedron by placing ABC and ABD in the same plane with C and D on the same side of AB and then rotating ABD about AB to obtain a new triangle ABD'. Clearly, any tetrahedron with adjacent sides congruent to ABC and ABD will be generated this way.

Let $a = |AC|$, $b = |AD|$, $c = |BC|$, $d = |BD|$. It follows that the other two faces of the tetrahedron are ACD' and BCD' with sides a, b, x and c, d, x, respectively, where $x = |CD'|$. A solution to the problem requires ACD' and BCD' to have equal areas. Since the assumption that ABC is not congruent to ABD implies that a, b are not equal to (c, d) in some order, Lemma 3.1 applies, and there is a unique D' with the required property.

Since triangles ABC and ABD have equal areas, ABC and ABD have equal altitudes with respect to AB, and thus ACD and BCD must have equal areas as well. This implies that the initial position $D' = D$ gives the only possible solution to ACD' and ABD' having equal areas. Since these lie in the same plane, there is no three-dimensional solution.

Proof of Lemma 3.1. Let the two triangles be ACD and BCD with notation as above, i.e. $a = |AC|$, $b = AD$, $c = |BC|$, $d = |BD|$, $CD = x$. Let $\alpha = \angle CAD$ and $\beta = \angle CBD$. Then from elementary trigonometry (law of cosines) one has

$$a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta = x^2\,, \tag{18}$$

while equating areas gives

$$ab\sin\alpha = cd\sin\beta\,. \tag{19}$$

Using $\sin^2 z = 1 - \cos^2 z$, one transforms (19) into

$$a^2b^2 - c^2d^2 - a^2b^2\cos^2\alpha + c^2d^2\cos^2\beta = 0\,. \tag{20}$$

Equation (18) implies that

$$a^2b^2\cos^2\alpha = \left(\frac{x^2-a^2-b^2}{2}\right)^2, \quad c^2d^2\cos^2\beta = \left(\frac{x^2-c^2-d^2}{2}\right)^2.$$

Plugging this into (20) gives

$$a^2b^2 - \left(\frac{x^2-a^2-b^2}{2}\right)^2 - c^2d^2 + \left(\frac{x^2-c^2-d^2}{2}\right)^2 = 0\,,$$

which leads to

$$x^2\left(a^2+b^2-c^2-d^2\right) = \frac{\left(a^2-b^2\right)^2 - \left(c^2-d^2\right)^2}{2}\,. \tag{21}$$

Now if $a^2+b^2 \neq c^2+d^2$, then there is at most one positive value of x satisfying (21), and the statement of the lemma follows. On the other hand, if $a^2+b^2 = c^2+d^2$, then (18) implies that $ab\cos\alpha = cd\cos\beta$. Combining this with (19), e.g. by using $\sin^2 z + \cos^2 z = 1$, one obtains $ab = cd$. One therefore gets

$$a^2 + 2ab + b^2 = c^2 + 2cd + d^2, \quad a^2 - 2ab + b^2 = c^2 - 2cd + d^2.$$

This implies that

$$a + b = c + d \quad \text{and} \quad a - b = \pm(c - d)\,,$$

and one concludes that a, b are equal to c, d in some order, contradicting the hypothesis. □

Problem 4. *The prime decompositions of different integers m and n involve the same primes. The integers $m+1$ and $n+1$ also have this property. Is the number of such pairs (m,n) finite or infinite?*

Answer. The number of such pairs is infinite.

Proof. Let $m = 2^k - 2\,,\ \ n = (m+1)^2 - 1\,,$ for $k = 2, 3, 4, \ldots$. Then

$$n + 1 = (m+1)^2\,,$$

so $n+1$ and $m+1$ have the same prime factors. Moreover,

$$n = (m+1)^2 - 1 = m\,(m+2)\,.$$

Since $m+2$ is a power of 2 and m is already even, it follows that m and n also have the same prime factors.

Remark 4.1. One can ask whether there are infinitely many pairs not of this form. This does not appear to be an easy question and even finding one other pair is non trivial. A computer search revealed that

$$m = 75 = 3 \cdot 5^2, \quad \text{and} \quad n = 1215 = 3^5 \cdot 5$$

also satisfy this condition since

$$m + 1 = 2^2 \cdot 19 \quad \text{and} \quad n + 1 = 2^6 \cdot 19\,.$$

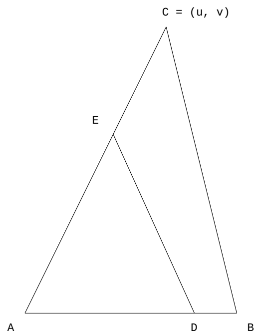
In fact, this is a special case of a well known problem of Erdős and Woods in number theory and logic, see the Notes.

Problem 5. *Draw a straight line that halves the area and perimeter of a triangle.*

Solution. Let the triangle be ABC, let $p = a+b+c$ be the perimeter, and, without loss of generality, assume that $b \geq a \geq c$. On AB, let D be such that the length of AD is

$$t_0 = \frac{p - \sqrt{p^2 - 8bc}}{4} \,,$$

and on AC, let E be such that the length of AE is $\dfrac{bc}{2t_0}$. Then the line DE splits ABC into two parts with equal areas and perimeters.



Proof. One can think of the triangle ABC as being in the xy-plane with the origin at A, $B=(c\,,0)$ and $C=(u\,,v)$, where $c\,,u\,,v > 0$. Under these assumptions, let D on AB be such that the length of AD is t and $c/2 \leq t \leq c$. One will construct E on AC such that DE divides the triangle into two equal areas. In fact, let E be such that the length of AE is $\dfrac{bc}{2t_0}$. Then the area of the triangle ADE is

$$\frac{cv}{2t}\,\frac{t}{2} = \frac{1}{2}\,\frac{cv}{2}\,,$$

so E satisfies this property. Note that for $t = c$, one gets $E = C/2$, and for $t = c/2$ one gets $E = C$.

Now the perimeter contribution of AD and AE is $t + \dfrac{bc}{2t}$, so one needs to solve the equation

$$t + \frac{bc}{2t} = \frac{a+b+c}{2}\,,$$

which has solutions $t = \frac{p \pm \sqrt{p^2 - 8bc}}{2}$. I will show that

$$t_0 = \frac{p - \sqrt{p^2 - 8bc}}{4}$$

satisfies all necessary conditions to give an actual solution, i.e.

$$p^2 \geq 8bc, \quad t_0 \leq c, \quad t_0 \geq \frac{c}{2}\,. \tag{22}$$

To prove the first inequality, note that it follows from

$$(-a+b-c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac \geq 0\,,$$

which implies

$$p^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \geq 8bc\,.$$

The second inequality in (22) is equivalent to

$$(a+b-3c)^2 \leq (a+b+c)^2 - 8bc\,,$$

which reduces to $8ac \geq 0$. Finally, the third inequality in (22) is equivalent to

$$(a+b-c)^2 \geq (a+b+c)^2 - 8bc\,,$$

which reduces to $b \geq a$, which is true by assumpion.

Remark 5.1. Since t_0 is a composition of additions, subtractions, divisions, and square roots of the sides of ABC, it follows that DE can be "drawn" with ruler and compass.

Problem 6. *Show that* $\displaystyle \frac{1}{\sin^2 x} \leq \frac{1}{x^2} + 1 - \frac{4}{\pi^2}\,, \quad 0 < x < \frac{\pi}{2}\,.$

Solution. The question can be rewritten as

$$\frac{1}{x^2} - \frac{1}{\sin^2 x} + 1 - \frac{4}{\pi^2} \geq 0\,, \quad 0 < x < \frac{\pi}{2}\,. \tag{23}$$

In order to prove this, one begins by showing that

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\frac{1}{3}\,.$$

This can be done either by expanding into a power series about $x=0$, or by L'Hôpital's rule, as follows.

$$\frac{1}{\sin^2 x}-\frac{1}{x^2}=\frac{x^2-\sin^2 x}{x^2\sin^2 x}\to\frac{0}{0}\,,\qquad x\to 0\,,$$

so the limit equals the limit of the derivative of the numerator divided by the derivative of the denominator. Iterating this process yields

$$\frac{2x-\sin 2x}{2x\sin^2 x+x^2\sin 2x}\to\frac{0}{0}\,,$$

$$\frac{1-\cos 2x}{\sin^2 x+2x\sin 2x+x^2\cos 2x}\to\frac{0}{0}\,,$$

$$\frac{2\sin 2x}{3\sin 2x+6x\cos 2x-2x^2\sin 2x}\to\frac{0}{0}\,,$$

$$\frac{2\cos 2x}{6\cos 2x-8x\sin 2x-x^2\cos 2x}\to\frac{1}{3}\,.$$

Since $1-\frac{4}{\pi^2}>\frac{1}{3}$, it follows that there is such $\delta>0$, for which strict inequality holds in (23) for all $0<x<\delta$.

Next, one rewrites (23) as

$$\frac{\sin x}{\sqrt{1-a\sin^2 x}}\ge x\,, \tag{24}$$

where $a=1-\frac{4}{\pi^2}$. Clearly, this is an equality for $x=0$, and a computation shows that it also holds for $x=\pi/2$. To show that the inequality is strict for $0<x<\frac{\pi}{2}$, one takes the second derivative of

$$f(x)=\frac{\sin x}{\sqrt{1-a\sin^2 x}}\,,$$

which is easily found as

$$f'(x)=\frac{\cos x}{(1-a\sin^2 x)^{3/2}}\,,\quad f''(x)=\frac{(a-1+2a\cos^2 x)\,\sin x}{(1-a\sin^2 x)^{5/2}}\,.$$

Since $a>\frac{1}{3}$, it follows that $f''(x)>0$ for $0<x<x_0$, where x_0 is the unique solution of $f''(x_0)=0$ in $\left(0\,,\frac{\pi}{2}\right)$ (that x_0 exists and is unique is immediate from the form of $f''(x)$). In other words, $f(x)$ is concave in $(0\,,x_0)$. Now $f(0)=0$, and by the first part, it is true that (24) holds for $0<x<\delta$, so the strict concavity of $f(x)$ implies that $f(x)>x$ for $0<x\le x_0$.

Since $f''(x)$ has only one zero in $\left(0\,,\frac{\pi}{2}\right)$ and

$$f''\left(\frac{\pi}{2}\right)=-\left(\frac{\pi}{2}\right)^3<0\,,$$

it follows that $f(x)$ is convex in $\left(x_0\,,\frac{\pi}{2}\right)$. Since $f(x_0)>x_0$ and $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$, convexity implies that $f(x)>x$ for $x_0<x<\frac{\pi}{2}$.

Problem 7. *Choose a point on each edge of a tetrahedron. Show that the volume of at least one of the resulting tetrahedrons is not greater than* 1/8 *of the volume of the initial tetrahedron.*

Solution. The most natural interpretation of "resulting tetrahedrons" is the tetrahedrons formed by each original corner and the points on edges that are adjacent to the corner, see Remark 7.3.

Lemma 7.1. *If the angles of a vertex of a tetrahedron are fixed, then the volume of the tetrahedron is proportional to the lengths of the sides adjacent to this vertex.*

Proof. One assumes the well known facts that the area of a triangle is proportional to the base times height, and that the volume of a tetrahedron is proportional to base times height. Let A be the vertex with fixed angles, and B,C,D the other vertices. Let C and D be fixed and let B vary. If one considers ABC to be the base of the tetrahedron, then the height remains fixed as B varies. Similarly, if one considers AB to be the base of ABC, then its height remains fixed. It follows that the volume of the tetrahedron is proportional to AB. By symmetry, this holds for B and C, proving the result.

Alternatively, if one lets α be the angle CBA and β the angle that AD makes with ABC, then the volume of the tetrahedron is simply

$$\frac{1}{6}\,|AB|\cdot|AC|\cdot|AD|\sin\alpha\sin\beta\,. \quad \square$$

Now let the tetrahedron be T with vertices $A_1\,,\ldots\,,A_4$. One then picks a point on each edge so that P_{ij} lies on A_iA_j, $1\le i\ne j\le 4$, with the convention that $P_{ij}=P_{ji}$. The resulting tetrahedrons are then $T_i\,,\ i=1\,,\ldots\,,4\,,$ where T_i has vertices A_i and $P_{ij}\,,\ j\ne i$. Let $v(R)$ be the volume of a three-dimensional region R, then the problem is to show that one of $v(T_i)/v(T)\le 1/8$.

In order to do this, let

$$r_{ij}\equiv\frac{|A_iP_{ij}|}{|A_iA_j|}$$

be the ratio of the distance between P_{ij} and A_i to the length of A_iA_j. Since the angles at the corner A_i of T_i remain fixed as the P_{ij}'s vary, one can apply Lemma 7.1 to get

$$\frac{v(T_i)}{v(T)}=\prod_{j\ne i} r_{ij}\,.$$

Multiplying all these quantities together gives

$$\prod_{i=1}^{4}\frac{v(T_i)}{v(T)}=\prod_{\substack{1\le i,j\le 4\\ i\ne j}} r_{ij}\,.$$

Since $r_{ij}=1-r_{ji}$, it follows that

$$\prod_{i=1}^{4}\frac{v(T_i)}{v(T)}=\prod_{1\le i<j\le 4} r_{ij}(1-r_{ij})\,.$$

Now it is easily shown that $x(1-x)\le\frac{1}{4}$ for $0<x<1$, so it follows that

$$\prod_{i=1}^{4}\frac{v(T_i)}{v(T)}\le\frac{1}{4^6}\,. \tag{25}$$

This implies that not all of the factors on the left of (25) can be greater than $(1/4^6)^{1/4} = 1/8$. The result follows.

Remark 7.1. This argument generalizes verbatim to n dimensions. Thus, let S be an n-dimensional simplex, i.e. the set

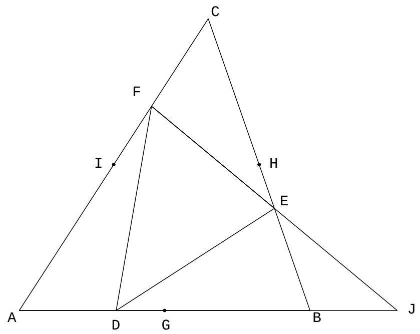
$$S = \{\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1} : \lambda_1, \ldots, \lambda_{n+1} \geq 0, \ \lambda_1 + \ldots + \lambda_{n+1} = 1\},$$

where the vertices $A_1, \ldots, A_{n+1} \in \mathbf{R}^n$ have the property that removing any one results in a linearly independent set. The generalization is the following. *Pick a point on each edge of an n-dimensional simplex, then the volume of one of the simplexes obtained by taking an original vertex and points that lie on edges adjacent to it must be not greater than $1/2^n$ of the volume of the original simplex.* The generalization of this to arbitrary polyhedrons is left as a problem for the reader.

Remark 7.2. This problem is frustrating because natural geometric arguments that prove the two-dimensional analogue do not seem to generalize well to three dimensions. As an example, a simple geometric argument is given for the two-dimensional case.

Proposition 7.1. *Pick a point on each edge of a triangle. Then one of the triangles formed by a vertex of the original triangle and points on the two adjacent edges has an area which is not greater than $1/4$ of the area of the original triangle.*

Proof. If some vertex, say A, has two edge points, say D, F, at least as close to A as to the other vertices, then the area of ADF is less than $1/4$ of that of the original triangle. The only other possibility is that each vertex has only one edge point closer to it. Let us say that D is on AB and $AD \leq DB$, that E is on BC and $BE \leq EC$, and that F is on CA and $CF \leq FA$. Moreover, let G be the midpoint of AB, H the midpoint of BC, and I the midpoint of CA.



One now shows that the area of DEF is not less than the area of GHI. Since AB and IH are parallel, it follows that FE is either parallel to AB or meets AB at J such that AJF is an acute angle. Now, let $x = |GD|$. Then in the first case, as x increases, the area of DEF remains constant. In the second case, as x increases, the distance from D to FJ increases, since AJF is acute. Since this equals the distance from D to FE, it follows that as x increases, the area of DEF increases.

One concludes that the area of DEF is non-decreasing in x. Since this argument holds for $|HE|$ and $|IF|$, it follows that the area of DEF is not less than the area of GHI, as claimed. Since the area of GHI is exactly $1/4$ of the area of ABC, the sum of the three remaining triangles is not greater than $3/4$ of the area of ABC, and thus one of the triangles ADF, BDE, CEF must have area less than $1/4$ of the area of ABC. $\square$

Generalizing this to three dimensions would entail finding a lower bound on the middle piece of the tetrahedron, i.e. what remains after the four tetrahedrons have been removed. However, in three dimensions, the volume of this piece is no longer a monotonic function of the edge points, as was the case in the above argument. Following through with this argument is left as a problem for the reader.

Remark 7.3. If one interprets "resulting tetrahedrons" as any tetrahedrons formed by joining one of the edge points to a vertex, then the solution is simple: pick a vertex A of the tetrahedron. If the points on the three edges from A all lie closer to A than to the other vertex on the edge, then the tetrahedron formed by these edge points and A is clearly not greater in volume than $1/8$ of the volume of the original tetrahedron. Otherwise, there is an edge point E such that the distance from E to BCD is less than or equal to half the distance from A to BCD. Now clearly, one of the four triangles formed by the edge points at the base has area which not greater than $1/4$ of the area of the base. It then follows that the tetrahedron formed by joining this triangle to E has volume which is not greater than $1/8$ of the original volume.

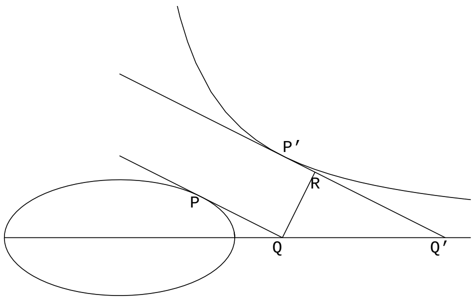
Problem 8. *We are told that* $a^2 + 4b^2 = 4$, $cd = 4$. *Show that*

$$(a-d)^2 + (b-c)^2 \geq 1.6\,.$$

Solution. The lower bound is found as follows: the form

$$(a-d)^2 + (b-c)^2$$

is the square of the Euclidean distance between (a,b) and (d,c), so the question reduces to finding the minimum distance between the curves $x^2 + 4y^2 = 4$, and $xy = 4$. The first of these is an ellipse with axes' lengths 2 and 1, while the second is a hyperbola. Clearly, this problem is symmetric with respect to the line $x = -y$, so one can restrict oneself to $x \geq -y$, and, as a consequence, to the component of the hyperbola with $c, d > 0$. Thus, in the following, "the hyperbola" will mean points (x,y) satisfying $xy = 4$ and $x, y > 0$. The main idea is the following simple observation.



Lemma 8.1. *If a convex curve* C *has a tangent line* L *and a concave*

curve C' has a tangent line L' such that L' is parallel to L and neither C nor C' lie between L and L', then the minimum distance between C and C' is greater than the distance between L and L'.

Proof. Note that if P is a point on C and P' is a point on C', then given the Lemma assumptions, PP' must cross L and L', so that $|PP'|$ is greater than the distance between L and L'. □

In order to use this, one must find points on the ellipse and on the hyperbola whose tangents have the same slope. In order to do this, one must first compute the slope of the tangents to these curves.

Thus, consider a point (x, y) on the ellipse. Differentiating

$$x^2 + 4y^2 = 4$$

gives

$$2x\,dx + 8y\,dy = 0\,,$$

so that

$$\frac{dy}{dx} = -\frac{x}{4y}$$

is the slope of the tangent line at (x, y). Similarly, the slope of the tangent line at the point (x, y) of the hyperbola is $-4/x^2$.

One now appeals to a "trick" which consists in taking

$$P = \left(\sqrt{2}\,, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad P' = \left(2\sqrt{2}\,, \sqrt{2}\right).$$

It turns out that P lies on the ellipse and P' on the hyperbola, and that the slopes of the tangent lines at P and P' are both equal to $-1/2$, as can be checked using the previous paragraph. A further simple computation shows that if L is the tangent line at P and L' the tangent line at Q, then these lines are given by the equations

$$L:\ y = -\frac{x}{2} + \sqrt{2}\,, \quad L':\ y = -\frac{x}{2} + 2\sqrt{2}\,.$$

To compute the distance between L and L', consider their intersections at the x-axis. Thus, L intersects the x-axis at $Q = (2\sqrt{2}\,, 0)$, and L' at $Q' = (4\sqrt{2}\,, 0)$. Now, let us draw a line from Q perpendicular to L meeting L' at R. Since L' has a slope of $-1/2$ with respect to the x-axis, it follows that the x-axis (pointing to $-\infty$) has a slope of 1/2 with respect to L' (pointing in the positive y direction). This implies that

$$\frac{|QR|}{|RQ'|} = \frac{1}{2}, \quad \text{so that} \qquad \frac{|QR|}{|QQ'|} = \frac{1}{\sqrt{5}},$$

by the Pythagorean theorem. Since $|QQ'| = 2\sqrt{2}$, one gets

$$|QR| = \frac{2\sqrt{2}}{\sqrt{5}}.$$

Since QR is perpendicular to L and L', it follows that $|QR|$ equals the distance between L and L'. One concludes that

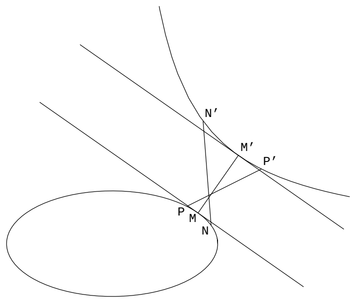
$$(a-d)^2 + (b-c)^2 \geq |QR|^2 = \frac{8}{5} = 1.6.$$

Remark 8.1. *The minimum value of*

$$(a-d)^2 + (b-c)^2 \quad \text{is} \quad 1.77479583276941567010\dots\,.$$

Proof. Unlike the other material in this section, the proof uses a computer algebra system, as this question does not appear to have a closed form solution.

One proceeds along the lines used in the solution to the problem. The idea is that Lemma 8.1 clearly implies that if M and M' lying on C and C', respectively, are such that MM' is orthogonal to the tangents at M and M', then $|MM'|$ is the minimum distance between C and C'. One therefore finds two such points.



In order to do this, one starts with a given value of α and finds points on the ellipse and the hyperbola where tangents to both have a slope equal to $-\alpha$. On the ellipse, let the tangent at $M=(x_0\,,y_0)$ have the slope $-\alpha$ so that $-\dfrac{x_0}{4y_0}=-\alpha$. Then,

$$x_0^2+4y_0^2=4 \qquad \Rightarrow \qquad x_0=\frac{4\alpha}{\sqrt{4\alpha^2+1}}\,, \quad y_0=\frac{1}{\sqrt{4\alpha^2+1}}\,.$$

Similarly, let $M'=(x_1\,,y_1)$ be a point on the hyperbola whose tangent has slope $-\alpha$. Then $-\dfrac{4}{x_1^2}=-\alpha$, and

$$x_1y_1=4 \qquad \Rightarrow \qquad x_1=\frac{2}{\sqrt{\alpha}}\,, \quad y=2\sqrt{\alpha}\,.$$

In order for MM' to be orthogonal to the tangents one must have

$$\frac{y_1-y_0}{x_1-x_0}=\frac{1}{\alpha}\,.$$

This gives

$$\frac{2\sqrt{\alpha}-\dfrac{1}{\sqrt{4\alpha^2+1}}}{\dfrac{2}{\sqrt{\alpha}}-\dfrac{4\alpha}{\sqrt{4\alpha^2+1}}}=\frac{2\alpha\sqrt{4\alpha^2+1}-\sqrt{\alpha}}{2\sqrt{4\alpha^2+1}-4\alpha\sqrt{\alpha}}=\frac{1}{\alpha}\,,$$

which simplifies to the equation

$$16\,\alpha^6-28\alpha^4-9\alpha^3+8\alpha^2+4=0\,. \tag{26}$$

This equation was examined using the computer algebra system *Mathematica*. The polynomial on the left hand side of (26) is irreducible over the rationals and the computer algebra system was unable to express the roots using radicals. Approximate roots are

$$-0.979691\pm0.34843\,i\,,\quad -0.04401\pm0.493223\,i\,,\quad 0.699695\,,\quad 1.34771\,.$$

One can eliminate all but the last two possibilities. Carrying out the above argument using the last root fails, as it ends up giving points N and N' with a negative slope for $N-N'$, see the figure, which means that it cannot be orthogonal to the tangent, since it would then have slope $1/\alpha>0$. The relevant root is therefore the second to last which, to twenty digits, is

$$\alpha_0=0.69969482002339060183\ldots\,.$$

Following the above argument, one lets

$$M = \left(\frac{4\alpha_0}{\sqrt{4\alpha_0^2+1}} \,,\, \frac{1}{\sqrt{4\alpha_0^2+1}} \right)$$

$$= (1.62722713282531988425\ldots \,,\, 0.58140602383297697452\ldots)\,,$$

$$M' = \left(\frac{2}{\sqrt{\alpha_0}} \,,\, 2\sqrt{\alpha_0} \right)$$

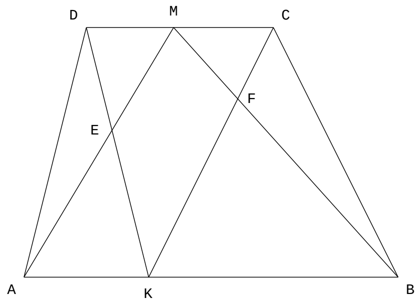
$$= (2.39097847459882936932\ldots \,,\, 1.67295525346422891327\ldots)\,.$$

One therefore gets the minimum value of $(a-d)^2 + (b-c)^2$ to be

$$\left|M - M'\right|^2 = 1.77479583276941567010\ldots \,.$$

Problem 9. *We are given a point K on the side AB of a trapezoid $ABCD$. Find a point M on the side CD that maximizes the area of the quadrangle which is the intersection of the triangles AMB and CDK.*

Answer. It is not clear from the statement of the problem whether AB is one of the parallel sides of the trapezoid. Since this interpretation seems more natural, I will treat this possibility only and leave the other case to the reader. The answer in this case is as follows. *If AB and CD are parallel, then M is chosen such that* $|DM| \cdot |AB| = |AK| \cdot |CD|\,.$



Proof. Let t be the area of the trapezoid, h the distance between the parallel lines AB and CD, and let q be the area of the quadrangle in question. The first observation is that the area of AMB plus the area of DKC equals t. To see this, one notes that the areas of AMB and DKC are independent of M and K since AB and CD are parallel. One can therefore take $K = A$ and $M = C$, in which case $ABCD$ is the disjoint union of AMB and DKC.

One then uses

$$\mathrm{area}\,(AMK \cup DKC) = \mathrm{area}\,(AMK) + \mathrm{area}\,(DKC) - q\,,$$

to get

$$q = t - \mathrm{area}\,(AMK \cup DKC)\,.$$

Now let E be the intersection of DK and AM, and F the intersection of MB and KC. One has

$$\begin{aligned}\mathrm{area}\,(AMK \cup DKC) = {} & \mathrm{area}\,(AEK) + \mathrm{area}\,(DEM) \\ & + \mathrm{area}\,(KFB) + \mathrm{area}\,(MFC) + q\,,\end{aligned}$$

so

$$q = \frac{t}{2} - \frac{\mathrm{area}\,(AEK) + \mathrm{area}\,(DEM) + \mathrm{area}\,(KFB) + \mathrm{area}\,(MFC)}{2}\,.$$

But since AB is parallel to DC, it follows that triangle AEK is similar to triangle MED and triangle KFB is similar to CFM. Now let h_1 be the altitude of DEM, i.e. the distance from E to DM, and h_2 the altitude of AEK, i.e. the distance from E to AK. One then has

$$h_2 = h_1\,\frac{|AK|}{|DM|}\,.$$

Moreover, since AB and CD are parallel, one also has $h_1 + h_2 = h$. One concludes that

$$h_1 = h\,\frac{|DM|}{|DM| + |AK|}\,.$$

This then implies that

$$\mathrm{area}\,(AEK) + \mathrm{area}\,(DEM) = \frac{h}{2}\,\frac{|AK|^2 + |DM|^2}{|AK| + |DM|}\,.$$

One similarly gets

$$\text{area}\,(KFB) + \text{area}\,(MFC) = \frac{h}{2}\,\frac{|KB|^2 + |MC|^2}{|KB| + |MC|}\;.$$

Let $x = |DM|$, so that $|MC| = |DC| - x$, then one wants to maximize

$$\frac{t}{2} - \frac{h}{4}\left(\frac{x^2 + |AK|^2}{x + |AK|} + \frac{(|DC| - x)^2 + |KB|^2}{|DC| - x + |KB|}\right)\,,$$

which is equivalent to minimizing

$$\begin{aligned} f(x) &= \frac{x^2 + |AK|^2}{x + |AK|} + \frac{(|DC| - x)^2 + |KB|^2}{|DC| - x + |KB|} \\ &= x - |AK| + \frac{2|AK|^2}{x + |AK|} - |KB| - x + \frac{2|KB|^2}{|DC| - x + |KB|} \\ &= -|AB| + \frac{2|AK|^2}{x + |AK|} + \frac{2|KB|^2}{|DC| - x + |KB|}\;. \end{aligned}$$

So, $f'(x) = 0$ gives

$$\frac{|AK|^2}{(x + |AK|)^2} = \frac{|KB|^2}{(|DC| - x + |KB|)^2}\,,$$

and since all quantities are positive, one can take positive square roots to obtain $|AK| \cdot |CD| = x\,|AB|$, which is exactly the expression claimed above.

To complete the proof, one checks that this gives a minimum of $f(x)$. But this follows from the fact that

$$f''(x) = \frac{4|AK|^2}{(x + |AK|)^3} + \frac{4|KB|^2}{(|DC| - x + |KB|)^3} > 0\,. \quad \square$$

Problem 10. *Can one cut a three-faced angle by a plane so that the intersection is an equilateral triangle?*

Answer. In general, one cannot cut a three-faced angle by a plane so that the intersection is an equilateral triangle.

Proof. One can take “three-faced” angle to mean the set of points

$$\{xU + yV + zW :\; x, y, z \geq 0\}\,,$$

where U, V, and W are unit vectors that do not all lie in a plane. The problem is therefore to find $x, y, z > 0$ such that

$$\|zW - xU\| = \|zW - yV\| = \|xU - yV\|\,.$$

Clearly, one can assume that $z = 1$, so the problem is equivalent to finding $x, y > 0$ such that

$$\begin{aligned}(W - xU)\cdot(W - xU) &= (W - yV)\cdot(W - yV)\,,\\ (W - xU)\cdot(W - xU) &= (xU - yV)\cdot(xU - yV)\,.\end{aligned} \tag{27}$$

A counter example is given by

$$U = (1\,,0\,,0)\,,\quad V = (0\,,1\,,0)\,,\quad W = \left(0\,,\frac{1}{\sqrt{2}}\,,\frac{1}{\sqrt{2}}\right).$$

Thus, we assume that there exist x, y that satisfy (27) for this choice of U, V, W. Then the first of equations (27) gives

$$x^2 + 1 = \left(y - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\,,$$

so that

$$y = \frac{1}{\sqrt{2}} \pm \sqrt{x^2 + \frac{1}{2}}\,.$$

Since one must have $x, y > 0$, the only solution is

$$y = \frac{1}{\sqrt{2}} + \sqrt{x^2 + \frac{1}{2}}\,. \tag{28}$$

The second of equations (27) implies that $x^2 + y^2 = x^2 + 1$. However, Eq. (28) implies that $y \geq \sqrt{2} > 1$, so there is a contradiction, and this proves the result.

Remark 10.1. One can give some general conditions for the existence of a solution (a complete characterization is left to the reader). In particular, it can be shown that there is a solution if either

$$\text{(a)}\ \ U\cdot V, U\cdot W, V\cdot W < 1/2\,,\qquad \text{or}\qquad \text{(b)}\ \ U\cdot V, U\cdot W, V\cdot W > 1/2\,.$$

Proof. The idea is to solve the first of equations (27) by finding a solution $y = f(x)$, $x \geq 0$, such that $y = f(x) > 0$ for $x > 0$, and

$f(x)$ is continuous. One then defines

$$g(x) = \big(W - xU\big)\cdot\big(W - xU\big)\,, \qquad h(x) = \big(xU - f(x)V\big)\cdot\big(xU - f(x)V\big)\,,$$

so that a solution exists when $g(x) = h(x)$. In order to do this, one finds a value $x_1 > 0$ such that $g(0) - h(0)$ and $g(x_1) - h(x_1)$ have opposite sign. Continuity will then imply that there is an $x_0 > 0$ for which $g(x_0) = h(x_0)$.

To prove (a), let us assume, without loss of generality, that

$$V \cdot W \geq U \cdot W\,.$$

One now solves the first of equations (27) to get

$$y = V \cdot W + \sqrt{(x - (U \cdot W))^2 + (V \cdot W)^2 - (U \cdot W)^2}\,.$$

Let $f(x)$ be the right hand side of the last equation. Note that

$$f(x) > 0 \quad \text{for} \quad x > 0\,,$$

since by assumption,

$$U \cdot W < 0 \quad \text{when} \quad V \cdot W < 0\,.$$

Now if $V \cdot W \leq 0$, then $f(0) = 0$, and otherwise $f(0) = 2(V \cdot W)$. Furthermore, $g(0) = 1$, while

$$h(0) = 0 \quad \text{if} \quad V \cdot W = 0\,,$$

and

$$h(0) = 4(V \cdot W)^2 \quad \text{if} \quad V \cdot W > 0\,.$$

By assumption, $4\,(V \cdot W)^2 < 1$, so in either case, one has $h(0) < g(0)$.

Now, it is clear that

$$\frac{f(x)}{x} \to 1 \;\text{ as }\; x \to \infty\,,$$

so

$$g(x) = x^2 + 1 - 2x(U \cdot W) \sim x^2,$$

while

$$\begin{aligned} h(x) &= x^2 + \big[f(x)\big]^2 - 2xf(x)(U \cdot V) \\ &\sim (2 - 2\,(U \cdot V))\,x^2 = (1 + \varepsilon)\,x^2, \quad \varepsilon > 0\,, \end{aligned}$$

since it was assumed that $U \cdot V < 1/2$. It follows that there is $x_1 > 0$, for which $g(x_1) < h(x_1)$ and the proof follows as outlined above.

(b) Again, assume that $W \cdot U \le W \cdot V$. In this case, one solves the second of equations (27) but this time one takes the solution

$$y = V \cdot W - \sqrt{(x - (U \cdot W))^2 + (V \cdot W)^2 - (U \cdot W)^2}\,.$$

Let $f(x)$ be the right hand side of this equation. It follows that

$$f(0) = 0\,, \quad f(2(U \cdot W)) = 0\,,$$

and

$$f(x) > 0\,, \quad \text{for } \; 2(U \cdot W) > x > 0\,.$$

Now $g(0) = 1$ and $h(0) = 0$, so $g(0) > h(0)$. On the other hand,

$$g\big(2(U \cdot W)\big) = 1\,,$$

while

$$h\big(2(U \cdot W)\big) = 4\big(U \cdot W\big)^2 > 1\,,$$

so that $h\big(2(U \cdot W)\big) > g\big(2(U \cdot W)\big)$, and the result follows by continuity.

Problem 11. *Let H_1, H_2, H_3, H_4, be the altitudes of a triangular pyramid. Let O be an interior point of the pyramid and let h_1, h_2, h_3, h_4 be the perpendiculars from O to the faces. Show that*

$$H_1^4 + H_2^4 + H_3^4 + H_4^4 \ge 1024\, h_1 \cdot h_2 \cdot h_3 \cdot h_4. \tag{29}$$

Solution. Let $ABCD$ be the tetrahedron and let its faces be F_1, F_2, F_3, and F_4, with areas f_1, f_2, f_3, and f_4, respectively. Let H_i be the altitude to F_i, and if P is an interior point of the tetrahedron, let $h_i = h_i(P)$ be the distance of P to F_i.

Recall that the volume of a tetrahedron is $\frac{1}{3}$ base $\times$ height (knowledge of the exact constant $\frac{1}{3}$ is not important here). Thus, letting V be the volume of the tetrahedron, one has $H_i = 3V/f_i$, where $i = 1, \ldots, 4$. Moreover, P divides $ABCD$ into 4 non-overlapping tetrahedrons $PABC$, $PABD$, $PACD$, and $PBCD$.

These tetrahedrons have volumes $h_i f_i/3$ in some order, so one also gets the identity $h_1 f_1 + h_2 f_2 + h_3 f_3 + h_4 f_4 = 3V$.

Now both sides of (29) are homogeneous of degree 4, so without loss of generality one can normalize the tetrahedron to have volume equal to $1/3$. It follows that

$$H_i = \frac{1}{f_i}\,, \; i = 1\,, \dots\,, 4\,, \quad h_1 f_1 + h_2 f_2 + h_3 f_3 + h_4 f_4 = 1\,. \tag{30}$$

One next finds an upper bound for $h_1 h_2 h_3 h_4$ by maximizing

$$\alpha(y_1\,, y_2\,, y_3\,, y_4) = y_1\, y_2\, y_3\, y_4\,,$$

given the constraints

$$y_1 f_1 + y_2 f_2 + y_3 f_3 + y_4 f_4 = 1\,, \quad y_i \geq 0\,.$$

(Whether this maximum is attained by an actual interior point of the tetrahedron is left as a problem for the reader.)

One observes that there will be a maximum with $y_i > 0$ since $\alpha(y_1, y_2, y_3, y_4)$ vanishes if any of the y_i's is zero. This implies that the maximum will be a local maximum, and one applies the following principle. Let S be a smooth surface of dimension of $n-1$ in Euclidean n-space and γ a real valued smooth function on S. Then at a local maximum s_0 of γ, the vector $\nabla\gamma = \left(\frac{\partial\gamma}{\partial x_1}\,, \dots\,, \frac{\partial\gamma}{\partial x_n}\right)$ is a multiple of the normal to S at s_0. To see why this should be true, recall that $\nabla\gamma$ points in the direction of maximum growth of γ, so if s_0 is a local maximum, then moving away from s_0 along S, i.e. locally orthogonally to the normal vector at s_0, should never increase γ.

One now lets

$$y_i = \frac{1}{4f_i} + x_i\,, \quad i = 1\,, \dots\,, 4\,,$$

and notes that maximizing α is equivalent to maximizing $\beta = \log\alpha$. This reduces the problem to maximizing

$$\beta(x_1, x_2, x_3, x_4) = \log\left[\prod_{i=1}^{4}\left(\frac{1}{4f_i} + x_i\right)\right],$$

given that

$$x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 = 0\,. \tag{31}$$

One easily computes

$$\nabla\beta(x_1,x_2,x_3,x_4) = \left(\frac{1}{\frac{1}{4f_1}+x_1}\,, \frac{1}{\frac{1}{4f_2}+x_2}\,, \frac{1}{\frac{1}{4f_3}+x_3}\,, \frac{1}{\frac{1}{4f_4}+x_4} \right),$$

while (31) defines a plane with normal vector $\mathbf{n} = (f_1, f_2, f_3, f_4)$. Equating $\nabla\beta = t\,\mathbf{n}$, for $t \neq 0$, results in

$$x_i = \frac{1}{f_i}\left(\frac{1}{t} - \frac{1}{4}\right), \quad i = 1,\ldots,4\,.$$

Applying (31) shows that in fact $x_1 = x_2 = x_3 = x_4 = 0$. It follows that the maximum of α occurs at

$$y_1 = \frac{1}{4f_1}\,, \quad y_2 = \frac{1}{4f_2}\,, \quad y_3 = \frac{1}{4f_3}\,, \quad y_4 = \frac{1}{4f_4}\,,$$

and thus the maximal value of α is $\frac{1}{2^8 f_1 f_2 f_3 f_4}$. This implies that

$$h_1h_2h_3h_4 \le \frac{1}{2^8 f_1 f_2 f_3 f_4}\,.$$

On the other hand, (30) implies that

$$H_1^4 + H_2^4 + H_3^4 + H_4^4 = \frac{1}{f_1^4} + \frac{1}{f_2^4} + \frac{1}{f_3^4} + \frac{1}{f_3^4}\,.$$

The final result will therefore follow from the inequality

$$z_1z_2z_3z_4 \le \frac{z_1^4 + z_2^4 + z_3^4 + z_4^4}{4}\,. \tag{32}$$

To prove this, one starts with $(a-b)^2 \ge 0$ which implies

$$ab \le \frac{a^2+b^2}{2}\,. \tag{33}$$

One therefore has

$$z_1z_2z_3z_4 \le \left(\frac{z_1^2+z_2^2}{2}\right)\left(\frac{z_3^2+z_4^2}{2}\right) \le \frac{1}{4}\,\frac{(z_1^2+z_2^2)^2 + (z_3^2+z_4^2)^2}{2}$$

$$= \frac{1}{4}\,\frac{z_1^4 + 2z_1^2z_2^2 + z_2^4 + z_3^4 + 2z_3^2z_4^2 + z_4^4}{2}\,,$$

which leads to (32) upon applying (33) to $z_1^2z_2^2$ and $z_3^2z_4^2$.

Problem 12. *Solve the system of equations.*

$$\begin{cases} y(x+y)^2 = 9 \\ y(x^3-y^3) = 7 \end{cases}$$

Answer. The only real solution is $x = 2$, $y = 1$.

Proof. Clearly this is a solution. To show that this is the only one, let $x = ty$, then the system becomes

$$\begin{cases} y^3(t+1)^2 = 9 \\ y^4(t^3-1) = 7\,. \end{cases} \tag{34}$$

Taking the first equation to the 4th power and cubing the second and dividing yields

$$\frac{(t+1)^8}{(t^3-1)^3} = \frac{9^4}{7^3}\,,$$

which reduces to finding the roots of

$$f(t) = 9^4\,(t^3-1)^3 - 7^3\,(t+1)^8\,.$$

Any real positive root t_0 of $f(t)$ will yield a solution x_0, y_0 by letting

$$y_0 = \left(\tfrac{9^4}{(t+1)^8}\right)^{1/12} \quad \text{and} \quad x_0 = t_0\,y_0\,.$$

Conversely, the above shows that every solution of (34) yields a positive real root of $f(t)$.

Clearly, $t = 2$ is a root of $f(t)$, and this corresponds to the solution $x = 2$, $y = 1$. By the previous argument, one only has to show that $f(t)$ has no other positive real root. This can be done by directly computing

$$\begin{aligned} \frac{f(t)}{t-2} &= 6561t^8 + 12779t^7 + 22814t^6 + 16341t^5 \\ &\quad + 13474t^4 + 2938t^3 + 6351t^2 + 3098t + 3452\,, \end{aligned}$$

and noting that all the coefficients are positive so there is no other positive real root. This computation can be done in a straightforward way by expanding

$$\begin{aligned} f(t) = 6561t^9 &- 343t^8 - 2744t^7 - 29287t^6 - 19208t^5 \\ &- 24010t^4 + 475t^3 - 9604t^2 - 2744t - 6904\,, \end{aligned}$$

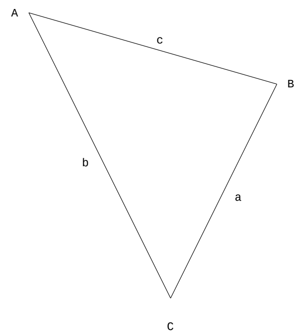
and then doing a long division by $t-2$. Such a computation was achieved in full during a train ride from IHES to Paris. Moreover, the fact that division by $t-2$ must leave a zero remainder provides an internal check for the computation.

Problem 13. *Show that if* a, b, *and* c *are the sides of a triangle and* A, B, *and* C *are its angles, then*

$$\frac{a+b-2c}{\sin(C/2)} + \frac{b+c-2a}{\sin(A/2)} + \frac{a+c-2b}{\sin(B/2)} \geq 0\,.$$

Solution. By collecting terms, one can rewrite the expression as

$$(a-b)\left(\frac{1}{\sin(B/2)} - \frac{1}{\sin(A/2)}\right) + (a-c)\left(\frac{1}{\sin(C/2)} - \frac{1}{\sin(A/2)}\right) \\ + (b-c)\left(\frac{1}{\sin(C/2)} - \frac{1}{\sin(B/2)}\right).$$



One now observes that each summand is non-negative.

In fact, consider a triangle ABC with sides $a = BC$, $b = AC$, $c = AB$. Then $b \geq a$ if and only if $\angle B \geq \angle A$, and since $A, B < 180°$, if and only if $\sin(B/2) \geq \sin(A/2)$. It follows that

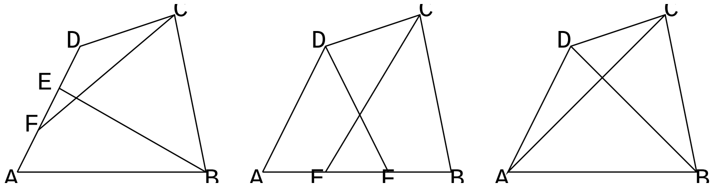
$$(a-b)\left(\frac{1}{\sin(B/2)} - \frac{1}{\sin(A/2)}\right) \geq 0\,.$$

Since this is true for any two sides and corresponding vertices, it holds for the other terms and the result follows.

Problem 14. *In how many ways can one represent a quadrangle as the union of two triangles?*

Answer. If the quadrangle is convex then there are exactly two ways, and if it not convex then the number of representations is infinite.

Proof. First consider a convex quadrangle $ABCD$ and assume that $ABCD$ is the union of two triangles T_1 and T_2.



First one shows that one of the triangles must contain three vertices. For if this were not the case, then each of the triangles would contain exactly two vertices leading to two cases.

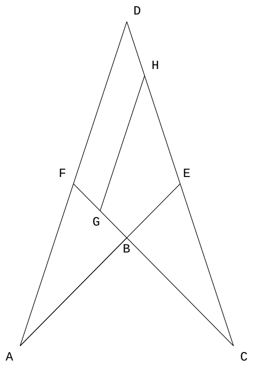
In the first case, each triangle contains two vertices on the same edge of the quadrangle. Without loss of generality, assume that T_1 contains AB and T_2 contains CD. Since a triangle is a closed convex set, and DA is not contained completely in T_1, there is a point E in the interior of AD for which $AE \subset T_1$ but $ED - \{E\} \not\subset T_1$. This condition implies that E is a vertex of T_1. Since a triangle is convex and the quadrangle is convex none of the sides AE, BE, or AB of T_1 can be extended outside of $ABCD$, so it follows that $T_1 = ABE$. Similarly, there is an F in the interior of AD such that $T_2 = CDF$. However, this shows that $T_1 \cup T_2$ does not include the interior of BC, so this case is not possible.

In the second case, each triangle contains opposite vertices of the quadrangle. Without loss of generality, assume that T_1 contains

A and C and T_2 contains B and D. By assumption, AB is not completely contained in T_1 or T_2, so as above, there is a point E in the interior of AB which is a vertex of T_1 and $T_1 = AEC$. Similarly, there is a point F in the interior of AB such that $T_2 = BDF$. This again implies that $T_1 \cup T_2$ does not contain the interior of BC, so this case is not possible either.

It follows that one of the triangles contains three vertices. Without loss of generality, assume that this is T_1 and that the vertices are A, B, C. Since the quadrangle is convex, none of the edges AB, AC, or BC can be extended and still remain in T_1 so $T_1 = ABC$. It follows that $BDA \subset T_2$, and since none of the edges of BDA can be extended, one has that $BDA = T_2$.

Thus, each choice of three vertices of $ABCD$ yields a partition into two triangles. There are 4 such choices, but two of these are equal by symmetry, so there are two choices: $ABC \cup BDA$ and $ABD \cup BCD$.



Consider the case where the quadrangle is not convex. Let the quadrangle be $ABCD$, where the angle B is greater than 180°. Now extend AB so that it cuts CD at E and extend CB so that it cuts AD at F. Also, let G be any point on BF and H be any point on ED. For any such choice, the quadrangle is the union of the triangles AED and CGH.

Remark 14.1. It seems clear that the examiners only meant *non-overlapping* triangles (note that the two triangles can *never* be disjoint). The "trap", which some students actually fell into [9], was to only consider convex quadrangles , but the examiners were in fact trapped by failing to consider the most general case.

In the non-convex case, there are three ways to represent the quadrangle as a union of non-overlapping triangles which, using the above notation, are $ABD \cup CBD$, $AED \cup CBE$, and $ABF \cup CFD$. The argument is similar to the convex case and is left to the reader.

Problem 15. *Show that*

$$\sum_{n=1}^{1000} \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{4}\,.$$

Solution. The factorization $n^3 + 3n^2 + 2n = n(n+1)(n+2)$ leads to the partial fraction expansion

$$\frac{1}{n^3 + 3n^2 + 2n} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) + \frac{1}{n+1} - \frac{1}{n+2}\,.$$

Now let $N > 3$, e.g. $N = 1000$, then

$$\sum_{n=1}^{N} \frac{1}{n^3 + 3n^2 + 2n} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+2}\right) - \sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)\,.$$

Each sum reduces by telescopic summation and this gives

$$\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) - \left(\frac{1}{2} - \frac{1}{N+2}\right)$$

$$= \frac{1}{4} + \frac{1}{2(N+2)} - \frac{1}{2(N+1)} = \frac{1}{4} - \frac{1}{2(N+1)(N+2)} < \frac{1}{4}\,.$$

Problem 16. *Solve the equation*

$$x^4 - 14x^3 + 66x^2 - 115x + 66.25 = 0\,.$$

Answer. The roots of $x^4 - 14x + 66x^2 - 115x + 66.25$ are

$\frac{7+i}{2} + \sqrt{4+2i}\,,\quad \frac{7+i}{2} - \sqrt{4+2i}\,,\quad \frac{7-i}{2} + \sqrt{4-2i}\,,\quad \frac{7-i}{2} - \sqrt{4-2i}\,,$

where $i = \sqrt{-1}$.

Proof. Let

$$f(x) = x^4 - 14x + 66x^2 - 115x + 66.25\,.$$

Substituting $x = \frac{y}{2}$ we rewrite $f(x) = 0$ as $g(y) = 0$, where

$$g(y) = y^4 - 28y^3 + 264y^2 - 920y + 1060\,.$$

One removes the cubic term by substituting $y = z + 7$ so that $g(y) = 0$ transforms into $h(z) = 0$, where

$$h(z) = z^4 - 30z^2 + 32z + 353\,.$$

Now $h(z) \equiv z^4 - z - 1 \pmod 3$, which is easily seen to be irreducible modulo 3. It follows that $f(x)$ is irreducible over the rationals, so there is no very easy solution to this problem. However, since this is an examination problem (one conjectures that students were not expected to be familiar with the solution to the general quartic), there might still be an "easy" solution. In particular, one could hope that $f(x)$ factors over a quadratic extension of the rational numbers. With this in mind, one writes

$$z^4 - 30z^2 + 32z + 353 = (z^2 + a\sqrt{D}\,z + b + c\sqrt{D})\,(z^2 - a\sqrt{D}\,z + b - c\sqrt{D})\,, \tag{35}$$

where a, b, c, D are integers (more generally, b and c could be half-integers) and D is squarefree. Equating terms in (35) one gets the conditions

$$\text{(I) } 2b - a^2D = -30, \quad \text{(II) } -2acD = 32\,, \quad \text{(III) } b^2 - c^2D = 353\,.$$

From (II) one concludes that D must be one of $-2, 2, -1$. If $D = -2$, then (III) has the solution $b = \pm 15$, $c = \pm 8$. But then, (I) would imply that a is divisible by 15, which is inconsistent with (II). If $D = 2$, then (III) has the solution $b = \pm 19$, $c = \pm 2$ (other solutions can be easily excluded). But then $a = \pm 4$ which is inconsistent with (I). Finally, if $D = -1$, then (III) has the solution $b = \pm 17$, $c = \pm 8$, and (II) implies that $a = \pm 2$. Trying out all the possible sign combinations, one eventually finds that $a = -2$, $c = -8$, $b = -17$ solves (I), (II), and (III). One therefore has the factorization

$$z^4 - 30z^2 + 32z + 353 = (z^2 - 2iz - 17 - 8i)\,(z^2 + 2iz - 17 + 8i). \tag{36}$$

Applying the quadratic formula to each term yields roots

$$z = i \pm 2\sqrt{4+2i} \quad \text{and} \quad z = -i \pm 2\sqrt{4-2i}$$

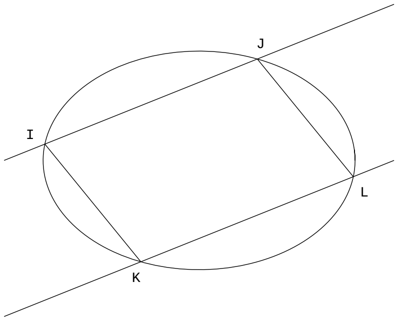
for the left and right factor of (36), respectively. The final answer follows on substituting $x = (z+7)/2$.

Problem 17. *Can a cube be inscribed in a cone so that 7 vertices of the cube lie on the surface of the cone?*

Answer. It is not possible to inscribe a cube in a cone so that 7 vertices of the cube lie on the cone.

Proof. If this were possible, then there would be a face $ABCD$ with all vertices on the cone, and the parallel face $EFGH$ would have at least 3 vertices on the cone. Now the face $ABCD$ lies on a plane which cuts the cone in a conic section, i.e. either in a hyperbola, parabola, ellipse, or two intersecting lines. Only an ellipse can circumscribe a square at 4 points, therefore, the intersection is an ellipse, say E_1. Since $EFGH$ is parallel to $ABCD$, it lies on a plane which also intersect the cone at an ellipse, say E_2.

Now $ABCD$ is symmetric with respect to E_1, i.e. its sides are parallel to the major or minor axes of E_1. Since $ABCD$, $EFGH$ are parallel and E_1 and E_2 are parallel, it follows that $EFGH$ is symmetric with respect to E_2. This implies that the vertices of $EFGH$ can meet E_2 at either 0, 2, or 4 points. This implies that $EFGH$ meets E_2 at 4 points. However, it is clear that there is a unique square that is inscribed symmetrically in an ellipse. Since E_1 and E_2 are parallel, they are similar, i.e. the ratio of their axes is the same, so the fact that they inscribe the same square implies that they are equal. This is clearly impossible, as different parallel sections of a cone must be different. It should also be noted that this also implies that E_1 cannot be a circle. The only other possibility is that $ABCD$ is not symmetric with respect to E_1. However, this cannot happen as every inscribed square in an ellipse must have its sides parallel to the major or minor axes.



In order to prove this, without loss of generality, consider an ellipse

$$x^2 + \frac{y^2}{a^2} = 1\,, \quad a > 0\,,$$

and a line

$$y = mx + b\,, \quad m \neq 0\,.$$

If these intersect at (x, y), then

$$(a^2 + m^2)x^2 - 2mbx + b^2 - a^2 = 0\,,$$

so the intersection points are

$$I = \left(\frac{-mb - a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}\,, \frac{ba^2 - am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right),$$

$$J = \left(\frac{-mb + a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}\,, \frac{ba^2 + am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right).$$

The length of the chord IJ is then

$$\frac{2a\sqrt{(1 + m^2)\,(a^2 + m^2 - b^2)}}{a^2 + m^2}\,.$$

It follows that given a and m, the only way to get two equal chords is to take $y = mx + b$ and $y = mx - b$ (this corresponds to the symmetry of the ellipse). The solutions corresponding to $-b$ are

$$K = \left(\frac{mb - a\sqrt{a^2+m^2-b^2}}{a^2+m^2}, \frac{-ba^2 - am\sqrt{a^2+m^2-b^2}}{a^2+m^2} \right),$$

$$L = \left(\frac{mb + a\sqrt{a^2+m^2-b^2}}{a^2+m^2}, \frac{-ba^2 + am\sqrt{a^2+m^2-b^2}}{a^2+m^2} \right).$$

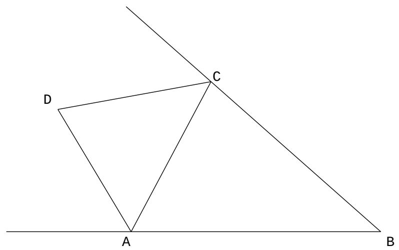
If these 4 points are to lie on a square then the angle KLJ must be 90°, in other words, the slope of LJ must be $-1/m$ since the slope of KL is m. Since

$$L - J = \left(\frac{2mb}{a^2+m^2}, \frac{-2ba^2}{a^2+m^2} \right),$$

this slope is $-a^2/m$. It follows that $a^2 = 1$, which implies that the ellipse is a circle. However, this contradicts the assumption that the ellipse circumscribes the square asymmetrically, and completes the proof.

Problem 18. *The angle bisectors of the exterior angles A and C of a triangle ABC intersect at a point of its circumscribed circle. Given the sides AB and BC, find the radius of the circle.*
[From Ref. 2: "The condition is incorrect: this doesn't happen."]

Solution. As indicated by A. Shen, the statement is incorrect. In fact, the following is true. *In a triangle ABC, the angle bisectors of the exterior angles of A and C cannot meet on the circumscribed circle of ABC.*



Proof. Let the exterior angle bisectors of A and C meet at the point D. If D were to lie on the circumscribed circle of ABC, then

$ABCD$ would be a cyclic quadrilateral. One appeals to the fact that in a cyclic quadrilateral the sum of opposite angle is 180° (this result seems well known and the easy proof is left to the reader). One now observes that the angle BAD is equal to

$$\frac{180^\circ - A}{2} + A = 90^\circ + \frac{A}{2}$$

and similarly the angle BCD is $90^\circ + C/2$. It follows that the sum of the angles BAD and BCD is $180^\circ + (A + C)/2 > 180^\circ$, which is a contradiction.

Problem 19. *A regular tetrahedron $ABCD$ with edge a is inscribed in a cone with a vertex angle of 90° in such a way that AB is on a generator of the cone. Find the distance from the vertex of the cone to the straight line CD.*

Answer. The statement of the problem is incorrect as the tetrahedron cannot be inscribed in the cone. Inscribing a tetrahedron in the cone means that all its vertices lying on the cone and that, apart from its vertices, it lies entirely inside a connected component of $\mathbf{R}^3$ minus the cone. As will be seen below, this is not possible. If one takes “inscribe” to mean only that the tetrahedron has all its vertices on the (double) cone, then the answer is $\sqrt{34}\,a/8$. However, this interpretation would imply that a cube could be inscribed in a cone, contradicting the result of Problem **17**.

Proof. Without loss of generality, one can take the cone to be given by the equation $x^2 + y^2 = z^2$ and the generator to be the line $x = 0$, $y = z$. Moreover, one can take the tetrahedron to have side length equal 1, so that for a tetrahedron of side a, the answer will be a times the answer for this case. Since A and B lie on the generator, one can assume, without loss of generality, that

$$A = \Big(0\,, t\,, t\Big)\,, \quad B = \Big(0\,, t + \frac{1}{\sqrt{2}}\,, t + \frac{1}{\sqrt{2}}\Big)\,.$$

Letting

$$C = \left(-\frac{1}{2}\,, t - \frac{1}{2} + \frac{1}{2\sqrt{2}}\,, t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)\,,$$

$$D = \left(\frac{1}{2}\,, t - \frac{1}{2} + \frac{1}{2\sqrt{2}}\,, t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)\,,$$

it is easily checked that $ABCD$ is a regular tetrahedron. In order for C and D to lie on the cone, one has to have

$$\frac{1}{4} + \left(t - \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^2 = \left(t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^2 ,$$

which has the unique solution $t_0 = \frac{1}{8} - \frac{1}{\sqrt{8}}$. It follows that the tetrahedron must have vertices

$$A = \left(0\,, \frac{1}{8} - \frac{1}{\sqrt{8}}\,, \frac{1}{8} - \frac{1}{\sqrt{8}}\right), \qquad B = \left(0\,, \frac{1}{8} + \frac{1}{\sqrt{8}}\,, \frac{1}{8} + \frac{1}{\sqrt{8}}\right),$$

$$C = \left(-\frac{1}{2}\,, -\frac{3}{8}\,, \frac{5}{8}\right), \qquad D = \left(\frac{1}{2}\,, -\frac{3}{8}\,, \frac{5}{8}\right).$$

Since $1/8 < 1/\sqrt{8}$, it follows that the interior of AB and the interior of CD lie on two different connected components of $\mathbf{R}^3$ minus the cone, so that $ABCD$ is not strictly inscribed in the double cone. In any case, the midpoint of CD is $(0, -3/8, 5/8)$ so that the distance from the vertex $(0, 0, 0)$ to CD is $\sqrt{34}/8$.

One must also consider the possibility of inscribing the tetrahedron asymmetrically by rotating it about $y = z$. However, the intersection of the cone with the possible rotations of the C and D about $y = z$ form a circle which lies in a plane orthogonal to the generator of the cone. Since the cone has vertex angle 90°, the intersection of this plane with the cone is a parabola. A circle and a parabola can intersect at two points at most, so this implies that any intersection point must be symmetric, and there are no other solutions.

Remark 19.1. The following shows how one was led to the original construction of C and D. One begins with a simple result about regular tetrahedrons.

Lemma 19.1. *Let $ABCD$ be a regular tetrahedron and φ be the angle between AB and $AC + AD$, i.e. the angle that a side from the base to the summit makes with the base, then* $\varphi = \arccos\bigl(1/\sqrt{3}\bigr)$.

Proof. This is clearly equivalent to showing that for a regular tetrahedron of side 1, the distance of the summit to the base is $\sqrt{2/3}$.

Thus, let the points be

$$A = \Big(0\,, \frac{\sqrt{3}}{2}\,, 0\Big)\,, \quad B = \Big(0\,, y\,, z\Big)\,, \quad C = \Big(-\tfrac{1}{2}\,, 0\,, 0\Big)\,, \quad D = \Big(\tfrac{1}{2}\,, 0\,, 0\Big)\,.$$

One has $\|C - B\| = 1$ and $\|A - B\| = 1$, so that

$$y^2 + z^2 = \tfrac{3}{4}\,, \qquad \Big(x - \frac{\sqrt{3}}{2}\Big)^2 + z^2 = 1\,.$$

Clearly $B = \Big(0\,, \dfrac{1}{2\sqrt{3}}\,, \sqrt{\tfrac{2}{3}}\Big)$ solves these equations, and this proves the Lemma 19.1. $\square$

In order to inscribe the tetrahedron in the cone, one translates it by $\big(0\,, -\frac{\sqrt{3}}{2}\,, 0\big)$ so that it has vertices

$$A = \Big(0\,, 0\,, 0\Big)\,, \qquad B = \Big(0\,, -\frac{1}{\sqrt{3}}\,, \sqrt{\tfrac{2}{3}}\Big)\,,$$

$$C = \Big(-\tfrac{1}{2}\,, -\frac{\sqrt{3}}{2}\,, 0\Big)\,, \qquad D = \Big(\tfrac{1}{2}\,, -\frac{\sqrt{3}}{2}\,, 0\Big)\,.$$

In this position, B has angle $\pi - \varphi$ in the yz-plane and one wants it to have angle $\pi/4$, so one rotates the tetrahedron by $\psi = \varphi - 3\pi/4$ degrees with respect to the x-axis, then slides it up the cone by adding $(0, k, k)$. Let $R_{x,\psi}$ be the rotation, then one is solving for k such that $R_{x,\psi}(C) + (0, k, k)$ lies on $x^2 + y^2 = z^2$.

It only remains to compute $R_{x,\psi}(C)$, which follows from

$$R_{x,\psi}(C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2\sqrt{2}} \\ \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix}\,,$$

using

$$\cos\psi = \cos\frac{3\pi}{4}\,\cos\varphi + \sin\frac{3\pi}{4}\,\sin\varphi = -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\,,$$

$$\sin\psi = \cos\frac{3\pi}{4}\,\sin\varphi - \sin\frac{3\pi}{4}\,\cos\varphi = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\,.$$

Problem 20. *Compare*

$$\log_3 4 \cdot \log_3 6 \cdot \ldots \cdot \log_3 80 \qquad \text{and} \qquad 2\log_3 3 \cdot \log_3 5 \cdot \ldots \cdot \log_3 79\,.$$

Answer.

$$\log_3 4 \cdot \log_3 6 \cdot \ldots \cdot \log_3 80 > 2\log_3 3 \cdot \log_3 5 \cdot \ldots \cdot \log_3 79\,.$$

Proof. Since there are the same number of $\log_3 \ldots$ terms on each side, the base 3 in the logarithm can be canceled out and the above is equivalent to

$$\log 4 \cdot \log 6 \cdot \ldots \cdot \log 80 > 2\,\log 3 \cdot \log 5 \cdot \ldots \cdot \log 79\,.$$

Taking logarithms of both sides, leads to the equivalent statement

$$\sum_{k=2}^{40} \log\log(2k) > \log 2 + \sum_{k=2}^{40} \log\log(2k-1)\,. \tag{37}$$

The proof of this will rely on two basic facts: that $\log\log x$ is concave for $x \geq 3$ and that

$$\int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \log 2\,. \tag{38}$$

To see that $\log\log x$ is concave for $x \geq 3$, note that

$$\frac{d^2}{dx^2}\log\log x = \frac{1}{x^2}\left(1 - \frac{1}{\log x}\right) > 0\,, \quad x > e\,.$$

In order to prove (38), note that

$$\int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \left.\frac{\log\log(2x-1)}{2}\right|_2^{41}$$

$$= \frac{\log\log 81 - \log\log 3}{2} = \log 2\,.$$

From the concavity of $\log\log x$ one has for $k \geq 2$,

$$\log\log(2k) > \frac{\log\log(2k+1) + \log\log(2k-1)}{2}\,.$$

Subtracting $\log\log(2k-1)$ from each side gives

$$\log\log(2k) - \log\log(2k-1) > \frac{\log\log(2k+1) - \log\log(2k-1)}{2}\,.$$

But the right hand side of this is exactly equal to

$$\frac{1}{2}\,\log\log(2x-1)\Big|_k^{k+1} = \int_k^{k+1} \frac{dx}{(2x-1)\log(2x-1)}\,.$$

It follows that

$$\sum_{k=2}^{40}\big[\log\log(2k) - \log\log(2k-1)\big] > \int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \log 2\,,$$

by (38). This last inequality is exactly (37) and the result follows.

Problem 21. *A circle is inscribed in a face of a cube of side a. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles.*

Answer. The minimum distance is

$$\frac{a}{\sqrt{20+8\sqrt{6}}}\,.$$

Proof. It is sufficient to treat the case of $a = 2$, as the solution is linear in a. One can thus consider that the cube has vertices at $(\pm1, \pm1, \pm1)$ and that the inscribed and circumscribed circles are given, respectively, by

$$(\cos t\,, \sin t\,, 1)\,,\ 0 \le t < 2\pi$$

and

$$(1\,, \sqrt{2}\,\sin u\,, \sqrt{2}\,\cos u)\,,\ 0 \le u < 2\pi\,,$$

The minimum distance will therefore be the minimum of

$$\sqrt{(\cos t - 1)^2 + (\sin t - \sqrt{2}\,\sin u)^2 + (1 - \sqrt{2}\,\cos u)^2}\,.$$

One therefore minimizes

$$\begin{aligned} &(\cos t - 1)^2 + (\sin t - \sqrt{2}\,\sin u)^2 + (1 - \sqrt{2}\,\cos u)^2 \\ &= 5 - 2\cos t - 2\sqrt{2}\sin t\sin u - 2\sqrt{2}\cos u\,. \end{aligned} \tag{39}$$

This is equivalent to maximizing

$$\cos t + \sqrt{2}\sin t \sin u + \sqrt{2}\cos u\,. \tag{40}$$

One does this by first considering u to be constant, and maximizing over t, and then maximizing over u. This requires the following lemma.

Lemma 21.1. *Let α and β be real numbers, then*

$$\max_{t\in[0,2\pi)}(\alpha\cos t + \beta\sin t) = \sqrt{\alpha^2+\beta^2}\,.$$

Proof. Let $\gamma = \sqrt{\alpha^2+\beta^2}$, then there is a φ such that $\alpha/\gamma = \sin\varphi$ and $\beta/\gamma = \cos\varphi$. It follows that $\alpha\cos t+\beta\sin t = \gamma\sin(t+\varphi)$, which immediately implies the result. $\square$

Continuing with the proof, Lemma 21.1 shows that the maximum of (40) is

$$\sqrt{1+2\sin^2 u} + \sqrt{2}\,\cos u\,,$$

which can be rewritten as

$$\sqrt{3-2\cos^2 u} + \sqrt{2}\,\cos u\,.$$

Since $\cos u$ takes on all values in $[-1,1]$, maximizing this last form over u is equivalent to maximizing

$$\sqrt{3-2x^2} + \sqrt{2}\,x, \quad x\in[-1,1]\,. \tag{41}$$

One checks for critical points by taking the derivative and setting it equal to zero. This gives

$$\sqrt{2} - \frac{2x}{3-2x^2} = 0\,,$$

so that $x = \sqrt{3}/2$, and the resulting value in (41) is $\sqrt{6}$. Since there is only one critical point in $[-1,1]$ the only other possible maxima are at $x=\pm1$, but these give $1\pm\sqrt{2}$ which are both smaller than $\sqrt{6}$.

It follows that the maximum value of (40) is $\sqrt{6}$ and plugging this back into (39) gives the minimum value

$$5-2\sqrt{6} = \frac{1}{5+2\sqrt{6}}\,.$$

The result then follows by substitution.

Remark 21.1. A. Shen notes [9] that there is an elegant solution to the problem that follows. Consider two spheres with center at the center of the cube with each containing one of the circles mentioned in the problem. Clearly, the distance between the circles cannot be less than the distances between the spheres. On the other hand, it is easy to see that there is a ray from the center that intersects both circles. It follows that this distance is minimal.

Problem 22. *Given k segments in a plane, show that the number of triangles all of whose sides belong to the given set of segments is less than $C\,k^{3/2}$, for some constant $C > 0$.*

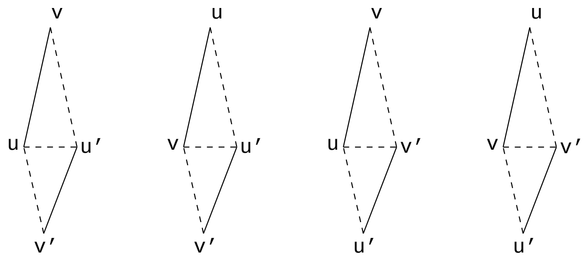
Solution. One has to interpret this as asking for triangles whose edges exactly belong to the set of given segments, see Section 4.

The problem is equivalent to bounding the number of triples $\{a,b\}$, $\{b,c\}$, $\{c,a\}$, where $\{a,b\}$, $\{b,c\}$, $\{c,a\}$ correspond to the endpoints of 3 distinct segments. Under this formulation it becomes clear that the fact that e_i are line segments is unimportant and that the problem rests on the fact that each e_i joins its 2 endpoints. In other words, one is really considering a (combinatorial) graph V with vertices the endpoints of the segments and edges the e_i's. The problem can therefore be restated as follows. *Let V be a graph, then the number of triangles in the graph is not greater than $C\,k^{3/2}$, where k is the number of edges in the graph.* Note that a triangle in a graph is a simply a set of 3 vertices that is completely connected. The main idea is the following.

Lemma 22.1. *Given a graph with k edges, the number of unordered pairs of distinct triangles, which have a common edge, is not greater than $2k^2$.*

Proof. Let $e_1, \ldots, e_k$ be the edges. To each unordered pair of edges (e, e'), where $e = \{u,v\}$ and $e' = \{u',v'\}$, one associates the 4 pairs of triangles

$$\{uvu', u'v'u\}, \quad \{vuu', u'v'v\}, \quad \{uvv', v'u'u\}, \quad \{vuv', v'u'v\}\,.$$



Thus each unordered pair of edges gives rise to at most 4 pairs of triangles with a common edge. Moreover, it is clear that any pair of triangles with a common edge will be generated in this way. It follows that the number of pairs of triangles with a common edge is not more than 4 times the number of unordered pairs of edges. Since the number of unordered pairs of edges is $k(k-1)/2 \leq k^2/2$, the result follows. $\square$

Now, given a graph V, let T be the total number of triangles, and for each edge e, let t_e be the number of triangles containing e. For each edge e, the number of pairs of triangles having e as a common edge is $t_e\,(t_e-1)/2$. Since distinct triangles cannot have more than one edge in common, the estimate of Lemma 22.1 implies

$$\sum_e \frac{t_e\,(t_e-1)}{2} \leq 2\,k^2\,.$$

A simple computation shows that

$$t \leq \frac{t(t-1)}{2} \quad \text{for} \quad t \geq 3\,,$$

so

$$\begin{aligned}\sum_e t_e^2 \;&\leq\; 4k^2 + \sum_e t_e \;\leq\; 4k^2 + \sum_e \frac{t_e(t_e-1)}{2} + 2\sum_e 1 \\ &\leq\; 6k^2 + 2k \leq\; 7k^2\,,\end{aligned} \tag{42}$$

since $\sum_e 1 = k$ is the number of edges (it is assumed that $k \geq 3$, otherwise there are no triangles). Now

$$\left(\sum_e t_e\right)^2 \le 2\sum_{t_e}\sum_{t_{e'}\le t_e} t_e\, t_{e'} \le 2k\sum_{t_e} t_e^2 \le 14\,k^3\,, \tag{43}$$

by (42). One concludes, by noting that each triangle contains exactly 3 edges, that

$$\sum_e t_e = 3T\,.$$

Plugging this into (43), one obtains

$$T \le \frac{\sqrt{14}}{3}\,k^{3/2}\,,$$

which gives the result with $C = \dfrac{\sqrt{14}}{3}\,.$

Remark 22.2. Ofer Gabber has noted that the above method can be improved to give the optimal constant $C = \sqrt{2}/3\,.$ This can be done using almost exactly the same techniques as follows (Ofer Gabber used a different approach). As before, one begins with a lemma.

Lemma 22.2. *Given a graph with k edges, the number of ordered pairs of triangles which have a common edge is not greater than $2\,k(k-1)$.*

Proof. One considers *ordered* pairs of distinct edges. Thus, let $(uv\,, u'v')$ be an ordered pair of edges. First assume that none of $u\,, v$ equals u', v'. Then, as in the above, one can make at most 4 ordered pairs of triangles with a common edge, where one always lets the triangle containing uv be the first component of the pair. However, any such pairs of triangles, if they exist, will be counted twice in total. For example, if the triangles uvu' and $u'v'u$ exist, then $u'v$ and uv' are edges, so that the ordered pair $(uvu', u'v'u)$ will also be counted by $(u'v\,, uv')$.

Next assume that one of u, v equals one of u', v', without loss of generality, say $u' = u$. Then the possible pairs of triangles one can construct are $\{uvv', uvv'\}$, i.e. the two triangles are equal, and pairs $\{uvw\,, v'uw\}$, where w is another vertex unequal to $u\,, v\,, v'$. This last possibility will already have been counted twice above by the ordered pairs $(uv\,, v'w)$ and $(vw\,, v'u)$ with distinct vertices, so it can be left out of this count. In the first case, the pair of $\{uvv', uvv'\}$, if it exists, will be counted 6 times: once by each of $(uv\,, uv')$, (uv', uv), $(vu\,, vv')$, (vv', vu), $(vv', v'u)$, $(v'u\,, vv')$.

Now let M be the number of ordered pairs of edges with no common vertex and N be the number of distinct ordered pairs of edges with a common vertex. Once again, one let t_e be the number of triangles containing the edge e. The above shows that

$$2\sum_e t_e(t_e-1) \le 4M\,, \quad \tfrac{6}{3}\sum_e t_e \le N\,.$$

The second inequality follows from the above argument and the fact that $\sum_e t_e$ counts each triangle 3 times. It follows that

$$\sum_e t_e^2 \le 2M + 2\,N \le 2k(k-1)\,, \tag{44}$$

since $M + N = k(k-1)$ is the number of ways of choosing distinct pairs of edges. $\square$

One applies the improved inequality (better known as the Cauchy-Schwarz inequality)

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right), \tag{45}$$

and (44) to get

$$\left(\sum_e t_e\right)^2 \le 2k^2(k-1)\,,$$

and the result follows as before.

The fact that the value $C = \sqrt{2}/3$ is optimal is proved by considering the complete graph with n vertices which has $\varepsilon_n = n(n-1)/2$ edges and $\tau_n = n(n-1)(n-2)/6$ triangles, so that

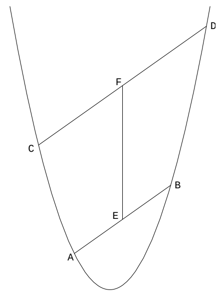
$$\frac{\tau_n}{\varepsilon_n^{3/2}} \to \frac{\sqrt{2}}{3}\,, \quad \text{as} \quad n \to \infty\,.$$

Remark 22.2. The argument directly generalizes to show that for each n, there is a constant C_n such that the number of n-gons all of whose sides belong to the segments is not greater than $C_n\,k^{n/2}$. In fact, Lemma 22.1 already proves this for $n = 4$.

Remark 22.3. Igor Rivin has proved all the above results using algebraic methods, i.e. using the spectral theory of the adjacency matrix [11]. Moreover, his paper proves the analogous optimal results for higher length cycles.

Problem 23. *Use ruler and compasses to construct the coordinate axes from the parabola $y = x^2$.*

Solution. Let $A = \big(a, a^2\big)$ and $B = \big(b, b^2\big)$ be two points on the parabola.



One can draw the line segment joining AB with ruler and compass. This line has slope

$$\frac{b^2 - a^2}{b - a} = a + b\,.$$

Let $C = (c, c^2)$ be a point on the parabola unequal to A or B. One can then use ruler and compass to draw a line L through C parallel to AB and say that this line meets the parabola at $D = (d, d^2)$. Since CD has the same slope as AB, it follows that $a + b = c + d$. One can then use ruler and compass to construct E, the midpoint of AB, and F, the midpoint of CD. It follows that

$$E = \left(\tfrac{a+b}{2}\,, \tfrac{a^2+b^2}{2}\right) \quad \text{and} \quad F = \left(\tfrac{c+d}{2}\,, \tfrac{c^2+d^2}{2}\right).$$

Using ruler and compass one constructs the line segment EF.

Since $a+b=c+d$, it follows that EF is parallel to the y-axis. Using ruler and compass, one constructs a line L' perpendicular to EF through E. Let L' meet the parabola at points G and H. Using ruler and compass, one constructs I, the midpoint of GH, and then draws through I a line L'' parallel to EF. It follows that L'' is the y-axis. Let L'' intersect the parabola at J. Using ruler and compass, one constructs the line L''' through J which is perpendicular to L''. Then L''' is the x-axis.

Problem 24. *Find all a such that for all $x<0$ we have the inequality*

$$ax^2 - 2x > 3a - 1\,.$$

Answer. The condition is that $0 \le a \le 1/3$.

Proof. By letting $x \mapsto -x$, the condition is equivalent to characterizing a for which

$$ax^2 + 2x > 3a - 1, \quad x > 0, \tag{46}$$

holds. If $0 \le a \le 1/3$, then $3a-1 \le 0$, so the right side of (46) is non-positive, while the left side is positive, so the inequality holds. On the other hand, if $a > 1/3$, then $3a-1>0$, so there is a small positive value of x for which (46) fails. Thus, if $1 > a > 1/3$, then one can take $x = \frac{3a-1}{3}$, since

$$ax^2 + 2x < 3x < 3a - 1\,.$$

If $a \ge 1$, then one can take $x = \dfrac{1}{3\sqrt{a}}$ since

$$ax^2 + 2x = \frac{1}{9} + \frac{2}{3\sqrt{a}} < 1 < 3a - 1\,.$$

Finally, if $a < 0$, then $ax^2 + 2x \to -\infty$ as $x \to \infty$, i.e. Eq. (46) fails for all sufficiently large x.

Problem 25. *Let A, B, C be the angles and a, b, c the sides of a triangle. Show that*

$$60^\circ \le \frac{aA + bB + cC}{a+b+c} \le 90^\circ\,.$$

Solution. Since $A+B+C=180°$, the statement can be rewritten as

$$\frac{A+B+C}{3} \leq \frac{aA+bB+cC}{a+b+c} \leq \frac{A+B+C}{2}\,. \tag{47}$$

To prove the right hand inequality, one multiplies by $2(a+b+c)$ to get the equivalent statement

$$Ab+Ac+Ba+Bc+Ca+Cb-Aa-Bb-Cc \geq 0\,.$$

Collecting terms, this can be rewritten as

$$A(b+c-a)+B(a+c-b)+C(a+b-c) \geq 0\,.$$

One now observes that each summand is positive. This follows from the triangle inequality which implies that

$$b+c>a\,, \quad a+c>b\,, \quad a+b>c\,.$$

One can therefore conclude that the inequality on the right of (47) is strict.

To prove the left hand inequality, one multiplies by $3(a+b+c)$ to get the equivalent statement

$$2aA+2bB+2cC-aB-aC-bA-bC-cA-cB \geq 0\,.$$

Upon collecting terms, this becomes

$$(A-B)(a-b)+(A-C)(a-c)+(B-C)(b-c) \geq 0\,.$$

One now observes that, as in Problem 13, each of these terms is non-negative. For example, $a \geq b$ if and only if $A \geq B$, so

$$(A-B)(a-b) \geq 0\,,$$

and similarly for the other terms.

3. Notes

Problem 1. This problems appears to be a standard result in elementary geometry, "the butterfly theorem" [12] (French edition only), [10,13–15]. However, this result was not included in any standard geometry textbooks used by the candidates [9]. In Ref. 10, it is stated that a proof was given in 1815 by W.G. Horner (of Horner's method for polynomials) and that the shortest proof depends on projective geometry [14]. Marcel Berger [12] has stated that the butterfly theorem is a good example of a deceptive result. In particular, it is a statement about circles and lengths which lead one to look for a metric proof. However, as is seen above such arguments are quite awkward, whereas the correct point of view is projective. A projective generalization is given in Refs. 13 and 14. Using the notation of the problem, this can be stated as follows. *Let AB be a chord of a conic section and let MN, ST be chords whose intersection does not lie on AB. If MN and ST both intersect AB at K and SN intersects AB at Q and MT intersects AB at Q, then K has the same harmonic conjugate with respect to P and Q and with respect to A and B.*

The proofs of Pavol Severa and David Ruelle both seem to be candidates "for the book" [16].

The first solution found by the author proceeded by converting it into a purely algebraic framework (the same is true for Problem **2**. This has the advantage of almost guaranteeing a solution, even if one has missed the "idea" of the intended solution (this is confirmed by the fact that this in fact worked). Moreover, the "conceptual" solution of problem **1** used intermediate results, e.g. Lemma 1.2, which seemed to be as subtle as the original statement, whereas the algebraic proof was fairly direct.

However, algebraic methods have the disadvantage that they require much algebraic computation in which any slight error destroys any possibility of obtaining the solution. Moreover, in order to keep the computations at a manageable level, one must be somewhat clever in setting up the algebraic formulation, as well as deciding how to proceed with the computation, e.g. see the solution to problem **2**. On the other hand, these considerations vanish almost completely if one allows oneself the use of a computer algebra system. Using such a system, the answer follows almost immediately from the algebraic formulation. One can argue that such proofs are more in the nature of verifications, in particular, they may not reveal how the result was originally discovered. These issues are discussed in Refs. 17 and 18.

Problem 3. This question appears as problem 10.13.11 in Ref. 12.

Problem 4. This question is a special case of a problem of Erdős and Woods [19, 20]. Thus, for an integer $k \geq 2$, one considers m, n for which $m+i, n+i$ have the same prime divisors for $i = 0, \ldots, k-1$. The problem in question is $k = 2$ for which all the known examples are given in the above solution. It is conjectured that there exists $k > 2$ such that if $m+i, n+i$ have the same prime divisors for $i = 0, \ldots, k-1$, then $m = n$. This conjecture has applications to logic [20]. This question and its generalizations has been studied by Balasubramanian, Langevin, Shorey, and Waldschmidt [21, 22].

Problem 5. In the formulation of Ref. 2, the word "perimeter" is given as "circumference."

Problem 6. In Ref. 2, the condition $0 < x < \pi/2$ is omitted which renders the condition invalid. To see this, note that the left hand side of (23) equals zero when $x = \pi/2$, but the derivative of the left hand side at $x = \pi/2$ is $-16/\pi^3 < 0$, so there is a small $\varepsilon > 0$ for which the left hand side of (23) is negative in $(\pi/2, \pi/2+\varepsilon)$.

Problem 8. In Ref. 2, the condition is incorrectly given as

$$a^2 + b^2 = 4, \quad cd = 4,$$

and the corresponding statement is false. In fact, the minimum value of $(a-d)^2 + (b-c)^2$ is $4 \cdot (3 - 2\sqrt{2})$ which is smaller than 1.6 and is actually smaller than 1, since

$$4 \cdot (3 - 2\sqrt{2}) = \frac{4}{3 + 2\sqrt{2}} < 1.$$

The minimum value in this case is found as in the solution of Problem **8** given above: one is computing the distance between the curves $x^2 + y^2 = 4$ and $xy = 4$. The first of these is a circle of radius 2, while the second is a hyperbola.

Let L be the line $x + y = 2\sqrt{2}$, then this is clearly a tangent to the circle since it meets the circle at the point $(\sqrt{2}, \sqrt{2})$ and is perpendicular to the radius. Likewise, the line L' given by $x+y = 4$ is tangent to $xy = 4$ since it meets it at $(2, 2)$ and the slope of $y = 4/x$

at $x = 2$ is

$$-\frac{4}{2^2} = -1\,.$$

Lemma 8.1 implies that the minimum distance between the two curves is not less than the distance between L and L'. However, the line joining $(\sqrt{2}\,,\sqrt{2}\,)$ and $(2\,,2)$ has slope 1 so is perpendicular to L and L', thus the distance between these two points will be the actual minimum distance between the curves. One then computes the minimum distance to be

$$\sqrt{2\cdot(2-\sqrt{2})^2} = \sqrt{2}\cdot(2-\sqrt{2}) = 2\sqrt{2}-2\,,$$

so that the minimum of $(a-d)^2+(b-c)^2$ is

$$(2\sqrt{2}-2)^2 = 4\cdot(3-2\,\sqrt{2})\,,$$

as claimed.

Problem 14. In Ref. 2, the examiners were incorrectly given as Ugol'nikov and Kibkalo.

Problem 15. In Ref. 2, the examiners were incorrectly given as Ugol'nikov and Kibkalo.

Problem 21. In Ref. 2, this is given as an example of a "murderous" problem, as it was the most difficult problem of the second round of the All-Union Olympiad in 1985. It was solved by 6 participants, partly solved by 3 people, and not solved by 91.

Problem 22. In Ref. 2, the original formulation was the following. Given k segments in the plane, give an upper bound for the number of triangles all of whose sides belong to the given set of segments. [Numerical data were given, but in essence one was asked to prove the estimate $O(k^{15})$.]

This formulation has the typographical error $O(k^{15})$ for $O(k^{1.5})$.

Parting thought. The following is my own suggestion for the type of problem considered in this paper and I leave it as an exercise for the reader. *Consider two triangles whose perimeters add up to a constant. What is the minimum value of the sum of the squares of* *the lengths of the edges of their symmetric difference (points which belong to exactly one triangle)?*

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SOLUTIONS TO THE YEAR 2000 INTERNATIONAL MATHEMATICAL OLYMPIAD

ILAN VARDI

This article is meant to complement my earlier paper on Mekhmat entrance examinations [1]. In that paper I stated that the Mekhmat problems appeared to be at the level of Olympiad problems which were an appropriate standard for comparison. In order to verify this claim, I decided to solve under similar conditions (without use of outside references or help) all the problems for a given Olympiad year. Doing *all* problems in a given year appears to be the fairest test of problem solving, as one cannot simply choose the problems one is best at. Finally, since many people work on Olympiad problems, one can compare one's solution with the best that has been found (this was not always possible with the Mekhmat problems).

My conclusion is that the IMO problems are generally much more challenging than the Mekhmat problems. In particular, it took me about 6 weeks to complete all the Olympiad problems and 6 weeks for all the Mekhmat problems. However, there were 6 Olympiad problems versus 25 Mekhmat problems.

On the other hand, the Olympiad problems were in general less interesting than the Mekhmat problems. In fact, I only found Olympiad Problems 3 and 6 to be interesting. Problem 3 required an interesting idea, which took me a long time to discover (3 weeks!), and it also took me a long time to find an idea for Problem 6, which follows from a result of some independent interest (Lemma 6.3).

The other problems were solved in a relatively short time without too much difficulty. If there was an idea in Problem 2 then I missed it as my solution consists of a completely unmotivated algebraic manipulation. Problem 5 is stated confusingly and has an alternate, possibly intractable interpretation. The lack of clarity might have been intentional in order not to give away the solution, see Remark 5.1.

The statements of the geometry Problems 1 and 6 are of Byzantine complexity, and this appears to be typical of the geometry questions posed in the IMO. Moreover, Problem 1 is in some sense deceptive. In comparison, the Mekhmat geometry problems were very elegant and in general interesting. Mekhmat Problems 1 and 2 were of comparable difficulty to the geometry problems posed here.

Since the source of the Mekhmat problems [2] only included one number theory question and one combinatorial question, it is less meaningful to compare the other problems to the Mekhmat questions.

I would like to thank K. S. Sarkaria and Harry Tamvakis for helpful comments. Alternate solutions appear on John Scholes' IMO web site [3]. Further IMO problems are given in Refs. [4–7].

Problems

Day 1

1. Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1\,.$$

3. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a *move* as follows.

1) Choose any two fleas, at points A and B, with A to the left of B.
2) Let the flea at A jump to a point C on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Day 2

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime numbers, and 2^n+1 is divisible by n.

6. Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The circle inscribed in the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the inscribed circle of the triangle ABC.

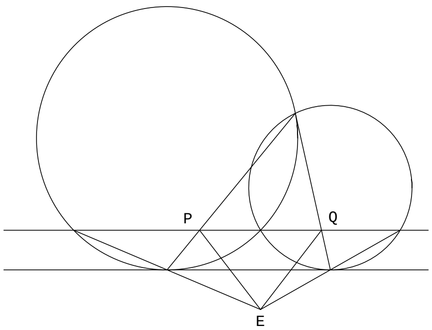
Solutions

Problem 1

Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D.

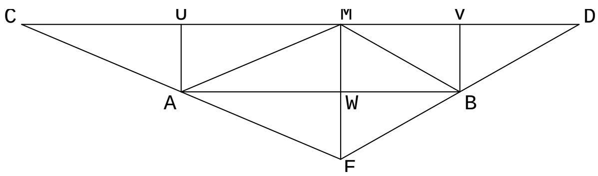
Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q.

Show that $EP = EQ$.



The somewhat complicated formulation of this problem is deceptive, and it is easily seen to be a combination of two simple results.

One starts by looking at a subproblem which can be analyzed by consideration of similar triangles, that is, the points A, B, C, D, E, M. In order to do this, one introduces 3 new points U, V, W, which are the midpoints of CM, MD and AB, respectively.



Since CM is a chord of the circle Γ_1 and A lies on Γ_1, it follows that $AC = AM$. Thus, $\angle MCA = \angle CMA$. Since CM is parallel to AB, $\angle CMA = \angle BAM$.

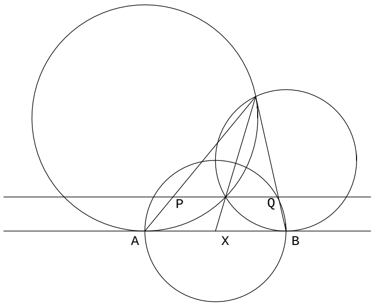
It follows that $CU = UM = AW$. Similarly, $DV = VM = BW$. One therefore sees that

$$CD = 2CU + 2DV = 2AB\,.$$

But, since CD and AB are parallel, the triangle CDA is similar to the triangle ABE, so it follows that $CE = 2\,AE$, in other words, that $CA = AE$. Finally, since $\angle BAE = \angle BAM$, one sees that EM is perpendicular to AB. The first result is therefore a lemma that follows.

Lemma 1.1. *Under the assumptions of Problem 1, EM is perpendicular to AB.*

This result clarifies the situation considerably since it is now clear that $EP = EQ$ if and only if $MP = MQ$, since CD is also perpendicular to EM.

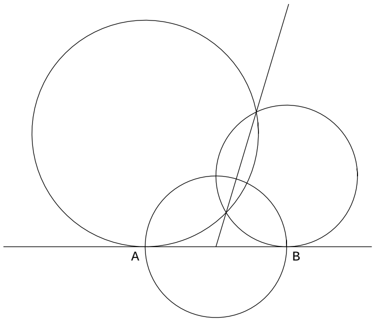


One therefore turns to the proof of $MP = MQ$. Extend the line MN to intersection AB at X. Since CD is parallel to AB, it follows that triangle NPM is similar to triangle NAX and that triangle NQM is similar to triangle NBX. The result therefore follows from the following interesting fact.

Lemma 1.2. *Let a line be tangent to two distinct circles at two different points A and B. Then the line joining the intersections of the two circles (or the tangent line if they touch once) bisects the line segment AB.*

Proof. The proof uses some elementary properties of inversion [8]. Let the notation be as in Problem 1, i.e. let Γ_1, Γ_2 be the circles, M and N be the intersection points and X the midpoint of AB.

Consider the circle Γ with center at X and radius AX. One does an inversion with respect to this circle. Since Γ_1 and Γ_2 are orthogonal to Γ (this is obvious at A and B), they are preserved. Since the line containing MN passes through infinity, it is transformed into a line or circle passing through X.



However, since Γ_1 and Γ_2 are preserved, M is mapped to N and N is mapped to M. Finally, let the line MN intersect Γ at Y, then Y is fixed, since it lies on Γ. Thus, the line MN goes to a line or circle containing X, M, N, and Y. However, M, N, and Y lie on a line, therefore MN goes to a straight line, in particular to itself. One concludes that the line MN passes through X, which is the midpoint of AB and the result follows. The case in which the circle are tangent is left as an exercise.

Remark 1.1. Harry Tamvakis has noted that Lemma 1.2 follows easily using the *power of a point with respect to a circle*. In other words, it is easily shown that if the point X is external to a circle Γ, then for any line XMN touching the circle at M and N, then $XM \times MN$ is a constant (in the case where the line XA is tangent to the circle, one takes $XA \times XA$).

Problem 2

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1\,.$$

The solution relies on identity of Lemma 2.1.

Lemma 2.1. *Let a, b, c be positive real numbers such that $abc = 1$, then*

$$\begin{aligned} 1 - \left(a - 1 + \tfrac{1}{b}\right)\left(b - 1 + \tfrac{1}{c}\right)\left(c - 1 + \tfrac{1}{a}\right) \\ = \left(c + \tfrac{1}{c} - 2\right)\left(a + \tfrac{1}{b} - 1\right) + \left(1 - \tfrac{1}{a}\right)(1 - b)\,. \end{aligned} \tag{1}$$

This identity easily implies the result. To see this, note that if $abc = 1$, then at least one of a, b, c is ≥ 1 and at least one is ≤ 1. It follows that at least one of the ordered pairs $(a\,, b)$, $(b\,, c)$, or $(c\,, a)$ has its first element ≥ 1 and the second ≤ 1. One can, without loss of generality, take this ordered pair to be $(a\,, b)$. The reason is that Lemma 2.1 will be applied, and its result also holds with the variables on the right hand side of (1) interchanged as $(a, b, c) \mapsto (c, a, b)$ or $(a, b, c) \mapsto (b, c, a)$, since the left hand side of (1) is invariant under these cyclic shifts.

One now applies Lemma 2.1. First the arithmetic-geometric inequality is used, which gives

$$\frac{1}{2}\left(c + \frac{1}{c}\right) \geq \sqrt{c \cdot \frac{1}{c}} = 1\,,$$

so that

$$c + \frac{1}{c} - 2 \geq 0\,.$$

Since $a \geq 1$ and $b \leq 1$, one has

$$a + \frac{1}{b} - 1 \geq 0\,, \quad 1 - \frac{1}{a} \geq 0\,, \quad 1 - b \geq 0\,.$$

It follows that each term on the right hand side of (1) is nonnegative which proves the result.

Proof of Lemma 2.1. Using the transformation rules

$$ab \mapsto \frac{1}{c}\,, \quad ac \mapsto \frac{1}{b}\,, \quad bc \mapsto \frac{1}{a}\,,$$

one expands

$$\left(a - 1 + \tfrac{1}{b}\right)\left(b - 1 + \tfrac{1}{c}\right)\left(c - 1 + \tfrac{1}{a}\right).$$

Thus,

$$\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)=\frac{1}{a}-b+\frac{b}{a}-c+1-\frac{1}{a}+1-\frac{1}{c}+b$$
$$=\frac{b}{a}+2-c-\frac{1}{c}\,.$$

One continues with

$$\left(a-1+\frac{1}{b}\right)\left(\frac{b}{a}+2-c-\frac{1}{c}\right)$$
$$=b+2a-\frac{1}{b}-\frac{a}{c}-\frac{b}{a}-2+c+\frac{1}{c}+\frac{1}{a}+\frac{2}{b}-\frac{c}{b}-a$$
$$=a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}-\frac{a}{c}-\frac{c}{b}-\frac{b}{a}-2\,.$$

Thus, the left hand side of (1) equals

$$3+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}-a-\frac{1}{a}-b-\frac{1}{b}-c-\frac{1}{c}\,. \tag{2}$$

One now computes

$$\begin{aligned}\left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right)&=\frac{1}{b}+\frac{c}{b}-c+\frac{a}{c}+a-\frac{1}{c}-2a-\frac{2}{b}+2\\&=2+\frac{a}{c}+\frac{c}{b}-a-\frac{1}{b}-c-\frac{1}{c}\,,\end{aligned} \tag{3}$$

and

$$\left(1-\frac{1}{a}\right)(1-b)=1-b-\frac{1}{a}+\frac{b}{a}\,. \tag{4}$$

It is seen that the right hand sides of (3) and (4) add up to the expression (2), which proves Lemma 2.1.

Remark 2.1. This proof appears to be completely unmotivated. In particular, there does not seem to be a conceptual explanation for the identity of Lemma 2.1. However, this identity fits into the general principle that inequalities should follow from identities, e.g. from the positivity of perfect squares [9].

Remark 2.2. A similar argument was found by Robin Chapman [3] using the identity

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right) = b\left(a^2 - \left(1 - \frac{1}{b}\right)^2\right).$$

This shows that the left hand side is $\leq b\,a^2$, so the product of the three such identities is not greater than $b\,a^2\,c\,b^2\,ac^2 = 1$, which yields the square of the desired inequality. This works if all terms in the product are positive, but it is easily shown that at most one such term can be negative, in which case the inequality also holds.

Remark 2.3. One can try the general method of Lagrange multipliers to solve this problem. However, this method is usually considered beyond the scope of the High School curriculum.

Problem 3

Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a move as follows:

1) *Choose any two fleas, at points A and B, with A to the left of B.*
2) *Let the flea at A jump to a point C on the line to the right of B with $BC/AB = \lambda$.*

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Answer. All such values of λ are given by $\lambda \geq \dfrac{1}{n-1}$.

(a) Let us assume that $\lambda \geq \dfrac{1}{n-1}$. Let us define a *big move* as a move involving the leftmost and the rightmost fleas. Then the fleas will all go to $+\infty$ using a succession of big moves.

First, let us consider the case where the fleas are all at different positions and let the positions be $x_n < x_{n-1} < \ldots < x_1$. Let m be the minimum distance between consecutive fleas.

One now performs a big move which puts the fleas at new positions $x'_n < x'_{n-1} < \ldots < x'_1$, where $x'_n = x_{n-1}, \ldots, x'_2 = x_1$, and $x'_1 = x_1 + \lambda(x_1 - x_n)$.

Let m' be the minimum between consecutive fleas in their new positions. Since $x'_n, \ldots, x'_2$ are the same as initially, it follows that the new minimum satisfies

$$m' \geq \min\left(m\,, x'_1 - x'_2\right) .$$

However,

$$x'_1 - x'_2 = \lambda(x_1 - x_n) \quad \text{and} \quad x_1 - x_n \geq (n-1)m\,.$$

It follows that

$$m' \geq \min\left(m\,, \lambda(n-1)m\right) \geq m\,,$$

since $\lambda(n-1) \geq 1$, by assumption.

It follows that the minimum distance between consecutive fleas is nondecreasing. Thus, let $M > 0$ be a lower bound for all these minimum distances. It follows that $x'_n \geq x_n + M$, i.e. the position of the rightmost flea is increased at least by M upon every big move. This shows that all the fleas go to $+\infty$.

If the fleas are not all distinct, then, since at least two fleas are distinct, it is clear that after at most $n-1$ iterations of big moves, all fleas will become distinct, which reduces to the previous case.

(b) Let us assume that $\lambda < \frac{1}{n-1}$ now. This case seems to require some kind of physical reasoning, as direct methods such as those in part (a) do not seem to work, see Remark 3.1.

Thus, one sees that there is a tradeoff between the position of the rightmost flea and the distance between the fleas. In other words, a greater distance to the right incurs a penalty in the distance between the fleas. Thus, one is led to the consideration of an *energy* to each placement of the fleas, where the position of the fleas is the *kinetic energy,* and the distances between the fleas is the *potential energy.*

By careful examination of the case of two fleas, one is led to the following definition.

Definition. Let the position of the fleas be $x = (x_1\,, x_2\,, \ldots\,, x_n)$, where $x_n \leq \ldots \leq x_2 \leq x_1$, then the kinetic energy $K(x)$ and the potential energy $P_\alpha(x)$ are

$$K(x) = x_1, \qquad P_\alpha(x) = \alpha \sum_{i=2}^{n} (x_1 - x_i)\,,$$

where α is a constant. The *total energy* is $E_\alpha(x) = K(x) + P_\alpha(x)$. This leads to the following Lemma.

Lemma 3.1. *If* $\alpha = \frac{\lambda}{1-(n-1)\lambda}$, *then the total energy is never increased by a move.*

The main result follows directly from Lemma 3.1. Indeed, one has

$$\frac{\lambda}{1-(n-1)\lambda} > 0\,,$$

since $\lambda < \frac{1}{n-1}$. Then the potential energy

$$P(x) = P_{\frac{\lambda}{1-(n-1)\lambda}}(x)\,.$$

is always positive. The Lemma shows that $K(x) + P(x) \leq E$ for some E independent of x, thus, for all x, $K(x) \leq E - P(x) \leq E$ and the result follows.

Proof of Lemma 3.1. First consider the case of a big move. Thus, the fleas are initially at $x_n \leq \cdots \leq x_1$ and that at x_n jumps to

$$x_1' = x_1 + \lambda(x_1 - x_n)\,,$$

while the other fleas remain at the same positions, i.e.

$$x_n' = x_{n-1}\,,\ \ldots\,,\ x_2' = x_1\,.$$

The change in potential energy is

$$\begin{aligned}
P_\alpha(x') - P_\alpha(x) &= \alpha \sum_{i=2}^{n} (x_1' - x_i') - \alpha \sum_{i=2}^{n} (x_1 - x_i) \\
&= \alpha \sum_{i=1}^{n-1} \big[(x_1' - x_1) + (x_1 - x_i)\big] - \alpha \sum_{i=2}^{n} (x_1 - x_i) \\
&= (n-1)\,\alpha\,(x_1' - x_1) - \alpha(x_1 - x_n) \\
&= \alpha\big[(n-1)\lambda - 1\big](x_1 - x_n)\,.
\end{aligned}$$

Moreover, the change in kinetic energy is

$$K(x') - K(x) = x_1' - x_1 = \lambda(x_1 - x_n)\,.$$

Thus, the difference in total energy is

$$\begin{aligned} E_\alpha(x') - E_\alpha(x) &= K(x') - K(x) + P_\alpha(x') - P_\alpha(x) \\ &= \lambda(x_1 - x_n) + \alpha\big[(n-1)\lambda - 1\big]\,(x_1 - x_n) \\ &= (x_1 - x_n)\Big(\lambda + \alpha\big[(n-1)\lambda - 1\big]\Big)\,, \end{aligned}$$

which is clearly zero when $\alpha = \frac{\lambda}{1-(n-1)\lambda}$. One has therefore proved the following lemma.

Lemma 3.2. *If* $\alpha = \frac{\lambda}{1-(n-1)\lambda}$ *, then the total energy is preserved by a big move.*

One now considers other types of moves. The first possibility is if $x_1' = x_1$, i.e. the position of the rightmost flea does not change. In this case, the kinetic energy stays the same, while the potential energy decreases, for any $\alpha > 0$. This is obvious, since

$$P_\alpha(x') - P_\alpha(x) = \alpha(x_i - x_j') < 0\,,$$

where the move consists in flea x_i jumping to x_j', and $x_j' > x_i$.

Finally, one considers the case where a flea jumps to $x_1' > x_1$ but from $x_j > x_n$, i.e. it is not a big move. One performs the same computation as with a big move. The change in potential energy is

$$\begin{aligned} P_\alpha(x') - P_\alpha(x) &= \alpha \sum_{i=2}^{n} (x_1' - x_i') - \alpha \sum_{i=2}^{n} (x_1 - x_i) \\ &= \alpha \sum_{\substack{1 \le i \le n \\ i \ne j}} \big[(x_1' - x_1) + (x_1 - x_i)\big] - \alpha \sum_{i=2}^{n} (x_1 - x_i) \\ &= (n-1)\,\alpha\,(x_1' - x_1) - \alpha(x_1 - x_j) \\ &\le \alpha\big[(n-1)\lambda - 1\big](x_1 - x_j)\,, \end{aligned}$$

since $x_1' - x_1 \le \lambda(x_1 - x_j)$, as x_j could at most jump over x_1. Similarly, the change in kinetic energy is

$$K(x') - K(x) = x_1' - x_1 \le \lambda(x_1 - x_j)\,,$$

so that the total change in energy is

$$E(x') - E(x) \le \Big(\lambda + \alpha\big[(n-1)\lambda - 1\big]\Big)\big(x_1 - x_j\big) = 0\,,$$

if

$$\alpha = \tfrac{\lambda}{1-(n-1)\lambda}\,.$$

This shows that the total energy is nonincreasing, which finishes the proof of Lemma 3.1.

Remark 3.1. This type of solution was previously discovered by Gerhard Woeginger [3].

Remark 3.2. It can be shown directly that the strategy used in part (a), i.e. using only big moves, does not take the fleas to $+\infty$. In fact, a big move is a linear transformation $x' = Ax$, where A is the matrix

$$A = \begin{pmatrix} 1+\lambda & 0 & \cdots & 0 & -\lambda \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

This matrix has characteristic polynomial of the form

$$\big(x-1\big)\big(x^{n-1} - \lambda(x^{n-2} + x^{n-3} + \ldots + 1)\big)\,,$$

which has one eigenvalue 1 while all other eigenvalues have absolute value strictly less than 1, when $\lambda < \frac{1}{n-1}$.

Using this method, or equivalent methods, it is simple to show that $x_1 - x_n \to 0$ exponentially, i.e. the distance between the rightmost flea and the leftmost flea goes to zero exponentially. Since, in a simple iteration of big moves, the position of the rightmost flea is essentially the sum of these distances, one gets that the position of the rightmost flea is bounded above.

However, this argument seems to break down if other types of moves are allowed. In particular, the distance between the rightmost and leftmost fleas no longer has to go to zero, e.g., if the leftmost flea never moves. Moreover, for moves other than big moves, the relabelling of the fleas is no longer unique, as a flea may land between any two others, to that the choice of matrix is no longer unique. Finally, even without this problem, the asymptotics of random matrix products is a nontrivial question.

Problem 4

A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Answer. There are 12 ways. If the boxes are denoted R, W, B, then the 2 basic configurations are

$$R = \{1, 4, 7, \ldots, 100\}, \quad W = \{2, 5, 8, \ldots, 98\}, \quad B = \{3, 6, 9, \ldots, 99\},$$

and

$$R = \{1\}, \quad W = \{2, 3, 4, \ldots, 99\}, \quad B = \{100\},$$

and the 12 solutions consists of the 6 permutations of each basic configuration.

Solution. Let the boxes be A, B, C. One first notes that $n \mapsto 101-n$ preserves $\{1, 2, \ldots, 100\}$, and in the following, the expression "by symmetry" will include this map. One begins the proof with the following useful results.

Lemma 4.1. *If A, B, C satisfy the conditions of the problem, and a and $a+k$ are in A, then it is not possible to have b in B such that*

$b+k$ *is in* C.

Proof. In fact, it is easily seen that this is equivalent to the conditions of the problem. Thus, if $b \in B$ and $c+k \in C$, then $a+b+k$ is in $A+B$ and in $A+C$, which is not allowed. $\square$

Lemma 4.2. *If* A, B, C *satisfy the conditions of the problem, and* A *contains* a *and* $a+k$ *and box* B *contains* b *and* $b+k$, *then if* c *is in* C *and* $1 \le c+k \le 100$, *then* $c+k$ *is in* C.

Proof. One applies Lemma 4.1 twice. Thus, applying it with respect to A shows that $c+k$ cannot be in B, and applying it with respect to B shows that $c+k$ cannot be in A. The result follows. $\square$

Lemma 4.3. *Let* A, B, C *satisfy the conditions of the problem , then it is not possible to have* a , $a+k \in A$ *and* b , $b+k \in B$, *where* $k=1$ *or* $k=2$.

Proof. Assume that such a and b exist. Then C has an element c such that

$$1 \le c+k \le 100 \quad \text{or} \quad 1 \le c-k \le 100\,,$$

so there exists c' such that $c'+k \in C$. It follows from Lemma 4.2 that

$$1 \le a \pm rk \le 100\,,\ r \in \mathbf{Z}\,,$$

are all in A and similarly,

$$1 \le b \pm rk \le 100\,,\ r \in \mathbf{Z}\,, \quad \text{and} \quad 1 \le c \pm rk \le 100\,,\ r \in \mathbf{Z}\,,$$

are all in B and C, respectively. If $k=1$, then each set has 100 elements while if $k=2$, each set has 50 elements, so A, B, C do not form a partition of $\{1,2,3,\ldots,100\}$. This contradiction proves the result. $\square$

The proof will now be split up according to the minimum of all the differences of two elements which are in the same box.

(a) The minimum is 1. Without loss of generality, one can assume that $a\,,a+1 \in A$. Now either B and C are of the form $\{1\}$ and $\{100\}$, or there exist (without loss of generality) $b \in B$ such that $1 < b < 100$ and (by the symmetry $n \mapsto 101-n$) $c \in C$ such that $c < 100$.

By Lemma 4.3, $b+1$ and $c+1$ are in A. It follows that $b+c+1$ is in $A+B$ and in $A+C$, which is not allowed. This possibility is therefore excluded unless B and C are of the form $\{1\}$ and $\{100\}$.

(b) The minimum is 2. Without loss of generality, one can assume that $a\,, a+2 \in A$. Since the minimum is 2, $a+1$ is not in A and thus, without loss of generality, $a+1 \in B$. Now either $a+3 \le 100$ or $a-1 \ge 1$. By symmetry ($n \mapsto 101-n$), assume that $a+3 \le 100$. By Lemma 4.1, one cannot have $a+3 \in C$. By Lemma 4.2, one cannot have $a+3 \in B$. From part (a), it follows that one cannot have $a+3 \in A$, since $a+2 \in A$. This case is therefore excluded.

(c) The minimum is 3. One of the sets A, B, C must have at least 34 members, since they form a partition of $\{1\,, 2\,, \ldots\,, 100\}$, otherwise one would have $|A|+|B|+|C| \le 33+33+33 = 99$. Without loss of generality, let this set be A. Now split up A into 33 bins $\{1\,,2\,,3\}$, $\{4\,,5\,,6\}\,, \ldots\,, \{97\,,98\,,99\}$. Since the minimum difference of elements of A is 3, each bin can have at most one element. But since there are more than 33 elements, and only 100 is left over, each bin has one element, thus exactly one element. Since there are at least 34 elements, it follows that 100 is in A. Since $100 \in A$, one cannot have 98 or 99 in A, and therefore $97 \in A$. Similarly, one has $94\,, \ldots\,, 7\,, 4\,, 1 \in A$. It follows that $A = \{1\,, 4\,, 7\,, \ldots\,, 100\}$.

From a similar argument, one has that B and C are $\{2\,, 5\,, \ldots\,, 98\}$ and $\{3\,, 6\,, \ldots\,, 99\}$ in some order.

(d) The minimum is greater than 3. This is not possible, as one of the boxes has at least 34 elements, and thus minimum difference ≤ 3, by the argument used in (c).

Remark 4.1. Harry Tamvakis has found the following simpler argument. It begins with considering the general problem with cards numbered 1 to n. One also shows that there are only two types of solutions, namely,

$$\{3a+k\}\,,\ k = 0\,,1\,,2\,, \quad \text{and} \quad \{1\}\,, \{2\,, \ldots\,, n-1\}\,, \{n\}\,,$$

but the main point is that the first type of solution is *stable* under $n \mapsto n+1$,(by adding the card numbered $n+1$ to the solution with n cards), while the second is not. This observation yields a simple induction argument that follows. Assume that the result holds for n, then given a legal configuration with $n+1$ cards, remove card $n+1$ from the boxes. This is a solution for n cards and therefore of first type, unless removing card $n+1$ left one box empty. In the first case, one easily shows that card $n+1$ must fit into the correct arithmetic progression modulo 3, yielding the first type of solution. In the second case, one similarly removes card 1 from the boxes which shows that its box only contained this one card, and one gets the second solution.

Problem 5

Determine whether or not there exists a positive integer n divisible by exactly 2000 different prime numbers, and 2^n+1 is divisible by n.

Answer. Yes, there exists such a number. In fact, for any positive integer k, there exists a positive integer n_k divisible by exactly k different prime numbers such that n_k divides $2^{n_k}+1$.

Solution. The proof is by induction. We begin the induction with $k=1$ and $n_1=3$, since 3 divides $9=2^3+1$.

Let us assume now that n_k has been constructed with n_k dividing $2^{n_k}+1$, and n_k having exactly k different prime factors. One uses the following steps to construct n_{k+1} (for ease of notation, let $N=n_k$).

a) *If p^α is the largest power of p dividing N and p^β is the largest power of p dividing 2^N+1, then $p^{\alpha+\beta}$ is the largest power of p dividing $2^{N^2}+1$.*
b) *N^2 divides $2^{N^2}+1$.*
c) *There is a prime q not dividing N but dividing $2^{N^2}+1$.*
d) *Let $n_{k+1}=N^2q$, then n_{k+1} satisfies the above conditions.*

Proof of a). Let p be any prime dividing N with p^α and p^β as in a). One can therefore write $2^N = A\,p^\beta - 1$ for some integer A not divisible by p. One now uses the binomial theorem to compute

$$\left(2^N\right)^p = \left(A\,p^\beta - 1\right)^p = A'\,p^{\beta+2} + A\,p^{\beta+1} - 1 = A_1\,p^{\beta+1} - 1\,,$$

for some integer $A_1 = A'\,p + A$ not divisible by p. Iterating this gives $(2^N)^{p^\alpha} = A_\alpha\, p^{\alpha+\beta} - 1$, for some integer A_α not divisible by p. Similarly,

$$2^{N^2} = \left((2^N)^{p^\alpha}\right)^{\frac{N}{p^\alpha}} = \left(A_\alpha\, p^{\alpha+\beta} - 1\right)^{\frac{N}{p^\alpha}}$$

$$= A''\, p^{\alpha+\beta+1} + A_\alpha \left(\frac{N}{p^\alpha}\right) p^{\alpha+\beta} - 1\,.$$

Since A_α is not divisible by p and, by definition, N/p^α is not divisible by p, it follows that $2^{N^2}+1$ is divisible by $p^{\alpha+\beta}$, but by no higher power, as claimed.

Proof of b). Note that, by assumption, $\beta \geq \alpha$, so one has that $p^{2\alpha}$ divides $2^{N^2}+1$. Since this holds for every p dividing N, it follows that N^2 divides $2^{N^2}+1$.

Proof of c). Let M be the largest divisor of 2^N+1 such that M is divisible only by primes that divide N. Similarly, let M' be the largest divisor of $2^{N^2}+1$, such that M' is divisible only by primes that divide N. It follows from part a) that $M' = MN$, since part a) proves this for each separate prime power dividing N. Since M and N both divide 2^N+1 it follows that both M and N are smaller or equal to 2^N+1. Therefore

$$M' = MN \leq \left(2^N+1\right)^2 \leq \left(2^{N+1}\right)^2 = 2^{2N+2}\,.$$

On the other hand, using $N \geq 3$,

$$2^{N^2} = \left(2^{2N}\right)^{\frac{N}{2}} = 2^{2N}\left(2^{2N}\right)^{\frac{N}{2}-1} \geq 2^{2N+3}\,,$$

since $\dfrac{N}{2} - 1 \geq \dfrac{3}{2}$, and thus $2N\Big(\dfrac{N}{2}-1\Big) \geq 3\,.$

It follows that $M' < 2^{N^2}+1$, and thus $2^{N^2}+1$ must be divisible by a prime q that does not divide N.

Proof of d). Let $n_{k+1} = N^2 q$. Then by b) one has

$$2^{N^2} \equiv -1 \left(\operatorname{mod} N^2\right),$$

so that

$$2^{n_{k+1}} \equiv \left(-1\right)^q = -1 \left(\operatorname{mod} N^2\right),$$

recalling that q must be odd since it divides a power of 2 plus 1. By the choice of q, one also has that

$$2^{N^2} \equiv -1 \pmod{q},$$

so that

$$2^{n_{k+1}} \equiv (-1)^q = -1 \pmod{q}.$$

The result follows from the Chinese Remainder Theorem.

Remark 5.1. The statement of the problem can be interpreted to mean: "n is divisible by exactly 2000 different prime numbers, each with multiplicity one." In other words, n is assumed to be squarefree. It is conceivable that a more precise form of the question was avoided in order not to give away the idea of the solution, which depends on having primes with high multiplicities.

The alternate question appears to be intractable, in the nature of resolving whether there are an infinite number of primes of the form $2^p - 1$. See Ref. [10] for examples of these types of unsolved problems.

Problem 6

Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The circle inscribed into the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines L_1, L_2, and L_3 be the reflections of the lines H_2H_3, H_3H_1, and H_1H_2 in the lines T_2T_3, T_3T_1, and T_1T_2, respectively.

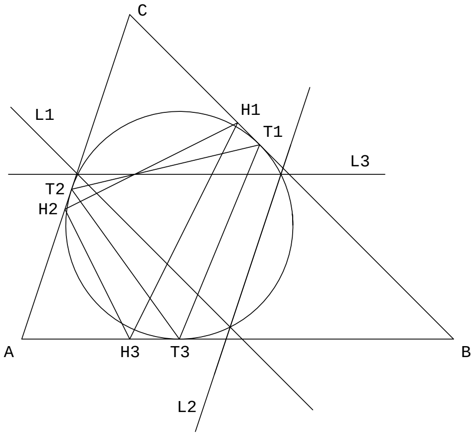
Prove that L_1, L_2, L_3 determine a triangle whose vertices lie on the circle inscribed into the triangle ABC.

The solution requires some basic results from triangle geometry which are easy to prove, but which will only be quoted here.

Theorem 6.1. *Let the notations be as in the statement of Problem 6. Then the following statements hold.*

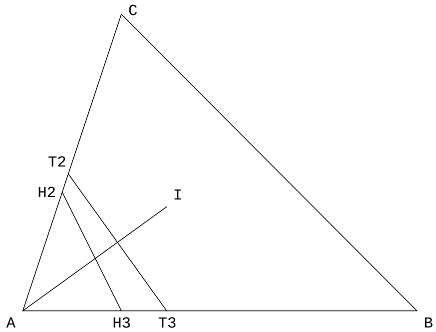
a) The center of the inscribed circle is the intersection of the bisectors of angles A, B, C.
b) The center of the circumscribed circle is the intersection of the right bisectors of the sides AB, BC, CA.

c) The triangle AH_2H_3 is similar to the triangle ABC, i.e. corresponding angles are equal. Similar statements hold for BH_3H_1 and CH_1H_2 .



These results immediately allow some progress.

Lemma 6.1. *Let the notations be as in the statement of Problem 6, then L_1, L_2 , L_3 determine a triangle (possibly a point) similar to triangle ABC.*



Proof. Let I be the center of the inscribed circle. Since AH_2H_3 is similar to ABC, and AI bisects the angle BAC, reflecting H_2H_3 about AI makes it parallel to BC. Since T_2T_3 is perpendicular to AI, it follows that reflecting H_2H_3 in T_2T_3 also makes it parallel to BC, since a sequence of two reflections along two perpendicular axes is equivalent to one rotation by 180°. Thus, L_1 is parallel to BC. Similarly, L_2 is parallel to AC and L_3 is parallel to AB. Note also that L_1 is opposite to A, L_2 is opposite to B, and L_3 is opposite to B. $\square$

In order to proceed, one introduces coordinates for points inside a triangle by their distance from the sides. Thus, a point P inside the triangle ABC will be denoted by $(x\,,y\,,z)$, where x, y, z are the distances from P to BC, AC, and AB, respectively (since the triangle is assumed to be acute, the question of signs does not arise). Note that P is uniquely determined by any two of $x\,,y\,,z$.

Remark 6.1. This parametrization corresponds to the usual barycentric coordinates since it is easy to show that, with h_1 the length of AH_1, etc., one has

$$P = \frac{x}{h_1}\,A + \frac{y}{h_2}\,B + \frac{z}{h_3}\,C\,, \quad \text{where} \quad \frac{x}{h_1} + \frac{y}{h_2} + \frac{z}{h_3} = 1\,.$$

The advantage of this parametrization is that it characterizes the centers of the inscribed and the circumscribed circles from knowledge of the sides alone.

Lemma 6.2. *Let the notation be as in the statement of Problem 6, and P a point in the triangle with coordinates (x, y, z).*

a) *If $(x, y, z) = (r, r, r)$, then P is the center and r the radius of the circle inscribed into the triangle.*
b) *If $(x, y, z) = (R\cos A\,, R\cos B\,, R\sin C)\,,$ where $A = \angle BAC$ etc., then P is the center and R the radius of the circle circumscribed around the triangle.*

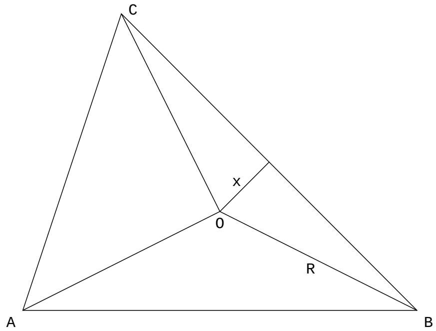
Proof. Part a) is trivial. To prove part b), let O be the center of the circumscribed circle. Since triangle ABC is acute, O lies inside the triangle, and $R = |OB|$ is the radius of the circumscribed circle. Let

$$\alpha \equiv \angle OBC = \angle OCB\,,$$

so that $x = R\,\sin\alpha$. Furthermore, let

$$\beta \equiv \angle OCA = \angle OAC \quad \text{and} \quad \gamma \equiv \angle OAB = \angle OBA\,.$$

Since $2\,(\alpha+\beta+\gamma) = 180°$ it follows that $\alpha = 90° - (\beta+\gamma)$. But also, one has $\beta + \gamma = \angle A$, so that $\sin\alpha = \cos A$. The result follows. $\square$



The final result is obtained directly from the following lemma.

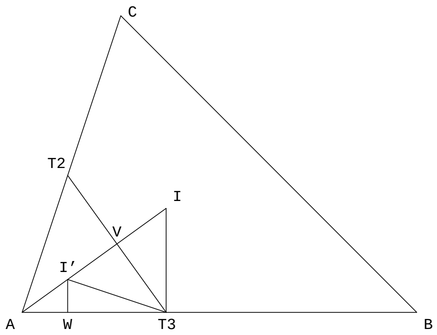
Lemma 6.3. *Let the notations be as in the statement of Problem 6, and let I' be the reflection of I, the center of the inscribed circle, in the line T_2T_3. Then I' is the center of the inscribed circle of AH_2H_3.*

Indeed, assume that Lemma 6.3 holds. Then the distance between I' and the line H_2H_3 is the radius of the inscribed circle of the triangle AH_2H_3. By Theorem 6.1c, this triangle is similar to the triangle ABC, so the radius of its inscribed circle is that of ABC multiplied by the factor of similarity. By Theorem 6.1c, the factor of similarity is

$$\frac{|AH_3|}{|AC|} = \cos A\,,$$

since CH_3 is an altitude. It follows that the distance from I' to the line H_2H_3 is $r\cos A$.

One now notes that, the distance from I', the reflection of I in T_2T_3, to the line H_2H_3 is exactly the same as the distance between I and the line L_1, the reflection of H_2H_3 in T_2T_3. One concludes that the distance from I to L_1 is $r\cos A$. Similarly, the distance from I to L_2 is $r\cos B$ and the distance from I to L_3 is $r\cos C$. Lemma 6.1 and Lemma 6.2 b show that I is the center of the circumscribed circle of the triangle with sides L_1, L_2, L_3, and that this circle has radius r. The result is therefore proved, given that Lemma 6.3 holds.



Proof of Lemma 6.3. In order to prove this, one uses the fact just proved above that the triangle AH_2H_3 has ratio of similarity $\cos A$ with respect to triangle ABC, so the radius of the inscribed circleis $r\cos A$, and then apply Lemma 6.2a.

One therefore needs to compute the distance from I' to AB. Thus, let W be the point on AB such that AWI' is a right angle. The axis of reflection T_2T_3 cuts the line AI perpendicularly, so that

$$r = |T_3I| = |T_3I'| \,.$$

It follows that

$$|I'W| = r\cos\big(\angle T_3I'W\big) .$$

Let $\delta = \angle BAI$. Since AWI' is a right angle,

$$\angle AI'W = \angle AIT_3 = 90^\circ - \delta \,.$$

Since $|T_3I| = |T_3I'|$, one also has $\angle T_3I'I = 90^\circ - \delta$. It follows that

$$\angle T_3I'W = 2\delta$$

and thus

$$|I'W| = r\cos(2\delta)\,.$$

By Theorem 6.1a, the line AI bisects $\angle A$, so it follows that $|I'W| = r\cos A$. Moreover, the fact that this line bisects $\angle BAC$ implies that the distance from I' to AC is also $r\cos A$. Since two distances to the sides determine a point uniquely and the radius of the inscribed circle is $r\cos A$, it follows that I' is the center of the inscribed circle, as claimed. $\square$

Remark 6.2. The above proof relies on two results: Lemma 6.1 and Lemma 6.3. The first is "soft" in the sense that it follows from known results without any computation. The second is the heart of the technical argument and appears to be of independent interest.

Remark 6.3. The above argument was discovered only after using a brute force algebraic method to prove the result. The following is a brief outline of this method.

Without loss of generality, one can assume that

$$A = (0\,,0)\,,\ B = (1\,,0)\,,\ C = (m\,,n),\ \text{where } 0 < m < 1 \text{ and } n > 0\,.$$

Furthermore, let

$$a = \sqrt{(1-m)^2 + n^2} \quad \text{and} \quad b = \sqrt{m^2 + n^2}$$

be the lengths of the sides opposite to A and B, respectively, and

$$p = 1 + b + c$$

the perimeter of the triangle.

Once again, let the center of the inscribed circle be I, then it is easily computed that I has coordinates

$$I\left(\tfrac{m+b}{p}\,,\,\tfrac{n}{p}\right), \quad \text{and so} \quad r = \tfrac{n}{p}\,.$$

If V is once again the intersection of T_2T_3 with AI, then

$$V = \tfrac{m+b}{2b}\, I\,.$$

The reflection of a point U in T_2T_3 is given by

$$U \mapsto U + 2\left(1 - \tfrac{U\cdot V}{\|V\|^2}\right) V\,.$$

Thus, if H_2' and H_3' are the images of H_2 and H_3 under a reflection in T_2T_3, then

$$H_2' = -\tfrac{m}{b}\, B + 2V \quad \text{and} \quad H_3' = -\tfrac{m}{b}\, C + 2V.$$

A computation shows that line L_1 is given by equation

$$y = \tfrac{n}{m-1}\, x + \tfrac{n}{bp\,(m-1)}\, \big(ma - b\,(b+1)\big)\,.$$

We now derive the equation of L_2 by using the symmetry $x \mapsto 1-x$, which provides $a \mapsto b$ and $m \mapsto 1-m$. It follows that L_2 is given by equation

$$y = \frac{n}{m}\, x - \frac{n}{m} - \frac{n}{apm}\big((1-m)b - a(a+1)\big)\,.$$

Fairly straightforward algebraic computation eventually yields the coordinates (x', y') of the intersection of L_1 and L_2, which are

$$x' = \frac{b^2(m-1)^2 - a^2m^2}{abp} + \frac{b+m}{p}\,, \quad y' = \frac{n\left(b^2(m-1) - a^2m\right)}{abp} + \frac{n}{p}\,.$$

Since

$$I = \left(\frac{b+m}{p}, \frac{n}{p}\right),$$

it is then easy to check that (x', y') is at distance $r = \frac{n}{p}$ from I. Since this holds for any two vertices, the result is proved.

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MY ROLE AS AN OUTSIDER

ILAN VARDI'S EPILOGUE

> *However, the majority of our contemporary academics disguise what is true with what is false and never surpass the limits of scientific fraud and posturing, using the science they have at their command only for vile material interest. And when they meet a man who sincerely searches for the truth, refusing falseness and lies, and rejecting ostentation and fraud, they take him for a fool and ridicule him.*

The above was written by Omar Khayyam (1048–1131) in his book *Treatise on Algebra*. In the intervening millenium things have certainly improved, but the phenomenon certainly persists, and nowhere is this more true than in the process of career advancement where admission to the best universities is a key factor. I have been continually amazed to see mathematicians who pride themselves on their rigour and objectivity, when it comes to their research, discard this mental discipline, when political advancement is concerned. Perhaps it is for this reason that many of the regretable stories recounted in this book do not surprise or shock me. And yet, these events are compelling because they feature a unique chapter in the history of mathematics – the only example of mathematics itself being used as a political tool. Indeed, much of the other excesses of the committees remind one of the shady dealings of many a hiring committee, the only difference being that in almost all such cases, victims are singled for purely personal reasons, whereas the events recounted in this book have a racial motivation affecting a large class of people.

It is the unique involvement of mathematics itself that made this book worth compiling and it is for this reason that it is worth reading. Indeed, I have just finished reading the article *Intellectual Genocide*[a] de-

[a]This essay written by B. Kanevsky and V. Senderov is published in Part 2 of the

scribing these events in detail, and its mathematical content convinces me more than ever that my work, as an outsider – I have never been to Russia and I don't speak a word of Russian – is an essential part of this story. The thing that made me realise this was the following problem given to Jewish students: it asks for the solutions of the equation $\sqrt{x+1}(4-x^2) = 0$, and one finds out that two students were cheated by the examiners using the trivial semantic point of the two wordings "-1 and 2" versus "-1 or 2". That professional mathematicians resorted to such tactics is shocking. But much more shocking to me is the fact that the solution $x = -2$ is never mentioned. I could not understand why this was, and it had to be explained to me by Editor Mikhail Shifman: In the period in which these problems were given, neither the professors nor the students would ever mention complex numbers in a problem solution, and a student would be immediately excluded if he did so. To me, it seems completely incredible that a professional mathematician would refuse to accept complex numbers as valid solutions to a problem. Confirmation that my position is far removed from that period is given by the fact that over the last 24 years, it had never occured to anyone reporting about the problems to even mention this supplementary solution (note that all these people were involved in that system in one way or another). And yet, -2 is also a solution, to paraphrase Galileo, an analogy which highlights the unscientific nature of the period. The title *Intellectual Genocide* is quite appropriate, but I believe that it should be applied to *all* parties involved.

The above example is far from isolated. The number of alternate solution not found by professors or students at the time, or by writers describing the events is quite significant. Moreover, the description of the difficulty of the problems is most often inaccurate. The description of some *killer problems* given in the article *Intellectual Genocide* appears to be largely incorrect; the only problems which are accurately appraised are the ones which the authors actually worked through completely. The moral of the story: If you want to understand the problems, then solve the problems, don't just talk about them. It should not come as too much of a surprise that many of the people who do the talking and not

present Collection. –Editor's note

the solving are the politicians. And this is exactly how I got interested in this project.

It is now five years since I completed my paper on the Mekh–Mat problems and it is finally appearing in book form. This would not have been possible without the tireless energy of Mikhail Shifman who was also able to include the original articles I drew on as well as some important personal accounts of the period in question, these previously only available in limited distribution as typewritten notes in Russian.

I first became aware of these problems while I was visiting the IHES[b] (one of the world's leading mathematics research institutes) in the summer of 1999. A visiting professor was attempting to prevent two Russian mathematicians from coming to the institute due to their involvement in anti-Semitic Mekh–Mat practices. Since I am not interested in politics but interested in mathematics, I was immediately intrigued by the problems themselves and began working on them continuously until I solved all 25 in the list in a period of six weeks. The paper is a result of this effort.

As for the political question, I was opposed to any attempt at retaliation, but referred the director to Anatoly Vershik who had written one of the original articles. He was also vehemently opposed to any action blocking the visits of the two Russian mathematicians, and this convinced the IHES administration to let them visit.

In subsequent years, my work on the Mekh–Mat Problems put me in the position of arbiter for such conflicts. On at least two occasions, I had to intervene on the behalf of Russian mathematicians whose visits to universities were in jeopardy due to their previous political activity. I am glad to say that my efforts were successful.

I hope that this will convince anyone looking at this book that its purpose is not retaliation against former wrongs. On the contrary, the authors of this work have been instrumental in protecting the rights of the perpetrators of wrongs described in this work. Even Kanevsky and Senderov, the authors of *Intellectual Genocide,* admit that their evidence is anectodal, so their work should only be taken as a record of

[b]Institut des Hautes Études Scientifiques in Bures-sür-Yvette (France), an institute of advanced research in mathematics and theoretical physics. –Editor's note

a highly regretable period in recent history.

Ilan Vardi

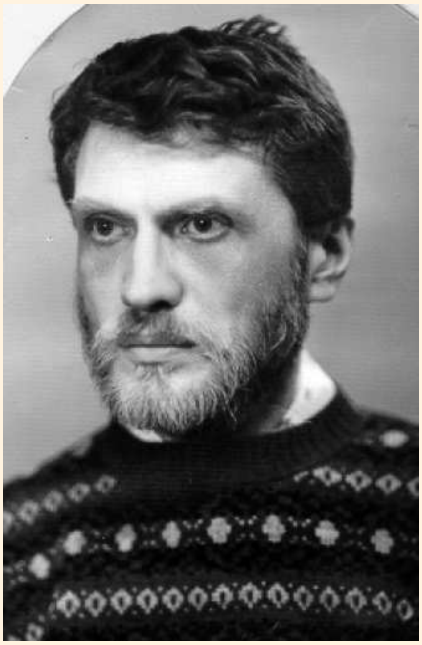
Paris, November 29, 2004

Part 2

Boris Kanevsky

Born in 1944. Graduated from Moscow University. In 1978-82 participated, with Senderov and Subbotovskaya, in Jewish People's University. Arrested in 1982 for "anti-Soviet slander." Released in 1985, in the very beginning of perestroika. Moved to Israel in 1987. Boris Kanevsky now lives near Jerusalem and teaches mathematics at a Jerusalem high school and Hebrew and Tel-Aviv Universities.

Valery Senderov
(photograph circa 1980)
Born in 1945. Graduated from Moscow Institute of Physics and Technology. Authored several dozen papers on functional analysis. In the 1980, together with Bella Subbotovskaya and Boris Kanevsky, was instrumental in the making of the "Jewish People's University." In 1982 was arrested for the anti-Soviet agitation and propaganda. The essay *Intellectual Genocide* the English translation of which is published below was used as an important part of incriminating evidence. He was found guilty and sentenced to seven years in hard-labor camps, with the subsequent five-year exile. Released in 1987 in connection with *perestroika.* Currently Valery Senderov lives in Moscow and continues his scientific and pedagogical activities in mathematics. He is also a free-lance writer on historical, philosophical and sociological topics. His articles appear on a regular basis in Novy Mir, Posev, Voprosy Philosofii, and other journals. A member of the International Human Rights Society.



The following article required extensive commentary. Footnotes were provided by the Editor, while comments on mathematical aspects, appended at the end of the article, were written by Ilan Vardi.

INTELLECTUAL GENOCIDE

Entrance Examinations for Jews at MGU, MFTI, MIFI[a]

B. I. KANEVSKY, V. A. SENDEROV

Second extended edition,[b] *1980*

> "... no cause can succeed without trampling a few delicate national flowers. But nothing in history was achieved without violence and implacable cruelty."
>
> Karl Marx and Friedrich Engels

Introduction

As far as Moscow is concerned, the most prestigious science and engineering institutions are the departments of natural sciences of Moscow University, the Moscow Institute for Physics and Technology and the Moscow Institute for Engineering and Physics. These institutes train a significant proportion of the country's researchers.

According to Soviet law every USSR citizen must have a passport[c] indicating his or her ethnic origin. If both parents are Jews, the passport must show "Jew" in the ethnic origin entry. Children of mixed marriages are allowed to choose either one of their parents' ethnic origin.

Admission committees require applicants to fill in forms stating their

[a] These acronyms stand for: MGU = Moscow State University (the most prestigious university in Russia), MFTI = the Moscow Institute for Physics and Technology and MIFI = Moscow Institute for Engineering and Physics. The latter two institutes are Russian analogues of MIT. – Editor's note

[b] This was a *samizdat* typewritten "publication." People retyped these publications using mechanical type-writers and carbon paper. Typewritten copies of this article obtained from various archives have slight differences. – Editor's note

[c] Soviet "passport" was in fact an internal ID card carrying extensive information such as home address, marital status, etc. Ethnicity — which in the Soviet parlance was referred to as "nationality" — was entry # 5. Entry # 5 became a euphemism for "Jewish." – Editor's note

ethnicity, but not those of their parents.[d] However, each applicant must submit his or her parent's first name, patronymic, family name, and place of work, allowing the admission committee to ascertain easily the ethnic origins of parents and grandparents.

In the present paper, any applicant having at least one Jewish grandparent will be referred to as a "Jew," following the standard admission committee practice, see Document No. 112 of the Moscow Helsinki group.

1 Mekhmat of MGU [Department of Mechanics and Mathematics]

In 1980, entrance examinations were held at the MFTI and MIFI in July, as usual. However, MGU examinations were were held in August and September, at the same time as other Moscow higher-education institutions, which was one reason why there was practically no competition for entrance to the Mekhmat among Muscovites. (The competition was held separately for applicants from Moscow and those from other cities because of space limitations in student hostels for non–Muscovites.) In fact, the number of applications practically coincided with the number of students to be admitted.

Among graduates of the leading Moscow physical and mathematical high schools [e] (there were more than 400 graduates from the best schools: No. 2, 7, 57, 91, and 179), not a single one with passport entry stamped "Jew" applied to the Mekhmat, accurately reflecting the hopelessness of this endeavor. However, some Jews with no identifying stamp — as a rule, recipients of awards and commendations for their physics and mathematics olympiads accomplishments — did apply hoping that, because of the lack of competition, they would not be subject to discriminatory selection. Also applying were Jews from less prestigious schools not as well connected to mathematical circles and where

[d] No rule is without exception. In 1980, MIFI applicants were required to indicate their parents' ethnic origin in an attached biography.

[e] In the 1960's special schools were established for gifted students in science and technology. They were allowed to transfer to a school or special class having an advanced curriculum in the area of their choice.

the scale of Mekhmat anti–Semitism was not fully understood.

Since the total number of Jews from physics and mathematics high schools applying to Mekhmat was not large it is not reasonable to compile tables and perform statistical analysis — the sample is too small. Instead, we will consider several Jewish applicants' "Mekhmat entrance examinations experiences" which should provide a clear picture of what went on.

That year, eight graduates of mathematics school No. 91 applied to the Mekhmat: Andreeva, Ermolaeva, Kontsevich, Pantaev were Russians, and Grilli, Gundobin, Krichevskii, Lavrovskii were Jews. The first four were admitted. The treatment of the four others was openly discriminatory. Krichevskii and Lavrovskii got a '2' in the first exam,[f] a written test, see below. Gundobin got a '2' in the second exam, an oral test. One of the problems given to him was to prove a complicated theorem not in the school curriculum, see problem # 26 from the *samizdat* compilation "Selected Problems from Oral Math Examinations, Mekhmat MGU, 1980." Every mathematician knows this theorem; therefore, the possibility that his examiners suggested it by mistake can be completely ruled out. The fourth Jew, Grilli, got a '2' in composition.[g] The grader's comment was standard for such cases: "The theme is not worked out." In the course of his appeal Grilli was denied access to his composition and the examiner's remarks. With no examiners' remarks in his hands, Grilli could provide no arguments in his favor to the Departmental Committee. Subsequent complaints were all in vain.

[f] The grading system in Russia is based on a five-point scale. The highest grade is 5 (corresponding to an American A), the next is 4 (corresponding to B), and so on. A '2' means that the student failed, barring him from any further exam. The "barely passed" level starts at '3'. –Editor's note

[g] Entrance examinations to Soviet institution always included a literary essay on topics suggested by the admissions committee. Usually, these topics were ideologically charged. –Editor's note

Written exam

Five problems were given and graded on a "pure plus system" meaning that if, in the examiners' opinion, a solution had any flaw, then it would not be counted. Two problems marked by pure plusses guaranteed a passing grade ('3' or higher).

Jewish applicants aware of this grading system would often select the two or three simplest problems and spend all the examination time trying to work out perfect solutions. Let us have a closer look, how far this strategy could go.

Sergei Krichevskii

In 1978, Krichevskii was one of the winners of the Moscow Mathematical Olympiad and of the Moscow branch of the All–Union Mathematical Olympiad. He solved three problems. We will limit our discussion to two of them as we do not have full information regarding the third one. Note again that two pure pluses would give Krichevskii a passing grade allowing him to get to the oral exam.

Problem 1.

Solve the equation $\sqrt{x+1}(4-x^2)=0\,.$

Krichevskii's solution[h] that $x=-1$ or $x=2$ was declared flawed.[i] Another flaw was writing the answer as $\{x\}=\{-1;2\}$.

[h] In some other works, for example, in that of Trutnev, see below, writing $x=-1$ and $x=2$ was declared incorrect.

[i] The reader may be surprised to know why neither Krichevskii nor Kanevsky and Senderov mention the solution $x=-2$. The Soviet high school curriculum at that time defined the function $\sqrt{x}$ as a function well-defined only for non-negative real values of x. Those students who knew that, in fact, it could be defined in the complex plane, were supposed to conceal their knowledge. Anybody who answered at the entrance examination to any university in the USSR that $\sqrt{x+1}=\pm i$ at $x=-2$ would be given a failing grade right away, and be barred from further examinations with no possibility of appeal. High school students were taught that in solving the equation $(4-x^2)\sqrt{x+1}=0$ the first thing to do was to write down that "the domain of definition" is $x\geq -1$. Formally, the examination requirements all over the country were standardized. –Editor's note

Problem 2.

Solve the equation $\sin 2x = \sqrt{3}\sin x$.

The only "flaw" was a slip of the pen — "n" instead of "π".

One of Krichevskii's "flaws" in solving the third problem was that he denoted an angle using the symbol "$\angle$" which the examiners told him was incomprehensible. In his appeal Krichevskii wrote that examiners not understanding school textbook notation was not his fault. This appeal was not accepted for consideration as it showed "contempt for the examiners."

Krichevskii's grade of '2' was left unchanged. "Even if you were right we would not have given you '3' anyway," they "consoled" Krichevskii.

Dmitrii Lavrovskii

Problem 1.

Solve the equation $\sqrt{2-x}\,(9-x^2)=0$.

Lavrovskii divided the solution of the problem into parts. In the first part, he defined the domain of admissible values $x \leq 2$. In the second part, Lavrovskii passed to an equivalent (within the domain) set of equations $9-x^2=0$ or $2-x=0$, which the examiners called a mistake.

A remark of a similar nature was made regarding the solution to the second problem. In the appeal, the examiners removed the remark, yet did not accept Lavrovskii's objection to the remarks pertaining to the first problem's solution. The '2' grade was left unchanged.

Apparently, the tactic of perfect solutions to two or three problems does not bring success. As will be seen, solving all the problems does not help undesirable applicants either.

Alexander Trutnev[j]

Trutnev solved all five problems yet the grade for his work was '2'. Trutnev's appeal was rejected on the grounds that there were no two "cleanly

[j] Based on his score in the 1980 MGU Mathematical Olympiad, Trutnev received a personal invitation to study at the Mekhmat.

solved" problems in his examination work. A typical reason for refusal was given by Professor A.S. Mishchenko, the Chair of the Appeals Committee, who, in considering the first problem whose answer was written: "$x = -1$ and $x = 2$", stated that this was a gross error because "and" does not have the same meaning as "or", i.e., had Alexander written the answer as "$x = -1$ or $x = 2$", the solution would have counted. Note that in the Krichevskii case above, the same Professor Mishchenko told the applicant that one was not supposed to use the word "or" here.

Oral exam

A number of techniques for "evaluating knowledge" were applied to Jewish applicants but one can single out two main ones. At the start is an exhausting two hour long cross–examination and during the third, fourth, and fifth hours of examination are given the so–called killer problems. Not more than 20 minutes is allocated to solving each problem; we are aware of no case when a student got more than half an hour to prepare an answer. This resulted in the failure of even the most capable and best-prepared of the undesired candidates; occasions when students got a '3' are rare. (We know of one exception to this rule this year.)

Let us now turn to oral examination stories.

Igor Averbakh

The year he applied, Averbakh had graduated with a gold medal from Chelyabinsk school No. 121. In the previous four years Igor had been winner of the Chelyabinsk Mathematical Olympiad; in the 8th and 9th grades he had participated in the All–Union Mathematical Olympiads; in 1978, he was one of the winners of the All–Russia and All–Union Mathematical Olympiads. In July of 1980, *he passed* the entrance examinations at the MFTI *with all 5's* but *was not admitted.*

"Grades received at entrance examinations do not play a decisive role for admission to our Institute" — from the MFTI prospectus.

Igor Averbakh applied to Moscow University and in August tried to pass the Mekhmat entrance examinations. His grade in the written math test was '3'.

His grade in oral math was also '3'. His examiners were Filimonov and Proshkin. The problems given at his oral examination were:

Problem 1.

The equation $x^2 - A = |x - B|$ is given.

For any non–negative integer n, find a set M_n of pairs (A, B) of parameters, for which the equation considered has exactly n solutions. Plot these sets on the (A, B) plane.[k]

Problem 2.

Solve the equation

$$\sin^3 x \cos\frac{x}{2} + \frac{1}{2}\sin x \sin\frac{x}{2}\left(1 + 2\cos\frac{x}{2}\right) = 1 + 6\sin^2\frac{x}{2}\ .$$

Problem 3.

A point K on the base AB of the trapezoid $ABCD$ is given. Find a point M on CD such that the area of a quadrangle formed by the intersection of triangles AMB and CDK will be the smallest.

A version of the latter problem — with the question of finding the the largest area — was given to 9th and 10th grade students at a special round of the 1973 MFTI Olympiad for special physics and mathematical schools of Moscow, where six problems were given in five hours. This same problem was given to 10th graders in the final round of the 1978 Kiev Mathematical Olympiad. There five problems were given in four hours. Only two students solved the problem at the Kiev Olympiad.

[k] In 1979, this problem was suggested to applicant Leonid Polterovich at the oral exam. Polterovich submitted an appeal, in which he complained, in particular, about the cumbersome nature of the solution to the problem. In considering the appeal, examiner Vavilov, trying to prove the pertinence of this problem for an oral math examination, got confused in the derivation of the solution. As a result, the Appeal Committee qualified this problem as unfit for oral examinations. Leonid Polterovich was admitted to the Mekhmat after numerous complaints and appeals.

The two versions of the problem are approximately equal in complexity level. Two solutions of Problem 3 obtained by the present authors use the largest area problem as a lemma.

After the examination, Igor Averbakh wrote the following appeal.

"At the oral examination in mathematics, I gave full and correct answers to both required questions as well as to several supplementary questions. (Three or four of them were recorded.) Two hours after receiving the required questions, I got three additional problems. I ran out of time on the first problem, the solution of which is very time consuming, and was stopped by the examiner. I correctly solved the second problem and got "plus" for it. I was then given a third problem, after the three hour long examination session. The third problem was definitely at Olympiad level, for example, it was given at the 1978 Kiev Mathematical Olympiad where it was the most difficult, in judges' opinion. Thus, I was forced to solve an olympiad problem under the condition of extreme fatigue, which I informed my examiner. The examination was conducted with gross violations of the Directive Letter No. 21 issued by the Ministry of Higher Education of the USSR on May 22, 1980, as well as of the internal rules of Mekhmat:

1) *Many additional questions that I was asked, were not recorded by the examiners, despite the guidelines of the above Directive Letter.*

2) *The examination session lasted longer than 3.5 hours.*

3) *Despite the Mekhmat rules (see "Examiner's Memo"), I was given an Olympiad problem, and this after a three hour session.*

The standards applied to my written work are beyond the level of school coursebooks. Many examiners' remarks were not correct in their essence. I consider my grades for the written and oral math examinations to be unfairly low and request the revision of both grades.

08.27.1980 Averbakh."

Both this and all subsequent complaints were declined. See below on how complaints are considered by the Mekhmat.

Dilyara Vegrina

In 1979 Dilyara graduated from Mathematical School No. 2, the best in Moscow, with all 5's in mathematics. In July 1979 she made an attempt to enter MGU's Mekhmat. Although Vegrina received prizes at the Moscow Mathematical Olympiads in the eighth and ninth grades, and in the tenth grade she was awarded a prize at the All-Union Mathematical Olympiad, she got a '2' at the very first examination (written mathematics). Her appeals and complaints proved to be in vain.

In August of the same year, Dilyara Vegrina applied to the Moscow Institute of Electronic Machine Building. Her grades at the entrance examinations were: written mathematics – '3'; oral mathematics – '3'; oral physics – '2'. Upon appeal, the grade was left unchanged.

In September of 1979, she enrolled in the correspondence school section of the Kalinin University Mathematics Department[l] and completed the two year curriculum in one year. In August of 1980, Vegrina made a second attempt to enter MGU's Mekhmat. Her written math grade was '3'.

Here are the problems Vegrina was given in her mathematics oral exam with examiners Pobedrya and Proshkin:

Problem 1.

Solve the inequality $3^y \log_3(9-x^2) \leq 1 + 3^{2y}$.

Problem 2.

The same as Problem 1 given to Averbakh.

Problem 3.

Can a plane intersect a trihedral angle in such a way that the cross section is an equilateral triangle? [m]

[l] Currently, the Tver University. –Editor's note

[m] This is one of the problems recommended by the Central Organizing Committee of the 1976 All–Union Olympiad for its third round. There, four hours were allocated for solving five problems. The concept of trihedral angle is absent from the *The Entrance Examinations Program for Applicants to USSR Higher Education Institutions*

Problem 4.

Solve the equation $\cot x = \sin(x + \frac{\pi}{4})$, where $0 < x < \pi$.

Problem 5.

A function $f(x)$ is continuous on the interval $[0, 1]$. Numbers A and B are such that $0 < A \le f(x) \le B$. Prove the inequality[n]

$$AB \int_0^1 \frac{dx}{f(x)} \le A + B - \int_0^1 f(x)dx\,.$$

Vegrina solved Problems 1 and 4. She was not allowed to finish Problem 2. Her math skills were evaluated as unsatisfactory, and she received a '2'.

"To the Appeal Committee of Mekhmat of MGU from the applicant D. Vegrina, Examination registration 110404 (110464?)

Statement

I request a change in my oral math examination grade. I answered all required examination questions and two of five additional problems were evaluated as solved by my examiner. The third additional problem was in fact recommended for the National Mathematical Olympiad while the second and fifth problems are also of Olympiad level. My examination session lasted for more than five hours (from 1:20 p.m. till 6:50 p.m.). At the end of the session, I was unable to answer examiner's question.

I consider my grade '2' — not passed —as unfair since four questions out of seven were answered correctly, and that the length of the examination session is obviously not within reasonable human endurance.

Vegrina"

in 1980.

[n] Solving this problem requires integration of the inequality. The school curriculum does not include integration of inequalities.

This letter as well as further appeals failed to change the examination result. Vegrina's appeals will be considered in more detail in a subsequent section.

Mikhail Temchin

Based on results of the 1980 Mekhmat Olympiad, Temchin received a personal invitation to the Mekhmat. He got a '3' in the written math examination after having solved four problems. The "flaw" in the first problem coincided with the one attributed to Lavrovskii in a corresponding problem (see above). The "flaw" in the second problem was the "unsubstantiated inequality" $\frac{\sqrt{6}}{2} > 1$ used by Temchin in his solution.

The oral examination started at 10 a.m. and, after 40 minutes of preparation, Temchin asked for permission to start giving his solutions. One of the examiners agreed but, after a closer look at the examination registration, asked for a delay. This procedure was repeated several times and only at 1 a.m. was Temchin called to answer, i.e., three hours after the beginning of the examination session. Temchin's answers to the required questions were the subject of some interesting remarks by the examiners. For example, he used the term "asymptote" in answering a question on the properties of the function $y = \frac{k}{x}$. This incited the following remark from the examiner: "Don't fool me!" The definition of an angle routinely found in school textbooks was first declared completely wrong but was accepted after some clarification. After all required questions were answered, the examiners told Temchin that his answers "were right but only formally." (What this means is unclear.)

We present here the supplementary problems given to Temchin at his oral examination.

Problem 1.

A function $f(x)$ is given, such that $f(0)=0$; $f(1)=1$; $f(88)=\sqrt{2}$. Prove that there exists a natural number k and real numbers x and y

such that the inequalities

$$\begin{cases} |x-y| \le 4\,, \\ \big(f(x+1)-f(x)\big)\,\big|f(y+2^k)-f(y)\big| > 0\,. \end{cases}$$

are valid.

Problem 2.

Identical to Problem # 2 given to Averbakh.

Problem 3.

Let P be an arbitrary point P inside an equilateral hexagon of unit side. Prove that the sum of distances from P to all corners is not greater than $2/(2-\sqrt{3})$.

Temchin did not solve the supplementary problems, and got the grade '2'.

When examiners hesitate to pass, the standard practice is to let last question determine the grade: '3' (barely passed) or '2' (not passed). This means that Problem # 3 given to Temchin should be considered a "barely passed/not passed" question. Let us dwell on this problem in more detail. Its solution requires the following:

1. Consideration of a function of two variables — let us denote it as $f(x,y)$.
2. Reducing it to a function of one variable, $\varphi_{x_0}(y) = f(x_0\,,y)$.
3. Investigation of the function $\varphi(y)$
   a) either with the aid of the second derivative,
   b) or using convexity of this function.

It is practically impossible for this problem to be solved by a student at the entrance examination for two reasons. First, all concepts necessary in point 3 of the solution are absent from the High School curriculum; it contains neither function analysis based on second derivatives, nor the concept of convex function; the very concept of partial derivative is not in the curriculum.

The second reason is as follows. The line of reasoning $1 \to 3$a or $1 \to 3$b, standard in calculus, is not in spirit of High School mathematics; nor does it belong there ideologically. Only the rudiments of calculus are studied in High School and derivations similar to that outlined above are unfamiliar to the students.

The exam took place on August 26. On August 28 Temchin's father died. Appeals of the oral examination results failed (see below).

Dmitrii Markhashov

Dmitrii Markhashov graduated from Moscow High School No. 21. His written math grade was '3' and his oral grade was '2'. After Markhashov successfully answered all required oral examination questions, the following additional problems were given to him.

Problem 1.

A point O is on the base of the triangular pyramid $MABC$. Prove that the sum of angles formed by the line OM with the edges MA, MB, and MC, is smaller than the sum of the plane angles at the top M, and greater than half this sum.[o]

Problem 2.

Solve the equation $(\sin x)^{\frac{11}{7}} + (\cos x)^{\frac{19}{11}} = \sqrt{\frac{19}{7}}$.

Problem 3.

Prove the inequality $\sqrt[3]{3 - \sqrt[3]{3}} + \sqrt[3]{3 + \sqrt[3]{3}} < 2\sqrt[3]{3}$.

[o]The first inequality in this problem can be found in the book by L. M. Lopovok, *Compilation of Problems in Stereometry*, (Moscow, Uchpedgiz, 1959), Problem # 261*. This is a problem "with asterisk", which means of increased difficulty. Immediately after the formulation of the problem in the above book, there is a hint which is in essence a full proof of this inequality. It is based on properties of plane angles of a trihedral angle. As has been already mentioned, the concept of the trihedral angle is not in the The Entrance Examinations Program.

Let us consider the last problem in more detail. It should be reminded that it was considered as a "barely passed/not passed" threshold problem — see comments to Temchin's problems.

Solution 1. The problem's assertion follows directly from convexity (upward) of the function $f(x) = \sqrt[3]{x}$,

$$\frac{f(3+\sqrt[3]{3}) + f(3-\sqrt[3]{3})}{2} < f\left(\frac{(3+\sqrt[3]{3}) + (3-\sqrt[3]{3})}{2}\right).$$

However, since the concept of convex functions is not taught in High Schools, let us consider other solutions.

Solution 2. Using the derivative, it is seen that the function

$$f(x) = \sqrt[3]{3+x} + \sqrt[3]{3-x}$$

attains its maximum value at $x=0$ and only at this point. Consequently, $f(0) > f(\sqrt[3]{3})$.

Regarding this solution, we note that proving inequalities using derivatives is not included in the school curriculum. Recall that applicants are familiar only with the rudiments of calculus. Therefore, discovering the above technique in twenty minutes during an examination seems unreasonable.

Solution 3. Let us rewrite the inequality to be proved in form

$$\sqrt[3]{1+\frac{\sqrt[3]{3}}{3}} + \sqrt[3]{1-\frac{\sqrt[3]{3}}{3}} < 2\,.$$

The inequality

$$\sqrt[3]{1+x} < 1 + \frac{x}{3}$$

is seen to hold for $x > -9$. Using this inequality twice, we derive the assertion of the problem.

From a formal point of view, this solution only uses elementary techniques. However, from an ideological standpoint, it is more complicated, to our mind, than the first two solutions. Namely, this solution requires exploiting an auxiliary inequality $\sqrt[3]{1+x} < 1 + \frac{x}{3}$. The most straightforward way to arrive at this auxiliary inequality is using Taylor's expansion so the simplicity of the means is illusory.

Solution 4. Let us denote

$$x_0 = \sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}}.$$

We have

$$x_0^3 = 6 + 3\sqrt[3]{9-\sqrt[3]{9}}\,x_0 .$$

Thus, x_0 is a root of the polynomial

$$\mathcal{P}_3(x) = x^3 - 3\sqrt[3]{9-\sqrt[3]{9}}x - 6 .$$

But

$$\mathcal{P}_3(0) < 0 , \quad \mathcal{P}_3'(0) < 0 .$$

Therefore, to prove the inequality $x_0 < 2\sqrt[3]{3}$, it is sufficient to prove that $\mathcal{P}_3(2\sqrt[3]{3}) > 0$. The latter can be directly verified.

The idea to use a polynomial to prove an inequality is undoubtedly nontrivial. It certainly is not in the High School curriculum. First, the solution under consideration is based on this idea and, second, it uses methods of calculus. As for the latter, see comments to Solution 2. (Such algebraic considerations associated with Vieta's formulas are hindered by the fact that the Vieta formulas are not studied in High School.)

For these reasons, expecting such a solution in an examination and in no more than twenty minutes is unreasonable.

Solution 5.

Lemma

The numbers $(a+b+c)$, where at least two terms are unequal, and $(a^3+b^3+c^3-3abc)$ are of the same sign.

Proof

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc) .$$

It is easy to see that the polynomial in the right hand brackets of the right hand side of this equation is always positive (except the trivial case $a = b = c$), proving the lemma.

Solution

By the Lemma, the inequality from the problem is equivalent to

$$6 - 24 + 3\sqrt[3]{9 - \sqrt[3]{9}} \cdot 2 \cdot \sqrt[3]{3} < 0\,,$$

or

$$3 > \sqrt[3]{3} \cdot \sqrt[3]{9 - \sqrt[3]{9}}\,, \quad \text{i.e.}$$

$$27 > 27 - 3 \cdot \sqrt[3]{9}\,.$$

Quod Erat Demonstrandum. The complexity of the above solution requires no comment.

Solution 6.

Lemma

Let $a + b \geq 2\,,\ a \neq 1\,.$ Then $a^3 + b^3 > 2$.

Proof

Let $a = 1 + t$. Then $b \geq 1 - t$. Consequently,

$$a^3 + b^3 \geq (1 + t)^3 + (1 - t)^3 > 2\,.$$

The lemma is proved.

Solution

We rewrite the inequality as

$$\sqrt[3]{1 + \frac{\sqrt[3]{3}}{3}} + \sqrt[3]{1 - \frac{\sqrt[3]{3}}{3}} < 2\,.$$

Let $\sqrt[3]{1 + \dfrac{\sqrt[3]{3}}{3}} = a$, $\sqrt[3]{1 - \frac{\sqrt[3]{3}}{3}} = b$. Let us assume that $a + b \geq 2\,.$ Then, by the lemma, $a^3 + b^3 > 2$. But $a^3 + b^3 = 2$. The inequality is proved.

We make only one comment regarding this solution: It seems easy to single out the number 2 as a "reference number" which, in turn, implies the above lemma, based on convexity considerations on the function $f(x) = x^3$ for $x > 0$. See our comments to the first solution.

One can certainly continue searching for further artificial solutions. However, what we have already written seems sufficient to characterize the nature of this "barely passed/not passed" threshold problem.

"To the Appeals Committee of the MGU Mekhmat
from applicant D. Markhashov,
Examination registration 120341

Statement

I request reconsideration of my oral math examination results. I answered fully and correctly all required questions and extra questions were raised en route. *Neither the extra questions nor my answers were recorded by the examiners, which is a violation of Directive Letter No. 21 dated May 22, 1980, which establishes a procedure for entrance examinations to higher education institutions. My examination session lasted longer than 4 hours and, as a result, I was very tired by the end.*

After solving the required problems, I was given three supplementary problems, which I did not solve. It has to be noted that the difficulty level of these problems is significantly higher than that of problems usually proposed at MGU oral math examinations, see, for example, the book by A. Tooma, V. Gutenmakher, N. Vasil'ev, and E. Rabbot.

The first problem given to me is listed in Lopovok's problem compilation as a problem of increased difficulty, where its full solution is given in the form of a note. The last problem given to me is based on the theory of convex functions, which is beyond the scope of the High School curriculum. During the examination session, the examiner rudely told me that I was thinking too slowly, which broke my concentration exactly when most needed to solve these difficult problems.

Therefore, I consider my grade unfairly low.

08.27.1980. Markhashov"

This written appeal and all subsequent ones submitted by Markhashov were rejected. But when, by the order of the Central Admission Committee (CAC), the Mekhmat provided Markhashov with solutions to the three supplementary problems he had been given during the session, Dmitrii discovered errors in two out of the three Mekhmat solutions, and informed CAC of these errors.

Two things then happened: first, the examiners told Markhashov that their mistakes were inconsequential and second, they changed Markhashov's oral math grade to '3–'.

The next day was devoted to composition for which Markhashov got a '4'. The following day, he took physics examination for which he got a '2'. All appeals proved futile.

For comparison, we present a set of problems for which a "desirable" applicant to the Mekhmat received a '5' at the 1980 oral math examination.

Problem 1.

Prove the inequality $\sin x + \cos x \leq \sqrt{2}\,.$

Problem 2.

Simplify the expression

$$\frac{1}{\frac{1}{\log_a x} + \frac{1}{\log_b x}}\,.$$

Problem 3.

Plot y vs. x where $y(x)$ is given by

$$|y| + |x - 1| = 1\,.$$

Problem 4.

Solve the equation

$$\cos x = \cos 2x\,.$$

Despite all this, some Jewish students still managed to pass examinations with all '3's. Usually, this outcome was welcome by the Mekhmat Admission Committee because these applicants would find themselves at the bottom of the list and would not be admitted due to the limit on total number of admissions. However, in 1980, there was *no* competition among Muscovites applying to the Mekhmat. Nevertheless, the Admission Committee set the passing total at 18.5, in violation of the principle of competitive selection.

Igor Averbakh, whom we have already mentioned, registered himself using his uncle's Moscow address and managed to take the examination competing within the Muscovite quota. He was not admitted to the Mekhmat. The Admission Committee announced to his father that part of Moscow's quota had suddenly been reserved for contestants originating from peripheral Soviet republics. Later on, evidence surfaced that 30 places of the total Mekhmat enrollment were transferred by the Mekhmat to other departments which could then recruit students having failed the competitive selection in these departments.

The discrimination against Jewish applicants at the Mekhmat cannot be concealed. At times, it emerges with comic spontaneity. For example, senior examiner, Professor A. S. Mishchenko, told S. Krichevskii (see above) that he — Krichevskii — *had acted discourteously*, by stressing in his appeals exactly those examiner's comments where he — Krichevskii — was *most certainly right* (!).

However, the Admission Committee carefully covers up all traces of attempts to prevent the dissemination of information by removing or destroying written evidence.

For instance, on August 25, the Deputy Secretary in charge of the Admission Committee, one Tatarinov, forbade applicants to talk to each other while writing appeals regarding the written mathematics examination. (They were writing about their answers and trying to compare their grades.) The appeals themselves were subject to double *censorship* which was disguised as "reviewing." At first, *a member of the examination commission*, who remains unknown, *made decisions as to whether to let* an appeal *through to Tatarinov for a review.* Then there was Tatarinov's review: *he selected texts that he did not like* suggesting that applicants rewrite them, "if they want to complain." Tatarinov announced to applicants that *he could bar any appeal he disliked from being considered.*

The same is true for the oral examination. Complaints about examiner rudeness, unfairly low grades, violations of rules and instructions, had no chances of being considered by the Appeals Committee and, thus, make it into the applicant's personal file. Sure enough, D. Markhashov and I. Averbakh's complaints discussed above were not even accepted for consideration. Averbakh's appeal provoked *the rage of Professor A. S. Mishchenko*, who tried to intimidate the applicant by aggressively asking how he dared to refer to the Ministry's Directive Letter, and demanding to know who made the *Examiners' Memo* available to him, thereby committing malfeasance (!). In other words, the Admission Committee treats the existing rules and provisions on the entrance examination procedure as "classified information," hiding them from applicants.

Consideration of complaints by appeal committees

The MGU's Admission Committee introduced a rule, according to which appeals on written and oral math examinations had to be submitted within one hour of the end of the examination session. We will not dwell on this rule requiring exhausted applicants, to compose a well thought document immediately after their oral exam session — a hard task in itself. Instead, let us turn to the appeals process *per se*. The simplest way is again to present examples.

I. Averbakh

The first appeal. "No, it is not '3"' — Professor Mishchenko exclaims after reading Igor's complaint. Then, after a pause. — "It is '2'! The examiner who gave you '3' for such an answer deserves a reprimand. Give me your examination form!" Mishchenko takes Averbakh's examination form and is about to cross out '3' and replace it with '2'. Someone from the Appeals Committee says: "We should discuss this first." They ask the applicant to leave the room. After approximately 10 minutes, they call him back and inform him that they decided not to change the grade '3' to '2' by a one vote margin.

The second appeal (to the Central Admission Committee). Among the members are the Mekhmat Dean O. B. Lupanov, Professor

A. S. Mishchenko, already mentioned, and others.

They ask Averbakh: "Is it possible to solve the problems given to you by methods included in the school curriculum?" "It is possible" — answers Igor. "Then what are you complaining about?" They would not, however, listen to what Averbakh had complained about.

D. Vegrina

The text of her appeal was quoted above. In response to this appeal, Dilyara was informed that she was supposed to solve problems, and she could not. As for the duration of the examination session: "The longer, the better for you," they told her, "you have more time to think." This ended consideration of Vegrina's appeal. [p]

A. Trutnev

Quite often, appeals to the Central Appeals Committee (CAC), to which applicants' complaints about Mekhmat's Appeal Committee were sent, were actually given to Mekhmat representatives for consideration. For instance, Professor Sadovnichii, a Mekhmat faculty member,[q] (conceal-

[p] As a correspondence student of Kalinin University, as mentioned above, D. Vegrina tried to transfer to the Mekhmat leading to the following events. On August 28, Mel'nikov (Associate Dean?) told Dilyara's father that she had no right to apply to Moscow University. Let us note that barring a student of a higher educational institution to apply to Moscow University is not mentioned anywhere in the admission rules. "Then," continued Mel'nikov, "we will investigate whether her documents are fake and whether a criminal case should be opened on the charge of possible falsification of documents." They never followed up with this, however. They "simply" refused to transfer Dilyara.

[q] Currently, Professor Viktor Sadovnichii is the Rector of Moscow University. In a conversation which took place in 2003, Valery Senderov told me that, in his mind, Professor Sadovnichii was not an ideological anti-Semite, but rather used the prevailing currents to propel himself to higher positions. It is curious to note that on December 9 2004, the Russian media outlet *Echo Moskvy* reported that Professor Sadovnichii vehemently opposed the idea of admission to Moscow University based on the results of a standardized test analogous to SAT. "We must protect a multivariant admission procedure in our University so that no talented kid is left behind. We should look for talents, in particular, at Olympiads and scholarly competitions," declared Professor Sadovnichii. –Editor's note

ing his position at Mekhmat) told Trutnev's parents that "CAC has complete trust in Mekhmat experts, the more so as CAC *per se* is not competent in such matters". Sadovnichii suggested that Trutnev should send his appeal back to Mekhmat (recall that this appeal contained complaints about Mekhmat actions).

M. Temchin

The case of applicant M. Temchin was considered by the CAC as a special case. Since Temchin's father died two days after he received '2' on the oral math examination, he could not appeal within the allowed time.

In early September, when he addressed the CAC with a request to reconsider the result of his oral math examination, he was told that, due to special circumstances and despite the late complaint, his case would be reopened. They notified M. Temchin to appear a week later. The next week, they repeated this notification, and when he came to the CAC for the third time, he was informed that his appeal was denied as it was submitted past the deadline.

Non–University Levels

Information for 1980, the year in which this document was compiled, is quite scarce so far because the bureaucratic details drag on for months. However, experience of previous years indicates that appeals and complaints regarding the Mekhmat's actions are routinely handled by forwarding them back to the Mekhmat.

This year everything is being repeated: both bureaucratic red tape in response to complaints as well as the responses themselves : "Your statement is forwarded to the MGU for consideration. MGU will inform the complainant of the results."

D. Vegrina and Trutnev's mother have already received such formal replies. It took the Ministry of Higher and Secondary Special Education of the USSR about a month to give the very same reply to Trutneva.

2 MIFI and MFTI

The number of Jewish applicants to the MIFI and MFTI is sufficient large that an illustrative table can be made.[r]

The graduates of schools No. 2, 7, 57, 91, and 179, who applied to the MFTI and MIFI in 1980, were divided by the authors of this document into two groups.

Group 1 consists of non–Jewish applicants, and Group 2 of Jewish applicants.

In 1980, of the 83 graduates from the above schools applying to the MIFI, 54 were non–Jewish (Group 1) and 29 Jewish (Group 2). Of the former 54, there were 36 admitted and of the latter 29 only 3 were admitted.

In 1980, of the 85 graduates from the above schools applying to the MFTI, 53 were in Group 1 and 32 in Group 2. Of the former 53 there were 39 admitted, and from the latter 32, there were 4 admitted.

Thus, the 107 applicants to both institutes from Group 1 yielded 75 admissions, while there were only 7 admissions from the 61 of Group 2. Applying standard methods of mathematical statistics shows that the probability of bias against Jewish applicants exceeds a 99.9% confidence level.

We should note that 3 out of 7 Jews admitted to these institutes were close relatives of scientists working there.

We further note that we encountered natural technical difficulties while gathering information on the MFTI and MIFI (which, in combination have more than 10 departments). Therefore, our table may contain some errors. However, in every case where our information was incomplete we gave the benefit of the doubt to the Admission Committee of the corresponding Institute. For example, we assigned all failed applicants on whom we did not have precise data to Group 1.

We also considered the Medical Evaluation Committees of the MFTI and MIFI as one of the elements of the selection process, subject to the

[r]The table containing the names of the 182 graduates, their ethnic origins and examination grades is appended to the Russian original of this article. We omit it in the English translation. – Editor's note

same statistical regularities as the Examination Committees. Therefore, applicants who were rejected on grounds of their medical status are also included in our table.

In column 5, the plus sign indicates that the applicant was admitted, with the minus sign signifying the contrary. Columns 3 and 4 indicate the applicant's ethnicity. A '2' in column 3 of the Table means that the ethnicity entry in the applicant's passport is "Jew", while '0' is entered in all other cases.

In column 4 the notation is as follows: '0' indicates the absence of Jewish grandparents, 1/2 stands for exactly one grandparent Jewish, 1 for exactly one parent Jewish, 2 for both parents Jewish.

Moscow, 1980

Translated from Russian by
Nodira Dadabayeva and Alexei Kobrinskii

REMARKS

ILAN VARDI

Page 5, Problem 1 given to Krichevskii: As often happens in these accounts, everyone overlooked an additional solution, namely $x = -2$. Apparently, it was assumed that only real numbers could be involved, since the solution $x = -2$ gives rise to the complex number $\sqrt{-1}$. In any case, the fact that the examiners, who were professional mathematicians, did not use this mathematical technicality to trap applicants instead of arguing trivial semantics provides a good insight into their lack of mathematical culture.

Page 8, Problem 1 given to Averbakh: The statement of this problem is somewhat confusing. Apparently, for each of $n = 0, 1, 2, \ldots$, one must generate pairs (A_k, B_k), $k = 1, \ldots, m$, such that the total number of solutions to

$$x^2 - A_k = |x - B_k| \,, \quad k = 1, \ldots, m \,,$$

is exactly n. Before giving a solution, one should note that there is clearly no unique solution to the problem just stated, and that even if there were, one could not plot an infinite set.

Solution: This solution is very straightforward, and the only case requiring some thought is when $n = 1$, which will therefore be left for the end. In fact, since the solution is the set $\{(1, 0), (2, 0), ..., (N, 0)\}$, except for the cases $n = 0, 1$, one cannot call this solution "cumbersome" (unless, of course, the interpretation of the problem is incorrect).

Case $n = 0$: The problem asks to find parameters for which there is no solution at all. This is clearly satisfied by the pair $(-1, 0)$ since the graphs of $y = x^2 + 1$ and $y = |x|$ do not cross. The set can be taken to be $\{(-1, 0)\}$.

Case $n = 2$: This is the simplest case. For $A > 0$, the parabola $y = x^2 - A$ runs below the x–axis, therefore crosses $y = |x|$ exactly twice. The set can be taken to be $\{(1, 0)\}$.

Case $n = 3$: This is also easy. Since the equation $x^2 = |x|$ has exactly 3 solutions $x = -1, 0, 1$, the set can be taken to be $\{(0, 0)\}$.

Case $n = 2r$, $r > 1$: By the above, one can use the set $\{(1,0),(2,0),\ldots,(r,0)\}$.

Case $n = 2r + 3$, $r > 0$: By the above, one can use the set $\{(0,0),(1,0),(2,0),\ldots,(r,0)\}$.

Case $n = 1$: This case is slightly harder, as one must find a pair (A, B) with only one solution. This is conceptually easy: One lifts the parabola $y = x^2 - A$ up from the x–axis (so that it no longer intersects $y = |x|$) and moves the V–shaped graph of $y = |X - B|$ to the right until it is tangent to the parabola on the left side. This is clearly sufficient to prove existence, but if one insists on an explicit solution, one can do so as follows *without using derivatives, only elementary High school techniques.*

By the geometric argument, one sees that A will be negative and B will be positive, and that one considers the intersection of $y = x^2 - A$ with $y = -x + B$ for $x < B$. This will happen when the equation $x^2 + x - A - B = 0$ holds, and there will be a unique solution exactly when the discriminant $1 + 4(A + B) = 0$, that is, when $B = -1/4 - A$. One can choose $A = -1$ and the set can be taken to be $\{(-1, 3/4)\}$.

Page 8, Comment on Problem 3 given to Averbakh: This question of finding the maximum is exactly Problem 9 in the Chapter "Mekhmat Entrance Examination Problems" of this book.

Page 10, Problem 3 given to Vegrina: This is Problem 10 in the Chapter "Mekh–mat Entrance Examination Problems" of this book. Interestingly, it appears that a complete answer to this question is an open problem (that is, not completely solved), so the problem can be considered difficult.

Page 11, Problem 5 given to Vegrina: In retrospect this problem seems easier than "hard" killer problems or Olympiad problems and the comment "Solving this problem requires integration of the inequality" is definitely incorrect – one only needs to know that the definite integral is the area between the graph of the curve given in the integrand, the x–axis, and the two vertical lines defined the upper and lower values on the definite integral symbol.

One first proves that $AB/y + y \leq A + B$ for all $A \leq y \leq B$. This inequality is completely trivial to prove, but it I only found this proof after having read enough of this article to understand that my original method using a convexity argument and the characterisation of the critical points of the graph was taboo in the High School curiculum (which goes to show that triviality can sometimes be hard, which can be interpreted as evidence of the tricky nature of these problems). In any case, since $y \geq A$ and $B - y \geq 0$ one has

$$\frac{A(B-y)}{y} \leq B - y\,.$$

The left side can be rewritten into two terms yielding

$$\frac{AB}{y} - A \leq B - y\,,$$

which gives the result after A and y exchange sides.

This shows that the graph of $AB/f(x) + f(x)$ is always on or below the line $y = A + B$, for $0 \leq x \leq 1$. Therefore, the area under this graph between $x = A$ and $x = A$ (which, *by definition,* is the definite integral $\int_0^1 AB/f(x) + f(x)dx$) is less than $A + B$. Note that the condition that $f(x)$ be continuous is completely superfluous and the result can even be made to hold for *any* function satisfying the inequalities, e.g., if one generalises the integral with the outer measure of the set between the graph and the x–axis. Once again, this is a testimony of the examiners' mathematical culture.

Page 12, Supplementary Problem 1 given to Temchin: The complexity of the problem's statement may put one off, but the solution is actually fairly straightforward and corresponds to the binary expansion $88 = 2^6 + 2^4 + 2^3$: First note that if $f(64) \neq 0$ then $x = 0$, $y = 0$, $k = 6$ is a solution. Similarly, if $f(65) \neq 0$ then $x = 0$, $y = 1$, $k = 6$ is a solution. So assume that $f(64) = 0$ and $f(65) = 1$. Now if $f(80) \neq 0$ then $x = 64$, $y = 64$, $k = 4$ is a solution. Similarly, if $f(81) \neq 1$ then $x = 64$, $y = 65$, $k = 4$ is a solution. So assume that $f(80) = 0$ and $f(81) = 1$. Since $f(88) \neq 0$, by assumption, it follows that $x = 80$,

$y = 80$, $k = 3$ is a solution. This answer shows that the result holds with the more restrictive upper bound $|x - y| \leq 1$.

Page 13, Appraisal of supplementary Problem 3 given to Temchin: This apraisal of the problem appears to be completely incorrect. I should first note that I found the problem itself to be challenging and at the same level as an Olympiad problem (note that geometry is my weak point). When I posted this problem on the French mathematics electronic newsgroup `fr.sci.maths`, solutions were posted within 24 hours by two users, and they were essentially similar to my solution given below. Though the difficulty of this problem is debatable, the following definitely shows that every single statement made above about the solution's requirements is false. In particular, the following solution uses on elementary High School techniques, and the first part of the solution is very intuitive and requires no special formulas, while the technical details of the other parts (trigonometric formulas) can be replaced with simple elementary geometry arguments, as was done by Patrick Coilland, one of the solvers on the internet newsgroup, see below.

The first part of the proof uses the following obvious fact:

Lemma. *Consider a triangle ABC and a point P in its interior. Then the perimeter of the triangle ABC is greater than the perimeter of the triangle PBC.*

Proof: This is the simplest case of the famous Archimedes axiom "If two convex curves have the same two endpoints and one lies entirely inside the other, then the outside curve (with respect to the line segment connecting the endpoints) has greater length." This simplest case has a direct proof: Consider the line L which bisects the angle BPC, and, without loss of generality, intersects AC at D. The angles BPD and CPD are both greater than 90 degrees, therefore $BD > BP$ and $CD > CP$. Moreover, by the triangle inequality (possibly degenerate), $BA + AD \geq BD$. The result follows.

Proof of result: Let the vertices of the hexagon be A, B, C, D, E, F in counterclockwise order, the center K and let M be the midpoint of AB.

When considering a point lying in the hexagon, one can, by symmetry, limit it to the triangle AMK, so consider a point P lying in this triangle.

Claim 1: If P is a point lying strictly inside the triangle AMK and Q is the point on MA such that PQ is perpendicular to MA, then the sum of the distances to the vertices is greater at Q than it is at P.

To see this, consider the triangle XQY, where (X, Y) is a pair of diametrically opposed vertices, that is, (A, D), (B, E), or (C, F). Then it is clear that P lies inside XQY. The Lemma therefore shows that the sum of the distances from Q to any pair of diametrically opposed vertices is greater than the corresponding distance from P. This proves the claim.

Claim 2: If Q is a point on MA, then $|QC| + |QF| \leq |AC| + |AF|$.

The proof uses basic trigonometry. Let θ be the angle CQF and let $c = |QC|$, $f = |QF|$, $d = |CF|$. Instead of directly considering the sum $c + f$, look at its square

$$(c + f)^2 = c^2 + f^2 + 2cf\,.$$

The law of cosines states that

$$c^2 + f^2 = d^2 + 2cf\cos\theta\,,$$

so

$$(c + f)^2 = d^2 + 2cf(1 + \cos\theta)\,.$$

Moreover, one has the elementary formula for the area Δ of the triangle CQF

$$\Delta = \frac{1}{2}\,cf\sin\theta\,,$$

which therefore gives

$$(c + f)^2 = d^2 + 4\Delta\frac{1 + \cos\theta}{\sin\theta}\,.$$

Note that for all Q on MA, the values of d and Δ remain constant. On the other hand, if $Q \neq A$ then the angle θ is greater than 90 degrees. (To be convinced of this, note that the angle CRF is equal to 90 degrees for all R on the circle defined by ACF and that Q lies strictly inside this circle.) The claim follows on noting that the virtually trivial fact that

$$\frac{1+\cos\theta}{\sin\theta} < 1$$

when θ is strictly greater than 90 degrees and less than 180 degrees, and that this function has value $= 1$ when θ equals 90 degrees at $Q = A$.

Claim 2: If Q is a point on MA, then $|QD| + |QE| \leq |AD| + |AE|$.

The proof uses the same ideas as the previous one. A similar trigonometric formula for $(|QD|+|QE|)^2$ can be derived, and the claim follows from the observation that the angle DQE is less than or equal to 90 degrees and the simple observation that

$$\frac{1+\cos\theta}{\sin\theta}$$

is increasing as θ decreases from 90 degrees to zero degrees.

A much simpler proof for both Claim 1 and Claim 2 was given by Patrick Coilland based on the following result:

Lemma PC. *Consider a line segment AB and a parallel line segment CD lying symmetrically above AB, i.e., the line joining the midpoints of AB and CD is perpendicular to AB and to CD. Then the distance $|AP| + |PB|$ when P is on CD is maximal for $P = C$ or $P = D$.*

Proof: Without loss of generality, one can consider the point P to lie on AM, where M is the midpoint of AB. Now let B' be the point which is the mirror image of B with respect to the line CD, and let E be the intersection of the lines AC and $B'P$. By symmetry, one has

$$|BC| + |CE| = |B'C| + |CE|,$$

and

$$|B'C| + |CE| \geq |B'E|,$$

by the triangle inequality. By symmetry, one has

$$|B'E| = |B'P| + |PE| = |BP| + |PE|.$$

One therefore gets the inequality

$$(*)\qquad |BC| + |CE| \geq |BP| + |PE| \,.$$

By the triangle inequality, one has

$$|PE| + |EA| \geq |PA| \,.$$

Adding this to $(*)$ gives

$$|BC| + |CE| + |EA| + |PE| \geq |BP| + |PA| + |PE| \,,$$

and the result of the Lemma follows on noting that $|CE|+|EA| = |AC|$ and cancelling out $|PE|$.

It is clear that the solution consisting of the two Lemmas and Claims 1, 2, 3 uses no concept from calculus and not only does it not appeal to any concept or technique not in the High School curiculum, but it does not even appeal to any concept or technique not in the Middle School (up to 14–15 years of age) curiculum. One could criticise this solution for not giving a rigourous proof in Claim 1 that the point P lies strictly inside the triangle XQY, but this point can be easily fixed up. The reader is invited to find other elementary solutions.

Page 19, Problems given to desirable applicants: For the record, here are solutions to problems 1, 2, 4:

Problem 1. It is sufficient to show that $(\sin x + \cos x)^2 \leq 2$. The most basic trigonometric formulas give

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x \leq 2.$$

Problem 2. By definition, $\log_c x = (\log x)/(\log c)$, so one has

$$\frac{1}{\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x}} = \frac{\log x}{\log a + \log b} = \log_{ab} x \,.$$

This formula directly generalises to a larger sum of terms, and multiplying both sides of the identity by the number of terms yields the interesting formulation: The harmonic mean of the logarithms of a given number to different bases is the logarithm of the number to the base equal to the geometric mean of the bases.

Problem 4. Basic trigonometric formulas show that

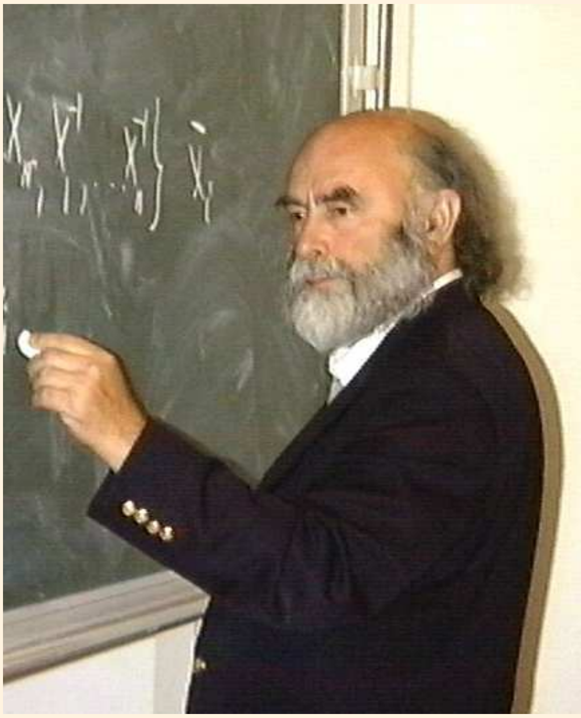
$$\cos 2x - \cos x = \cos^2 x - \sin^2 x - \cos x = 2\cos^2 x - \cos x - 1\,.$$

The question is to determine for which x this equals zero. Let $y = \cos x$, then

$$2y^2 - y - 1 = (2y+1)(y-1) = 0$$

holds exactly when $y = 1$ or $y = -1/2$. The corresponding values for x are: x equals zero or x equals 120 degrees or x equals 240 degrees. To be completely precise, one should include all values of x which differ from these by an integer multiple of 360 degrees (I avoid semantic questions regarding to the use of the words 'and' and 'or' in the previous sentence).

These solutions give convincing evidence that these problems are much easier than the ones given above.



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SCIENCE AND TOTALITARIANISM[a]

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Let us turn to the 1970s and more recent years. This was a time of gradual disintegration of the Soviet system. After erroneous hopes of some intelligentsia about liberalization of the regime in the 1950s and 1960s, people came to the understanding (especially after Czechoslovakia[b]) that the system was already incapable of change. The authorities made a desperate attempt to preserve the general features of the regime without changing anything essential. State-organized terror in the scientific community manifested itself at all levels of preparation and work of scientists: admission to universities and graduate studies, defense of dissertations, work recruitment, degree and title awarding, travels abroad, and contacts with world science were under the most severe control by the authorities. A new tendency omnipresent at that time was full mutual understanding between the Communist party and personnel-controlling bodies, i. e. KGB.[c] A de facto merging of the so-called "public organizations" and Secret Service was completed by the end of the 1960s and 70s. During Stalin's years, if Stalin so desired, he could nominate obviously (or outwardly) nonorthodox but competent specialists to important scientific and organizational positions (up to the President of the Academy of Sciences of USSR). Among such nominees were people who were not members of the Communist party, a practice totally abandoned under Brezhnev's rule. During Brezhnev's years, data from application questionnaires[d] became the only criteria.

[a]This is the English translation of Part 2 of A. Vershik's article published in Russian in 1998 in the magazine *Zvezda*, # 8, p. 181.

[b]The author refers to the invasion of Czechoslovakia by the Warsaw pact countries led by the USSR in August 1968. –Editor's note

[c]The Soviet Secret Police. –Translators note

[d]Such questionnaires contained a few dozen entries, including ethnic origin, membership in the Communist party, detailed data on close and not so close relatives, trips to foreign countries and contacts with foreign organizations, service in the Soviet

Professional qualities played no role.

A direct consequence of these policies was expulsion of talented youth; because of "inappropriate" data in their application questionnaires or based on other grounds, many young people could not get adequate positions in academia and were driven into insignificant second-rate institutions[e] where selection was not so strict. Only a few mathematicians who brought glory to their science during the 1960s, 70s and 80s, belonged to the officially supported establishment, which was quite loyal to the authorities. It is sufficient to compare the lists of those who were officially nominated and supported by the National Committee of Soviet Mathematicians to those whose reputations were indeed high. The latter list included mostly people who were marginalized, denied promotion, not allowed to travel abroad, who worked in some kinds of *sharashka*[f] or played secondary roles at universities, who did not follow or did not want to follow the Soviet standards.

A special role in establishing such a climate in mathematics in the 1970s and 1980s belonged to big shots – Academicians I. M. Vinogradov, L. S. Pontryagin and others. These scientists, who had formerly been leaders in their fields and did not belong — at least superficially — to the Communist party/Soviet state elite, actually made an unwritten deal with the authorities which allowed them to set policy in mathematics. The power of this clique in mathematics was unlimited. They provided directors to the Institute of Mathematics of the Soviet Academy of Sciences, for many years; all significant administrative bodies — VAK[g] (responsible for confirmation of academic degrees and titles), National

Army, and so on. –Editor's note.

[e]In the Russian original, "boxes." Here Prof. Vershik refers to numerous institutions conducting classified research for the military. Since many of them had no official unclassified names, they were known as PO box number such and such. In Soviet newspeak "post-office box" or just "box" became synonymous to a low-efficient technical research institute with a monstrous number of personnel and virtually no scientific output. –Editor's note

[f]Approximately the same as the "post office box", see the previous footnote. –Editor's note

[g]Abbreviation for *Supreme Attestation Committee.* PhD and other academic degrees in the Soviet Union were not considered valid until they were "confirmed" by VAK. –Editor's note

Committee of Soviet Mathematicians (supervising Soviet participation in mathematical congresses abroad and international conferences), editorial boards of the leading journals, editorial committees (issuing permits for book publication), and, of course, elections to the Academy of Sciences (awarding prizes and many other issues) — all of the above were in their hands.

Scandals associated with the rejection of excellent dissertations, as well as impediments to publications and international travel to conferences and congresses became the norm of mathematical life for the “undesirables” in those years.

Needless to say, their policies were highly anti-Semitic; but yet more important for them was to distinguish “their folks” from “aliens” — the latter included not only Jews but a significant number of active mathematicians in general. Gradually, these aging rulers of the mathematical community, as well as the supreme rulers of the country, lost all contact with both science and reality. On the other hand, their work discredited the official science and caused it moral damage, which did not seemingly concern the authorities at all. By the last years of Brezhnev’s rule, far-sighted scientists from Western countries had already realized that, rather than dealing with Soviet officials and establishment, one should deal directly with scientists. The long-term effects of these policies are yet to be understood. In any case, the situation in Soviet science in the 1970s outlined above played a considerable role in brain drain and emigration of mathematicians.

There is one more important circumstance. Among the social ills of Soviet society, the one most difficult to cure was its moral degradation, manifesting itself, in particular, in the character of relationships between people, scientists, and collectives. As soon as the principle of “clean personal data” became a major factor in staff selection and a mandatory regulation for admission to universities and prestigious institutions, award nominations, international travel and other similar issues, it was turned into a powerful weapon against scientific opponents and competitors whose academic advancement could be blocked with ease.

It would be instructive to take a respectable university, for example, Leningrad University, or a research institute of the Soviet Academy of Sciences, and carefully trace consequences of the above policy: gradual expulsion of talent, decay and death of excellent scientific schools, appointment of slimeballs — protegés of the Communist Party apparatus and KGB — to administrative positions, creation of sham scientific reputations, triumph of mediocrity, and, most importantly, mental mutilation of youth who were trained to adhere to subservience and doublethink. What could they learn from everyday examples of fake competitions, mock admission examinations aimed at discriminating against "undesirable applicants" who had no chance to be admitted?

One can give dozens of examples demonstrating how the full-blown perverted system of relations gave to mediocre administrators and *partkoms*[h] huge powers over scientists, putting them (scientists) into a humiliating position. Here is one of them, referring to the Moscow University in the 1970s. A departmental official learned that the outstanding mathematician, V.I. Arnold, agreed to serve on the editorial board of a foreign journal without getting appropriate permission (which, by the way, was not necessary in the given case because the journal's sole task was publication of articles from the Soviet Union translated from Russian in English). This was sufficient grounds for not granting him permission for participation in the international congress of mathematicians in Warsaw, where he was invited as one of the major speakers. Let me mention another example, this time at the Department of Mathematics at Leningrad University. V.A. Rokhlin (1919–1984), a world caliber mathematician, who created a topology seminar at Leningrad University, one of the best in the world, during his 20 year-long tenure in the Department, was not allowed to keep any of his students in the Department. By doing so, the university deprived itself of the opportunity to create a brilliant school, members of which now represent the elite of world science. They emigrated in the 1970s and 1980s and are presently faculty in the best Western universities.

The fate of V.A. Rokhlin is a separate story; I would just like to mention that the university nobodies forced him into retirement at the age of 60; they could not forgive this wonderful scientist his independent

[h]Local Committees of the Communist Party. –Editor's note

personality, his *gulag* past, and the emigration of his son. I remember that when I organized a meeting devoted to his 60^{th} anniversary in 1979, this caused a flurry at the *partkom*; the party secretary had to seek advice from the *raikom*[i] on how to handle this matter. Only later I learned the reason for their concern: coincidentally, our meeting was held on December 10, the human rights day, and watchful members of the *partkom* saw in this coincidence a dangerous plot of political enemies.

Almost none of my best students were accepted to graduate studies in the 1970s and 80s; even if rarely one would slip into grad school with incredible difficulty, after his or her PhD there were no employment opportunities. The doors of universities or other research institutions were closed for my students. Many of them asked me (prior to emigration) if there was any hope at all that I could hire him or her in one of my laboratories, or at the Math Department, or any other place, for continuation of scientific work. My reply was always honest; I had no such hope. Moreover, when I tried to intervene to help a bright schoolboy or schoolgirl get admission to our Mathematical Department, this always had negative consequences; the admission was never granted. It was the same story with graduate students. For a certain period of time I was forbidden to have graduate students at all.

The example of our university exhibits in a clear way how the criteria of loyalty at that time led to rejection of active scientists. For instance, a person could not get a professorship unless he or she completed a course at the so-called university of Marxism-Leninism. Volunteering for the "public benefit" was mandatory for getting positive references (necessary for all sorts of appointments, foreign travel permission, and so on). "Public benefit" was exclusively defined as ideology-motivated activities. For a long time, I was handling the day-to-day running of the Leningrad (now St. Petersburg) Mathematical Society, a public organization dedicated to the advancement of science and independent of Leningrad University. According to rational thinking, this should be called public service; however, a university official told me: "How can it be public service if, in fact, you enjoy it?"

The influence of the KGB in science was destructive not only in the

[i] Regional Communist Party Committee. –Editor's note

issue of research personnel selection. The system of the so-called "expert certification" of scientific publications was an obstacle for publication of scientific papers because it was very difficult to get such a"certification" for a person who was not an employee of an institution officially designated to carry out research in the given area.

In the 1970s and 80s, the technique of rejecting talented students, especially at Moscow University, became quite contrived: special people assembled problem sets for entrance examinations which could not be solved[j]; "undesired" applicants were assigned to separate groups intended for "special treatment," sophisticated methods were developed, effectively barring "undesired" applicants and their parents from a fair appeal process. The motivations of people who carried out "the order of the party"[k] were quite versatile: some were blackmailed for past misdeeds; others did it for their own pleasure. At Leningrad University, the way the "party order" was implemented was simpler. Here the applicants to the Math Department were flunked by physicists or literature instructors, the graders of literary essays.[l] Of course, all this was orchestrated by the *partkom* and the department of personnel which controlled the work of all admission committees. In many cases, the cynicism of the admission committees was so obnoxious that they did not consider it necessary to disguise what they did. For example, a perfectly correct solution to an exam problem would be declared wrong, and so on.

The question of why all this was done, in my opinion, does not have a rational answer. Certainly, the 1970's were the years of omnipresent corruption. Preferential treatment for children of functionaries and corrupted officials was common. Guaranteed admission of such children to highly respected educational institutions meant denial of admission to truly deserving aspiring students. But this is only a part of the story. It is known that TsK[m] issued a [secret] decree (1975–76) on personnel policies where, I was told, it was stated that admission to prestigious

[j] The infamous killer problems. –Translators note

[k] The author means the Communist Party of the Soviet Union, the only one that existed in the country. –Translators note

[l] Compositions or essays on Soviet literature were mandatory as a part of entrance examinations to all departments. –Editor's note

[m] The Central Committee of the Communist Party. Translators' note

higher-education institutions as well as career advancement of individuals whose "nationality[n] related to countries conducting anti-Soviet policy," had to be limited. The essence here is not *only* anti-Semitism — the above statement covers also ethnic Germans, Greeks, Chinese and other ethnic groups — or discrimination on ethnic and political grounds. The main aspect here is that the decree unleashed, and even instructed, to carry out selections. Why? It was done for the purpose of getting an obedient and loyal work force whose professional qualifications were not necessarily important. Talented people had to be restrained by the system as they were a potential source of disobedience. This short-sighted and destructive policy finally led to a crisis in all spheres of life. The absence of professional skills and work ethics[o] caused destructive trends in the economy, politics, and science. One of the glaring examples of non-professionalism in science policy was a relocation of Leningrad University from the city center to Petergof (more accurately, the village Martyshkino), which resulted in ruining scientific life and the educational process in all relocated departments of the University.

I would like to compose a "White Book" of all these misdeeds. Almost ten years of openness have elapsed — have we ever heard of these issues? Functionaries and scientific persecutors of those years live nicely alongside us and even hold the same or sometimes higher positions. In 1987, I tried to publish my thoughts on the admission practices in Soviet universities in the liberal weekly *Moskovskie Novosti* [*Moscow News*]. My attempt was stonewalled. Recently, I co-authored (with A. Shen) an article *On Admission to Mathematical Departments in the 1970s and 80s.* It was published in a popular international journal, *The Mathematical Intelligencer.*[p] This article gives numerous examples, mainly taken from Moscow University practice, showing how "undesirable" applicants

[n]In Russian the word "nationality" is used for ethnic origin. What is known as nationality in the West is called citizenship. First and foremost this decree was aimed against Jews who were considered to be associated with Israel, which, in turn, was considered to be one of the main enemies of the USSR. –Translators note

[o]The author uses here the word "Chernobylism," a self-constructed noun describing the wide-spread slovenliness of the type which led to the Chernobyl nuclear disaster in April 1986. –Editor's note

[p]Also reprinted in the present Collection. –Editor's note

were flunked. We gave examples of killer problems, which probably would not have been solved by many professors, as well as the names of the examiners and organizers of what we called "the gas chambers for undesirable applicants." The names include the current Rector of Moscow University (in the past, an active party functionary), the recent secretary of the *partkom* at Leningrad University, and many other former activist executioners of the party line on the issues of personnel. It seems that they are not concerned about the past; they hold highly visible positions and are engaged in "building" a new Russia. Their fear of public exposure faded, there is no reason to confess and nothing to regret about.

Up to now the public has not been aware of many important details related to the practices of those years, for example, the above-mentioned secret instruction of TsK on personnel policies. It is interesting to note that if they need to (for instance, during election campaigns) the current authorities willingly talk about the crimes of the Soviet government in the 1930s and 40s. As for the more recent past — the activities of the Communist Party in the 1960s, 70s and 80s, which were not as bloody but no less destructive with respect to our own country than the tragedy of the 1930s — the authorities keep silent.

When I write this, in no way do I have in mind a witch hunt, but we cannot free ourselves from the stinking traits of the Soviet past without an open discussion of all aspects of this past. On the other hand, this is a question of confidence in the new power, which cannot exist if a former activist of the Brezhnev epoch becomes a distinguished functionary in the 1990s and expertly talks about democracy, organizing meetings commemorating the regime victims, or consecrating a church. Unfortunately, the victims themselves are in no rush to step forward and tell their stories. This silence is no evidence of forgiveness or modesty and does not contribute to the purification of the moral climate in the country. Nor does it help to guarantee that past misdeeds will not be repeated in the future.

I should add that there were noble people, too, who risked their careers and did not fear to raise their voice against vileness at their institutes and universities. Their names should became known too.

S. P. Merkuryev, the late Rector of St. Petersburg University, with whom I discussed the possibility of studying the history of the university admission practices and other aspects of activities of the party fuctionaries, was ready to assist in this investigation. But he warned me that I would find hardly anyone who would be willing to conduct a professional investigation; people continue to fear big shots of the past as they still have power now. And Merkuryev was right. As I found out shortly after our conversation, former communists who retained power, slightly disguising themselves in the clothes of democracy-builders, managed to intimidate a few decent historians, whom I addressed. I am afraid that the appropriate moment and the opportunity for such an investigation have been missed. Evidently, it will be the scholars of the XXI century who will engage in the historical studies of the humiliation of science.

Translated from Russian by
Nodira Dadabayeva and Alexey Kobrinskii

ADMISSION TO THE MATHEMATICS DEPARTMENTS IN RUSSIA IN THE 1970s AND 1980s[a]

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For a number of years now, the universities of the former Soviet Union have been free of [Communist] party committees (*partkom*'s). These committees were made up of individuals whose job was to see that the party line was followed, and above all to watch over the purity of the cadres — the loyalty of professors and students, the purity of their curricula vitae, and preferment to the necessary people. Now the offices of party committees have been allotted to computer centers, centers for "intellectual investigations," and so on. Many of their present occupants occupied them in the past, but now they investigate problems of the interaction of science and religion, they criticize Marxism, they invite new-wave politicians and psychics, they talk about their past difficulties at work.

The one thing they do not talk about is their cadre work during the period of stagnation. The top partkom secretary of one of the finest Russian universities (Leningrad), who very carefully carried out party directives about the purity of the cadres, is now the director of a cultural center where he organizes evenings of Jewish culture. The vice-rector of another university (Moscow), once extremely active in all official campaigns and purges and in the organization of "selections" in university admissions, has now become an ardent democrat and an organizer of the most progressive projects.

Of course this is wonderful — only, one may still ask, "why, gentlemen, are you silent about how things were done, how you managed education, admission to universities, selection of cadres?" It would be useful for the educational community to know how and why the science lost hundreds, and possibly thousands, of undoubtedly talented individuals, potential leaders, hard workers profoundly dedicated to learning,

[a]Reprinted from *The Mathematical Intelligencer*, Vol. 16, No. 4, p. 4, 1994.

whose lives have been distorted, often irreparably so.

One of the most important objectives of cadre politics at the leading universitites, particularly in the capital, seems to have been to limit the admission of Jews and members of certain other ethnic minorities. Of course, this was not the only objective. It was important not to admit political pariahs (and their children!) as students, graduate students, researchers, and professors. Likewise, it was important to help children and relatives of the *nomenclatura* (party and government officials, KGB) who were [falsely] classified as children of workers. And workers, in the "proletarian" state, enjoyed a mandatory quota of admissions.

Once emigration was permitted there was, so to say, an official pretext for not accepting Jews or not assigning them to prestigious work, for not providing incentives, awarding degrees, etc. Thus, one killed two birds with one stone: the country got rid of some of the disaffected, and at the same time one restricted them at home.

But there was one more objective, perhaps the most important one, that one never talked openly about, namely *holding down the number of talented people*. The mediocrity of official Soviet Union during the era of mature socialism did not just happen; it was imposed from above and readily accepted below. It was in full accord with the lack of talent in the whole leadership, mitigated only by isolated fluctuations.

To this date, we do not know the details of the secret instruction of the early 1970s which (I was told) was more or less to the following effect: restrict or delay the admission to certain postsecondary schools of individuals with ties to states whose politics are hostile to the USSR. Apparently, these could be only Jews, Germans, Koreans, Greeks, and possibly Taiwanese Chinese.

Many of us know quite a few concrete stories. I could tell how much unbelievable was my admission to the Leningrad Mekh-mat (Faculty of Mathematics and Mechanics) in 1951, at the height of Stalin's war against the cosmopolites; how crudely they used to fail capable students whom I tried to help enter the university in the 1970s by recommending them to the then dean; how I was prevented from hiring talented graduate students and how these very same students eventually managed to find positions at the most prestigious Western universities; and finally, how in 1985, almost in the time of *perestroika*, my daughter, with a scholarly paper accepted for publication, was not admitted to the Philological Department of Leningrad University.

It is surprising that so few testimonies of the hundreds of victims and witnesses have so far appeared in print. All we have is G. Freiman's *It Seems I am a Jew*[b] with some problems and remarks by A. D. Sakharov, and materials collected by B. Kanevsky and V. Senderov.

It is just as surprising that so far, to the best of my knowledge, none of the hundreds of people from the departments of personnel, from partkoms, from the lecturers who conducted purges at the examinations — no one from "the other side" has provided testimony. After all, not all of these people are naive, and not all are absolute scoundrels. Some of them were victims of circumstances. They have hardly any reason to fear revenge, even less court action. All echoes of these tragedies are fading; justice demands confessions. But no — they are silent. Some have become democrats, some profess love of Jews, and some propose to emigrate and ask people whom they had earlier slighted for recommendations. Some maintain that nothing wrong took place. And some do not deny that it all happened but insist all was done "correctly."

There are very few documents left. The perpetrators realized that it would not do any good to leave traces.

When I approached S. P. Merkuryev, Rector of St. Petersburg University (he died a short time ago) and asked him if it is possible to see the archives of the party committee that dealt with these matters, he offered to help me but warned that I should not overestimate the change since the putsch; almost all the organizers of these things have retained not only their former positions but also power at the University, and, for example, he was unable to remove one of the particularly odious deans.

I soon saw a confirmation. When I attempted to encourage two historians — who had earlier been expelled from the University partly because they tried to object to scandalous practices of the kind I describe here — to work in the archives, they refused, saying, "We are afraid that 'they' will get us."

In 1987 I brought an article about a case of admission to the pro-

[b]Southern Illinois University Press, 1980.

gressive weekly *Moscow News*. The head of the department told me, "We cannot print the article dealing with this topic. There will be a flood of angry letters."

But I hope that the conspiracy of silence will not last forever. I am glad that I was able to persuade Alexander Shen, who has worked a lot with the university and secondary students, to write of the materials he has collected.

Mathematical audiences (not only in the West) will find it interesting to learn some details and solve the little problems that a school graduate was supposed to solve in a few minutes. Try to imagine a young boy or girl who has made a commitment to learning, who may have good basis for this decision (participation in olympiads, math circles, and so on), and who faces an examiner who has his instructions and his arsenal of killer problems. These examiners and admissions chairs were generally boorish and treated the school graduates shamefully. As is often the case, we know the names of those who carried out the instructions (the examiners) but not those who gave them. It would make sense to list secretaries of admissions, deans, and so on, who knew of the scandal and covered it up, right to the top of the party–KGB structure. Even these names are not such a deep secret.

We have used here materials only on admissions to Mekh-mat at Moscow University and only from the 1980s and, in part, the 1970s. There are other departments, other universities, and institutes. And there are questions of defenses of dissertations (VAK), of employment of young scholars, and many others.

Is there anything surprising about the drain of Russian science, emigration, apathy, and the low prestige of official institutes and academies? All of this was predictable from what was done.

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ENTRANCE EXAMINATION TO THE MEKHMAT [a]

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Preliminaries

Some time ago, discrimination against Jews in entrance examinations to the leading postsecondary institutions, especially Mekhmat at Moscow State University (MGU), was a fiercely debated subject. I think that we can now afford to look more calmly at the events and see their role in the history of Russian mathematics.

This kind of discrimination was sometimes talked about as if it were the main, and virtually the only, blemish on the otherwise spotless reputation of the national party. This tone was sometimes understandable (for example, one had to talk this way in complaints about Mekhmat submitted to the Committee of Party Control of the Central Committee of the Communist Party). In reality, of course, this was just one of many injustices, some far worse.

I entered the Mekhmat in 1974, began my graduate studies in 1979, and completed them in 1982. I have worked in mathematical schools from 1977 until today. I will write mostly about things I have had direct contact with. Let us hope my account will be supplemented by others.

In many countries, including Russia, the proportion of Jews is appreciably greater among scholars than in the whole population. In entrance to mathematical classes and schools (with equal requirements for all applicants), the proportion of Jews among those who passed the examinations (and among those taking them) is significantly higher than in the population as a whole. Whatever the meaning of this phenomenon,

[a] Reprinted from *The Mathematical Intelligencer*, Vol. 16, No. 4, p. 6, 1994.

it has to be kept in mind.

Elimination of Undesirable School Graduates

After certain events in 1967 (the well-known letter of 99 mathematicians in defense of Esenin-Volpin[b]) and especially in 1968 (mathematicians protesting the intervention in Czechoslovakia), the situation at the Mekhmat worsened significantly. I. G. Petrovskii ("the last non-party rector of MGU"), who had done many good things, died in 1973. His successor, R. V. Khokhlov ("the last decent rector of MGU"), perished in 1977. By 1973, the "special program" of elimination of undesirable graduates, especially Jews, was in full swing. The category of "undesirables" included the (small) group of those who did not belong to the Komsomol.[c] From that time on and until 1989–1990, when this practice was halted, the situation stayed much the same. The number of victims did change: in later years, the potential victims, aware of the barriers, did not try to apply. Also, in the mid-80s there was a time when Mekhmat students — unlike students at other institutions — were drafted. This reduced the number of applicants to Mekhmat.

Yet another form of discrimination began in 1974. It was open but no less unjust. It involved separate quotas for Muscovites and non-Muscovites (the same number of places were reserved for each group although non-Muscovites were more numerous). The ostensible reason was the shortage of rooms. An applicant who did not ask for a place in a hostel (but had no close relations in Moscow) was, however, also classified as a non-Muscovite. The harm from this discrimination was offset by the lower level of the competition for non-Muscovites.

During the period of anti-Jewish discrimination the following people were among the responsible officers of the admissions committee (in various capacities): Lupanov (current dean of the Mekhmat), Sadovnichii (current rector of MGU), Maksimov, Proshkin, Sergeev, Chasovskikh,

[b] Alexander Esenin-Volpin, a prominent mathematician and one of the pioneers of the human rights movement in Soviet Union. Esenin-Volpin was arrested on charges of anti-Soviet agitation and held in a psychiatric hospital. —Editor's note

[c] Young Communist League. —Editor's note

Tatarinov, Shidlovskii, Fedorchuk, I. Melnikov, Aleshin, Vavilov, and Chubarikov.

How Things Were Done: The Procedure

Direct discrimination was a natural concomitant of the shabby conduct of the examinations. The written part of the examination in mathematics consisted of a few simple problems that required only computational accuracy, and one or two very involved and artificial problems (the last problem was usually of this kind). Only "pure pluses" were counted. A flaw in the solution (sometimes invented and sometimes due to the examiner's failure to understand the work) meant loss of most of the credit for the problem. As a result, most of the applicants got threes and twos (out of five); the examination was almost totally uninformative.

Now we come to the oral part of the examination in mathematics. Even if there were no discernible discrimination, it is virtually impossible for all examiners to make the same demands on applicants. The required questions are very general and imprecise, and the requirements of the examiners are necessarily non-comparable; all the more so because, as a rule, the examiners had no school contact with the students.

The examination included writing a composition and passing an oral test in physics. The physics exam was given by members of the MGU Physics Department. It was not a particularly brilliant department, and the task of giving examinations was assigned to its less brilliant members.

Some Examples. In 1980 no credit was given for the solution of a problem (an equation in x) because the answer was in the form "$x = 1; 2$" and the required answer was "$x = 1$ or $x = 2$" (the school graduate was Krichevskii, the senior examiner in mathematics was Mishchenko; source: B. I. Kanevsky, V. A. Senderov, *Intellectual Genocide*, Moscow, Samizdat, 1980). In 1988, during an oral examination, a student who defined a circle as "a set of points equidistant — that is, at a given distance — from a given point" was told that his answer was incorrect because he hadn't stipulated that the distance was not zero (the textbook had no such stipulation). The graduates's name was Arkhipov and the names of the examiners were Kovalev and Ambroladze. The 1974 examination in physics included the question: What is the direction of the pressure at the vertical side of the a glass of water. The answer "perpendicular to the side" was declared to be incorrect (pressure is not a a vector and is not directed anywhere — graduate Muchnik).

Procedural Points. Sometimes the questioning began a few hours after the distribution of examination questions (school graduate Temchin, 1980, waited three hours). The questioning could last for hours (5.5 in the case of the graduate Vegrina; examiners Filimonov and Proshkin, 1980; cited by B. T. Polyak, letter to *Pravda*, Samisdat, 1980). Parents and teachers of the graduates were not allowed to see the student's papers (letter 05-02/27, 31 July 1988, secretary of the admissions committee L. V. Yakovenko). An appeal could be lodged only within an hour after the oral examination. The hearing involved in the appeal was extremely hostile (in 1980, A. S. Mishchenko faulted graduate Krichevskii at the hearing for appealing against precisely those remarks of the examiners where he (Krichevskii) was clearly in the right; Kanevsky and Senderov, *op. cit.*).

How It Was Done: "Killer" Problems

An important tool (in addition to procedural points and pickiness) was the choice of problems. The readers who are mathematicians can evaluate the level of difficulty of the problems below by themselves. We can assure non-mathematical readers that the level of difficulty of the "killer" problems is comparable to that of the All-Union Mathematical Olympiads, and many of them *are* olympiad problems. (For example, problem No.2 of Smurov and Balsanov turned out to be the most difficult problem of the second round of the All-Union Olympiad in 1985. It was solved by 6 participants, partly solved by 3, and not solved by 91.)

For comparison, we adduce first typical ordinary problems (from the mid-1980s). Grades quoted are out of 5 (5 is excellent).

First variant (those who solve both parts get "5").

1. Show that in a triangle the sum of the altitudes is less than the

perimeter.

2. The number p is a prime, $p \geq 5$. Show that $p^2 - 1$ is divisible by 24.

Second variant (those who solve the first two parts get "4").

1. Draw the graphs of $y = 2x + 1$, $y = |2x + 1|$, $y = 2|x| + 1$.

2. Determine the signs of the coefficients of a quadratic trinominal from its graph.

3. x and y are vectors such that $x + y$ and $x - y$ have the same length. Show that x and y are perpendicular.

Now the "killer" problems. The names of the examiners and the years of the examination are given in parentheses.

1. K is the midpoint of a chord AB. MN and ST are chords that pass through K. MT intersects AK at a point P and NS intersects KB at a point Q. Show that $KP = KQ$.

2. A quadrangle in space is tangent to a sphere. Show that the points of tangency are coplanar. (Maksimov, Falunin, 1974)

1. The faces of a triangular pyramid have the same area. Show that they are congruent.

2. The prime decompositions of different integers m and n involve the same primes. The integers $m+1$ and $n+1$ also have this property. Is the number of such pairs (m, n) finite or infinite? (Nesterenko, 1974)

1. Draw a straight line that halves the area and circumference of a triangle.

2. Show that $(1/\sin^2 x) \leq (1/x^2) + 1 - 4/\pi^2$.

3. Choose a point on each edge of a tetrahedron. Show that the volume of at least one of the resulting tetrahedrons is $\leq 1/8$ of the volume of initial tetrahedron. (Podkolzin, 1978)

We are told that $a^2 + b^2 = 4$, $cd = 4$. Show that $(a-d)^2 + (b-c)^2 \geq 1.6$. (Sokolov, Gashkov, 1978)

We are given a point K on the side AB of a trapezoid $ABCD$. Find a point M on the side CD that maximizes the area of quadrangle which is the intersection of the triangles AMB and CDK. (Fedorchuk, 1979; Filimonov, Proshkin, 1980)

Can one cut a three-faced angle by a plane so that the intersection is an equilateral triangle? (Pobedrya, Proshkin, 1980)

1. Let H_1, H_2, H_3, H_4 be the attitudes of a triangular pyramid. Let O be an interior point of the pyramid and let h_1, h_2, h_3, h_4 be the perpendiculars from O to the faces. Show that $H_1^4 + H_2^4 + H_3^4 + H_4^4 \geq 1024\ h_1 \cdot h_2 \cdot h_3 \cdot h_4$.

2. Solve the system of equation $y(x+y)^2 = 9$, $y(x^3 - y^3) = 7$. (Vavilov, Ugol'nikov, 1981)

Show that if a, b, c are the sides of a triangle and A, B, C are its angles, then

$$\tfrac{a+b-2c}{\sin(C/2)} + \tfrac{b+c-2a}{\sin(A/2)} + \tfrac{a+c-2b}{\sin(B/2)} \geq 0$$

(Dranishnikov, Savchenko, 1984)

1. In how many ways can one represent a quadrangle as the union of two triangles?

2. Show that the sum of the numbers $1/(n^3 + 3n^2 + 2n)$ for n from 1 to 1000 is $< 1/4$. (Ugol'nikov, Kibkalo, 1984)

1. Solve the equation $x^4 - 14x^3 + 66x^2 - 115x + 66.25 = 0$

2. Can a cube be inscribed in a cone so that 7 vertices of the cube lie on the surface of the cone? (Evtushik, Lyubishkin, 1984)

1. The angle bisectors of the exterior angles A and C of a triangle ABC intersect at a point of its circumscribed circle. Given the sides AB and BC, find the radius of the circle. [The condition is incorrect: this does not happen — A. Shen.]

2. A regular tetrahedron $ABCD$ with edge a is inscribed in a cone with a vertex angle of $90°$ in such a way that AB is on a generator of

the cone. Find the distance from the vertex of the cone to the straight line CD. (Evtushik, Lyubishkin, 1986)

1. Let $\log(a,b)$ denote the logarithm of b to a base a. Compare the numbers $\log(3,4)\cdot\log(3,6)\cdot\ldots\cdot\log(3,80)$ and $2\log(3,3)\cdot\log(3,5)\cdot\ldots\cdot\log(3,79)$.
2. A circle is inscribed in a face of a cube of side a. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles. (Smurov, Balsanov, 1986)

Given k segments in a plane, give an upper bound for the number of triangles all of whose sides belong to the given set of segments. (Andreev, 1987) [Numerical data were given, but in essence one was asked to prove the estimate $O(k^{15})$. A. Shen.]

Use ruler and compasses to construct, from the parabola $y = x^2$, the coordinate axes. (Kisilev, Ocheretyankinskii, 1988)

Find all a such that for all $x < 0$ we have the inequality

$$ax^2 - 2x > 3a - 1.\ \text{(Tatarinov, 1988)}$$

Given the graph of a parabola, construct the axes. (Krylov E. S., Kozlov K. L., 1989) [These examiners told a graduate that an extremum is defined as a point at which the derivative is zero. They also reproached another graduate for not saying "the set of ALL points" when he defined a circle as the set of points at a given distance from a given point.]

Let A, B, C be angles and a, b, c the sides of a triangle. Show that

$$60^\circ \leq \tfrac{aA+bB+cC}{a+b+c} \leq 90^\circ.$$

(Podol'skii, Aliseichik, 1989)

Statistics — The Mekhmat at MGU and Other Institutions

The most detailed data on graduates of mathematical schools were obtained in 1979 by Kanevsky and Senderov. They divided the graduates of schools 2, 7, 19, 57, 179, and 444 who intended to enter the Mekhmat into two groups. One group of 47 consisted of students whose parents and grandparents were not Jews. Another group of 40 consisted of students with a Jewish parent or a grandparent. The results of olympiads (see the Table below) show that the graduates were well prepared, but when it comes to admission, the results are noticeably different.

Mekhmat at MGU

	First group	Second group
Total graduates	47	40
Olympiad winners	14	26
Multiple winners	4	11
Total olympiad prizes	26	48
Admitted	40	6

Kanevskii and Senderov give figures also for two other institutions:

MIFI

	First group	Second group
Total graduates	54	29
Admitted	26	3

MFTI

	First group	Second group
Total graduates	53	32
Admitted	39	4

Of course, the character of the entrance examinations became known to school graduates, and those suspected of Jewishness began to apply to other places, for the most part to departments of applied mathematics where there was no discrimination. (One very well-known place was the “kerosinka” — the Gubkin Oil and Gas Institute.)

Mathematical Schools and Olympiads

When we talk about mathematical schools, we exclude the boarding school #18 at MGU. Proximity to the Mekhmat unavoidably leaves its imprint. In the remaining schools, discrimination by nationality was mostly insignificant.

As a rule, selection of students for a particular class depended largely on the teachers of mathematics and was controlled by the administration to a minor extent. In 1977, in school #91, the administration was presented with a list of students in the math class and did not make any changes. In 1982, in school #57, the situation was more complicated because the school was subject to district administration, and the class list had to be acceptable to the district committee. So, some students favored by the district authorities were accepted outside the competition. In 1987, in school #57, "wartime resourcefulness" was successfully applied: Russian names picked at random were added to the list of students sent for approval to the district committee (which did not check which of the students on the list later attended). It seems that after that there were no problems (*perestroika*!).

One could speculate that discrimination in admissions to the Mekhmat (very well-known to both teachers and students of math classes) and the large percentage of Jews among teachers and students could give rise to a problem of "interethenic relations" (injustice often gives rise to injustice in reverse). I have often heard such speculations, but I am convinced that in most mathematical classes (and the best ones) no such things ever happened.

As for the olympiads, the Moscow city olympiad was quite a long time relatively independent from official departments. But in the late 1970s, after Mishchenko's letter to the *partkom* (it is amusing that recently Mischenko asserted publicly that he was not in the least involved, but he did not challenge the authenticity of his letter), control of the olympiads was given to Mekhmat — and, to a large extent, to the very same people who controlled the entrance examinations. It seems to me that the result was not so much discrimination as plain incompetence. (For example, in 1989, after my conversation with the people who managed the olympiad, it became clear that a large bundle of papers got lost. Following urgent requests, it was found. I was even permitted to see the papers of the students in the class in which I lecture. A significant portion of these papers were improperly graded.)

General Remarks, History

It seems that now practically no one denies there was discrimination in entrance examinations (that is, no one except possibly university administrators — but then they are the people least able to shift responsibility). In particular, Shafarevich mentions this kind of discrimination in his article in the collection *Does Russia have a Future?*

This discrimination causes two kinds of harm. First, many gifted students have been turned down or have not tried to enter the Mekhmat. In addition to this direct harm, there is also an indirect kind: participation in entrance examinations has become a means of checking the loyalty of graduate students and co-workers, and a criterion for the selection of co-workers. Many distinguished people (regardless of nationality) who refused to be accomplices have not been employed by the Mekhmat.

The situation has brought protests whose form depended on the circumstances and the courage of the protesters. I probably know only some of the incidents.

In 1979, document #112 of the Moscow group for implementing the Helsinki agreements, entitled "Discrimination against Jews entering the university," was signed by E. Bonner, S. Kallistratova, I. Kovalev, M. Landa, N. Meiman, T. Osipova and Yu. Yarem-Ageev. Included in this document were the statistical data collected by B. I. Kanevsky and V. A. Senderov.

On the basis of the 1980 admission figures, Kanevsky and Senderov wrote, and distributed through Samizdat, the paper "Intellectual Genocide: examinations for Jews at MGU, MFTI and MIFI."

I remember well my reaction, at that time, to the activities of Kanevsky and Senderov (which, I now realize, was largely a form of cowardice): the result of their collecting data will be that students of math schools will be rejected just like Jews. (This did not happen, although there were such attempts.)

Also, Kanevsky, Senderov, mathematics teachers in math schools, former graduates of math schools, and others, helped students and their parents to write appeals and complaints. Incidentally, this activity was sometimes criticized in the following terms: "By inciting students to fight injustice you are using others to fight your war with the Soviet authorities, and you are subjecting children and their parents to nervous stresses." In some cases, the plaintiffs succeeded (by threatening to cause an international scandal or by taking advantage of a blunder of an examiner), but an overwhelming majority of complaints were without effect.

There were attempts to help some very capable students (Jews or those who could be taken for Jews) by behind-the-scene negotiations. I myself took part in such attempts twice, in 1980 and in 1984. In one case it was possible to convince the Admission Committee that the graduate was not Jewish, that his name just sounded Jewish-like; and in the second case they closed their eyes to the Jewishness of the graduate's father. It was not a simple matter to find a chain of people, ending with a person who was a member of the Admission Committee, each of whom could talk to the next one about such a delicate topic. (In one case I know of, one member of such a chain was A. N. Kolmogorov.[d] To this day I have second thoughts about the morality of these activities of ours.

In 1979–1982, on the initiative of B. A. Subbotovskaya and with the active support of B. I. Kanevsky, mathematics instruction was organized for those not going to the Mekhmat: once a week, every Saturday afternoon, lectures on basic mathematical subjects were presented to interested students. These sessions took place at the "kerosinka" or at the humanities building of MGU (of course, without the knowledge of the administration — we simply took advantage of the available empty rooms). Xerox copies of the lectures were given out to the students. These studies were referred to as "courses for improving the qualifications of lecturers in evening mathematical schools," but the participants called them "the Jewish People's university." This went on for a number of years, until one of the participants, and Kanevsky and Senderov, were arrested for anti-Soviet activities; after an interrogation at the KGB, Bella Subbotovskaya died in a car accident under unclear circumstances. It should be noted that some of the participants in these

[d] A famous Russian mathematician. – Editor's note

studies who were not Mekhmat students (some *were* Mekhmat students) were very gifted, but very few of them became professional mathematicians.

I remember my reaction, at the time, to the arrest of Senderov and others: well, instead of teaching mathematics they engaged in anti-Soviet agitation, and because of them (!) now everyone has been caught in the act.

Other attempted protests: in 1980 and 1981 B. T. Polyak wrote to *Pravda* about scandalous practices (without bringing in the issue of anti-Semitism — he must have hoped that he could influence the Mekhmat within the existing system).

Perestroika began in 1988 and one could openly and safely write about anti-Semitism (even to the Committee of Party Control, then still in existence). Some people, including Senderov, then released from prison, went to various departments, including the city *partkom* and the city Department of Education, trying in some way to influence the Mekhmat. "The dialogue with the opposition" took more concrete forms and there were no accusations of anti-Soviet agitation, but the only positive result was that one of the graduates involved was allowed a special examination. After that, the discussion continued inside the university (at the meetings of the Scientific Council of the Mekhmat, in wall newspapers, and so on). It died off gradually, because discrimination in entrance examinations ceased, and many of the participants in the discussion scattered all over the world.

Part 3

Katherine Tylevich

born in 1983.

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FREE EDUCATION AT THE HIGHEST PRICE:
A Brief Glimpse at Soviet Realities, Bella Abramovna Subbotovskaya and "The Jewish People's University"

KATHERINE TYLEVICH

This article is devoted to a unique page of Soviet history and mathematics; it discusses the "Jewish People's University," a fascinating institution without walls that serves as alma mater for hundreds of today's leading physicists, mathematicians, professors and researchers who, in their youth, were unjustly denied access to traditional Soviet universities. In its short existence in the late '70s and early '80s, "The Jewish People's University" delivered a rich intellectual and emotional stimulus to hundreds of students and their professors, both Jewish and non, who sought to pursue the study of math and physics in an academic, rather than a politically overshadowed, environment.

Between 1978 and 1982, "The People's University," as it became widely known, reputedly rivaled even Mekh-Mat (the Harvard of the Soviet Union, so to speak) in terms of academics. Unlike some students at Moscow's leading university, those who studied at "The People's University" had no intentions of avoiding army draft; they had no such luxury. They were there for the purpose of learning in its purest form.

If the "Jewish People's University" had a formal agenda, then it was certainly to offer a first-rate, advantageous education to those Jewish students to whom higher university administrators and Soviet politics unjustly closed the door. But off the record, "The People's University" was a powerful symbolic blow against Soviet anti-Semitism, and against the Soviet system in general. Despite what we know or remember of the late Soviet regime today, it is still difficult to believe that the figurative fight waged by a select few in the late '70s and early '80s would essentially result in two "prisoners of war." It is even more difficult to believe that this figurative fight would result in a literal death.

The fate of Bella Abramovna Subbotovskaya, coordinator and mastermind of the highly unusual university, is mysterious to say the least. Reading less like non-fiction and more like a psychological thriller, the alleged circumstances surrounding Subbotovskaya's death involve a dark, quiet night, an abandoned street, one unobservant or possibly crazed driver going at high speed, and an unreliable, perhaps even malleable witness. Officially, Bella Abramovna Subbotovskaya met her death as the result of a careless driver. The sole witness reported a second car that paused beside Subbotovskaya's body, minutes following the collision. The ambulance came immediately.

Versions of reality may have changed in times of stress or pressure, but authorized records remain the same to this day. These records claim that the second car was the one to end Subbotovskaya's life. Of course, just as there is unofficial truth behind most sanctioned lies of the former Soviet government, there is an unofficial, but widely believed explanation for Subbotovskaya's untimely death; an explanation that, although simple to understand, is hard to digest: Subbotovskaya was purposefully killed by the KGB as the result of her unapologetic safeguarding of "The Jewish People's University."

Paranoia and suspicion are easy to succumb to, especially in a culture where the unpredictable, the unbelievable and the uncalled for are very much a part of public consciousness. Surely, skeptics may doubt the "conspiracy theory" behind Subbotovskaya's death. After all, the tragedy occurred the night of September 23, 1982 — long after the death of Stalin, in an altogether different social and political period in the Soviet Union. Notwithstanding, it is difficult to overlook the frightful, yet essential details left out of the "closed case." Carried to this day by members of Subbotovskaya's social, intellectual and familial circles, are bits of information that do not simply imply, but provide what many see as proof of premeditated murder. After all, devoted students and dedicated professors were not the only regulars at this not-so-underground university; members of the KGB frequented the sessions as if they, too, stood to gain from the study of math and physics.

Politics were a strictly taboo subject at "The People's University," regardless of whether a member of the KGB was thought to be present at a lecture. In an effort to protect their students, Subbotovskaya and her two colleagues, Valery Senderov and Boris Kanevsky, never strayed from the institution's blatant mission: To give those students — particularly those Jewish students — who crave it, the opportunity to study math and physics at an advanced level. But if the frequent congregation of mostly young, Jewish scholars was not enough to attract the attention of the KGB, then the connection of Kanevsky and Senderov to these assemblies certainly was. The two men were, after all, known and active Soviet dissidents. In fact, the year "The People's University" began to truly develop, 1979, was the same year that the two mathematicians orchestrated and executed a study that publicized the existence of methodical anti-Semitic discrimination at all levels of entrance to Soviet universities.[a]

Essentially, Kanevsky and Senderov used numbers — science — to prove a point that was highly emotive. Their study followed 87 aspirants seeking admission to Moscow University's leading mathematics faculty. The candidates had a lot in common: all were recent graduates of specialized math and physics high schools in Moscow, many of them were nationally renowned in mathematics Olympiads. 40 of the candidates, however, gave "undesirable" information on their entrance forms. 40 of them were Jewish either by passport or "by trace." Entrance forms required that students state their nationality alongside the names and patronymics of their parents. Even an "officially" Russian student, suspected of having even one Jewish grandparent, could be placed in a group of undesirables.

The study clearly showed that Jewish candidates were methodically forced outside the gates of the prestigious university, even though their credentials were similar to, or better than those of other applicants. Of the 47 aspirants who were not Jewish, 40 were accepted after taking the entrance exam. Of the 40 candidates who had at least one Jewish grandparent, all but six were rejected. To add insult to injury, Kanevsky and Senderov also cite one case when examiners wrongly thought that one applicant was Jewish and lowered his grades. After the applicant's mother proved that their family had no Jewish lineage, however, administrators immediately improved his grades and admitted him into the university.

[a]Kanevsky and Senderov "published" their results in *Samizdat* in 1980, see *Intellectual Genocide"* in Part 2 of the present collection. –Editor's note

Kanevsky and Senderov confirmed the ends of a means that had been know for years: Examiners made it virtually impossible for Jewish students to receive high enough marks to enter the elite grounds of Moscow University. Jewish students were, after all, tested with problems that took professors — expert mathematicians — hours, even days to solve. Some problems had no solutions at all. The consequences of such practice were not just inevitable, they were deliberate: As the study shows, when Moscow University admitted 475 students to the mathematics faculty in 1979, only 10 of them were Jewish. This was no coincidence, but rather an example of common Soviet practice. In bringing the idea of "The Jewish People's University" to Subbotovskaya, Kanevsky and Senderov went against that practice. And in bringing "The Jewish People's University" to life as organizer, mediator and supporter, Subbotovskaya ultimately sacrificed herself for the sake of knowledge and justice.

In the summer of 1982, an arrest of Kanevsky, Senderov and a student of "The People's University" intensified the KGB's suspicion of the university to a most-undesirable level of scrutiny and investigation. Although the charges of anti-Soviet activity for which the men were detained had no relation to "The People's University," the connection of the suspects involved inescapably and unfortunately led KGB investigators to the university's figurative gates.

Bella Subbotovskaya provided the KGB with an easy scapegoat: herself. While her two colleagues and one of her students faced imprisonment, Subbotovskaya faced the questions of the KGB. She did so heroically — to a point where several versions of the investigation and of her testimony have morphed to legendary proportions. One version claims that after the KGB demanded that Subbotovskaya cease her underground teaching, they asked her to present a written statement of her purpose in upholding "The Jewish People's University." When she wrote them that her intention was "to give Jewish children the opportunity to learn math," the KGB commanded that she remove the word "Jewish" from her statement.

Yet, there exists an even more striking version: When asked personally, "What is the purpose of The People's University,'" Subbotovskaya reportedly answered without pause. “To give Jewish children the opportunity to learn math,” she said. KGB members never wrote down her answer.

Admirably, bravely, after years of teaching students, Subbotovskaya took it upon herself to “teach” the KGB. As “legend” has it, she did the unthinkable. Equipped with statistics and facts, Subbotovskaya personally requested a meeting with the notorious intelligence organization. She was going to prove why anti-Semitic discrimination by universities is a crime to the KGB. Everybody waited to hear news of her arrest, but news of her death came instead ... and with it, whispers of premeditated murder.

Subbotovskaya visited her mother regularly — a well-known fact. On the night of September 23, 1982, she was leaving her mother’s apartment; it was after 11:00 p.m., when there was virtually no traffic, no pedestrians, no movement on the streets. Bella Abramovna Subbotovskaya always called her mother upon safe arrival home, so when 12:30 a.m. came without a phone ring, her mother called the police. She received the news immediately: A terrible tragedy had occurred. An accident.

The police had broken convention and legal code in a most suspicious manner. It is atypical for members of the police force to deliver such news to a caller ... atypical for them to do it so promptly.

Her funeral was a silent one. Amidst Subbotovskaya’s students, colleagues, friends, family, and admirers, stood several unwelcome guests — several members of the KGB. Nobody volunteered to eulogize Subbotovskaya; nobody made a sound except for her mother. The elderly Rebecca Yevseyevna finally cried out: “Why won’t anybody pronounce one word?” Bella Abramovna’s husband quickly escorted the aged woman out of the funeral home.

A period of hushed judgments followed the mute memorial service. Subbotovskaya’s family and friends all discussed in low voices; all thought quietly as to why Subbotovskaya’s wounds did not match the apparent cause of death. An unspoken consensus developed into a softly spoken understanding. At the age of 44, Bella Abramovna Subbotovskaya became the first real victim of a seemingly non-combative, officially nonexistent, fight.

Charges of anti-Soviet activity and propaganda landed Boris Kanevsky and Valery Senderov behind bars, for 5 and 7 years respectively. Senderov was to serve 5 additional years in exile. They received the sentences the same year that Subbotovskaya lost her life. Officially, the imprisonment and charges were unrelated to the existence of "The Jewish People's University." But officially, like so many words in this essay, almost always belongs in quotation marks. Apparently, when Kanevsky and Senderov faced interrogation by the KGB, they largely answered to questions regarding "The Jewish People's University."

For a year following Subbotovskaya's death, "The Jewish People's University" continued to exist, but not to thrive. As was evident even while she was living, Subbotovskaya was irreplaceable as the foundation and the construction of "The People's University." The university had no walls, and it was up to Subbotovskaya to build them before each meeting. Initially, she opened her own home to students and professors, and when the demand for knowledge grew bigger, she reserved any meeting space that could hold hundreds of starving minds — sometimes class names were invented for reservation of large auditoriums. She copied notes for all of the students; she even kept them fed. And all for free. In the end, the only people to pay a cost for the existence of "The People's University" were Subbotovskaya, Senderov and Kanevsky.

"The Jewish People's University" began as a small gathering of 14 people in Subbotovskaya's two-room apartment. A month later, it developed into a meeting of over 30. At the end of 1979, 110 students were "enrolled." And by the time "The Jewish People's University" finally closed its doors, it boasted well over 350 alumni — talented young men and women, most of them victims of discrimination, many of them future professors of nationally ranking universities, many of them future famous mathematicians and physicists. They learned from the best. Professors at "The Jewish People's University" included D. Fuchs, A. Shen, A. Sosinsky, B. Feigin, M. Marinov, among others; it even included famous "visiting" Princeton University professor, John Milnor.

Bella Abramovna Subbotovskaya and her colleagues Kanevsky and Senderov did a completely selfless act — in part for justice, in part for the unadulterated sake of education. The "Jewish People's University" gave Jewish children the opportunity to learn math — it is as simple as that. It gave Jewish children the opportunity to focus on their studies, instead of their ethnicity. In doing so, "The People's University" defied an entire social system.

Bella Abramovna Subbotovskaya
(circa 1980)
Courtesy of A. Kanel-Belov and I. Muchnik

Circa 1961
Courtesy of I. Muchnik

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JEWISH UNIVERSITY[a]

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To the Memory of
Bella Subbotovskaya

In the summer of 1980, I received at my home two guests with whom I had not been previously acquainted — Valera Senderov and Borya Kanevsky, both not yet in prison at that time. They came to me to make the arrangements concerning my participation in the endeavor, which proved to be quite successful for the last two years: parallel lecture courses to the MekhMat curriculum for young people who had been unfairly denied admission by the Moscow University Admission Committee. The names "Jewish University," "Jewish People's University," and even the acronym ENU[b] appeared later,[c] although what is true is true: The majority of the victims of the scoundrels from the examination and the Admission Committee in Moscow University were sinful in the fifth point.[d] (Actually, the students selected in 1980, whom I taught, were victims only peripherally: this was the year of the Olympic Games, and the privilege of taking July exams for early admission to the University and MIFI[e] was canceled, thereby depriving potential victims of a "safety net" provided by "reserve" institutes where they could ap-

[a] Translated from the Russian by Roman K. Kovalev, The College of New Jersey, Department of History, Ewing, NJ 08628, USA; e-mail: kovalev@tcnj.edu. Footnotes marked by BKR belong to A. Belov-Kanel and A. Reznikov.

[b] Еврейский Народный Университет.

[c] An "official" name did exist: "Courses for the Upgrading of Professional Qualification for Instructors of the Evening Mathematical Schools," but, of course, it was not used by the students. –BKR

[d] Ethnicity — which in the Soviet parlance was referred to as "nationality" — was entry # 5 in the Soviet passport. Entry # 5 became a euphemism for "Jewish." – Editor's note

[e] MIFI is the acronym for Moscow Institute for Physics and Engineering. Regular entrance examinations in all Soviet Universities begin on August 1. –Editor's note.

ply following the flunking of exams at MekhMat. In the absence of the safety net, they, circumventing the University, sought admission to various institutes of the "kerosinka" type.[f] No one wished to risk the possibility of being drafted and sent to Afghanistan.

The "professorial" staff was quickly assembled. The team was robust: Alyosha Sossinsky taught algebra (Borya Feigin later replaced him), Andrei Zelevinsky taught lessons on analysis, and I was left with analytical geometry and linear algebra. I recall Andrei's first lecture: he wrote on the chalkboard the formula

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

and said that this formula contained all the mathematical wisdom: integral, differential, radical, e, π and infinity. In general, Andrei, lacking experience in teaching calculus (neither had he experience in research in analysis), delivered a brilliant course; I recall asymptotic series with the convergence sectors of which I had no clue previously. My course was more standard, although I recall that I introduced functors in the third or fourth lesson (I attempted to explain something that I still do not understand despite the fact that in every textbook it is written: the isomorphism between a finite-dimensional vector space and its conjugate depends on its basis while the isomorphism with the second conjugate does not depend on anything). All this came after the organizational meeting at Bella Subbotovskaya's apartment.

Several words should be said specifically about Bella. I studied with her in the same group at MekhMat, and we had known each other since 1955. We were not particularly great friends, since it was not easy to be friends with Bella. Nervous, loud, unusually demanding of everyone, she did not fit into the usual posse. Our class was very strong (Serezha Novikov, Vitya Palamodov, Galya Tyurina, Sasha Olevsky, Volodya Zorich, Sasha Vinogradov were all in our class); we showed off to one another and never suspected that the awkward, noisy Bella was one of the best mathematicians amongst us. I can best recall the

[f] *Kerosinka* is a nickname for the Gubkin Oil and Gas Institute. The students of the institute are known as *kerosinshchiks*. – Editor's note

various amusing stories. In the summer of 1957, packed into freight train cars, several hundred university students were sent to work to *tselina*.[9] Farewells — everyone was excited. And suddenly — who is this? — but it is Bella, believe it or not, with a bald, shaved head. Her mother is with her — Rebekka Evseevna — I knew her (her father was killed in the War). But, mothers were saying their goodbyes to many, including myself. Her mother was actually going *with us* to the Virgin Lands! On some small way-station, a grim-looking *komsomol* worker rolled up to our car and asked: "And why is your mother going?" In response, Rebekka Evseevna took out of her purse a *komsomol* pass, all is written there: she is going to the Virgin Lands by the calling of her heart. The *komsomol* guy moved off. We ended up working in different places at the Virgin Lands. I met Bella only once and received from her a scolding for the strike we had initiated. But, the guys who worked in the same team with the mother and daughter of Subbotovskayas spoke endlessly about the dinners prepared by Rebekka Evseevna.

When we graduated from the University, our ways parted. I found out later that she studied with Lupanov (the present chair of MekhMat; I do not wish to speak of him) in graduate school, married, had a daughter, published (under the name Muchnik) several outstanding studies, and defended her dissertation. Then, some sort of a rupture — I do not know the details. There was a divorce with a reversion to her maiden name, illness, and a return to life in the form of a teacher of elementary classes in an ordinary Moscow secondary school. The only thing that remained from her former life was the chamber orchestra of Moscow University, where Bella played the viola until her last days (and the orchestra bus later took her to the cemetery). Bella never taught at our University. Her functions were strictly organizational — such was her choice.

[9] *Tselina* means virgin soil in Russian. In 1954 Nikita Khrushchev initiated the "Virgin Lands Campaign" to open up vast tracts of unused (virgin) steppe in the northern Kazakhstan and the Altai region of Russia. With all this new land, it was necessary to bring there a vast amount of people from all over the Soviet Union. The Komsomol (Young Communist League) was charged with providing appropriate workforce. More than 300000 people — in particular, hundreds of thousands of soldiers and students — were involved. –Editor's note

We worked selflessly with our students, wrote synopses of our lectures (which were photocopied somewhere on copiers[h]), and arranged home consultations. It was difficult for the students: technical drawing and other technical disciplines, material sciences of all sorts during the day, and contrived mathematics in the evening. Out of the students of my course (and there were over 70 of them[i]), only a handful became professional mathematicians: Vitia Ginzburg, Alyosha Belov-Kanel, Fedya Malikov, Sasha Odessky, Andrei Reznikov, Borya and Misha Shapiro — who else? But the benefit of the studies was brought to many, I am sure. We changed our meeting places, assembled at Bella's school, at the Oil and Gas Institute ("Kerosinka"), at the Humanities building and later at the Chemistry Department of Moscow University. In the majority of cases, our "seminar" was more or less "legal" (I myself went to the Chemistry Department to ask for permission from the assistant chair; and at the "Kerosinka," while I cannot be fully sure, I heard that one of the students received a reprimand from the komsomol bureau for missing my lectures). Bella collected the fivers[j] and brought mounds of sandwiches (later KGB made attempts to present these fivers as evidence of her guilt).

I forgot to mention, Borya and Valera came to one of the first classes to question the kids concerning where they and their classmates had applied, and where they had been accepted (they were collecting this data to document evidence of what all knew anyway, but many pretended not to know — about the discrimination against Jews in the admission to Moscow University). A year passed and we moved on to the second year; Borya and Valera took on to pull the second assortment of classes. (Incidentally, Valera Senderov was an instructor of the highest quality). I switched over to differential geometry, while Borya Feigin lectured about Lie algebras and D-modules. Our seminar was working. In March 1982, Jack Milnor, André Haefliger, Bob MacPherson, and Duza McDuff came to Moscow for a private visit, as it is presently

[h] B. Kanevsky was able to make photocopies, a task totally impossible in those days, in a "box"; see footnote e in Vershik's article *Science and Totalitarianism.* –BKR

[i] 120 students came to the first lesson! –BKR

[j] Five-ruble notes. –Translator's note

called. Milnor, a great mathematician and lecturer, gave a talk specifically for our students (Alyosha Sossinsky translated). There were many people who were not our own — we gave room to all.

The academic year ended and suddenly all hell broke loose. I will attempt to reconstruct the development of these dramatic events.

In June 1982, Serezha Lvovsky (whom I did not know at that time) came to the Laboratory Building of Moscow University where I worked. He took me outside. This was the news: Senderov and Kanevsky (and a student from their section) had been arrested.[k] He explained the reason for the arrests: In April, someone dispersed leaflets against *subbotniks*[l] (can you imagine — *subbotniks*!). The left-over leaflets were not discarded but were retained for the next year (oh, God!). And so they were caught during the search (there were always enough informers) and had the list of their class in the "Jewish University" confiscated. Information about us, supposedly, did not surface — although who knows ...

We met with Andrei to discuss what we should say in the event we were called to questioning (we were not). We phoned Bella and decided not to resume the courses in the autumn (we had dreamed about the third year).[m] Stressful months passed by, and in August (Brezhnev had several months more to live), the KGB called on Bella. After she had gone there, we met (this was our last meeting; we spoke once more later by telephone). I will lay out the story as she told it to me.

On the morning of that day, her phone rings and a male voice states: "Bella Abramovna, I — such-and-such (not clear), would like to meet with you." Bella took him for someone else, since she was expecting a call. I, she said, am very busy today; I will be going to various places, so let us meet up at the subway station — Kolomenskaya Station, first car of the downtown-bound train. At the designated hour Bella is there. She looks around — no one. But, there he is, a wide-shouldered person

[k]Senderov and Geltser were arrested on June 17, Kanevsky on June 21. -BKR

[l]From Russian *Subbota* — Saturday. In Soviet times *subbotnik* was allegedly enthusiastic and allegedly volunteer unpaid work of the population on Saturdays supervised by the Communist Party officials and organized a few times a year. In fact, it was neither enthusiastic nor volunteer. –Editor's note

[m]The classes, although in a different form, continued until the spring of 1983. –BKR

with a bull-like neck. “Clearly, not a hero of my novel,” Bella told me.

The train passes, and then another. Bella waits, and suddenly this person walks up to her: “Bella Abramovna?” Bella, with a smile — “so it is you who is waiting for me? I need to go to Kuznetsky Most now, so let us go there together.”

“No,” — he answers — “we will go to a different address.” And then he takes Bella by her arm. They go upstairs and there is a car waiting there for them. They drive not to the Lubyanka but somewhere else; Bella told me this address, but I do not remember it. They led her into an office to a young person wearing shoulder-straps — a senior lieutenant or a captain. In Moscow, he says, a gang of “tutors” is operating, who rob people under the guise of preparing them for exams. And there, he slaps down Senderov’s list on the desk. Bella, of course, began to flirt with him (an incorrigible female fortune!). But, how can you, she says, these people are already students and we instructed them in mathematics for free (and the fivers? — and slap — another list — of those who gave the fivers — also in Senderov’s class — but these fivers are for snacks).

“And so you can also come — you probably also like mathematics.” It was clear that she took a liking to the KGB officer. They spoke about this and that — about mathematics. Well, goodbye and here is your pass (for exiting — without it, one could not leave). And here is a protocol — sign it. Bella reads the protocol — no, I did not say this, so I will not sign. He begins to entreat her, saying that this is just a formality that is expected of him. But, no, she entrenched (she was always stubborn). Well, fine, do not sign it, but think about it some more. If you change your mind, come back to us. He gave her his name, internal telephone, and that was all.

Bella tells me all of this and suddenly says: I have decided to go there again; he did call on me after all. I said: “under no circumstance should you go!” Well, she said, I shall chat with him; he, it seems to me, understands everything and this may be of benefit to our boys (who had been arrested).

Just try to dissuade her! It would have been great to just tie her up and not let her go ...

6

The next day brings the last call from Bella. Well, did you go? Yes, but he did not receive me. He said rather dryly: I do not need anything else from you. Several days later,[n] Bella died under the wheels of a truck in a quiet and deserted alley where even a bicyclist rarely passes each hour. An account of an eyewitness: In a whirlwind, a truck ran along the alley and struck Bella, who walked on a sidewalk or next to it, and disappeared. Bella was delivered directly to the morgue. It was difficult to recognize her in the coffin: her battered head had been poorly reconstructed with clay at the morgue. There was a mass of people at the funeral, but the conversations concerning the circumstances of her death were limited — it was not the time. I do not know, has the time come, has the truck been found and its driver; has the chain of events been reconstructed from that side, that of the KGB? "Jewish University" ceased to exist, and with *perestroika*, mass emigration and the lifting of limitations on admission of Jews to Moscow University, the problem itself ceased to exist. And here ends my story.

[n] It occurred on the night of September 23, 1982. Bella Subbotovskaya was 44. –BKR

Andrei Zelevinsky

Born in 1953.

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REMEMBERING BELLA ABRAMOVNA[a]

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It was Dmitry Borisovich Fuchs who introduced me to Bella Abramovna Subbotovskya. This apparently occurred during the summer or early autumn of 1980. She proposed that I participate in the work of the"People's University" and I recall that it did not take long for me to consent. The risk of participating was apparent (even for me, in my youthful thoughtlessness at that time); hence, the quick decision was not so trivial. I had two reasons: an immediate sense of "rightness" of the whole endeavor and that absolute feeling of trust which I felt towards Bella Abramovna, something that never left me during the course of our acquaintance and contact (which unfortunately was not very long).

A few words should be said about background. In those years, the atmosphere of deep absurdity reigning in the Soviet society was so apparent for the people of my circle that there was no need to discuss it. The most cannibalistic era of the Soviet regime was in the past and there were few who seriously accepted the official ideology. But, open dissent was still punished. Official anti-Semitism was flourishing and was promoted at all levels in conjunction with general distrust towards the *intelligentsia* and culture. Since the majority of the population had formed during the period of Soviet power, the regime appeared unshakeable and eternal, while active dissidents seemed as quixotic idealists (as the later developments had shown, in reality they proved to have more foresight than my friends and I).

However, let us move closer to the matter. The individuals that were responsible for admission to MekhMat of Moscow University by then lost the last shred of human decency. The lightest suspicion of Jewish origins was enough to make the admission practically impossible. And in addition, for greater absurdity, many of the strong students who

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had graduated from the leading mathematical schools — often having proven themselves at mathematical olympiads of various levels — were weeded out regardless of nationality (apparently being "socially alien").

Although cadre policy based on the same principles led to a dramatic decline of the instructors' level at MekhMat, there were still relatively many mathematicians and instructors of high caliber, remains of the past. One of the greatest virtues of MekhMat was the traditional system of fundamental mathematical education at the lower level courses. Without access to this system, for many of the most capable and seriously involved math students, the road to professional mathematics was, if not totally closed, then at least greatly hampered.

Bella Abramovna's and her like-minded people's idea was humane and simple: attempt to at least partially restore fairness by offering students who were seriously interested in mathematics the possibility of receiving that fundamental mathematical education which the administrators of MekhMat deprived them. This idea could not but evoke a response from me, not only based on moral grounds, but also because, being myself Jewish and a graduate of Moscow Mathematical School No. 2, known at the time for its free-thinking spirit, I easily identified myself with my future students (although I was lucky, and my journey to mathematics was much easier).

Among the organizers of the People's University, PU for short,[b] aside from Bella Abramovna, I also met Boris Kanevsky and Valery Senderov. I had no doubts that they all, in addition to organizing our classes, were involved in other "illegal" activities. According to an unwritten agreement, I never talked with them about these subjects, assuming (apparently naively), that this may serve as a defense in the case of KGB's (Committee for State Security; I explain this to those lucky ones for whom this dreadful acronym does not mean much) interest in my persona: "well, I know nothing, they asked me to deliver a couple of lectures on mathematics for young people, but why and for what reason, I had no idea ..." I suspect that many of my colleagues at PU shared a similar "ostrich-like" position with me. This agreement was observed with great

[b]A commonly accepted name, I believe, it did not have; among the other names used I recall "Open University" and "Jewish University."

tact from the side of Bella Abramovna and Boris Kanevsky, with whom I mainly dealt (Senderov, as far as I recall, appeared at our classes not very often and was not involved in their day-to-day running. Perhaps, this was different in other sections). The only exception that I can now recall was an evening meeting with a bard-dissident, Petr Starchik, at Bella Abramovna's apartment, where she invited my wife and myself together with several students and instructors of PU. The evening, incidentally, was wonderful; the reader can acquaint him/herself with a biography of Starchik and his songs, for example on the webpage *http://www.bard.ru*.

Several words should be said about the organization of lessons during those two years (1980–81 and 1981–82) when I taught at PU. The lessons were given once a week on Saturdays at various places: most commonly at the Gubkin Oil and Gas Institute (the famous "kerosinka"[c]), where many of our students had studied. Boris Kanevsky, in addition to running recitation sessions in my calculus course, photocopied and distributed to the students lecture notes and handouts with exercises (now it is almost impossible to imagine what a serious crime the Soviet state considered the unsanctioned use of the photocopying machine; in accordance with the above-mentioned agreement, I never asked him how he gained access to the photocopier and what other printed material he created on it). The rest of the practical organization lay on Bella Abramovna's shoulders, who in my eyes was the soul of our cause. She composed lists of students, led the count of enrollment, arranged places for class meetings, informed all about any possible changes in scheduling, made sure that classes met and adjourned on time, brought all the materials necessary for classes (for instance, chalk), and even made delicious sandwiches, which we all consumed during breaks. She accomplished all these tasks with a smile and without obvious efforts. In general, it always seemed to me that her mere presence at lessons and breaks created a wonderfully pleasant, warm, and homely environment.

[c]Kerosinka is a nickname for the Gubkin Oil and Gas Institute. A Russian kerosinka is a kerosene-burning cooking device, low-tech but efficient in Russian conditions. The students of the institute are known as kerosinshchiks. – Editor's note

She took care of all practical everyday problems of all the instructors. By the way, it goes without saying that no one received any money for their work (I am not sure, perhaps a little contribution was collected from the students for photocopies and such expenses).

During my two years of work at PU, I taught a lecture course on calculus with elements of functional analysis. Fuchs, at the same time, taught geometry, and algebra was at first taught by Aleksei Bronislavovich Sosinsky, and then my old friend and classmate, Boris Feigin.

It took me some time to choose the program of my course. On the one hand, the general idea was to explain basics of calculus, without delving too much into more advanced topics. On the other hand, the majority of our students studied full time at the applied mathematics departments of decent technical schools, and thus already had some knowledge of calculus, especially on a “technological” level. Therefore, I did not wish to develop a course on the basis of a standard MekhMat curriculum for freshmen: I was afraid that the students would quickly lose interest, thinking that I am not telling them anything new. The way I resolved this dilemma was by attempting to offer traditional ideas in new packaging. The form of this packaging included ideas from several, particularly French sources: “Foundations of Modern Analysis” by J. Dieudonne, “Differential Calculus and Differential Forms” by H. Cartan, and even “Functions of a Real Variable” by N. Bourbaki (may V.I. Arnold please forgive me). With such an approach, the elements of topology and functional analysis were introduced rather early, providing an opportunity to put forth the principles of differential and integral calculus working with functions taking values in the Banach spaces. Thus, even familiar standard facts were treated in a new light, offering students an opportunity to better appreciate and feel the logic of the arguments. It is not up to me to judge the success of this attempt. In any event, it seemed to me that students received my course with interest and understanding.

I am proud of the fact that a number of students who attended my course overcame all the obstacles and became highly successful professional mathematicians: Aleksei Belov-Kanel, Arkady Berenstein, Viktor Ginzburg, Feodor Malikov, Andrei Reznikov, Mikhail Shapiro (I ask for forgiveness if I have omitted to mention someone). I hope that in their success there has been a grain of my input; but, unquestionably, they owe a lot more to Bella Abramovna.

The studies at PU continued without obstruction for several years, until the coming of the merciless hounding. Several people connected to PU, including Kanevsky and Senderov, were arrested in June 1982, and on the 23$^{\text{rd}}$ of September of that same year Bella Abramovna died tragically. As far as I know, the circumstances of her death (assassination?) have still not been uncovered. I can only say that every person with whom I have discussed the matter, none of my friends and colleagues had the slightest of doubts that the KGB arranged her assassination. Why? If the authorities wanted to shut down PU as soon as possible and without any extra noise, then extinguishing Bella Abramovna, presenting her death as an accident, was the simplest means of achieving this goal. As I said earlier, everything depended on her.

Unfortunately, I knew very little about Bella Abramovna (and found out little during the course of our brief acquaintance), besides that she finished MekhMat and was Fuchs's classmate. Her warmth, kindheartedness, and optimism immediately made one predisposed towards her and feel at ease with her. She showed motherly affection to PU's students and, as far as I can tell, evoked equally warm feelings in response. The organization of PU demanded of her great courage and resolve, and the support of its continuation demanded incessant efforts; but in her behavior there was no sign of self-importance or "showing off." In the general atmosphere of "phoniness" — the most common feature of Soviet society of those years — the very fact of precise and continuous functioning of PU, provided by Bella Abramovna's efforts, gave students (and also the instructors) a significant lesson in professionalism and responsibility.

I am grateful to fate for the acquaintance and cooperation with this remarkable woman. For me she will always remain a moral compass, and my work at PU — a subject of pride and wonderful recollections.

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BELLA ABRAMOVNA SUBBOTOVSKAYA[a]

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It somehow worked out that, while I lived together with Bella for ten years and then remained in close contact with her for another twelve years after the divorce, we had no opportunity to tell one another about our childhood and days of youth. This is a major oversight that limits my description of Bella as a person, who was the soul of the People's University. However, there is one detail of her childhood I can relate. Beginning with first grade, but perhaps even earlier, Bella had fallen in love with mathematics. She read any book on mathematics that fell into her hands and solved all mathematical problems and exercises in her textbooks. Just imagine Berezanskii's exercise book for 5th Grade — a huge text containing several thousand problems. Bella received this book in September and in October she had a pile of notebooks filled with solutions to all of the exercises found in the text.

How do I know this? — From her own words. She attempted to convey to me her vision of the beauty of mathematical problems. "A problem cannot be uninteresting, it can only be simple or complicated" — this was her saying. Bella was interested in mathematical logic, but in every talk we attended together she found something beautiful — even when the topic was topology, theory of measures, or computational grids for solving differential equations in partial derivatives. After a seminar where a rather weak paper had been presented, Bella usually put forward some sort of a remark suggesting how she would alter the structure of the question, or how she would continue the inquiry posed in the paper. She would make a completely different problem, and I quite honestly could not find any relation between the two; I would ask,

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puzzled, what connection there was between the problem she posed and the one we had heard about during the talk. Usually, she would point to a curious association provoked by the topic of the presentation. Her main interests lay not so much in the formulation of the question, but in seeking connections between seemingly unrelated issues. She loved the aphorism: "There will always be more questions than answers in mathematics, because every answer gives birth to several new questions." It is a shame that I cannot recall who was the first to say this beloved phrase. Even an answer to a not-so-interesting question provides a plethora of new questions, amongst which one may very often encounter at least one that is interesting. In this way, mathematics hides beauty, she loved to say.

The most wonderful thing is that Bella did not just perceive everything beautiful in mathematics; she had the ability to convey her perception to the most varied types of people. It should be said that listening to her stories about the beauty in mathematics was a learning experience. It was the case especially when Bella spoke of some "infinitely complex and unattractive" formulas. Derivations of such formulas presented in themselves an endless row of technical calculations, but Bella had the gift of being able to describe unique peculiarities of the formulas so that they became beautiful. They did not become simple, but their complexities were understood after her explanation, and seemed natural. "People" — Bella loved to say — " found out that no one is protected against complexities of the world through mathematics, after they had created it." Before then, people did not think much about the complexities of life, did not notice, and did not remember them. Mathematics permitted the recording of everyday interactions in the form of complex, but beautiful constructions. It is precisely this beauty that permits us to hold these constructions in the field of our attention and manipulate them. Bella considered "complex" as something that could be observed through a chain of a hundred or a thousand simple things. The larger part of everyday relations, she believed, was not perceived by humanity — it was left unnoticed. It could leave an impression as an immaterial feeling, which is quickly washed away, and then the construction ceases to exist for us.

I have no specific intentions of characterizing Bella as a philosopher, although she was unquestionably a person of acute thinking. With the above statements I simply wished to observe that Bella's love of beauty in mathematics was astonishingly harmonized with her critical evaluation of her own abilities in understanding the complexities of things. I shall risk further by saying that Bella very keenly felt the principal limitations of a human in comprehending complex constructions. In this regard, it may seem paradoxical that Bella considered mathematics the simplest of sciences, since the constructions studied in mathematics are most precisely determined by and connected with logical trappings. On many occasions, she brought to my attention that the most complex issues, specifically those that are insurmountable, we encounter not in mathematics, but in connection to nature, art, and, particularly, in our human relationships.

Perhaps, she would say, there is utility in cybermodels of art, but one should be aware that the insight provided by these models is totally different from the kind of understanding which comes, for instance, from listening directly to music. This direct, "instant" understanding cannot in any way be mathematical because it is very personal. Its reception cannot be recorded so as to imprint it into the memory of others. It lives inside a person for so long as they live, and dies with them. The unique advantage of mathematics is that it has the ability to capture something significantly beautiful. Of course, it is far from everything, but still something important. And it is this part that can be passed from one individual to another.

I would like to relate an episode from our lives — how we met and married. I believe it reveals the character of her nature which overwhelmed everyone she encountered — her liberated sense of accepting all of her surroundings.

I met Bella for the first time at A.A. Lyapunov's seminar on cybernetics in 1960. Zaripov presented a paper on how he composed music on the computer. She and I were very excited by this lecture, particularly by the melodies composed by a machine, which not only did not seem machine-like but also were simply interesting. Bella sat just in front of me, and we often exchanged winks and nods during the course of the presentation. After the lecture, we wandered through university corridors and discussed various possibilities of computer-generated music. Somehow we immediately departed from the idea of composing music and came to discuss the issue of how one might study musical compositions with the help of a computer.

I was a very sassy character. With no musical background, half a year prior to that seminar, I came up with the idea that I must study the statistics of musical phrases in Jewish folksongs, and that this had to be done using a computer. I had a large collection of these songs — more than 800, some in various renditions. The main thought that preoccupied me was to find an automatic method of distinguishing those melody fragments in the songs that could be examined as independent musical phrases.

So, it was this issue that Bella and I began to discuss after the seminar. But, we had very different interests in our discourse. I attempted to discuss the question that preoccupied me and seek ideas in Zaripov's presentation that could help me resolve this question. Bella, however, was concerned with something entirely different. She suddenly understood that with me she could sit, listen and understand songs that were of great interest to her. Computer capabilities concerned her little. She just immersed herself in Jewish folk melody.

To imagine for oneself the entire ridiculousness of our discussion, it is imperative to note that Bella, at that time, was a "real cyber-guru," knowing a great deal and already possessing published results on the comparative complexity of various bases of algebraic logic. In addition, by that time she had already finished ten years of music school at the conservatory, playing the violin. Besides, being a 5th-year student at Moscow University, she was also a 1st-year student at the Gnessinykh Musical Institute[b] in vocal performance. It seemed as though she would have been preoccupied mainly with questions of cyberscience in relation to music. She had all the necessary knowledge to be involved in such

[b]Государственное музыкальное училище имени Гнесиных, a famous school of music in Moscow. It was founded at the height of a new wave of Russian enlightenment at the end of the nineteenth century by the sisters Eugenia, Helena and Maria Gnessins. –Editor's note

issues.

Everything turned out to the contrary. At that time, I lived in Gorky.[c] We agreed to meet regularly. It was a good thing that I could arrange frequent trips to Moscow and my travel expenses were reimbursed. Bella promised to inform me in advance of planned seminars that touched on subjects related to computers in music. We also agreed to correspond on the topic of how to simplify the study of Jewish folksongs, so that it would be possible to begin the interesting work as soon as possible.

We corresponded for several months very intensively and, although the project took the lead in our letters, we wrote much about ourselves to one another and about other subjects which were on our minds. At the end of March, Bella became seriously ill and landed in the hospital. I came to see her in early April and for one week we promenaded about the hospital yard, often escaping over the fence and strolling through the large park next to the hospital. The sensations were terrific and, at the same time, it seemed that we were both in prison. It was then that I proposed to Bella that we run away from this "prison" somewhere far, far away, for instance, to the city of Zhukovsky,[d] where I had an acquaintance.

During the course of this flight we decided to get married. This occurred in the summer of 1961. We settled in a little wooden house, in a 6-meter room[e] with a stove-heater, and an outhouse in the yard. But, this was no ordinary yard. Around it was a "beehive" of similar houses. In each lived three to four Jewish families with two or three children, grandmothers and grandfathers who spoke Yiddish quite well. Our wedding was held in this yard. (None of our neighbors had a room large enough.) All the inhabitants of the surrounding houses attended the wedding and sang many Jewish songs in Yiddish. Bella accompanied them on the violin. Thereafter, three marvelous people came to the

[c]Before 1917 and after 1990, Nizhny Novgorod. The city lies at the confluence of the Volga and Oka rivers, 260 miles (420 km) east of Moscow. –Editor's note

[d]This remark is ironic. Zhukovsky is a small town located just 22 miles (35 km) southeast of Moscow. It is known due to the Central Aero-Hydrodynamic Institute located there. –Editor's note

[e]Approximately 60 square feet. –Editor's note

wedding: Reveka Boyarskaya — a famous composer of Jewish songs in the 1930s, her husband — a critic of Jewish Yiddish literature (who had been just rehabilitated in 1960 after spending years in prison), and Ovsey Driz — a wonderful Jewish poet, who became our close friend for many years.

Driz brought with him the score of a brand new song, written by Reveka Boyarskaya, based on his poems that were dedicated to the memory of Baby Yar.[f] Driz sung it. Everyone wept, including the children, who had not understood a single word but heard the endless tragedy of the melody. And although this was a wedding, everyone asked Driz to sing it again, and Bella and I were the first to ask. Bella also proposed to accompany him on the violin. Following the performance, everyone kissed her and Driz. This was the first performance of the marvelous song to the people. How thankful Bella was to Boyarskaya and Driz for the gift of this song! She was all around Boyarskaya, touching her, gazing on her as if gazing on a living miracle. For half an hour she forgot about the wedding and about me, and then later begged for understanding that she had seen a treasure of the most brilliant kind in Boyarskaya's soul and had been unable to tear herself away. Several years later, when the song became famous worldwide, thanks to the concerts of the wonderful performer of Jewish folksongs Nechama Lifshitsaite, Bella found a record in a store with the song (officially, the song was entitled "Mother's Song"). She bought two hundred copies of the record to give to friends.

Love and grandiose creative plans permitted us to easily overcome the mounting inconveniences of our existence in almost total absence of money. Bella worked with A.A. Lyapunov on problems of optimization. Concurrently, she entered graduate school where she continued comparative studies of the complexity of various bases of the algebraic logic functions. I was engaged in biocybernetics at the A.A. Vishnevsky Institute of Surgery with a wonderful psychiatrist and thinker S.N. Braines.

We put our project of analyzing Jewish folksongs on the computer

[f] Also spelled Babiy Yar or Babi Yar. A large ravine on the northern edge of the city of Kiev in Ukraine, the site of a mass grave of at least 100000 Jews, whom Nazi German SS squads murdered in 1941. –Editor's note

to the side, since Bella had to also continue her vocal studies at the Gnesinykh Musical Institute, where she had already completed her first year. But, she did not study there much the second year. In January 1962, she became engrossed in a totally different idea, which caused her to sacrifice her studies at the Gnesinykh Institute. Just next to our house, a school for adults was opened on the Preobrazhenka and all the neighbors and teachers asked her to help get it started, since they had no math teacher. Beginning with the first classes, Bella was quite taken by the possibility and idea of teaching mathematics. Prior to that, I had not heard from her or from any of her friends about her love of teaching. I am almost certain that getting so seriously involved was unexpected for her, too. Soon, teaching at the school for working youth (as all evening schools were called at the time) came to occupy the main role in the discussions between Bella and myself. Most of her students were workers ranging from 25 to 40 years of age. She explained to me that the key difficulty with teaching such students was that they could not and did not want to do homework. For this reason, all curriculum had to be covered in class. She came up with the idea of grouping students into bunches of two–three people according to their relative knowledge and preparing specially designed exercises for each group. Six to eight such groups were formed. She found time to solve all the problems with each group. Towards the end of each lesson, all the exercises were solved in each group. Knowing that the majority of the students would not do home assignments, Bella still gave out homework, although these problems were very similar to those solved by the students in the previous class. Sometimes, she hit the target straight on as some students did do their homework. Such cases she viewed as great achievements, something that made it all worth the while to prepare homework assignments.

Her preparation at home for teaching was enormous. But, while knowing how quickly she could do everything, I could not understand how she managed to work in class with six to eight groups of students all at once. Only after some two to three months did she show me her "secrets." It turned out that she did not just simply invent the exercises. She also provided each problem visually in the form of a "hint." Bella loved to share her impressions of her students. She spoke of many as talented, but who had been tired by life. Quite often she noted that if these students had had the opportunity to study in childhood, they would have been able to organize their lives quite differently.

During February and March 1962, when we were expecting a baby, Bella, continuing to teach her adults, began to discuss the problem of "how to properly teach mathematics to the very young." And, with time, when our daughter was born, this question became the chief question for Bella. We moved many times, and in every part of Moscow where we lived, she was able to convince the school principals that it was imperative to organize an additional class in mathematics for the 1st–2nd graders. She called this class "Mathematics for the Curious."

The class was taught regularly, twice a week, with homework. This was not at all a little math club for resolving brain–rattling problems and puzzles, although she did utilize in her lessons games found in books, or of her own invention. This was a totally new program in the making.

Her friend, Olga Belyanina, who had married a Frenchman, signed her up for a subscription to a monthly magazine dealing with the issue of teaching children five to eight years of age. We bought books on mathematical games. Bella believed that physical manipulation which leads to solutions of logical and set-theoretical problems is a great mechanism for remembering emotional and spatial elements which acquaint small children with abstract constructs. Therefore, she included motion exercises in her lessons. Quite often, the space where she conducted her classes tuned into a field for games as children solved mathematical problems by moving from place to place.

Bella discussed with the editors of the journal that Olga had subscribed to for her, her ideas on the role of game manipulation in teaching children abstract concepts. In preparing her teaching programs, she consulted with A.N. Kolmogorov, A.A. Markov, S.P. Novikov,[g] with a few well-known French mathematicians, and her previous classmates who had become teachers.

The most important thing I remembered was the special attention she gave to the development of children's abstract thinking in connec-

[g]Russian mathematicians of the world caliber. –Editor's note

tion with binary relations and their properties. She familiarized children with the basics of set theory and described relations as graphs. She organized the lessons so that each mathematical construction was first introduced in a "wrapper" made of surrounding objects and their interconnections. For instance, each child had to indicate who, among those present, had been at his or her birthday celebration. As a result, a graph of preferences emerged, and based on this graph it was determined that the class had two leaders and that there were two girls and one boy who were practically never invited to birthdays, while at the same time, they had invited many to theirs. Bella explained to the children that next time they should not forget to invite specifically those three children to their birthdays.

Afterwards, she proposed to resolve another problem: "Let us find all the children who were invited by both leaders. Then, let us note those who were also invited by the three 'forgotten' children. The noted children comprise a certain part of the graph. It includes ... (calls out children's names). A part of the graph is also a graph. To underscore that this graph is part of a larger graph, it is called a subgraph of the larger graph. It should be noted, that this subgraph contains many arrows inside and many arrows coming from the parent (large) graph, which do not belong to the subgraph. On the other hand, the number of arrows that originate in the subgraph and go into the large graph is small. Count the number of arrows lying inside the subgraph, those that come in the subgraph, and those that leave it."

The concept of the subgraph and arrows that enter the subgraph was first presented in a blackboard picture of the graph. This picture she made herself, asking for assistance from the students. Immediately after such a concrete conceptualization, Bella taught more exact notions such as subsets dual to a given subset, oriented cycles in the graphs, etc. She possessed an amazing ability to bring a child into the sphere of complex formal relations through concrete "living" situations, and then abstract from such situations general formal properties. I could not believe that children five to eight years old could understand and remember these abstract concepts. I especially could not understand how lessons that addressed and elucidated these concepts could be interesting to such small children. To show me how all this worked, Bella invited me to her classes several times. All the children worked hard, with interest, as if nothing were more fascinating. These were the most ordinary of children. Every 10 minutes, Bella called for a 5-minute break, so that the lesson took 30 minutes of the 45-minute class. Furthermore, Bella divided the class of approximately twenty pupils into three groups: two small groups consisting of five children each, and one large one comprised of ten. She permitted one of the small groups to play games while the children in the second small group acted as her assistants. Along with her, they drew pictures, counted, drilled the basics, and learned to recognize their correct solutions in various situations. The main secret was that these children–helpers, together with Bella, taught the pupils in the larger group. Being the teacher's assistants was held in great esteem, but the helpers did not receive any points for correct answers or actions. Points were given only to the children of the large group. Bella re-configured the groups after each break and by transferring functions from one group to another and by controlling who, in each group, performed various tasks during the lesson, she made sure that, on average, each child participated uniformly. Perhaps this is not particularly well stated, but I perceived these lessons of hers as a well-directed play, performed at a good tempo to the great pleasure of the actors (children).

In addition to teaching at school, Bella was preparing a project on the larger program of teaching mathematics to groups of older kindergartners, since our daughter was still attending kindergarten. Unfortunately, Bella had no opportunity to test this program in actual lessons. These materials were lost. I am sure that today they would be in demand, especially her wonderful collection of exercises and games for the very young. They were constructed differently than those for the schoolchildren, although they were also directed to the comprehension of basic concepts of set theory, the main operations with sets, and binary relations.

While she was providing regular lessons in mathematical logic and pedagogy of mathematics for the young, Bella held positions at various technical research institutes, the last of which was the institute associated with the Frunze Machine-Building Factory. Bella made a living only from this applied work. Although sometimes she was able to find an interesting combinatorial problem even in this job, for the most part she was involved in programming and numeric computations. Bella did not like this engineering work, but performed it very carefully. She was most disturbed when she had to write work plans, which were already known to be impossible to fulfill by the designated time.

Our daughter Masha studied in a wonderful mathematics–oriented school, the soul of which was Boris Geidman, the renowned, in my opinion, teacher of mathematics. Bella, by want and not, came to be caught up in the affairs of this school. Above all, she became actively involved with the issue of where the children graduating from this school would go to pursue further study. Given that the majority of the kids were Jewish, with known limitations on entering an educational establishment of higher learning, this was a real problem.

Once (I do not recall now when it was) Bella approached me and offered the following: she would fully resign from her engineering job and would not receive her salary. Instead, she would teach children who had finished school, to prepare for the mathematics entrance exams to educational establishments of higher learning, especially to the Mekh-Mat (Department of Mechanics and Mathematics) at Moscow University. In addition, she decided to organize a public control over the procedure for entrance of Jewish children to Mekh-Mat. She also took it upon herself to help those who had "flunked" to put together petitions for reconsideration. But, most importantly, she came up with the idea that those who were denied entry to Mekh-Mat she would put through her own high-quality preparatory program, taught by her at home and/or at some public organization, disguised as a "math circle." She asked me, whether I agreed or not, to take on the financing of her and our daughter on some minimal level, as well as some additional minor expenses which were necessary for her to institute this program. Specifically, the latter involved the printing of teaching materials and food for the children during recess (Bella believed it was imperative for children to have a snack between classes). Of course, I agreed. Quite quickly, she found enthusiastic assistants among the students of Moscow University and other institutions of higher learning, as well as among teachers and professors. In this way, "People's University" began its work at building # 15/2, apt. 28 on Nametkin Street in Moscow. This activity fully engaged Bella and she was happy.

Bella took seriously and undertook with great zeal the preparation of the program that would provide a high-quality education, comparable to the level of the Mekh-Mat. She tapped into all of her acquaintances among eminent teachers, to create a program of the highest order and, at the same time, one that was compressed and realistic — one that would allow her to realize the program in the home environment and with a nominal workforce of enthusiasts. And this she was able to achieve. The University functioned for three years and prepared almost 100 mathematicians.[h]

But, in the summer of 1982, people from the KGB[i] asked her to dissolve it. They arrested V. Senderov, one of the key instructors at the university, and began to call on Bella as a witness to his case. I saw her off every time she went to the KGB. During this time we were practically inseparable. She told me about the conversations on Lubyanka[j] and shared her surprise at the fact that she felt no fear, but I was scared for her. She could easily anger KGB officers, and to make a criminal out of a witness was a common practice in Soviet Union. I thought she was out of her mind when she gave them replies such as these: "Do you truly need my answer to your question?" (to the question of whether Senderov taught mathematics); "I do not think you are interested in my children" (to the question regarding what she fed her children); "I do

[h] According to A. Belov-Kanel and A. Reznikov who wrote an introductory article for the 2005 Almanac of *Matematicheskoye Prosveshcheniye* the class of 1978 consisted of 14 students, while *each* of the classes of 1979–81 included over 100 students. The peak was in 1980 when 120 new students began studies in "People's University." For obvious reasons not all students managed to graduate. Belov-Kanel and Reznikov estimate that the actual number of graduates was in the ballpark of 100. Twenty of them became professional mathematicians and physicists at various universities and research labs all over the world. –Editor's note

[i] Soviet Secret Police. –Editor's note

[j] Lubyanka Square in Moscow is the site of the KGB headquarters. In Soviet times it was called Dzerzhinsky Square; Lubyanka became a euphemism for KGB. –Editor's note

not know why you are forcing me, for the fourth time, to acknowledge that I do have *propiska* on Nametkin Street."[k]

I had only two pieces of advice for her, which I repeated without end: not to go there if at all possible (if they were calling her by telephone) and to simply be silent, without remarks, when the questions were derisive. Bella did not agree. She viewed her answers as a means of retaining her composure and not saying anything unnecessary.

The collapse of the university did not break her. She began to compose songs to Berns' and Driz's poems; she compiled a plan for the composition of music to the large cycle of poems by Lorka. Her friends organized a small concert where they sang her songs. Bella managed to develop a program for the comparative study of infinite-basis algebraic logic, and also obtained the first results in this new direction of her mathematical explorations. It is difficult for me to imagine what kind of a new life Bella would have made for herself, but it is easy to imagine how happy she would have been with her grandson who is growing now in our daughter's family. She would have been just bursting with joy and would have found free time to play with him, despite being preoccupied by a thousand interesting things.

Bella's life ended suddenly and tragically on October 24, 1982. She was 44. Her motivation to live with a purpose did not come from great "lovers of humanity." It lived in her from birth. Her piercing desire to be useful was felt, it seems to me, by everyone who met her at least several times. I think that those several hundred people who knew her through the "People's University," or others who met her before then (for instance, during her student years), can recall that unique feeling for life with which Bella Abramovna Subbotovskaya inspired us all.

I was a witness to the way she designed her teaching of mathematics in the evening school for working youth; how she also accomplished this task with 1st- and 2nd-grade schoolchildren; and, finally, how she developed and realized the program of teaching an entire complex of courses in mathematics at the university level. In all three of these

[k]In Russian, прописка, a kind of a residence permit which, in effect, eliminated the freedom of movement inside the country. It was impossible for a non-Muscovite to settle in Moscow since the residence permit was never granted. –Editor's note

directions, she found novel opportunities in her preparations. They were put through her vision of mathematics and her amazing commitment to this discipline. She was absorbed by this work and very generously gave it her life.

About the Editor:

M. Shifman is Professor of Physics at the University of Minnesota. After receiving his PhD (1976) from the Institute of Theoretical and Experimental Physics in Moscow he went through all stages of the academic career there. In 1990 he moved to the USA. He has published over 260 research papers in particle physics and field theory and written and edited several books. In 1997 he was elected as a Fellow of the American Physical Society. He has had the honor of receiving the 1999 Sakurai Prize for Theoretical Particle Physics.

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If the answer to any of the above questions is yes, this is the book for you. Two essays written by the Canadian mathematician Ilan Vardi constitute its core. The first essay presents a thorough analysis of contrived problems suggested to "undesirable" applicants to the Department of Mathematics of Moscow University. His second essay gives an in-depth discussion of solutions to the Year 2000 International Mathematical Olympiad, with emphasis on the comparison of the olympiad problems to those given at the Moscow University entrance examinations.

The second part of the book provides a historical background around a unique phenomenon in mathematics, which flourished in the 1970s and 80s in the USSR. Specially designed math problems were used not to test students' ingenuity and creativity but, rather, as "killer problems," to deny access to higher education to "undesirable" applicants. The focus of this part is the 1980 essay *Intellectual Genocide* written by B. Kanevsky and V. Senderov. It is being published for the first time.