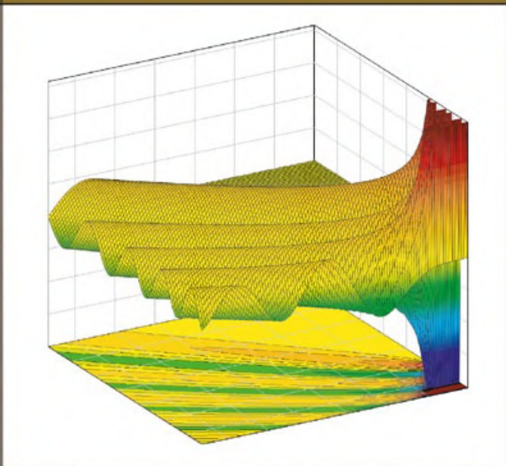
KEITH OLDHAM, JAN MYLAND,
& JEROME SPANIER

AN ATLAS OF FUNCTIONS

SECOND EDITION

WITH *EQUATOR*, THE
ATLAS FUNCTION CALCULATOR



CD-ROM

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PREFACE

The majority of scientists, mathematicians and engineers must consult reference books containing information on a variety of functions. This is because all but the most mundane quantitative work involves relationships that are best described by mathematical functions of various complexities. Of course, the need will depend on the user, but most will require information about the general behavior of the function in question and its mathematical properties, as well as its numerical values at a number of arguments.

The first edition of *An Atlas of Functions*, the product of collaboration between a mathematician and a chemist, appeared during an era when the programmable calculator was the workhorse for the numerical evaluation of functions. That role has now been taken over by the omnipresent computer, and therefore the second edition delegates this duty to *Equator, the Atlas function calculator*. This is a software program that, as well as carrying out other tasks, will calculate values of over 200 functions, mostly with 15 digit precision. There are numerous other improvements throughout this new edition but the objective remains the same: to provide the reader, regardless of his or her discipline, with a succinct compendium of information about all the common mathematical functions in use today.

While relying on *Equator* to generate exact numerical values, the *Atlas of Functions* describes each function graphically and gives ready access to the most important definitions, properties, expansions and other formulas that characterize it, and its relationship to other functions. As well, the utility of the *Atlas* is enhanced by the inclusion of sections that briefly discuss important topics related to specific functions; the new edition has many more such sections. The book is organized into 64 chapters, each of which is devoted to one function or to a family of closely related functions; these appear roughly in order of increasing complexity. A standard format has been adopted for each chapter to minimize the effort needed to locate a sought item of information. A description of how the chapters are sectioned is included as Chapter 0. Several appendices, a bibliography and two comprehensive indices complete the volume.

In addition to the traditional book format, an electronic version of *An Atlas of Functions* has also been produced and may even be available through your library or other information center. The chapter content of the paper and electronic editions is identical, but *Equator, the Atlas function calculator* is not included in the latter. The *Equator* CD is included with the print version of the book, and a full description of the software will be found in Appendix C. Because *Equator* is such a useful adjunct to the *Atlas*, stand-alone copies of the *Equator* CD have been made widely available, through booksellers and elsewhere, primarily for the benefit of users of the electronic version of the *Atlas*.

Though the formulas in the *Atlas* and the routines in *Equator* have been rigorously checked, errors doubtless remain. If you encounter an obscurity or suspect a mistake in either the *Atlas* or *Equator*, please let us know at koldham@trentu.ca, jmyland@trentu.ca or jspanier@uci.edu. An *Errata* of known errors and revisions will be found on the publisher's website; please access www.springer.com/978-0-387-48806-6 and follow the links. This will be updated as and if new errors are detected or clarifications are found to be needed. Use of the *Atlas of Functions* or *Equator, the Atlas function calculator* is at your own risk. The authors and the publisher disclaim liability for any direct or consequential damage resulting from use of the *Atlas* or *Equator*.

It is a pleasure to express our gratitude to Michelle Johnston, Sten Engblom, and Trevor Mace-Brickman for their help in the creation of the *Atlas* and *Equator*. The frank comments of several reviewers who inspected an early version of the manuscript have also been of great value. We give sincere thanks to *Springer*, and particularly to Ann Kostant and Oona Schmid, for their commitment to the lengthy task of carrying the concept of *An Atlas of Functions* through to reality with thoroughness, enthusiasm, skill, and even some humor. Their forbearance in dealing with the authors is particularly appreciated.

We hope you will enjoy using *An Atlas of Functions* and *Equator*, and that they will prove helpful in your work or studies.

January 2008

Keith B. Oldham
Jan C. Myland
Jerome Spanier

CONTENTS

Every chapter has sections devoted to: notation, behavior, definitions, special cases, intrarelationships, expansions, particular values, numerical values, limits and approximations, operations of the calculus, complex argument, generalizations, and cognate functions. In addition, each chapter has the special features itemized below its title.

CHAPTER 0

GENERAL CONSIDERATIONS

Functions are operators that accept numbers as input and generate other numbers as output. The simplest kinds receive one number, usually named the *argument*, and produce another number, called the *value* of the function

0:0:1
$$\textit{argument} \xrightarrow{\text{input}} \boxed{\text{function}} \xrightarrow{\text{output}} \textit{value}$$

Functions that need only one argument to trigger an output are *univariate* functions. *Bivariate* functions require two input numbers; one of these two *variables* generally retains the name "argument", whereas the second variable goes by another name, such as *order, index*, *modulus*, *coefficient*, *degree*, or *parameter*. There are also *trivariate*, *quadrivariate*, and even *multivariate* functions. Likewise certain functions may give multiple outputs; the number of such output values may be finite or infinite. Such *multivalued* functions are often conventionally restricted to deliver a single output, a so-called *principal value*, and this is the standard used in the *Atlas*.

In this chapter are collected some considerations that relate to all, or most, functions. The general organization of the *Atlas* is also explained here. Thus, this could be a good starting point for the reader. However, the intent of the authors is that the information in the *Atlas* be immediately available to an unprepared reader. There are no special codes that must be mastered in order to use the book, and the only conventions that we adopt are those that are customary in scientific writing.

Each chapter in the *Atlas* is devoted to a single function or to a small number of intimately related functions. The preamble to the chapter exposes any such relationships and introduces special features of the subject function.

0:1 NOTATION

The *nomenclature* and *symbolism* of mathematical functions are bedeviled by ambiguities and inconsistencies. Several names may attach to a single function, and one symbol may be used to denote several functions. In the first section of each chapter the reader is alerted to such sources of possible confusion.

For the sake of standardization, we have imposed certain conventions relating to symbols. Though this has meant sometimes adopting unfamiliar notation, each function in the *Atlas* has its own unique symbol, as listed in the *Symbol Index*. We have eschewed boldface and similar typographical niceties for symbolizing functions on the grounds that they are difficult to reproduce by pencil on paper. We reserve the use of italics to represent numbers (such as function arguments, x (or y); constants, c; and coefficients, a_1, a_2, a_3, $\cdots$, a_n) and avoid their use in symbolizing functions. When a variable is necessarily an integer it is represented by n (or m), rather than x, and

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often appears subscript following the function's symbol, instead of within parentheses. Variables that are often integers, but are not necessarily so, are represented by ν (or μ). Arguments that are frequently interpreted as angles may be represented by θ (or ϕ). Occasionally, as in Chapter 64, some symbol other than x is used unexpectedly to represent the variable, x being reserved to serve as the argument of a more general function. For the same reason, we may avoid using "argument" as the name of the variable in such cases. Notice that a roman f symbol is used to represent an arbitrary function, or as a stand-in for a group of specified function symbols, but the italic f is employed to signify the numerical value of $\mathrm{f}(x)$ corresponding to a specific x. Thus the axes of cartesian graphs may be labeled x and f, rather than the customary x,y found in texts dealing with analytical geometry.

We generally avoid the use of primes to represent differentiation but, where they have become part of the established symbolism, as in Section 52:7 and Chapter 56, this usage is followed. Elsewhere a notation such as f' merely connotes "another f".

In Sections 46:14 and 46:15, the z symbol serves as a cartesian coordinate. Elsewhere the symbol z is reserved to denote a *complex variable* equal to $x+iy$. All other variables are implicitly real, unless otherwise noted.

The names of most functions end with the word "function" (the error function, the Hurwitz function), but three other terminal words are commonly encountered. Some univariate functions that accept, exclusively or primarily, integer arguments are called "numbers" (Fibonacci numbers, lambda numbers); a few bivariate functions are named similarly (Stirling numbers). Functions defined as power series of finite length generally take the name "polynomials" (Chebyshev polynomials, exponential polynomial). The word "integral" often ends the name of functions that are defined as integrals (hyperbolic cosine integral, Dawson's integral). Yet other functions have unique names that don't fit into the general pattern (dilogarithm, binomial coefficient). Adjectives relate some functions to a parent function (associated Laguerre function, incomplete gamma function, auxiliary Fresnel integral). An index of function symbols will be found following the appendices, while the names of all our functions are included in the Subject Index that concludes the *Atlas*.

0:2 BEHAVIOR

This section reveals how the function changes in value as its variables change, thereby exposing the general "shape" of the function. This information is conveyed by a verbal description, supplemented by graphics.

There are several styles of figure that amplify the text in Sections 2 and elsewhere. The first is a cartesian line-graph of the function's values f plotted straightforwardly versus its argument x. Frequently there are several lines, representing different functions, plotted in different colors on the same graph. The second style of figure, particularly suitable for bivariate functions, is a three-dimensional orthographic view of the surface, showing how the function varies in magnitude as each of the variables changes over a restricted range. Pairs of such graphics are often used in Sections 11 to represent the real and imaginary parts of complex-valued functions. Not infrequently, complex-valued functions are inherently multivalued and, to convert such a function into a single-valued counterpart, it is necessary to "cut" the surface; such a cut appears as a grey "cliff" on the three-dimensional figure. Our three-dimensional graphics are colored, but the color plays only a subsidiary role. Some bivariate functions have *discontinuities,* such as a sudden change in value from $+\infty$ to $-\infty$, and when there are several of these, three-dimensional figures become so confused as to be unhelpful. In these circumstances, we sometimes resort to a third style of graphic, that we call a *projection graph*. In this perspective representation, a three-dimensional image is combined with a two-dimensional display, the axes of which correspond to two variables, with color being used to indicate the magnitude of the function at each point in the rectangular space. With trivariate and quadrivariate functions, graphical representation ceases to be useful and the behaviors of such functions may be described in this *Atlas* without the aid of graphics. You will encounter figures of other kinds, too, each designed to be helpful in the local context.

Some functions are defined for all values of their variable(s), from $-\infty$ to $+\infty$. For other functions there are restrictions, such as $-1 < x \le 1$ or $n = 1,2,3,\cdots$, on those values that specify the *domain* of each variable and thereby the *range* of the function. Likewise, the function itself may be restricted in range and may be real valued, complex valued, or each of these in different domains. Such considerations are discussed in the second section of each chapter. In this context, the idea of *quadrants* is sometimes useful. Borrowed from the graphical representation of the function f, this concept facilitates separate discussion of the properties of a function f according to the signs of x and f, as in the table.

Quadrant	x	f
first	+	+
second	−	+
third	−	−
fourth	+	−

0:3 DEFINITIONS

Often there are several formulas relating a function to its variable(s), although they may not all apply over the entire range of the function. These various interrelationships are listed in the third section of each chapter under the heading "Definitions" even though, from a strictly logical viewpoint, some might prefer to select one as the unique definition and cite the others as "equivalences" or "representations".

Several types of definition are encountered in Sections 3. For example, a function may be defined:

(a) by an equation that explicitly defines the function in terms of simpler functions and algebraic operations;

(b) by a formula relating the function to its variable(s) through a finite or an infinite number of arithmetic or algebraic operations;

(c) as the derivative or indefinite integral of a simpler function;

(d) as an *integral transform* of the form

0:3:1 $$f(x) = \int_{t_0}^{t_1} g(x,t)\,dt$$

where g is a function having one more variable than f, t_0 and t_1 being specified limits of integration;

(e) through a *generating function*, G(x,t), that defines a family of functions $f_j(x)$ via the expansion

0:3:2 $$G(x,t) = \sum_j f_j(x)\, g_j(t)$$

where $g_j(t)$ is a simpler set of functions such as t^j;

(f) as the *inverse* of another function F(x) so that the implicit equation

0:3:3 $$F\big(f(x)\big) = x$$

is used to define f(x) [this is graphically equivalent to reflecting the function F(x) in a straight line of unity slope through the origin, as elaborated in Section 14:15];

(g) as a special case or a limiting case of a more general function;

(h) parametrically through a pair of equations that separately relate the function f(x) and its argument x to a third variable;

(i) implicitly via a differential equation [Section 24:14], the solution (or one of the solutions) of which is the subject function;

(j) through concepts borrowed from geometry or trigonometry; and

(k) by *synthesis*, the application of a sequence of algebraic and differintegration [Section 12:14] operations applied to a simpler function, as described in Section 43:14.

0:4 SPECIAL CASES

If the function reduces to a simpler function for special values of the variable(s), this is noted in the fourth section of each chapter.

0:5 INTRARELATIONSHIPS

An equation linking the two functions f(x) and g(x) is an *interrelationship* between them. In contrast, one speaks of an *intrarelationship* if there is a formula that provides a link between instances of a single function at two or more values of one of its variables, for example, between f(x_1) and f(x_2). In this *Atlas* intrarelationships will be found in Section 5 of each chapter, interrelationships mainly in Sections 3 and 12.

An equation expressing the relationship between f($-x$) and f(x) is called a *reflection formula*. Less commonly there exist reflection formulas relating f($a-x$) to f($a+x$) for nonzero values of a.

A second class of intrarelationships are *translation formulas*; these relate f($x+a$) to f(x). The most general translation formula, in which a is free to vary continuously, becomes an *argument-addition formula* that relates f($x+y$) to f(x) and f(y). However, many translation formulas are restricted to special values of a such as $a = 1$ or $a = n\pi$; the relationships are then known as *recurrence relations* or *recursion formulas*. Such relationships are common in bivariate functions; a recursion formula then normally relates f(ν, x) to f($\nu-1$, x) or to both f($\nu-1$, x) and f($\nu-2$, x). A very general argument-addition formula is provided by the *Taylor expansion* (Brook Taylor, English mathematician and physicist, 1685 – 1731):

0:5:1
$$\mathrm{f}(y \pm x) = \mathrm{f}(y) \pm x\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}x}(y) + \frac{x^2}{2!}\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(y) \pm \frac{x^3}{3!}\frac{\mathrm{d}^3\mathrm{f}}{\mathrm{d}x^3}(y) + \cdots$$

Expressions for the remainder after this series is truncated to a finite number of terms are provided by Abramowitz and Stegun [Section 3.6], and by Jeffrey [page 79].

A third class of intrarelationships are *argument-multiplication formulas* that relate f(nx) to f(x). More rarely there exist *function-multiplication formulas* or *function-addition formulas* that provide expressions for f(x) f(y) and f(x) + f(y), respectively.

Yet other intrarelationships are those provided by finite and infinite series. With bivariate and multivariate functions there may be a great number of such formulas, and functions other than f may be involved.

0:6 EXPANSIONS

The sixth section of each chapter is devoted to ways in which the function(s) may be expressed as a finite or infinite array of terms. Such arrays are normally series, products, or continued fractions.

Notation such as

0:6:1
$$\mathrm{f}(x) = \sum_{j=0}^{\infty} \mathrm{g}_j(x)$$

is used to represent a *convergent infinite series*, where g is a function of j and x. Unless otherwise qualified, 0:6:1 implies that, for values of x in a specified range, the numerical value of the finite sum

0:6:2
$$\mathrm{g}_0(x) + \mathrm{g}_1(x) + \mathrm{g}_2(x) + \cdots + \mathrm{g}_j(x) + \cdots + \mathrm{g}_J(x)$$

can be brought indefinitely close to f(x) by choosing J to be a large enough integer.

Frequently encountered are convergent series whose successive terms, for sufficiently large j, decrease in magnitude and alternate in sign. We shall loosely call such series *alternating series*. A valuable property of such alternating series enables the remainder after a finite number of terms are summed to be estimated in terms of the first omitted term. When $\sum(-)^j g_j(x)$ is used to represent an alternating series, this result:

0:6:3
$$\left|\sum_{j=0}^{\infty}(-)^j g_j(x) - \sum_{j=0}^{J}(-)^j g_j(x)\right| < \left|g_{J+1}(x)\right|$$

plays an important role in the design of many algorithms.

In contrast to 0:6:1, the symbolism

0:6:4
$$f(x) \sim \sum_j g_j(x) \qquad j = 0,1,2,\cdots,J \qquad x \to \infty$$

which is reserved for *asymptotic series*, implies that, for every J, the numerical value of 0:6:2 can be brought indefinitely close to f(x) by making x, not J, sufficiently large. It is this restriction on the magnitude of x that makes an asymptotic expansion, though of great utility in many applications, rather treacherous for the incautious user [see Hardy].

If the function $g_j(x)$ in 0:6:1 or 0:6:4 can be written as the product $c_j x^{\alpha+\beta j}$, where c_j is independent of x while α and β are constants, then the expansions 0:6:1 and 0:6:4 are called *Frobenius series* (Ferdinand Georg Frobenius, Prussian mathematician, 1849−1917). In the case of an asymptotic series, β is often negative. When $\alpha = 0$ and $\beta = 1$, the name *power series* [Section 10:13] is used if the series is infinite, or polynomial [Chapter 17] if it is finite.

The *infinite product* notation

0:6:5
$$f(x) = \prod_{j=0}^{\infty} g_j(x)$$

implies that the numerical value of the finite product

0:6:6
$$g_0(x)g_1(x)g_2(x)\cdots g_J(x)$$

approaches f(x) indefinitely closely as J takes larger and larger integer values.

The notation

0:6:7
$$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\,\frac{\alpha_4}{\beta_4 + \cdots}$$

is a standard abbreviation for the *continued fraction*

0:6:8
$$\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cfrac{\alpha_3}{\beta_3 + \cfrac{\alpha_4}{\beta_4 + \cdots}}}}$$

in which each α_j and β_j may denote constants or variables. A continued fraction may serve as a representation of some function f(x). Continued fractions may be infinite, as denoted in 0:6:7, or finite (or "terminated"):

0:6:9
$$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\cdots\frac{\alpha_{J-1}}{\beta_{J-1} +}\,\frac{\alpha_J}{\beta_J}$$

though the former are most common in this *Atlas*. Of great utility in working with continued fractions is the equivalence

0:6:10
$$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\cdots\frac{\alpha_n}{\beta_n} = \beta_0 + \frac{\gamma_1\alpha_1}{\gamma_1\beta_1 +}\,\frac{\gamma_1\gamma_2\alpha_2}{\gamma_2\beta_2 +}\,\frac{\gamma_2\gamma_3\alpha_3}{\gamma_3\beta_3 +}\cdots\frac{\gamma_{n-1}\gamma_n\alpha_n}{\gamma_n\beta_n}$$

In what we shall call the "standard" form of a continued fraction, the variable x appears only in the numerators, that is, only in the α portions of 0:6:9. However, other forms exist in which x is part of β, or both α and β, as in the left-hand side of the identity

0:6:11 $$\frac{1}{\gamma_0-}\,\frac{\gamma_0 x}{\gamma_1+x-}\,\frac{\gamma_1 x}{\gamma_2+x-}\,\frac{\gamma_2 x}{\gamma_3+x-}\cdots\frac{\gamma_{n-1}x}{\gamma_n+x}=\frac{1}{\gamma_0}+\frac{x}{\gamma_0\gamma_1}+\frac{x^2}{\gamma_0\gamma_1\gamma_2}+\cdots+\frac{x^n}{\gamma_0\gamma_1\gamma_2\cdots\gamma_n}$$

which demonstrates the interchangeability of continued fractions and polynomials. Lozenge diagrams [Section 10:14] can facilitate such as interchange.

0:7 PARTICULAR VALUES

If certain values of the variable(s) of a function generate noteworthy function values, these are cited in the seventh section of each chapter, often as a table. The entry "−∞|+∞" in such a table, or elsewhere in the *Atlas*, or in the output of *Equator*, implies that the function has a discontinuity and, moreover, that at an argument slightly more negative than the argument in question, the function's value is large and negative; whereas, at an argument slightly more positive, the function is large and positive. Entries such as "+∞|+∞" similarly provide information about the sign of the function's value on either side of a discontinuity.

In Section 7 of many chapters we include information about those arguments that lead to inflections, minima, maxima, and particularly zeros of the subject function f(x). The term *extremum* is used to mean either a local maximum or a local minimum.

An *inflection* of a function occurs at a value of its argument at which the second derivative of the function is zero; that is:

0:7:1 $$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(x_\mathrm{i})=0 \qquad \mathrm{f}(x_\mathrm{i})=\text{inflection of }\mathrm{f}(x)$$

A local *minimum* and a local *maximum* of a function are characterized respectively by

0:7:2 $$\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}(x_\mathrm{m})=0,\quad \frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(x_\mathrm{m})>0 \qquad \mathrm{f}(x_\mathrm{m})=\text{minimum of }\mathrm{f}(x)$$

and

0:7:3 $$\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}(x_\mathrm{M})=0,\quad \frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(x_\mathrm{M})<0 \qquad \mathrm{f}(x_\mathrm{M})=\text{maximum of }\mathrm{f}(x)$$

A *zero* of a function is a value of its argument at which the function vanishes; that is, if

0:7:4 $$f(r)=0 \quad \text{then} \quad r=\text{a zero of }\mathrm{f}(x)$$

Equivalent to the phrase "a zero of f(x)" is "a *root* of the equation f(x) = 0." A *double zero* or a *double root* occurs at a value r of the argument such that

0:7:5 $$\mathrm{f}(r)=\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}(r)=0 \qquad r=\text{a double zero of }\mathrm{f}(x)$$

The concept extends to multiple zeros or repeated roots; thus, if

0:7:6 $$\mathrm{f}(r)=\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}(r)=\cdots=\frac{\mathrm{d}^n\mathrm{f}}{\mathrm{d}x^n}(r)=0 \qquad r=\text{a zero of }\mathrm{f}(x)\text{ of multiplicity }n+1$$

A value r of x that satisfies 0:7:4 but not 0:7:5 corresponds to a *simple zero* or a *simple root*. The graphical significance of a root, a maximum, a minimum, and an inflection, is evident from Figure 0-1.

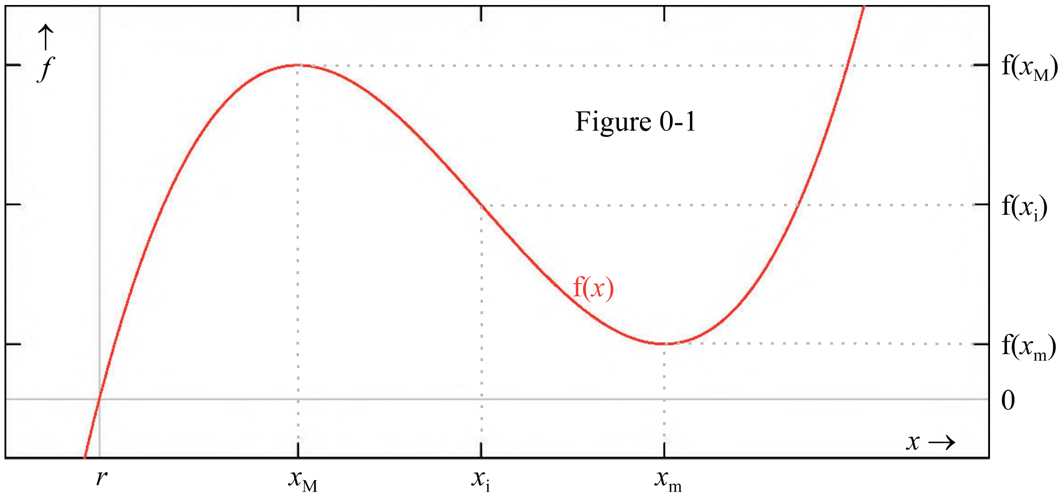

Figure 0-1

0:8 NUMERICAL VALUES

Very few tables of numerical values are found in the *Atlas* because the compact disk that accompanies the print edition of this book is designed to obviate that need. As described in Appendix C, the disk provides access to *Equator, the Atlas function calculator*. As well as carrying out certain other tasks, *Equator* is able to calculate the numerical values of over two hundred mathematical functions.

The computational methods employed by *Equator* are so diverse and interconnected that it is impractical to present the code or algorithms. Nevertheless, the mathematical basis of the calculations is explained in Section 8 of the relevant chapter. Generally, Section 8 reveals the domain(s) of the variable(s) within which *Equator* operates but the user must appreciate that not all of the variable combinations that lie within these domains will necessarily generate a function value. For a variety of reasons, including overflow or underflow during a computation, or an inadequacy of residual precision at some stage of the calculation, no numerical output may be possible. Our goal is that any answer generated be significant to the number of digits cited in the output. See Appendix C for further information about *Equator*.

0:9 LIMITS AND APPROXIMATIONS

Often, as the argument or another variable of the function approaches a particular number, such as zero or infinity, its behavior comes to approximate that of some simpler function as a limit. Such instances are noted in Sections 9, either verbally or with the help of an equation. The symbol ≈ indicates approximate equality.

Limiting behaviors can often serve as valuable approximations, and these may be presented in Sections 9. Whenever some approximation, not necessarily arising from a limit, is particularly noteworthy, it is reported in this section, too. The symbol → is used to indicate approach.

0:10 OPERATIONS OF THE CALCULUS

Some of the most important properties of functions are associated with their behavior when subjected to the various operations of the calculus. Accordingly, the tenth is often one of the largest sections of a chapter.

For most functions defined for a continuous range of argument x, the *derivative*

0:10:1 $$\frac{\mathrm{d}f}{\mathrm{d}x}$$

exists and is reported in Sections 10. The derivatives of function combinations such as f(x)g(x), f(x)/g(x), or f(g(x)) are seldom reported because they may be readily evaluated via the *product rule*:

0:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x) = f(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f}{\mathrm{d}x}$$

the *quotient rule*:

0:10:3 $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{1}{g(x)}\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{f(x)}{g^2(x)}\frac{\mathrm{d}g}{\mathrm{d}x}$$

or the *chain rule*:

0:10:4 $$\frac{\mathrm{d}}{\mathrm{d}x}f\left(g(x)\right) = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} \qquad g = g(x)$$

Derivatives of multivariate functions are often denoted $\partial f/\partial x$ to draw attention to the fact that the variables other than the chosen one, here x, are being kept constant.

If f and g are two functions of the same variable, then, in addition to the elementary rule

0:10:5 $$\frac{\mathrm{d}g}{\mathrm{d}f} = \left(\frac{\mathrm{d}f}{\mathrm{d}g}\right)^{-1}$$

for derivative inversion, the less familiar formulas

0:10:6 $$\frac{\mathrm{d}^2g}{\mathrm{d}f^2} = -\left(\frac{\mathrm{d}f}{\mathrm{d}g}\right)^{-3}\left(\frac{\mathrm{d}^2f}{\mathrm{d}g^2}\right)$$

and

0:10:7 $$\frac{\mathrm{d}^3g}{\mathrm{d}f^3} = \left(\frac{\mathrm{d}f}{\mathrm{d}g}\right)^{-5}\left[3\left(\frac{\mathrm{d}^2f}{\mathrm{d}g^2}\right)^2 - \left(\frac{\mathrm{d}f}{\mathrm{d}g}\right)\left(\frac{\mathrm{d}^3f}{\mathrm{d}g^3}\right)\right]$$

are sometimes useful. Similar formulas for higher derivatives may be derived.

If f(x) is differentiated n times, the nth derivative

0:10:8 $$\frac{\mathrm{d}^nf}{\mathrm{d}x^n}$$

is generated, and expressions for this result are sometimes reported. The nth derivative of a product is given by the *Leibniz theorem* (Gottfried Wilhelm von Leibniz, German mathematician, 1646 – 1716):

0:10:9 $$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)g(x) &= f(x)\frac{\mathrm{d}^ng}{\mathrm{d}x^n} + \frac{n}{1!}\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}^{n-1}g}{\mathrm{d}x^{n-1}} + \frac{n(n-1)}{2!}\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\frac{\mathrm{d}^{n-2}g}{\mathrm{d}x^{n-2}} + \cdots + \frac{n}{1!}\frac{\mathrm{d}^{n-1}f}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}^nf}{\mathrm{d}x^n}g(x)\\ &= \sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^{n-j}f}{\mathrm{d}x^{n-j}}\frac{\mathrm{d}^jg}{\mathrm{d}x^j}\end{aligned}$$

where $\binom{n}{j}$ denotes a binomial coefficient [Chapter 6] and $n!$ the factorial function [Chapter 2]. Another useful operation bearing the name of Leibniz gives a rule for *differentiating an integral*:

0:10:10 $$\frac{\mathrm{d}}{\mathrm{d}x}\int_{t_0(x)}^{t_1(x)} \mathrm{f}(x,t)\,\mathrm{d}t = \frac{\mathrm{d}t_1}{\mathrm{d}x}\mathrm{f}\big(x,t_1(x)\big) - \frac{\mathrm{d}t_0}{\mathrm{d}x}\mathrm{f}\big(x,t_0(x)\big) + \int_{t_0(x)}^{t_1(x)} \frac{\partial}{\partial x}\mathrm{f}(x,t)\,\mathrm{d}t$$

One application of differentiation is in attaching meaning to the quotient f(x)/g(x) of two functions at an argument value, say $x = a$, at which they are both zero. Then, according to *L'Hôpital's rule* (Guillaume François, Marquis de L'Hôpital, 1661−1705, French aristocrat and mathematician, author of the first textbook on the calculus)

0:10:11 $$\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right)_{x=a} = \frac{(\mathrm{df}/\mathrm{d}x)_{x=a}}{(\mathrm{dg}/\mathrm{d}x)_{x=a}} \qquad \text{if} \quad \mathrm{f}(a) = \mathrm{g}(a) = 0$$

If both derivatives are themselves zero, the rule can be applied a second time; and so on.

Reference books often express the results of *indefinite integration* in a form such as

0:10:12 $$\int \mathrm{f}(x)\,\mathrm{d}x = c + \mathrm{F}(x)$$

where F(x) is a function that gives f(x) on differentiation (that is, dF/dx = f(x)), and c is an arbitrary constant. To achieve closer unity with the representation

0:10:13 $$\int_{x_0}^{x_1} \mathrm{f}(t)\,\mathrm{d}t$$

of a *definite integral*, the *Atlas* adopts the formulation

0:10:14 $$\int_{x_0}^{x} \mathrm{f}(t)\,\mathrm{d}t = \mathrm{F}(x) - \mathrm{F}(x_0)$$

for indefinite integration. In 0:10:13 and 0:10:14, x_0 and x_1 are specified lower and upper limits, x_0 in 0:10:14 usually being chosen to make F(x_0) vanish. Of course, the information contained in 0:10:12 is also present in 0:10:14. An invaluable tool for integration is the formula for *integration by parts*:

0:10:15 $$\int_{x_0}^{x_1} \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t = \int_{t=x_0}^{t=x_1} \mathrm{f}(t)\,\mathrm{dG}(t) = \mathrm{f}(x_1)\mathrm{G}(x_1) - \mathrm{f}(x_0)\mathrm{G}(x_0) - \int_{x_0}^{x_1} \frac{\mathrm{df}}{\mathrm{d}t}(t)\,\mathrm{G}(t)\,\mathrm{d}t$$

where g is the derivative of G. See Section 37:14 for further benefits of integration by parts.

Integration, definite or indefinite, can lead to a infinite result; such integrals are not listed in the *Atlas*. However, an integral may be finite even if the integrand encounters an infinity at one of its limits. Even when the infinity is within the integration bounds, symmetry of the integrand may allow a finite so-called *principal value* or *Cauchy limit* [Sections 7:10 and 37:3] to be ascribed to the integral.

Where no analytical formulation is known, it may be necessary to evaluate an integral numerically. Algorithms for *numerical integration*, or *quadrature* as it is often called, will be found in Sections 4:14, 24:15, and 62:15. A change in variable, for example

0:10:16 $$\int_{x_0}^{x_1} \mathrm{f}(t)\,\mathrm{d}t \to \int_0^1 \mathrm{g}(u)\,\mathrm{d}u$$

may be beneficial prior to quadrature and is usually mandatory if either of the limits is infinite. The adjacent table gives some

x_0	x_1	g(u)
> 0	∞	$\dfrac{x_0}{u^2}\mathrm{f}\left(\dfrac{x_0}{u}\right)$
> -1	∞	$\dfrac{1+x_0}{(1-u)^2}\mathrm{f}\left(\dfrac{u+x_0}{1-u}\right)$
$-\infty$	< 0	$\dfrac{-x_1}{u^2}\mathrm{f}\left(\dfrac{x_1}{u}\right)$
$-\infty$	< 1	$\dfrac{-(1+x_1)}{u^2}\mathrm{f}\left(\dfrac{1+x_1-u}{u}\right)$
$-\infty$	∞	$\dfrac{2u^2-2u+1}{u^2(1-u)^2}\mathrm{f}\left(\dfrac{2u-1}{u-u^2}\right)$

suggestions for replacement variables. Judicious partitioning of the integrand, as in the example

0:10:17 $$\int_0^\pi \frac{\sin\left(\sqrt{t}\right)}{t}\mathrm{d}t = \int_0^\pi \frac{1}{\sqrt{t}}\mathrm{d}t - \int_0^\pi \left\{\frac{1}{\sqrt{t}} - \frac{\sin\left(\sqrt{t}\right)}{t}\right\}\mathrm{d}t = 2\sqrt{\pi} - \int_0^\pi \frac{\sqrt{t}-\sin\left(\sqrt{t}\right)}{t}\mathrm{d}t$$

can be a valuable prelude to numerical integration.

Sections 10 sometimes include formulas for the *semiderivative* or *semiintegral* of a function. These are the simplest fractional cases of the generalized *differintegral* $\mathrm{d}^\nu f/\mathrm{d}x^\nu$ [Section 12:14]. Other fractional calculus results may be displayed too.

Some important integral transforms involving a chapter's function(s) are listed in Section 10, some symbol being used as the "dummy" variable arising by transformation of a function of the real variable t. The most common transformation encountered in this *Atlas* is the *Laplace transformation*

0:10:18 $$\int_0^\infty f(t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{f(t)\} = \overline{f}(s)$$

discussed in Section 26:15. The dummy variable s of Laplace transformation is often a complex variable, but no recognition of this appears in Sections 10. Nor are the restrictions that often exist on the domain of s made explicit. The notations $\mathfrak{L}\{f(t)\}$ or $\overline{f}(s)$ mean the Laplace transform of f(t).

0:11 COMPLEX ARGUMENT

Other than in the eleventh section of each chapter, the argument of a function is generally treated as real. Moreover we usually restrict the range of the function to ensure that function values are real.

In Sections 11, however, the effect of replacing the real argument x by the complex variable $z = x + iy$ may be explored. Often a pair of three-dimensional diagrams shows the real Re[f(z)] and imaginary Im[f(z)] parts, of the complex-valued function f(z). Not infrequently, complex-valued functions are inherently multivalued and, to convert such a function into a single-valued counterpart, it is necessary to "cut" the surface that defines its values. Such a cut appears as a grey "cliff" on the graphic.

The most general route used to invert a Laplace transform, thereby converting a function of s back into a function of t, involves contour integration in the complex plane via the *Bromwich integral* [Section 26:15]. Accordingly, such inversions appear in Section 11 of the chapter devoted to the image function $\overline{f}(s)$. The notation

0:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \overline{f}(s)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\overline{f}(s)\right\} = f(t)$$

is adopted to display the relationship between the image function and its inverse transform f(t). The operation is known as *inverse Laplace transformation*.

0:12 GENERALIZATIONS

Functions can be arranged in a hierarchy in which lower members are subsumed by more general higher members. In the chapter dealing with a particular f(x) function, special cases of f(x) – that is, functions lower in the hierarchy – are reported in Section 4. Conversely, functions that occur higher than f(x) in some hierarchy are reported in Section 12; that is, the functions reported here are ones of which f(x) is a special case.

0:13 COGNATE FUNCTIONS

The thirteenth section of each chapter is devoted to citing functions – other than those reported in Sections 4 and 12 – that are closely related to the subject function, and to exploring the properties of some of them.

0:14 RELATED TOPICS

An application, or some other feature relevant to the function, is elaborated in the final section of many chapters. Such applications often amount to brief discourses on important topics in science, engineering, and applied mathematics. For example, certain topics in curve-fitting, integral transforms, geometric properties of various sorts, and coordinate transformations, are to be found among the many items exposed in Sections 14. Occasionally, two sections are needed to report related topics; the extra section is then numbered 15. The Table of Contents identifies these topics along with other special features of each chapter.

CHAPTER 1

THE CONSTANT FUNCTION *c*

Lacking dependence on even a single variable, the constant function is the simplest, and an almost trivial, function.

1:1 NOTATION

Constants are also known as *invariants* and are represented by a variety of symbols, mostly letters drawn from early members of the Latin and Greek alphabets. In this chapter, we mostly employ c to represent an arbitrary constant.

1:2 BEHAVIOR

Figure 1-1 is a graphical representation of the constant function $f(x) = c$, a horizontal line extending to $x = \pm\infty$, reflecting the fact that f takes the same value for all x.

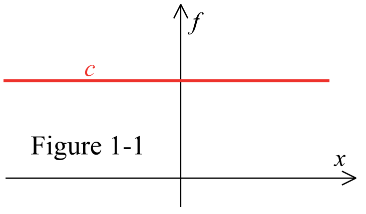

Figure 1-1

1:3 DEFINITIONS

The constant function is defined for all values of its argument x and has the same value, c, irrespective of x.

1:4 SPECIAL CASES

When c is zero, the constant function is sometimes termed the *zero function*. Likewise, the function $f(x) = c = 1$ is sometimes known as the unit function or *unity function*.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_2, © Springer Science+Business Media, LLC 2009

1:5 INTRARELATIONSHIPS

Being relations between function values at different values of the argument, intrarelationships are of no consequence for the constant function.

1:6 EXPANSIONS

A constant may be represented as a finite sum by utilizing the formulas for an *arithmetic series*:

1:6:1
$$c = \alpha + (\alpha+\delta) + (\alpha+2\delta) + \cdots + (\alpha+J\delta) = \sum_{j=0}^{J}(\alpha + j\delta)$$
$$\alpha = \frac{c}{J+1} - \frac{J\delta}{2} \qquad \text{or} \qquad c = (J+1)\left(\alpha + \frac{J\delta}{2}\right)$$

a *geometric series*:

1:6:2
$$c = \alpha + \alpha\beta + \alpha\beta^2 + \cdots + \alpha\beta^J = \sum_{j=0}^{J}\alpha\beta^j$$
$$\alpha = c\frac{\beta-1}{\beta^{J+1}-1} \qquad \text{or} \qquad c = \alpha\frac{\beta^{J+1}-1}{\beta-1}$$

or an *arithmetic-geometric series*:

1:6:3
$$c = \alpha + \beta(\alpha+\delta) + \beta^2(\alpha+2\delta) + \cdots + \beta^J(\alpha+J\delta) = \sum_{j=0}^{J}\beta^j(\alpha + j\delta)$$
$$\alpha = \frac{c(\beta-1) - J\delta\beta^{J+1} + \delta\beta(\beta^J-1)/(\beta-1)}{\beta^{J-1}-1} \qquad \text{or} \qquad c = \left(\beta^{J+1}-1\right)\left[\beta(\alpha+\delta)-\alpha\right] + J(\beta-1)\beta^{J+1}\delta$$

In these formulas β and δ are arbitrary and J may be any positive integer.

Any constant greater than ½ may be expanded as the infinite geometric sum

1:6:4
$$c = 1 + \left(\frac{c-1}{c}\right) + \left(\frac{c-1}{c}\right)^2 + \left(\frac{c-1}{c}\right)^3 + \cdots = \sum_{j=1}^{\infty}\left(\frac{c-1}{c}\right)^j \qquad c > \tfrac{1}{2}$$

or as the infinite product

1:6:5
$$c = \left[1 + \left(\frac{c-1}{c}\right)\right]\left[1 + \left(\frac{c-1}{c}\right)^2\right]\left[1 + \left(\frac{c-1}{c}\right)^4\right]\cdots = \prod_{j=0}^{\infty}\left[1 + \left(\frac{c-1}{c}\right)^{2^j}\right] \qquad c > \tfrac{1}{2}$$

A constant is expansible as the infinite continued fraction

1:6:6
$$c = \frac{\alpha}{\beta+}\frac{\alpha}{\beta+}\frac{\alpha}{\beta+}\cdots$$

in the variety of ways indicated in the table, which lists three alternative assignments of the terms α and β, any one of which validates expansion 1:6:6.

α	β	constraint
c	$1-c$	$-1 \le c < 1$
1	$\frac{1-c^2}{c}$	$0 < c^2 < 1$
$c^2 + c$	1	$c \ge -\tfrac{1}{2}$

1:7 PARTICULAR VALUES

Certain constants occur frequently in the theory of functions. Four of these – *Archimedes's constant*, *Catalan's constant*, the *base of natural logarithms* and *Euler's constant* – are important irrational numbers. There are many formulations of these four constants other than the ones we present here; see Gradshteyn and Ryzhik [Chapter 0] for some of these.

Archimedes (Archimedes of Syracuse, Greek philosopher, 287 – 212 BC) himself was content merely to bracket his constant by (223/71) < π < (22/7). It was the sixteenth-century Frenchman François Viète ("Vieta") who discovered the first formula

1:7:1 $$\pi = \frac{2}{\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \times \cdots} = 3.1415\ 92653\ 58979$$

for *Archimedes's constant*, also known simply as *pi*. It may also be defined by the infinite sum

1:7:2 $$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = 3.1415\ 92653\ 58979$$

discovered by Gregory (James Gregory, Scottish mathematician, 1638 – 1675), as the infinite product

1:7:3 $$\pi = 2 \times \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \cdots = 2\prod_{j=1}^{\infty} \frac{j^2}{j^2 - \frac{1}{4}} = 3.1415\ 92653\ 58979$$

and in numerous other ways. The definition of *Catalan's constant* (Eugène Charles Catalan, Belgian mathematician 1814 – 1894) is similar to 1:7:2

1:7:4 $$G = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\ 55941\ 77219$$

The *base of natural logarithms* may be defined as a sum of all reciprocal factorial functions [Chapter 2]

1:7:5 $$e = 1 + \frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!} = 2.7182\ 81828\ 45905$$

or by the limit operation

1:7:6 $$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182\ 81828\ 45905$$

A limit operation also defines *Euler's constant*

1:7:7 $$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right) = \lim_{n\to\infty}\left(\sum_{j=1}^{n} \frac{1}{j} - \ln(n)\right) = 0.57721\ 56649\ 01533$$

The latter is also known as *Mascheroni's constant* (Lorenzo Mascheroni, Italian priest, 1750 – 1800) and is often denoted by C. Confusingly, authors who employ C to represent Euler's constant may use γ to represent e^C.

Also of widespread occurrence throughout the *Atlas* is the *Gauss's constant*

1:7:8 $$g = \frac{1}{\mathrm{mc}\left(1, \sqrt{2}\right)} = 0.83462\ 68416\ 74073$$

where mc denotes the common, or arithmeticogeometric, mean [Section 61:14]. It is related to the *ubiquitous constant* U through $Ug = 1/\sqrt{2}$. Other named constants are Apéry's constant Z [Section 3:7] and the golden section υ [Section 23:14].

A very important family of constants are the *integers*, $\cdots,-3,-2,-1,0,1,2,3,\cdots$ and especially the *natural numbers*, $1,2,3,\cdots$ discussed in Section 1:14. Other families that occur principally in coefficients of series expansions are the factorials $n!$ [Chapter 2], Bernoulli numbers B_n [Chapter 4], and Euler numbers E_n [Chapter 5]. Fibonacci numbers are discussed in Section 23:14.

1:8 NUMERICAL VALUES

Equator provides values of the constants π, G, e, γ, g, Z, and υ, exact to 15 digits. Simply type the corresponding keyword, which is **pi**, **catalan**, **ebase**, **euler**, **gauss**, **apery**, or **golden**. These keywords may be freely used in "constructing" the variable(s) of any other *Equator* function, as explained in Appendix C. As well as these seven mathematical constants, many physical constants are available through *Equator*: see Appendix A for these.

1:9 LIMITS AND APPROXIMATIONS

Approximations are seldom needed for constants, but approximations as fractions are available through *Equator*'s rational approximation routine (keyword **rational**) [Section 8:13].

1:10 OPERATIONS OF THE CALCULUS

Differentiation gives

1:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}c = 0$$

while indefinite and definite integration produce

1:10:2 $$\int_0^x c\,\mathrm{d}t = cx$$

and

1:10:3 $$\int_{x_0}^{x_1} c\,\mathrm{d}t = c\left(x_1 - x_0\right)$$

respectively. The result

1:10:4 $$\int_0^\infty c\,\exp(-st)\,\mathrm{d}t = \mathcal{L}\{c\} = \frac{c}{s}$$

describes the Laplace transformation of a constant.

The results of semidifferentiation and semiintegration [Section 12:14] with a lower limit of zero are

1:10:5 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}c = \frac{c}{\sqrt{\pi x}}$$

and

1:10:6 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}c = 2c\sqrt{\frac{x}{\pi}}$$

Differintegration [Section 12:14] with a lower limit of zero yields

1:10:7 $$\frac{d^{v}}{dx^{v}}c=\frac{cx^{-v}}{\Gamma(1-v)}$$

where Γ is the gamma function [Chapter 43]. In fact, equations 1:10:1, 1:10:2, 1:10:5, and 1:10:6 are the $v = 1, -1$, ½ and −½ instances of 1:10:7.

1:11 COMPLEX ARGUMENT

A *complex constant* can be expressed in terms of two real constants in either rectangular or polar notation

1:11:1 $$c=\begin{cases}\alpha+i\beta & \text{where} \quad \alpha=\rho\cos(\theta) \quad \text{and} \quad \beta=\rho\sin(\theta)\\ \rho\exp(i\theta) & \text{where} \quad \rho=\sqrt{\alpha^{2}+\beta^{2}} \quad \text{and} \quad \theta=\arctan(\beta/\alpha)+\pi\left[1-\mathrm{sgn}(\alpha)\right]/2\end{cases}$$

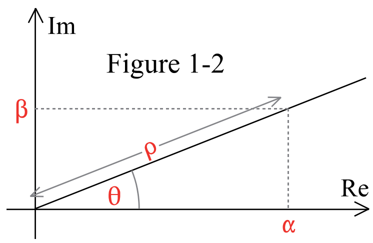

Figure 1-2

with $i=\sqrt{-1}$. The names *real part*, *imaginary part*, *modulus*, and *phase* are accorded to α, β, ρ, and θ. Figure 1-2 shows how α, β, ρ and θ are related. The expression $c = \alpha + i\beta$ is the more useful in formulating the rules for the addition or subtraction of two complex constants:

1:11:2 $$c_1 \pm c_2=(\alpha_1+i\beta_1)\pm(\alpha_2+i\beta_2)=(\alpha_1\pm\alpha_2)+i(\beta_1\pm\beta_2)$$

whereas $c = \rho\exp(i\theta)$ is the more convenient to formulate the multiplication

1:11:3 $$c_1c_2=[\rho_1\exp(i\theta_1)][\rho_2\exp(i\theta_2)]=\rho_1\rho_2\exp\{i(\theta_1+\theta_2)\}$$

or division

1:11:4 $$\frac{c_1}{c_2}=\frac{\rho_1\exp(i\theta_1)}{\rho_2\exp(i\theta_2)}=\frac{\rho_1}{\rho_2}\exp\{i(\theta_1-\theta_2)\}$$

of two complex numbers, or in the raising of a complex number to a real power

1:11:5 $$c^{v}=[\rho\exp(i\theta)]^{v}=\rho^{v}\exp(iv\theta)$$

If v is not an integer, this exponentiation operation gives rise to a multivalued complex number [see, for example, Section 13:14]. The raising of a real number to a complex-valued power is handled by the expression

1:11:6 $$v^{\alpha+i\beta}=v^{\alpha}\exp\{i\beta\ln(v)\}$$

provided that v is positive.

The inverse Laplace transform of the constant c is a Dirac function [Chapter 9], of magnitude c, located at the origin

1:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty}c\frac{\exp(ts)}{2\pi i}ds=\mathfrak{I}\{c\}=c\delta(t)$$

1:12 GENERALIZATIONS

A constant is a member of the polynomial function family, other members of which are discussed in Chapters 19−25. The constant function is the special $b = 0$ case of the linear function discussed in Chapter 7.

1:13 COGNATE FUNCTIONS

Whereas the constant function has the same value for all x, the related *pulse function* is zero at values of the argument outside a "window" of width h, and is a nonzero constant, c, within this window. The concept of a general "window function" is discussed in Section 9:13. The pulse function in Figure 1-3 takes the value c in the range $a-(h/2) < x < a+(h/2)$ but equals zero elsewhere. The value of the a parameter establishes the location of the pulse, while c and h are termed the *pulse height* and *pulse width* respectively. The pulse function may be represented by

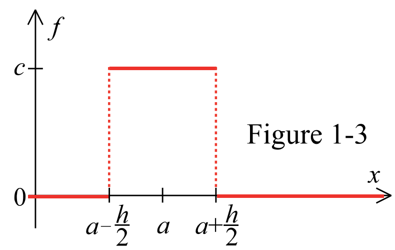

Figure 1-3

1:13:1 $$c\left[\mathrm{u}\left(x-a+\frac{h}{2}\right)-\mathrm{u}\left(x-a-\frac{h}{2}\right)\right]$$

in terms of the Heaviside function [Chapter 9].

The addition of a number of pulse functions, having various locations, heights, and widths, produces a function whose map consists of horizontal straight line segments. Such a function, known as a *piecewise-constant function*, may be used to approximate a more complicated or incompletely known function. It is the approximation recorded, for example, whenever a varying quantity is measured by a digital instrument.

1:14 RELATED TOPIC: the natural numbers

The *natural numbers*, $1,2,3,\cdots$ are ubiquitous in mathematics and science. We record here several results for finite sums of their powers:

1:14:1 $$1+2+3+\cdots+n=\sum_{j=1}^{n} j=\frac{n(n+1)}{2} \qquad n=1,2,3,\cdots$$

1:14:2 $$1^2+2^2+3^2+\cdots+n^2=\sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6} \qquad n=1,2,3,\cdots$$

1:14:3 $$1^3+2^3+3^3+\cdots+n^3=\sum_{j=1}^{n} j^3=\frac{n^2(n+1)^2}{4} \qquad n=1,2,3,\cdots$$

Similarly, the sums of fourth and fifth powers of the first n natural numbers are $n(n+1)(2n+1)(3n^2+3n-1)/30$ and $n^2(n+1)^2(2n^2+2n-1)/12$, respectively. The general case is

1:14:4 $$1^m+2^m+3^m+\cdots+n^m=\sum_{j=1}^{n} j^m=\frac{\mathrm{B}_{m+1}(n+1)-\mathrm{B}_{m+1}}{m+1} \qquad n,m=1,2,3,\cdots$$

where B_m denotes a Bernoulli number [Chapter 4] and $\mathrm{B}_m(x)$ denotes a Bernoulli polynomial [Chapter 19]. If m is not an integer, summation 1:14:4 may be evaluated generally by equation 12:5:5. The sum of the reciprocals of the first n natural numbers is

1:14:5 $$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\sum_{j=1}^{n}\frac{1}{j}=\gamma+\psi(n+1) \qquad n=1,2,3,\cdots$$

where γ is Euler's constant [Section 1:7] and $\psi(x)$ denotes the digamma function [Chapter 44]. When continued indefinitely, the sum 1:14:5 defines the divergent *harmonic series*.

The corresponding expressions when the signs alternate are

1:14:6 $$1-2+3-4+\cdots\pm n=-\sum_{j=1}^{n}(-)^j j=\begin{cases}(n+1)/2 & n=1,3,5,\cdots\\ -n/2 & n=2,4,6,\cdots\end{cases}$$

1:14:7 $$1^2-2^2+3^2-4^2+\cdots\pm n^2=-\sum_{j=1}^{n}(-)^j j^2=\begin{cases}n(n+1)/2 & n=1,3,5,\cdots\\ -n(n+1)/2 & n=2,4,6,\cdots\end{cases}$$

1:14:8 $$1^3-2^3+3^3-4^3+\cdots\pm n^3=-\sum_{j=1}^{n}(-)^j j^3=\begin{cases}(2n^3+3n^2-1)/4 & n=1,3,5,\cdots\\ -n^2(2n+3)/4 & n=2,4,6,\cdots\end{cases}$$

1:14:9 $$1^m-2^m+3^m-4^m+\cdots\pm n^m=-\sum_{j=1}^{n}(-)^j j^m=-\frac{\mathrm{E}_m(0)}{2}-\frac{(-)^n\mathrm{E}_m(n+1)}{2} \qquad n,m=1,2,3,\cdots$$

where $\mathrm{E}_m(x)$ denotes an Euler polynomial [Chapter 20], and

1:14:10 $$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{n}=-\sum_{j=1}^{n}\frac{(-1)^j}{j}=\begin{cases}\psi(n+1)-\psi\left(\dfrac{n+1}{2}\right) & n=1,3,5,\cdots\\ \psi(n+1)-\psi\left(\dfrac{n}{2}+1\right) & n=2,4,6,\cdots\end{cases}$$

Note that, whereas the $n=\infty$ version of the harmonic series 1:14:5 does not converge, series 1:14:10 approaches the limit $\ln(2)$ as $n\to\infty$.

The numbers $2,4,6,\cdots$ are called the *even numbers*. Sums of their powers are easily found by using the identity

1:14:11 $$2^m+4^m+6^m+\cdots+n^m=2^m\left[1^m+2^m+3^m+\cdots+\left(\frac{n}{2}\right)^m\right] \qquad n=2,4,6,\cdots$$

in conjunction with equations 1:14:1−1:14:5. Likewise, use of these equations, together with the identity

1:14:12 $$1^m+3^m+5^m+\cdots+n^m=\left[1^m+2^m+3^m+\cdots+n^m\right]-2^m\left[1^m+2^m+3^m+\cdots+\left(\frac{n-1}{2}\right)^m\right] \qquad n=1,3,5,\cdots$$

permits sums of powers of the *odd numbers*, $1,3,5,\cdots$, to be evaluated.

For the *infinite* sums $\sum j^{-\nu}$ where j runs from 1 to ∞, see Chapter 3. The same chapter also addresses the related infinite sums $\sum(-)^j j^{-\nu}$, $\sum(2j-1)^{-\nu}$, and $\sum(-)^j(2j-1)^{-\nu}$. For other sums of numerical series, see Sections 44:14 and 64:6.

CHAPTER 2

THE FACTORIAL FUNCTION *n*!

The factorial function occurs widely in function theory; especially in the denominators of power series expansions of many transcendental functions. It also plays an important role in combinatorics [Section 2:14]. Because they too arise in the context of combinatorics, Stirling numbers of the second kind are discussed in Section 2:14 [those of the first kind find a home in Chapter 18].

Double and triple factorial functions are described in Section 2:14.

2:1 NOTATION

The factorial function of n, also spoken of as "n factorial", is generally given the symbol $n!$. It is represented by $\lfloor\!\underline{n}$ in older literature. The symbol $\Pi(n)$ is occasionally encountered.

2:2 BEHAVIOR

The factorial function is defined only for nonnegative integer argument and is itself a positive integer. Because of its explosive increase, a plot of $n!$ versus n is not very informative. Figure 2-1 is a graph of the logarithm (to base 10) of $n!$ versus n. Note that $70! \approx 10^{100}$.

2:3 DEFINITIONS

The factorial function of the positive integer n equals the product of all positive integers up to and including n:

2:3:1 $$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n = \prod_{j=1}^{n} j \qquad n = 1, 2, 3, \cdots$$

This definition is supplemented by the value

2:3:2 $$0! = 1$$

conventionally accorded to zero factorial.

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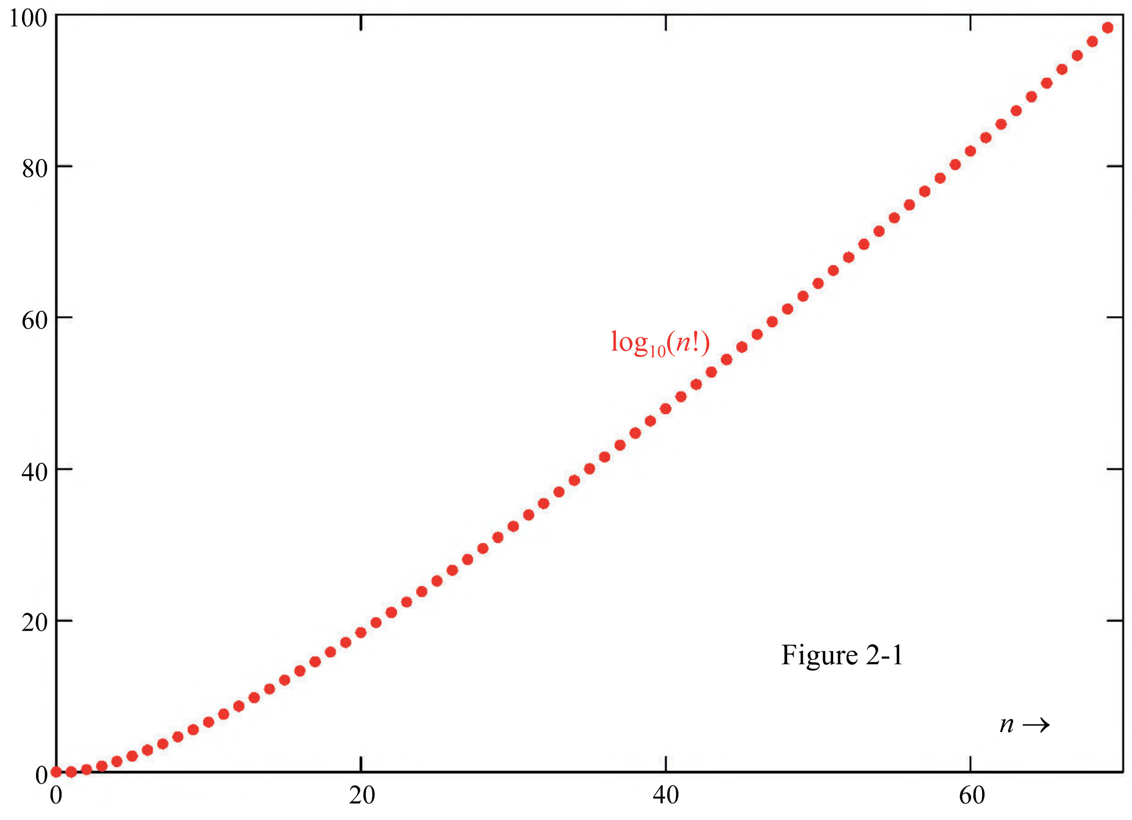


The exponential function [Chapter 26] is a generating function [Section 0:3] for the reciprocal of the factorial function

2:3:3 $$\exp(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$

2:4 SPECIAL CASES

There are none.

2:5 INTRARELATIONSHIPS

The most important property of the factorial function is its recurrence

2:5:1 $$(n+1)! = n!(n+1) \qquad n = 0,1,2,\cdots$$

which may be iterated to produce the argument-addition formula

2:5:2 $$(n+m)! = n!(n+1)(n+2)\cdots(n+m) = n!(n+1)_m \qquad n,m = 0,1,2,\cdots$$

where $(n+1)_m$ is a Pochhammer polynomial [Chapter 18]. Formula 2:5:2 leads to an expression for the ratio of two factorials. An alternative expression is

2:5:3 $$\frac{n!}{(n-m)!} = n(n-1)(n-2)\cdots(n-m+1) = (-)^m(-n)_m \qquad n > m$$

Setting $m = n$ in equation 2:5:2 provides a *duplication formula* for the factorial function, enabling $(2n)!$ to be expressed with the help of a Pochhammer polynomial. Alternative duplication formulas are available from equations 2:12:3 and 2:12:4, which may be rewritten as

2:5:4
$$n! = \begin{cases} 2^n \left(\tfrac{1}{2}\right)_{n/2} \left(\tfrac{1}{2}n\right)! & n = 2,4,6,\cdots \\ 2^n \left(\tfrac{1}{2}\right)_{(n+1)/2} \left(\tfrac{1}{2}n - \tfrac{1}{2}\right)! & n = 1,3,5,\cdots \end{cases}$$

There are analogous triplication formulas that can be developed from the equations in 2:12:5.

The frequent occurrence of factorials as coefficients of power series permits the summation of such series as

2:5:5
$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!} = \exp(1) = 2.7182\ 81828\ 45905$$

and

2:5:6
$$\frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \cdots = \sum_{j=0}^{\infty} \frac{1}{(j!)^2} = \mathrm{I}_0(2) = 2.2795\ 85302\ 33607$$

where I_0 is the modified Bessel function [Chapter 49]. The corresponding series with alternating signs sum similarly to $\exp(-1)$ and to the particular value $\mathrm{J}_0(2)$ of the zero-order Bessel function [Chapter 52]. There is even the intriguing asymptotic result [see equation 37:13:4]

2:5:7
$$0! - 1! + 2! - \cdots = \sum_{j=0}^{\infty} (-)^j j! \sim \int_0^{\infty} \frac{\exp(-t)}{1+t} \mathrm{d}t = 0.59634\ 73623\ 23194$$

Moreover, the series $\Sigma(-1)^n/(2n)!$ sums to $\cos(1)$ and there are several analogous summations.

2:6 EXPANSIONS

Stirling's formula [see also Section 43:6]

2:6:1
$$n! \sim \sqrt{2\pi n}\, \exp(-n) n^n \left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots \right] \qquad n \to \infty$$

provides an expansion for the factorial function. Though they are technically asymptotic [Section 0:6], this expansion and a similar one for the logarithm of the factorial function

2:6:2
$$\ln(n!) \sim \ln\sqrt{2n\pi} + n \left[\ln(n) - 1 + \sum_{j=1} \frac{\mathrm{B}_j}{j(j-1)} \left(\frac{1}{n}\right)^j \right]$$
$$= n\ln(n) - n + \frac{\ln(2n\pi)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots \qquad n \to \infty$$

are remarkably accurate, even for small n. B_j is the j^{th} Bernoulli number [Chapter 4].

2:7 PARTICULAR VALUES

0!	1!	2!	3!	4!	5!	6!	7!	8!	9!	10!	11!	12!
1	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600

2:8 NUMERICAL VALUES

The decimal integer representing $n!$ has exactly $\mathrm{Int}(n/5)+\mathrm{Int}(n/25)+\mathrm{Int}(n/125)+\cdots$ terminal zeros; for example 31! ends with seven zeros. This rule is useful in calculating exact numerical values of large factorials. Int is the integer-value function describe in Chapter 8.

Equator's factorial function routine (keyword **!**) provides values of $n!$. For integer input in the range $0 < n \le 170$, a simple algorithm based on recursion 2:5:1, followed by rounding, is used to compute $n!$. Exact output is reported up to $20! = 2.4329\ 02008\ 17664\ \mathrm{E}{+}18$. For $21 \le n \le 170$, *Equator* provides a floating point approximation of $n!$ precise to 15 digits.

Separately, values of the natural and decadic logarithms, $\ln(n!)$ and $\log_{10}(n!)$, are provided by the logarithmic factorial function and logarithm to base 10 of the factorial function routines (**ln!** and **log10!**). Such logarithmic values are useful when n is large because $n!$ itself is then prohibitively huge. For input up to $n = 170$, $\ln(n!)$ is computed by simply taking the logarithm of the output from the routine described above. For integer input in the range $171 \le n \le 1\mathrm{E}305$, *Equator* uses Stirling's formula in the truncated and concatenated form

2:8:1
$$\ln(n!) = \frac{\ln(2n\pi)}{2} - n\left(1 - \ln(n) - \frac{1}{12n^2}\left(1 - \frac{1}{30n^2}\left(1 - \frac{2}{7n^2}\right)\right)\right)$$

Division by 2.3025 85092 99405 generates $\log_{10}(n!)$.

2:9 LIMITS AND APPROXIMATIONS

For large argument, the limiting formula

2:9:1
$$n! \to \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad n \to \infty$$

applies, e being the base of natural logarithms [Section 1:7]. The related approximation

2:9:2
$$n! \approx \mathrm{Round}\left\{(1+12n)\sqrt{\frac{\pi}{72n}}\left(\frac{n}{e}\right)^n\right\}$$

is surprisingly good, and even exact, for small positive integers. Round is the rounding function, described in Section 8:13.

2:10 OPERATIONS OF THE CALCULUS

No operations of the calculus are possible on a function such as $n!$ that is defined only for discrete arguments.

2:11 COMPLEX ARGUMENT

In view of relation 2:12:1, the gamma function formulas given in Section 43:11 may be used to ascribe meaning to $(n+im)!$.

2:12 GENERALIZATIONS

The factorial function is a special case of the gamma function [Chapter 43]

2:12:1 $$n! = \Gamma(n+1) \qquad n = 0,1,2,\cdots$$

and of the Pochhammer polynomial [Chapter 18]

2:12:2 $$n! = (1)_n \qquad n = 0,1,2,\cdots$$

The latter identity permits us to write

2:12:3 $$(2n)! = 4^n (\tfrac{1}{2})_n (1)_n$$

and

2:12:4 $$(2n+1)! = 4^n (1)_n (\tfrac{3}{2})_n$$

Similarly

2:12:5 $$(3n)! = 3^{3n}\left(\tfrac{1}{3}\right)_n\left(\tfrac{2}{3}\right)_n(1)_n, \qquad (3n+1)! = 3^{3n}\left(\tfrac{2}{3}\right)_n(1)_n\left(\tfrac{4}{3}\right)_n, \qquad (3n+2)! = 3^{3n}(1)_n\left(\tfrac{4}{3}\right)_n\left(\tfrac{5}{3}\right)_n$$

and so on.

2:13 COGNATE FUNCTIONS: multiple factorials

See 6:3:4 for the close relationship between the factorial function and binomial coefficients.

The *double factorial* or *semifactorial function* is defined by

2:13:1 $$n!! = \begin{cases} 1 & n = -1,0 \\ n\times(n-2)\times(n-4)\times\cdots\times 5\times 3\times 1 & n = 1,3,5,\cdots \\ n\times(n-2)\times(n-4)\times\cdots\times 6\times 4\times 2 & n = 2,4,6,\cdots \end{cases}$$

For even argument it reduces to

2:13:2 $$n!! = 2^{n/2}(n/2)! \qquad n = 0,2,4,\cdots$$

while for odd n it may be expressed in terms of factorials, or as a gamma function [Chapter 43] or as a Pochhammer polynomial [Chapter 18]

2:13:3 $$n!! = \frac{n!}{2^{(n-1)/2}\left(\frac{n-1}{2}\right)!} = 2^{n/2}\sqrt{\frac{2}{\pi}}\,\Gamma\left(1+\frac{n}{2}\right) = 2^{(n+1)/2}\left(\frac{1}{2}\right)_{(n+1)/2} \qquad n = 1,3,5,\cdots$$

Equivalent to the last equation is

2:13:4 $$(2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

These formulas are used by *Equator*'s double factorial function routine (keyword **!!**) to compute values of $n!!$ for integers in the range $-1 \le n \le 300$. Of course

2:13:5 $$n!!(n-1)!! = n! \qquad n = 0,1,2,\cdots$$

Some early members of the double-factorial family are listed below.

$(-1)!!$	$0!!$	$1!!$	$2!!$	$3!!$	$4!!$	$5!!$	$6!!$	$7!!$	$8!!$	$9!!$	$10!!$	$11!!$	$12!!$	$13!!$	$14!!$
1	1	1	2	3	8	15	48	105	384	945	3840	10395	46080	135135	645120

Note that, apart from $0!! = 1$, the double factorial $n!!$ shares the parity of n. Also notice that, to accord with the $n = -1$ instance of the general recursion formula

2:13:6 $$(n+2)!! = (n+2)n!!$$

$(-1)!!$ is assigned the value of unity. With a similar rationale, one sometimes encounters the values $(-3)!! = -1$, $(-5)!! = 1/3$, etc.

Of frequent occurrence [for example in Sections 6:4, 32:5, 61:6 and 62:12] is the ratio $(n-1)!!/n!!$ of the double factorials of consecutive integers. For odd n, the ratio is expressible by the integral

2:13:7 $$\frac{(n-1)!!}{n!!} = \frac{2^{n-1}}{n!}\left[\left(\frac{n-1}{2}\right)!\right]^2 = \int_0^{\pi/2} \sin^n(t)\,\mathrm{d}t \qquad n = 1,3,5,\cdots$$

while for even n it is given by *Wallis's formula* (John Wallis, English mathematician and cryptographer, 1616–1703)

2:13:8 $$\frac{(n-1)!!}{n!!} = \frac{n!}{2^n\left[(n/2)!\right]^2} = \frac{2}{\pi}\int_0^{\pi/2} \sin^n(t)\,\mathrm{d}t \qquad n = 0,2,4,\cdots$$

This important ratio has the asymptotic expansion

2:13:9 $$\frac{(n-1)!!}{n!!} \sim \begin{cases} \sqrt{\dfrac{2}{\pi n}}\left[1 - \dfrac{1}{4n} + \dfrac{1}{32n^2} - \cdots\right] & \text{even } n \to \infty \\ \sqrt{\dfrac{\pi}{2n}}\left[1 - \dfrac{1}{4n} + \dfrac{1}{32n^2} - \cdots\right] & \text{odd } n \to \infty \end{cases}$$

Finite sums of some such ratios obey the simple rule

2:13:10 $$1 + \frac{1}{2} + \frac{3}{8} + \frac{15}{48} + \cdots + \frac{(2n-1)!!}{(2n)!!} = \sum_{j=0}^{n} \frac{(2j-1)!!}{(2j)!!} = \frac{(2n+1)!!}{(2n)!!} \qquad n = 0,1,2,\cdots$$

and there is the related infinite summation due to Ross:

2:13:11 $$\frac{1}{2} + \frac{3}{16} + \frac{15}{144} + \frac{105}{1536} + \cdots = \sum_{j=1}^{\infty} \frac{(2j-1)!!}{j(2j)!!} = \ln(4)$$

The *triple factorial* is defined analogously

2:13:12 $$n!!! = \begin{cases} 1 & n = -2,-1,0 \\ n\times(n-3)\times(n-6)\times\cdots\times 7\times 4\times 1 & n = 1,4,7,\cdots \\ n\times(n-3)\times(n-6)\times\cdots\times 8\times 5\times 2 & n = 2,5,8,\cdots \\ n\times(n-3)\times(n-6)\times\cdots\times 9\times 6\times 3 & n = 3,6,9,\cdots \end{cases}$$

and finds application in connections with Airy functions [Chapter 56]. Some early values are:

$(-2)!!!$	$(-1)!!!$	$0!!!$	$1!!!$	$2!!!$	$3!!!$	$4!!!$	$5!!!$	$6!!!$	$7!!!$	$8!!!$	$9!!!$	$10!!!$	$11!!!$	$12!!!$
1	1	1	1	2	3	4	10	18	28	80	162	280	880	1944

The extension to a *quadruple factorial* $n!!!!$ is obvious; it is useful in Sections 43:4 and 59:7.

2:14 RELATED TOPIC: combinatorics and Stirling numbers of the second kind

The factorial function appears very often in applications involving *combinatorics*. For example, the number of *permutations* (arrangements) of n objects, all different, is $n!$. If not all of the n objects are different, the number of permutations is reduced to

2:14:1 $$\frac{n!}{(n_1)!(n_2)!\cdots(n_J)!} \qquad n = n_1 + n_2 + \cdots + n_J$$

where there are n_1 samples of object 1, n_2 samples of object 2, ···, n_J samples of object J, so that $\Sigma n_j = n$.

If from a group of n objects, all different, one withdraws m objects, one at a time, the number of *variations* (possible withdrawal sequences) is

2:14:2 $$\frac{n!}{(n-m)!} \qquad m \le n$$

If one ignores the order of withdrawal, 2:14:2 is reduced to

2:14:3 $$\frac{n!}{(n-m)!m!} \qquad m \le n$$

which then represents the number of ways in which m objects can be chosen from among n, all different, and is known as the number of *combinations*. Expression 2:14:3 is, in fact, the binomial coefficient addressed in Chapter 6.

The number of *partitions* (different ways in which n distinct objects may be placed in m identical boxes so that each box contains at least one object) is given by a *Stirling number of the second kind*. There is no standardized notation for such functions; this *Atlas* uses the symbol $\sigma_n^{(m)}$. Clearly, no partitioning is possible if $m = 0$ or if $n < m$ and accordingly the second Stirling number is zero in such circumstances. Otherwise, a general formula is

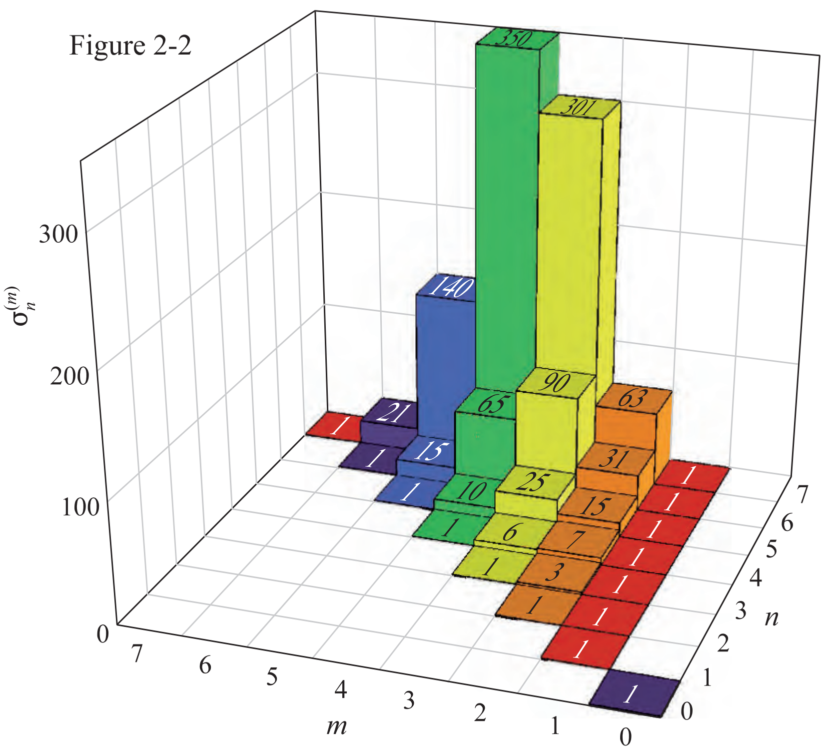
2:14:4 $$\sigma_n^{(m)} = \sum_{j=0}^{m} \frac{(-)^{m-j}\, j^n}{(m-j)!\,j!}$$

Non-zero values of Stirling numbers of the second kind are, of course, positive integers and some are shown in Figure 2-2. Others may be calculated via the recursion formula

2:14:5 $$\sigma_n^{(m)} = m\sigma_{n-1}^{(m)} + \sigma_{n-1}^{(m-1)}$$

for $m = 1,2,3,\cdots$ and $n = 1,2,3,\cdots$. This recursion forms the basis of *Equator*'s Stirling number of the second kind routine (keyword **sigmanum**). First, $\sigma_0^{(1\to m)}$ and $\sigma_{1\to n}^{(0)}$ are initialized to zero while $\sigma_0^{(0)}$ is set equal to 1. Then $\sigma_1^{(1\to m)}, \sigma_2^{(1\to m)}, \cdots, \sigma_n^{(1\to m)}$ are calculated via recursion 2:14:5. Because Stirling numbers are integers, rounding ensures that the 15 digits that *Equator* generates are exact.

Figure 2-2



CHAPTER 3

THE ZETA NUMBERS AND RELATED FUNCTIONS

As detailed in Section 3:14, the four number families $\zeta(n)$, $\lambda(n)$, $\eta(n)$, and $\beta(n)$ occur as coefficients in many power series expansions. In this context it is only positive integer orders, $n = 1, 2, 3, \cdots$, that are encountered, and this chapter therefore emphasizes these cases. However, one is able to extend the definitions of all four functions to accept noninteger and negative orders, and these possibilities are also addressed here.

The first three functions are interrelated by the simple proportionalities

3:0:1 $$\frac{\zeta(\nu)}{2^\nu} = \frac{\lambda(\nu)}{2^\nu - 1} = \frac{\eta(\nu)}{2^\nu - 2}$$

and by the consequential identity

3:0:2 $$\zeta(\nu) + \eta(\nu) = 2\lambda(\nu)$$

but there are no corresponding relationships involving $\beta(\nu)$. Because they are so easily related to $\zeta(\nu)$ via 3:0:1, few formulas for $\lambda(\nu)$ and $\eta(\nu)$ are exhibited in this chapter.

3:1 NOTATION

The numbers $\zeta(n)$, $\lambda(n)$, $\eta(n)$, and $\beta(n)$ do not appear to have acquired definitive names; we shall call them *zeta numbers*, *lambda numbers*, *eta numbers*, and *beta numbers*. When the order is unrestricted, the symbolism $\zeta(\nu)$, $\lambda(\nu)$, $\eta(\nu)$ and $\beta(\nu)$ and the names *zeta function*, *lambda function*, *eta function* and *beta function* are commonplace, though in this *Atlas* the "number" terminology is employed generally, whether the argument is an integer or not. The last is also known as *Catalan's beta function*, which minimizes the possibility of confusion with the (complete) beta function of Section 43:13 or the incomplete beta function of Chapter 58, to which it is totally unrelated. Nevertheless "beta number" is the name used here.

The symbols σ and L have been used, respectively, in place of η and β. Riemann's name (Georg Friedrich Bernhard Riemann, German mathematician, 1826 − 1866) is associated with the zeta number, which is often known as "Riemann's function" or "Riemann's zeta function". We avoid these names to lessen confusion with the bivariate function of Chapter 64, with which Riemann's name is also commonly associated. You may also encounter Dirichlet's name associated with these functions.

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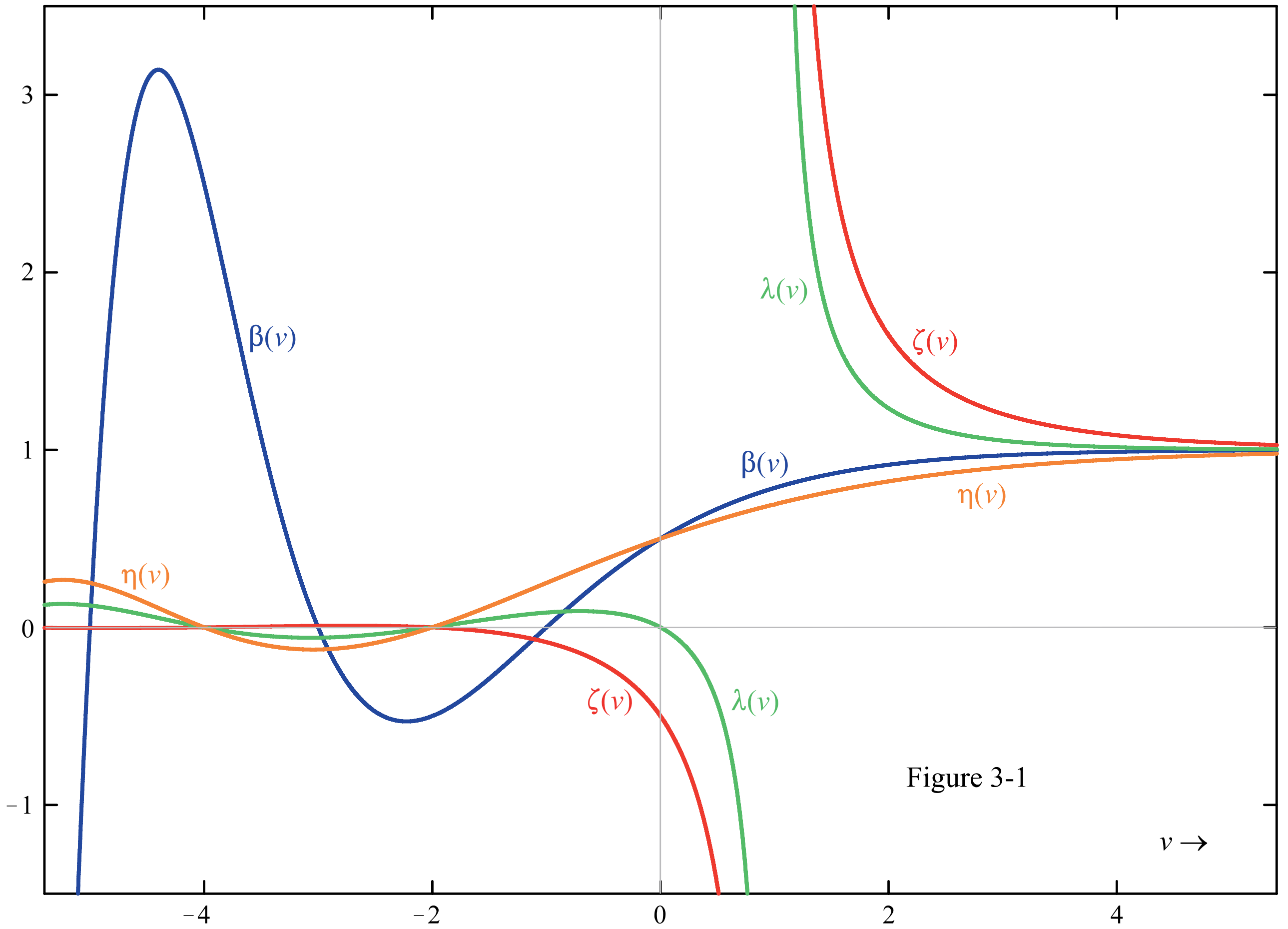


Figure 3-1

3:2 BEHAVIOR

Figure 3-1 maps the four functions. The zeta and lambda numbers develop infinite discontinuities of the $-\infty|+\infty$ type at $v=1$, whereas $\eta(v)$ and $\beta(v)$ are finite for all values of the order v. All four functions approach unity rapidly as the order v increases. For negative orders, the four functions are oscillatory and quasiperiodic [Section 36:13], with a constant period of 4 and with ever-increasing amplitudes as $v \to -\infty$. The zeros of $\zeta(v)$, $\lambda(v)$ and $\eta(v)$ occur when v is an even negative integer, whereas $\beta(v)=0$ at $v=-1,-3,-5,\cdots$.

3:3 DEFINITIONS

The four functions may be defined by the definite integrals

3:3:1 $$\zeta(v)=\frac{1}{\Gamma(v)}\int_0^\infty \frac{t^{v-1}}{\exp(t)-1}\,\mathrm{d}t \qquad v>1$$

3:3:2 $$\lambda(v)=\frac{1}{2\Gamma(v)}\int_0^\infty t^{v-1}\operatorname{csch}(t)\,\mathrm{d}t \qquad v>1$$

3:3:3 $$\eta(\nu) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}}{\exp(t)+1} \mathrm{d}t \qquad \nu > 0$$

and

3:3:4 $$\beta(\nu) = \frac{1}{2\Gamma(\nu)} \int_0^\infty t^{\nu-1} \operatorname{sech}(t) \mathrm{d}t \qquad \nu > 0$$

involving functions discussed in Chapters 12, 26, 29 and 43.

The most commonly encountered definitions of zeta, lambda, eta, and beta numbers employ the infinite series 3:6:1−3:6:4. These series, which may be reformulated as limits, for example

3:3:5 $$\zeta(\nu) = \lim_{J\to\infty}\left\{1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \cdots + \frac{1}{(J-1)^\nu} + \frac{1}{J^\nu}\right\} = \lim_{J\to\infty}\sum_{j=1}^{J} j^{-\nu} \qquad \nu > 1$$

apply equally to noninteger orders, provided that $\nu > 1$ for $\zeta(n)$ and $\lambda(n)$, or $\nu > 0$ for $\eta(n)$ and $\beta(n)$. Moreover, when a subsidiary series, involving Pochhammer and Euler polynomials [Chapters 18 and 20], is added into definition 3:3:5, to give

3:3:6 $$\begin{aligned}\zeta(\nu) &= \lim_{J\to\infty}\left\{1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \cdots + \frac{1}{(J-1)^\nu} + \frac{1}{(\nu-1)J^{\nu-1}} + \frac{1}{2J^\nu} + \frac{\nu}{12J^{\nu+1}} - \frac{\nu(\nu+1)(\nu+2)}{720J^{\nu+3}} + \cdots\right\} \\ &= \lim_{J\to\infty}\left\{\sum_{j=1}^{J} j^{-\nu} + \sum_{k=0}^{K} \frac{(\nu)_{k-1}\mathrm{B}_k}{k!J^{k+\nu-1}}\right\} \qquad \nu \neq 1\end{aligned}$$

the limit provides a definition of the zeta number for any order whatsoever (except $\nu = 1$): positive, zero, or negative, integer or noninteger. The summand of the k-summation involves functions from Chapters 2, 4, and 18. The upper limit K of this second summation must be at least $\mathrm{Int}(1-\nu)$, but may be larger with beneficial effect on convergence. The corresponding definition of the beta number is

3:3:7 $$\begin{aligned}\beta(\nu) &= \lim_{J\to\infty}\left\{1 - \frac{1}{3^\nu} + \frac{1}{5^\nu} - \cdots - \frac{(-1)^J}{2(2J-1)^\nu} - \frac{(-)^J\nu}{12J^{\nu+1}} - \frac{(-)^J\nu(\nu+1)(\nu+2)}{6(2J-1)^{\nu+3}} + \cdots\right\} \\ &= \lim_{J\to\infty}\left\{\sum_{j=1}^{J}\frac{(-1)^{j+1}}{(2j-1)^\nu} + \sum_{k=1}^{K}\frac{2^{k-1}(2^k-1)(\nu)_{k-1}\mathrm{B}_k}{k!(2J-1)^{k+\nu-1}}\right\}\end{aligned}$$

Because $\mathrm{B}_k = 0$ for $k = 2,4,6,\cdots$, many of the terms in the k-summations in the last two formulas are zero. Equations 3:3:6 and 3:3:7 have their origins in the Euler-Maclaurin formula [Section 4:14], and are therefore technically asymptotic. Nevertheless, with a suitable choice of K, they converge excellently.

3:4 SPECIAL CASES

When their argument is an integer, the zeta, lambda, eta, and beta numbers often equal rational numbers, or are related to well-known mathematical constants, as detailed in Section 3:7.

3:5 INTRARELATIONSHIPS

The zeta and beta numbers satisfy the reflection formulas

3:5:1 $$\zeta(1-v) = \frac{2\Gamma(v)\cos(v\pi/2)}{(2\pi)^v}\zeta(v) \qquad v \neq 0,-1,-2,\cdots$$

and

3:5:2 $$\beta(1-v) = \frac{\Gamma(v)\sin(v\pi/2)}{(\pi/2)^v}\beta(v) \qquad v \neq 0,-1,-2,\cdots$$

involving the gamma function [Chapter 43] and the functions of Chapter 32. Thereby either function of order less than ½ may be related to one with order greater than ½.

With f(n) representing any one of the four numbers $\zeta(n)$, $\lambda(n)$, $\eta(n)$, or $\beta(n)$, one may sum the infinite series $\Sigma(-)^n\mathrm{f}(n)/n$, as well as the series of complements $\Sigma[1-\mathrm{f}(n)]$, $\Sigma(-)^n[1-\mathrm{f}(n)]$, $\Sigma[1-\mathrm{f}(n)]/n$, and $\Sigma(-)^n[1-\mathrm{f}(n)]$. With the lower summation limit taken as $n = 2$, these sums are tabulated below. The sums involve the logarithmic function [Chapter 25] of various constants [Section 1:7]. Also listed in the table are the sums $\Sigma(-)^n\mathrm{f}(n)$. Strictly, these particular series do not converge, the tabulated entries being the limits

3:5:3 $$\lim_{J\to\infty}\left\{\mathrm{f}(2)-\mathrm{f}(3)+\mathrm{f}(4)-\cdots\mp\mathrm{f}(J-1)\pm\frac{1}{2}\mathrm{f}(J)\right\}$$

which do indeed converge and whose values may be associated with $\Sigma(-)^n\mathrm{f}(n)$.

	$\sum_{n=2}^{\infty}(-)^n\mathrm{f}(n)$	$\sum_{n=2}^{\infty}(-)^n\frac{\mathrm{f}(n)}{n}$	$\sum_{n=2}^{\infty}[1-\mathrm{f}(n)]$	$\sum_{n=2}^{\infty}(-)^n[1-\mathrm{f}(n)]$	$\sum_{n=2}^{\infty}\frac{1-\mathrm{f}(n)}{n}$	$\sum_{n=2}^{\infty}(-)^n\frac{1-\mathrm{f}(n)}{n}$
$\mathrm{f}=\zeta$	1	γ	-1	$\frac{-1}{2}$	$\gamma-1$	$1-\ln(2)-\gamma$
$\mathrm{f}=\lambda$	$\ln(2)$	$\frac{\gamma}{2}+\ln\left(\frac{2}{\sqrt{\pi}}\right)$	$\ln(2)-1$	$\frac{1}{2}-\ln(2)$	$\frac{\gamma}{2}+\ln\left(\sqrt{\pi}\right)-1$	$1-\ln\left(\frac{4}{\sqrt{\pi}}\right)-\frac{\gamma}{2}$
$\mathrm{f}=\eta$	$\ln(4)-1$	$\ln\left(\frac{4}{\pi}\right)$	$\ln(4)-1$	$\frac{3}{2}-\ln(4)$	$\ln(\pi)-1$	$1-\ln\left(\frac{8}{\pi}\right)$
$\mathrm{f}=\beta$	$\frac{\pi}{4}-\ln\left(\sqrt{2}\right)$	$\frac{\pi}{4}-\ln(2g)$	$\frac{\pi}{4}+\ln\left(\sqrt{2}\right)-1$	$\frac{1}{2}+\ln\left(\sqrt{2}\right)-\frac{\pi}{4}$	$\frac{\pi}{4}+\ln\left(\frac{\pi g}{2}\right)-1$	$1+\ln(g)-\frac{\pi}{4}$

3:6 EXPANSIONS

The series

3:6:1 $$\zeta(v) = 1+\frac{1}{2^v}+\frac{1}{3^v}+\cdots = \sum_{j=1}^{\infty} j^{-v} \qquad v>1$$

3:6:2 $$\lambda(v) = 1+\frac{1}{3^v}+\frac{1}{5^v}+\cdots = \sum_{j=1}^{\infty}(2j-1)^{-v} \qquad v>1$$

3:6:3 $$\eta(v) = 1-\frac{1}{2^v}+\frac{1}{3^v}-\cdots = \sum_{j=1}^{\infty}(-)^{j+1}j^{-v} \qquad v>0$$

3:6:4 $$\beta(v) = 1 - \frac{1}{3^v} + \frac{1}{5^v} - \cdots = \sum_{j=1}^{\infty}(-)^{j+1}(2j-1)^{-v} \qquad v > 0$$

are the most useful representations of the four functions and serve as definitions of the zeta and lambda numbers, $\zeta(n)$ and $\lambda(n)$, for $n = 2,3,4,\cdots$, as well as for the eta and beta numbers, $\eta(n)$ and $\beta(n)$, for $n = 1,2,3,\cdots$.

Zeta and lambda numbers are expansible as the infinite products

3:6:5 $$\zeta(v) = \frac{1}{1-2^{-v}}\frac{1}{1-3^{-v}}\frac{1}{1-5^{-v}}\frac{1}{1-7^{-v}}\cdots = \prod_{j=1}^{\infty}\frac{1}{1-\pi_j^{-v}} \qquad v>1$$

3:6:6 $$\lambda(v) = \frac{1}{1-3^{-v}}\frac{1}{1-5^{-v}}\frac{1}{1-7^{-v}}\frac{1}{1-11^{-v}}\cdots = \prod_{j=2}^{\infty}\frac{1}{1-\pi_j^{-v}} \qquad v>1$$

where π_j is the jth prime number.

Useful for orders v close to 1 are the expansions

3:6:7 $$\zeta(v) = \frac{1}{v-1} + \sum_{j=0}^{\infty}\frac{(-)^j\gamma_j}{j!}(v-1)^j$$

3:6:8 $$\eta(v) = \frac{1-2^{1-v}}{v-1} - \sum_{j=1}^{\infty}\frac{(1-v)^j}{j!}\sum_{k=0}^{j-1}\gamma_k\binom{j}{k}\ln^{j-k}(2)$$

where γ_0 is Euler's constant and the other γ_j are the so-called *Stieltjes constants* [Wolfram].

3:7 PARTICULAR VALUES

Below are listed values of the zeta, lambda, eta and beta numbers for $n = -5,-4,\cdots,3,4$ and ∞. In this listing G is Catalan's constant [Section 1:7] and Z is Apéry's constant (Roger Apéry, French mathematician, 1916 – 1994) which takes the value

3:7:1 $$Z = \zeta(3) = 1.2020\ 56903\ 15959$$

	$n=-5$	$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=\infty$
$\zeta(n)$	$\frac{-1}{252}$	0	$\frac{1}{120}$	0	$\frac{-1}{12}$	$\frac{-1}{2}$	$-\infty\|+\infty$	$\frac{\pi^2}{6}$	Z	$\frac{\pi^4}{90}$	1
$\lambda(n)$	$\frac{31}{252}$	0	$\frac{-7}{120}$	0	$\frac{1}{12}$	0	$-\infty\|+\infty$	$\frac{\pi^2}{8}$	$\frac{7Z}{8}$	$\frac{\pi^4}{96}$	1
$\eta(n)$	$\frac{1}{4}$	0	$\frac{-1}{8}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\ln(2)$	$\frac{\pi^2}{12}$	$\frac{3Z}{8}$	$\frac{7\pi^4}{720}$	1
$\beta(n)$	0	$\frac{5}{2}$	0	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\pi}{4}$	G	$\frac{\pi^3}{32}$	$\beta(4)$	1

For $n = 2,4,6,\cdots$ all zeta, lambda, and eta numbers equal π^n multiplied by a proper fraction, the fraction being related by equation 3:13:1 to the Bernoulli number B_n of Chapter 4. Similarly, for $n = 1,3,5,\cdots$, $\beta(n)$ is proportional to π^n, the proportionality constant being a proper fraction related to the Euler number E_{n-1} [Chapter 5] by equation 3:13:2. For negative integer orders, the zeta and beta numbers are related much more simply to the Bernoulli and

Euler numbers; thus:

3:7:2
$$\zeta(-n) = \frac{-\mathrm{B}_{n+1}}{n+1} \qquad n = 1,2,3,\cdots$$

and

3:7:3
$$\beta(-n) = \frac{\mathrm{E}_n}{2} \qquad n = 0,1,2,\cdots$$

3:8 NUMERICAL VALUES

Equator's zeta number routine (keyword **zetanum**) calculates $\zeta(v)$, using equation 3:6:7 for $0.995 \le v \le 1.01$, equation 3:3:6 for other values in excess of -0.5 and up to 169, and formula 3:5:1 for $-169 \le v < -0.5$. For the most part, the lambda number and eta number routines (keywords **lambdanum** and **etanum**) use equation 3:0:1, and the corresponding zeta number, but 3:6:8 is substituted in the immediate vicinity of $v = 1$. Equations 3:3:7 and 3:5:2 form the basis of *Equator*'s beta number routine (keyword **betanum**), the latter being employed for $-169 \le v \le 0$.

3:9 LIMITS AND APPROXIMATIONS

All four functions approach unity as the order v approaches infinity, the limiting behaviors being evident from equations 3:6:1−4.

As v itself approaches unity, from either direction, equation 3:6:7 shows that the zeta number exhibits the limiting behavior

3:9:1
$$\zeta(v) \to \frac{1}{v-1} + \gamma \qquad v \to 1$$

where γ is Euler's constant [Section 1:7]. Likewise, the $v \to 1$ limit of $\eta(v)$ follows from 3:6:8.

As the order approaches $-\infty$, the following limits are attained by the zeta number

3:9:2
$$\zeta(v) \to \sqrt{\frac{-2v}{\pi}} \left(\frac{\pi e}{-v}\right)^{v} \sin\left(\frac{v\pi}{2}\right) \qquad v \to -\infty$$

and by the beta number

3:9:3
$$\beta(v) \to \sqrt{\frac{-8v}{\pi}} \left(\frac{\pi e}{-2v}\right)^{v} \cos\left(\frac{v\pi}{2}\right) \qquad v \to -\infty$$

The implications of these limits for the zeta and beta numbers of large negative integer order are:

3:9:4
$$\zeta(n) \approx \begin{cases} 0 & n \text{ negative, large, and even} \\ (-)^{(1-n)/2}\sqrt{-2n/\pi}\left(-\pi e/n\right)^{n} & n \text{ negative, large, and odd} \end{cases}$$

3:9:5
$$\beta(n) \approx \begin{cases} (-)^{n/2}\sqrt{-8n/\pi}(-\pi e/2n)^{n} & n \text{ negative, large, and even} \\ 0 & n \text{ negative, large, and odd} \end{cases}$$

Based on truncating definition 3:6:5, the underestimate

3:9:6
$$\zeta(n) \approx \frac{1}{(1-2^{-n})(1-3^{-n})(1-5^{-n})(1-7^{-n})}$$

finds application in Section 4:8. Similarly, the overestimate

3:9:7 $$\beta(n) \approx \frac{1+5^{-n}-7^{-n}}{1+3^{-n}}$$

which is in error by less that 0.2% for $n \geq 2$, is useful in Section 5:8.

3:10 OPERATIONS OF THE CALCULUS

The derivatives and indefinite integrals of the four functions may be expressed as infinite series, for example

3:10:1 $$\frac{\mathrm{d}}{\mathrm{d}v}\zeta(v) = -\sum_{j=2}^{\infty}\frac{\ln(j)}{j^{v}} \qquad v>1$$

and

3:10:2 $$\int_0^v \beta(t)\,\mathrm{d}t = v + \sum_{j=1}^{\infty}\frac{(-1)^j}{\ln(2j+1)}\left[1-(2j+1)^{-v}\right] \qquad v \geq 0$$

but not as established functions. At $v = 0$ the derivative of the zeta number equals $-\ln\left(\sqrt{2\pi}\right)$.

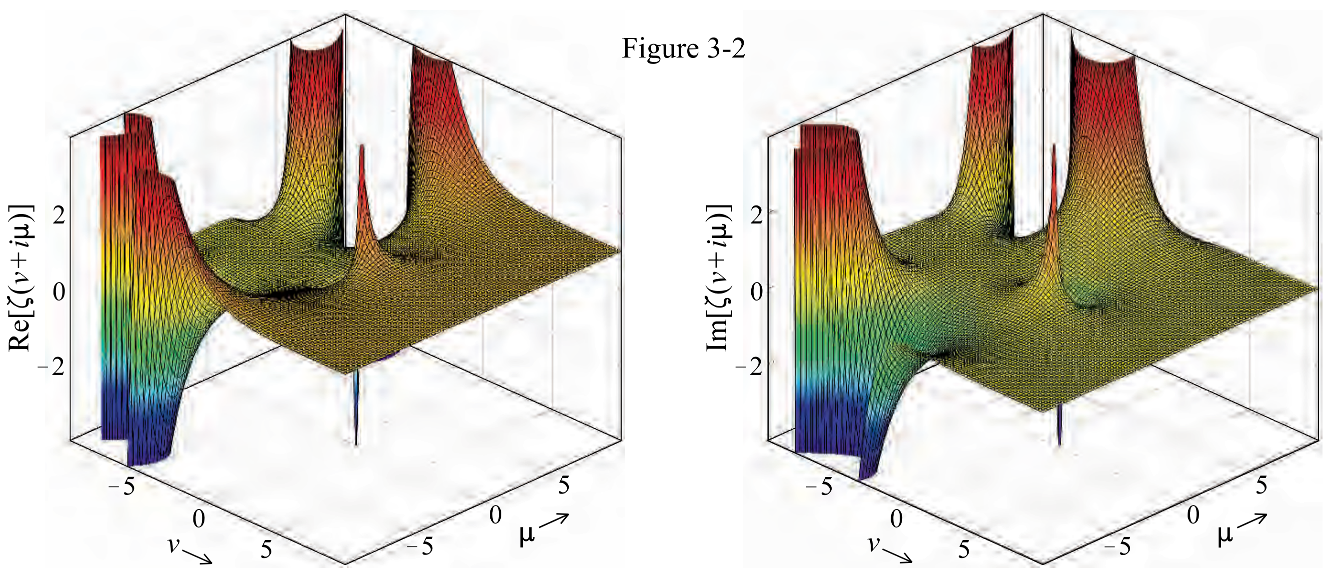
3:11 COMPLEX ARGUMENT

In terms of its real and imaginary parts, the zeta number of complex argument is given by

3:11:1 $$\zeta(v+i\mu) = \sum_{k=1}^{\infty}\frac{\cos\{\mu\ln(k)\}}{k^{v}} - i\sum_{k=2}^{\infty}\frac{\sin\{\mu\ln(k)\}}{k^{v}} \qquad v>1$$

with similar formulas serving the other three functions. Note that 3:11:1 is restricted to $v > 1$, but Figure 3-2 depicts the behavior in a more widespread region of the complex plane. Notice that the discontinuity encountered at $v = 1$ along the real line does not extend beyond that line. Not evident in the figure is that, in accord with *Riemann's hypothesis*, there exists a series of complex zeros (that is, points in the complex plane where both the real and the imaginary parts are zero) along the line $v = ½$.

Figure 3-2



3:12 GENERALIZATIONS

The four functions of this chapter are special cases of the Hurwitz function of Chapter 64. Thus:

3:12:1 $$\zeta(v)=\zeta(v,1)$$

3:12:2 $$\lambda(v)=2^{-v}\zeta\left(v,\tfrac{1}{2}\right)$$

3:12:3 $$\eta(v)=2^{1-v}\zeta\left(v,\tfrac{1}{2}\right)-\zeta(v,1)=\eta(v,1)$$

3:12:4 $$\beta(v)=4^{-v}\left[\zeta\left(v,\tfrac{1}{4}\right)-\zeta\left(v,\tfrac{3}{4}\right)\right]=2^{-v}\eta(v,\tfrac{1}{2})$$

The last pair of equations shows that the bivariate eta function [Section 64:13] is also a generalization of the eta and beta numbers. See Section 64:12 for the representation of the beta number as a Lerch function.

The zeta, eta, lambda, and beta numbers are also generalized respectively by the functions

3:12:5 $$\sum_{j=1}^{\infty}\frac{\cos(jx)}{j^n},\qquad -\sum_{j=1}^{\infty}\frac{(-)^j\cos(jx)}{j^n},\qquad \sum_{j=0}^{\infty}\frac{\cos\{(2j+1)x\}}{(2j+1)^n},\quad \text{and}\quad \sum_{j=0}^{\infty}\frac{(-)^j\cos\{(2j+1)x\}}{(2j+1)^n}$$

and by similar sums in which sin replaces cos. See Section 32:14 for these functions.

3:13 COGNATE FUNCTIONS

When n is even, $\zeta(n)$ is related to the absolute value of the Bernoulli number B_n [Chapter 4] by

3:13:1 $$\zeta(n)=\frac{(2\pi)^n\,|B_n|}{2n!}\qquad n=2,4,6,\cdots$$

For odd n the zeta number is related to the Bernoulli polynomial [Chapter 19] via the integral

3:13:2 $$\zeta(n)=\frac{(2\pi)^n}{2n!}\left|\int_0^1 B_n(t)\cot(\pi t)\,dt\right|\qquad n=1,3,5,\cdots$$

The beta number of odd argument n is related to the Euler number E_{n-1} [Chapter 5] by

3:13:3 $$\beta(n)=\left(\frac{\pi}{2}\right)^n\frac{|E_{n-1}|}{2(n-1)!}\qquad n=1,3,5,\cdots$$

while for even n the relationship is to the integral of an Euler polynomial [Chapter 20]

3:13:4 $$\beta(n)=\frac{\pi^n}{4(n-1)!}\left|\int_0^1 E_{n-1}(t)\sec(\pi t)\,dt\right|\qquad n=2,4,6,\cdots$$

3:14 RELATED TOPIC: trigonometric and hyperbolic expansions

The four number families occur as coefficients in power series expansions of certain circular and hyperbolic functions of argument πx or $\pi x/2$:

3:14:1 $$\cot(\pi x)=\frac{1}{\pi x}-\frac{2}{\pi x}\sum_{n=1}^{\infty}\zeta(2n)x^{2n}\qquad -1<x<1$$

3:14:2 $$\csc(\pi x) = \frac{1}{\pi x} + \frac{2}{\pi x}\sum_{n=1}^{\infty}\eta(2n)x^{2n} \qquad -1 < x < 1$$

3:14:3 $$\tan\left(\frac{\pi x}{2}\right) = \frac{4}{\pi x}\sum_{n=1}^{\infty}\lambda(2n)x^{2n} \qquad -1 < x < 1$$

3:14:4 $$\sec\left(\frac{\pi x}{2}\right) = \frac{4}{\pi x}\sum_{n=1}^{\infty}\beta(2n-1)x^{2n-1} \qquad -1 < x < 1$$

3:14:5 $$\coth(\pi x) = \frac{1}{\pi x} - \frac{2}{\pi x}\sum_{n=1}^{\infty}(-)^n\zeta(2n)x^{2n} \qquad -1 < x < 1$$

3:14:6 $$\mathrm{csch}(\pi x) = \frac{1}{\pi x} + \frac{2}{\pi x}\sum_{n=1}^{\infty}(-)^n\eta(2n)x^{2n} \qquad -1 < x < 1$$

3:14:7 $$\tanh\left(\frac{\pi x}{2}\right) = \frac{-4}{\pi x}\sum_{n=1}^{\infty}(-)^n\lambda(2n)x^{2n} \qquad -1 < x < 1$$

and

3:14:8 $$\mathrm{sech}\left(\frac{\pi x}{2}\right) = \frac{-4}{\pi x}\sum_{n=1}^{\infty}(-)^n\beta(2n-1)x^{2n-1} \qquad -1 < x < 1$$

Similar series represent the logarithms of certain trigonometric functions

3:14:9 $$\ln\{\csc(\pi x)\} = -\ln\{\sin(\pi x)\} = -\ln(\pi x) + \sum_{n=1}^{\infty}\frac{\zeta(2n)}{n}x^{2n} \qquad -1 < x < 1$$

3:14:10 $$\ln\left\{\tan\left(\frac{\pi x}{2}\right)\right\} = -\ln\left\{\cot\left(\frac{\pi x}{2}\right)\right\} = \ln\left(\frac{\pi x}{2}\right) + \sum_{n=1}^{\infty}\frac{\eta(2n)}{n}x^{2n} \qquad -1 < x < 1$$

3:14:11 $$\ln\left\{\sec\left(\frac{\pi x}{2}\right)\right\} = -\ln\left\{\cos\left(\frac{\pi x}{2}\right)\right\} = \sum_{n=1}^{\infty}\frac{\lambda(2n)}{n}x^{2n} \qquad -1 < x < 1$$

Notice that it is invariably the zeta, eta, and lambda numbers of even argument, and the beta numbers of odd argument, that appear in such expansions.

3:15 RELATED TOPIC: Debye functions

When v is the integer $n+1$, equation 3:3:1 may be rewritten

3:15:1 $$n!\,\zeta(n+1) = \int_0^{\infty}\frac{t^n}{\exp(t)-1}\,\mathrm{d}t$$

For $n = 1,2,3,\cdots$, the incomplete version of this integral

3:15:2 $$\int_0^{x}\frac{t^n}{\exp(t)-1}\,\mathrm{d}t$$

is important in the theory of heat capacities and is known as the nth *Debye function* (Peter Joseph William Debye, Dutch then U.S. physical chemist, 1884 – 1966). Values of these functions are calculable from the series

3:15:3 $$\int_0^x \frac{t^n}{\exp(t)-1}\mathrm{d}t = \frac{x^n}{n} - \frac{x^{n+1}}{2(n+1)} + \frac{x^{n+2}}{12(n+2)} - \frac{x^{n+4}}{720(n+4)} + \cdots = \sum_{j=0}^{\infty}\frac{\mathrm{B}_j x^{n+j}}{j!(n+j)} \qquad |x| < 2\pi$$

where B_j is a Bernoulli number [Chapter 4]. For large x it is better to evaluate the sum

3:15:4 $$\int_x^{\infty} \frac{t^n}{\exp(t)-1}\mathrm{d}t = n!\sum_{j=1}^{\infty}\left[\frac{x^n}{n!\,j} + \frac{x^{n-1}}{(n-1)!\,j^2} + \frac{x^{n-2}}{(n-2)!\,j^3} + \cdots + \frac{1}{j^{n+1}}\right]\exp(-jx) \qquad x > 0$$

and then subtract this sum from expression 3:15:1. *Equator*'s Debye function routine (keyword **Debye**) uses equation 3:15:3 to provide values of these functions for $|x| \le 5$ and $1 \le n \le 200$. For $x > 5$ and $1 \le n \le 9$, equations 3:15:1 and 3:15:4 are employed.

CHAPTER 4

THE BERNOULLI NUMBERS B_n

Bernoulli numbers are *rational numbers* (that is, they are expressible as l/m, where l and m are integers) that arise as coefficients in the power-series expansions of certain hyperbolic and trigonometric functions [Sections 6 of Chapters 28−34, 43, and 44]. They are also important in the context of the Euler-Maclaurin formula, an important tool that is discussed in Section 4:14.

Value	*Atlas* system	Rival system
1	B_0	
$-1/2$	B_1	
$1/6$	B_2	B_1
0	B_3	
$-1/30$	B_4	$-B_2$
0	B_5	
$1/42$	B_6	B_3
0	B_7	
$-1/30$	B_8	$-B_4$

4:1 NOTATION

There are two systems for indexing and assigning signs to Bernoulli numbers. Unfortunately, both systems are in widespread use, although the one we adopt is more generally used than its rival and, importantly, it is the system that is more compatible with the definition of Bernoulli polynomials [Chapter 19]. The table compares the two systems. Some authors employ both systems, using B_n for one set of Bernoulli numbers and some modified symbolism such as B^*_n or $\bar{B}_n$ for the other. These latter are sometimes called *auxiliary Bernoulli numbers*. Convert them to the *Atlas* system via

4:1:1
$$\bar{B}_n = (-)^{n+1} B_{2n}$$

4:2 BEHAVIOR

In our notation, all Bernoulli numbers with odd degree, except B_1, are zero. All B_n for which the index n is a multiple of 4, except B_0, are negative rational numbers. All other Bernoulli numbers of even degree are positive rational numbers.

The magnitudes of the Bernoulli numbers are modest up to about $n = 15$, but beyond this $|B_n|$ burgeons rapidly. For this reason, we have chosen a logarithmic scale for Figure 4-1, which depicts the magnitudes of all non-zero Bernoulli numbers up to $n = 116$. Red points correspond to positive Bernoulli numbers and blue to negative. Note the alternation of positive and negative signs, and that, because their Bernoulli numbers are zero, no points appear

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_5, © Springer Science+Business Media, LLC 2009

for odd n, except when $n = 1$.

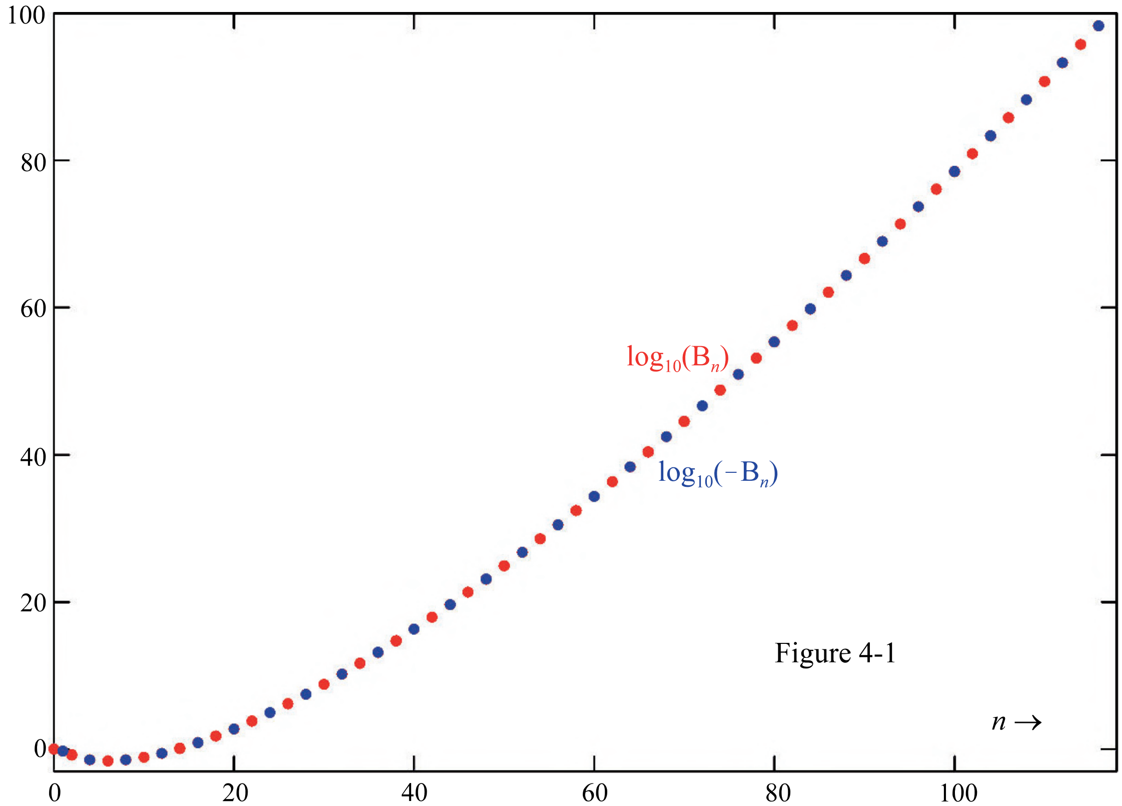


Figure 4-1

B_0 is the only Bernoulli number that is a nonzero integer. However

4:2:1 $$2(n+1)!!B_n = \text{an odd integer} \qquad n = 2,4,6,\cdots$$

where $n!!$ is the double factorial function [Section 2:13].

4:3 DEFINITIONS

The Bernoulli numbers are defined through the generating function

4:3:1 $$\frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

Moreover, equation 4:5:1 serves as a definition of all Bernoulli numbers beyond $n = 1$.

Integral representations of the nonzero Bernoulli numbers of even degree include

4:3:2 $$B_n = -\left(\frac{-1}{\pi^2}\right)^{n/2} \int_0^{\infty} t^n \operatorname{csch}^2(t)\,dt \qquad n = 2,4,6,\cdots$$

where csch denotes the hyperbolic cosecant function [Chapter 29]. For other integral representations, see Erdélyi et al. [*Higher Transcendental Functions*, Volume 1, pages 38−39].

4:4 SPECIAL CASES

When n is an even integer, the Bernoulli number B_n is expressible as a zeta number [Chapter 3]

4:4:1 $$B_n = (-)^{(n+2)/2}\frac{2n!}{(2\pi)^n}\zeta(n) \qquad n = 2,4,6,\cdots$$

4:5 INTRARELATIONSHIPS

The formula

4:5:1 $$B_n = -n!\sum_{j=0}^{n-1}\frac{B_j}{j!(n+1-j)!} \qquad n = 1,2,3,\cdots$$

allows any Bernoulli number (except B_0, of course) to be calculated from all of its predecessors. Equivalent to this equation is the more compact expression

4:5:2 $$\sum_{j=0}^{n}\binom{n+1}{j}B_j = 0 \qquad n = 1,2,3,\cdots$$

where $\binom{n}{j}$ is a binomial coefficient [Chapter 6].

4:6 EXPANSIONS

Even-indexed Bernoulli numbers are expansible as

4:6:1 $$B_n = (-)^{(n+2)/2}\frac{2n!}{(2\pi)^n}\left[1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right] = (-)^{(n+2)/2}2n!\sum_{j=1}^{\infty}(2j\pi)^{-n} \qquad n = 2,4,6,\cdots$$

If the multiplier $(-1)^{(n+2)/2}$ is replaced by $-\cos(n\pi/2)$, the expansion is valid for all values of n except 0 and 1.

4:7 PARTICULAR VALUES

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$	0	$\frac{7}{6}$	0	$\frac{-3617}{510}$	0	$\frac{43867}{798}$

4:8 NUMERICAL VALUES

With keyword **Bnum**, *Equator*'s Bernoulli number routine for B_n, directly returns the values 1 or $-½$ when $n = 0$ or 1; and 0 when $n = 3,5,7,\cdots$. Below is described the procedure adopted when n is an even positive integer from 2 through 170. Of course, *Equator* returns Bernoulli numbers as decimals, not as fractions. If fractions are sought, then *Equator*'s rational approximation routine [keyword **rational**, Section 8:13] may be employed to find the integers l and m such that $B_n = l/m$.

From 4:2:1, it follows that

4:8:1 $$(n+1)!!B_n - \tfrac{1}{2} = \text{an integer}$$

Thereby a sufficiently good approximation $\hat{B}_n$, to the Bernoulli number B_n, permits an *exact* value to be calculated via the correction formula

4:8:2 $$B_n = \frac{\text{Round}\left\{(n+1)!!\hat{B}_n - \tfrac{1}{2}\right\} + \tfrac{1}{2}}{(n+1)!!}$$

where Round is the rounding function [Section 8:13]. Equation 3:9:6 provides a close approximation to the zeta number $\zeta(n)$ and it therefore follows from 4:4:1 that the expression

4:8:3 $$B_n \approx \frac{-2n!}{(-\pi^2)^{n/2}(2^n-1)(1-3^{-n})(1-5^{-n})(1-7^{-n})} \qquad n = 2,4,6,\cdots$$

approximates Bernoulli numbers closely. This is the source of the approximation, $\hat{B}_n$, used by *Equator* in formula 4:8:2.

4:9 LIMITS AND APPROXIMATIONS

As n approaches infinity through even values, the Bernoulli numbers approach the limit

4:9:1 $$B_n \to (-)^{(n+2)/2} \frac{2n!}{(2\pi)^n} \qquad \text{even } n \to \infty$$

After incorporation of the limiting expression [equation 2:9:1] for the factorial function, this becomes

4:9:2 $$B_n \to (-)^{(n+2)/2} \left(\frac{n}{2\pi e}\right)^n \sqrt{8\pi n} \qquad \text{even } n \to \infty$$

From limit 4:9:1 it is evident that the ratio of two Bernoulli numbers, the indices of which are even and differ by 2, is approximated by the formula

4:9:3 $$\frac{B_n}{B_{n-2}} \approx \frac{-n(n-1)}{4\pi^2} \qquad n = 2,4,6,\cdots$$

which becomes increasingly accurate as $n \to \infty$.

4:10 OPERATIONS OF THE CALCULUS

Neither differentiation nor integration may be applied to discretely defined functions such as B_n.

4:11 COMPLEX ARGUMENT

The definition of the Bernoulli number B_n assumes n to be real.

4:12 GENERALIZATIONS

Bernoulli numbers are the values adopted by the corresponding Bernoulli polynomial $B_n(x)$ [Chapter 20] at zero argument, 4:12:1 $$B_n = B_n(0) \qquad n = 0,1,2,\cdots$$

and, with the sole exception of the $n = 1$ instance, also at an argument of unity

4:12:2 $$B_n = B_n(1) \qquad n = 0,2,3,4,5,\cdots$$

4:13 COGNATE FUNCTIONS

Bernoulli numbers are connected through equation 4:4:1 to zeta numbers [Chapter 3] and have much in common with Euler numbers [Chapter 5], to which they are related through

4:13:1 $$\frac{4^n(4^n-1)}{(2n)!}B_{2n} = \frac{E_{2n-2}}{1!(2n-2)!} + \frac{E_{2n-4}}{3!(2n-4)!} + \frac{E_{2n-6}}{5!(2n-6)!} + \cdots + \frac{E_0}{(2n-1)!0!}$$

4:14 RELATED TOPIC: the Euler-Maclaurin formula

Bernoulli numbers occur as coefficients in the *Euler-Maclaurin formula*. Let $x_0, x_1, x_2, \cdots, x_J$ be uniformly spaced arguments of the function $f(x)$, with $x_{j+1} - x_j = h$ and $j = 0, 1, 2, \cdots, J-1$. Then

4:14:1 $$h\sum_{j=0}^{J-1} f(x_j) - \int_{x_0}^{x_J} f(t)\,dt \sim -\frac{h}{2}\big[f(x_J) - f(x_0)\big] + \frac{h^2}{12}\left[\frac{df}{dx}(x_J) - \frac{df}{dx}(x_0)\right] - \frac{h^4}{720}\left[\frac{d^3f}{dx^3}(x_J) - \frac{d^3f}{dx^3}(x_0)\right]$$
$$+\frac{h^6}{30240}\left[\frac{d^5f}{dx^5}(x_J) - \frac{d^5f}{dx^5}(x_0)\right] + \cdots \quad = \quad \sum_{n=1} \frac{h^n B_n}{n!}\left[\frac{d^{n-1}f}{dx^{n-1}}(x_J) - \frac{d^{n-1}f}{dx^{n-1}}(x_0)\right]$$

provided that the function $f(x)$ is continuous and sufficiently differentiable. This formula is extremely useful in calculating (either analytically or numerically, exactly or approximately) an integral from a sum, or vice versa. It may be written more succinctly as

4:14:2 $$\frac{f(x_0) + f(x_J)}{2} + \sum_{j=1}^{J-1} f(x_j) - \frac{1}{h}\int_{x_0}^{x_J} f(t)\,dt \sim \sum_{m=1,3} \frac{h^m}{D_m} \frac{d^m f}{dx^m}\bigg|_{x_0}^{x_J} \qquad x_J = x_0 + Jh$$

where the divisors D_m, listed below, are equal to $(m+1)!/B_{m+1}$; they increase in magnitude so fast that convergence of the right-hand side of 4:14:2 is often extremely rapid. This speedy convergence means that, though these series are technically asymptotic, this is seldom of practical concern.

D_1	D_3	D_5	D_7	D_9	D_{11}	D_{13}	D_{15}
12	−720	30240	−1209600	479001600	−1892437580.31838	74724249600	−148387707891953.

There is a second Euler-Maclaurin formula. Again, it describes the relationship between an integral and a sum, but now the nodes to which the sum relates lie midway between those used in the first formula. In our succinct notation, the *second Euler-Maclaurin formula* is

4:14:3 $$\sum_{j=0}^{J-1} f(x_j + \tfrac{1}{2}h) - \frac{1}{h}\int_{x_0}^{x_J} f(t)\,dt \sim -\sum_{m=1,3} \frac{(1-2^{-m})h^m}{D_m} \frac{d^m f}{dx^m}\bigg|_{x_0}^{x_J} \qquad x_J = x_0 + Jh$$

With their right-hand sides approximated by zeros, equations 4:14:2 and 4:14:3 provide means of crudely approximating an integral by a sum. When used in this way, they are known as the *trapezoidal approximation* and the *midpoint approximation*, respectively. The right-hand members of these equations then represent correction terms useful in improving the approximations. Notice the difference in signs of the right-hand sides of the two Euler-Maclaurin formulas, which implies that if the trapezoidal approximation overestimates the integral then the midpoint formula underestimates it. A compromise between the two should therefore perform better than either. The familiar *Simpson's Rule* for numerical integration arises by just such an argument. See Section 62:15 for further information on Simpson's approximation and on Romberg integration, which exploits equation 4:14:2 in a more refined fashion.

CHAPTER 5

THE EULER NUMBERS E_n

Euler numbers occur as coefficients in the power series expansions of the secant [Chapter 33], hyperbolic secant [Chapter 29], and gudermannian [Section 33:14] functions.

Value	*Atlas* system	Rival system
1	E_0	E_0
0	E_1	
−1	E_2	$-E_1$
0	E_3	
5	E_4	E_2
0	E_5	
−61	E_6	$-E_3$
0	E_7	
1385	E_8	E_4

5:1 NOTATION

As with Bernoulli numbers [Section 4:1], there are two notational systems in use for indexing Euler numbers. The table exemplifies these two systems. Some authors use both systems and introduce supplementary notation such as E_n^* or $\bar{E}_n$ to distinguish one from the other. Numbers in the less favored system are sometimes called *auxiliary Euler numbers*.

5:2 BEHAVIOR

Unlike Bernoulli numbers, the Euler numbers are integers: they may be positive, negative or zero. E_n is defined for all nonnegative integer degree n.

Euler numbers of odd index are invariably zero. Euler numbers of even degree are positive integers, or negative integers, according as n is, or is not, a multiple of 4. In Figure 5-1, either $\log_{10}(E_n)$, red points, or $\log_{10}(-E_n)$, blue points, is plotted versus even values of n. Note the very rapid increase in the absolute value $|E_n|$ with increasing even n, which dictated the logarithmic presentation in the figure.

Curiously, the least significant digit of each negative Euler number is a 1, so that

5:2:1 $$\mathrm{frac}\left(\frac{-E_n}{10}\right) = \frac{1}{10} \qquad n = 2,6,10,\cdots$$

whereas the least significant digit of each positive Euler number is a "5", that is

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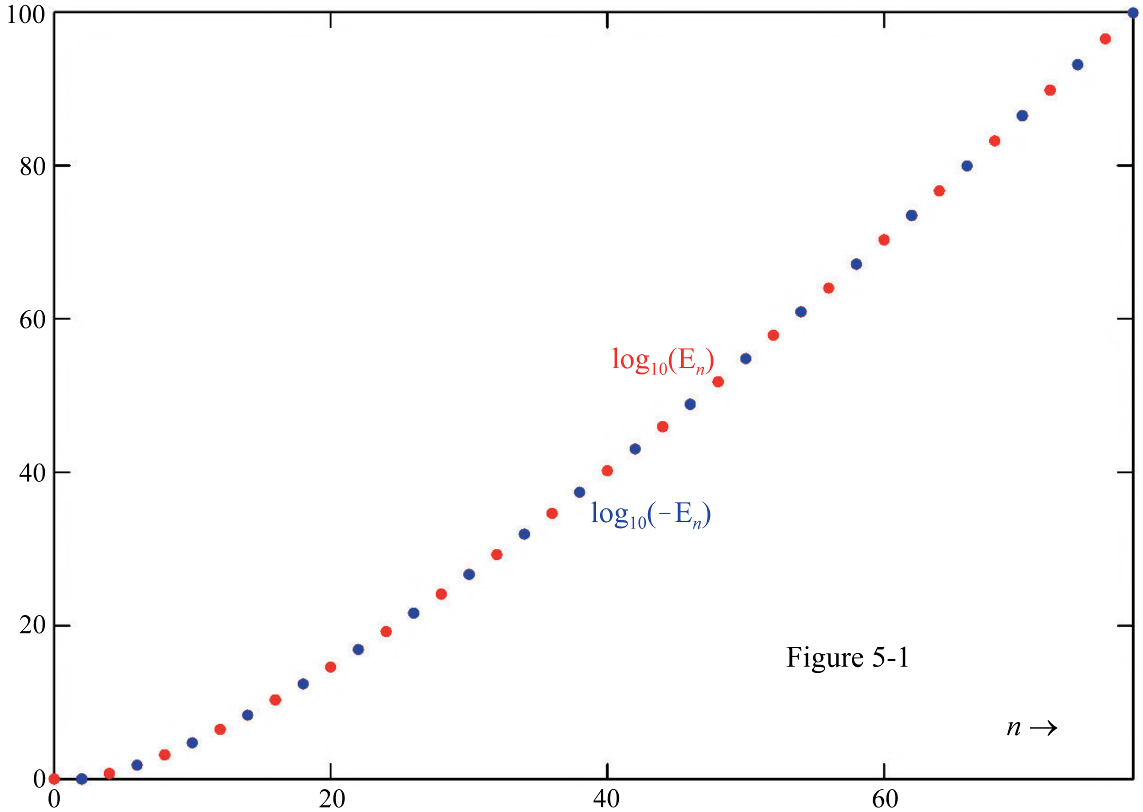

Figure 5-1

5:2:2 $$\text{frac}\left(\frac{E_n}{10}\right) = \frac{5}{10} \qquad n = 4,8,12,\cdots$$

where frac denotes the fractional-value function [Chapter 8].

5:3 DEFINITIONS

The generating function

5:3:1 $$\text{sech}(t) = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

may be used to define the Euler numbers, though the summation is slow to converge for larger t values. Here sech is the hyperbolic secant discussed in Chapter 29.

An integral definition for Euler numbers of even degree is

5:3:2 $$E_n = (-)^{n/2}\left(\frac{2}{\pi}\right)^{n+1}\int_0^{\infty} t^n \text{sech}(t)\,\mathrm{d}t \qquad n = 0,2,4,\cdots$$

With E_0 specified as unity, equation 5:5:2 sometimes serves as a definition of other Euler numbers of even degree.

5:4 SPECIAL CASES

The Euler number of even degree is related by the formula

5:4:1 $$E_n = (-)^{n/2} 2\left(\frac{2}{\pi}\right)^{n+1} n!\,\beta(n+1) \qquad n = 0,2,4,\cdots$$

to the beta number discussed in Chapter 3.

5:5 INTRARELATIONSHIPS

The compact expression

5:5:1 $$\sum_{j=0}^{n}\binom{2n}{2j} E_{2j} = 0 \qquad n = 1,2,3,\cdots$$

where $\binom{2n}{2j}$ is the binomial coefficient treated in Chapter 6, gives rise to the formula

5:5:2 $$E_n = -n!\sum_{j=0}^{(n-2)/2} \frac{E_{2j}}{(n-2j)!(2j)!} \qquad n = 2,4,6,\cdots \qquad E_0 = 1$$

by which any Euler number of even degree may be calculated from its predecessors.

5:6 EXPANSIONS

It follows from equations 5:4:1 and 3:6:4 that Euler numbers of even degree may be expanded as

5:6:1 $$E_n = (-)^{n/2} 2n!\left(\frac{2}{\pi}\right)^{n+1}\left[1 - \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} - \cdots\right] = (-)^{n/2}\frac{2n!}{\pi^{n+1}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+\frac{1}{2})^{n+1}} \qquad n = 0,2,4,\cdots$$

5:7 PARTICULAR VALUES

E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}
1	0	−1	0	5	0	−61	0	1385	0	−50521	0	2702765	0	−199360981

5:8 NUMERICAL VALUES

Equations 5:2:1 and 5:2:2 are equivalent, respectively, to

5:8:1 $$\frac{9 - E_n}{10} = \text{a positive integer} \qquad n = 2,6,10,\cdots$$

and

5:8:2 $$\frac{5 + E_n}{10} = \text{a positive integer} \qquad n = 4,8,12,\cdots$$

It follows that, if $\hat{E}_n$ approximates E_n well enough, then the *exact* value of the Euler number can be found from

5:8:3
$$\mathrm{E}_n = \begin{cases} 9-10\mathrm{Round}\left(\dfrac{9-\hat{\mathrm{E}}_n}{10}\right) & n=2,6,10,\cdots \\ 10\mathrm{Round}\left(\dfrac{5+\hat{\mathrm{E}}_n}{10}\right)-5 & n=4,8,12,\cdots \end{cases}$$

where Round is the rounding function discussed in Section 8:13. Amply accurate for this purpose is the approximation

5:8:4
$$\hat{\mathrm{E}}_n \approx (-)^{n/2}2\left(\frac{2}{\pi}\right)^{n+1} n!\left(\frac{1+5^{-n-1}-7^{-n-1}}{1+3^{-n-1}}\right) \qquad n=2,4,6,\cdots$$

derived by combining equation 5:4:1 with approximation 3:9:7. Equations 5:8:3 and 5:8:4 form the basis for *Equator*'s Euler number routine (keyword **Enum**) for even degree up to $n = 170$. The output is exact through $\mathrm{E}_{20} = 370\ 371\ 188\ 237\ 525$. For $n \geq 22$, a floating point answer rounded to 15 significant digits is provided.

5:9 LIMITS AND APPROXIMATIONS

The limit

5:9:1
$$\mathrm{E}_n \to (-)^{n/2}2\left(\frac{2}{\pi}\right)^{n+1} n! \qquad \text{even } n\to\infty$$

holds as n approaches infinity through even values.

5:10 OPERATIONS OF THE CALCULUS

Neither differentiation nor integration may be applied to discretely defined functions such as E_n.

5:11 COMPLEX ARGUMENT

Euler numbers are encountered only with real nonnegative integer degrees.

5:12 GENERALIZATIONS

Euler polynomials $\mathrm{E}_n(x)$ [Chapter 20] are a generalization of Euler numbers, to which they are related by

5:12:1
$$\mathrm{E}_n = 2^n\mathrm{E}_n(1/2) \qquad n=0,1,2,\cdots$$

5:13 COGNATE FUNCTIONS

Euler numbers have much in common with the beta numbers [Chapter 3] and with Bernoulli numbers [Chapter 4], to which they are related by equations 5:4:1 and 4:13:1, respectively.

CHAPTER
6

THE BINOMIAL COEFFICIENTS $\binom{\nu}{m}$

Binomial coefficients occur widely throughout mathematics; for example in the expansions discussed in Section 6:14 and in the Leibniz theorem, equation 0:10:6.

Binomial coefficients are so named because they are the numerical multipliers that arise when a two-term sum such as $(a + b)$, a so-called *binomial*, is raised to a power. The corresponding numbers that arise in expansions of powers of such extended sums as $(a + b + c + d)$ are termed multinomial coefficients and are discussed briefly in Section 6:12.

6:1 NOTATION

We refer to ν and m, respectively, as the upper index and the lower index of the binomial coefficient. The lower index of the binomial coefficient $\binom{\nu}{m}$ is invariably a nonnegative integer, whereas the upper index may take any real value. Positive integer values of the upper index are frequent, however, and in that circumstance the notation $\binom{n}{m}$ is used. Alternative to $\binom{n}{m}$ are the notations ${}_nC_m$ and $C_m^{(n)}$ which have their origins in the role played by binomial coefficients in expressing the number of combinations of m objects selected from a group of n different objects [Section 2.14].

6:2 BEHAVIOR

When it is not zero, the binomial coefficient $\binom{n}{m}$ is invariably a positive integer. The values of such integers are shown in Figure 6-1 for small values of the indices. For a given n, the maximal value of $\binom{n}{m}$ occurs when $m = n/2$ if n is even, or jointly at $m = (n \pm 1)/2$ when n is odd.

When ν is not restricted to a nonnegative integer, the behavior of the binomial coefficient $\binom{\nu}{m}$ is illustrated in Figure 6-2, and (except when $m = 0$) includes positive, negative, and zero values. The magnitude of these values is modest for $-1 < \nu < m+1$ but increases towards $\pm\infty$ outside this range.

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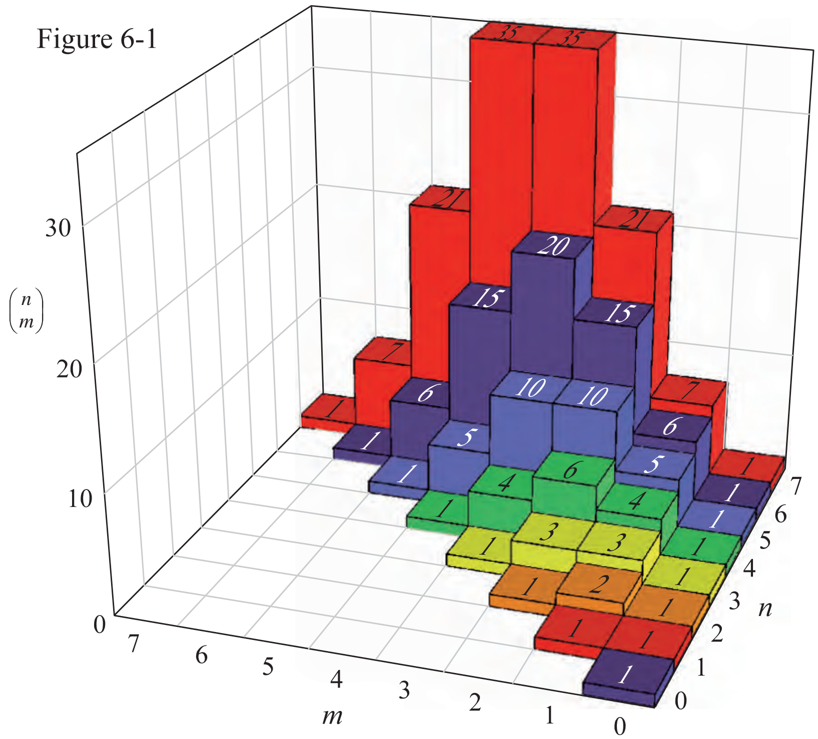
Figure 6-1



6:3 DEFINITIONS

The binomial coefficient is defined as the m-fold product

6:3:1
$$\binom{\nu}{m} = \left(\frac{\nu-m+1}{1}\right)\left(\frac{\nu-m+2}{2}\right)\left(\frac{\nu-m+3}{3}\right)\cdots\left(\frac{\nu}{m}\right) = \prod_{j=0}^{m-1}\frac{\nu-j}{m-j}$$

As is standard for empty products, this definition includes unity as the definition of $\binom{\nu}{0}$. The generating function [Section 0:3]

6:3:2
$$(1+t)^{\nu} = \sum_{m=0}^{\infty}\binom{\nu}{m}t^m \qquad -1<t<1$$

also defines the binomial coefficient.

A definition in terms of Pochhammer polynomials [Chapter 18]

6:3:3
$$\binom{\nu}{m} = \frac{(\nu-m+1)_m}{m!} = \frac{(-)^m(-\nu)_m}{m!}$$

is possible generally, whereas a definition exclusively in terms of the factorial function [Chapter 2]

6:3:4
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

requires that the upper index be a nonnegative integer.

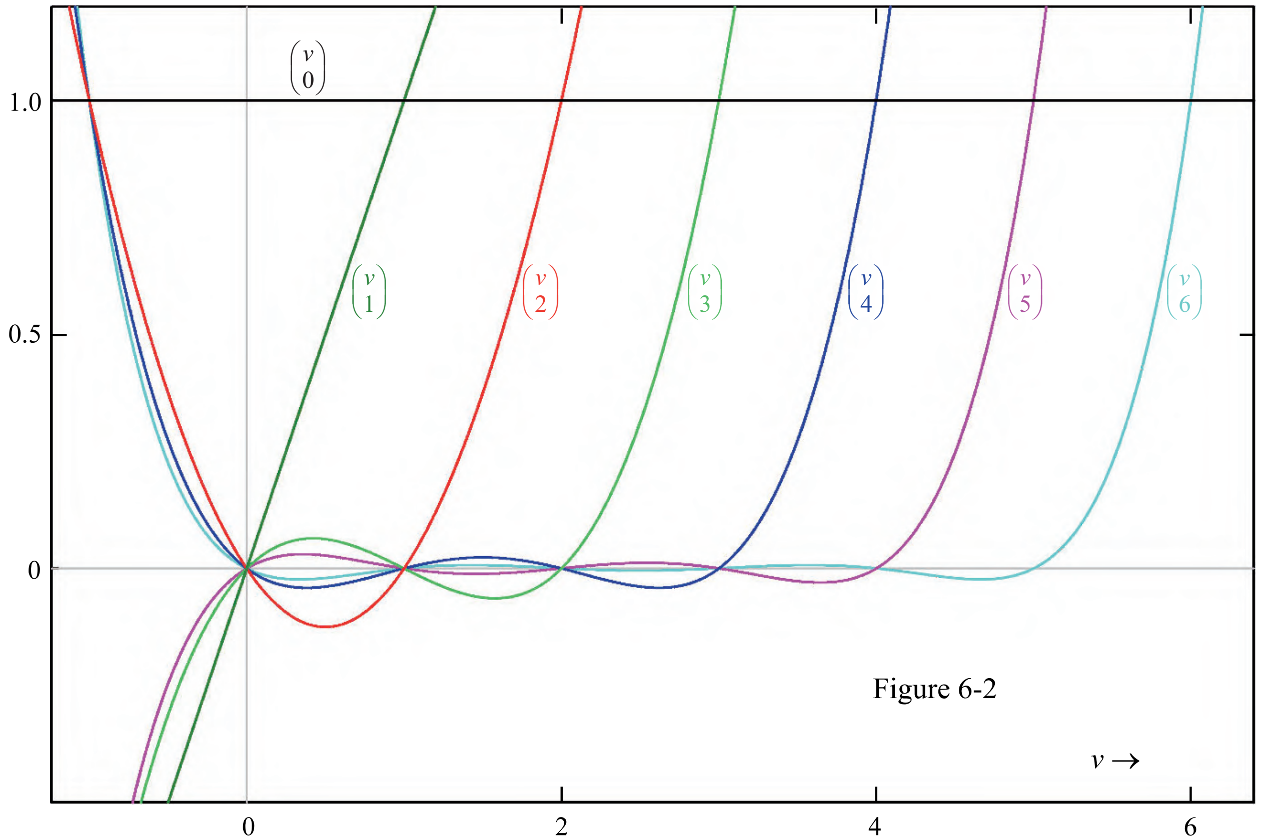


Figure 6-2

6:4 SPECIAL CASES

Reduction to an expression involving the double factorial [Section 2:13] occurs when the upper index is equal to twice the lower index

6:4:1 $$\binom{2m}{m} = \frac{4^m(2m-1)!!}{(2m)!!} = \frac{(2m)!}{(m!)^2} \qquad m = 0,1,2,\cdots$$

or differs by unity from twice the lower index

6:4:2 $$\binom{2m-1}{m-1} = \binom{2m-1}{m} = \frac{2^{2m-1}(2m-1)!!}{(2m)!!} = \frac{(2m)!}{2(m!)^2} \qquad m = 1,2,3,\cdots$$

6:5 INTRARELATIONSHIPS

There exist reflection formulas for the upper

6:5:1 $$\binom{-\nu}{m} = (-)^m \binom{\nu+m-1}{m}$$

and lower

6:5:2 $$\binom{n}{n-m} = \binom{n}{m}$$

indices, as well as recursion formulas

6:5:3 $$\binom{v+1}{m} = \binom{v}{m} + \binom{v}{m-1}$$

6:5:4 $$\binom{v}{m+1} = \frac{v-m}{m+1}\binom{v}{m}$$

for each index. The addition formula

6:5:5 $$\binom{v+\mu}{m} = \sum_{j=0}^{m}\binom{v}{j}\binom{\mu}{m-j}$$

known as *Vandermonde's convolution*, applies to the upper index.

There are a number of intrarelationships involving sums of binomial coefficients, including

6:5:6 $$\sum_{j=0}^{n-m}\binom{j+m}{m} = \binom{n+1}{m+1} \qquad n > m$$

6:5:7 $$\sum_{j=0}^{m}(-)^j\binom{v}{j} = (-)^m\binom{v-1}{m}$$

and

6:5:8 $$\sum_{j=0}^{J}\binom{n}{j}\binom{n}{n-J+j} = \binom{2n}{J} \qquad n \geq J$$

Formula 6:5:5 provides an expression for a sum of products of binomial coefficients; a second such formula is

6:5:9 $$\sum_{j=0}^{J}\binom{j}{m}\binom{j+v}{m} = \binom{m+v}{m}\binom{J+1+v}{J-m} \qquad J \geq m$$

When the upper index is an integer, finite, or infinite series of binomial coefficients frequently have simple expressions; examples include

6:5:10 $$\sum_{j=0}^{J}\binom{n}{j} = 2^n \qquad J \geq n$$

6:5:11 $$\sum_{j=0}^{J}(-)^j\binom{n}{j} = 0 \qquad J \geq n$$

6:5:12 $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{J} = 2^{n-1} \qquad n \leq J = 2,4,6,\cdots$$

6:5:13 $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{J} = 2^{n-1} \qquad n \leq J = 1,3,5,\cdots$$

and

6:5:14 $$\sum_{j=0}^{J} j\binom{n}{j} = 2^{n-1}n \qquad J \geq n$$

Similarly, the sum of squares of binomial coefficients gives

6:5:15 $$\sum_{j=0}^{J}\binom{n}{j}^2=\binom{2n}{n} \qquad J \ge n$$

6:6 EXPANSIONS

A binomial coefficient may be expanded as a power series in its upper index by the formula

6:6:1 $$\binom{v}{m}=\frac{1}{m!}\sum_{j=0}^{m}\mathrm{S}_m^{(j)}v^j$$

in which $\mathrm{S}_m^{(j)}$ is a Stirling number of the first kind [Section 18:6]. Additionally, definition 6:3:1 constitutes the expansion of $\binom{v}{m}$ as a finite product.

6:7 PARTICULAR VALUES

	$\binom{-1}{m}$	$\binom{-1/2}{m}$	$\binom{0}{m}$	$\binom{1/2}{m}$	$\binom{1}{m}$	$\binom{2m-1}{m}$	$\binom{2m}{m}$	$\binom{2m+1}{m}$
$m=0$	1	1	1	1	1	1	1	1
$m=1$	-1	$-1/2$	0	$1/2$	1	1	2	3
$m=2$	1	$3/8$	0	$-1/8$	0	3	6	10
$m=2,3,\cdots$	$(-1)^m$	$\frac{(-)^m(2m-1)!!}{(2m)!!}$	0	$\frac{(-)^{m+1}(2m-3)!!}{(2m)!!}$	0	$\frac{4^m(2m-1)!!}{2(2m)!!}$	$\frac{4^m(2m-1)!!}{(2m)!!}$	$\frac{4^{m+1}(2m-1)!!}{2(2m+2)!!}$

In addition to the cases tabulated above, there are the particular values

6:7:1 $$\binom{v}{0}=1 \qquad \text{and} \qquad \binom{v}{1}=v$$

which apply for all values of the upper index.

6:8 NUMERICAL VALUES

Binomial coefficients of integer indices may be arranged in the triangular arrangement shown on the right, in which each entry is the sum of the two above. This is commonly called *Pascal's triangle* (Blaise Pascal, French physicist and philosopher, 1623–1662), though it was described much earlier by the twelfth century mathematician Yanghui.

Equator's binomial coefficient routine (keyword **bincoef**) for calculating accurate values of $\binom{v}{m}$ is based on equation 6:3:1. The largest value of v for which all $\binom{v}{m}$ are calculable is 1029.3.

6:9 LIMITS AND APPROXIMATIONS

The limits

6:9:1 $$\binom{\nu}{m} \to \frac{\nu^m}{m!}\left[1-\frac{m(m-1)}{2\nu}+\frac{m(m-1)(m-2)(3m-1)}{24\nu^2}-\cdots\right] \qquad \nu\to\pm\infty$$

and

6:9:2 $$\binom{\nu}{m} \to \frac{(-1)^m}{m^{\nu+1}\Gamma(-\nu)}\left[1+\frac{\nu(\nu-1)}{2m}+\frac{\nu(\nu-1)(\nu-2)(3\nu-1)}{24m^2}+\cdots\right] \qquad m\to\infty$$

govern the behavior of the binomial coefficient when one index, but not both, is large. Of course limit 6:9:2, though universally valid, is useful only when ν is not a nonnegative integer; otherwise $\binom{\nu}{m}$ is zero for all $m>\nu$.

If both indices are large and positive, and especially when m lies in the vicinity of $\nu/2$, the binomial coefficient is well approximated by the *Laplace-de Moivre formula*

6:9:3 $$\binom{\nu}{m} \approx 2^\nu\sqrt{\frac{2}{\nu\pi}}\exp\left\{\frac{-2}{\nu}\left(m-\frac{\nu}{2}\right)^2\right\} \qquad \nu \text{ large}$$

which finds statistical applications [Sokolnikoff and Redheffer pages 623–626]. Even for ν as small as 10, the approximation leads to small absolute errors, as shown by Figure 6-3, in which the points are $\binom{10}{m}$ with $m=0, 1, 2, \cdots,$ 11 and the line is based on approximation 6:9:3 with $\nu=10$ and m treated as a continuous variable.

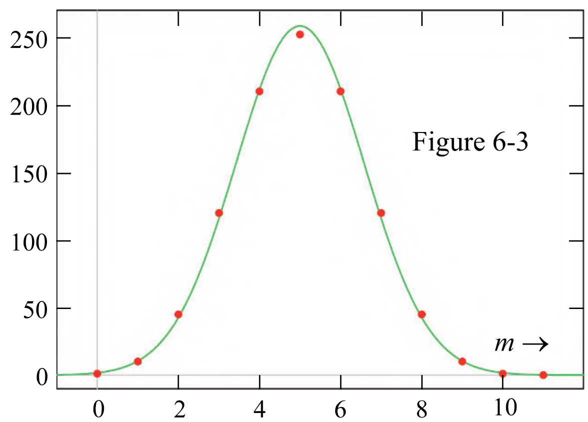


Figure 6-3

6:10 OPERATIONS OF THE CALCULUS

Differentiation with respect to the upper index gives a derivative involving a difference of two digamma functions [Chapter 44]:

6:10:1 $$\frac{\partial}{\partial\nu}\binom{\nu}{m}=\binom{\nu}{m}\left[\frac{1}{\nu}+\frac{1}{\nu-1}+\frac{1}{\nu-2}+\cdots+\frac{1}{\nu-m+1}\right]=\binom{\nu}{m}\left[\psi(-\nu)-\psi(m-\nu)\right]$$

6:11 COMPLEX ARGUMENT

The equivalence presented in equation 6:13:1 permits the formulas of Section 43:11 to be used to evaluate a binomial coefficient when one or both of the indices are imaginary or complex.

6:12 GENERALIZATIONS

The restriction that the lower index of a binomial coefficient be an integer may be relaxed by making use of the identity

6:12:1 $$\binom{v}{m} = \frac{1}{m\,\mathrm{B}(m,v-m+1)} \qquad m \neq 0$$

involving the complete beta function B [Section 43:13]. Thus the quantity $1/[\mu\mathrm{B}(\mu,v-\mu+1)]$ can be regarded as a generalized binomial coefficient, where μ, replacing *m*, is not necessarily an integer.

A generalization in a different direction is provided by *multinomial coefficients*. These are the coefficients that arise in such expansions as

6:12:2 $$\begin{aligned}(a+b+c)^5 &= \left[a^5+b^5+c^5\right]+5\left[a^4b+a^4c+ab^4+ac^4+b^4c+bc^4\right]\\ &+10\left[a^3b^2+a^3c^2+a^2b^3+a^2c^3+b^3c^2+b^2c^3\right]+20\left[a^3bc+ab^3c+abc^3\right]+30\left[a^2b^2c+a^2bc^2+ab^2c^2\right]\end{aligned}$$

When only integer powers are considered, all multinomial coefficients are integers. The general expression is

6:12:3 $$(a_1+a_2+a_3+\cdots+a_n)^N = \sum \mathrm{M}(N,m_1,m_2,m_3,\cdots,m_n)\left[a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}+a_1^{m_2}a_2^{m_1}\cdots a_n^{m_n}+\cdots\right]$$

where the summation embraces all combinations of the nonnegative integer *m* parameters that satisfy

6:12:4 $$m_1+m_2+m_3+\cdots+m_n = N$$

These parameters are not necessarily all distinct. In the 6:12:2 example, where $n = 3$ and $N = 5$, there are five summands because of the five ways (5+0+0 , 4+1+0 , 3+2+0 , 3+1+1 , 2+2+1) in which the number 5 can be composed by addition of three nonnegative integers. The multinomial coefficients themselves are given by

6:12:5 $$\mathrm{M}(N,m_1,m_2,m_3,\cdots,m_n) = \frac{N!}{(m_1)!(m_2)!(m_3)!\cdots(m_n)!}$$

For example, the second right-hand term in 6:12:2 has the multinomial (specifically a trinomial) coefficient of 5, equal to 5!/(4!1!0!). See Abramowitz and Stegun (pages 831−832) for a lengthy listing of multinomial coefficients. *N* need not be positive; see equation 15:6:1 for a counterexample.

6:13 COGNATE FUNCTIONS

The Pochhammer polynomial [Chapter 18], the factorial function [Chapter 2], and the (complete) beta function [Section 43:13] are all allied to the binomial coefficient, to which they are related through equations 6:3:3, 6:3:4, and 6:12:1. Because $\Gamma(v)$ generalizes $(v-1)!$, the gamma function [Chapter 43] too is related through

6:13:1 $$\frac{\Gamma(v+1)}{\Gamma(m+1)\Gamma(v-m+1)} = \binom{v}{m}$$

to the binomial coefficient.

6:14 RELATED TOPIC: binomial expansions

The *binomial theorem*, also known as *Newton's formula*, permits a binomial to be raised to any power:

6:14:1 $$(a+b)^v = \begin{cases} a^v\left[1+\binom{v}{1}\dfrac{a}{b}+\binom{v}{2}\dfrac{a^2}{b^2}+\binom{v}{3}\dfrac{a^3}{b^3}+\cdots\right] = \displaystyle\sum_{m=0}^{\infty}\binom{v}{m}a^{v-m}b^m & -b<a<b \\ b^v\left[1+\binom{v}{1}\dfrac{b}{a}+\binom{v}{2}\dfrac{b^2}{a^2}+\binom{v}{3}\dfrac{b^3}{a^3}+\cdots\right] = \displaystyle\sum_{m=0}^{\infty}\binom{v}{m}a^m b^{v-m} & -a<b<a \end{cases}$$

This is a special case of the Taylor expansion, equation 0:5:1. The resulting series, known as a *binomial series*, terminates if v is a positive integer (in which case there is no restriction on the a/b ratio), but is infinite otherwise.

This chapter concludes with a catalog of some important binomial series. Frequently, binomials consist of a variable paired with a constant; the simplest of such pairings, $1\pm x$, serve in this role in the following examples.

6:14:2 $$(1\pm x)^n = 1\pm nx + \frac{n(n-1)}{2}x^2 \pm \frac{n(n-1)(n-2)}{6}x^3 + \cdots + n(\pm x)^{n-1} + (\pm x)^n = \sum_{m=0}^{n}\frac{n!}{m!(m-n)!}(\pm x)^n$$

6:14:3 $$(1\pm x)^4 = 1\pm 4x + 6x^2 \pm 4x^3 + x^4$$

6:14:4 $$(1\pm x)^3 = 1\pm 3x + 3x^2 \pm x^3$$

6:14:5 $$(1\pm x)^2 = 1\pm 2x + x^2$$

6:14:6 $$(1\pm x)^{3/2} = 1\pm\frac{3}{2}x + \frac{3}{8}x^2 \mp \frac{1}{16}x^3 + \frac{3}{128}x^4 \mp \frac{3}{256}x^5 + \frac{7}{1024}x^6 \mp \frac{9}{2048}x^7 + \cdots = 3\sum_{m=0}^{\infty}\frac{(2m-5)!!}{m!}\left(\frac{\mp x}{2}\right)^m$$

6:14:7 $$(1\pm x)^{1/2} = 1\pm\frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \frac{7}{256}x^5 - \frac{21}{1024}x^6 \pm \frac{33}{2048}x^7 - \cdots = -\sum_{m=0}^{\infty}\frac{(2m-3)!!}{m!}\left(\frac{\mp x}{2}\right)^m$$

6:14:8 $$(1\pm x)^{-1/2} = 1\mp\frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \frac{35}{128}x^4 \mp \frac{63}{256}x^5 + \frac{231}{1024}x^6 \mp \frac{429}{2048}x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(2m)!}{(m!)^2}\left(\frac{\mp x}{4}\right)^m$$

6:14:9 $$(1\pm x)^{-1} = 1\mp x + x^2 \mp x^3 + x^4 \mp x^5 + x^6 \mp x^7 + x^8 \mp x^9 + x^{10} \mp x^{11} + x^{12} \mp \cdots = \sum_{m=0}^{\infty}(\mp x)^m$$

6:14:10 $$(1\pm x)^{-3/2} = 1\mp\frac{3}{2}x + \frac{15}{8}x^2 \mp \frac{35}{16}x^3 + \frac{315}{128}x^4 \mp \frac{693}{256}x^5 + \frac{2003}{1024}x^6 \mp \frac{6435}{2048}x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(2m+1)!}{(m!)^2}\left(\frac{\mp x}{4}\right)^m$$

6:14:11 $$(1\pm x)^{-2} = 1\mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + 7x^6 \mp 8x^7 + 9x^8 \mp 10x^9 + 11x^{10} \mp \cdots = \sum_{m=0}^{\infty}(m+1)(\mp x)^m$$

6:14:12 $$(1\pm x)^{-3} = 1\mp 3x + 6x^2 \mp 10x^3 + 15x^4 \mp 21x^5 + 28x^6 \mp 36x^7 + 45x^8 \mp \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}(\mp x)^m$$

6:14:13 $$(1\pm x)^{-4} = 1\mp 4x + 10x^2 \mp 20x^3 + 35x^4 \mp 56x^5 + 84x^6 \mp 120x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)(m+3)}{6}(\mp x)^m$$

6:14:14 $$(1\pm x)^{-n} = 1\mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)\ldots(m+n-1)}{(n-1)!}(\mp x)^m$$

See Section 2:13 for the double factorial function ( )!!. Series 6:14:6−14 converge if $-1 < x < 1$; sometimes they are also convergent if $x = 1$ or if $x = -1$. The general expression, valid for any value of v may be written in terms of the Pochhammer polynomial as

6:14:15 $$(1\pm x)^{-v} = \sum_{m=0}^{\infty}\frac{(v)_m}{m!}(\mp x)^m \qquad -1 < x < 1$$

Sometimes useful is the finite series plus remainder

6:14:16 $$(1\pm x)^v = \sum_{j=0}^{J-1}\frac{(-v)_j}{j!}(\mp x)^j + \frac{(-v)_j}{(J-1)!}(\mp x)^J\int_0^1(1-t)^{J-1}(1+xt)^{v-J}\,\mathrm{d}t \qquad -1 < x < 1$$

CHAPTER 7

THE LINEAR FUNCTION $bx + c$ AND ITS RECIPROCAL

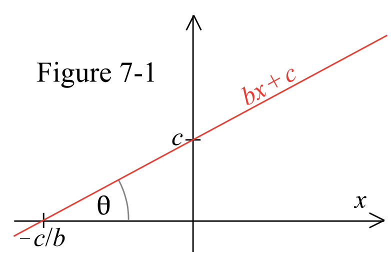
Many relationships in science and engineering take the form $f(x) = bx + c$, and many others can be cast into this form by a redefinition of the variables. The analysis of experimental results can thereby often be reduced to the determination of the coefficients b and c from paired x, $f(x)$ data. The *linear regression* or *least squares* method of performing this analysis is exposed in Section 7:14.

7:1 NOTATION

The linear function $bx + c$, or its special case $1 + x$, is sometimes referred to as a "binomial function" but confusingly it is also termed a "monomial function". Neither name is used in this *Atlas*.

The constant c is termed the *intercept*, whereas b is known as the *slope* or *gradient*. These names derive, of course, from the observation that, if the linear function $bx + c$ is plotted versus x, its graph is a straight line that intersects the vertical axis at altitude c and whose inclination from the horizontal is characterized by the number b, which measures the rate at which the function's value increases with x. The b coefficient is also the tangent [Chapter 34] of angle θ shown in Figure 7-1. The term "slope" is occasionally associated with the angle θ itself, but it more usually means $\tan(\theta)$, being negative if θ is an obtuse angle ($90^\circ < \theta < 180^\circ$). The letter m often replaces b as a symbol for the slope. In the figures, but not elsewhere in the chapter, b and c are assumed positive.

Figure 7-1



The name "inverse linear function" is sometimes given to the function $1/(bx + c)$. Throughout this *Atlas*, however, we reserve the phrase "inverse function" for the relationship described in equation 0:3:3. The unambiguous name *reciprocal linear function* is used here.

The name *rectangular hyperbola* may also be associated with the function $1/(bx + c)$. However, this name is also applicable to other functions, described in Section 7:13, that share the same shape as, but possess a different orientation to, the reciprocal linear function illustrated in Figure 7-2.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_8, © Springer Science+Business Media, LLC 2009

7:2 BEHAVIOR

The linear function $bx + c$ is defined for all values of the argument x and (unless $b = 0$) itself assumes all values. The same is true of the reciprocal linear function which, however, displays an infinite discontinuity at $x = -c/b$, as illustrated in Figure 7-2. A graph of the reciprocal linear function $\mathrm{f}(x) = 1/(bx + c)$ has a high degree of symmetry [Section 14:15], being inversion symmetric about the point $x = -c/b$, $f = 0$ and displaying mirror symmetry towards reflections in the lines $-x\mathrm{sgn}(b)-(c/b)$ and $x\mathrm{sgn}(b)+(c/b)$. Here sgn is the signum function [Chapter 8].

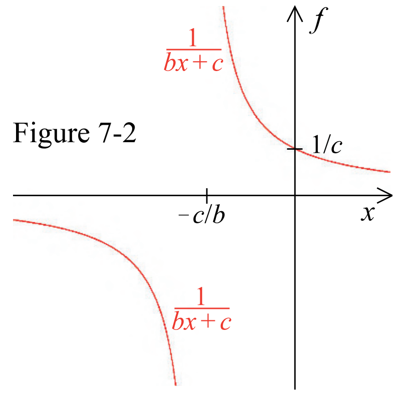


Figure 7-2

7:3 DEFINITIONS

The arithmetic operations of multiplication by b and addition of c fully define $\mathrm{f}(x) = bx + c$. The same operations, followed by division into unity, define the reciprocal linear function.

The linear function is completely characterized when its values, f_1 and f_2, are known at two (distinct) arguments, x_1 and x_2. The slope and intercept may be found from the formula

7:3:1
$$bx + c = \left(\frac{f_2 - f_1}{x_2 - x_1}\right)x + \frac{x_2 f_1 - x_1 f_2}{x_2 - x_1}$$

7:4 SPECIAL CASES

When $b = 0$, the linear function and its reciprocal reduce to a constant [Chapter 1]. When $c = 0$, the linear function is proportional to its argument x.

7:5 INTRARELATIONSHIPS

Both the linear function and its reciprocal obey the reflection formula

7:5:1
$$\mathrm{f}\left(-x - \frac{c}{b}\right) = -\mathrm{f}\left(x - \frac{c}{b}\right) \qquad \mathrm{f}(x) = bx + c \quad \text{or} \quad \frac{1}{bx + c}$$

Two linear functions that share the same b parameter represent straight lines that are *parallel*; the distance separating these lines is $|c_2 - c_1|/\sqrt{1 + b^2}$. If the slopes of two straight lines satisfy the relation $b_1 b_2 = -1$, the lines are mutually perpendicular. intersecting at the point $x = (c_2-c_1)/(b_1-b_2)$. The inverse of the linear function $\mathrm{f}(x) = bx + c$, defined by $\mathrm{F}\{\mathrm{f}(x)\} = x$, is another linear function $\mathrm{F}(x) = (x/b)-(c/b)$; graphically, these two straight lines usually cross at $x = c(1+b^2)/(1-b^2)$.

The sum or difference of two linear functions is a third linear function $(b_1 \pm b_2)x + c_1 \pm c_2$ and this property extends to multiple components. The product of two, three, or many linear functions is a quadratic function [Chapter 15], a cubic function [Chapter 16] or a higher polynomial function [Chapter 17]. Unless $b_2 c_1 = b_1 c_2$, the quotient of two linear functions is the infinite power series:

7:5:2
$$\frac{b_1x+c_1}{b_2x+c_2}=\begin{cases}\dfrac{c_1}{c_2}+\left(\dfrac{c_1}{c_2}-\dfrac{b_1}{b_2}\right)\displaystyle\sum_{j=1}^{\infty}\left(\dfrac{-b_2x}{c_2}\right)^j & |x|<\left|\dfrac{c_2}{b_2}\right| \\ \dfrac{b_1}{b_2}-\left(\dfrac{c_1}{c_2}-\dfrac{b_1}{b_2}\right)\displaystyle\sum_{j=1}^{\infty}\left(\dfrac{-c_2}{b_2x}\right)^j & |x|>\left|\dfrac{c_2}{b_2}\right|\end{cases}$$

The sum or difference of two reciprocal linear functions is a rational function [Section 17:13] of numeratorial and denominatorial degrees of 1 and 2 respectively. Finite series of certain reciprocal linear functions may be summed in terms of the digamma function [Chapter 44]

7:5:3
$$\frac{1}{c}+\frac{1}{x+c}+\frac{1}{2x+c}+\cdots+\frac{1}{Jx+c}=\sum_{j=0}^{J}\frac{1}{jx+c}=\frac{1}{x}\left[\psi\left(J+1+\frac{c}{x}\right)-\psi\left(\frac{c}{x}\right)\right]$$

or in terms of Bateman's G function [Section 44:13]

7:5:4
$$\frac{1}{c}-\frac{1}{x+c}+\frac{1}{2x+c}-\cdots\pm\frac{1}{Jx+c}=\sum_{j=0}^{J}\frac{(-1)^j}{jx+c}=\frac{1}{2x}\left[\mathrm{G}\left(\frac{c}{x}\right)\pm\mathrm{G}\left(J+1+\frac{c}{x}\right)\right]$$

In formula 7:5:4, the upper/lower signs are taken depending on whether J is even or odd. The corresponding infinite sum is

7:5:5
$$\frac{1}{c}-\frac{1}{x+c}+\frac{1}{2x+c}-\cdots=\sum_{j=0}^{\infty}\frac{(-1)^j}{jx+c}=\frac{1}{2x}\mathrm{G}\left(\frac{c}{x}\right)$$

See Section 44:14 for further information on this topic.

7:6 EXPANSIONS

The linear function may be expanded as a infinite series of Bessel functions [Chapter 52]

7:6:1
$$bx+c=c+2\,\mathrm{J}_1(bx)+6\,\mathrm{J}_3(bx)+10\,\mathrm{J}_5(bx)+\cdots=c+2\sum_{j=1}^{\infty}(2j-1)\,\mathrm{J}_{2j-1}(bx)$$

though this representation is seldom employed.

The reciprocal linear function is expansible as a *geometric series* in alternative forms

7:6:2
$$\frac{1}{bx+c}=\begin{cases}\dfrac{1}{c}-\dfrac{bx}{c^2}+\dfrac{b^2x^2}{c^3}-\dfrac{b^3x^3}{c^4}+\cdots=\dfrac{1}{c}\displaystyle\sum_{j=0}^{\infty}\left(\dfrac{-bx}{c}\right)^j & |x|<\left|\dfrac{c}{b}\right| \\ \dfrac{1}{bx}-\dfrac{c}{b^2x^2}+\dfrac{c^2}{b^3x^3}-\dfrac{c^3}{b^4x^4}+\cdots=\dfrac{1}{bx}\displaystyle\sum_{j=0}^{\infty}\left(\dfrac{-c}{bx}\right)^j & |x|>\left|\dfrac{c}{b}\right|\end{cases}$$

according to the magnitude of the argument x compared to that of the ratio c/b. Likewise there are two alternatives when the reciprocal linear function is expanded as an infinite product

7:6:3
$$\frac{1}{bx+c}=\begin{cases}\dfrac{c-bx}{c^2}\displaystyle\prod_{j=1}^{\infty}\left[1+\left(\dfrac{bx}{c}\right)^{2^j}\right] & -1<\dfrac{bx}{c}<1 \\ \dfrac{bx-c}{b^2x^2}\displaystyle\prod_{j=1}^{\infty}\left[1+\left(\dfrac{c}{bx}\right)^{2^j}\right] & \left|\dfrac{bx}{c}\right|>1\end{cases}$$

For example, if $|x| < 1$

7:6:4
$$\frac{1}{1 \pm x} = (1 \mp x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots$$

7:7 PARTICULAR VALUES

As Figure 7-1 shows, the linear function $bx+c$ equals c when $x = 0$ and equals zero when $x = -c/b$. The reciprocal linear function has neither an extremum nor a zero, but it incurs a discontinuity at $x = -c/b$ [Figure 7-2].

7:8 NUMERICAL VALUES

These are easily calculated by direct substitution. The construction feature of *Equator* enables a linear function to be used as the argument of another function.

7:9 LIMITS AND APPROXIMATIONS

The reciprocal linear function approaches zero asymptotically as $x \to \pm\infty$.

7:10 OPERATIONS OF THE CALCULUS

The rules for differentiation of the linear function and its reciprocal are

7:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}(bx+c) = b$$

and

7:10:2
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{bx+c}\right) = \frac{-b}{(bx+c)^2}$$

while those for indefinite integration are

7:10:3
$$\int_0^x (bt+c)\mathrm{d}t = \frac{bx^2}{2} + cx$$

and

7:10:4
$$\int_0^x \frac{1}{bt+c}\,\mathrm{d}t = \frac{1}{b}\ln\left(\left|\frac{bx+c}{c}\right|\right)$$

If $0 < -c/b < x$, the integrand in 7:10:4 encounters an infinity; in this event, the integral is to be interpreted as a *Cauchy limit*. This means that the ordinary definition of the integral is replaced by

7:10:5
$$\lim_{\varepsilon\to 0}\left\{\int_0^{-(c/b)-\varepsilon} \frac{1}{bt+c}\,\mathrm{d}t + \int_{-(c/b)+\varepsilon}^{x} \frac{1}{bt+c}\,\mathrm{d}t\right\}$$

Several other integrals in this section, including those that follow immediately, may also require interpretation as Cauchy limits, but mention of this will not always be made

7:10:6 $$\int_0^x \frac{1}{(Bt+C)(bt+c)}\,\mathrm{d}t = \frac{1}{bC-Bc}\ln\left(\left|\frac{C(bx+c)}{c(Bx+C)}\right|\right) \qquad Bc \neq bC$$

7:10:7 $$\int_0^x \frac{Bt+C}{bt+c}\,\mathrm{d}t = \frac{Bx}{b} + \frac{bC-Bc}{b^2}\ln\left(\left|\frac{bx+c}{c}\right|\right)$$

7:10:8 $$\int_0^x \frac{t^n}{bt+c}\,\mathrm{d}\,x = \frac{(-c)^n}{b^{n+1}}\left[\ln\left(\left|\frac{bx+c}{c}\right|\right) + \sum_{j=1}^{n}\frac{1}{j}\left(\frac{-bx}{c}\right)^j\right] \qquad n = 0,1,2,\cdots$$

Formulas for the semidifferentiation and semiintegration [Section 12:14] of the linear function are

7:10:9 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}(bx+c) = \frac{2bx+c}{\sqrt{\pi x}}$$

and

7:10:10 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}(bx+c) = \sqrt{\frac{x}{\pi}}\left[\frac{4bx}{3} + 2c\right]$$

when the lower limit is zero. The table below shows the semiderivatives and semiintegrals of the reciprocal linear functions $1/(bx+c)$ and $1/(bx-c)$, when the lower limit is zero, and when it is $-\infty$. In this table, but not necessarily elsewhere in the chapter, b and c are positive.

	$\mathrm{f}(x) = \dfrac{1}{bx+c}$		$\mathrm{f}(x) = \dfrac{1}{bx-c}$	
$\dfrac{\mathrm{d}^{½}\mathrm{f}}{\mathrm{d}x^{½}}$	$\dfrac{\sqrt{bx+c} - \sqrt{bx}\,\mathrm{arsinh}\left(\sqrt{bx/c}\right)}{\sqrt{\pi x(bx+c)^3}}$	$x > 0$	$\dfrac{-\sqrt{c-bx} - \sqrt{bx}\,\mathrm{arcsin}\left(\sqrt{bx/c}\right)}{\sqrt{\pi x(c-bx)^3}}$	$0 < x < \dfrac{c}{b}$
$\dfrac{\mathrm{d}^{-½}\mathrm{f}}{\mathrm{d}x^{-½}}$	$\dfrac{2\mathrm{arsinh}\left(\sqrt{bx/c}\right)}{\sqrt{\pi b(bx+c)}}$	$x > 0$	$\dfrac{-2\mathrm{arcsin}\left(\sqrt{bx/c}\right)}{\sqrt{\pi b(c-bx)}}$	$0 < x < \dfrac{c}{b}$
$\left.\dfrac{\mathrm{d}^{½}}{\mathrm{d}t^{½}}\mathrm{f}(t)\right\|_{-\infty}^{x}$	$\dfrac{1}{2}\sqrt{\dfrac{\pi b}{(-bx-c)^3}}$	$x < \dfrac{-c}{b}$	$\dfrac{-1}{2}\sqrt{\dfrac{\pi b}{(c-bx)^3}}$	$x < \dfrac{c}{b}$
$\left.\dfrac{\mathrm{d}^{-½}}{\mathrm{d}t^{-½}}\mathrm{f}(t)\right\|_{-\infty}^{x}$	$-\sqrt{\dfrac{\pi}{b(-bx-c)}}$	$x < \dfrac{-c}{b}$	$-\sqrt{\dfrac{\pi}{b(c-bx)}}$	$x < \dfrac{c}{b}$

See Sections 12:14 and 64:14 for the definitions and symbolism of semidifferintegrals with various lower limits.

The Laplace transforms of the linear and reciprocal linear functions are

7:10:11 $$\int_0^\infty (bt+c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{bt+c\} = \frac{cs+b}{s^2}$$

and

7:10:12
$$\int_0^\infty \frac{1}{bt+c}\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{1}{bt+c}\right\} = \frac{-1}{b}\exp\left(\frac{cs}{b}\right)\mathrm{Ei}\left(\frac{-cs}{b}\right)$$

the latter involving an exponential integral [Chapter 37]. A general rule for the Laplace transformation of the product of any transformable function f(t) and a linear function is

7:10:13
$$\int_0^\infty (bt+c)\,\mathrm{f}(t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{(bt+c)\mathrm{f}(t)\} = c\mathfrak{L}\{\mathrm{f}(t)\} - b\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}\{\mathrm{f}(t)\}$$

Requiring a Cauchy-limit interpretation, the integral transform

7:10:14
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{f}(t)\frac{\mathrm{d}t}{t-y}$$

is called a *Hilbert transform* (David Hilbert, German mathematician, 1862 – 1943). The Hilbert transforms of many functions, mostly piecewise-defined functions [Section 8:4], are tabulated by Erdélyi, Magnus, Oberhettinger and Tricomi [*Tables of Integral Transforms*, Volume 2, Chapter 15]. For example, the Hilbert transform of the pulse function [Section 1:13] is

7:10:15
$$\frac{1}{\pi}\int_{-\infty}^{\infty} c\left[u\left(t-a+\frac{h}{2}\right)-u\left(t-a-\frac{h}{2}\right)\right]\frac{\mathrm{d}t}{t-y} = \frac{c}{\pi}\ln\left(\frac{2(a-y)+h}{2(a-y)-h}\right)$$

A valuable feature of Hilbert transformation is that the inverse transform is identical in form to the forward transformation, apart from a sign change.

7:11 COMPLEX ARGUMENT

The linear function of complex argument, and its reciprocal, split into the real and imaginary parts

7:11:1
$$bz+c = (bx+c)+iby$$

and

7:11:2
$$\frac{1}{bz+c} = \frac{bx+c}{(bx+c)^2+b^2y^2} - i\frac{by}{(bx+c)^2+b^2y^2}$$

if b and c are real.

The inverse Laplace transformation of the linear and reciprocal linear functions leads to functions from Chapters 10 and 26

7:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty}(bs+c)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\{bs+c\} = b\delta'(t)+c\delta(t)$$

and

7:11:4
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{bs+c}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{bs+c}\right\} = \frac{1}{b}\exp\left(\frac{-ct}{b}\right)$$

The Laplace inversion of a function of $bx+c$ is related to the inverse Laplace transform of the function itself through the general formula

7:11:5
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\mathrm{f}(bs+c)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\{\mathrm{f}(bs+c)\} = \frac{1}{b}\exp\left(\frac{-ct}{b}\right)\mathfrak{I}\{\bar{\mathrm{f}}(s)\}$$

7:12 GENERALIZATIONS

The linear and reciprocal linear functions are the $v = \pm 1$ cases of the more general function $(bx + c)^v$, two other instances of which are addressed in Chapter 11. More broadly, all the functions of Chapters 10−14 are particular examples of a wide class of algebraic functions generalized by the formula $(bx^n + c)^v$.

The linear function is an early member of a hierarchy in which the quadratic and cubic functions [Chapters 15 and 16] are higher members and which generalizes to the polynomial functions discussed Chapter 17. All the "named" polynomials [Chapters 18−24] have a linear function as one member of their families.

7:13 COGNATE FUNCTIONS

A frequent need in science and engineering is to approximate a function whose values, $f_0, f_1, f_2, \cdots, f_n$, are known only at a limited number of arguments x_0, x_1, x_2, $\cdots$, x_n, the so-called data points. The simplest way of constructing a function that fits all the known data, but one that is adequate in many applications, is by using a *piecewise-linear function*. In graphical terms, this implies simply "connecting the dots". For any argument lying between two adjacent data points, the interpolation

7:13:1 $$\mathrm{f}(x) = \frac{\left(x_{j+1} - x\right) f_j + \left(x - x_j\right) f_{j+1}}{x_{j+1} - x_j} \qquad x_j \le x \le x_{j+1}$$

applies. Usually a piecewise-linear function has discontinuities at all the interior data points. This defect is overcome by the "sliding cubic" and "cubic spline" interpolations exposed in Section 17:14.

The simplest reciprocal linear functions $1/(1 \pm x)$ serve as prototypes, or basis functions, for all $L = K$ hypergeometric functions [Section 18:14]. All the functions in Tables 18-1 and 18-2, as well as many others, may be "synthesized" [Section 43:14] from $1/(1+x)$ or $1/(1-x)$.

The reciprocal linear function is related to the function addressed in Section 15:4 because the shape of each is a *rectangular hyperbola*. Thus, clockwise rotation [Section 14:15] through an angle of $\pi/4$ of the curve $1/(bx + c)$ about the point $x = -c/b$ on the x-axis produces a new function,

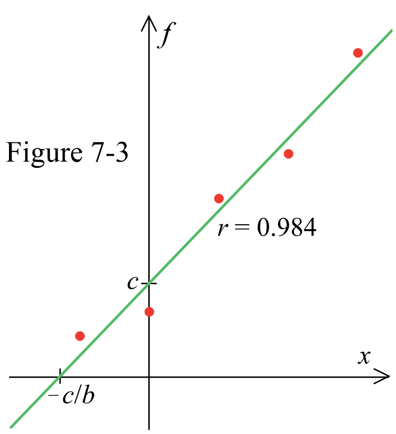
7:13:2 $$\pm\sqrt{\left(x + \frac{c}{b}\right)^2 + \frac{2}{b}}$$

that is a rectangular hyperbola of the class discussed in Section 15:4.

7:14 RELATED TOPIC: linear regression

Frequently experimenters collect data that are known, or believed, to obey the equation $f = \mathrm{f}(x) = bx + c$ but which incorporate errors. From the data, which consists of the n pairs of numbers (x_1, f_1), (x_2, f_2), (x_3, f_3), $\cdots$, (x_n, f_n), the scientist needs to find the b and c coefficients of the *best straight line* through the data, as in Figure 7-3. If the errors obey, or are assumed to obey, a Gaussian distribution [Section 27:14] and are entirely associated with the measurement of f (that is, the x values are exact), then the adjective "best" implies minimizing the sum of the squared deviations, $\sum(bx + c - f)^2$. The

Figure 7-3



procedure for finding the coefficients that achieve this minimization is known as *linear regression* or *least squares* and leads to the formulas

7:14:1
$$b=\frac{n\sum xf-\sum x\sum f}{n\sum x^2-\left(\sum x\right)^2}=\frac{6\left[2\sum jf-(n+1)\sum f\right]}{n(n^2-1)h}$$

and

7:14:2
$$c=\frac{\sum x^2\sum f-\sum x\sum xf}{n\sum x^2-\left(\sum x\right)^2}=\frac{\sum f-b\sum x}{n}=\frac{\sum f}{n}-\frac{b(x_1+x_n)}{2}$$

An abbreviated notation, exemplified by

7:14:3
$$\sum xf=\sum_{j=1}^{n}x_jf_j$$

is used in the formulas of this section. Evaluation of these formulas simplifies considerably in the common circumstance in which data are gathered with equal spacing, this is when $x_2-x_1=x_3-x_2=\cdots=x_n-x_{n-1}=h$. The simplified formulas appear in red in equations 7:14:1, 7:14:2, 7:14:4 and 7:14:8.

A measure of how well the data obey the linear relationship is provided by the *correlation coefficient*, given by

7:14:4
$$r=\frac{n\sum xf-\sum x\sum f}{\sqrt{\left[n\sum x^2-\left(\sum x\right)^2\right]\left[n\sum f^2-\left(\sum f\right)^2\right]}}=b\sqrt{\frac{n\sum x^2-\left(\sum x\right)^2}{n\sum f^2-\left(\sum f\right)^2}}=\frac{nhb}{6}\sqrt{\frac{3(n^2-1)}{n\sum f^2-\left(\sum f\right)^2}}$$

Values close to ±1 imply a good fit of the data to the linear function, whereas r will be close to zero if there is little or no correlation between f and x. Sometimes r^2 is cited instead of r.

Commonly there is a need to know not only what the best values are of the slope b and the intercept c but also what uncertainties attach to these best values. Quoting their *standard errors* [Section 40:14] in the format

7:14:5
$$\text{slope}=b\pm\Delta b \qquad \text{where} \quad \Delta b=\text{standard error in } b$$

and

7:14:6
$$\text{intercept}=c\pm\Delta c \qquad \text{where} \quad \Delta c=\text{standard error in } c$$

is a succinct way of reporting the uncertainties associated with least squares determinations. the significance to be attached to these statements is that the probability is approximately 68% that the true slope will lie between $b-\Delta b$ and $b+\Delta b$. Similarly, there is a 68% probability that the true intercept lies in the range $c\pm\Delta c$. The formulas giving these standard errors are

7:14:7
$$\Delta b=\sqrt{\frac{n\sum f^2-\left(\sum f\right)^2}{n\sum x^2-\left(\sum x\right)^2}-\left[\frac{n\sum xf-\sum x\sum f}{n\sum x^2-\left(\sum x\right)^2}\right]^2}=b\sqrt{\frac{1}{r^2}-1}$$

and

7:14:8
$$\Delta c=\Delta b\sqrt{\frac{\sum x^2}{n}}=\frac{\Delta b}{2}\sqrt{\left(x_1+x_n\right)^2+\frac{(n^2-1)h^2}{3}}$$

A related, but simpler, problem is the construction of the best straight line through the points (x_1,f_1), (x_2,f_2), (x_3,f_3), $\cdots$, (x_n,f_n), with the added constraint that the line *must* pass through the point (x_0,f_0). In practical problems this obligatory point is often the $x=0$, $f=0$ origin. Equations 7:14:1 and 7:14:2 should *not* be used in these circumstances, though they often are. The appropriate replacements are

7:14:9 $$b = \frac{\sum (x - x_0)(f - f_0)}{\sum (x - x_0)^2}$$

and

7:14:10 $$c = f_0 - bx_0$$

Equations 7:14:1−7:14:8 are based on the assumption that all data points are known with equal reliability, a condition that is not always valid. Variable reliability can be treated by assigning different *weights* to the points. If the value f_1 is more reliable than f_2, then a larger weight w_1 is assigned to the data pair (x_1, f_1) than the weight w_2 assigned to point (x_2, f_2). The weights then appear as multipliers in all summations, leading to the formulas

7:14:11 $$b = \frac{\sum w \sum wxf - \sum wx \sum wf}{\sum w \sum wx^2 - \left(\sum wx\right)^2}$$

and

7:14:12 $$c = \frac{\sum wf - b\sum wx}{\sum w}$$

for the slope and intercept. Only the *relative* weights are of import; the absolute values of w_1, w_2, w_3, $\cdots$, w_n have no significance beyond this. In practice, one attempts to assign a weight w_j to the jth point that is inversely proportional to the square of the uncertainty in f_j. Notice that formulas 7:14:1 and 7:14:2 are the special cases of 7:14:11 and 7:14:12 in which all w's are equal. Similarly, the formulas in 7:14:9 and 7:14:10 result from setting the weight of one point, (x_0, f_0), to be overwhelmingly greater than all the other weights, which are uniform.

CHAPTER

8

MODIFYING FUNCTIONS

The functions primarily addressed in this chapter are the *signum function* sgn(x), the *absolute-value function* $|x|$, the *integer-value function* Int(x), the *fractional-value function* frac(x), the *integer-part function* Ip(x), and the *fractional-part function* Fp(x). Only the first four of these find more than occasional use elsewhere in the *Atlas*. Section 8:12 is devoted to the *modulo function* v(mod μ), while the *rounding function* Round(x) is addressed in Section 8:13.

Most of the functions in this *Atlas* operate on variable(s) to produce an entirely new value: the output number bears no obvious resemblance to the input number(s). This is not the case for the six functions of the present chapter; here the functions of interest leave part of the number, or at least one of its attributes, unchanged. For instance, the integer-value function, Ip(x), retains all of the content of a number that lies to the left of its decimal point. Because these functions dismember their arguments, the pieces may be reassembled, as the equations:

8:0:1 $$|x|\,\mathrm{sgn}(x) = x$$

8:0:2 $$\mathrm{Int}(x) + \mathrm{frac}(x) = x$$

and

8:0:3 $$\mathrm{Ip}(x) + \mathrm{Fp}(x) = x$$

demonstrate.

8:1 NOTATION

The signum function sgn(x) is also called the sign function and may be symbolized sign(x) or sg(x).

An alternative name for the absolute value $|x|$ of x is the *magnitude* of x, or, especially when the argument is complex, the *modulus*. There is a danger of confusing modulus with "modulo" [Section 8:12], especially as both words are commonly abbreviated to "mod". The symbol ABS(x) or abs(x) sometimes replaces $|x|$, especially in computer applications.

Because the values they generate are often identical, the distinction between the integer-value and integer-part functions is not always recognized; the same confusion applies to the two fractional functions. The terms "ceiling", "floor", "fix", and "truncation", are encountered in computer terminology, the last three sometimes being synonymous with the operation carried out by the Ip function. The *ceiling function* of x, sometimes denoted $\lceil x \rceil$,

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is the largest integer not exceeding x: the companion notation for the *floor function* is $\lfloor x \rfloor$. The notation $[x]$ finds use as an alternative to Int(x).

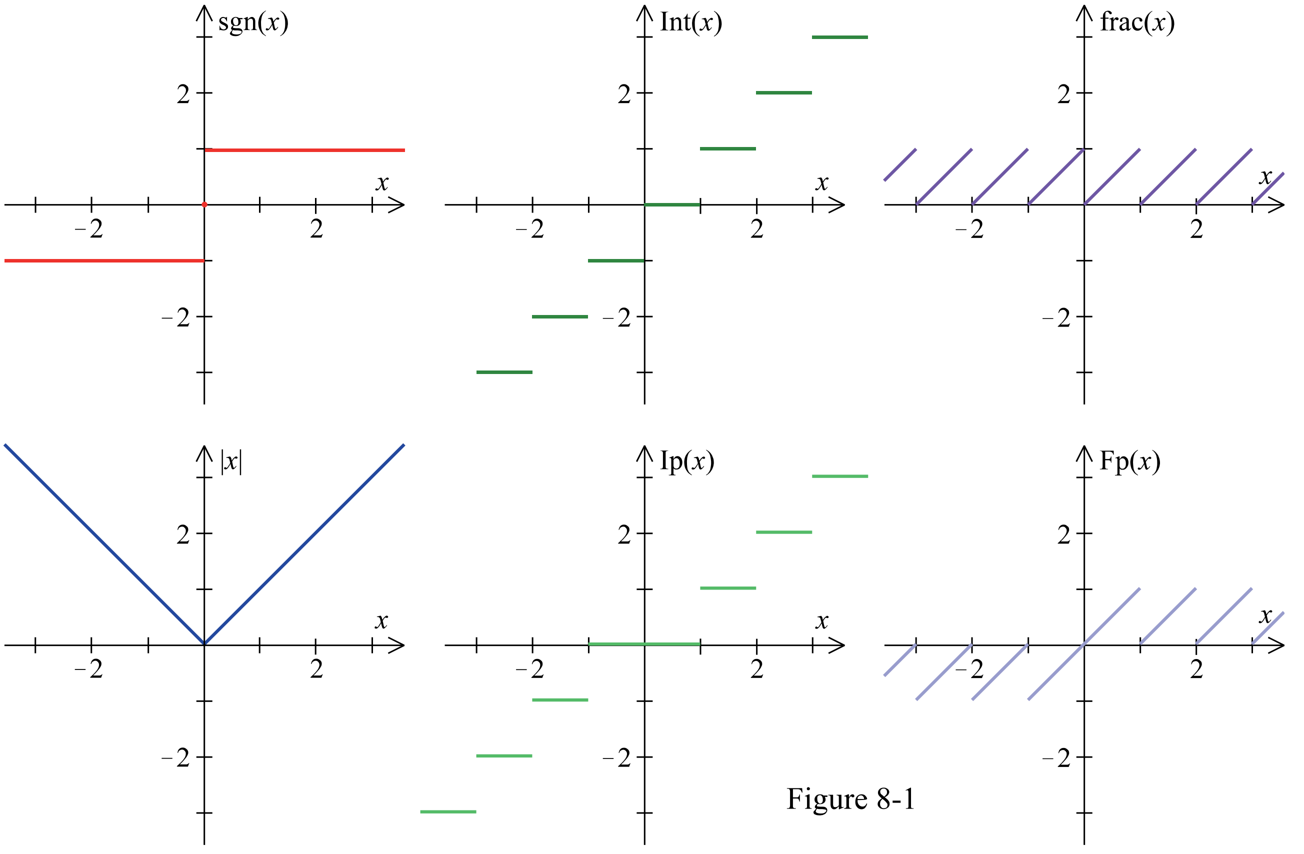

Figure 8-1

8:2 BEHAVIOR

When applied to numbers, the effect of each modifying function is immediately evident from its definition or from the diagrams in Figure 8-1. This figure shows the result of applying each of the modifying functions to the argument x. However, for the most part, the modifying functions are not encountered in isolation, but associated with another function, the properties of which are thereby transformed. The modification may be to the argument of a function, as in f{frac(x)} or to the function value, as in frac{f(x)}. We speak of the first instance as *internal modification* of the f function, while the latter is an instance of *external modification*. Either type of modification applied to a continuous function generally introduces one or more discontinuities, examples of which will be found in Section 8:4.

Modifying functions may destroy preexisting symmetries, or may create new symmetries. For example, the absolute-value function, applied externally to the function x^3, destroys the preexisting inversion symmetry [Section 14:15], but creates a new mirror symmetry. Or, stated differently but equivalently, the odd function x^3 is turned into an even function $|x^3|$.

8:3 DEFINITIONS

As its name implies, the signum function extracts the sign of its argument:

8:3:1
$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

An interesting integral also generates the signum function

8:3:2
$$\mathrm{sgn}(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(xt)}{t}\,\mathrm{d}t$$

The absolute-value function equals its argument, or the negative of its argument, depending on whether the argument is positive or negative

8:3:3
$$|x| = \begin{cases} x & x \ge 0 \\ -x & x \le 0 \end{cases}$$

The integer-value function extracts from its argument the largest integer that does not exceed the argument. The fractional-value function is the difference between its argument and the largest integer that does not exceed its argument

8:3:4
$$\mathrm{Int}(x) = n$$

8:3:5
$$\mathrm{frac}(x) = x - n$$

$$\left.\begin{matrix} \mathrm{Int}(x) = n \\ \mathrm{frac}(x) = x - n \end{matrix}\right\} \quad n \le x < n+1 \qquad n = 0, \pm1, \pm2, \cdots$$

The integer-part of a decimal number is that part of the number to the left of the decimal point, including the sign of the number. The fractional-part is the part to the right of the decimal point, again including the number's sign. For example

8:3:6
$$\mathrm{Ip}(-2.34) = -2 \qquad \text{and} \qquad \mathrm{Fp}(-2.34) = -0.34$$

so that relationship 8:0:3 holds. For nonnegative arguments, the integer-value function and the integer-part function yield the same value, as do the fractional-value and fractional-part pair

8:3:7
$$\mathrm{Ip}(x) = \mathrm{Int}(x) \qquad \mathrm{Fp}(x) = \mathrm{frac}(x) \qquad x \ge 0$$

The relationships for negative arguments are, however

8:3:8
$$\mathrm{Ip}(x) = \mathrm{Int}(x) + 1 \qquad \mathrm{Fp}(x) = \mathrm{frac}(x) - 1 \qquad x < 0$$

8:4 SPECIAL CASES

Piecewise-defined functions are constructed by specifying different definitions in separate ranges of their argument. A unitary definition is, however, often possible by employing modifying functions; for example, the signum function sgn(x) describes a step without the need to separately describe the behavior in two ranges.

Piecewise-defined functions that have a repetitive character are often called *waveforms*; some are illustrated in Figure 8-1 and others in Section 36:14. The function $(-1)^{\mathrm{Int}(x)}$ is a *square waveform*. frac(x) is a *sawtooth waveform*. The function $|2\mathrm{frac}(x/2)-1|$, involving two modifying functions, describes a *triangular waveform*. $|\sin(x)|$ is a fully rectified sinusoidal waveform [Section 36:13]. The function Int(x) is a *staircase waveform* with unity treads and unity risers. Ip(x) is a similar staircase, but one of its treads is of double width.

8:5 INTRARELATIONSHIPS

Many relationships between different members of the modifying function family are given in Sections 8:0, 8:3 and 8:8. The only noteworthy intrarelationships of individual members are their reflection properties. The absolute-value function is an *even function*

8:5:1 $$|-x| = |x|$$

whereas the signum, integer-part and fractional-part functions are *odd functions*

8:5:2 $$\mathrm{sgn}(-x) = -\mathrm{sgn}(x) \qquad \mathrm{Ip}(-x) = -\mathrm{Ip}(x) \qquad \mathrm{Fp}(-x) = -\mathrm{Fp}(x)$$

Neither designation applies to the integer-value or fractional-value functions.

8:6 EXPANSIONS

In general, modifying functions cannot be distributed through a sum of arguments, for instance $\mathrm{Int}(x+y) \neq \mathrm{Int}(x) + \mathrm{Int}(y)$. A consequence is that, even though a function $f(x)$ may itself be expansible, this expansibility is not generally preserved by the action of modifying functions, whether they are applied internally or externally; for example $f(|x|)$ and $|f(x)|$ do not generally have series expansions even if $f(x)$ does.

Functions such as $\mathrm{frac}(x)$, $\mathrm{Int}\{\sin(x)+1\}$ and $(-1)^{\mathrm{Int}(x)}$ are periodic and are therefore subject to Fourier expansion as discussed in Chapter 36. Not only is the square wave function expressible as a series of sine waves, but the converse is also the case, as the equation

8:6:1 $$\sin(\omega t) = \frac{\pi}{4}\sum_{j=1,3}^{\infty}\frac{m_j}{j}(-1)^{\mathrm{Int}(j\omega t)}$$

demonstrates. Note that j is restricted to odd natural numbers. Here m_j is a *Möbius coefficient* (August Ferdinand Möbius, German mathematician and astronomer, 1790 – 1868), defined as follows

8:6:2 $$m_j = \begin{cases} 0 \text{ if } j \text{ is divisible by the square of any odd integer other than unity (e.g. } 9,25,27,45,49,63,\cdots) \\ 1 \text{ if } j \text{ is unity or the product of an even number of distinct primes (e.g. } 1,15,21,33,35,39,\cdots) \\ -1 \text{ if } j \text{ is a prime or the product of an odd number of distinct primes (e.g. } 3,5,7,11,13,17,\cdots) \end{cases}$$

8:7 PARTICULAR VALUES

Whereas each of the modifying functions may receive any real number as an argument, the accessible range of output values is seriously restricted. At most, only three values, $0, \pm1$, are available to a signum function. Negative values are inaccessible to absolute-value functions. Integer-value and integer-part functions can adopt only integer values. All fractional-value functions lie in the range $0 \leq \mathrm{frac}\{f(x)\} < 1$, whereas a somewhat wider range, $-1 < \mathrm{Fp}\{f(x)\} < 1$, is available to the values of fractional-part functions.

8:8 NUMERICAL VALUES

Programming languages incorporate access to at least two of the modifying functions: $|x|$ and $\mathrm{Ip}(x)$, and others may be provided too. If there is an "integer" function, ascertain whether it is Int or Ip if you intend to supply negative arguments. By using the formulas in Section 8:0, you can create Fp(x) as $x-$Ip(x) and (except for $x=0$) sgn(x) as $|x|/x$. The integer- and fractional-value functions are also accessible from Ip(x) via the formulas:

8:8:1 $$\mathrm{Int}(x)=\mathrm{Ip}(x)+\frac{|x|-x}{2x} \qquad x\neq 0$$

8:8:2 $$\mathrm{frac}(x)=x+\frac{x-|x|}{2x}-\mathrm{Ip}(x) \qquad x\neq 0$$

Equator's integer-part function, fractional-part function, integer-value function, and fractional-value function routines (keywords **Ip**, **Fp**, **Int**, and **frac**) provide values of Ip(x), Fp(x), Int(x), and frac(x).

8:9 LIMITS AND APPROXIMATIONS

There are none that warrant inclusion.

8:10 OPERATIONS OF THE CALCULUS

Though each instance should be carefully examined in terms of the definition of the modifying function, discontinuities introduced into functions by internal or external modification need not inhibit the application of differential or integral operators. When differentiating, the discontinuities should be avoided, as in

8:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}\left(|x|\right)=\mathrm{sgn}(x)\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}x}\left(|x|\right) \qquad x\neq 0$$

No integration should be carried out across a discontinuity. The example

8:10:2 $$\int_{-1}^{1}\mathrm{f}\left(|t|\right)\mathrm{d}t=\int_{-1}^{0}\mathrm{f}\left(-t\right)\mathrm{d}t+\int_{0}^{1}\mathrm{f}\left(t\right)\mathrm{d}t=2\int_{0}^{1}\mathrm{f}\left(t\right)\mathrm{d}t$$

illustrates how the prohibition can be circumvented. The Cauchy-limit "trick" [Section 7:10] is sometimes useful.

Modifying functions, despite their inherent discontinuities, are often converted to continuous functions under the action of the Laplace transform. For example

8:10:3 $$\int_{0}^{\infty}\mathrm{sgn}(t-c)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{sgn}(t-c)\}=\frac{2\exp(-cs)-1}{s} \qquad c>0$$

8:10:4 $$\int_{0}^{\infty}\mathrm{Int}(t)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{Int}(t)\}=\frac{\coth(s/2)-1}{2s}$$

8:10:5 $$\int_{0}^{\infty}\mathrm{frac}(t)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{frac}(t)\}=\frac{s+2-s\coth(s/2)}{2s^2}$$

8:10:6 $$\int_{0}^{\infty}\left|\,\mathrm{frac}(t)-\tfrac{1}{2}\right|\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\left|\,\mathrm{frac}(t)-\tfrac{1}{2}\right|\right\}=\frac{2s-\tanh(s)}{4s^2}$$

8:10:7 $$\int_{0}^{\infty}(-1)^{\mathrm{Int}(t)}\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{(-1)^{\mathrm{Int}(t)}\right\}=\frac{\tanh(s/2)}{s}$$

Chapter 30 is devoted to the tanh and coth functions.

8:11 COMPLEX ARGUMENT

The concept of "sign" has no meaning for a complex number and therefore neither of definitions 8:3:1 or 8:3:2 can be carried over into the complex plane. Instead, one defines the absolute value (or "modulus") of a complex number by

8:11:1 $$|z| = |x+iy| = \sqrt{x^2+y^2}$$

Notice that this definition is little different from that for a real number, because $|x| = \sqrt{x^2}$ can serve as an alternative to 8:3:3.

Likewise, because the statements $z_1 < z_2$ and $z_1 > z_2$ are meaningless, functions such as Int and Fp cannot be applied to complex numbers. Of course these modifying functions can be used in complex algebra with the real or imaginary parts of complex numbers as their arguments.

8:12 GENERALIZATIONS: including the modulo function

The signum function is a special case of the Heaviside function [Chapter 9]

8:12:1 $$\mathrm{sgn}(x) = 2\mathrm{u}(x) - 1$$

except possibly when $x = 0$.

The bivariate *modulo function* is a generalization of the fractional-value function inasmuch as

8:12:2 $$v(\mathrm{mod}\ 1) = \mathrm{frac}(v)$$

The notation $v(\mathrm{mod}\ \mu)$ is standard for the modulo function but is dangerous, in that, for example, $2v(\mathrm{mod}\ \mu)$ does not necessarily have twice the value of $v(\mathrm{mod}\ \mu)$. Especially when the variables are positive integers, the modulo function is also known as the *remainder function*. "Modulus" and "cycle length" are encountered as names of the μ variable. We allow v and μ to adopt any real value, except that the latter may not be zero.

In words, the modulo function $v(\mathrm{mod}\ \mu)$ is defined by the operation of starting with v, and then subtracting μ repeatedly until the remainder is smaller than the modulus μ. For example

8:12:3 $$4.3(\mathrm{mod}\ 2.1) = 0.1$$

Symbolically, this definition is equivalent to

8:12:4 $$v(\mathrm{mod}\ \mu) = \mu\,\mathrm{frac}\left(\frac{v}{\mu}\right)$$

though numerical implementation via

8:12:5 $$v(\mathrm{mod}\ \mu) = v\left[1 - \frac{\mathrm{Int}(\nu/\mu)}{\nu/\mu}\right]$$

preserves precision better. The verbal definition given above applies only if v and μ are positive, whereas 8:12:4 validly defines the modulo function irrespective of the signs of its two variables. Thus, for example, in contrast to 8:12:3, this *Atlas* considers

8:12:6 $$-4.3(\mathrm{mod}\ 2.1) = 2.0 \qquad 4.3(\mathrm{mod}\ -2.1) = -2.0 \qquad \text{and} \qquad -4.3(\mathrm{mod}\ -2.1) = -0.1$$

However, there is no unanimity among authors over the outcome of the modulo operation when either or both of v and μ is negative.

Precision loss is a grave problem in evaluating the modulo function, especially when v is much larger than μ. *Equator*'s modulo function routine (keyword **mod**) for calculating $v(\mathrm{mod}\ \mu)$ is based on formula 8:12:5 and suffers no loss of precision until the ratio v/μ exceeds 10^{12} or μ becomes less than 10^{-14}.

8:13 COGNATE FUNCTIONS

A *trailing digit* is a decimal digit that lies to the right of the decimal point. The operation of *rounding* means truncating a number by removing one, several, or all of the rightmost trailing digits, while minimizing the difference between the rounded number and the original. This caveat may require the adjustment of one or more of the residual digits. For example, 7.140 is the result of rounding 7.13962 to three trailing (four significant) digits.

The so-called *rounding function*, more descriptively termed the *nearest-integer function*, removes all trailing digits. It is simply

8:13:1 $$\mathrm{Round}(x) = \mathrm{Int}(x + \tfrac{1}{2})$$

The symbols $\langle x \rangle$, $[x]$, and nint(x) are encountered as synonyms of Round(x). Beware of the symbol "rnd(x)"; it may represent the rounding operation, but it is also used in computer languages to summon a random number.

The motive for rounding a number may be to decrease the space it occupies (in a computer or on a page) or to make the number more realistically match the quantity it represents. Alternatively, the goal of economy without too great a loss of precision can often be met by replacing the decimal number by a (proper or improper) fraction. One might even consider the replacement of a decimal number by an almost-equivalent fraction as a sort of rounding.

The conversion of a fraction into a decimal number is readily accomplished by division, but the converse conversion is less straightforward. Of course, a decimal number x is rarely replaceable exactly by a fraction, so the pertinent problem devolves into finding a numeratorial integer n and a denominatorial integer d, along with the fractional error ε that accompanies the approximation

8:13:2 $$x \approx \frac{n}{d} \qquad \varepsilon = \left| \frac{n}{xd} - 1 \right|$$

Equator has a rational approximation routine (keyword **rational**) that produces a sequence of increasingly accurate fractional replacements for a decimal number x, each output being accompanied by the associated fractional error. All such approximations are listed, provided that each new approximation is at least 10% better than the prior approximation, up to the smaller of n or d being 10^6. Output can be halted by pressing the Esc key.

8:14 RELATED TOPIC: base conversion

The integer-value and fractional-value functions lie at the heart of such devices as analog-to-digital convertors that digitize measured signals of various kinds. They also play roles in the conversion of a number x from one number system to another; from the decimal system to the binary, for example, or from the hexadecimal to the decimal.

Any positive number can be represented as

8:14:1 $$x = N_0 \bullet N_{-1} N_{-2} N_{-3} \cdots \times \beta^n$$

where n is an integer and each N_j takes one of the integer values $0,1,2,\cdots,(\beta-1)$, where β is the *base* of the number system and where $N_0 \neq 0$. Such a representation is termed *floating point* or, if $\beta = 10$, *scientific notation*. An alternative is the *fixed point* representation

8:14:2 $$x = N_n N_{n-1} \cdots N_0 \bullet N_{-1} N_{-2} \cdots$$

where, as before, each N_j is an integer drawn from the set $0,1,2,\cdots,(\beta-1)$ except $N_n \neq 0$. Changing from one base system to another, say from decimal to binary or from hexadecimal to decimal, is easily accomplished.

A number that is exact in one number system may be incapable of finite representation in another number system. For example, many decimal numbers, such as 1.7, cannot be expressed in other than an infinite number of binary digits. This has consequences in the computation of function values by computers, which operate in binary arithmetic. For reasons explained in Section C:10, *Equator* provides a nearest binary approximant routine (keyword **bin**) to maximize the precision by which certain function values may be computed. This routine returns the nearest binary number having no more than 15 decimal digits.

CHAPTER 9

THE HEAVISIDE u($x-a$) AND DIRAC δ($x-a$) FUNCTIONS

The functions of this chapter occur primarily as multipliers of other functions, which they thereby modulate. The Dirac function is the derivative of the Heaviside function

9:0:1 $$\delta(x-a)=\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{u}(x-a)$$

Strictly speaking, the Dirac function is not a function at all, because it violates conditions that respectable functions obey; nevertheless the great utility of this "function", notably in the fields of classical and quantum mechanics, warrants its inclusion in the *Atlas*.

9:1 NOTATION

The names of these functions recognize the achievements of two English innovators, Oliver Heaviside (electrical engineer, 1859 – 1925) and Paul Adrian Maurice Dirac (nuclear physicist, 1902 – 1984).

Synonyms of "Heaviside function" include *unit-step function*, *Heaviside theta function*, and *Heaviside's step function*; the symbols $\theta(x-a)$, $\mathrm{H}(x-a)$, and $\mathrm{S}_a(x)$ are encountered. The Dirac function has the alternative names *unit-impulse function*, *impulse function*, *delta function*, and *Dirac's delta function*. The last variant stresses the distinction from Kronecker's delta function [Section 9:13].

The salient property of each of these functions occurs when its argument is zero, and this is the reason for the unusual representation of the argument of these functions. In truth, these functions are bivariate, but it is only the difference, $x-a$, between the two variables that affects the functions' values. This also explains why the notations $\mathrm{u}(x)$ and $\delta(x)$ replace $\mathrm{u}(x-a)$ and $\delta(x-a)$ when a is zero.

9:2 BEHAVIOR

As illustrated in Figure 9-1, the Heaviside function $\mathrm{u}(x-a)$ adopts the value zero when x is less than a and the

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value unity when $x > a$, so that the alternative name "unit-step function" is, indeed, appropriate. The value of $\mathrm{u}(x-a)$, when $x = a$, is usually regarded as ½, but this is seldom of consequence.

The Dirac function cannot be graphed; $\delta(x-a)$ is zero for all x, except at $x = a$, where it is infinite.

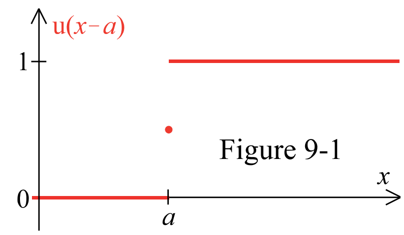

Figure 9-1

9:3 DEFINITIONS

The Heaviside function is defined by

9:3:1 $$\mathrm{u}(x-a) = \begin{cases} 0 & x < a \\ \tfrac{1}{2} & x = a \\ 1 & x > a \end{cases}$$

and therefore the effect of multiplying any function $\mathrm{f}(x)$ by $\mathrm{u}(x-a)$ is to nullify the function for arguments less than a, but to preserve $\mathrm{f}(x)$ unchanged for $x > a$.

Equation 9:0:1 provides one definition of the Dirac function. It may be defined as a limit in several ways, including

9:3:2 $$\delta(x-a) = \lim_{v\to\infty}\left[\sqrt{\frac{v}{\pi}}\exp\left\{-v(x-a)^2\right\}\right]$$

and

9:3:3 $$\delta(x-a) = \lim_{\varepsilon\to 0}\left[\frac{1}{2\varepsilon}\operatorname{sech}^2\left(\frac{x-a}{\varepsilon}\right)\right]$$

in terms of functions discussed in Chapters 27 and 29. The Dirac function may also be considered as the limiting case of a pulse function [Section 1:13] in which the pulse width is progressively diminished, while preserving the product (pulse width)×(pulse height) equal to unity. All these definitions – and many others – describe a function peaked at $x = a$ that, as the limit is approached, becomes infinitely high and infinitesimally wide but whose area remains constant and equal to unity. Figure 9-2 illustrates progress towards the limit in the case of definition 9:3:3.

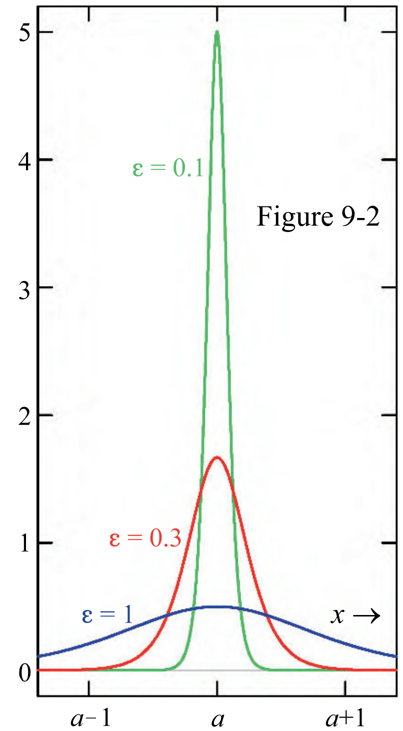

Figure 9-2

Yet another representation is as the definite integral

9:3:4 $$\delta(x-a) = \int_{-\infty}^{\infty}\cos\{2\pi(x-a)t\}\,\mathrm{d}t$$

9:4 SPECIAL CASES

The signum function [Chapter 8] is an adaptation of the Heaviside function

9:4:1 $$\mathrm{sgn}(x) = 2\mathrm{u}(x) - 1$$

9:5 INTRARELATIONSHIPS

The Heaviside function satisfies the reflection formula

9:5:1 $$u(a-x) = 1 - u(x-a)$$

whereas, for the Dirac function

9:5:2 $$\delta(a-x) = \delta(x-a)$$

Other intrarelationships obeyed by the Dirac function include the multiplication property

9:5:3 $$\delta\{v(x-a)\} = \frac{\delta(x-a)}{v} \qquad v > 0$$

and

9:5:4 $$\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right]$$

9:6 EXPANSIONS

There are none.

9:7 PARTICULAR VALUES

	$x < a$	$x = a$	$x > a$
u(x−a)	0	½	1
δ(x−a)	0	∞	0

9:8 NUMERICAL VALUES

Values of these functions require no computation.

9:9 LIMITS AND APPROXIMATIONS

The discontinuous Heaviside and Dirac functions may be approximated by continuous functions in many ways. For example

9:9:1 $$u(x-a) \approx \frac{1 + \tanh\{v(x-a)\}}{2} \qquad \text{very large } v$$

or

9:9:2 $$\delta(x-a) \approx \frac{v\exp\left\{-v^2(x-a)^2\right\}}{\sqrt{\pi}} \qquad \text{very large } v$$

9:10 OPERATIONS OF THE CALCULUS

As equation 9:0:1 states, the derivative of the Heaviside function is the Dirac function. If the Dirac function is itself differentiated, the unit-moment function $\delta^{(1)}(x-a)$, mentioned in Section 9:12, results. Integration of the two functions yields:

9:10:1 $$\int_{x_0}^{x} \mathrm{u}(t-a)\,\mathrm{d}t = [x-a]\mathrm{u}(x-a) \qquad x_0 < a < x$$

9:10:2 $$\int_{x_0}^{x} \delta(t-a)\,\mathrm{d}t = \begin{cases} 1 & x_0 < a < x \\ 0 & a < x_0 \quad \text{or} \quad a > x \end{cases}$$

The integration of the product of an arbitrary function f(x) with the functions of this chapter produces interesting and useful results:

9:10:3 $$\int_{x_0}^{x} \mathrm{u}(t-a)\mathrm{f}(t)\,\mathrm{d}t = \int_{a}^{x} \mathrm{f}(t)\,\mathrm{d}t \qquad x_0 < a < x$$

9:10:4 $$\int_{x_0}^{x} \delta(t-a)\mathrm{f}(t)\,\mathrm{d}t = \mathrm{u}(x-a)\mathrm{f}(a) \qquad x_0 < a < x$$

A special case of the last equation constitutes what is known as the *sifting property*

9:10:5 $$\int_{-\infty}^{\infty} \delta(t-a)\mathrm{f}(t)\,\mathrm{d}t = \mathrm{f}(a)$$

of the Dirac function: multiplying a function f(x) by δ(x−a) and integrating “sifts out” the value of f at $x = a$.

Another way in which the Dirac function finds use is in being “convolved” with another function. The notation f(x)✻g(x) represents the *convolution* of the f and g functions, defined by

9:10:6 $$\mathrm{f}(x)\ast\mathrm{g}(x) = \int_{-\infty}^{\infty} \mathrm{f}(t)\mathrm{g}(x-t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{f}(x-t)\mathrm{g}(t)\,\mathrm{d}t$$

The convolution of two Dirac functions obeys the rule

9:10:7 $$\delta(x-a)\ast\delta(x-b) = \delta(x-a-b)$$

Laplace transformation of the Heaviside and Dirac functions leads to an exponentially decaying function [Chapter 26] of the dummy variable

9:10:8 $$\int_{0}^{\infty} \mathrm{u}(t-a)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{u}(t-a)\} = \frac{\exp\{-as\}}{s} \qquad a > 0$$

9:10:9 $$\int_{0}^{\infty} \delta(t-a)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\delta(t-a)\} = \exp\{-as\} \qquad a > 0$$

The latter transform exemplifies the sifting property.

9:11 COMPLEX ARGUMENT

A complex number is zero only if its real and imaginary parts are *both* zero. Thus the Dirac function of complex argument δ(x+iy−a−bi) is nonzero only when $x = a$ and $y = b$. It obeys

9:11:1
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x+iy-a-ib)\,\mathrm{d}x\,\mathrm{d}y=1$$

as well as other relations that parallel its behavior as a function of a real argument.

9:12 GENERALIZATIONS

The functions $\mathrm{u}(x-a)$, $\delta(x-a)$ and $\delta^{(1)}(x-a)$ may be regarded as the $\nu=0$, $\nu=1$, and $\nu=2$ cases of a continuum of functions defined by the differintegral [Section 12:14]

9:12:1
$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}}\mathrm{u}(x-a) \qquad a>0$$

which evaluates to

9:12:2
$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}}\mathrm{u}(x-a)=\mathrm{u}(x-a)\frac{(x-a)^{-\nu}}{\Gamma(1-\nu)}$$

The gamma function, Γ, is addressed in Chapter 43. When $\nu=1,2,3,\cdots$, the gamma function $\Gamma(1-\nu)$ is infinite, so that formula 9:12:2 evaluates to zero, except at $x=a$.

The $\nu=2$ case of 9:12:1, the *unit-moment function*, may be defined by limiting operations analogous to 9:3:2 and 9:3:3, for example

9:12:3
$$\delta^{(1)}(x-a)=\lim_{\varepsilon\to 0}\left[\frac{-1}{\varepsilon^{2}}\mathrm{sech}^{2}\left(\frac{x-a}{\varepsilon}\right)\tanh\left(\frac{x-a}{\varepsilon}\right)\right]$$

Alternatively, it may be considered as the limit of two pulse functions: first a positive-going pulse, immediately followed by a negative-going replica. As the pulses individually approach Dirac functions, the second on the heels of the first, the combination becomes $\delta^{(1)}(x-a)$. The unit-moment function, which is alternatively symbolized $\delta'(x-a)$, has a sifting property too, but it sifts out the derivative of the $\mathrm{f}(x)$ function:

9:12:4
$$\int_{-\infty}^{\infty}\delta^{(1)}(t-a)\mathrm{f}(t)\,\mathrm{d}t=-\frac{\mathrm{df}}{\mathrm{d}t}(a)$$

9:13 COGNATE FUNCTIONS

A *window function* contains a segment of some function $\mathrm{f}(x)$, whose definition is preserved within the argument range $x_0<x<x_1$, but which is zero otherwise. Two Heaviside functions switch the function on and off in the following formula

9:13:1
$$\mathrm{f}(x)\left[\mathrm{u}(x-x_0)-\mathrm{u}(x-x_1)\right] \qquad x_1>x_0$$

that implements the windowing. The pulse function [Section 1:13] is the simplest example.

The Dirac function is a bivariate function of two variables, x and a, each of which can adopt any real value; it is zero unless these two variables are equal. The *Kronecker function* or *Kronecker delta function*, $\delta_{n,m}$, (Leopold Kronecker, German mathematician, 1823 −1891) is an analogous bivariate function but its two variables are restricted to integer values. It is defined by

9:13:2 $$\delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

The *comb function* or *shah function*, comb(x,P), generates a sequence of Dirac functions whenever x is a multiple of P. For example, comb(x,1) has the property of being zero except for integer x, when it is infinite.

9:14 RELATED TOPIC: Green's functions

One application of the Dirac function arises in the context of *Green's function*, named for the English applied mathematician George Green (1793 – 1841). Of course, Green did not use the term "Dirac function": he died long before Dirac's birth.

The concept of a Green function is at once simple, yet profound [see, for example, Morse and Feshbach, Chapter 7]. These functions are employed in studies of physical situations in which a source of something (radiation, heat, electric field, diffusing chemicals, etc.) makes its presence felt at some remote site. The source may be of any shape and it may be of constant or varying intensity. The idea is that the source may be dissected into an infinite array of Dirac functions, in space and/or time; then, knowing the remote effect of one of these elements, suitable integrations will lead to knowledge of the effect of the source as a whole. The Green function is the contribution of one such element. Consider the example of a uniform cartesian plane containing a source, one element of which is at location (x', y') emitting diffusant at time t'. The effect of that element at some other point (x, y), at some subsequent time t, is given by a Green function, which takes the form

9:14:1 $$\frac{Q(x',y',t')}{4\pi\kappa[t-t']}\exp\left\{\frac{-[x-x']^2-[y-y']^2}{4\kappa[t-t']}\right\}$$

if the plane is infinite in both spatial dimensions. κ is a characteristic constant, the diffusivity. The Q term is the intensity of the source element. The emission from that element is represented by

9:14:2 $$Q(x',y',t')\delta(x-x')\delta(y-y')\delta(t-t')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'$$

and involves three Dirac functions.

CHAPTER 10

THE INTEGER POWERS x^n AND $(bx+c)^n$

With $n = 0, \pm1, \pm2, \cdots$ this chapter concerns the function $(bx + c)^n$ and its special $b = 1$, $c = 0$ case. The powers $1, x, x^2, \cdots$ and the reciprocal powers $1, x^{-1}, x^{-2}, \cdots$ are the units from which power series are built. Such expansions and their applications are addressed in Section 10:13. Section 10:14 provides a brief exposition on the intriguing and useful lozenge diagrams.

10:1 NOTATION

The two formulas $(bx + c)^{-n}$ and $1/(bx + c)^n$ are equivalent in all respects. The powers x^2 and x^3 are termed the *square* and the *cube* of x respectively, and the special properties of functions containing these units are addressed in Chapters 15 and 16.

In the general notation β^α, β is known as the *base* and α as the *power* or *exponent*. In this chapter [and in Chapter 12] the family of functions in which the base is the primary variable is treated, with the exponent held constant. In contrast, Chapter 26 is concerned with functions in which the exponent varies and the base is held constant. The instance in which both the base and the power are the same variable is touched on briefly in Sections 26:2 and 26:13.

10:2 BEHAVIOR

The power function is defined for all values of x and for all integer n except that x^n is undefined when both x and n are zero. Figures 10-1 and 10-2 illustrate the behavior of x^n for $n = 0, \pm1, \pm2, \pm3, \pm4, \pm7$ and ±12. Notice the contrasting behavior of the positive and negative powers. Note also how the reflection properties depend on whether n is even or odd.

If n is positive, $(bx + c)^n$ has a zero of multiplicity n [Section 0:7] at $x = -c/b$. This value of x is the site of an infinite discontinuity in $(bx + c)^n$ when n is negative.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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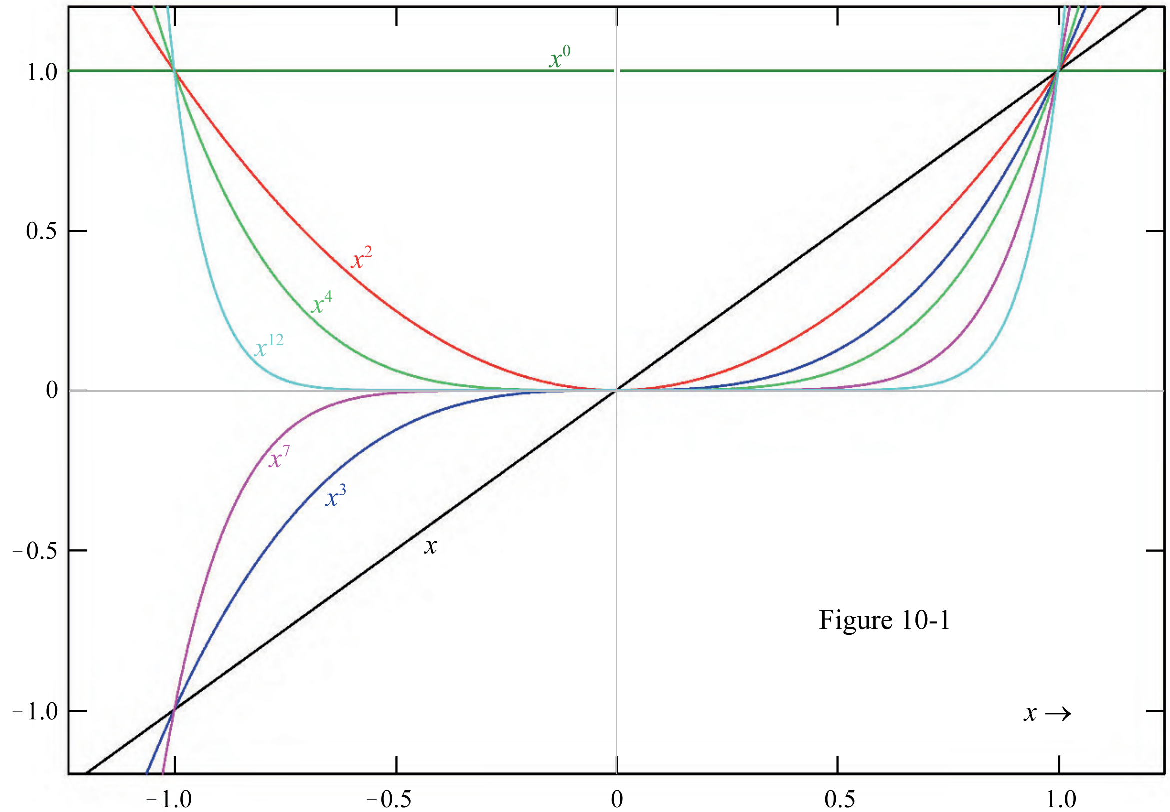


Figure 10-1

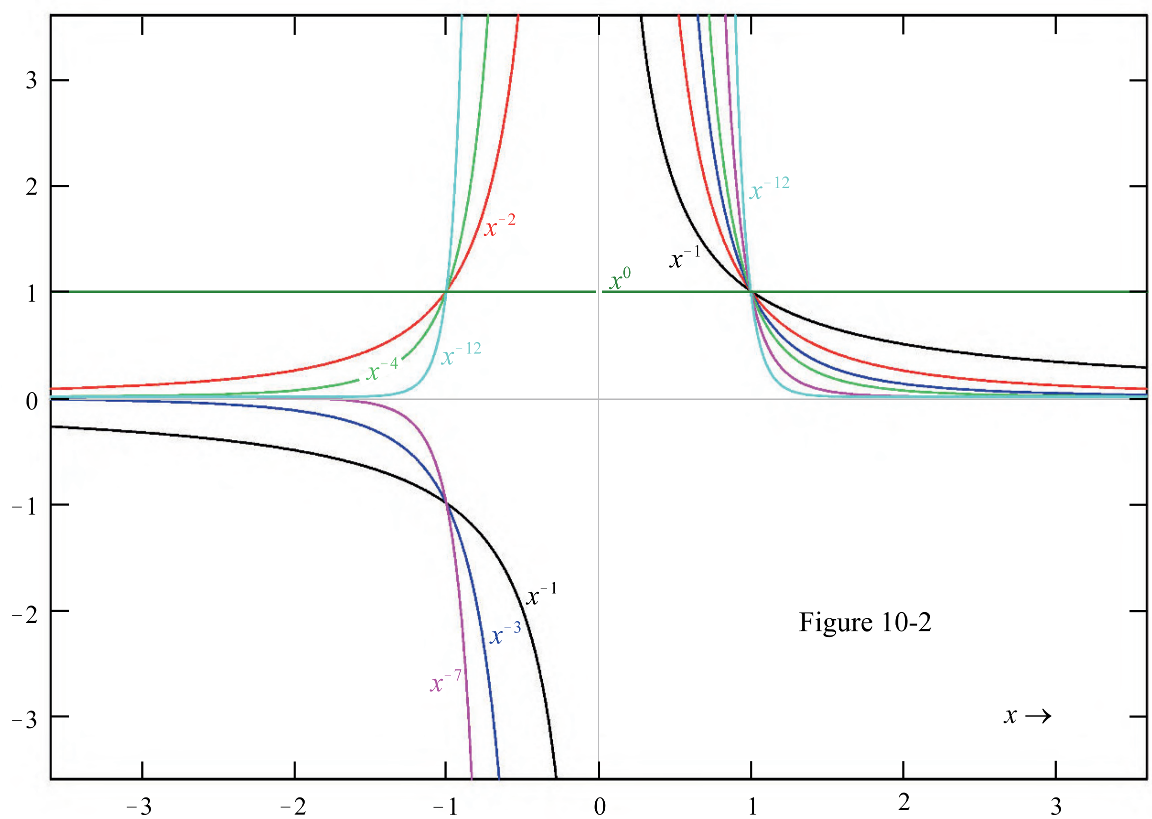


Figure 10-2

10:3 DEFINITIONS

The function $(bx + c)^n$ is defined by:

10:3:1
$$(bx+c)^n = \begin{cases} \prod_{j=1}^{-n} \frac{1}{bx+c} & n=-1,-2,-3,\cdots \\ 1 & n=0 \\ \prod_{j=1}^{n}(bx+c) & n=1,2,3,\cdots \end{cases}$$

10:4 SPECIAL CASES

When $n = 0$

10:4:1
$$(bx+c)^n = 1 \qquad n=0$$

and when $n = \pm 1$, reduction occurs to the functions treated in Chapter 7. 0^0 is generally undefined, though it may be ascribed a value of either 0 or 1 in certain contexts.

When $b = 0$, $(bx + c)^n$ reduces to a constant [Chapter 1] for all values of c and n.

10:5 INTRARELATIONSHIPS

The function x^n obeys the simple reflection formula

10:5:1
$$(-x)^n = \begin{cases} x^n & n=0,\pm 2,\pm 4,\cdots \\ -x^n & n=\pm 1,\pm 3,\pm 5,\cdots \end{cases}$$

For the $(bx + c)^n$ functions, reflection occurs about $x = -c/b$

10:5:2
$$\left[b\left(\frac{-c}{b}-x\right)+c\right]^n = (-)^n\left[b\left(\frac{-c}{b}+x\right)+c\right]^n$$

The recurrences

10:5:3
$$(bx+c)^n = (bx+c)(bx+c)^{n-1}$$

and

10:5:4
$$(bx+c)^{-n} = \frac{(bx+c)^{-n+1}}{bx+c}$$

apply, as do the *laws of exponents*

10:5:5
$$(bx+c)^n(bx+c)^m = (bx+c)^{n+m}$$

10:5:6
$$\frac{(bx+c)^n}{(bx+c)^m} = (bx+c)^{n-m}$$

and

10:5:7
$$\left[(bx+c)^n\right]^m = (bx+c)^{nm}$$

The simplest instances

10:5:8 $$x^2 - y^2 = (x-y)(x+y)$$

10:5:9 $$x^3 \pm y^3 = (x \pm y)\left(x^2 \mp xy + y^2\right)$$

10:5:10 $$x^4 - y^4 = (x-y)(x+y)\left(x^2 + y^2\right)$$

10:5:11 $$x^4 + y^4 = \left(x^2 - \sqrt{2}xy + y^2\right)\left(x^2 + \sqrt{2}xy + y^2\right)$$

of *function-subtraction* and *function-addition formulas* for integer powers generalize to formulas involving the cosine function [Chapter 32]:

10:5:12 $$x^n \pm y^n = (x+y)\prod_{j=1}^{(n-1)/2}\left[x^2 \pm 2xy\cos\left(\frac{2j\pi}{n}\right) + y^2\right] \qquad n = 1,3,5,\cdots$$

10:5:13 $$x^n + y^n = \prod_{j=1}^{n/2}\left[x^2 + 2xy\cos\left(\frac{2j\pi - \pi}{n}\right) + y^2\right] \qquad n = 2,4,6,\cdots$$

10:5:14 $$x^n - y^n = (x+y)(x-y)\prod_{j=1}^{(n-2)/2}\left[x^2 - 2xy\cos\left(\frac{2j\pi}{n}\right) + y^2\right] \qquad n = 2,4,6,\cdots$$

As elaborated in Section 17:7, $x^n \pm y^n$ may always be expressed as the product of n factors, possibly complex; for example, 10:5:11 becomes the product of four factors, each of which is subsumed in $x \pm (1 \pm i)y/\sqrt{2}$.

Finite series of positive or negative integer powers may be summed as *geometric series*:

10:5:15 $$1 + x + x^2 + \cdots + x^{n-1} + x^n = \frac{1 - x^{n+1}}{1 - x} \qquad n = 1,2,3,\cdots$$

10:5:16 $$1 + x^{-1} + x^{-2} + \cdots + x^{1-n} + x^{-n} = \frac{x - x^{-n}}{x - 1} \qquad n = 1,2,3,\cdots$$

The corresponding infinite series are summable only for restricted ranges of x, as discussed in Section 10:13.

10:6 EXPANSIONS

If n is positive, $(bx + c)^n$ may be expanded binomially as the finite series

10:6:1 $$(bx+c)^n = c^n + nc^{n-1}bx + \frac{n(n-1)}{2!}c^{n-2}b^2x^2 + \cdots + ncb^{n-1}x^{n-1} + b^nx^n = c^n\sum_{j=0}^{n}\binom{n}{j}\left(\frac{bx}{c}\right)^j \qquad n = 0,1,2,\cdots$$

for all x, where $\binom{n}{j}$ is the binomial coefficient of Chapter 6. If n is negative, the series is infinite and takes the form

10:6:2 $$(bx+c)^n = c^n + nc^{n-1}bx + \frac{n(n-1)}{2!}c^{n-2}b^2x^2 + \cdots = c^n\sum_{j=0}^{\infty}\binom{j-n-1}{j}\left(\frac{-bx}{c}\right)^j \qquad n = -1,-2,-3,\cdots \qquad |x| < \left|\frac{c}{b}\right|$$

or

10:6:3 $$(bx+c)^n = b^nx^n + ncb^{n-1}x^{n-1} + \cdots = b^nx^n\sum_{j=0}^{\infty}\binom{j-n-1}{j}\left(\frac{-c}{bx}\right)^j \qquad n = -1,-2,-3,\cdots \qquad |x| > \left|\frac{c}{b}\right|$$

depending on the magnitude of x. Section 6:14 presents some of the specific examples.

Positive integer powers may be expanded in terms of Pochhammer polynomials [Chapter 18]

10:6:4 $$x^n = \sum_{j=0}^{n} \sigma_n^{(j)} (x-j+1)_j = \sum_{j=0}^{n} (-)^j \sigma_n^{(j)} (-x)_j \qquad n=0,1,2,\cdots$$

where $\sigma_n^{(j)}$ is a Stirling number of the second kind [Section 2:14], or in terms of Chebyshev polynomials of the first kind [Chapter 22]:

10:6:5 $$x^n = \gamma_n^{(n)} \mathrm{T}_n(x) + \gamma_{n-2}^{(n)} \mathrm{T}_{n-2}(x) + \gamma_{n-4}^{(n)} \mathrm{T}_{n-4}(x) + \cdots + \begin{cases} \gamma_0^{(n)} \mathrm{T}_0(x) & n=0,2,4,\cdots \\ \gamma_1^{(n)} \mathrm{T}_1(x) & n=1,3,5,\cdots \end{cases}$$

For example, $x^5 = \frac{5}{8}\mathrm{T}_1(x) + \frac{5}{16}\mathrm{T}_3(x) + \frac{1}{16}\mathrm{T}_5(x)$. The coefficients $\gamma_j^{(n)}$ are zero whenever n and j are of unlike parity or j exceeds n. $\gamma_0^{(0)} = \gamma_1^{(1)} = 1$. Other $\gamma_j^{(n)}$ are positive rational numbers that are calculable by sufficient applications of the recursion formulas $\gamma_0^{(n)} = \frac{1}{2}\gamma_1^{(n-1)}$, $\gamma_1^{(n)} = \gamma_0^{(n-1)} + \frac{1}{2}\gamma_2^{(n-1)}$, and for $j \ge 2$, $\gamma_j^{(n)} = \frac{1}{2}\gamma_{j+1}^{(n-1)} + \frac{1}{2}\gamma_{j-1}^{(n-1)}$. Expansions similar to 10:6:5 exist for each orthogonal polynomial family [Chapters 21−24].

10:7 PARTICULAR VALUES

For b not equal to zero, $(bx+c)^n$ adopts the particular values:

	$x = -(1+c)/b$	$x = -c/b$	$x = (1-c)/b$
$n<0$	$(-1)^n$	∞	1
$n=0$	1	undef	1
$n>0$	$(-1)^n$	0	1

10:8 NUMERICAL VALUES

Equator's power function routine (keyword **power**) can calculate integer (or non-integer) powers. Additionally, the "variable construction" feature of *Equator* [Appendix C] allows t^p, or $wt^p + k$, to be used as the argument of another function.

10:9 LIMITS AND APPROXIMATIONS

The limiting behavior of x^n is evident from Figures 10-1 and 10-2. Note that the discontinuity suffered by x^{-n} at $x = 0$ is of the $+\infty|+\infty$ variety when $n = 2,4,6,\cdots$ but is $-\infty|+\infty$ for $n = 1,3,5,\cdots$.

10:10 OPERATIONS OF THE CALCULUS

The rule for differentiation

10:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(bx+c)^n = nb(bx+c)^{n-1}$$

is the $m = 1$ case of the multiple-differentiation formula

10:10:2
$$\frac{d^m}{dx^m}(bx+c)^n = \begin{cases} (n)_m b^m (bx+c)^{n-m} & m \le n \\ 0 & m > n \end{cases}$$

that employs the Pochhammer notation [Chapter 18]. General formulas for indefinite and definite integration are:

10:10:3
$$\int_{-c/b}^{x} (bt+c)^n \, dt = \begin{cases} \dfrac{(bx+c)^{n+1}}{(n+1)b} & n = 0,1,2,\cdots \\ \infty & n = -1,-2,-3,\cdots \end{cases}$$

10:10:4
$$\int_{x}^{\infty} (bt+c)^n \, dt = \begin{cases} \infty & n = -1,0,1,2,\cdots \\ \dfrac{(bx+c)^{n+1}}{(-n-1)b} & n = -2,-3,-4,\cdots \end{cases}$$

10:10:5
$$\int_{x_0}^{x_1} (bt+c)^n \, dt = \begin{cases} \dfrac{(bx_1+c)^{n+1} - (bx_0+c)^{n+1}}{(n+1)b} & n = 0,1,\pm 2,\pm 3,\cdots \\ \dfrac{1}{b}\ln\left(\dfrac{bx_1+c}{bx_0+c}\right) & n = -1 \end{cases}$$

Differintegrals [Section 12:14] of the nonnegative power x^n are given by the formula

10:10:6
$$\frac{d^\mu x^n}{dx^\mu} = \frac{n! x^{x-\mu}}{\Gamma(n-\mu+1)} \qquad n = 0,1,2,\cdots \qquad x > 0$$

where Γ is the gamma function [Chapter 43].

On Laplace transformation, a nonnegative integer power obeys the simple formula

10:10:7
$$\int_0^\infty t^n \exp(-st) \, dt = \mathfrak{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad n = 0,1,2,\cdots$$

Negative integer powers cannot be transformed, but powers of the reciprocal linear function transform as follows, provided $c \neq 0$:

10:10:8
$$\int_0^\infty (bt+c)^n \exp(-st) \, dt = \mathfrak{L}\{(bt+c)^n\} = \frac{b^n}{(-n-1)!(-s)^{n+1}}\left[-\exp\left(\frac{cs}{b}\right)\mathrm{Ei}\left(\frac{-cs}{b}\right) + \sum_{j=1}^{-n-1}(j-1)!\left(\frac{-b}{cs}\right)^j\right]$$

for $n = -1,-2,-3,\cdots$. The transform generates a product of the exponential [Chapter 26] and exponential integral [Chapter 37] functions.

10:11 COMPLEX ARGUMENT

Via a binomial expansion [Section 6:14], integer powers of the complex variable z, may be expressed as a pair of power series. For example, if n is positive

10:11:1
$$z^n = (x+iy)^n = x^n \sum_{j=0}^{\mathrm{Int}(n/2)} \binom{n}{2j}\left(\frac{-y^2}{x^2}\right)^j + ix^{n-1}y \sum_{j=0}^{\mathrm{Int}\{(n-1)/2\}} \binom{n}{2j+1}\left(\frac{-y^2}{x^2}\right)^j$$

while, for a negative power, one can split the function into real and imaginary components as

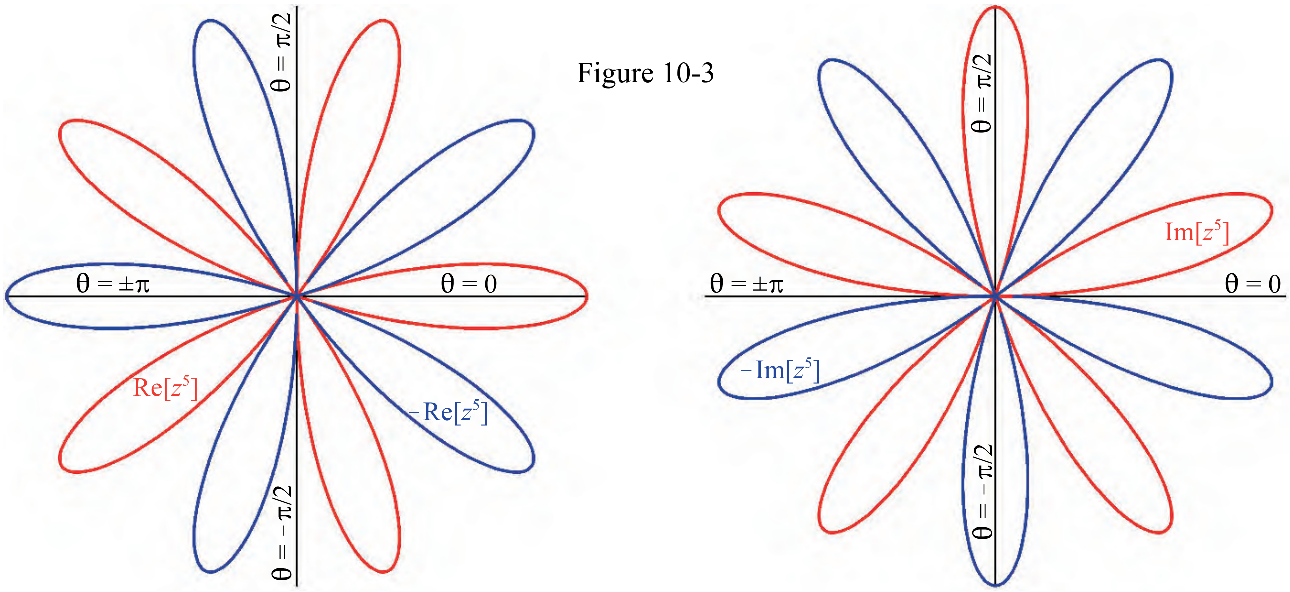


Figure 10-3

10:11:2 $$\frac{1}{z^n}=\frac{(x-iy)^n}{(x^2+y^2)^n}=\left(\frac{x}{x^2+y^2}\right)^n\sum_{j=0}^{\mathrm{Int}(n/2)}\binom{n}{2j}\left(\frac{-y^2}{x^2}\right)^j-\frac{ix^{n-1}y}{(x^2+y^2)^n}\sum_{j=0}^{\mathrm{Int}[(n-1)/2]}\binom{n}{2j+1}\left(\frac{-y^2}{x^2}\right)^j$$

In these two equations the rectangular representation $z = x + iy$ of a complex variable has served as the vehicle for expressing the properties of the power function when its argument is complex. For this function, however, it is more rewarding to use the polar representation $z = \rho\exp(i\theta)$ of a complex number. With this approach, *de Moivre's theorem* [Section 32:11] leads to

10:11:3 $$z^n=\rho^n\exp\left(ni\theta\right)=\rho^n\left[\cos(n\theta)+i\sin(n\theta)\right]$$

irrespective of the sign of n (there is a pole at the origin when n is negative). The real part is

10:11:4 $$\mathrm{Re}[z^n]=\rho^n\cos(n\theta)\qquad\text{where}\quad \rho=(x^2+y^2)^{n/2}\quad\text{and}\quad\theta=\arctan(y/x)+\pi\left[1-\mathrm{sgn}(x)\right]/2$$

with the expression for $\mathrm{Im}[z^n]$ having sin replace cos, but being otherwise similar. Figure 10-3 is a polar graph illustrating equation 10:11:4 and its imaginary counterpart for the case $n = 5$. In this representation the parts (real or imaginary) are zero at the center, with the red and blue "petals" respectively representing positive and negative excursions. At the edges of their petals, the parts adopt the value $\pm\rho^n$. *Equator*'s complex number raised to a real power routine (keyword **compower**) uses equation 10:11:4, and its congener $\mathrm{Im}[z^n] = \rho^n\sin(n\theta)$, to compute the real and imaginary parts of $(x+iy)^n$.

The reciprocal power s^{-n} undergoes Laplace inversion to give $t^{n-1}/(n-1)!$ and this generalizes to

10:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty}(bs+c)^n\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{(bs+c)^n\right\}=\frac{b^nt^{-n-1}}{(-n-1)!}\exp\left(\frac{-ct}{b}\right)\qquad n=-1,-2,-3,\cdots$$

10:12 GENERALIZATIONS

The restriction that the power be an integer is removed in Chapter 12. Quadratic functions [Chapter 15], cubic functions [Chapter 16], and polynomials [Chapters 17−24] are weighted finite sums of nonnegative integer powers.

10:13 COGNATE FUNCTIONS: power series

An infinite sum of weighted positive powers of a variable is a *power series*

10:13:1 $$a_0 + a_1 x + a_2 x^2 + \cdots + a_j x^j + \cdots = \sum_{j=0}^{\infty} a_j x^j$$

The a's, which are generally functions of j, but not of x, are the *coefficients* of the series, while $a_j x^j$ is the typical *term*. A similar series in which the general term is $a_j x^{\alpha+\beta j}$ is a *Frobenius series*; by redefining the variable to be x^β and by withdrawing a factor of x^α, such a series can be converted to a power series.

Some special cases of power series in which the coefficients are drawn from the set $(1,0,-1)$ include:

10:13:2 $$1 \pm x + x^2 \pm x^3 + x^4 \pm \cdots = \frac{1}{1 \mp x} \qquad -1 \le x < 1$$

10:13:3 $$x + x^3 + x^5 + x^7 + x^9 + \cdots = \frac{1}{2}\left[\frac{1}{1-x} - \frac{1}{1+x}\right] = \frac{x}{1-x^2} \qquad -1 < x < 1$$

to which many others could be appended. These are summable series of integer powers whose exponents increase linearly, but one may also sum similar series in which the exponents increase quadratically, as follows

10:13:4 $$1 - x + x^4 - x^9 + x^{16} - \cdots = \frac{1}{2}\left[\theta_4\left(0, \frac{-\ln(x)}{\pi^2}\right) + 1\right] \qquad 0 < x < 1$$

10:13:5 $$1 + x + x^4 + x^9 + x^{16} + \cdots = \frac{1}{2}\left[\theta_3\left(0, \frac{-\ln(x)}{\pi^2}\right) + 1\right] \qquad 0 < x < 1$$

10:13:6 $$x + x^9 + x^{25} + x^{49} + x^{81} + \cdots = \frac{1}{2}\theta_2\left(0, \frac{-4\ln(x)}{\pi^2}\right) \qquad 0 < x < 1$$

in terms of exponential theta functions [Section 27:13] of zero parameter. The quantity $\{-\ln(x)\}/\pi^2$ that appears in these formulas is closely related to the *nome function* discussed in Section 61:15.

Addition and subtraction of power series is straightforward. Thus if A, B and C are the power series $\sum a_j x^j$, $\sum b_j x^j$ and $\sum c_j x^j$, then if $A \pm B = C$ one has $c_j = a_j \pm b_j$. The rules for exponentiation and multiplication are

10:13:7 $$A^n = C \quad \text{where} \quad c_0 = a_0^n \quad \text{and} \quad c_j = \frac{n}{ja_0}\sum_{k=1}^{j}(j+k)a_{j-k+1}c_{k-1} \qquad \text{for } j = 1,2,3,\cdots$$

and

10:13:8 $$AB = C \quad \text{where} \quad c_j = \sum_{k=0}^{j} a_k b_{j-k}$$

but when $C = A/B$, the expression for c_j is too elaborate to be generally useful. In an operation known as *reversion of series*, a power series A in the variable x is converted into a power series for x, with a normalized A as the variable.

10:13:9 $$x = a_1 \sum_{k=1}^{\infty} d_k \left(\frac{A - a_0}{a_1^2}\right)^k \quad \text{where} \quad \begin{aligned} &d_1 = 1,\ d_2 = -a_2,\ d_3 = 2a_2^2 - a_1a_3,\ d_4 = 5a_2(a_1a_3 - a_2^2) - a_1^2a_4, \\ &d_5 = 7a_2^2(2a_2^2 - 3a_1a_3) + 3a_1^2(a_3^2 + 2a_2a_4) - a_1^3a_5, \\ &d_6 = 7[6a_2^3(2a_1a_3 - a_2^2) + a_1^3(a_3a_4 + a_2a_5) - 4a_1^2a_2(a_3^2 + a_2a_4)] - a_1^4a_6 \end{aligned}$$

There is no general formula for the d coefficients. The operations of differentiation and integration may be carried out term-by-term and generate other power series. Differintegration generally produces a Frobenius series. Operations on convergent power series do not necessarily preserve convergence.

As Sections 6 of most of the chapters in this *Atlas* will attest, almost all mathematical functions may be expanded via the *Maclaurin series* (Colin Maclaurin, 1698–1746, a Scottish mathematical prodigy, who defended his master's thesis at the age of 14)

10:13:10 $$\mathrm{f}(x) = \mathrm{f}(0) + x\frac{\mathrm{df}}{\mathrm{d}x}(0) + \frac{x^2}{2}\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(0) + \frac{x^3}{6}\frac{\mathrm{d}^3\mathrm{f}}{\mathrm{d}x^3}(0) + \cdots = \sum_{j=0}^{\infty}\frac{x^j}{j!}\frac{\mathrm{d}^j\mathrm{f}}{\mathrm{d}x^j}(0)$$

This is the special $y = 0$ case of the Taylor series 0:5:1. This formula permits most functions that can be repeatedly differentiated to be expressed as power series. Accordingly, a truncated version of such a power series is commonly used as a source of the (approximate) numerical value of a function for some specified value of the argument x:

10:13:11 $$\mathrm{f}(x) \approx \sum_{j=0}^{J} a_j x^j \qquad a_j = \frac{1}{j!}\frac{\mathrm{d}^j\mathrm{f}}{\mathrm{d}x^j}(0) \qquad \text{large } J$$

One may steadily increase J, calculating these *partial sums* the while. *Equator* frequently employs this tactic, ceasing the incrementation when three consecutive partial sums are identical (to the precision of the computation). Unfortunately, many series are not sufficiently convergent to yield adequate numerical approximations even when J has the large values accessible with speedy computers. Other numerical problems, in the form of *rounding errors* and *precision loss*, arise from the finite number of significant digits carried by most computer programs. These difficulties are mostly encountered when the terms in the Maclaurin series alternate in sign. Alternative methods are then sought, or a careful check is kept of the significance lost, the precision of the final answer being adjusted accordingly.

One simple remedy that is often effective is to convert the truncated power series into a *concatenation* (or "nested sum") that may be summed "backwards"

10:13:12 $$\mathrm{f}(x) \approx \Bigg(\bigg(\Big(\cdots\big(\big(\tfrac{1}{2}b_J x + 1\big)b_{J-1}x + 1\big)b_{J-2}x + \cdots + 1\Big)b_2 x + 1\bigg)b_1 x + 1\Bigg)a_0 \qquad b_j = a_j / a_{j-1}$$

Notice that this formula incorporates the ruse, useful only when the series alternates in sign, of halving the final summed term. A similar, and often helpful, stratagem is to convert the series to the continued fraction [see 0:6:12]

10:13:13 $$\mathrm{f}(x) \approx \frac{a_0}{1-}\,\frac{b_1 x}{b_1 x-}\,\frac{b_2 x}{b_2 x-}\cdots\frac{b_{J-2}x}{b_{J-2}x-}\,\frac{b_{J-1}x}{b_{J-1}x-}\,\frac{b_J x}{2}$$

but there are no guarantees in this field, which is as much art as science.

There exist more radical techniques for finding numerical values of f(x). Though often classified under the "summation of series" rubric, these approaches actually abandon the 10:13:11 expansion of f(x) in favor of some other representation, such as a rational function [Section 14:13], a standard continued fraction [Section 0:6] or a non-Maclaurin series. In Section 10:14, some of these techniques will be discussed in the context of lozenge diagrams, but a simpler transformation, due to Euler, will be exposed here.

The *Euler transformation* replaces the power series f(x) = $\sum a_j x^j$ by

10:13:14 $$\mathrm{f}(x) = \frac{1}{x}\left[e_1\frac{x}{1+x} + e_2\left(\frac{x}{1+x}\right)^2 + e_3\left(\frac{x}{1+x}\right)^3 + \cdots\right] = \frac{1}{x}\sum_{k=1}^{\infty} e_k\left(\frac{x}{1+x}\right)^k$$

By equating coefficients, one easily finds that $e_1 = a_0$, $e_2 = a_0+a_1$, $e_3 = a_0 + 2a_1 + a_2$, and generally

10:13:15 $$e_k = \sum_{j=0}^{k-1}\binom{k-1}{j}a_j$$

The transformed series frequently has much improved convergence, so that a truncated version of 10:13:14 may provide an acceptable numerical approximation.

For very large values of the argument x, most of these summation strategies fail to deliver useful numerical values. Fortunately, for most functions, f, there exist power series expansions of f($1/x$). These are generally *asymptotic series*, so that increasing the number of summed terms improves the approximation only up to a certain (x-dependent) point. An asymptotic relationship is indicated by the symbol "~" replacing the usual "=". For example, the asymptotic expansion [see 41:6:6]

10:13:16 $$\sqrt{\frac{\pi}{x}}\exp\left(\frac{1}{x}\right)\operatorname{erfc}\left(\frac{1}{\sqrt{x}}\right)\sim 1-\frac{x}{2}+\frac{3x^2}{4}-\frac{15x^3}{8}+\cdots=\sum_{j=0}(2j-1)!!\left(\frac{-x}{2}\right)^j$$

yields a power series, values of early partial sums of which, for $x=0.15$, are shown in Figure 10-4 as black dots. These initially converge towards the correct result ($0.937597\cdots$), shown by the blue line, but then wander away. The ruses and transformations discussed above and in the next section remain useful, and are doubly necessary because wantonly increasing the number of terms is not an option with asymptotic series. Thus, halving the last term in a partial sum leads to the red points in Figure 10-4. Clearly these ameliorate the difficulty without overcoming the asymptoticity.

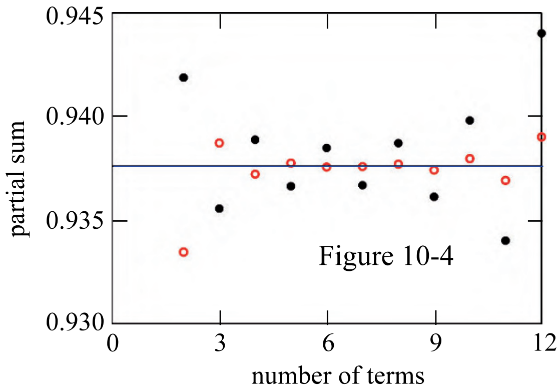
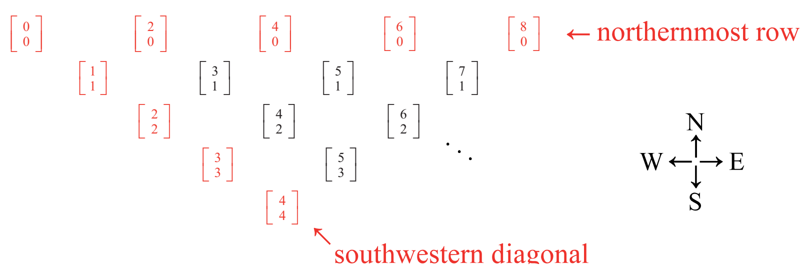


Figure 10-4

10:14 RELATED TOPIC: lozenge diagrams

A *lozenge diagram* is a two-dimensional array of numbers (or symbols representing numbers) arranged in a fashion that aids the conceptualization of certain useful operations performed on power series. Each element of the lozenge diagram is characterized by two integer indices, n and m, each index taking nonnegative integer values. The element itself is denoted $\left[{n \atop m}\right]$, but only those elements in which the indices have like parities appear. The arrangement of the elements, as shown below, is such that most occupy a vertex of at least one rhombus, and as many as four.



The lozenge diagram extends indefinitely to the right and downwards. In most applications, numbers or symbols are entered into the "northernmost", $m=0$, row. A specific *propagation rule* is then applied to create new entries in the diagram. The sequencing of propagation may proceed row by row downwards or, often more conveniently, by creating new entries in the order $\left[{1 \atop 1}\right]$, $\left[{3 \atop 1}\right]$, $\left[{2 \atop 2}\right]$, $\left[{5 \atop 1}\right]$, $\left[{4 \atop 2}\right]$, $\left[{3 \atop 3}\right]$, $\left[{7 \atop 1}\right]$, and so on. The propagation rules vary according to the operation for which the lozenge diagram is being used, but in all cases the element $\left[{n \atop m}\right]$ is computed, by arithmetic operations, from the adjacent elements $\left[{n \atop m-2}\right]$, $\left[{n-1 \atop m-1}\right]$, and $\left[{n+1 \atop m-1}\right]$. Often, the final output information appears in the "southwestern" diagonal, that is, in the elements in which n and m are equal. Four applications of the lozenge diagram will be elaborated in this section [see Wimp for details]. In the first, a Padé table [Section 17:12] is created from a power series. In the second and third, power series are transformed and thereby summed numerically. In the fourth, a power series is converted into a continued fraction.

Successive partial sums of the standard power series $\sum a_j x^j$ may be fed into the northernmost row of the lozenge diagram, so that

10:14:1 $$\left[\begin{smallmatrix} 2n \\ 0 \end{smallmatrix}\right] = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad n = 0,1,2,\cdots$$

In this application, the propagation rule used to form successive elements in the lozenge diagram is the *Schmidt-Wynn transformation*, or the *ε-algorithm*. It is

10:14:2 $$\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] = \begin{cases} \dfrac{1}{\left[\begin{smallmatrix} n+1 \\ m-1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right]} & \text{when} \quad m = 1 \\ \left[\begin{smallmatrix} n \\ m-2 \end{smallmatrix}\right] + \dfrac{1}{\left[\begin{smallmatrix} n+1 \\ m-1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right]} & \text{otherwise} \end{cases} \qquad \text{or} \qquad S = \begin{cases} \dfrac{1}{E-W} & \text{first row} \\ N + \dfrac{1}{E-W} & \text{other rows} \end{cases}$$

The second alternative in 10:14:2 provides a convenient mnemonic based on the points of the compass viewed from the center of each rhombus. A portion of the lozenge diagram derived in this way from the power series for the exponential function $\exp(x)$ follows:

$$\begin{array}{ccccccccc}
1 & & 1+x & & 1+x+\frac{1}{2}x^2 & & 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3 & & 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4 \\
& \dfrac{1}{x} & & \dfrac{2}{x^2} & & \dfrac{6}{x^3} & & \dfrac{24}{x^4} & \\
& & \dfrac{1+\frac{1}{2}x}{1-\frac{1}{2}x} & & \dfrac{1+\frac{2}{3}x+\frac{1}{6}x^2}{1-\frac{1}{3}x} & & \dfrac{1+\frac{3}{4}x+\frac{1}{4}x^2+\frac{1}{24}x^3}{1-\frac{1}{4}x} & & \\
& & & \dfrac{-2}{x}+\dfrac{12}{x^2}-\dfrac{12}{x^3} & & \dfrac{-6}{x^2}+\dfrac{48}{x^3}-\dfrac{72}{x^4} & & \ddots & \\
& & & & \dfrac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2} & & & &
\end{array}$$

As a comparison with the table in Section 17:12 shows, not all the entries in the lozenge diagram are members of the Padé table, but those for which m and n are even, shown in red, are. And not all members of the Padé table can be generated by this propagation rule; the others, however, can be found similarly by starting with the reciprocal of the power series for the reciprocal of the function, in this case $1/\exp(-x)$, as the input. All the expressions shown in red are valid approximations to $\exp(x)$. Among these rational functions, the most useful often are those that lie on the southwestern diagonal, which are 1, $(1+\frac{1}{2}x)/(1-\frac{1}{2}x)$, $(1+\frac{1}{2}x+\frac{1}{12}x^2)/(1-\frac{1}{2}x+\frac{1}{12}x^2), \cdots$ in this case. For most functions these *diagonal Padé approximants* are better approximations, and in some cases phenomenally better approximations, to the function, than are the partial sums of the truncated power series.

It is evident from the burgeoning complexity of the scheme that as a means of constructing, algebraically by hand, diagonal approximants of ever-larger order, the Schmidt-Wynn procedure soon becomes prohibitively tedious. However, it is simple to program the propagation rule to process numbers rather than symbols. In this way, a sequence of numerical values of the partial sums of power series may be converted arithmetically into a sequence of numerical values of the diagonal rational functions. For the case of the exponential function, the diagonal approximants (equal to 1, 3, 2.71429, 2.71831, ··· when $x = 1$) do not converge to the true value (2.71828 to six digits) very much faster than do the partial sums (1, 2, 2.5, 2.66667, 2.70833, 2.71667, ···) themselves. Consider, however, the $x = 1$ instance of the function [Section 37:6] that has the asymptotic power series expansion

10:14:3 $$\mathrm{f}(x) \sim \sum_{j=0} j!(-x)^j = 1 - x + 2!x^2 - 3!x^3 + 4!x^4 - 5!x^5 + \cdots$$

The sequence of partial sums is shown in red as the northernmost row in the lozenge diagram below, together with early results of Schmidt-Wynn transformation

$$\begin{array}{ccccccccccccc}
1 & & 0 & & 2 & & -4 & & 20 & & -100 & & 620 \\
& -1 & & \tfrac{1}{2} & & -\tfrac{1}{6} & & \tfrac{1}{24} & & -\tfrac{1}{120} & & \tfrac{1}{720} & \\
& & \tfrac{2}{3} & & \tfrac{1}{2} & & \tfrac{4}{5} & & 0 & & \tfrac{20}{7} & & \\
& & & -\tfrac{11}{2} & & \tfrac{19}{6} & & -\tfrac{29}{24} & & \tfrac{41}{120} & & & \\
& & & & \tfrac{8}{13} & & \tfrac{4}{7} & & \tfrac{20}{31} & & \ddots & & \\
& & & & & -\tfrac{235}{12} & & \tfrac{593}{48} & & & & & \\
& & & & & & \tfrac{44}{73} & & & & & &
\end{array}$$

In this application, it is again only every second element in the southwestern diagonal that provides a useful output. Even though the series 10:14:3 is atrociously divergent when $x = 1$, the red diagonal sequence takes the values 1.0000, 0.6667, 0.6154, 0.6027, that soon converge towards the "correct" value 0.5963 [equation 2:5:7] of f(1). *Equator* frequently uses this so-called ε-*transformation* to evaluate function values from poorly convergent power series. Note that, in this particular application of the lozenge diagram, it is advantageous to enter the negative of successive terms of the original power series directly into the second ($m = 1$) row, rather than calculating them from the northernmost ($m = 0$) row. By so doing, one avoids the significance loss that comes from subtracting two partial sums that may be nearly equal.

Another procedure, named the η-*transformation*, is a somewhat similar application of a lozenge diagram. Again the northernmost row represents power series 10:14:3, but with the difference that each element is now the numerical value of a *term* in the series, rather than being its partial sum. Keeping with the $x = 1$ instance of function 10:14:3 as our example, the lozenge diagram for the η-algorithm is

$$\begin{array}{ccccccccccccc}
1 & & -1 & & 2 & & -6 & & 24 & & -120 & & 720 \\
& -\tfrac{1}{2} & & \tfrac{2}{3} & & -\tfrac{3}{2} & & \tfrac{24}{5} & & -20 & & \tfrac{720}{7} & \\
& & \tfrac{1}{6} & & -\tfrac{1}{6} & & \tfrac{3}{10} & & -\tfrac{4}{5} & & \tfrac{20}{7} & & \\
& & & -\tfrac{2}{21} & & \tfrac{3}{26} & & -\tfrac{8}{35} & & \tfrac{20}{31} & & & \\
& & & & \tfrac{4}{91} & & -\tfrac{4}{91} & & \tfrac{16}{217} & & \ddots & & \\
& & & & & -\tfrac{6}{221} & & \tfrac{16}{511} & & & & & \\
& & & & & & \tfrac{18}{1241} & & & & & &
\end{array}$$

In compass-point format, the propagation rule used by the η-algorithm is

10:14:4
$$S = \begin{cases} \dfrac{1}{(1/E)-(1/W)} & \text{when} \quad S = \left[{n \atop 1}\right] \\ N+E-W & \text{when} \quad S = \left[{n \atop 2}\right], \left[{n \atop 4}\right], \left[{n \atop 6}\right] \cdots \\ \dfrac{1}{(1/N)+(1/E)-(1/W)} & \text{when} \quad S = \left[{n \atop 3}\right], \left[{n \atop 5}\right], \left[{n \atop 7}\right] \cdots \end{cases}$$

For the η-algorithm, *all* the elements in the southwestern diagonal are useful: they represent successive terms in a numerical series corresponding to the $x = 1$ version of 10:14:3. From entries as far as $n = m = 6$, one has

10:14:5
$$\mathrm{f}(1) \approx 1 - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{2}{21} + \tfrac{4}{91} - \tfrac{6}{221} + \tfrac{18}{1241} = \tfrac{44}{73}$$

Note the identity of this result with that obtained from the $\left[{6 \atop 6}\right]$ result of the ε-transformation.

Our final application of lozenge diagrams is that developed by Heinz Rutishauser (Swiss mathematician, 1918 – 1970). Any power series may be written as a concatenation (or nested sum):

10:14:6
$$\mathrm{f}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_j x^j + \cdots = a_0 + b_1 x\Big(1 + b_2 x\Big(1 + b_3 x\Big(1 + \ldots + b_j x\big(1+\cdots\big)\Big)\Big)\Big)$$

where $b_j = a_j/a_{j-1}$. Note that for an alternating series, all b's are negative. It is the b_jx terms that are fed into the northernmost row of a lozenge diagram in the *Rutishauser transformation* [see Acton]. Notice that b_1x enters $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ b_2x enters $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and generally $\left[\begin{smallmatrix}2j-2\\0\end{smallmatrix}\right] = b_jx$. The propagation rule for this algorithm is simple but bizarre:

10:14:7
$$S = \begin{cases} -E+W & \text{when} \quad S=\left[\begin{smallmatrix}n\\1\end{smallmatrix}\right] \\ NE/W & \text{when} \quad S=\left[\begin{smallmatrix}n\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}n\\4\end{smallmatrix}\right],\left[\begin{smallmatrix}n\\6\end{smallmatrix}\right]\cdots \\ N-E+W & \text{when} \quad S=\left[\begin{smallmatrix}n\\3\end{smallmatrix}\right],\left[\begin{smallmatrix}n\\5\end{smallmatrix}\right],\left[\begin{smallmatrix}n\\7\end{smallmatrix}\right]\cdots \end{cases}$$

Note that the a_0 term, that leads the series in 10:14:6, plays no part in the algorithm. The output from the transformation does not relate directly to the power series that was input, but to an equivalent continued fraction. If, for the elements on the southwestern diagonal, we adopt the nomenclature $c_1x = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = b_1x$, $c_2x = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $c_3x = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ and generally $c_jx = \left[\begin{smallmatrix}j-1\\j-1\end{smallmatrix}\right]$, then the continued fraction in question is

10:14:8
$$\mathrm{f}(x) = \frac{a_0}{1-}\,\frac{c_1x}{1+}\,\frac{c_2x}{1-}\,\frac{c_3x}{1+}\,\frac{c_4x}{1-}\,\frac{c_5x}{1+}\,\frac{c_6x}{1-\cdots}$$

Thus the a coefficients of the original series have been converted, via the concatenation b coefficients, to the continued fraction c constants. The Rutishauser algorithm fails when applied to the $x = 1$ case of 10:14:3, the example treated previously. As an alternative, we reconsider the exponential series

10:14:9
$$\mathrm{f}(x) = \exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \qquad a_j = \frac{1}{j!} \qquad b_jx = \frac{x}{j}$$

this time with x unspecified, and construct the following lozenge diagram:

$$\begin{array}{ccccccccccccc}
x & & \frac{1}{2}x & & \frac{1}{3}x & & \frac{1}{4}x & & \frac{1}{5}x & & \frac{1}{6}x & & \frac{1}{7}x \\
& \frac{1}{2}x & & \frac{1}{6}x & & \frac{1}{12}x & & \frac{1}{20}x & & \frac{1}{30}x & & \frac{1}{42}x & \\
& & \frac{1}{6}x & & \frac{1}{6}x & & \frac{3}{20}x & & \frac{2}{15}x & & \frac{5}{42}x & & \\
& & & \frac{1}{6}x & & \frac{1}{10}x & & \frac{1}{15}x & & \frac{1}{21}x & & & \\
& & & & \frac{1}{10}x & & \frac{1}{10}x & & \frac{2}{21}x & & \ddots & & \\
& & & & & \frac{1}{10}x & & \frac{1}{21}x & & & & & \\
& & & & & & \frac{1}{14}x & & & & & &
\end{array}$$

It follows that

10:14:10
$$\exp(x) = \frac{1}{1-}\,\frac{x}{1+}\,\frac{\frac{1}{2}x}{1-}\,\frac{\frac{1}{6}x}{1+}\,\frac{\frac{1}{6}x}{1-}\,\frac{\frac{1}{10}x}{1+}\,\frac{\frac{1}{10}x}{1-}\,\frac{\frac{1}{14}x}{1+\cdots}$$

The occurrence of zeros or infinities in lozenge propagation calculations can disable the procedure. Sometimes it is possible to proceed without penalty by replacing the infinity or zero, respectively, by a very large or a very small number, such as the $10^{\pm 99}$ used by *Equator* for this purpose during ε-transformations.

CHAPTER 11

THE SQUARE-ROOT FUNCTION $\sqrt{bx+c}$ AND ITS RECIPROCAL

Functions involving noninteger powers are known as *algebraic functions*. The *square-root function* $\sqrt{x}$ and the *reciprocal square-root function* $1/\sqrt{x}$ are the simplest algebraic functions. For the most part, this chapter considers these functions with their arguments generalized to $bx + c$. These functions have the shape of a parabola; some geometric properties of the parabola are detailed in Section 11:14.

11:1 NOTATION

Especially in computer applications, $\sqrt{x}$ is sometimes denoted SQRT(x) or SQR(x). The notation $\sqrt[2]{x}$ is also encountered.

The notations $x^{½}$ and $\sqrt{x}$ are often interpreted as equivalent, as are $x^{-½}$ and $1/\sqrt{x}$, but this *Atlas* makes a distinction. If the argument x is real and positive, $x^{½}$ has two alternative values, one positive and one negative. The square-root function $\sqrt{x}$, however, is single valued and equal to the positive of the two $x^{½}$ values. Accordingly, $\sqrt{bx+c}$ is equivalent to $|(bx + c)^{½}|$ and $(bx + c)^{-½}$ is equivalent to $\pm 1/\sqrt{bx+c}$.

Graphing $\pm\sqrt{bx+c}$ versus x generates a curve known as a *parabola* and therefore the name *semiparabolic function* may be applied appropriately to the $\sqrt{bx+c}$ function. The word "parabola" refers to the *shape* of the curve, irrespective of its orientation or its placement in the cartesian plane. A parabola that has its axis of symmetry along the x-axis will be identified as a *horizontal parabola*; in rectangular coordinates, it is described by the formula $f = (bx + c)^{½}$.

11:2 BEHAVIOR

Figure 11-1 maps the functions $\sqrt{bx+c}$ and $1/\sqrt{bx+c}$ under conditions in which b and c are positive. Of course, the graphical orientation reverses if b is negative and the placement of the semiparabola along the x-axis depends on the magnitudes of both coefficients, b and c. Neither the square-root function nor its reciprocal is defined as a real number for arguments less than $-c/b$ if b is positive, or greater than $-c/b$ if b negative. In other words, the

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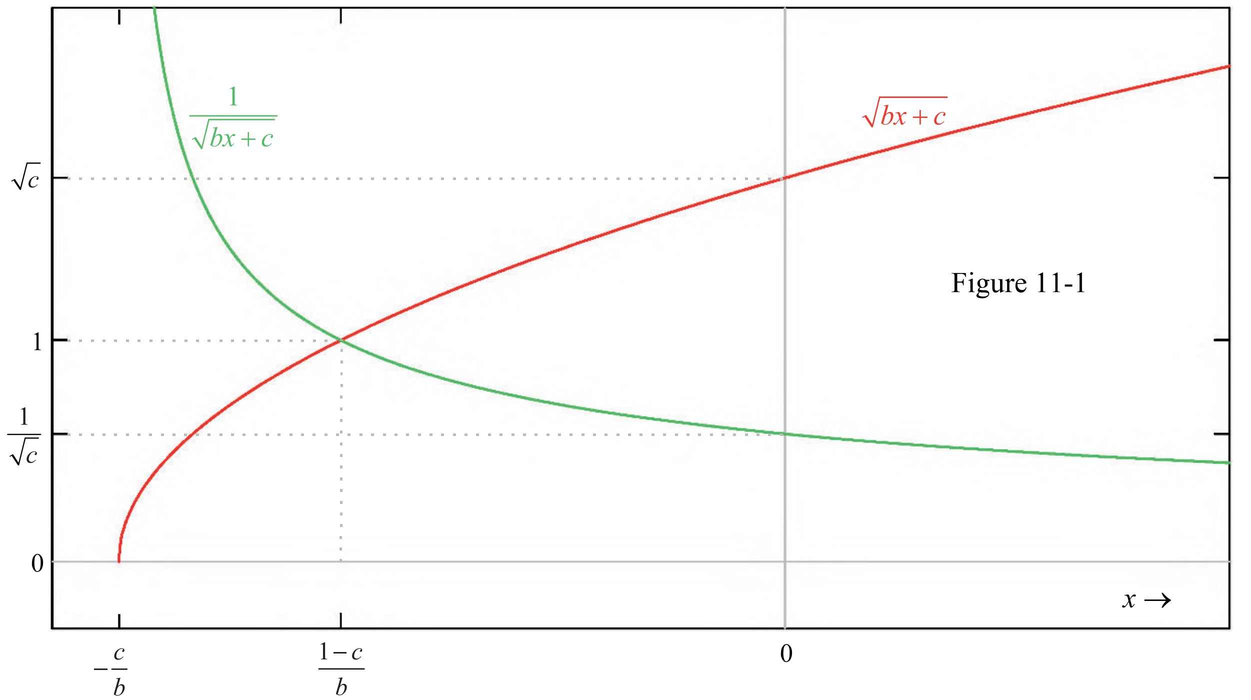


Figure 11-1

functions occupy a *semiinfinite domain*. The range of each function extends over all nonnegative values. Both functions have an infinite slope at $x = -c/b$, at which point $\sqrt{bx+c}$ is zero and its reciprocal is infinite.

11:3 DEFINITIONS

The square-root function is defined as the positive version of the inverse function [Section 0:3] of the square function; it "undoes" the operation of squaring:

11:3:1 $$\sqrt{(bx+c)^2} = |bx+c|$$

A parabola is defined geometrically as constituting all points P whose distance PF from a point F, called the *focus* of the parabola, equals the shortest distance D′P from point P to the straight line DD‴, called the *directrix*. Figure 11-2 illustrates this definition. For the horizontal parabola $f = \pm\sqrt{bx+c}$, the focus lies at the point whose rectangular coordinates are $(x, f) = ((b/4)-(c/b), 0)$ and the directrix is the line $x = -(b/4)-(c/b)$. The *apex* A of the parabola lies at the midpoint of the line segment D″F.

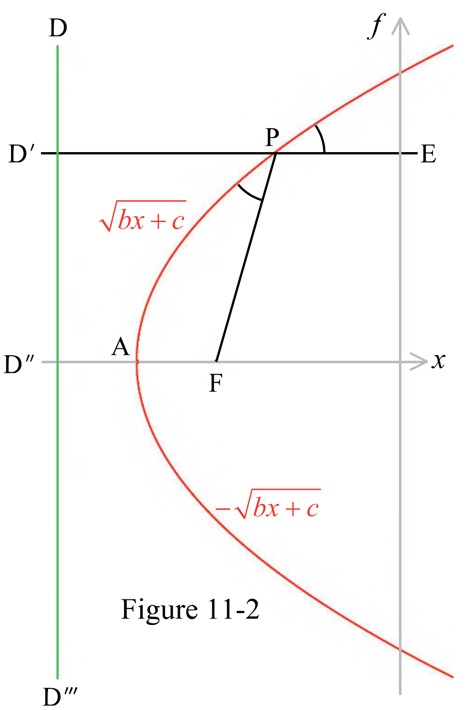


Figure 11-2

11:4 SPECIAL CASES

When $b = 0$, both functions reduce to constants.

11:5 INTRARELATIONSHIPS

If b_1 and b_2 share the same sign, the two square-root functions $\sqrt{b_1x+c_1}$ and $\sqrt{b_2x+c_2}$, as well as their reciprocals, coexist over a semiinfinite domain, either from the lesser of $-c_1/b_1$ or $-c_2/b_2$ to $+\infty$ or from $-\infty$ to the greater of $-c_1/b_1$ or $-c_2/b_2$. Within this domain the product of these functions obeys the rule:

11:5:1 $$\sqrt{b_1x+c_1}\sqrt{b_2x+c_2}=\sqrt{b_1b_2x^2+\left(b_1c_2+b_2c_1\right)x+c_1c_2}$$

and produces a root-quadratic function [Section 15:15]. The quotient $\sqrt{b_1x+c_1}/\sqrt{b_2x+c_2}$ generates the same root-quadratic function divided by the linear function b_2x+c_2. If b_1 and b_2 have opposite signs, and

11:5:2 $$\frac{c_1}{|b_1|}+\frac{c_2}{|b_2|}>0$$

then there is a finite domain of overlap between the two square-root functions; therein their product is a semielliptic function, akin to those discussed in Chapter 13

11:5:3 $$\sqrt{b_1x+c_1}\sqrt{b_2x+c_2}=\sqrt{-b_1b_2}\sqrt{\left(\frac{c_1}{2b_1}-\frac{c_2}{2b_2}\right)^2-\left(x+\frac{c_1}{2b_1}+\frac{c_2}{2b_2}\right)^2}\qquad b_1b_2<0$$

If the inequality 11:5:2 is violated, the two functions have no domain in common.

11:6 EXPANSIONS

Binomial expansion [Section 6:14] leads to the series

11:6:1 $$\sqrt{bx+c}=\begin{cases}\sqrt{c}+\dfrac{bx}{2\sqrt{c}}-\dfrac{b^2x^2}{8\sqrt{c^3}}+\dfrac{b^3x^3}{16\sqrt{c^5}}-\dfrac{5b^4x^4}{128\sqrt{c^7}}+\cdots=\sqrt{c}\displaystyle\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_j}{j!}\left(\frac{-bx}{c}\right)^j & |bx|<|c| \\ \sqrt{bx}+\dfrac{c}{2\sqrt{bx}}-\dfrac{c^2}{8\sqrt{b^3x^3}}+\dfrac{c^3}{16\sqrt{b^5x^5}}-\cdots=\sqrt{bx}\displaystyle\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_j}{j!}\left(\frac{-c}{bx}\right)^j & |bx|>|c|\end{cases}$$

The binomial expansion of the reciprocal square-root function has a similar dichotomy:

11:6:2 $$\frac{1}{\sqrt{bx+c}}=\begin{cases}\dfrac{1}{\sqrt{c}}-\dfrac{bx}{2\sqrt{c^3}}+\dfrac{3b^2x^2}{8\sqrt{c^5}}-\dfrac{5b^3x^3}{16\sqrt{c^7}}+\dfrac{35b^4x^4}{128\sqrt{c^9}}-\cdots=\dfrac{1}{\sqrt{c}}\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j}{j!}\left(\frac{-bx}{c}\right)^j & |bx|<|c| \\ \dfrac{1}{\sqrt{bx}}-\dfrac{c}{2\sqrt{b^3x^3}}+\dfrac{3c^2}{8\sqrt{b^5x^5}}-\dfrac{5c^3}{16\sqrt{b^7x^7}}+\cdots=\dfrac{1}{\sqrt{bx}}\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j}{j!}\left(\frac{-c}{bx}\right)^j & |bx|>|c|\end{cases}$$

The square-root function can also be expanded as a ratio of two exponential series [Section 27:13]:

11:6:3 $$\sqrt{x}=\frac{\frac{1}{2}+\exp\left(\frac{-\pi}{x}\right)+\exp\left(\frac{-4\pi}{x}\right)+\exp\left(\frac{-9\pi}{x}\right)+\cdots}{\frac{1}{2}+\exp(-\pi x)+\exp(-4\pi x)+\exp(-9\pi x)+\cdots}=\frac{\frac{1}{2}+\displaystyle\sum_{j=1}^{\infty}\exp\left(\frac{-j^2\pi}{x}\right)}{\frac{1}{2}+\displaystyle\sum_{j=1}^{\infty}\exp\left(-j^2\pi x\right)}$$

both of which converge very rapidly.

11:7 PARTICULAR VALUES

The tabulated values apply if both b and c are positive.

	$x=-c/b$	$x=0$	$x=(1-c)/b$	$x=+\infty$
$\sqrt{bx+c}$	0	$\sqrt{c}$	1	$+\infty$
$1/\sqrt{bx+c}$	$+\infty$	$1/\sqrt{c}$	1	0

11:8 NUMERICAL VALUES

Equator provides a square-root function routine (keyword **sqrt**). This is designed to handle the complex number $x+iy$ but, with the default $y=0$ retained, the square root $\sqrt{x}$ (real or imaginary) of any real number x is output. The magnitudes of x and y must lie between 10^{-150} and 10^{150}.

Alternatively, you may find the square root of a positive real number x by using *Equator*'s power function routine [Section 12:8, keyword **power**] and setting $\nu = ½$. Inputting $\nu = -½$ gives the reciprocal square root. Although each of $x^{½}$ and $x^{-½}$ has two values, only the positive option is returned by **power**.

The arithmetic function [Appendix, Section C:10, keyword **arith**] is yet another way of finding a square root.

Via the "variable construction" feature [Appendix, Section C:4], the quantity $wt^{\pm½}+k$ may serve as a variable of any *Equator* function.

11:9 LIMITS AND APPROXIMATIONS

If ρ is an approximate value of $\sqrt{x}$, then $(\rho^2+x)/2\rho$ is a better approximation. This is the basis of *Newton's method* for calculating square roots.

11:10 OPERATIONS OF THE CALCULUS

The formulas

11:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{bx+c}=\frac{b}{2\sqrt{bx+c}}$$

and

11:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{bx+c}}=\frac{-b}{2\sqrt{(bx+c)^3}}$$

describe the differentiation of the square-root and reciprocal square-root functions. The corresponding indefinite integrals are

11:10:3 $$\int_{-c/b}^{x}\sqrt{bt+c}\,\mathrm{d}t=\frac{2}{3b}\sqrt{(bx+c)^3}$$

and

11:10:4
$$\int_{-c/b}^{x} \frac{1}{\sqrt{bt+c}}\,\mathrm{d}t = \frac{2}{b}\sqrt{bx+c}$$

Related indefinite integrals include

11:10:5
$$\int_{-c/b}^{x} t\sqrt{bt+c}\;\mathrm{d}t = \frac{6bx-4c}{15b^2}\sqrt{(bx+c)^3}$$

and

11:10:6
$$\int_{-c/b}^{x} \frac{1}{t\sqrt{bt+c}}\,\mathrm{d}t = \frac{2}{\sqrt{c}}\operatorname{arcsec}\left(\sqrt{\frac{bx}{c}}\right) \qquad x > \frac{c}{b} > 0$$

These are just two examples drawn from a large class of integrals in which the integrand is $\sqrt{(b_1t+c_1)^n(b_2t+c_2)^m}$, at least one of the integers n and m being odd. A long list of such integrals will be found in Gradshteyn and Rhyzik [Sections 2.21−2.24].

With a lower limit of zero, the semiderivatives and semiintegrals [Section 12:14] of the square-root function and its reciprocal are given by the following formulas:

11:10:7
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sqrt{bx+c} = \sqrt{\frac{c}{\pi x}} + \sqrt{\frac{|b|}{\pi}}\,\phi$$

11:10:8
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\sqrt{bx+c} = \sqrt{\frac{cx}{\pi}} + \frac{bx+c}{\sqrt{\pi|b|}}\,\phi$$

11:10:9
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\frac{1}{\sqrt{bx+c}} = \frac{1}{bx+c}\sqrt{\frac{c}{\pi x}}$$

11:10:10
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\frac{1}{\sqrt{bx+c}} = \frac{2\phi}{\sqrt{\pi|b|}}$$

$$\phi = \begin{cases} \arctan\left(\sqrt{bx/c}\right) & b>0 \quad x>-c/b \\ \operatorname{artanh}\left(\sqrt{-bx/c}\right) & b<0 \quad x<-c/b \end{cases}$$

Among important Laplace transforms involving square roots are

11:10:11
$$\int_0^\infty \sqrt{t}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2\sqrt{s^3}}$$

11:10:12
$$\int_0^\infty \frac{1}{\sqrt{t}}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$$

11:10:13
$$\int_0^\infty \sqrt{t^{2n-1}}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\sqrt{t^{2n-1}}\right\} = \left(\tfrac{1}{2}\right)_n\sqrt{\frac{\pi}{s^{2n+1}}} \qquad n=0,1,2,\cdots$$

11:10:14
$$\int_0^\infty \sqrt{bt+c}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\sqrt{bt+c}\right\} = \frac{\sqrt{c}}{s} + \sqrt{\frac{\pi b}{4s^3}}\exp\left(\frac{cs}{b}\right)\operatorname{erfc}\left(\sqrt{\frac{cs}{b}}\right)$$

and

11:10:15
$$\int_0^\infty \frac{1}{\sqrt{bt+c}}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\frac{1}{\sqrt{bt+c}}\right\} = \sqrt{\frac{\pi}{bs}}\exp\left(\frac{cs}{b}\right)\operatorname{erfc}\left(\sqrt{\frac{cs}{b}}\right)$$

Functions from Chapters 18 and 41 are generated by these transformations.

11:11 COMPLEX ARGUMENT

As with a real number, the square root of a complex number z has two values. In rectangular and polar notation [Section 1:11] these are

11:11:1 $$z^{1/2} = \pm\left[\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\,\mathrm{sgn}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right] = \sqrt{\rho}\exp\left\{i\left(\frac{\theta-\pi}{2}\pm\frac{\pi}{2}\right)\right\}$$

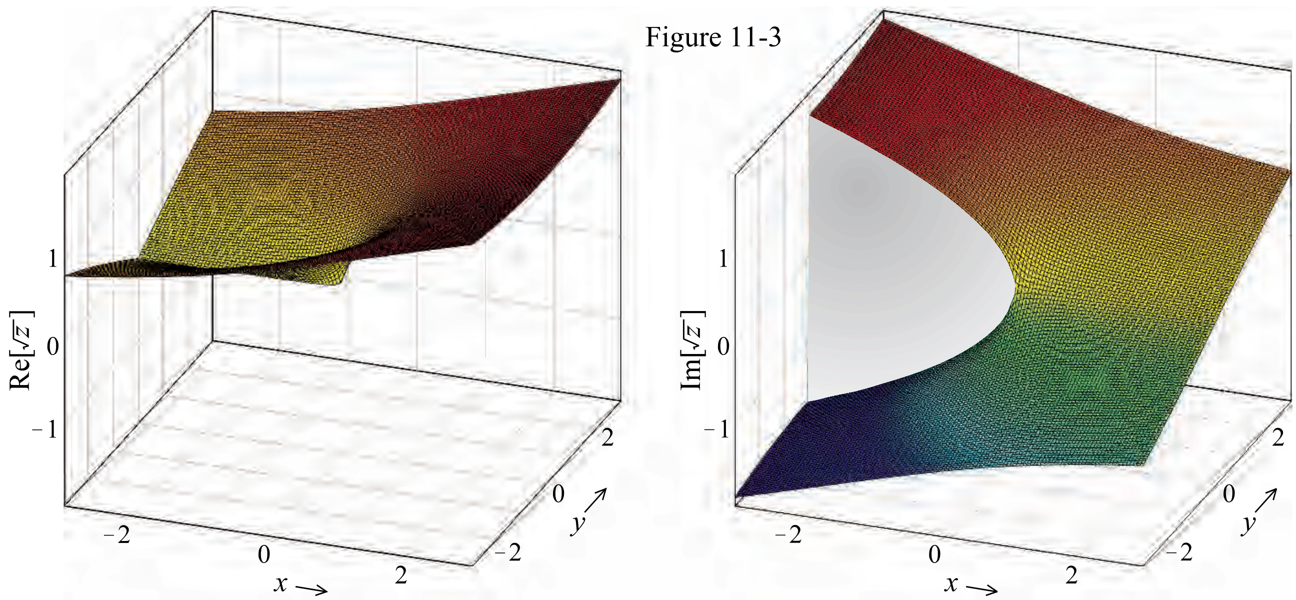
when $z = x+iy = \rho\exp(i\theta)$. However, our definition of $\sqrt{z}$ is single-valued and equal to the positive option of $z^{½}$. Similarly

11:11:2 $$z^{-1/2} = \pm\left[\sqrt{\frac{\sqrt{x^2+y^2}+x}{2(x^2+y^2)}} - i\,\mathrm{sgn}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2(x^2+y^2)}}\right] = \frac{1}{\sqrt{\rho}}\exp\left\{i\left(\frac{\pi-\theta}{2}\pm\frac{\pi}{2}\right)\right\}$$

It should be noted that the "rule" $\sqrt{z_1}\sqrt{z_2} = \sqrt{z_1 z_2}$ may be violated, but $\sqrt{z_1}\sqrt{z_2} = \pm\sqrt{z_1 z_2}$ is always correct.

The values of the square root of a complex number, $\sqrt{x+iy}$ is calculable through *Equator*'s **sqrt** routine [Section 11:8] or via its **compower** routine [Section 12:11]. The latter routine can also generate values of $\sqrt{1/(x+iy)}$. These values have the real and imaginary parts shown in Figures 11-3 and 11-4. Notice that Im[$\sqrt{z}$] displays a discontinuity, colored gray on the diagram, along the negative x-axis. Poles are present at the origin for both $\mathrm{Re}[1/\sqrt{z}]$ and $\mathrm{Im}[1/\sqrt{z}]$, and the latter displays discontinuity as well, again along the x-axis.

Figure 11-3



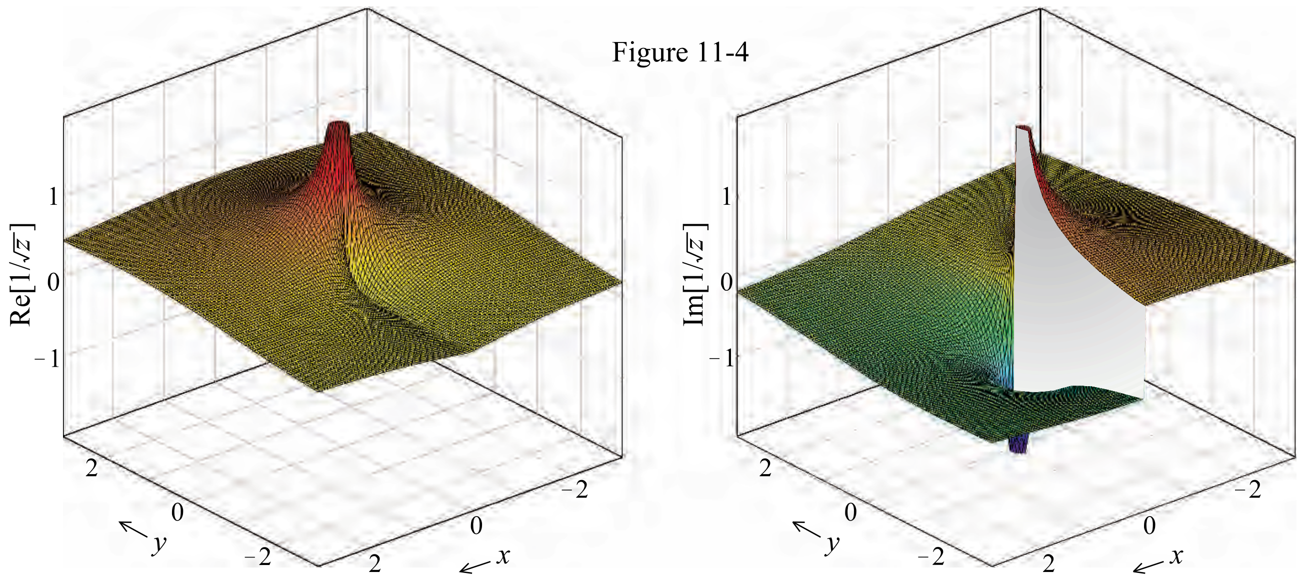
In addition to $1^{½} = \pm 1$, the following special cases are noteworthy:

11:11:3 $$(-1)^{1/2} = \pm i = \exp\left(\frac{\pm i\pi}{2}\right) = \pm\exp\left(\frac{i\pi}{2}\right)$$

11:11:4 $$i^{1/2} = \frac{\pm(1+i)}{\sqrt{2}} = \exp\left(\frac{i(\pm 2\pi-\pi)}{4}\right)$$

11:11:5 $$(-i)^{1/2} = \frac{\pm(1-i)}{\sqrt{2}} = \exp\left(\frac{i(\pi\pm 2\pi)}{4}\right)$$

Figure 11-4



The following inverse Laplace transforms involve the square-root function or its reciprocal:

11:11:6
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt{s}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{\sqrt{s}}\right\}=\frac{1}{\sqrt{\pi t}}$$

11:11:7
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt{s^{2n+1}}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{\sqrt{s^{2n+1}}}\right\}=\frac{1}{(\frac{1}{2})_n}\sqrt{\frac{t^{2n-1}}{\pi}}\qquad n=0,1,2,\cdots$$

11:11:8
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\sqrt{bs+c}}{s}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\sqrt{bs+c}}{s}\right\}=\sqrt{\frac{b}{\pi t}}\exp\left(\frac{-ct}{b}\right)+\sqrt{c}\,\mathrm{erf}\left(\sqrt{\frac{ct}{b}}\right)$$

11:11:9
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\sqrt{bs+c}}{\sqrt{s}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\sqrt{bs+c}}{\sqrt{s}}\right\}=\sqrt{b}\delta(t)+\frac{c}{2\sqrt{b}}\exp\left(\frac{-ct}{2b}\right)\left[\mathrm{I}_0\left(\frac{ct}{2b}\right)+\mathrm{I}_1\left(\frac{ct}{2b}\right)\right]$$

11:11:10
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\sqrt{s}}{\sqrt{bs+c}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\sqrt{s}}{\sqrt{bs+c}}\right\}=\frac{\delta(t)}{\sqrt{b}}-\frac{c}{2\sqrt{b^3}}\exp\left(\frac{-ct}{2b}\right)\left[\mathrm{I}_0\left(\frac{ct}{2b}\right)-\mathrm{I}_1\left(\frac{ct}{2b}\right)\right]$$

11:11:11
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt{bs+c}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{\sqrt{bs+c}}\right\}=\frac{1}{\sqrt{\pi bt}}\exp\left(\frac{-ct}{b}\right)$$

11:11:12
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt{s}\sqrt{bs+c}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{\sqrt{s}\sqrt{bs+c}}\right\}=\frac{1}{\sqrt{b}}\exp\left(\frac{-ct}{2b}\right)\mathrm{I}_0\left(\frac{ct}{2b}\right)$$

11:11:13
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s\sqrt{bs+c}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{s\sqrt{bs+c}}\right\}=\frac{1}{\sqrt{c}}\mathrm{erf}\left(\sqrt{\frac{ct}{b}}\right)$$

The functions resulting from these inversions include the error function erf [Chapter 40], the Dirac function δ [Chapter 9] and the hyperbolic Bessel functions I_0 and I_1 [Chapter 49].

11:12 GENERALIZATIONS

The $x^{\pm 1/2}$ functions are the simplest fractional powers. Other members of the x^{v} family are addressed in Chapter 12. The root-quadratic function and its reciprocal [Section 15:13] are generalizations in a different direction.

11:13 COGNATE FUNCTIONS

The functions $\sqrt{(bx+c)^{\pm 3}}$, $\sqrt{(bx+c)^{\pm 5}}$, etc. have properties similar to those of the square-root function and its reciprocal.

11:14 RELATED TOPIC: geometric properties of the parabola

A useful property of the parabola may be illustrated by reference to Figure 11-2 on an earlier page. If, as shown, the horizontal line D′P is extrapolated to E, then the lines EP and PF make equal angles with the parabolic curve at point P. Thus, if the parabola represents a mirror and EP is a ray of light, the ray will be reflected and reach the focus F. The same is true of any ray parallel to EP. Moreover, this "focusing" property is duplicated in the parabola's three-dimensional counterpart, the *paraboloid*; it lies behind the paraboloidal design of such devices as searchlights, telescopes and satellite dishes.

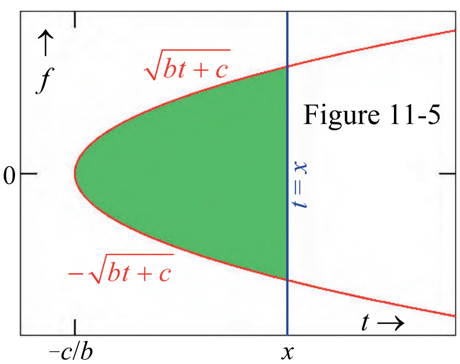


Figure 11-5

Figure 11-5 shows a shaded region bounded by the horizontal parabola $f(t) = \pm\sqrt{bt+c}$ and the ordinate $t = x$. The area of this region may be found with the aid of integral 11:10:3 as

11:14:1
$$\text{shaded area} = 2\int_{-c/b}^{x} \sqrt{bt+c}\,\mathrm{d}t = \frac{4\sqrt{(bx+c)^3}}{3b}$$

The curved portion of the perimeter of the shaded region has a length that may be found by application of formula 39:14:3

11:14:2
$$\text{curved perimeter} = 2\int_{-c/b}^{x} \sqrt{1+\left(\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{bt+c}\right)^2}\,\mathrm{d}t = \frac{2}{b}\sqrt{(bx+c)\left(bx+c+\frac{b^2}{4}\right)} + \frac{b}{2}\operatorname{arsinh}\left(\frac{2\sqrt{bx+c}}{b}\right)$$

This length is to be incremented by $2\sqrt{bx+c}$ to give the total length of the perimeter of the shaded area in Figure 11-5.

CHAPTER 12

THE NONINTEGER POWERS x^ν

Relationships in science and engineering frequently involve fractional exponents. In this chapter we address the power function x^ν, where ν is real but otherwise unrestricted. As is clarified in Section 12:2, the function x^ν is often undefined as a real number when x is negative.

12:1 NOTATION

The symbol x^ν represents the argument x raised to power ν; both these quantities are real except in Section 12:11. The symbols $x^{-\nu}$ and $1/x^\nu$ represent identical functions.

When ν equals $1/n$, where $n = 2, 3, 4, \cdots$, the symbol $\sqrt[n]{x}$ sometimes replaces $x^{1/n}$, though this symbolism is seldom used in the *Atlas* [but see Section 16:4 for the distinction we draw there between $x^{1/3}$ and $\sqrt[3]{x}$]. The names *square root* of x, *cube root* of x, *fourth root* of x and n^{th} *root* of x are given to $x^{1/2}$, $x^{1/3}$, $x^{1/4}$, and $x^{1/n}$.

12:2 BEHAVIOR

With both x and ν restricted to real values, it is instructive to examine the behavior of the x^ν function in the context of quadrants, as defined in Section 0:2. In all quadrants other than the first, radically different behaviors are exhibited according to the properties of the number ν.

The power function x^ν exists in the first quadrant irrespective of the magnitude and rationality of ν. It acquires the value unity at $x = 1$ for all ν. If ν is positive, x^ν is zero at $x = 0$ and increases indefinitely as x takes ever-larger positive values, as is evident from several of the examples illustrated in Figure 12-1. If ν is negative, x^ν is infinite at $x = 0$, but adopts finite values that diminish as x increases, approaching zero in the $x \to \infty$ limit.

If ν is irrational (that is incapable, like π or $-\pi/4$, of being expressed as a ratio m/n of two integers), then x^ν exists only in the first quadrant as illustrated by the black curves in Figure 12-1.

There are three classes of rational numbers, according as the parities of m/n are even/odd, odd/odd, or odd/even. Of course, there is no even/even class because such fractions can always be reduced to one of the other classes by sufficient divisions of m and n by 2.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_13, © Springer Science+Business Media, LLC 2009

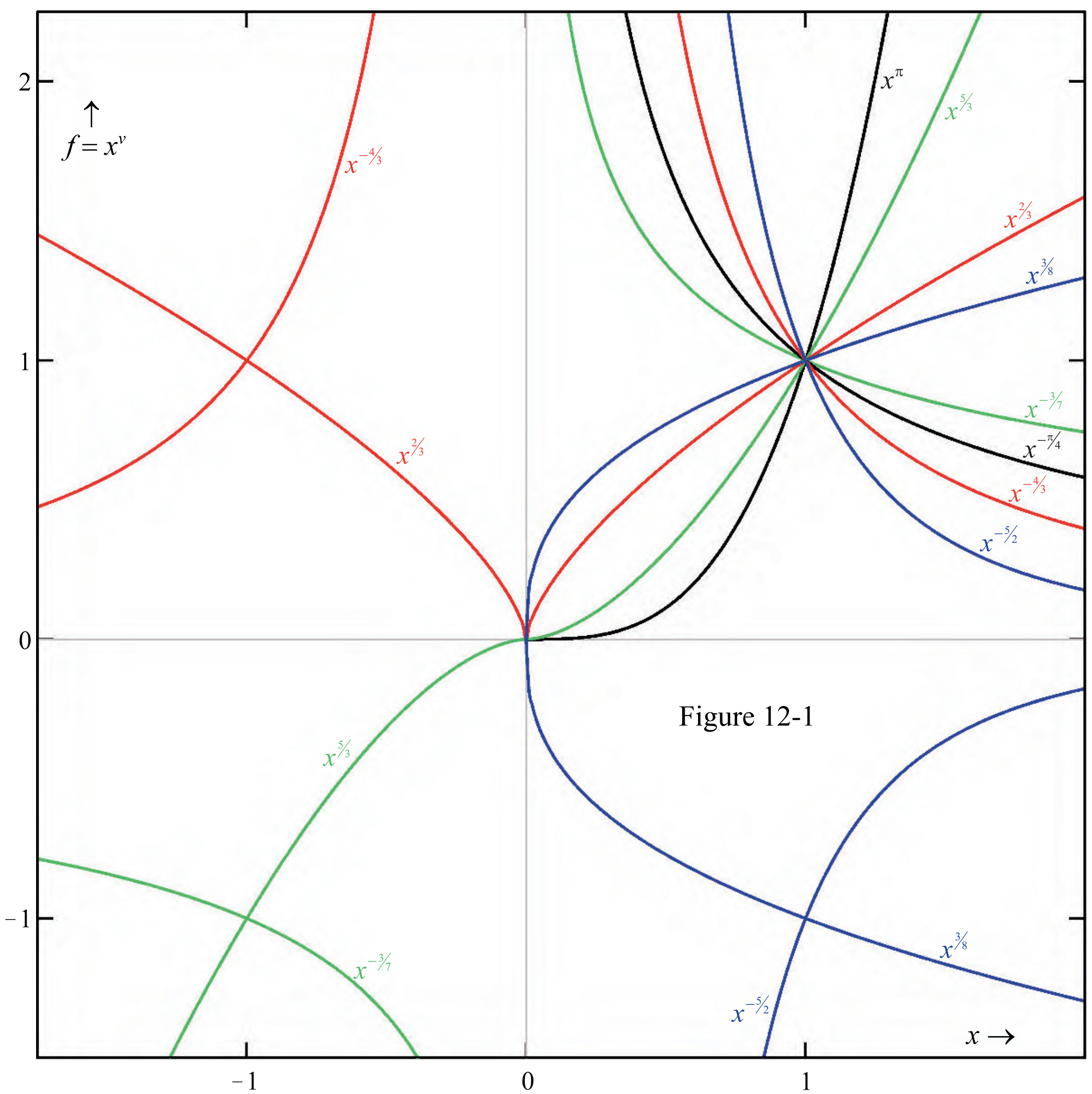


Figure 12-1

When ν is a rational number of the even/odd class, such as $\nu = m/n = 2/3$ or $-4/3$, the x^ν function exists in the first and second quadrants only. The function itself is even in these cases and takes only nonnegative values, as illustrated by the red curves in Figure 12-1.

It is the first and third quadrants that are occupied by the x^ν function when ν is a rational number, positive or negative, of the odd/odd class. The power function is then an odd function, conforming to the $(-x)^\nu = -x^\nu$ rule, as exemplified by the green curves in the figure.

When ν falls in the odd/even class, x^ν is defined as a real number only for nonnegative x and is two-valued, with plus-or-minus options (except perhaps at $x = 0$). For example $2^{3/8} = \pm 1.2968\cdots$. The blue curves in Figure 12-1 show the $\nu = \frac{3}{8}$ and $\frac{-5}{2}$ examples, which lie in the first and fourth quadrants only.

The behavior of the power function x^ν close to $x = 0$ is of interest. Notice that, though the function itself may be continuous there, the slope of x^ν suffers a discontinuity if $0 < \nu < 1$. This is true irrespective of the parities of m and n, though not all instances are illustrated in Figure 12-1.

12:3 DEFINITIONS

When ν is the reciprocal of a positive integer n greater than zero, the definition of x^ν relies on the concept of an inverse function [Section 0:3]. Thus the n^{th} root, $x^{1/n}$ is defined as the real number f such that

12:3:1 $$f^n = x \qquad \text{whence} \qquad f = x^{1/n} \qquad n = 2,3,4,\cdots$$

Thus there is a plus-or-minus option in $x^{1/n}$ if n is even, but positivity is mandatory if n is odd. Any other rational, noninteger power of x is then defined by raising $x^{1/n}$ to the appropriate integer power, positive or negative:

12:3:2 $$\left(x^{1/n}\right)^m \qquad \text{where} \qquad \nu = \frac{m}{n} \qquad m = \pm1,\pm2,\pm3,\cdots$$

The plus-or-minus option is thereby lost if m is even, but not otherwise.

When ν is irrational, x^ν require definition as a limit. Let m_1/n_1, m_2/n_2, m_3/n_3, ··· be progressively better rational approximations to ν. Then

12:3:3 $$\lim_{j\to\infty}\left\{x^{m_j/n_j}\right\} = x^\nu$$

provides the required definition. For example, x^π could be defined as the limit of the sequence x^3, $x^{31/10}$, $x^{314/100}$, $x^{3142/1000}$, $x^{31416/10000}$,···, or some similar sequence.

In view of equation 12:10:8, the function x^ν may be defined as the result of the differintegration operation [Section 12:14] applied to the function $\mathrm{f}(x) = x$

12:3:4 $$x^\nu = \Gamma(1+\nu)\frac{\mathrm{d}^{1-\nu}}{\mathrm{d}x^{1-\nu}}x \qquad x>0$$

for all ν except negative integers. Γ is the (complete) gamma function [Chapter 43].

12:4 SPECIAL CASES

Powers with exponents $\nu = \pm1$ and $\nu = \pm\frac{1}{2}$ are addressed in Chapters 7 and 11. Cases in which ν is an integer are the subject of Chapter 10.

12:5 INTRARELATIONSHIPS

No reflection formula holds when ν is irrational or a member of the odd/even family of rational powers. Otherwise the power function is either even or odd according to the parity of m:

12:5:1 $$(-x)^{m/n} = (-)^m x^{m/n} \qquad m = \pm1,\pm2,\pm3,\cdots \qquad n = 1,3,5,\cdots$$

The following *laws of exponents* apply for all values of μ and ν:

12:5:2 $$x^\mu x^\nu = x^{\mu+\nu} \qquad \frac{x^\mu}{x^\nu} = x^{\mu-\nu} \qquad \text{and} \qquad \left(x^\mu\right)^\nu = x^{\mu\nu}$$

but care is needed to ensure that all quantities remain real.

Useful for large J, the series of powers of the natural numbers has a sum given asymptotically by

12:5:3 $$1+2^\nu+\cdots+J^\nu = \sum_{j=1}^{J} j^\nu \sim \zeta(-\nu) - \sum_{k=0}^{\infty}\frac{(-\nu)_{k-1}\mathrm{B}_k}{k!J^{k-1-\nu}} = \zeta(-\nu) + \frac{J^{1+\nu}}{1+\nu} + \frac{J^\nu}{2} + \frac{\nu J^{\nu-1}}{12} - \frac{\left(\nu^3-3\nu^2+2\nu\right)J^{\nu-3}}{720} + \cdots$$

and involves functions from Chapters 2, 3, 4, and 18. The summation is invalid for $\nu = 0$ or -1, and terminates if

ν is a positive integer. Finite and infinite sums of the related series

12:5:4 $$u^\nu + (1+u)^\nu + (2+u)^\nu + (3+u)^\nu \cdots = \sum_{j=0} (j+u)^\nu$$

and

12:5:5 $$u^\nu + (1+u)^\nu x + (2+u)^\nu x^2 + (3+u)^\nu x^3 \cdots = \sum_{j=0} (j+u)^\nu x^j$$

may, in favorable circumstances, be obtained by exploiting the properties of Hurwitz and Lerch functions [Chapter 64 and Section 64:12].

12:6 EXPANSIONS

The x^ν function may be expanded as a power series either binomially [Section 6:14] in $(x-1)$

12:6:1 $$x^\nu = 1 + \nu(x-1) + \frac{\nu(\nu-1)}{2!}(x-1)^2 + \frac{\nu(\nu-1)(\nu-2)}{3!}(x-1)^3 + \cdots = \sum_{j=0}^{\infty}\binom{\nu}{j}(x-1)^j$$

or via the Euler transformation [Section 10:13], in $(x-1)/x$

12:6:2 $$x^\nu = \frac{1}{x} + (1+\nu)\frac{x-1}{x^2} + \left(1 + \tfrac{3}{2}\nu + \tfrac{1}{2}\nu^2\right)\frac{(x-1)^2}{x^3} + \cdots = \frac{1}{x-1}\sum_{k=1}^{\infty}\left(\frac{x-1}{x}\right)^k \sum_{j=0}^{k-1}\binom{k-1}{j}\binom{\nu}{j}$$

Equation 53:14:4 provides an expansion of x^ν in terms of Bessel functions.

12:7 PARTICULAR VALUES

		$x=-\infty$	$x=-1$	$x=0$	$x=1$	$x=\infty$
irrational	$\nu<0$	undef	undef	$+\infty$	1	0
irrational	$\nu>0$	undef	undef	0	1	$+\infty$
even/odd	$\nu<0$	0	1	$+\infty\|+\infty$	1	0
even/odd	$\nu>0$	$+\infty$	1	0	1	$+\infty$
odd/odd	$\nu<0$	0	-1	$-\infty\|+\infty$	1	0
odd/odd	$\nu>0$	$-\infty$	-1	0	1	$+\infty$
odd/even	$\nu<0$	undef	undef	$\pm\infty$	$+1$	0
odd/even	$\nu>0$	undef	undef	0	±1	$\pm\infty$

In the table, in which the colors are keyed to Figure 12-1, "undef" means that x^ν is not defined as a real function for the argument in question. The particular values arising when $x = e = 2.71828\cdots$ might also be mentioned, for then $x^\nu = \exp(\nu)$ [Chapter 26].

12:8 NUMERICAL VALUES

Most computer languages provide numerical access to noninteger powers via the coding x^v or, occasionally, x**v. Alternatively, the simple algorithm

12:8:1 $$x^\nu = \exp\{\nu \ln(x)\}$$

involving functions from Chapters 26 and 25, may be used when x is positive.

If ν is an integer, *Equator*'s bivariate power function routine (keyword **power**) returns a correctly signed value of x^ν, for either sign of x. Moreover, *Equator* provides values of x^ν for any positive values of x, whatever value ν might have. But when ν is a rational number of the odd/even class, only the positive option of x^ν is returned, this being the principal value [Section 0:0]. When x is negative and ν is an noninteger, *Equator* returns the message "complex" even though, in some cases, there exists a real number that would be an appropriate answer. The example $(-32)^{0.2} = -2$ is such an instance. To obtain a real answer, in a case such as this, type the ν input in the format "integer/integer"; a real answer will be provided by *Equator* whenever one exists.

Equator's complex number raised to a real power routine (keyword **compower**) [Section 10:11] allows either real or complex numbers to be raised to a real power, integer or noninteger, positive or negative. The output is generally a complex number, unlike that of the power function routine. The significance of the output is described in Section 12:11.

Equator's variable construction feature (Appendix, Section C:4) permits the use of a power as the argument of another function.

12:9 LIMITS AND APPROXIMATIONS

The table in Section 12:7 shows the limits approached by x^ν as $x \to 0$, $x \to +\infty$ or $x \to -\infty$.

When x is close to unity and the magnitude of ν is not too large, the linear approximation

12:9:1 $$x^\nu \approx 1 - \nu + \nu x$$

is good, but

12:9:2 $$x^\nu \approx \frac{x+1+\nu(x-1)}{x+1-\nu(x-1)}$$

is better.

For small ν, the approximation

12:9:3 $$x^\nu = \exp\{\nu \ln(x)\} \approx 1 + \nu \ln(x)$$

is useful.

12:10 OPERATIONS OF THE CALCULUS

Differentiation gives

12:10:1 $$\frac{d}{dx} x^\nu = \nu x^{\nu-1}$$

Indefinite integration of the power function requires different limits according to the value of ν:

12:10:2 $$\int_0^x t^\nu \, dt = \frac{x^{\nu+1}}{\nu+1} \qquad \nu > -1 \qquad x > 0$$

12:10:3 $$\int_1^x t^\nu \,\mathrm{d}t = \ln(x) \qquad \nu=-1 \qquad \text{all } x$$

12:10:4 $$\int_x^\infty t^\nu \,\mathrm{d}t = \frac{-x^{1+\nu}}{1+\nu} \qquad \nu<-1 \qquad x>0$$

The reflection property [equation 12:5:1] may be used to adapt 12:10:2 or 12:10:4 to negative x. A general expression for integration of the product of a power function with a linear function raised to a power is

12:10:5 $$\int_0^x t^\nu (bt+c)^\mu \,\mathrm{d}t = \begin{cases} \dfrac{c^{1+\nu+\mu}}{b^{1+\nu}} \mathrm{B}\left(1+\nu, -1-\nu-\mu, \dfrac{bx}{bx+c}\right) & b>0,\ 0<bx<c \\ \dfrac{c^{1+\nu+\mu}}{(-b)^{1+\nu}} \mathrm{B}\left(1+\nu, 1+\mu, \dfrac{-bx}{c}\right) & b<0,\ 0<-bx<c \end{cases} \quad \left.\begin{matrix} c>0 \\ \nu>-1 \end{matrix}\right\}$$

the integral being in terms of the trivariate incomplete beta function [Chapter 58].

Formulas for semidifferentiation and semiintegration of the power function, with lower limit zero, involve the gamma function [Chapter 43]:

12:10:6 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+\nu)} x^{\nu-1/2} \qquad \nu>-1,\ \ x>0$$

12:10:7 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(\frac{3}{2}+\nu)} x^{\nu+1/2} \qquad \nu>-1,\ \ x>0$$

These are just the $\mu = \pm\frac{1}{2}$ cases of the general rule for differintegration [Section 12:14] of a power function:

12:10:8 $$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1-\mu+\nu)} x^{\nu-\mu} \qquad \nu>-1,\ \ x>0$$

and equations 12:10:1 and 12:10:2 display the $\mu = \pm 1$ instances. The corresponding formula for differintegration with a lower limit of $-\infty$ is

12:10:9 $$\left.\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} x^\nu \right|_{-\infty} = \frac{\Gamma(\mu-\nu)}{\Gamma(-\nu)} (-x)^{\nu-\mu} \qquad \nu<\mu,\ \ x<0$$

The transformation

12:10:10 $$\int_0^\infty \mathrm{f}(t) t^{\nu-1} \mathrm{d}t$$

creates a possibly-complex-valued function of the ν variable, known as the *Mellin transform* (Robert Hjalmar Mellin, Finnish mathematician, 1854 – 1933) of the function f(t). A tabulation of over 250 Mellin transforms is given by Erdélyi, Magnus, Oberhettinger, and Tricomi [*Tables of Integral Transforms*, Volume 1, Chapter 6], together with some general formulas and a listing of inverse Mellin transforms. Thus the following definite integrals may be regarded as Mellin transforms, as may that in 12:10:16.

12:10:11 $$\int_0^\infty \frac{t^\nu}{(bt+c)^\mu} \mathrm{d}t = \frac{c^{1+\nu-\mu}}{b^{1+\nu}} \mathrm{B}(1+\nu,\ \mu-\nu-1) \qquad -1<\nu<\mu-1$$

12:10:12 $$\int_0^\infty \frac{t^\nu}{t^2+a^2} \mathrm{d}t = \frac{\pi a^{\nu-1}}{2} \sec\left(\frac{\nu\pi}{2}\right) \qquad -1<\nu<1$$

12:10:13 $$\int_0^\infty t^v \ln(1+a^2t)\mathrm{d}t = \frac{-\pi}{(1+v)a^{2+2v}}\csc(v\pi) \qquad -2<v<-1$$

12:10:14 $$\int_0^\infty t^v \sin(\omega t)\mathrm{d}t = \frac{\Gamma(1+v)}{\omega^{1+v}}\cos\left(\frac{v\pi}{2}\right) \qquad -2<v<0 \qquad v\neq -1$$

12:10:15 $$\int_0^\infty t^v \arctan(t)\,\mathrm{d}t = \frac{\pi}{2(1+v)}\csc\left(\frac{v\pi}{2}\right) \qquad -2<v<-1$$

The Γ function in 12:10:14 and the B function in 12:10:11 are the complete gamma function [Chapter 43] and the (bivariate) complete beta function [Section 43:13]. Note that the domain of v required to validate these transforms is often severely restricted.

The following Laplace transforms involve the complete [Chapter 43] and incomplete [Chapter 45] gamma functions

12:10:16 $$\int_0^\infty t^v \exp(-st)\mathrm{d}t = \mathfrak{L}\{t^v\} = \frac{\Gamma(v+1)}{s^{v+1}} \qquad v>-1$$

12:10:17 $$\int_0^\infty (bt+c)^v \exp(-st)\,\mathrm{d}t = \mathfrak{L}\{(bt+c)^v\} = \frac{b^v}{s^{v+1}}\exp\left(\frac{cs}{b}\right)\Gamma\left(v+1,\frac{cs}{b}\right) \qquad v>-1$$

12:11 COMPLEX ARGUMENT

If z is complex and v real, different portions of the chain of equalities

12:11:1 $$(x+iy)^v = z^v = [\rho\exp(i\theta)]^v = \rho^v\exp(iv\theta) = \rho^v[\cos(\theta)+i\sin(\theta)]^v = \rho^v[\cos(v\theta)+i\sin(v\theta)]$$

are identified by different authorities as *de Moivre's theorem* (Abraham de Moivre, French mathematician, 1667 – 1754). Here ρ is the modulus $|z|$ of the complex variable, equal to $\sqrt{x^2+y^2}$, and θ is its phase, equal to $\arctan(y/x)$ when x is positive and to $\pi + \arctan(y/x)$ when x is negative. In implementing this formula, one must recognize that θ may be augmented by $2k\pi$, where k is any integer, without changing its import.

If v is real and equal to m/n, the complex power z^v may be represented as $\left(z^{1/n}\right)^m$, that is, as the complex variable $z^{1/n}$ raised to an integer power, as in Section 10:11. Hence, in the case of a rational v, ascribing significance to z^v devolves into identifying the properties of $z^{1/n}$, the so-called *nth root* of the complex z. By de Moivre's theorem:

12:11:2 $$z^{1/n} = \left|\rho^{1/n}\right|\left[\cos\left(\frac{\theta}{n}\right)+i\sin\left(\frac{\theta}{n}\right)\right]$$

To illustrate this formula, consider the $n=3$ case when the complex variable z has real and imaginary parts of 0.94 and 1.29 respectively. Its modulus and phase are $\rho = \sqrt{(0.94)^2+(1.29)^2} = 1.60$ and $\theta = \arctan(1.29/0.94) = 0.941 = 54^\circ$. Thus, in Figure 12-2, the point labeled Z represents z. One calculates $|\rho^{1/3}| = 1.17$ and $\theta/3 = 18^\circ$. The following possibilities must be considered:

12:11:3 $$z^{1/3} = 1.17\left[\cos\left(18^\circ+\frac{k}{3}360^\circ\right)+i\sin\left(18^\circ+\frac{k}{3}360^\circ\right)\right] \qquad k=0,\pm1,\pm2,\cdots$$

but it turns out that, whatever value is chosen for k, there are only three distinct answers, that we may take to arise from the choices $k = 0, \pm1$. They are:

12:11:4 $$z^{1/3}=\begin{cases}1.17\left[\cos\left(18^\circ\right)+i\sin\left(18^\circ\right)\right]\\ \qquad =1.11+0.36i\\ 1.17\left[\cos\left(18^\circ+120^\circ\right)+i\sin\left(18^\circ+120^\circ\right)\right]\\ \qquad =-0.87+0.78i\\ 1.17\left[\cos\left(18^\circ-120^\circ\right)+i\sin\left(18^\circ-120^\circ\right)\right]\\ \qquad =-0.24-1.14i\end{cases}$$

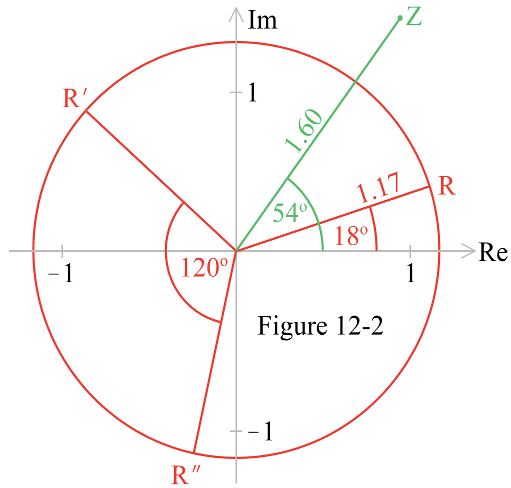


Figure 12-2

and they are represented on the diagram by the points R, R′, and R″, which are equally spaced around a circle of radius 1.17.

It is generally true that there are n complex roots, not only of $z^{1/n}$ but also of $z^{m/n}$. Two or one of these will be real if z is real, depending on whether n is even or odd. When ν is irrational, the range of k must be restricted so that $-\pi<\theta+2k\pi/\nu\le\pi$; the number of roots will lie between $(2\pi/\nu)-1$ and $(2\pi/\nu)+1$. *Equator*'s complex number raised to a real power routine (keyword **compower**) may be used to evaluate the real and imaginary parts of $(x+iy)^\nu$; however, it provides only one such pair of numbers, the principal value, defined as $\exp\{\nu\ln(x)\}$ [Section 25:11]. Note that, when the input to this routine has $y=0$, the principal value of x^ν is returned by *Equator*. However, this is not always what might be expected. For example, $(-8)^{1/3}$ is not -2, but $1+\sqrt{3}\,i$.

Some important inverse Laplace transforms involving the arbitrary power function are

12:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{s^\nu\right\}=\frac{1}{\Gamma(-\nu)t^{\nu+1}}\qquad \nu<0$$

12:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} (s+a)^\nu\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{(s+a)^\nu\right\}=\frac{\exp(-at)}{\Gamma(-\nu)t^{\nu+1}}\qquad \nu<0$$

12:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \left(\sqrt{s}+\sqrt{a}\right)^\nu\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{(\sqrt{s}+\sqrt{a})^\nu\right\}=\frac{-\nu\exp(at/2)}{\sqrt{2^{1+\nu}\pi t^{2+\nu}}}\mathrm{D}_{\nu-1}\left(\sqrt{2at}\right)\qquad \nu<0$$

12:11:8 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu\exp(-as)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{s^\nu\exp(-as)\right\}=\frac{\mathrm{u}(t-a)}{\Gamma(-\nu)(t-a)^{\nu+1}}\qquad \nu<0$$

12:11:9 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu\exp\left(-a\sqrt{s}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{s^\nu\exp\left(-a\sqrt{s}\right)\right\}=\frac{\exp(-a^2/8t)}{\sqrt{2^{2\nu+1}\pi t^{\nu+1}}}\mathrm{D}_{2\nu-1}\left(\frac{a}{\sqrt{2t}}\right)\qquad \nu<0$$

12:11:10 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu\exp(-a/s)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{s^\nu\exp(-a/s)\right\}=\left(\frac{a}{t}\right)^{(\nu+1)/2}\mathrm{J}_{-\nu-1}\left(2\sqrt{at}\right)\qquad \nu<0$$

12:11:11 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu\exp(a/s)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{s^\nu\exp(a/s)\right\}=\left(\frac{a}{t}\right)^{(\nu+1)/2}\mathrm{I}_{-\nu-1}\left(2\sqrt{at}\right)\qquad \nu<0$$

Among the functions appearing in these inverse transforms are the parabolic cylinder function D_ν of Chapter 46, the Bessel function J_ν [Chapter 53], and the modified Bessel function I_ν [Chapter 50]. Some important special cases of 12:11:5 are summarized in the following panel, in which g represents Gauss's constant.

$\dfrac{1}{s^{1/4}}$	$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{s^{3/4}}$	$\dfrac{1}{s}$	$\dfrac{1}{\sqrt{s^3}}$	$\dfrac{1}{s^2}$	$\dfrac{1}{s^{5/2}}$	$\dfrac{1}{s^3}$	$\dfrac{1}{s^{7/2}}$	$\dfrac{1}{s^n}$	$\dfrac{1}{s^{n+\frac{1}{2}}}$
$\dfrac{1}{\sqrt{g}\,(2\pi t)^{3/4}}$	$\dfrac{1}{\sqrt{\pi t}}$	$\sqrt{g}\left(\dfrac{2}{\pi t}\right)^{1/4}$	1	$2\sqrt{\dfrac{t}{\pi}}$	t	$\dfrac{4}{3}\sqrt{\dfrac{t^3}{\pi}}$	$\dfrac{t^2}{2}$	$\dfrac{8t^{5/2}}{15\sqrt{\pi}}$	$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{4^n n!\,t^{n-\frac{1}{2}}}{(2n)!\sqrt{\pi}}$

12:12 GENERALIZATIONS

The functions $(bx+c)^\nu$ and $(ax^2+bx+c)^\nu$ generalize x^ν and their properties may sometimes be deduced by appropriate substitutions. The important $\nu = \pm½$ instances of these two generalizations are discussed in Chapter 11 and Section 15:15.

12:13 COGNATE FUNCTIONS

There is a multitude of functions of the forms $\mathrm{f}(x^\nu)$ and $[\mathrm{f}(x)]^\nu$, involving arbitrary powers, though the $\nu = \pm½$ instances are the most important noninteger cases. The $(a^2-x^2)^{\pm½}$ and $(x^2\pm a^2)^{\pm½}$ functions are the subjects of the next two chapters.

12:14 RELATED TOPIC: the fractional calculus

It is conventional to represent the operations of double, triple, and n-fold differentiation of the function $\mathrm{f}(x)$ by the notation

12:14:1
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) \qquad \frac{\mathrm{d}^3}{\mathrm{d}x^3}\mathrm{f}(x) \qquad \text{and} \qquad \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{f}(x)$$

Inasmuch as integration and differentiation are, to an extent, inverse processes, it is not unreasonable to use a symbolism similar to those in 12:14:1, but with negative superscripts, to represent single and multiple integrations. Thus one may define

12:14:2
$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\mathrm{f}(x) \equiv \int_0^x \mathrm{f}(x')\,\mathrm{d}x' \qquad \frac{\mathrm{d}^{-2}}{\mathrm{d}x^{-2}}\mathrm{f}(x) \equiv \int_0^x\!\int_0^{x'} \mathrm{f}(x'')\,\mathrm{d}x''\,\mathrm{d}x' \qquad \text{etc.}$$

In this way, we have created a unified notation

12:14:3
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{f}(x)$$

that encompasses repeated differentiation when the order μ is $2, 3, \cdots, n, \cdots$, and single or multiple integration when $\mu = -1, -2, \cdots, -n, \cdots$. Also, of course, the $\mu = 1$ and $\mu = 0$ versions can represent single differentiation and $\mathrm{f}(x)$ itself, so that all integer values are covered. The term *differintegration* is used to describe the hybrid operation: *differintegral* describes the resulting function.

The mission of the fractional calculus is to extend the meaning of 12:14:3 to include noninteger values of μ and find utility for the resulting operation. There are several ways to define a differintegral so that reduction occurs to established definitions of differentiations and integration when μ is an integer. Probably the most general of these is the limit definition

12:14:4
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{f}(x) \equiv \lim_{J\to\infty}\left\{\left(\frac{J}{x}\right)^\mu \sum_{j=0}^{J-1}\frac{(-\mu)_j}{j!}\mathrm{f}\left(\frac{J-j}{J}x\right)\right\}$$

due to Grünwald and written here in terms of the Pochhammer polynomial [Chapter 18]. Of more utility, however, is the definition as an integral transform, originating in the work of Riemann and Liouville (Joseph Liouville, French mathematician, 1809 – 1882):

12:14:5
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{f}(x) \equiv \frac{1}{\Gamma(-\mu)}\int_0^x \frac{f(x')}{(x-x')^{1+\mu}}\mathrm{d}x' \qquad \mu<0$$

which, however, applies only to negative orders of differintegration. Here Γ is the gamma function [Chapter 43]. Extension to positive orders relies on classical differentiation through the formula

12:14:6
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{f}(x) \equiv \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{\frac{\mathrm{d}^{\mu-n}}{\mathrm{d}x^{\mu-n}}\mathrm{f}(x)\right\} \qquad n>\mu>0$$

where n is any integer greater than μ, so that the embraced quantity in 12:14:6 has a negative order and can be defined by 12:14:5. For other definitions, see Oldham and Spanier [Chapter 3].

In formulating the equations above, a lower limit of zero was assumed and this is the most usual choice. Other alternatives may be selected, however. It may seem counterintuitive to refer to a lower limit in the context of differentiation but, in fact, a lower limit must be specified for *all* instances of differintegration, except when $\mu = 0, 1, 2, 3, \cdots$. The most general case has an arbitrary lower limit, say a. An appropriate notation, and the corresponding *Riemann-Liouville definitions* are

12:14:7
$$\left.\frac{\mathrm{d}^\mu}{\mathrm{d}t^\mu}\mathrm{f}(t)\right|_a^x \equiv \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_a^x \frac{\mathrm{f}(t)}{(x-t)^{1+\mu}}\mathrm{d}t & \mu<0 \\ \dfrac{1}{\Gamma(n-\mu)}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\displaystyle\int_a^x \frac{\mathrm{f}(t)}{(x-t)^{1+\mu-n}}\mathrm{d}t & n>\mu>0 \end{cases}$$

A popular choice of lower limit is $-\infty$ and these instances, termed *Weyl differintegrals*, are the subject of Section 64:14.

There are many practical applications of the fractional calculus [Hilfer]. In some fields, μ is an adjustable parameter; in others it is fixed. In the latter case, the most important values of μ are $\pm\frac{1}{2}$, these operations being known as *semidifferentiation* or *semiintegration*, and collectively as *semidifferintegration*. Frequently in the *Atlas*, examples of zero-lower-limit semidifferintegrals are included in Section 10 of the chapter devoted to the target function, and sometime Weyl differintegrals and more general results are listed too [as in Section 10 of the present chapter]. In Section 43:14, it is explained how the fractional calculus serves to provide a facile method of "synthesizing" one function from another. The important *composition rule*

12:14:8
$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left\{\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{f}(x)\right\} = \frac{\mathrm{d}^{\nu+\mu}}{\mathrm{d}x^{\nu+\mu}}\mathrm{f}(x)$$

of which 12:14:6 is an instance, plays the crucial role in these syntheses. There are some exceptions to the composition "rule" (for example when $\mathrm{f}(x) = x^2$, $\mu = 3$ and $\nu = -2$), but it applies in most instances.

CHAPTER 13

THE SEMIELLIPTIC FUNCTION $(b/a)\sqrt{a^2-x^2}$ AND ITS RECIPROCAL

The $(b/a)\sqrt{a^2-x^2}$ function is closely associated with the geometry of the ellipse, which is addressed in Section 13:14. The semicircle corresponds to the special $b = a$ instance and its geometry is the subject of Section 13:15. Whenever the b/a multiplier is of no importance, as in Sections 13:6, 13:10, and 13:11, it is omitted.

13:1 NOTATION

With $|x| < |a|$, a cartesian graph of the function pair $(\pm b/a)\sqrt{a^2-x^2}$ versus x is an ellipse and therefore *semielliptic function* is an appropriate name for $(b/a)\sqrt{a^2-x^2}$, with *semicircular function* being apposite for $\sqrt{a^2-x^2}$. Following the convention explained in Section 11:1, the notation $(b/a)[a^2-x^2]^{1/2}$ is equivalent to $(\pm b/a)\sqrt{a^2-x^2}$.

The parameters a and b, both positive, are the *semiaxes*, the larger being the *major semiaxis* (or "semimajor axis") and the smaller the *minor semiaxis*. Primarily, concern will be for the case in which $a \geq b$ and the ellipse to which this relates will be termed a *horizontal ellipse*, on account of the orientation of its major axis when the function is graphed. Conversely, when $b > a$, we speak of a *vertical ellipse*. The ratio, b/a, of the semiaxes of a horizontal ellipse is represented by k' in discussions of the elliptic family of functions [Chapters 61−63]. The quantity known as the *eccentricity*, or sometimes as the *ellipticity*, is

13:1:1
$$k = \sqrt{1-(k')^2} = \frac{\sqrt{a^2-b^2}}{a}$$

Eccentricities lie in the range $0 \leq k < 1$ for horizontal ellipses and for the functions that describe them. Zero eccentricity corresponds to a semicircle. Eccentricities of unity or more correspond to other functions and the entire class – the conic sections – is addressed in Section 15:15.

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13:2 BEHAVIOR

The functions $(b/a)\sqrt{a^2-x^2}$ and $a/\left(b\sqrt{a^2-x^2}\right)$ acquire real values only in the domain $-a \le x \le a$ of their argument. The range of the semielliptic function is between zero and b, whereas its reciprocal lies between $1/b$ and infinity. Figure 13-1 shows one graph of the reciprocal $a/\left(b\sqrt{a^2-x^2}\right)$ and two graphs of $(b/a)\sqrt{a^2-x^2}$, one of the latter corresponding to a vertical semiellipse with b greater than a, and the other to the more canonical horizontal semiellipse with b less than a.

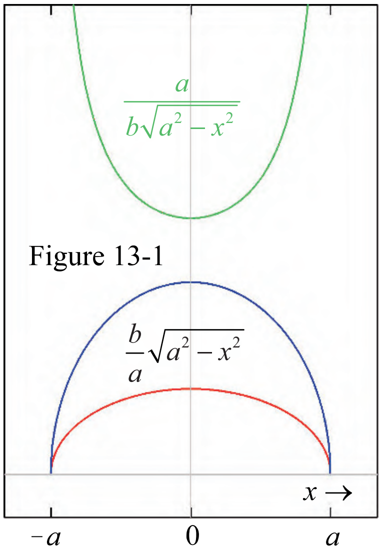


Figure 13-1

13:3 DEFINITIONS

The algebraic operations of squaring [Chapter 10] and taking the square root [Chapter 11], together with arithmetic operations, fully define the semielliptic function and its reciprocal.

Multiplying two related square-root functions [Chapter 11] is another route to the definition of a semielliptic function:

13:3:1 $$\sqrt{\frac{b}{a}x+b}\sqrt{\frac{-b}{a}x+b}=\frac{b}{a}\sqrt{a^2-x^2}$$

A parametric definition of the $f(x)=f=(\pm b/a)\sqrt{a^2-x^2}$ function pair is in terms of two trigonometric functions [Chapter 32]:

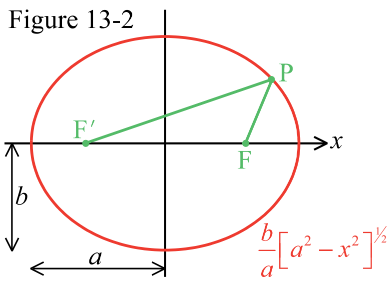
13:3:2 $$f=b\sin(t), \quad x=a\cos(t)$$

An ellipse may be defined geometrically in two distinct ways. One of these is explained in Section 15:15; the other is illustrated in Figure 13-2. The ellipse is defined as the locus of all points P such that the sum of the distances from P to two fixed points F and F′ obeys the simple relationship

13:3:3 $$\mathrm{PF}+\mathrm{PF}'=\text{a constant}$$

Each of the fixed points is termed a *focus* of the ellipse. The *interfocal separation*, the distance FF′ between the two foci, must, of course, be less than the constant in 13:3:3. If both foci lie on the x-axis, equidistant from the origin, then the ellipse is our standard horizontal ellipse, the foci have the coordinates $\left(\pm\sqrt{a^2-b^2},0\right)$, the interfocal separation is $2ka$, and the constant in equation 13:3:3 is $2a$.

Figure 13-2



13:4 SPECIAL CASES

When $b=a$, the semielliptical function becomes the semicircular function $\sqrt{a^2-x^2}$ and the common value of the two semiaxes is known as the *radius*. The geometrical definition of a circle is as the locus of all points lying at a constant distance a from a fixed point, the *center* of the circle. Other geometric properties of the semicircle are described in Section 13:15.

As $b \to 0$, the ellipse degenerates towards a straight line segment of length $2a$.

13:5 INTRARELATIONSHIPS

The $\mathrm{f}(x)=(b/a)\sqrt{a^2-x^2}$ function is an even function, f($-x$) = f(x), as is its reciprocal. The formula

13:5:1 $$\mathrm{f}(vx)=\frac{b}{(v/a)}\sqrt{\left(\frac{a}{v}\right)^2-x^2}$$

shows that the multiplication of the argument by a constant creates another semielliptic function, one semiaxis being rescaled, the other remaining unchanged. Multiplying the argument by $v = a/b$ creates a semicircle of radius b.

The inverse function [Section 0:3] of the horizontal $(b/a)\sqrt{a^2-x^2}$ function is the vertical $(a/b)\sqrt{b^2-x^2}$ function. These two semielliptic functions are sometimes said to be *conjugates* of each other.

13:6 EXPANSIONS

A binomial expansion of the semicircular function

13:6:1 $$\sqrt{a^2-x^2}=a\left[1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}-\frac{x^6}{16a^6}-\frac{5x^8}{128a^8}-\frac{7x^{10}}{256a^{10}}-\cdots\right]=a\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\left(\frac{-x^2}{a^2}\right)^j=a\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^j$$

is valid provided $-a\le x\le a$. Each coefficient in the series is expressible either as a binomial coefficient [Chapter 6] or as a ratio of Pochhammer polynomials [Chapter 18], or indeed in several other ways. The similar expansion of the reciprocal semicircular function

13:6:2 $$\frac{1}{\sqrt{a^2-x^2}}=\frac{1}{a}\left[1+\frac{x^2}{2a^2}+\frac{3x^4}{8a^4}+\frac{5x^6}{16a^6}+\frac{35x^8}{128a^8}+\frac{63x^8}{256a^8}+\cdots\right]=\frac{1}{a}\sum_{j=0}^{\infty}\binom{\frac{-1}{2}}{j}\left(\frac{-x^2}{a^2}\right)^j=\frac{1}{a}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^j$$

is restricted to $-a<x<a$.

Trigonometric substitution creates the series [see equations 32:6:2 and 33:6:2]

13:6:3 $$\sqrt{a^2-x^2}=a\sin(\theta)=a\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right]$$

13:6:4 $$\frac{1}{\sqrt{a^2-x^2}}=\frac{\csc(\theta)}{a}=\frac{1}{a}\left[\frac{1}{\theta}+\frac{\theta}{6}+\frac{7\theta^3}{360}+\frac{31\theta^5}{15120}+\cdots\right]$$

$$\theta=\arccos\left(\frac{x}{a}\right)$$

See Section 21:15 for expansion of the semielliptic function and its reciprocal in terms of Legendre polynomials.

13:7 PARTICULAR VALUES

The a and b parameters are positive in the following table:

	$x=-a$	$x=0$	$x=\sqrt{a^2-b^2}$	$x=a$
$\frac{b}{a}\sqrt{a^2-x^2}$	0	b	$\frac{b^2}{a}$	0
$\frac{a}{b\sqrt{a^2-x^2}}$	∞	$\frac{1}{b}$	$\frac{a}{b^2}$	∞

13:8 NUMERICAL VALUES

It is straightforward to calculate values of the semielliptic function and its reciprocal. For example, *Equator*'s power function routine (keyword **power**) may be used with $v=\pm 0.5$ after first using the variable construction feature [Appendix, Section C:4] with $k=b^2$, $w=-b^2/a^2$, and $p=2$ to adjust the argument.

13:9 LIMITS AND APPROXIMATIONS

As x approaches a, the semielliptic function comes to approximate a square-root function,

13:9:1 $$\frac{b}{a}\sqrt{a^2-x^2}\approx b\sqrt{\frac{2(a-x)}{a}} \qquad x\to a>0$$

The limiting form as x approaches $-a$ is $b\sqrt{2(a+x)/a}$.

13:10 OPERATIONS OF THE CALCULUS

Throughout this section, the argument x is restricted to a range between $-a$ and a, a being positive. The following formulas describe differentiation and indefinite integration:

13:10:1 $$\frac{d}{dx}\sqrt{a^2-x^2}=\frac{-x}{\sqrt{a^2-x^2}}$$

13:10:2 $$\frac{d}{dx}\frac{1}{\sqrt{a^2-x^2}}=\frac{x}{\sqrt{\left(a^2-x^2\right)^3}}$$

13:10:3 $$\int_0^x\sqrt{a^2-t^2}\,dt=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)$$

13:10:4 $$\int_0^x\frac{dt}{\sqrt{a^2-t^2}}=\arcsin\left(\frac{x}{a}\right)$$

13:10:5 $$\int_0^x t\sqrt{a^2-t^2}\,dt=\frac{a^3-\sqrt{(a^2-x^2)^3}}{3}$$

and

13:10:6 $$\int_x^a\frac{1}{t\sqrt{a^2-t^2}}\,dt=\frac{1}{a}\operatorname{arsech}\left(\frac{x}{a}\right) \qquad 0<x<a$$

The last two integrals are simple examples of a general class of indefinite integral $\int t^n\left(\sqrt{a^2-t^2}\right)^m dt$, where n is any integer and m is an odd integer. Such integrals evaluate to algebraic expressions, or may contain an arcsin [Chapter 35] or arsech [Chapter 31] term. Gradshteyn and Ryzhik [Section 2.27] list more than fifty such integrals, including some general formulas.

Among other important integrals are

13:10:7
$$\int_0^x \frac{\mathrm{d}t}{\sqrt{a^2-t^2}\sqrt{a^2-k^2t^2}} = \frac{1}{a}\mathrm{F}\left\{k,\ \arcsin\left(\frac{x}{a}\right)\right\}$$

and

13:10:8
$$\int_0^x \frac{\sqrt{a^2-k^2t^2}}{\sqrt{a^2-t^2}}\mathrm{d}t = a\,\mathrm{E}\left\{k,\ \arcsin\left(\frac{x}{a}\right)\right\}$$

which serve as definitions of the incomplete elliptic integrals [Chapter 62] of the first and second kinds.

Semidifferentiation or semiintegration [Section 12:14], with lower limit zero, leads to:

13:10:9
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sqrt{a^2-x^2} = \sqrt{\frac{a}{2\pi}}\left[2\mathrm{E}(k,\varphi) - \mathrm{F}(k,\varphi) - \frac{2x-a}{\sqrt{2ax}}\right]$$

13:10:10
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\frac{1}{\sqrt{a^2-x^2}} = \frac{\sqrt{a/2\pi}}{a+x}\left[\frac{2\mathrm{E}(k,\varphi)}{a-x} - \frac{\mathrm{F}(k,\varphi)}{a} + \sqrt{\frac{2}{ax}}\right]$$

13:10:11
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\sqrt{a^2-x^2} = \sqrt{\frac{8a}{\pi}}\frac{x}{3}\left[2\mathrm{E}(k,\varphi) + \left(\frac{a}{x}-1\right)\mathrm{F}(k,\varphi) - \frac{2x-a}{\sqrt{2ax}}\right]$$

13:10:12
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\frac{1}{\sqrt{a^2-x^2}} = \sqrt{\frac{2}{\pi a}}\mathrm{F}(k,\varphi)$$

$$0<x<a \qquad k=\sqrt{\frac{a+x}{2a}} \qquad \varphi = \arcsin\left(\sqrt{\frac{2x}{a+x}}\right)$$

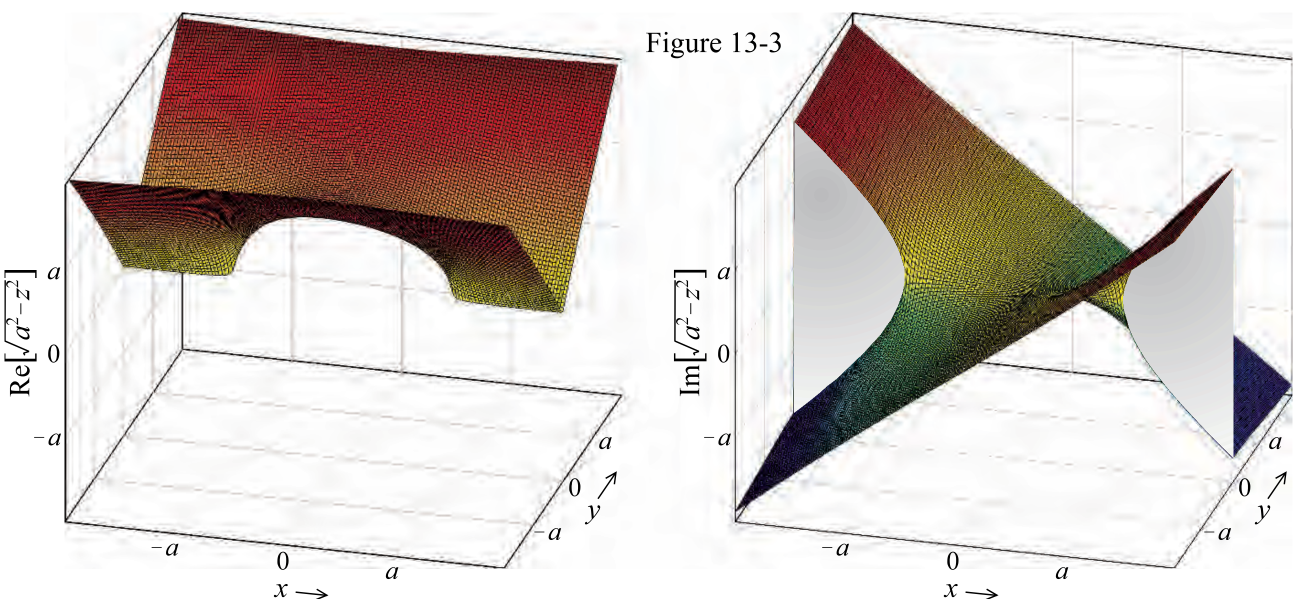
13:11 COMPLEX ARGUMENT

Figure 13-3 shows the real and imaginary components of the semicircular function of complex argument:

13:11:1
$$\sqrt{a^2-(x+iy)^2} = \sqrt{\frac{A+a^2-x^2+y^2}{2}} - i\,\mathrm{sgn}(xy)\sqrt{\frac{A-a^2+x^2-y^2}{2}}$$

where $A=\sqrt{(a^2+x^2+y^2)^2-4a^2x^2}$ and sgn is the signum function [Chapter 8].

Figure 13-3

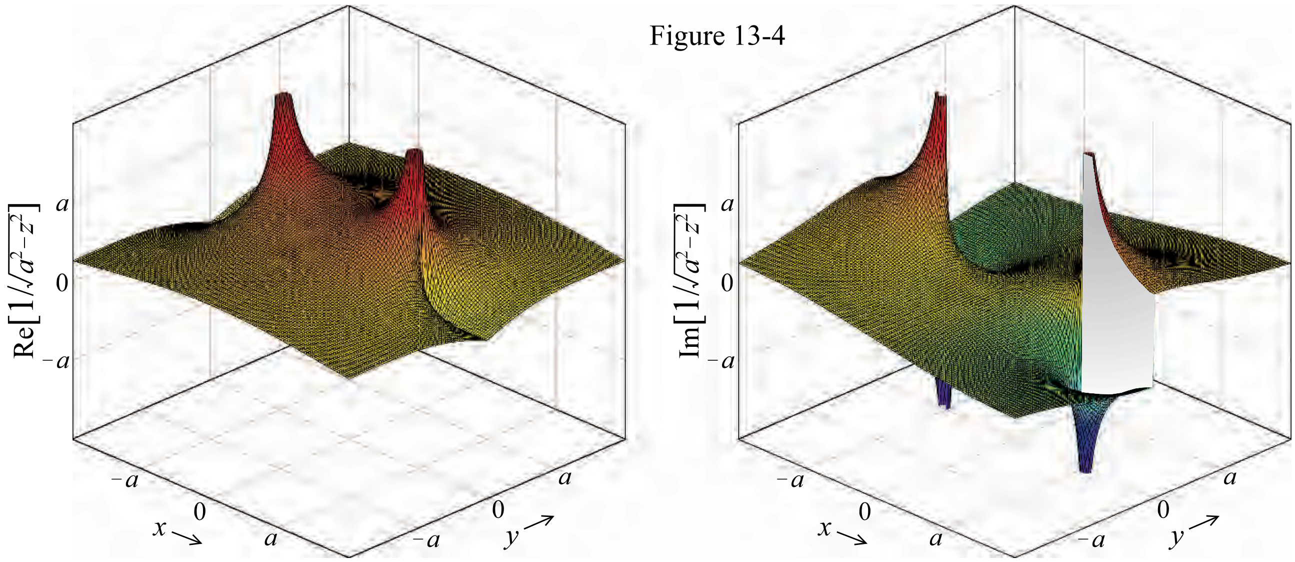


The corresponding parts of the reciprocal semicircular function are

13:11:2 $$\frac{1}{\sqrt{a^2-(x+iy)^2}}=\sqrt{\frac{A+a^2-x^2+y^2}{2A^2}}+i\,\mathrm{sgn}(xy)\sqrt{\frac{A-a^2+x^2-y^2}{2A^2}}$$

as illustrated in Figure 13-4. Note the poles on the real axis at $x=\pm a$.

Figure 13-4



If the argument is purely imaginary, the semielliptic function becomes real

13:11:3 $$(b/a)\sqrt{a^2-(iy)^2}=(b/a)\sqrt{y^2+a^2}$$

and corresponds to a vertical semihyperbolic function, as described in the following chapter.

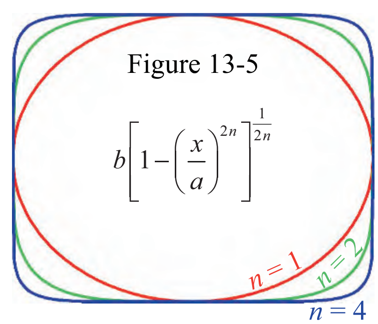
13:12 GENERALIZATIONS

The *root-quadratic function* $\sqrt{ax^2+bx+c}$ is a generalization of the semielliptic function. See Section 15:15 for the conditions under which the root-quadratic function becomes semielliptic.

The elliptic function may be regarded as the special $n=1$ case of the more general function

13:12:1 $$b\left[1-\left(\frac{x}{a}\right)^{2n}\right]^{\frac{1}{2n}} \qquad n=1,2,3,\cdots$$

The curves obtained by plotting these functions, the n = 1, 2, and 4 cases of which are included in Figure 13-5, have been called *superellipses*. As $n\to\infty$ the curve approaches a rectangle.



13:13 COGNATE FUNCTIONS

The functions of Chapter 14 are related to the semielliptic function in essentially the same way that the functions of Chapters 28−31 are related to those of Chapters 32−35. That is, members of one group of functions can be obtained from the other by replacement of x by ix, with perhaps a minor adjustment, such as a sign change. The word "modified" is used when this replacement is applied to Bessel functions [Chapters 52−54] to generate the functions of Chapters 49−51 and, in that sense, the functions of Chapter 14 are "modified semielliptic functions".

Like the standard semielliptic function pair $(\pm b/a)\sqrt{a^2-x^2}$, the function pair

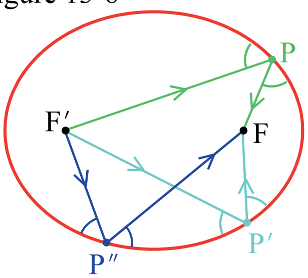
13:13:1 $$\frac{(a^2-b^2)x \pm ab\sqrt{2a^2+2b^2-4x^2}}{a^2+b^2}$$

describes an ellipse, centered at the origin, the major and minor semiaxes being of lengths a and b respectively. This ellipse, however, is oriented so that its major axis is at 45° to the x-axis. Note that it may be resolved into a linear function $(a^2-b^2)x/(a^2+b^2)$ plus a standard horizontal ellipse with semiaxes of lengths $\sqrt{(a^2+b^2)/2}$ and $\sqrt{2/(a^2+b^2)}$. A similar resolution is possible for an ellipse located anywhere in the cartesian plane and with any orientation.

13:14 RELATED TOPIC: geometric properties of the ellipse

The two foci, F′ and F, of the horizontal ellipse $(b/a)[a^2-x^2]^{\frac{1}{2}}$ are located on the x-axis at $x=\pm\sqrt{a^2-b^2}$. If P is any point on the ellipse, the lines PF′ and PF make equal angles with the ellipse, as Figure 13-6 indicates. This means that any wave motion radiating from point F′ will be reflected from the ellipse and arrive at point F from a variety of directions. The radiation is said to have been "focused" at F. As explained in Section 13-3, the path lengths via points P, P′ and P″ are all equal (to $2a$), and so the journey times are also equal and the radiation arrives in synchrony. This property of the ellipse is maintained in its three-dimensional counterpart, the *ellipsoid*, and is exploited in furnace design and in several acoustic and optical devices.

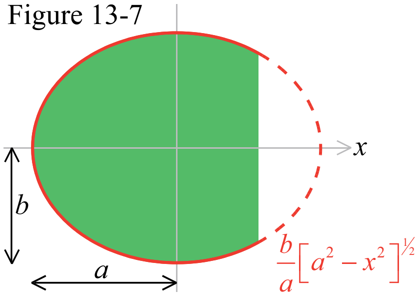
Figure 13-6



With a and b positive, the total area of an ellipse is πab. The area of the segment of an ellipse defined by the abscissal range $-a$ to x, and shown shaded in Figure 13-7 is

13:14:1 $$\begin{matrix}\text{green}\\ \text{area}\end{matrix} = \frac{2b}{a}\int_{-a}^{x}\sqrt{a^2-t^2}\,\mathrm{d}t = ab\left[\frac{x}{a^2}\sqrt{a^2-x^2}+\arcsin\left(\frac{x}{a}\right)+\frac{\pi}{2}\right]$$

Figure 13-7



The length of the curved portion of the boundary of the shaded region is

13:14:2 $$2\int_{-a}^{x}\sqrt{1+\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{b}{a}\sqrt{a^2-t^2}\right)^2}\,\mathrm{d}t = 2a\left[\mathrm{E}(k)+\mathrm{E}\left(k,\arcsin\left(\frac{x}{a}\right)\right)\right]$$

where k is the eccentricity of the ellipse, $\sqrt{1-(b/a)^2}$. The entire perimeter of the ellipse has a length of $4a\mathrm{E}(k)$. $\mathrm{E}(k)$ denotes the complete elliptic integral of the second kind of modulus k and $\mathrm{E}(k,\varphi)$ denotes the incomplete elliptic integral of the second kind of modulus k and amplitude φ. These functions are addressed in Chapters 61 and 62 respectively.

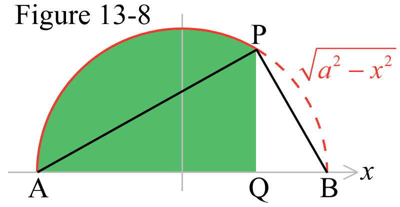
Together with the parabola [Chapter 11] and the hyperbola [Chapter 14], the ellipse and the circle constitute *curves of second degree*, also known as *conic sections*; their shared properties are the subject of Section 15:15.

13:15 RELATED TOPIC: geometry of the semicircle

Formula 13:10:3 shows that the area, shaded in Figure 13-8, of a segment of a semicircle is

13:15:1 $$\frac{a^2}{2}\left[\frac{\pi}{2}+\arcsin\left(\frac{x}{a}\right)\right]+\frac{x}{2}\sqrt{a^2-x^2}$$

Figure 13-8



irrespective of the sign of x. The total area of the semicircle is $\pi a^2/2$. The length of the curved boundary of the shaded region is

13:15:2 $$a\left[\frac{\pi}{2}+\arcsin\left(\frac{x}{a}\right)\right]$$

the total semicircumference being of length πa.

An important property of a semicircle is that the angle APB in Figure 13-8 is a right angle, for any point P on the perimeter. Hence the triangles APB, AQP and PQB are similar and right-angled, permitting pythagorean and trigonometric relationships [Chapters 32−34] to be applied.

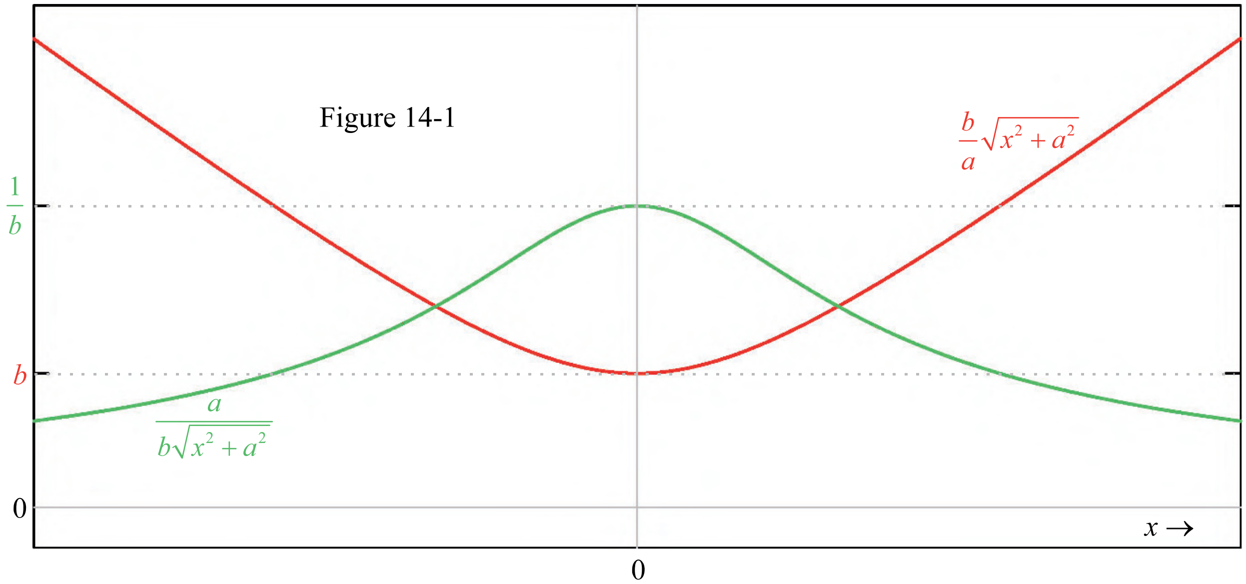
CHAPTER 14

THE SEMIHYPERBOLIC FUNCTIONS $(b/a)\sqrt{x^2 \pm a^2}$ AND THEIR RECIPROCALS

The functions of this chapter are closely associated with the geometry of the hyperbola, a topic addressed in Section 14:14. The graphical representations of the $(b/a)\sqrt{x^2+a^2}$ and $(b/a)\sqrt{x^2-a^2}$ functions are interconvertible by scaling and rotation operations; these, and other operations, are the subject of Section 14:15.

14:1 NOTATION

The $(b/a)\sqrt{x^2+a^2}$ function, shown in Figure 14-1, corresponds to one-half of a hyperbola, and the $(b/a)\sqrt{x^2-a^2}$ function, illustrated in Figure 14-2, corresponds to two-quarters of a different hyperbola. For this reason, these two functions are called *semihyperbolic functions*. The constants *a* and *b* are the *parameters*; they are



K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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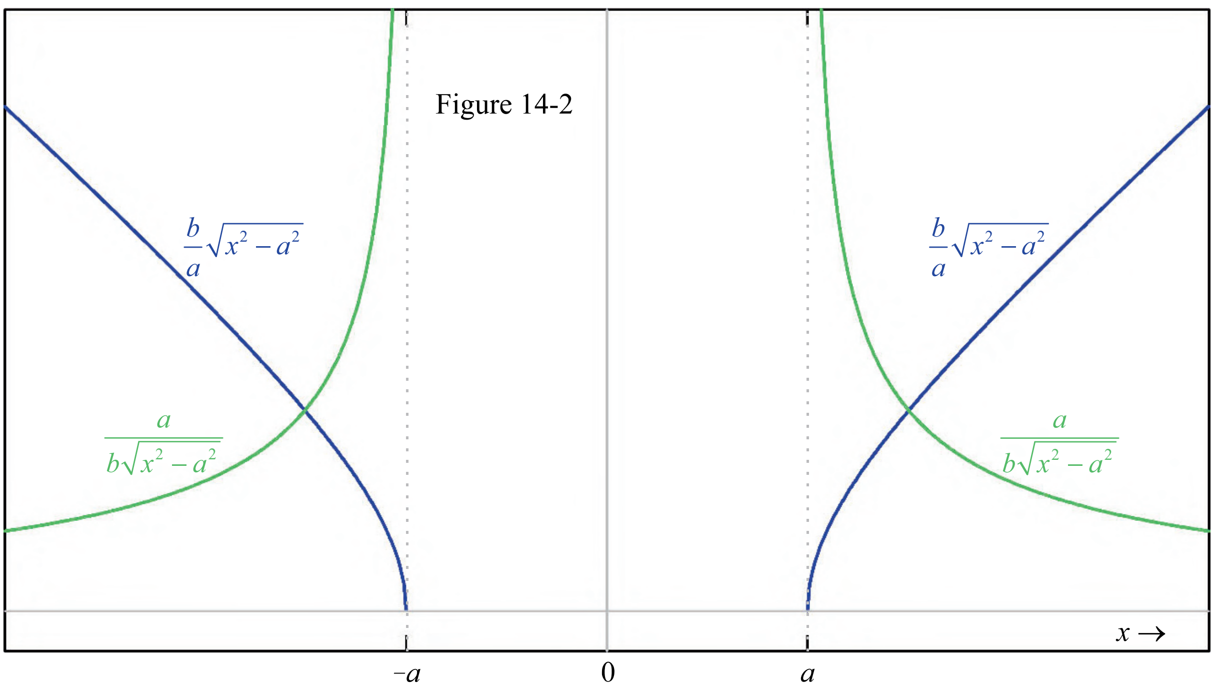
regarded as positive throughout the chapter. The adjectives "vertical" and "horizontal" will be used to distinguish between the $(b/a)\sqrt{x^2+a^2}$ and $(b/a)\sqrt{x^2-a^2}$ functions, in recognition of their graphical orientation. These two functions are said to be *conjugates* of each other.

14:2 BEHAVIOR

The red curve in Figure 14-1 depicts a typical vertical semihyperbolic function. It accepts any argument x; its range lies between b and ∞. The reciprocal vertical semihyperbolic function, shown in green in the same figure, is also defined for all x; it adopts values between zero and $1/b$.

The behaviors of the horizontal semihyperbolic function and its reciprocal are evident in Figure 14-2. Neither of these functions adopts real values in the $-a < x < a$ gap. Outside this forbidden zone, both functions adopt positive values ranging between zero and infinity.



14:3 DEFINITIONS

The algebraic operations of squaring [Chapter 10] and taking the square root [Chapter 11], together with arithmetic operations, fully define both varieties of semihyperbolic function and their reciprocals.

One way of defining a horizontal semihyperbolic function is as the product of two closely related square-root functions [Chapter 11]:

14:3:1 $$\sqrt{\frac{bx}{a}+b}\sqrt{\frac{bx}{a}-b} = \frac{b}{a}\sqrt{x^2-a^2}$$

but no corresponding definition (from real functions) exists for the vertical version.

The semihyperbolic functions, of both the vertical and horizontal varieties, are expansible hypergeometrically [Section 18:14], as in equations 14:6:1 and 14:6:3. The same is true of the reciprocal semihyperbolic functions, whose expansions are given in 14:16:2 and 14:16:3. These expansions open the way to definition via synthesis [Section 43:14] from simpler functions.

A parametric definition [Section 0:3] of the vertical semihyperbolic function is in terms of the hyperbolic sine and cosine functions [Chapter 28]:

14:3:2 $$\mathrm{f} = b\cosh(t), \quad x = a\sinh(t): \qquad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 + a^2}$$

The roles are reversed for the horizontal version

14:3:3 $$\mathrm{f} = b\sinh(t), \quad x = a\cosh(t): \qquad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 - a^2}$$

A hyperbola, and hence the semihyperbolic functions, may be defined geometrically in two distinct ways. One of these is detailed in Section 14:14, the other in Section 15:15.

14:4 SPECIAL CASES

When $b = a$, the horizontal and vertical semihyperbolic functions become $\sqrt{x^2 - a^2}$ and $\sqrt{x^2 + a^2}$ respectively. Shown in Figure 14-3, they are termed *rectangular semihyperbolic functions*. The horizontal rectangular semihyperbolic function may be transformed into its vertical cohort on rotation about the origin by an angle of $\pi/2$. This can be established by setting $\theta = \pi/2$ in the formulas [Section 14:15]

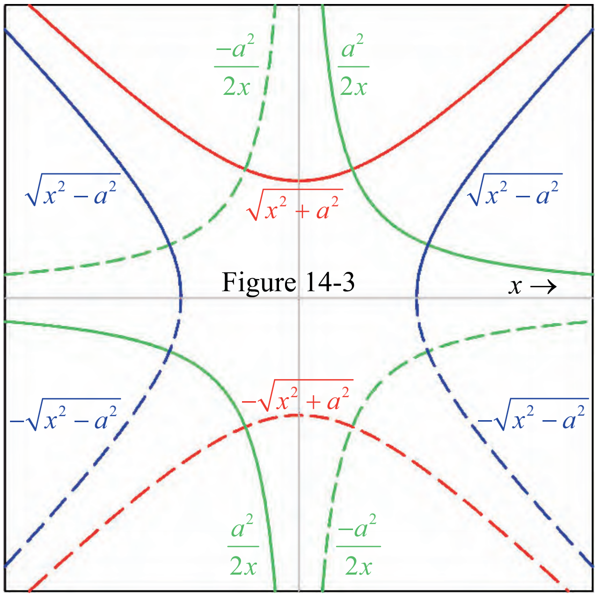


Figure 14-3

14:4:1 $$x_{\mathrm{n}} = x_{\mathrm{o}}\cos(\theta) - f_{\mathrm{o}}\sin(\theta)$$
$$f_{\mathrm{n}} = f_{\mathrm{o}}\cos(\theta) + x_{\mathrm{o}}\sin(\theta)$$

for rotation counterclockwise through an angle θ about the origin. Here the subscript "$_{\mathrm{o}}$" denotes an old (pre-rotation) coordinate, whereas "$_{\mathrm{n}}$" signifies the new (post-rotation) equivalent. More interesting than rotation by a right-angle, however, is the effect of rotation by an angle of $\pi/4$ applied to the horizontal rectangular semihyperbolic function, $f_{\mathrm{o}} = \sqrt{x_{\mathrm{o}}^2 - a^2}$. Then, because $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, one finds

14:4:2 $$\left.\begin{aligned} x_{\mathrm{n}} &= \frac{x_{\mathrm{o}} - f_{\mathrm{o}}}{\sqrt{2}} \\ f_{\mathrm{n}} &= \frac{f_{\mathrm{o}} + x_{\mathrm{o}}}{\sqrt{2}} \end{aligned}\right\} x_{\mathrm{n}} f_{\mathrm{n}} = \frac{x_{\mathrm{o}}^2 - f_{\mathrm{o}}^2}{2} = \frac{a^2}{2} \qquad \text{whence} \qquad f_{\mathrm{n}} = \frac{a^2}{2x_{\mathrm{n}}}$$

Thus the rotated function, which could be considered a *diagonal semihyperbolic function*, is a special case of a reciprocal linear function, as in Chapter 7, or an integer power function [Chapter 10]. A further rotation of $a^2/2x$ by $\pi/4$ produces $\sqrt{x^2 + a^2}$. Still more rotations by 45° lead successively to the various functions shown in Figure 14-3, all of which are branches of rectangular semihyperbolic functions.

14:5 INTRARELATIONSHIPS

Semihyperbolic functions are even functions, as are their reciprocals

14:5:1 $$f(-x) = f(x) \qquad f(x) = \frac{b}{a}\sqrt{x^2 \pm a^2} \quad \text{or} \quad \frac{a}{b\sqrt{x^2 \pm a^2}}$$

Multiplication of the argument of a semihyperbolic function f(x) by a constant leads to another semihyperbolic function

14:5:2 $$f(vx) = \frac{b}{a}\sqrt{(vx)^2 \pm a^2} = \frac{b}{(a/v)}\sqrt{x^2 \pm (a/v)^2}$$

the b parameter being unaffected. In Section 14:15, this is termed an "argument scaling operation".

Apart from an interchange of the a and b parameters, the inverse function [Section 0:3] of a semihyperbolic function is its conjugate

14:5:3 $$F(x) = \frac{a}{b}\sqrt{x^2 \mp b^2} \qquad \text{where} \qquad F\big(f(x)\big) = x \quad \text{and} \quad f(x) = \frac{b}{a}\sqrt{x^2 \pm a^2}$$

with interchanged parameters.

14:6 EXPANSIONS

The horizontal semihyperbolic function may be expanded binomially

14:6:1 $$\frac{b}{a}\sqrt{x^2 - a^2} = b\left[\frac{x}{a} - \frac{a}{2x} - \frac{a^3}{8x^3} - \frac{a^5}{16x^5} - \frac{5a^7}{128x^7} - \cdots\right] = b\sum_{j=0}^{\infty}(-)^j\binom{\frac{1}{2}}{j}\frac{a^{2j-1}}{x^{2j-1}} = \frac{bx}{a}\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^{-j}$$

Of course, this expansion is invalid in the region $|x| < a$, where the real function does not exist. There are several alternative ways of expressing the coefficients of such series, in addition to the binomial coefficient [Chapter 6] or Pochhammer polynomials [Chapter 18] employed here. The similar expansion of the reciprocal horizontal semihyperbolic function

14:6:2 $$\frac{a}{b\sqrt{x^2 - a^2}} = \frac{1}{b}\left[\frac{a}{x} + \frac{a^3}{2x^3} + \frac{3a^5}{8x^5} + \frac{5a^7}{16x^7} + \frac{35a^9}{128x^9} + \cdots\right] = \frac{1}{b}\sum_{j=0}^{\infty}(-)^j\binom{\frac{-1}{2}}{j}\frac{a^{1+2j}}{x^{1+2j}} = \frac{a}{bx}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^{-j}$$

is again restricted to $-a < x < a$. However, for the vertical semihyperbolic function, and its reciprocal, there are no restrictions because alternative binomial expansions exist. The first version of each equation below is applicable when $|x| \le a$, the second when $|x| > a$.

14:6:3 $$\frac{b}{a}\sqrt{x^2 + a^2} = \begin{cases} b\left[1 + \frac{x^2}{2a^2} - \frac{x^4}{8a^4} + \frac{x^6}{16a^6} - \frac{5x^8}{128a^8} + \frac{7x^{10}}{256a^{10}} - \cdots\right] = b\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\frac{x^{2j}}{a^{2j}} = b\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^{j} \\ b\left[\frac{x}{a} + \frac{a}{2x} - \frac{a^3}{8x^3} + \frac{a^5}{16x^5} - \frac{5a^7}{128x^7} + \frac{7a^9}{256x^9} - \cdots\right] = b\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\frac{a^{2j-1}}{x^{2j-1}} = \frac{bx}{a}\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^{-j} \end{cases}$$

14:6:4
$$\frac{a}{b\sqrt{x^2+a^2}} = \begin{cases} \dfrac{1}{b}\left[1-\dfrac{x^2}{2a^2}+\dfrac{3x^4}{8a^4}-\dfrac{5x^6}{16a^6}+\dfrac{35x^8}{128a^8}-\dfrac{63x^{10}}{256a^{10}}+\cdots\right] = \dfrac{1}{b}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\dfrac{x^{2j}}{a^{2j}} = \dfrac{1}{b}\sum_{j=0}^{\infty}\dfrac{(\frac{1}{2})_j}{(1)_j}\left(\dfrac{-x^2}{a^2}\right)^j \\ \dfrac{1}{b}\left[\dfrac{a}{x}-\dfrac{a^3}{2x^3}+\dfrac{3a^5}{8x^5}-\dfrac{5a^7}{16x^7}+\dfrac{35a^9}{128x^9}-\dfrac{63a^{11}}{256x^{11}}+\cdots\right] = \dfrac{1}{b}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\dfrac{a^{2j+1}}{x^{2j+1}} = \dfrac{a}{bx}\sum_{j=0}^{\infty}\dfrac{(\frac{1}{2})_j}{(1)_j}\left(\dfrac{-x^2}{a^2}\right)^{-j} \end{cases}$$

More rapidly convergent series may result when hyperbolic functions are substituted

14:6:5
$$\frac{b}{a}\sqrt{x^2-a^2} = b\left[t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\cdots\right]$$

14:6:6
$$\frac{a}{b\sqrt{x^2-a^2}} = \frac{1}{b}\left[\frac{1}{t}-\frac{t}{6}+\frac{7t^3}{360}-\frac{31t^5}{15120}+\cdots\right] \qquad x = a\cosh(t)$$

14:6:7
$$\frac{b}{a}\sqrt{x^2+a^2} = b\left[1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\cdots\right]$$

14:6:8
$$\frac{a}{b\sqrt{x^2+a^2}} = \frac{1}{b}\left[1-\frac{t^2}{2}+\frac{5t^4}{24}-\frac{61t^6}{720}+\cdots\right] \qquad x = a\sinh(t)$$

See Chapter 28 and 29 for the bases of these equations.

14:7 PARTICULAR VALUES

	$x=-\infty$	$x=-\sqrt{a^2+b^2}$	$x=-a$	$x=0$	$x=a$	$x=\sqrt{a^2+b^2}$	$x=\infty$
$\frac{b}{a}\sqrt{x^2-a^2}$	$+\infty$	$\frac{b^2}{a}$	0	undef	0	$\frac{b^2}{a}$	$+\infty$
$\frac{b}{a}\sqrt{x^2+a^2}$	$+\infty$	$\frac{b\sqrt{2a^2+b^2}}{a}$	$\sqrt{2}\,b$	b	$\sqrt{2}\,b$	$\frac{b\sqrt{2a^2+b^2}}{a}$	$+\infty$

14:8 NUMERICAL VALUES

These are readily calculated, for example by *Equator*'s x^ν power function routine (keyword **power**) with $\nu = \pm 1/2$, after the variable construction feature [Appendix, Section C:4] is first used with $w = b^2/a^2$, $p = 2$, and $k = \pm b^2$.

14:9 LIMITS AND APPROXIMATIONS

Both semihyperbolic functions approach the linear functions $x = \pm bx/a$ as $x \to \pm\infty$. These lines are known as the *asymptotes* of the corresponding hyperbolas. Specifically

14:9:1
$$\left.\begin{array}{r}\lim\limits_{x\to\infty}\left\{(b/a)\sqrt{x^2\pm a^2}\right\}\\ \lim\limits_{x\to-\infty}\left\{(-b/a)\sqrt{x^2\pm a^2}\right\}\end{array}\right\} = \frac{bx}{a}$$

and

14:9:2
$$\left.\begin{array}{r}\lim\limits_{x\to-\infty}\left\{(b/a)\sqrt{x^2\pm a^2}\right\}\\ \lim\limits_{x\to\infty}\left\{(-b/a)\sqrt{x^2\pm a^2}\right\}\end{array}\right\} = \frac{-bx}{a}$$

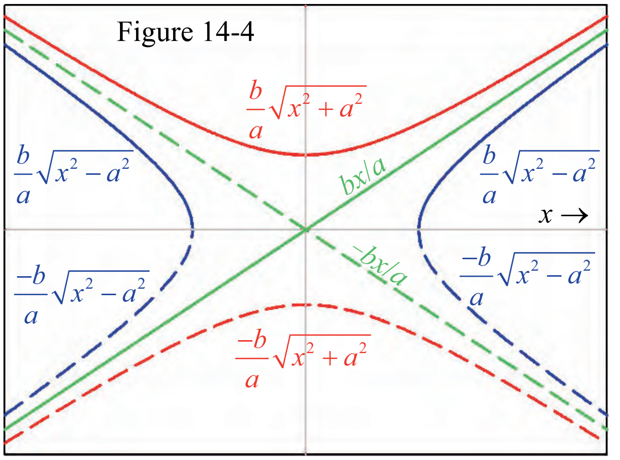


Figure 14-4 shows these asymptotes and also nicely illustrates the relationships between the four semihyperbolic functions diagrammed there.

Correspondingly, as $x \to \pm\infty$, the reciprocal semihyperbolic functions approach zero as reciprocal linear functions $\pm a/bx$.

Near its positive apex, $x = a$, the horizontal semihyperbolic function approximates a square-root function:

14:9:3
$$\frac{b}{a}\sqrt{x^2-a^2} \approx \sqrt{\frac{2b^2}{a}}\sqrt{x-a} \qquad x-a \text{ small}$$

14:10 OPERATIONS OF THE CALCULUS

The b/a multiplier will be omitted in this section. Formulas for differentiation and integration are

14:10:1
$$\frac{d}{dx}\sqrt{x^2\pm a^2} = \frac{x}{\sqrt{x^2\pm a^2}}$$

14:10:2
$$\frac{d}{dx}\frac{1}{\sqrt{x^2\pm a^2}} = \frac{-x}{\sqrt{\left(x^2\pm a^2\right)^3}}$$

14:10:3
$$\int_0^x \sqrt{t^2+a^2}\,dt = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\operatorname{arsinh}\left(\frac{x}{a}\right)$$

14:10:4
$$\int_a^x \sqrt{t^2-a^2}\,dt = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\operatorname{arcosh}\left(\frac{x}{a}\right)$$

14:10:5
$$\int_0^x \frac{1}{\sqrt{t^2+a^2}}\,dt = \operatorname{arsinh}\left(\frac{x}{a}\right)$$

14:10:6
$$\int_a^x \frac{1}{\sqrt{t^2-a^2}}\,dt = \operatorname{arcosh}\left(\frac{x}{a}\right)$$

The last two integrals serve as definitions of the inverse hyperbolic sine and cosine functions [Chapter 31]. A long list of indefinite integrals of the form $\int t^n(t^2\pm a^2)^{m+\frac{1}{2}}dt$, where n and m are integers, will be found in Gradshteyn and Ryzhik [Section 2.27]; one example, generating a function from Chapter 35, is

14:10:7 $$\int_a^x \frac{\sqrt{t^2-a^2}}{t}\,dt = \sqrt{x^2-a^2} - a\,\mathrm{arcsec}\left(\frac{x}{a}\right)$$

Pages 219 − 284 of the same book are an invaluable source of hundreds of results, such as that required in14:14:6, for the integration of functions involving several terms of the form $\sqrt{\pm x \pm c}$.

The Laplace transform of the reciprocal vertical semihyperbolic function is given by

14:10:8 $$\int_0^\infty \frac{1}{\sqrt{t^2+a^2}}\exp(-st)\,dt = \mathcal{L}\left\{\frac{1}{\sqrt{t^2+a^2}}\right\} = \frac{\pi}{2}\left[\mathrm{h}_0(as) - \mathrm{Y}_0(as)\right]$$

where the functions appearing in the transform are the Struve function [Chapter 57] and the Neumann function [Chapter 54] of zero order.

14:11 COMPLEX ARGUMENT

With imaginary argument, a horizontal semihyperbolic function becomes an imaginary vertical semihyperbolic function

14:11:1 $$(b/a)\sqrt{(iy)^2-a^2} = (ib/a)\sqrt{y^2+a^2}$$

The converse is true only in part, because the vertical semihyperbolic function becomes a real semielliptic function [Chapter 13] only for a range of magnitudes of its imaginary argument

14:11:2 $$(b/a)\sqrt{(iy)^2+a^2} = \begin{cases} (b/a)\sqrt{a^2-y^2} & |y| < a \\ (ib/a)\sqrt{y^2-a^2} & |y| > a \end{cases}$$

With $z = x + iy$, the real and imaginary parts of the vertical semihyperbolic function of complex argument are given by

14:11:3 $$\frac{b}{a}\sqrt{z^2+a^2} = \frac{b}{\sqrt{2}a}\sqrt{x^2-y^2+a^2+\sqrt{A+B}} + \frac{ib\,\mathrm{sgn}(xy)}{\sqrt{2}a}\sqrt{\sqrt{A+B}-x^2+y^2-a^2}$$

where $A = a^4 + (x^2+y^2)^2$ and $B = 2a^2(x^2-y^2)$; sgn is the signum function [Chapter 8] equal to ±1 according to the sign of its argument, or to zero if its argument is zero. The corresponding formula for the horizontal semihyperbolic function of complex argument is

14:11:4 $$\frac{b}{a}\sqrt{z^2-a^2} = \frac{b}{\sqrt{2}a}\sqrt{\sqrt{A-B}+x^2-y^2-a^2} + \frac{ib\,\mathrm{sgn}(xy)}{\sqrt{2}a}\sqrt{-x^2+y^2+a^2+\sqrt{A-B}}$$

14:12 GENERALIZATIONS

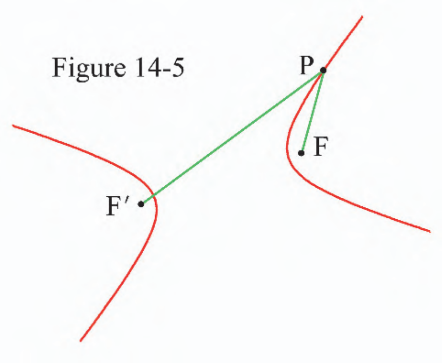
Semihyperbolic functions are instances of the root-quadratic function discussed in Section 15:13. They are also conic sections [Section 15:15].

14:13 COGNATE FUNCTIONS

For n = 3, 4, 5, ⋯, the functions $(b/a)\left[x^n \pm a^n\right]^{1/n}$ have shapes very similar to hyperbolas, especially if n is even. The straight line $f(x) = bx/a$ is an asymptote for all these functions, as is $f(x) = -bx/a$ if n is even.

14:14 RELATED TOPIC: geometry of the hyperbola

There are two distinct geometric definitions of a hyperbola, one of which is addressed in Section 15:15. The second, illustrated in Figure 14-5, is based on two points, F and F′, each of which is termed a *focus* of the hyperbola. A hyperbola is defined as the locus of all points P such that the distance from P to the more remote focus exceeds that to the nearer focus by a constant:



14:14:1 $$\left|\mathrm{PF}' - \mathrm{PF}\right| = \text{a constant} = 2a$$

The eccentricity k of the hyperbola, which necessarily exceeds unity and equals $\sqrt{2}$ for a rectangular hyperbola, is defined as

14:14:2 $$\frac{\left|\mathrm{FF}'\right|}{\left|\mathrm{PF}' - \mathrm{PF}\right|} = k > 1$$

where FF′ is the *interfocal separation*, the distance between the two foci, equal to $2ka$. The two parameters of the hyperbola (sometimes called its *semiaxes*) are a, defined in 14:14:1 and b, given by

14:14:3 $$b = a\sqrt{k^2 - 1} \qquad \text{whence} \qquad k = \frac{\sqrt{a^2 + b^2}}{a}$$

The b parameter may have a magnitude smaller than, equal to, or greater than a. As Figure 14-5 shows, the hyperbola has two branches, separated from each other (by $2a$ at their closest approach). The definition in this paragraph covers both branches equally.

If the two foci are equidistant from the origin and on a line perpendicular to the x-axis through the origin, then the equations

14:14:4 $$\mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 + a^2} \qquad \text{and} \qquad \mathrm{f}(x) = \frac{-b}{a}\sqrt{x^2 + a^2}$$

describe the upper and lower branches of the hyperbola respectively. This is the hyperbola that we call a vertical hyperbola. In view of the distinction [Section 12:1] between the symbols $\sqrt{t}$ and $t^{½}$, the vertical hyperbola in its entirety is described by $(b/a)[x^2 + a^2]^{½}$.

If the two foci lie on the x-axis, equidistant from the origin, then the hyperbola is described as a horizontal hyperbola and it is described by the formula $(b/a)[x^2 - a^2]^{½}$. The upper half of *each* branch of this hyperbola is described by $(b/a)\sqrt{x^2 - a^2}$ while the lower half of each branch is covered by the formula $(-b/a)\sqrt{x^2 - a^2}$.

Conjugate hyperbolas, that is, vertical and horizontal hyperbolas sharing the same a and b parameters, also share the same asymptotes.

The area enclosed by the rightmost branch of a horizontal hyperbola, $\pm(b/a)\sqrt{x^2 - a^2}$ and the ordinate $\mathrm{f}(x) = x$, as illustrated in Figure 14-6, may be evaluated by recourse to integral 14:10:4 and is

14:14:5 $$\begin{matrix}\text{shaded}\\ \text{area}\end{matrix} = 2\int_a^x \frac{b}{a}\sqrt{t^2 - a^2}\,\mathrm{d}t = \frac{bx}{a}\sqrt{x^2 - a^2} - ab\,\mathrm{arcosh}\left(\frac{x}{a}\right)$$

The curved perimeter of the shaded area bounded by graphs of the function $\mathrm{f}(x) = \pm(b/a)\sqrt{x^2 - a^2}$ has a length given in terms of incomplete elliptic integrals F and E [Chapter 62] by

14:14:6 $$2\int_a^x \sqrt{1 + \left(\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2k\int_a^x \sqrt{\frac{t^2 - (a/k)^2}{t^2 - a^2}}\,\mathrm{d}t = 2ka\left[\frac{k^2 - 1}{k^2}\mathrm{F}\left(\frac{1}{k}, \varphi\right) - \mathrm{E}\left(\frac{1}{k}, \varphi\right) + \frac{x}{a}\sin(\varphi)\right]$$

where $k = \sqrt{a^2 + b^2}/a$ and $\varphi = \arctan\{(k/b)\sqrt{x^2 - a^2}\}$.

Also shown in Figure 14-6 are the asymptotes OS and OT of the hyperbola and the tangent MN to the hyperbolic branch at an arbitrary point P (M and N being the points at which the tangent meets the asymptotes, as depicted in Figure 14-6). Two remarkable properties of the hyperbola are that P bisects the line MN, so that MP = PN, and that the area of the triangle MNO equals $|ab|$ independent of the position of P on the hyperbola.

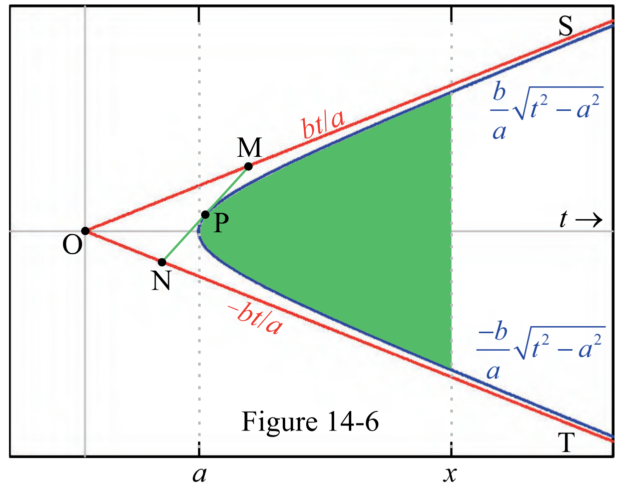


Figure 14-6

14:15 RELATED TOPIC: graphical operations

Because two- or three-dimensional graphs are generally helpful in appreciating the properties of functions, many are scattered throughout this *Atlas*.

For a univariate function f(x), the common graphical representation is as a cartesian graph in which the argument x and the value f of the function at that argument serve as the rectangular coordinates (x, f); Figures 14-1 and 14-2 are examples. Beyond mere visualization, graphs can be useful in revealing relationships between functions; for example, in Section 14:4 it is shown how an operation – rotation about the origin in that case – could convert a rectangular semihyperbolic function into the simpler $a^2/2x$ function. In this section we catalog five operations that change the shape, the location, or the orientation of a graph, and show how this affects the formula of the function. The original function (x_o, f_o) transforms to a new function (x_n, f_n) on subjection to some specified operation. Note that the axes are treated as fixed; it is the function that changes. Figure 14-7 shows a fragment of a representative function, in black. In each of five other colors is shown the result of a specified operation.

Perhaps the simplest operation is *scaling*, of which there are two versions. In *function scaling*, all function values are multiplied by a *scaling factor*, here λ. The equations describing function scaling are $x_n = x_o$ and $f_n = \lambda f o$. The result is a function that has been altered by expansion or contraction of its vertical dimension by a factor of λ, as illustrated in red in Figure 14-7, for the $\lambda = 2$ case. There is also *argument scaling* in which it is x that is multiplied by a scaling factor v, leading to $f_n = \mathrm{f}(vx_o)$. This stretches or compresses the curve horizontally, but is not illustrated in Figure 14-7.

Translation affects the location of a function without changing its shape, size or orientation. The equation pair

14:15:1 $$x_n = x_o + x_P \quad \text{and} \quad f_n = f_o + f_P$$

describes the operation. Here (x_P, f_P) are the coordinates of the point P which corresponds, in the new location, to the old origin, as illustrated in blue in Figure 14-7 for the $x_P = 3, f_P = 1$ case.

By *rotation about a point* R is meant that every point on the original graph retains its original distance from point R, but the line joining the two points rotates counterclockwise through an angle θ. The transformation equations are

14:15:2 $$x_n = x_R - (x_R - x_o)\cos(\theta) + (f_R - f_o)\sin(\theta) \quad \text{and} \quad f_n = f_R - (f_R - f_o)\cos(\theta) - (x_R - x_o)\sin(\theta)$$

For the point $(x_R, f_R) = (-5/4, 5/4)$ and $\theta = 135°$, the result of this operation is shown in green. Commonly, point R is the origin, in which case $x_R = f_R = 0$ and the equation pair 14:15:3 reduces to equations 14:4:1. The operations described in this section may be applied sequentially; one application of this concept establishes a relationship between the vertical semihyperbolic function and the horizontal semihyperbolic function:

14:15:3 $$\frac{b}{a}\sqrt{x^2 - a^2} \xrightarrow[\lambda = a/b]{\text{scale with}} \sqrt{x^2 - a^2} \xrightarrow[\text{with } \theta = \pi/2]{\text{rotate about origin}} \sqrt{x^2 + a^2} \xrightarrow[\lambda = b/a]{\text{scale with}} \frac{b}{a}\sqrt{x^2 + a^2}$$

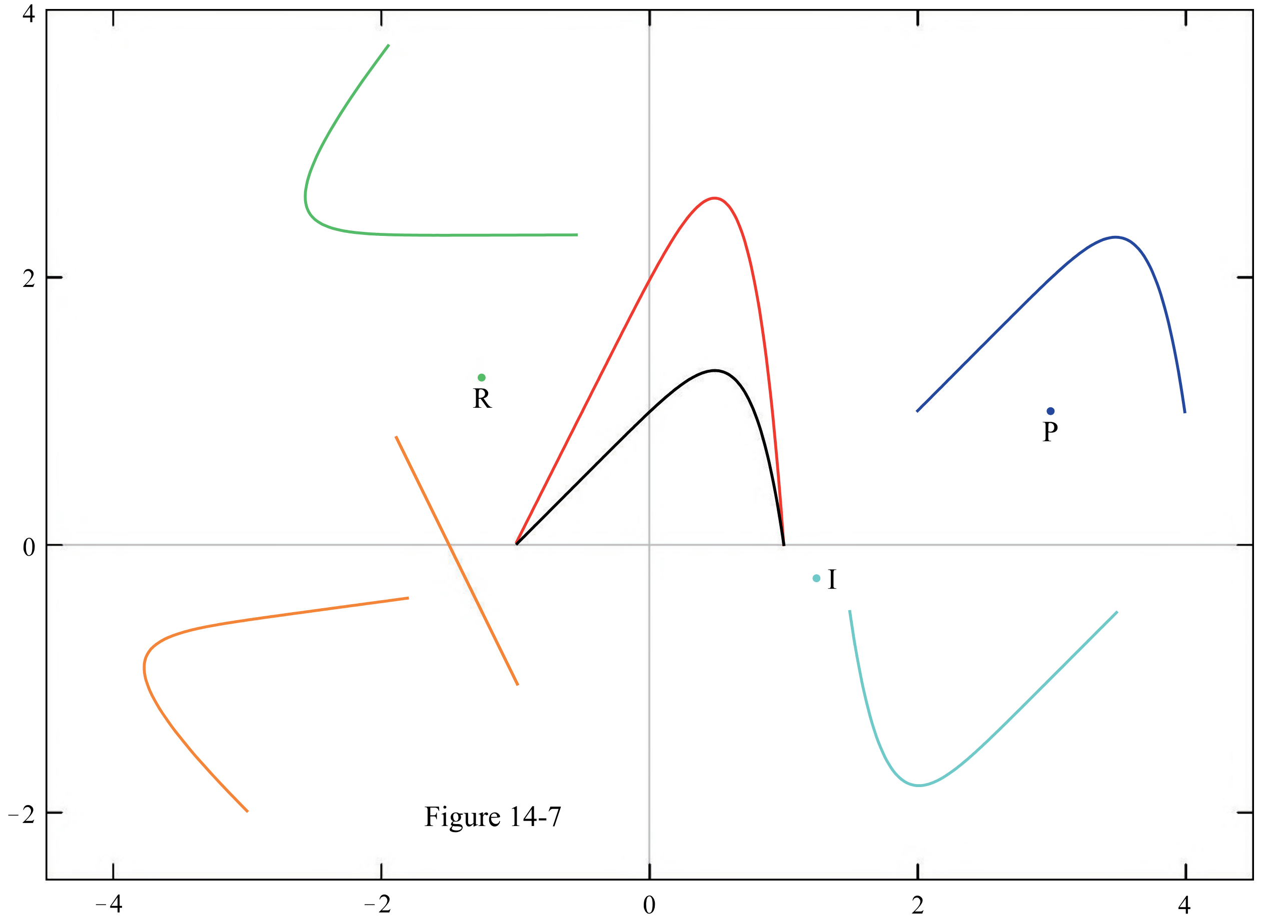


Figure 14-7

Reflection in the line bx+c implies that, from each point on the original graph, a perpendicular is dropped onto the line and then extrapolated to a new point that is as far from the line as was the line from the original point. The formulas governing this operation, are

14:15:4 $$x_{\mathrm{n}} = \frac{(1-b^2)x_{\mathrm{o}} + 2b(f_{\mathrm{o}} - c)}{1+b^2} \quad \text{and} \quad f_{\mathrm{n}} = \frac{(b^2-1)f_{\mathrm{o}} + 2bx_{\mathrm{o}} + 2c}{b^2+1}$$

With the line $f = -2x-3$ serving as the "mirror", the transformation is illustrated in orange. Reflection in the line $x = 0$ simply alters the sign of the argument, $(x_{\mathrm{n}}, f_{\mathrm{n}}) = (-x_{\mathrm{o}}, f_{\mathrm{o}})$; a characteristic of *even* functions is that reflection in the line $x = 0$ leaves the function unchanged. The property of being unaffected by reflection is termed *mirror symmetry*. Reflection in the line $f = x$ causes an interchange of the function's value with its argument, $(x_{\mathrm{n}}, f_{\mathrm{n}}) = (f_{\mathrm{o}}, x_{\mathrm{o}})$; that is, it generates the *inverse function* [Section 0:3].

The final operation that will be mentioned is named *inversion* though, confusingly, this is unconnected with inverse functions. *Inversion through a point* I means constructing the straight line that joins each point on the original graph to I, extrapolating this line and then creating a new point on the extrapolate an equal distance beyond. The formulas

14:15:5 $$x_{\mathrm{n}} = 2x_{\mathrm{I}} - x_{\mathrm{o}} \quad \text{and} \quad f_{\mathrm{n}} = 2f_{\mathrm{I}} - f_{\mathrm{o}}$$

describe the operation of inversion, in the present sense. The result of an inversion through point $(x_{\mathrm{I}}, f_{\mathrm{I}}) = (\frac{5}{4}, \frac{-1}{4})$ is shown in turquoise in Figure 14-7. Inversion through the origin changes the sign of both coordinates $(x_{\mathrm{n}}, f_{\mathrm{n}}) = (-x_{\mathrm{o}}, -f_{\mathrm{o}})$; a characteristic of *odd* functions is that they are unchanged by inversion through the origin. *Inversion symmetry* is the name given to the property of being unaffected by inversion.

CHAPTER 15

THE QUADRATIC FUNCTION $ax^2 + bx + c$ AND ITS RECIPROCAL

As a cartesian graph, the shape of ax^2+bx+c is that of a parabola and, in this respect, the quadratic function resembles the square-root function of Chapter 11. The root-quadratic function, addressed in Section 15:13 and 15:14, may also adopt the shape of a parabola.

15:1 NOTATION

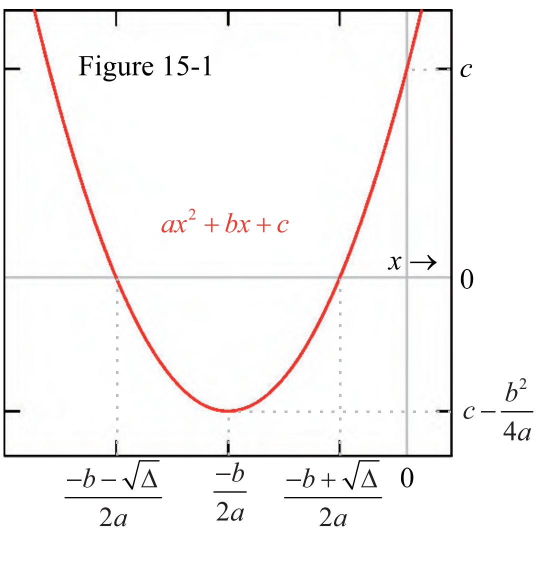
The constants a, b, and c, that, together with the argument x, compose the quadratic function, are called *coefficients*. In the graphs of this chapter, a is taken to be positive, though the formulas are valid for either sign. The sign of the quantity

15:1:1 $$\Delta = b^2 - 4ac$$

known as the *discriminant* of the quadratic function, influences several of the function's properties. Some authors define the discriminant as the negative of the quantity specified in 15:1:1, as it was in the first edition of this *Atlas*.

15:2 BEHAVIOR

Irrespective of the values of its coefficients, the quadratic function adopts real values for any real argument; however, it has a limited range, extending (for positive a) only over $-\Delta/4a \le ax^2 + bx + c \le \infty$. At $x = -b/2a$, the function experiences an extremum: a minimum or a maximum according as a is positive or negative. It is the sign of the discriminant that determines whether the quadratic function adopts the value zero. In drawing Figure 15-1, both a and Δ are treated as positive, so that the quadratic function crosses the x-axis twice.



K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_16, © Springer Science+Business Media, LLC 2009

The behavior of the reciprocal quadratic function is even more affected by the sign of the discriminant. If Δ is negative, the $1/(ax^2+bx+c)$ function is a contiguous function, adopting values between zero and $-4a/\Delta$, as illustrated in blue in Figure 15-2. However, when the discriminant is positive, the reciprocal quadratic function has the three branches, shown in red in the figure, with discontinuities at $x=(\sqrt{\Delta}-b)/2a$ and $x=-(\sqrt{\Delta}+b)/2a$.

Both the quadratic function and its reciprocal have mirror symmetry about the line $x=-b/2a$, irrespective of the value of the discriminant.

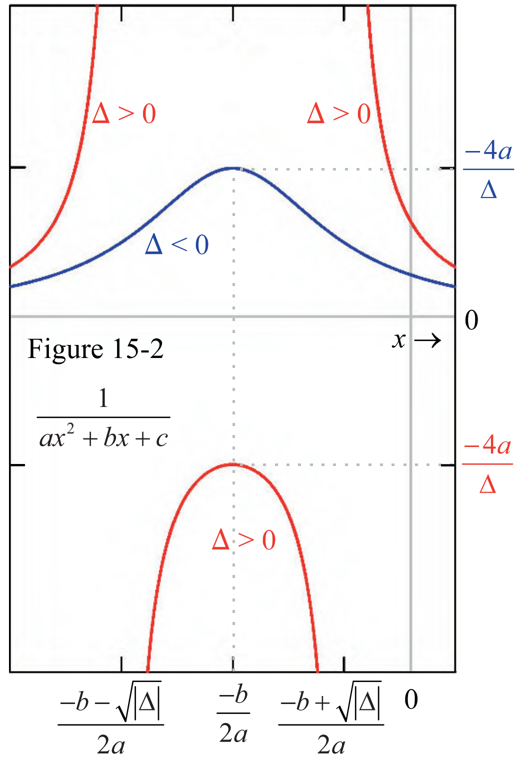


Figure 15-2 $\dfrac{1}{ax^2+bx+c}$

15:3 DEFINITIONS

Writing the quadratic functions as $c+x(b+ax)$ shows that the operations of multiplication and addition suffice to provide a definition. It may also be defined as the product of two linear functions:

$$ax^2+bx+c=\left(ax+\frac{b-\sqrt{\Delta}}{2}\right)\left(x+\frac{b+\sqrt{\Delta}}{2a}\right) \qquad 15:3:1$$

The cartesian graph of the $f=ax^2+bx+c$ function is a parabola with its *focus* at the point $(x,f)=(-b/2a,\ c+(b^2-1)/4a)$ and its *directrix* as the horizontal line $f=c-(b^2-1)/4a$. Thus the function may be defined by recourse to the definition of a parabola given in Section 11:3. Yet another definition is as the inverse function of a translated [Section 14:15] square-root function [Chapter 11]

$$\mathrm{F}(x)=ax^2+bx+c \qquad \mathrm{f}(x)=\sqrt{\frac{x}{a}+\frac{\Delta}{4a^2}}-\frac{b}{2a} \qquad \mathrm{F}\big(\mathrm{f}(x)\big)=\mathrm{f}\big(\mathrm{F}(x)\big)=x \qquad 15:3:2$$

If the discriminant is positive, the reciprocal quadratic function can be defined as the difference between two reciprocal linear functions [Chapter 7]:

$$\frac{1}{ax^2+bx+c}=\frac{1}{\sqrt{\Delta}\,x+\dfrac{b\sqrt{\Delta}-\Delta}{2a}}-\frac{1}{\sqrt{\Delta}\,x+\dfrac{b\sqrt{\Delta}+\Delta}{2a}} \qquad \Delta>0 \qquad 15:3:3$$

15:4 SPECIAL CASES

A linear function [Chapter 7] is the special $a=0$ case of the quadratic function. When $b=2\sqrt{ac}$, so that the discriminant is zero, the quadratic function reduces to $[\sqrt{a}\,x+\sqrt{c}\,]^2$, a square function [Chapter 10].

When the discriminant is zero, the reciprocal quadratic function has the unusual property of encountering a infinite discontinuity of the $+\infty|+\infty$ type at $x=-\sqrt{c/a}$.

In the special case when $b^2=4(ac-\pi^2)$, equation 15:10:4 shows the total area under the $1/(ax^2+bx+c)$ curve to be unity. In this circumstance, one can rewrite the normalized formula as

15:4:1 $$\frac{1}{ax^2+bx+c}=\frac{\pi/a}{\pi\left[\{\pi/a\}^2+\{x+(b/2a)\}^2\right]}$$

This is the equation that describes a Lorenz distribution [see the table in Section 27:14], which is therefore a special case of the reciprocal quadratic function.

15:5 INTRARELATIONSHIPS

Both the quadratic function and its reciprocal obey the reflection formula

15:5:1 $$\mathrm{f}\left(x+\frac{b}{2a}\right)=\mathrm{f}\left(-x-\frac{b}{2a}\right) \qquad \mathrm{f}=ax^2+bx+c \quad \text{or} \quad \frac{1}{ax^2+bx+c}$$

The sum or difference of two quadratic functions is generally another quadratic function, while their product is invariably a quartic function [Section 16:13]. Provided that the discriminant, Δ, of the denominatorial function is positive, the quotient of two quadratic functions can be expressed in terms of a constant and two reciprocal linear functions, as follows:

15:5:2 $$\frac{a'x^2+b'x+c'}{ax^2+bx+c}=\frac{a'}{a}-\frac{a'r_+^2+b'r_++c'}{\sqrt{\Delta}(x-r_+)}+\frac{a'r_-^2+b'r_-+c'}{\sqrt{\Delta}(x-r_-)} \qquad r_\pm=\left(-b\pm\sqrt{\Delta}\right)/2a$$

where r_+ and r_- are the zeros [Section 15:7] of the denominatorial quadratic function.

15:6 EXPANSIONS

Trinomial expansions [Section 6:12] for the reciprocal quadratic function exist, though they are of limited utility:

15:6:1 $$\frac{1}{ax^2+bx+c}=\begin{cases}\dfrac{1}{c}-\dfrac{b}{c^2}x+\dfrac{b^2-ac}{c^3}x^2-\dfrac{b^3-2abc}{c^4}x^3+\dfrac{b^4-3ab^2c+a^2c^2}{c^5}x^4-\cdots=\dfrac{1}{c}\displaystyle\sum_{j=0}^{\infty}\left(\frac{ax^2+bx}{-c}\right)^j \\ \dfrac{1}{ax^2}-\dfrac{b}{a^2x^3}+\dfrac{b^2-ac}{a^3x^4}-\dfrac{b^3-2abc}{a^4x^5}+\dfrac{b^4-3ab^2c+a^2c^2}{a^5x^6}-\cdots=\dfrac{1}{ax^2}\displaystyle\sum_{j=0}^{\infty}\left(\frac{bx+c}{-ax^2}\right)^j\end{cases}$$

The first expansion is valid for small argument (that is, when $|ax^2+bx|<|c|$), the second for large argument (when $|ax^2|>|bx+c|$).

15:7 PARTICULAR VALUES

The zeros of the ax^2+bx+c function, and the discontinuities of its reciprocal, are given by the well-known formula

15:7:1 $$r_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To preserve significance it is better (if b is positive), to calculate r_- first and then r_+ as c/ar_-. There is a double zero at $-b/2a$ if the discriminant vanishes, and the zeros are complex if the discriminant is negative. *Equator* treats the zeros as the quadrivariate function

15:7:2 $$\mathrm{r}_2(a,b,c,n)=\frac{-b+n\sqrt{b^2-4ac}}{2a}=\mathrm{Re}[\mathrm{r}_2(a,b,c,n)]+i\,\mathrm{Im}[\mathrm{r}_2(a,b,c,n)] \qquad n=-1,+1$$

and its quadratic zeros routine (keyword **r2**) outputs both the real and imaginary parts, the latter being 0 unless the discriminant is negative. By default, *Equator* generates a short table giving both zeros.

The following table is applicable whether the discriminant is positive or negative, but the a coefficient is assumed positive.

		$x=-\infty$	$x=\dfrac{-b-\sqrt{\lvert\Delta\rvert}}{2a}$	$x=\dfrac{-b}{2a}$	$x=0$	$x=\dfrac{-b+\sqrt{\lvert\Delta\rvert}}{2a}$	$x=\infty$
ax^2+bx+c	$\Delta>0$	$+\infty$	0	$\dfrac{-\Delta}{4a}$ (minimum)	c	0	$+\infty$
	$\Delta<0$	$+\infty$	$-\Delta/2a$		c	$-\Delta/2a$	$+\infty$
$\dfrac{1}{ax^2+bx+c}$	$\Delta>0$	0	$+\infty\vert-\infty$	$\dfrac{-4a}{\Delta}$ (maximum)	$1/c$	$-\infty\vert+\infty$	0
	$\Delta<0$	0	$-2a/\Delta$		$1/c$	$-2a/\Delta$	0

15:8 NUMERICAL VALUES

These are easily calculated, for example with *Equator*'s quadratic function routine (keyword **quadratic**).

15:9 LIMITS AND APPROXIMATIONS

Limiting expressions for the reciprocal quadratic function could be derived from 15:6:1, but they are seldom used.

15:10 OPERATIONS OF THE CALCULUS

The following formulas address the differentiation and integration of the quadratic function and its reciprocal

15:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(ax^2+bx+c)=2ax+b$$

15:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{ax^2+bx+c}\right)=\frac{-2ax-b}{(ax^2+bx+c)^2}$$

15:10:3 $$\int_0^x (at^2+bt+c)\mathrm{d}t=\frac{2ax^3+3bx^2+6cx}{6}$$

15:10:4 $$\int_{-b/2a}^{x}\frac{1}{at^2+bt+c}\mathrm{d}t=\begin{cases}\dfrac{-2}{\sqrt{\Delta}}\mathrm{artanh}\left(\dfrac{2ax+b}{\sqrt{\Delta}}\right) & \Delta>0 \qquad x<\dfrac{\sqrt{\Delta}-b}{2a}\\[2ex] \dfrac{2}{\sqrt{-\Delta}}\arctan\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) & \Delta<0\end{cases}$$

The 15:10:4 integral is infinite if $\Delta = 0$, but if the lower limit is changed to zero, it equals $2/(2ax+b)$. Another important integral is

15:10:5
$$\int_0^x \frac{t}{at^2+bt+c}\mathrm{d}t = \frac{1}{2a}\ln\left(\frac{ax^2+bx+c}{c}\right) - \begin{cases} \dfrac{b}{a\sqrt{\Delta}}\operatorname{artanh}\left(\dfrac{x\sqrt{\Delta}}{bx+2c}\right) & \Delta > 0 \\ \dfrac{b}{a\sqrt{-\Delta}}\arctan\left(\dfrac{x\sqrt{-\Delta}}{bx+2c}\right) & \Delta < 0 \end{cases}$$

and many related integrals of the general form $\int t^n(at^2+bt+c)^m\,\mathrm{d}t$, where n and m are integers, will be found listed by Gradshteyn and Ryzhik [Section 2.17].

The Laplace transform of the quadratic function is straightforward

15:10:6
$$\int_0^\infty (at^2+bt+c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{at^2+bt+c\right\} = \frac{2a+bs+cs^2}{s^3}$$

but that of its reciprocal is elaborate

15:10:7
$$\mathfrak{L}\left\{\frac{1}{at^2+bt+c}\right\} = \begin{cases} \dfrac{\exp(b/2a)}{\sqrt{\Delta}}\left[\exp\left(\dfrac{\sqrt{\Delta}}{2a}\right)\mathrm{Ei}\left(\dfrac{\sqrt{\Delta}+b}{2a}\right) - \exp\left(\dfrac{-\sqrt{\Delta}}{2a}\right)\mathrm{Ei}\left(\dfrac{\sqrt{\Delta}-b}{2a}\right)\right] & \Delta > 0 \\ \dfrac{2}{b} + \dfrac{s}{a}\exp\left(\dfrac{bs}{2a}\right)\mathrm{Ei}\left(\dfrac{-bs}{2a}\right) & \Delta = 0 \\ \dfrac{2}{\sqrt{-\Delta}}\left[\left\{\dfrac{\pi}{2} - \mathrm{Si}\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\right\}\cos\left(\dfrac{\sqrt{-\Delta}}{2a}s\right) + \mathrm{Ci}\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\sin\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\right] & \Delta < 0 \end{cases}$$

and involves functions from Chapters 26, 32, 36, and 37.

15:11 COMPLEX ARGUMENT

The real and imaginary parts of the quadratic function and its reciprocal when the argument is $z = x+iy$ are

15:11:1
$$az^2+bz+c = \left[a(x^2-y^2)+bx+c\right] + i\left[2axy+by\right]$$

and

15:11:2
$$\frac{1}{az^2+bz+c} = \frac{a(x^2-y^2)+bx+c}{\left[a(x^2+y^2)+bx+c\right]^2+y^2\Delta} - i\frac{2axy+by}{\left[a(x^2+y^2)+bx+c\right]^2+y^2\Delta}$$

Important inverse Laplace transforms include

15:11:3
$$\mathfrak{I}\left\{\frac{1}{as^2+bs+c}\right\} = \frac{\exp(r_+t)-\exp(r_-t)}{a(r_+-r_-)} = \begin{cases} (2/\sqrt{\Delta})\exp(-bt/2a)\sinh(\sqrt{\Delta}t/2a) & \Delta > 0 \\ (t/a)\exp(-bt/2a) & \Delta = 0 \\ (2/\sqrt{-\Delta})\exp(-bt/2a)\sin(\sqrt{-\Delta}t/2a) & \Delta < 0 \end{cases}$$

where, $r_\pm$ are given in 15:7:1 and, as before, $\Delta = b^2-4ac$.

15:12 GENERALIZATIONS

A quadratic function is a member of all the polynomial families [Chapters 17−24].

15:13 COGNATE FUNCTION: the root-quadratic function

The *root-quadratic function* $\sqrt{ax^2+bx+c}$ and its reciprocal are functions of some importance; they are clearly generalizations of the functions addressed in Chapters 11, 13, and 14. Some unifying properties of the root-quadratic function are presented in Section 15:15.

Several valuable integrals involving the reciprocal root-quadratic function are

15:13:1
$$\int_{-b/2a}^{x}\frac{1}{\sqrt{at^2+bt+c}}\mathrm{d}t=\begin{cases}\dfrac{-1}{\sqrt{-a}}\operatorname{arcsin}\left(\dfrac{b+2ax}{\sqrt{\Delta}}\right) & \Delta>0 \quad a<0 \quad \dfrac{b-\sqrt{\Delta}}{-2a}\le x\le\dfrac{b+\sqrt{\Delta}}{-2a}\\ \dfrac{1}{\sqrt{a}}\operatorname{arsinh}\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) & \Delta<0 \quad a>0\end{cases}$$

15:13:2
$$\int_{x_0}^{x}\frac{1}{\sqrt{at^2+bt+c}}\mathrm{d}t=\frac{1}{\sqrt{a}}\operatorname{arcosh}\left(\frac{2ax+b}{\sqrt{\Delta}}\right) \qquad \Delta>0 \qquad a>0 \qquad x_0=\frac{\sqrt{\Delta}-b}{2a}$$

15:13:3
$$\int_{-2c/b}^{x}\frac{1}{t\sqrt{at^2+bt+c}}\mathrm{d}t=\begin{cases}\dfrac{1}{\sqrt{-c}}\operatorname{arcsin}\left(\dfrac{bx+2c}{x\sqrt{\Delta}}\right) & \Delta>0 \quad c<0 \quad \dfrac{-2c}{b+\sqrt{\Delta}}\le x\le\dfrac{-2c}{b-\sqrt{\Delta}}\\ \dfrac{-1}{\sqrt{c}}\operatorname{arsinh}\left(\dfrac{bx+2c}{x\sqrt{-\Delta}}\right) & \Delta<0 \quad c>0\end{cases}$$

and

15:13:4
$$\int_{x_0}^{x}\frac{1}{t\sqrt{at^2+bt+c}}\mathrm{d}t=\frac{-1}{\sqrt{c}}\operatorname{arcosh}\left(\frac{bx+2c}{x\sqrt{\Delta}}\right) \qquad \Delta>0 \qquad c>0 \qquad x_0=\frac{2c}{\sqrt{\Delta}-b}$$

Others are given by Gradshteyn and Ryzhik [Section 2.26] and by Jeffrey [Section 4.3.4].

15:14 RELATED TOPIC: the trajectory of a projectile

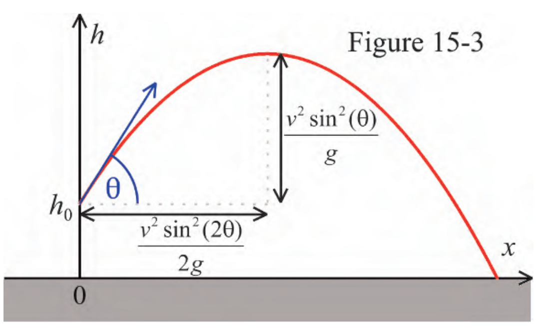


Figure 15-3

Heavy projectiles journey through the air following a parabolic course that is best described by a quadratic function. Neglecting the effect of air resistance, the object travels with a constant speed in the horizontal direction, while experiencing a constant acceleration (or force) vertically downwards. If the projectile is launched from a height h_0 with an initial velocity v at an angle θ to the horizontal, the equation describing its trajectory gives its height h, at a distance x downrange, as

15:14:1
$$h=\frac{-g\sec^2(\theta)}{2v^2}x^2+x\tan(\theta)+h_0$$

where g is the gravitational acceleration [see Appendix, Section A:6]. As Figure 15-3 will confirm, the greatest height is attained by launching at $\theta = \pi/2$, but the greatest range requires that $\theta = \pi/4$. The projectile remains airborne for the time interval

15:14:2
$$\frac{v}{g}\left[\sin(\theta) + \sqrt{\frac{2gh_0}{v^2} + \sin^2(\theta)}\right]$$

Multiply this expression by $v\cos(\theta)$ to find the total range.

15:15 RELATED TOPIC: conic sections

Figure 15-4 shows that there is a unity between the geometries of the functions discussed in Chapters 11, 13, and 14 that is not apparent when these geometries – those of the horizontal parabola, ellipse, and hyperbola – are described by the canonical formulations used in their respective chapters. However, if the curves are moved along the x-axis, so that one of the foci falls at $x = 0$, the three horizontal geometries come to be described by the single root-quadratic equation

15:15:1
$$\mathrm{f}(x) = \left[f_0^{\,2} + 2kf_0x + (k^2 - 1)x^2\right]^{1/2}$$

Whereas three *different* equations are normally used to describe the ellipse, the parabola and the hyperbola, this

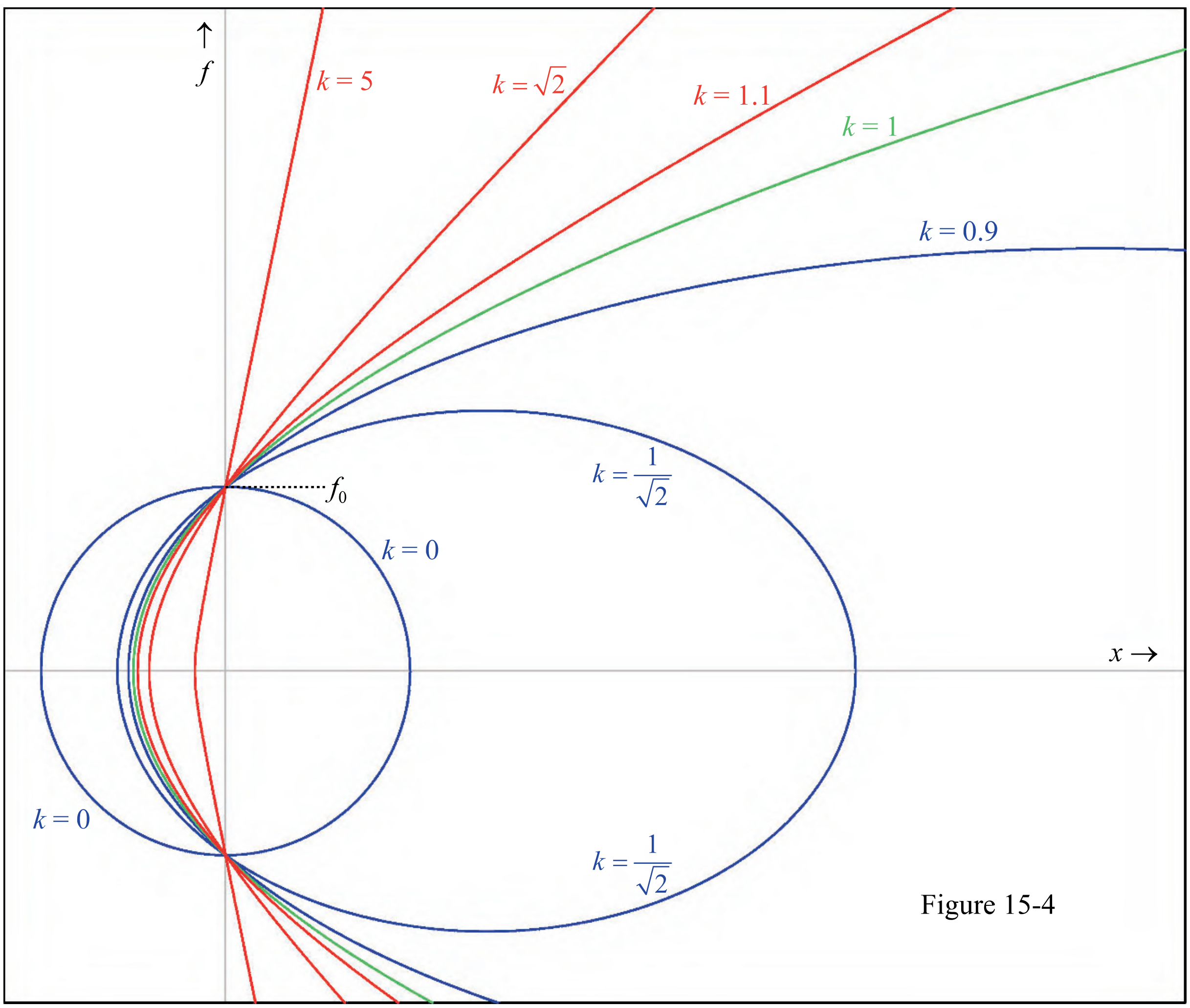


Figure 15-4

equation accommodates all three! If the eccentricity k lies between 0 and 1, equation 15:15:1 describes an ellipse; if $k=1$, the equation is that of a parabola; and if k exceeds 1, a hyperbola is represented. The quantity f_0 is the value of the function at $x=0$, that is, at the focus. Figure 15-4 was drawn using a range of k values, but a single f_0. The circle is the $k=0$ case and $k=\sqrt{2}$ corresponds to a rectangular hyperbola [Section 14:4]. The $k=1$ parabola separates the elliptical curves from those corresponding to hyperbolas (for clarity only one branch of each hyperbola is shown, though the equation describes both). For large k, the hyperbola is virtually a pair of straight lines, which is evident from equation 15:15:1 because, when k is so large that $k^2-1\approx k^2$, the equation becomes $\mathrm{f}(x)\approx\pm(kx+f_0)$.

Collectively, all these curves are called *conic sections*, or simply *conics*, because each can be generated by intersecting a cone with an appropriately oriented plane. More formally, they are known as *curves of the second degree*. Equation 15:15:1 can be considered the defining equation of any *horizontal conic*. Conics possess certain features in common. With the exception of the parabola, they each have two axes of mirror symmetry: one is the x-axis, the other being the line $x=kf_0/(1-k^2)$. In general they have two foci, with an interfocal separation of $2kf_0/(1-k^2)$, but this is zero for the circle and infinite for the parabola. Both foci lie on the x-axis with one at $x=0$. In the context of Figure 15-4, the second focus of the ellipses lies to the right of the origin whereas it lies to the left for the hyperbolas.

By rewriting equation 15:15:1 as the square root of the product of two linear functions

15:15:2 $$\mathrm{f}(x)=\left[\{f_0+(k-1)x\}\{f_0+(k+1)x\}\right]^{1/2}$$

one may identify the domain of the real function as

15:15:3 $$\frac{-f_0}{1+k}\le x\le\frac{f_0}{1-k}\qquad\text{when}\qquad 0\le k\le 1$$

15:15:4 $$-\infty\le x\le\frac{-f_0}{k-1}\quad\text{and}\quad\frac{-f_0}{k+1}\le x\le+\infty\qquad\text{when}\qquad k\ge 1$$

This conforms with the property that the ellipse is a contiguous curve, whereas each hyperbola has two branches (the left-hand branches are not shown in Figure 15-4).

Whereas equation 15:15:1 serves as a definition only of horizontal conics, there is a geometric definition that applies to a conic anywhere in the cartesian plane. Let F be a point in Figure 15-5 that will serve as a focus of the conic, and DD″ be a straight line, called the *directrix*, positioned anywhere in the plane and with any orientation. The conic is uniquely defined once the locations of the point and the line are selected, and a nonnegative constant k is chosen. Then the conic is defined as the locus of all points P such that

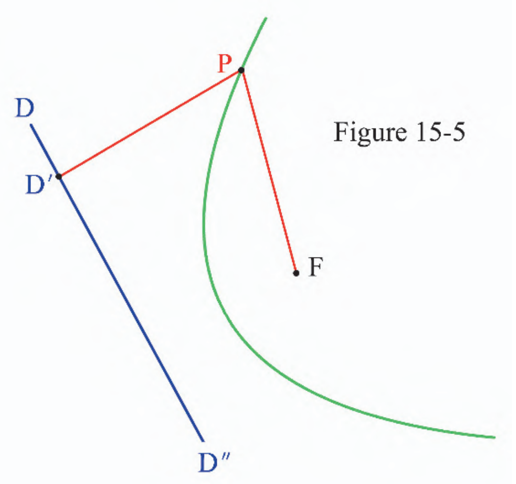


Figure 15-5

15:15:5 $$\frac{\mathrm{PF}}{\mathrm{PD}'}=k$$

where D′ is the nearest point on the directrix to P. The constant k is, of course, the *eccentricity*, so that

15:15:6 $$\text{If}\quad\frac{\mathrm{PF}}{\mathrm{PD}'}\begin{cases}<1, & \text{the conic is an ellipse}\\ =1, & \text{the conic is a parabola}\\ >1, & \text{the conic is a hyperbola}\end{cases}$$

So this simple property serves as a definition of all three types of curve. If the conic obeys equation 15:15:1, and the focus in question is that positioned at the origin, then the equation of the directrix is $x=-f_0/k$. Since, apart from the special cases of the circle and the parabola, each conic has two foci, so it has two directrices.

CHAPTER 16

THE CUBIC FUNCTION $x^3 + ax^2 + bx + c$

The cubic function is a polynomial function of degree 3, and accordingly the general properties of polynomials [Chapter 17] are applicable. Cubic functions find frequent application in data interpolation, a topic addressed in Section 16:14.

16:1 NOTATION

The most general formulation of a cubic function is $a_3x^3 + a_2x^2 + a_1x + a_0$, with four *coefficients*. However, it is a simple matter to factor out the leading coefficient and accordingly this chapter mostly addresses the function

16:1:1 $$\mathrm{f}(x) = x^3 + ax^2 + bx + c$$

The following quantities, that we term *parameters*, are important in determining the properties of the cubic function.

16:1:2 $$P = \frac{a^2 - 3b}{9}$$

16:1:3 $$Q = \frac{ab}{6} - \frac{c}{2} - \frac{a^3}{27}$$

and

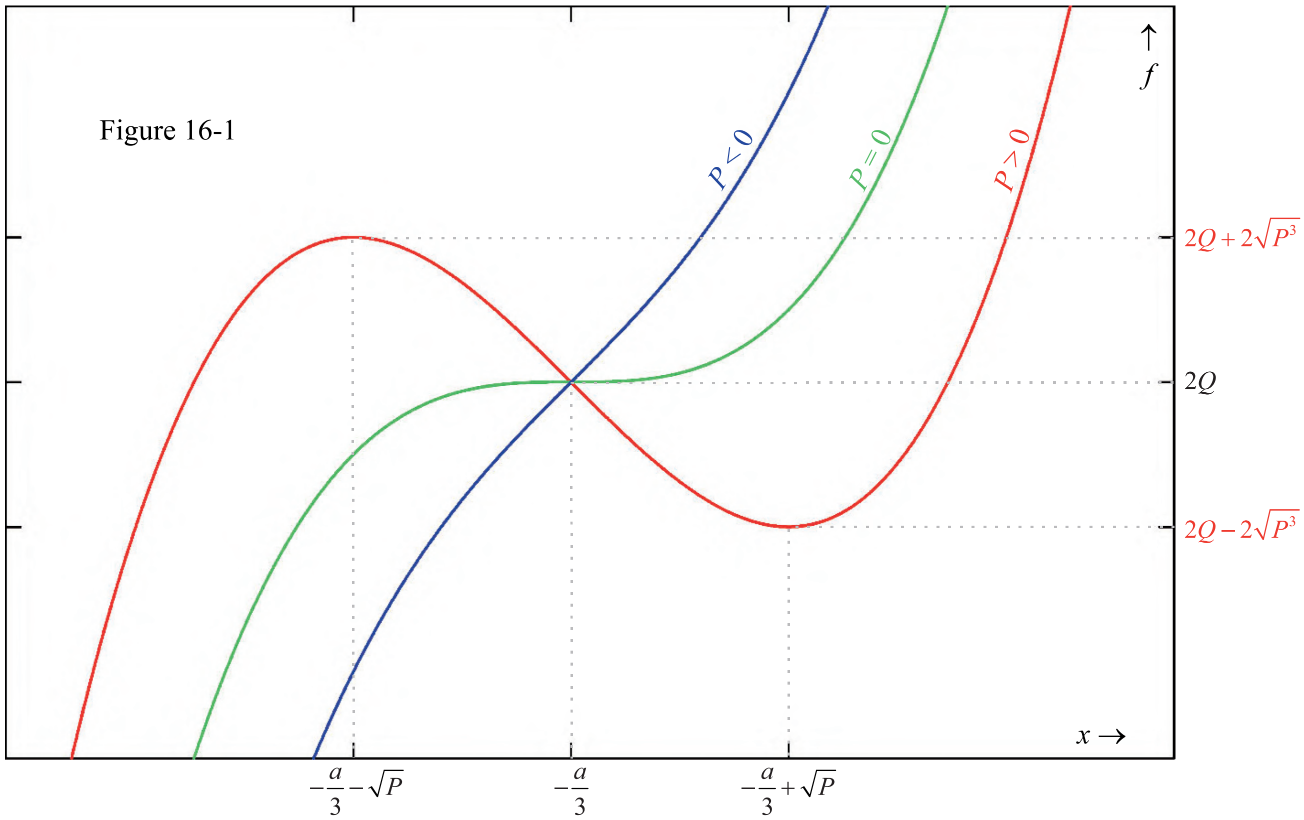
16:1:4 $$D = P^3 - Q^2$$

The last, or sometimes its negative, is known as the *discriminant* of the cubic function.

16:2 BEHAVIOR

Irrespective of the values of its coefficients, the range and domain of the cubic function are unrestricted. A cartesian graph of the cubic function $f = \mathrm{f}(x) = x^3 + ax^2 + bx + c$ has inversion symmetry [Section 14:15] through the point with rectangular coordinates

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_17, © Springer Science+Business Media, LLC 2009



16:2:1
$$(x, f) = \left(\frac{-a}{3}, 2Q\right)$$

which is also a point of inflection, as Figure 16-1 illustrates. This diagram shows graphs of the cubic function for representative values of P and demonstrates that the sign of this parameter determines whether the cubic function possesses extrema. If a^2 exceeds $3b$, so that P is positive, then there is a maximum at $x = -(a/3) - \sqrt{P}$ and a minimum at $x = -(a/3) + \sqrt{P}$. Otherwise the cubic function is a *monotonic function*; that is, its slope never changes sign.

Figure 16-1 does not locate $f = 0$, but it is clear that its location will determine the number of real zeros that the cubic function possesses. In fact, the existence of three distinct zeros requires that both P and D be positive.

16:3 DEFINITIONS

Writing the cubic functions as the *concatenation*

16:3:1
$$\mathrm{f}(x) = c + x\big(b + x(a + x)\big)$$

confirms that the arithmetic operations of addition and multiplication suffice to define the cubic function.

The product of three linear functions creates a cubic function:

16:3:2
$$(x - r_0)(x - r_1)(x - r_{-1}) = x^3 - (r_0 + r_1 + r_{-1})x^2 + (r_0 r_1 + r_0 r_{-1} + r_1 r_{-1})x - r_0 r_1 r_{-1}$$

but not all cubic functions can be defined in this way, unless two of the r's are sometimes allowed to assume complex values. The r quantities, the *zeros* of the cubic function, are addressed in Section 16:7. *Every* cubic function may, however, be defined as the product of a linear function and a quadratic function:

16:3:3 $$(x-r)\left(x^2+(a+r)x-\frac{c}{r}\right)=x^3+ax^2+bx+c$$

with r being the real zero, or one of the real zeros.

16:4 SPECIAL CASES

The cubic equation $\mathrm{f}(x)=x^3+ax^2+bx+c$ factors straightforwardly when the c coefficient, or both of the other coefficients, or any one of the parameters [Section 16:1] equals zero; thus;

16:4:1 $$c=0 \qquad \mathrm{f}(x)=x\left[x^2+ax+b\right]$$

16:4:2 $$a=b=0 \qquad \mathrm{f}(x)=\left[x+c^{1/3}\right]\left[x^2-c^{1/3}x+c^{2/3}\right]$$

16:4:3 $$P=0 \qquad \mathrm{f}(x)=\left[y-\sqrt[3]{2Q}\right]\left[y^2+\sqrt[3]{2Q}\,y+\sqrt[3]{4Q^2}\right] \qquad y=x+\frac{a}{3},\quad 2Q=\frac{a^3}{27}-c$$

16:4:4 $$Q=0 \qquad \mathrm{f}(x)=y\left[y^2-3P\right] \qquad y=x+\frac{a}{3},\quad 3P=\frac{a^2}{9}-\frac{3c}{a}$$

16:4:5 $$D=0 \qquad \mathrm{f}(x)=\left[y+\sqrt{P}\right]^2\left[y-2\sqrt{P}\right] \qquad y=x+\frac{a}{3},\quad \sqrt{P}=\frac{\sqrt{a^2-3b}}{3}$$

By $\sqrt[3]{2Q}$ we imply the *real* cube root of $2Q$.

If all the coefficients are zero, the cubic reduces to a power function $\mathrm{f}(x) = x^3$. If all the parameters are zero, reduction occurs to another power function: $\mathrm{f}(x) = [x + (a/3)]^3$.

16:5 INTRARELATIONSHIPS

The cubic function $\mathrm{f}(x)=x^3+ax^2+bx+c$ obeys the reflection formula

16:5:1 $$\mathrm{f}\left(\frac{-a}{3}-x\right)=-4Q-\mathrm{f}\left(\frac{-a}{3}+x\right)$$

where the parameter Q is defined in 16:1:2.

Setting $y = x + (a/3)$ converts one cubic function to another:

16:5:2 $$x^3+ax^2+bx+c=y^3-3Py-2Q$$

This transformation represents a simplification because the new argument appears only twice in the new formulation. A further contraction to a form in which there is a single appearance of the argument is also possible. The form of the new argument depends on the sign of the P parameter and, if P is positive, also on the magnitude of $y/\sqrt{P}$. For negative P

16:5:3 $$\mathrm{f}(x)=2\left[\sqrt{(-P)^3}\,\sinh(t)-Q\right] \qquad \text{where} \qquad t=3\,\mathrm{arsinh}\left(\frac{3x+a}{6\sqrt{-P}}\right) \qquad P<0$$

When P is positive and larger than $[(3x + a)/6]^2$

16:5:4 $$\mathrm{f}(x)=2\left[\sqrt{P^3}\,\cos(\theta)-Q\right] \qquad \text{where} \qquad \theta=3\arccos\left(\frac{3x+a}{6\sqrt{P}}\right) \qquad P>\frac{(3x+a)^2}{36}$$

whereas if P is positive but smaller than $[(3x+a)/6]^2$

16:5:5 $$\mathrm{f}(x)=2\left[\sqrt{P^3}\,\mathrm{sgn}\left(\frac{3x+a}{6\sqrt{P}}\right)\cosh(t)-Q\right] \qquad \text{where} \qquad t=3\,\mathrm{arcosh}\left(\left|\frac{3x+a}{6\sqrt{P}}\right|\right)$$

The keys to deriving these formulas lie in equations 28:5:6, 32:5:5, and 28:5:5.

The inverse function of the cubic is multivalued if $P>0$, but can be shown from 16:5:3 to be

16:5:6 $$\frac{-a}{3}+2\sqrt{-P^3}\,\sinh\left\{\frac{1}{3}\mathrm{arsinh}\left(\frac{x+2Q}{2\sqrt{-P^3}}\right)\right\} \qquad P<0$$

for a cubic with a negative P parameter.

16:6 EXPANSIONS

The expansions discussed in Section 17:6 apply, but they are of little utility for the cubic function.

16:7 PARTICULAR VALUES

The cubic function x^3+ax^2+bx+c has an inflection at $x=-a/3$, irrespective of the other two coefficients. As Figure 16-1 shows, a maximum and a minimum are exhibited only if the parameter P is positive.

Equator's notation for the three zeros of the cubic functions is $\mathrm{r}_3(a,b,c,n)$, with $n=0,\pm1$. If any one of c, P, Q, or D are zero, or if both a and b are zero, then the zeros may be found straightforwardly from the special-case equations in Section 16:4. Otherwise the zeros are calculable by the procedure outlined in the following paragraph. One of these zeros, $\mathrm{r}_3(a,b,c,0)$ will be real invariably, but the other two, $\mathrm{r}_3(a,b,c,+1)$ and $\mathrm{r}_3(a,b,c,-1)$, will be complex (or imaginary) unless both P and D are positive. When two complex zeros exist, they always occur as a *conjugate pair*; that is, they have identical real parts and their imaginary parts are equal in magnitude but opposite in sign.

One real zero and two complex zeros exist when $P<0$, irrespective of the value of D; they are:

16:7:1 $$\left.\begin{aligned}\mathrm{r}_3(a,b,c,0)&=\frac{-a}{3}+2\sqrt{-P}\,\sinh(t)\\ \mathrm{r}_3(a,b,c,\pm1)&=\frac{-a}{3}-\sqrt{-P}\,\sinh(t)\pm i\sqrt{-3P}\,\cosh(t)\end{aligned}\right\} \quad t=\frac{1}{3}\mathrm{arsinh}\left(\frac{Q}{\sqrt{(-P)^3}}\right)$$

For $P>0$ and $D>0$, there are three real zeros:

16:7:2 $$\mathrm{r}_3(a,b,c,n)=\frac{-a}{3}+2\sqrt{P}\cos\left(\frac{2n\pi+\arccos\left(Q/\sqrt{P^3}\right)}{3}\right) \qquad n=0,\pm1$$

For $P>0$ and $D<0$, there is one real zero and two complex zeros:

16:7:3 $$\left.\begin{aligned}\mathrm{r}_3(a,b,c,0)&=\frac{-a}{3}+u+v\\ \mathrm{r}_3(a,b,c,\pm1)&=\frac{-a}{3}-\frac{u+v}{2}\pm i\sqrt{3}\,\frac{u-v}{2}\end{aligned}\right\} \quad \left\{\begin{aligned}u&=\sqrt[3]{Q-\sqrt{-D}}\\ v&=\sqrt[3]{Q+\sqrt{-D}}\end{aligned}\right.$$

By $\sqrt[3]{}$ we imply the *real* cube root. These formulas originate from equations 16:5:3−5 and are used by *Equator* in its cubic zeros routine (keyword **r3**), though simpler methods are employed for the special cases enumerated in Section 16:4. *Equator* treats the zeros as a complex-valued quadrivariate function

16:7:4 $$\mathrm{r}_3(a,b,c,n) = \mathrm{Re}\left[\mathrm{r}_3(a,b,c,n)\right] + i\,\mathrm{Im}\left[\mathrm{r}_3(a,b,c,n)\right] \qquad n = -1, 0, +1$$

and outputs both real and imaginary parts, the latter being 0 whenever the zero is real. By default, *Equator* generates a short table giving all three zeros. The algorithm is exact but, because large losses of significance can occasionally occur, check answers carefully if precision is an issue.

16:8 NUMERICAL VALUES

These are easily calculated, for example through *Equator*'s cubic function routine (keyword **cubic**).

16:9 LIMITS AND APPROXIMATIONS

The cubic function is dominated by its x^3 term when its argument is of large magnitude.

There is seldom a need to approximate a cubic function; on the contrary, cubic functions are themselves often used to approximate more complicated functions, as explained in Section 16:14.

16:10 OPERATIONS OF THE CALCULUS

As with all polynomials, the operations of the calculus may be carried out on the cubic function term by term:

16:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + ax^2 + bx + c) = 3x^2 + 2ax + b$$

16:10:2 $$\int_0^x (t^3 + at^2 + bt + c)\,\mathrm{d}t = \frac{3x^4 + 4ax^3 + 6bx^2 + 12cx}{12}$$

16:10:3 $$\int_0^\infty (t^3 + at^2 + bt + c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{t^3 + at^2 + bt + c\right\} = \frac{cs^3 + bs^2 + 2as + 6}{s^4}$$

Integrals of $\mathrm{f}(t)/\left(t^3 + at^2 + bt + c\right)$ can often be evaluated by following the procedure described in Section 17:13. Indefinite integrals of such functions as $1/\sqrt{t^3 + at^2 + bt + c}$ or $t/\sqrt{t^3 + at^2 + bt + c}$ are the subject of Section 62:14.

16:11 COMPLEX ARGUMENT

When the argument is $z = x + iy$, the real and imaginary parts of the cubic function are

16:11:1 $$z^3 + az^2 + bz + c = \left[x^3 + a(x^2 - y^2) + bx + c - 3xy^2\right] + iy\left[3x^2 - y^2 + 2axy + by\right]$$

The cubic function of complex argument encounters no poles or other discontinuities, other than at infinity.

16:12 GENERALIZATIONS: including zeros of the quartic function

A cubic function is a polynomial of degree 3. As such, it is the third member in a hierarchy of which the linear function and quadratic functions are lower members and the quartic, quintic, etc. are higher members. The *Atlas* treats these higher members as a general family in the next chapter. However, one aspect of *quartic* functions is addressed here because the properties of cubic functions are relevant.

If the coefficients of the quartic

16:12:1 $$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

are real, the quartic's zeros may be calculated by first finding any zero r of the so-called *cubic resolvent function*

16:12:2 $$x^3 + ax^2 + bx + c \qquad \text{where} \quad \begin{cases} a = -a_2 \\ b = a_1a_3 - 4a_0 \\ c = 4a_0a_2 - a_1^2 - a_0a_3^2 \end{cases}$$

using the method of Section 16:7. Then the four zeros of 16:12:1 can usually be found from *Ferrari's solution* [Lodovico Ferrari, Italian mathematician, 1522−1565]

16:12:3 $$\mathrm{r}_4(a_3,a_2,a_1,a_0) = \left\{\begin{matrix} (-a_3 + s \pm q)/4 \\ \text{and} \\ (-a_3 - s \pm p)/4 \end{matrix}\right\} \qquad \text{where} \quad \begin{cases} s = \sqrt{4r - 4a_2 + a_3^2} \\ q = \sqrt{2a_3^2 - 4a_2 - 4r + (8a_2a_3 - 2a_3^3 - 16a_1)/s} \\ p = \sqrt{2a_3^2 - 4a_2 - 4r - (8a_2a_3 - 2a_3^3 - 16a_1)/s} \end{cases}$$

If s should equal zero, the term $(8a_2a_3 - 2a_3^3 - 16a_1)/s$ is to be replaced by $8\sqrt{r^2 - 4a_0}$.

Ferrari's solution is the basis of *Equator*'s quartic zeros routine (keyword **r4**). However, if the chosen cubic zero leads to a small value of s, serious precision loss may occur. To counter this, *Equator* selects the cubic zero that leads to the largest s.

16:13 COGNATE FUNCTIONS

The *reciprocal cubic function* $1/(x^3 + ax^2 + bx + c)$ is of some interest and provides an exemplary model of reciprocal polynomial functions [Section 17:13] in general.

Partial fractionation is a method of expanding reciprocal polynomials. Equation 16:3:3 may be used to suggest the splitting of the reciprocal cubic as follows

16:13:1 $$\mathrm{f}(x) = \frac{1}{x^3 + ax^2 + bx + c} = \frac{1}{(x-r)\left(x^2 + (a+r)x - \dfrac{c}{r}\right)} = \frac{\alpha}{x-r} + \frac{\beta x + \gamma}{x^2 + (a+r)x - \dfrac{c}{r}}$$

where r is a real zero, the constants α, β, and γ being initially unknown. They may be determined, however, by first multiplying 16:13:1 by the cubic to remove the denominators, which leads to

16:13:2 $$1 = [\alpha + \beta]x^2 + [\alpha(a+r) - \beta r + \gamma]x + \left[\alpha\frac{c}{r} - \gamma r\right]$$

This is an identity and therefore one may equate coefficients of like terms from each side of equation 16:13:2. The three simultaneous equations $\alpha + \beta = 0$, $\alpha(a + r) - \beta r + \gamma = 0$, and $(-\alpha c/r) - \gamma r = 1$ that emerge may then be solved, leading eventually to

16:13:3 $$\mathrm{f}(x) = \frac{r}{r^2 + ar - \frac{c}{r}} \left[\frac{1}{x-r} - \frac{x+a+2r}{x^2 + (a+r)x - \frac{c}{r}} \right] = \frac{-r}{br+2c} \left[\frac{1}{x-r} - \frac{x+a+2r}{x^2 + (a+r)x - \frac{c}{r}} \right]$$

The second equality is a consequence of $r^3 + ar^2 + br + c = 0$. Further expansion of the final term into the sum of two reciprocal linear terms is possible, but these may be complex.

Partial fractionation is often used as a prelude to an operation of the calculus. It permits, for example, the integral of a reciprocal cubic function to be evaluated, via equation 16:13:3, with the aid of formulas 7:10:4, 15:10:4, and 15:10:5. It is used abundantly in Laplace inversion [Section 26:15].

16:14 RELATED TOPICS: the sliding cubic and the cubic spline

x	$\mathrm{f}(x)$
x_0	f_0
x_1	f_1
$\vdots$	$\vdots$
x_{j-1}	f_{j-1}
x_j	f_j
x_{j+1}	f_{j+1}
x_{j+2}	f_{j+2}
$\vdots$	$\vdots$
x_{J-1}	f_{J-1}
x_J	f_J

Technologists and engineers commonly collect extensive lists of values f of a function f(x) without knowing the form of the relationship between f(x) and its argument x. The table shows fragments of such a list. A frequent need is to present these data graphically, or use them to estimate a value of the function at an argument where no measurement was made. Two situations arise in this setting. In the first, the tabular data are regarded as exact and the problem is one of *interpolation*. In this case, the task is the selection of a relationship that is satisfied locally or globally, that relationship then being assumed to apply equally well between measurement points. In the second scenario, error is assumed to contaminate the f data and a (usually rather simple) relationship is sought that does not exactly reproduce the measured f_j values, but comes close. Such a procedure is known as *regression*; Section 7:14 is devoted to the simplest kind of regression, in which the function to which the data are fitted is a straight line, and the use of more complicated fitting functions is explored in Section 17:14.

Polynomials are commonly used for both interpolation and regression. The remainder of this section addresses two ways in which *piecewise-cubic functions* are employed in interpolation. The first, which provides a satisfactory interpolation without undue complexity, is the *sliding cubic* or *Lagrange four-point interpolate*. The idea is that a cubic function is fitted so as to pass through a quartet of adjacent data pairs: (x_{j-1}, f_{j-1}), (x_j, f_j), (x_{j+1}, f_{j+1}), and (x_{j+2}, f_{j+2}), but is used to represent the data only between the middle two points of the quartet. The cubic that has this property is, for $x_j \le x \le x_{j+1}$,

16:14:1 $$\hat{\mathrm{f}}_j(x) = \sum_{k-j=-1}^{+2} \frac{(x-x_l)(x-x_m)(x-x_n)}{(x_k-x_l)(x_k-x_m)(x_k-x_n)} f_k$$

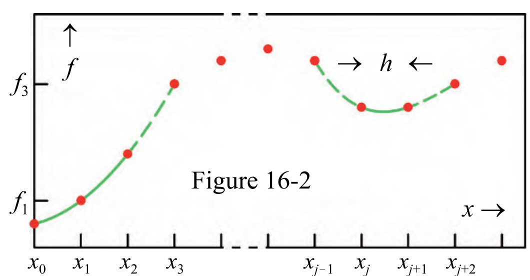


Figure 16-2

where l, m, and n are the three integers other than k from the set $(j-1, j, j+1, j+2)$. In the common case, illustrated in Figure 16-2, in which the data are evenly spaced so that $x_{j+2} - x_{j+1} = x_{j+1} - x_j = x_j - x_{j-1} = h$, equation 16:14:1 becomes

16:14:2 $$\hat{\mathrm{f}}_j(x) = a_{3j}\left(\frac{x-x_j}{h}\right)^3 + a_{2j}\left(\frac{x-x_j}{h}\right)^2 + a_{1j}\frac{x-x_j}{h} + a_{0j} \qquad \begin{cases} a_{3j} = \frac{1}{6}f_{j+2} - \frac{1}{2}f_{j+1} + \frac{1}{2}f_j - \frac{1}{6}f_{j-1} \\ a_{2j} = \frac{1}{2}f_{j+1} - f_j + \frac{1}{2}f_{j-1} \\ a_{1j} = -\frac{1}{6}f_{j+2} + f_{j+1} - \frac{1}{2}f_j - \frac{1}{3}f_{j-1} \\ a_{0j} = f_j \end{cases}$$

This is the interpolating cubic fitted over $x_{j-1}\le x\le x_{j+2}$ but used only for the argument range from x_j to x_{j+1}, as shown at the right-hand side in Figure 16-2. As x reaches and passes x_{j+1}, the cubic "slides" to a next quartet. Of course, no quartet is available for the end regions $x_0\le x\le x_1$ or $x_{J-1}\le x\le x_J$; and so the interpolating cubics from the penultimate internodal zones $x_1\le x\le x_0$ and $x_{J-2}\le x\le x_{J-1}$ are taken to apply to the end zones too. This is illustrated at the left in Figure 16-2.

The curve produced by the sliding cubic interpolation is continuous, but there is a small (and often visually undetectable) discontinuity in slope at each node. This defect is overcome in the *cubic spline* which not only has no discontinuity in the slope (that is, in the first derivative of f) at the nodal points, but no discontinuity in the second derivative either! The equation describing the interpolated spline between the nodes x_j and x_{j+1} is the cubic function

16:14:3 $$\hat{\mathrm{f}}_j(x)=a_{3j}\left(\frac{x-x_j}{h}\right)^3+a_{2j}\left(\frac{x-x_j}{h}\right)^2+a_{1j}\frac{x-x_j}{h}+a_{0j}\qquad\begin{cases}a_{3j}=\frac{1}{6}g_{j+1}-\frac{1}{6}g_j\\ a_{2j}=\frac{1}{2}g_j\\ a_{1j}=f_{j+1}-f_j-\frac{1}{3}g_j-\frac{1}{6}g_{j+1}\\ a_{0j}=f_j\end{cases}$$

when the data are separated evenly by h. The g terms are proportional to the second derivatives of the spline at its nodes; for example

16:14:4 $$g_j=h^2\left.\frac{\mathrm{d}^2\hat{\mathrm{f}}}{\mathrm{d}x^2}\right|_{x=x_j}$$

These terms are unknown a priori; however, the recursion formula

16:14:5 $$g_{j+1}=f_{j+1}-2f_j+f_{j-1}-4g_j-g_{j-1}$$

interrelates three consecutive g values. There are $J-1$ recursions linking the g_0, g_1, g_2, $\cdots$, g_{J-1}, g_J terms. These recursion equations may be solved simultaneously if g_0 and g_J are taken to be zero. Thereby a *natural cubic spline* may be created. A natural spline is one that is linear at its extremities. The description of splines given by Chapra and Canale [pages 495−505] is very readable and their book provides formulas for unequally spaced data. Hamming [Section 20.9], another excellent source, discusses "unnatural" splines and shows how to set up a tridiagonal matrix to solve the simultaneous equations.

There is a heavy computational burden in the creation of a cubic spline but the result is extremely smooth. Because the fitting is "global", there is a disconcerting dependence of the shape at one end of the fitted curve upon data at the other end. Such an effect is, of course, entirely absent in the sliding cubic and other "local" interpolations.

CHAPTER 17

POLYNOMIAL FUNCTIONS

Polynomial functions find application in modeling other functions that are of a more complex, or of unknown, form. Much attention in this chapter is directed towards the zeros of polynomials because, in applications, these are often the crucial feature of a polynomial function.

17:1 NOTATION

A polynomial function is often simply called a *polynomial*; another synonym is *integral function*.

This *Atlas* uses the general notation

17:1:1 $$\mathrm{p}_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{j=0}^{n} a_j x^j \qquad a_n \neq 0$$

to denote a polynomial function of argument x and degree n. The a constants, treated as real in this chapter, are the *coefficients* of the polynomial: there are $n+1$ of them, some of which (but not a_n) may be zero. Each unit in 17:1:1, typically $a_j x^j$, is described as a *term*. Special names are given to polynomials of degrees 5, 4, 3, 2, 1, or 0; they are known as *quintic functions*, *quartic functions*, *cubic functions*, *quadratic functions*, *linear functions*, or *constant functions*.

The values of the argument x that cause the polynomial $\mathrm{p}_n(x)$ to become zero are termed the *zeros* of the polynomial. There are n such values that will be denoted $\rho_1, \rho_2, \cdots, \rho_j, \cdots, \rho_n$ and they may be real or complex. They are also known as the *roots* of the equation $\mathrm{p}_n(x) = 0$.

17:2 BEHAVIOR

A polynomial function is defined for all values, $-\infty < x < \infty$, of its argument. A polynomial of odd degree has an unrestricted range, whereas the range is semiinfinite if n is even. A *semiinfinite* range is one that extends to either $+\infty$ or $-\infty$, but not to both.

A polynomial may have real zeros and/or minima and/or maxima and/or inflections. The number of these occurrences, if any, of these features depends in a complicated way on the coefficients. In the general case, all that

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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can be said is that the numbers of the various features are bracketed as tabulated here.

	$n = 3, 5, 7, \cdots$	$n = 2, 4, 6, \cdots$	
		$a_n > 0$	$a_n < 0$
N_r, the number of real zeros	$1 \le N_\mathrm{r} \le n$	$0 \le N_\mathrm{r} \le n$	$0 \le N_\mathrm{r} \le n$
N_m, the number of minima	$0 \le N_\mathrm{m} \le (n-1)/2$	$1 \le N_\mathrm{m} \le (n/2)$	$N_\mathrm{m} = N_\mathrm{M} - 1$
N_M, the number of maxima	$N_\mathrm{M} = N_\mathrm{m}$	$N_\mathrm{M} = N_\mathrm{m} - 1$	$1 \le N_\mathrm{M} \le (n/2)$
N_i, the number of inflections	$1 \le N_\mathrm{i} \le n-2$	$0 \le N_\mathrm{i} \le n-2$	$0 \le N_\mathrm{i} \le n-2$

17:3 DEFINITIONS

Writing the polynomial function in the *concatenated form*

17:3:1 $$\mathrm{p}_n(x) = a_0 + x\Big(a_1 + x\Big(a_2 + x\Big(a_3 + \cdots + x(a_{n-2} + x\big(a_{n-1} + xa_n\big)\big)\cdots\Big)\Big)\Big)$$

shows that the arithmetic operations of multiplication and addition suffice to define a polynomial function. A concatenation is also described as *nested sum.*

The product of n linear functions produces a polynomial of degree n

17:3:2 $$\prod_{j=1}^{n}\left(bx + c_j\right) = \mathrm{p}_n(x) \qquad b = (a_n)^{1/n}$$

but, with c_j real, not every polynomial function may be defined in this way. However, *every* polynomial of degree n may be defined as the n-fold product

17:3:3 $$\mathrm{p}_n(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_j)\cdots(x - r_n) = a_n\prod_{j=1}^{n} x - r_j$$

where each r is either a real zero or a complex zero. These zeros are often referred to as *roots*, because each zero of the polynomial $\mathrm{p}_n(x)$ is also a root of the equation $\mathrm{p}_n(x) = 0$. Complex zeros occur in *conjugate pairs* that share the same real part, but have imaginary parts that sum to zero. Thus if r_j and r_{j+1} are such a conjugate pair

17:3:4 $$(x - r_j)(x - r_{j+1}) = \big(x - (\rho_j + i\iota_j)\big)\big(x - (\rho_j - i\iota_j)\big) = x^2 - 2\rho_j x + \rho_j^2 + \iota_j^2$$

where ρ_j and ι_j denote the real and imaginary parts, $\mathrm{Re}\big[r_j\big]$ and $\mathrm{Im}\big[r_j\big]$, of the jth zero. An expansion equivalent to 17:3:3, but avoiding complex or imaginary quantities is

17:3:5 $$\mathrm{p}_n(x) = \prod_{j=1}^{N_\mathrm{r}}\left(x - r_j\right)\prod_{k}(x^2 - 2\rho_k x + \rho_k^2 + \iota_k^2) \qquad k = (N_\mathrm{r}+1), (N_\mathrm{r}+3), \cdots, (n-1)$$

where N_r is the number of real zeros. This representation shows why there are two classes of factor – "linear zeros" and "quadratic zeros" – from either or both of which any polynomial may be constructed.

Interest in equation 17:3:3 is predominantly in the direction of finding the zeros from a known polynomial, rather than in construction a polynomial from known zeros. This matter is taken up in Section 17:7.

17:4 SPECIAL CASES

The $n = 0$, 1, 2, and 3 instances of the polynomial function $\mathrm{p}_n(x)$ are addressed in Chapters 1, 7, 15, and 16. The zeros of $n = 4$ polynomials are the subject of Section 16:12. Power series [Section 10:13] are polynomials of infinite degree.

If the coefficients a_j of a polynomial function obey the relationship

17:4:1 $$\frac{a_{n-j}}{a_n} = \binom{n}{j}\left[\frac{a_0}{a_n}\right]^{j/n} \qquad j = 0,1,2,\cdots,n-1,n$$

then the polynomial $\sum a_j x^j$ reduces to the power function $(bx+c)^n$ of Chapter 11 with $b = (a_n)^{1/n}$ and $c = (a_0)^{1/n}$.

Each of Chapters 18−24 addresses a polynomial family with special coefficients.

17:5 INTRARELATIONSHIPS

Polynomial functions may be added and subtracted term by term. The multiplication of two polynomials $\mathrm{p}_n(x)$ and $\mathrm{p}_m(x)$ gives a polynomial function of degree $n+m$. The quotient of two polynomials is, in general, a rational function [Section 17:13].

The argument-multiplication formula

17:5:1 $$\mathrm{p}_n(vx) = \sum_{j=0}^{n}\left(v^j a_j\right)x^j \qquad \text{where} \qquad \mathrm{p}_n(x) = \sum_{j=0}^{n} a_j x^j$$

shows that multiplying the argument of a polynomial by a constant generates another polynomial of the same degree but with a revised set of coefficients. Another valuable transformation that produces one polynomial from another arises if the argument x is replaced by $1/y$. One finds that

17:5:2 $$a_n\left(\frac{1}{y}\right)^n + a_{n-1}\left(\frac{1}{y}\right)^{n-1} + \cdots + a_1\left(\frac{1}{y}\right) + a_0 = y^{-n}\left[a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1}y + a_n\right]$$

so that the order in which the coefficients occur has been reversed. An important consequence of this is that the *zeros of the transformed polynomial are the reciprocals of the zeros of the original polynomial*.

A *factor* of the polynomial $\mathrm{p}_n(x)$ is any polynomial of degree less than n such that the division of $\mathrm{p}_n(x)$ by this factor creates a third polynomial, leaving no remainder, irrespective of x. Thus, if

17:5:3 $$\frac{\mathrm{p}_n(x)}{\mathrm{p}_m(x)} = \mathrm{p}_{n-m}(x) \qquad m < n$$

then both $\mathrm{p}_m(x)$ and $\mathrm{p}_{n-m}(x)$ are factors of $\mathrm{p}_n(x)$. The division prescribed in 17:5:3 is a step in the process of finding the zeros of a polynomial, as addressed in Section 17:7. The procedure is called *synthetic division* or *polynomial deflation*. The most important instances are when $m = 1$ or 2; that is, when the divisor is a linear or a quadratic function. If the factor is the linear function $x-r$, r being real, then the new (deflated) coefficients resulting from the operation

17:5:4 $$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x-r} = a'_{n-1}x^{n-1} + a'_{n-2}x^{n-2} + \cdots + a'_1 + a'_0$$

can be found by the recursive algorithm

17:5:5
$$a'_j = \begin{cases} 0 & \text{for} \quad j = n \\ a_{j+1} & \text{for} \quad j = n-1 \\ a_{j+1} + ra'_{j+1} & \text{for} \quad j = n-2,\, n-3, \cdots, 2, 1, 0 \end{cases}$$

Notice that this procedure does not involve the argument x; in effect x has been set to unity. Nor is the final coefficient a_0 involved. This coefficient is, however, present in the "remainder" of the division, which is the quantity that would have been assigned to a'_{-1} had algorithm 17:5:5 been executed one step further than necessary; that is:

17:5:6
$$\text{remainder} = a'_{-1} = a_0 + ra'_0$$

Of course if $x-r$ is indeed a factor, this remainder will be zero (or a very small quantity arising from arithmetic roundoff), providing a valuable check. When the factor is quadratic, say $x^2 + px + q$, deflation decreases the degree by 2:

17:5:7
$$\frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{x^2 + px + q} = a'_{n-2} x^{n-2} + a'_{n-3} x^{n-3} + \cdots + a'_1 + a'_0$$

The algorithm for the new coefficients is again recursive:

17:5:8
$$a'_j = \begin{cases} 0 & \text{for} \quad j = n, n-1 \\ a_{j+2} & \text{for} \quad j = n-2 \\ a_{j+2} - pa'_{j+1} & \text{for} \quad j = n-3 \\ a_{j+2} - pa'_{j+1} - qa'_{j+2} & \text{for} \quad j = n-4,\, n-5, \cdots, 2, 1, 0 \end{cases}$$

but this time there are two "remainders"

17:5:9
$$\text{remainders} = \begin{cases} a'_{-1} = a_1 - pa'_0 - qa'_1 \\ a'_{-2} = a_0 - qa'_0 \end{cases}$$

both of which should be numerically negligible if $x^2 + px + q$ is indeed an exact factor.

17:6 EXPANSIONS

In addition to that shown in 17:3:1, the alternative form

17:6:1
$$\sum_{j=0}^{n} a_j x^j = a_0 + b_1 x\Big(1 + b_2 x\Big(1 + b_3 x\Big(1 + \cdots + b_{n-1} x\big(1 + b_n x\big)\cdots\Big)\Big)\Big) \qquad b_j = \frac{a_0\, a_j}{a_{j-1}}$$

is sometimes a more convenient concatenated expansion of the polynomial function. Conversion of this representation to a continued fraction in the

17:6:2
$$\frac{a_0}{1-}\,\frac{c_1 x}{1+}\,\frac{c_2 x}{1-}\,\frac{c_3 x}{1+}\,\frac{c_4 x}{1-}\cdots\frac{c_{n-2}}{1\pm}\,\frac{c_{n-1}}{1 \mp c_n}$$

format is possible through the Rutishauser transformation [Section 10:14]. Another style of continued fraction expansion of the polynomial $p_n(x)$, namely

17:6:3
$$\sum_{j=0}^{n} a_j x^j = \frac{a_0}{1-}\,\frac{a_1 x}{a_0 + a_1 x -}\,\frac{a_0 a_2 x}{a_1 + a_2 x -}\,\frac{a_1 a_3 x}{a_2 + a_3 x -}\cdots\frac{a_{n-2} a_n x}{a_{n-1} + a_n x}$$

is given in Section 0:6. The important expansions as products are shown as equations 17:3:3 and 17:3:4.

17:7 PARTICULAR VALUES

The particular values of a polynomial include its zeros, maxima, minima, and inflections. The extrema and the inflections are the zeros of the polynomials derived, respectively, by differentiating the original polynomial function once or twice. Therefore, locating zeros lies at the heart of finding all these particular values. It is only for polynomials of degrees 1, 2, 3, and 4 that analytical expressions for zeros exist in the general case. Formulas for finding the zeros of polynomials of degree 2, 3, and 4 are reported in Sections 15:7, 16:7, and 16:12.

The polynomial function $p_n(x)$ has n zeros and, typically, some will be real and some will be complex. The mathematical and computational aspects of finding the zeros of an arbitrary polynomial is a problem that has occupied the attention of scholars for many years, but the definitive solution, if one exists, has yet to be discovered. Some of the pitfalls are exposed in the excellent commentaries of Acton [Chapter 6] and Hamming [Chapter 7]. The problem is tricky in that certain zeros may be elusive, and computationally intricate, inasmuch as several different operations are involved in the overall process. If you need to find the zeros of many polynomials, or if your particular polynomial is of large degree or is known to be "awkward", you will probably choose to employ the services of the purpose-designed software available in such packages as Excel®, Maple®, Mathematica®, or Matlab®, or you may build your own algorithm using the procedures advocated by IMSL (the International Mathematical and Statistical Library), by Press et al., or by Chapra and Canale [see Bibliography for details of these sources]. Accordingly, this *Atlas* provides only the following overview of a pedestrian approach suitable for locating the zeros of polynomials that lack "awkwardness". Awkward polynomials are ones with zeros (i) six or more of which are complex; (ii) that are of high multiplicity [Section 0:7]; (iii) that, while not being identical, are discomfitingly close together; or (iv) that are evenly spaced.

The problem is to find all the real zeros, the r's, together with all the $(2\rho,\rho^2+\iota^2)$ pairs, these latter being the quantities defined in 17:3:4. First, inspect for "obvious" zeros. If $a_0 = 0$, $r = 0$ must be a zero. If n is odd and the coefficients sum to zero, $r = 1$ will be a zero. You may choose to divide the polynomial by a_n; this won't change the zeros or help you to find them, but it will give you one less number to worry about.

After these preliminaries, compute the values of the polynomial at many (say 51) equally spaced x-values that embrace the range between -1 and $+1$ (say at $x = -1.02, -0.98, \cdots, 0.98, 1.02$). While doing that, you may as well let your computer also find values of the polynomial's first and second derivatives

17:7:1 $$\mathrm{p}_n(x)=\sum_{j=0}^{n}a_jx^j \qquad \frac{\mathrm{d}\,\mathrm{p}_n}{\mathrm{d}x}(x)=\sum_{j=0}^{n-1}(j+1)a_{j+1}x^j \qquad \frac{\mathrm{d}^2\,\mathrm{p}_n}{\mathrm{d}x^2}(x)=\sum_{j=0}^{n-2}(j+1)(j+2)a_{j+2}x^j$$

Inspect a table of these data, looking primarily for changes in sign. If the polynomial changes sign between one value of x and the next, but nothing dramatic happens to its derivatives, it is likely that a single real zero r lies in this interval. If it is a double zero that occurs within the interval, $\mathrm{dp}/\mathrm{d}x$ will change sign but p_n and $\mathrm{d}^2\mathrm{p}_n/\mathrm{d}x^2$ will not, though p_n will be small in magnitude. If p_n and $\mathrm{d}^2\mathrm{p}_n/\mathrm{d}x^2$ change sign, but not $\mathrm{dp}_n/\mathrm{d}x$, this signals the likelihood that a triple zero lies within, or close to, the interval, though the awkward possibility of one double zero and one single zero or even three single zeros cannot be excluded. No information concerning the quadratic zeros will be provided by this sign-change survey. Moreover it will, of course, only locate those real zeros that lie within the $-1 \le x \le 1$ range. To find the r's lying outside this range, replace the argument of the polynomial by its reciprocal, and survey the $-1 \le x \le 1$ range as before. In accord with transformation 17:5:2, the reciprocals of the zeros found in this second survey should be appended to those found in the first.

You now know, crudely, where the real zeros are located. Assess whether your findings are complete: the parity of N_r, the number of real zeros, should be that of n. Should you suspect the presence of closely spaced zeros, you may want to calculate additional data at strategic sites.

Choose the real zero r in which you have the most confidence, and refine your knowledge of its location. There are many standard methods for doing this methodically: the *Newton-Raphson procedure* [Section 52:15] is probably the most popular. Next, use the refined r in algorithm 17:5:5 to deflate the polynomial to degree $n-1$, checking by means of 17:5:6 that the remainder is zero or small. Now, use this deflated polynomial as the basis on which to refine the second real zero, then deflate once more. Proceed in this way until all the real roots are established and deflation has produced a polynomial of even degree, $n-N_r$.

Then turn to the quadratic zeros. Refining them is trickier because there are two parameters, 2ρ and $(\rho^2+\iota^2)$, to be optimized each time. Again, details of the techniques used – Bairstow's method is a favorite – will not be given here but are available in the sources cited earlier. If you are lucky, $n-N_r$ will equal 2 or 4 and no refinement will be needed. In these cases, exact values of the complex zeros are available by solving the quadratic or quartic polynomial by the standard methods of Section 15:7 or 16:12.

As a check, confirm that the sum of all your zeros, real plus complex, satisfies the rule

17:7:2
$$\frac{-a_{n-1}}{a_n} = r_0 + r_1 + r_3 \cdots + r_{n-1} + r_n = \sum r$$

and has no (or only a small) imaginary component. Better still, use equations 17:7:2, 17:7:3, etc. to recreate the original polynomial. Equation 17:7:2 is the first of a set of so-called *elementary symmetric relations* that express the coefficients of the polynomial in terms of its zeros. The second, third, and last (the nth) of these relations are

17:7:3
$$\frac{a_{n-1}}{a_n} = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + r_2r_4 + \cdots + r_2r_n + r_3r_4 + \cdots + r_{n-1}r_n$$

17:7:4
$$\frac{-a_{n-3}}{a_n} = r_1r_2r_3 + r_1r_2r_4 + \cdots + r_1r_2r_n + r_1r_3r_4 + \cdots + r_1r_3r_n + \cdots + r_{n-2}r_{n-1}r_n$$

17:7:5
$$\frac{(-)^n a_0}{a_n} = r_1r_2r_3 \cdots r_{n-1}r_n$$

The pattern, with its alternating signs, will be clear. Collectively, these equations are known as *Vieta's theorem*. Equation 16:3:2 is an example.

17:8 NUMERICAL VALUES

Formula 17:3:1 provides a convenient way of computing values of a polynomial. If, instead, you use 17:1:1, beware of the danger of loss of significance when x is large, especially if a_n and a_{n-1} have opposite signs.

17:9 LIMITS AND APPROXIMATIONS

A polynomial $\mathrm{p}'_m(x)$ may be constructed to approximate a polynomial $\mathrm{p}_n(x)$ of higher degree, over a specified interval, so that

17:9:1
$$a'_m x^m + a'_{m-1}x^{m-1} + \cdots + a'_1 + a'_0 \approx a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \qquad x_0 < x < x_1 \qquad m < n$$

by a process known as *economization* or *telescoping*, which sacrifices precision for brevity. Economization is often used when $n = \infty$, that is to abbreviate a power series, because then one has no option but to curtail the original expansion when seeking to use it numerically. Often the choice of m is determined by the largest absolute error ε that can be tolerated, that is

17:9:2
$$\left|\mathrm{p}'_n(x) - \mathrm{p}_m(x)\right| \le \varepsilon \qquad x_1 \le x \le x_0$$

The key principle behind economization is that the Chebyshev polynomials $\mathrm{T}_j(y)$ [Chapter 22] never exceed unity in the range $-1 \le y \le 1$ and therefore, for any collection of c's

17:9:3 $$\sum_j c_j \mathrm{T}_j(y) \le \sum_j |c_j||\mathrm{T}_j(y)| \le \sum_j |c_j| \qquad -1 \le y \le 1$$

So, if m is the smallest integer that meets the inequality

17:9:4 $$\sum_{j=m+1}^{n} |c_j| \le \varepsilon$$

for a given n and a given set of c's, it follows that, to the precision defined by 17:9:2,

17:9:5 $$\sum_{j=0}^{m} c_j \mathrm{T}_j(y) \approx \sum_{j=0}^{n} c_j \mathrm{T}_j(y)$$

Now, each Chebyshev polynomial of degree j may be replaced by a sum of power functions up to and including y^j. The equivalence is expressed in formula 22:6:5 and implementing it leads to

17:9:6 $$\sum_{j=0}^{m} b'_j y^j \approx \sum_{j=0}^{n} b_j y^j \qquad \text{where} \qquad b'_j = \sum_{k=0}^{m} \tau_k^{(j)} c_j \quad \text{and} \quad b_j = \sum_{k=0}^{n} \tau_k^{(j)} c_j$$

the τ terms being those discussed in Section 22:6. Our actual problem occupies the region $x_0 \le x \le x_J$ not $-1 \le y \le 1$. However, one can convert between x and y by a simple transformation, whereby it follows from 17:9:6 that

17:9:7 $$\sum_{j=0}^{m} b'_j \left(1 - 2\frac{x - x_0}{x_J - x_0}\right)^j \approx \sum_{j=0}^{n} b_j \left(1 - 2\frac{x - x_0}{x_J - x_0}\right)^j$$

Binomial expansion and collection of terms will now allow a final solution matching equation 17:9:1.

The previous paragraph explains the procedure in principle. In practice, the algorithm would be numerical and it could not proceed as indicated above, because the c terms are not known a priori. Numerical implementation actually proceeds in the order a's $\to$ b's $\to$ c's $\to m \to b'$'s $\to a'$'s.

Historically, series economization played a valuable role in approximating infinite series by a small number of terms that could be summed by hand. With the advent of electronic devices that can sum thousands of terms in fractions of a second, as well as being able to test the extent of convergence, the procedure is on its way to becoming obsolete.

17:10 OPERATIONS OF THE CALCULUS

Because operations of the calculus can be distributed through a sum, any such operation that can be performed on an isolated integer power function may be applied to a polynomial. Thus differentiation, integration, and Laplace transformation can be carried out through extensions of the formulas in Section 16:10.

17:11 COMPLEX ARGUMENT

With real coefficients, the polynomial function of complex argument $z = x + iy$ has real and imaginary parts as follows:

17:11:1 $$\mathrm{Re}[\mathrm{p}_n(z)] = \sum_k (-)^{k/2} \sum_{j=k}^{n} \binom{j}{k} a_j x^{j-k} y^k \qquad k = 0, 2, 4, \cdots, (n \text{ or } n-1)$$

17:11:2 $$\mathrm{Im}[\mathrm{p}_n(z)] = \sum_k (-)^{(k-1)/2} \sum_{j=k}^{n} \binom{j}{k} a_j x^{j-k} y^k \qquad k = 1,3,5,\cdots,(n \text{ or } n-1)$$

Polynomials have no poles or other discontinuities in the complex plane.

17:12 GENERALIZATIONS: including rational functions

The ratio of two polynomial functions is a *rational function*, for which this *Atlas* adopts a special symbol

17:12:1 $$\mathrm{R}_n^m(x) = \frac{\mathrm{p}'_m(x)}{\mathrm{p}_n(x)} = \frac{a'_m x^m + a'_{m-1}x^{m-1} + \cdots + a'_1 x + a'_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$$

If $m \geq n$, the rational function is said to be "improper"; synthetic division [Section 17:5] by the factors of the denominatorial polynomial will reduce such a function to a polynomial plus a proper rational function (for which $m < n$) plus a polynomial. In the division, terms representing the polynomial appear in the remainder. For example

17:12:2 $$\frac{4x^5 - 3x^4 + 2x^3 - x^2 - 1}{x^3 - x^2 + 2} = 4x^2 + x + 3 - \frac{6x^2 + 2x + 7}{x^3 - x^2 + 2} = 4x^2 + x + 3 - \frac{\frac{11}{5}}{x+1} - \frac{\frac{19}{5}x + \frac{13}{5}}{x^2 - 2x + 2}$$

The final step in 17:12:2 provides an example of a proper rational fraction being split into partial fractions.

A standard method of approximating a transcendental function [Section 25:0] $\mathrm{f}(x)$ is by truncating its Maclaurin series [Section 10:13]

17:12:3 $$\mathrm{f}(x) \approx \hat{\mathrm{f}}(x) = \mathrm{f}(0)\left[1 + \frac{1}{\mathrm{f}(0)} \sum_{j=1}^{J} \frac{\mathrm{d}^j \mathrm{f}}{\mathrm{d}x^j}(0) \frac{x^j}{j!}\right]$$

but an alternative, and usually better, approximation is as a rational function

17:12:4 $$\mathrm{f}(x) \approx \hat{\mathrm{f}}(x) = \mathrm{f}(0)\left[\frac{1 + a'_1 x + a'_2 x^2 + \cdots + a'_m x^m}{1 + a_1 x + a_2 x^2 + \cdots + a_n x^n}\right]$$

Such an approximation is known as a *Padé approximant* (Henri Eugène Padé, French mathematician, 1863 – 1953) and is simply a rational function with the leading terms adjusted to be equal. The coefficients may be found by matching the terms in 17:12:4 with those in 17:12:3, or by the lozenge-diagram procedure discussed in Section 10:14. Of course there are Padé approximants corresponding to each m,n pair and they may be assembled into a so-called *Padé table*. A portion of such a table for the exponential function [$\exp(x)$, Chapter 26] is shown below. Notice that the top row contains the partial sums of the Maclaurin expansion, whereas the left-hand column is the reciprocal of the expansion of $\exp(-x)$

$\mathrm{R}_0^0(x) = 1$	$\mathrm{R}_1^0(x) = 1 + x$	$\mathrm{R}_2^0(x) = 1 + x + \frac{1}{2}x^2$	$\mathrm{R}_3^0(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$
$\mathrm{R}_1^0(x) = \dfrac{1}{1-x}$	$\mathrm{R}_1^1(x) = \dfrac{1+\frac{1}{2}x}{1-\frac{1}{2}x}$	$\mathrm{R}_1^2(x) = \dfrac{1+\frac{2}{3}x+\frac{1}{6}x^2}{1-\frac{1}{3}x}$	$\mathrm{R}_1^3(x) = \dfrac{1+\frac{3}{4}x+\frac{1}{4}x^2+x^3}{1-\frac{1}{4}x}$
$\mathrm{R}_2^0(x) = \dfrac{1}{1-x+\frac{1}{2}x^2}$	$\mathrm{R}_2^1(x) = \dfrac{1+\frac{1}{3}x}{1-\frac{2}{3}x+\frac{1}{6}x^2}$	$\mathrm{R}_2^2(x) = \dfrac{1+\frac{1}{2}x+\frac{1}{12}x^2}{1-\frac{1}{2}x+\frac{1}{12}x^2}$	$\mathrm{R}_2^3(x) = \dfrac{1+\frac{3}{5}x+\frac{3}{20}x^2+\frac{1}{60}x^3}{1-\frac{2}{5}x+\frac{1}{20}x^2}$
$\mathrm{R}_3^0(x) = \dfrac{1}{1-x+\frac{1}{2}x^2-\frac{1}{6}x^3}$	$\mathrm{R}_3^1(x) = \dfrac{1+\frac{1}{4}x}{1-\frac{3}{4}x+\frac{1}{4}x^2-\frac{1}{24}x^3}$	$\mathrm{R}_3^2(x) = \dfrac{1+\frac{2}{5}x+\frac{1}{20}x^2}{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}$	$\mathrm{R}_3^3(x) = \dfrac{1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{120}x^3}{1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3}$

All these expressions can be considered as generalizations of the polynomial function.

Sums such as the following may be converted to polynomial functions, or to polynomials of lower degree, 17:12:5 $$a_{-3}x^{-3}+a_{-2}x^{-2}+a_{-1}x^{-1}+a_0+a_1x+a_2x^2$$

17:12:6 $$x^{3/2}+2x^{5/2}-5x^{7/2}+x^{9/2}-x^{11/2}$$

17:12:7 $$a_0+a_1x^5+a_2x^{10}+a_3x^{15}+a_4x^{20}$$

17:12:8 $$ax^{-1/3}-b^2+a^3x^{1/3}-b^4x^{2/3}+a^5x$$

either by withdrawing a power factor, or redefining the argument, or both.

17:13 COGNATE FUNCTIONS: reciprocal polynomials

The reciprocal polynomial function $1/\mathrm{p}_n(x)$ occurs frequently. Because [Section 17:5] factoring the polynomial into linear and quadratic terms, with real coefficients, is always possible, any reciprocal polynomial function can be expressed as

17:13:1 $$\frac{1}{\mathrm{p}_n(x)}=\frac{1}{a_n(x-r_1)(x-r_2)\cdots(x-r_J)(x^2+p_1+q_1)(x^2+p_2+q_2)\cdots(x^2+p_K+q_K)}$$

where $J+2K=n$. Furthermore, such an expression may be split into $(J+K)$ *partial fractions* [Section 16:13]. Provided that each $1/(x-r_j)$ factor is distinct (that is, no other r has the same value as r_j), then the partial fraction arising from the factor is $\alpha_j/(x-r_j)$. With a similar proviso, each $1/(x^2+p_k+q_k)$ gives a partial fraction of $(\beta_k+\gamma_k)/(x^2+px+q)$. Thus

17:13:2 $$\frac{a_n}{p_n(x)}=\sum_{j=1}^{J}\frac{\alpha_j}{x-r_j}+\sum_{k=1}^{K}\frac{\beta_kx+\gamma_k}{x^2+p_kx+q_k}$$

If the factors are *not* distinct, that is if they represent a multiple zero, then expansion 17:13:2 must be modified. If, for example, $r_2=r_3$, then in the first summation

17:13:3 $$\frac{\alpha_2}{x-r_2}+\frac{\alpha_3}{x-r_3}\quad\text{should be replaced by}\quad\frac{\alpha_2}{x-r_2}+\frac{\alpha_3}{(x-r_2)^2}$$

In the case $r_2=r_3=r_4$, then the replacement would include a term $\alpha_4/(x-r_2)^3$, and likewise for higher multiplicities. Similarly, if $p_2=p_3$ and $q_2=q_3$, then in the second summation in 17:13:2

17:13:4 $$\frac{\beta_2x+\gamma_2}{x^2+p_2x+q_2}+\frac{\beta_3x+\gamma_3}{x^2+p_3x+q_3}\quad\text{should be replaced by}\quad\frac{\beta_2x+\gamma_2}{x^2+p_2x+q_2}+\frac{\beta_3x+\gamma_3}{(x^2+p_2x+q_2)^2}$$

Replacements when quadratic zeros have a multiplicity greater than 2 follow the same pattern as their linear congeners.

The *method of undetermined coefficients* is used to find the α, β, and γ constants that appear in the partial fraction expansion. An $n=3$ example is presented in Section 16:13. The process devolves into solving a set of simultaneous algebraic equations and seldom causes difficulty, though it may be daunting. Fortunately, when repeated zeros are absent, or have been removed by deflation, an "automated" method of determining undetermined coefficient exists. Consider a typical reciprocal polynomial and its partial fractions:

17:13:5 $$\frac{a_n}{\mathrm{p}_n(x)}=\prod_{k=1}^{n}\frac{1}{x-r_k}=\sum_{j=1}^{n}\frac{\alpha_j}{x-r_j}$$

Notice that we have chosen to distinguish the product index from the summation index. Now multiply by *all n* of the $x-\rho$ factors. This converts the product into unity. In the summation, however, only one of the terms cancels in each summand. Hence

17:13:6 $$1=\sum_{j=1}^{n}\alpha_j\prod_{k=1\cdots n}^{k\neq j}x-r_k$$

This is an identity, valid for all x, so let us set x equal to one particular r_j; all the summands in 17:13:6 will thereby vanish, except for the chosen j, so that

17:13:7 $$1=\alpha_j\prod_{k=1\cdots j}^{k\neq j}r_j-r_k$$

This provides a convenient expression for α_j, but an even more elegant result is possible. Differentiate the original polynomial,

17:13:8 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{p}_n(x)=a_n\frac{\mathrm{d}}{\mathrm{d}x}\prod_{k=1}^{n}(x-r_k)=a_n\sum_{j=1}^{n}\prod_{k=1\cdots n}^{k\neq j}(x-r_k)$$

then set the argument to r_j. In this milieu, too, all the summands disappear except for the chosen j:

17:13:9 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{p}_n(r_j)=a_n\prod_{k=1\cdots n}^{k\neq j}(r_j-r_k)$$

Combination of equations 17:13:7 and 17:13:9 now leads to a succinct expression for α_j, which can be inserted into 17:13:5, to produce

17:13:10 $$\frac{1}{\mathrm{p}_n(x)}=\sum_{j=1}^{n}\frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{p}_n(r_j)}\left(\frac{1}{x-r_j}\right)$$

The coefficient of the jth partial fraction of a reciprocal polynomial is seen to be the reciprocal of the polynomial's derivative, evaluated at the jth zero. This formula applies equally to real and complex zeros, but only when repeated zeros are absent.

Partial fractionation is frequently performed on reciprocal polynomial functions as a prelude to operations of the calculus. In this context, the following formulas are useful:

17:13:11 $$\int_{1+r}^{x}\frac{1}{t-r}\mathrm{d}t=\ln(x-r)$$

17:13:12 $$\int_{x}^{\infty}\frac{1}{(t-r_1)(t-r_2)}\mathrm{d}t=\frac{1}{r_2-r_1}\ln\left(\frac{x-r_1}{x-r_2}\right)\qquad r_1\neq r_2$$

17:13:13 $$\int_{x}^{\infty}\frac{1}{(t-r)^2}\mathrm{d}t=\frac{-1}{x-r}$$

17:13:14 $$\int_{-p/2}^{x}\frac{1}{t^2+pt+q}\mathrm{d}t=\frac{2}{\sqrt{4q-p^2}}\arctan\left(\frac{2x+p}{\sqrt{4q-p^2}}\right)\qquad 4q>p^2$$

17:13:15 $$\int_{0}^{\infty}\frac{1}{t-r}\exp(-st)\mathrm{d}t=\mathcal{L}\left\{\frac{1}{t-r}\right\}=-\exp(-rs)\mathrm{Ei}(rs)$$

17:13:16 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s-r}\frac{\exp(st)}{2\pi i}\mathrm{d}s=\mathcal{I}\left\{\frac{1}{s-r}\right\}=\exp(rt)$$

17:13:17
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2+ps+q}\frac{\exp(st)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{s^2+ps+q}\right\}=\begin{cases}\dfrac{\exp(-pt/2)}{\sqrt{q-\frac{1}{4}p^2}}\sin\left(\sqrt{q-\frac{1}{4}p^2}\,t\right) & 4q>p^2\\ t\exp(-pt/2) & 4q=p^2\\ \dfrac{\exp(-pt/2)}{\sqrt{\frac{1}{4}p^2-q}}\sinh\left(\sqrt{\frac{1}{4}p^2-q}\,t\right) & 4q<p^2\end{cases}$$

17:14 RELATED TOPIC: polynomial fitting

Section 7:14 addresses linear regression, the exercise in which a linear function is chosen which "best" represents a data set resembling that shown as red dots in Figure 7-3. A geometric interpretation is that the data represent the rectangular coordinates of J+1 points in the cartesian plane and the linear function is the straight line $bx+c$ that comes closest to the points in the sense that the sum of the squares of the distances between the points and the line is the least possible. In the present section, the goal is similar but, instead of a straight line, it is the best polynomial of degree K that is computed. The procedure is named *polynomial fitting* or *polynomial regression* or *least-squares regression*. Only evenly spaced data are addressed here. Of course K is a positive integer and if it lies in the range $1 \le K \le J$, polynomial regression results. If $K = J$, the fitting is exact and the calculated polynomial passes exactly through all the (J+1) points.

The procedure has similarities to that described in Section 17:9 and, as there, the properties of Chebyshev polynomials provide the key to polynomial fitting. Here, however, it is the *discrete* Chebyshev polynomials [Section 22:13] that serve this role. The input data are in the form of (J+1) paired values, (x_0,f_0), (x_1,f_1), $\cdots$, (x_J,f_J), and the sought approximating polynomial will be denoted $\hat{\mathrm{p}}_K(x)$. The first step in the procedure is to scale the independent variable, which presently occupies the $x_0 \le x \le x_J$ domain, to fall between -1 and 1 by defining

17:14:1
$$y=\frac{2x-x_0-x_j}{x_J-x_0}$$

Secondly, after choosing K, the K+1 members of a set of coefficients are calculated by the formula

17:14:2
$$a_k=\frac{(J+1-k)_k}{(J+1)_{k+1}}[2k+1]\sum_{j=0}^{J}f_j\,\mathrm{t}_k^{(J)}(y_j) \qquad k=0,1,\cdots,K$$

in which Pochhammer polynomials [next chapter] play a role. Here, $\mathrm{t}_k^{(J)}(y)$ denotes the appropriate kth discrete Chebyshev polynomial of argument y, where "appropriate" means "designed to cater to J+1 equally spaced data", as intimated by the $^{(J)}$ superscript. The definitions and properties of these polynomials are addressed in Section 22:13. Do not make the easy mistake of imagining that the a_k values are the coefficients of the fitting function $\hat{\mathrm{p}}_K$. Each a must first be multiplied by the corresponding discrete Chebyshev polynomial. This operation yields a term in the sought fitting polynomial. Rescaling back to the original x variable gives the fitted polynomial

17:14:3
$$\hat{\mathrm{p}}_K=\sum_{k=0}^{K}a_k\mathrm{t}_k^{(J)}(y)=\sum_{k=0}^{K}a_k\mathrm{t}_k^{(J)}\left(\frac{2x-x_0-x_J}{x_J-x_0}\right)$$

As a simple example, refer to Figure 17-1, which shows 5 data points spaced evenly between $x_0 = 1$ and $x_J = x_4 = 9$. We seek the quadratic function that "best" fits these data. First use formula 17:14:1, which in this case is $y_j = (x_j-5)/4$, to change the domain to that embraced by the fourth column of the data table overleaf.

We next seek the three coefficients a_0, a_1, and a_2. From information in Section 22:13, the first three discrete

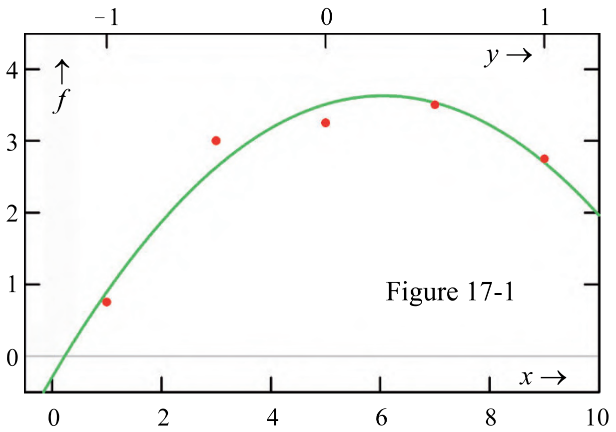


Figure 17-1

j	x	f	y	$\hat{p}_2$
0	1	0.75	−1.0	0.89
1	3	3.00	−0.5	2.63
2	5	3.25	0	3.51
3	7	3.5	0.5	3.53
$4 = J$	9	2.75	1.0	2.70

Chebyshev polynomials are $t_0^{(4)}(y) = 1$, $t_1^{(4)}(y) = y$, and $t_2^{(4)}(y) = 2y^2 - 1$. In turn, each of these may be inserted into equation 17:14:2, which in this case reads

17:14:4
$$a_k = \frac{(5-k)_k}{(5)_{k+1}}(2k+1)\sum_{j=0}^{4} f_j t_k^{(4)}(y_j) \qquad k = 0,1,2$$

and used to calculate $a_0 = 2.650$, $a_1 = 0.900$ and $a_2 = -0.857$. The sought polynomial is therefore

17:14:5
$$\hat{p}_2(y) = a_0 + a_1 y + a_2(2y^2 - 1) = -1.714y^2 + 0.900y + 3.507$$

or

17:14:6
$$\hat{p}_2(x) = a_0 + a_1\frac{x-5}{4} + a_2\left[2\left(\frac{x-5}{4}\right)^2 - 1\right] = -0.107x^2 + 1.296x - 0.296$$

The green curve in Figure 17-1 shows the fitted quadratic function. The corresponding values at the data points are listed in the table.

17:15 RELATED TOPIC: polynomial families

Throughout this chapter thus far, the coefficients a_j of the polynomial function $p_n(x)$ have been treated as arbitrary constants. However, the next seven chapters address polynomials in which the coefficients, far from being arbitrary, are fully determinate. These polynomial functions are grouped into "families", the members of which differ in their degree. There is a limitless number of family members $p_0(x)$, $p_1(x)$, $p_2(x)$, $\cdots$, $p_n(x)$, $\cdots$ and generally each family is known by the name of a famous mathematician. The ordinal and cardinal numbers are used to sequence members of a named polynomial family and to indicate the degree of each member. For example $B_6(x)$ is the sixth Bernoulli polynomial function and is a polynomial of degree 6, with 7 terms.

There is a specific rule by which the coefficient of x^j may be calculated from the integers j and n. One family differs from another in the nature of this rule. Note the dependence of the coefficient, not only on j, but also on n.

The seven following chapters fall into two groups. The first three are not, but the final four are, *orthogonal polynomials*. The meaning and significance of orthogonality is explained in Section 21:14.

CHAPTER 18

THE POCHHAMMER POLYNOMIALS $(x)_n$

There is little interest in the Pochhammer polynomials in their own right; however, their simple recursion properties enable these functions to play a valuable role in the algebra of other functions, especially the hypergeometric functions discussed in Section 18:14.

18:1 NOTATION

These polynomial functions, in which x is the argument and n the degree, were studied in 1730 by Stirling and later by Appell, who used the symbol (x,n). The name "Pochhammer polynomial" recognizes Leo August Pochhammer (German mathematician, 1841−1920) who introduced the now conventional $(x)_n$ notation. Alternative names are *shifted factorial function*, *rising factorial*, and *upper factorial*. The alternative overbarred symbol $x^{\overline{n}}$ is occasionally encountered.

18:2 BEHAVIOR

The Pochhammer polynomial is defined for all real x and all nonnegative integer n values (though see Section 18:12 for a generalization to negative n). In common with other polynomials, it has an unrestricted range when n is odd, but a semiinfinite range for $n = 2, 4, 6, \cdots$.

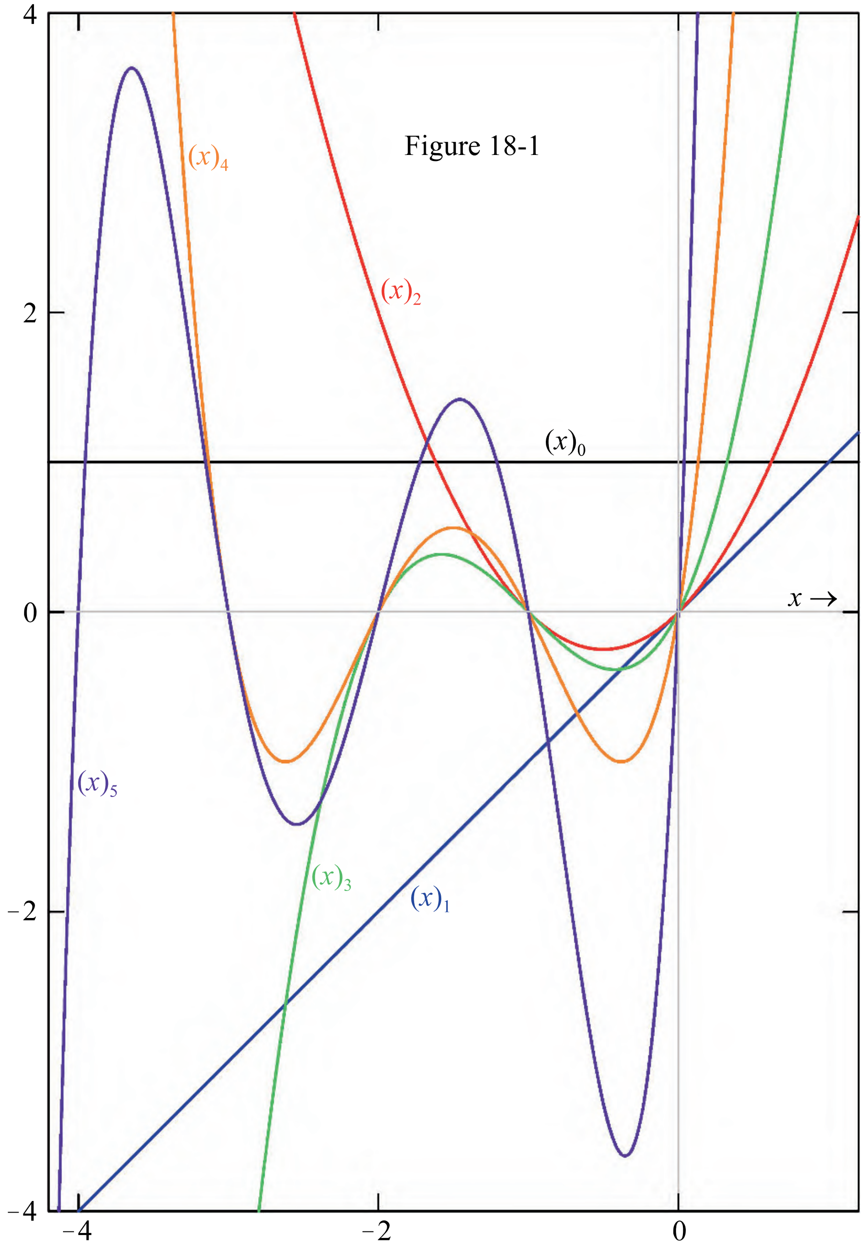
Figure 18-1 shows graphs of early members of the Pochhammer polynomial family; note that $(x)_n$ has exactly $\mathrm{Int}\left(\frac{n-1}{2}\right)$ maxima, $\mathrm{Int}\left(\frac{n}{2}\right)$ minima and n zeros, the latter occurring at $x = 0, -1, -2, \cdots, (1-n)$.

18:3 DEFINITIONS

The Pochhammer polynomial is defined by the n-fold product

18:3:1 $$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \prod_{j=0}^{n-1}(x+j)$$

DOI 10.1007/978-0-387-48807-3_19,



Empty products are generally interpreted as unity and this is the case for the Pochhammer polynomial of zero degree:

18:3:2 $$(x)_0 = 1$$

An equivalent definition

18:3:3 $$(x)_n = n!\binom{x+n-1}{n}$$

expresses $(x)_n$ in terms of a factorial function and a binomial coefficient [Chapters 2 and 6, respectively]. Equation 18:12:1 can also serve as a definition.

A generating function [Section 0:3] for the Pochhammer polynomial is

18:3:4
$$\frac{1}{(1-t)^{v}}=\sum_{n=0}^{\infty}(v)_n\frac{t^n}{n!}$$

and it is also generated by repeatedly differentiating a power of which $-v$ is the exponent:

18:3:5
$$x^{v}\frac{d^n}{dx^n}x^{-v}=(v)_n\left(\frac{-1}{x}\right)^n$$

18:4 SPECIAL CASES

$(x)_0$	$(x)_1$	$(x)_2$	$(x)_3$	$(x)_4$	$(x)_5$	$(x)_6$
1	x	x^2+x	x^3+3x^2+2x	$x^4+6x^3+11x^2+6x$	$x^5+10x^4+35x^3+50x^2+24x$	$x^6+15x^5+85x^4+225x^3+274x^2+120x$

18:5 INTRARELATIONSHIPS

Pochhammer polynomials obey the reflection formula

18:5:1
$$(-x)_n=(-)^n(x-n+1)_n$$

Equivalently

18:5:2
$$\left(\frac{n-1}{2}-x\right)_n=(-)^n\left(x-\frac{n-1}{2}\right)_n$$

which explains the even or odd symmetry about $x=(1-n)/2$ evident in Figure 18-1.

The argument-duplication formulas

18:5:3
$$(2x)_n=\begin{cases}2^n(x)_{n/2}(x+\tfrac{1}{2})_{n/2} & n=0,2,4,\cdots\\ 2^n(x)_{(n+1)/2}(x+\tfrac{1}{2})_{(n-1)/2} & n=1,3,5,\cdots\end{cases}$$

have analogs in expressions for $(3x)_n$, $(4x)_n$, and generally for $(mx)_n$, where m is a positive integer. Equation 18:5:3 may be reformulated into a degree-duplication formula

18:5:4
$$(x)_{2n}=4^n\left(\frac{x}{2}\right)_n\left(\frac{1+x}{2}\right)_n$$

and similarly

18:5:5
$$(x)_{2n+1}=4^n x\left(\frac{1+x}{2}\right)_n\left(1+\frac{x}{2}\right)_n=2^{2n+1}\left(\frac{x}{2}\right)_{n+1}\left(\frac{1+x}{2}\right)_n$$

Similar formulas for $(x)_{3n}$, $(x)_{3n+1}$, $(x)_{3n+2}$, $(x)_{4n}$, etc. may be derived readily.

Simple recursion formulas exist for both the argument

18:5:6
$$(x+1)_n=\left[1+\frac{n}{x}\right](x)_n$$

and the degree

18:5:7
$$(x)_{n+1}=[n+x](x)_n=x(x+1)_n$$

of Pochhammer polynomial functions. There are many useful formulas expressing the quotient of two Pochhammer polynomials:

18:5:8
$$\frac{(x)_n}{(x)_m} = \begin{cases} (x+m)_{n-m} & n \ge m \\ \dfrac{1}{(x+n)_{m-n}} & n \le m \end{cases}$$

18:5:9
$$\frac{(x+m)_n}{(x)_n} = \frac{(x+n)_m}{(x)_m} \qquad m = 0,1,2,\cdots$$

18:5:10
$$\frac{(x-m)_n}{(x)_n} = \frac{(x-m)_m}{(x-m+n)_m} = \frac{(1-x)_m}{(1-n-x)_m} \qquad m = 0,1,2,\cdots$$

Addition formulas exist for both the argument and the degree of a Pochhammer polynomial. The expression

18:5:11
$$(x+y)_n = \sum_{j=0}^{n} \binom{n}{j} (x)_j (y)_{n-j}$$

which closely resembles the binomial theorem [equation 6:14:1], is known as *Vandermonde's theorem* (Alexandre-Théophile Vandermonde, French violinist and mathematician, 1735 – 1795). The rule

18:5:12
$$(x)_{n+m} = (x)_n (x+n)_m$$

is a simple consequence of definition 18:3:1.

18:6 EXPANSIONS: Stirling numbers of the first kind

Of course, the Pochhammer polynomial is expansible as the product 18:3:1. As a sum, its expansion involves the absolute values of the numbers $S_n^{(m)}$, known as the *Stirling numbers of the first kind*.

18:6:1 $$(x)_n = (-)^n \sum_{m=1}^{n} S_n^{(m)} (-x)^m = \sum_{m=0}^{n} \left| S_n^{(m)} \right| x^m$$

These numbers are negative whenever $n+m$ is odd and $0 < m < n$. Figure 18-2 shows the absolute values of early Stirling numbers of the first kind and more can be calculated via the recursion formula

18:6:2
$$S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}$$
$$n = 0,1,2,\cdots \qquad m = 1,2,3,\cdots$$

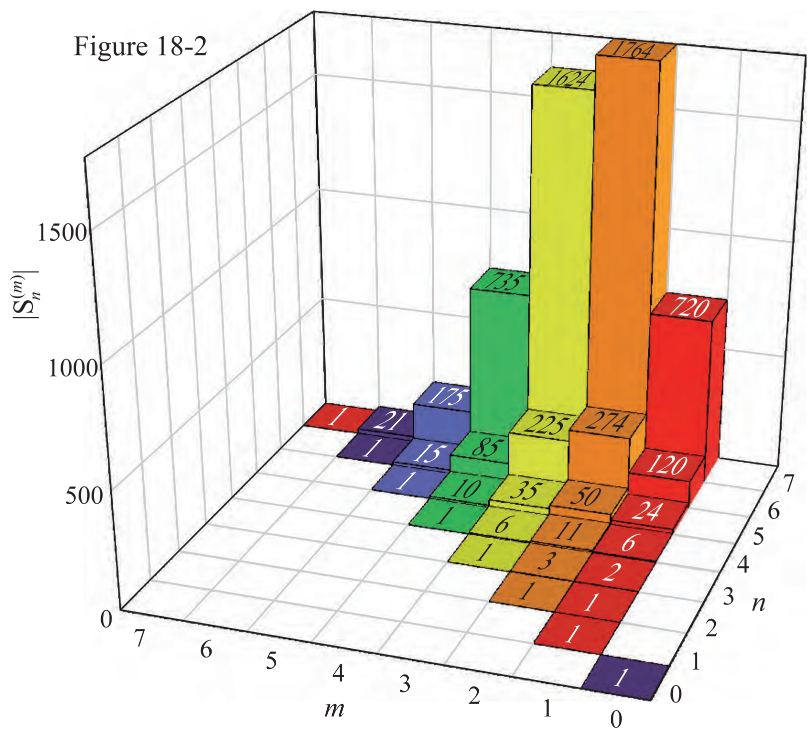
This formula is the basis of *Equator*'s Stirling number of the first kind (keyword **Snum**) routine. The numbers satisfy the following summations

18:6:3
$$\sum_{m=1}^{n} S_n^{(m)} = 0 \qquad n = 2,3,4,\cdots$$

18:6:4
$$\sum_{m=0}^{n} \left| S_n^{(m)} \right| = n! \qquad n = 0,1,2,\cdots$$

It is sometimes useful to expand a

Figure 18-2



reciprocal Pochhammer polynomial as partial fractions [Section 17:13]. The result is

18:6:5 $$\frac{1}{(x)_n}=\sum_{j=0}^{n-1}\frac{(-1)^j}{j!(n-j-1)!}\frac{1}{x+j}=(n-1)!\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(-1)^j}{x+j}$$

18:7 PARTICULAR VALUES

	$(-n)_n$	$(\frac{1-n}{2})_n$	$(-m)_n$ $m=1,2,\cdots,n-1$	$(\frac{-1}{2})_n$	$(0)_n$	$(\frac{1}{2})_n$	$(1)_n$	$(2)_n$	$(n)_n$
$n=0$	1	1	1	1	1	1	1	1	1
$n=1,3,5,\cdots$	$-n!$	0	0	$\frac{-(2n)!}{4^n(2n-1)n!}$	0	$\frac{(2n)!}{4^n n!}$	$n!$	$(n+1)!$	$\frac{(2n)!}{2n!}$
$n=2,4,6,\cdots$	$n!$	$\frac{(-)^{n/2}(n!)^2}{4^n(\frac{n}{2}!)^2}$	0	$\frac{-(2n)!}{4^n(2n-1)n!}$	0	$\frac{(2n)!}{4^n n!}$	$n!$	$(n+1)!$	$\frac{(2n)!}{2n!}$

As the table shows, the Pochhammer polynomial of an integer can be expressed as a factorial function or as the quotient of two factorials

18:7:1 $$(1)_n=n! \qquad (2)_n=(n+1)! \qquad (3)_n=\frac{(n+2)!}{2} \qquad (m)_n=\frac{(n+m-1)!}{(m-1)!}$$

Similarly, the Pochhammer polynomial of half an odd integer is related to double-factorials [Section 2:13]

18:7:2 $$(1/2)_n=\frac{(2n-1)!!}{2^n} \qquad (3/2)_n=\frac{(2n+1)!!}{2^n} \qquad (m/2)_n=\frac{(2n+m-2)!!}{2^n(m-2)!!} \qquad m=1,3,5,\cdots$$

18:8 NUMERICAL VALUES

Equator can provide accurate values of $(x)_n$ by its Pochhammer polynomial routine (keyword **Poch**).

18:9 LIMITS AND APPROXIMATIONS

As $x\to+\infty$, $(x)_n$ approaches $+\infty$ smoothly and rapidly. As x becomes increasingly negative, $(x)_n$ passes through $(n-1)$ extrema before heading rapidly towards $+\infty$, if n is even, or $-\infty$ if n is odd. By use of equation 18:12:1, the limiting behavior of the Pochhammer polynomial can be deduced from those of the gamma function, as discussed in Section 43:9. Thus, when n is large, x remaining modest, the asymptotic expansion

18:9:1 $$(x)_n \sim \frac{n^{x-1}n!}{\Gamma(x)}\left[1+\frac{x(x-1)}{2n}+\frac{x(x-1)(x-2)(3x-1)}{24n^2}+\cdots\right] \qquad n\to\infty$$

holds and shows, for example, that

18:9:2 $$\left(\tfrac{1}{2}\right)_n \to \frac{n!}{\sqrt{\pi n}} \qquad n \to \infty$$

On the other hand, the Stirling approximation [equation 43:6:6], coupled with 18:12:1, leads to

18:9:3 $$\left(\tfrac{1}{2}\right)_n \to \sqrt{2}\left(\frac{n}{e}\right)^n \qquad n \to \infty$$

The coexistence of limits 18:9:2 and 18:9:3 provides an interesting link between what are probably the three most important irrational numbers: π, e, and $\sqrt{2}$.

For large n, and x close to $-n/2$, the Pochhammer polynomial approximates a sine function [Chapter 32].

18:9:4 $$(x)_n \approx 2\left(\frac{n}{2e}\right)^n \sin(\pi x) \qquad x + \frac{n}{2} << \sqrt{n} \quad \text{large positive } n$$

The development of this sinusoidal behavior is evident in Figure 18-1, even for n as small as 4.

18:10 OPERATIONS OF THE CALCULUS

Linear operators such as differentiation and indefinite integration may be applied term by term to all polynomials, including $(x)_n$. Differentiation and integration of the Pochhammer polynomial give

18:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(x)_n = (x)_n \sum_{j=0}^{n-1} \frac{1}{x+j} = (x)_n \left[\psi(n+x) - \psi(x)\right]$$

18:10:2 $$\int_0^x (t)_n \, \mathrm{d}t = \sum_{j=1}^{n+1} \left| \mathrm{S}_n^{(j-1)} \right| \frac{x^j}{j}$$

The ψ function is the digamma function [Chapter 44] and $\mathrm{S}_n^{(j-1)}$ represents a Stirling number from Section 18:6.

18:11 COMPLEX ARGUMENT

If values of the Pochhammer polynomial with complex argument are needed, which they seldom are, they are available by combining equation 18:6:1 and 17:11:1.

18:12 GENERALIZATIONS

Pochhammer polynomials may be expressed as a ratio of two gamma functions [Chapter 43]

18:12:1 $$(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}$$

This representation opens the door to a generalization in which the degree n is not necessarily an integer.

A less profound generalization is to maintain n as an integer, but allow it to adopt negative values. This is possible by basing the definition of such Pochhammer polynomials on recursion 18:5:7 and leads to the conclusion that

18:12:2 $$(x)_{-1} = \frac{1}{x-1}$$

and generally

18:12:3 $$(x)_{-n} = \frac{1}{(x-1)(x-2)\cdots(x-n)} = \frac{(-)^n x}{[x-n](-x)_n}$$

18:13 COGNATE FUNCTIONS

Factorial functions [Chapter 2], binomial coefficients [Chapter 6], the gamma function [Chapter 43] and the (complete) beta function [Section 43:13] are all closely related to the Pochhammer polynomial.

A *factorial polynomial*, as defined by Tuma [Section 1.03] is

18:13:1 $$x_h^{(n)} = x(x-h)(x-2h)\cdots(x-nh+h)$$

but another function given the same name is

18:13:2 $$x^{[n]} = x(x-1)(x-2)\cdots(x-n+1)$$

This latter function also goes by the names *falling factorial* and *lower factorial* and may be symbolized $x^{\underline{n}}$ or, unfortunately, $(x)_n$. Yet another confusing symbolism, due to Kramp, is

18:13:3 $$x^{n/c} = x(x+c)(x+2c)\cdots(x+nc-c)$$

None of the notations in this paragraph is employed in the *Atlas*.

18:14 RELATED TOPIC: hypergeometric functions

Pochhammer polynomials occur in the coefficients of the special kind of power series known as a *hypergeometric function*. The most general representation of such a function is as the sum

18:14:1 $$\sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j \cdots (a_K)_j}{(c_1)_j (c_2)_j (c_3)_j \cdots (c_L)_j} x^j$$

where x is the argument, a_1, a_2, $\cdots$, a_K are prescribed *numeratorial parameters*, and c_1, c_2, $\cdots$, c_L are prescribed *denominatorial parameters.* Any real number is permissible as a parameter, except that nonpositive integers are problematic. If such an integer is one of the *a* parameter, series 18:14:1 will generally terminate, thus representing a polynomial. The only circumstance in which a nonpositive integer is legitimate as a denominatorial *c* parameter, is if another nonpositive integer of smaller magnitude (that is, a less negative integer) occurs in the numerator. In such cases the series terminates. Of course, the same Pochhammer term may not be in both the numerator and the denominator: they would cancel.

The argument x may have either sign but its permissible range is determined by the *numeratorial order* K and the *denominatorial order* L. These K and L orders are nonnegative integers, usually small ones. If $L > K$, the hypergeometric series necessarily converges for all finite values of x. If $L = K$, convergence is generally limited to the argument range $|x| < 1$. If $L < K$ the series diverges (unless it terminates) for all nonzero arguments, but it may nevertheless usefully represent a function asymptotically for small values of $|x|$ [37:6:5 provides an example].

The name "hypergeometric function" arises because 18:14:1 can be regarded as an extension of the geometric series (equation 1:6:4 or 6:14:9), to which it reduces when $L = K = 0$. Choosing suitable values of the *a*'s and *c*'s often gives rise to well-known functions when L and K are small. As well, a number of generic functions, such as the Kummer function [Chapter 47] the Gauss hypergeometric function [Chapter 60], and the Claisen functions [equation 18:14:5] are instances of hypergeometric functions in which the *a*'s and *c*'s are largely unrestricted. The so-called *generalized hypergeometric function*, or *extended hypergeometric function*, often denoted ${}_pF_q(a_1,\cdots,a_p;c_1,\cdots,c_q;x)$ is a hypergeometric function in which one of the denominatorial parameters is constrained to be unity:

18:14:2
$$ {}_pF_q\left(a_1,a_2,a_3,\cdots,a_p;c_1,c_2,c_3,\cdots,c_q;x\right)=\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j(a_3)_j\cdots(a_p)_j}{(c_1)_j(c_2)_j(c_3)_j\cdots(c_q)_j(1)_j}x^j $$

so that $p = K$ but q and L differ by unity. Other notations include

18:14:3
$$ {}_pF_q\left(\begin{matrix}a_1,a_2,a_3,\cdots,a_p\\ c_1,c_2,c_3,\cdots,c_q\end{matrix}\Big| x\right) \quad\text{and}\quad \left[x\;\frac{a_1-1,a_2-1,a_3-1,\cdots,a_K-1}{c_1-1,c_2-1,c_3-1,\cdots,c_L-1}\right] $$

Some of these notations imply a phantom denominatorial $(1)_j$. In this *Atlas*, we adopt no special notation for hypergeometric functions, preferring to spell out the series explicitly as in 18:14:1. If a $(1)_j$ is present in the denominator, it is shown there.

As the tables in this section attest, a very large fraction of the functions discussed in the *Atlas* may be expressed hypergeometrically. Moreover, in the terminology of Section 43:14, almost all of these functions may be synthesized from a basis function, such as the ones listed in equations 43:14:1−4. Do not be misled into imagining that the only hypergeometric functions are those in the tables. In fact, subject to possible limitations on the argument x, almost any assignment of a's and c's leads to a valid hypergeometric function. It is just that most such assignments do not correspond to functions that have been glorified by special names and symbols.

Table 18-1

a	c	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}(-x)^j$
ν	1	$(1-x)^{-\nu}$	$(1+x)^{-\nu}$
ν	2	$\frac{1-(1-x)^{1-\nu}}{(1-\nu)x}$	$\frac{(1+x)^{1-\nu}-1}{(1-\nu)x}$
$1+\nu$	ν	$\frac{\nu+(1-\nu)x}{\nu(1-x)^2}$	$\frac{\nu-(1-\nu)x}{\nu(1+x)^2}$
1	2	$\frac{-\ln(1-x)}{x}$	$\frac{\ln(1+x)}{x}$
1	3	$\frac{2}{x^2}\left[x+(1-x)\ln(1-x)\right]$	$\frac{2}{x^2}\left[(1+x)\ln(1+x)-x\right]$
1	$3/2$	$\frac{\arcsin\left(\sqrt{x}\right)}{\sqrt{x(1-x)}}$	$\frac{\mathrm{arsinh}\left(\sqrt{x}\right)}{\sqrt{x(1+x)}}$
1	$1/2$	$\frac{1}{1-x}+\frac{\sqrt{x}\arcsin\left(\sqrt{x}\right)}{\sqrt{(1-x)^3}}$	$\frac{1}{1+x}-\frac{\sqrt{x}\,\mathrm{arsinh}\left(\sqrt{x}\right)}{\sqrt{(1+x)^3}}$
$-1/2$	$1/2$	$1-\sqrt{x}\,\mathrm{artanh}\left(\sqrt{x}\right)$	$1+\sqrt{x}\arctan\left(\sqrt{x}\right)$
$1/2$	$3/2$	$\frac{\mathrm{artanh}(\sqrt{x})}{\sqrt{x}}$	$\frac{\arctan(\sqrt{x})}{\sqrt{x}}$
ν	$1+\nu$	$\nu\Phi(x,1,\nu)$	$\nu\Phi(-x,1,\nu)$
a	c	$\frac{(c-1)\mathrm{B}(c-1,a-c+1,x)}{(1-x)^{a-c+1}x^{c-1}}$	

Hypergeometric functions in which $L = K$ have the common feature of being amenable to synthesis, ultimately from one or other of the $1/(1\pm x)$ functions. Table 18-1 lists examples of $L = K = 1$ hypergeometric functions, while Table 18-2 similarly lists $L = K = 2$ hypergeometrics. There is a plethora of functions that are expressible as $L = K = 2$ hypergeometric functions; entries in Table 18-2 have been chosen as representative, rather than exhaustive. See Section 60:4 for details of the ways in which an associated Legendre function may be represented as a Gauss hypergeometric function; that is, formulated as an $L = K = 2$ hypergeometric. $L = K = 3$ cases, include the class of *Claisen functions*, important in hydrodynamics and described by

Table 18-2

a_1	a_2	c_1	c_2	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c_1)_j(c_2)_j}(-x)^j$
$v-\frac{1}{2}$	v	1	$2v$	$\left[\left(1+\sqrt{1-x}\right)/2\right]^{1-2v}$	$\left[\left(1+\sqrt{1+x}\right)/2\right]^{1-2v}$
$-v$	$\frac{1}{2}-v$	½	1	$\frac{1}{2}\left[\left(1+\sqrt{x}\right)^{2v}+\left(1-\sqrt{x}\right)^{2v}\right]$	$(1+x)^{v}\cos\left\{2v\arctan\left(\sqrt{x}\right)\right\}$
$-v$	v	½	1	$\cos\left\{2v\arcsin\left(\sqrt{x}\right)\right\}$	$\frac{1}{2}\left[\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}+\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}\right]$
$\frac{1}{2}-v$	$1-v$	1	3⁄2	$\frac{\left(1+\sqrt{x}\right)^{2v}-\left(1-\sqrt{x}\right)^{2v}}{4v\sqrt{x}}$	$\frac{(1-x)^{-v}}{2v\sqrt{x}}\sin\left\{2v\arctan(\sqrt{x})\right\}$
$\frac{1}{2}-v$	$\frac{1}{2}+v$	1	3⁄2	$\frac{\sin\left\{2v\arcsin\left(\sqrt{x}\right)\right\}}{2v\sqrt{x}}$	$\frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}-\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}}{4v\sqrt{x}}$
$1-v$	$1+v$	1	3⁄2	$\frac{\sin\left\{2v\arcsin\left(\sqrt{x}\right)\right\}}{2v\sqrt{x(1-x)}}$	$\frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}-\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}}{4v\sqrt{x(1+x)}}$
1	3⁄2	2	2	$\frac{-4}{x}\left[\ln\left(\frac{\sqrt{x}}{2}\right)+\mathrm{arcosh}\left(\frac{1}{\sqrt{x}}\right)\right]$	$\frac{-4}{x}\left[\ln\left(\frac{\sqrt{x}}{2}\right)+\mathrm{arsinh}\left(\frac{1}{\sqrt{x}}\right)\right]$
1	1	2	2	$-\left[\mathrm{diln}(1-x)\right]/x$	$\left[\mathrm{diln}(1+x)\right]/x$
½	½	1	3⁄2	$\left[\arcsin\left(\sqrt{x}\right)\right]/\sqrt{x}$	$\left[\mathrm{arsinh}\left(\sqrt{x}\right)\right]/\sqrt{x}$
½	½	3⁄2	2	$\frac{2}{x}\left[\sqrt{x}\arcsin\left(\sqrt{x}\right)-1+\sqrt{1-x}\right]$	$\frac{2}{x}\left[\sqrt{x}\,\mathrm{arcsinh}\left(\sqrt{x}\right)+1-\sqrt{1+x}\right]$
$-n$	n+1	1	1	$\mathrm{P}_n(1-2x)$	$\mathrm{P}_n(1+2x)$
$-n$	n	½	1	$\mathrm{T}_n(1-2x)$	
$-n$	$\frac{1}{2}-n$	½	1	$(1-x)^n\,\mathrm{T}_n\left\{(1+x)/(1-x)\right\}$	
$-n$	n+2	1	3⁄2	$\frac{1}{n+1}\mathrm{U}_n(1-2x)$	$\frac{1}{n+1}\mathrm{U}_n(1+2x)$
n	n+v	1	$\frac{1}{2}+\frac{v}{2}$	$\frac{n!}{(v)_n}\mathrm{C}_n^{(v/2)}(1-2x)$	$\frac{n!}{(v)_n}\mathrm{C}_n^{(v/2)}(1+2x)$
½	½	1	1	$\frac{2}{\pi}\mathrm{K}\left(\sqrt{x}\right)$	$\frac{2}{\pi\sqrt{1+x}}\mathrm{K}\left(\sqrt{\frac{x}{1+x}}\right)$
−½	½	1	1	$\frac{2}{\pi}\mathrm{E}\left(\sqrt{x}\right)$	$\frac{2\sqrt{1+x}}{\pi}\mathrm{E}\left(\sqrt{\frac{x}{1+x}}\right)$
a	b	1	c	F(a,b;c;x)	$\left(1+x\right)^{-a}\mathrm{F}\left(a,c-b;c;\frac{x}{1+x}\right)$

Table 18-3

c	$\sum_{j=0}^{\infty}\frac{1}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{1}{(c)_j}(-x)^j$
$-\frac{1}{2}$	$1-2x-2\sqrt{\pi x^3}\exp(x)\operatorname{erf}\left(\sqrt{x}\right)$	$1+2x-4\sqrt{x^3}\operatorname{daw}\left(\sqrt{x}\right)$
$\frac{1}{2}$	$1+\sqrt{\pi x}\exp(x)\operatorname{erf}\left(\sqrt{x}\right)$	$1-2\sqrt{x}\operatorname{daw}\left(\sqrt{x}\right)$
1	$\exp(x)$	$\exp(-x)$
$\frac{3}{2}$	$\sqrt{\frac{\pi}{4x}}\exp(x)\operatorname{erf}\left(\sqrt{x}\right)$	$\frac{1}{\sqrt{x}}\operatorname{daw}\left(\sqrt{x}\right)$
2	$\frac{\exp(x)-1}{x}$	$\frac{1-\exp(-x)}{x}$
$\frac{5}{2}$	$\frac{3}{4x^2}\left[\sqrt{\pi x}\exp(x)\operatorname{erf}\left(\sqrt{x}\right)-2x\right]$	$\frac{3}{2x}\left[1-\frac{1}{\sqrt{x}}\operatorname{daw}\left(\sqrt{x}\right)\right]$
3	$\frac{2}{x^2}\left[\exp(x)-1-x\right]$	$\frac{2}{x^2}\left[x-1+\exp(-x)\right]$
n	$\frac{(n-1)!}{x^{n-1}}\left[\exp(x)-\mathrm{e}_{n-2}(x)\right]$	$\frac{(n-1)!x}{(-x)^n}\left[\mathrm{e}_{n-2}(-x)-\exp(-x)\right]$
ν	$\Gamma(\nu)\exp(x)\gamma\mathrm{n}(\nu-1,x)$	$\Gamma(\nu)\exp(-x)\gamma\mathrm{n}(\nu-1,-x)$

Table 18-4

a	c_1	c_2	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j}(-x)^j$
$-\nu$	1	1	$\mathrm{L}_\nu(x)$	$\mathrm{L}_\nu(-x)$
$-n$	1	μ	$\frac{n!}{(\mu)_n}\mathrm{L}_n^{(\mu-1)}(x)$	$\frac{n!}{(\mu)_n}\mathrm{L}_n^{(\mu-1)}(-x)$
$-n$	$\frac{1}{2}$	1	$\frac{(-1)^n}{4^n\left(\frac{1}{2}\right)_n}\mathrm{H}_{2n}\left(\sqrt{x}\right)$	
$\frac{1}{2}$	$\frac{3}{2}$	2		$\left\{\sqrt{\pi}\left[\sqrt{x}+\operatorname{ierfc}\left(\sqrt{x}\right)-1\right]\right\}/x$
ν	$\frac{1}{2}$	1	$\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{2^{1-\nu}\sqrt{\pi}}\exp\left(\frac{x}{2}\right)\sum_{\pm}\mathrm{D}_{-2\nu}\left(\pm\sqrt{2x}\right)$	$\frac{\Gamma(1-\nu)}{2^\nu\sqrt{2x}}\exp\left(\frac{-x}{2}\right)\sum_{\pm}\mathrm{D}_{2\nu-1}\left(\pm\sqrt{2x}\right)$
$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{\sqrt{x}}\exp(x)\operatorname{daw}\left(\sqrt{x}\right)$	$\sqrt{\frac{\pi}{4x}}\operatorname{erf}\left(\sqrt{x}\right)$
$\frac{3}{2}$	1	$\frac{5}{2}$	$\frac{3\exp(x)}{2x}\left[1-\frac{\operatorname{daw}\left(\sqrt{x}\right)}{\sqrt{x}}\right]$	$\frac{3}{2x}\left[\sqrt{\frac{\pi}{4x}}\operatorname{erf}\left(\sqrt{x}\right)-\exp(-x)\right]$
1	2	2	$[-\mathrm{Ein}(-x)]/x$	$[\mathrm{Ein}(x)]/x$
ν	1	μ	$\mathrm{M}(\nu,\mu,x)$	$\exp(x)\mathrm{M}(\mu-\nu,\mu,x)$
ν	1	$\nu+1$		$\nu x^{-\nu}\gamma(\nu,x)$

18:14:4
$$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j(a_3)_j}{(c_1)_j(c_2)_j(1)_j}x^j$$

of which an example is

18:14:5
$$\sum_{j=0}^{\infty}\frac{(2\mu)_j(2\nu-2\mu-\frac{1}{2})_j(\nu-\frac{1}{2})_j}{(\nu)_j(2\nu-1)_j(1)_j}x^j=\left[\mathrm{F}(\mu,\nu-\mu-\tfrac{1}{2},\nu,x\right]^2$$

Please refer to the Symbol Index for the meaning of any unfamiliar symbol. Equation 18:14:6 provides a non-Claisen example of a $L=K=3$ hypergeometric function.

Table 18-5

c_1	c_2	$\sum_{j=0}^{\infty}\frac{1}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{1}{(c_1)_j(c_2)_j}(-x)^j$
$-1/2$	$1/2$	$1-2\pi x\,\mathrm{l}_{-1}\left(2\sqrt{x}\right)$	$1+2\pi x\mathrm{h}_{-1}\left(2\sqrt{x}\right)$
$1/2$	1	$\cosh\left(2\sqrt{x}\right)$	$\cos\left(2\sqrt{x}\right)$
$1/2$	$3/2$	$\frac{\pi}{2}\mathrm{l}_{-1}\left(2\sqrt{x}\right)$	$\frac{\pi}{2}\mathrm{h}_{-1}\left(2\sqrt{x}\right)$
$2/3$	1	$\frac{3^{2/3}\Gamma(2/3)}{2}\left[\frac{\mathrm{Bi}\left(3^{2/3}x^{1/3}\right)}{\sqrt{3}}+\mathrm{Ai}\left(3^{2/3}x^{1/3}\right)\right]$	
$3/4$	$5/4$		$\frac{\sqrt{\pi}}{2x^{1/4}}\left[\frac{\sin\left(2\sqrt{x}+\pi/4\right)}{\sqrt{2}}-\mathrm{Gres}\left(\sqrt{2}x^{1/4}\right)\right]$
1	1	$\mathrm{I}_0\left(2\sqrt{x}\right)$	$\mathrm{J}_0\left(2\sqrt{x}\right)$
1	$4/3$	$\frac{\Gamma(1/3)}{(24x)^{1/3}}\left[\frac{\mathrm{Bi}\left(3^{2/3}x^{1/3}\right)}{\sqrt{3}}-\mathrm{Ai}\left(3^{2/3}x^{1/3}\right)\right]$	
1	$3/2$	$\frac{\sinh\left(2\sqrt{x}\right)}{2\sqrt{x}}$	$\frac{\sin\left(2\sqrt{x}\right)}{2\sqrt{x}}$
1	μ	$\Gamma(\mu)x^{(1-\mu)/2}\mathrm{I}_{\mu-1}\left(2\sqrt{x}\right)$	$\Gamma(\mu)x^{(1-\mu)/2}\mathrm{J}_{\mu-1}\left(2\sqrt{x}\right)$
$3/2$	$3/2$	$\frac{\pi}{4\sqrt{x}}\mathrm{l}_0\left(2\sqrt{x}\right)$	$\frac{\pi}{4\sqrt{x}}\mathrm{h}_0\left(2\sqrt{x}\right)$
$5/4$	$7/4$		$\frac{3\sqrt{\pi}}{8x^{3/4}}\left[\mathrm{Fres}\left(\sqrt{2}x^{1/4}\right)-\frac{\cos\left(2\sqrt{x}+\pi/4\right)}{\sqrt{2}}\right]$
$3/2$	2	$\frac{1}{x}\sinh^2\left(\sqrt{x}\right)$	$\frac{1}{x}\sin^2\left(\sqrt{x}\right)$
$3/2$	μ	$\frac{\sqrt{\pi}\,\Gamma(\mu)}{2}x^{(1-2\mu)/4}\mathrm{l}_{(2\mu-3)/2}\left(2\sqrt{x}\right)$	$\frac{\sqrt{\pi}\,\Gamma(\mu)}{2}x^{(1-2\mu)/4}\mathrm{h}_{(2\mu-3)/2}\left(2\sqrt{x}\right)$

18:14:6
$$\sum_{j=0}^{\infty}\frac{(1)_j\left(\frac{3}{2}\right)_j\left(\frac{3}{2}\right)_j}{(2)_j(2)_j\left(\frac{5}{2}\right)_j}x^j=\frac{12}{x}\left[1+\ln\left(\frac{2}{1+\sqrt{1-x}}\right)-\frac{\arcsin\left(\sqrt{x}\right)}{\sqrt{x}}\right]\qquad 0<x\le 1$$

The exponential function is the prototype $L = K+1$ hypergeometric function

18:14:7
$$\exp(\pm x)=\sum_{j=0}^{\infty}\frac{1}{(1)_j}(\pm x)^j$$

All other hypergeometric functions that have one more denominatorial than numeratorial parameter may be synthesized from it. Tables 18-3 and 18-4 respectively are listings of some examples of $L = K+1 = 1$ and $L = K+1 = 2$ hypergeometric functions. An example of an $L = K+1 = 3$ hypergeometric is

18:14:8
$$\sum_{j=0}^{\infty}\frac{(2)_j(2)_j}{(1)_j(3)_j(3)_j}(-x)^j=\frac{4}{x^2}\left[\mathrm{Ein}(x)+\exp(-x)-1\right]$$

The starting point for the synthesis of $L = K+2$ hypergeometric functions is the zero-order modified Bessel function $\mathrm{I}_0\left(2\sqrt{x}\right)$ or the corresponding (circular) Bessel function $\mathrm{J}_0\left(2\sqrt{x}\right)$. Examples of $L = K+2 = 3$ hypergeometrics are assembled in Tables 18-5 and 18-6. There are rather few instances of $L = K+2 = 4$ hypergeometrics, but one is

18:14:9
$$\sum_{j=0}^{\infty}\frac{(\nu)_j\left(\nu+\frac{1}{2}\right)_j}{(1)_j(1+\nu-\mu)_j(\mu+\nu)_j(2\nu)_j}x^j=\frac{\Gamma(\mu+\nu)\Gamma(1+\nu-\mu)}{(x/4)^{(2\nu-1)/2}}\mathrm{I}_{\mu+\nu-1}\left(\sqrt{x}\right)\mathrm{I}_{\nu-\mu}\left(\sqrt{x}\right)$$

Table 18-6

a	c_1	c_2	c_3	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j(c_3)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j(c_3)_j}(-x)^j$
$\frac{1}{2}$	$1-\nu$	1	$1+\nu$	$\nu\pi\csc(\nu\pi)\mathrm{I}_{-\nu}\left(\sqrt{x}\right)\mathrm{I}_{\nu}\left(\sqrt{x}\right)$	$\nu\pi\csc(\nu\pi)\mathrm{J}_{-\nu}\left(\sqrt{x}\right)\mathrm{J}_{\nu}\left(\sqrt{x}\right)$
$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2\sqrt{x}}\mathrm{Shi}\left(2\sqrt{x}\right)$	$\frac{1}{2\sqrt{x}}\mathrm{Si}\left(2\sqrt{x}\right)$
$\frac{3}{4}$	1	$\frac{3}{2}$	$\frac{7}{4}$		$\frac{3\sqrt{\pi}}{4x^{3/4}}\mathrm{S}\left(\sqrt{2}\,x^{1/4}\right)$
1	$\frac{3}{2}$	2	2	$\frac{1}{x}\mathrm{Chin}\left(2\sqrt{x}\right)$	$\frac{1}{x}\mathrm{Cin}\left(2\sqrt{x}\right)$
2	1	$\frac{5}{2}$	3	$\frac{3}{x^2}\left[\frac{\sqrt{x}\sinh\left(2\sqrt{x}\right)}{2}-\sinh^2\left(\sqrt{x}\right)\right]$	
ν	$\frac{1}{2}$	1	$1+\nu$		$\frac{2\nu}{(4x)^{\nu}}\left[\mathrm{C}(2\nu,0)-\mathrm{C}\left(2\nu,2\sqrt{x}\right)\right]$
ν	1	$\nu+\frac{1}{2}$	2ν		$\left(\frac{4}{x}\right)^{(2\nu-1)/2}\left[\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{J}_{(2\nu-1)/2}\left(\sqrt{x}\right)\right]^2$

For some obscure reason, hypergeometric functions in which the denominatorial order exceeds the numeratorial order by 3 seldom correspond to named functions, one a rare exception appearing in equation 53:11:3 and another being

18:14:10
$$\sum_{j=0}^{\infty}\frac{1}{\left(\frac{1}{3}\right)_j\left(\frac{2}{3}\right)_j(1)_j}x^j=\frac{1}{3}\exp\left(3x^{1/3}\right)+\frac{2}{3}\exp\left(-3/2x^{1/3}\right)\cos\left(3\sqrt{3}x^{1/3}/2\right)$$

which is an example of a Mittag-Leffler function [Section 45:14]. In contrast, named cases of $L = K+4 = 4$ hypergeometric functions are quite abundant, an instance being the Kelvin function [Chapter 55]

18:14:11
$$\sum_{j=0}^{\infty}\frac{1}{\left(1/2\right)_j\left(1/2\right)_j(1)_j(1)_j}(-x)^j=\mathrm{ber}\left(4x^{1/4}\right)$$

Table 18-7

a	$\sum_{j=0}^{\infty}(a)_j x^j$	$\sum_{j=0}^{\infty}(a)_j(-x)^j$
$-n$	$n!(-x)^n\mathrm{e}_n\left(\frac{-1}{x}\right)$	$n!x^n\mathrm{e}_n\left(\frac{1}{x}\right)$
$-1/2$	$1-\sqrt{x}\,\mathrm{daw}\left(\frac{1}{\sqrt{x}}\right)$	$1+\frac{\sqrt{\pi x}}{2}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right)$
$1/2$	$\frac{2}{\sqrt{x}}\mathrm{daw}\left(\frac{1}{\sqrt{x}}\right)$	$\sqrt{\frac{\pi}{x}}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right)$
1	$\frac{1}{x}\exp\left(\frac{-1}{x}\right)\mathrm{Ei}\left(\frac{1}{x}\right)$	$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right)$
$3/2$		$\frac{2\sqrt{\pi}}{x}\exp\left(\frac{1}{x}\right)\mathrm{ierfc}\left(\frac{1}{\sqrt{x}}\right)$
ν		$x^{-\nu}\exp\left(\frac{1}{x}\right)\Gamma\left(1-\nu,\frac{1}{x}\right)$

Table 18-8

a_1	a_2	c	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c)_j}(-x)^j$
$1/2-\nu$	$1/2+\nu$	1	$\sqrt{\frac{\pi}{x}}\exp\left(\frac{-1}{2x}\right)\mathrm{I}_\nu\left(\frac{1}{2x}\right)$	$\frac{1}{\sqrt{\pi x}}\exp\left(\frac{1}{2x}\right)\mathrm{K}_\nu\left(\frac{1}{2x}\right)$
$\frac{-n}{2}$	$\frac{1-n}{2}$	1	$2-\left(\frac{\sqrt{x}}{2}\right)^n\mathrm{H}_n\left(\frac{1}{\sqrt{x}}\right)$	
$1/6$	$5/6$	1	$\sqrt{\pi}\left(\frac{3}{4x}\right)^{1/6}\exp\left(\frac{-1}{2x}\right)\mathrm{Bi}\left\{\left(\frac{3}{4x}\right)^{2/3}\right\}$	$\sqrt{\pi}\left(\frac{48}{x}\right)^{1/6}\exp\left(\frac{1}{2x}\right)\mathrm{Ai}\left\{\left(\frac{3}{4x}\right)^{2/3}\right\}$
ν	$\nu+1/2$	1		$\left(\frac{2}{x}\right)^\nu\exp\left(\frac{1}{2x}\right)\mathrm{D}_{-2\nu}\left(\sqrt{\frac{2}{x}}\right)$
ν	μ	1		$\frac{1}{x^\nu}\mathrm{U}\left(\nu,1+\nu-\mu;\frac{1}{x}\right)$

Examples of hypergeometric functions of the $L = K-1 = 0$ and $L = K-1 = 1$ families are listed in Tables 18-7 and 18-8. Of course, these correspond to asymptotic series. With even worse convergence properties are the $L = K-2 = 0$ hypergeometrics of which a few are shown in Table 18-9. The series

18:14:12 $$\sum_{j=0}^{\infty}\frac{(\nu-\tfrac{1}{2})_j(\tfrac{1}{2})_j(\nu+\tfrac{1}{2})_j}{(1)_j}x^j = \frac{2}{\sqrt{x}}\mathrm{I}_\nu\left(\frac{1}{\sqrt{x}}\right)\mathrm{K}_\nu\left(\frac{1}{\sqrt{x}}\right)$$

is an example of an $L = K-2 = 1$ hypergeometric function. Two important $L = K-2 = 2$ hypergeometric functions occur in Section 53:6.

Table 18-9

a_1	a_2	$\sum_{j=0}^{\infty}(a_1)_j(a_2)_j(-x)^j$
ν	$\tfrac{1}{2}$	$\frac{\sqrt{\pi}\Gamma(1-\nu)}{x^{(1+2\nu)/4}}\left[\mathrm{h}_{(1-2\nu)/2}\left(\frac{2}{\sqrt{x}}\right)-\mathrm{Y}_{(1-2\nu)/2}\left(\frac{2}{\sqrt{x}}\right)\right]$
$\tfrac{1}{4}$	$\tfrac{3}{4}$	$\frac{2\sqrt{\pi}}{x^{1/4}}\mathrm{Fres}\left(\frac{2}{x^{1/4}}\right)$
$\tfrac{1}{2}$	1	$\frac{2}{\sqrt{x}}\mathrm{fi}\left(\frac{2}{\sqrt{x}}\right)$
$\tfrac{3}{4}$	$\tfrac{5}{4}$	$\frac{8\sqrt{\pi}}{x^{3/4}}\mathrm{Gres}\left(\frac{2}{x^{1/4}}\right)$
1	$\tfrac{3}{2}$	$\frac{4}{x}\mathrm{gi}\left(\frac{2}{\sqrt{x}}\right)$
ν	$\nu+\tfrac{1}{2}$	$\left(\frac{4}{x}\right)^\nu\left[\cos\left(\frac{2}{\sqrt{x}}\right)\mathrm{S}\left(1-2\nu,\frac{2}{\sqrt{x}}\right)-\sin\left(\frac{2}{\sqrt{x}}\right)\mathrm{C}\left(1-2\nu,\frac{2}{\sqrt{x}}\right)\right]$

Let G_j denote the following abbreviation

18:14:13 $$G_j = \frac{(a_1+j)(a_2+j)(a_3+j)\cdots(a_K+j)}{(c_1+j)(c_2+j)(c_3+j)\cdots(c_L+j)}$$

then any hypergeometric function is given by

18:14:14 $$1 \pm G_0x + G_0G_1x^2 \pm G_0G_1G_2x^3 + \cdots(\pm)^J G_0G_1G_2\cdots G_{J-1}x^J + R_J$$

where R_J is the remainder if the summation is halted after the Jth term. Ignoring R_J, a convenient method of calculating the hypergeometric function is via the concatenation

18:14:15 $$\left(\left(\cdots\left(G_{J-1}x\pm1\right)G_{J-2}x\pm\cdots\pm1\right)G_1x\pm1\right)G_0x+1$$

In discussing the general properties of hypergeometric functions, use will be made of a collapsed notation exemplified by the replacement of $(a_1)_j(a_2)_j\cdots(a_K)_j$ by $(a_{1\to K})_j$. Likewise $(a_{1\to K}+1)_j$ implies the K-fold product $(a_1+1)_j(a_2+1)_j\cdots(a_K+1)_j$.

The recursion relation

18:14:16 $$\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}+1\right)_j}{\left(c_{1\to L}+1\right)_j}(\pm x)^j=\frac{\pm c_{1\to L}}{a_{1\to K}x}\left[-1+\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm x)^j\right]$$

is satisfied by any hypergeometric function. Furthermore, any hypergeometric function can be split into two others with an inflated parameter set:

18:14:17 $$\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}+1\right)_j}{\left(c_{1\to L}+1\right)_j}(\pm x)^j=\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a_{1\to K}\right)_j\left(\frac{1}{2}+\frac{1}{2}a_{1\to K}\right)_j}{\left(\frac{1}{2}c_{1\to L}\right)_j\left(\frac{1}{2}+\frac{1}{2}c_{1\to L}\right)_j}\left(\frac{x^2}{4^{L-K}}\right)^j\pm\frac{a_{1\to K}x}{c_{1\to L}}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}+\frac{1}{2}a_{1\to K}\right)_j\left(1+\frac{1}{2}a_{1\to K}\right)_j}{\left(\frac{1}{2}+\frac{1}{2}c_{1\to L}\right)_j\left(1+\frac{1}{2}c_{1\to L}\right)_j}\left(\frac{x^2}{4^{L-K}}\right)^j$$

Of course, this result may become invalid if it creates new denominatorial parameters that are nonnegative integers. Replacing x in this formula by ix shows that a hypergeometric function of imaginary argument has real and imaginary parts that are themselves hypergeometric functions.

Embodying the fractional calculus [Section 12:14], a formula of very wide applicability is

18:14:18 $$\frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}}\left\{x^{\mu}\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm x)^j\right\}=\frac{\Gamma(\mu+1)x^{\mu-\nu}}{\Gamma(\mu-\nu+1)}\sum_{j=0}^{\infty}\frac{(\mu+1)_j\left(a_{1\to K}\right)_j}{(\mu-\nu+1)_j\left(c_{1\to L}\right)_j}(\pm x)^j$$

where ν and μ are not necessarily integers. This formula is invalid if either μ or $\mu-\nu$ is a negative integer; if they are *both* negative integers, it fails if ν is negative. Examples of the $\mu=0$ version include semidifferentiation

18:14:19 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm x)^j\right\}=\frac{1}{\sqrt{\pi x}}\sum_{j=0}^{\infty}\frac{(1)_j\left(a_{1\to K}\right)_j}{\left(\frac{1}{2}\right)_j\left(c_{1\to L}\right)_j}(\pm x)^j$$

semiintegration

18:14:20 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm x)^j\right\}=2\sqrt{\frac{x}{\pi}}\sum_{j=0}^{\infty}\frac{(1)_j\left(a_{1\to K}\right)_j}{\left(\frac{3}{2}\right)_j\left(c_{1\to L}\right)_j}(\pm x)^j$$

and integration

18:14:21 $$\int_0^x\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm t)^j\right\}\mathrm{d}t=x\sum_{j=0}^{\infty}\frac{(1)_j\left(a_{1\to K}\right)_j}{(2)_j\left(c_{1\to L}\right)_j}(\pm x)^j$$

The formula for ordinary differentiation

18:14:22 $$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm x)^j\right\}=\frac{a_{1\to K}}{c_{1\to L}}\sum_{j=0}^{\infty}\frac{(2)_j\left(a_{1\to K}\right)_j}{(1)_j\left(c_{1\to L}\right)_j}(\pm x)^j$$

also follows from 18:14:18, but only after a preliminary step based on recursion 18:14:16. Notice that all the formulas 18:14:16−18:14:22 maintain the $L-K$ difference. Laplace transformation, however, decreases this difference

18:14:23 $$\int_0^{\infty}\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm t)^j\right\}\exp(-st)\mathrm{d}t=\mathfrak{L}\left\{\sum_{j=0}^{\infty}\frac{\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm t)^j\right\}=\frac{1}{s}\sum_{j=0}^{\infty}\frac{(1)_j\left(a_{1\to K}\right)_j}{\left(c_{1\to L}\right)_j}(\pm s)^j$$

worsening the convergence properties of the hypergeometric function.

Specific to the hypergeometric 1:1 functions, are the reflection formula

18:14:24 $$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j=\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a)}\frac{(1-x)^{c-a-1}}{x^{c-1}}-\frac{c-1}{a-c+1}\sum_{j=0}^{\infty}\frac{(a)_j}{(a-c+2)_j}(1-x)^j$$

and the following rule

18:14:25 $$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j=\frac{(c-a)_n}{(c)_n}\left(\frac{x}{x-1}\right)^n\sum_{j=0}^{\infty}\frac{(a)_j}{(c+n)_j}x^j+\frac{1}{1-x}\sum_{j=0}^{n-1}\frac{(1-c)_j}{(a-c+1)_j}\left(\frac{x}{x-1}\right)^j$$

which permits the denominatorial parameter to be incremented by an integer, at the expense of an additional polynomial function.

CHAPTER 19

THE BERNOULLI POLYNOMIALS $B_n(x)$

Brothers Jacques (or Jakob) Bernoulli (1654–1705) and Jean (or Johann) Bernoulli (1667–1748) were talented Swiss mathematicians. Brother Jacques was responsible for these polynomials. Jean's son, Daniel Bernoulli (1700–1782), is known for his work on fluid dynamics.

19:1 NOTATION

$B_n(x)$ is the usual symbol for a Bernoulli polynomial function of argument x and degree n, though $\overline{B}_n(x)$ is sometimes used by authors who adopt the "rival" notation [Section 4:1] for Bernoulli numbers. The name "Bernoulli polynomial" and the symbol $\Phi_n(x)$ has been used to represent the quantity that this *Atlas* represents by $B_n(x)-B_n$, where B_n is the nth Bernoulli number [Chapter 4].

19:2 BEHAVIOR

Bernoulli polynomials are defined for all nonnegative integer n and all real argument x, though $0 \le x \le 1$ is the most important range and the one on which Figure 19-1 concentrates.

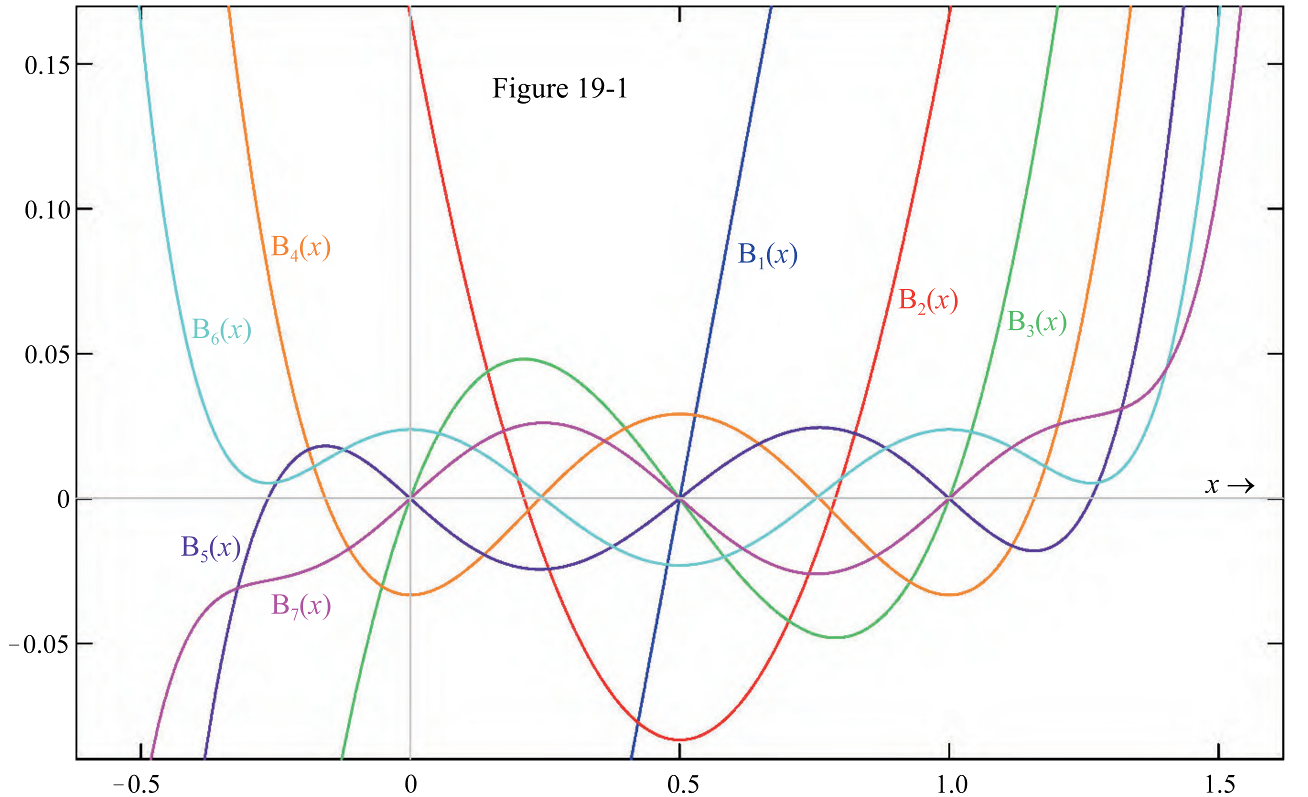
The magnitude of $B_n(x)$ for $n = 2,3,\cdots,10,11$ and $0 \le x \le 1$ is small, never exceeding $1/6$. Except for the $n = 0$ and $n = 1$ cases, the Bernoulli polynomial equals the Bernoulli number B_n at each end of the $0 \le x \le 1$interval. At the $x = 1/2$ center of this interval $B_n(x)$ displays an extremum when n is even, but equals zero when n is odd.

19:3 DEFINITIONS

The Bateman manuscript [Erdélyi et al., *Higher Transcendental Functions*, pages 38-39] gives several integral representations of the Bernoulli polynomials but the usual definition is provided by the generating function

19:3:1
$$\frac{t\exp(xt)}{\exp(t)-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_20,



Another definition,

19:3:2 $$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} B_j$$

expresses Bernoulli polynomials as a sum with coefficients comprised of Bernoulli numbers [Chapter 4] and binomial coefficients [Chapter 6].

19:4 SPECIAL CASES

$B_0(x)$	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$	$B_6(x)$
1	$x-\frac{1}{2}$	$x^2-x+\frac{1}{6}$	$x^3-\frac{3}{2}x^2+\frac{1}{2}x$	$x^4-2x^3+x^2-\frac{1}{30}$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x$	$x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}$

The coefficients of all Bernoulli polynomials up to $B_{15}(x)$ are listed by Abramowitz and Stegun [page 509].

19:5 INTRARELATIONSHIPS

According to the parity of their degree, Bernoulli polynomials have even or odd symmetry about $x = \frac{1}{2}$

19:5:1 $$B_n\left(\tfrac{1}{2}-x\right) = (-)^n B_n\left(\tfrac{1}{2}+x\right) \qquad n = 0,1,2,\cdots$$

They obey the reflection formula

19:5:2 $$(-)^n \mathrm{B}_n(-x) = \mathrm{B}_n(x) + nx^{n-1} \qquad n = 0,1,2.\cdots$$

The general argument-addition and argument-multiplication formulas

19:5:3 $$\mathrm{B}_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} \mathrm{B}_j(x) y^{n-j}$$

and

19:5:4 $$\mathrm{B}_n(mx) = m^{n-1} \sum_{j=0}^{m-1} \mathrm{B}_n\left(x + \frac{j}{m}\right) \qquad n = 0,1,2,\cdots \qquad m = 1,2,3,\cdots$$

have important special cases:

19:5:5 $$\mathrm{B}_n(x+1) = \sum_{j=0}^{n} \binom{n}{j} \mathrm{B}_j(x) = \mathrm{B}_n(x) + nx^{n-1}$$

which represents an argument-recursion formula, and the argument-duplication formula

19:5:6 $$\mathrm{B}_n(2x) = 2^{n-1}\left[\mathrm{B}_n(x) + \mathrm{B}_n\left(x + \tfrac{1}{2}\right)\right]$$

From 19:5:5 one may also derive the summation formula

19:5:7 $$\sum_{j=0}^{n} \binom{n+1}{j} \mathrm{B}_j(x) = (n+1)x^n$$

19:6 EXPANSIONS

Power series for the Bernoulli polynomial $\mathrm{B}_n(x)$ generally involve the Bernoulli numbers B_n [Chapter 4]. In using these formulas, recognize that all Bernoulli numbers with odd degrees, except B_1, are zero. The nth Bernoulli polynomial may be expanded as a power series in x

19:6:1 $$\mathrm{B}_n(x) = x^n - \frac{n}{2}x^{n-1} + \frac{n(n-1)}{12}x^{n-2} - \cdots + nx\mathrm{B}_{n-1} + \mathrm{B}_n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}\, \mathrm{B}_j$$

or in $(x-1)$

19:6:2 $$\mathrm{B}_n(x) = (x-1)^n + \frac{n}{2}(x-1)^{n-1} + \cdots - (-)^n n(x-1)\mathrm{B}_{n-1} + (-)^n \mathrm{B}_n = \sum_{j=0}^{n} \binom{n}{j} \frac{\mathrm{B}_j}{(x-1)^{j-n}}$$

but, because of the symmetry of $\mathrm{B}_n(x)$ about $x = ½$, the most concise expansion is in terms of $x - ½$. One finds, for $n = 0, 1, 2, \cdots$, that

19:6:3 $$\mathrm{B}_n(x) = \left(x-\tfrac{1}{2}\right)^n - \frac{n(n-1)}{24}\left(x-\tfrac{1}{2}\right)^{n-1} + \cdots + \left\{\begin{matrix} [2^{1-n}-1]\mathrm{B}_n \\ \text{or} \\ n\left[2^{-n}-1\right]\left(x-\tfrac{1}{2}\right)\mathrm{B}_{n-1} \end{matrix}\right\} = \sum_{j=0}^{\mathrm{Int}(n/2)} \binom{n}{2j} \frac{\left[2^{1-2j}-1\right]\mathrm{B}_{2j}}{\left(x-\tfrac{1}{2}\right)^{2j-n}}$$

where the form of the term in braces depends on whether n is even (upper alternative) or odd.

For $n \geq 3$, notice in Figure 19-1 that, in the interval $0 < x < 1$, each Bernoulli polynomial closely resembles a sinusoid [Chapter 32] of unity period. Such a sinusoid is, in fact, the first component in the Fourier expansion [Section 36:6] of these polynomials:

19:6:4 $$\mathrm{B}_n(x) = -2n! \sum_{j=1}^{\infty} \frac{\cos\left(2j\pi x - \frac{1}{2}n\pi\right)}{(2j\pi)^n} = \frac{-2n!}{(2\pi)^n} \sum_{j=1}^{\infty} \left\{\begin{matrix} (-)^{n/2} j^{-n} \cos(2j\pi x) & n = 2,4,6,\cdots \\ (-)^{(n-1)/2} j^{-n} \sin(2j\pi x) & n = 3,5,7,\cdots \end{matrix}\right. \qquad 0 \leq x \leq 1$$

The sums in this expansion are closely related to those addressed in Section 32:14.

19:7 PARTICULAR VALUES

	$B_n(-1)$	$B_n(-\frac{1}{2})$	$B_n(0)$	$B_n(\frac{1}{4})$	$B_n(\frac{1}{2})$	$B_n(\frac{3}{4})$	$B_n(1)$	$B_n(\frac{3}{2})$
$n=2,4,\cdots$	$n+B_n$	$\dfrac{n+B_n}{2^{n-1}}-B_n$	B_n	$\dfrac{B_n}{2^n}-\dfrac{2B_n}{4^n}$	$\dfrac{2B_n}{2^n}-B_n$	$\dfrac{B_n}{2^n}-\dfrac{2B_n}{4^n}$	B_n	$\dfrac{n+B_n}{2^{n-1}}-B_n$
$n=3,5,\cdots$	$-n$	$\dfrac{-n}{2^{n-1}}$	0	$\dfrac{-nE_{n-1}}{4^n}$	0	$\dfrac{nE_{n-1}}{4^n}$	0	$\dfrac{n}{2^{n-1}}$

19:8 NUMERICAL VALUES

Explicit formulas, such as those listed in Section 19:4 are used by *Equator*'s Bernoulli polynomials routine (keyword **Bpoly**) for $n=0,1,2,\cdots,7$. For other degrees up to $n=170$, recursion 19:5:5 is used either to decrease or increase the argument until it lies in the interval $0\le x<1$. Thereafter – apart from the very smallest arguments when approximation 19:9:1 is adopted – the truncated sum

19:8:1 $$B_n(x)\approx\frac{-2n!}{(2\pi)^n}\sum_{j=99,98}^{1}\frac{\cos\left\{\pi\left(2jx-\frac{1}{2}n\right)\right\}}{j^n}\qquad 10^{-8}<x\le 1$$

which is based on formula 19:6:4, is used, implemented via the reperiodized cosine [Section 32:8].

Bernoulli polynomials of large degree have many zeros. With patient searching [for example through the Newton-Raphson procedure, Section 52:15] these zeros are easily located numerically to 15 digits, but do not expect to find an argument value (other than 0, ½, or 1 for odd n) at which B_n is *exactly* zero, because the derivative of a Bernoulli polynomial of large degree is often huge close to its zeros.

19:9 LIMITS AND APPROXIMATIONS

As $x\to\pm\infty$, the Bernoulli polynomial $B_n(x)$ becomes dominated by its leading term x^n in its expansion and therefore (unless $n=0$) tends to $-\infty$ if n is odd and x negative, or to $+\infty$ otherwise.

As x approaches zero, the approximation

19:9:1 $$B_n(x)\approx B_n+nxB_{n-1}\qquad |x|\text{ small}$$

progressively improves.

19:10 OPERATIONS OF THE CALCULUS

19:10:1 $$\frac{d}{dx}B_n(x)=nB_{n-1}(x)$$

19:10:2 $$\int_0^x B_n(t)\,dt=\frac{B_{n+1}(x)-B_{n+1}}{n+1}$$

19:10:3
$$\int_0^1 \mathrm{B}_n(t)\,\mathrm{d}t = \begin{cases} 1 & n=0 \\ 0 & n=1,2,3,\cdots \end{cases}$$

19:10:4
$$\int_x^{x+1} \mathrm{B}_n(t)\,\mathrm{d}t = x^n \qquad n \neq 0$$

19:10:5
$$\int_0^1 \mathrm{B}_m(t)\mathrm{B}_n(t)\,\mathrm{d}t = \left\{ \begin{array}{ll} \dfrac{m!n!}{(m+n)!}\mathrm{B}_{m+n} & m,n \text{ odd} \\ 0 & m+n \text{ odd} \\ \dfrac{-m!n!}{(m+n)!}\mathrm{B}_{m+n} & m,n \text{ even} \end{array} \right\} \quad m,n=1,2,3,\cdots$$

This last formula establishes that, though Bernoulli polynomials are not generally orthogonal [Section 21:14], they do possess this property when $(m+n)$ is odd.

19:11 COMPLEX ARGUMENT

Bernoulli polynomials are rarely encountered with complex arguments.

19:12 GENERALIZATIONS

A class of functions defined by the generating function

19:12:1
$$\frac{t^m \exp(xt)}{[\exp(t)-1]^m} = \sum_{n=0}^{\infty} \mathrm{B}_n^{(m)}(x)\frac{t^n}{n!}$$

is termed [Korn and Korn, page 824] the *Bernoulli polynomials of order m and degree n*. A more general set of *higher order Bernoulli polynomials* is described by Erdélyi et al. [*Higher Transcendental Functions*, page 39].

Yet another generalization [Hilfer, page 60], which removes the restriction that the degree be an integer, stems from equation 19:6:5. Replacement of n by v, which is not necessarily an integer, leads to the definition

19:12:2
$$\mathrm{B}_v(x) = -2\Gamma(1+v)\sum_{j=1}^{\infty}\frac{\cos\left(2j\pi x - \frac{1}{2}vx\right)}{(2j\pi)^v} \qquad 0 \le x \le 1$$

None of these generalizations will be pursued in the *Atlas*.

19:13 COGNATE FUNCTIONS

Bernoulli polynomials are closely related to the Euler polynomials of the following chapter. Equations 20:3:3 and 20:3:4 make the connection explicit.

19:14 RELATED TOPIC: sums of powers

From integrals 19:10:2 and 19:10:4, it follows that

19:14:1 $$x^n+(1+x)^n+(2+x)^n+\cdots+(m-1+x)^n=\frac{B_{n+1}(m+x)-B_{n+1}(x)}{n+1}$$

One important application of this equation is derived by setting $x = 1$, which leads to

19:14:2 $$1^n+2^n+3^n+\cdots+m^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1} \qquad m,n=1,2,3,\cdots$$

This result, and its special cases when n takes integer values up to 5, are reported in Section 1:14.

Another application of formula 19:14:1 follows from setting $x = ½$, for then one finds

19:14:3 $$1^n+3^n+5^n+\cdots+(2m-1)^n=\frac{B_{n+1}(2m)-2^nB_{n+1}(m)+[2^n-1]B_{n+1}}{n+1} \qquad m,n=1,2,3,\cdots$$

A summation with alternating signs can be obtained by subtracting equation 19:14:2 (with m replaced by $2m-1$) from twice 19:14:3. The result is

19:14:4 $$1^n-2^n+3^n-\cdots+(2m-1)^n=\frac{B_{n+1}(2m)-2^{n+1}B_{n+1}(m)+[2^{n+1}-1]B_{n+1}}{n+1} \qquad m,n=1,2,3,\cdots$$

Note that the finite series in 19:14:2−4 correspond to the infinite series discussed in Chapter 3.

CHAPTER 20

THE EULER POLYNOMIALS $E_n(x)$

Named for gifted Swiss mathematician Leonhard Euler (1707 − 1783), these polynomial functions have much in common with Bernoulli polynomials. Both these polynomial families are useful in summing series involving quantities raised to integer powers, as explained in Sections 19:14 and 20:14.

20:1 NOTATION

An Euler polynomial of degree n and argument x is generally denoted $E_n(x)$, although the symbol $\overline{E}_n(x)$ is encountered occasionally. The symbol $E_n(x)$ is also used to represent the unrelated Schlömilch function, mentioned in Section 37:14 but not used elsewhere in the *Atlas*. The unsubscripted $E(x)$ denotes the unrelated elliptic integral discussed in Chapter 61.

20:2 BEHAVIOR

Though the Euler polynomials are defined for all nonnegative integer degrees and all real arguments, the domain embracing $0 \le x \le 1$, and occupying the center of Figure 20-1, is the most important. All Euler polynomials of even degree, apart from $E_0(x)$, become zero at each end of this interval and exhibit an extremum at $x = ½$. Conversely, all $E_n(x)$ of odd degree are zero at $x = ½$ and, apart from $E_1(x)$, display extrema at both $x = 0$ and $x = 1$. Some zeros and extrema lie outside $0 \le x \le 1$.

20:3 DEFINITIONS

Euler polynomials are defined by the generating function

20:3:1 $$\frac{2\exp(xt)}{1+\exp(t)} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$$

or, in terms of Euler numbers [Chapter 5],

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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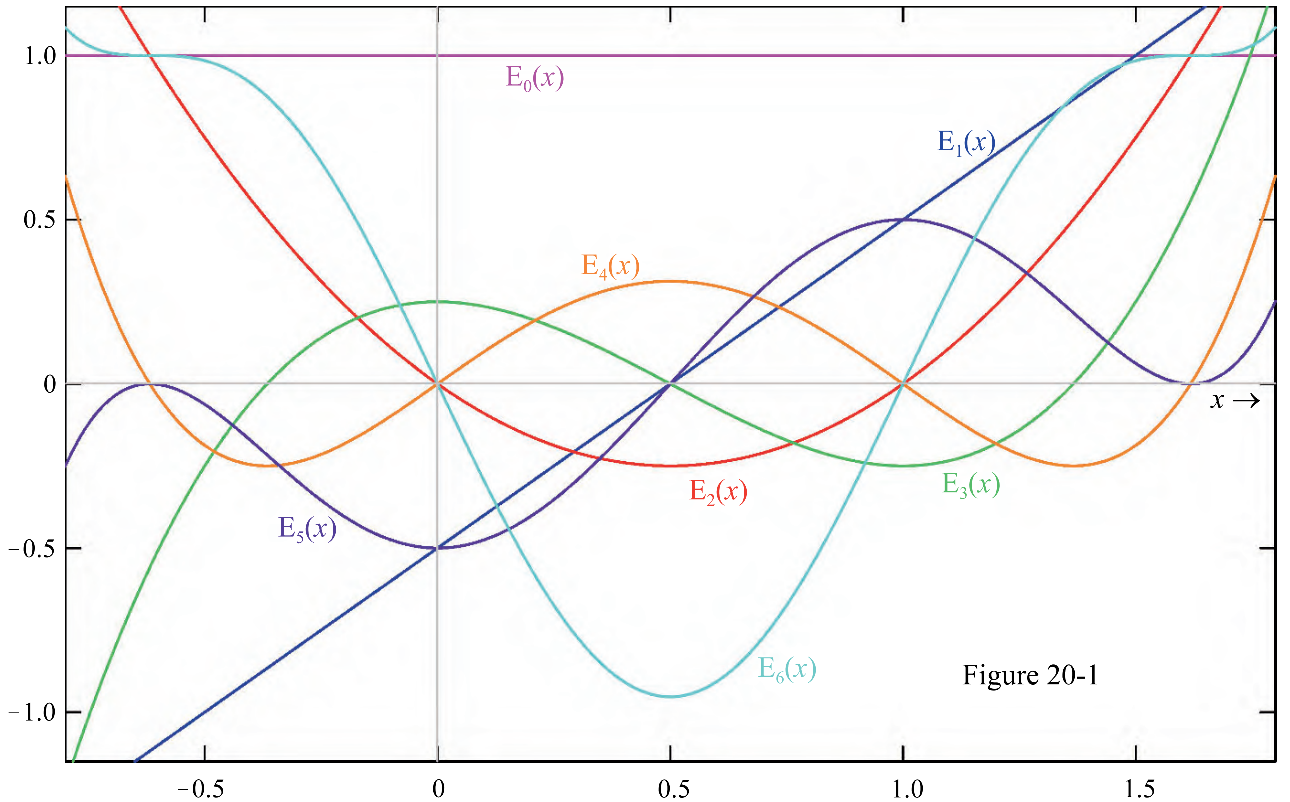


Figure 20-1

20:3:2 $$E_n(x) = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}[2x-1]^{n-j}E_j$$

Because all odd Euler numbers E_j are zero, the summation in 20:3:2 has only 1+Int(n/2) nonzero terms.

The formulas

20:3:3 $$E_n(x) = \frac{2}{n+1}\left[B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right]$$

and

20:3:4 $$E_n(x) = \frac{2^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right]$$

among others, define Euler polynomials in terms of Bernoulli polynomials [Chapter 19].

20:4 SPECIAL CASES

$E_0(x)$	$E_1(x)$	$E_2(x)$	$E_3(x)$	$E_4(x)$	$E_5(x)$	$E_6(x)$	$E_7(x)$
1	$x-\frac{1}{2}$	x^2-x	$x^3-\frac{3}{2}x^2+\frac{1}{4}$	x^4-2x^3+x	$x^5-\frac{5}{2}x^4+\frac{5}{2}x^2-\frac{1}{2}$	$x^6-3x^5+5x^3-3x$	$x^7-\frac{7}{2}x^6+\frac{35}{4}x^4-\frac{21}{2}x^2+\frac{17}{8}$

The coefficients of all Euler polynomials up to E_{15} are listed by Abramowitz and Stegun [page 809]. Note that the coefficients are invariably integers for even degree, but that coefficients are frequently fractional when n is odd. The denominator of the fraction is invariably a power of 2. Notice also that, even though many powers are missing (their coefficients are zero), the terms present always alternate in sign.

20:5 INTRARELATIONSHIPS

According to the parity of their degree, Euler polynomials have even or odd symmetry about $x = ½$

20:5:1 $$E_n\left(\tfrac{1}{2}-x\right) = (-)^n E_n\left(\tfrac{1}{2}+x\right)$$

or equivalently

20:5:2 $$E_n(-x) = (-)^n E_n(1+x)$$

and, additionally, they obey the reflection formula

20:5:3 $$(-)^n E_n(-x) = 2x^n - E_n(x)$$

about $x = 0$. Combination of the two prior results leads to

20:5:4 $$E_n(1+x) = 2x^n - E_n(x)$$

which serves as an argument-recursion formula and is exploited in Section 20:14.

The general argument-addition formula

20:5:5 $$E_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}E_j(x)y^{n-j} = \sum_{j=0}^{n}\binom{n}{j}E_j(y)x^{n-j}$$

is the key to deriving a number of important results. Thus, because $2^nE_n(½)$ equals the Euler number E_n, setting $y = ½$ gives

20:5:6 $$E_n\left(\tfrac{1}{2}+x\right) = x^n\sum_{j=0}^{n}\binom{n}{j}\frac{E_j}{(2x)^j}$$

while setting $y = 1$ leads to the summation formula

20:5:7 $$\sum_{j=0}^{n}\binom{n}{j}E_j(x) = E_n(1+x)$$

With $y = 0$, and with help from the formula [Section 20:7] that relates the Euler polynomial of zero argument to a Bernoulli number, one finds

20:5:8 $$E_n(x) = \sum_{j=0}^{n}\binom{n}{j}E_j(0)x^{n-j} = -2x^n\sum_{j=0}^{n}\frac{2^{j+1}-1}{j+1}\binom{n}{j}\frac{B_{j+1}}{x^j}$$

which provides a power-series expansion for the Euler polynomial. Finally, setting $y = x$ generates the argument-duplication formula

20:5:9 $$E_n(2x) = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}E_j(x)$$

However, setting $m = 2$ in 20:5:10 provides a simpler expression for $E_n(2x)$.

The argument-multiplication formula for Euler polynomials adopts alternative forms according to the parity of the multiplier

20:5:10 $$E_n(mx) = \begin{cases} m^n \sum_{j=0}^{m-1} (-)^j E_n\left(x + \frac{j}{m}\right) & m = 1,3,5,\cdots \\ \frac{-2m^n}{n+1} \sum_{j=0}^{m-1} (-)^j B_{n+1}\left(x + \frac{j}{m}\right) & m = 2,4,6,\cdots \end{cases}$$

Definition 20:3:4 is a special case of this formula.

20:6 EXPANSIONS

Equation 20:5:7 provides a polynomial expansion, in powers of x, for $E_n(x)$, of which the early and late terms are

20:6:1 $$E_n(x) = x^n - \frac{nx^{n-1}}{2} + \frac{n(n-1)(n-2)x^{n-3}}{24} - \cdots - (2^{n-1}-1)nB_{n-1}x^2 - (2^{n+1}-2)B_n x - \frac{2^{n+2}-2}{n+1}B_{n+1}$$

and, via the reflection formulas, this may be restated as a series in $(x-1)$:

20:6:2 $$E_n(x) = (x-1)^n + \frac{n(x-1)^{n-1}}{2} - \cdots + (2^{n-1}-1)nB_{n-1}(x-1)^2 + (2^{n+1}-2)B_n(x-1) + \frac{2^{n+2}-2}{n+1}B_{n+1}$$

However, the most succinct expansion is in terms of $(x-\frac{1}{2})$:

20:6:3 $$E_n(x) = \left(x-\tfrac{1}{2}\right)^n - \frac{n^2-n}{8}\left(x-\tfrac{1}{2}\right)^{n-2} + \cdots + \binom{n}{j}\frac{E_j}{2^j}\left(x-\tfrac{1}{2}\right)^{n-j} + \cdots + \frac{nE_{n-1}}{2^{n-1}}\left(x-\tfrac{1}{2}\right) + \frac{E_n}{2^n}$$

which is equivalent to definition 20:3:2.

A Fourier expansion [Section 36:6] of an Euler polynomial leads to the formula

20:6:4 $$E_n(x) = \frac{4n!}{\pi^{n+1}} \sum_{j=1,3,5}^{\infty} \frac{f(x)}{j^{n+1}} \qquad f(x) = \sin\left(j\pi x - \frac{n\pi}{2}\right) = \begin{cases} (-)^{(n+1)/2}\cos(j\pi x) & n = 1,3,5,\cdots \\ (-)^{n/2}\sin(j\pi x) & n = 2,4,6,\cdots \end{cases}$$

which is valid for $0 \le x \le 1$. Note that the summation index takes odd values only.

20:7 PARTICULAR VALUES

	$E_n(-1)$	$E_n(-\frac{1}{2})$	$E_n(0)$	$E_n(\frac{1}{6})$	$E_n(\frac{1}{3})$	$E_n(\frac{1}{2})$	$E_n(\frac{2}{3})$	$E_n(\frac{5}{6})$	$E_n(1)$	$E_n(\frac{3}{2})$	$E_n(2)$
$n = 1,3,\cdots$	$2\beta_n - 2$	$\frac{-1}{2^{n-1}}$	$-2\beta_n$		$\frac{1-3^n}{3^n}\beta_n$	0	$\frac{3^n-1}{3^n}\beta_n$		$2\beta_n$	$\frac{1}{2^{n-1}}$	$2-2\beta_n$
$n = 2,4,\cdots$	2	$\frac{2-E_n}{2^n}$	0	$\frac{3^n+1}{2^{n+1}3^n}E_n$		$\frac{E_n}{2^n}$		$\frac{3^n+1}{2^{n+1}3^n}E_n$	0	$\frac{2-E_n}{2^n}$	2

The abbreviation $\beta_n = (2^{n+1}-1)B_{n+1}/(n+1)$ is used in the table.

Notice near the left-hand frame of Figure 20-1 that the argument (about -0.61803) at which $E_2(x)$ and $E_6(x)$ equal unity coincides with that at which $E_4(x)$ and $E_5(x)$ equal zero, as well as that at which $E_5(x)$ peaks and $E_6(x)$ inflects. This argument is, in fact, $-1/\upsilon$, where υ is the *golden section* cited in Section 23:14. Similar coincidences occur at argument $x = \upsilon = 1.6180$.

20:8 NUMERICAL VALUES

For $0 \le x \le 1$, *Equator*'s Euler polynomial routine (keyword **Epoly**) uses whichever of the trio of equations 20:6:1, 20:6:2, or 20:6:3 makes the leading term closest to zero. For $x > 1$, the formula

20:8:1 $$\mathrm{E}_n(x) = (-)^{\mathrm{Int}(x)}\mathrm{E}_n\big(\mathrm{frac}(x)\big) - 2\sum_{j=1}^{\mathrm{Int}(x)} (-)^j (x-j)^n \qquad x > 1$$

is used to bring the argument into the range of one of the trio. The formula

20:8:2 $$\mathrm{E}_n(x) = (-)^{\mathrm{Int}(x)}\left[\mathrm{E}_n\big(\mathrm{frac}(x)\big) + 2\sum_{j=1}^{-\mathrm{Int}(x)} (-)^j \big(\mathrm{frac}(x)-j\big)^n\right] \qquad x < 0$$

performs a similar task when $x < 0$. Each of these formulas is a consequence of 20:5:4.

20:9 LIMITS AND APPROXIMATIONS

Except when $n = 0$, Euler polynomials rapidly approach $+\infty$ (or $-\infty$ if n is odd and x is negative) as $x \to \pm\infty$. For large n and $0 \le x \le 1$, $\mathrm{E}_n(x)$ is approximated by a sinusoid [Chapter 36]

20:9:1 $$\mathrm{E}_n(x) \approx \frac{4n!}{\pi^{n+1}}\sin\left(\pi x - \frac{n\pi}{2}\right) \qquad \text{large } n \text{ and } 0 \le x \le 1$$

20:10 OPERATIONS OF THE CALCULUS

20:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{E}_n(x) = n\mathrm{E}_{n-1}(x)$$

20:10:2 $$\int_{x_0}^{x} \mathrm{E}_n(t)\,\mathrm{d}t = \frac{\mathrm{E}_{n+1}(x)}{n+1} \qquad \begin{cases} n = 0,2,4,\cdots & x_0 = \tfrac{1}{2} \\ n = 1,3,5,\cdots & x_0 = 0 \end{cases}$$

20:10:3 $$\int_0^1 \mathrm{E}_n(t)\,\mathrm{d}t = \begin{cases} \dfrac{4(2^{n+2}-1)}{(n+1)(n+2)}\mathrm{B}_{n+2} & n = 0,2,4,\cdots \\ 0 & n = 1,3,5,\cdots \end{cases}$$

20:10:4 $$\int_0^1 \mathrm{E}_m(t)\mathrm{E}_n(t)\,\mathrm{d}t = \left\{ \begin{array}{ll} \dfrac{-4(2^{m+n+2}-1)m!n!}{(m+n+2)!}\mathrm{B}_{m+n+2} & m,n \text{ odd} \\ 0 & m+n \text{ odd} \\ \dfrac{4(2^{m+n+2}-1)m!n!}{(m+n+2)!}\mathrm{B}_{m+n+2} & m,n \text{ even} \end{array} \right\} \qquad m,n = 0,1,2,\cdots$$

20:10:5 $$\int_0^1 \mathrm{E}_n(t)\sec(\pi t)\,\mathrm{d}t = \begin{cases} 0 & n = 0,2,4,\cdots \\ \dfrac{(-)^{(n+1)/2}4n!}{\pi^{n+1}}\beta(n) & n = 1,3,5,\cdots \end{cases}$$

The functions occurring in the final integral are the beta number [Chapter 3] and the secant function [Chapter 33].

20:11 COMPLEX ARGUMENT

Applications of Euler polynomials are primarily confined to real arguments.

20:12 GENERALIZATIONS

Analogs of the generalizations discussed in Section 19:12 can be constructed for Euler polynomials, but these are not discussed in the *Atlas*.

20:13 COGNATE FUNCTIONS

Euler and Bernoulli polynomials are closely allied; they are linked, for example, through equations 20:3:3 and 20:3:4.

20:14 RELATED TOPIC: sums of alternating power series

Iteration of equation 20:5:3 leads to

20:14:1
$$x^n - (1+x)^n + (2+x)^n - \cdots \begin{cases} +(m-1+x)^n = \left[E_n(x) + E_n(m+x)\right]/2 & m = 1,3,5,\cdots \\ -(m-1+x)^n = \left[E_n(x) - E_n(m+x)\right]/2 & m = 2,4,6,\cdots \end{cases}$$

Setting $x = 1$ yields the useful result

20:14:2
$$1^n - 2^n + 3^n - \cdots \mp (m-1)^n \pm m^n = \frac{2^{n+1}-1}{n+1}B_{n+1} \pm \frac{E_n(m+1)}{2} \qquad m,n = 1,2,3\cdots$$

which is an alternative to 19:14:4 and equivalent to equation 1:14:9, having the special cases reported as 1:14:6−1:14:9. Setting $x = ½$ in 20:14:1 leads to

20:14:3
$$1^n - 3^n + 5^n - \cdots \mp (2m-3)^n \pm (2m-1)^n = \frac{E_n \pm 2^n E_n\left(m+\tfrac{1}{2}\right)}{2}$$

CHAPTER 21

THE LEGENDRE POLYNOMIALS $P_n(x)$

Legendre polynomial functions provide solutions to differential equations that govern important physical phenomena. Named for the prolific French mathematician Adrien Marie Legendre (1752−1833), these polynomials constitute a family that is one of the simplest that exhibits orthogonality, a property described in Section 21:14.

21:1 NOTATION

The symbol $P_n(x)$ is standard for the Legendre polynomial of degree n and argument x, though the P is often italicized. The name *spherical polynomial* is also encountered, *zonal surface harmonic function* [Section 59:14] being yet another name. When orthogonality is important, *normalized* or *orthonormal* Legendre polynomials are often specified; this implies multiplication of $P_n(x)$ by $\sqrt{n+\frac{1}{2}}$, but some authors do not make the presence of this factor explicit.

The symbol $P_\nu(x)$, where ν is unrestricted, is given to a class of functions discussed in Chapter 59 and named *Legendre functions of the first kind*. The degree of a Legendre *polynomial*, on the other hand, is usually restricted to nonnegative integers. The two P functions are identical for integer ν of either sign because

21:1:1
$$P_\nu(x) = \begin{cases} P_n(x) & \nu = 0,1,2,\cdots \qquad n=\nu \\ P_{-n-1}(x) & \nu = -1,-2,-3\cdots \qquad n=-1-\nu \end{cases}$$

Thus the functions of this chapter are just special instances of Legendre functions of the first kind and are sometimes not regarded as distinct from the more general function.

Provided that it does not exceed unity in magnitude, the argument x of a Legendre polynomial is often replaced by the cosine of a subsidiary variable

21:1:2
$$P_n\big(\cos(\theta)\big) = P_n\big(x\big) \qquad \theta = \arccos(x) \qquad -1 \le x \le 1$$

This reflects the manner in which Legendre polynomials often arise in scientific and engineering applications.

The *shifted Legendre polynomials*, distinguished by an asterisk, have a changed argument

21:1:3
$$P_n^*(x) = P_n(2x-1)$$

so that the orthogonality interval [Section 21:14] becomes 0 to 1 instead of −1 to 1. Another modified Legendre polynomial is

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_22, © Springer Science+Business Media, LLC 2009

21:1:4
$$P^n(x) = \frac{(2n)!}{2^n} P_n(x)$$

Neither of these modifications is used in the *Atlas*. See Sections 21:12 and 22:12, respectively, for explanations of the $P_\nu^{(\mu)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ notations.

21:2 BEHAVIOR

We allow the argument x to adopt any real value in this chapter, though its most important domain is $-1 \le x \le 1$ and restriction to this domain is obligatory when the argument is replaced by $\cos(\theta)$. As Figure 21-1 illustrates, the range of the Legendre polynomial reflects its argument and degree as follows

21:2:1
$$\begin{cases} 1 \le P_n(x) < \infty & x \ge 1, & n = 0,1,2,\cdots \\ -1 \le P_n(x) < 1 & -1 \le x \le 1, & n = 0,1,2,\cdots \\ 1 \le P_n(x) < \infty & x \le -1, & n = 0,2,4,\cdots \\ -\infty < P_n(x) \le -1 & x \le -1, & n = 1,3,5\cdots \end{cases}$$

The polynomial $P_n(x)$ has exactly n zeros and $(n-1)$ extrema; they all lie within the $-1 < x < 1$ zone.

Apart from the $n = 0$ case, $P_n(x)$ increases in magnitude without limit outside the $|x| < 1$ region, as $|x|$ increases, the rate of increase being greater the larger n is.

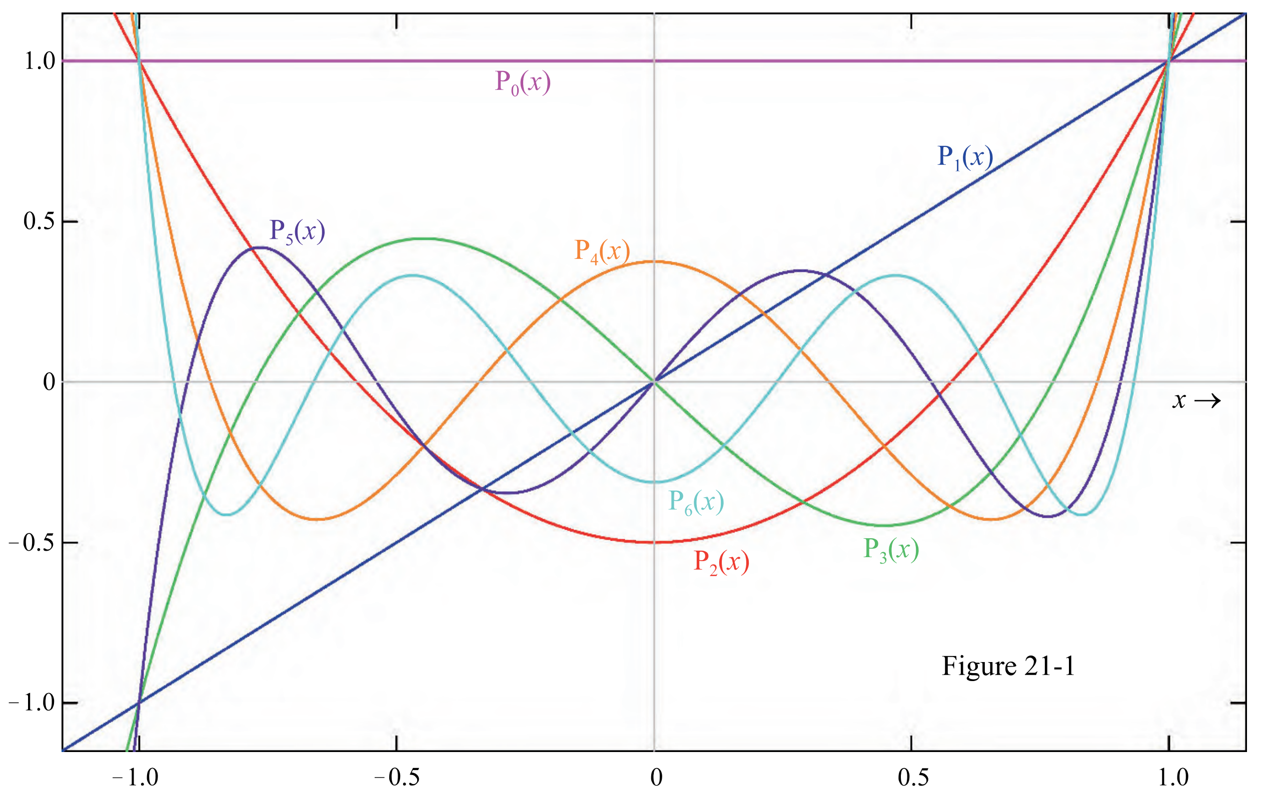


Figure 21-1

21:3 DEFINITIONS

Legendre polynomials may be defined by the generating function

21:3:1
$$\frac{1}{\sqrt{1-2xt+t^2}} = \left\{ \begin{array}{ll} \sum_{n=0}^{\infty} \mathrm{P}_n(x) t^n & -1<t<1 \\ \sum_{n=0}^{\infty} \mathrm{P}_n(x) t^{-n-1} & |t|>1 \end{array} \right\} -1 \le x \le 1$$

or by *Rodrigues's formula* (Benjamin Olinde Rodrigues, French mathematician, 1794 – 1851)

21:3:2
$$\mathrm{P}_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^2-1\right)^n$$

One of many integral representations of Legendre polynomials is

21:3:3
$$\mathrm{P}_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + \sqrt{x^2-1}\cos(t) \right]^n \mathrm{d}t \qquad x>1$$

known as *Laplace's representation*.

Because the Legendre polynomial is a hypergeometric function [Section 18:14]

21:3:4
$$\mathrm{P}_n(1-2x) = \sum_{j=0}^{\infty} \frac{(-n)_j (n+1)_j}{(1)_j (1)_j} x^j$$

it may be synthesized [Section 43:14] by the two-step process

21:3:5
$$\frac{1}{1-x} \xrightarrow[1]{-n} (1-x)^n \xrightarrow[1]{1+n} \mathrm{P}_n(1-2x)$$

from the basis function $1/(1-x)$.

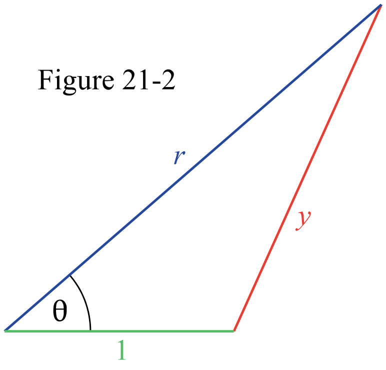
The Legendre polynomial $\mathrm{f}(x) = \mathrm{P}_n(x)$ satisfies *Legendre's differential equation*

21:3:6
$$(1-x^2)\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} + n(n+1)\mathrm{f} = 0$$

The most general solution of this equation is $\mathrm{f}(x) = w_1\mathrm{P}_n(x)+w_2\mathrm{Q}_n(x)$ where the w's are arbitrary weights and $\mathrm{Q}_n(x)$ is the function addressed in Section 21:13. Alternatively, the general solution of 21:3:6 may be written as the arbitrarily weighted sums of two series, namely those given by equation 21:6:1.

Figure 21-2 illustrates a geometric definition that explains how Legendre polynomials arise in certain applications. The triangle shown has two sides, one of unity length and one of length r, enclosing an angle θ. By the *law of cosines* [Section 34:14], the length y of the third side equals $\sqrt{1-2r\cos(\theta)+r^2}$. In some physical problems, it is necessary to expand $1/y$ as a power series in reciprocal powers of r. Definition 21:3:1 shows that the appropriate series is

Figure 21-2



21:3:7
$$\frac{1}{y} = \frac{\mathrm{P}_0(\cos(\theta))}{r} + \frac{\mathrm{P}_1(\cos(\theta))}{r^2} + \frac{\mathrm{P}_2(\cos(\theta))}{r^3} + \cdots$$

if $r > 1$.

21:4 SPECIAL CASES

$P_0(x)$	$P_1(x)$	$P_2(x)$	$P_3(x)$	$P_4(x)$	$P_5(x)$	$P_6(x)$
1	x	$\frac{3}{2}x^2-\frac{1}{2}$	$\frac{5}{2}x^3-\frac{3}{2}x$	$\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}$	$\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$	$\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}$

When an angular argument is adopted, it is often more convenient to use the cosines of multiple angles, rather than powers, as the expansion terms; for example

21:4:1 $$P_4(\cos(\theta)) = [9+20\cos(2\theta)+35\cos(4\theta)]/64$$

Gradshteyn and Ryzhik [Section 8.91] and Jeffrey [Section 18.2.4.2] list the corresponding expressions for other degrees.

21:5 INTRARELATIONSHIPS

Legendre polynomials are even or odd

21:5:1 $$P_n(-x) = (-)^{\left|n+\frac{1}{2}\right|-\frac{1}{2}}P_n(x)$$

according to the parity of n. The recursion formula

21:5:2 $$P_n(x) = \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x) \qquad n=2,3,4,\cdots$$

relates three polynomials of consecutive degrees.

Section 21:4 lists expressions for $P_n(x)$ functions as finite series in powers of x. The converse is also possible: powers may be expressed as finite series of Legendre polynomials. For example

21:5:3 $$x^4 = \tfrac{1}{5}P_0(x) + \tfrac{4}{7}P_2(x) + \tfrac{8}{35}P_4(x)$$

The general formula, which employs Pochhammer polynomials, is

21:5:4 $$x^n = \begin{cases} \displaystyle\sum_{j=0}^{n/2}\left[\frac{4j+1}{n+1}\right]\frac{\left(\frac{n+2}{2}-j\right)_j}{\left(\frac{n+3}{2}\right)_j}P_{2j}(x) & n=0,2,4,\cdots \\ \displaystyle\sum_{j=0}^{(n-1)/2}\left[\frac{4j+3}{n+2}\right]\frac{\left(\frac{n+1}{2}-j\right)_j}{\left(\frac{n+4}{2}\right)_j}P_{2j+1}(x) & n=1,3,5,\cdots \end{cases}$$

Not only powers, but a large number of functions f(x) of x can be expressed as series of Legendre polynomials by exploiting the technique described in Section 21:15.

Among summation formulas for Legendre polynomials are

21:5:5 $$P_0(x)+3P_1(x)+\cdots+(2n-3)P_{n-2}(x)+(2n-1)P_{n-1}(x) = \frac{n[P_{n-1}(x)-P_n(x)]}{1-x}$$

and

21:5:6 $$P_0(x)-3P_1(x)+\cdots\pm(2n-3)P_{n-2}(x)\mp(2n-1)P_{n-1}(x) = \frac{\mp n[P_{n-1}(x)+P_n(x)]}{1+x}$$

21:6 EXPANSIONS

There are numerous power-series expansions of Legendre polynomials, all of which terminate; for example

21:6:1
$$P_n(x) = \begin{cases} \dfrac{(-)^{n/2}(n-1)!!}{n!!}\left[1 - \dfrac{n(n+1)}{2!}x^2 + \dfrac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots \right. & \left. n = 0,2,4,\cdots \right] \\ \dfrac{(-)^{(n-1)/2}n!!}{(n-1)!!}\left[x - \dfrac{(n-1)(n+2)}{3!}x^3 + \dfrac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \right. & \left. n = 1,3,5,\cdots \right] \end{cases}$$

There are many ways in which the Legendre polynomials may be written as Gauss hypergeometric functions [Chapter 60]; the simplest is known as *Murphy's formula*, which leads to a terminating power series in $1-x$:

21:6:2
$$P_n(x) = F\left(-n, 1+n; 1; \frac{1-x}{2}\right) = 1 - \frac{n(n+1)}{2}(1-x) + \frac{(n-1)n(n+1)(n+2)}{16}(1-x)^2 - \cdots + \frac{(2n)!}{(n!)^2}\left(\frac{x-1}{2}\right)^n$$

More are listed by Gradshteyn and Ryzhik [Section 8.91], or may be found by specializing formulas from Chapter 59 to integer degree.

The infinite Fourier series

21:6:3
$$P_n(\cos(\theta)) = \frac{4}{\pi}\frac{(2n)!!}{(2n+1)!!}\sum_{j=0}^{\infty}\frac{(2j-1)!!(n+1)_j}{(2j)!!(n+\frac{3}{2})_j}\sin((n+2j+1)\theta)$$

involves coefficients that are quotients of Pochhammer polynomials [Chapter 18] and double factorials [Section 2:13].

21:7 PARTICULAR VALUES

Legendre polynomials in the $n = 2,6,10,\cdots$ family display a minimum at $x = 0$, whereas a maximum is exhibited there if n is a multiple of 4. Away from $x = 0$, we know of no general formulas for the zeros or extrema of $P_n(x)$.

	$P_n(-1)$	$P_n(0)$	$P_n(1)$
$n = 0,2,4,\cdots$	1	$\dfrac{(-)^{n/2}(n-1)!!}{n!!}$	1
$n = 1,3,5,\cdots$	-1	0	1

21:8 NUMERICAL VALUES

Equator's Legendre P polynomial routine (keyword **Ppoly**) uses recursion formula 21:5:2, initialized by $P_0(x) = 1$ and $P_1(x) = x$.

21:9 LIMITS AND APPROXIMATIONS

The approximation [Lebedev]

21:9:1
$$P_n(\cos(\theta)) \approx \sqrt{\frac{2}{n\pi\sin(\theta)}}\sin\left\{\left(n+\frac{1}{2}\right)\theta + \frac{\pi}{4}\right\}$$

holds for large n when $0 < \theta < \pi$.

21:10 OPERATIONS OF THE CALCULUS

Differentiation and integration of a Legendre polynomial give

21:10:1
$$\frac{d}{dx}P_n(x) = \frac{n}{1-x^2}\left[P_{n-1}(x) - xP_n(x)\right]$$

and

21:10:2
$$\int_{-1}^{x} P_n(t)\,dt = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} \qquad n = 1,2,3,\cdots$$

Equation 21:10:1 is just one of a number of formulas relating the Legendre polynomials to its derivative. Others include

21:10:3
$$x\frac{d}{dx}P_n(x) - \frac{d}{dx}P_{n-1}(x) = nP_n(x) \qquad n = 1,2,3,\cdots$$

and

21:10:4
$$\frac{d}{dx}\left\{P_{n+1}(x) - P_{n-1}(x)\right\} = (2n+1)P_n(x) \qquad n = 1,2,3,\cdots$$

There are many useful definite integrals involving Legendre polynomials. These include

21:10:5
$$\int_{-1}^{1} P_n(t)\,dt = 0 \qquad n = 1,2,3,\cdots$$

21:10:6
$$\int_{-1}^{1} \frac{P_n(t)}{\sqrt{1-t}}\,dt = \frac{\sqrt{8}}{2n+1} \qquad n = 0,1,2,\cdots$$

21:10:7
$$\int_{-1}^{1} \frac{P_n(t)}{\sqrt{1-t^2}}\,dt = \begin{cases} \pi\left[(n-1)!!/n!!\right]^2 & n = 0,2,4,\cdots \\ 0 & n = 1,3,5,\cdots \end{cases}$$

and

21:10:8
$$\int_{0}^{1} t^{\mu}P_n(t)\,dt = \frac{(\mu-n+2)(\mu-n+4)(\mu-n+6)\cdots(\mu-\frac{1}{2}\pm\frac{1}{2})}{(\mu+n+1)(\mu+n-1)(\mu+n-3)\cdots(\mu+\frac{3}{2}\mp\frac{1}{2})} \qquad \mu > \pm\tfrac{1}{2} - \tfrac{3}{2}$$

where, in the final example, the upper/lower signs apply according as n is even/odd. Gradshteyn and Ryzhik devote several pages [Sections 7.22−7.25] to a listing of further integrals involving Legendre polynomials. The integral, between −1 and +1, of the product of a Legendre polynomial and some other function may often be evaluated by recourse to Rodrigues's definition 21:3:2, with help from parts integration [Section 0:10].

Legendre functions are orthogonal [Section 21:14] on the interval between −1 and +1 with a weight function of unity:

21:10:9
$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases}$$

Laplace transforms involving the Legendre polynomial include

21:10:10
$$\int_{0}^{\infty} P_n(1+t)\exp(-st)\,dt = \mathcal{L}\left\{P_n(1+t)\right\} = \sqrt{\frac{2}{\pi s}}\exp(s)K_{(2n+1)/2}(s)$$

21:10:11
$$\int_0^\infty \mathrm{P}_n(1-t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{P}_n(1-t)\} = \sum_{j=0}^{n}\frac{(-n)_j(1+n)_j}{(1)_j}\left(\frac{1}{2s}\right)^j = \frac{1}{\pi s}\exp\left(\frac{1}{2s}\right)\mathrm{i}_n\left(\frac{1}{2s}\right)$$

and

21:10:12
$$\int_0^\infty (t+1)^n \mathrm{P}_n\left(\frac{t-1}{t+1}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{(t+1)^n\mathrm{P}_n\left(\frac{t-1}{t+1}\right)\right\} = \frac{n!}{s^{n+1}}\mathrm{L}_n(s)$$

Respectively, functions from Chapter 51, Section 28:13, and Chapter 23 appear in these three transforms. The Hilbert transform [Section 7:10] of a Legendre polynomial

21:10:13
$$\frac{1}{\pi}\int_{-1}^{1}\mathrm{P}_n(t)\frac{\mathrm{d}t}{t-y} = \frac{-2}{\pi}\mathrm{Q}_n(y)$$

generates a Legendre function of the second kind [Section 21:13] of the same integer degree, a result known as *Neumann's formula*.

21:11 COMPLEX ARGUMENT

Although most applications of Legendre polynomials require a real argument, there is some interest in $\mathrm{P}_n(z)$ where z is a complex variable. One definition, the *Schläfli representation* (Ludwig Schläfli, Swiss theologian and mathematician, 1814 – 1895) employs the contour integral

21:11:1
$$\mathrm{P}_n(z) = \frac{1}{2^n}\oint\frac{\left(t^2-1\right)^n}{\left(t-z\right)^{n+1}}\frac{\mathrm{d}t}{2\pi i}$$

where the contour encloses the point z once.

Two formulas for Laplace inversion are

21:11:2
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s}\mathrm{P}_n\left(1-\frac{a}{s}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{s}\mathrm{P}_n\left(1-\frac{a}{s}\right)\right\} = \frac{1}{a}\sum_{j=0}^{n}\frac{(-n)_j(1+n)_j}{(1)_j(1)_j(1)_j}\left(\frac{at}{2}\right)^j$$

and

21:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^{n+1}}\mathrm{P}_n\left(1-\frac{a}{s}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{s^{n+1}}\mathrm{P}_n\left(1-\frac{a}{s}\right)\right\} = \frac{t^n}{2^n n! a}\mathrm{L}_n\left(\frac{at}{2}\right)$$

21:12 GENERALIZATIONS

A modest generalization of the $\mathrm{P}_n(x)$ polynomial is to admit negative integer degrees by the degree-reflection formula

21:12:1
$$\mathrm{P}_{-n}(x) = \mathrm{P}_{n-1}(x)$$

which is consistent with 21:1:1. More radical is the generalization of the degree to any real number, thereby creating the Legendre function of the first kind, $\mathrm{P}_\nu(x)$, discussed in Chapter 59. A still wider generalization is to the associated Legendre functions of the first kind, $\mathrm{P}_\nu^{(\mu)}(x)$, which are the subject of Section 59:13.

21:13 COGNATE FUNCTIONS

The $Q_n(x)$ functions are mentioned in the context of equations 21:3:6 and 21:10:13. They are sometimes called "*Legendre polynomials of the second kind*" but this is an unfortunate name because they are not polynomial functions. They are the integer-degree instances of *Legendre functions of the second kind*, addressed in more detail in Chapter 59. Some special cases, applicable when $-1 < x < 1$, are tabulated

$Q_0(x)$	$Q_1(x)$	$Q_2(x)$	$Q_3(x)$
$\mathrm{artanh}(x)$	$x\,\mathrm{artanh}(x)-1$	$\left(\frac{3}{2}x^2-\frac{1}{2}\right)\mathrm{artanh}(x)-\frac{3}{2}x$	$\left(\frac{5}{2}x^3-\frac{3}{2}x\right)\mathrm{artanh}(x)-\frac{5}{2}x^2+\frac{2}{3}$

and the general formula is

21:13:1
$$Q_n(x)=P_n(x)\mathrm{artanh}(x)-2\sum_{j=1,3,5}^{J}\frac{2n-2j+1}{j(2n-j+1)}P_{n-j}(x)\qquad J=1+2\mathrm{Int}\left(\frac{n-1}{2}\right)$$

Here, $\mathrm{artanh}(x)$ is the inverse hyperbolic tangent function [Chapter 31], being equivalent to $\ln\left(\sqrt{(1+x)/(1-x)}\right)$. These definitions apply only for $|x| < 1$; outside this range replace $\mathrm{artanh}(x)$ by $\mathrm{arcoth}(x)$.

In distinction from $P_n(x)$, $Q_n(x)$ has a discontinuity of the $+\infty|+\infty$ variety at $x=1$, another discontinuity at $x=-1$, and, as Figure 21-3 suggests, an approach towards zero as $x\to\pm\infty$. Moreover, in contrast to $P_n(x)$, $Q_n(x)$ is an even function when n is odd, and vice versa. Nonetheless, the two functions do have much in common: equations 21:5:2 and 21:10:1−4 retain their validity when Q replaces P. Each satisfies Legendre's differential equation 21:3:7.

$Q_n(z)$ is multivalued in the complex plane, unless suitably cut. Usually a branch cut is applied along the real axis between -1 and $+1$. The average of the values on either side of the cut is assigned to $Q_n(x)$, while the difference of these values is $-\pi i P_n(x)$. See Erdélyi et al [*Higher Transcendental Functions*, Volume 2, Page 181].

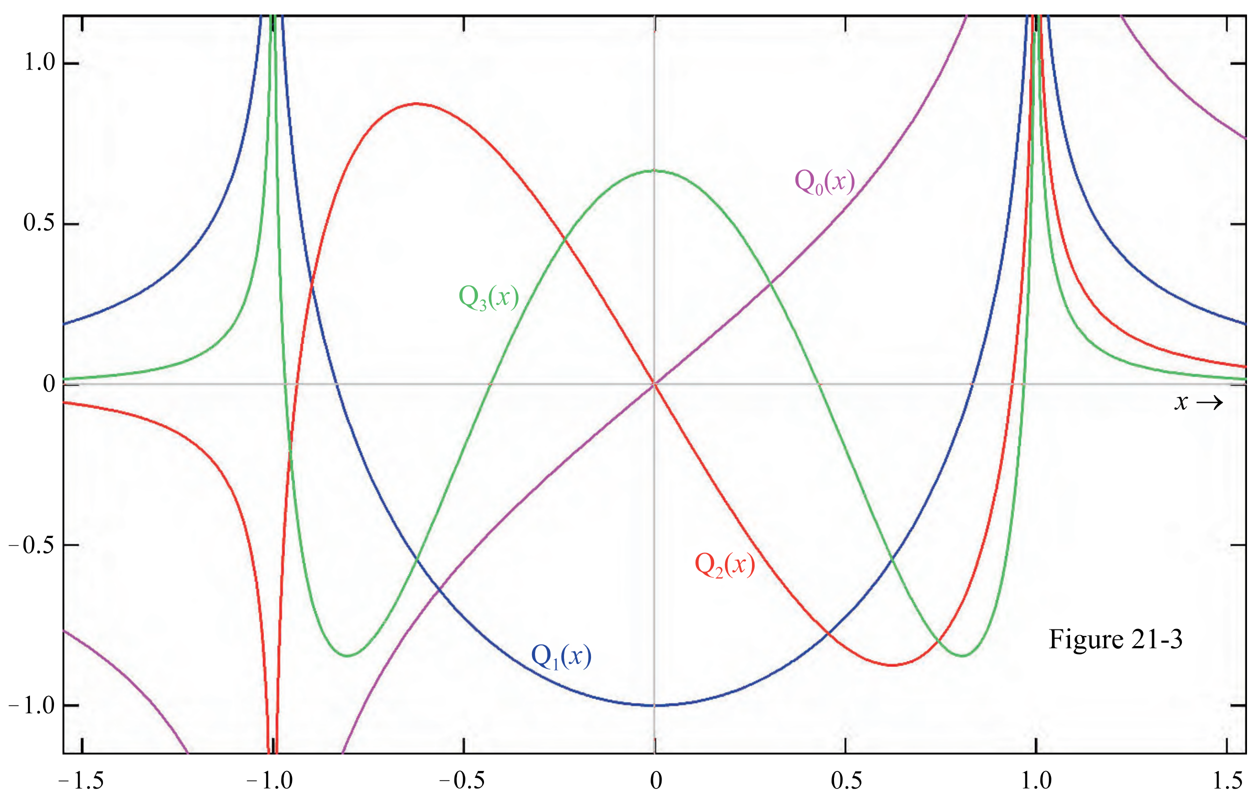


Figure 21-3

21:14 RELATED TOPIC: orthogonality

The integral, between distinct limits, of the square f × f of a real function f is never zero (unless the function is the zero function); in fact it must be positive. The definite integral of the product of two functions f × g is rarely zero

21:14:1 $$\int_{x_0}^{x_1} f(t)g(t)\,dt = 0 \qquad f \neq g$$

but this unusual property is possessed by any two Legendre polynomials, $P_n(x)$ and $P_m(x)$, if $x_0 = -1$ and $x_1 = 1$. Specifically, to repeat equation 21:10:10

21:14:2 $$\int_{-1}^{1} P_n(t)P_m(t)\,dt = \begin{Bmatrix} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{Bmatrix} \quad m,n = 0,1,2,\cdots$$

The two Legendre polynomials are said to be mutually "orthogonal on the interval x_0 and x_1".

If a carefully chosen third function, called a *weight function* $w(t)$ is introduced into the integrand, then the integral is more likely to vanish. If it does, that is if

21:14:3 $$\int_{x_0}^{x_1} w(t)f(t)g(t)\,dt = 0 \qquad f \neq g$$

then one says that "f and g are orthogonal on the interval x_0 to x_1 with respect to the weight function w". The weight function is required to be nonnegative throughout the x_0 to x_1 interval. Legendre polynomials are orthogonal on the interval −1 to +1 with a weight function of unity.

Many polynomial functions, including those in Chapters 21 through 24 of this *Atlas*, constitute *orthogonal families*. The accompanying table shows some of the orthogonality properties of the Legendre, Chebyshev, Jacobi, Gegenbauer, Laguerre, and Hermite polynomials.

$\Psi_n(t)$	x_0 to x_1	$w(t)$	Ω_n^2
$P_n(t)$	−1 to 1	1	$2/(2n+1)$
$T_n(t)$	−1 to 1	$1/\sqrt{1-t^2}$	$\pi/2$ (or π if $n=0$)
$U_n(t)$	−1 to 1	$\sqrt{1-t^2}$	$\pi/2$
$P_n(2t-1)$	0 to 1	$1/\sqrt{1-t^2}$	$1/(2n+1)$
$T_n(2t-1)$	0 to 1	$1/\sqrt{1-t^2}$	$\pi/2$ (or π if $n=0$)
$U_n(2t-1)$	0 to 1	$\sqrt{t-t^2}$	$\pi/8$
$P_n^{(\nu,\mu)}(t)$	−1 to 1	$\dfrac{(1-t)^\nu}{(1+t)^\mu}$	$\dfrac{2^{\nu+\mu+1}\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{(2n+\nu+\mu+1)n!\Gamma(n+\nu+\mu+1)}$
$C_n^{(\lambda)}(t)$	−1 to 1	$(1-t^2)^{(2\lambda-1)/2}$	$\dfrac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n+\lambda)n!\Gamma^2(\lambda)}$
$L_n(x)$	0 to ∞	$\exp(-t)$	1
$H_n(x)$	−∞ to ∞	$\exp(-t^2)$	$2^n n!\sqrt{\pi}$

A general definition is that the family of functions $\Psi_0(x)$, $\Psi_1(x)$, $\Psi_2(x)$, ⋯ are orthogonal when, for all choices of nonnegative integers n and m

21:14:4
$$\int_{x_0}^{x_1} w(t)\Psi_n(t)\Psi_m(t)\,dt = \begin{cases} 0 & m \ne n \\ \Omega_n^2 \ne 0 & m = n \end{cases}$$

where x_0, x_1, and w are specified. An alternative way of formulating this equation is

21:14:5
$$\int_{x_0}^{x_1} w(t)\left[\frac{\Psi_n(t)}{\Omega_n}\right]\left[\frac{\Psi_m(t)}{\Omega_m}\right]dt = \delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker delta function [Section 9:13]. The Ω terms are normalizing factors and $\Psi_n(x)/\Omega_n$ is the *normalized* or *orthonormal* version of the nth orthogonal polynomial. The normalized version of the Legendre polynomial is $\sqrt{n+\frac{1}{2}}\,P_n(x)$.

Orthogonality is not restricted to polynomials. How the sine and cosine functions group into orthogonal families is explained in Section 32:10.

From a weighted sum of orthogonal functions one may extract the coefficient of one particular, say the mth, summand by the integration procedure

21:14:6
$$\int_{x_0}^{x_1}\left[\sum_n a_n\Psi_n(t)\right]\Psi_m(t)w(t)\,dt = a_m\Omega_m^2$$

This is one of the prime benefits conferred by orthogonality.

21:15 RELATED TOPIC: expansions in Legendre functions

The orthogonality property of the orthogonal polynomials $\Psi_0(x)$, $\Psi_1(x)$, $\Psi_2(x)$, $\cdots$ permits their use as a *basis set* for the expansion

21:15:1
$$f(x) = c_0\Psi_0(x) + c_1\Psi_1(x) + c_2\Psi_2(x) + \cdots$$

for a wide range of f(x) functions. Such an expansion, which is called a *orthogonal series* or a *generalized Fourier expansion* is an alternative to, and in some circumstances an improvement on, the most usual basis set, the powers x^0, x^1, x^2, $\cdots$. The c's are the coefficients of the expansion. Multiplying equation 21:15:1 by $w(x)\Psi_n(x)$ and integrating shows that the coefficients may be found from the integral

21:15:2
$$c_n = \frac{1}{\Omega_n^2}\int_{x_0}^{x_1} f(t)w(t)\Psi_n(t)\,dt$$

Any orthogonal family can be used in this context, but the Legendre polynomials are often the most convenient because their weight function is unity. For this basis set,

21:15:3
$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)\,dt$$

Two examples of series constructed in this way are

21:15:4
$$\sqrt{a^2-x^2} = \frac{\pi a}{2}\left[\frac{1}{2} - \sum_{j=2,4}^{\infty}(2j+1)\frac{(j-3)!!(j-1)!!}{j!!(j+1)!!}P_j\left(\frac{x}{a}\right)\right]$$

and

21:15:5
$$\frac{1}{\sqrt{a^2-x^2}} = \frac{\pi}{2a}\sum_{j=0,2}^{\infty}(2j+1)\left[\frac{(j-1)!!}{j!!}\right]^2 P_j\left(\frac{x}{a}\right)$$

while a third is provided by equation 35:6:5. In these equations, the coefficients are written as double factorials.

CHAPTER 22

THE CHEBYSHEV POLYNOMIALS $T_n(x)$ AND $U_n(x)$

Named for the Russian mathematician Pafnuty Lvovich Chebyshev (1821−1894), these two kinds of polynomial function are interrelated in the many ways detailed in Section 22:5 and by several of the integrals in Section 22:10. $T_n(x)$ is extensively used in fitting procedures of various kinds. Gegenbauer and Jacobi polynomials are also briefly addressed in this chapter, as are discrete Chebyshev polynomials.

22:1 NOTATION

Alternative transliterations from the Cyrillic alphabet lead to spelling variants ranging from Chebyshev to Tschebischeff.

Unfortunately, mathematicians differ in their definitions of Chebyshev polynomials, and there is not even unanimity on which of the two kinds constitutes the "first" family and which the "second". The notations $T_n(x)$ and $U_n(x)$ may be encountered with meanings equivalent to the $T_n(x)/2^{n-1}$ and $nU_n(x)$ of this *Atlas*. Moreover, the symbol $U_n(x)$ and the name "Chebyshev polynomial of the second kind" is commonly applied to a set of functions defined by $\arcsin(x)$ when $n = 0$, and by our $\sqrt{1-x^2}\,U_{n-1}(x)$ when $n = 1,2,3,\cdots$, despite these not being polynomials at all.

Several supplementary notations may be encountered, but none of these is employed in the *Atlas*. Thus $T_n^*(x)$ and $U_n^*(x)$ are often used to symbolize the *shifted Chebyshev polynomials* $T_n(2x-1)$ and $U_n(2x-1)$, cited in the table in Section 21:14. The symbols $C_n(x)$ and $S_n(x)$ have been used for $2T_n(x/2)$ and $U_n(x/2)$, respectively. Under the name *Chebyshev functions*, $\overline{T}_\nu(x)$ and $\overline{U}_\nu(x)$ symbolize functions resembling $T_n(x)$ and $U_n(x)$ but in which the degree is not restricted to integer values.

Here, we shall name $T_n(x)$ the first, and $U_n(x)$ the second, Chebyshev polynomial family. The degree n takes the nonnegative integer values $0,1,2,\cdots$, though $U_0(x)$ is sometimes excluded from the second family when orthogonality is an issue. The argument x is unrestricted, but interest is concentrated in the $-1 \le x \le 1$ range.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_23,

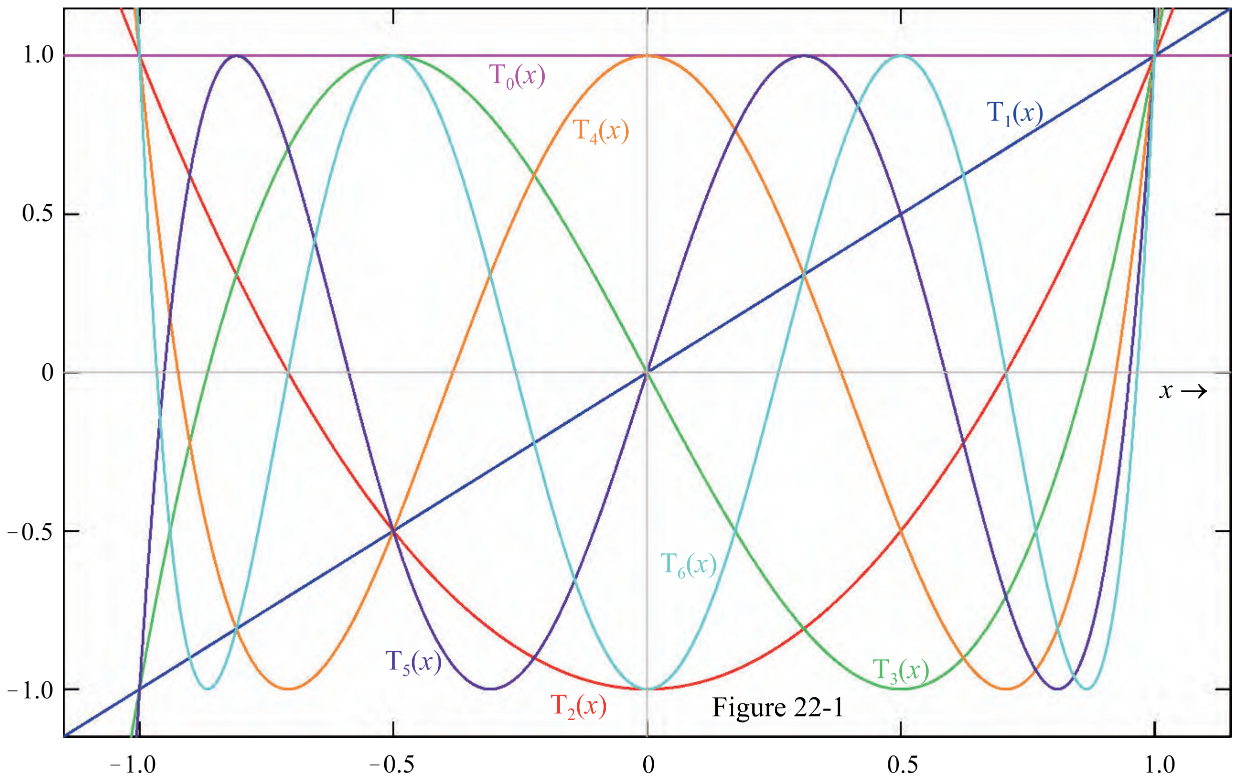


Figure 22-1

22:2 BEHAVIOR

Figures 22-1 and 22-2 portray $T_n(x)$ and $U_n(x)$ respectively. Each kind of Chebyshev polynomial has precisely n zeros, Int$(n/2)$ minima and Int$\{(n-1)/2\}$ maxima, all these features lying within $-1 < x < 1$. The locations of the zeros are given in Section 22:7. All the maxima of $T_n(x)$ take the value +1, whereas $T_n(x)$ equals −1 at all the minima; no comparable rules apply to $U_n(x)$. For $-1 \le x \le 1$ range, the ranges of the two polynomials are

22:2:1 $$-1 \le T_n(x) \le 1$$

22:2:2 $$-n-1 \le U_n(x) \le n+1$$

$$\left.\right\} \quad \left\{ \begin{array}{l} -1 \le x \le 1 \\ n = 1, 2, 3, \cdots \end{array} \right.$$

22:3 DEFINITIONS

Some, but not all, of these definitions apply outside the $-1 \le x \le 1$ range of interest.

Purely algebraic definitions of the polynomial families are

22:3:1 $$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right]$$

22:3:2 $$U_n(x) = \frac{1}{2i\sqrt{1-x^2}}\left[\left(x + i\sqrt{1-x^2}\right)^{n+1} - \left(x - i\sqrt{1-x^2}\right)^{n+1}\right]$$

but the definitions most usually encountered are trigonometrically based:

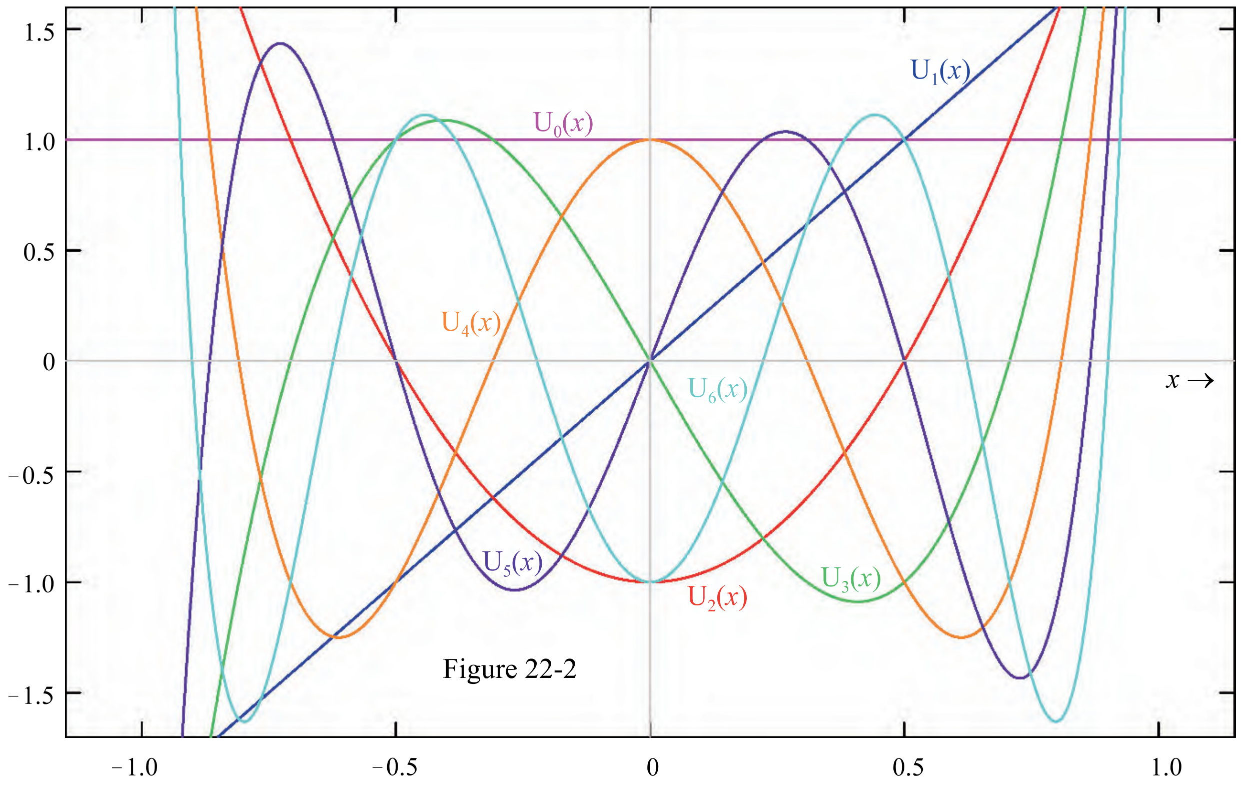


Figure 22-2

22:3:3 $$T_n(x) = \cos(n\theta)$$

22:3:4 $$U_n(x) = \csc(\theta)\sin\{(n+1)\theta\}$$

$$\left.\right\} \theta = \arccos(x) \qquad -1 \le x \le 1$$

The Chebyshev polynomials may be written as Gauss hypergeometric functions [Chapter 60], or equivalently as the straightforward hypergeometric functions

22:3:5 $$T_n(1-2x) = \sum_{j=0}^{n} \frac{(-n)_j (n)_j}{(\tfrac{1}{2})_j (1)_j} x^j$$

and

22:3:6 $$U_n(1-2x) = (n+1)\sum_{j=0}^{n} \frac{(-n)_j (n+2)_j}{(1)_j (\tfrac{3}{2})_j} x^j$$

They may be synthesized by routes analogous to that in 21:3:5.

Chebyshev polynomials may be defined by the generating functions

22:3:7 $$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n$$

22:3:8 $$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n$$

or by the *Rodrigues's formulas*

22:3:9 $$T_n(x) = \frac{(-)^n\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n}\left(1-x^2\right)^{(2n-1)/2}$$

22:3:10 $$U_n(x) = \frac{(-)^n(n+1)}{(2n-1)!!\sqrt{1-x^2}}\frac{d^n}{dx^n}\left(1-x^2\right)^{(2n+1)/2}$$

For $n = 1,2,3,\cdots$, a general solution to *Chebyshev's differential equation*

22:3:11 $$(1-x^2)\frac{d^2f}{dx^2} - x\frac{df}{dx} + n^2 f = 0$$

is

22:3:12 $$f(x) = w_1 T_n(x) + w_2\sqrt{1-x^2}\,U_{n-1}(x)$$

where the w's are arbitrary. For $n = 0$, the solution is $f(x) = w_1 + w_2 \arcsin(x)$.

22:4 SPECIAL CASES

$T_0(x)$	$T_1(x)$	$T_2(x)$	$T_3(x)$	$T_4(x)$	$T_5(x)$	$T_6(x)$	$T_7(x)$
1	x	$2x^2-1$	$4x^3-3x$	$8x^4-8x^2+1$	$16x^5-20x^3+5x$	$32x^6-48x^4+18x^2-1$	$64x^7-112x^5+56x^3-7x$

$U_0(x)$	$U_1(x)$	$U_2(x)$	$U_3(x)$	$U_4(x)$	$U_5(x)$	$U_6(x)$	$U_7(x)$
1	$2x$	$4x^2-1$	$8x^3-4x$	$16x^4-12x^2+1$	$32x^5-32x^3+6x$	$64x^6-80x^4+24x^2-1$	$128x^7-192x^5+80x^3-8x$

22:5 INTRARELATIONSHIPS

Chebyshev polynomials are even or odd

22:5:1 $$f_n(-x) = (-)^n f_n(x) \qquad f = T \text{ or } U$$

according to the parity of their degree. The recursion formula

22:5:2 $$f_n(x) = 2x f_{n-1}(x) - f_{n-2}(x) \qquad n = 2,3,4,\cdots$$

also applies equally to both kinds of Chebyshev polynomial.

The formulas

22:5:3 $$T_n(x) = U_n(x) - xU_{n-1}(x) \qquad n = 1,2,3,\cdots$$

and

22:5:4 $$U_n(x) = \frac{T_n(x) - xT_{n+1}(x)}{1-x^2} \qquad n = 0,1,2,\cdots$$

permit one kind of Chebyshev polynomial to be expressed in terms of the other. There are also formulas expressing the product of two Chebyshev polynomials as the sum of other Chebyshev polynomials:

22:5:5 $$T_n(x)T_m(x) = \frac{T_{n+m}(x) + T_{n-m}(x)}{2} \qquad n \ge m$$

22:5:6 $$U_n(x)U_m(x) = \frac{T_{n-m}(x) - T_{n+m+2}(x)}{2(1-x^2)} \qquad n \ge m$$

22:5:7
$$T_n(x)U_m(x) = \begin{cases} \tfrac{1}{2}U_{n+m}(x) + \tfrac{1}{2}U_{n-m}(x) & n \le m \\ \tfrac{1}{2}U_{n+m}(x) & n = m+1 \\ \tfrac{1}{2}U_{n+m}(x) - \tfrac{1}{2}U_{n-m-2}(x) & n \ge m+2 \end{cases}$$

Series of Chebyshev polynomials of the first kind have the sums

22:5:8
$$T_0(x) + T_2(x) + T_4(x) + \cdots + T_n(x) = \tfrac{1}{2} + \tfrac{1}{2}U_n(x) \qquad n = 0,2,4,\cdots$$

and

22:5:9
$$T_1(x) + T_3(x) + T_5(x) + \cdots + T_n(x) = \tfrac{1}{2}U_n(x) \qquad n = 1,3,5,\cdots$$

Clearly, these two expressions may be combined additively or subtractively to produce other useful sums. The corresponding series for Chebyshev polynomials of the second kind are

22:5:10
$$U_0(x) + U_2(x) + U_4(x) + \cdots + U_n(x) = \frac{1 - T_{n+2}(x)}{2(1-x^2)} \qquad n = 0,2,4,\cdots$$

and

22:5:11
$$U_1(x) + U_3(x) + U_5(x) + \cdots + U_n(x) = \frac{x - T_{n+2}(x)}{2(1-x^2)} \qquad n = 1,3,5,\cdots$$

Positive integer powers of x can be expressed as finite series of Chebyshev polynomials of either kind. For example

22:5:12
$$x^3 = \tfrac{3}{4}T_1(x) + \tfrac{1}{4}T_3(x) = \tfrac{1}{4}U_1(x) + \tfrac{1}{8}U_3(x)$$

Restricting attention to the polynomials of the first kind, one has the general expression

22:5:13
$$x^n = \sum_{j=0}^{n} \gamma_j^{(n)} T_j(x) = \gamma_n^{(n)} T_n(x) + \gamma_{n-2}^{(n)} T_{n-2}(x) + \gamma_{n-4}^{(n)} T_{n-4}(x) + \cdots + \begin{cases} \gamma_0^{(n)} T_0(x) & n = 0,2,4,\cdots \\ \gamma_1^{(n)} T_1(x) & n = 1,3,5,\cdots \end{cases}$$

Note that $\gamma_j^{(n)} = 0$ if j and n have unlike parities, or if $j > n$. Values of these coefficients are provided by *Equator*'s Chebyshev gamma coefficient routine (keyword **Chebygamma**) which uses the following recursive algorithm, executed sequentially for $n = 0$, $n = 1$, $n = 2$, $n = 3$, $n = 4$, $\cdots$.

22:5:14
$$\begin{cases} n = 0,1 & \gamma_n^{(n)} = 1 \\ n = 2,4,6,\cdots & \begin{cases} \gamma_0^{(n)} = \tfrac{1}{2}\gamma_1^{(n-1)} \\ \gamma_j^{(n)} = \tfrac{1}{2}\gamma_{j-1}^{(n-1)} + \tfrac{1}{2}\gamma_{j+1}^{(n-1)} \quad \text{for } j = 2,4,6,\cdots,(n-2) \\ \gamma_n^{(n)} = \tfrac{1}{2}\gamma_{n-1}^{(n-1)} \end{cases} \\ n = 3,5,7,\cdots & \begin{cases} \gamma_1^{(n)} = \gamma_0^{(n-1)} + \tfrac{1}{2}\gamma_2^{(n-1)} \\ \gamma_j^{(n)} = \tfrac{1}{2}\gamma_{j-1}^{(n-1)} + \tfrac{1}{2}\gamma_{j+1}^{(n-1)} \quad \text{for } j = 3,5,7,\cdots,(n-2) \\ \gamma_n^{(n)} = \tfrac{1}{2}\gamma_{n-1}^{(n-1)} \end{cases} \end{cases}$$

Extensive tables of these, and similar, coefficients can be found in Abramowitz and Stegun [Chapter 22]. Here we list only coefficients of $T_0(x)$, $T_1(x)$, $T_2(x)$ and $T_3(x)$ in the expansions of x^0, x^1, x^2, $\cdots$, x^{13}.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$
$\gamma_0^{(n)}$	1	0	$1/2$	0	$3/8$	0	$5/16$	0	$35/128$	0	$63/256$	0	$231/1024$	0
$\gamma_1^{(n)}$	0	1	0	$3/4$	0	$5/8$	0	$35/64$	0	$63/128$	0	$231/512$	0	$429/1024$
$\gamma_2^{(n)}$	0	0	$1/2$	0	$1/2$	0	$15/32$	0	$7/16$	0	$105/256$	0	$99/256$	0
$\gamma_3^{(n)}$	0	0	0	$1/4$	0	$5/16$	0	$21/64$	0	$21/64$	0	$165/512$	0	$1287/4096$

22:6 EXPANSIONS

Explicit power-series expressions for the two Chebyshev polynomials are

22:6:1 $$\mathrm{T}_n(x)=\frac{n}{2}\sum_{j=0}^{\mathrm{Int}(n/2)}\frac{(-)^j}{n-j}\binom{n-j}{j}(2x)^{n-2j} \qquad n=1,2,3,\cdots$$

and

22:6:2 $$\mathrm{U}_n(x)=\sum_{j=0}^{\mathrm{Int}(n/2)}(-)^j\binom{n-j}{j}(2x)^{n-2j} \qquad n=0,1,2,\cdots$$

but these may also be written in a binomial-expansion-like format as

22:6:3 $$\mathrm{T}_n(x)=\binom{n}{0}x^n-\binom{n}{2}x^{n-2}(1-x^2)+\binom{n}{4}x^{n-4}(1-x^2)^2-\cdots$$

and

22:6:4 $$\mathrm{U}_{n-1}(x)=\binom{n}{1}x^{n-1}-\binom{n}{2}x^{n-3}(1-x^2)+\binom{n}{3}x^{n-5}(1-x^2)^2-\cdots$$

Furthermore, series expansions may be developed from the hypergeometric representations 22:3:5 and 22:3:6.

Equation 22:6:3 may be redrafted as

22:6:5 $$\mathrm{T}_n(x)=\sum_{k=n,n-2,n-4}^{0\text{ or }1}\tau_k^{(n)}x^k$$

Early values of the coefficients $\tau_k^{(n)}$ are

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$
$\tau_0^{(n)}$	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0
$\tau_1^{(n)}$	0	1	0	−3	0	5	0	−7	0	9	0	−11	0	13
$\tau_2^{(n)}$	0	0	2	0	−8	0	18	0	−32	0	50	0	−72	0
$\tau_3^{(n)}$	0	0	0	4	0	−20	0	56	0	−120	0	220	0	−364

Note that if n and k are of unlike parity, or if $k>n$, the $\tau_k^{(n)}$ coefficient is zero. For computational purposes, non-zero values of these coefficients in the power-series expansion of the first kind of Chebyshev polynomial are most easily found from the formula

22:6:6 $$\tau_k^{(n)} = \frac{(-)^{(n-k)/2}2^{k-1}n\left(\frac{1}{2}n+\frac{1}{2}k-1\right)!}{\left(\frac{1}{2}n-\frac{1}{2}k\right)!k!} = \frac{(-)^{(n-k)/2}2^k n}{n+k}\binom{\frac{1}{2}n+\frac{1}{2}k}{k}$$

except that $\tau_0^{(0)}=1$. This is the formula used by *Equator*'s Chebyshev tau coefficient routine (keyword **Chebytau**) to compute the coefficients of x^k in the power-series expansion of $T_n(x)$.

22:7 PARTICULAR VALUES

	$T_n(-1)$	$T_n(0)$	$T_n(1)$
$n=0,2,4,\cdots$	1	$(-1)^{n/2}$	1
$n=1,3,5,\cdots$	-1	0	1

	$U_n(-1)$	$U_n(0)$	$U_n(1)$
$n=0,2,4,\cdots$	$n+1$	$(-1)^{n/2}$	$n+1$
$n=1,3,5,\cdots$	$-n-1$	0	$n+1$

The zeros of the Chebyshev polynomials occur at arguments given by

22:7:1 $$r_j = \cos\left\{\frac{(2j-1)\pi}{2n}\right\} \qquad j=1,2,3,\cdots,n \qquad \text{where} \qquad T_n(r_j)=0$$

and

22:7:2 $$r_j = \cos\left\{\frac{j\pi}{n+1}\right\} \qquad j=1,2,3,\cdots,n \qquad \text{where} \qquad U_n(r_j)=0$$

22:8 NUMERICAL VALUES

Equator's Chebyshev polynomial of the first kind and Chebyshev polynomial of the second kind routines (keywords **Tpoly** and **Upoly**) are both based on recursion relation 22:5:2. The sole difference between the routines is that the T routine is initialized by $f_0=1$, $f_1=x$, whereas the U routine uses $f_0=1$, $f_1=2x$.

22:9 LIMITS AND APPROXIMATIONS

In the limits as $x\to\pm\infty$

22:9:1 $$T_n(x)\to 2^{n-1}x^n \qquad n=1,2,3,\cdots$$

and

22:9:2 $$U_n(x)\to 2^n x^n \qquad n=0,1,2,\cdots$$

22:10 OPERATIONS OF THE CALCULUS

Luke [pages 298−302] gives a plethora of information respecting differentiation and indefinite integration of the Chebyshev polynomials. Some of the simplest results are:

22:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) = nU_{n-1}(x) = \frac{n\left[T_{n-1}(x) - xT_n(x)\right]}{1-x^2} \qquad n=1,2,3,\cdots$$

22:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}U_n(x) = \frac{nxU_n(x) - (n+1)T_{n+1}(x)}{1-x^2} \qquad n=1,2,3,\cdots$$

22:10:3 $$\int_0^x T_n(t)\,\mathrm{d}t = \frac{n}{n^2-1}\left[T_{n+1}(x) + \sin\left(\frac{n\pi}{2}\right)\right] - \frac{x}{n-1}T_n(x) \qquad n=2,3,4,\cdots$$

22:10:4 $$\int_0^x U_n(t)\,\mathrm{d}t = \frac{T_{n+1}(x) + \sin(n\pi/2)}{n+1} \qquad n=0,1,2,\cdots$$

22:10:5 $$\int_x^1 U_n(t)\,\mathrm{d}t = \frac{1-T_{n+1}(x)}{n+1} \qquad n=0,1,2,\cdots$$

The definite integrals

22:10:6 $$\int_{-1}^1 \frac{T_n(t)T_m(t)}{\sqrt{1-t^2}}\mathrm{d}t = \begin{cases} 0 & m\neq n \\ \pi & m=n=0 \\ \pi/2 & m=n=1,2,3,\cdots \end{cases}$$

and

22:10:7 $$\int_{-1}^1 \sqrt{1-t^2}\,U_n(t)U_m(t)\,\mathrm{d}t = \begin{cases} 0 & m\neq n \\ \pi/2 & m=n=0,1,2,\cdots \end{cases}$$

establish the orthogonality of the Chebyshev polynomials [Section 21:14]. Other definite integrals include

22:10:8 $$\int_{-1}^1 T_n^2(t)\,\mathrm{d}t = \frac{4n^2-2}{4n^2-1} \qquad n=0,1,2,\cdots$$

22:10:9 $$\int_0^1 \frac{\cos(bt)T_n(t)}{\sqrt{1-t^2}}\mathrm{d}t = \frac{(-)^{n/2}\pi}{2}J_n(b) \qquad b>0 \qquad n=0,2,4,\cdots$$

22:10:10 $$\int_0^1 \frac{\sin(bt)T_n(t)}{\sqrt{1-t^2}}\mathrm{d}t = \frac{(-)^{(n-1)/2}\pi}{2}J_n(b) \qquad b>0 \qquad n=1,3,5,\cdots$$

and others are listed by Gradshteyn and Ryzhik [Sections 7.34−7.36]. In these integrals J_n denotes the Bessel function of order n [Chapter 51].

Examples of Laplace transforms of Chebyshev polynomials are

22:10:11 $$\int_0^\infty \frac{T_n(1-2t)}{\sqrt{t}}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\frac{T_n(1-2t)}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\sum_{j=0}^\infty \frac{(-n)_j(n)_j}{(1)_j}\left(\frac{1}{s}\right)^j$$

and

22:10:12 $$\int_0^\infty \sqrt{t}\,U_n(1-2t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\sqrt{t}U_n(1-2t)\right\} = (-)^{n/2}\frac{n}{2}\sqrt{\frac{\pi}{s^3}}\sum_{j=0}^\infty \frac{(1-n)_j(n+1)_j}{(1)_j}\left(\frac{1}{s}\right)^j$$

Though written as $L = K-1 = 1$ hypergeometric functions, the transforms are themselves polynomials (they are sometimes called *Rainville polynomials*). Each kind of Chebyshev polynomial, multiplied by its weight function, converts, on Hilbert transformation [Section 7:10], to the other kind:

22:10:13 $$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}}\frac{dt}{t-y} = U_{n-1}(y) \qquad -1<y<1$$

22:10:14 $$\frac{1}{\pi}\int_{-1}^{1}U_n(t)\sqrt{1-t^2}\frac{dt}{t-y} = T_{n+1}(y) \qquad -1<y<1$$

These last two integrals must be interpreted as their Cauchy limits [equation 7:10:5].

22:11 COMPLEX ARGUMENT

Chebyshev polynomials of complex argument are sometimes encountered. The real and imaginary parts of early $T_n(x+iy)$ polynomials are

	$T_0(x+iy)$	$T_1(x+iy)$	$T_2(x+iy)$	$T_3(x+iy)$	$T_4(x+iy)$
Re	1	x	$2x^2-2y^2-1$	$4x^3-12xy^2-3x$	$8x^4-48x^2y^2+8y^4-8x^2+8y^2+1$
Im	0	y	$4xy$	$12x^2y-4y^3-3y$	$32x^3y-32xy^3-16xy$

and the corresponding U polynomials are

	$U_0(x+iy)$	$U_1(x+iy)$	$U_2(x+iy)$	$U_3(x+iy)$	$U_4(x+iy)$
Re	1	$2x$	$4x^2-4y^2-1$	$8x^3-24xy^2-4x$	$16x^4-96x^2y^2+16y^4-12x^2+12y^2+1$
Im	0	$2y$	$8xy$	$24x^2y-8y^3-4y$	$64x^3y-64xy^3-24xy$

Applicable when $n = 2, 3, 4, \cdots$ and to both T and U, the recursion formulas

22:11:1 $$\mathrm{Re}\{f_n(x+iy)\} = 2x\,\mathrm{Re}\{f_{n-1}(x+iy)\} - 2y\,\mathrm{Im}\{f_{n-1}(x+iy)\} - \mathrm{Re}\{f_{n-2}(x+iy)\}$$

and

22:11:2 $$\mathrm{Im}\{f_n(x+iy)\} = 2y\,\mathrm{Re}\{f_{n-1}(x+iy)\} + 2x\,\mathrm{Im}\{f_{n-1}(x+iy)\} - \mathrm{Im}\{f_{n-2}(x+iy)\}$$

permit any Chebyshev polynomial of complex argument to be constructed.

Inverse Laplace transformation of Chebyshev polynomials leads to the Hermite polynomials [Chapter 24]

22:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^n}T_n\left(1-\frac{1}{s}\right)\frac{\exp(ts)}{2\pi i}ds = \mathfrak{I}\left\{\frac{1}{s^n}T_n\left(1-\frac{1}{s}\right)\right\} = \frac{(-t)^n}{2(2n-1)!t}H_{2n}\left(\sqrt{\frac{t}{2}}\right)$$

and

22:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^{n+1}}U_n\left(\frac{1}{s}-1\right)\frac{\exp(ts)}{2\pi i}ds = \mathfrak{I}\left\{\frac{1}{s^{n+1}}U_n\left(\frac{1}{s}-1\right)\right\} = \frac{t^n}{(2n-1)!\sqrt{2t}}H_{2n-1}\left(\sqrt{\frac{t}{2}}\right)$$

22:12 GENERALIZATIONS: Gegenbauer and Jacobi polynomials

The bivariate Chebyshev polynomial generalizes to the trivariate *Gegenbauer polynomial* (Leopold Bernhard Gegenbauer, Austrian mathematician, 1849 – 1903) or *ultraspherical polynomial* $C_n^{(\lambda)}(x)$. One of several ways of defining this polynomial family is through their generating function

22:12:1 $$\frac{1}{\left(1-2tx+t^2\right)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n \qquad \lambda \neq 0$$

from which it is follows that the Legendre and Chebyshev polynomials are the special cases

22:12:2 $$P_n(x) = C_n^{(1/2)}(x)$$

22:12:3 $$T_n(x) = \frac{n}{2}\lim_{\lambda\to 0}\left\{\frac{C_n^{(\lambda)}(x)}{\lambda}\right\}$$

and

22:12:4 $$U_n(x) = C_n^{(1)}(x)$$

of the Gegenbauer polynomials. The limiting operation in 22:12:3 is taken as a definition of $C_n^{(0)}(x)$, which is thereby seen to equal $(2/n)T_n(x)$, except $C_0^{(0)}(x) = 1$. The $\lambda = 0$ case is not covered by the table.

$C_0^{(\lambda)}(x)$	$C_1^{(\lambda)}(x)$	$C_2^{(\lambda)}(x)$	$C_3^{(\lambda)}(x)$	$C_4^{(\lambda)}(x)$
1	$2\lambda x$	$-\lambda+2\lambda(1+\lambda)x^2$	$2\lambda(1+\lambda)\left[-x+\frac{2}{3}(2+\lambda)x^3\right]$	$\lambda(1+\lambda)\left[\frac{1}{2}-2(2+\lambda)x^2+\frac{2}{3}(2+\lambda)(3+\lambda)x^4\right]$

More expressions may be added through the recursion formula

22:12:5 $$C_n^{(\lambda)}(x) = \frac{2n+2\lambda-2}{n}xC_{n-1}^{(\lambda)}(x) - \frac{n+2\lambda-2}{n}C_{n-2}^{(\lambda)}(x)$$

which applies also to the $\lambda = 0$ case. This recursion formula is the basis of *Equator*'s Gegenbauer polynomial routine (keyword **Cpoly**).

The trivariate Gegenbauer polynomial family is itself a special case

22:12:6 $$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) \qquad \frac{-1}{2} < \lambda \neq 0$$

of the quadrivariate *Jacobi polynomial*. The value of a Jacobi polynomial is determined by: the argument x, a real number with interest concentrated in the $-1 \le x \le 1$ domain; the nonnegative integer degree n; and two distinct real parameters, v and μ. The conventional notation for the Jacobi polynomial is $P_n^{(v,\mu)}(x)$ and its generating function is

22:12:7 $$\frac{2^{v+\mu}}{\sqrt{1-2xt+t^2}\left[1-t+\sqrt{1-2xt+t^2}\right]^{v}\left[1+t+\sqrt{1-2xt+t^2}\right]^{\mu}} = \sum_{n=0}^{\infty} P_n^{(v,\mu)}(x)t^n$$

The Jacobi polynomial is the grandparent of many other orthogonal polynomial families, inasmuch as the Legendre, Chebyshev, Laguerre, associated Laguerre and Hermite polynomials are all special cases:

22:12:8 $$P_n(x) = P_n^{(0,0)}(x)$$

22:12:9 $$T_n(x)=\frac{n!}{\left(\tfrac{1}{2}\right)_n}P_n^{(-1/2,-1/2)}(x)$$

22:12:10 $$U_n(x)=\frac{(n+1)!}{\left(\tfrac{3}{2}\right)_j}P_n^{(1/2,1/2)}(x)$$

22:12:11 $$L_n(x)=\lim_{\mu\to\infty}P_n^{(0,\mu)}\left(1-\frac{2x}{\mu}\right)$$

22:12:12 $$L_n^{(\nu)}(x)=\lim_{\mu\to\infty}P_n^{(\nu,\mu)}\left(1-\frac{2x}{\mu}\right)$$

22:12:13 $$H_n(x)=\begin{cases}(-)^{n/2}\,n!!L_{n/2}^{(-1/2)}(x^2) & n=0,2,4,\cdots\\ (-)^{(n-1)/2}\,2^{n/2}(n-1)!!xL_{(n-1)/2}^{(1/2)}(x^2) & n=1,3,5,\cdots\end{cases}$$

For this reason, many authors prefer to concentrate on the Jacobi polynomials, educing the properties of other orthogonal polynomials therefrom. *Applications* of orthogonal polynomials, on the other hand, are predominantly based on the simpler polynomial families.

Jacobi polynomials are sometimes known as *hypergeometric polynomials* because they may be represented, in two ways, as the product of a binomial coefficient and a Gauss hypergeometric function [Chapters 6 and 60]:

22:12:14 $$P_n^{(\nu,\mu)}(x)=\binom{n+\nu}{n}\sum_{j=0}^{\infty}\frac{(-n)_j(n+\nu+\mu+1)_j}{(1)_j(\nu+1)_j}\left(\frac{1-x}{2}\right)^j \qquad -3\le x\le 1$$

22:12:15 $$P_n^{(\nu,\mu)}(x)=\left(\frac{x+1}{2}\right)^n\binom{n+\nu}{n}\sum_{j=0}^{\infty}\frac{(-n)_j(-n-\mu)_j}{(1)_j(\nu+1)_j}\left(\frac{x-1}{x+1}\right)^j \qquad x\ge 0$$

Jacobi polynomials are even/odd according as their degree is even/odd. *Equator*'s Jacobi polynomial routine (keyword **Jacobipoly**) uses $P_0^{(\nu,\mu)}(x)=1$ and $P_1^{(\nu,\mu)}(x)=\frac{1}{2}(\nu+\mu+2)x+\frac{1}{2}(\nu-\mu)$ to initialize the recursion formula

22:12:16 $$P_n^{(\nu,\mu)}(x)=\frac{(2n+\nu+\mu-1)\left[(2n+\nu+\mu-2)(2n+\nu+\mu)x+\nu^2-\mu^2\right]}{2n(n+\nu+\mu)(2n+\nu+\mu-2)}P_{n-1}^{(\nu,\mu)}(x) - \frac{(n+\nu-1)(n+\mu-1)(2n+\nu+\mu)}{n(n+\nu+\mu)(2n+\nu+\mu-2)}P_{n-2}^{(\nu,\mu)}(x)$$

This formula breaks down when $\nu+\mu+1$ equals a negative integer but, nevertheless, *Equator* delivers accurate values of the Jacobi polynomial when the parameters sum to a value close to -2, -3, etc.

For $\nu>-1$ and $\mu>-1$, the orthogonality of the Jacobi polynomials is manifested by

22:12:17 $$\int_{-1}^{1}(1-t)^{\nu}(1+t)^{\mu}P_m^{(\nu,\mu)}(t)P_n^{(\nu,\mu)}(t)\,dt=\begin{cases}0 & m\ne n\\ \dfrac{2^{\nu+\mu+1}\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{(2n+\nu+\mu+1)n!\,\Gamma(n+\nu+\mu+1)} & m=n\end{cases}$$

22:13 COGNATE FUNCTIONS: the discrete Chebyshev polynomials

The orthogonality property, addressed in Section 21:14, may be used to construct continuous functions from suitable polynomial families, as is demonstrated in Section 21:15. There is, however, another variety of orthogonality that relates to functions whose values are known only at a set of discrete points, say $y_0, y_1, \cdots, y_j, \cdots, y_J$. These functions, too, are generally polynomials, but the orthogonality relationship that they satisfy is a summation rather than an integration:

22:13:1
$$\sum_{j=0}^{J} \mathrm{w}(y)\Psi_n(y)\Psi_m(y) = \begin{cases} 0 & m \neq n \\ \Omega_n^2 & m = n \end{cases}$$

Such Ψ functions are known as *discrete orthogonal polynomials*.

The simplest family of discrete orthogonal polynomial functions are those with a weight function $\mathrm{w}(y)$ of unity and have been named *discrete Chebyshev polynomials*. The notation $\mathrm{t}_0^{(J)}(y), \mathrm{t}_1^{(J)}(y), \cdots, \mathrm{t}_J^{(J)}(y)$ is used for the $(J+1)$-member set. If y is restricted to the $-1 \le y \le 1$ range, and the data are evenly spaced on this interval, then the orthogonality condition is

22:13:2
$$\sum_{j=0}^{J} \mathrm{t}_m^{(J)}(y_j)\mathrm{t}_n^{(J)}(y_j) = \begin{cases} 0 & m \neq n \\ \Omega_n^2 & m = n \end{cases} \qquad \text{where} \qquad \Omega_n^2 = \frac{(J+1)_{n+1}}{[2n+1](J-n+1)_n}$$

The t polynomials depend on J, as well as on n, and some early members of the family are

$\mathrm{t}_0^{(J)}(y)$	$\mathrm{t}_1^{(J)}(y)$	$\mathrm{t}_2^{(J)}(y)$	$\mathrm{t}_3^{(J)}(y)$	$\mathrm{t}_4^{(J)}(y)$
1	y	$\dfrac{3Jy^2 - J - 2}{2(J-1)}$	$\dfrac{5J^2y^3 - (3J^2+6J-4)y}{2(J-1)(J-2)}$	$\dfrac{35J^3y^4 - 10J(3J^2+6J-10)y^2 + 3(J+4)(J^2-4)}{8(J-1)(J-2)(J-3)}$

Others can be calculated by the recursion formula

22:13:3
$$\mathrm{t}_n^{(J)}(y) = \frac{(2n-1)Jy\,\mathrm{t}_{n-1}^{(J)}(y) - (n-1)(J+n)\mathrm{t}_{n-2}^{(J)}(y)}{n(J-n+1)}$$

as employed by *Equator*'s discrete Chebyshev polynomial routine (keyword **discCheby**). An application of these polynomials will be found in Section 17:14.

Despite their name, discrete Chebyshev polynomials have less in common with Chebyshev polynomials than they have with Legendre functions, with which they share a unity weight function and to which they reduce as J approaches infinity

22:13:4
$$\lim_{J\to\infty} \mathrm{t}_n^{(J)}(y) = \mathrm{P}_n(y)$$

CHAPTER 23

THE LAGUERRE POLYNOMIALS $L_n(x)$

Laguerre polynomials (Edmond Nicolas Laguerre, French mathematician, 1834−1886) are orthogonal [Section 21:14] on the interval 0 to ∞ with a weight function of $\exp(-x)$. They arise in a number of scientific problems, as in solutions to the wave equation.

23:1 NOTATION

Though the $L_n(x)$ symbol is used for both, Laguerre polynomials are defined in two distinct ways; many authors write $L_n(x)$ for what the *Atlas* represents by $n!L_n(x)$. The name "Laguerre polynomial" may be applied to the general class of orthogonal polynomials discussed in Section 23:12 under the name *associated Laguerre polynomials*.

Though this *Atlas* avoids the ambiguity, $L_n(x)$ is commonly employed to denote the unrelated hyperbolic Struve function [Section 57:13].

23:2 BEHAVIOR

While interest is concentrated on positive arguments, the Laguerre polynomial is defined for all real argument x and all nonnegative integer degree n. As Figure 23-1 shows, the first few Laguerre polynomials are quite simple functions. As n increases, the polynomial displays an oscillatory behavior, the amplitude and period of the oscillations increasing with x, until, beyond r_n, the last zero, the oscillations cease abruptly and the function heads rapidly towards $(-)^n\infty$. Within its oscillatory ambit, the function displays exactly $\mathrm{Int}(n/2)$ minima, $\mathrm{Int}\{(n-1)/2\}$ maxima and n zeros. All the zeros occur between $x = 0$ and an imperfectly known upper value

23:2:1 $$0 < r_1 < r_2 < \cdots < r_m < \cdots < r_n < 2n + 1 + \sqrt{4n^2 + 4n + \tfrac{5}{4}} \qquad L_n(r_m) = 0$$

23:3 DEFINITIONS

All orthogonal polynomials, including Laguerre polynomials, can be defined by a generating function

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_24,

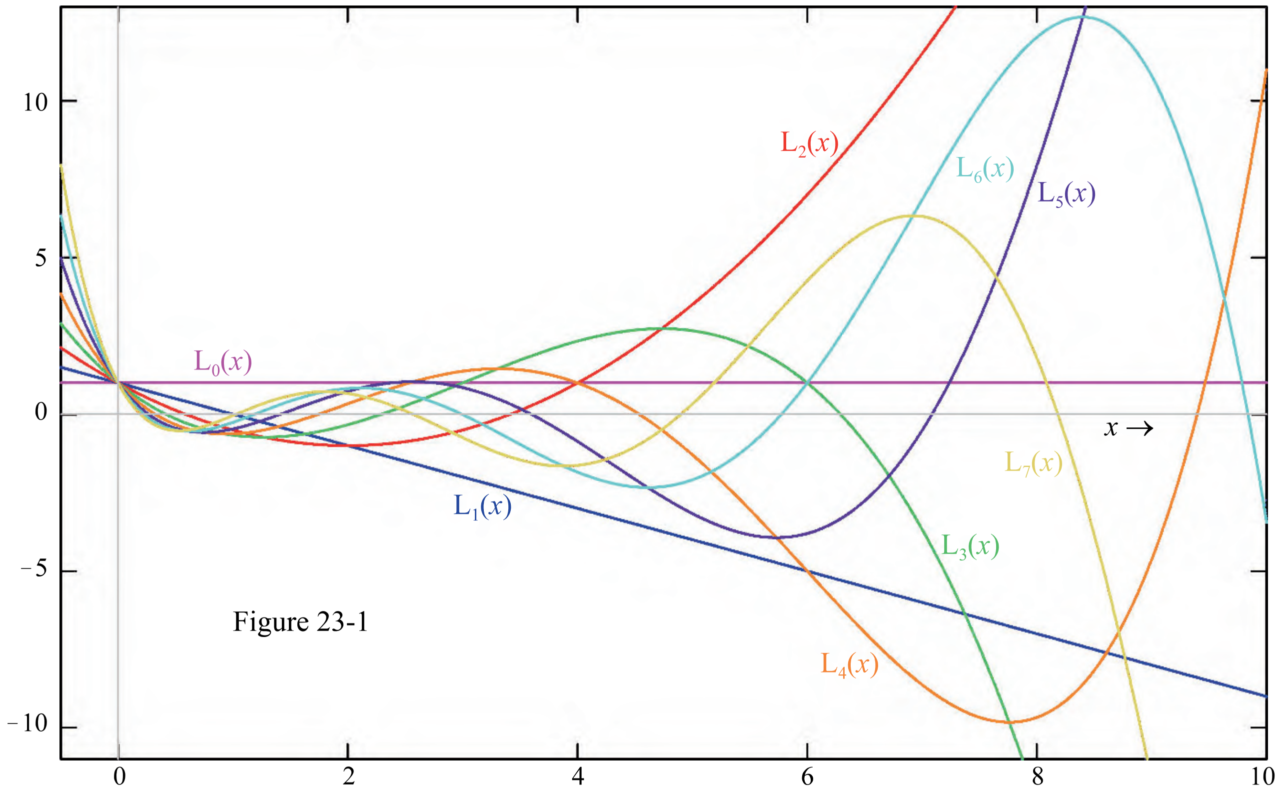


Figure 23-1

23:3:1 $$\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty}\mathrm{L}_n(x)t^n \qquad -1<t<1$$

and by a *Rodrigues's formula*

23:3:2 $$\mathrm{L}_n(x)=\frac{\exp(x)}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^n\exp(-x)\right\}$$

It may also be defined in a manner resembling a binomial expansion:

23:3:3 $$\mathrm{L}_n(x)=\sum_{j=0}^{n}\binom{n}{j}\frac{(-x)^j}{j!}$$

and by formula 22:12:10 as a limiting case of a Jacobi polynomial. An unusual definition, establishing contact with a Bessel function [Chapter 52] is

23:3:4 $$\mathrm{L}_n(x)=\frac{\exp(x)}{n!}\int_0^{\infty}t^n\exp(-t)\mathrm{J}_0\left(2\sqrt{xt}\right)\mathrm{d}t$$

Definitions as $L=K+1=2$ hypergeometric functions

23:3:5 $$\mathrm{L}_n(x)=\sum_{j=0}^{\infty}\frac{(-n)_j}{(1)_j(1)_j}x^j$$

23:3:6 $$\mathrm{L}_n(x)=\exp(x)\sum_{j=0}^{\infty}\frac{(1+n)_j}{(1)_j(1)_j}(-x)^j$$

imply the possibility of synthesis [Section 43:14] from an exponential function, thus:

23:3:7 $$\exp(x) \xrightarrow[1]{-n} L_n(x)$$

23:3:8 $$\exp(-x) \xrightarrow[1]{n+1} \exp(-x)L_n(x)$$

One solution of Laguerre's differential equation

23:3:9 $$x\frac{d^2 f}{dx^2} + (1-x)\frac{df}{dx} + nf = 0$$

is

23:3:10 $$f(x) = wL_n(x)$$

where the weight w is arbitrary. The second solution is considerably more complicated [see Murphy].

23:4 SPECIAL CASES

$L_0(x)$	$L_1(x)$	$L_2(x)$	$L_3(x)$	$L_4(x)$	$L_5(x)$
1	$-x+1$	$\frac{1}{2}x^2 - 2x + 1$	$-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1$	$\frac{1}{24}x^4 - \frac{2}{3}x^3 + 3x^2 - 4x + 1$	$-\frac{1}{120}x^5 + \frac{5}{24}x^4 - \frac{5}{3}x^3 + 5x^2 - 5x + 1$

23:5 INTRARELATIONSHIPS

The recursion

23:5:1 $$L_n(x) = \frac{2n-1-x}{n}L_{n-1}(x) - \frac{n-1}{n}L_{n-2}(x) \qquad n = 2,3,4,\cdots$$

relates a Laguerre polynomial to two earlier members of the family. The following argument-addition and argument-multiplication formulas apply to Laguerre polynomial functions:

23:5:2 $$L_n(x+y) = \frac{1}{n!}\left(\frac{-1}{4}\right)^n \sum_{j=0}^{n} \binom{n}{j} H_{2j}\left(\sqrt{x}\right) H_{2n-2j}\left(\sqrt{y}\right)$$

23:5:3 $$L_n(bx) = (1-b)^n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{b}{1-b}\right)^j L_j(x)$$

whereas function-multiplications play a leading role in the *Christoffel-Darboux formula*

23:5:4 $$\sum_{j=0}^{n} L_j(x)L_j(y) = (n+1)\frac{L_n(x)L_{n+1}(y) - L_{n+1}(x)L_n(y)}{x-y}$$

23:6 EXPANSIONS

An explicit expression of the power-series expansion for the Laguerre polynomials is

23:6:1 $$L_n(x) = (-)^n \left[\frac{x^n}{n!} - \frac{n}{1!(n-1)!}x^{n-1} + \frac{n(n-1)}{2!(n-2)!}x^{n-2} - \cdots - (-)^n nx + (-1)^n\right]$$

This expansion is equivalent to 23:3:3.

It is evident from Section 23:4, that the nth Laguerre polynomial $L_n(x)$ is built from a weighted sum of all the powers of x up to x^n. It follows that the first n such expansions can be solved simultaneously to eliminate all the powers $x^0, x^1, x^2, \cdots, x^{n-1}$ and produce an expression for x^n of the form

23:6:2
$$x^n = a_0 L_0(x) + a_1 L_1(x) + \cdots + a_j L_j(x) + \cdots + a_n L_n(x)$$

where the a's are numerical coefficients that can be found by solving the simultaneous equations, or from the integrations

23:6:3
$$a_j = \int_0^\infty x^n \exp(-x) L_j(x)\, dx$$

For example, one finds

23:6:4
$$x^4 = 24L_0(x) - 96L_1(x) + 144L_2(x) - 96L_3(x) + 24L_4(x)$$

The symmetry seen in this example is general. Abramowitz and Stegun [Table 22.10] list coefficients of all such expansions up to that of x^{12}.

23:7 PARTICULAR VALUES

Beyond the first few, no formulas for the exact locations of the zeros of the Laguerre polynomials are known, but see equation 23:9:3. Nor are formulas for the locations or the values of the extrema known to us.

23:8 NUMERICAL VALUES

Equator's Laguerre polynomial routine (keyword **Lpoly**) employs recursion 23:5:1, initiated by $L_0(x) = 1$ and $L_1(x) = 1 - x$.

23:9 LIMITS AND APPROXIMATIONS

For positive argument, the Laguerre polynomials are bounded by

23:9:1
$$-\exp\left(\frac{x}{2}\right) \le L_n(x) \le \exp\left(\frac{x}{2}\right) \qquad x > 0$$

The asymptotic form [Lebedev] adopted by the Laguerre polynomial for large degree is

23:9:2
$$L_n(x) \sim \frac{1}{\sqrt{\pi\sqrt{nx}}} \exp\left(\frac{x}{2}\right) \cos\left(2\sqrt{nx} - \frac{\pi}{4}\right)$$

The mth zero of the Laguerre polynomial is bracketed by

23:9:3
$$\frac{j_{0,m}^2}{4n+2} < r_m < \frac{2m+1}{2n+1}\left[2m + 1 + \sqrt{4m^2 + 4m + \tfrac{5}{4}}\right]$$

where $j_{0,m}$ is the mth zero of the Bessel J_0 function [Section 53:7].

23:10 OPERATIONS OF THE CALCULUS

Differentiation of a Laguerre polynomial generates a difference of two Laguerre polynomials of like argument or, more succinctly, an associated Laguerre polynomial [Section 23:12] of unity order

23:10:1 $$\frac{d}{dx}L_n(x) = \frac{n}{x}\left[L_n(x) - L_{n-1}(x)\right] = -L_{n-1}^{(1)}(x)$$

and the latter iterates to the multiple-differentiation formula

23:10:2 $$\frac{d^m}{dx^m}L_n(x) = (-)^m L_{n-1}^{(m)}(x)$$

An associated Laguerre polynomial of unity order is also generated by integration:

23:10:3 $$\int_0^x L_n(t)\,dt = L_n(x) - L_{n+1}(x) = \frac{x}{n+1}L_{n-1}^{(1)}(x)$$

The orthogonality [Section 22:14] of Laguerre polynomials is established by the integral

23:10:4 $$\int_0^\infty \exp(-t)L_n(t)L_m(t)\,dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

See Section 24:15 for an application to numerical integration. Other important integrals include

23:10:5 $$\int_x^\infty \exp(-t)L_n(t)\,dt = \exp(-x)\left[L_n(x) - L_{n-1}(x)\right] = \frac{-n\exp(-x)}{x}L_{n-1}^{(1)}(x)$$

and

23:10:6 $$\int_0^x L_n(t)L_m(x-t)\,dt = \int_0^x L_{n+m}(t)\,dt = \frac{x}{n+m+1}L_{n+m}^{(1)}(x)$$

Many others are listed by Gradshteyn and Ryzhik [Sections 7.41 and 7.42].

Examples of Laplace transforms involving Laguerre polynomials are:

23:10:7 $$\int_0^\infty L_n(t)\exp(-st)\,dt = \mathfrak{L}\{L_n(t)\} = \frac{(s-1)^n}{s^{n+1}}$$

23:10:8 $$\int_0^\infty t^\nu L_n^{(m)}(t)\exp(-st)\,dt = \mathfrak{L}\{t^\nu L_n^{(m)}(t)\} = \frac{\Gamma(\nu+1)}{n!s^{\nu+1}}P_n^{(m,\nu-n-m)}\left(1-\frac{2}{s}\right) \qquad \nu > -1$$

Transform 23:10:8, which generates a Jacobi polynomial, relates to an *associated* Laguerre polynomial [Section 23:12]; merely set $m = 0$ to obtain the formula for a simple Laguerre polynomial.

23:11 COMPLEX ARGUMENT

Applications of Laguerre polynomials generally employ real arguments.

Inverse Laplace transformation of an *associated* Laguerre polynomial generates a Jacobi polynomial.

23:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L_n^{(m)}(s)}{s^\nu}\frac{\exp(ts)}{2\pi i}\,ds = \mathfrak{I}\left\{\frac{L_n^{(m)}(s)}{s^\nu}\right\} = \frac{t^{\nu-n-1}(1+t)^n}{\Gamma(\nu)}P_n^{(m,\nu-n-1)}\left(\frac{t-1}{t+1}\right)$$

Set $m = 0$ to obtain the formula for a simple Laguerre polynomial.

23:12 GENERALIZATIONS: including associated Laguerre polynomials

The Laguerre polynomial is the $m = 0$ case of the *associated Laguerre polynomial* $L_n^{(m)}(x)$

23:12:1 $$L_n(x) = L_n^{(0)}(x)$$

Most of the definitions of the Laguerre polynomial, given in Section 23:3, need only minor adjustment to provide a definition of the associated Laguerre polynomial of degree n and order m. For example

23:12:2 $$\frac{1}{(1-t)^{m+1}}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(m)}(x)t^n \qquad -1<t<1$$

23:12:3 $$L_n^{(m)}(x) = \frac{\exp(x)}{n!x^m}\frac{d^n}{dx^n}\left\{x^{n+m}\exp(-x)\right\}$$

23:12:4 $$L_n^{(m)}(x) = \sum_{j=0}^{n}\binom{n+m}{n-j}\frac{(-x)^j}{j!}$$

23:12:5 $$L_n^{(m)}(x) = \frac{\exp(x)}{n!x^{m/2}}\int_0^\infty t^{(2n+m)/2}\exp(-t)J_m\left(2\sqrt{tx}\right)dt$$

23:12:6 $$L_n^{(m)}(x) = \frac{(m+1)_n}{n!}\sum_{j=0}^{\infty}\frac{(-n)_j}{(1)_j(m+1)_j}x^j$$

and

23:12:7 $$x\frac{d^2 f}{dx^2} + (m+1-x)\frac{df}{dx} + nf = 0 \qquad \text{where} \qquad f(x) = wL_n^{(m)}(x)$$

Caution is necessary because the associated Laguerre polynomials are sometimes defined as

23:12:8 $$n!\frac{d^m}{dx^m}L_n(x)$$

This definition leads to properties radically different from those of the polynomials defined by equations 23:12:2−7.

The recursion property, 23:5:1, of the Laguerre polynomial generalizes to

23:12:9 $$L_n^{(m)}(x) = \frac{2n+m-1-x}{n}L_{n-1}^{(m)}(x) - \frac{n+m-1}{n}L_{n-2}^{(m)}(x)$$

which is useful in the construction or extension of the following table, which lists examples of associated Laguerre polynomials.

	$L_0^{(m)}(x)$	$L_1^{(m)}(x)$	$L_2^{(m)}(x)$	$L_3^{(m)}(x)$	$L_4^{(m)}(x)$
$m=0$	1	$-x+1$	$\frac{1}{2}x^2-2x+1$	$-\frac{1}{6}x^3+\frac{3}{2}x^2-3x+1$	$\frac{1}{24}x^4-\frac{2}{3}x^3+3x^2-4x+1$
$m=1$	1	$-x+2$	$\frac{1}{2}x^2-3x+3$	$-\frac{1}{6}x^3+2x^2-6x+4$	$\frac{1}{24}x^4-\frac{5}{6}x^3+5x^2-10x+5$
$m=2$	1	$-x+3$	$\frac{1}{2}x^2-4x+6$	$-\frac{1}{6}x^3+\frac{5}{2}x^2-10x+10$	$\frac{1}{24}x^4-x^3+\frac{15}{2}x^2-20x+15$
$m=3$	1	$-x+4$	$\frac{1}{2}x^2-5x+10$	$-\frac{1}{6}x^3+3x^2-15x+20$	$\frac{1}{24}x^4-\frac{7}{6}x^3+\frac{21}{2}x^2-35x+35$
$m=4$	1	$-x+5$	$\frac{1}{2}x^2-6x+15$	$-\frac{1}{6}x^3+\frac{7}{2}x^2-21x+35$	$\frac{1}{24}x^4-\frac{4}{3}x^3+14x^2-56x+70$

The differential and orthogonality properties of associated Laguerre polynomials are revealed in the following:

23:12:10 $$\frac{\mathrm{d}}{\mathrm{d}x}L_n^{(m)}(x) = \frac{n}{x}L_n^{(m)}(x) - \frac{n+m}{x}L_{n-1}^{(m)}(x) = -L_{n-1}^{(m+1)}$$

23:12:11 $$\int_0^\infty t^m \exp(-t)L_n^{(m)}(t)L_{n'}^{(m)}(t)\,\mathrm{d}t = \begin{cases} 0 & n \neq n' \\ (m+n)!/n! & n = n' \end{cases}$$

To this point it has been assumed that the order m of the associated Laguerre polynomials is a nonnegative integer, and this is frequently the case. However, many of the definitions remain valid when m is a negative integer, or a noninteger of either sign, and, indeed, *Equator*'s associated Laguerre polynomial routine (keyword **assocLpoly**), which employs 23:12:9, initialized by $L_0^{(m)}(x) = 1$ and $L_1^{(m)}(x) = 1 + m - x$, does not require m to be an integer. The name *generalized Laguerre polynomial* has been used to describe associated Laguerre polynomials that lack the integer restriction on their order. See equation 45:6:6 for a connection to the incomplete gamma function. Two simple instances may be considered as Hermite polynomials [Chapter 24]:

23:12:12 $$L_n^{(1/2)}(x) = \frac{1}{2}\left(\frac{-1}{4}\right)^n \frac{H_{2n+1}\left(\sqrt{x}\right)}{n!\sqrt{x}} \qquad \text{and} \qquad L_n^{(-1/2)}(x) = \left(\frac{-1}{4}\right)^n \frac{H_{2n}\left(\sqrt{x}\right)}{n!}$$

The generalized Laguerre polynomial is an instance of the Kummer function [Chapter 47]:

23:12:13 $$L_n^{(\mu)}(x) = \binom{n+\mu}{n} M(-n, \mu+1, x)$$

Yet another generalization arises by allowing the degree of the Laguerre polynomial (*not* the associated version), to adopt noninteger values. The resulting function is no longer a polynomial, being called a *Laguerre function*. It, too, is a simple example of a Kummer function and an $L = K+1 = 2$ hypergeometric function

23:12:14 $$L_\nu(x) = M(-\nu, 1, x) = \sum_{j=0}^{\infty} \frac{(-\nu)_j}{(1)_j (1)_j} x^j$$

Confusingly, the name "Laguerre function" is also applied to the solution of the radial portion of the *Schrödinger equation* [Section 46:14] for the hydrogen atom. A good deal more complicated than 23:12:14, this is

23:12:15 $$(n+1)!\exp\left(\frac{-x}{2}\right) x^\ell \frac{\mathrm{d}^{2\ell+1}}{\mathrm{d}x^{2\ell+1}} L_{n+1}(x)$$

in the terminology of the *Atlas*, n and ℓ being the appropriate quantum numbers.

Perhaps the ultimate generalization of the Laguerre polynomial is when neither the degree nor the order need be an integer. With suitable replacements for factorial functions and Pochhammer polynomials, equations 23:12:3, and 23:12:5−7 can provide definitions of such a *generalized Laguerre function* $L_\nu^{(\mu)}(x)$. It is related to the Kummer function of Chapter 47 and the gamma function of Chapter 43 by

23:12:16 $$L_\nu^{(\mu)}(x) = \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu)\Gamma(1+\mu)} M(-\nu, 1+\mu, x)$$

23:13 COGNATE FUNCTIONS

Hermite polynomials, the subject of the next chapter, have many similarities to Laguerre polynomials and there are several linkages between them, including 23:12:12. Both are orthogonal with an exponential function as the weight function; they differ in that the orthogonality interval is semiinfinite for the Laguerre polynomial, but (doubly) infinite for the Hermite version.

23:14 RELATED TOPIC: Fibonacci numbers

In common with many other named polynomial functions, such as the Chebyshev and Hermite polynomials [Chapters 22 and 24], Laguerre polynomials obey a simple three-term recursion − formula 23:5:1 in the Laguerre case. Two other named polynomial functions, not addressed elsewhere in this *Atlas*, obey the similar, but even simpler, three-term recursion

23:14:1 $$f_{n+1}(x) = x\,f_n(x) + f_{n-1}(x)$$

These are the Lucas polynomials and the Fibonacci polynomials, but only the latter will be considered here.

We adopt the symbol $\mathrm{Fib}_n(x)$ for *Fibonacci polynomials*, which are defined by recursion 23:14:1 initialized by $f_0(x) = 0$, $f_1(x) = 1$. Early members are

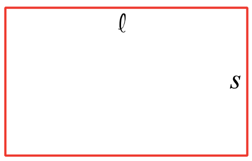
$\mathrm{Fib}_0(x)$	$\mathrm{Fib}_1(x)$	$\mathrm{Fib}_2(x)$	$\mathrm{Fib}_3(x)$	$\mathrm{Fib}_4(x)$	$\mathrm{Fib}_5(x)$	$\mathrm{Fib}_6(x)$	$\mathrm{Fib}_7(x)$	$\mathrm{Fib}_8(x)$
0	1	x	x^2+1	x^3+2x	x^4+3x^2+1	x^5+4x^3+3x	$x^6+5x^4+6x^2+1$	$x^7+6x^5+10x^3+4x$

Initialized by $\mathrm{Fib}_0(x)$ and $\mathrm{Fib}_1(x)$, recursion 23:14:1 is used by *Equator*'s Fibonacci polynomial routine, keyword **Fibpoly**.

Fibonacci numbers, are the $x = 1$ values of the eponymous polynomial, each such number being the sum of its two predecessors. Important in a variety of contexts, these numbers even appear in Dan Brown's popular novel *The Da Vinci Code*. They are named for Leonardo Fibonacci ("Leonardo of Pisa" 1175−1230), but were known to the ancients in connection with the *golden section* [Newman, page 98], which is the positive number υ that exceeds its reciprocal by unity, that is

23:14:2 $$\upsilon = \frac{1+\upsilon}{\upsilon} \qquad \text{whence} \qquad \upsilon = \frac{\sqrt{5}+1}{2} = 1.6180\,33988\,74989$$

The "golden" sobriquet has its origin in the claim that a rectangle has the most aesthetically appealing shape when the ratio ℓ/s of the longer to the shorter side equals the ratio of the semiperimeter to the longer side, so that



23:14:3 $$\frac{\ell}{s} = \frac{\ell+s}{\ell} = \upsilon$$

More obscurely, υ is the radius of the circle that exscribes a regular decagon of unity side.

The ratio $\mathrm{Fib}(n)/\mathrm{Fib}(n-1)$ of a Fibonacci number to its predecessor is alternately less than and greater than the golden section and this ratio converges rapidly to υ. This convergence is incorporated into the exact formula

23:14:4 $$\mathrm{Fib}(n) = \mathrm{Fib}_n(1) = \mathrm{Round}\left(\frac{\upsilon^n}{\sqrt{5}}\right) \qquad n = 0,1,2,\cdots$$

This is how *Equator*'s Fibonacci number routine (keyword **Fibnum**) primarily operates, generating exact integers for values of n through 73. For $74 \le n \le 1476$, a 15-digit approximation is provided. The first eighteen Fibonacci numbers are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathrm{Fib}(n)$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597

CHAPTER 24

THE HERMITE POLYNOMIALS $H_n(x)$

Named for the Frenchman, Charles Hermite (1822−1901) these polynomials are orthogonal [Section 21:14] on the infinite interval $-\infty < x < \infty$ with a weight function of $\exp(-x^2)$. They arise in physics, as in the solution of Schrödinger's differential equation for a simple harmonic oscillator [Section 24:13]. This differential equation belongs to a broad class of second-order differential equations, many members of which may be solved by the approach exposed in Section 24:14.

24:1 NOTATION

The notation $H_n(x)$ is in general use for the Hermite polynomial of degree n and argument x, however it is occasionally defined with sign opposite to that adopted here. An *alternative Hermite function* is also in common use. Unfortunately it, too, is sometimes symbolized $H_n(x)$, but more often $He_n(x)$ is preferred. The *Atlas* uses the $H_n(x)$ variant exclusively. The relationship between the two varieties of Hermite function is

24:1:1 $$\mathrm{He}_n(x) = \frac{1}{2^{n/2}} \mathrm{H}_n\left(\frac{x}{\sqrt{2}}\right)$$

The conventional notation for the unrelated Struve function [Chapter 57] is also $H_n(x)$. To avoid ambiguity, the *Atlas* denotes the Struve function by $h_n(x)$.

24:2 BEHAVIOR

The Hermite polynomial $H_n(x)$ has exactly n zeros, $\mathrm{Int}(n/2)$ minima and $\mathrm{Int}((n-1)/2)$ maxima. These features are symmetrically disposed about $x = 0$, and all occur in the zone $-\sqrt{2n} < x < \sqrt{2n}$. Outside this domain $|H_n(x)|$ increases monotonically, except for $H_0(x)$, remaining bounded globally by

24:2:1 $$|\mathrm{H}_n(x)| \le \sqrt{2^n n! \exp(x^2)}$$

The range spanned by the oscillations of $H_n(x)$ increases so dramatically with increasing n that Figure 24-1 portrays the behavior of $H_n(x)/n!$ rather than that of the Hermite polynomials themselves.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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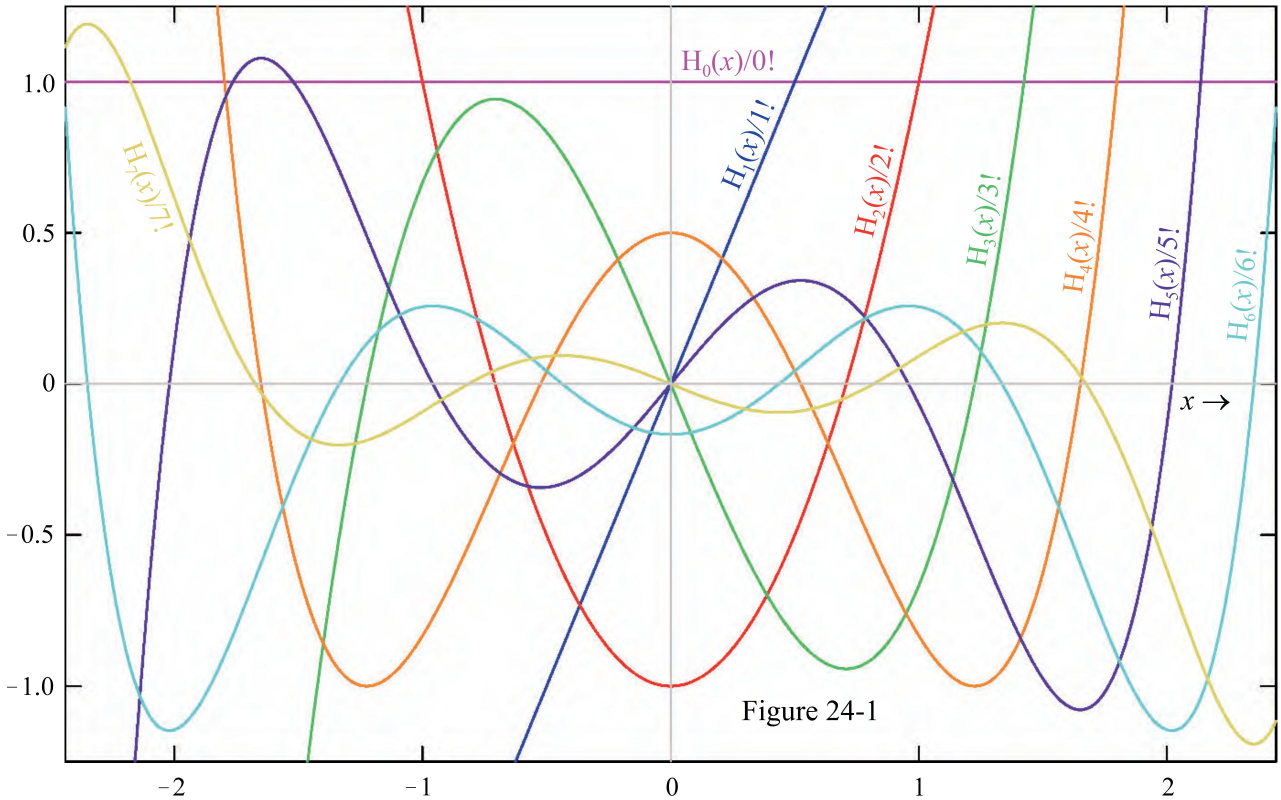


Figure 24-1

24:3 DEFINITIONS

In common with all orthogonal polynomials, there is a generating function

24:3:1
$$\exp\left(2xt-t^2\right)=\sum_{n=0}^{\infty}\mathrm{H}_n(x)\frac{t^n}{n!}$$

and a *Rodrigues's formula*

24:3:2
$$\mathrm{H}_n(x)=(-)^n\exp\left(x^2\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(-x^2\right)$$

that can serve as definitions of the Hermite polynomial. As well, it can be defined as the limiting operation 22:12:12 applied to the associated Laguerre polynomial, or through the integral representation

24:3:3
$$\mathrm{H}_n(x)=\frac{2^{n+1}}{\sqrt{\pi}}\exp(x^2)\int_0^{\infty}t^n\exp(-t^2)\cos\left(2xt-\frac{n\pi}{2}\right)\mathrm{d}t$$

There are two formulas that go by the name of *Hermite's differential equation*. One is

24:3:4
$$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}-2x\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}+2n\mathrm{f}=0$$

and it is solved by $\mathrm{f}=w\mathrm{H}_n(x)$, w being an arbitrary constant. The second version lacks both of the 2's present in 24:3:4, and its solution is the function defined in 24:1:1.

Routes by which a Hermite polynomial may be synthesized [Section 43:14] are

24:3:5 $$\exp(x) \xrightarrow[1/2]{-n/2} \frac{H_n\left(\sqrt{x}\right)}{(-2)^{n/2}(n-1)!!} \qquad n=0,2,4,\cdots$$

and

24:3:6 $$\exp(x) \xrightarrow[3/2]{(1-n)/2} \frac{-H_n\left(\sqrt{x}\right)}{(-2)^{(n+1)/2}n!!\sqrt{x}} \qquad n=1,3,5,\cdots$$

Expansions 24:6:4 and 24:6:5 lie at the heart of these syntheses.

24:4 SPECIAL CASES

$H_0(x)$	$H_1(x)$	$H_2(x)$	$H_3(x)$	$H_4(x)$	$H_5(x)$	$H_6(x)$
1	$2x$	$4x^2-2$	$8x^3-12x$	$16x^4-48x^2+12$	$32x^5-160x^3+120x$	$64x^6-480x^4+720x^2-120$

24:5 INTRARELATIONSHIPS

The reflection formula

24:5:1 $$H_n(-x)=(-)^n H_n(x)$$

establishes that a Hermite polynomial is even or odd in accordance with the parity of its degree. The recursion formula

24:5:2 $$H_n(x)=2x\,H_{n-1}(x)-(2n-2)\,H_{n-2}(x) \qquad n=2,3,4,\cdots$$

links three consecutive members of the family.

There is an argument-addition formula for Hermite polynomials

24:5:3 $$H_n\left(x+y\right)=\frac{1}{2^{n/2}}\sum_{j=0}^{n}\binom{n}{j}H_j\left(\sqrt{2}x\right)H_{n-j}\left(\sqrt{2}y\right)$$

and also an example of the *Christoffel-Darboux formulas*

24:5:4 $$\sum_{j=0}^{n}\frac{H_j(x)H_j(y)}{2^j\,j!}=\frac{H_{n+1}(x)H_n(y)-H_n(x)H_{n+1}(y)}{2^{n+1}n!(x-y)}$$

Two infinite series of Hermite polynomials have the following sums:

24:5:5 $$\frac{H_0(x)}{0!}-\frac{H_2(x)}{2!}+\frac{H_4(x)}{4!}-\cdots=e\cos(2x)$$

24:5:6 $$\frac{H_1(x)}{1!}-\frac{H_3(x)}{3!}+\frac{H_5(x)}{5!}-\cdots=e\sin(2x)$$

where e is the base of natural logarithms [Chapter 1]. Series akin to these in which the signs are uniformly positive sum to $(1/e)\cosh(2x)$ and $(1/e)\sinh(2x)$ respectively and consequently

24:5:7 $$\frac{H_0(x)}{0!}+\frac{H_1(x)}{1!}+\frac{H_2(x)}{2!}+\cdots=\exp(2x-1)$$

which, alternatively, follows directly from the generating function 24:3:1.

24:6 EXPANSIONS

The power-series expansion of the Hermite polynomials can be written in several equivalent ways:

24:6:1 $$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots \begin{cases} +(-2)^{n/2}(n-1)!! & n = 0,2,4,\cdots \\ -(-2)^{(n+1)/2}n!!x & n = 1,3,5,\cdots \end{cases}$$

24:6:2 $$H_n(x) = n!\sum_{j=0}^{\mathrm{Int}(n/2)} \frac{(-1)^j}{j!(n-2j)!}(2x)^{n-2j}$$

24:6:3 $$H_n(x) = (2x)^n \sum_{j=0}^{\mathrm{Int}(n/2)} \frac{\left(\frac{-n}{2}\right)_j \left(\frac{1-n}{2}\right)_j}{(1)_j}\left(\frac{-1}{x^2}\right)^j$$

24:6:4 $$H_n(x) = (-2)^{n/2}(n-1)!!\sum_{j=0}^{n/2} \frac{\left(\frac{-n}{2}\right)_j}{\left(\frac{1}{2}\right)_j (1)_j} x^{2j} \qquad n = 0,2,4,\cdots$$

24:6:5 $$H_n(x) = -(-2)^{(n+1)/2} n!! \sum_{j=0}^{(n-1)/2} \frac{\left(\frac{1-n}{2}\right)_j}{(1)_j \left(\frac{3}{2}\right)_j} x^{2j+1} \qquad n = 1,3,5,\cdots$$

24:7 PARTICULAR VALUES

At $x = 0$, Hermite polynomials of even degree have an integer value, while those of odd degree are zero:

24:7:1 $$H_n(0) = \begin{cases} (-2)^{n/2}(n-1)!! & n = 0,2,4,\cdots \\ 0 & n = 1,3,5,\cdots \end{cases}$$

Apart from these, formulas for the zeros are known only for the lowest degrees. The two zeros of $H_2(x)$ are $\pm 1/\sqrt{2}$, the three zeros of $H_3(x)$ are 0, $\pm\sqrt{3/2}$, and the four zeros of $H_4(x)$ are

24:7:2 $$H_4(r) = 0 \qquad r = \pm\sqrt{\frac{3 \pm \sqrt{6}}{2}} = \begin{cases} \pm 0.52464\ 76232\ 75290 \\ \pm 1.6506\ 80123\ 88578 \end{cases}$$

Because of the simplicity of formula 24:10:1, these zeros of H_4 are the locations of extrema of H_5 and inflections of H_6.

24:8 NUMERICAL VALUES

Equator's Hermite polynomial routine (keyword **Hpoly**) uses recursion 24:5:2, initialized by $H_0(x) = 1$ and $H_1(x) = 2x$, to calculate exact values of $H_n(x)$.

24:9 LIMITS AND APPROXIMATIONS

A Hermite polynomial of large degree is approximated by the oscillatory function

24:9:1 $$H_n(x) \approx \begin{cases} (-2)^{n/2}(n-1)!!\exp\left(x^2/2\right)\cos\left(\sqrt{2n+1}\;x\right) & \text{even } n \to \infty \\ (-2)^{(n-1)/2}n!!\exp\left(x^2/2\right)\sin\left(\sqrt{2n+1}\;x\right) & \text{odd } n \to \infty \end{cases}$$

This approximation is best for small $|x|$ and fails hopelessly for $|x| > \sqrt{2n}$, where the polynomial ceases to oscillate. Approximation 24:9:1 predicts that the zeros of the Hermite polynomial lie at

24:9:2 $$r \approx \frac{j\pi}{2\sqrt{2n+1}} \qquad \text{where} \begin{cases} j = \pm1, \pm3, \pm5, \cdots & \text{if } n \text{ large and even} \\ j = 0, \pm2, \pm4, \cdots & \text{if } n \text{ large and odd} \end{cases}$$

As expected, this approximation has least error for zeros close to $x = 0$.

24:10 OPERATIONS OF THE CALCULUS

The following simple results hold:

24:10:1 $$\frac{d}{dx}H_n(x) = 2n\,H_{n-1}(x) \qquad n = 1, 2, 3, \cdots$$

24:10:2 $$\frac{d}{dx}\left\{\exp(-x^2)\,H_n(x)\right\} = -\exp(-x^2)\,H_{n+1}(x)$$

24:10:3 $$\int_0^x H_n(t)\,dt = \begin{cases} \dfrac{H_{n+1}(x)}{2n+2} & n = 0, 2, 4, \cdots \\ \dfrac{H_{n+1}(x) - (-2)^{(n+1)/2}\,n!!}{2n+2} & n = 1, 3, 5, \cdots \end{cases}$$

and

24:10:4 $$\int_0^x \exp\left(-t^2\right)H_n(t)\,dt = \begin{cases} -\exp\left(-x^2\right)H_{n-1}(x) & n = 0, 2, 4, \cdots \\ (-2)^{(n-1)/2}(n-2)!! - \exp\left(-x^2\right)H_{n-1}(x) & n = 1, 3, 5, \cdots \end{cases}$$

Among the many listed by Gradshteyn and Ryzhik [Sections 7.37 and 7.38], the following definite integrals are of particular interest

24:10:5 $$\int_{-\infty}^{\infty} \exp(-t^2)H_n(bt)\,dt = \begin{cases} \sqrt{\pi}\,n!(b^2-1)^{n/2}/(\tfrac{n}{2})! & n = 0, 2, 4, \cdots \\ 0 & n = 1, 3, 5, \cdots \end{cases}$$

24:10:6 $$\int_{-\infty}^{\infty} t\exp(-t^2)H_n(bt)\,dt = \begin{cases} 0 & n = 0, 2, 4, \cdots \\ \sqrt{\pi}\,n!!b(2b^2-2)^{(n-1)/2} & n = 1, 3, 5, \cdots \end{cases}$$

24:10:7 $$\int_{-\infty}^{\infty} \exp\left(-(t-x)^2\right)H_n(t)\,dt = \begin{cases} \sqrt{\pi}\,(2x)^n & n = 2, 4, 6, \cdots \\ 0 & n = 1, 3, 5, \cdots \end{cases}$$

24:10:8 $$\int_{-\infty}^{\infty} t^n\exp(-t^2)H_n(bt)\,dt = \sqrt{\pi}\,n!P_n(b)$$

24:10:9 $$\int_0^\infty x^\nu \exp\left(\frac{-x^2}{\mu^2}\right)H_n(x)\,dx = \begin{cases} \dfrac{(-4)^{n/2}}{2}\left(\dfrac{1}{2}\right)_{n/2}\Gamma\left(\dfrac{1+\nu}{2}\right)\mu^{1+\nu}\,F\left(\dfrac{-n}{2},\dfrac{1+\nu}{2},\dfrac{1}{2},\mu^2\right) & \nu>-1 \quad n=2,4,6,\cdots \\ (-4)^{(n-1)/2}\,2\left(\dfrac{1}{2}\right)_{(n-1)/2}\Gamma\left(\dfrac{1+\nu}{2}\right)\mu^{2+\nu}\,F\left(\dfrac{1-n}{2},1+\dfrac{\nu}{2},\dfrac{3}{2},\mu^2\right) & \nu>-2 \quad n=1,3,5,\cdots \end{cases}$$

The P, Γ, and F functions in the previous two equations are the Legendre polynomial [Chapter 21], the gamma function [Chapter 43] and the Gauss hypergeometric function [Chapter 60].

The orthogonality of Hermite polynomials is established by

24:10:10 $$\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)H_m(t)\,dt = \begin{cases} 0 & m\neq n \\ \sqrt{\pi}\,2^n n! & m=n \end{cases}$$

Term-by term Laplace transformation of formulas 24:6:5 and 24:6:6 leads to

24:10:11 $$\mathcal{L}\{H_n(t)\} = \begin{cases} \dfrac{(-2)^{n/2}(n-1)!!}{s}\displaystyle\sum_{j=0}^{n/2}\left(\tfrac{-n}{2}\right)_j\left(\frac{4}{s^2}\right)^j = \dfrac{8^{n/2}(n-1)!!\frac{n}{2}!}{s^{n+1}}e_{n/2}\left(\dfrac{-s^2}{4}\right) & n=0,2,4,\cdots \\ \dfrac{-(-2)^{(n+1)/2}n!!}{s^2}\displaystyle\sum_{j=0}^{(n-1)/2}\left(\tfrac{1-n}{2}\right)_j\left(\frac{4}{s^2}\right)^j = \dfrac{8^{(n+1)/2}n!!\frac{n-1}{2}!}{s^{n+1}}e_{(n-1)/2}\left(\dfrac{-s^2}{4}\right) & n=1,3,5,\cdots \end{cases}$$

in which $e_n(x)$ represents the exponential polynomial [Section 26:13].

24:11 COMPLEX ARGUMENT

Applications of Hermite polynomials with complex argument are rare.

An example of an inverse Laplace transform involving a Hermite polynomial is

24:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{H_{2n}\left(\sqrt{s}\right)}{s^{(2n+1)/2}}\frac{\exp(ts)}{2\pi i}\,ds = \mathcal{I}\left\{\frac{H_{2n}\left(\sqrt{s}\right)}{s^{(2n+1)/2}}\right\} = \frac{4^n}{\sqrt{\pi t}}(1+t)^n\,T_n\left(\frac{1-t}{1+t}\right)$$

24:12 GENERALIZATIONS

Hermite polynomials are special cases of the parabolic cylinder function [Chapter 46]. Those parabolic cylinder functions whose order ν is a nonnegative integer n are related through

24:12:1 $$D_n(x) = \frac{1}{2^{n/2}}\exp\left(\frac{-x^2}{4}\right)H_n\left(\frac{x}{\sqrt{2}}\right)$$

to the corresponding Hermite polynomial. The parabolic cylinder function itself generalizes to the Kummer and Tricomi functions [Chapters 47 and 48].

24:13 COGNATE FUNCTIONS: the Hermite functions

By analogy with other orthogonal polynomials, one might expect that the Hermite function would be $H_\nu(x)$ and arise by relaxing the requirement that the degree of the Hermite polynomial be an integer. This is not so. The name *Hermite function of degree n* is actually given to the product

24:13:1 $$f_n(x) = (-)^n \exp\left(-x^2/2\right) H_n(x) \qquad n = 0,1,2,\cdots$$

This function is the Hermite polynomial, multiplied by the square root of its weight function, so that it satisfies an orthogonality relationship without any further weighting:

24:13:2 $$\int_{-\infty}^{\infty} f_n(t) f_m(t)\,dt = \begin{cases} 0 & m \neq n \\ 2^n \sqrt{\pi}\, n! & m = n \end{cases}$$

The Hermite function may be defined by the repeated derivative

24:13:3 $$f_n(x) = \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-x^2\right) \qquad n = 0,1,2,\cdots$$

or, in operator notation [Boas, Section 12.22], as

24:13:4 $$f_n(x) = \left(\frac{d}{dx} - x\right)^n \exp\left(\frac{-x^2}{2}\right) \qquad n = 0,1,2,\cdots$$

For the harmonic oscillator, it satisfies the *Schrödinger equation*

24:13:5 $$\frac{d^2 f}{dx^2} + (2n + 1 - x^2) f = 0$$

of atomic physics, and is related to the parabolic cylinder function of Chapter 46 by

24:13:6 $$f_n(x) = 2^{n/2} \exp\left(x^2/2\right) D_n\left(\sqrt{2}\, x\right)$$

24:14 RELATED TOPIC: solving differential equations

Equation 24:3:4 and 24:13:5 are examples drawn from the large and important class of differential equations that may be written in the general form

24:14:1 $$\frac{d^2}{dx^2} f(x) - 2B(x) \frac{d}{dx} f(x) + C(x) f(x) = 0$$

The algebraic functions B and C may or may not involve the independent variable x and may even be zero. For reasons that need not detain us, such a differential equation is described as linear, homogeneous, second-order, and ordinary. It, and its inhomogeneous counterpart, to be considered later, are the only differential equations that will be addressed in this section. The solution is invariably of the form

24:14:2 $$f = w_1 f_1(x) + w_2 f_2(x)$$

where w_1 and w_2 are arbitrary weighting factors and f_1, f_2 are functions either of which separately satisfy the differential equation. To provide a complete solution, f_1 and f_2 must be "linearly independent", a condition that will be identified later.

There are many approaches to solving 24:14:1 [see Murphy for a concise description]. One that has merit requires that the so-called *invariant function*, defined by

24:14:3 $$I(x) = C(x) - B^2(x) + \frac{d}{dx} B(x)$$

first be determined. Then the solution of the differential equation 24:14:1 is

24:14:4 $$\mathrm{f} = w_1\,\mathrm{f}_1(x) + w_2\mathrm{f}_2(x) = \exp\left(\int \mathrm{B}(x)\,\mathrm{d}x\right)\left[w_1\mathrm{F}_1(x) + w_2\mathrm{F}_2(x)\right]$$

where the integration is indefinite and where the identities of the functions F_1 and F_2 depend solely on the invariant function I. The table that follows gives the two F functions (or sometimes just one of them) that correspond to the listed standard forms of the invariant I(x) function.. The numbers in brackets reference the relevant chapter or section. Do not make the easy mistake of regarding the F's as the solutions to the differential equation: they require multiplication by $\exp\left(\int \mathrm{B}(x)\,\mathrm{d}x\right)$ first, which often simplifies the expression.

I(x)	$F_{1,2}(x)$	Ref.
$-4a^2x^{p-2}$	$\sqrt{x}\,\mathrm{I}_{\pm\frac{1}{p}}\left(\frac{a}{p}x^{p/2}\right)$ or, if $1/p$ is the integer n, $\sqrt{x}\,\mathrm{I}_n\left(anx^{1/2n}\right)$, $\sqrt{x}\,\mathrm{K}_n\left(anx^{1/2n}\right)$	[50]
$4a^2x^{p-2}$	$\sqrt{x}\,\mathrm{J}_{\pm\frac{1}{p}}\left(\frac{a}{p}x^{p/2}\right)$ or, if $1/p$ is the integer n, $\sqrt{x}\,\mathrm{J}_n\left(anx^{1/2n}\right)$, $\sqrt{x}\,\mathrm{Y}_n\left(anx^{1/2n}\right)$	[53]
$-64a^2x^2$	$\sqrt{x}\,\mathrm{I}_{\pm\frac{1}{4}}\left(ax^2\right)$	[50]
$(v+½)a^2-(a^4x^2/4)$	$\mathrm{D}_v(\pm ax)$	[46]
$\dfrac{(2v+1)a^2-x^2}{a^4}$	$\exp\left(\frac{-x^2}{2a^2}\right)\mathrm{M}\left(\frac{-v}{2},\frac{1}{2},\frac{x^2}{a^2}\right)$, $x\exp\left(\frac{-x^2}{2a^2}\right)\mathrm{M}\left(\frac{1-v}{2},\frac{3}{2},\frac{x^2}{a^2}\right)$	[47]
$(2n+1)a^2-a^4x^2$	$\exp(-a^2x^2/2)\mathrm{H}_n(ax)$, $\exp\left(-a^2x^2/2\right)\mathrm{M}\left(-n/2,1/2,a^2x^2\right)$	[24]
$-(2n+1)a^2-a^4x^2$	$\exp(a^2x^2/2)\mathrm{i}^n\mathrm{erfc}(\pm ax)$	[40:13]
$64a^2x^2$	$\sqrt{x}\,\mathrm{J}_{\pm\frac{1}{4}}\left(ax^2\right)$	[53]
a^3x	$\mathrm{Ai}(ax)$, $\mathrm{Bi}(ax)$	[56]
a^3x+a^2b	$\mathrm{Ai}(ax+b)$, $\mathrm{Bi}(ax+b)$	[56]
$-a^2$	$\exp(\pm ax)$	[26]
0	1, x	
a^2	$\sin(ax)$, $\cos(ax)$	[32]
$a^2/4x$	$\sqrt{x}\,\mathrm{J}_1\left(a\sqrt{x}\right)$, $\sqrt{x}\,\mathrm{Y}_1\left(a\sqrt{x}\right)$	[52,54]
$-a^2/4x$	$\sqrt{x}\,\mathrm{I}_1\left(a\sqrt{x}\right)$, $\sqrt{x}\,\mathrm{K}_1\left(a\sqrt{x}\right)$	[49,51]
$(¼-a^2)/x^2$	$x^{½\pm a}$	[12]
$1/4x^2$	$\sqrt{x}$, $\sqrt{x}\ln(x)$	[25]
$(¼+a^2)/x^2$	$\sqrt{x}\sin(a\ln x)$, $\sqrt{x}\cos(a\ln x)$	[32]
$a^2+(1/4x^2)$	$\sqrt{x}\,\mathrm{J}_0(ax)$, $\sqrt{x}\,\mathrm{Y}_0(ax)$	[52,54]
$-a^2+(1/4x^2)$	$\sqrt{x}\,\mathrm{I}_0(ax)$, $\sqrt{x}\,\mathrm{K}_0(ax)$	[49,51]
$a^2-\dfrac{\nu^2-\frac{1}{4}}{x^2}$	$\sqrt{x}\,\mathrm{J}_{\pm\nu}(ax)$ or, if ν is the integer n $\sqrt{x}\,\mathrm{J}_n(ax)$, $\sqrt{x}\,\mathrm{Y}_n(ax)$	[52-54]

$I(x)$	$F_{1,2}(x)$	Ref.
$-a^2-\dfrac{\nu^2-\frac{1}{4}}{x^2}$	$\sqrt{x}\,I_{\pm\nu}(ax)$ or, if ν is the integer n $\sqrt{x}\,I_n(ax)$, $\sqrt{x}\,K_n(ax)$	[49,51]
$\dfrac{-ax-\nu^2+1}{4x^2}$	$\sqrt{x}\,I_\nu\left(\sqrt{ax}\right)$, $\sqrt{x}\,K_\nu\left(\sqrt{ax}\right)$	[50,51]
$\dfrac{3a^2-(8n+4)ax-4x^2}{16a^2x^2}$	$x^{1/4}\exp\left(\dfrac{x}{2a}\right)\mathrm{i}^n\mathrm{erfc}\left(\pm\sqrt{\dfrac{x}{a}}\right)$	[40:13]
$[1+(4n+2)ax-a^2x^2]/4x^2$	$\sqrt{x}\exp(-ax/2)L_n(ax)$	[23]
$\dfrac{1-4\mu^2+4\nu ax-a^2x^2}{4x^2}$	$M_{\nu,\mu}(ax)$, $W_{\nu,\mu}(ax)$	[48:13]
$-2/(a^2+x^2)$	(a^2+x^2), $ax+(a^2+x^2)\arctan(x/a)$	[35]
$\dfrac{-2a}{x(a-x)^2}$	$\dfrac{x}{a-x}$, $\dfrac{a+x}{a}+\dfrac{2x}{a-x}\ln\left(\dfrac{x}{a}\right)$	[25]
$\dfrac{a^2+3x^2}{4x^2(a^2-x^2)}$	$\sqrt{x}\,E\left(\dfrac{x}{a}\right)$, $\sqrt{x}\left[E\left(\dfrac{\sqrt{a^2-x^2}}{a}\right)-K\left(\dfrac{\sqrt{a^2-x^2}}{a}\right)\right]$	[61]
$\dfrac{\nu^2+\nu+1}{a^2-x^2}+\dfrac{x^2}{\left(a^2-x^2\right)^2}$	$\sqrt{a^2-x^2}\,P_\nu\left(\dfrac{x}{a}\right)$, $\sqrt{a^2-x^2}\,Q_\nu\left(\dfrac{x}{a}\right)$ polynomials if $\nu = n$	[59]
$\dfrac{(n^2+\frac{1}{2})}{a^2-x^2}+\dfrac{\frac{3}{4}x^2}{\left(a^2-x^2\right)^2}$ $n\neq 0$	$(a^2-x^2)^{1/4}\,T_n\left(\dfrac{x}{a}\right)$, $(a^2-x^2)^{3/4}\,U_{n-1}\left(\dfrac{x}{a}\right)$	[22]
$\dfrac{a^4+2a^2x^2-15x^4}{4x^2(a^2-x^2)^2}$	$\sqrt{x(a^2-x^2)}\,K\left(\dfrac{x}{a}\right)$, $\sqrt{x(a^2-x^2)}\,K\left(\dfrac{\sqrt{a^2-x^2}}{a}\right)$	[61]
$\dfrac{n^2+2n\lambda+\lambda+\frac{1}{2}}{a^2-x^2}+\dfrac{(\frac{3}{4}+\lambda-\lambda^2)x^2}{\left(a^2-x^2\right)^2}$	$\left(a^2-x^2\right)^{(1+2\lambda)/4}C_n^{(\lambda)}\left(\dfrac{x}{a}\right)$	[22:12]
$-\dfrac{(\alpha+\beta)^2-1}{4(a-x)^2}-\dfrac{\alpha\beta}{x(a-x)}-\dfrac{(4\gamma^2-1)a}{4x^2(a-x)}$	$x^{\frac{1}{2}\pm\gamma}(a-x)^{(\alpha+\beta+1)/2}F\left(\alpha+\frac{1}{2}\pm\gamma,\beta+\frac{1}{2}\pm\gamma;1\pm 2\gamma;\dfrac{x}{a}\right)$	[60]
a^2/x^4	$x\exp(\pm a/x)$	[27]
$(-a^2-2x^2)/x^4$	$x(x\pm a)\exp(\mp a/x)$	[27]
$(a^2-2x^2)/x^4$	$x^2\cos(a/x)+ax\sin(a/x)$, $x^2\sin(a/x)-ax\cos(a/x)$	[32]

By setting $B(x)$ equal to zero, one can see that F_1 and F_2 are nothing but solutions of the differential equation

24:14:5 $$\frac{d^2}{dx^2}F(x)+I(x)F(x)=0 \qquad F(x)=w_1F_1(x)+w_2F_2(x)$$

Another common type of differential equation is the *inhomogeneous* variant of 24:14:1, 24:14:6 $$\frac{d^2}{dx^2}f(x)-2B(x)\frac{d}{dx}f(x)+C(x)f(x)=R(x)$$

with a right-hand member, R(x), that is either a constant or a function of x. To solve this equation, first ignore the right-hand member and solve the corresponding homogeneous equation, 24:14:1. Then, knowing f_1 and f_2, the complete solution of 24:14:6 is

24:14:7 $$f=f_1(x)\left[w_1-\int_{x_0}^{x}\frac{R(t)f_2(t)}{W(t)}dt\right]+f_2(x)\left[w_2+\int_{x_0}^{x}\frac{R(t)f_1(t)}{W(t)}dt\right]$$

with x_0 arbitrary, and where

24:14:8 $$W(t)=f_1(t)\frac{d}{dt}f_2(t)-f_2(t)\frac{d}{dt}f_1(t)=f_1^2(t)\frac{d}{dt}\frac{f_2(t)}{f_1(t)}$$

Thus, at least in principle, if the solution of 24:14:1 can be found, so can that of 24:14:6. The integrals in 24:14:7 are known as *particular integrals*. The quantity W(t) is the *Wronskian* (Josef-Maria Hoëné de Wronski, 1778–1853, Polish, then French, mathematician and philosopher) of the two functions f_1 and f_2. These two solutions are linearly independent only if the Wronskian is nonzero. See Section 26:3 for a simple example.

24:15 RELATED TOPICS: Gauss-Hermite integration and other Gaussian quadratures

Suppose one needs to integrate a function of the form f(t)exp($-t^2$) numerically, between limits of $\pm\infty$. A powerful way of doing this uses the orthogonality properties [Section 21:14] of Hermite polynomials to generate the approximation

24:15:1 $$\int_{-\infty}^{\infty}f(t)\exp(-t^2)\,dt\approx\frac{2^{n-1}n!\sqrt{\pi}}{n^2}\sum_{j=1}^{n}\frac{f(r_j)}{H_{n-1}^2(r_j)}\qquad n=2,3,4,\cdots$$

where the set $r_1, r_2, \cdots, r_n$ includes all the zeros of the Hermite $H_n(t)$ polynomial. Choosing $n=4$ as a simple example, and using result 24:7:2, one finds

24:15:2 $$\int_{-\infty}^{\infty}f(t)\exp(-t^2)dt\approx 12\sqrt{\pi}\left\{\frac{f\left(-\sqrt{\frac{3}{2}+\sqrt{\frac{3}{2}}}\right)}{H_3^2\left(-\sqrt{\frac{3}{2}+\sqrt{\frac{3}{2}}}\right)}+\frac{f\left(-\sqrt{\frac{3}{2}-\sqrt{\frac{3}{2}}}\right)}{H_3^2\left(-\sqrt{\frac{3}{2}-\sqrt{\frac{3}{2}}}\right)}+\frac{f\left(\sqrt{\frac{3}{2}-\sqrt{\frac{3}{2}}}\right)}{H_3^2\left(\sqrt{\frac{3}{2}-\sqrt{\frac{3}{2}}}\right)}+\frac{f\left(\sqrt{\frac{3}{2}+\sqrt{\frac{3}{2}}}\right)}{H_3^2\left(\sqrt{\frac{3}{2}+\sqrt{\frac{3}{2}}}\right)}\right\}$$

More usually, one would need to precalculate the zeros [by the Newton-Raphson method of Section 52:15, for example] or use tabulated values [such as those in Chapter 25 of Abramowitz and Stegun].

The Gauss-Hermite integration formula 24:15:2 is just one of a large number of integration schemes known collectively as *Gauss integration*, or *Gaussian quadrature*. They share the characteristic of being weighted sums of the values of the function f evaluated at the zeros of an orthogonal polynomial, the weights often being inversely proportional to the square of the polynomial of one order lower. Some other examples are *Gauss-Laguerre integration*

24:15:3 $$\int_{0}^{\infty}f(t)\exp(-t)dt\approx\frac{1}{n^2}\sum_{j=1}^{n}\frac{r_j f(r_j)}{L_{n-1}^2(r_j)}\qquad n=2,3,4,\cdots$$

and *Gauss-Legendre integration*

24:15:4 $$\int_{-1}^{1} f(t)\,dt \approx \frac{2}{n^2}\sum_{j=1}^{n}\frac{(1-r_j^2)\,f(r_j)}{P_{n-1}^2(r_j)} \qquad n=2,3,4,\cdots$$

The weights are uniform for *Gauss-Chebyshev integration*

24:15:5 $$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,dt \approx \frac{\pi}{n}\sum_{j=1}^{n} f(r_j) \qquad n=2,3,4,\cdots$$

in which r_j is the jth zero of the Chebyshev $T_n(t)$ polynomial.

The derivation of the Gaussian quadrature formulas [Matthews and Walker, Section 13-2] treats the f function as a polynomial in t. The closer that is to being the truth, and the larger n is, the better the approximation. If f is, in fact, a polynomial of degree $2n-1$ or less, the formulas are exact.

CHAPTER 25

THE LOGARITHMIC FUNCTION ln(*x*)

The functions introduced in Chapters 10−24 are *algebraic*, that is, they may be defined by a finite set of addition, subtraction, multiplication, division, and exponentiation operations. *Transcendental functions*, in contrast, cannot be expressed as a finite combination of algebraic terms. All the functions, from this point onwards in the *Atlas* are transcendental. The logarithmic function is among the simplest of these.

Prior to the advent of electronic calculators and computers, logarithms [especially decadic logarithms, Section 25:14] were used extensively as aids to arithmetic computation, in the form of “log tables” and slide rules.

In addition to the logarithmic function itself, this chapter briefly addresses the logarithmic integral function li(x), the dilogarithm diln(x), and polylogarithms.

25:1 NOTATION

The name *logarithm* is used synonymously with *logarithmic function* to describe ln(x). Alternative notations are log(x) and $\log_e(x)$, though the former is also used for the decadic logarithm [Section 25:14]. The initial letter of the symbol is sometimes written in script type, ℓn(x) to avoid possible confusion with the numeral “one”. See Section 25:11 for the capitalized Ln(x).

To emphasize the distinction from logarithms to other bases [Section 25:14], ln(x) is variously referred to as the *logarithm to base e*, the *natural logarithm*, the *hyperbolic logarithm*, or the *Naperian logarithm* [John Napier, 1550−1617, Scottish mathematician].

25:2 BEHAVIOR

As a real-valued function, ln(x) is defined only for positive x; it takes values of $-\infty$, 0, and $+\infty$ when $x = 0$, 1, and ∞. As Figure 25-1 shows, it is a monotonic function, with a positive slope that decreases steadily as the argument x increases. Even though the logarithmic function does eventually acquire an infinite value, its approach to infinity is slower than that of any positive power of x, so that:

25:2:1 $$\frac{\ln(x)}{x^\nu} \to 0 \qquad x \to \infty \qquad \text{all } \nu > 0$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_26, © Springer Science+Business Media, LLC 2009

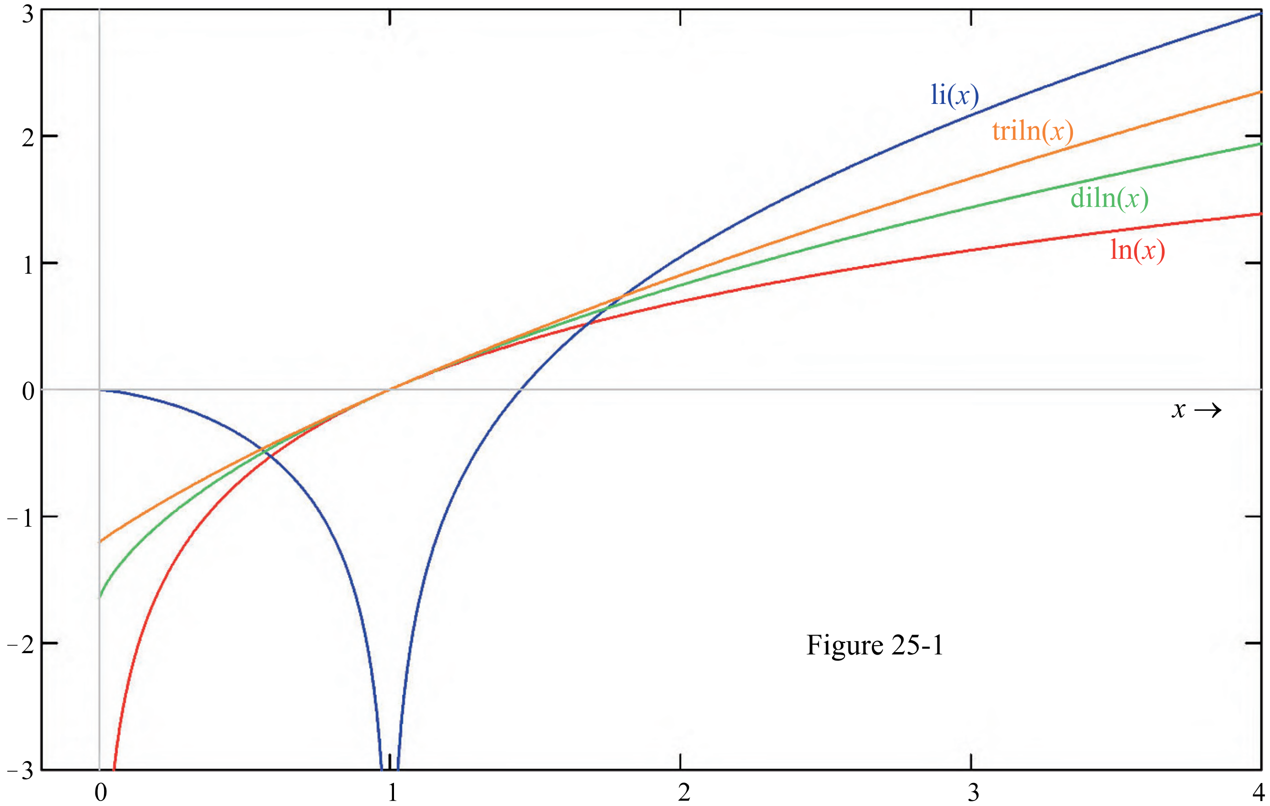


Figure 25-1

Some authors enforce evenness on the logarithmic function by decreeing that $\ln(-x) = \ln(x)$, but the *Atlas* does not follow this practice.

25:3 DEFINITIONS

The logarithmic function is defined through the integral [Figure 25-2]

25:3:1 $$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t \qquad x > 0$$

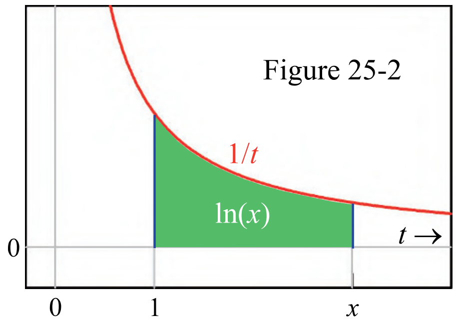


Figure 25-2

or by the limiting operation

25:3:2 $$\ln(x) = \lim_{v\to 0}\left\{\frac{x^v - 1}{v}\right\} \qquad x > 0, \quad v > 0$$

The logarithm may also be defined as the inverse function of the exponential exp(x) function [Chapter 26]

25:3:3 $$\ln\left(\exp(x)\right) = x$$

or by synthesis [Section 43:14] from the simplest basis hypergeometric function

25:3:4 $$\frac{1}{1-x} \quad \xrightarrow[2]{1} \quad \frac{-\ln(1-x)}{x}$$

25:4 SPECIAL CASES

There are none of great importance. However, there are some formulas, for example 31:5:8 and 31:5:9, that apply only to logarithms of integer argument.

25:5 INTRARELATIONSHIPS

The logarithmic function of an argument in the interval $0 \le x \le 1$ is related to one with argument in the $1 \le x \le \infty$ domain by the simple reciprocation formula

25:5:1 $$\ln\left(\frac{1}{x}\right) = -\ln(x) \qquad x > 0$$

Provided that x and y are both positive, the logarithms of products, quotients and powers are given by

25:5:2 $$\ln(xy) = \ln(x) + \ln(y)$$

25:5:3 $$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

and

25:5:4 $$\ln\left(x^{\nu}\right) = \nu \ln(x)$$

Thus, on taking logarithms, the operations of multiplication, division, and exponentiation are replaced by addition, subtraction, and multiplication. It is the simplification afforded by these transformations that was responsible for the widespread use of logarithms in computation, before calculators became ubiquitous. Operation 25:5:2 generalizes to permit the replacement of a product by a summation:

25:5:5 $$\ln\left(\prod_j x_j\right) = \sum_j \ln(x_j)$$

where the product, and therefore the summation, may be finite or infinite (provided, in the infinite case, that the series converges).

The difference of two logarithms whose arguments differ by unity, can be expressed as an inverse hyperbolic cotangent [Chapter 31]

25:5:6 $$\ln(x+1) - \ln(x) = 2\,\mathrm{arcoth}(2x+1)$$

25:6 EXPANSIONS

The logarithmic function may be expanded as a power series in a variety of ways, of which the following are representative:

25:6:1 $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = -\sum_{j=1}^{\infty} \frac{(-x)^j}{j} \qquad -1 < x \le 1$$

25:6:2 $$\ln(x) = \frac{x-1}{x} + \frac{(x-1)^2}{2x^2} + \frac{(x-1)^3}{3x^3} + \cdots = \sum_{j=1}^{\infty} \frac{(x-1)^j}{jx^j} \qquad x \ge \frac{1}{2}$$

25:6:3 $$\ln(x) = \frac{x^2-1}{x^2+1} + \frac{(x^2-1)^3}{3(x^2+1)^3} + \frac{(x^2-1)^5}{5(x^2+1)^5} + \cdots = \sum_{j=0}^{\infty} \frac{1}{2j+1}\left(\frac{x^2-1}{x^2+1}\right)^{2j+1} \qquad x > 0$$

As well, the logarithm is expansible through the continued fractions

25:6:4 $$\ln(1+x) = 1 - \frac{1}{1+}\,\frac{x}{1-x+}\,\frac{x}{2-x+}\,\frac{4x}{3-2x+}\,\frac{9x}{4-3x+}\,\frac{16x}{5-4x+}\cdots$$

and

25:6:5 $$\frac{\ln(1+x)}{x} = \frac{1}{1+}\,\frac{x}{2+}\,\frac{x}{3+}\,\frac{4x}{4+}\,\frac{4x}{5+}\,\frac{9x}{6+}\,\frac{9x}{7+}\,\frac{16x}{8+\cdots}$$

25:7 PARTICULAR VALUES

$\ln(0)$	$\ln(1/e)$	$\ln(1)$	$\ln(e)$	$\ln(e^n)$	$\ln(\infty)$
$-\infty$	-1	0	1	n	$+\infty$

25:8 NUMERICAL VALUES

Equator's logarithmic function routine (keyword **ln**) can compute logarithms of any positive number in the range 10^{-308} to 10^{308}.

25:9 LIMITS AND APPROXIMATIONS

For arguments close to unity, the approximation

25:9:1 $$\ln(x) \approx \frac{x-1}{\sqrt{x}} \qquad |x-1| \text{ small}$$

is valid. The limit

25:9:2 $$\ln(n) \to -\gamma + \sum_{j=1}^{n} \frac{1}{j} \qquad n \text{ is a large integer}$$

governs the logarithm of a large integer, γ being Euler's constant [Section 1:4]. Also see 25:2:1.

25:10 OPERATIONS OF THE CALCULUS

Simple and multiple differentiation give

25:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\ln(bx+c) = \frac{b}{bx+c}$$

25:10:2 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ln(bx+c) = -(n-1)!\left(\frac{-b}{bx+c}\right)^n \qquad n = 1,2,3,\cdots$$

while indefinite integration yields the following results:

25:10:3
$$\int_0^x \ln(t)\,\mathrm{d}t = x[\ln(x)-1] = x\ln\left(\frac{x}{e}\right)$$

25:10:4
$$\int_{(1-c)/b}^x \ln(bt+c)\,\mathrm{d}t = \left(x+\frac{c}{b}\right)[\ln(bx+c)-1]+\frac{1}{b}$$

25:10:5
$$\int_1^x \ln^n(t)\,\mathrm{d}t = (-)^n n!x\sum_{j=0}^n \frac{[-\ln(x)]^j}{j!} \qquad n=0,1,2,\cdots$$

25:10:6
$$\int_1^x t^\nu \ln(t)\,\mathrm{d}t = \begin{cases} \dfrac{x^{\nu+1}}{\nu+1}\left[\ln(x)-\dfrac{1}{\nu+1}\right] & \nu\neq -1 \\ \dfrac{\ln^2(x)}{2} & \nu=-1 \end{cases}$$

25:10:7
$$\int_{-c/b}^x \frac{1}{\ln(bt+c)}\,\mathrm{d}t = \frac{\mathrm{li}(bx+c)}{b}$$

25:10:8
$$\int_0^x \frac{t^\nu}{\ln(t)}\,\mathrm{d}t = \begin{cases} \mathrm{li}(x^{\nu+1}) & \nu\neq -1 \\ \ln(\ln(x)) & \nu=-1 \end{cases}$$

The li function in the last two formulas is the logarithmic integral [Section 25:13].

The following definite integrals are of interest and lead to important constants [Section 1:7]:

25:10:9
$$\int_1^\infty \frac{\ln(\ln(t))}{t^2}\,\mathrm{d}t = \int_0^1 \ln(-\ln(t))\,\mathrm{d}t = -\gamma$$

25:10:10
$$\int_1^\infty \frac{\ln(t)}{t^2\pm t}\,\mathrm{d}t = -\int_0^1 \frac{\ln(t)}{1\pm t}\,\mathrm{d}t = (3\mp 1)\frac{\pi^2}{24}$$

25:10:11
$$\int_1^\infty \frac{\ln(t)}{t^2+1}\,\mathrm{d}t = -\int_0^1 \frac{\ln(t)}{t^2+1}\,\mathrm{d}t = G$$

25:10:12
$$\int_1^\infty \frac{\ln(t)}{t^2-1}\,\mathrm{d}t = \int_0^1 \frac{\ln(t)}{t^2-1}\,\mathrm{d}t = \frac{\pi^2}{8}$$

25:10:13
$$\int_0^1 \ln(t)\ln(1\pm t)\,\mathrm{d}t = \begin{cases} 2-\ln(4)-\frac{1}{12}\pi^2 \\ 2-\frac{1}{6}\pi^2 \end{cases}$$

Others are given by Gradshteyn and Ryzhik [Sections 4.2–4.4].

Semidifferentiation and semiintegration [Section 12:14], with a lower limit of zero, yield

25:10:14
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\ln(bx) = \frac{\ln(4bx)}{\sqrt{\pi x}}$$

25:10:15
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\ln(bx) = 2\sqrt{\frac{x}{\pi}}\,[\ln(4bx)-2]$$

Euler's constant γ [Section 1:7] enters the Laplace transforms [Section 26:15] of the logarithmic function. This transform, and two others, are:

25:10:16 $$\int_0^\infty \ln(t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\ln(t)\} = \frac{-\gamma-\ln(s)}{s}$$

25:10:17 $$\int_0^\infty \ln(bt+c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\ln(bt+c)\} = \frac{\ln(c)}{s} - \frac{1}{s}\exp\left(\frac{cs}{b}\right)\mathrm{Ei}\left(\frac{-cs}{b}\right)$$

25:10:18 $$\int_0^\infty t^\nu \ln(t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{t^\nu \ln(t)\} = \frac{\Gamma(\nu+1)\left[\psi(\nu+1)-\ln(s)\right]}{s^{\nu+1}}$$

Here, Ei is the exponential integral function [Chapter 37] and $\psi(\nu+1)$ is a digamma function [Chapter 44]. Note that, for positive integer argument, $\psi(n+1)$ equals $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\gamma$, the sum of the reciprocals of the first n natural numbers, less Euler's constant.

25:11 COMPLEX ARGUMENT

When its argument is the complex variable $z=\rho\exp(i\theta)=x+iy$, the logarithm is a multi-valued function denoted, in this *Atlas* and elsewhere, by Ln(z). In the complex Cartesian plane, it may be expressed in either polar or rectangular coordinates. In the former it is simply

25:11:1 $$\mathrm{Ln}(z) = \ln(\rho) + i\theta$$

which shows immediately that the real part depends only on the radial coordinate ρ whereas the imaginary part is dependent only on the angle θ. It is the fact that θ is unlimited that leads to the multivalued nature of the Ln function. To establish a principal value, θ is constrained to lie within $-\pi < \theta \le \pi$ and the complex plane is cut along the real axis between $x = 0$ and $x = -\infty$. When this restriction is applied, the Ln( ) function becomes ln( ).

The dependence of the real and imaginary parts of ln($x+iy$) on the rectangular coordinates x and y is illustrated in Figure 25-3. The real part is seen to have perfect rotational symmetry centered on a singularity at the origin, confirming that only the distance from the origin is significant:

25:11:2 $$\mathrm{Re}\left[\ln(x+iy)\right] = \ln\left(\sqrt{x^2+y^2}\right) = \ln(\rho)$$

Conversely, the imaginary part, depicted in the second diagram of Figure 25-3, depends only on the angle θ. In rectangular coordinates, this imaginary part is best described by

25:11:3 $$\mathrm{Im}\left[\ln(x+iy)\right] = \mathrm{sgn}(y)\,\mathrm{arccot}\left(x/|y|\right) = \theta$$

If the argument is purely imaginary

25:11:4 $$\ln(iy) = \ln(|y|) + \mathrm{sgn}(y)\frac{i\pi}{2}$$

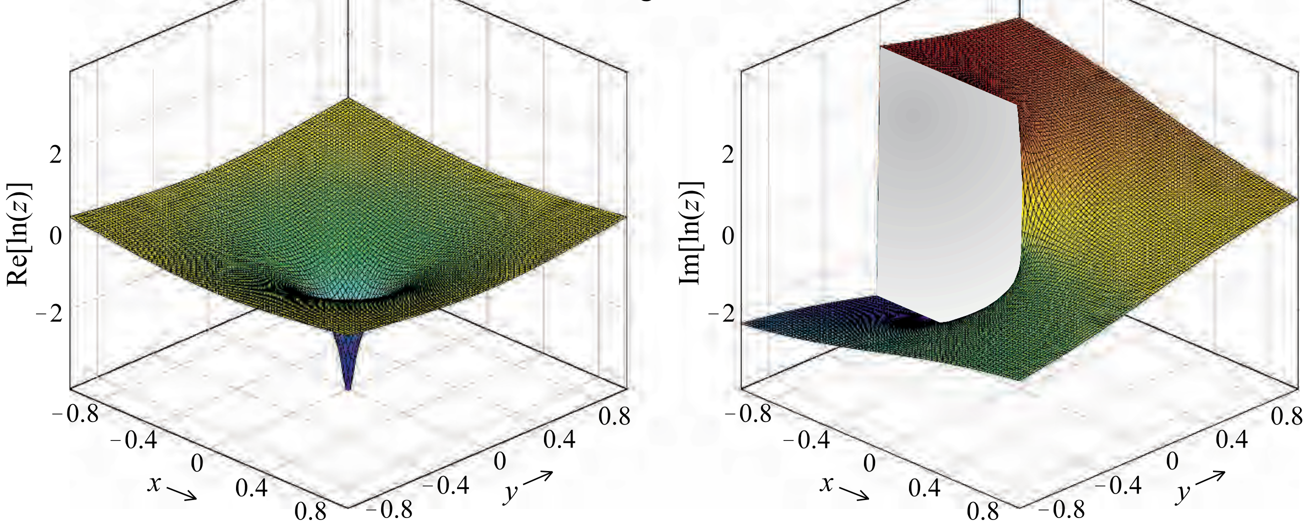
For example $\ln(i) = i\pi/2$ and $\ln(-i) = -i\pi/2$. When $y = 0$ and $x = -1$, we are dealing with a point that actually lies on the cut. If y is very slightly positive, then $\ln(-1+\delta i) = +i\pi$, whereas a slightly negative y leads to $\ln(-1-\delta i) = -i\pi$. In such circumstances it is usual to assign a value equal to the average on each side of the cut, in this case zero.

Examples of inverse Laplace transforms involving the logarithmic function are:

25:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\ln(s)}{s^{n+1}}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{\ln(s)}{s^{n+1}}\right\} = \frac{t^n}{n!}\left[\psi(n+1)-\gamma-\ln(t)\right] \qquad n=0,1,2,\cdots$$

25:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\ln^2(s)}{s}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{\ln^2(s)}{s}\right\} = \left[\ln(t)+\gamma\right]^2-\frac{\pi^2}{6}$$

Figure 25-3



25:12 GENERALIZATIONS: generalized logarithm and polylogarithms

One generalization of the logarithmic function is towards other bases. This avenue is explored in Section 25:14.

Another generalization, reducing to the standard logarithm when $\nu = 1$, has been termed the *generalized logarithmic function* [Oldham and Spanier, Section 10.5] and is related to the incomplete beta function [Chapter 58] by

25:12:1 $$\ln_\nu(x) = \sum_{j=0}^{\infty}\frac{1}{j+\nu}\left(\frac{x-1}{x}\right)^{j+\nu} = \mathrm{B}\left(\nu,0,\frac{x-1}{x}\right)$$

In the remainder of the present section we discuss *polylogarithms*, which represent a generalization of the logarithm in a third direction. Polylogarithms are themselves special cases of Lerch's function [Section 64:12]. Also known as *Jonquière's functions* (Ernest Jean Philippe Fauque de Jonquières, 1820 − 1901, French naval officer and mathematician), they appear in the *Feynman diagrams* of particle physics.

Via expansion 25:6:1, the logarithm may be defined, for a limited range of argument, by

25:12:2 $$\ln(x) = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots = -\sum_{j=1}^{\infty}\frac{(1-x)^j}{j} \qquad 0<x\le 2$$

A natural extension of this definition is to the *dilogarithm*

25:12:3 $$\mathrm{diln}(x) = -\sum_{j=1}^{\infty}\frac{(1-x)^j}{j^2}$$

the *trilogarithm*

25:12:4 $$\mathrm{triln}(x) = -\sum_{j=1}^{\infty} \frac{(1-x)^j}{j^3}$$

and generally, the *polylogarithm* of order v

25:12:5 $$\mathrm{polyln}_v(x) = -\sum_{j=1}^{\infty} \frac{(1-x)^j}{j^v}$$

where v is usually, but not necessarily, a positive integer. Be aware, however, that this is not the way in which dilogarithms, trilogarithms and polylogarithms are customarily defined. Erdélyi et al. [Higher Transcendental Functions, page 30−31, using the notations $F(x,v)$ and $L_n(x)$], and *Mathematica* adopt definitions equivalent to our $-\mathrm{polyln}_v(1-x)$, as does Thompson [Page 182, using the notation $\mathrm{Li}_n(x)$]. These rival definitions do not reduce to the ordinary logarithmic function when the order is unity and are not, therefore, true generalizations of the logarithm. Abramowitz and Stegun [Section 27.7, using the notation $f(x)$] adopt a definition of the dilogarithm similar to ours, but of the opposite sign. Particular values of important polylogarithms, as we define them, are given in the table that follows. In this table, ζ and η are functions from Chapter 3 and Z is Apéry's constant given in equation 3:7:1.

	$x = 0$	$x = ½$	$x = 1$	$x = 2$
$\ln(x)$	$-\infty$	$-\ln(2)$	0	$\ln(2)$
$\mathrm{diln}(x)$	$\frac{-\pi^2}{6}$	$\frac{-\pi^2}{12} - \frac{\ln^2(2)}{2}$	0	$\frac{-\pi^2}{12}$
$\mathrm{triln}(x)$	$-Z$	$\frac{-7Z}{8} - \frac{\pi^2 \ln^2(2)}{12} - \frac{\ln^3(2)}{6}$	0	$\frac{3Z}{4}$
$\mathrm{polyln}_n(x)$	$-\zeta(n)$		0	$\eta(n)$

Figure 25-1 includes a graph of the dilogarithm, diln(x) which, being a hypergeometric function [Table 18-2], may be synthesized in the two-step process

25:12:6 $$\frac{1}{1-x} \xrightarrow{\frac{1}{2}} \frac{-\ln(1-x)}{x} \xrightarrow{\frac{1}{2}} \frac{-\mathrm{diln}(1-x)}{x}$$

As an alternative to the definition as series 25:12:3, the dilogarithm [Figure 25-4] is also defined by *Spence's integral* (William Spence, Scottish mathematician, 1777−1815)

25:12:7 $$\mathrm{diln}(x) = \int_1^x \frac{\ln(t)}{t-1}\,\mathrm{d}t \qquad x \ge 0$$

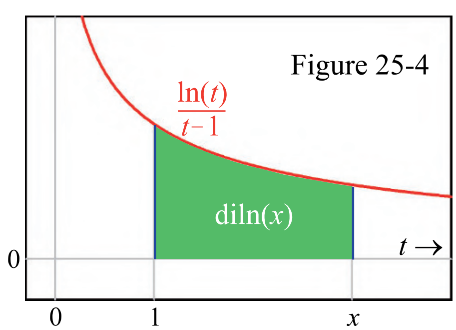


Figure 25-4

The summation 25:12:3 is rapidly convergent for $½ \le x < 1$ and advantage of this is taken by *Equator*'s dilogarithm routine (keyword **diln**). For arguments in the $0 < x < ½$ range, *Equator* exploits the reflection formula

25:12:8 $$\mathrm{diln}(x) = \ln(x)\ln(1-x) - \mathrm{diln}(1-x) - \frac{\pi^2}{6} \qquad 0 \le x \le 1$$

Those in the range $1 < x \le 2$ are accessed by the reciprocation formula

25:12:9
$$\mathrm{diln}(x) = \frac{\ln^2(x)}{2} - \mathrm{diln}\left(\frac{1}{x}\right) \qquad x > 0$$

and for values of x exceeding 2 this formula is replaced by

25:12:10
$$\mathrm{diln}(x) = \mathrm{diln}\left(\frac{x-1}{x}\right) + \ln(x)\ln\left(\frac{x-1}{\sqrt{x}}\right) - \frac{\pi^2}{6} \qquad x > 0$$

The dilogarithm is relevant to Fermi-Dirac and Bose-Einstein distributions [Section 27:14], as is the trilogarithm. The latter may be defined, in addition to the series definition 25:12:4, as the integral

25:12:11
$$\mathrm{triln}(x) = \int_1^x \frac{\mathrm{diln}(t)}{t-1}\,\mathrm{d}t \qquad x \geq 0$$

or hypergeometrically [Section 18:14]

25:12:12
$$\mathrm{triln}(x) = (x-1)\sum_{j=0}^{\infty} \frac{(1)_j(1)_j(1)_j}{(2)_j(2)_j(2)_j}(1-x)^j \qquad 0 < x \leq 2$$

A graph of triln(x) is included in Figure 25-1 and *Equator*'s trilogarithm routine (keyword **triln**) calculates values for nonnegative arguments. This routine uses formula 25:12:4 for arguments in the domain ½ < x ≤ 2, but exploits the relation

25:12:13
$$\mathrm{triln}(x) = -\mathrm{triln}(1-x) - \mathrm{triln}\left(\frac{1}{x}\right) - Z - \frac{\pi^2}{6}\ln(x) + \frac{1}{2}\ln^2(x)\ln(1-x) - \frac{1}{6}\ln^3(x) \qquad x > 0$$

for x values between 0 and ½. Likewise, the formula

25:12:14
$$\mathrm{triln}(x) = \mathrm{triln}\left(\frac{x}{x-1}\right) + \frac{\pi^2}{6}\ln(x-1) + \frac{1}{6}\ln^3(x-1)$$

usefully extends the domain of easily accessible values to arguments larger than 2.

25:13 COGNATE FUNCTIONS: inverse hyperbolic functions and the logarithmic integral

The *inverse hyperbolic functions* [Chapter 31] are logarithms of modified argument.

The *logarithmic integral* li(x), a function with importance in number theory, is defined by the indefinite integral [Figure 25-5]

25:13:1
$$\mathrm{li}(x) = \int_0^x \frac{1}{\ln(t)}\,\mathrm{d}t$$

Some authorities regard the logarithmic integral as being defined only for arguments exceeding unity, but, in this *Atlas*, li(x) exists also in $0 \leq x < 1$, with a Cauchy limit definition

25:13:2
$$\mathrm{li}(x) = \lim_{\varepsilon\to 0}\{\mathrm{li}(1-\varepsilon)\} + \int_{1+\varepsilon}^{x} \frac{1}{\ln(t)}\,\mathrm{d}t \qquad x \geq 0 \qquad \varepsilon > 0$$

serving to extend the definition across the discontinuity at x = 1. The definite integral

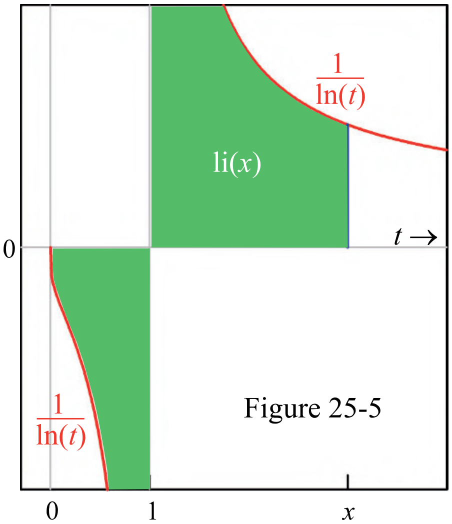


Figure 25-5

25:13:3 $$\mathrm{li}(x)=\int_0^1\frac{x}{\ln(x)+\ln(t)}\,\mathrm{d}t$$

also provides a definition. The zero of the logarithmic integral occurs at

25:13:4 $$r=1.4513\,69234\,88338 \qquad \mathrm{li}(r)=0$$

The identity

25:13:5 $$\mathrm{li}(x)=\mathrm{Ei}\big(\ln(x)\big)$$

where Ei is the exponential integral function [Chapter 37] may be used to reveal many properties of this function and is the basis of *Equator*'s logarithmic integral routine (keyword **li**) to compute numerical values of li(x).

Operations of the calculus yield the following results for the logarithmic integral function:

25:13:6 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{li}(x^{v})=\frac{x^{v-1}}{\ln(x)} \qquad v\neq 0$$

25:13:7 $$\int_0^x \mathrm{li}(bt)\,\mathrm{d}t=x\,\mathrm{li}(bx)-\frac{1}{b}\mathrm{li}(b^2x^2) \qquad b>0$$

25:13:8 $$\int_0^1 t^{v}\,\mathrm{li}(t)\,\mathrm{d}t=\frac{-\ln(2+v)}{1+v} \qquad v>-2$$

25:13:9 $$\int_1^\infty t^{v}\,\mathrm{li}(t)\,\mathrm{d}t=\frac{\ln(-2-v)}{1+v} \qquad v<-2$$

25:14 RELATED TOPICS: logarithms to other bases

Just as ln(x) can be defined as the inverse function of exp(x) or e^x where e is the number $2.7182\,81828\,45905\cdots$, so can a function be defined as the inverse of β^x, where β is any positive number other than unity. A function so defined is called the *logarithm to base* β of argument x and it is denoted $\log_\beta(x)$. Such a logarithm is directly proportional to the logarithmic function ln(x):

25:14:1 $$\log_\beta(x)=\log_\beta(e)\ln(x)=\frac{\ln(x)}{\ln(\beta)} \qquad x>0,\quad 0<\beta\neq 1$$

and *Equator*'s logarithm to any base routine (keyword **loganybase**) utilizes this relationship.

Logarithms to base 2 are called *binary logarithms*

25:14:2 $$\log_2(x)=1.4426\,95040\,88896\,\ln(x)=\frac{\ln(x)}{0.69314\,71805\,59945}$$

Logarithms to base 10 are called *decadic logarithms*, *common logarithms* or *Briggsian logarithms* (Henry Briggs, English mathematician, 1561−1630):

25:14:3 $$\log_{10}(x)=0.43429\,44819\,03252\,\ln(x)=\frac{\ln(x)}{2.3025\,85092\,99405}$$

The subscript is often omitted, so that $\log_{10}(x)$ may be written log(x) or even lg(x). *Equator* provides a decadic logarithm routine (keyword **log10**) The decadic logarithm of a number in scientific notation [Section 8:14] is easily found because

25:14:4 $$\log_{10}\left(N_0{\bullet}N_{-1}N_{-2}N_{-3}\cdots\times 10^n\right) = n + \log_{10}\left(N_0{\bullet}N_{-1}N_{-2}N_{-3}\cdots\right)$$

The first right-hand term n, an integer, is the *characteristic* of the decadic logarithm. The value of the second right-hand term, which necessarily lies in the range $0{\bullet}0000\cdots$ through $0{\bullet}9999\cdots$, is called the *mantissa* of the logarithm. Chemists use a "p" notation

25:14:5 $$\mathrm{p}x = -\log_{10}(x)$$

For example pH = $-\log_{10}(H)$, where H is the activity (or molar concentration) of hydrogen ions. The name *cologarithm* of x is sometimes applied to $-\log_{10}(x)$.

CHAPTER 26

THE EXPONENTIAL FUNCTION $\exp(\pm x)$

Many natural and manmade assemblies – bacterial populations, investments, radioactivity, and light-bulb arrays – increase or decrease at a rate proportional to their size. This characteristic leads to *exponential growth* or *exponential decay*, with a functional dependence expressed by $\exp\left\{\pm t\ln(2)/t_{1/2}\right\}$, t representing time and $t_{1/2}$ the *half-time*, the constant interval that it takes for the assembly to double (or halve) in size.

In this chapter we address the *exponential functions* $\exp(\pm x)$, and sometimes generalize these to $\exp(bx)$ or $\exp(bx+c)$. Consideration of the exponential functions of arguments more complicated than these is deferred to the next chapter. The self-exponential function, the exponential polynomial function, and some other functions that are related to the exponential function are briefly addressed in this chapter, as is the important technique of Laplace transformation.

26:1 NOTATION

The process of raising a constant to a power is known as "exponentiation" and accordingly β^x is sometimes called an "exponential function". The *Atlas*, however, adopts a more restricted interpretation of this name and applies it only to cases in which the base β is the number $e = 2.7182\,81828\,45905\cdots$, the base of natural logarithms [Section 1:7]. Moreover, mainly for typographical convenience, we mostly write $\exp(x)$ instead of e^x, though the two notations are equivalent in every respect.

Colloquially, exponential functions are referred to as "exponentials". Rarely the name *natural antilogarithm* and the symbol $\ln^{-1}(x)$ are encountered.

26:2 BEHAVIOR

The exponential function accepts real arguments of any magnitude and either sign, but itself adopts only positive values. As illustrated in Figure 26-1, $\exp(x)$ increases monotonically with x, approaching zero as $x \to -\infty$ and increasing rapidly towards infinity at large positive x. The converse applies to $\exp(-x)$: it becomes limitlessly large as x become ever more negative and decays rapidly towards zero at large positive x.

The exponential function $\exp(x)$ has two remarkable properties: that it is its own derivative [equation 26:10:1] and that changing the sign of its argument is equivalent to reciprocation [equation 26:5:1].

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_27,

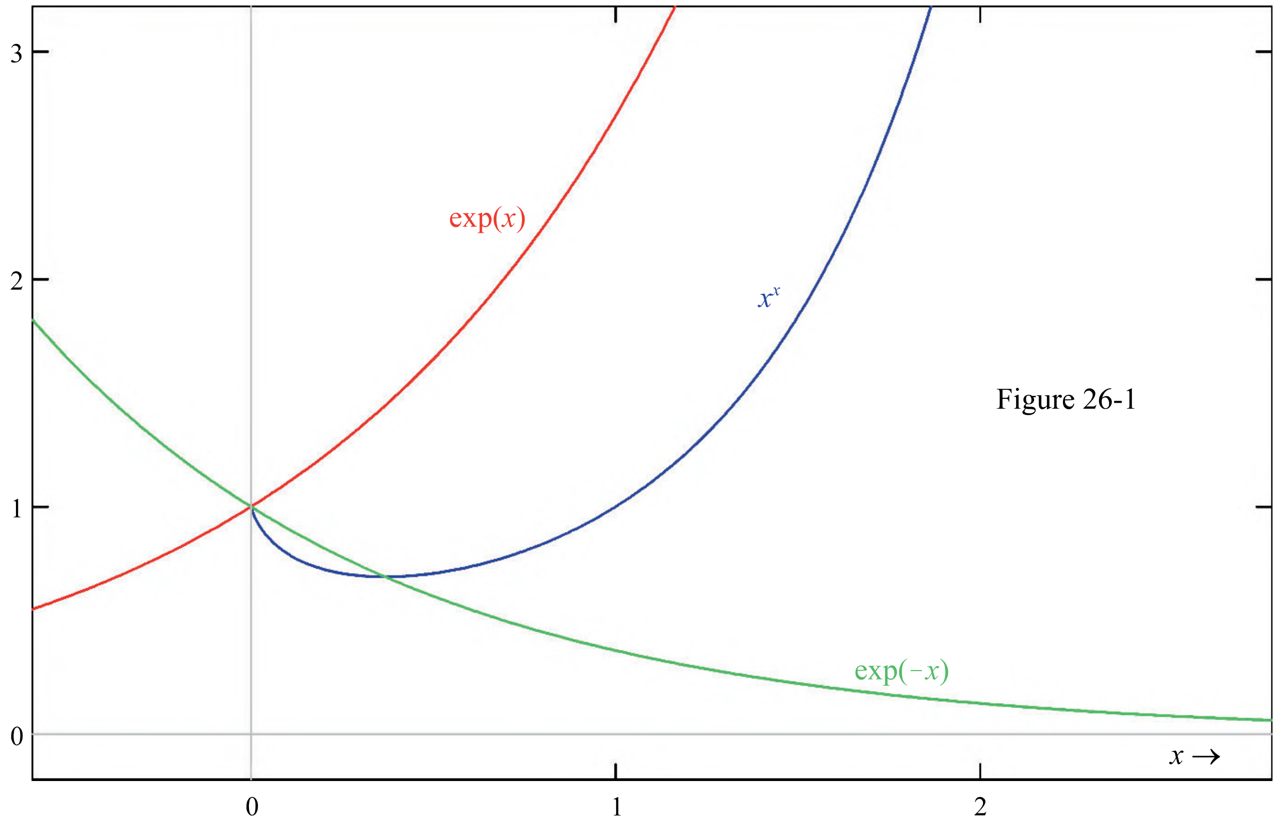


Figure 26-1

26:3 DEFINITIONS

The exponential function may be defined as the inverse of the logarithmic function:

26:3:1 $$\exp(\ln(x)) = x$$

or as the result of exponentiating the number e

26:3:2 $$\exp(x) = e^x = (2.7182\,81828\,45905\cdots)^x$$

The series

26:3:3 $$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{x^j}{j!} = \exp(x)$$

can provide another definition and a fourth, albeit not a very useful one in practice, is as the limit

26:3:4 $$\exp(x) = \lim_{v\to\infty}\left(1 + \frac{x}{v}\right)^v$$

One of the simplest first-order differential equations, and its solution, are

26:3:5 $$\frac{\mathrm{df}}{\mathrm{d}x} = b\mathrm{f} \qquad \mathrm{f} = w\exp(bx)$$

The solution can also be written exp($bx+c$), where $w = \exp(c)$ and c is arbitrary. The exponential function also solves a large number of other differential equations; an example, and its solution, are

26:3:6 $$\frac{d^2 f}{dx^2} = b^2 f \qquad f(x) = w_1 \exp(bx) + w_2 \exp(-bx)$$

Moreover, if B, C, and R are constants, with $B^2 > C$, the second-order differential equation

26:3:7 $$\frac{d^2 f}{dx^2} - 2B\frac{df}{dx} + Cf = R$$

has the solution [Section 24:4]

26:3:8 $$f = \frac{R}{C} + w_1 \exp(b_+ x) + w_2 \exp(b_- x) \qquad \text{where} \quad b_\pm = B \pm \sqrt{B^2 - C}$$

in which the w's are arbitrary constants.

26:4 SPECIAL CASES

There are none.

26:5 INTRARELATIONSHIPS

The simplicity of the reflection, addition, subtraction, and involution formulas

26:5:1 $$\exp(-x) = \frac{1}{\exp(x)}$$

26:5:2 $$\exp(x+y) = \exp(x)\exp(y)$$

26:5:3 $$\exp(x-y) = \frac{\exp(x)}{\exp(y)}$$

26:5:4 $$\exp^{\nu}(x) = \exp(\nu x)$$

partly explains the widespread utility of exponential functions. The congruence of these formulas with those for operations on powers [compare with equations 12:5:2−4] confirms that the argument of an exponential function may be regarded as an exponent, validating the representation of $\exp(x)$ as e^x.

Several infinite series of exponential functions may be summed geometrically [equation 6:14:9] and then rewritten as hyperbolic functions [Chapters 29 and 30]:

26:5:5 $$\exp(-x) + \exp(-2x) + \exp(-3x) + \cdots = \frac{1}{\exp(x)-1} = \frac{1}{2}\coth\left(\frac{x}{2}\right) - \frac{1}{2} \qquad x > 0$$

26:5:6 $$\exp(-x) - \exp(-2x) + \exp(-3x) - \cdots = \frac{1}{\exp(x)+1} = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2}\right) \qquad x > 0$$

26:5:7 $$\exp(-x) + \exp(-3x) + \exp(-5x) + \cdots = \frac{\exp(x)}{\exp(2x)-1} = \frac{1}{2}\operatorname{csch}(x) \qquad x > 0$$

26:5:8 $$\exp(-x) - \exp(-3x) + \exp(-5x) - \cdots = \frac{\exp(x)}{\exp(2x)+1} = \frac{1}{2}\operatorname{sech}(x) \qquad x > 0$$

Similar sums in which the arguments are multiples of the squares of the natural (or the odd) numbers lead to exponential theta functions [Section 27:13]

26:5:9 $$\exp(-x)+\exp(-4x)+\exp(-9x)+\cdots=\sum_{j=1}^{\infty}\exp(-j^2x)=\frac{1}{2}\theta_3\left(0,\frac{x}{\pi^2}\right)-\frac{1}{2} \qquad x>0$$

26:5:10 $$\exp(-x)-\exp(-4x)+\exp(-9x)-\cdots=-\sum_{j=1}^{\infty}(-)^j\exp(-j^2x)=\frac{1}{2}-\frac{1}{2}\theta_4\left(0,\frac{x}{\pi^2}\right) \qquad x>0$$

26:5:11 $$\exp(-x)+\exp(-9x)+\exp(-25x)+\cdots=\sum_{j=1}^{\infty}\exp\left[-(2j-1)^2x\right]=\frac{1}{2}\theta_2\left(0,\frac{4x}{\pi^2}\right) \qquad x>0$$

26:6 EXPANSIONS

The expansion

26:6:1 $$\exp(bx+c)=1+\frac{bx+c}{1!}+\frac{(bx+c)^2}{2!}+\frac{(bx+c)^3}{3!}+\cdots=\sum_{j=0}^{\infty}\frac{(bx+c)^j}{j!}$$

is valid for all arguments. After factoring out $\exp(c)$, it may be rewritten as the concatenation

26:6:2 $$\exp(bx+c)=\exp(c)\left[1+\frac{bx}{1}\left(1+\frac{bx}{2}\left(1+\frac{bx}{3}\left(1+\frac{bx}{4}(1+\cdots)\right)\right)\right)\right]$$

An expansion in terms of modified Bessel functions [Chapter 49] is

26:6:3 $$\exp(\pm x)=\mathrm{I}_0(x)\pm 2\mathrm{I}_1(x)+2\mathrm{I}_2(x)\pm 2\mathrm{I}_3(x)+\cdots$$

There are several ways in which the exponential function may be expanded as a continued fraction:

26:6:4 $$\exp(x)=\frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{x}{2-}\,\frac{x}{5+}\,\frac{x}{2-}\,\frac{x}{7+\cdots}$$

26:6:5 $$\exp(x)=1+\frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{x}{2+}\,\frac{x}{5-}\,\frac{x}{2+}\,\frac{x}{7-\cdots}$$

26:6:6 $$\exp(x)=1+\frac{1}{1-}\,\frac{\frac{1}{2}x}{1+}\,\frac{\frac{1}{6}x}{1-}\,\frac{\frac{1}{6}x}{1+}\,\frac{\frac{1}{10}x}{1-}\,\frac{\frac{1}{10}x}{1+}\,\frac{\frac{1}{14}x}{1-}\,\frac{\frac{1}{14}x}{1+}\,\frac{\frac{1}{18}x}{1-\cdots}$$

The construction of the third of these continued fractions is discussed in Section 10:14.

26:7 PARTICULAR VALUES

$\exp(-\infty)$	$\exp(-1)$	$\exp(0)$	$\exp(1)$	$\exp(\infty)$
0	$1/e$	1	e	∞

26:8 NUMERICAL VALUES

Equator's exponential function routine (keyword **exp**) can evaluate $\exp(x)$ for $-710\le x\le 710$.

26:9 LIMITS AND APPROXIMATIONS

For positive integer argument, the limit

26:9:1
$$\exp(n) \to \sqrt{2\pi n}\,\frac{n^n}{n!} \qquad n \to \infty$$

exists and the approximation that it represents may be improved by elaboration via Stirling's formula 2:6:1. For noninteger argument, replace n by x and $n!$ by $\Gamma(x+1)$ [Chapter 43]. The limit as the argument approaches $-\infty$ is available through 26:5:1. Close to $x = 0$ the "*compound interest*" approximation

26:9:2
$$\exp(x) \approx \left(1+\frac{x}{N}\right)^N \qquad |x| \text{ small} \qquad N \text{ a large number}$$

based on limit 26:3:4, could be useful.

By setting $\exp(x) \approx \mathrm{e}_n(x)$ or $\exp(x) \approx 1/\mathrm{e}_n(-x)$, with large enough n, the exponential polynomial [Section 26:12] can provide a good approximation if $|x|$ is not too large, but a rational function [Section 17:12] approximation will generally be better.

26:10 OPERATIONS OF THE CALCULUS

Because $\exp(bx+c) = \exp(c)\exp(bx)$, any calculus operation applied to $\exp(bx+c)$ is simply equal to $\exp(c)$ multiplied by the result of that operation applied to $\exp(bx)$.

The well-known rules for the differentiation

26:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(bx) = b\exp(bx)$$

multiple differentiation

26:10:2
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(bx) = b^n\exp(bx)$$

and integration

26:10:3
$$\int_{-\infty}^{x}\exp(bt)\,\mathrm{d}t = \frac{1}{b}\exp(bx)$$

generalize to the formula for Weyl differintegration [that is, differintegration with a lower limit of $-\infty$, Section 12:14]

26:10:4
$$\left.\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\exp(bx)\right|_{-\infty} = b^\nu\exp(bx) \qquad b > 0$$

For example, Weyl semidifferentiation and semiintegration ($\nu = \pm\tfrac{1}{2}$) yield $\sqrt{b}\,\exp(bx)$ and $\left(1/\sqrt{b}\right)\exp(bx)$ respectively, though only if $b > 0$. See Section 64:14 for more information about Weyl differintegration.

Formulas 26:10:1 and 26:10:2 are special cases of the general rule

26:10:5
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\exp(bx) = \frac{\exp(bx)}{x^\mu}\gamma\mathrm{n}(-\mu, bx)$$

for differintegration with lower limit of zero. Here $\gamma\mathrm{n}(\ ,\ )$ is the entire incomplete gamma function described in Chapter 45. The special $\mu = -1$, $\mu = -\tfrac{1}{2}$ and $\mu = \tfrac{1}{2}$ cases of formula 26:10:5 are

26:10:6
$$\frac{d^{-1}}{dx^{-1}}\exp(bx)=\int_0^x \exp(bt)\,dt=\frac{1}{b}[\exp(bt)-1]$$

26:10:7
$$\frac{d^{-1/2}}{dx^{-1/2}}\exp(bx)=\begin{cases}\left(1/\sqrt{b}\right)\exp(bx)\,\mathrm{erf}\left(\sqrt{bx}\right) & b>0\\ \left(2/\sqrt{-\pi b}\right)\mathrm{daw}\left(\sqrt{-bx}\right) & b<0\end{cases}$$

and

26:10:8
$$\frac{d^{1/2}}{dx^{1/2}}\exp(bx)=\frac{1}{\sqrt{\pi x}}\begin{cases}+\sqrt{b}\exp(bx)\,\mathrm{erf}\left(\sqrt{bx}\right) & b>0\\ -2\sqrt{-b/\pi}\;\mathrm{daw}\left(\sqrt{-bx}\right) & b<0\end{cases}$$

where erf and daw denote the error function [Chapter 40] and Dawson's integral [Chapter 42] respectively. Of course, when $\mu = 1$ or n, equation 26:10:5 reduces to 26:10:1 or 26:10:2.

The following formulas are special cases of the indefinite integration of the product of a power and an exponential function:

26:10:9
$$\int_0^x t^n \exp(bt)\,dt=\frac{(-)^n n!}{b^{n+1}}[\exp(bx)e_n(-bx)-1] \qquad n=0,1,2,\cdots$$

26:10:10
$$\int_0^x \sqrt{t}\exp(bt)\,dt=\frac{\sqrt{x}}{b}\exp(bx)+\begin{cases}-b^{-3/2}\exp(bx)\mathrm{daw}\left(\sqrt{bx}\right) & b>0\\ \sqrt{-\pi/4b^3}\;\mathrm{erf}\left(\sqrt{-bx}\right) & b<0\end{cases}$$

26:10:11
$$\int_0^x \frac{\exp(bt)}{\sqrt{t}}\,dt=\begin{cases}\left(2/\sqrt{b}\right)\exp(bx)\mathrm{daw}\left(\sqrt{bx}\right) & b>0\\ \sqrt{-\pi/b}\;\mathrm{erf}\left(\sqrt{-bx}\right) & b<0\end{cases}$$

26:10:12
$$\int_{x_0}^x \frac{\exp(bt)}{t}\,dt=\mathrm{Ei}(bx) \qquad x_0=0.36250\;74107\;81367$$

26:10:13
$$\int_{x_0}^x \frac{\exp(bt)}{t^n}\,dt=\frac{b^{n-1}}{(n-1)!}\left[\mathrm{Ei}(bx)-\exp(bx)\sum_{j=1}^{n-1}\frac{(j-1)!}{(bx)^j}+\exp(bx_0)\sum_{j=1}^{n-1}\frac{(j-1)!}{(bx_0)^j}\right] \qquad n=1,2,3,\cdots$$

In the two preceding integrals, Ei( ) is the exponential integral [Chapter 37] and x_0 is its zero. A Cauchy interpretation of these integrals [Section 0:10] may be needed for some values of b and x. Some important transcendental functions are defined by indefinite integration of such products; examples are:

26:10:14
$$\int_0^x t^\nu \exp(bt)\,dt=\frac{x^{\nu+1}}{\nu+1}\mathrm{M}\left(\nu+1,\nu+2,bx\right) \qquad \nu>-1$$

26:10:15
$$\int_0^x t^\nu \exp(-bt)\,dt=b^{-\nu-1}\gamma(\nu+1,bx) \qquad \nu>-1 \qquad b>0$$

26:10:16
$$\int_x^\infty t^\nu \exp(-bt)\,dt=b^{-\nu-1}\Gamma(\nu+1,bx) \qquad \nu\le-1 \qquad b>0$$

In these formulas M(, ,) is the Kummer function [Chapter 47], while γ( ) and Γ( ) are two varieties of incomplete gamma function [Chapter 45]. A table in Section 37:14 lists instances of $\int t^{\nu}\exp(t)\mathrm{d}t$ for many commonly encountered values of ν.

Some other important indefinite integrals include

26:10:17 $$\int_0^x \frac{\mathrm{d}t}{a+\exp(bt+c)} = \frac{x}{a} - \frac{1}{ab}\ln\left(\frac{a+\exp(bx+c)}{a+\exp(c)}\right)$$

and

26:10:18 $$\int_{-\infty}^x \frac{\mathrm{d}t}{\exp(bt)+a^2\exp(-bt)} = \frac{1}{ab}\arctan\left(\frac{\exp(bx)}{a}\right) \qquad a>0$$

and

26:10:19 $$\int_x^{\infty} \frac{\mathrm{d}t}{\exp(bt)-a^2\exp(-bt)} = \frac{1}{ab}\operatorname{arcoth}\left(\frac{\exp(bx)}{a}\right) \qquad a>0 \qquad x>\frac{\ln(a)}{b}$$

The mutually complementary indefinite integrals

26:10:20 $$\int_0^x \frac{t^n}{\exp(t)-1}\mathrm{d}t \qquad \text{and} \qquad \int_x^{\infty} \frac{t^n}{\exp(t)-1}\mathrm{d}t \qquad n=1,2,3,\cdots$$

are addressed in Section 3:15, the former defining the *Debye functions*. Their sum is contained in the formula

26:10:21 $$\int_0^{\infty} \frac{t^n}{\exp(t)\pm 1}\mathrm{d}t = \begin{cases} n!\,\eta(n+1) \\ n!\,\zeta(n+1) \end{cases} \qquad n=0,1,2,\cdots$$

where the η and ζ functions are those of Chapter 3.

The definite integrals

26:10:22 $$\int_1^{\infty} \frac{\exp(-xt)}{t^n}\mathrm{d}t \qquad \int_1^{\infty} t^n \exp(-xt)\,\mathrm{d}t \qquad \text{and} \qquad \int_{-1}^{1} t^n\exp(-xt)\,\mathrm{d}t$$

are discussed in Section 37:13 and related integrals are tabulated in Section 37:14. Some thirty pages are devoted by Gradshteyn and Ryzhik [Sections 3.3 and 3.4] to definite integrals of exponential functions.

When a function $\mathrm{f}(t)$ is multiplied by $\exp(-st)$ and integrated over $0 \le t \le \infty$, the integral, if it exists, is a function known as the *Laplace transform* of $\mathrm{f}(t)$ and denoted as

26:10:23 $$\int_0^{\infty} \mathrm{f}(t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{f}(t)\} = \text{ a function of } s$$

Section 26:15 addresses Laplace transformation in some detail. Laplace transforms involving the function f are listed towards the end of Section 10 of the chapter in this *Atlas* that is devoted to the function f. Thus, we here present the Laplace transform of the exponential function itself

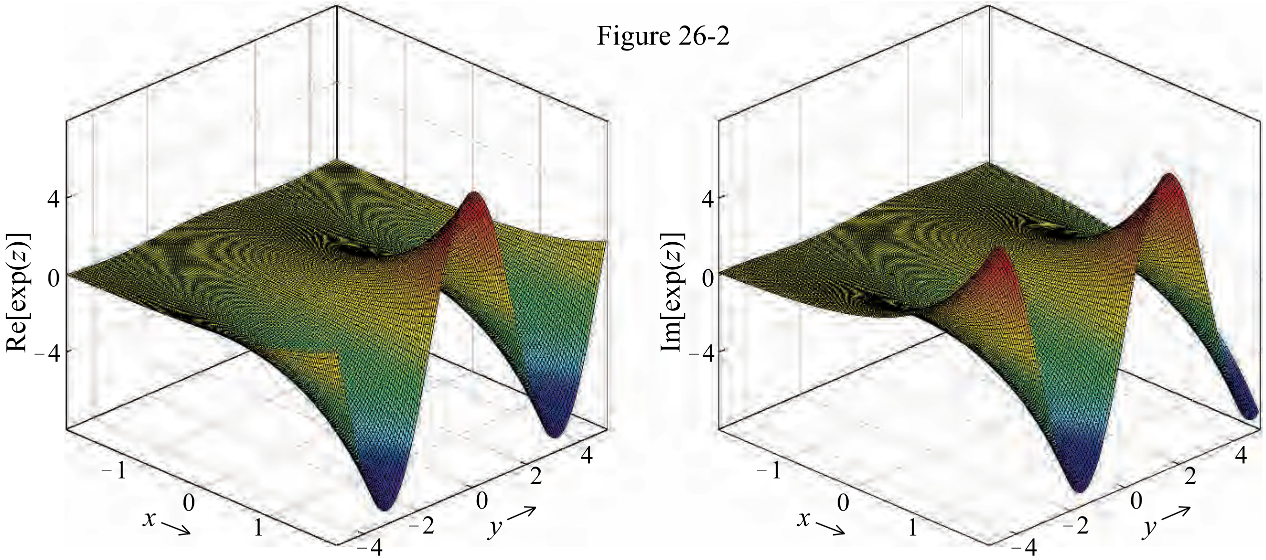
26:10:24 $$\int_0^{\infty} \exp(-bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\exp(-bt)\} = \frac{1}{s+b} \qquad b \ge 0$$

and its product with an arbitrary power

26:10:25 $$\int_0^{\infty} t^{\nu}\exp(-bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\{t^{\nu}\exp(-bt)\} = \frac{\Gamma(1+\nu)}{(s+b)^{1+\nu}} \qquad b \ge 0 \qquad \nu > -1$$

The gamma function [Chapter 43] appears in the latter transform.

Figure 26-2



26:11 COMPLEX ARGUMENT

When its argument is the complex number $z = x+iy$, the exponential function is a simple single-valued function lacking discontinuities. In fact, exp(z) is an *analytic function*, which means that it possesses a convergent power series, akin to 26:6:1, for all values of x and y. Figure 26-2 shows the real and imaginary parts of exp(x+iy), which are described by the formulas

26:11:1 $$\mathrm{Re}\left[\exp(x+iy)\right]=\exp(x)\cos(y) \qquad \text{and} \qquad \mathrm{Im}\left[\exp(x+iy)\right]=\exp(x)\sin(y)$$

The real and imaginary parts appear very similar; indeed, the difference is only in phase and could be nullified by a $\pi/2$ translation along the imaginary axis. The sinusoidal behavior contributed by the cosine and sine terms is present throughout the complex plane; in the diagrams it appears more accentuated for positive x because of the presence of the exp(x) multiplier. The equation

26:11:2 $$\exp(z)=\exp(x)\left[\cos(y)+i\sin(y)\right]$$

is known as *Euler's formula*; it establishes that the exponential function of purely imaginary argument is a periodic function of period 2π, with the following particular values:

$\exp(-i\pi)$	$\exp\left(\frac{-3}{4}i\pi\right)$	$\exp\left(\frac{-1}{2}i\pi\right)$	$\exp\left(\frac{-1}{4}i\pi\right)$	$\exp(0i\pi)$	$\exp\left(\frac{1}{4}i\pi\right)$	$\exp\left(\frac{1}{2}i\pi\right)$	$\exp\left(\frac{3}{4}i\pi\right)$	$\exp(i\pi)$
-1	$(-1-i)/\sqrt{2}$	$-i$	$(1-i)/\sqrt{2}$	1	$(1+i)/\sqrt{2}$	i	$(-1+i)/\sqrt{2}$	-1

Equator uses equation 26:11:2 in its exponential function of complex argument routine (keyword **complexp**) to evaluate the exponential function of the complex number $x+iy$.

Roberts and Kaufman devote many pages to inverse Laplace transforms that involve exponential functions; here we report only three. Each member of this trio gives rise to a discontinuous function from Chapter 9

26:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\exp(-bs)}{s}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s=\mathscr{I}\left\{\frac{\exp(-bs)}{s}\right\}=\mathrm{u}(t-b)$$

26:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\exp(-bs)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\{\exp(-bs)\}=\delta(t-b)$$

26:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\exp(-bs)}{s+a}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\exp(-bs)}{s+a}\right\}=\mathrm{u}(t-b)\exp\big(-a(t-b)\big)$$

In these formulas, which are special cases of the more general inversion formula in 26:15:11, b is restricted to nonnegative values.

26:12 GENERALIZATIONS

Inasmuch as they all reduce to $\exp(x)$ for certain values of their variables, the antilogarithm β^x, the exponential polynomial $\mathrm{e}_n(x)$, the incomplete gamma function $\Gamma(\nu,x)$, the Kummer function $\mathrm{M}(a,c,x)$, and the Tricomi function $\mathrm{U}(a,c,x)$, may all be regarded as generalizations of the exponential function, as may all hypergeometric functions [Section 18:14] for which $L=K+1$. The first two of these will be addressed briefly here, discussion of the others being postponed to Chapters 45, 47, and 48.

The function β^x, where β is a positive constant other than unity, is called an *antilogarithm* or a *generalized exponential function* (though that name is also given to the Mittag-Leffler function, Section 45:14). It is equivalent to an exponential function with a changed argument:

26:12:1 $$\beta^x=\exp(bx)\qquad\text{where}\qquad b=\ln(\beta)$$

Values of β^x can be calculated through *Equator*'s power function routine (keyword **power**) by placing β in the "x" box. The number 10 is the most commonly encountered base, 10^x being known as the *common* or *decadic antilogarithm* of x.

The term "exponential polynomial" is used with two meanings. One is as an alternative name for the *Bell polynomial*, important in combinatorics, but not further considered here. In the *Atlas*, the *exponential polynomial* is derived from the power-series expansion, equation 26:3:3, of the exponential function by truncation after the nth term:

26:12:2 $$\mathrm{e}_n(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}=\sum_{j=0}^{n}\frac{x^j}{j!}$$

$\mathrm{e}_0(x)$	$\mathrm{e}_1(x)$	$\mathrm{e}_2(x)$	$\mathrm{e}_3(x)$	$\mathrm{e}_4(x)$	$\mathrm{e}_5(x)$
1	$1+x$	$1+x+\frac{1}{2}x^2$	$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$	$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$	$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5$

Values may be found from *Equator*'s exponential polynomial routine (keyword **epoly**), which uses either equation 26:12:2 or a suitably truncated version of the relationship

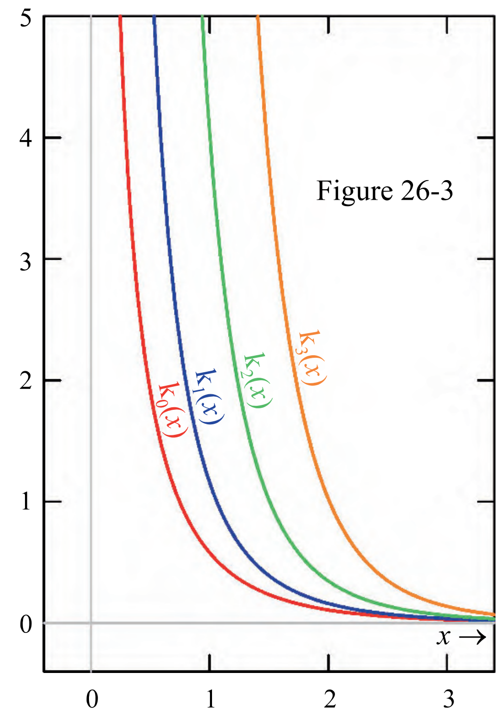
26:12:3 $$\mathrm{e}_n(x)=\exp(x)-\sum_{j=n+1}^{\infty}\frac{x^j}{j!}$$

Table 18-7 reveals a hypergeometric route to the exponential polynomial. For large n, the asymptotic approximation

26:12:4 $$\mathrm{e}_n(x)\sim\exp(x)-\frac{(n+2)x^{n+1}}{(n+1)!(n+2-x)}\qquad\text{large } n$$

is valid.

26:13 COGNATE FUNCTIONS



All the functions mentioned in Tables 18-3 and 18-4, as well as many others, can be synthesized from the exponential function. Thus any function with either of the hypergeometric representations shown may be synthesized as follows:

26:13:1 $$\exp(\pm x) \xrightarrow[c_1]{1} \sum_{j=0}^{\infty} \frac{(\pm x)^j}{(c_1)_j} \xrightarrow[c_2]{a} \sum_{j=0}^{\infty} \frac{(a)_j}{(c_1)_j (c_2)_j} (\pm x)^j$$

where the symbolism is explained in Section 43:14.

Those Macdonald functions [Chapter 51] $K_\nu(x)$ of orders that are multiples of half an odd positive integer are related in a very simple way to $\exp(-x)$. The name *spherical Macdonald function* and the symbol $k_n(x)$ are given to the function

26:13:2 $$k_n(x) = \sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x) \qquad n = 0, 1, 2, \cdots$$

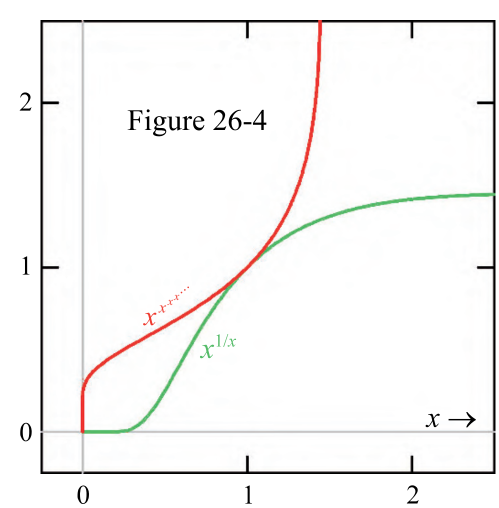
The first few members are mapped in Figure 26-3 and tabulated below and more can be found from the recursion

26:13:3 $$k_n(x) = \frac{2n-1}{x} k_{n-1}(x) + k_{n-2}(x)$$

$k_0(x)$	$k_1(x)$	$k_2(x)$	$k_3(x)$	$k_4(x)$
$\frac{\pi\exp(-x)}{2x}$	$\frac{\pi\exp(-x)}{2x}\left[1+\frac{1}{x}\right]$	$\frac{\pi\exp(-x)}{2x}\left[1+\frac{3}{x}+\frac{3}{x^2}\right]$	$\frac{\pi\exp(-x)}{2x}\left[1+\frac{6}{x}+\frac{15}{x^2}+\frac{15}{x^3}\right]$	$\frac{\pi\exp(-x)}{2x}\left[1+\frac{10}{x}+\frac{45}{x^2}+\frac{105}{x^3}+\frac{105}{x^4}\right]$

The properties of these functions follow from those of the Macdonald function. Their numerical values are calculable by *Equator*'s spherical Macdonald function routine (keyword **k**), which relies on equation 26:13:3.

26:14 RELATED TOPIC: the self-exponential function



An interesting function is the *self-exponential function* x^x. It is defined as a real function only for positive x and takes the value unity at $x = 0$, as its graph in Figure 26-1 illustrates. From its derivative

26:14:1 $$\frac{d}{dx} x^x = x^x [1 + \ln(x)]$$

one learns that its slope is $-\infty$ at $x = 0$ and that x^x has a minimum value of $\exp(-1/e)$ at $x = 1/e$. Its value and its derivative are both unity at $x = 1$ and it increases extremely rapidly at large x, being well approximated by the formula

26:14:2 $$x^x \approx \sqrt{\frac{x}{2\pi}} \Gamma(x) \exp(x) \qquad x \text{ large}$$

A related function, $x^{1/x}$ is mapped in green in Figure 26-4. It equals its argument at $x = 0$ and 1 and passes through a gentle maximum of magnitude $\exp(1/e)$ at $x = e$ before decaying very slowly towards unity. The inverse function [Section 0:3] of $x^{1/x}$, mapped in red in Figure 26-4, is the limit of the sequence

26:14:3 $$x,\ x^x,\ x^{x^x},\ x^{x^{x^x}},\cdots \qquad 0 < x < e^{1/e}$$

In implementing this sequence, a new base is added at each step. For the $x = 0.5$ example, one first calculates $0.5^{0.5} = 0.70710\cdots$, then $0.5^{0.70710\cdots} = 0.61254\cdots$, then $0.5^{0.61254\cdots} = 0.65404\cdots$, and so on until $0.5^{0.64119\cdots} = 0.64119\cdots$ is reached. This example resolves the ambiguity that otherwise exists in 26:14:3.

26:15 RELATED TOPIC: Laplace transformation

A foundation laid by French mathematicians Pierre Simon de Laplace (1749 – 1827) and Augustin Louis Cauchy (1759 – 1857) has been built upon, mostly by engineers, to construct one of the most powerful tools for solving problems of practical importance. The procedure is especially useful in solving systems of coupled ordinary differential equations and in embedding boundary conditions into the solutions of partial differential equations. Two distinct processes, *Laplace transformation* and *inverse Laplace transformation* – that we shall often abbreviate to "transformation" and "inversion" in what follows – are involved in all applications, the motivation for which is evident from the adjacent scheme. The phrase "in *Laplace space*" is used to describe manipulations involving the s variable.

$$\begin{tikzcd}
\boxed{\text{The original problem involving the } t \text{ variable}} \arrow[r, "\text{Laplace}", "\text{transformation}"'] \arrow[d, "\text{difficult}"] & \boxed{\text{Transformed problem involving the } s \text{ variable}} \arrow[d, "\text{facile}"] \\
\boxed{\text{The final solution involving the } t \text{ variable}} & \boxed{\text{Solution to transformed problem in the } s \text{ variable}} \arrow[l, "\text{Laplace}"', "\text{inversion}"]
\end{tikzcd}$$

If $\mathrm{f}(t)$ and $\mathrm{g}(t)$ represent two functions of the variable t, then we shall use the notations $\overline{\mathrm{f}}(s)$ and $\overline{\mathrm{g}}(s)$ to denote their Laplace transforms. Moreover, we adopt the operator symbols $\mathscr{L}$ and $\mathscr{I}$ to signify the operations of transformation and inversion; thus:

26:15:1 $$\mathscr{L}\{\mathrm{f}(t)\} = \overline{\mathrm{f}}(s) \qquad \text{and} \qquad \mathscr{I}\{\overline{\mathrm{f}}(s)\} = \mathrm{f}(t)$$

The s variable goes by the name of *transform variable*. There is a reciprocity between t and s in the sense that if, as is often the case in applications, t represents a time (with "second" as its unit) then s resembles a frequency (in reciprocal seconds or "hertz").

Not every function is transformable or invertible. For $\mathrm{f}(t)$ to have a Laplace transform, it must be defined at all points on the semiinfinite range $0 \le t < \infty$. To make them submissive to transformation, functions are often "windowed" [Section 9:13], that is, they are artificially set to zero outside a range in which they have natural variability. There is no requirement that $\mathrm{f}(t)$ be finite throughout $0 \le t < \infty$, but it must not be "too infinite"; thus $\mathrm{f}(0)$ may be infinite provided that $t\mathrm{f}(t) \to 0$ as $0 \leftarrow t$. The one salient requirement for $\overline{\mathrm{f}}(s)$ to be invertible is that $\overline{\mathrm{f}}(s) \to 0$ as $s \to \infty$. That a function be Laplace transformable or invertible does not imply that its transform or inverse can be found in terms of named functions. This is emphatically the case for inversion. When an attempted application fails to yield to the Laplace approach, it is usually because the inversion step is insurmountable.

Mathematically, the operations of Laplace transformation and inversion are defined by the integral transforms

26:15:2 $$\mathscr{L}\{\mathrm{f}(t)\} = \int_0^\infty \mathrm{f}(t)\exp(-st)\,\mathrm{d}t \qquad \text{and} \qquad \mathscr{I}\{\overline{\mathrm{f}}(s)\} = \int_{\alpha-i\infty}^{\alpha+i\infty} \overline{\mathrm{f}}(s)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s$$

the latter being known as the *Bromwich integral* (Thomas John l'Anson Bromwich, English mathematician, 1875 – 1929). The α in the Bromwich integral is arbitrary, provided that the integration path in the complex plane is chosen to the right of any singularity in the integrand. In practice, these formulas are seldom used to find transforms or inverse transforms, tables or computer searches being used instead. The book by Roberts and Kaufman and that by Oberhettinger and Badii each has thousands of entries systematically listed so as to facilitate the location of both transforms and inverse transforms. In this *Atlas*, transforms will be found towards the end of Section 10 of each chapter. To locate $\mathfrak{I}\{\bar{\mathrm{f}}(s)\}$ look in Section 11 of the chapter dealing with the function f. We have placed these latter entries in the sections dealing with complex argument, because the Bromwich formula requires integration in the complex plane. However, the complex nature of the s variable is seldom of practical concern.

The transformation and inversion operations are linear and homogeneous, which means that

26:15:3 $$\mathfrak{L}\{w_1\,\mathrm{f}(t)+w_2\,\mathrm{g}(t)\}=w_1\,\bar{\mathrm{f}}(s)+w_2\,\bar{\mathrm{g}}(s) \qquad \text{and} \qquad \mathfrak{I}\{w_1\,\bar{\mathrm{f}}(s)+w_2\,\bar{\mathrm{g}}(s)\}=w_1\,\mathrm{f}(t)+w_2\,\mathrm{g}(t)$$

The *scaling properties*

26:15:4 $$\mathfrak{L}\{\mathrm{f}(bt)\}=\frac{1}{b}\bar{\mathrm{f}}\left(\frac{s}{b}\right) \qquad \text{and} \qquad \mathfrak{I}\{\bar{\mathrm{f}}(bs)\}=\frac{1}{b}\mathrm{f}\left(\frac{t}{b}\right)$$

generalize to

26:15:5 $$\mathfrak{L}\{\mathrm{f}(bt+c)\}=\frac{1}{b}\exp\left(\frac{cs}{b}\right)\left[\bar{\mathrm{f}}\left(\frac{s}{b}\right)-\int_0^c \mathrm{f}(t)\exp\left(\frac{-st}{b}\right)\mathrm{d}t\right] \qquad \text{and} \qquad \mathfrak{I}\{\bar{\mathrm{f}}(bs+c)\}=\frac{1}{b}\exp\left(\frac{-ct}{b}\right)\mathrm{f}\left(\frac{t}{b}\right)$$

A frequent need is to transform the product $\mathrm{g}(t)\mathrm{f}(t)$, or to invert the product $\bar{\mathrm{g}}(s)\bar{\mathrm{f}}(s)$, of two functions. If the g function is t itself, or an integer power thereof, the following formulas apply

26:15:6 $$\mathfrak{L}\{t\,\mathrm{f}(t)\}=-\frac{\mathrm{d}\bar{\mathrm{f}}}{\mathrm{d}s}(s) \qquad \text{and} \qquad \mathfrak{I}\{s\,\bar{\mathrm{f}}(s)\}=\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}t}(t)+\mathrm{f}(0)\delta(t)$$

26:15:7 $$\mathfrak{L}\{t^n\,\mathrm{f}(t)\}=(-)^n\frac{\mathrm{d}^n\,\bar{\mathrm{f}}}{\mathrm{d}s^n}(s) \qquad \text{and} \qquad \mathfrak{I}\{s^n\,\bar{\mathrm{f}}(s)\}=\frac{\mathrm{d}^n\,\mathrm{f}}{\mathrm{d}t^n}(t)+\sum_{j=0}^{n-1}\frac{\mathrm{d}^j\,\mathrm{f}}{\mathrm{d}t^j}(0)\frac{\mathrm{d}^j\,\delta}{\mathrm{d}t^j}(t)$$

26:15:8 $$\mathfrak{L}\left\{\frac{\mathrm{f}(t)}{t}\right\}=\int_x^\infty \bar{\mathrm{f}}(s)\,\mathrm{d}s \qquad \text{and} \qquad \mathfrak{I}\left\{\frac{\bar{\mathrm{f}}(s)}{s}\right\}=\int_0^t \mathrm{f}(t)\,\mathrm{d}t$$

26:15:9 $$\mathfrak{L}\left\{\frac{\mathrm{f}(t)}{t^n}\right\}=\int_x^\infty\cdots\int_x^\infty \bar{\mathrm{f}}(s)[\mathrm{d}s]^n \qquad \text{and} \qquad \mathfrak{I}\left\{\frac{\bar{\mathrm{f}}(s)}{s^n}\right\}=\int_0^t\cdots\int_0^t \mathrm{f}(t)[\mathrm{d}t]^n$$

The quantity $(\mathrm{d}^j\delta/\mathrm{d}t^j)(t)$ in 26:15:7 is the jth derivative of the Dirac function [Section 9:12] at $t=0$. The terms in which these latter functions occur have influence only at $t=0$ and may frequently be ignored entirely. When the multiplicative g function is an exponential function, one has

26:15:10 $$\mathfrak{L}\{\exp(bt)\,\mathrm{f}(t)\}=\bar{\mathrm{f}}(s-b) \qquad \text{and, for } b>0 \qquad \mathfrak{I}\{\exp(-bs)\,\bar{\mathrm{f}}(s)\}=\mathrm{u}(t-b)\,\mathrm{f}(t-b)$$

The second of these formulas, in which u( ) is the Heaviside function [Chapter 9] illustrates what is sometimes known as the *delay property* of Laplace inversion. Figure 26-5 exemplifies the relationship of $\mathrm{u}(t-b)\mathrm{f}(t-b)$ to $\mathrm{f}(t)$ itself. Formulas 26:15:10 can be developed to provide a method of transforming the product $\mathrm{g}(t)\mathrm{f}(t)$ or inverting the product $\bar{\mathrm{g}}(s)\bar{\mathrm{f}}(s)$ when g( ) or $\bar{\mathrm{g}}(\;)$ is any function, such as sinh, cos, or coth, that can be expressed as a weighted sum (finite or infinite) of exponentials.

Though it lacks a transformation analog, there exists a formula for the inversion of the product $\bar{\mathrm{g}}(s)\bar{\mathrm{f}}(s)$ of *any* pair of functions. This is the *convolution property* of Laplace inversion:

26:15:11 $$\mathcal{I}\left\{\overline{\mathrm{g}}(s)\,\overline{\mathrm{f}}(s)\right\} = \int_0^t \mathrm{g}(t-t')\,\mathrm{f}(t')\,\mathrm{d}t' = \int_0^t \mathrm{g}(t')\,\mathrm{f}(t-t')\,\mathrm{d}t'$$

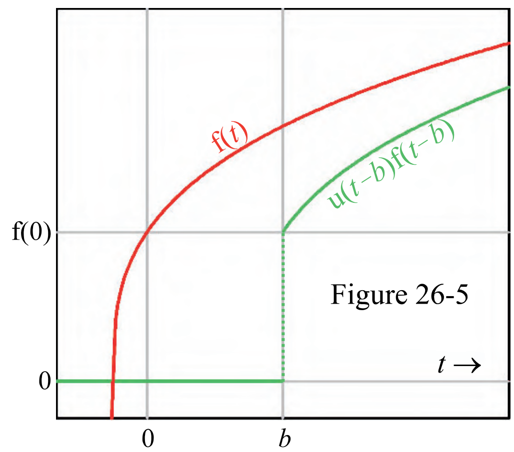


Figure 26-5

Though an integration remains to be carried out, this is a valuable means of leaving Laplace space. Each of the integrals in 26:15:11 is known as a *convolution* of the g(t) and f(t) functions and is symbolized g(t)✻f(t). Each function is said to be "convolved" with the other.

Equations in 26:15:6 − 10 may be employed "the other way around", to find the Laplace transform or inverse of a function subjected to an operation of the calculus. For example, transformation or inversion of an indefinite integral yields

26:15:12 $$\mathcal{L}\left\{\int_0^t \mathrm{f}(t)\,\mathrm{d}t\right\} = \frac{\overline{\mathrm{f}}(s)}{s} \qquad \text{and} \qquad \mathcal{I}\left\{\int_s^\infty \overline{\mathrm{f}}(s)\,\mathrm{d}s\right\} = \frac{\mathrm{f}(t)}{t}$$

In fact, these formulas may be generalized as follows

26:15:13 $$\mathcal{L}\left\{\frac{\mathrm{d}^\nu \mathrm{f}}{\mathrm{d}t^\nu}(t)\,\mathrm{d}t\right\} = s^\nu\,\overline{\mathrm{f}}(s) - \sum_{j=1}^{-\mathrm{Int}(-\nu)} s^{j-1}\frac{\mathrm{d}^{\nu-j}\,\overline{\mathrm{f}}}{\mathrm{d}s^{\nu-j}}(0) \qquad \text{and} \qquad \mathcal{I}\left\{\frac{\mathrm{d}^\nu\,\overline{\mathrm{f}}}{\mathrm{d}(-s)^\nu}(s)\bigg|_{-\infty}\right\} = t^\nu\,\mathrm{f}(t)$$

to any order of differintegration [Section 12:14]. The summation is empty, and therefore equal to zero, if $\nu \le 0$.

The technique of partial fractionation [Section 16:13] is frequently used in Laplace inversion and, in this context, equations 17:13:14 − 17 are useful. A powerful inversion tool, known as the *Heaviside expansion formula*, is a development of the "automated" partial fractionation discussed in Section 17:13. If $\overline{\mathrm{f}}(s)$ is the ratio of two polynomial functions $\mathrm{p}_m(s)/\mathrm{p}_n(s)$ where the degree m of the numeratorial polynomial is less than that, n, of the denominatorial polynomial, then

26:15:14 $$\mathrm{f}(t) = \mathcal{I}\left\{\frac{\mathrm{p}_m(s)}{\mathrm{p}_n(s)}\right\} = \sum_{j=1}^{n}\frac{\mathrm{p}_m(r_j)\exp(r_j t)}{\dfrac{\mathrm{d}\,\mathrm{p}_n}{\mathrm{d}s}(r_j)}$$

where $r_1, r_2, \cdots, r_j, \cdots, r_n$ are the distinct zeros of the denominatorial polynomial. This rule can be applied if m is as small as 0 or 1, and n is as large as ∞. If some of the zeros are not distinct, then use

26:15:15 $$\mathcal{I}\left\{\frac{\mathrm{p}_m(s)}{(s-r)^n}\right\} = t^{n-1}\exp(rt)\sum_{j=0}^{n-1}\frac{t^{-j}}{(n-j-1)!\,j!}\frac{\mathrm{d}^j\,\mathrm{p}_m}{\mathrm{d}s^j}(r)$$

CHAPTER 27

EXPONENTIALS OF POWERS $\exp(\pm x^{\nu})$

Though the functions in this chapter are just composites of those discussed in Chapters 10, 12, and 26, the practical importance of exponential functions of powers of the argument x makes it appropriate to devote a chapter to them. For example, the temperature-dependence of many physical properties obeys an $\exp\{-(\text{constant})/T\}$ law, while random events, a topic discussed in Section 27:14, frequently involve the function $\exp(-x^2)$. Because of their applications, this chapter concentrates on cases of $\exp(\pm x^{\nu})$ in which ν is an integer or half-integer.

27:1 NOTATION

No special notation is needed beyond that introduced in Sections 10:1, 12:1, and 26:1. Note that $x^{-\nu}$ is interchangeable with $1/x^{\nu}$ and that $\exp\{\mathrm{f}(x)\}$ is frequently represented as $e^{\mathrm{f}(x)}$.

27:2 BEHAVIOR

As with the functions discussed in Chapter 12, the range of $\exp(\pm x^{\nu})$ depends in a detailed way on the nature of the number ν when ν is not an integer. The complications mostly disappear when x is positive, and this will be assumed throughout this chapter, except when ν is an integer or half-integer.

Figures 27-1 and 27-2 show maps of the functions $\exp(\pm 1/x)$, $\exp\left(\pm 1/\sqrt{x}\right)$, $\exp\left(\pm\sqrt{x}\right)$, and $\exp(\pm x^2)$. The important graph of $\exp(-x^2)$, discussed further in Section 27:14, is known as a *Gaussian* or *Gauss curve*; it has a maximum value of unity at $x = 0$ and points of inflection [Section 0:7] at $x = \pm 1/\sqrt{2}$. The $\exp(-1/x)$ function inflects at $x = ½$.

27:3 DEFINITIONS

With x replaced by $\pm x^{\nu}$, the definitions 26:3:1 to 26:3:4 can serve to define $\exp(\pm x^{\nu})$.

Provided $p \neq -1$, a first-order differential equation and its solution are

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_28,

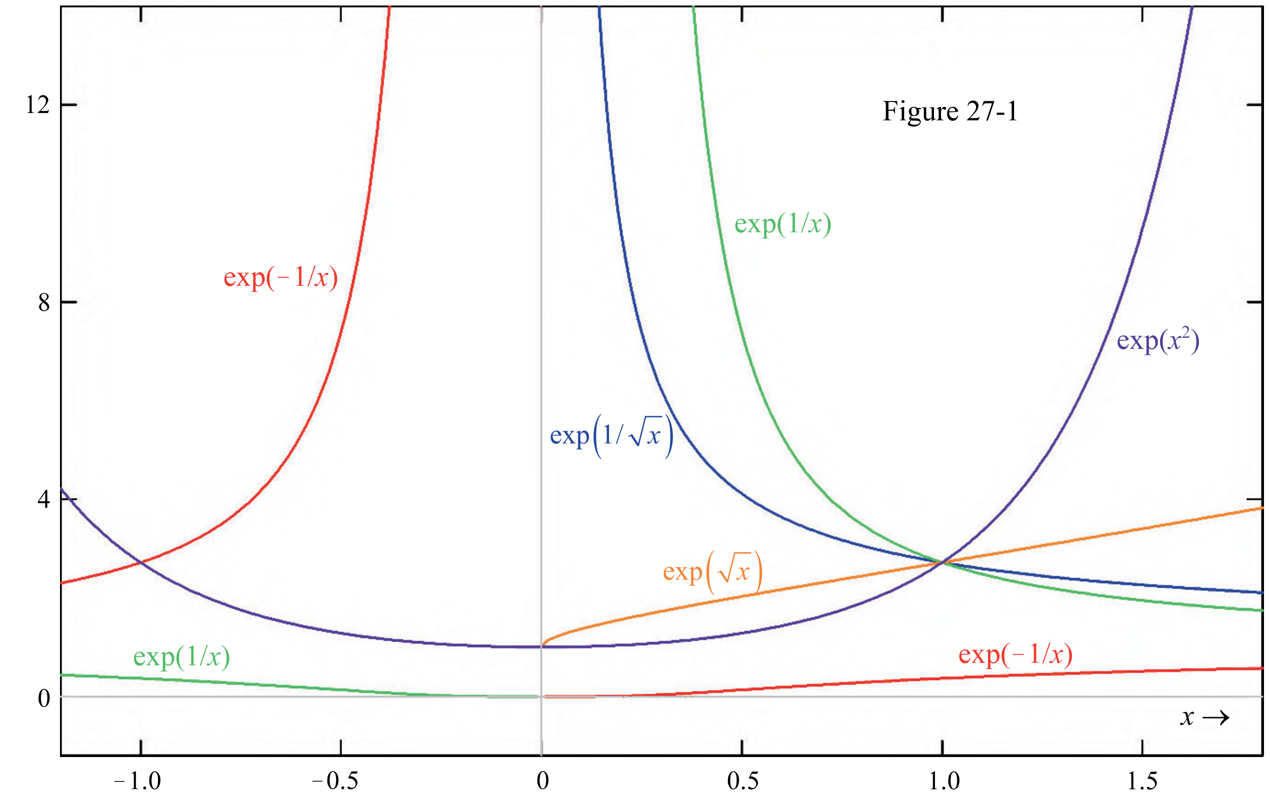


Figure 27-1
exp(−1/x)
exp(1/x)
exp(x²)
exp(1/√x)
exp(√x)
exp(1/x)
exp(−1/x)
x →
12
8
4
0
−1.0
−0.5
0
0.5
1.0
1.5

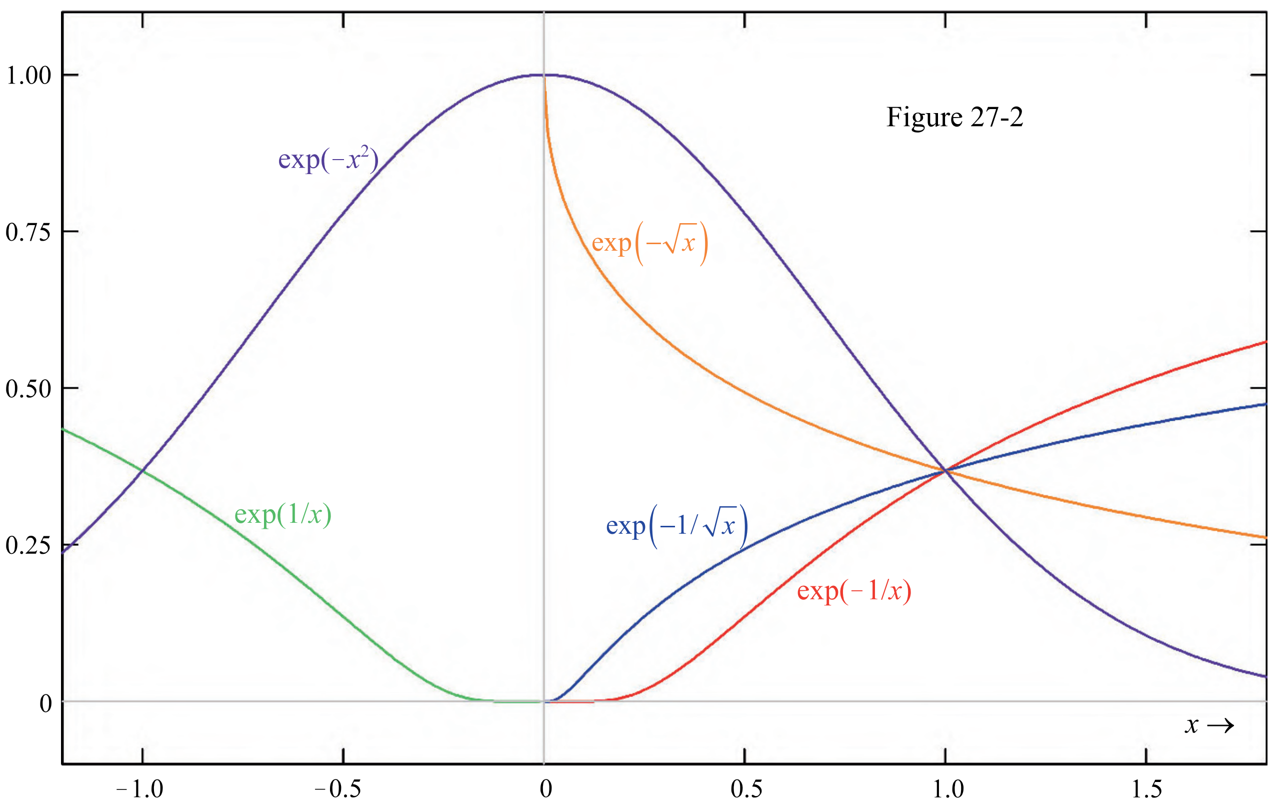


Figure 27-2
exp(−x²)
exp(−√x)
exp(1/x)
exp(−1/√x)
exp(−1/x)
x →
1.00
0.75
0.50
0.25
0
−1.0
−0.5
0
0.5
1.0
1.5

27:3:1 $$\frac{\mathrm{d}f}{\mathrm{d}x} = bx^\nu f \qquad f = w\exp\left(\frac{bx^{1+\nu}}{1+\nu}\right)$$

where w is arbitrary.

The *Laplace-de Moivre formula* 6:9:3 can be used to define the Gaussian function as a limit involving a binomial coefficient

27:3:2 $$\exp\left(-x^2\right) = \lim_{\nu\to\infty}\left\{\frac{\sqrt{\pi\nu}}{4^\nu}\binom{2\nu}{\nu + x\sqrt{\nu}}\right\}$$

27:4 SPECIAL CASES

When $\nu = 1$, the functions $\exp(\pm x^\nu)$ reduce to those of Chapter 26. When $\nu = 0$, the functions become constant.

27:5 INTRARELATIONSHIPS

If care is taken to respect the attendant restrictions, the x and y arguments in equations 26:5:1−12 may be replaced by powers of x and y.

There exist interesting relationships between sums of exponentials of $-\pi/x$ and similar sums of exponentials of $-\pi x$. For example, if x is positive

27:5:1 $$\frac{1}{2} + \exp\left(\frac{-\pi}{x}\right) + \exp\left(\frac{-4\pi}{x}\right) + \exp\left(\frac{-9\pi}{x}\right) + \cdots = \sqrt{x}\left[\frac{1}{2} + \exp(-\pi x) + \exp(-4\pi x) + \exp(-9\pi x) + \cdots\right]$$

This identity arises from the properties of the exponential theta-three function [Section 27:13] of zero parameter. This and similar relations may be derived from equations 27:13:17−19.

27:6 EXPANSIONS

The series expansion

27:6:1 $$\exp\left(\pm x^\nu\right) = 1 \pm \frac{x^\nu}{1!} + \frac{x^{2\nu}}{2!} \pm \frac{x^{3\nu}}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{(\pm 1)^j}{j!} x^{j\nu}$$

holds whenever x^ν is well defined.

The continued fraction expansion

27:6:2 $$\exp\left(\sqrt{x}\right) = 1 + \cfrac{2\sqrt{x}}{2-\sqrt{x}+}\;\frac{x/3}{2+}\;\frac{x/15}{2+}\;\frac{x/35}{2+}\cdots\frac{x/(4j^2-1)}{2+\;\cdots}$$

is rapidly convergent.

27:7 PARTICULAR VALUES

		$x=0$	$x=1$	$x=\infty$
$\exp(x^\nu)$	$\nu<0$	∞	e	1
	$\nu>0$	1	e	∞
$\exp(-x^\nu)$	$\nu<0$	0	$1/e$	1
	$\nu>0$	1	$1/e$	0

27:8 NUMERICAL VALUES

By using the variable construction feature [Appendix C:4], *Equator*'s exponential function routine (keyword **exp**) can provide numerical values of the exponentials of powers.

27:9 LIMITS AND APPROXIMATIONS

Though crude, the approximation

27:9:1 $$\exp(-x^2) \approx \begin{cases} 1-\dfrac{|x|}{\sqrt{\pi}} & -\sqrt{\pi} \le x \le \sqrt{\pi} \\ 0 & |x| \ge \sqrt{\pi} \end{cases}$$

is surprisingly good. Moreover, the areas under the Gauss curve and this triangular approximant are identical.

27:10 OPERATIONS OF THE CALCULUS

Differentiation gives

27:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\alpha x^\nu) = -\alpha\nu x^{\nu-1}\exp(-\alpha x^\nu)$$

Indefinite integration of the general expressions $\exp(-\alpha t^\nu)$ and $t^w\exp(-\alpha t^\nu)$ can be accomplished by making the substitution $-\alpha t^\nu = \pm y$ and utilizing the general formulas contained in equations 26:10:13−16. Moreover, the table in Section 37:14 can be useful in this regard. Below are listed some important indefinite integrals:

27:10:2 $$\int_0^x \exp(t^2)\,\mathrm{d}t = \exp(x^2)\,\mathrm{daw}(x)$$

27:10:3 $$\int_1^x \exp(\sqrt{t})\,\mathrm{d}t = 2(\sqrt{x}-1)\exp(\sqrt{x}) \qquad x>0$$

27:10:4 $$\int_{x_0}^x \exp\left(\frac{1}{\sqrt{t}}\right)\mathrm{d}t = (x+\sqrt{x})\exp\left(\frac{1}{\sqrt{x}}\right) - \mathrm{Ei}\left(\frac{1}{\sqrt{x}}\right) - (x_0+\sqrt{x_0})\exp\left(\frac{1}{\sqrt{x_0}}\right) \qquad x>0 \quad x_0 = 7.2065\,95823\,66350$$

27:10:5 $$\int_{x_0}^{x}\exp\left(\frac{1}{t}\right)\mathrm{d}t = x\exp\left(\frac{1}{x}\right) - \mathrm{Ei}\left(\frac{1}{x}\right) - x_0\exp\left(\frac{1}{x_0}\right) \qquad x>0 \quad x_0 = 2.6845\,10350\,82071$$

27:10:6 $$\int_0^x \exp\left(-t^2\right)\mathrm{d}t = \frac{\sqrt{\pi}}{2}\mathrm{erf}\left(x\right)$$

27:10:7 $$\int_x^\infty \exp\left(-\sqrt{t}\right)\mathrm{d}t = 2\left(\sqrt{x}+1\right)\exp\left(-\sqrt{x}\right) \qquad x>0$$

27:10:8 $$\int_0^x \exp\left(\frac{-1}{\sqrt{t}}\right)\mathrm{d}t = \left(x-\sqrt{x}\right)\exp\left(\frac{-1}{\sqrt{x}}\right) - \mathrm{Ei}\left(\frac{-1}{\sqrt{x}}\right) \qquad x>0$$

27:10:9 $$\int_0^x \exp\left(\frac{-1}{t}\right)\mathrm{d}t = x\exp\left(\frac{-1}{x}\right) + \mathrm{Ei}\left(\frac{-1}{x}\right) \qquad x>0$$

27:10:10 $$\int_x^\infty \exp\left(-at^2-bt-c\right)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-4ac}{4a}\right)\mathrm{erfc}\left(\frac{2ax+b}{2\sqrt{a}}\right) \qquad a>0$$

Chapters 42, 37, and 40 are devoted to the daw, Ei, and erfc functions, respectively.

Some important definite integrals include

27:10:11 $$\int_0^\infty t^\omega \exp\left(-\alpha t^v\right)\mathrm{d}t = \frac{\Gamma\left(\frac{\omega+1}{v}\right)}{v\alpha^{(\omega+1)/v}} \qquad \frac{\omega+1}{v}>0 \qquad \alpha>0$$

and

27:10:12 $$\int_0^\infty \exp\left(-\alpha t^2 - \frac{\beta}{t^2}\right)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\exp\left(-2\sqrt{\alpha\beta}\right) \qquad \alpha>0<\beta$$

where Γ is the gamma function [Chapter 43].

The Laplace transform

27:10:13 $$\int_0^\infty t^v \exp\left(-\frac{\alpha}{t}-st\right)\mathrm{d}t = \mathfrak{L}\left\{t^v\exp\left(\frac{-\alpha}{t}\right)\right\} = 2\left(\frac{\alpha}{s}\right)^{(v+1)/2}\mathrm{K}_{v+1}\left(2\sqrt{\alpha s}\right)$$

in which K is the Macdonald function [Chapter 51], adopts the simpler form $\sqrt{\pi/s}\exp\left(-2\sqrt{\alpha s}\right)$ when $v = -½$. Similarly, the general result

27:10:14 $$\int_0^\infty t^v \exp\left(-\alpha\sqrt{t}-st\right)\mathrm{d}t = \mathfrak{L}\left\{t^v\exp\left(-\alpha\sqrt{t}\right)\right\} = \frac{\Gamma(2v+2)}{2^v s^{v+1}}\exp\left(\frac{\alpha^2}{8s}\right)\mathrm{D}_{-2v-2}\left(\frac{\alpha}{\sqrt{2s}}\right)$$

in which D is the parabolic cylinder function [Chapter 48], simplifies to $\sqrt{\pi/s}\exp\left(\alpha^2/4s\right)\mathrm{erfc}\left(\alpha/2\sqrt{s}\right)$ when $v=-½$. The α multiplier must be positive and the power v must exceed -1 in all these transforms, and also in

27:10:15 $$\int_0^\infty t^v \exp\left(-\alpha t^2-st\right)\mathrm{d}t = \mathfrak{L}\left\{t^v\exp\left(-\alpha t^2\right)\right\} = \frac{\Gamma(v+1)}{(2\alpha)^{(v+1)/2}}\exp\left(\frac{s^2}{8\alpha}\right)\mathrm{D}_{-v-1}\left(\frac{s}{\sqrt{2\alpha}}\right)$$

This last transform reduces to $\sqrt{\pi/4\alpha}\exp\left(s^2/4\alpha\right)\mathrm{erfc}\left(s/\sqrt{4\alpha}\right)$ for the Gaussian case in which $v=0$. The function erfc is the error function complement [Chapter 40].

27:11 COMPLEX ARGUMENT

The real and imaginary parts of the Gaussian function of complex argument $z = x + iy$ are contained in the formula

27:11:1
$$\exp\left(-z^2\right) = \exp\left(y^2 - x^2\right)\left[\cos(2xy) - i\sin(2xy)\right]$$

The corresponding formula for the exponential of a reciprocal complex variable is

27:11:2
$$\exp\left(\frac{1}{z}\right) = \exp\left(\frac{x}{x^2+y^2}\right)\left[\cos\left(\frac{y}{x^2+y^2}\right) - i\sin\left(\frac{y}{x^2+y^2}\right)\right] \qquad y \neq \pm x$$

For $\nu > 0$, some inverse Laplace transforms involving exponentials of powers are:

27:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty} s^\nu \exp\left(-\alpha\sqrt{s} + ts\right)\frac{ds}{2\pi i} = \mathfrak{I}\left\{s^\nu \exp\left(-\alpha\sqrt{s}\right)\right\} = \sqrt{\frac{2}{\pi}}\frac{\exp\left(-\alpha^2/8t\right)}{(2t)^{\nu+1}}\mathrm{D}_{2\nu+1}\left(\frac{\alpha}{\sqrt{2t}}\right)$$

27:11:4
$$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{-\nu} \exp\left(\frac{\alpha}{s} + ts\right)\frac{ds}{2\pi i} = \mathfrak{I}\left\{\frac{\exp(\alpha/s)}{s^\nu}\right\} = (t/\alpha)^{(\nu-1)/2}\,\mathrm{I}_{\nu-1}\left(2\sqrt{\alpha t}\right) \qquad \alpha > 0$$

where D is a parabolic cylinder function [Chapter 48] and I is a modified Bessel function [Chapter 50]. The inverse transform 27:11:3 simplifies to $\left\{\alpha/\left(2\pi^{3/2}\right)\right\}\exp\left(-\alpha^2/4t\right)$ when $\nu = 0$.

27:12 GENERALIZATIONS

The exponential function of powers may be generalized "to other bases" in a similar way to that described in Section 26:12.

With $\nu > 2$, the functions

27:12:1
$$\mathrm{f}(x) = \frac{\nu\alpha^{1/\nu}}{2\Gamma(1/\nu)}\exp\left(-\alpha|x|^\nu\right) \qquad \alpha > 0$$

have been called *supergauss functions*. They share with the Gaussian function $\left(1/\sqrt{\pi\alpha}\right)\exp\left(-\alpha x^2\right)$ the properties of being symmetrically peaked at $x = 0$ and enclosing an area of unity. In 27:12:1, Γ is the gamma function [Chapter 43]. Though the moniker "supergauss" no longer applies, the $\nu = 1$ and $\nu = 2$ cases of function 27:12:1 are respectively the Laplace and normal probability distribution functions [Section 27:14] of zero mean. With ν an adjustable positive parameter, function 27:12:1 is employed in statistics as the probability function of the *Weibull distribution* (Waloddi Weibull, Swedish engineer, 1887 – 1979).

27:13 COGNATE FUNCTIONS: exponential theta functions

The bivariate *exponential theta-one*, *theta-two*, *theta-three* and *theta-four functions* are defined by

27:13:1
$$\theta_1(\nu,x) = \frac{1}{\sqrt{\pi x}}\sum_{j=-\infty}^{+\infty}(-)^j\exp\left(\frac{-(\nu-\frac{1}{2}+j)^2}{x}\right)$$

27:13:2
$$\theta_2(\nu,x) = \frac{1}{\sqrt{\pi x}}\sum_{j=-\infty}^{+\infty}(-)^j\exp\left(\frac{-(\nu+j)^2}{x}\right) = \theta_1\left(\nu+\tfrac{1}{2},x\right)$$

27:13:3 $$\theta_3(v,x)=\frac{1}{\sqrt{\pi x}}\sum_{j=-\infty}^{+\infty}\exp\left(\frac{-(v+j)^2}{x}\right)$$

and

27:13:4 $$\theta_4(v,x)=\frac{1}{\sqrt{\pi x}}\sum_{j=-\infty}^{+\infty}\exp\left(\frac{-(v+\frac{1}{2}+j)^2}{x}\right)=\theta_3\left(v+\tfrac{1}{2},x\right)$$

but the reader should be alert to the wide variety of theta functions in use, and the confused symbolism. We refer to v and x as the *periodic variable* and *aperiodic variable*, respectively. The subscript n in $\theta_n(v,x)$ has no numerical significance; it merely distinguishes one theta function from the others. Summations which, like those in 27:13:1−4, run from $-\infty$ to $+\infty$ are called *Laurent series* (Pierre Alphonse Laurent, French mathematician, 1813−1854). The annotated summation symbol implies that j takes *all* integer values: negative, zero, and positive. Thus an alternative representation of each exponential theta function is

27:13:5 $$\sqrt{\pi x}\,\theta_n(v,x)=(0\text{th summand})+\sum_{j=1}^{\infty}(j\text{th summand})+\sum_{j=-\infty}^{-1}(j\text{th summand})\qquad n=1,2,3,4$$

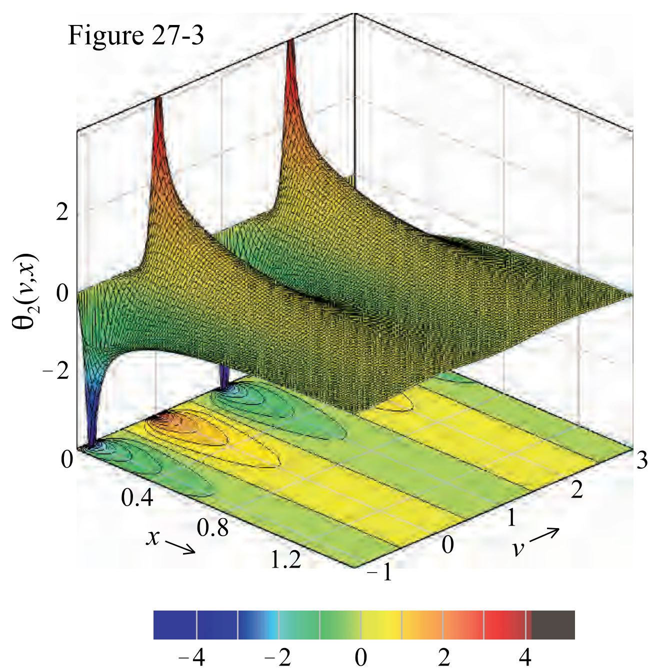
There exists another quartet of exponential theta functions, known as *modified exponential theta functions*, and distinguished by having the theta symbol “hatted”. The summands in the definitions of these modified functions are exactly like those in 27:13:5, but there is a sign change when j is negative:

27:13:6 $$\sqrt{\pi x}\,\hat{\theta}_n(v,x)=(0\text{th summand})+\sum_{j=1}^{\infty}(j\text{th summand})-\sum_{j=-\infty}^{-1}(j\text{th summand})$$

for n = 1,2,3, and 4. We will not discuss modified exponential theta functions elsewhere in this section. They arise frequently in inverting Laplace transforms; for example, they are encountered in Section 28:11.

It is clear from definitions 27:13:1−4 that incrementing v by unity causes no change whatsoever in the theta-three and theta-four functions, and merely changes the sign of the theta-one and theta-two functions. This implies that the exponential theta functions are periodic functions [Chapter 36] of the variable v, with a period of 2 for θ_1 and θ_2 and of 1 for θ_3 and θ_4. Moreover, because $\theta_2(v, x) = \theta_1(v+½, x)$, the theta-one and theta-two functions differ only in phase, with a similar kinship between θ_3 and θ_4. The theta-two case is illustrated in Figure 27-3. This periodicity is brought out more clearly in the following equivalent representations

Figure 27-3



27:13:7 $$\theta_1\left(v,x\right)=2\sum_{j=0}^{\infty}(-)^j\exp\left\{-(j+\tfrac{1}{2})^2\pi^2x\right\}\sin\left\{2(j+\tfrac{1}{2})v\pi\right\}$$

27:13:8 $$\theta_2\left(v,x\right)=2\sum_{j=0}^{\infty}\exp\left\{-(j+\tfrac{1}{2})^2\pi^2x\right\}\cos\left\{2(j+\tfrac{1}{2})v\pi\right\}$$

27:13:9
$$\theta_3(v,x) = 1 + 2\sum_{j=1}^{\infty} \exp\{-j^2\pi^2 x\}\cos\{2jv\pi\}$$

27:13:10
$$\theta_4(v,x) = 1 + 2\sum_{j=1}^{\infty} (-)^j \exp\{-j^2\pi^2 x\}\cos\{2jv\pi\}$$

of the exponential theta functions.

These theta functions satisfy the intriguing *quadruplication formulas*

27:13:11
$$2\theta_1(2v,4x) = \theta_3(v-\tfrac{1}{4},x) - \theta_4(v-\tfrac{1}{4},x)$$

27:13:12
$$2\theta_2(2v,4x) = \theta_3(v,x) - \theta_4(v,x)$$

27:13:13
$$2\theta_3(2v,4x) = \theta_3(v,x) + \theta_4(v,x)$$

27:13:14
$$2\theta_4(2v,4x) = \theta_3(v-\tfrac{1}{4},x) + \theta_4(v-\tfrac{1}{4},x)$$

that may be redrafted in several alternative ways.

The second derivative with respect to the periodic variable v of, for example, the exponential theta-three function

27:13:15
$$\frac{\partial^2}{\partial v^2}\theta_3(v,x) = -8\pi^2\sum_{j=1}^{\infty} j^2 \exp\{-j^2\pi^2 x\}\cos\{2jv\pi\}$$

is proportional to its first derivative with respect to the aperiodic variable x

27:13:16
$$\frac{\partial}{\partial x}\theta_3(v,x) = -2\pi^2\sum_{j=1}^{\infty} j^2 \exp\{-j^2\pi^2 x\}\cos\{2jv\pi\}$$

This means that a theta-three function (and the same is true of the others) can provide solutions to partial differential equations of the form

27:13:17
$$\frac{\partial^2}{\partial y^2}\mathrm{f}(y,t) = (\text{constant})\frac{\partial}{\partial t}\mathrm{f}(y,t)$$

which occur frequently in descriptions of such physical phenomena as heat transport and diffusion.

Also of importance are the special cases of the exponential theta-two, theta-three and theta-four functions in which the v is zero. These *exponential theta functions of zero periodic variable* may be formulated in two distinct ways as series of exponential functions

27:13:18
$$\frac{1}{\sqrt{\pi x}}\left[1 + 2\sum_{j=1}^{\infty}(-)^j \exp\left\{\frac{-j^2}{x}\right\}\right] = \theta_2(0,x) = 2\sum_{j=0}^{\infty}\exp\left\{\frac{-(2j+1)^2\pi^2 x}{4}\right\} \qquad x>0$$

27:13:19
$$\frac{1}{\sqrt{\pi x}}\left[1 + 2\sum_{j=1}^{\infty}\exp\left\{\frac{-j^2}{x}\right\}\right] = \theta_3(0,x) = 1 + 2\sum_{j=1}^{\infty}\exp\{-j^2\pi^2 x\} \qquad x>0$$

27:13:20
$$\frac{2}{\sqrt{\pi x}}\sum_{j=0}^{\infty}\exp\left\{\frac{-(2j+1)^2}{4x}\right\} = \theta_4(0,x) = 1 + 2\sum_{j=1}^{\infty}(-)^j\exp\{-j^2\pi^2 x\} \qquad x>0$$

Apart from a simple constant, the three theta functions of zero periodic variable coalesce when $x = 1/\pi$, leading to the particular values

27:13:21
$$\theta_2\left(0,\frac{1}{\pi}\right) = 2^{-1/4}\theta_3\left(0,\frac{1}{\pi}\right) = \theta_4\left(0,\frac{1}{\pi}\right) = \sqrt{g} = 0.91357\,91381\,56117$$

where g is Gauss's constant [Section 1:7].

Laplace transformation of exponential theta functions generates hyperbolic functions [Chapters 28–30]:

27:13:22 $$\int_0^\infty \theta_1(v,t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\theta_1(v,t)\} = \mathfrak{L}\{\theta_2(v+\tfrac{1}{2},t)\} = -\sinh\{2v\sqrt{s}\}\frac{\operatorname{sech}(\sqrt{s})}{\sqrt{s}}$$

27:13:23 $$\int_0^\infty \theta_4(v,t)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\theta_4(v,t)\} = \mathfrak{L}\{\theta_3(v-\tfrac{1}{2},t)\} = \cosh\{2v\sqrt{s}\}\frac{\operatorname{csch}(\sqrt{s})}{\sqrt{s}}$$

The following inverse Laplace transforms yield infinite sums of sine functions [Chapter 32]

27:13:24 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\theta_2(v,s)}{s}\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{\theta_2(v,s)}{s}\right\} = \mathfrak{I}\left\{\frac{\theta_1(v+\frac{1}{2},s)}{s}\right\} = \frac{1}{\pi}\sum_{j=-\infty}^{\infty}\frac{(-)^j\sin\{2(v+j)\sqrt{t}\}}{v+j}$$

27:13:25 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\theta_3(v,s)}{s}\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{\theta_3(v,s)}{s}\right\} = \mathfrak{I}\left\{\frac{\theta_4(v-\frac{1}{2},s)}{s}\right\} = \frac{1}{\pi}\sum_{j=-\infty}^{\infty}\frac{\sin\{2(v+j)\sqrt{t}\}}{v+j}$$

When the periodic variable v is zero, these sums correspond to interesting discontinuous functions. For example, the transform $(1/s)\theta_4(0,s)$ inverts to a function that takes the value +1 when $0<t<\pi^2$, $4\pi^2<t<9\pi^2$, $16\pi^2<t<25\pi^2$, and so on, but equals −1 for other positive values of t. See Roberts and Kaufman for many other instances of the Laplace transformation and inversion of exponential theta functions.

The series in 27:13:1–4 converge so rapidly if $x \le 1/\pi$, that very few terms need be summed to produce accurate values of these theta functions. Moreover, if $x > 1/\pi$, series 27:13:7–10 converge even more rapidly! Summing these series is the method used by *Equator*'s exponential theta-one function, exponential theta-two function, exponential theta-three function, and exponential theta-four function routines. Their keywords are **theta1**, **theta2**, **theta3**, and **theta4**.

Though the four theta functions we have discussed thus far – the *exponential theta functions* – are the most transparent, they are not the ones most commonly encountered. Another quartet goes by the name *Jacobi theta functions*, *elliptic theta function* or, unfortunately, just *theta functions*. They occur particularly in connection with the functions that the *Atlas* addresses in Chapter 61–63. We shall call them *elliptic theta functions*. The relationship between the exponential and elliptic theta functions

27:13:26 $$\vartheta_n\left(\pi v,\exp\left(-\pi^2 x\right)\right) = \theta_n(v,x) \qquad n=1,2,3,4$$

involves radical changes in the variables, but otherwise no change in properties. Note our use of changed typography to distinguish the two kinds of theta function. Almost invariably the aperiodic variable used in connection with elliptic theta functions is the *nome* q [Section 61:15]. Thus, with x now representing the *periodic* variable, we have

27:13:27 $$\vartheta_n(x,q) = \theta_n\left(\frac{x}{\pi},\frac{-\ln(q)}{\pi^2}\right) \qquad n=1,2,3,4$$

Because of the easy interconversion, *Equator* does not provide a routine for elliptic theta functions. However, the related *Neville's theta functions*, defined and discussed in Section 61:15, are directly accessible through *Equator*.

27:14 RELATED TOPIC: distributions

When a measurement or observation is repeated a large number N of times, it frequently happens that the values found are not identical. We say that the measured quantity x has a *distribution*. Sometimes (as in rolling dice) only a finite set of discrete values is accessible to x. Here, however, we consider only continuous cases (exemplified by the sizes of raindrops) in which possible x values are limited in proximity only by the discrimination of the measuring device.

Let the measurements be arranged in order of size along the line $x_0 \le x \le x_1$, where x_0 and x_1 are the smallest and largest values open to x. (Usually $x_1 = \infty$ and $x_0 = 0$ or $-\infty$). Then, as $N \to \infty$, the density of data along the line comes to describe a function $\mathrm{P}_{\mathrm{dist}}(x)$ characteristic of the particular distribution. Among the names by which this function goes are *probability function* (the *Atlas* choice), *distribution function*, *density function*, and *frequency function*. Each probability function has a corresponding *cumulative function* $\mathrm{F}_{\mathrm{dist}}(x)$ defined by

27:14:1 $$\mathrm{F}_{\mathrm{dist}}(x) = \int_{x_0}^{x} \mathrm{P}_{\mathrm{dist}}(x)\,\mathrm{d}x$$

Clearly $\mathrm{F}_{\mathrm{dist}}(x_0) = 0$ and, because the totality of the probability must be unity, $\mathrm{F}_{\mathrm{dist}}(x_1) = 1$. The significance of $\mathrm{F}_{\mathrm{dist}}(x)$ is that this function expresses the probability that an individual measurement will not exceed x. The many diagrams in this section display examples of probability functions and cumulative functions.

In discussing distributions, statisticians speak of *percentiles* or *percentage points*. The pth percentile is the value of x at which $\mathrm{F}_{\mathrm{dist}}(x)$ equals $p/100$. The fiftieth percentile, at which the cumulative function acquires the value ½, is also called the *median* of the distribution.

If one temporarily thinks of x as a function of P, rather than vice versa, then it follows that the average value of x, known as the *mean* μ of the distribution, is given by

27:14:2 $$\mu = \frac{\int_{x_0}^{x_1} x\mathrm{P}(x)\,\mathrm{d}x}{\int_{x_0}^{x_1} \mathrm{P}(x)\,\mathrm{d}x} = \frac{\int_{x_0}^{x_1} x\mathrm{P}(x)\,\mathrm{d}x}{\mathrm{F}(x_1)} = \int_{x_0}^{x_1} x\mathrm{P}(x)\,\mathrm{d}x$$

In addition to the mean, a distribution is also characterized by its *moments*, the jth of which is defined by the integral $\int (x-\mu)^j \mathrm{P}(x)\,\mathrm{d}x$. The second moment, symbolized σ^2, is particularly important:

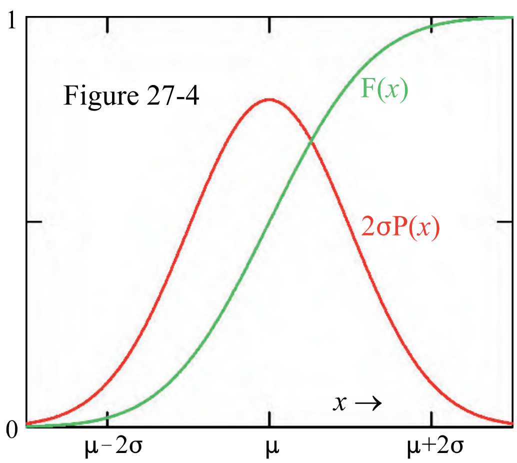
27:14:3 $$\sigma^2 = \int_{x_0}^{x_1} (x-\mu)^2 \mathrm{P}(x)\,\mathrm{d}x$$

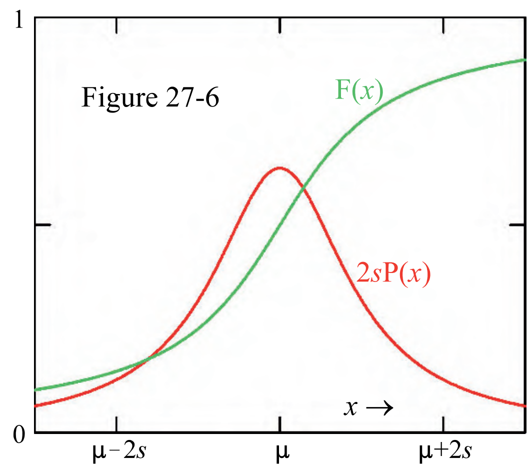
It is named the *variance* of the distribution and its square root is the *standard deviation* σ.

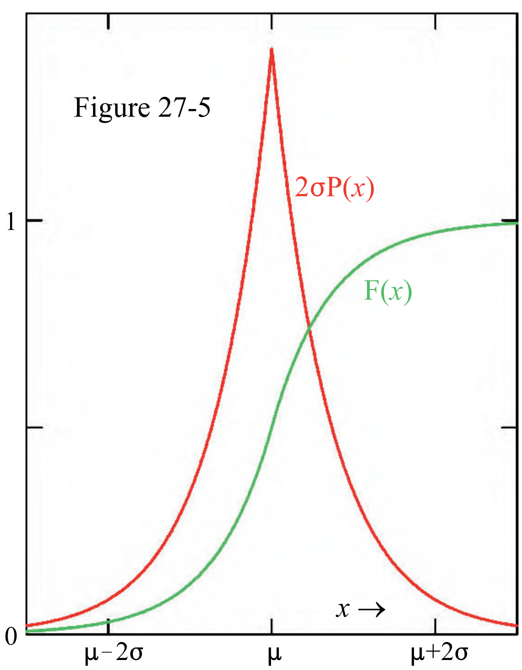
Mathematical functions from many *Atlas* chapters serve as the probability and cumulative functions used to model phenomena in the natural and social sciences. In the tables below we present a few of these. Very many other distributions cater to specialized tasks in probability theory and in statistics; for these we refer you elsewhere, to the books by Johnson, for example. *Equator* provides routines for all the illustrated functions, under such names as probability function for a normal distribution, or cumulative function for a Maxwell distribution. The keyword for an *Equator* distribution has **P** or **F** as its initial letter, according to whether the probability or cumulative function is sought, followed by the distribution's name, so that **PLorentz** and **Flogistic** are typical keywords.

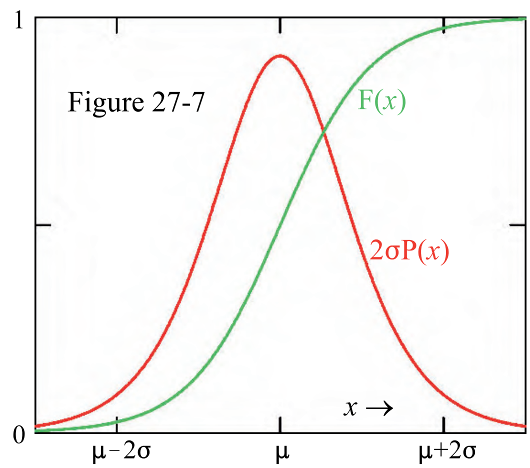
The distributions displayed in the four figures on the facing page all have $x_0 = -\infty$, $x_1 = \infty$. They are all symmetrical about their means, which implies $\mathrm{F}(\mu) = ½$, so that the mean, the median and the peak (also called the *mode*, or *most probable value*) coincide. As its name suggests, the *normal distribution* (also known as the Gauss or *Gaussian distribution*) is the one in most frequent use, though often inappropriately. Aspects of the normal distribution are discussed also in Section 40:14. Three distributions are encountered in chemical separatory techniques: the peaks for chromatography are normal, electrochemical peaks are logistic, while those in nuclear-resonance-spectroscopy follow a *Lorentz distribution* (Henrik Antoon Lorentz, Dutch physicist, 1853 – 1928). For the Lorentz (also known as the *Cauchy distribution*), there is no standard deviation, because integral 27:14:3 does not converge for this $\mathrm{P}(x)$ function. Another parameter s has therefore been chosen to play the role of characterizing the width-to-height ratio of the peak.

distribution	probability function $\mathrm{P}(x)$	cumulative function $\mathrm{F}(x)$
normal [Figure 27-4]	$\dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\dfrac{-(x-\mu)^2}{2\sigma^2}\right)$	$\dfrac{1}{2}+\dfrac{1}{2}\mathrm{erf}\left(\dfrac{x-\mu}{\sqrt{2}\,\sigma}\right)$
Laplace [Figure 27-5]	$\dfrac{1}{\sqrt{2}\,\sigma}\exp\left(\dfrac{-\sqrt{2}}{\sigma}\lvert x-\mu\rvert\right)$	$\dfrac{1}{2}+\dfrac{\mathrm{sgn}(x-\mu)}{2}\left[1-\exp\left(\dfrac{-\sqrt{2}}{\sigma}\lvert x-\mu\rvert\right)\right]$
Lorentz [Figure 27-6]	$\dfrac{s}{\pi\left[s^2+(x-\mu)^2\right]}$	$\dfrac{1}{2}+\dfrac{1}{\pi}\arctan\left(\dfrac{x-\mu}{s}\right)$
logistic [Figure 27-7]	$\dfrac{\pi}{\sqrt{48}\,\sigma}\mathrm{sech}^2\left(\dfrac{\pi(x-\mu)}{\sqrt{12}\,\sigma}\right)$	$\dfrac{1}{2}+\dfrac{1}{2}\tanh\left(\dfrac{\pi(x-\mu)}{\sqrt{12}\,\sigma}\right)$









distribution	probability function $\mathrm{P}(x)$	cumulative function $\mathrm{F}(x)$
Boltzmann [Figure 27-8] $\sigma = \mu$	$\dfrac{1}{\mu}\exp\left(\dfrac{-x}{\mu}\right)$	$1-\exp\left(\dfrac{-x}{\mu}\right)$
Rayleigh [Figure 27-9] $\sigma = \sqrt{(4-\pi)/\pi}\;\mu$	$\dfrac{\pi x}{2\mu^2}\exp\left(\dfrac{-\pi x^2}{4\mu^2}\right)$	$1-\exp\left(\dfrac{-\pi x^2}{4\mu^2}\right)$
log-normal [Figure 27-10] $\alpha = 2\ln\left\{\left(\mu^2+\sigma^2\right)/\mu^2\right\}$	$\sqrt{\dfrac{\mu}{\pi\alpha x^3}}\exp\left\{\dfrac{-1}{\alpha}\ln^2\left(\dfrac{x}{\mu}\right)-\dfrac{\alpha}{16}\right\}$	$\dfrac{1}{2}+\dfrac{1}{2}\mathrm{erf}\left\{\dfrac{1}{\sqrt{\alpha}}\ln\left(\dfrac{x}{\mu}\right)+\dfrac{\sqrt{\alpha}}{4}\right\}$
Maxwell [Figure 27-11] $\sigma = \sqrt{(3\pi/8)-1}\;\mu$	$\dfrac{32x^2}{\pi^2\mu^3}\exp\left(\dfrac{-4x^2}{\pi\mu^2}\right)$	$\mathrm{erf}\left(\dfrac{2x}{\sqrt{\pi}\mu}\right)-\dfrac{4x}{\pi\mu}\exp\left(\dfrac{-4x^2}{\pi\mu^2}\right)$
Fermi-Dirac (upper signs) Bose-Einstein (lower signs)	$\dfrac{\pm\alpha\beta}{\ln(1-\beta)\left[\exp(\alpha x)\pm\beta\right]}$	$1-\dfrac{\ln\{\exp(\alpha x)\pm\beta\}-\alpha x}{\ln(1\pm\beta)}$

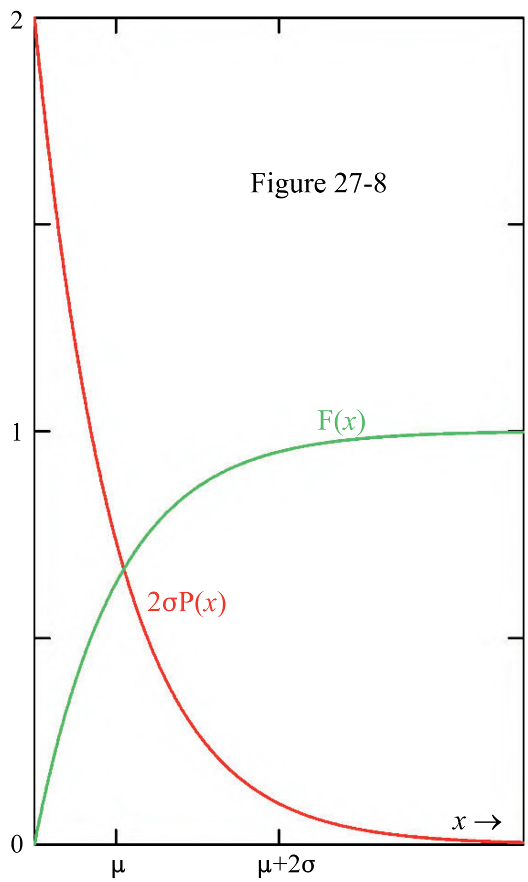


Figure 27-8

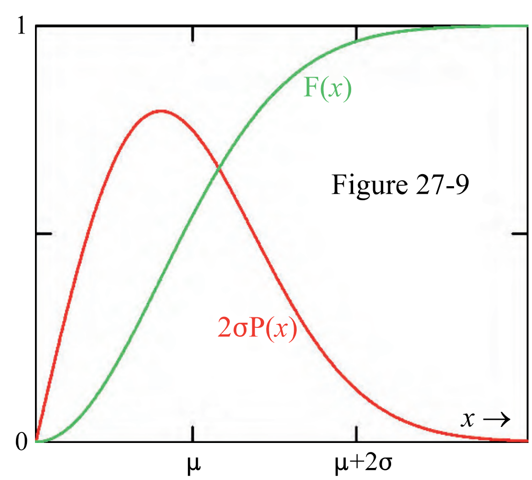


Figure 27-9

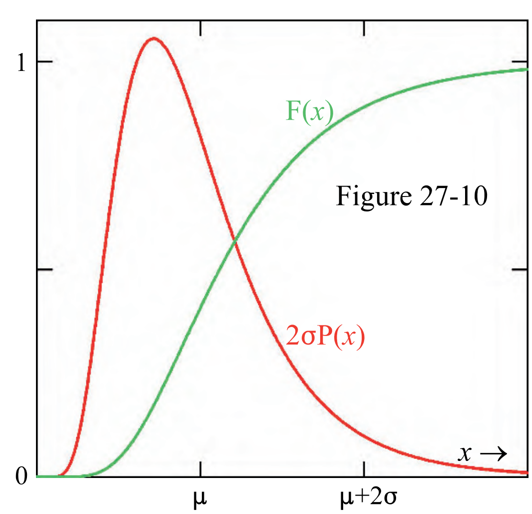


Figure 27-10

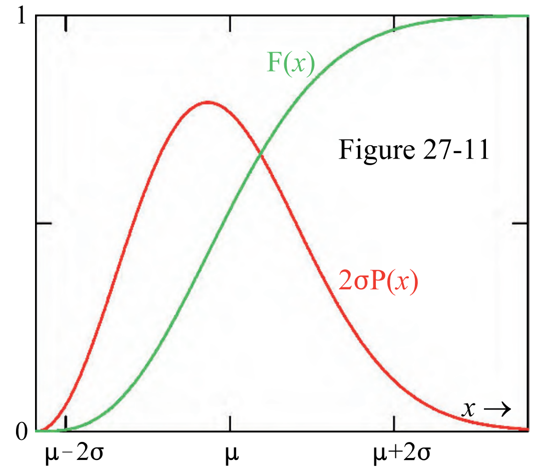


Figure 27-11

The vertical scale in all eight figures serves both the probability and the cumulative curves, providing values of both F(x) and 2σP(x). The figures are scaled uniformly to permit intercomparison, with the areas beneath all the red peaks representing unity in all cases.

Many semiinfinite distributions (with $x_0 = 0$, $x_1 = \infty$) that occur in physics are listed in the table on the facing page and are displayed in the accompanying graphs. The *Boltzmann distribution* (Ludwig Boltzmann, Austrian theoretical physicist, 1844 − 1906) describes the falloff in atmospheric pressure with height, as well as many other phenomena in which a force is opposed by thermal agitation. The expression for the radial analogue of normal distribution is attributed to the English chemist/physicist John William Strutt, 1842 − 1919, who became Lord Rayleigh on his father's death. Also related to the normal distribution is the *log-normal distribution* which is often used to model the size distribution of granular materials. On analyzing the motion of gas molecules, the Scottish mathematical physicist James Clerk Maxwell (1831 − 1879) found their speeds obeyed the distribution that now bears his name. The *Fermi-Dirac* and *Bose-Einstein distributions* govern the behaviors of subatomic particles.

Notice that in the Boltzmann, Rayleigh, and Maxwell distributions there is only one parameter, because the mean μ and the standard deviation σ are proportional to each other. The μ and σ parameters are mutually independent for the log-normal distribution; μ = 2σ was arbitrarily chosen for Figure 27-10. The α and β parameters in the formulas for the Fermi-Dirac and Bose-Einstein distributions are related implicitly to the mean and variance through the equations

27:14:4
$$\mu = \frac{\mathrm{diln}(1\pm\beta)}{\alpha\ln(1\pm\beta)}$$

and

27:14:5
$$\sigma^2 = \frac{2\ln(1\pm\beta)\mathrm{triln}(1\pm\beta) - \mathrm{diln}^2(1\pm\beta)}{\alpha^2\ln^2(1\pm\beta)}$$

which involve functions from Chapter 25.

CHAPTER 28

THE HYPERBOLIC COSINE cosh(*x*) AND SINE sinh(*x*) FUNCTIONS

This chapter and the next two chapters address the six so-called *hyperbolic functions*. The present chapter deals with the two most important of the six: the *hyperbolic cosine* and the *hyperbolic sine*. These two functions are interrelated by

28:0:1 $$\cosh^2(x) - \sinh^2(x) = 1$$

and by each being the derivative of the other [equations 28:10:1 and 28:10:2].

28:1 NOTATION

The names of these functions arise because of their complex algebraic relationship [Section 32:11] to the cosine and sine functions. Their association with the hyperbola is explained in Section 28:3.

The notations ch(x) and sh(x) sometimes replace cosh(x) and sinh(x). Although they cause confusion, the capitalized symbolism Cos(x) and Sin(x) is occasionally encountered.

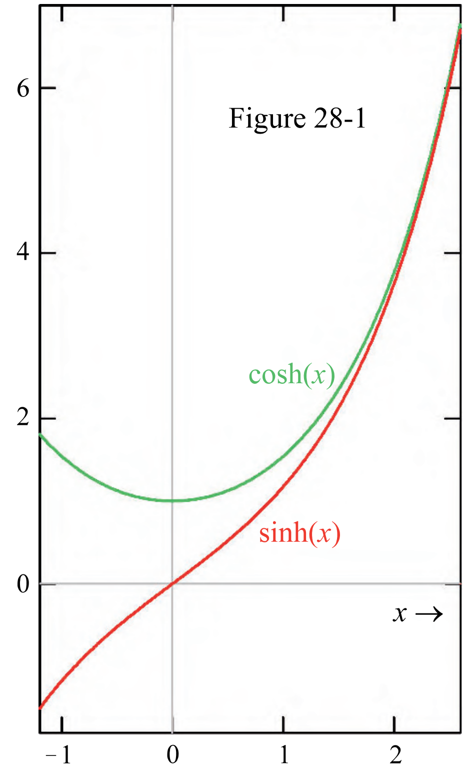


Figure 28-1

28:2 BEHAVIOR

Both functions are defined for all arguments but, whereas the hyperbolic sine adopts all values, the range of the hyperbolic cosine is restricted to cosh(x) ≥ 1. As Figure 28-1 illustrates, both functions tend exponentially towards ±∞ as the argument acquires large magnitudes of either sign.

A curve representing the hyperbolic cosine function f = cosh(x) has the property that its slope df/dx at any point is equal to the length, [Section 37:14] measured along the curve, from (x = 0, f = 1) to that point.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_29, © Springer Science+Business Media, LLC 2009

It is this property that is responsible for the fact, established in Section 28:14, that a freely suspended rope or chain adopts a shape described by a hyperbolic cosine.

28:3 DEFINITIONS

The hyperbolic cosine and sine functions are defined in terms of the exponential function by

28:3:1 $$\cosh(x) = \frac{\exp(x)+\exp(-x)}{2}$$

and

28:3:2 $$\sinh(x) = \frac{\exp(x)-\exp(-x)}{2}$$

Because the two functions may be represented hypergeometrically as the sums

28:3:3 $$\cosh(x) = \sum_{j=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_j (1)_j} \left(\frac{x^2}{4}\right)^j$$

and

28:3:4 $$\sinh(x) = x\sum_{j=0}^{\infty} \frac{1}{(1)_j \left(\frac{3}{2}\right)_j} \left(\frac{x^2}{4}\right)^j$$

they may be synthesized from the $L = K+2 = 2$ basis hypergeometric function [Section 43:14], as follows

28:3:5 $$\mathrm{I}_0\left(2\sqrt{x}\right) \xrightarrow[\frac{1}{2}]{1} \cosh\left(2\sqrt{x}\right)$$

28:3:6 $$\mathrm{I}_0\left(2\sqrt{x}\right) \xrightarrow[\frac{3}{2}]{1} \frac{1}{2\sqrt{x}}\sinh\left(2\sqrt{x}\right)$$

A second-order differential equation and its solution are

28:3:7 $$\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} = b^2\,\mathrm{f}(x) \qquad \mathrm{f} = w_1\cosh(bx) + w_2\sinh(bx)$$

were b, w_1, and w_2 are constants.

The (circular) sine and cosine functions [Chapter 32] are closely related to the circle, but it is less straightforward to demonstrate the relationship of the hyperbolic sine and cosine to the hyperbola. However, consider the two diagrams in Figure 28-2, which show cartesian graphs of the horizontal rectangular hyperbola [Section 14:4] $y = \pm\sqrt{x^2-1}$ and the circle $y = \pm\sqrt{1-x^2}$. Imagine the lines shown in blue to have started as horizontal diameters and pivoted about the origins, these rotating vectors having come to rest in the angled positions shown. In so doing, each has swept out an area shown in green. Let this area (both segments) be *ar* in the hyperbolic case and *ar′* in the case of the circle. Then the points P and P′, where the rotors terminate on their respective curves in the first quadrants, have rectangular coordinates related to the areas, as follows

28:3:8 $$x_{\mathrm{P}} = \mathrm{OQ} = \cosh(ar) \qquad \text{and} \qquad y_{\mathrm{P}} = \mathrm{PQ} = \sinh(ar)$$

28:3:9 $$x_{\mathrm{P'}} = \mathrm{O'Q'} = \cos(ar') \qquad \text{and} \qquad y_{\mathrm{P'}} = \mathrm{P'Q'} = \sin(ar')$$

These relationship serve as geometric definitions of the circular and hyperbolic cosines and sines, and demonstrate the kinship between the two families of functions.

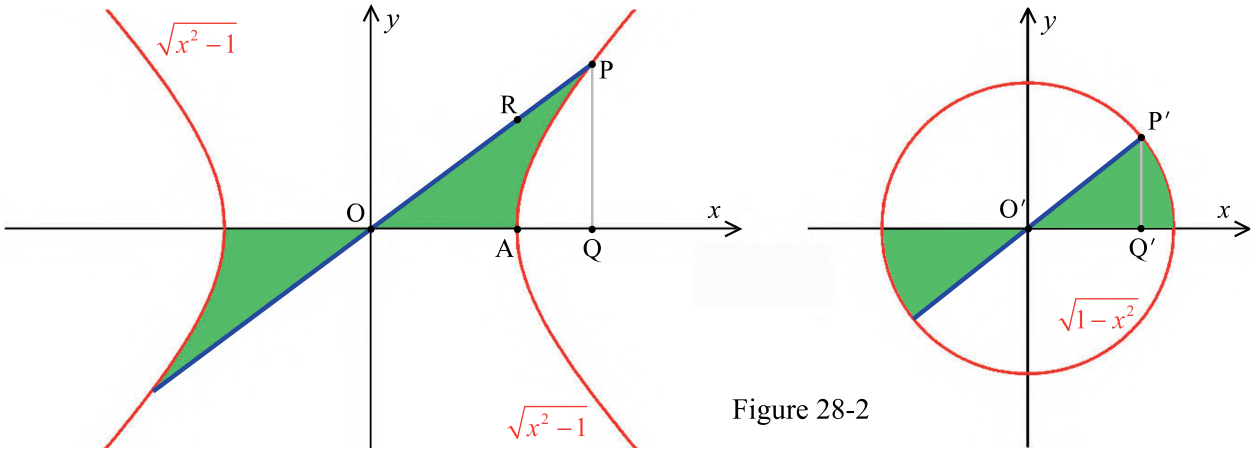


Figure 28-2

28:4 SPECIAL CASES

There are none.

28:5 INTRARELATIONSHIPS

The hyperbolic cosine function is even

28:5:1 $$\cosh(-x) = \cosh(x)$$

whereas the hyperbolic sine is an odd function

28:5:2 $$\sinh(-x) = -\sinh(x)$$

The duplication and triplication formulas

28:5:3 $$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x)$$

28:5:4 $$\sinh(2x) = 2\sinh(x)\cosh(x) = 2\sinh(x)\sqrt{1+\sinh^2(x)}$$

28:5:5 $$\cosh(3x) = 4\cosh^3(x) - 3\cosh(x)$$

28:5:6 $$\sinh(3x) = 4\sinh^3(x) + 3\sinh(x) = \sinh(x)\left[4\cosh^2(x) - 1\right]$$

generalize to

28:5:7 $$\cosh(nx) = \mathrm{T}_n\{\cosh(x)\} = \sum_{k=0}^{n} \tau_k^{(n)} \cosh^k(x)$$

and

28:5:8 $$\sinh(nx) = \sinh(x)\mathrm{U}_{n-1}\{\cosh(x)\} = \frac{1}{\sinh(x)} \sum_{k=0}^{n} \left[\tau_k^{(n)} \cosh^{k+1}(x) - \tau_k^{(n-1)} \cosh^k(x)\right]$$

where T_n and U_n are Chebyshev polynomials [Chapter 22]. The Chebyshev tau coefficients $\tau_k^{(n)}$ are discussed in Section 22:6. The hyperbolic version of *de Moivre's theorem* [Section 12:11]

28:5:9 $$\cosh(nx) \pm \sinh(nx) = \left[\cosh(x) \pm \sinh(x)\right]^n = \exp(\pm nx)$$

is also useful.

Equations 28:5:3 and 28:5:4 may be regarded as special cases of the argument-addition formulas

28:5:10 $$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

28:5:11 $$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

From 28:5:3 one may derive the expressions

28:5:12 $$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}$$

and

28:5:13 $$\sinh\left(\frac{x}{2}\right) = \mathrm{sgn}(x)\sqrt{\frac{\cosh(x)-1}{2}}$$

for the hyperbolic functions of half argument, as well as the formulas

28:5:14 $$\cosh^2(x) = \frac{\cosh(2x)+1}{2}$$

and

28:5:15 $$\sinh^2(x) = \frac{\cosh(2x)-1}{2}$$

for the squares. These latter may be generalized to the expressions

28:5:16 $$\cosh^n(x) = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{j=0}^{(n-1)/2} \binom{n}{j} \cosh\{(n-2j)x\} & n = 1,3,5,\cdots \\ \dfrac{(n-1)!!}{n!!} + \dfrac{1}{2^{n-1}} \displaystyle\sum_{j=0}^{(n/2)-1} \binom{n}{j} \cosh\{(n-2j)x\} & n = 2,4,6,\cdots \end{cases}$$

and

28:5:17 $$\sinh^n(x) = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{j=0}^{(n-1)/2} (-)^j \binom{n}{j} \sinh\{(n-2j)x\} & n = 1,3,5,\cdots \\ \dfrac{(-)^{n/2}(n-1)!!}{n!!} + \dfrac{1}{2^{n-1}} \displaystyle\sum_{j=0}^{(n/2)-1} (-)^j \binom{n}{j} \cosh\{(n-2j)x\} & n = 2,4,6,\cdots \end{cases}$$

for any positive integer power of the hyperbolic cosine or sine.

The function-addition formulas

28:5:18 $$\cosh(x) \pm \sinh(x) = \exp(\pm x)$$

28:5:19 $$\cosh(x) + \cosh(y) = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

28:5:20 $$\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

28:5:21 $$\sinh(x) \pm \sinh(y) = 2\sinh\left(\frac{x \pm y}{2}\right)\cosh\left(\frac{x \mp y}{2}\right)$$

and the function-multiplication formulas

28:5:22 $$\sinh(x)\sinh(y) = \frac{1}{2}\cosh(x+y) - \frac{1}{2}\cosh(x-y)$$

28:5:23 $$\sinh(x)\cosh(y) = \frac{1}{2}\sinh(x+y) + \frac{1}{2}\sinh(x-y)$$

28:5:24 $$\cosh(x)\cosh(y) = \frac{1}{2}\cosh(x+y) + \frac{1}{2}\cosh(x-y)$$

complete our listing of intrarelationships between these most malleable functions.

28:6 EXPANSIONS

The hyperbolic cosine and sine functions may be expanded as infinite series

28:6:1 $$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!}$$

28:6:2 $$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}$$

or as infinite products

28:6:3 $$\cosh(x) = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{9\pi^2}\right)\left(1 + \frac{4x^2}{25\pi^2}\right)\cdots = \prod_{j=1}^{\infty}\left(1 + \frac{x^2}{(j-\frac{1}{2})^2\pi^2}\right)$$

28:6:4 $$\sinh(x) = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\cdots = x\prod_{j=1}^{\infty}\left(1 + \frac{x^2}{j^2\pi^2}\right)$$

Each of these products may be written as $\prod\{1-(x/r_j)\}$ where the r's are the complex zeros of the function. In fact, these zeros are mostly imaginary and equal $\pm i\pi/2$, $\pm 3i\pi/2$, $\pm 5i\pi/2, \cdots$ in the case of cosh(x) and 0, $\pm i\pi$, $\pm 2i\pi$, $\pm 3i\pi, \cdots$ for sinh(x).

The hyperbolic cosine and sine may also be expanded as series of modified Bessel functions [Chapter 49]:

28:6:5 $$\cosh(x) = \mathrm{I}_0(x) + 2\mathrm{I}_2(x) + 2\mathrm{I}_4(x) + 2\mathrm{I}_6(x) + \cdots$$

28:6:6 $$\sinh(x) = 2\mathrm{I}_1(x) + 2\mathrm{I}_3(x) + 2\mathrm{I}_5(x) + \cdots$$

28:7 PARTICULAR VALUES

	$x=-\infty$	$x=-1$	$x=\ln(\sqrt{2}-1)$	$x=0$	$x=\ln(\sqrt{2}+1)$	$x=1$	$x=\infty$
cosh(x)	$+\infty$	$\frac{e^2+1}{2e}$	$\sqrt{2}$	1	$\sqrt{2}$	$\frac{e^2+1}{2e}$	$+\infty$
sinh(x)	$-\infty$	$\frac{1-e^2}{2e}$	-1	0	1	$\frac{e^2-1}{2e}$	$+\infty$

28:8 NUMERICAL VALUES

Equator's hyperbolic cosine function and hyperbolic sine function routines (keywords **cosh** and **sinh**) return accurate values for arguments with magnitudes as large as 710.

28:9 LIMITS AND APPROXIMATIONS

Definitions 28:3:1 and 28:3:2 show that as $x \to \infty$, both functions approach $[\exp(x)]/2$. As $x \to -\infty$, $\cosh(x)$ approaches $[\exp(-x)]/2$, whereas $\sinh(x)$ approaches $[-\exp(-x)]/2$.

28:10 OPERATIONS OF THE CALCULUS

Differentiation and indefinite integration of $\cosh(bx)$ and $\sinh(bx)$ give

28:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(bx) = b\,\sinh(bx)$$

28:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(bx) = b\,\cosh(bx)$$

28:10:3 $$\int_0^x \cosh(bt)\mathrm{d}t = \frac{\sinh(bx)}{b}$$

28:10:4 $$\int_0^x \sinh(bt)\mathrm{d}t = \frac{\cosh(bx)-1}{b}$$

The general formulas

28:10:5 $$\int_0^x \cosh^n(t)\mathrm{d}t = \begin{cases} \dfrac{(n-1)!!\sinh(x)}{n!!}\displaystyle\sum_{j=0}^{(n-1)/2}\frac{(2j-1)!!}{(2j)!!}\cosh^{2j}(x) & n=1,3,5,\cdots \\ \dfrac{(n-1)!!}{n!!}\left[x+\sinh(x)\displaystyle\sum_{j=0}^{(n/2)-1}\frac{(2j)!!}{(2j+1)!!}\cosh^{2j+1}(x)\right] & n=2,4,6,\cdots \end{cases}$$

28:10:6 $$\int_0^x \sinh^n(t)\mathrm{d}t = \begin{cases} \dfrac{(-)^{(n-1)/2}(n-1)!!}{n!!}\left[-1+\cosh(x)\displaystyle\sum_{j=0}^{(n-1)/2}\frac{(-)^j(2j-1)!!}{(2j)!!}\sinh^{2j}(x)\right] & n=1,3,5,\cdots \\ \dfrac{(-)^{n/2}(n-1)!!}{n!!}\left[x-\cosh(x)\displaystyle\sum_{j=0}^{(n/2)-1}\frac{(-)^j(2j)!!}{(2j+1)!!}\sinh^{2j+1}(x)\right] & n=2,4,6,\cdots \end{cases}$$

permit indefinite integration of integer powers of the hyperbolic sine and cosine functions. Alternative expressions may be derived by integration of equations 28:5:16 and 28:5:17. Many integrals, such as those of $t^n\cosh(bt)$ and $t^n\sinh(bt)$ may be evaluated by first breaking the hyperbolic functions into their $\exp(x)$ and $\exp(-x)$ components and using formulas from Sections 26:10, 27:10, or 37:14. Integrals of $(1/t)\cosh(bt)$ and $(1/t)\sinh(bt)$ are the subject of Chapter 38. The indefinite integrals of $\cosh^\omega(t)$ and $\sinh^\omega(t)$, where ω is an arbitrary power, are discussed in Section 58:14.

A large number of other indefinite and definite integrals of the hyperbolic cosine and hyperbolic sine functions exist; see Sections 2.4 and 3.5 of Gradshteyn and Ryzhik for these.

As with integrals, many quantities involving cosh or sinh terms may be Laplace transformed by first decomposing the hyperbolic functions into their exponential components. The first three example below may be established by that route

28:10:7 $$\int_0^\infty \cosh(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\cosh(bt)\} = \frac{s}{s^2-b^2}$$

28:10:8 $$\int_0^\infty \sinh(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\sinh(bt)\} = \frac{b}{s^2-b^2}$$

28:10:9 $$\int_0^\infty \frac{\cosh(bt)}{t^\nu}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\frac{\cosh(bt)}{t^\nu}\right\} = \frac{\Gamma(1-\nu)}{2}\left[(s-b)^{\nu-1}+(s+b)^{\nu-1}\right] \qquad \nu<1$$

28:10:10 $$\int_0^\infty \sinh^\nu(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\sinh^\nu(bt)\} = \frac{1}{2^{\nu+1}b}\mathrm{B}\left(\frac{s-\nu b}{2b},\nu+1\right) \qquad \nu>-1$$

The fourth involves the incomplete beta function from Chapter 58.

28:11 COMPLEX ARGUMENT

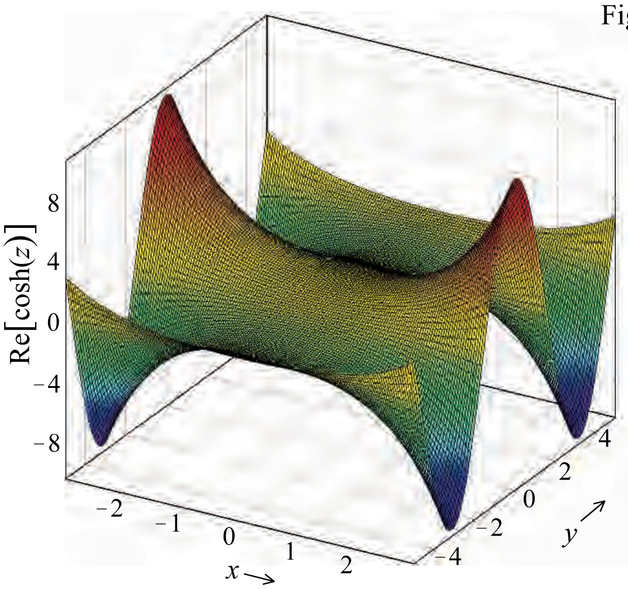
Figures 28-3 and 28-4 show, respectively, the behaviors of the hyperbolic cosine and hyperbolic sine of argument $x+iy$. The real and imaginary parts of these functions of complex argument are

28:11:1 $$\mathrm{Re}\{\cosh(x+iy)\} = \cosh(x)\cos(y) \quad \text{and} \quad \mathrm{Im}\{\cosh(x+iy)\} = \sinh(x)\sin(y)$$

28:11:2 $$\mathrm{Re}\{\sinh(x+iy)\} = \sinh(x)\cos(y) \quad \text{and} \quad \mathrm{Im}\{\sinh(x+iy)\} = \cosh(x)\sin(y)$$

Along the imaginary axis (that is, when $x = 0$) the three-dimensional figures here and overleaf demonstrate the development of sinusoidal behavior in the real part, for the hyperbolic cosine, and in the imaginary part, for the hyperbolic sine. This is as expected because the $x = 0$ versions of the above equations are

Figure 28-3



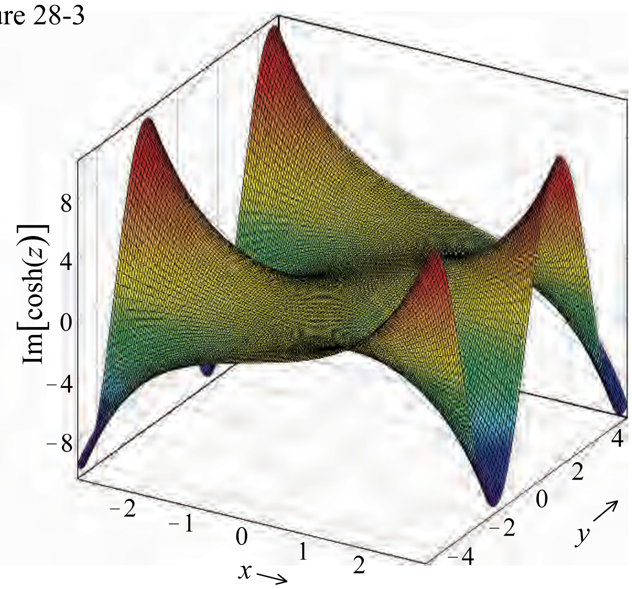
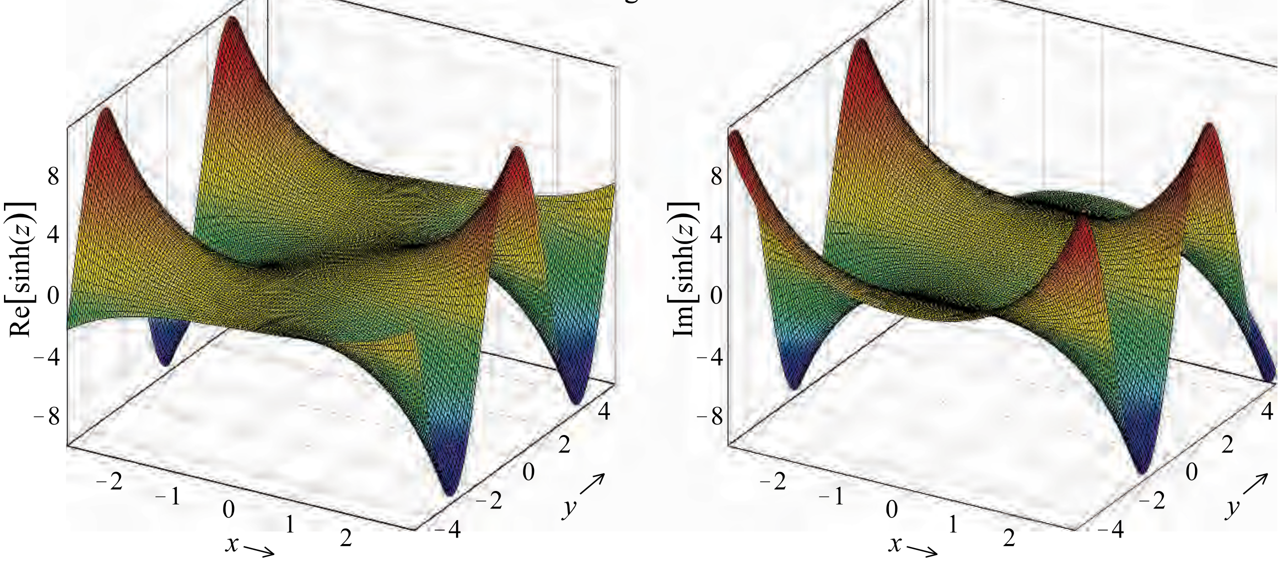


Figure 28-4



28:11:3 $$\cosh(iy) = \cos(y)$$

and

28:11:4 $$\sinh(iy) = i\sin(y)$$

Inverse Laplace transforms involving the hyperbolic cosine and sine functions can frequently be deduced by splitting the hyperbolic function into a pair of exponential functions. Others that cannot be treated in that way often lead to exponential theta function or modified exponential theta functions [Section 27:14], or their derivatives or integrals. Examples include

28:11:5 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{\cosh\left(v\sqrt{s}\right)}{\sqrt{s}\sinh\left(b\sqrt{s}\right)} \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\frac{\cosh\left(v\sqrt{s}\right)}{\sqrt{s}\sinh\left(b\sqrt{s}\right)}\right\} = \frac{1}{b}\theta_4\left(\frac{v}{2b}, \frac{t}{b^2}\right) \qquad -b \le v \le b$$

28:11:6 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{\cosh\left(v\sqrt{s}\right)}{\sqrt{s}\cosh\left(b\sqrt{s}\right)} \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\frac{\cosh\left(v\sqrt{s}\right)}{\sqrt{s}\cosh\left(b\sqrt{s}\right)}\right\} = \frac{-1}{b}\hat{\theta}_1\left(\frac{v}{2b}, \frac{t}{b^2}\right) \qquad -b \le v \le b$$

and

28:11:7 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{\cosh\left(v\sqrt{s}\right)}{\sinh\left(b\sqrt{s}\right)} \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\frac{\cosh\left(v\sqrt{s}\right)}{\sinh\left(b\sqrt{s}\right)}\right\} = \frac{-1}{b}\frac{\partial}{\partial v}\theta_1\left(\frac{v}{2b}, \frac{t}{b^2}\right) \qquad -b < v < b$$

See Roberts and Kaufman for others.

28:12 GENERALIZATIONS

The Jacobian elliptic functions nc(k,x) and nd(k,x) may be regarded as generalizations of cosh(x), to which they reduce when $k = 1$. Likewise, sc(k,x) and sd(k,x) reduce to sinh(x) when $k = 1$ and therefore generalize the hyperbolic sine. See Chapter 63 for all these Jacobian elliptic functions.

28:13 COGNATE FUNCTIONS

The expressions

28:13:1 $$\cosh(x)=\sqrt{1+\sinh^2(x)}=\frac{1}{\mathrm{sech}(x)}=\frac{\sqrt{1+\mathrm{csch}^2(x)}}{|\mathrm{csch}(x)|}=\frac{1}{\sqrt{1-\tanh^2(x)}}=\frac{|\coth(x)|}{\sqrt{\coth^2(x)-1}}$$

28:13:2 $$\sinh(x)=\mathrm{sgn}(x)\sqrt{\cosh^2(x)-1}=\frac{\mathrm{sgn}(x)\sqrt{1-\mathrm{sech}^2(x)}}{\mathrm{sech}(x)}=\frac{1}{\mathrm{csch}(x)}=\frac{\tanh(x)}{\sqrt{1-\tanh^2(x)}}=\frac{\mathrm{sgn}(x)}{\sqrt{\coth^2(x)-1}}$$

relate the hyperbolic cosine and sine to the other hyperbolic functions [Chapters 29 and 30].

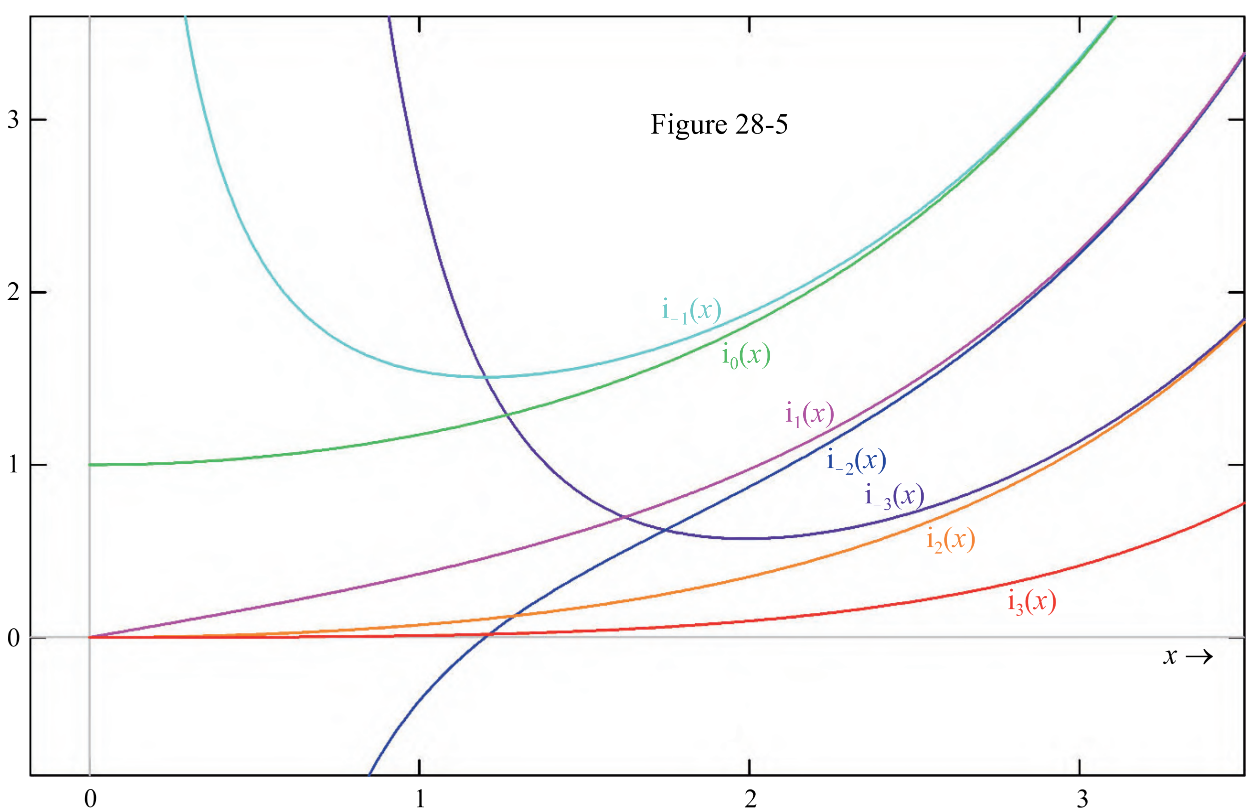
Though they are defined for positive argument as modified Bessel functions [Chapter 50] of half-odd order, the so-called *modified* (or *hyperbolic*) *spherical Bessel functions of the first kind*

28:13:3 $$\mathrm{i}_n(x)=\sqrt{\frac{\pi}{2x}}\,\mathrm{I}_{n+\frac{1}{2}}(x)$$

are closely related to the hyperbolic cosine and hyperbolic sine functions. Early members of the family are

$\mathrm{i}_0(x)$	$\mathrm{i}_1(x)$	$\mathrm{i}_2(x)$	$\mathrm{i}_3(x)$
$\dfrac{\sinh(x)}{x}$	$\dfrac{\cosh(x)}{x}-\dfrac{\sinh(x)}{x^2}$	$\dfrac{\sinh(x)}{x}\left[1+\dfrac{3}{x^2}\right]-\dfrac{3\cosh(x)}{x^2}$	$\dfrac{\cosh(x)}{x}\left[1+\dfrac{15}{x^2}\right]-\dfrac{\sinh(x)}{x^2}\left[6+\dfrac{15}{x^2}\right]$

and are shown in Figure 28-5. Others may be constructed via the recursion



28:13:4 $$\mathrm{i}_n(x)=\frac{1-2n}{x}\mathrm{i}_{n-1}(x)+\mathrm{i}_{n-2}(x)$$

The recursion may be written "backwards"

28:13:5 $$\mathrm{i}_{-n}(x)=\frac{3-2n}{x}\mathrm{i}_{1-n}(x)+\mathrm{i}_{2-n}(x)$$

to construct a family of which the initial members are

$\mathrm{i}_{-1}(x)$	$\mathrm{i}_{-2}(x)$	$\mathrm{i}_{-3}(x)$	$\mathrm{i}_{-4}(x)$
$\dfrac{\cosh(x)}{x}$	$\dfrac{\sinh(x)}{x}-\dfrac{\cosh(x)}{x^2}$	$\dfrac{\cosh(x)}{x}\left[1+\dfrac{3}{x^2}\right]-\dfrac{3\sinh(x)}{x^2}$	$\dfrac{\sinh(x)}{x}\left[1+\dfrac{15}{x^2}\right]-\dfrac{\cosh(x)}{x^2}\left[6+\dfrac{15}{x^2}\right]$

and which is related by $\mathrm{i}_{-n}(x)=\sqrt{\pi/2x}\,\mathrm{I}_{(½)-n}(x)$ to modified Bessel functions. This family goes by the name *modified* (or *hyperbolic*) *spherical Bessel functions of the second kind* and examples are also shown in Figure 28-5. Numerical values from either family may be obtained via *Equator*'s modified spherical Bessel function routine (keyword **i**).

28:14 RELATED TOPIC: the catenary

If a flexible rope of length L and weight W is suspended at two points of equal height, separated by a chasm of width X, the rope adopts a characteristic shape known as a *catenary* and illustrated by the red curve in Figure 28-6. Here we demonstrate that, in terms of rectangular coordinates erected as shown in the figure, the catenary conforms to the equation

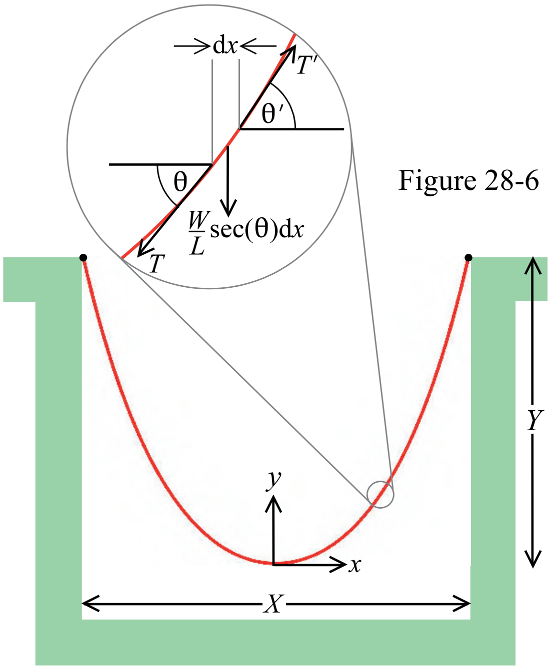


Figure 28-6

28:14:1 $$\frac{Wy}{LT_0}=\cosh\left(\frac{Wx}{LT_0}\right)-1$$

where T_0 is the tension in the rope at its lowest point. The parameter W/T_0 can be found by solving the implicit equation

28:14:2 $$\frac{W}{2T_0}=\sinh\left(\frac{WX}{2LT_0}\right)$$

and this parameter, through the equation

28:14:3 $$Y=L\left[\sqrt{\frac{T_0^2}{W^2}+\frac{1}{4}}-\frac{T_0}{W}\right]$$

also determines the sag in the rope, the vertical distance Y between the suspension points and the rope's lowest point. For example, if a rope of 20 m length is suspended over a chasm of 10 m width, equation 28:14:2 is satisfied by $W/T_0 \approx 8.709$ and the sag is $Y = 7.964$ m.

Consider a small segment of the rope of length as in the exploded inset of Figure 28-6. There, the rope makes an angle θ to the horizontal, so that the segment's length is dx/cos(θ) and its weight is (W/L)sec(θ)dx. The segment is acted upon by three forces as represented by the three arrows. In addition to its weight, the other two forces are the tensions T and T' in the rope, below and above the segment.

Because the rope is in equilibrium, the horizontal components of the forces must balance; that is

28:14:4
$$T\cos(\theta) = T'\cos(\theta') = (T+\mathrm{d}T)\cos(\theta+\mathrm{d}\theta) = (T+\mathrm{d}T)\big[\cos(\theta)-\sin(\theta)\mathrm{d}\theta\big]$$

where θ and θ' (equal to θ+dθ) are the angles so marked on the diagram and dT is the increment $T'-T$ of tension along the segment's length. The $T\cos(\theta)$ term cancels and the term dTdθ can be ignored. It follows that

28:14:5
$$\mathrm{d}T = T\tan(\theta)\,\mathrm{d}\theta$$

which integrates to

28:14:6
$$T = T_0\sec(\theta)$$

where the constant of integration has been identified as T_0, the tension in the rope at its nadir.

Likewise, the vertical components of the forces must balance and therefore,

28:14:7
$$\frac{W}{L}\sec(\theta)\,\mathrm{d}x + T\sin(\theta) = T'\sin(\theta') = (T+\mathrm{d}T)\sin(\theta+\mathrm{d}\theta) = (T+\mathrm{d}T)\big[\sin(\theta)+\cos(\theta)\mathrm{d}\theta\big]$$

which, arguing as before, simplifies to $(W/L)\sec(\theta)\mathrm{d}x = \sin(\theta)\mathrm{d}T + T\cos(\theta)\mathrm{d}\theta$ and combines with 28:14:5 into the remarkably simple result formulated as the first equality in

28:14:8
$$W\,\mathrm{d}x = LT\,\mathrm{d}\theta = LT_0\sec(\theta)\,\mathrm{d}\theta = LT_0\,\mathrm{d}\{\mathrm{invgd}(\theta)\}$$

The second equality in 28:14:8 follows from 28:14:6 and the third is a consequence of the definition [Section 33:15] of the inverse gudermannian function.

Because W, L and T_0 are all constants, integration of equation 28:14:8 yields $Wx/LT_0 = \mathrm{invgd}(\theta)$, which inverts to give an expression for the angle θ and thence for the slope dy/dx of the rope:

28:14:9
$$\mathrm{gd}\left(\frac{Wx}{LT_0}\right) = \theta = \arctan\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Here gd is the gudermannian function [Section 33:15], one of the properties of which is that $\tan\{\mathrm{gd}(t)\} = \sinh(t)$. It follows that

28:14:10
$$\mathrm{d}y = \sinh\left(\frac{Wx}{LT_0}\right)\mathrm{d}x$$

which integrates straightforwardly to 28:14:1, proving that a catenary has a shape matching a hyperbolic cosine.

The total length of the rope can be found by the integration [Section 39:14]

28:14:11
$$L = \int_{-X/2}^{X/2}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{-X/2}^{X/2}\cosh\left(\frac{Wx}{LT_0}\right)\mathrm{d}x = \frac{2LT_0}{W}\sinh\left(\frac{WX}{2LT_0}\right)$$

from which equation 28:14:2 follows.

CHAPTER 29

THE HYPERBOLIC SECANT sech(*x*) AND COSECANT csch(*x*) FUNCTIONS

The two functions treated here, which are interrelated by the formulas

29:0:1 $$\mathrm{csch}^2(x) = \frac{\mathrm{sech}^2(x)}{1-\mathrm{sech}^2(x)}$$

and

29:0:2 $$\mathrm{sech}^2(x) = \frac{\mathrm{csch}^2(x)}{1+\mathrm{csch}^2(x)}$$

are perhaps the least frequently encountered members of the six hyperbolic functions.

29:1 NOTATION

The notation cosech(x) is sometimes used for the *hyperbolic cosecant*. Some authors admit only four hyperbolic functions, using 1/cosh(x) and 1/sinh(x) to represent the *hyperbolic secant* and cosecant.

29:2 BEHAVIOR

Figure 29-1 provides maps of the two functions. The hyperbolic secant sech(x) accepts any real argument x but its values are confined to the narrow range $0 \le \mathrm{sech}(x) \le 1$. In contrast, both the domain and the range of the hyperbolic cosecant are unlimited. The csch(x) function encounters a discontinuity of the $-\infty|+\infty$ variety at $x = 0$.

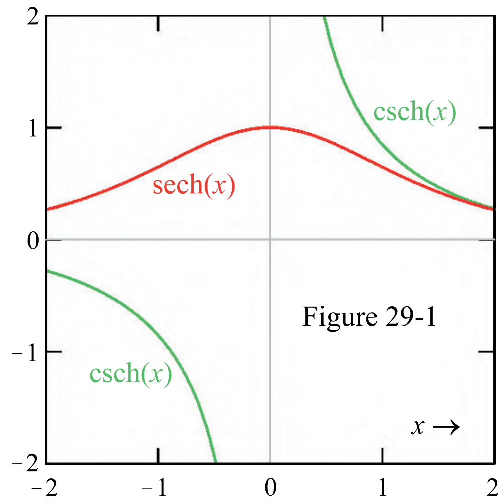


Figure 29-1

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_30,

29:3 DEFINITIONS

The hyperbolic secant and hyperbolic cosecant are the reciprocals of the functions of Chapter 28 and this is how they are usually defined:

29:3:1 $$\mathrm{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{\exp(x)+\exp(-x)}$$

29:3:2 $$\mathrm{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{\exp(x)-\exp(-x)}$$

Two differential equations and their solutions are

29:3:3 $$\frac{d\mathrm{f}}{dx} + \mathrm{f}\sqrt{b^2 \pm \mathrm{f}^2} = 0 \qquad \mathrm{f} = b\,{}^{\mathrm{csch}}_{\mathrm{sech}}(bx)$$

29:4 SPECIAL CASES

There are none.

29:5 INTRARELATIONSHIPS

The hyperbolic secant and cosecant obey the reflection formulas

29:5:1 $$\mathrm{sech}(-x) = \mathrm{sech}(x)$$

29:5:2 $$\mathrm{csch}(-x) = -\mathrm{csch}(x)$$

and the duplication formulas

29:5:3 $$\mathrm{sech}(2x) = \frac{\mathrm{sech}^2(x)}{2-\mathrm{sech}^2(x)}$$

29:5:4 $$\mathrm{csch}(2x) = \frac{\mathrm{sech}(x)\,\mathrm{csch}(x)}{2}$$

Other relationships may be derived via the equations of Section 28:5, but these are generally more complicated and less useful than are the intrarelationships of the hyperbolic cosine or hyperbolic sine.

29:6 EXPANSIONS

The functions sech(x) and csch(x) may be expanded as power series

29:6:1 $$\mathrm{sech}(x) = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = \frac{4}{\pi}\sum_{j=0}^{\infty}\beta(2j+1)\left(\frac{-4x^2}{\pi^2}\right)^j = \sum_{j=0}^{\infty}\frac{\mathrm{E}_{2j}\,x^{2j}}{(2j)!} \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

and

29:6:2 $$\mathrm{csch}(x) = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} + \frac{2}{x}\sum_{j=1}^{\infty}\eta(2j)\left(\frac{-x^2}{\pi^2}\right)^j = \sum_{j=0}^{\infty}\frac{(2-4^j)}{(2j)!}\mathrm{B}_{2j}\,x^{2j-1} \qquad -\pi < x < \pi$$

The β and η numbers are defined in Chapter 3 and the Euler E and Bernoulli B numbers in Chapters 5 and 4 respectively.

Expansions as exponentials take the forms

29:6:3 $$\mathrm{sech}(x) = 2\exp(-|x|) - 2\exp(-3|x|) + 2\exp(-5|x|) - \cdots = 2\sum_{j=0}^{\infty}(-)^j \exp\left(-(2j+1)|x|\right) \qquad x \neq 0$$

and

29:6:4 $$\mathrm{csch}(x) = 2\,\mathrm{sgn}(x)\left[\exp(-|x|) + \exp(-3|x|) + \exp(-5|x|) + \cdots\right] = 2\,\mathrm{sgn}(x)\sum_{j=1}^{\infty}\exp\left((1-2j)|x|\right) \qquad x \neq 0$$

As well, the hyperbolic secant and hyperbolic cosecant can be expanded as the partial fractions

29:6:5 $$\mathrm{sech}(x) = \frac{4\pi}{\pi^2+4x^2} - \frac{12\pi}{9\pi^2+4x^2} + \frac{20\pi}{25\pi^2+4x^2} - \cdots = \sum_{j=0}^{\infty}(-)^j \frac{(2j+1)\pi}{\left(j+\frac{1}{2}\right)^2\pi^2 + x^2}$$

29:6:6 $$\mathrm{csch}(x) = \frac{1}{x} - \frac{2x}{\pi^2+x^2} + \frac{2x}{4\pi^2+x^2} - \frac{2x}{9\pi^2+x^2} + \cdots = \sum_{j=-\infty}^{\infty}(-)^j \frac{x}{j^2\pi^2+x^2}$$

29:7 PARTICULAR VALUES

	$x=-\infty$	$x=-1$	$x=\ln\left(\sqrt{2}-1\right)$	$x=0$	$x=\ln\left(\sqrt{2}+1\right)$	$x=1$	$x=\infty$
sech(x)	0	$\frac{2e}{e^2+1}$	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	$\frac{2e}{e^2+1}$	0
csch(x)	0	$\frac{-2e}{e^2-1}$	-1	$-\infty \vert +\infty$	1	$\frac{2e}{e^2-1}$	0

29:8 NUMERICAL VALUES

Equator's hyperbolic secant function and hyperbolic cosecant function routines (keywords **sech** and **csch**) return exact values of the sech(x) and csch(x) functions for all arguments of magnitude not greater than 709.

29:9 LIMITS AND APPROXIMATIONS

At large arguments of either sign, both functions decay exponentially in magnitude:

29:9:1 $$\mathrm{sech}(x) \to 2\exp\left(-|x|\right) \qquad x \to \pm\infty$$

29:9:2 $$\mathrm{csch}(x) \to 2\,\mathrm{sgn}(x)\exp\left(-|x|\right) \qquad x \to \pm\infty$$

As its argument declines in magnitude, the hyperbolic cosecant obeys the limiting formula

29:9:3 $$\lim_{x\to 0}\left\{x\,\mathrm{csch}(x)\right\} = 1$$

irrespective of the sign of x.

29:10 OPERATIONS OF THE CALCULUS

Differentiation of the hyperbolic secant or hyperbolic cosecant gives

29:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x) = -\mathrm{sech}(x)\sqrt{1-\mathrm{sech}^2(x)}$$

29:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{csch}(x) = -\mathrm{csch}(x)\coth(x) = -\mathrm{csch}(x)\sqrt{1+\mathrm{csch}^2(x)}$$

The indefinite integration of these functions yields rather complicated results:

29:10:3 $$\int_0^x \mathrm{sech}(t)\,\mathrm{d}t = \arctan\big(\sinh(x)\big) = \mathrm{gd}(x)$$

where gd is the gudermannian function discussed in Section 33:14, and

29:10:4 $$\int_x^\infty \mathrm{csch}(t)\,\mathrm{d}t = \ln\left(\coth\left(\frac{x}{2}\right)\right) \qquad x>0$$

but their squares integrate more simply as

29:10:5 $$\int_0^x \mathrm{sech}^2(t)\,\mathrm{d}t = \tanh(x)$$

and

29:10:6 $$\int_x^\infty \mathrm{csch}^2(t)\,\mathrm{d}t = \coth(x)-1 \qquad x>0$$

Indefinite integration of the square roots of the hyperbolic secant and cosecant function generates special cases of the incomplete elliptic integral of the first kind [Chapter 62] in which the modulus equals $1/\sqrt{2}$.

29:10:7 $$\int_0^x \sqrt{\mathrm{sech}(t)}\,\mathrm{d}t = \sqrt{2}\ \mathrm{F}\left(\frac{1}{\sqrt{2}};\varphi\right) \qquad \sin(\varphi) = \sqrt{1-\mathrm{sech}(x)}$$

29:10:8 $$\int_0^x \sqrt{\mathrm{csch}(t)}\,\mathrm{d}t = \mathrm{F}\left(\frac{1}{\sqrt{2}};\varphi\right) \qquad \cos(\varphi) = \frac{\mathrm{csch}(x)-1}{\mathrm{csch}(x)+1}$$

For a generalization to arbitrary power, see Section 58:14.

Useful definite integrals include

29:10:9 $$\int_0^\infty t^n\,\mathrm{sech}(t)\,\mathrm{d}t = 2n!\beta(n+1) \qquad n=0,1,2,\cdots$$

and

29:10:10 $$\int_0^\infty t^n\,\mathrm{csch}(t)\,\mathrm{d}t = 2n!\lambda(n+1) \qquad n=1,2,3,\cdots$$

where Chapter 3 describes the beta and lambda numbers.

Whereas the definite integral over $0 \le x \le \infty$ of csch(x) diverges, this is not true of sech(x):

29:10:11 $$\int_0^\infty \mathrm{sech}(t)\,\mathrm{d}t = \frac{\pi}{2}$$

Between the same limits, the integrals of the square roots of both functions converge:

29:10:12 $$\int_0^\infty \sqrt{\mathrm{sech}(t)}\,\mathrm{d}t = \pi g$$

and

29:10:13 $$\int_0^\infty \sqrt{\mathrm{csch}(t)}\,\mathrm{d}t = \sqrt{2}\,\pi g$$

where g is Gauss's constant [Section 1:7]. These three integrals are the "complete" versions of the indefinite integrals 29:10:3, 29:10:7, and 29:10:8.

The digamma function [Chapter 44] appears in expressions for the Laplace transforms of the hyperbolic secant and its square:

29:10:14 $$\int_0^\infty \mathrm{sech}(bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\mathrm{sech}(bt)\} = \frac{1}{2b}\left[\psi\left(\frac{s+3b}{4b}\right) - \psi\left(\frac{s+b}{4b}\right)\right]$$

29:10:15 $$\int_0^\infty \mathrm{sech}^2(bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\mathrm{sech}^2(bt)\} = \frac{s}{2b^2}\left[\psi\left(\frac{s+2b}{4b}\right) - \psi\left(\frac{s}{4b}\right)\right] - \frac{1}{b}$$

29:11 COMPLEX ARGUMENT

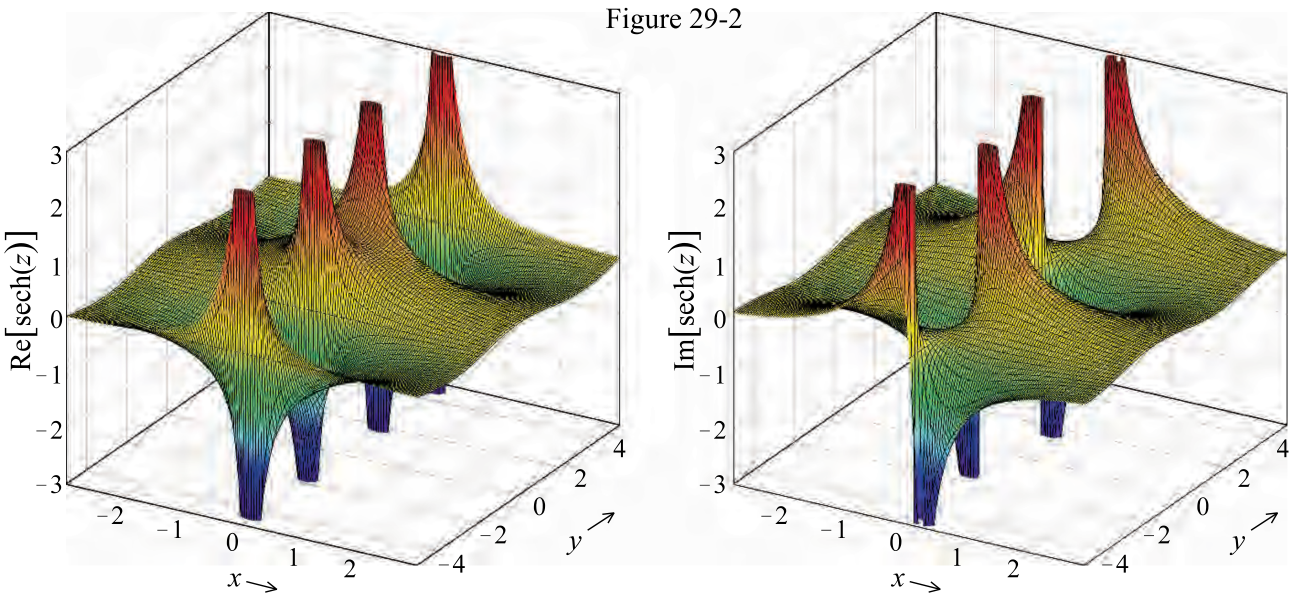
The real and imaginary parts of the hyperbolic secant of complex argument, displayed in Figure 29-2, are contained in the formula

29:11:1 $$\mathrm{sech}(x+iy) = \frac{2\cosh(x)\cos(y) - 2i\sinh(x)\sin(y)}{\cosh(2x)+\cos(2y)}$$

Note that despite the discontinuities that occur on the imaginary axis at $y = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$, the imaginary part of sech(0+iy) is zero elsewhere. For purely imaginary argument

29:11:2 $$\mathrm{sech}(iy) = \frac{2\cos(y)}{1+\cos(2y)}$$

Figure 29-2



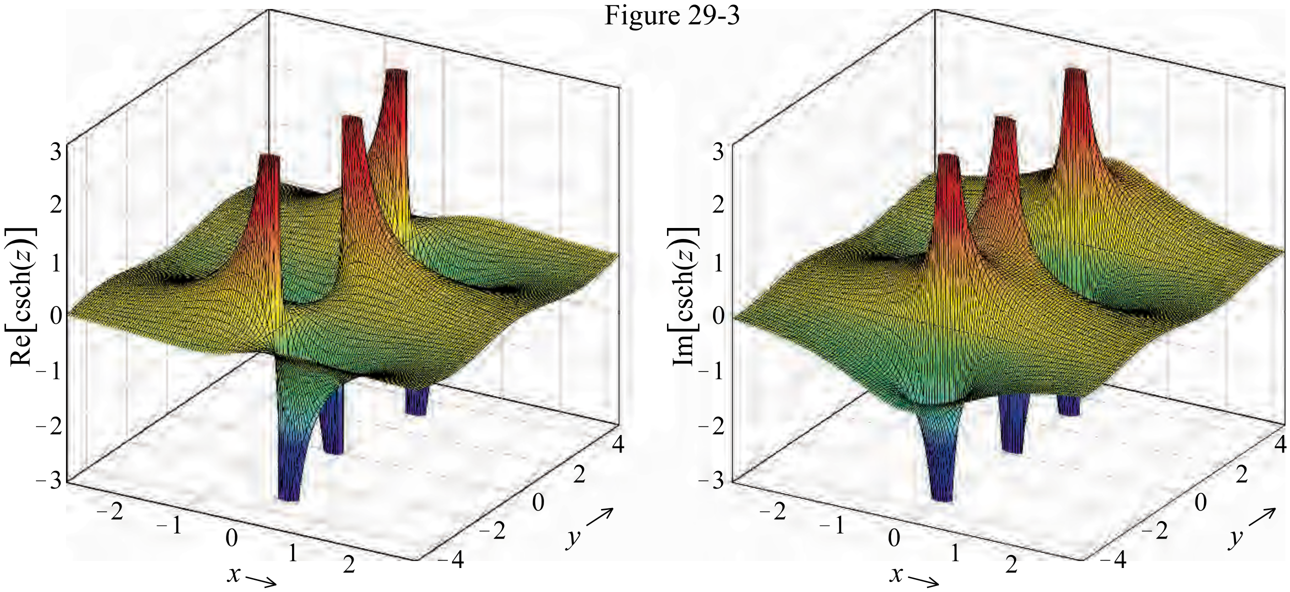
The real and imaginary parts of the hyperbolic cosecant of complex argument, displayed in Figure 29-3, are contained in the formula

29:11:3 $$\mathrm{csch}(x+iy)=\frac{2\sinh(x)\cos(y)-2i\cosh(x)\sin(y)}{\cosh(2x)-\cos(2y)}$$

Discontinuities occur on the imaginary axis at $y=0,\pm\pi,\pm2\pi,\pm3\pi,\cdots$, but the real part of csch(0+iy) is zero elsewhere. For purely imaginary argument

29:11:4 $$\mathrm{csch}(iy)=\frac{-2i\sin(y)}{1-\cos(2y)}$$

Figure 29-3



Some inverse Laplace transforms relating to the hyperbolic secant and hyperbolic cosecant yield exponential theta functions [Section 27:14] and variants thereof:

29:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\mathrm{sech}(v\sqrt{s})}{\sqrt{s}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{G}\left\{\frac{\mathrm{sech}(v\sqrt{s})}{\sqrt{s}}\right\}=\frac{1}{v}\hat{\theta}_2\left(\frac{1}{2},\frac{t}{v^2}\right)$$

29:11:6 $$\mathfrak{G}\left\{\mathrm{csch}(v\sqrt{s})\right\}=\frac{-1}{v^2}\left|\frac{\partial}{\partial\lambda}\theta_4\left(\frac{\lambda}{2},\frac{t}{v^2}\right)\right|_{\lambda=0}$$

Others lead to piecewise-linear functions of t as exemplified in Figure 29-4.

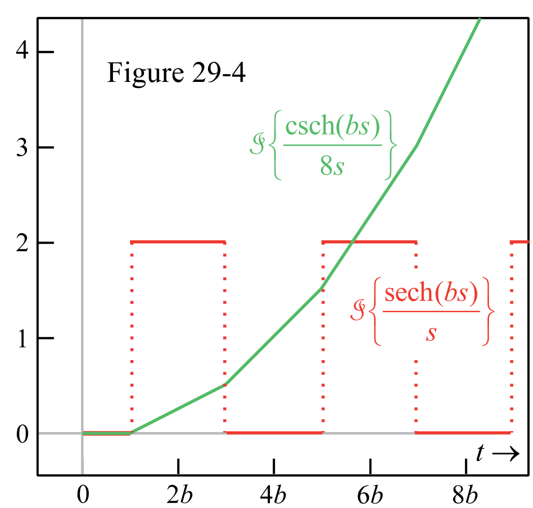


Figure 29-4

29:12 GENERALIZATIONS

The Jacobian elliptic functions cn(k,x) and dn(k,x), discussed in Chapter 63, are generalizations of sech(x), while cs(k,x) and ds(k,x) similarly generalize csch(x).

29:13 COGNATE FUNCTIONS

The following expressions relate the hyperbolic secant and cosecant to the other hyperbolic functions [Chapters 28 and 30]:

29:13:1
$$\mathrm{sech}(x)=\frac{1}{\sqrt{1+\sinh^2(x)}}=\frac{1}{\cosh(x)}=\frac{|\mathrm{csch}(x)|}{\sqrt{1+\mathrm{csch}^2(x)}}=\sqrt{1-\tanh^2(x)}=\frac{\sqrt{\coth^2(x)-1}}{|\coth(x)|}$$

29:13:2
$$\mathrm{csch}(x)=\frac{1}{\sinh(x)}=\frac{\mathrm{sgn}(x)}{\sqrt{\cosh^2(x)-1}}=\frac{\mathrm{sgn}(x)\,\mathrm{sech}(x)}{\sqrt{1-\mathrm{sech}^2(x)}}=\frac{\sqrt{1-\tanh^2(x)}}{\tanh(x)}=\mathrm{sgn}(x)\sqrt{\coth^2(x)-1}$$

29:14 RELATED TOPIC: representation of hyperbolic functions through triangles

Figure 29-5 depicts three right-angled triangles that are *similar* to each other; that is, of the same shape but different sizes. If the sides marked "1" are all of unity length, then the remaining six sides all have lengths corresponding to the six hyperbolic functions [Chapters 28, 29, 30]. With the aid of this figure, hyperbolic functions can be interrelated either by similarity arguments or by the well-known theorem of Pythagoras (though it appears to have been applied a millennium earlier, Pythagoras of Samos, 569−475 B.C., provided the first proof). By ratioing the green and blue sides of the first two triangles, for example, similarity makes it evident that

29:14:1
$$\frac{\tanh(x)}{\mathrm{sech}(x)}=\frac{\sinh(x)}{1}$$

By applying Pythagoras's theorem to the third triangle, one easily arrives at

29:14:2
$$\mathrm{csch}^2(x)=\coth^2(x)-1$$

In fact, all the information needed to construct equations 28:13:1, 28:13:2, 29:13:1, 29:13:2, 30:13:1, and 30:13:2 is available by inspection of Figure 29-5.

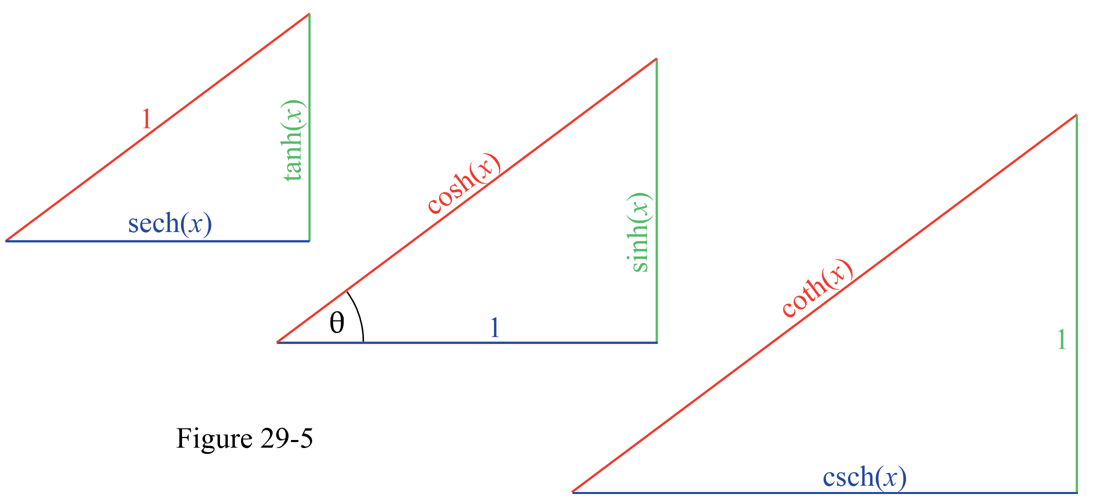


Figure 29-5

The figure raises the question "What is x?" In the corresponding diagram [Section 33:14] for circular functions, the argument x equals the angle marked θ, but for the sextet of hyperbolic functions there is no triangular feature that corresponds directly to the argument x and the angles. There is, however, an indirect relationship to the angle θ via the *inverse gudermannian function* [Section 33:14]; in fact

29:14:3 $$x = \mathrm{invgd}(\theta)$$

For example, when $\theta = \pi/4$,

29:14:4 $$x = \mathrm{invgd}\left(\frac{\pi}{4}\right) = \ln\left(1+\sqrt{2}\right) \approx 0.88137$$

CHAPTER

30

THE HYPERBOLIC TANGENT tanh(*x*) AND COTANGENT coth(*x*) FUNCTIONS

The two functions of this chapter are the reciprocals of each other

30:0:1 $$\tanh(x)\coth(x) = 1$$

and are closely related to the other hyperbolic functions [Chapters 28 and 29].

30:1 NOTATION

The symbolism th(x) sometimes replaces tanh(x) and cth(x) or ctnh(x) sometimes replaces coth(x). The misleading capitalized notations Tan(x) and Cot(x) are occasionally encountered.

30:2 BEHAVIOR

The *hyperbolic tangent* and *hyperbolic cotangent* functions are defined for all real values of their arguments, but each is restricted in its range. The hyperbolic tangent adopts values only within $-1 \le \tanh(x) \le 1$, whereas the coth(x) function assumes all values ≤ -1 and $\ge +1$.

As shown in Figure 30-1, both functions lie exclusively in the first and third quadrants and both approach sgn(x) as $x \to \pm\infty$.

30:3 DEFINITIONS

The most usual definitions of the hyperbolic tangent and hyperbolic cotangent functions are in terms of the functions of Chapter 28 or their exponential equivalents:

30:3:1 $$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\exp(2x)-1}{\exp(2x)+1}$$

DOI 10.1007/978-0-387-48807-3_31,

and

30:3:2 $$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{\exp(2x)+1}{\exp(2x)-1} = \frac{1}{\tanh(x)}$$

The hyperbolic tangent may be defined with respect to the geometry of a *rectangular hyperbola*. Refer to Figure 28-2. With the shaded area *ar* serving as argument, the hyperbolic tangent is the length of the straight line connecting points R and A:

30:3:3 $$\mathrm{RA} = \tanh(ar)$$

In the diagram, point A is the apex of the rectangular hyperbola and R is the point on the rotating vector vertically above A.

Each of the tanh(x) and coth(x) functions satisfies the differential equation

30:3:4 $$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 - f^2$$

providing yet another definition.

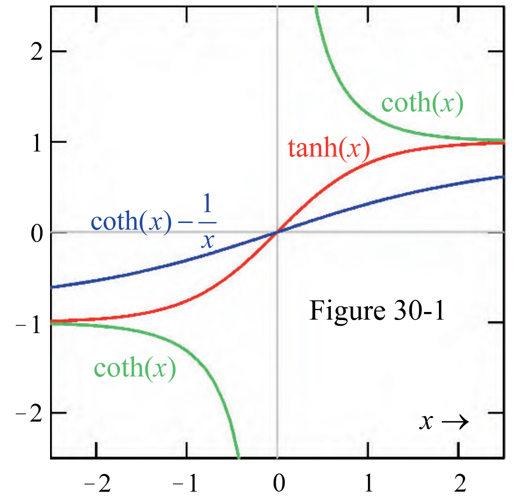


Figure 30-1

30:4 SPECIAL CASES

There are none.

30:5 INTRARELATIONSHIPS

Both functions are odd:

30:5:1 $$f(-x) = -f(x) \qquad f = \tanh \text{ or } \coth$$

The duplication formulas

30:5:2 $$\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)} = \frac{2}{\tanh(x)+\coth(x)}$$

30:5:3 $$\coth(2x) = \frac{\coth^2(x)+1}{2\coth(x)} = \frac{\coth(x)+\tanh(x)}{2}$$

are special cases of the argument-addition expressions

30:5:4 $$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

and

30:5:5 $$\coth(x \pm y) = \frac{1 \pm \coth(x)\coth(y)}{\coth(x) \pm \coth(y)}$$

The equations in this paragraph may be used to build formulas for tanh($3x$), coth($4x$), etc.

The half-argument formulas

30:5:6 $$\tanh\left(\frac{x}{2}\right) = \frac{1-\sqrt{1-\tanh^2(x)}}{\tanh(x)} = \coth(x) - \mathrm{sgn}(x)\sqrt{\coth^2(x)-1} = \coth(x) - \mathrm{csch}(x)$$

and

30:5:7 $$\coth\left(\frac{x}{2}\right)=\coth(x)+\mathrm{sgn}(x)\sqrt{\coth^2(x)-1}=\frac{1+\sqrt{1-\tanh^2(x)}}{\tanh(x)}=\coth(x)+\mathrm{csch}(x)$$

are useful.

The sums and differences of the functions of this chapter may be expressed in terms of hyperbolic sines and hyperbolic cosines:

30:5:8 $$\tanh(x)\pm\tanh(y)=\frac{\sinh(x\pm y)}{\cosh(x)\cosh(y)}$$

30:5:9 $$\coth(x)\pm\tanh(y)=\frac{\cosh(x\pm y)}{\sinh(x)\cosh(y)}$$

30:5:10 $$\coth(x)\pm\coth(y)=\frac{\sinh(x\pm y)}{\sinh(x)\sinh(y)}$$

30:6 EXPANSIONS

The hyperbolic tangent and hyperbolic cotangent functions may be expanded as power series

30:6:1 $$\tanh(x)=x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\cdots=\frac{-2}{x}\sum_{j=1}^{\infty}\lambda(2j)\left(\frac{-4x^2}{\pi^2}\right)^j=\frac{1}{x}\sum_{j=1}^{\infty}\frac{(4^j-1)\mathrm{B}_{2j}}{(2j)!}(4x^2)^j \qquad \frac{-\pi}{2}<x<\frac{\pi}{2}$$

30:6:2 $$\coth(x)=\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}-\cdots=\frac{1}{x}-\frac{2}{x}\sum_{j=1}^{\infty}\zeta(2j)\left(\frac{-x^2}{\pi^2}\right)^j=\frac{1}{x}\sum_{j=0}^{\infty}\frac{\mathrm{B}_{2j}}{(2j)!}(4x^2)^j \qquad -\pi<x<\pi$$

Here, either the lambda and zeta numbers [Chapter 3] or the Bernoulli numbers [Chapter 4] can serve as coefficients.

The functions may be expanded as rapidly convergent Laurent series of exponentials [Section 27:13]:

30:6:3 $$\tanh(x)=2\,\mathrm{sgn}(x)\left[\tfrac{1}{2}-\exp\left(-2|x|\right)+\exp\left(-4|x|\right)-\exp\left(-6|x|\right)+\cdots\right]=\mathrm{sgn}(x)\sum_{j=-\infty}^{+\infty}(-)^j\exp\left(-2|jx|\right)$$

30:6:4 $$\coth(x)=2\,\mathrm{sgn}(x)\left[\tfrac{1}{2}+\exp\left(-2|x|\right)+\exp\left(-4|x|\right)+\exp\left(-6|x|\right)+\cdots\right]=\mathrm{sgn}(x)\sum_{j=-\infty}^{+\infty}\exp\left(-2|jx|\right)$$

The following partial-fraction expansions hold:

30:6:5 $$\tanh(x)=\frac{8x}{\pi^2+4x^2}+\frac{8x}{9\pi^2+4x^2}+\frac{8x}{25\pi^2+4x^2}+\cdots=\sum_{j=0}^{\infty}\frac{8x}{(2j+1)^2\pi^2+4x^2}$$

30:6:6 $$\coth(x)=\frac{1}{x}+\frac{2x}{\pi^2+x^2}+\frac{2x}{4\pi^2+x^2}+\frac{2x}{9\pi^2+x^2}+\cdots=\sum_{j=-\infty}^{\infty}\frac{x}{j^2\pi^2+x^2}$$

as well as the continued-fraction expansion

30:6:7 $$\tanh(x)=\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\,\frac{x^2}{7+\cdots}$$

30:7 PARTICULAR VALUES

	$x=-\infty$	$x=-1$	$x=\ln(\sqrt{2}-1)$	$x=\frac{-1}{2}$	$x=0$	$x=\frac{1}{2}$	$x=\ln(\sqrt{2}+1)$	$x=1$	$x=\infty$
tanh(x)	-1	$\frac{1-e^2}{1+e^2}$	$-\sqrt{2}$	$\frac{1-e}{1+e}$	0	$\frac{e-1}{e+1}$	$\sqrt{2}$	$\frac{e^2-1}{e^2+1}$	1
coth(x)	-1	$\frac{1+e^2}{1-e^2}$	$\frac{-1}{\sqrt{2}}$	$\frac{1+e}{1-e}$	$-\infty\mid+\infty$	$\frac{e+1}{e-1}$	$\frac{1}{\sqrt{2}}$	$\frac{e^2+1}{e^2-1}$	1

30:8 NUMERICAL VALUES

Equator's hyperbolic tangent function and hyperbolic cotangent function routines (keywords **tanh** and **coth**) can provide exact values for all arguments that can be input.

The hyperbolic tangent function rapidly approaches unity as its argument increases. Should accurate values of $1-\tanh(x)$ be needed for large x, use the formula

30:8:1
$$1-\tanh(x)=2\left[\exp(-2x)-\exp(-4x)+\exp(-6x)-\cdots\right]$$

30:9 LIMITS AND APPROXIMATIONS

Close to an argument of zero, one or more early terms in expansions 30:6:1 and 30:6:2 provide approximations. Likewise for arguments of large magnitude, early terms in 30:6:3 and 30:6:4 suffice.

30:10 OPERATIONS OF THE CALCULUS

Differentiation and indefinite integration give

30:10:1
$$\frac{d}{dx}\tanh(bx)=b\,\mathrm{sech}^2(bx)=b\left[1-\tanh^2(bx)\right]$$

30:10:2
$$\frac{d}{dx}\coth(bx)=-b\,\mathrm{csch}^2(bx)=b\left[1-\coth^2(bx)\right]$$

30:10:3
$$\int_0^x \tanh(bt)\,dt=\frac{1}{b}\ln\left(\cosh(bx)\right)$$

30:10:4
$$\int_{x_0}^x \coth(t)\,dt=\ln\left(\sinh(x)\right) \qquad x>0 \qquad x_0=\ln\left(1+\sqrt{2}\right)=0{\cdot}88137\,35870\,19543$$

30:10:5
$$\int_x^\infty \left[\coth(t)-1\right]dt=-\ln\left(1-\exp(-2x)\right) \qquad x>0$$

30:10:6 $$\int_x^\infty [1-\tanh(t)]\,\mathrm{d}t = \ln\big(1+\exp(-2x)\big)$$

30:10:7 $$\int_0^x \tanh^2(t)\,\mathrm{d}t = x - \tanh(x)$$

30:10:8 $$\int_1^x \coth^2(t)\,\mathrm{d}t = \frac{2}{e^2-1} + x - \coth(x)$$

Some Laplace transforms of the hyperbolic tangent and hyperbolic cotangent functions include

30:10:9 $$\int_0^\infty \tanh(bt)\exp(-st)\,\mathrm{d}t = \mathcal{L}\{\tanh(bt)\} = \frac{1}{2b}\left[\psi\left(\frac{s+2b}{4b}\right) - \psi\left(\frac{s}{4b}\right)\right] - \frac{1}{s}$$

30:10:10 $$\int_0^\infty t^v \coth(t)\exp(-st)\,\mathrm{d}t = \mathcal{L}\{t^v\coth(t)\} = \Gamma(1+v)\left[\frac{\zeta(1+v,\tfrac{1}{2}s)}{2^v} - \frac{1}{s^{1+v}}\right] \qquad v>0$$

and involve functions from Chapters 44, 43, and 64.

30:11 COMPLEX ARGUMENT

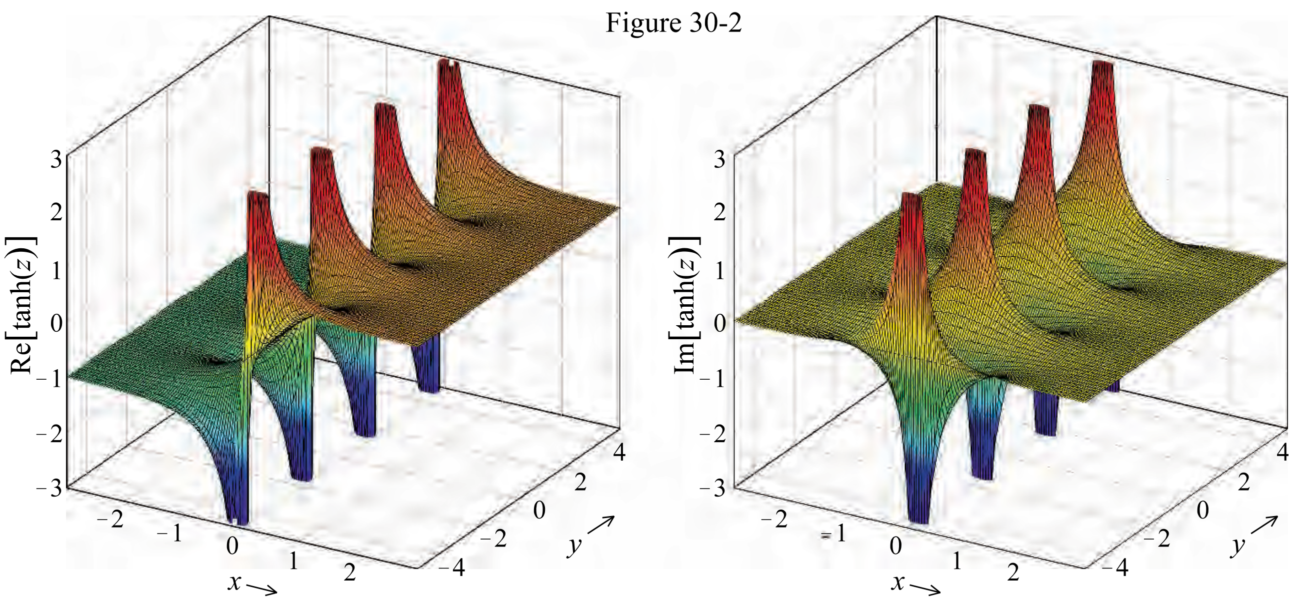
Figure 30-2 shows the real and imaginary parts of the hyperbolic tangent of a complex argument z, which in equation form are

30:11:1 $$\mathrm{Re}\{\tanh(x+iy)\} = \frac{\cosh(x)\sinh(x)}{\sinh^2(x)+\cos^2(y)} = \frac{\sinh(2x)}{\cosh(2x)+\cos(2y)}$$

and

30:11:2 $$\mathrm{Im}\{\tanh(x+iy)\} = \frac{\cos(y)\sin(y)}{\sinh^2(x)+\cos^2(y)} = \frac{\sin(2y)}{\cosh(2x)+\cos(2y)}$$

Figure 30-2



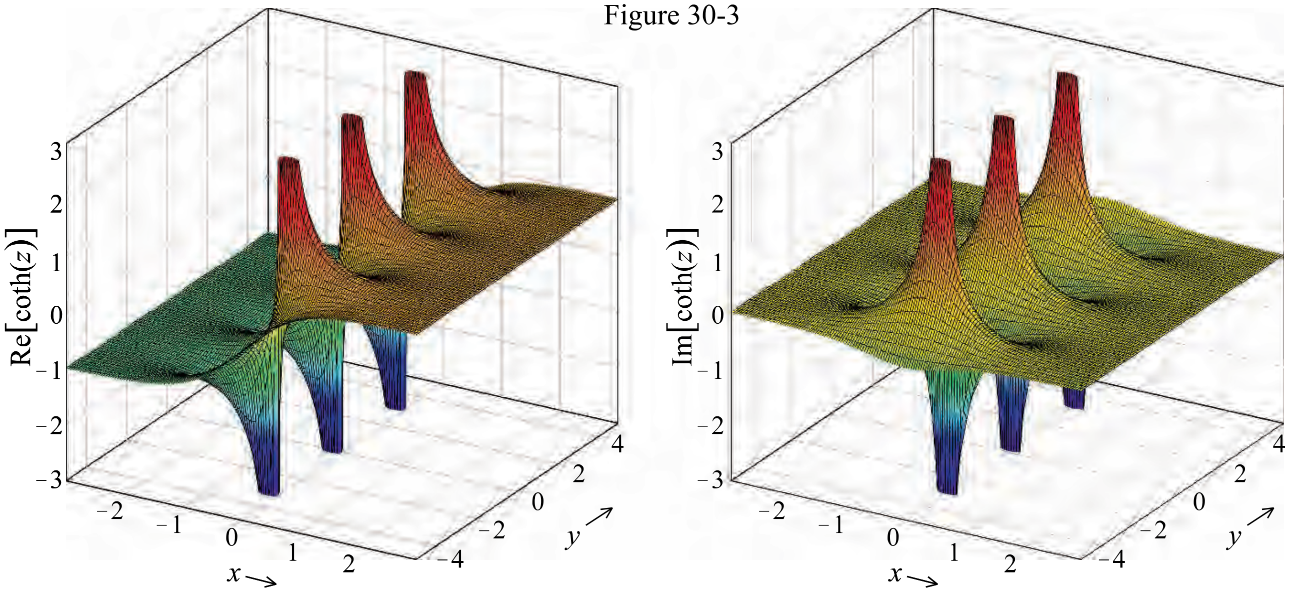


Figure 30-3

The corresponding formulas for the hyperbolic cotangent, shown in Figure 30-3, follow from

30:11:3 $$\coth(x+iy) = \frac{\cosh(x)\sinh(x) - i\cos(y)\sin(y)}{\sinh^2(x)+\sin^2(y)} = \frac{\sinh(2x)-i\sin(2y)}{\cosh(2x)-\cos(2y)}$$

These equations imply, and the figures confirm, that both functions are periodic in y, with a period of π. Moreover, comparison of Figures 30-2 and 30-3 suggests that tanh(z) and coth(z) differ only by translation along the imaginary axis and indeed

30:11:4 $$\coth(x+iy) = \tanh\left(x + i(y+\tfrac{1}{2}\pi)\right)$$

For a purely imaginary argument

30:11:5 $$\tanh(iy) = \frac{i\sin(2y)}{1+\cos(2y)}$$

and

30:11:6 $$\coth(iy) = \frac{-i\sin(2y)}{1-\cos(2y)}$$

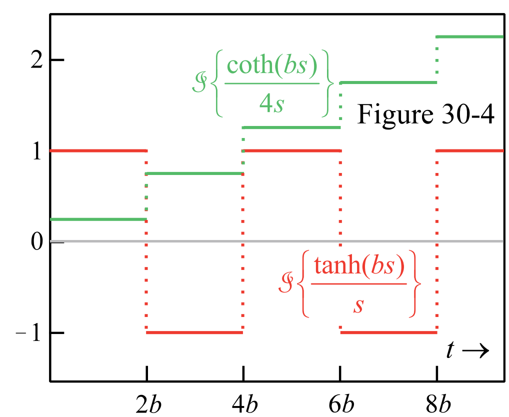


Figure 30-4

Some inverse Laplace transforms of the hyperbolic tangent function and the hyperbolic cotangent function give discontinuous functions such as those illustrated in Figure 30-4. Others yield exponential theta functions of zero period [Section 27:13]

30:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\tanh\left(b\sqrt{s}\right)}{\sqrt{s}} \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\frac{\tanh\left(b\sqrt{s}\right)}{\sqrt{s}}\right\} = \frac{1}{b}\theta_2\left(0, \frac{t}{b^2}\right)$$

30:11:8 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\coth\left(b\sqrt{s}\right)}{\sqrt{s}} \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\frac{\coth\left(b\sqrt{s}\right)}{\sqrt{s}}\right\} = \frac{1}{b}\theta_3\left(0, \frac{t}{b^2}\right)$$

30:12 GENERALIZATIONS

The Jacobian elliptic functions sn(k,x) and ns(k,x) [Chapter 63] may be regarded as generalizations of tanh(x) and coth(x) respectively. As $k \to 1$, $\mathrm{sn}(k,x) \to \tanh(x)$ and $\mathrm{ns}(k,x) \to \coth(x)$.

30:13 COGNATE FUNCTIONS

The expressions

30:13:1 $$\tanh(x) = \frac{\sinh(x)}{\sqrt{1+\sinh^2(x)}} = \frac{\sqrt{\cosh^2(x)-1}}{\mathrm{sgn}(x)\cosh(x)} = \mathrm{sgn}(x)\sqrt{1-\mathrm{sech}^2(x)} = \frac{\mathrm{sgn}(x)}{\sqrt{1+\mathrm{csch}^2(x)}} = \frac{1}{\coth(x)}$$

and

30:13:2 $$\coth(x) = \frac{\sqrt{1+\sinh^2(x)}}{\sinh(x)} = \frac{\mathrm{sgn}(x)\cosh(x)}{\sqrt{\cosh^2(x)-1}} = \frac{\mathrm{sgn}(x)}{\sqrt{1-\mathrm{sech}^2(x)}} = \mathrm{sgn}(x)\sqrt{1+\mathrm{csch}^2(x)} = \frac{1}{\tanh(x)}$$

relate the tangent and cotangent to other members of the hyperbolic family. Figure 29-5 is useful in expressing these relationships.

30:14 RELATED TOPIC: the Langevin function

The *Langevin function* (Paul Langevin, French physicist, 1872 – 1946)

30:14:1 $$\coth(x) - \frac{1}{x}$$

is important in the theory of dielectrics, where it arises as the averaging procedure

30:14:2 $$\frac{\int_0^\pi \exp\{x\cos(\theta)\}\sin(\theta)\cos(\theta)\mathrm{d}\theta}{\int_0^\pi \exp\{x\cos(\theta)\}\sin(\theta)\mathrm{d}\theta} = \frac{\int_{-x}^{x} t\exp(t)\mathrm{d}t}{x\int_{-x}^{x}\exp(t)\mathrm{d}t} = \coth(x) - \frac{1}{x} \qquad t = x\cos(\theta)$$

applied to dipoles in an electric or magnetic field. *Equator* has a Langevin function routine with keyword **Langevin**.

Figure 30-1 shows a graph of the Langevin function and equation 30:6:2 can be adapted to express its power series expansion in two equivalent ways, involving either the zeta or Bernoulli numbers. The continued fraction expression

30:14:3 $$\coth(x) - \frac{1}{x} = \frac{\frac{1}{3}x}{1+}\,\frac{\frac{1}{15}x^2}{1+}\,\frac{\frac{1}{35}x^2}{1+}\,\frac{\frac{1}{63}x^2}{1+}\,\frac{\frac{1}{99}x^2}{1+\cdots}$$

applies. The Langevin function itself, and its reciprocal, may be expanded as the partial fractions:

30:14:4 $$\coth(x) - \frac{1}{x} = 2x\sum_{j=1}^{\infty}\frac{1}{x^2 + j^2\pi^2}$$

30:14:5 $$\frac{1}{\coth(x) - (1/x)} = \frac{3}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2 + \mathrm{r}_n^2(1)}$$

In the latter expansion $\mathrm{r}_n(1)$ is the nth root of the equation $\tan(x) = x$ [Section 34:7].

Formulas for the indefinite integral and Laplace transform of the Langevin function are

30:14:6
$$\int_0^x \left[\coth(t) - \frac{1}{t}\right]\mathrm{d}t = \ln\left(\frac{\sinh(x)}{x}\right)$$

and

30:14:7
$$\int_0^\infty \left[\coth(t) - \frac{1}{t}\right]\exp(-st)\mathrm{d}t = \mathcal{L}\left\{\coth(t) - \frac{1}{t}\right\} = \ln\left(\frac{s}{2}\right) - \psi\left(\frac{s}{2}\right) - \frac{1}{s}$$

where ψ is the digamma function [Chapter 44].

CHAPTER 31

THE INVERSE HYPERBOLIC FUNCTIONS

There are six *inverse hyperbolic functions* – the *inverse hyperbolic sine*, the *inverse hyperbolic cosine*, the *inverse hyperbolic secant*, the *inverse hyperbolic cosecant*, the *inverse hyperbolic tangent*, and the *inverse hyperbolic cotangent*. Each of these functions can be replaced by any one of the other five, provided that the argument is changed in accord with the table that follows.

	f = arsinh	f = arcosh	f = arsech	f = arcsch	f = artanh	f = arcoth
arsinh(x) =	$\mathrm{f}(x)$	$\sigma\,\mathrm{f}\left(\sqrt{1+x^2}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$	$\mathrm{f}\left(\dfrac{1}{x}\right)$	$\mathrm{f}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$	$\mathrm{f}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$
arcosh(x) = $x \ge 1$	$\mathrm{f}\left(\sqrt{x^2-1}\right)$	$\mathrm{f}(x)$	$\mathrm{f}\left(\dfrac{1}{x}\right)$	$\mathrm{f}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$	$\mathrm{f}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$	$\mathrm{f}\left(\dfrac{x}{\sqrt{x^2-1}}\right)$
arsech(x) = $0 \le x \le 1$	$\mathrm{f}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$	$\mathrm{f}\left(\dfrac{1}{x}\right)$	$\mathrm{f}(x)$	$\mathrm{f}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$	$\mathrm{f}\left(\sqrt{1-x^2}\right)$	$\mathrm{f}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$
arcsch(x) = $x \ne 0$	$\sigma\,\mathrm{f}\left(\dfrac{1}{x}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$	$\mathrm{f}(x)$	$\sigma\,\mathrm{f}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$	$\sigma\,\mathrm{f}\left(\sqrt{1+x^2}\right)$
artanh(x) = $-1 \le x \le 1$	$\mathrm{f}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$	$\sigma\,\mathrm{f}\left(\sqrt{1-x^2}\right)$	$\mathrm{f}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$	$\mathrm{f}(x)$	$\mathrm{f}\left(\dfrac{1}{x}\right)$
arcoth(x) = $\|x\| > 1$	$\sigma\,\mathrm{f}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{x}{\sqrt{x^2-1}}\right)$	$\sigma\,\mathrm{f}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$	$\sigma\,\mathrm{f}\left(\sqrt{x^2-1}\right)$	$\mathrm{f}\left(\dfrac{1}{x}\right)$	$\mathrm{f}(x)$

In this table σ represents sgn(x).

The inverse hyperbolic sine, cosine, and tangent are the most commonly encountered of the six and, in some of the sections of this chapter, concentration is on these three to the exclusion of the others.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_32, © Springer Science+Business Media, LLC 2009

31:1 NOTATION

The prefix "ar" connotes "area" and its significance stems from the unity-radius rectangular hyperbola in Figure 28-2. In the text accompanying that diagram, it is shown that a particular length x could be defined, for example, as the hyperbolic cosine of an area *ar*.

31:1:1 $$x = \cosh(area) = \cosh(ar)$$

By virtue of the meaning of "inverse function" [Section 0:3], the area *ar* becomes the inverse of hyperbolic cosine of x and an appropriate notation for this inverse function is therefore "arcosh(x)", signifying the area associated with the hyperbolic cosine function:

31:1:2 $$ar = \text{inverse hyperbolic cosine of } x = \operatorname{arcosh}(x)$$

The notations argcosh(x), arccosh(x), arch(x), and $\cosh^{-1}(x)$ all find use as alternatives to arcosh(x), with corresponding usages for the other five inverse hyperbolic functions. Similar notations with a capitalized initial letter – Arcosh(x), Arch(x), $\mathrm{Cosh}^{-1}(x)$, etc. – are sometimes used synonymously with arcosh(x), but more often these notations refer to the multiple-valued functions addressed in Section 31:11.

31:2 BEHAVIOR

Although the six inverse hyperbolic functions, illustrated in Figure 31-1, have diverse behaviors, they all adopt real values only in the first and third quadrants [see Section 0:2 for the significance of "quadrant"].

The inverse hyperbolic sine function has the simplest behavior of the six. It is unlimited in its domain of x and itself adopts all values. The inverse hyperbolic cosecant, arcsch(x), has two branches as mapped in Figure 31:1; it shares the sign of its argument and has a discontinuity of the $-\infty|+\infty$ variety at $x = 0$.

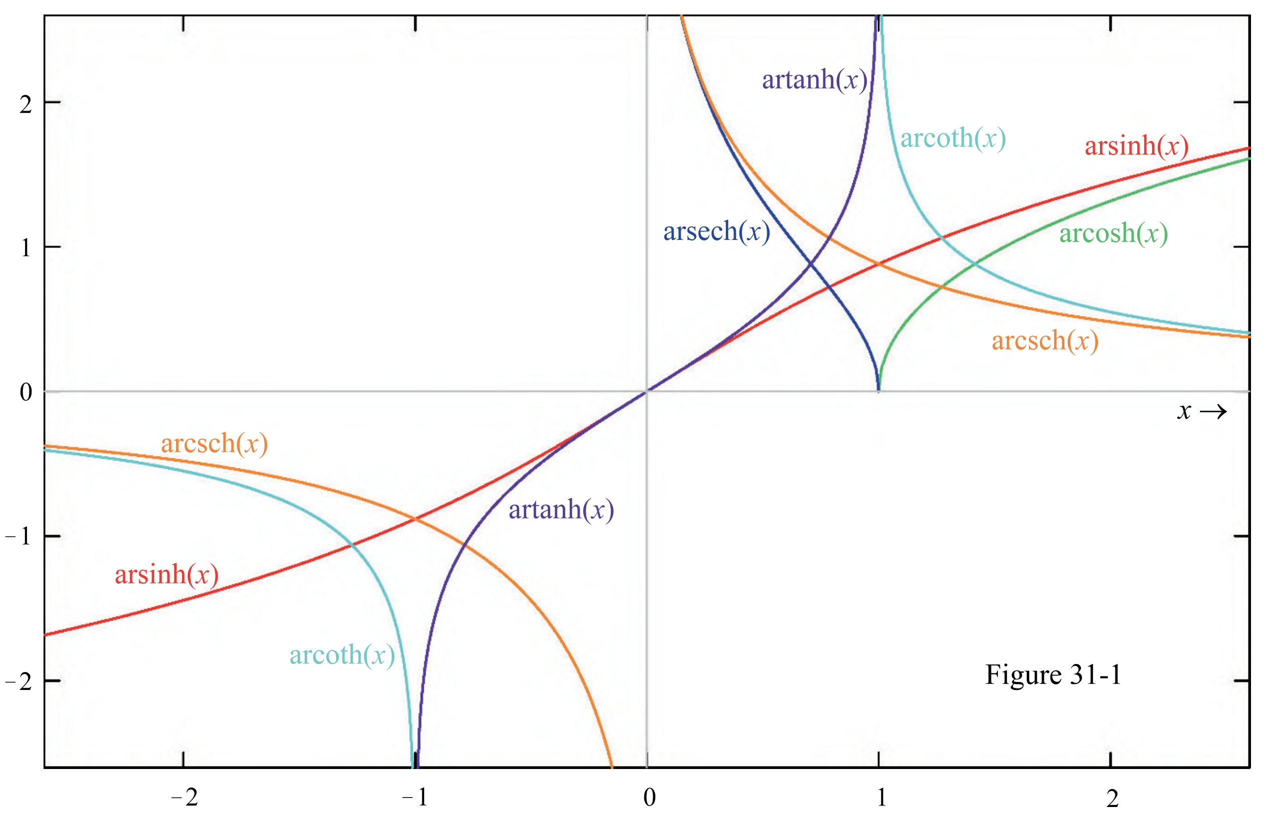


Figure 31-1

The inverse hyperbolic cosine is defined here for argument $x \ge 1$, although some authors extend its domain of definition to $|x| \ge 1$ via $\mathrm{arcosh}(-x) = \mathrm{arcosh}(x)$. Likewise the inverse hyperbolic secant is treated in this *Atlas* as having a domain of only $0 \le x \le 1$, though you may encounter its extension to $-1 \le x \le 0$ with $\mathrm{arsech}(-x) = \mathrm{arsech}(x)$. Both arcosh and arsech are invariably nonnegative.

The domain of the inverse hyperbolic tangent is $-1 \le x \le 1$ and the value of the function approaches $\pm\infty$ as $x \to \pm 1$. The inverse hyperbolic cotangent has two branches. For $1 \le x \le \infty$, $\mathrm{arcoth}(x)$ is positive, whereas it is negative when $-\infty \le x \le -1$, and undefined as a real function in the $-1 < x < 1$ gap.

31:3 DEFINITIONS

Definitions based on the geometry of the rectangular hyperbola are mentioned in Section 31:1, but the most usual definitions fall into three distinct categories: as inverses of the hyperbolic functions, as indefinite integrals of functions from Chapters 13–15, and as logarithms of modified argument.

Four of the six inverse hyperbolic function are defined straightforwardly as the inverse function of the corresponding hyperbolic functions:

31:3:1 $$\text{if} \quad x = \sinh(f) \quad \text{then} \quad f = \mathrm{arsinh}(x)$$

31:3:2 $$\text{if} \quad x = \mathrm{csch}(f) \quad \text{then} \quad f = \mathrm{arcsch}(x)$$

31:3:3 $$\text{if} \quad x = \tanh(f) \quad \text{then} \quad f = \mathrm{artanh}(x)$$

31:3:4 $$\text{if} \quad x = \coth(f) \quad \text{then} \quad f = \mathrm{arcoth}(x)$$

but, to avoid doubly valued functions, it is necessary to restrict the domain of the inverse hyperbolic cosine and secant functions:

31:3:5 $$\text{if} \quad x = \cosh(f) \quad \text{and} \quad x \ge 0 \quad \text{then} \quad f = \mathrm{arcosh}(x)$$

31:3:6 $$\text{if} \quad x = \mathrm{sech}(f) \quad \text{and} \quad x \ge 0 \quad \text{then} \quad f = \mathrm{arsech}(x)$$

Not all authorities apply such restrictions.

The following indefinite integrals, illustrated in the small figures, define the inverse hyperbolic functions

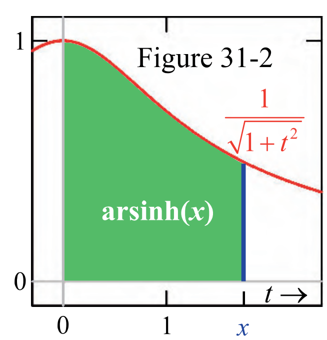


Figure 31-2

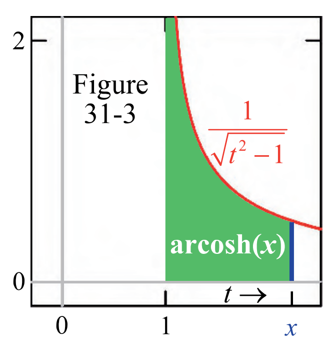


Figure 31-3

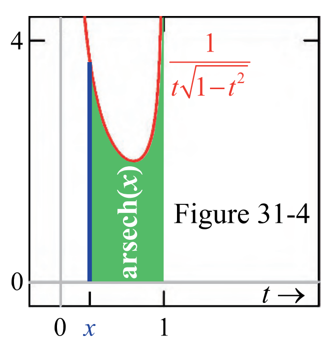


Figure 31-4

31:3:7 $$\mathrm{arsinh}(x) = \int_0^x \frac{\mathrm{d}t}{\sqrt{1+t^2}}$$

31:3:8 $$\mathrm{arcosh}(x) = \int_1^x \frac{\mathrm{d}t}{\sqrt{t^2-1}} \qquad x \ge 1$$

31:3:9 $$\mathrm{arsech}(x) = \int_x^1 \frac{\mathrm{d}t}{t\sqrt{1-t^2}} \qquad 0 \le x \le 1$$

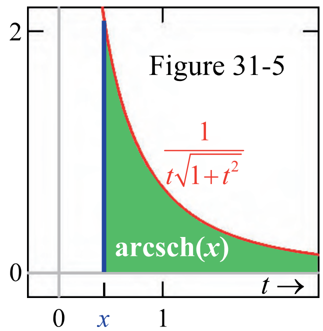


Figure 31-5

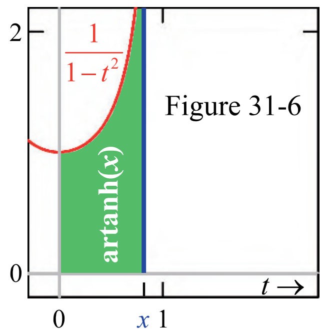


Figure 31-6

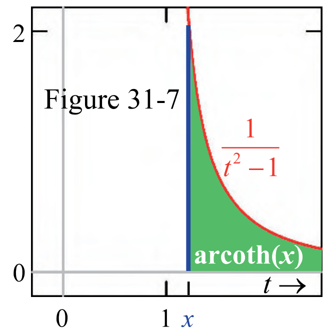


Figure 31-7

31:3:10
$$\operatorname{arcsch}(x) = \begin{cases} \displaystyle\int_x^\infty \frac{\mathrm{d}t}{t\sqrt{1+t^2}} & x \geq 0 \\ \displaystyle\int_{-\infty}^x \frac{\mathrm{d}t}{t\sqrt{1+t^2}} & x \leq 0 \end{cases}$$

31:3:11
$$\operatorname{artanh}(x) = \int_0^x \frac{\mathrm{d}t}{1-t^2} \qquad -1 \leq x \leq 1$$

31:3:12
$$\operatorname{arcoth}(x) = \begin{cases} \displaystyle\int_x^\infty \frac{\mathrm{d}t}{t^2-1} & x \geq 1 \\ \displaystyle -\int_{-\infty}^x \frac{\mathrm{d}t}{t^2-1} & x \leq -1 \end{cases}$$

Logarithmic functions of diverse algebraic arguments provide what are probably the most useful definitions of the inverse hyperbolic functions:

31:3:13
$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

31:3:14
$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geq 1$$

31:3:15
$$\operatorname{arsech}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad 0 \leq x \leq 1$$

31:3:16
$$\operatorname{arcsch}(x) = \ln\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$$

31:3:17
$$\operatorname{artanh}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 \leq x \leq 1$$

31:3:18
$$\operatorname{arcoth}(x) = \ln\left(\sqrt{\frac{x+1}{x-1}}\right) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad |x| \geq 1$$

Those inverse hyperbolic functions that are expansible hypergeometrically can be synthesized [Section 43:14], provided $0 \leq x \leq 1$. Examples include

31:3:19
$$\frac{1}{1-x} \xrightarrow[3/2]{1/2} \frac{\operatorname{artanh}\left(\sqrt{x}\right)}{\sqrt{x}}$$

31:3:20
$$\frac{1}{1+x} \xrightarrow[1]{1/2} \frac{1}{\sqrt{1+x}} \xrightarrow[3/2]{1/2} \frac{\operatorname{arsinh}\left(\sqrt{x}\right)}{\sqrt{x}}$$

31:3:21
$$\frac{1}{1-x} \xrightarrow[2]{1} \frac{-\ln(1-x)}{x} \xrightarrow[2]{3/2} \frac{4}{x}\left[\ln\left(\frac{2}{\sqrt{x}}\right)-\operatorname{arcosh}\left(\frac{1}{\sqrt{x}}\right)\right]$$

31:4 SPECIAL CASES

When its argument is an odd integer, the inverse hyperbolic cotangent reduces to half the difference of the logarithms of two consecutive integers. This rule, exemplified by $\operatorname{arcoth}(3)=\frac{1}{2}\ln(2)$, is

31:4:1
$$\operatorname{arcoth}\left(2n+1\right)=\ln\left(\sqrt{1+\frac{1}{n}}\right)=\frac{\ln(n+1)-\ln(n)}{2} \qquad n=1,2,3,\cdots$$

In fact, this formula, a consequence of definition 31:3:18, applies when n is replaced by any nonzero real number whatsoever.

31:5 INTRARELATIONSHIPS

The table in this chapter's preamble lists the multifarious relationships between different members of the inverse hyperbolic function family. Here we address instances of relationships between values of a single function at two dissimilar arguments.

Four of the six inverse hyperbolic functions are odd

31:5:1
$$\mathrm{f}(-x)=-\mathrm{f}(x) \qquad \mathrm{f}=\text{arsinh, arcsch, artanh, arcoth}$$

but no reflection formula applies to arcosh or arsech.

Several function-addition/subtraction formulas exist for the inverse hyperbolic functions. These include

31:5:2
$$\operatorname{arsinh}(x)\pm\operatorname{arsinh}(y)=\operatorname{arsinh}\left(x\sqrt{1+y^2}\pm y\sqrt{1+x^2}\right)$$

31:5:3
$$\operatorname{arsinh}(x)\pm\operatorname{arcosh}(y)=\operatorname{arsinh}\left(xy\pm\sqrt{\left(x^2+1\right)\left(y^2-1\right)}\right)$$

31:5:4
$$\operatorname{arcosh}(x)\pm\operatorname{arcosh}(y)=\operatorname{arcosh}\left(xy\pm\sqrt{\left(x^2-1\right)\left(y^2-1\right)}\right)$$

31:5:5
$$\operatorname{artanh}(x)\pm\operatorname{artanh}(y)=\operatorname{artanh}\left(\frac{x\pm y}{1\pm xy}\right)$$

31:5:6
$$\operatorname{artanh}(x)\pm\operatorname{arcoth}(y)=\operatorname{artanh}\left(\frac{1\pm xy}{x\pm y}\right)$$

and

31:5:7 $$\operatorname{arcoth}(x) \pm \operatorname{arcoth}(y) = \operatorname{arcoth}\left(\frac{1 \pm xy}{x \pm y}\right)$$

A great many more intrarelationships can be developed from these by using the equivalences tabulated in Section 31:0. Of course, all arguments must lie in acceptable domains for these formulas to be applicable.

Consequences of formula 31:4:1 are the expressions

31:5:8 $$2\sum_{j=n}^{m-1} \operatorname{arcoth}(2j+1) = \ln(m) - \ln(n) \qquad m = n+1, n+2, n+3, \cdots$$

and

31:5:9 $$2\sum_{j=n}^{nm-1} \operatorname{arcoth}(2j+1) = 2\sum_{j=n}^{nm-1} \operatorname{artanh}\left(\frac{1}{2j+1}\right) = \ln(m) \qquad n, m = 1, 2, 3, \cdots$$

for the sums of finite series of inverse hyperbolic cotangents of odd integers.

31:6 EXPANSIONS

The following power series apply:

31:6:1 $$\operatorname{artanh}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2j+1} = x\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j}{\left(\frac{3}{2}\right)_j}\left(x^2\right)^j \qquad |x| < 1$$

31:6:2 $$\operatorname{arsinh}(x) = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \cdots = \sum_{j=0}^{\infty} \frac{(-)^j (2j-1)!!x^{2j+1}}{(2j)!!(2j+1)} = x\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{(1)_j \left(\frac{3}{2}\right)_j}\left(-x^2\right)^j \qquad |x| < 1$$

31:6:3 $$\ln(2x) - \operatorname{arcosh}(x) = \frac{1}{4x^2} + \frac{3}{32x^4} + \frac{5}{96x^6} + \cdots = \sum_{j=1}^{\infty} \frac{(2j-1)!!}{2j(2j)!!x^{2j}} = \frac{1}{4x^2}\sum_{j=0}^{\infty} \frac{(1)_j \left(\frac{3}{2}\right)_j}{(2)_j (2)_j}\left(\frac{1}{x^2}\right)^j \qquad x > 1$$

31:6:4 $$\ln(2|x|) - \operatorname{sgn}(x)\operatorname{arsinh}(x) = \frac{-1}{4x^2} + \frac{3}{32x^4} - \cdots = \sum_{j=1}^{\infty} \frac{(-)^j (2j-1)!!}{2j(2j)!!x^{2j}} = \frac{-1}{4x^2}\sum_{j=0}^{\infty} \frac{(1)_j \left(\frac{3}{2}\right)_j}{(2)_j (2)_j}\left(\frac{-1}{x^2}\right)^j \qquad |x| \ge 1$$

Notice that the final expressions in 31:6:1−4 confirm that the left-hand members of these four equations are hypergeometric functions [Section 18:14]. Replacing x by $1/x$ in these four series provides expansions for $\operatorname{arcoth}(x)$, $\operatorname{arcsch}(x)$, $\ln(2/x) - \operatorname{arsech}(x)$, and $\ln(2/|x|) - \operatorname{sgn}(x)\operatorname{arcsch}(x)$.

The inverse hyperbolic tangent and inverse hyperbolic sine may be expanded as the continued fractions:

31:6:5 $$\operatorname{artanh}(x) = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{4x^2}{5-}\,\frac{9x^2}{7-}\,\frac{16x^2}{9-}\,\frac{25x^2}{11-\cdots}$$

31:6:6 $$\operatorname{arsinh}(x) = \frac{x}{1+}\,\frac{2x^2}{3+}\,\frac{2x^2}{5+}\,\frac{12x^2}{7+}\,\frac{12x^2}{9+}\,\frac{30x^2}{11+}\,\frac{30x^2}{13+}\,\frac{56x^2}{15+\cdots}$$

31:7 PARTICULAR VALUES

The values of certain pairs of inverse hyperbolic functions coalesce when the squares of their arguments equal the golden section υ [given by $(\sqrt{5}+1)/2$, see Section 23:14] or its reciprocal. Denoted in the table by α or β, the values acquired at these points of coalescence are

31:7:1 $$\alpha = \ln\left(\sqrt{\upsilon-1}+\sqrt{\upsilon}\right) = 0.72181\,77375\,89405$$

or

31:7:2 $$\beta = \ln\left(\sqrt{\upsilon}+\sqrt{\upsilon+1}\right) = 1.0612\,75061\,90504$$

See Section 35:7 for a connection to the "golden triangle".

	$x=-\infty$	$x=-\sqrt{\upsilon}$	$x=-1$	$x=\frac{-1}{\sqrt{\upsilon}}$	$x=0$	$x=\frac{1}{\sqrt{\upsilon}}$	$x=1$	$x=\sqrt{\upsilon}$	$x=\infty$
arsinh(x)	$-\infty$	$-\beta$	$\ln\left(\sqrt{2}-1\right)$	$-\alpha$	0	α	$\ln\left(\sqrt{2}+1\right)$	β	∞
arcosh(x)	undef	undef	undef	undef	undef	undef	0	α	∞
arsech(x)	undef	undef	undef	undef	$+\infty$	α	0	undef	undef
arcsch(x)	0	$-\alpha$	$\ln\left(\sqrt{2}-1\right)$	$-\beta$	$-\infty\|+\infty$	β	$\ln\left(\sqrt{2}+1\right)$	α	0
artanh(x)	undef	undef	$-\infty$	$-\beta$	0	β	$+\infty$	undef	undef
arcoth(x)	0	$-\beta$	$-\infty$	undef	undef	undef	$+\infty$	β	0

The entry "undef" in the table means that the function is not defined as a real quantity at the argument in question.

31:8 NUMERICAL VALUES

Equator's keywords for the inverse hyperbolic functions are simply the six letters of the function's symbol. For example, **arsinh** is the keyword for *Equator*'s inverse hyperbolic sine function routine. Generally, an algorithm based on equations 31:3:13−18 is used in each of these routines, but formulas based on series from Section 31:6 or 31:9 are substituted as the argument approaches the limits of its domain.

31:9 LIMITS AND APPROXIMATIONS

For large positive arguments, arsinh(x) and arcosh(x) both approach ln(2x) as a limit and are well approximated by

31:9:1 $$\operatorname{arsinh}(x) \approx \operatorname{sgn}(x)\left[\ln\left(2|x|\right)+\frac{1}{4x^2}+\frac{3}{32x^4}\right] \qquad \text{large } |x|$$

and

31:9:2 $$\operatorname{arcosh}(x) \approx \ln\left(2x\right)-\frac{1}{4x^2}+\frac{1}{16x^4} \qquad \text{large positive } x$$

In the same limit, arcoth(x) and arcsch(x) approach $1/x$ and corresponding approximations are

31:9:3 $$\mathrm{arcoth}(x) \approx \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} \qquad \text{large } |x|$$

and

31:9:4 $$\mathrm{arcsch}(x) \approx \frac{1}{x} - \frac{1}{6x^3} + \frac{3}{40x^5} \qquad \text{large } |x|$$

These approximations originate by truncating series from Section 31:6.

For small magnitudes of x, the following approximations apply:

31:9:5 $$\mathrm{arcsch}(x) \approx \mathrm{sgn}(x)\left[\ln\left(\frac{2}{|x|}\right) + \frac{x^2}{4} - \frac{3x^4}{32}\right] \qquad \text{small } |x|$$

and

31:9:6 $$\mathrm{arsech}(x) \approx \ln\left(\frac{2}{x}\right) - \frac{x^2}{4} - \frac{3x^4}{32} \qquad \text{small positive } x$$

Four of the inverse hyperbolic functions approach either zero or infinity as their arguments approach unity from one directions or the other, as Figure 31-1 makes clear. Similar limits afflict artanh(x) and arcoth(x) as x approaches -1. Useful approximations that hold during these approaches, becoming exact in the limits, are:

31:9:7 $$\mathrm{arcosh}(x) \approx (13 - x)\sqrt{\frac{x-1}{72}} \qquad (x-1) \text{ small and positive}$$

31:9:8 $$\mathrm{arsech}(x) \approx (17 - 5x)\sqrt{\frac{1-x}{72}} \qquad (1-x) \text{ small and positive}$$

31:9:9 $$\mathrm{artanh}(x) \approx \frac{-\mathrm{sgn}(x)}{2}\left[\ln\left(\frac{1-|x|}{2}\right) + \frac{1-|x|}{2}\right] \qquad \left(1-|x|\right) \text{ small}$$

and

31:9:10 $$\mathrm{arcoth}(x) \approx \frac{-\mathrm{sgn}(x)}{2}\left[\ln\left(\frac{|x|-1}{2}\right) - \frac{|x|-1}{2}\right] \qquad \left(|x|-1\right) \text{ small}$$

31:10 OPERATIONS OF THE CALCULUS

The derivatives of the six inverse hyperbolic functions are

	f = arsinh	f = arcosh	f = arsech	f = arcsch	f = artanh	f = arcoth
$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x) =$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{1}{\sqrt{x^2-1}}$	$\frac{-1}{x\sqrt{1-x^2}}$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$	$\frac{1}{1-x^2}$	$\frac{-1}{x^2-1}$

while their indefinite integrals are

31:10:1 $$\int_0^x \mathrm{arsinh}(t)\,\mathrm{d}t = x\,\mathrm{arsinh}(x) + 1 - \sqrt{1+x^2}$$

31:10:2 $$\int_1^x \operatorname{arcosh}(t)\,\mathrm{d}t = x\operatorname{arcosh}(x) - \sqrt{x^2-1} \qquad x \geq 1$$

31:10:3 $$\int_x^1 \operatorname{arsech}(t)\,\mathrm{d}t = \arccos(x) - x\operatorname{arsech}(x) \qquad 0 \leq x \leq 1$$

31:10:4 $$\int_1^x \operatorname{arcsch}(t)\,\mathrm{d}t = x\operatorname{arcsch}(x) + \left|\operatorname{arsinh}(x)\right| - \ln\left(3+\sqrt{8}\right) \qquad x \neq 0$$

31:10:5 $$\int_0^x \operatorname{artanh}(t)\,\mathrm{d}t = x\operatorname{artanh}(x) + \ln\left(\sqrt{1-x^2}\right) \qquad |x| < 1$$

and

31:10:6 $$\int_2^x \operatorname{arcoth}(t)\,\mathrm{d}t = x\operatorname{arcoth}(x) + \ln\left(\sqrt{\frac{x^2-1}{27}}\right) \qquad x > 1$$

The semiderivative [Section 12:14]

31:10:7 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\operatorname{artanh}\left(\sqrt{x}\right) = \frac{1}{2}\sqrt{\frac{\pi}{1-x}}$$

links the inverse hyperbolic tangent to a simple algebraic function.

Some definite integrals and Laplace transforms of the inverse hyperbolic functions include:

31:10:8 $$\int_0^1 \frac{\operatorname{artanh}(t)}{t}\mathrm{d}t = \int_1^\infty \frac{\operatorname{arcoth}(t)}{t}\mathrm{d}t = \frac{\pi^2}{8}$$

31:10:9 $$\int_0^\infty \operatorname{arsinh}(bt)\exp(-st)\,\mathrm{d}t = \mathscr{L}\{\operatorname{arsinh}(bt)\} = \frac{\pi}{2s}\left[\mathrm{h}_0\left(\frac{s}{b}\right) - \mathrm{Y}_0\left(\frac{s}{b}\right)\right]$$

and

31:10:10 $$\int_0^\infty \operatorname{arcosh}(1+bt)\exp(-st)\,\mathrm{d}t = \mathscr{L}\{\operatorname{arcosh}(1+bt)\} = \frac{1}{s}\exp\left(\frac{s}{b}\right)\mathrm{K}_0\left(\frac{s}{b}\right)$$

Zero-order instances of functions from Chapters 57, 54 and 51 occur in these transforms.

31:11 COMPLEX ARGUMENT

To indicate that the function is multivalued, the symbol of an inverse hyperbolic function is customarily capitalized, as in Arsinh($x+iy$), when its argument is complex. Expressing such a function as its real and imaginary values can be accomplished through the logarithmic equivalent [formulas 31:3:13−18], followed by recourse to equations 25:11:2 and 25:11:3. The results are excessively complicated and no example will be presented here. However, the principal values of the real and imaginary parts of Arsinh($x+iy$), Arcosh($x+iy$), and Artanh($x+iy$) are shown graphically in Figures 31-8, 31-9, and 31-10. Observe the discontinuities in the real part of the inverse hyperbolic sine function and in the imaginary parts of arcosh($x+iy$) and artanh($x+iy$). Note that the initial letter of the symbol is not capitalized when the principal value is being represented.

Figure 31-8

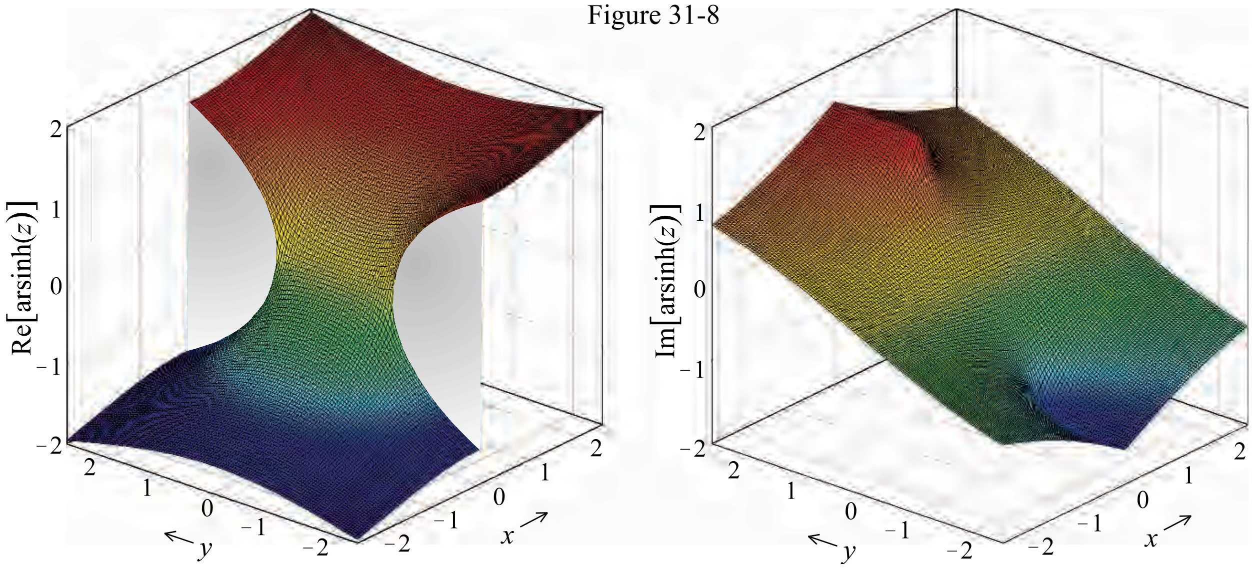


Figure 31-9

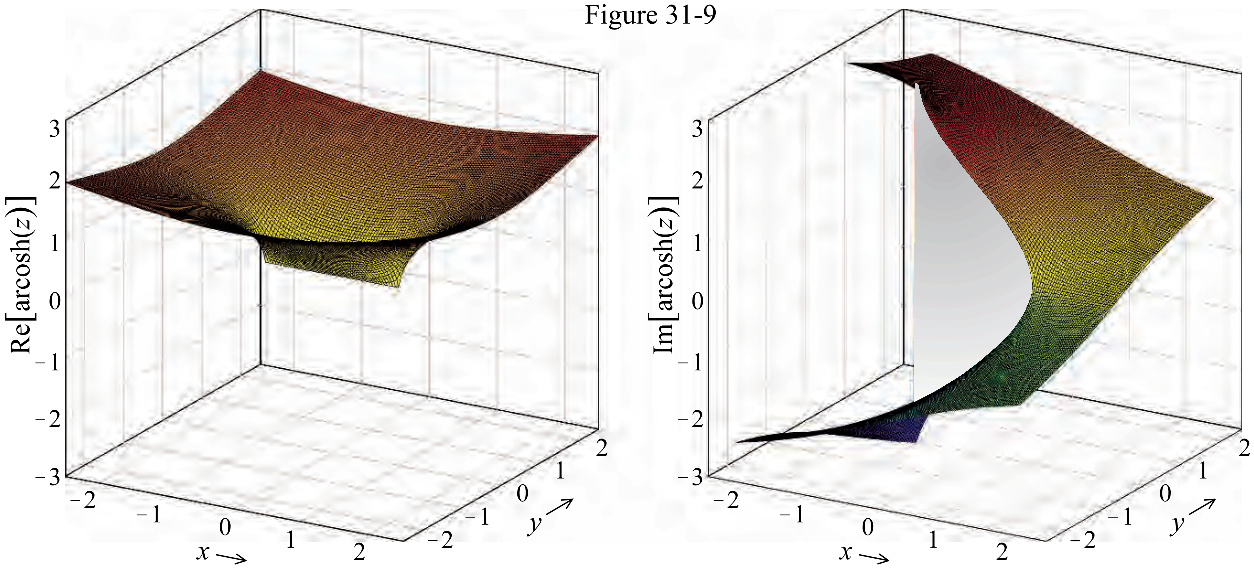
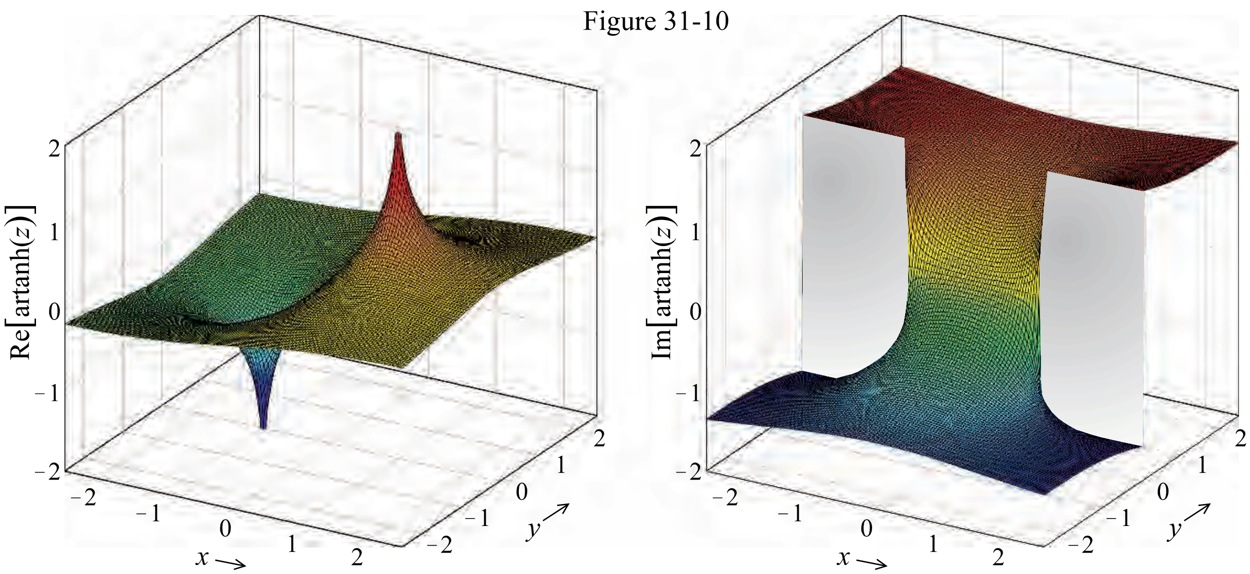


Figure 31-10



The inverse hyperbolic functions of purely imaginary argument, corresponding to a section through the figures along the imaginary axis (where $x = 0$), exhibit the behaviors shown in the panel below

arsinh(iy)	arcsch(iy)	arcosh(iy)	arsech(iy)	artanh(iy)	arcoth(iy)
$i\,\mathrm{arcsin}(y)$	$-\,i\,\mathrm{arccsc}(y)$	$\mathrm{arsinh}(y)+\frac{1}{2}i\pi$	$\mathrm{arcsch}(y)-\frac{1}{2}i\pi$	$i\,\mathrm{arctan}(y)$	$-\,i\,\mathrm{arccot}(y)$

In four cases, the result of the operation is an imaginary version of an *inverse circular function* [Chapter 35].

Inverse Laplace transformation is not commonly applied to inverse hyperbolic functions, but one interesting instance is

31:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \mathrm{arcoth}(bs)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\mathrm{arcoth}(bs)\} = \frac{\sin(t/b)}{t} = \frac{\pi}{b}\mathrm{sinc}\left(\frac{\pi t}{b}\right)$$

where sinc( ) is the *sampling function* [Section 32:13].

31:12 GENERALIZATIONS

The inverse hyperbolic tangent is a special case of the *generalized logarithm* [Section 25:12] and of the *incomplete beta function* [Chapter 58]

31:12:1 $$\mathrm{artanh}(x) = \frac{\mathrm{sgn}(x)}{2}\ln_{1/2}\left(\frac{1}{1-x^2}\right) = \frac{\mathrm{sgn}(x)}{2}\mathrm{B}\left(\tfrac{1}{2}, 0, x^2\right) \qquad -1 < x < 1$$

The *inverse hyperbolic sine* is an instance of the *Gauss hypergeometric function* [Chapter 60]

31:12:2 $$\mathrm{arsinh}(x) = x\,\mathrm{F}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, -x^2\right) \qquad -1 < x < 1$$

31:13 COGNATE FUNCTIONS

The functions of this chapter have much in common with logarithms and with the inverse circular (or inverse trigonometric) functions of Chapter 35.

CHAPTER 32

THE COSINE cos(x) AND SINE sin(x) FUNCTIONS

This chapter is one of the longest in the *Atlas*, befitting the paramount importance of the two functions addressed. Not only are these functions themselves periodic, but they are the units from which all other periodic functions [Chapter 36] may be built. The *cosine function* and *sine function* are interrelated by

32:0:1 $$\cos^2(x) + \sin^2(x) = 1$$

and by

32:0:2 $$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

and, apart from a sign change in one case, by each being the derivative of the other.

32:1 NOTATION

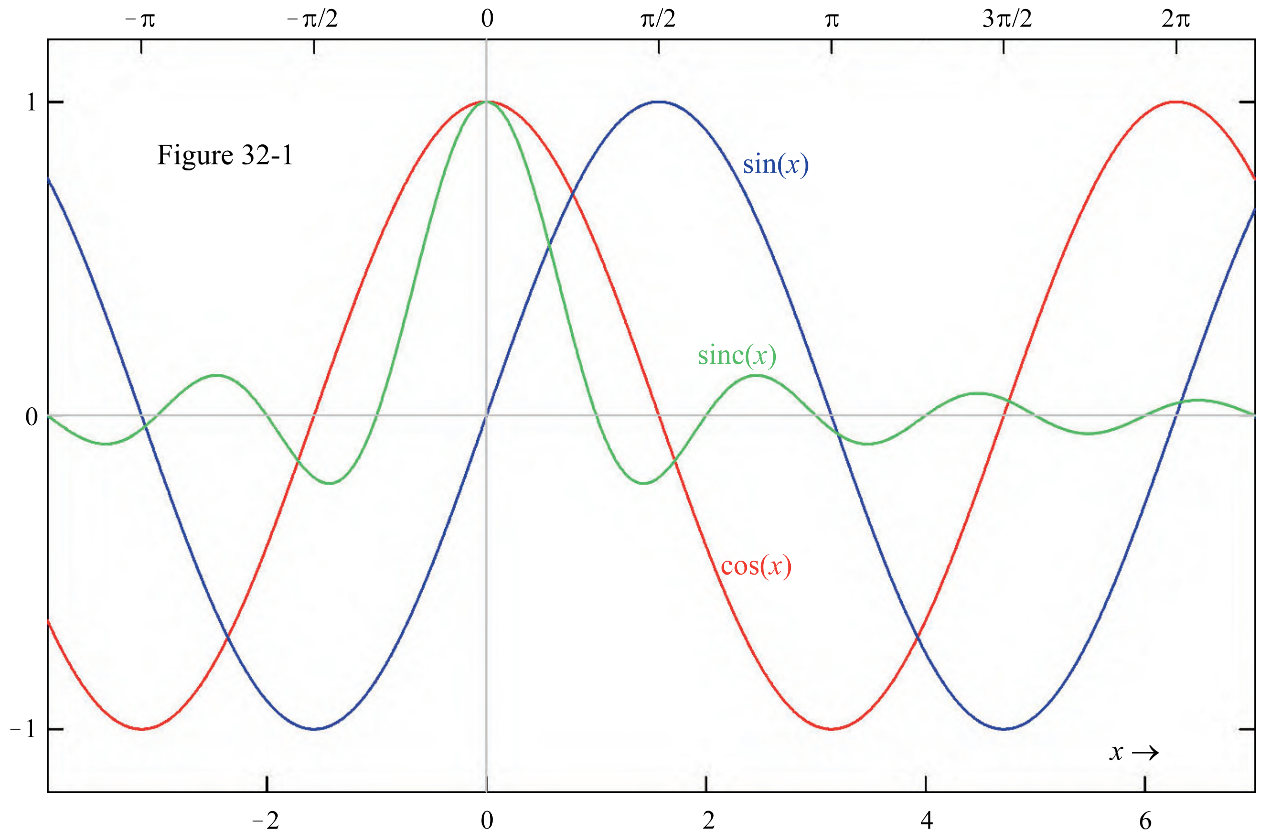
The symbols cos(x) and sin(x) are universal. In some computer codes, Cos and Sin substitute for cos and sin, but when these capitalized symbols are used in mathematics, they often refer to the functions of Chapter 28, not to those of the present chapter.

In this chapter, as throughout the *Atlas*, x is the standard symbol for a real argument. However, the arguments of the cosine and sine functions [and of the functions addressed in Chapters 33 and 34] are often regarded as angles, rather than as numbers to which no special significance is to be attached. We shall write cos(θ) and sin(θ) [as well as sec(θ), tan(θ), etc.] when we particularly wish to emphasize the angular interpretation of the argument.

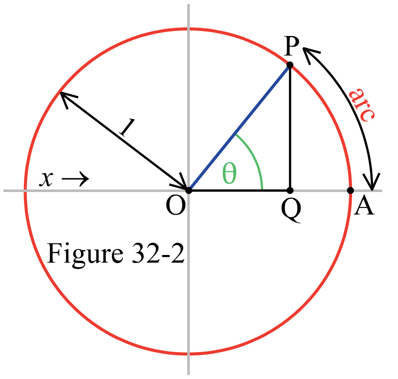
The cosine and sine functions are known collectively as *sinusoidal functions* or *sinusoids*, and this name also applies to any weighted sum, $w_1\cos(\theta) + w_2\sin(\theta)$ of the two. When such a weighted sum is written, as it always may be [equation 32:5:25], in the form $\sqrt{w_1^2 + w_2^2}\cos(\theta + \phi)$, the angle ϕ is known as the *phase* of the sinusoid and the root-mean-square $\sqrt{w_1^2 + w_2^2}$ as its *amplitude*.

Circular functions describes the functions of Chapters 32, 33, and 34 collectively, to distinguish them from the corresponding hyperbolic functions. They are also known as *trigonometric functions* because of their role in trigonometry [Section 34:14].

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_33, © Springer Science+Business Media, LLC 2009



32:2 BEHAVIOR



The sinusoids are periodic functions with a period of 2π; that is, their values at an argument of $x\pm2\pi$, $x\pm4\pi$, $x\pm6\pi$, etc. exactly equal their values at x. This is evident from Figure 32-1. The periodicity of the sine and cosine functions becomes self evident when the argument is regarded as an angle, as in Figure 32-2. Because a vector at angle θ is coincident with those at angles $\theta\pm2\pi$, $\theta\pm4\pi$, etc, it follows that

32:2:1 $$\cos(\theta\pm2\pi)=\cos(\theta\pm4\pi)=\cdots=\cos(\theta\pm2n\pi)=\cos(\theta)=\mathrm{OQ}$$

and similarly for the sine and, indeed, for all the circular functions. This being so, it suffices to consider angular arguments only in the range $0\le\theta<2\pi$ (an alternative is $-\pi<\theta\le\pi$). Within this range, a breakdown into four *quadrants* is often useful because of the symmetries exhibited by the circular functions. The first quadrant encompasses values of θ between 0 and $\pi/2$, with the ranges for the other quadrants as tabulated. This usage of the term "quadrant" is equivalent to that introduced in Section 0:2. The table also informs on the behavior of each function in the four quadrants.

First quadrant $0\le\theta\le\pi/2$	Second quadrant $\pi/2\le\theta\le\pi$	Third quadrant $\pi\le\theta\le3\pi/2$	Fourth quadrant $3\pi/2\le\theta\le2\pi$
$1\ge\cos(\theta)\ge0$	$0\ge\cos(\theta)\ge-1$	$-1\le\cos(\theta)\le0$	$0\le\cos(\theta)\le1$
$0\le\sin(\theta)\le1$	$1\ge\sin(\theta)\ge0$	$0\ge\sin(\theta)\ge-1$	$-1\le\sin(\theta)\le0$

In each period, $\cos(x)$ and $\sin(x)$ each display two zeros, one maximum and one minimum.

The values at the maxima are invariably +1, while the minimal values of the functions are invariably −1. The locations of the zeros and extrema are detailed in equations 32:7:1 and 32:7:2.

32:3 DEFINITIONS

The exponential function of imaginary argument [Section 26:11] provides definitions of the cosine and sine functions:

32:3:1
$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}$$

32:3:2
$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}$$

Several polynomial functions show sinusoidal behavior in the limit of large order and thereby enable definitions as limits. Equation 18:9:4 is one such example and the Euler polynomials [Chapter 20] provide two others

32:3:3
$$\cos(x) = \frac{1}{2}\lim_{n\to\infty}\left\{\frac{(-\pi^2)^n n}{(2n)!}\mathrm{E}_{2n-1}\left(\frac{x}{\pi}\right)\right\}$$

32:3:4
$$\sin(x) = \frac{\pi}{4}\lim_{n\to\infty}\left\{\frac{(-\pi^2)^n}{(2n)!}\mathrm{E}_{2n}\left(\frac{x}{\pi}\right)\right\}$$

A geometric definition of the two functions in terms of the area of a sector of the unity-radius circle is given in Section 28:3. However, the more usual geometric definitions equate the cosine and sine functions respectively to the lengths of the lines OQ and PQ shown in Figure 32-2. Lying on a circle of unit radius, the point P can be considered to have reached its present position, from its original location at point A, by rotation of the radial vector OP through angle θ. The argument of the cosine and sine functions can be regarded either as the angle θ or as the length of the curved *arc* AP

32:3:5
$$\mathrm{OQ} = \cos(\theta) = \cos(arc) \qquad \text{and} \qquad \mathrm{PQ} = \sin(\theta) = \sin(arc)$$

These definitions apply in all quadrants, and even if θ exceeds 2π or is negative.

With ω and *c* as constants, a second-order differential equation and its solution are

32:3:6
$$\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + \omega^2 \mathrm{f} = c \qquad \mathrm{f} = w_1 \sin(\omega x) + w_2 \cos(\omega x) + \frac{c}{\omega^2}$$

and involves an arbitrarily weighted sinusoid. Moreover, with ω, *b*, and *c* as constants, a first order differential equation and its solution are

32:3:7
$$\frac{\mathrm{df}}{\mathrm{d}x} = \sqrt{c + b\mathrm{f} - \omega^2\mathrm{f}^2} \qquad \mathrm{f} = \frac{\sqrt{b^2 + 4\omega^2 c}}{2\omega^2}\left[\sqrt{w}\sin(\omega x) - \sqrt{1-w}\cos(\omega x)\right] + \frac{b}{2\omega^2}$$

The solution involves a sinusoid in which the weight *w* may be zero, unity, or any value in between.

Because, as equations 32:6:1 and 32:6:2 confirm, the cosine and sine functions are hypergeometric, they may be synthesized [Section 43:14] from the corresponding basis function, the zero-order Bessel function:

32:3:8
$$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[\frac{1}{2}]{1} \cos\left(2\sqrt{x}\right)$$

32:3:9
$$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[\frac{3}{2}]{1} \frac{\sin\left(2\sqrt{x}\right)}{2\sqrt{x}}$$

32:4 SPECIAL CASES

There are none.

32:5 INTRARELATIONSHIPS

The cosine is an even function, whereas the sine is odd:

32:5:1 $$\cos(-x)=\cos(x) \qquad \sin(-x)=-\sin(x)$$

The duplication and triplication formulas

32:5:2 $$\cos(2x)=\cos^2(x)-\sin^2(x)=1-2\sin^2(x)=2\cos^2(x)-1$$

32:5:3 $$\sin(2x)=2\sin(x)\cos(x)=2\sin(x)\sqrt{1-\sin^2(x)}=\csc(x)\left[2\cos(x)-2\cos^3(x)\right]$$

32:5:4 $$\cos(3x)=4\cos^3(x)-3\cos(x)$$

and

32:5:5 $$\sin(3x)=3\sin(x)-4\sin^3(x)=\csc(x)\left[-1+5\cos^2(x)-4\cos^4(x)\right]$$

generalize to

32:5:6 $$\cos(nx)=\cos^n(x)\sum_{j=0,2,4}^{n\ or\ n-1}(-)^{n/2}\binom{n}{j}\cot^j(x)=\mathrm{T}_n\{\cos(x)\}$$

and

32:5:7 $$\sin(nx)=\cos^n(x)\sum_{j=1,3,5}^{n\ or\ n-1}(-)^{(n-1)/2}\binom{n}{j}\tan^j(x)=\sin(x)\,\mathrm{U}_{n-1}\{\cos(x)\}$$

where T_n and U_n are Chebyshev polynomials [Chapter 22]. There are also argument-multiplication formulas as products:

32:5:8 $$\cos(nx)=2^{n-1}\prod_{j=0}^{n-1}\cos\left(x+\frac{(1-n+2j)\pi}{2n}\right) \qquad n=1,2,3,\cdots$$

32:5:9 $$\sin(nx)=2^{n-1}\prod_{j=0}^{n-1}\sin\left(x+\frac{j\pi}{n}\right) \qquad n=1,2,3,\cdots$$

Equations 32:5:2 and 32:5:3 may be regarded as special cases of argument-addition formulas. These, and the corresponding argument-subtraction formulas, are

32:5:10 $$\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$$

and

32:5:11 $$\sin(x\pm y)=\sin(x)\cos(y)\pm\cos(x)\sin(y)$$

which, in turn, have the important special cases tabulated.

	$n=0,4,8,\cdots$	$n=1,5,9,\cdots$	$n=2,6,10,\cdots$	$n=3,7,11,\cdots$
$\sin\left(x\pm\frac{1}{2}n\pi\right)$	$\sin(x)$	$\pm\cos(x)$	$-\sin(x)$	$\mp\cos(x)$
$\cos\left(x\pm\frac{1}{2}n\pi\right)$	$\cos(x)$	$\mp\sin(x)$	$-\cos(x)$	$\pm\sin(x)$

From equation 32:5:2 one may derive the expressions

32:5:12 $$\cos\left(\frac{x}{2}\right)=(-)^{\mathrm{Int}\{(\pi+|x|)/2\pi\}}\sqrt{\frac{1+\cos(x)}{2}}$$

and

32:5:13 $$\sin\left(\frac{x}{2}\right)=(-)^{\mathrm{Int}\{|x|/2\pi\}}\sqrt{\frac{1-\cos(x)}{2}}$$

for the cosine and sine of half argument, as well as the formulas

32:5:14 $$\cos^2(x)=\frac{1+\cos(2x)}{2}$$

32:5:15 $$\sin^2(x)=\frac{1-\cos(2x)}{2}$$

for the squares. Note that the modifying function in formulas 32:5:12 and 32:5:13 is the integer *value* function, not the integer *part* function. The latter may give an erroneous answer when x is negative. Formulas 32:5:14 and 32:5:13 generalize to the following formulas for positive integer powers

32:5:16 $$\cos^n(x)=\begin{cases}\dfrac{1}{2^{n-1}}\displaystyle\sum_{j=0}^{(n-1)/2}\binom{n}{j}\cos\{(n-2j)x\} & n=1,3,5,\cdots\\ \dfrac{(n-1)!!}{n!!}+\dfrac{1}{2^{n-1}}\displaystyle\sum_{j=0}^{(n/2)-1}\binom{n}{j}\cos\{(n-2j)x\} & n=2,4,6,\cdots\end{cases}$$

and

32:5:17 $$\sin^n(x)=\begin{cases}\dfrac{(-1)^{(n-1)/2}}{2^{n-1}}\displaystyle\sum_{j=0}^{(n-1)/2}(-)^j\binom{n}{j}\sin\{(n-2j)x\} & n=1,3,5,\cdots\\ \dfrac{(n-1)!!}{n!!}+\dfrac{(-1)^{n/2}}{2^{n-1}}\displaystyle\sum_{j=0}^{(n/2)-1}(-)^j\binom{n}{j}\cos\{(n-2j)x\} & n=2,4,6,\cdots\end{cases}$$

in which there are alternatives according to the parity of the power.

The cosine and sine functions obey the function-addition/subtraction formulas

32:5:18 $$\cos(x)\pm\sin(x)=\sqrt{2}\sin\left(x\pm\tfrac{1}{4}\pi\right)=\sqrt{2}\cos\left(x\mp\tfrac{1}{4}\pi\right)$$

32:5:19 $$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

32:5:20 $$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

32:5:21 $$\sin(x)\pm\sin(y)=2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)$$

as well as the function-multiplication rules

32:5:22 $$\cos(x)\cos(y)=\tfrac{1}{2}\cos(x+y)+\tfrac{1}{2}\cos(x-y)$$

32:5:23 $$\cos(x)\sin(y)=\tfrac{1}{2}\sin(x+y)-\tfrac{1}{2}\sin(x-y)$$

32:5:24 $$\sin(x)\sin(y)=\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y)$$

Equation 32:5:18 is a special case of the important formula

32:5:25
$$w_1\sin(x)+w_2\cos(x)=\mathrm{sgn}(w_1)\sqrt{w_1^2+w_2^2}\,\sin\left\{x+\arctan\left(w_2/w_1\right)\right\}$$
$$=\mathrm{sgn}(w_2)\sqrt{w_1^2+w_2^2}\,\cos\left\{x-\mathrm{arccot}\left(w_2/w_1\right)\right\}$$

whereby any mixed sinusoid may be expressed as a sine or cosine with a nonzero phase. The arctan and arccot functions appear in Chapter 35. See Section 35:14 for an explanation of the need for the signum modifying functions.

Infinite series of the form

32:5:26
$$\tfrac{1}{2}c_0+c_1\cos(x)+c_2\cos(2x)+c_3\cos(3x)+\cdots=\tfrac{1}{2}c_0+\sum_{j=1}^{\infty}c_j\cos(jx)$$

or

32:5:27
$$s_1\sin(x)+s_2\sin(2x)+s_3\sin(3x)+\cdots=\sum_{j=1}^{\infty}s_j\sin(jx)$$

or sometimes of the combination $\tfrac{1}{2}c_0+\sum c_j\cos(jx)+s_j\sin(jx)$, with the c's and s's being appropriate constants, are termed *Fourier series* and are discussed in Chapter 36. Here we quote two examples with especial relevance to the present chapter. If v is a noninteger, then

32:5:28
$$\cos(vx)=\frac{2v}{\pi}\sin(v\pi)\left[\frac{1}{2v^2}+\frac{\cos(x)}{1-v^2}-\frac{\cos(2x)}{4-v^2}+\frac{\cos(3x)}{9-v^2}-\cdots\right]\qquad -\pi<x<\pi$$

and

32:5:29
$$\sin(vx)=\frac{2}{\pi}\sin(v\pi)\left[\frac{\sin(x)}{1-v^2}-\frac{2\sin(2x)}{4-v^2}+\frac{3\sin(3x)}{9-v^2}-\cdots\right]\qquad -\pi<x<\pi$$

32:6 EXPANSIONS

Maclaurin series exist for the two functions and for their logarithms. Those for the cosine

32:6:1
$$\cos(x)=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots=\sum_{j=0}^{\infty}\frac{(-x^2)^j}{(2j)!}=\sum_{j=0}^{\infty}\frac{1}{\left(\frac{1}{2}\right)_j(1)_j}\left(\frac{-x^2}{4}\right)^j$$

and the sine

32:6:2
$$\sin(x)=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}-\cdots=x\sum_{j=0}^{\infty}\frac{(-x^2)^j}{(2j+1)!}=x\sum_{j=0}^{\infty}\frac{1}{(1)_j\left(\frac{3}{2}\right)_j}\left(\frac{-x^2}{4}\right)^j$$

converge for all arguments but the logarithmic series have limited convergence domains:

32:6:3
$$\ln\left(\cos(x)\right)=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\cdots=-\sum_{j=1}^{\infty}\frac{\lambda(2j)}{j}\left(\frac{2x}{\pi}\right)^{2j}=\sum_{j=1}^{\infty}\frac{[1-4^j]\left|\mathrm{B}_{2j}\right|}{2j(2j)!}(2x)^{2j}\qquad |x|<\frac{\pi}{2}$$

32:6:4
$$\ln\left(\frac{\sin(x)}{x}\right)=-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\cdots=-\sum_{j=1}^{\infty}\frac{\zeta(2j)}{j}\left(\frac{x}{\pi}\right)^{2j}=-\sum_{j=1}^{\infty}\frac{\left|\mathrm{B}_{2j}\right|}{2j(2j)!}(2x)^{2j}\qquad |x|<\pi$$

The lambda, zeta, and Bernoulli numbers [Chapters 3 and 4] are involved in these expansions.

The cosine and sine functions are expansible as infinite products:

32:6:5 $$\cos(x) = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots = \prod_{j=1}^{\infty}\left(1-\frac{x^2}{\left(j-\frac{1}{2}\right)^2\pi^2}\right)$$

32:6:6 $$\sin(x) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots = x\prod_{j=1}^{\infty}\left(1-\frac{x^2}{j^2\pi^2}\right)$$

With r_j denoting the jth zero of f, both of the above formulas are embraced by the single infinite product expression

32:6:7 $$\mathrm{f}(x) = \prod_j\left(1-\frac{x}{r_j}\right) \qquad \mathrm{f}(x)=\cos(x) \text{ or } \frac{\sin(x)}{x}$$

where the product is over all the zeros, positive and negative. Another infinite product formula is

32:6:8 $$\sin(x) = x\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\cdots = x\prod_{j=1}^{\infty}\cos\left(\frac{x}{2^j}\right)$$

The cosine and sine functions may be expressed as infinite sums of Bessel functions [Chapter 52]:

32:6:9 $$\cos(x) = \mathrm{J}_0(x) - 2\,\mathrm{J}_2(x) + 2\,\mathrm{J}_4(x) - 2\,\mathrm{J}_6(x) + \cdots = \mathrm{J}_0(x) + 2\sum_{j=1}^{\infty}(-)^j\,\mathrm{J}_{2j}(x)$$

32:6:10 $$\sin(x) = 2\,\mathrm{J}_1(x) - 2\,\mathrm{J}_3(x) + 2\,\mathrm{J}_5(x) - 2\,\mathrm{J}_7(x) + \cdots = 2\sum_{j=0}^{\infty}(-)^j\,\mathrm{J}_{2j+1}(x)$$

32:7 PARTICULAR VALUES

θ	15°	18°	22½°	30°	36°	45°	54°	60°	67½°	72°	75°
x	$\pi/12$	$\pi/10$	$\pi/8$	$\pi/6$	$\pi/5$	$\pi/4$	$3\pi/10$	$\pi/3$	$3\pi/8$	$2\pi/5$	$5\pi/12$
cos	$\frac{\sqrt{3}+1}{\sqrt{8}}$	$\frac{\sqrt{5+\sqrt{5}}}{\sqrt{8}}$	$\frac{1}{\sqrt{4-\sqrt{8}}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3+\sqrt{5}}}{\sqrt{8}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5-\sqrt{5}}}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{4+\sqrt{8}}}$	$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{8}}$	$\frac{\sqrt{3}-1}{\sqrt{8}}$
sin	$\frac{\sqrt{3}-1}{\sqrt{8}}$	$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{8}}$	$\frac{1}{\sqrt{4+\sqrt{8}}}$	$\frac{1}{2}$	$\frac{\sqrt{5-\sqrt{5}}}{\sqrt{8}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3+\sqrt{5}}}{\sqrt{8}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{4-\sqrt{8}}}$	$\frac{\sqrt{5+\sqrt{5}}}{\sqrt{8}}$	$\frac{\sqrt{3}+1}{\sqrt{8}}$

The table shows particular values occurring in the range $0 < x < \pi/2$. The table in Section 32:5 can be used to expand this range. The most important particular values are those that correspond to zeros and extrema, namely:

32:7:1 $$\cos(v\pi) = \begin{cases} 1 & v = 0, \pm 2, \pm 4, \cdots \\ 0 & v = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots \\ -1 & v = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

32:7:2 $$\sin(v\pi) = \begin{cases} 1 & v = +\frac{1}{2}, -\frac{3}{2}, +\frac{5}{2}, -\frac{7}{2}, \cdots \\ 0 & v = 0, \pm 1, \pm 2, \cdots \\ -1 & v = -\frac{1}{2}, +\frac{3}{2}, -\frac{5}{2}, +\frac{7}{2}, \cdots \end{cases}$$

32:8 NUMERICAL VALUES

With keywords **cos** and **sin**, *Equator*'s cosine function and sine function routines offer the user a choice of arguments in radian or degree measure. Preserving significance for arguments of large magnitude is a challenge for the circular functions, but *Equator*'s $\cos(x)$ and $\sin(x)$ routines do not begin to suffer precision degradation until $|x|$ reaches about 1×10^{15}. Thereafter, fewer than 15 digits are reported.

Very frequently, and possibly more often than not, a user of the sine or cosine function needs to perform a multiplication by π before invoking the function. This poses an additional computational challenge because of the loss of precision when multiplying by a (necessarily inexact) π. To counter this problem, *Equator* provides a reperiodized cosine function $\cos(\pi x)$ routine and a reperiodized sine function $\sin(\pi x)$ routine, into each of which a very precise value of π is incorporated. These functions, for which the keywords are **cospi** and **sinpi**, should be used routinely in preference to premultiplying x by pi. *Equator* uses these reperiodized routines in calculating the values of circular functions when the user inputs the argument in degrees.

32:9 LIMITS AND APPROXIMATIONS

Close to a zero r of these functions, a cubic relationship, either

32:9:1 $$f(x) \approx x - r - \frac{(x-r)^3}{6} \qquad f = \sin \text{ or } \cos$$

or the negative of this, provides a good approximation. Similarly, the quadratic approximations

32:9:2 $$f(x) \approx 1 - \frac{(x-x_\mathrm{M})^2}{2} \qquad \text{and} \qquad f(x) \approx -1 + \frac{(x-x_\mathrm{m})^2}{2} \qquad f = \sin \text{ or } \cos$$

hold close to a maximum x_M or minimum x_m of each function. Approximation 32:9:2 is tantamount to asserting that, near their maxima and minima, the cosine and sine functions come to lie on a circle of unity radius centered on the x-axis [Figure 39-2].

32:10 OPERATIONS OF THE CALCULUS

The differentiation formulas

32:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(bx) = b\cos(bx) = b\sin\left(bx + \tfrac{1}{2}\pi\right)$$

32:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\cos(bx) = -b\sin(bx) = b\cos\left(bx + \tfrac{1}{2}\pi\right)$$

may be generalized to

32:10:3 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(bx) = b^n f\left(bx + \frac{n\pi}{2}\right) \qquad n = 0,1,2,\cdots \qquad f = \cos \text{ or } \sin$$

Indefinite integration gives

32:10:4 $$\int_0^x \cos(bt)\mathrm{d}t = \frac{\sin(bx)}{b}$$

and

32:10:5
$$\int_0^x \sin(bt)\,\mathrm{d}t = \frac{1-\cos(bx)}{b}$$

which are special cases of the more general formulas

32:10:6
$$\int_0^x \cos^n(t)\,\mathrm{d}t = \begin{cases} \dfrac{(n-1)!!}{n!!}\sin(x)\displaystyle\sum_{j=0}^{(n-1)/2}\frac{(2j-1)!!}{(2j)!!}\cos^{2j}(x) & n=1,3,5,\cdots \\ \dfrac{(n-1)!!}{n!!}\left[x+\sin(x)\displaystyle\sum_{j=0}^{(n-2)/2}\frac{(2j)!!}{(2j+1)!!}\cos^{2j+1}(x)\right] & n=2,4,6,\cdots \end{cases}$$

32:10:7
$$\int_0^x \sin^n(t)\,\mathrm{d}t = \begin{cases} \dfrac{(n-1)!!}{n!!}\left[1-\cos(x)\displaystyle\sum_{j=0}^{(n-1)/2}\frac{(2j-1)!!}{(2j)!!}\sin^{2j}(x)\right] & n=1,3,5,\cdots \\ \dfrac{(n-1)!!}{n!!}\left[x-\cos(x)\displaystyle\sum_{j=0}^{(n-2)/2}\frac{(2j)!!}{(2j+1)!!}\sin^{2j+1}(x)\right] & n=2,4,6,\cdots \end{cases}$$

The technique discussed in the context of equation 34:10:19 is useful for evaluating some integrals of the cosine and sine functions. Other important indefinite integrals include the following

32:10:8
$$\int_0^x \frac{1}{1+a\cos(bt)}\mathrm{d}t = \begin{cases} \dfrac{2}{b\sqrt{a^2-1}}\,\mathrm{artanh}\left(\sqrt{\dfrac{a+1}{a-1}}\cot\left(\dfrac{bx}{2}\right)\right) & a>1 \\ \dfrac{1}{b}\tan\left(\dfrac{bx}{2}\right) & a=1 \\ \dfrac{2}{b\sqrt{1-a^2}}\arctan\left(\sqrt{\dfrac{1-a}{1+a}}\tan\left(\dfrac{bx}{2}\right)\right) & |a|<1 \end{cases}$$

and

32:10:9
$$\int_0^x \frac{1}{1+a\sin(bt)}\mathrm{d}t = \begin{cases} \dfrac{2}{b\sqrt{a^2-1}}\left[\mathrm{arcosh}(a)-\mathrm{arcoth}\left(\dfrac{a+\tanh(bx/2)}{\sqrt{a^2-1}}\right)\right] & |a|>1 \\ \dfrac{1}{b}\left[\tan\left(\dfrac{bx}{2}\mp\dfrac{\pi}{4}\right)\pm 1\right] & a=\pm 1 \\ \dfrac{2}{b\sqrt{1-a^2}}\left[\arctan\left(\dfrac{a+\tan(bx/2)}{\sqrt{1-a^2}}\right)-\arcsin(a)\right] & |a|<1 \end{cases}$$

which utilize functions from Chapters 31, 34 and 35. The indefinite integral of the cosine or sine of a quadratic function may be expressed in terms of a Fresnel integral [Chapter 39]

32:10:10
$$\int_{-b/2a}^{x} \begin{matrix}\cos\\ \sin\end{matrix}\left(at^2+bt+c\right)\mathrm{d}t = \sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2-4ac}{4a}\right)\begin{matrix}\mathrm{C}\\ \mathrm{S}\end{matrix}\left(\frac{2ax+b}{2\sqrt{a}}\right)\pm\sin\left(\frac{b^2-4ac}{4a}\right)\begin{matrix}\mathrm{S}\\ \mathrm{C}\end{matrix}\left(\frac{2ax+b}{2\sqrt{a}}\right)\right]$$

if $a > 0$, as may

32:10:11
$$\int_0^x \frac{1}{\sqrt{t}} \begin{matrix}\cos \\ \sin\end{matrix}(bt)\,\mathrm{d}t = \sqrt{\frac{2\pi}{b}} \begin{matrix}\mathrm{C} \\ \mathrm{S}\end{matrix}\left(\sqrt{bx}\right)$$

Moreover, with Ci and Si symbolizing the cosine integral and sine integral from Chapter 38, one has

32:10:12
$$\int_0^x \frac{\sin(bt)}{t}\,\mathrm{d}t = \mathrm{Si}(bx)$$

and

32:10:13
$$\int_0^\infty \frac{\cos(bt)}{t}\,\mathrm{d}t = -\mathrm{Ci}(bx)$$

Refer to Section 39:12 for the indefinite integrals $\int \sin(t)\mathrm{d}t/t^{n/2}$ and $\int \cos(t)\mathrm{d}t/t^{n/2}$ where n is a positive integer.

Turning to definite integrals, there is the general rule

32:10:14
$$\int_0^\infty \mathrm{f}(at^\nu)\mathrm{d}t = \frac{\Gamma\{1+(1/\nu)\}}{a^{1/\nu}}\mathrm{f}\left(\frac{\pi}{2\nu}\right) \qquad \mathrm{f} = \sin \text{ or } \cos \qquad a>0 \qquad \nu>1$$

The integrals

32:10:15
$$\int_0^\infty \exp(-t^2)\cos(\omega t)\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}\exp\left(\frac{-\omega^2}{4}\right)$$

32:10:16
$$\int_0^\infty \exp(-t^2)\sin(\omega t)\,\mathrm{d}t = \mathrm{daw}\left(\frac{\omega}{2}\right)$$

[see Chapter 42 for daw] exemplify two very general integral transforms known collectively as *Fourier transforms*. The *cosine transformation* is defined by

32:10:17
$$\overline{\mathrm{f}}_{\mathrm{C}}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{f}(t)\cos(\omega t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{f}(t)\exp(i\omega t)\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{f}(t)\exp(-i\omega t)\mathrm{d}t$$

while the corresponding *sine transformation* is

32:10:18
$$\overline{\mathrm{f}}_{\mathrm{S}}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{f}(t)\sin(\omega t)\mathrm{d}t = \frac{-i}{\sqrt{2\pi}}\int_0^\infty \mathrm{f}(t)\exp(i\omega t)\mathrm{d}t + \frac{i}{\sqrt{2\pi}}\int_0^\infty \mathrm{f}(t)\exp(-i\omega t)\mathrm{d}t$$

In the first of their *Tables of Integral Transforms* volumes [Chapters I and II], Erdélyi, Magnus, Oberhettinger and Tricomi list many such transform pairs, although their definitions omit the $\sqrt{2/\pi}$ multiplier that the *Atlas*, along with many other authorities, employ. Transforms 32:10:17 and 32:10:18 share the convenient feature that transform and inversion formulas are almost identical. The final equalities in these equations show that the cosine and sine transforms are closely related to the Laplace transforms, and this feature is sometimes useful in determining expressions for $\overline{\mathrm{f}}_{\mathrm{C}}$ or $\overline{\mathrm{f}}_{\mathrm{S}}$. See Section 32:14 for other aspects of Fourier transformation.

With a lower limit of $-\infty$, differintegration [Section 12:14] of the sinusoidal functions maintains their sinusoidal behavior but (unless $\nu = 0, \pm 4, \pm 8, \cdots$) modifies the phase:

32:10:19
$$\left.\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\mathrm{f}(bx)\right|_{-\infty} = b^\nu\,\mathrm{f}\left(bx + \tfrac{1}{2}\nu\pi\right) \qquad \mathrm{f} = \cos \text{ or } \sin \qquad b>0$$

The amplitude of the sinusoid is either attenuated or amplified, depending on the sign of ν and whether b is greater or less than unity. Equations 32:10:1−3 are instances of this rule. Another example is the semiintegration formula

32:10:20
$$\left.\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\cos(bx)\right|_{-\infty} = \frac{1}{\sqrt{b}}\cos\left(bx-\frac{\pi}{4}\right) = \frac{\cos(bx)+\sin(bx)}{\sqrt{2b}}$$

The response of sinusoidal functions to differintegration with a lower limit of zero is similar to that expressed in 32:10:19 but, in addition to producing a modified sinusoid, a supplementary function is also generated; this decays as x increases, becoming negligible after several periods. For the semidifferentiation and semiintegration cases the phase shift is always 45° in one direction or the other:

32:10:21
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\cos(bx) = \sqrt{b}\cos\left(bx+\tfrac{1}{4}\pi\right) + \frac{1}{\sqrt{\pi x}} - \sqrt{2b}\,\mathrm{Fres}\left(\sqrt{bx}\right)$$

32:10:22
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sin(bx) = \sqrt{b}\sin\left(bx+\tfrac{1}{4}\pi\right) - \sqrt{2b}\,\mathrm{Gres}\left(\sqrt{bx}\right)$$

32:10:23
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\begin{matrix}\cos\\ \sin\end{matrix}(bx) = \frac{1}{\sqrt{b}}\begin{matrix}\cos\\ \sin\end{matrix}\left(bx-\tfrac{1}{4}\pi\right) \mp \sqrt{\frac{2}{b}}\begin{matrix}\mathrm{Gres}\\ \mathrm{Fres}\end{matrix}\left(\sqrt{bx}\right)$$

where the supplementary functions include auxiliary Fresnel integrals addressed in Chapter 39.

The definite integrals

32:10:24
$$\int_0^{2\pi}\cos(nt)\cos(mt)\mathrm{d}t = \left\{\begin{matrix} 0 & m\ne n \\ 2\pi & m=n=0 \\ \pi & m=n\ne 0\end{matrix}\right\} \quad \begin{matrix} n=0,1,2,\cdots \\ m=0,1,2,\cdots\end{matrix}$$

32:10:25
$$\int_0^{2\pi}\sin(nt)\sin(mt)\mathrm{d}t = \left\{\begin{matrix} 0 & m\ne n \\ \pi & m=n\end{matrix}\right\} \quad \begin{matrix} n=1,2,3,\cdots \\ m=1,2,3,\cdots\end{matrix}$$

and

32:10:26
$$\int_0^{2\pi}\cos(nt)\sin(mt)\mathrm{d}t = 0 \qquad n=0,1,2,\cdots \qquad m=1,2,3,\cdots$$

show that the set of cosine functions, cos(nx, $n=0,1,2,3,\cdots$), form an orthogonal family [Section 21:14] with a weight function of unity on the interval from 0 to 2π (or from $-\pi$ to π). The set of sines, sin(nx, $n=1,2,3,\cdots$), have the same property, as do the conjoined set of cosines and sines. The cosines or the sines are orthogonal also on the smaller interval from 0 to π, but the conjoined set is not.

Included among Laplace transforms of functions involving cosines or sines are:

32:10:27
$$\int_0^{\infty}\cos(\omega t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

32:10:28
$$\int_0^{\infty}\sin(\omega t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

32:10:29
$$\int_0^{\infty}t\cos(\omega t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{t\cos(\omega t)\} = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$$

32:10:30
$$\int_0^{\infty}t\sin(\omega t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{t\sin(\omega t)\} = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

32:10:31 $$\int_0^\infty \frac{\sin(\omega t)}{t}\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{\sin(\omega t)}{t}\right\} = \arctan\left(\frac{\omega}{s}\right)$$

and

32:10:32 $$\int_0^\infty \frac{\sin(\sqrt{\omega t})}{t}\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{\sin(\sqrt{\omega t})}{t}\right\} = \pi\,\mathrm{erf}\left(\sqrt{\frac{\omega}{4s}}\right)$$

the last two involving functions from Chapters 35 and 40.

32:11 COMPLEX ARGUMENT

For a complex argument $x+iy$, one has

32:11:1 $$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

and

32:11:2 $$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

These two relationships are illustrated in the accompanying three-dimensional graphs which show the real and imaginary parts of $\cos(x+iy)$ [Figure 32-3] and $\sin(x+iy)$ [Figure 32-4]. For purely imaginary argument the formulas become

32:11:3 $$\cos(iy) = \cosh(y)$$

and

32:11:4 $$\sin(iy) = i\sinh(y)$$

Useful relationships are embodied in *de Moivre's theorem*

32:11:5 $$[\cos(z) + i\sin(z)]^{v} = \exp(ivz) = \cos(vz) + i\sin(vz)$$

where z itself may be complex. Unless v is an integer, this theorem applies only if the real part of z lies between $-\pi$ and π.

Figure 32-3

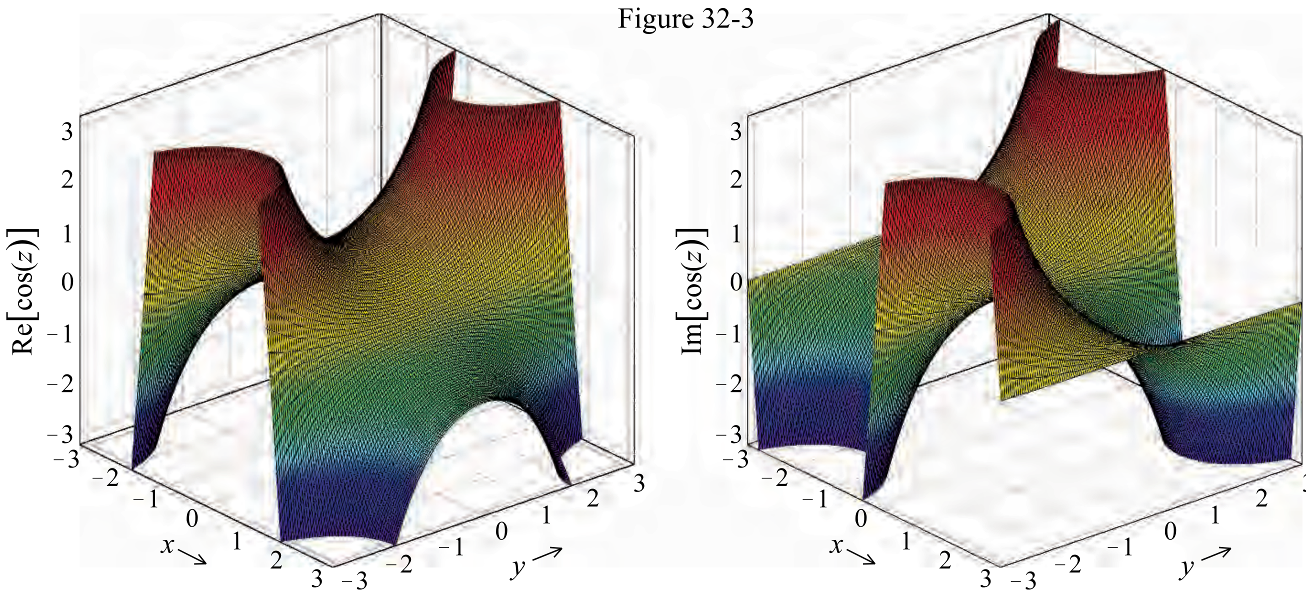
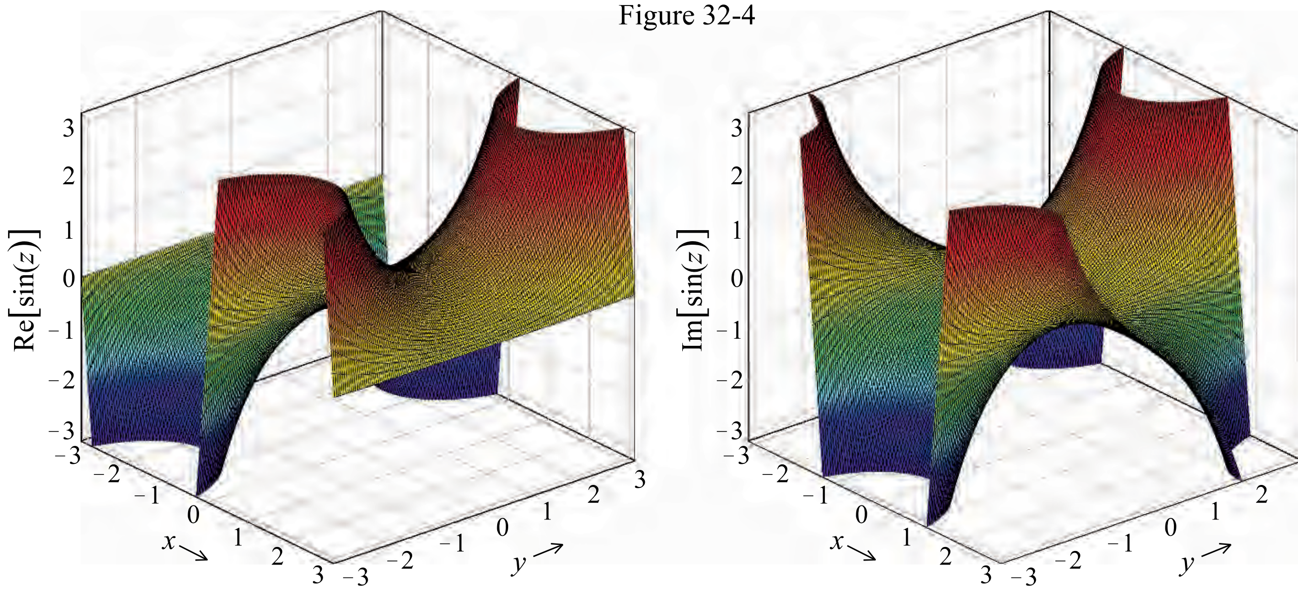


Figure 32-4



Inverse Laplace transformations include

32:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{b}{s}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{1}{s}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{b}{s}\right)\right\}=\begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}\left(2\sqrt{bt}\right)$$

in which Kelvin functions [Chapter 55] are generated. Among other inversion formulas is

32:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\exp\left(-\sqrt{bs}\right)}{\sqrt{s}}\begin{matrix}\cos\\ \sin\end{matrix}\left(\sqrt{bs}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\exp\left(-\sqrt{bs}\right)}{\sqrt{s}}\begin{matrix}\cos\\ \sin\end{matrix}\left(\sqrt{bs}\right)\right\}=\frac{1}{\sqrt{\pi t}}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{b}{2t}\right)$$

32:12 GENERALIZATIONS

Inasmuch as it may be resolved into a set of cosines and/or sines, any periodic function [Chapter 36] may be said to be a generalization of the sinusoidal functions.

The Jacobian elliptic functions $\mathrm{cn}(k,x)$ and $\mathrm{cd}(k,x)$ are generalizations of $\cos(x)$ to which they reduce when $k=0$

32:12:1 $$\mathrm{cn}(0,x)=\mathrm{cd}(0,x)=\cos(x)$$

Likewise, because

32:12:2 $$\mathrm{sn}(0,x)=\mathrm{sd}(0,x)=\sin(x)$$

the $\mathrm{sn}(k,x)$ and $\mathrm{sd}(k,x)$ Jacobian functions generalize the sine function. Jacobian elliptic functions are addressed in Chapter 63.

The sine function is a special case of the incomplete elliptic function of the second kind [Chapter 62]

32:12:3 $$\sin(x)=\mathrm{E}(x,1)$$

32:13 COGNATE FUNCTIONS

The cosine and sine functions are related to the other circular functions by

32:13:1 $$\cos(x)=\sigma_{12}\sqrt{1-\sin^2(x)}=\frac{1}{\sec(x)}=\frac{\sigma_{13}\sqrt{\csc^2(x)-1}}{\csc(x)}=\frac{\sigma_{14}}{\sqrt{1+\tan^2(x)}}=\frac{\sigma_{12}\cot(x)}{\sqrt{\cot^2(x)+1}}$$

32:13:2 $$\sin(x)=\sigma_{12}\sqrt{1-\cos^2(x)}=\frac{\sigma_{13}\sqrt{\sec^2(x)-1}}{\sec(x)}=\frac{1}{\csc(x)}=\frac{\sigma_{14}\tan(x)}{\sqrt{1+\tan^2(x)}}=\frac{\sigma_{12}}{\sqrt{\cot^2(x)+1}}$$

where the σ multipliers take the values +1 or −1 to reflect the quadrant [Section 32:2] in which x (interpreted as an angle) lies: for example $\sigma_{12}=+1$ only in the first and second quadrants, σ_{13} is positive in the first and third quadrants, and so on. The explicit formulas

32:13:3 $$\sigma_{12}=(-1)^{\mathrm{Int}(x/\pi)} \qquad \sigma_{13}=(-1)^{\mathrm{Int}(2x/\pi)} \qquad \sigma_{14}=(-1)^{\mathrm{Int}\{(2x+\pi)/\pi)\}}=\sigma_{12}\sigma_{13}$$

apply. Figure 33-2 provides a geometric interpretation of the interrelationships among circular functions.

The *versine function* vers(x), *coversine function* covers(x), and *haversine function* hav(x), defined by

32:13:4 $$\mathrm{vers}(x)=1-\cos(x) \qquad \mathrm{covers}(x)=1-\sin(x) \qquad \mathrm{hav}(x)=\frac{1-\cos(x)}{2}=\sin^2\left(\frac{x}{2}\right)$$

are archaic functions seldom encountered nowadays. The function sinc(x), sometimes known as the *sampling function*, is important in spectral theory. Figure 32-1 includes a graph of sinc(x), which is usually defined by

32:13:5 $$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

Be aware, however, that the definition sinc(x) = sin(x)/x is occasionally used. *Equator*'s sampling function routine (keyword **sinc**) uses equation 32:13:5. Its power series follows from 32:6:2 and its logarithm has the expansion

32:13:6 $$\ln\left(\mathrm{sinc}(x)\right)=-\sum_{j=1}^{\infty}\frac{\zeta(2j)}{j}x^{2j} \qquad x<1$$

in terms of zeta numbers [Chapter 3]. Equation 31:11:3 shows the Laplace transform of sinc(t) to be arcoth(πs).

With $n=0,\pm1,\pm2,\cdots$ and $x\ge0$, the symbols $\mathrm{j}_n(x)$ and $\mathrm{y}_n(x)$ and the names *spherical Bessel function* (*of the first kind*) and *spherical Neumann function* (or *spherical Bessel function of the second kind*) are given to the functions defined by

32:13:7 $$\mathrm{j}_n(x)=\sqrt{\frac{\pi}{2x}}\mathrm{J}_{n+\frac12}(x)=(-)^n\sqrt{\frac{\pi}{2x}}\mathrm{Y}_{-n-\frac12}(x)=\frac{x^n}{(2n+1)!!}\sum_{j=0}^{\infty}\frac{1}{(1)_j\left(\frac32+n\right)_j}\left(\frac{-x^2}{4}\right)^j \qquad n=0,1,2,\cdots$$

32:13:8 $$\mathrm{y}_n(x)=\sqrt{\frac{\pi}{2x}}\mathrm{Y}_{n+\frac12}(x)=(-)^{n+1}\sqrt{\frac{\pi}{2x}}\mathrm{J}_{-n-\frac12}(x)=\frac{-(2n-1)!!}{x^{n+1}}\sum_{j=0}^{\infty}\frac{1}{(1)_j\left(\frac12-n\right)_j}\left(\frac{-x^2}{4}\right)^j \qquad n=0,1,2,\cdots$$

in terms of Bessel [Chapter 53] and Neumann [Chapter 54] functions of half-odd order. These functions have acquired the adjective "spherical" because they solve Laplace's equation [Section 46:15] in spherical coordinates [Section 46:14]. In fact, $\mathrm{j}_n(r)$ and $\mathrm{y}_n(r)$ provide solutions of equation 59:14:2 whenever v is an integer. The series expansion in the above two equations are valid only for nonnegative integer n but, the interrelationships

32:13:9 $$\mathrm{j}_n(x)=(-)^n\mathrm{y}_{-n-1}(x) \qquad \text{and} \qquad \mathrm{y}_n(x)=(-)^{n+1}\mathrm{j}_{-n-1}(x)$$

permit extension to negative orders. The significance of the adjective "spherical" that attaches to these functions will be evident from Section 59:14. Though defined as cylinder functions, these functions are nevertheless closely related to the sine and cosine. Explicit polynomial formulas in terms of sinusoids are

32:13:10
$$\mathrm{j}_n(x)=\begin{cases}\dfrac{(-1)^{n/2}}{x}\left[\sin(x)\displaystyle\sum_{k=0,2}^{n}(-)^{k/2}\mathrm{A}_k^{(n)}(x)+\cos(x)\sum_{k=1,3}^{n-1}(-)^{(k+1)/2}\mathrm{A}_k^{(n)}(x)\right] & n \text{ even}\\ \dfrac{(-1)^{(n+1)/2}}{x}\left[\cos(x)\displaystyle\sum_{k=0,2}^{n-1}(-)^{k/2}\mathrm{A}_k^{(n)}(x)-\sin(x)\sum_{k=1,3}^{n}(-)^{(k+1)/2}\mathrm{A}_k^{(n)}(x)\right] & n \text{ odd}\end{cases}$$

and

32:13:11
$$\mathrm{y}_n(x)=\begin{cases}\dfrac{(-1)^{1-(n/2)}}{x}\left[\cos(x)\displaystyle\sum_{k=0,2}^{n}(-)^{k/2}\mathrm{A}_k^{(n)}(x)-\sin(x)\sum_{k=1,3}^{n-1}(-)^{(k+1)/2}\mathrm{A}_k^{(n)}(x)\right] & n \text{ even}\\ \dfrac{(-1)^{(1+n)/2}}{x}\left[\sin(x)\displaystyle\sum_{k=0,2}^{n-1}(-)^{k/2}\mathrm{A}_k^{(n)}(x)+\cos(x)\sum_{k=1,3}^{n}(-)^{(k+1)/2}\mathrm{A}_k^{(n)}(x)\right] & n \text{ odd}\end{cases}$$

where $\mathrm{A}_0^{(0)}(x)=1,\ \mathrm{A}_0^{(1)}(x)=1,\ \mathrm{A}_1^{(1)}(x)=-1/x$, and generally

32:13:12
$$\mathrm{A}_k^{(n)}(x)=\frac{(n+k)!}{(n-k)!k!(-2x)^k}$$

Early members of the j and y families are graphed in Figure 32-5 and formulated in the table overleaf. Both families are also accessible from the sine function through formulas involving either repeated differentiation

32:13:13
$$\left(-2\sqrt{x}\right)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}=\mathrm{j}_n\left(\sqrt{x}\right)=(-)^n\mathrm{y}_{1-n}\left(\sqrt{x}\right)\qquad n=0,1,2,\cdots$$

or repeated integration

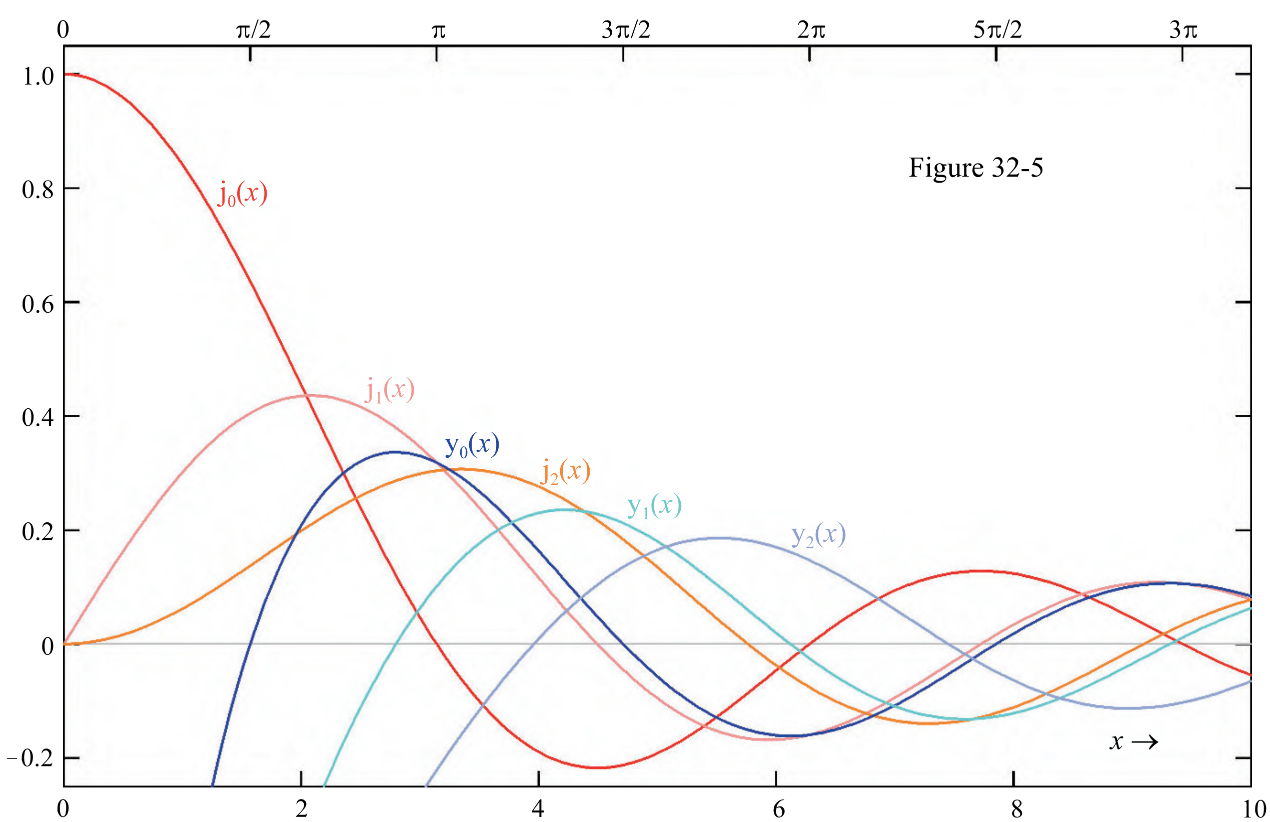


Figure 32-5

32:13:14
$$\frac{2}{\left(2\sqrt{x}\right)^{n+1}}\int_0^x\cdots\int_0^x \sin\left(\sqrt{t}\right)(\mathrm{d}t)^n = \mathrm{j}_n\left(\sqrt{x}\right) = (-)^n\,\mathrm{y}_{1-n}\left(\sqrt{x}\right) \qquad n=1,2,3,\cdots$$

or via the recursion

32:13:15
$$\mathrm{f}_n(x) = \frac{2n-1}{x}\mathrm{f}_{n-1}(x) - \mathrm{f}_{n-2}(x) \qquad \mathrm{f} = \mathrm{j} \text{ or } \mathrm{y}$$

	$n=0$	$n=1$	$n=2$	$n=3$
$\mathrm{j}_n(x)$	$\frac{\sin(x)}{x}$	$\frac{-\cos(x)}{x}+\frac{\sin(x)}{x^2}$	$\frac{-\sin(x)}{x}\left[1-\frac{3}{x^2}\right]-\frac{3\cos(x)}{x^2}$	$\frac{\cos(x)}{x}\left[1-\frac{15}{x^2}\right]-\frac{\sin(x)}{x^2}\left[6-\frac{15}{x^2}\right]$
$\mathrm{y}_n(x)$	$\frac{-\cos(x)}{x}$	$\frac{-\sin(x)}{x}-\frac{\cos(x)}{x^2}$	$\frac{\cos(x)}{x}\left[1-\frac{3}{x^2}\right]-\frac{3\sin(x)}{x^2}$	$\frac{\sin(x)}{x}\left[1-\frac{15}{x^2}\right]+\frac{\cos(x)}{x^2}\left[6-\frac{15}{x^2}\right]$

For more discussion of these functions, see Abramowitz and Stegun [Section 10.1].

With the keywords **j** and **y**, *Equator* provides values of these functions through the spherical Bessel function or spherical Neumann function routines, for $n \le 100$. These routines use recursion formula 32:13:15, coupled with explicit formulas for the $n = 0$ and $n = 1$ cases, or the power series 32:13:7. Both routines rely on 32:13:9 for negative n. As is usually the case, precision is diminished in the immediate vicinity of the zeros of these functions.

32:14 RELATED TOPIC: Clausen's integral and related series

With f denoting either cos or sin, and n a positive integer that we term the order, the four families of series

32:14:1
$$\sum_{j=1}^{\infty}\frac{\mathrm{f}(jx)}{j^n} \qquad 0 \le x \le 2\pi$$

32:14:2
$$-\sum_{j=1}^{\infty}\frac{(-)^j\,\mathrm{f}(jx)}{j^n} \qquad -\pi \le x \le \pi$$

32:14:3
$$\sum_{j=0}^{\infty}\frac{\mathrm{f}\{(2j+1)x\}}{(2j+1)^n} \qquad 0 \le x \le \pi$$

and

32:14:4
$$\sum_{j=0}^{\infty}\frac{(-)^j\,\mathrm{f}\{(2j+1)x\}}{(2j+1)^n} \qquad -\tfrac{1}{2}\pi \le x \le \tfrac{1}{2}\pi$$

are of interest in a number of contexts. They are examples both of Fourier series [Chapter 36] and of Hurwitz functions [Chapter 64]. Additionally, they represent generalizations of the functions of Chapter 3. Note that there is variety in the regions of convergence of these series, but all converge – though sometimes intolerably slowly – within at least $0 \le x \le \pi/2$. Here we shall mainly be concerned with the summation of these series and with an important integral that arises in that operation. Notice that the third sum is just the average of the first and second, irrespective of f or the order n. It is convenient first to address the $n = 1$ cases and then proceed to higher orders.

Some of the $n = 1$ members of each series may be summed to a constant or simple polynomial in x/π or as the inverse Gudermannian function [Section 33:14]. Otherwise, as shown in the following table, the sums may be expressed as the logarithm of a circular function.

	$\sum_{j=1}^{\infty}\frac{f(jx)}{j}$	$-\sum_{j=1}^{\infty}\frac{(-)^j f(jx)}{j}$	$\sum_{j=0}^{\infty}\frac{f\{(2j+1)x\}}{2j+1}$	$\sum_{j=0}^{\infty}\frac{(-)^j f\{(2j+1)x\}}{2j+1}$
f = cos	$\ln\left\{\frac{1}{2}\csc\left(\frac{x}{2}\right)\right\}$	$\ln\left\{2\cos\left(\frac{x}{2}\right)\right\}$	$\frac{1}{2}\ln\left\{\frac{1}{2}\cot\left(\frac{x}{2}\right)\right\}$	$\frac{\pi}{4}$
f = sin	$\frac{\pi-x}{2}$	$\frac{x}{2}$	$\frac{\pi}{4}$	$\frac{1}{2}\mathrm{invgd}(x)$

Progress from a sum of order n to one of order $n+1$ is accomplished by integration. One may integrate the cosine case to generate the higher-order sine sum. Taking series 32:14:1 as an example:

32:14:5 $$\sum_{j=1}^{\infty}\frac{\sin(jx)}{j^{n+1}}=\sum_{j=1}^{\infty}\int_0^x\frac{\cos(jt)}{j^n}\mathrm{d}t=\int_0^x\left\{\sum_{j=1}^{\infty}\frac{\cos(jt)}{j^n}\right\}\mathrm{d}t$$

whereas the procedure to construct the $(n+1)$th cosine member is exemplified by

32:14:6 $$\sum_{j=1}^{\infty}\frac{\cos(jx)}{j^{n+1}}=\sum_{j=1}^{\infty}\int_x^{\pi/2}\frac{\sin(jt)}{j^n}\mathrm{d}t=\left\{\int_0^{\pi/2}-\int_0^{x}\right\}\left\{\sum_{j=0}^{\infty}\frac{\sin(jt)}{j^n}\right\}\mathrm{d}t=\zeta(n+1)-\int_0^x\left\{\sum_{j=1}^{\infty}\frac{\sin(jt)}{j^n}\right\}\mathrm{d}t$$

Similar integrations cater to the other three series. Formula 32:14:5 shows that to evaluate $\Sigma\sin(jx)/j^2$ we need to integrate $\ln\left\{\frac{1}{2}\csc\left(\frac{1}{2}x\right)\right\}$. This integral cannot be expressed in terms of simpler functions and is known as *Clausen's integral* (Thomas Clausen, 1801−1885, Danish mathematician, astronomer, and geophysicist). It is symbolized Clausen(x) in this *Atlas*.

32:14:7 $$\sum_{j=1}^{\infty}\frac{\sin(jx)}{j^2}=\mathrm{Clausen}(x)=\int_0^x\ln\left\{\frac{1}{2}\csc\left(\frac{t}{2}\right)\right\}\mathrm{d}t$$

As the following table shows, Clausen's integral also appears in the order-two sums of other series.

	$\sum_{j=1}^{\infty}\frac{f(jx)}{j^2}$	$-\sum_{j=1}^{\infty}\frac{(-)^j f(jx)}{j^2}$	$\sum_{j=0}^{\infty}\frac{f\{(2j+1)x\}}{(2j+1)^2}$	$\sum_{j=0}^{\infty}\frac{(-)^j f\{(2j+1)x\}}{(2j+1)^2}$
f = cos	$\frac{\pi^2}{6}-\frac{\pi x}{2}+\frac{x^2}{4}$	$\frac{\pi^2}{12}-\frac{x^2}{4}$	$\frac{\pi^2}{8}-\frac{\pi x}{4}$	$G-\frac{1}{2}\sum_{j=0}^{\infty}\frac{\lvert E_j\rvert x^{2j+2}}{(2j+2)!}$
f = sin	Clausen(x)	Clausen($\pi-x$)	$\frac{1}{2}\mathrm{Clausen}(x)+\frac{1}{2}\mathrm{Clausen}(\pi-x)$	$\frac{\pi x}{4}$

Constructed with the aid of equation 32:14:5, 32:14:6 and their analogues, some order-three sums are:

	$\sum_{j=1}^{\infty}\frac{f(jx)}{j^3}$	$-\sum_{j=1}^{\infty}\frac{(-)^j f(jx)}{j^3}$	$\sum_{j=0}^{\infty}\frac{f\{(2j+1)x\}}{(2j+1)^3}$	$\sum_{j=0}^{\infty}\frac{(-)^j f\{(2j+1)x\}}{(2j+1)^3}$
f = cos	$Z-\int_0^x\mathrm{Clausen}(t)\,\mathrm{d}t$	$\frac{3Z}{4}-\int_{\pi-x}^{\pi}\mathrm{Clausen}(t)\,\mathrm{d}t$	average of the two entries to the left	$\frac{\pi^3}{32}-\frac{\pi x^2}{8}$
f = sin	$\frac{\pi^2 x}{6}-\frac{\pi x^2}{4}+\frac{x^3}{12}$	$\frac{\pi^2 x}{12}-\frac{x^3}{12}$	$\frac{\pi^2 x}{8}-\frac{\pi x^2}{8}$	$Gx-\frac{1}{2}\sum_{j=0}^{\infty}\frac{\lvert E_j\rvert x^{2j+3}}{(2j+3)!}$

Catalan's and Apéry's constants, *G* and *Z* [Chapters 1 and 3] appear in these tables, as do the Euler numbers E_j [Chapter 5].

Clausen's integral is defined in the interval $0 \le x \le \pi$, though some authorities extend this definition to 2π with $\text{Clausen}(\pi+x) = -\text{Clausen}(\pi-x)$. It obeys the duplication formula

32:14:8 $$\text{Clausen}(2x) = 2\left[\text{Clausen}(x) - \text{Clausen}(\pi - x)\right]$$

and, as Figure 32-6 shows, has a maximum value of close to unity at $x = \pi/3$. By using formulas 25:10:3 and 33:6:4, it may be expanded as the series

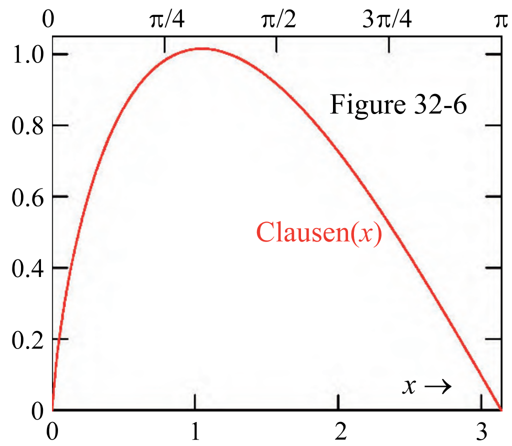


Figure 32-6

32:14:9 $$\text{Clausen}(x) = x - x\ln(x) + \frac{x^3}{72} + \frac{x^5}{14400} + \frac{x^7}{1270080} + \cdots = x\left[1 - \ln(x) - \sum_{j=1}^{\infty} \frac{(-x^2)^j B_{2j}}{2j(2j+1)!}\right]$$

where the B's are Bernoulli numbers [Chapter 4]. With keyword **Clausen**, *Equator* uses this equation, or

32:14:10 $$\text{Clausen}(\pi - x) = x\ln(2) - \frac{x^3}{24} - \frac{x^5}{960} - \frac{x^7}{20160} - \cdots = x\left[\ln(2) + \sum_{j=1}^{\infty} \frac{(4^j - 1)B_{2j}}{2j(2j+1)!}(-x^2)^j\right]$$

in its Clausen's integral routine.

32:15 RELATED TOPIC: Fourier transformation

As a purely mathematical operation applied to a function f of a (possibly complex) variable z, *Fourier transformation* (or *exponential Fourier transformation*) uses the integral transform

32:15:1 $$\int_{-\infty}^{\infty} f(z)\exp(-2i\pi\zeta z)\,dz = F(\zeta)$$

to generate a complex function of the variable ζ. To avoid ambiguity with what follows, $F(\zeta)$ is said to be a "continuous" Fourier transform. The inversion of the $F(\zeta)$ function back to $f(z)$ is accomplished by an analogous procedure,

32:15:2 $$\int_{-\infty}^{\infty} F(\zeta)\exp(2i\pi z\zeta)\,d\zeta = f(z)$$

that differs in form from the transformation formula only by a sign. In view of definitions 32:3:1 and 32:3:2, equation 32:15:1 may be rewritten

32:15:3 $$F(\zeta) = R(\zeta) - iI(\zeta) \qquad R(\zeta) = \int_{-\infty}^{\infty} f(z)\cos(2\pi\zeta z)\,dz, \qquad I(\zeta) = \int_{-\infty}^{\infty} f(z)\sin(2\pi\zeta z)\,dz$$

Though these adjectives are misleading, the $R(\zeta)$ and $I(\zeta)$ functions are often known as the "real" and "imaginary" Fourier transforms. The connection of this *continuous Fourier transform* to the Laplace transforms defined in equations 32:10:17 and 32:10:18 is apparent.

Few areas of mathematics have spawned as many powerful practical applications as Fourier transformation. However, the stage on which the majority of these applications is played is quite remote from the transforms defined above. In practice, the variables are generally time t and frequency ω, both real. The domain of the t variable, instead of being doubly infinite as in 32:15:1, is restricted to a finite range $0 \le t \le T$. And, most dissimilar of all, rather than being considered as a function f(t) of a continuous time variable, f is known experimentally only at a finite number N of equally spaced instants. The input data, a so-called *time series*, thus consists of a set of data that can be represented by $f_0, f_1, f_2, \cdots, f_n, \cdots, f_{N-1}$. The interval between consecutive data is T/N, or equivalently $2\pi/\omega N$. In this setting, the operation of Fourier transformation devolves into a replacement of the original N numbers, each representing the amplitude at a particular time, by another set of N numbers, each representing the intensity of the signal at a particular frequency. With $m = 0, 1, 2, \cdots, N-1$, the analogs of equations 32:15:1 and 32:15:3 are respectively

32:15:4
$$F_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n \exp\left(\frac{-2\pi inm}{N}\right)$$

and

32:15:5
$$R_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n \cos\left(\frac{2\pi mn}{N}\right) \qquad \text{and} \qquad I_m = \frac{-1}{N}\sum_{n=0}^{N-1} f_n \sin\left(\frac{2\pi mn}{N}\right)$$

Because F_m is complex, equations 32:15:5 are computationally more convenient. The R_m and I_m numbers are the *discrete Fourier transforms* of the time series, and again the unfortunate "real" and "imaginary" designators are generally used to distinguish between them. Notice that R_0 is simply the average value f_{ave} of the f's and that I_0 is zero, as is $I_{N/2}$ if N is even. It might appear as if the information in the original N data has been converted into $2N-1$ output numbers, but only N of these are distinct because R_{N-m} merely duplicates R_m, while $I_{N-m} = -I_m$. The original data may be regenerated from the transformed data by the *discrete Fourier inversion* formula

32:15:6
$$\begin{aligned} f_n &= R_0 + \sum_{m=1}^{N-1} R_m \cos\left(\frac{2\pi nm}{N}\right) - I_m \sin\left(\frac{2\pi nm}{N}\right) \\ &= f_{\text{ave}} + 2\sum_{m=1}^{\text{Int}(N/2)} R_m \cos\left(\frac{2\pi nm}{N}\right) - I_m \sin\left(\frac{2\pi nm}{N}\right) \end{aligned}$$

in which the second formulation takes account of the duplications.

Scientific and technological applications of Fourier transformation fall into two broad categories. In the first of these, transformation is followed by some type of processing of the transforms, the modified transforms being then inverted. A simple example is *Fourier smoothing*, in which the high frequency (large m) components of the transformed signal are nulled or attenuated prior to inversion back to modified data, which we represent by $\hat{f}$. *Demodulation* (removal of a high-frequency "carrier" wave) is another. In the second category, interest is in the transforms themselves, and often in their so-called *power spectrum*. This is

32:15:7
$$P_m = 2\left[R_m^2 + I_m^2\right]$$

and is frequently displayed as a series of vertical lines in a graph of P_m versus m.

Even with the advent of computers, calculating discrete Fourier transforms was a lengthy task until the perfection of the fast Fourier transform by Tukey and Cooley in 1965 [see Weaver's appendix for a program based on the FFT algorithm]. The speed of modern computers makes the fast Fourier transform, with its frequently built-in restriction that N must be a power of 2, unnecessary nowadays, though it has habituated. The view is commonplace that speedy Fourier transformation is an arcane process, involving binary arithmetic and unsuitable for user-programming. The *Atlas* includes the following description to demonstrate that this is not so. There are, at most, three procedures involved in discrete Fourier transformation: (a) conversion of f data to R data, (b) conversion of f data to I data, and (c) conversion of R and I data to $\hat{f}$ data. The description here addresses (c) only, but the simpler procedures required for (a) and (b) are so similar that a single routine can be written to perform all three.

The object is to carry out the process indicated in 32:15:6 expeditiously. This equation may be reformulated as

32:15:8 $$\frac{\hat{f}_n - f_{ave}}{2} = \sum_{m=2,4,\cdots} R_m \cos\left\{\frac{2\pi mn}{N}\right\} + I_m \sin\left\{\frac{2\pi mn}{N}\right\} + \sum_{m=1,3,\cdots} R_m \cos\left\{\frac{2\pi mn}{N}\right\} + I_m \sin\left\{\frac{2\pi mn}{N}\right\}$$

by simply splitting the summands into even-*m* and odd-*m* moieties. Upper limits have been omitted from the summations, it being understood that all nonzero summands are included. The odd-*m* moiety is now rewritten and expanded, via equations 32:5:10 and 32:5:11, as follows

32:15:9 $$\sum_{m=1,3,\cdots} R_m \cos\left\{\frac{2\pi(m+1)n}{N} - \frac{2\pi n}{N}\right\} + I_m \sin\left\{\frac{2\pi(m+1)n}{N} - \frac{2\pi n}{N}\right\} = \sum_{m=1,3,\cdots} [R_m c_n - I_m s_n]\cos\left\{\frac{2\pi(m+1)n}{N}\right\} + [R_m s_n + I_m c_n]\sin\left\{\frac{2\pi(m+1)n}{N}\right\}$$

where c_n and s_n are being used as abbreviations for $\cos(2\pi n/N)$ and $\sin(2\pi n/N)$ respectively. Next, the *m* in 32:15:9 is redefined as $m-1$, so that the summation now runs over $2,4,\cdots$, allowing the two summations to coalesce, whereby equation 32:15:8 becomes

32:15:10 $$\frac{\hat{f}_n - f_{ave}}{2} = \sum_{m=2,4,\cdots} [R_m + R_{m-1}c_n - I_{m-1}s_m]\cos\left\{\frac{2\pi mn}{N}\right\} + [I_m + R_{m-1}s_n + I_{m-1}c_n]\sin\left\{\frac{2\pi mn}{N}\right\}$$

Because only even values of *m* remain, this summation index may be halved, producing

32:15:11 $$\frac{\hat{f}_n - f_{ave}}{2} = \sum_{m=1,2} [R_{2m} + R_{2m-1}c_n - I_{2m-1}s_n]\cos\left\{\frac{4\pi mn}{N}\right\} + [I_{2m} + R_{2m-1}s_n + I_{2m-1}c_n]\sin\left\{\frac{4\pi mn}{N}\right\}$$

At this stage, in an actual algorithm, the terms in square brackets would be replaced by their numerical equivalents. A comparison of equations 32:15:6 and 32:15:11 shows that, at the expense of having to evaluate two new coefficients, the procedure has condensed the number of summed terms by a factor of (almost or exactly, depending on the parity of *N*) two. A careful analysis shows that if the condensation is iterated, for a total of *p* times, where $p = \text{Int}\{\log_2(N-1)\}$, then the final summation has just one term:

32:15:12 $$\frac{\hat{f}_n - f_{\text{ave}}}{2} = [number]\cos\left\{\frac{2^{p+1}\pi n}{N}\right\} + [number]\sin\left\{\frac{2^{p+1}\pi n}{N}\right\}$$

whence $\hat{f}$ is easily calculated. If *N* is huge, as it often is in technological applications, the procedure just described, which is essentially that of Sande and Tukey [see Chapra and Canale], leads to a massive saving in arithmetic.

CHAPTER 33

THE SECANT sec(*x*) AND COSECANT csc(*x*) FUNCTIONS

These functions are the reciprocals of those addressed in Chapter 32. The *secant function* and *cosecant function* are interrelated by

33:0:1 $$\frac{1}{\sec^2(x)}+\frac{1}{\csc^2(x)}=1$$

and by

33:0:2 $$\csc\left(x+\frac{\pi}{2}\right)=\sec(x)$$

33:1 NOTATION

To avoid possible confusion with the functions of Chapter 29, the names *circular secant* and *circular cosecant* may be used. Because of their applicability to triangles [Section 34:14] the functions of Chapters 32−34 are known collectively as *trigonometric functions*.

The symbol cosec(x) may be found in place of csc(x). As in Chapters 32 and 34, we replace the x symbol for the argument by θ whenever an angular interpretation is pertinent.

33:2 BEHAVIOR

The sec(x) and csc(x) functions are defined for all real values of their argument x and, as Figure 33-1 illustrates, the functions themselves adopt all values except those between −1 and +1. In accord with equation 33:0:2, the two functions are seen to differ only by an offset of $\pi/2$ in their arguments. The figure also shows the functions to be periodic with periods of 2π.

33:2:1 $$f(x)=f(x\pm 2\pi)=f(x\pm 4\pi)=f(x\pm 6\pi)=\cdots \qquad f=\sec \text{ or } \csc$$

As with the cosine and sine functions, this behavior is self-evident when the argument is regarded as an angle θ.

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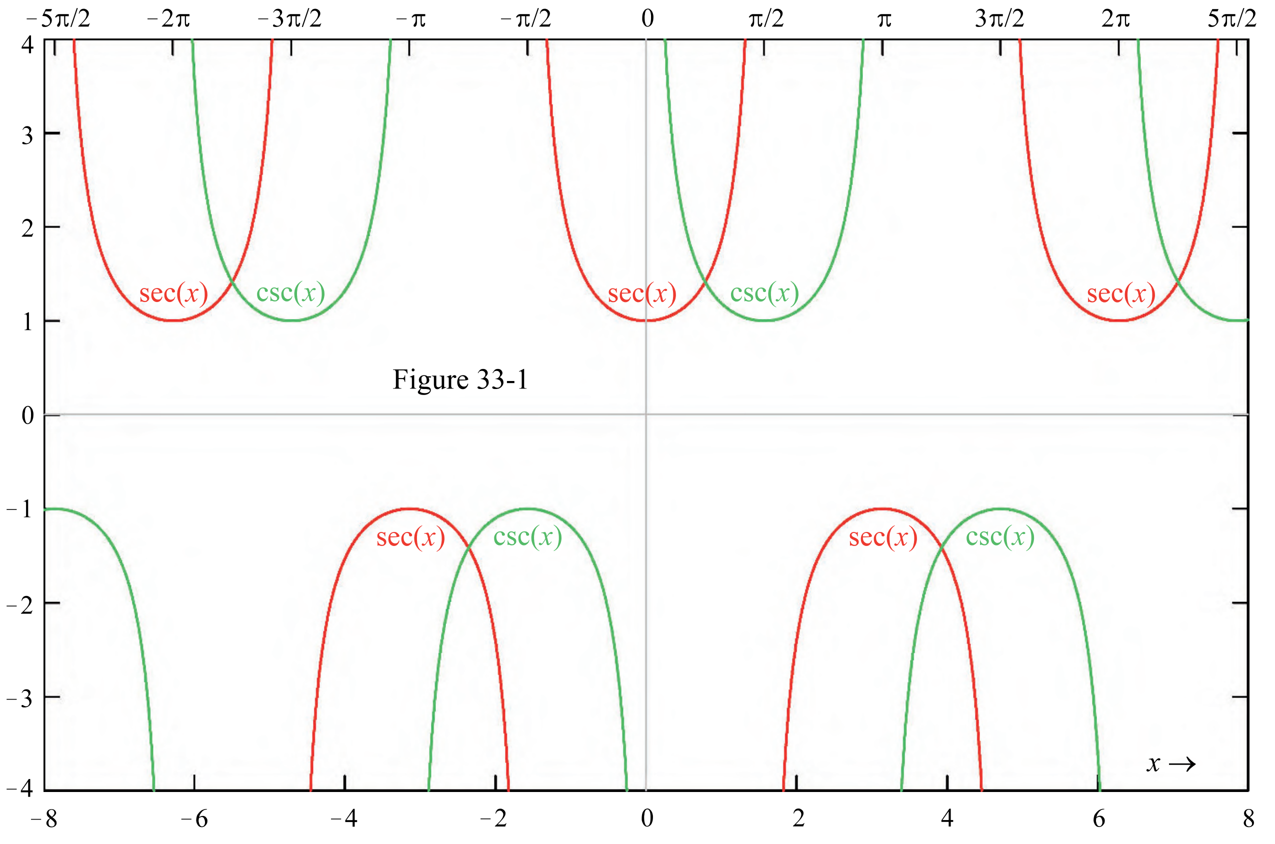


Figure 33-1

Because of the periodicity, it suffices to consider behaviors only within the $0 \le \theta \le 2\pi$ interval. In this range, the two functions adopt values as tabulated below:

First quadrant $0 \le \theta \le \pi/2$	Second quadrant $\pi/2 \le \theta \le \pi$	Third quadrant $\pi \le \theta \le 3\pi/2$	Fourth quadrant $3\pi/2 \le \theta \le 2\pi$
$1 \le \sec(\theta) \le +\infty$	$-\infty \le \sec(\theta) \le -1$	$-1 \ge \sec(\theta) \ge -\infty$	$+\infty \ge \sec(\theta) \ge 1$
$+\infty \ge \csc(\theta) \ge 1$	$1 \le \csc(\theta) \le +\infty$	$-\infty \ge \csc(\theta) \le -1$	$-1 \ge \csc(\theta) \ge -\infty$

Discontinuities are encountered by the secant function when its argument is an odd multiple of $\pi/2$. The cosecant experiences similar sign-changing discontinuities at every multiple of π.

33:3 DEFINITIONS

The secant and cosecant functions may be defined as the reciprocals of the functions of Chapter 32:

33:3:1
$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Equivalently, they may be defined in terms of exponentials of imaginary argument

33:3:2
$$\sec(x) = \frac{2\exp(ix)}{\exp(2ix)+1} \qquad \csc(x) = \frac{2i\exp(ix)}{\exp(2ix)-1}$$

The integral transforms

33:3:3 $$\sec(x)=\frac{2}{\pi}\int_0^\infty \frac{t^{2x/\pi}}{t^2+1}\,\mathrm{d}t \qquad \frac{-\pi}{2}<x<\frac{\pi}{2}$$

and

33:3:4 $$\csc(x)=\frac{1}{\pi}\int_0^\infty \frac{t^{x/\pi}}{t^2+t}\,\mathrm{d}t \qquad 0<x<\pi$$

provide other definitions of the secant and cosecant functions.

For arguments in the range $0<x<\pi/2$, Section 33:14 describes definitions in terms of triangles, not only of the secant and cosecant functions, but of all the functions – the *trigonometric* functions – of Chapters 32−24.

33:4 SPECIAL CASES

There are none.

33:5 INTRARELATIONSHIPS

Whereas the secant is an even function, the cosecant is odd:

33:5:1 $$\sec(-x)=\sec(x) \qquad \csc(-x)=-\csc(x)$$

The recursion formulas

33:5:2 $$\sec\left(x\pm\frac{n\pi}{2}\right)=\begin{cases}\sec(x) & n=0,4,8,\cdots\\ \mp\csc(x) & n=1,5,9,\cdots\\ -\sec(x) & n=2,6,10,\cdots\\ \pm\csc(x) & n=3,7,11,\cdots\end{cases}$$

and

33:5:3 $$\csc\left(x\pm\frac{n\pi}{2}\right)=\begin{cases}\csc(x) & n=0,4,8,\cdots\\ \pm\sec(x) & n=1,5,9,\cdots\\ -\csc(x) & n=2,6,10,\cdots\\ \mp\sec(x) & n=3,7,11,\cdots\end{cases}$$

parallel those of the cosine and sine, but there are no simple formulas, akin to the argument-addition formulas 32:5:10 and 32:5:11, to express $\sec(x\pm y)$ or $\csc(x\pm y)$. Such expressions – as well as those for $\sec^n(x)$, $\csc(x)\pm\csc(y)$, etc. – are best constructed from the formulas of Section 32:5 by making use of the identities in 33:3:1.

The secants and cosecants of double-argument and half-argument are given by the formulas

33:5:4 $$\sec(2x)=\frac{(-)^{\mathrm{Int}[(2|x|+\pi/2)/\pi]}\sec^2(x)}{2-\sec^2(x)}$$

33:5:5 $$\csc(2x)=\frac{\sec(x)\csc(x)}{2}=\frac{(-)^{\mathrm{Int}(2|x|/\pi)}\csc^2(x)}{2\sqrt{\csc^2(x)-1}}$$

33:5:6
$$\sec\left(\frac{x}{2}\right)=(-)^{\mathrm{Int}[(1+|x|/\pi)/2]}\sqrt{\frac{2\sec(x)}{1+\sec(x)}}$$

and

33:5:7
$$\csc\left(\frac{x}{2}\right)=(-)^{\mathrm{Int}[|x|/(2\pi)]}\sqrt{\frac{2\sec(x)}{\sec(x)-1}}$$

33:6 EXPANSIONS

The power-series expansion of the secant function

33:6:1
$$\sec(x)=1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots=\frac{4}{\pi}\sum_{j=0}^{\infty}\beta(2j+1)\left(\frac{4x^2}{\pi^2}\right)^j=\sum_{j=0}^{\infty}\frac{|\mathrm{E}_{2j}|}{(2j)!}x^{2j}\qquad |x|<\frac{\pi}{2}$$

can be expressed in terms of either the beta numbers from Chapter 3 or the Euler numbers [Chapter 5], whereas it is the eta numbers or Bernoulli numbers [Chapters 3 and 4] that play analogous roles for the cosecant

33:6:2
$$\csc(x)=\frac{1}{x}+\frac{x}{6}+\frac{7x^3}{360}+\frac{61x^5}{15120}+\cdots=\frac{2}{x}\sum_{j=0}^{\infty}\eta(2j)\left(\frac{x^2}{\pi^2}\right)^j=\sum_{j=0}^{\infty}\frac{|(4^j-2)\mathrm{B}_{2j}|}{(2j)!}x^{2j-1}\qquad |x|<\pi$$

Similar alternatives attend the power-series expansions of the logarithms of $\sec(x)$ and $x\csc(x)$

33:6:3
$$\ln\{\sec(x)\}=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\frac{17x^8}{2520}+\cdots=\sum_{j=1}^{\infty}\frac{\lambda(2j)}{j}\left(\frac{4x^2}{\pi^2}\right)^j=\sum_{j=1}^{\infty}\frac{(4^j-1)|\mathrm{B}_{2j}|}{2j(2j)!}(4x^2)^j\qquad |x|<\frac{\pi}{2}$$

and

33:6:4
$$\ln\{x\csc(x)\}=\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+\frac{x^8}{36800}+\cdots=\sum_{j=1}^{\infty}\frac{\zeta(2j)}{j}\left(\frac{x^2}{\pi^2}\right)^j=\sum_{j=1}^{\infty}\frac{|\mathrm{B}_{2j}|}{2j(2j)!}(4x^2)^j\qquad |x|<\pi$$

The secant and cosecant functions may be expanded as partial fractions

33:6:5
$$\sec(x)=\frac{4\pi}{\pi^2-4x^2}-\frac{12\pi}{9\pi^2-4x^2}+\frac{20\pi}{25\pi^2-4x^2}-\cdots=\pi\sum_{j=0}^{\infty}\frac{(-)^j(2j+1)}{(j+\frac{1}{2})^2\pi^2-x^2}$$

and

33:6:6
$$\csc(x)=\frac{1}{x}+\frac{2x}{\pi^2-x^2}-\frac{2x}{4\pi^2-x^2}+\frac{2x}{9\pi^2-x^2}-\cdots=\frac{1}{x}-2x\sum_{j=1}^{\infty}\frac{(-1)^j}{j^2\pi^2-x^2}$$

as may their squares

33:6:7
$$\sec^2(x)=\frac{4}{(\pi-2x)^2}+\frac{4}{(\pi+2x)^2}+\frac{4}{(3\pi-2x)^2}+\frac{4}{(3\pi+2x)^2}+\cdots=\sum_{j=-\infty}^{+\infty}\frac{1}{[(j+\frac{1}{2})\pi+x]^2}$$

and

33:6:8
$$\csc^2(x)=\frac{1}{x^2}+\frac{1}{(\pi-x)^2}+\frac{1}{(\pi+x)^2}+\frac{1}{(2\pi-x)^2}+\frac{1}{(2\pi+x)^2}+\cdots=\sum_{j=-\infty}^{+\infty}\frac{1}{[j\pi+x]^2}$$

The last pair of formulas are Laurent series [Section 27:13], in which the summation runs over all integers: positive, negative and zero.

33:7 PARTICULAR VALUES

The most important particular values are

33:7:1
$$\sec(v\pi) = \begin{cases} 1 & v = 0, \pm 2, \pm 4, \cdots \\ +\infty\vert-\infty & v = +\frac{1}{2}, -\frac{3}{2}, +\frac{5}{2}, -\frac{7}{2}, \cdots \\ -1 & v = \pm 1, \pm 3, \pm 5, \cdots \\ -\infty\vert+\infty & v = -\frac{1}{2}, +\frac{3}{2}, -\frac{5}{2}, +\frac{7}{2}, \cdots \end{cases}$$

and

33:7:2
$$\csc(v\pi) = \begin{cases} -\infty\vert+\infty & v = 0, -1, +2, -3, \cdots \\ 1 & v = \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \cdots \\ +\infty\vert-\infty & v = 1, -2, +3, -4, \cdots \\ -1 & v = \pm\frac{3}{2}, \pm\frac{7}{2}, \pm\frac{11}{2}, \cdots \end{cases}$$

Others in the range $0 < x < \pi/2$ are given in the table below. Use equations 33:5:2 or 33:5:3 to develop particular values outside this range.

θ	15°	18°	22½°	30°	36°	45°	54°	60°	67½°	72°	75°
x	$\pi/12$	$\pi/10$	$\pi/8$	$\pi/6$	$\pi/5$	$\pi/4$	$3\pi/10$	$\pi/3$	$3\pi/8$	$2\pi/5$	$5\pi/12$
sec	$\frac{\sqrt{8}}{\sqrt{3}+1}$	$\frac{\sqrt{8}}{\sqrt{5+\sqrt{5}}}$	$\sqrt{4-\sqrt{8}}$	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{8}}{\sqrt{3+\sqrt{5}}}$	$\sqrt{2}$	$\frac{\sqrt{8}}{\sqrt{5-\sqrt{5}}}$	2	$\sqrt{4+\sqrt{8}}$	$\frac{\sqrt{8}}{\sqrt{3-\sqrt{5}}}$	$\frac{\sqrt{8}}{\sqrt{3}-1}$
csc	$\frac{\sqrt{8}}{\sqrt{3}-1}$	$\frac{\sqrt{8}}{\sqrt{3-\sqrt{5}}}$	$\sqrt{4+\sqrt{8}}$	2	$\frac{\sqrt{8}}{\sqrt{5-\sqrt{5}}}$	$\sqrt{2}$	$\frac{\sqrt{8}}{\sqrt{3+\sqrt{5}}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{4-\sqrt{8}}$	$\frac{\sqrt{8}}{\sqrt{5+\sqrt{5}}}$	$\frac{\sqrt{8}}{\sqrt{3}+1}$

33:8 NUMERICAL VALUES

Equator's secant function and cosecant function routines (keywords **sec** and **csc**) provide exact values of sec(x) and csc(x). The algorithms simply reciprocate the corresponding values of cos(x) and sin(x).

33:9 LIMITS AND APPROXIMATIONS

Close to an argument value x_∞ at which the cosecant function suffers a discontinuity of the $-\infty\vert+\infty$ variety, the approximation

33:9:1
$$\csc(x) \approx \frac{1}{x-x_\infty} + \frac{x-x_\infty}{6} \qquad x \approx x_\infty = 0, \pm 2\pi, \pm 4\pi, \cdots$$

holds. Further terms are available from equation 33:6:2. The negative of this approximation is valid for arguments close to one of the $+\infty|-\infty$ discontinuities at $x_\infty = \pm\pi, \pm3\pi, \pm5\pi$, etc. Similarly, for the secant function,

33:9:2 $$\sec(x) \approx \frac{\pm1}{x-x_\infty} \pm \frac{x-x_\infty}{6} \qquad x \approx x_\infty = \frac{\pm\pi}{2}, \frac{\mp3\pi}{2}, \frac{\pm5\pi}{2}, \frac{\mp7\pi}{2}, \cdots$$

33:10 OPERATIONS OF THE CALCULUS

Differentiation and indefinite integration of the secant and cosecant functions lead to

33:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x) = \frac{\sec^2(x)}{\csc(x)}$$

33:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x) = \frac{-\csc^2(x)}{\sec(x)}$$

33:10:3 $$\int_0^x \sec(t)\,\mathrm{d}t = \ln\{\sec(x)+\tan(x)\} = \mathrm{invgd}(x) \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

and

33:10:4 $$\int_{\pi/2}^x \csc(t)\,\mathrm{d}t = \ln\left\{\tan\left(\frac{x}{2}\right)\right\} = \ln\{\csc(x)-\cot(x)\} \qquad 0 < x < \pi$$

The invgd function in 33:10:3 is the inverse Gudermannian function discussed in Section 33:15. The indefinite integrals of the squares of secant and cosecant functions yield the functions of Chapter 34:

33:10:5 $$\int_0^x \sec^2(t)\,\mathrm{d}t = \tan(x) \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

33:10:6 $$\int_{\pi/2}^x \csc^2(t)\,\mathrm{d}t = \cot(x) \qquad \frac{\pi}{2} < x < \pi$$

The definite integral

33:10:7 $$\int_0^{\pi/2} x\csc(x)\,\mathrm{d}x = 2G$$

generates twice Catalan's constant [Chapter 1].

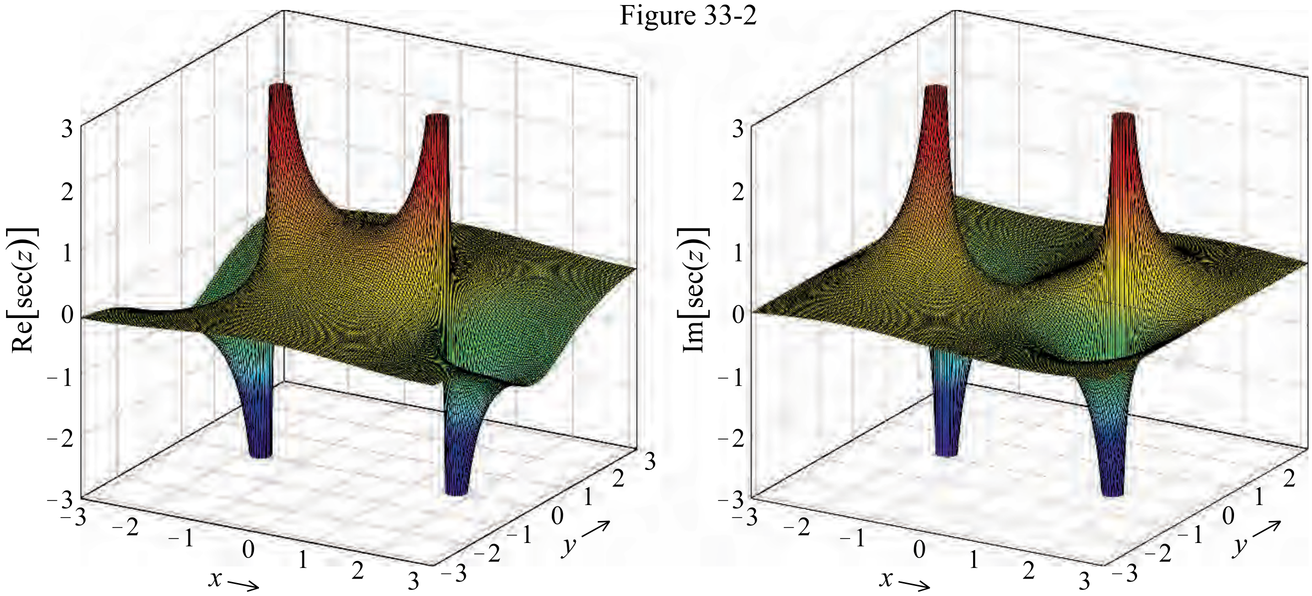
33:11 COMPLEX ARGUMENT

The secant of the complex variable $z = x + iy$ takes complex values given by the alternative formulas

33:11:1 $$\sec(x+iy) = \frac{\cos(x)\cosh(y) + i\sin(x)\sinh(y)}{\cos^2(x)+\sinh^2(y)} = \frac{2\cos(x)\cosh(y)+2i\sin(x)\sinh(y)}{\cos(2x)+\cosh(2y)}$$

From the denominator of the first expression it is clear that sec(z) is finite unless $y = 0$ and x equals one of the values $(2n+1)\pi/2$, where n is an integer. This is confirmed by Figure 33-2, the two diagrams of which depict the real and imaginary parts of sec(z). The corresponding diagrams for the cosecant of a complex variable are not shown, but they are very similar to Figure 33-2 except that the poles occur at $y = 0$, $x = n\pi$, as would be expected from the first

Figure 33-2



alternative in the expression

33:11:2 $$\csc(x+iy)=\frac{\sin(x)\cosh(y)-i\cos(x)\sinh(y)}{\sin^2(x)+\sinh^2(y)}=\frac{-2\sin(x)\cosh(y)+2i\cos(x)\sinh(y)}{\cos(2x)-\cosh(2y)}$$

When the argument is purely imaginary, equations 33:11:1 and 33:11:2 reduce respectively to

33:11:3 $$\sec(iy)=\mathrm{sech}(y)$$

and

33:11:4 $$\csc(iy)=-i\,\mathrm{csch}(y)$$

33:12 GENERALIZATIONS

The Jacobian elliptic functions [Chapter 63] $\mathrm{nc}(k,x)$ and $\mathrm{dc}(k,x)$ are generalizations of $\sec(x)$, to which they reduce as $k\to 0$. Similarly, $\csc(x)$ is the $k\to 0$ limit of the $\mathrm{ns}(k,x)$ and $\mathrm{ds}(k,x)$ functions.

33:13 COGNATE FUNCTIONS

The secant and cosecant functions are related to other circular functions through the formulas:

33:13:1 $$\sec(x)=\frac{\sigma_{14}}{\sqrt{1-\sin^2(x)}}=\frac{1}{\cos(x)}=\frac{\sigma_{13}\csc(x)}{\sqrt{\csc^2(x)-1}}=\sigma_{14}\sqrt{1+\tan^2(x)}=\frac{\sigma_{12}\sqrt{\cot^2(x)+1}}{\cot(x)}$$

33:13:2 $$\csc(x)=\frac{\sigma_{12}}{\sqrt{1-\cos^2(x)}}=\frac{1}{\sin(x)}=\frac{\sigma_{13}\sec(x)}{\sqrt{\sec^2(x)-1}}=\frac{\sigma_{14}\sqrt{1+\tan^2(x)}}{\tan(x)}=\sigma_{12}\sqrt{\cot^2(x)+1}$$

The multiplier σ_{1n} is equal to +1 in the first and nth quadrants, but −1 in the other two quadrants.

The secant and cosecant functions are related to hyperbolic functions in two distinct ways: through formulas 33:11:3 and 33:11:4, and via the Gudermannian function of Section 33:15.

Not appearing elsewhere in the *Atlas*, and not available through *Equator*, is the archaic *exsecant* function

33:13:3 $$\mathrm{exsec}(x) = \sec(x) - 1 = \tan(x)\tan\left(\tfrac{1}{2}x\right)$$

The identities

33:13:4 $$\pi\sec(\nu\pi) = \Gamma\left(\tfrac{1}{2}-\nu\right)\Gamma\left(\tfrac{1}{2}+\nu\right) \qquad \text{and} \qquad \pi\csc(\nu\pi) = \Gamma(\nu)\Gamma(1-\nu)$$

provide links to the gamma function of Chapter 43.

33:14 RELATED TOPIC: trigonometric interpretation of the circular functions

Figure 33-3 shows three *similar* right-angled triangles; that is, the triangles have the same shape but different sizes. If the so-marked side of each triangle is of unity length, then the lengths of the other six sides correspond to the six circular functions, as shown. The argument of the functions can be interpreted either as the angle marked θ or as the arc length x of the unity-radius circle subtended by that angle, as diagrammed. Besides serving to define the six functions trigonometrically, many interrelations between them may be derived by applying similarity and Pythagorean relationships to the triangles. For example, by equating the ratios of the green to the red sides in the first and second triangles, it follows that

33:14:1 $$\frac{\sin(\theta)}{1} = \frac{\tan(\theta)}{\sec(\theta)}$$

while

33:14:2 $$\csc^2(\theta) = 1 + \cot^2(\theta)$$

is a consequence of applying the theorem of Pythagoras [Section 34:15] to the third triangle.

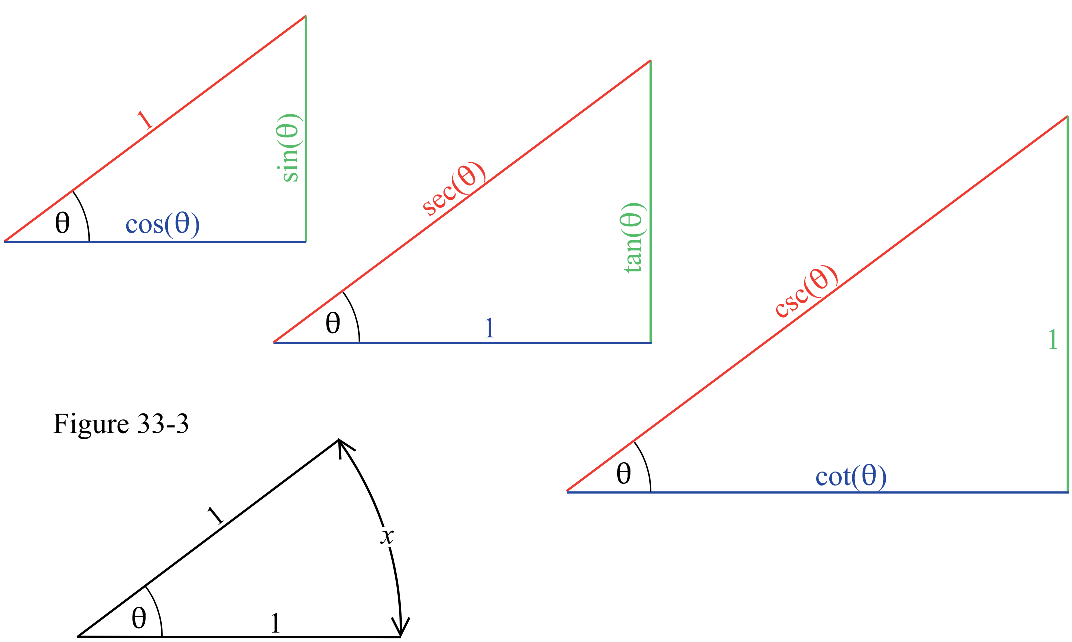


Figure 33-3

Because in trigonometry it is thought of as an angle, we have chosen θ as the argument of the functions marked in Figure 33-3 and as the variable in the above equations. With equal validity, θ may be replaced by x in the figure and interpreted as the marked arc length. Of course, the equations hold without the need to associate the variable with any geometric feature whatsoever.

33:15 RELATED TOPIC: the Gudermannian function and its inverse

Comparison of Figures 29-5 and 33-3 suggests that there should be a relationship between the six hyperbolic functions and the six circular functions. In fact, there is a one-to-one relationship

33:15:1
$$\left\{\begin{array}{c} \cos(\theta)=\mathrm{sech}(x) \\ \sin(\theta)=\tanh(x) \\ \sec(\theta)=\cosh(x) \\ \csc(\theta)=\coth(x) \\ \tan(\theta)=\sinh(x) \\ \cot(\theta)=\mathrm{csch}(x) \end{array}\right\} \qquad \frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$$

between pairs of functions, provided that the arguments are suitably related. That relationship involves the *Gudermannian function* (Christoph Gudermann, 1798 – 1852, German mathematician)

33:15:2
$$\theta=\mathrm{gd}(x)$$

or the *inverse Gudermannian function*

33:15:3
$$x=\mathrm{invgd}(\theta)$$

also denoted $\mathrm{gd}^{-1}(\theta)$. These functions are illustrated in Figure 33-4. Of course, it is not necessary that the argument of the circular functions be regarded as angles. See Section 28:14 for an application of these functions.

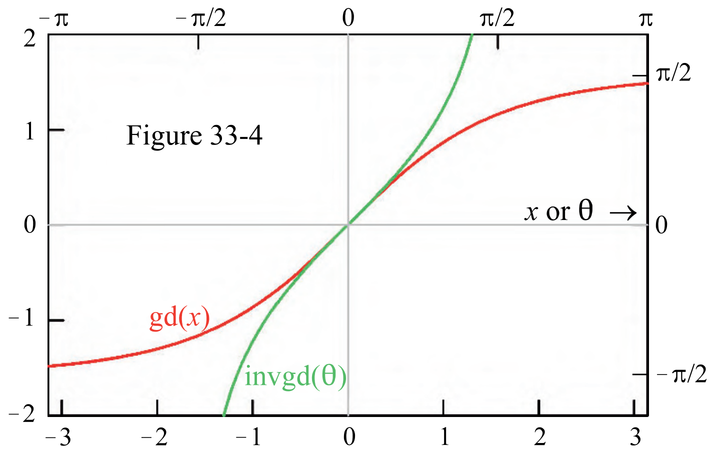


Figure 33-4

Respectively, the Gudermannian function and its inverse may be defined as the indefinite integral of the hyperbolic and circular secants:

33:15:4
$$\mathrm{gd}(x)=\int_0^x \mathrm{sech}(t)\,\mathrm{d}t$$

33:15:5
$$\mathrm{invgd}(\theta)=\int_0^\theta \sec(t)\,\mathrm{d}t$$

although there are many other relationships, including

33:15:6
$$\mathrm{gd}(x)=\arctan\{\sinh(x)\}=2\arctan\{\exp(x)\}-\frac{\pi}{2}=2\arctan\left\{\tanh\left(\frac{x}{2}\right)\right\}$$

and

33:15:7
$$\mathrm{invgd}(\theta)=\mathrm{arsinh}\{\tan(\theta)\}=\ln\{\sec(\theta)+\tan(\theta)\}=\ln\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}=\mathrm{F}(1,\theta)$$

that serve as alternative definitions. The final item in 33:15:7 is the incomplete elliptic integral of the first kind [Chapter 62] of unity modulus and argument θ. It is the first equality in each of formulas 33:15:6 and 33:15:7 that is exploited by *Equator* in its Gudermannian function and inverse Gudermannian function routines (keywords **gd** and **invgd**).

As the figure illustrates, both functions are odd. The domain of the inverse Gudermannian function is restricted to $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ while that of the Gudermannian functions is unrestricted. The power series for gd(x) and invgd(θ) involve Euler numbers [Chapter 5] and are remarkably similar:

33:15:8 $$\mathrm{gd}(x) = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{x^7}{5040} + \cdots = \sum_{j=0}^{\infty} \frac{\mathrm{E}_{2j}}{(2j+1)!} x^{2j+1} \qquad -1 < x < 1$$

33:15:9 $$\mathrm{invgd}(\theta) = \theta + \frac{\theta^3}{6} + \frac{\theta^5}{24} + \frac{\theta^7}{5040} + \cdots = \sum_{j=0}^{\infty} \frac{|\mathrm{E}_{2j}|}{(2j+1)!} \theta^{2j+1} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Another simple expansion is

33:15:10 $$\mathrm{invgd}(\theta) = 2\tau + \frac{2\tau^3}{3} + \frac{2\tau^5}{5} + \frac{2\tau^7}{7} + \cdots = 2\tau \sum_{j=0}^{\infty} \frac{\tau^{2j}}{2j+1} \qquad \tau = \tan\left(\frac{\theta}{2}\right)$$

[Other properties of τ are discussed in Section 34:14.] Several other expansions exist for the Gudermannian function and its inverse [Beyer, *Handbook of Mathematical Functions*, pages 323–325]. The Gudermannian function's approach to its limiting value of $\pi/2$ is described by

33:15:11 $$\mathrm{gd}(x) \approx \frac{\pi}{2} - 2\exp(-x) + \frac{2}{3}\exp(-3x) \qquad x \text{ large and positive}$$

and the approach of its inverse to infinity is described by

33:15:12 $$\mathrm{invgd}(\theta) \approx -\ln\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \qquad \left(\frac{\pi}{2} - \theta\right) \text{ small and positive}$$

CHAPTER 34

THE TANGENT tan(x) AND COTANGENT cot(x) FUNCTIONS

The functions of this chapter are the reciprocals of each other

34:0:1 $$\tan(x)\cot(x) = 1$$

and are also interrelated by

34:0:2 $$\cot(x) = \tan\left(\frac{\pi}{2} - x\right)$$

Together with the cosine, sine, secant, and cosecant functions, they constitute the family of *circular* functions. The six members of this family are also known as *trigonometric* functions because of the role they play in the mensuration of triangles, a topic addressed in Section 34:15.

34:1 NOTATION

The alternative notation tg(x) is occasionally encountered for the tangent function, while cotan(x) or ctg(x) sometimes replaces cot(x). To emphasize the distinction from their hyperbolic counterparts [Chapter 30], the names *circular tangent* and *circular cotangent* may be used. As elsewhere in this *Atlas*, this chapter often uses a symbol other than x to represent the argument of the tangent and cotangent functions when the intent is to focus on the angular interpretation of the argument.

34:2 BEHAVIOR

Like the other four circular functions, the tangent and cotangent are periodic but, unlike the others, the period is π, not 2π. This is clearly brought out in Figure 34:1. Notice that tan(x) encounters a discontinuity of the $+\infty|-\infty$ variety at $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2,\cdots$. Conversely, the discontinuities of cot(x) are of the $-\infty|+\infty$ type and occur at $x = 0, \pm\pi, \pm2\pi,\cdots$.

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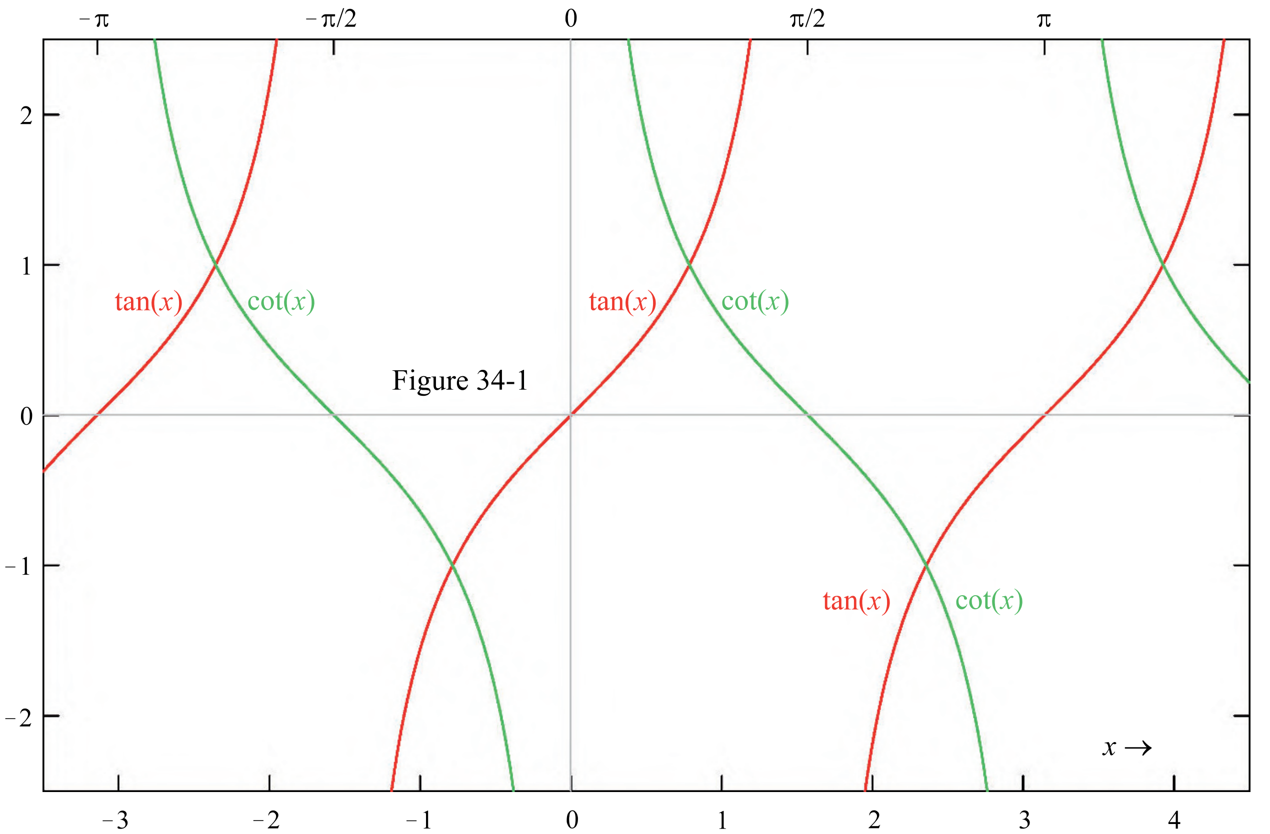


Figure 34-1

34:3 DEFINITIONS

The tangent and cotangent functions may be defined in terms of other circular functions in a variety of ways:

34:3:1 $$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cot(x)} = \sqrt{\sec^2(x) - 1} = \int_0^x \sec^2(t)\,\mathrm{d}t$$

34:3:2 $$\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)} = \sqrt{\csc^2(x) - 1} = \int_x^{\pi/2} \csc^2(t)\,\mathrm{d}t$$

There are alternative definitions via exponential functions of imaginary argument

34:3:3 $$\tan(x) = i\,\frac{1 - \exp(2ix)}{1 + \exp(2ix)}$$

34:3:4 $$\cot(x) = i\,\frac{\exp(2ix) + 1}{\exp(2ix) - 1}$$

or through the definite integral transforms

34:3:5 $$\tan(x) = \frac{2}{\pi}\int_0^\infty \frac{t^{2x/\pi} - 1}{t^2 - 1}\,\mathrm{d}t \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

34:3:6 $$\cot(x) = \frac{2}{\pi}\int_0^\infty \frac{t^{2x/\pi} - t}{t - t^3}\,\mathrm{d}t \qquad 0 < x < \pi$$

both of which require a Cauchy limit interpretation [Section 0:10].

Geometrically, a definition of the tangent in terms of an angle is provided by tan(θ) = (PQ)/(OQ) in Figure 32-2 or in terms of an area by tan(ar') = (P′Q′)(O′Q′) in Figure 28-2. The cotangent is simply the reciprocal of either of these ratios.

Another definition is provided by one solution of the following differential equation

34:3:7
$$\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}x} = a\mathrm{f}^2 + b\mathrm{f} + c \qquad \mathrm{f}(x) = \begin{cases} \dfrac{\sqrt{4ac-b^2}}{2a}\tan\left(\dfrac{x\sqrt{4ac-b^2}}{2}\right) - \dfrac{b}{2a} & b^2 < 4ac \\[2ex] \dfrac{-(1+bx)}{4ax} & b^2 = 4ac \\[2ex] \dfrac{-\sqrt{b^2-4ac}}{2a}\tanh\left(\dfrac{x\sqrt{b^2-4ac}}{2}\right) - \dfrac{b}{2a} & b^2 > 4ac \end{cases}$$

This solution may be redrafted in terms of the cot and coth functions.

34:4 SPECIAL CASES

There are none.

34:5 INTRARELATIONSHIPS

The tangent and cotangent functions are both odd

34:5:1
$$\mathrm{f}(-x) = -\mathrm{f}(x) \qquad \mathrm{f} = \tan \text{ or } \cot$$

and both obey the reflection formula

34:5:2
$$\mathrm{f}\left(\tfrac{1}{4}\pi + x\right) = \frac{1}{\mathrm{f}\left(\tfrac{1}{4}\pi - x\right)} \qquad \mathrm{f} = \tan \text{ or } \cot$$

The argument-addition formulas

34:5:3
$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$

and

34:5:4
$$\cot(x \pm y) = \frac{\cot(x)\cot(y) \mp 1}{\cot(y) \pm \cot(x)}$$

have the special cases

34:5:5
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} = \frac{2\cot(x)}{\cot^2(x)-1} = \frac{1}{\cot(2x)}$$

34:5:6
$$\tan(3x) = \frac{3\tan(x)-\tan^3(x)}{1-3\tan^2(x)} = \frac{3\cot^2(x)-1}{\cot^3(x)-3\cot(x)} = \frac{1}{\cot(3x)}$$

and generate the entries in the following table.

	$n = 0,4,8,\cdots$	$n = 1,5,9,\cdots$	$n = 2,6,10,\cdots$	$n = 3,7,11,\cdots$
$\tan\left(x \pm \frac{n\pi}{4}\right) =$	$\tan(x)$	$\frac{\tan(x) \pm 1}{1 \mp \tan(x)}$	$\frac{-1}{\tan(x)}$	$\frac{\tan(x) \mp 1}{1 \pm \tan(x)}$
$\cot\left(x \pm \frac{n\pi}{4}\right) =$	$\cot(x)$	$\frac{\cot(x) \mp 1}{1 \pm \cot(x)}$	$\frac{-1}{\cot(x)}$	$\frac{\cot(x) \pm 1}{1 \mp \cot(x)}$

The tangent of half argument

34:5:7 $$\tan\left(\frac{x}{2}\right) = (-)^{\mathrm{Int}(x/\pi)} \sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$$

is a convenient function in calculations, as discussed in Section 34:14.

The function-addition/subtraction formulas

34:5:8 $$\cot(x) \pm \tan(x) = 2 \,{}^{\csc}_{\cot}(2x)$$

34:5:9 $$\tan(x) \pm \tan(y) = \sin(x \pm y)\sec(x)\sec(y)$$

34:5:10 $$\cot(x) \pm \cot(y) = \sin(y \mp x)\csc(x)\csc(y)$$

and the function-multiplication formulas

34:5:11 $$\tan(x)\tan(y) = \frac{\cos(x-y) - \cos(x+y)}{\cos(x-y) + \cos(x+y)} = \frac{1}{\cot(x)\cot(y)}$$

34:5:12 $$\tan(x)\cot(y) = \frac{\sin(x+y) + \sin(x-y)}{\sin(x+y) - \sin(x-y)} = \frac{1}{\cot(x)\tan(y)}$$

complete our list of intrarelationships.

34:6 EXPANSIONS

Power series for the tangent and cotangent functions, as well as for their logarithms, may be written with coefficients that incorporate either Bernoulli numbers [Chapter 4] or functions from Chapter 3:

34:6:1 $$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots = \sum_{j=1}^{\infty} \frac{4^j(4^j-1)|\mathrm{B}_{2j}|}{(2j)!} x^{2j-1} = \frac{2}{x}\sum_{k=1}^{\infty} \lambda(2k)\left[\frac{2x}{\pi}\right]^{2k} \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$$

34:6:2 $$\cot(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots = \frac{1}{x} - \sum_{j=1}^{\infty} \frac{4^j|\mathrm{B}_{2j}|}{(2j)!} x^{2j-1} = \frac{1}{x} - \frac{2}{x}\sum_{k=1}^{\infty} \zeta(2k)\left[\frac{x}{\pi}\right]^{2k} \qquad -\pi < x < \pi$$

34:6:3 $$\left.\begin{matrix} -\ln\{x\cot(x)\} \\ \ln\{[\tan(x)/x]\} \end{matrix}\right\} = \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots = \sum_{j=1}^{\infty} \frac{(4^j-2)|\mathrm{B}_{2j}|}{2j(2j)!}(4x^2)^j = \sum_{k=1}^{\infty} \frac{\eta(2k)}{k}\left[\frac{4x^2}{\pi^2}\right]^k \qquad |x| < \frac{\pi}{2}$$

Either of the partial fraction expansions

34:6:4 $$\tan(x) = \frac{8x}{\pi^2 - 4x^2} + \frac{8x}{9\pi^2 - 4x^2} + \frac{8x}{25\pi^2 - 4x^2} + \cdots$$

or

34:6:5 $$\cot(x) = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} - \frac{2x}{9\pi^2 - x^2} - \cdots$$

may be written concisely as

34:6:6 $$\mathrm{f}(x) = \sum_j \frac{1}{x - (x_\infty)_j} \qquad \mathrm{f} = \cot \text{ or } -\tan$$

where the $(x_\infty)_j$ terms are the values of the argument that make f(x) infinite. The continued fraction expansion

34:6:7 $$\tan(x) = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\frac{x^2}{7-\cdots}$$

holds for the tangent function, provided that x is not an odd multiple of $\pi/2$.

34:7 PARTICULAR VALUES

The most important particular values adopted by the tangent and cotangent functions are

34:7:1 $$\tan(x) = \begin{cases} 0 & x = 0, \pm\pi, \pm 2\pi, \cdots \\ 1 & x = +\frac{1}{4}\pi, -\frac{3}{4}\pi, +\frac{5}{4}\pi, \cdots \\ +\infty \mid -\infty & x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots \\ -1 & x = -\frac{1}{4}\pi, +\frac{3}{4}\pi, -\frac{5}{4}\pi, \cdots \end{cases}$$

and

34:7:2 $$\cot(x) = \begin{cases} -\infty \mid +\infty & x = 0, \pm\pi, \pm 2\pi, \cdots \\ 1 & x = +\frac{1}{4}\pi, -\frac{3}{4}\pi, +\frac{5}{4}\pi, \cdots \\ 0 & x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots \\ -1 & x = -\frac{1}{4}\pi, +\frac{3}{4}\pi, -\frac{5}{4}\pi, \cdots \end{cases}$$

Others in the range $0 < x < \pi/2$ are included in the following table.

θ	15°	18°	22½°	30°	36°	45°	54°	60°	67½°	72°	75°
x	$\pi/12$	$\pi/10$	$\pi/8$	$\pi/6$	$\pi/5$	$\pi/4$	$3\pi/10$	$\pi/3$	$3\pi/8$	$2\pi/5$	$5\pi/12$
tan	$2-\sqrt{3}$	$\sqrt{1-\frac{2}{\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-\sqrt{20}}$	1	$\sqrt{1+\frac{2}{\sqrt{5}}}$	$\sqrt{3}$	$\sqrt{2}+1$	$\sqrt{5+\sqrt{20}}$	$2+\sqrt{3}$
cot	$2+\sqrt{3}$	$\sqrt{5+\sqrt{20}}$	$\sqrt{2}+1$	$\sqrt{3}$	$\sqrt{1+\frac{2}{\sqrt{5}}}$	1	$\sqrt{5-\sqrt{20}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$\sqrt{1-\frac{2}{\sqrt{5}}}$	$2-\sqrt{3}$

This table may be extended by the use of the reflection formula 34:1 and the table in Section 34:5.

There is an infinite set of values that satisfy the equation

34:7:3 $$\tan(x) = bx \qquad -\infty < b < \infty$$

and they arise in solving certain problems. The location of these values, known as the *roots* of equation 34:7:3, is illustrated by the dots in Figure 34-2. One of the roots is $x = 0$, but all the others depend on b. The positive roots will be denoted $\mathrm{r}_n(b)$. There are also negative roots, $r_{-n}(b) = -\mathrm{r}_n(b)$, occupying the left-hand side of the figure. The nth positive root lies in the range

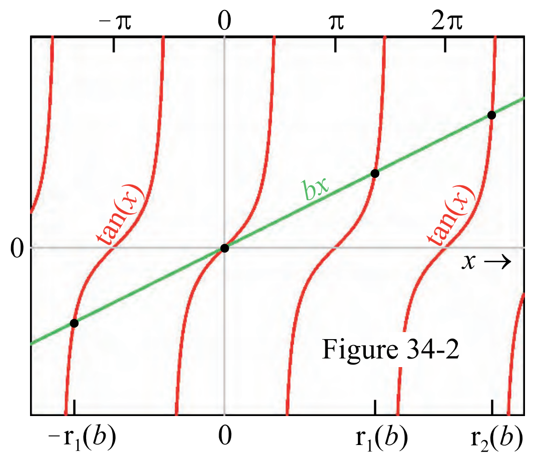


Figure 34-2

34:7:4 $$\left(n-\tfrac{1}{2}\right)\pi < \mathrm{r}_n(b) < \left(n+\tfrac{1}{2}\right)\pi \qquad n = 1,2,3,\cdots$$

irrespective of b. However, if $b > 1$, there is an additional positive root, denoted $\mathrm{r}_0(b)$, lying within $0 < x < \pi/2$. *Equator*'s tangent root routine (keyword **r**) provides values of $\mathrm{r}_n(b)$ for all real b and all integer n. It operates by inverting equation 34:7:3 to

34:7:5 $$\mathrm{r}_n(b) = \mathrm{Arctan}\left\{b\,\mathrm{r}_n(b)\right\} = n\pi + \arctan\left\{b\,\mathrm{r}_n(b)\right\}$$

in which arctan is the inverse tangent function [Chapter 35] and Arctan is the multivalued inverse tangent function [Section 35:12]. Equation 34:7:5 has the fortunate property that, if an inexact value of $\mathrm{r}_n(b)$ is inserted into its right-hand side, evaluation of the left-hand side produces a less inexact result. *Equator* starts with $\mathrm{r}_n(b) \approx (n+\frac{1}{4})\pi$ and exploits this property until successive answers no longer change.

The case $b = 1$ is especially important. Values of $\mathrm{r}_n(1)$ occur in the expansion of the Langevin function [Section 30:13] and correspond to the zeros of the Bessel functions of moiety order [Section 53:7]. For large n, the sum

34:7:6 $$\mathrm{r}_n(1) = N - \frac{1}{N} - \frac{2}{3N^3} - \frac{13}{15N^5} - \frac{146}{105N^7} - \frac{781}{315N^9} - \cdots \qquad \text{where} \quad N = \left(n+\tfrac{1}{2}\right)\pi$$

can be useful. Sums of reciprocal powers of $\mathrm{r}_n(1)$ occur in other problems and, in that context, the results

34:7:7 $$\sum_{n=1}^{\infty}\left[\mathrm{r}_n(1)\right]^{-2} = \frac{1}{10}, \qquad \sum_{n=1}^{\infty}\left[\mathrm{r}_n(1)\right]^{-4} = \frac{1}{350}, \qquad \text{and} \qquad \sum_{n=1}^{\infty}\left[\mathrm{r}_n(1)\right]^{-6} = \frac{1}{7875}$$

should be noted.

The roots of

34:7:8 $$\cot(x) = bx \qquad -\infty < b < \infty$$

the positive members of which we denote by $\rho_n(b)$, also arise in certain problems. Zero is not a root in this case, nor are there any $\rho_0(b)$ roots. The nth positive root lies in the range $(n-1)\pi < \rho_n(b) < n\pi$. *Equator*'s cotangent root routine (keyword **rho**) works in a fashion similar to that for the tangent root, with starting values of $(n-\frac{1}{4})\pi$ and 34:7:5 replaced by

34:7:9 $$\rho_n(b) = \mathrm{Arccot}\left\{b\rho_n(b)\right\} = (n-1)\pi + \mathrm{arccot}\left\{b\rho_n(b)\right\}$$

34:8 NUMERICAL VALUES

Equator's tangent function and cotangent function routines (keywords **tan** and **cot**) provide values of $\tan(x)$ and $\cot(x)$ by using the first equality in each of expressions 34:3:1 and 34:3:2.

34:9 LIMITS AND APPROXIMATIONS

When its argument is close to one of the values, x_∞, at which $\tan(x)$ has a discontinuity, then the limiting approximation

34:9:1 $$\tan(x) \to \frac{1}{x_\infty - x} - \frac{x_\infty - x}{3} \qquad x \to x_\infty = \frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \frac{\pm 5\pi}{2}, \cdots$$

holds. Similarly, for the cotangent

34:9:2 $$\cot(x) \to \frac{1}{x - x_\infty} - \frac{x - x_\infty}{3} \qquad x \to x_\infty = 0, \pm\pi, \pm 2\pi, \cdots$$

34:10 OPERATIONS OF THE CALCULUS

Formulas for differentiation and indefinite integration are

34:10:1 $$\frac{d}{dx}\tan(x) = \sec^2(x) = 1 + \tan^2(x)$$

34:10:2 $$\frac{d}{dx}\cot(x) = -\csc^2(x) = -1 - \cot^2(x)$$

34:10:3 $$\int_0^x \tan(t)\,dt = \ln\{\sec(x)\} = \tfrac{1}{2}\ln\{1 + \tan^2(x)\}$$

34:10:4 $$\int_x^{\pi/2} \cot(t)\,dt = \ln\{\csc(x)\} = \tfrac{1}{2}\ln\{1 + \cot^2(x)\}$$

Multiple differentiations of the cotangent function are discussed in Section 44:12.

Integer powers of the tangent and cotangent functions integrate to give:

34:10:5 $$\int_0^x \tan^n(t)\,dt = \frac{\tan^{n-1}(x)}{n-1} - \frac{\tan^{n-3}(x)}{n-3} + \frac{\tan^{n-5}(x)}{n-5} \cdots \begin{cases} -\tfrac{1}{2}\tan^2(x) + \ln\{\sec(x)\} & n = 5, 9, 13, \cdots \\ +\tan(x) - x & n = 2, 6, 10, \cdots \\ +\tfrac{1}{2}\tan^2(x) - \ln\{\sec(x)\} & n = 3, 7, 11, \cdots \\ -\tan(x) + x & n = 4, 8, 12, \cdots \end{cases}$$

34:10:6 $$\int_x^{\pi/2} \cot^n(t)\,dt = \frac{\cot^{n-1}(x)}{n-1} - \frac{\cot^{n-3}(x)}{n-3} + \frac{\cot^{n-5}(x)}{n-5} \cdots \begin{cases} -\tfrac{1}{2}\cot^2(x) + \ln\{\csc(x)\} & n = 5, 9, 13, \cdots \\ +\cot(x) + x & n = 2, 6, 10, \cdots \\ +\tfrac{1}{2}\cot^2(x) - \ln\{\csc(x)\} & n = 3, 7, 11, \cdots \\ -\cot(x) - x & n = 4, 8, 12, \cdots \end{cases}$$

Section 58:14 discusses the integration of the tangent or cotangent function raised to an arbitrary power.

Important definite integrals include

34:10:7 $$\int_0^{\pi/2} \tan^\nu(t)\,dt = \int_0^{\pi/2} \cot^\nu(t)\,dt = \frac{\pi}{2}\sec\left(\frac{\nu\pi}{2}\right) \qquad -1 < \nu < 1$$

34:10:8 $$\int_0^{\pi/4} \tan^\nu(t)\,dt = \int_{\pi/4}^{\pi/2} \cot^\nu(t)\,dt = \frac{1}{4}\mathrm{G}\left(\frac{1+\nu}{2}\right) \qquad \nu > -1$$

and

34:10:9 $$-\int_0^{\pi/4} \ln\{\tan(t)\}\,\mathrm{d}t = \int_{\pi/4}^{\pi/2} \ln\{\cot(t)\}\,\mathrm{d}t = G$$

leading to the secant function [Chapter 33], the Bateman's G function from Section 44:13, and Catalan's constant G [Section 1:7].

34:11 COMPLEX ARGUMENT

The tangent of the complex variable $z = x + iy$ takes complex values given by the formula

34:11:1 $$\tan(x+iy) = \frac{\sin(2x) + i\sinh(2y)}{\cos(2x) + \cosh(2y)}$$

The denominator of this expression makes it evident that tan(z) is finite unless $y = 0$ and x equals one of the values $(2n+1)\pi/2$, where n is an integer. This is confirmed by Figure 34-3, the two diagrams of which depict the real and imaginary parts of tan(z). The corresponding diagrams for the cotangent of a complex variable are not shown; the poles in that case occur at $y = 0$, $x = n\pi$, in conformity with the formula

34:11:2 $$\cot(x+iy) = \frac{\sin(2x) - i\sinh(2y)}{-\cos(2x) + \cosh(2y)}$$

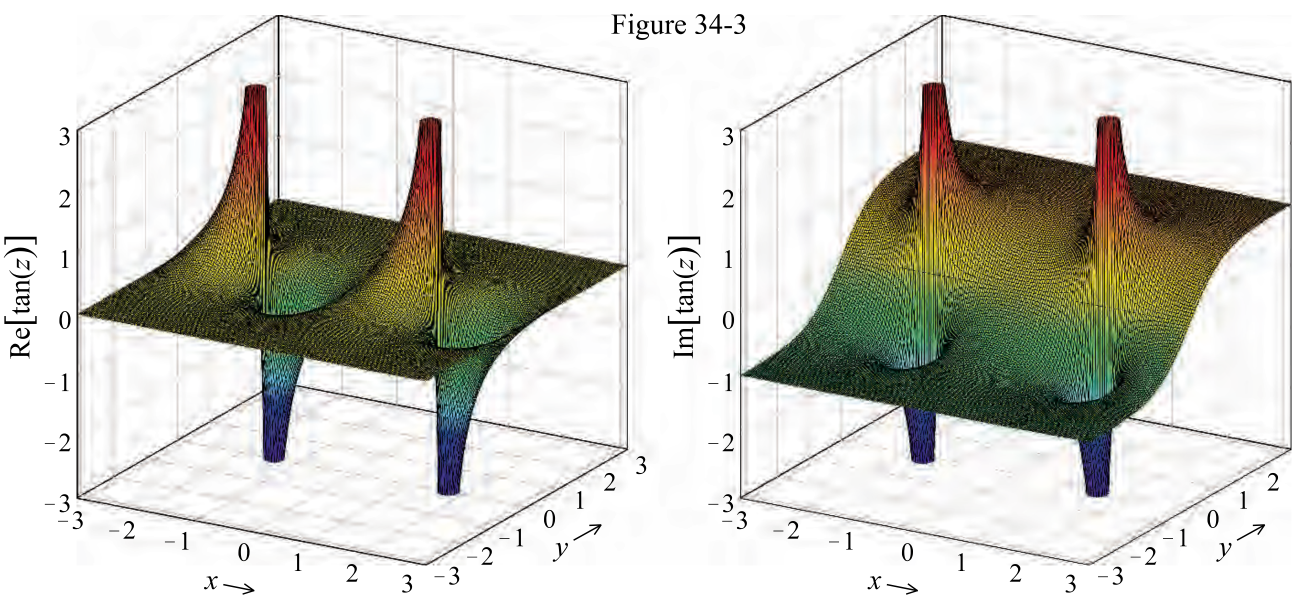
When the argument is purely imaginary, equations 34:11:1 and 34:11:2 reduce respectively to

34:11:3 $$\tan(iy) = i\tanh(y)$$

and

34:11:4 $$\cot(iy) = -i\coth(y)$$

Figure 34-3



34:12 GENERALIZATIONS

As periodic functions, the tangent and cotangent are special cases of the functions of Chapter 36.

Respectively, tan(x) and cot(x) generalize to the Jacobian elliptic functions sc(k,x) and cs(k,x) that are discussed in Chapter 63. When $k = 0$, one has

34:12:1 $$\mathrm{sc}(0,x) = \tan(x) \qquad \text{and} \qquad \mathrm{cs}(0,x) = \cot(x)$$

34:13 COGNATE FUNCTIONS

The tangent and cotangent functions are related to the other circular functions through the equivalences

34:13:1 $$\tan(x) = \frac{\sigma_{12}\sqrt{1-\cos^2(x)}}{\cos(x)} = \frac{\sigma_{14}\sin(x)}{\sqrt{1-\sin^2(x)}} = \sigma_{13}\sqrt{\sec^2(x)-1} = \frac{\sigma_{13}}{\sqrt{\csc^2(x)-1}} = \frac{1}{\cot(x)}$$

and

34:13:2 $$\cot(x) = \frac{\sigma_{12}\cos(x)}{\sqrt{1-\cos^2(x)}} = \frac{\sigma_{14}\sqrt{1-\sin^2(x)}}{\sin(x)} = \frac{\sigma_{13}}{\sqrt{\sec^2(x)-1}} = \sigma_{13}\sqrt{\csc^2(x)-1} = \frac{1}{\tan(x)}$$

where σ_{1n} equals +1 in the first and nth quadrants, but is −1 otherwise [also see 32:13:3].

The inverse tangent and cotangent functions are among those addressed in the next chapter.

34:14 RELATED TOPIC: the tangent of half argument

In general, algebraic functions are more easily manipulated than are the circular functions of Chapters 32-34. Thus methods of converting the circular functions temporarily into algebraic functions can be beneficial. The tangent of half argument, that we here abbreviate to τ,

34:14:1 $$\tan\left(\frac{x}{2}\right) = \tau \qquad -\pi < x < \pi$$

is convenient in this regard because all the circular functions are expressible easily in terms of τ, as are some commonly encountered combinations of circular functions. Some of the equivalences are:

cos(x)	sin(x)	sec(x)	csc(x)	tan(x)	cot(x)	sec(x) + tan(x)	sec(x) − tan(x)	csc(x) + cot(x)	csc(x) − cot(x)
$\frac{1-\tau^2}{1+\tau^2}$	$\frac{2\tau}{1+\tau^2}$	$\frac{1+\tau^2}{1-\tau^2}$	$\frac{1+\tau^2}{2\tau}$	$\frac{2\tau}{1-\tau^2}$	$\frac{1-\tau^2}{2\tau}$	$\frac{1+\tau}{1-\tau}$	$\frac{1-\tau}{1+\tau}$	$\frac{1}{\tau}$	τ

As an elementary illustration of the utility of this approach, consider a need to solve the equation

34:14:2 $$2\cos(x) + \sin(x) = 1$$

Substitution from the compendium above leads easily to the quadratic equation $3\tau^2 - 2\tau - 1 = 0$, whence $\tau = (1\pm2)/3$. Accordingly, there are two real solutions:

34:14:3 $$x = 2\arctan(1) = \frac{\pi}{2} \qquad \text{or} \qquad x = 2\arctan\left(-\tfrac{1}{3}\right) = -0.64350$$

The tangent of half-argument is also a useful aid to the evaluation of indefinite integrals involving circular functions. By using the expressions collected above, together with the differential identity

34:14:4 $$\mathrm{d}t=\frac{2\,\mathrm{d}\tau}{1+\tau^2} \qquad \text{where} \qquad \tau=\tan\left(\frac{t}{2}\right)$$

one can often evaluate integrals that are tricky to decipher otherwise. As a simple example

34:14:5 $$\int_0^x \frac{1}{[1+\cos(t)]^2}\mathrm{d}t=\frac{1}{2}\int_0^{\tan(x/2)}\left(1+\tau^2\right)\mathrm{d}\tau=\frac{1}{2}\left[\tau+\frac{\tau^3}{3}\right]_0^{\tan(x/2)}=\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{6}\tan^3\left(\frac{x}{2}\right)$$

34:15 RELATED TOPIC: triangles

Trigonometric (otherwise called "circular") functions [Chapters 32−34] and their inverses [Chapter 35] play an indispensable role in the mensuration of triangles. Historically, it was the need to determine the sides and angles of triangles that led to the invention of these functions.

Figure 34-4 shows a triangle in which angle C is a right angle, $C=\pi/2$. Note that side a lies opposite to angle A and similarly for the other sides and angles. The side c opposite the right angle is known as the *hypotenuse*. Angle C having been defined, there remain five undefined parameters: the lengths a, b, and c of the sides, and the angles A and B. If any two of these five parameters are specified, then the other three are calculable via the following table, provided that at least one of the specified parameters is the length of a side. Checkmarks ✓ indicate the given parameters in the table, which also has a column listing the triangle's area in terms of the givens.

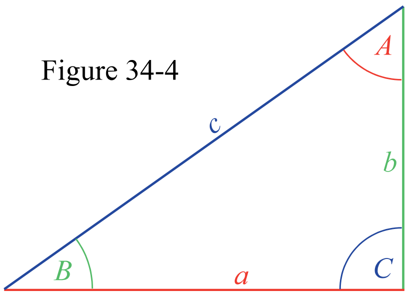
a	b	c	A	B	area
✓	✓	$\sqrt{a^2+b^2}$	$\arctan(a/b)$	$\arctan(b/a)$	$ab/2$
✓	$\sqrt{c^2-a^2}$	✓	$\arcsin(a/c)$	$\arccos(a/c)$	$\frac{1}{2}a\sqrt{c^2+a^2}$
✓	$a\cot(A)$	$a\csc(A)$	✓	$\frac{\pi}{2}-A$	$\frac{1}{2}a^2\cot(A)$
✓	$a\tan(B)$	$a\sec(B)$	$\frac{\pi}{2}-B$	✓	$\frac{1}{2}a^2\tan(B)$
$\sqrt{c^2-b^2}$	✓	✓	$\arccos(b/c)$	$\arcsin(b/c)$	$\frac{1}{2}b\sqrt{c^2-b^2}$
$b\tan(A)$	✓	$b\sec(A)$	✓	$\frac{\pi}{2}-A$	$\frac{1}{2}b^2\tan(A)$
$b\cot(B)$	✓	$b\csc(B)$	$\frac{\pi}{2}-B$	✓	$\frac{1}{2}b^2\cot(B)$
$c\sin(A)$	$c\cos(A)$	✓	✓	$\frac{\pi}{2}-A$	$\frac{1}{4}c^2\sin(2A)$
$c\cos(B)$	$c\sin(B)$	✓	$\frac{\pi}{2}-B$	✓	$\frac{1}{4}c^2\sin(2B)$

The most important, and well-known, relationship applicable to right-angled triangles is the *theorem of Pythagoras*

34:15:1 $$a^2 + b^2 = c^2$$

Sets of integers (a,b;c) that satisfy this relationship are known as *Pythagorean trios*; a few examples are (3,4;5), (5,12;13), (7,24;25), (8,15;17), (9,40;41), (13,84;85), (20,99;101), and (119,120;169).

Figure 34-4



Known parameters	Formulas for unknown parameters
The three sides, a, b, and c. No side may have a length that exceeds the sum of the lengths of the other two sides.	$A = 2\arctan\left(\sqrt{\left[a^2-(b-c)^2\right]/\left[(b+c)^2-a^2\right]}\right)$ $B = 2\arctan\left(\sqrt{\left[b^2-(c-a)^2\right]/\left[(c+a)^2-b^2\right]}\right)$ $C = 2\arctan\left(\sqrt{\left[c^2-(a-b)^2\right]/\left[(a+b)^2-c^2\right]}\right)$
Two sides and the angle opposite the longer of those two sides. For example: a, b, and A, where $a \ge b$.	$c = \sqrt{a^2 + b^2\cos(2A) - 2b\cos(A)\sqrt{a^2 - b^2\sin^2(A)}}$ $B = \arcsin\{(b/a)\sin(A)\}$ $C = \arccos\left\{(b/a)\sin^2(A) + \cos(A)\sqrt{1-(b/a)^2\sin^2(A)}\right\}$
Two sides and the angle opposite the shorter of those two sides. For example: a, b, and B, where $a \ge b$.	$c = \sqrt{a^2\cos(2B) + b^2 \pm 2a\cos(B)\sqrt{b^2 - a^2\sin^2(B)}}$ $A = (\pi/2) \pm \arccos\{(a/b)\sin(B)\}$ $C = \arccos\left\{(a/b)\sin^2(A) \mp \cos(B)\sqrt{1-(a/b)^2\sin^2(B)}\right\}$
Two sides and the angle between them. For example: a, b, and C.	$c = \sqrt{a^2 + b^2 - 2ab\cos(C)}$ $A = \arcsin\left\{a\sin(C)/\sqrt{a^2+b^2-2ab\cos(C)}\right\}$ $B = \arcsin\left\{b\sin(C)/\sqrt{a^2+b^2-2ab\cos(C)}\right\}$
Two angles and the side opposite one of them. For example: a, A, and B.	$b = a\csc(A)\sin(B)$ $c = a\left[\cot(A)\sin(B) + \cos(B)\right]$ $C = \pi - A - B$
Two angles and the side between them. For example: a, B, and C.	$b = a\csc(C)/\left[\cot(B) + \cot(C)\right]$ $c = a\csc(B)/\left[\cot(B) + \cot(C)\right]$ $A = \pi - B - C$

For a triangle that is not necessarily right angled, a number of general laws interrelate the six parameters a, b, c, A, B, and C. These laws include the *law of cosines*,

34:15:8 $$2bc\cos(A) = b^2 + c^2 - a^2$$

the *law of sines*

34:15:9 $$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

and the *law of tangents*

34:15:10 $$(a+b)\tan\left(\frac{A-B}{2}\right) = (a-b)\tan\left(\frac{A+B}{2}\right)$$

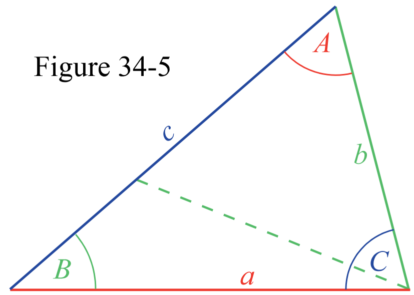


Figure 34-5

Of the six parameters, three (including at least one side) must usually be specified for the triangle to be fully determined. The table on the previous page presents all possible scenarios. For example, if the side a is of known length and the angles B and C are also given, then the other sides have lengths b and c, as well as angle A given by the appropriate entries in the table. Notice in this table that prescribing values of a, b and B does not fully delineate the triangle if a exceeds b. The line of length b has two alternative positions, in that case, these being illustrated in Figure 34-5 by the full green line and the dashed green line. In consequence, the values of c, A and C each have the two alternatives tabulated.

One of the following formulas for the area of an arbitrary triangle

34:15:11 $$area = \frac{bc}{2}\sin(A) = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{a+b+c}{2}$$

involves the *semiperimeter* of the triangle.

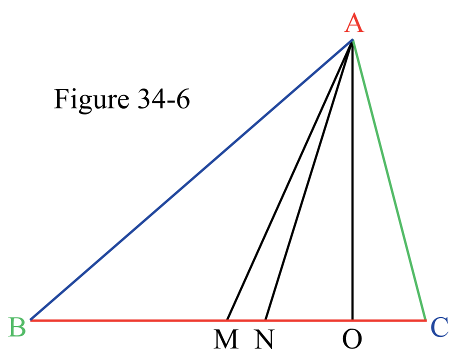


Figure 34-6

Three constructions that are frequently made to a triangle are illustrated in Figure 34-6. The *altitude* AO, that is, the line through A perpendicular to the side BC, has a length

34:15:12 $$\text{AO} = b\sin(C) = c\sin(B) \qquad \text{angle BOA} = \frac{\pi}{2}$$

A line through a vertex of the triangle that bisects the angle there is known as a *bisector*; the bisector of the angle BAC in Figure 34-6 has a length

34:15:13 $$\text{AN} = \frac{2bc\cos(A/2)}{b+c} \qquad (\text{angle BAN}) = (\text{angle NAC}) = \frac{(\text{angle BAC})}{2}$$

Line AM, which bisects the line BC, is termed a *median* of the triangle; its length is

34:15:14 $$\text{AM} = \tfrac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos(A)} \qquad \text{BM} = \text{MC} = \frac{a}{2}$$

and it cuts the triangle into two smaller triangles of equal areas.

CHAPTER 35

THE INVERSE CIRCULAR FUNCTIONS

This chapter's six functions – the *inverse cosine function* arccos(x), the *inverse sine function* arcsin(x), the *inverse secant function* arcsec(x), the *inverse cosecant function* arccsc(x), the *inverse tangent function* arctan(x) and the *inverse cotangent function* arccot(x) – are known collectively as the *inverse circular functions* or the *inverse trigonometric functions*. Only two of these, the arctan(x) and arccot(x) functions, take real values throughout the entire $-\infty < x < \infty$ domain of real arguments.

Four of the six functions – the inverse secant, the inverse cosecant, the inverse tangent and the inverse cotangent – adopt real values within at least the $x \geq 1$ domain, where they are linked by the following relationships:

35:0:1 $$\mathrm{arcsec}(x) = \frac{\pi}{2} - \mathrm{arccsc}(x) = \arctan\left(\sqrt{x^2-1}\right) = \mathrm{arccot}\left(\frac{1}{\sqrt{x^2-1}}\right) \qquad x \geq 1$$

35:0:2 $$\mathrm{arccsc}(x) = \frac{\pi}{2} - \mathrm{arcsec}(x) = \arctan\left(\frac{1}{\sqrt{x^2-1}}\right) = \mathrm{arccot}\left(\sqrt{x^2-1}\right) \qquad x \geq 1$$

35:0:3 $$\arctan(x) = \mathrm{arcsec}\left(\sqrt{x^2+1}\right) = \mathrm{arccsc}\left(\frac{\sqrt{x^2+1}}{x}\right) = \mathrm{arccot}\left(\frac{1}{x}\right) \qquad x \geq 1$$

35:0:4 $$\mathrm{arccot}(x) = \mathrm{arcsec}\left(\frac{\sqrt{x^2+1}}{x}\right) = \mathrm{arccsc}\left(\sqrt{x^2+1}\right) = \arctan\left(\frac{1}{x}\right) \qquad x \geq 1$$

The same four functions also coexist for arguments less than -1, but the interrelations there often differ from those given above, being:

35:0:5 $$\mathrm{arcsec}(x) = \frac{\pi}{2} - \mathrm{arccsc}(x) = \pi - \arctan\left(\sqrt{x^2-1}\right) = \mathrm{arccot}\left(\frac{-1}{\sqrt{x^2-1}}\right) \qquad x \leq -1$$

35:0:6 $$\mathrm{arccsc}(x) = \frac{\pi}{2} - \mathrm{arcsec}(x) = \arctan\left(\frac{-1}{\sqrt{x^2-1}}\right) = -\mathrm{arccot}\left(\sqrt{x^2-1}\right) \qquad x \leq -1$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_36, © Springer Science+Business Media, LLC 2009

35:0:7 $$\arctan(x) = -\mathrm{arcsec}\left(\sqrt{x^2+1}\right) = \mathrm{arccsc}\left(\frac{\sqrt{x^2+1}}{x}\right) = -\frac{\pi}{2} - \mathrm{arccot}(x) \qquad x \le -1$$

35:0:8 $$\mathrm{arccot}(x) = -\pi + \mathrm{arcsec}\left(\frac{\sqrt{x^2+1}}{x}\right) = -\mathrm{arccsc}\left(\sqrt{x^2+1}\right) = -\frac{\pi}{2} - \arctan(x) \qquad x \le -1$$

The next eight expressions address the $0 \le x \le 1$ and $-1 \le x < 0$ domains within which it is the inverse cosine, the inverse sine, the inverse tangent and the inverse cotangent functions that coexist. In those ranges, the four functions are interrelated by

35:0:9 $$\arccos(x) = \frac{\pi}{2} - \arcsin(x) = \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) = \mathrm{arccot}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad 0 \le x \le 1$$

35:0:10 $$\arcsin(x) = \frac{\pi}{2} - \arccos(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \mathrm{arccot}\left(\frac{\sqrt{1-x^2}}{x}\right) \qquad 0 \le x \le 1$$

35:0:11 $$\arctan(x) = \arccos\left(\frac{1}{\sqrt{1+x^2}}\right) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\pi}{2} - \mathrm{arccot}(x) \qquad 0 \le x \le 1$$

35:0:12 $$\mathrm{arccot}(x) = \arccos\left(\frac{x}{\sqrt{1+x^2}}\right) = \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{\pi}{2} - \arctan(x) \qquad 0 \le x \le 1$$

35:0:13 $$\arccos(x) = \frac{\pi}{2} - \arcsin(x) = \pi + \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) = \mathrm{arccot}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad -1 \le x < 0$$

35:0:14 $$\arcsin(x) = \frac{\pi}{2} - \arccos(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \mathrm{arccot}\left(\frac{\sqrt{1-x^2}}{x}\right) - \pi \qquad -1 \le x < 0$$

35:0:15 $$\arctan(x) = -\arccos\left(\frac{1}{\sqrt{1+x^2}}\right) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\pi}{2} - \mathrm{arccot}(x) \qquad -1 \le x < 0$$

35:0:16 $$\mathrm{arccot}(x) = \arccos\left(\frac{x}{\sqrt{1+x^2}}\right) = \pi - \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{\pi}{2} - \arctan(x) \qquad -1 \le x < 0$$

Despite their abundance, by no means are the relationships above the only ones linking the six functions.

The three equations

35:0:17 $$\left.\begin{array}{l} \arcsin(x) + \arccos(x) \\ \mathrm{arcsec}(x) + \mathrm{arccsc}(x) \\ \arctan(x) + \mathrm{arccot}(x) \end{array}\right\} = \frac{\pi}{2}$$

hold wherever the function pair in question is defined. With x replaced by the complex variable z, they hold globally.

35:1 NOTATION

The symbolism $\cos^{-1}(x)$, $\tan^{-1}(x)$, etc. frequently replaces arccos(x), arctan(x), etc. Variants such as arccosec(x), arctg(x), argsin(x), and arcctg(x) are encountered occasionally. The origin of the prefix "arc" is evident from Figure 32-2 and equation 32:3:5; because, for example, the length OQ is defined as the cosine of *arc*, it is natural to regard the *arc* itself as the arc of the cosine, or arccos. More usually, however, the arccosine is thought of as being associated with the angle (POQ in the figure), rather than the arc length.

The notation arccot(x) is sometimes used to denote a function defined, for negative x, somewhat differently than here, being equal to our arccot(x) − π. You may encounter similar discrepancies for other inverse circular functions.

The multivalued functions Arctan(x), Arcsin(x), etc., are discussed in Section 35:12 but, confusingly, these functions are often denoted by the uncapitalized arctan(x), arcsin(x), etc.

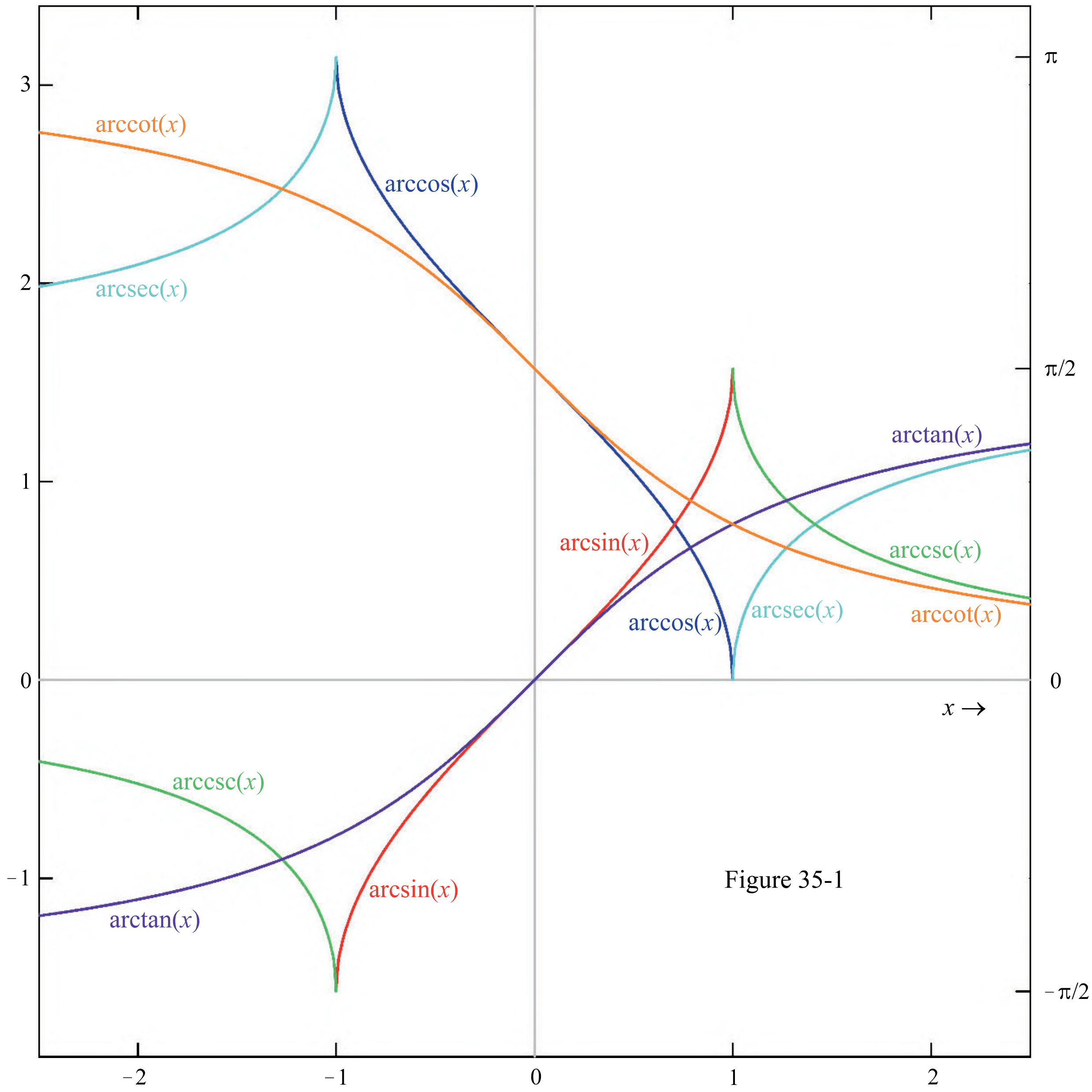


Figure 35-1

35:2 BEHAVIOR

Figure 35-1 shows the varied behaviors of the six functions. Notice that the inverse secant and inverse cosecant functions each have two branches. The following table details the real arguments that the six functions will accept and the values they themselves adopt, but appreciate that not all authorities agree on the definitions for negative arguments.

f(x)	Domain of x	Range of f
arctan(x)	$-\infty \le x \le \infty$	$-\frac{1}{2}\pi \le \arctan(x) \le \frac{1}{2}\pi$
arcsin(x)	$-1 \le x \le 1$	$-\frac{1}{2}\pi \le \arcsin(x) \le \frac{1}{2}\pi$
arccsc(x)	$-\infty \le x \le -1$ and $1 \le x \le \infty$	$-\frac{1}{2}\pi \le \mathrm{arccsc}(x) \le -1$ and $1 \le \mathrm{arccsc}(x) \le \frac{1}{2}\pi$
arccot(x)	$-\infty \le x \le \infty$	$0 \le \mathrm{arccot}(x) \le \pi$
arccos(x)	$-1 \le x \le 1$	$0 \le \arccos(x) \le \pi$
arcsec(x)	$-\infty \le x \le -1$ and $1 \le x \le \infty$	$\frac{1}{2}\pi \le \mathrm{arcsec}(x) \le \pi$ and $0 \le \mathrm{arcsec}(x) \le \frac{1}{2}\pi$

35:3 DEFINITIONS

As their names imply, the inverse circular functions are the inverses of the circular functions of Chapters 32−34. However, because the circular functions are periodic, it is necessary to restrict the range of their arguments so that, on inversion, single-valued functions are created. The restrictions customarily selected for this purpose are

35:3:1 $$f = \arctan(x) \quad \text{where} \quad x = \tan(f)$$

35:3:2 $$f = \arcsin(x) \quad \text{where} \quad x = \sin(f)$$

35:3:3 $$f = \mathrm{arccsc}(x) \quad \text{where} \quad x = \csc(f)$$

$$\left.\right\} \quad \frac{-\pi}{2} \le f \le \frac{\pi}{2}$$

35:3:4 $$f = \mathrm{arc}\cot(x) \quad \text{where} \quad x = \cot(f)$$

35:3:5 $$f = \arccos(x) \quad \text{where} \quad x = \cos(f)$$

35:3:6 $$f = \mathrm{arcsec}(x) \quad \text{where} \quad x = \sec(f)$$

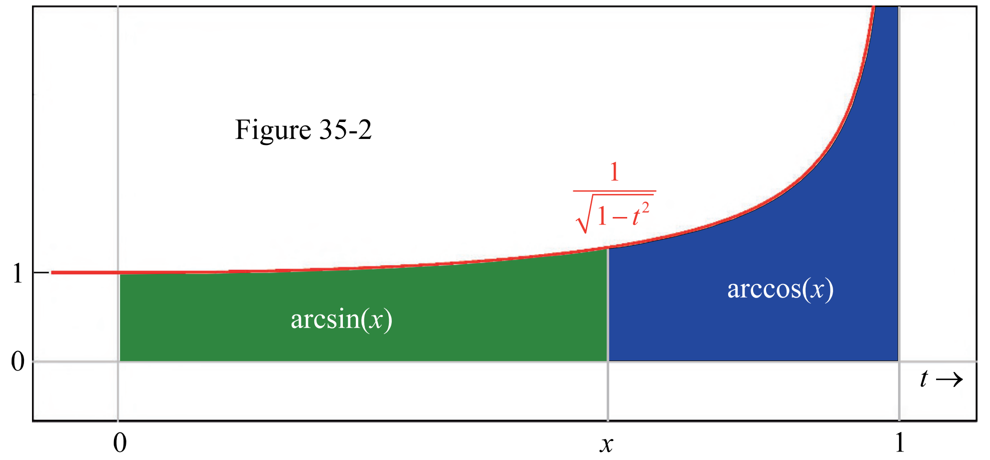
$$\left.\right\} \quad 0 \le f \le \pi$$

There are three definite integrals of algebraic functions, all of which evaluate to π/2:

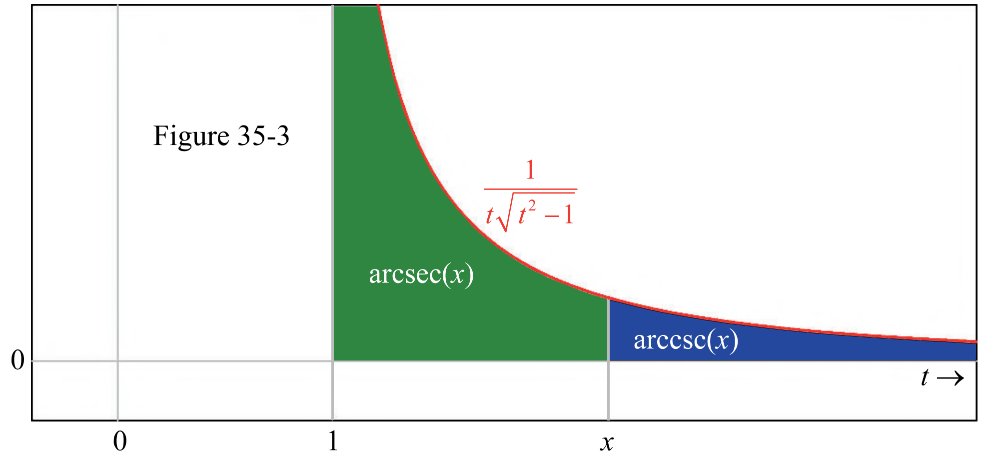
35:3:7 $$\int_0^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \int_1^\infty \frac{1}{t\sqrt{t^2-1}}\mathrm{d}t = \int_0^\infty \frac{1}{1+t^2}\mathrm{d}t = \frac{\pi}{2}$$

"Incomplete" versions of these integrals can serve as definitions of the six inverse circular functions, as illustrated in Figures 35-2 through 35-4:

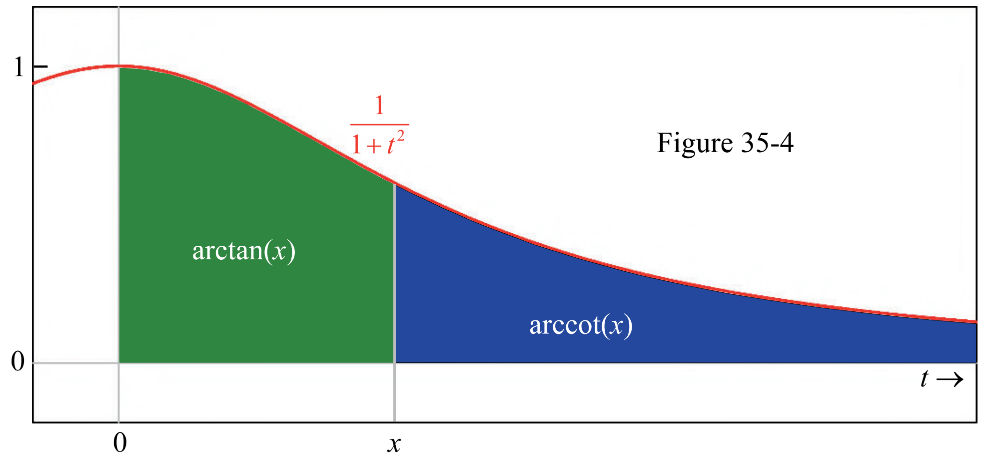
35:3:8 $$\int_0^x \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \arcsin(x) \qquad \text{and} \qquad \int_x^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \arccos(x) \qquad 0 < x < 1$$



35:3:9 $$\int_1^x \frac{1}{t\sqrt{t^2-1}}\,dt = \text{arcsec}(x) \qquad \text{and} \qquad \int_x^\infty \frac{1}{t\sqrt{t^2-1}}\,dt = \text{arccsc}(x) \qquad x>1$$



35:3:10 $$\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x) \qquad \text{and} \qquad \int_x^\infty \frac{1}{1+t^2}\,dt = \text{arccot}(x) \qquad x>0$$



Similar definitions serve to define the functions for negative x. Of course, formulas 35:0:17 are direct consequences of the complementarity of these definitions.

Figure 32-2 may be adapted easily to provide geometric definitions of the inverse circular functions.

As evident from equations 35:6:1 and 35:6:2, the inverse tangent and inverse sine are hypergeometric functions and, as such, they may be synthesized [Section 43:14] as follows

35:3:11 $$\frac{1}{1+x} \xrightarrow[3/2]{1/2} \frac{\arctan\left(\sqrt{x}\right)}{\sqrt{x}}$$

35:3:12 $$\frac{1}{1-x} \xrightarrow[1]{1/2} \frac{1}{\sqrt{1-x}} \xrightarrow[3/2]{1/2} \frac{\arcsin\left(\sqrt{x}\right)}{\sqrt{x}}$$

35:4 SPECIAL CASES

There are none.

35:5 INTRARELATIONSHIPS

The inverse tangent, sine and cosecant are odd functions:

35:5:1 $$\mathrm{f}(-x) = -\mathrm{f}(x) \qquad \mathrm{f} = \text{arctan, arcsin, or arccsc}$$

whereas the other three inverse circular functions obey the reflection formula

35:5:2 $$\mathrm{f}(-x) = \pi - \mathrm{f}(x) \qquad \mathrm{f} = \text{arccot, arccos, or arcsec}$$

Argument-reciprocation formulas for each inverse circular function are:

35:5:3 $$\arccos\left(\frac{1}{x}\right) = \mathrm{arcsec}(x) \qquad |x| \geq 1$$

35:5:4 $$\arcsin\left(\frac{1}{x}\right) = \mathrm{arccsc}(x) \qquad |x| \geq 1$$

35:5:5 $$\mathrm{arcsec}\left(\frac{1}{x}\right) = \arccos(x) \qquad |x| \leq 1$$

35:5:6 $$\mathrm{arccsc}\left(\frac{1}{x}\right) = \arcsin(x) \qquad |x| \leq 1$$

35:5:7 $$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}\mathrm{sgn}(x) - \arctan(x) = \mathrm{arccot}(x) - \frac{\pi}{2}\left[1 - \mathrm{sgn}(x)\right] \qquad x \neq 0$$

35:5:8 $$\mathrm{arccot}\left(\frac{1}{x}\right) = \frac{\pi}{2}\mathrm{sgn}(x) - \mathrm{arccot}(x) = \arctan(x) - \frac{\pi}{2}\left[1 - \mathrm{sgn}(x)\right] \qquad x \neq 0$$

If f is an inverse circular function, then the value of $\mathrm{f}(x) \pm \mathrm{f}(x)$ will not necessarily lie within the range of values of f itself. This range diversity is catered to in the following formulas, in which k takes one of the values -1, 0, or +1

$$\left.\begin{array}{ll} \text{35:5:9} & \arctan(x)\pm\arctan(y)=k\pi+\arctan\left(\dfrac{x\pm y}{1\mp xy}\right) \\[2ex] \text{35:5:10} & \arcsin(x)\pm\arcsin(y)=k\pi+\arcsin\left(x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right) \\[2ex] \text{35:5:11} & \operatorname{arccsc}(x)\pm\operatorname{arc\,csc}(y)=k\pi+\operatorname{arccsc}\left(\dfrac{xy}{\sqrt{y^2-1}\pm\sqrt{x^2+1}}\right) \end{array}\right\} \quad k=\operatorname{Int}\left(\frac{1}{2}+\frac{\mathrm{f}(x)\pm\mathrm{f}(y)}{\pi}\right)$$

$$\left.\begin{array}{ll} \text{35:5:12} & \operatorname{arccot}(x)\pm\operatorname{arccot}(y)=k\pi+\operatorname{arccot}\left(\dfrac{x\pm y}{1\mp xy}\right) \\[2ex] \text{35:5:13} & \arccos(x)\pm\arccos(y)=k\pi+\arccos\left(xy\mp\sqrt{1-y^2}\sqrt{1-x^2}\right) \\[2ex] \text{35:5:14} & \operatorname{arcsec}(x)\pm\operatorname{arcsec}(y)=k\pi+\operatorname{arccsc}\left(\dfrac{xy}{1\mp\sqrt{y^2-1}\sqrt{x^2+1}}\right) \end{array}\right\} \quad k=\frac{1}{2}-\operatorname{Int}\left(\frac{\mathrm{f}(x)\pm\mathrm{f}(y)}{\pi}\right)$$

A plethora of other formulas may be constructed by combining the formulas of this section with those of Section 35:0. Moreover, in this exercise, formulas from Chapters 32−35 may be invoked. For example, because $\sin(\theta/2) = \sqrt{\{1-\cos(\theta)\}/2}$, then

35:5:15
$$\arccos(x)=2\arcsin\left(\sqrt{\frac{1-x}{2}}\right) \qquad -1\le x\le 1$$

35:6 EXPANSIONS

The two fundamental series expansions of the inverse circular functions are

35:6:1
$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=x\sum_{j=0}^{\infty}\frac{\left(-x^2\right)^j}{2j+1}=x\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_j}{\left(\tfrac{3}{2}\right)_j}\left(-x^2\right)^j \qquad |x|\le 1$$

35:6:2
$$\arcsin(x)=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\cdots=\sum_{j=0}^{\infty}\frac{(2j-1)!!}{(2j)!!}\frac{x^{2j+1}}{2j+1}=x\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_j\left(\tfrac{1}{2}\right)_j}{(1)_j\left(\tfrac{3}{2}\right)_j}\left(x^2\right)^j \qquad |x|\le 1$$

It is the hypergeometric nature of these series that validates the syntheses portrayed in 35:3:11 and 35:3:12. These two basic formulas are easily adapted to provide expansions of the other inverse circular functions through their multifarious interrelationships. For example, in light of equation 35:5:15, replacement of x in equation 35:6:2 by $\sqrt{x/2}$ leads to the expansion

35:6:3
$$\arccos(1-x)=\sqrt{2x}\left[1+\frac{x}{12}+\frac{3x^2}{160}+\frac{5x^3}{896}+\cdots\right]=\sqrt{2x}\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_j\left(\tfrac{1}{2}\right)_j}{(1)_j\left(\tfrac{3}{2}\right)_j}\left(\frac{x}{2}\right)^j \qquad 0\le x\le 2$$

As a second example, whereas expansion 35:6:1 of the inverse tangent requires that the argument be less than unity, an alternative expansion, exploiting the first equality in 35:5:7, leads to

35:6:4 $$\frac{\pi}{2}-\arctan(x)=\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\cdots=\sum_{j=0}^{\infty}\frac{(-)^j x^{-2j-1}}{2j+1}=\frac{1}{x}\sum_{j=0}^{\infty}\frac{(\tfrac{1}{2})_j}{(\tfrac{3}{2})_j}\left(\frac{-1}{x^2}\right)^j \qquad x\geq 1$$

an expansion valid in the domain complementary to that covered by 35:6:1.

Continued fraction expansions of the inverse circular functions include

35:6:5 $$\arctan(x)=\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+}\,\frac{9x^2}{7+}\,\frac{16x^2}{9+\cdots}$$

and

35:6:6 $$\frac{\arcsin(x)}{\sqrt{1-x^2}}=\frac{x}{1-}\,\frac{2x}{3-}\,\frac{2x}{5-}\,\frac{12x}{7-}\,\frac{12x}{9-}\,\frac{30x}{11-}\,\frac{30x}{13-}\,\frac{56x}{15-\cdots}$$

35:7 PARTICULAR VALUES

The entry "undef" in the table below means that this *Atlas* regards the function to be undefined, as a real quantity, at the argument in question. A blank space indicates that the function has a value that is not noteworthy.

	$x=-\infty$	$x=-\sqrt{2}$	$x=-1$	$x=\frac{-1}{\sqrt{2}}$	$x=0$	$x=\frac{1}{\sqrt{2}}$	$x=1$	$x=\sqrt{2}$	$x=\infty$
arccos(x)	undef	undef	π	$\frac{3\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0	undef	undef
arcsin(x)	undef	undef	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	undef	undef
arcsec(x)	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	undef	undef	undef	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
arccsc(x)	0	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$	undef	undef	undef	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0
arctan(x)	$-\frac{\pi}{2}$		$-\frac{\pi}{4}$		0		$\frac{\pi}{4}$		$\frac{\pi}{2}$
arccot(x)	π		$\frac{3\pi}{4}$		$\frac{\pi}{2}$		$\frac{\pi}{4}$		0

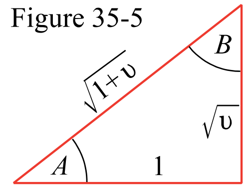
In Section 31:7 it is demonstrated that arguments of $\pm\sqrt{\upsilon}$ or $\pm 1/\sqrt{\upsilon}$, where υ is the "golden section" [Section 23:14], lead to coalescence of certain inverse hyperbolic functions. The same phenomenon occurs with the inverse circular functions. One has, for example:

35:7:1 $$\arctan\left(\sqrt{\upsilon}\right)=\arccos\left(\sqrt{\upsilon}\right)=\operatorname{arccot}\left(1/\sqrt{\upsilon}\right)=\operatorname{arcsec}\left(1/\sqrt{\upsilon}\right)=A=0.66623\;94324\;92515$$

35:7:2 $$\operatorname{arccot}\left(\sqrt{\upsilon}\right)=\arcsin\left(\sqrt{\upsilon}\right)=\arctan\left(1/\sqrt{\upsilon}\right)=\operatorname{arccsc}\left(1/\sqrt{\upsilon}\right)=B=0.90455\;68943\;02381$$

A and *B*, which sum to π/2, are the angles in the right-angled triangle shown in Figure 35-5. They are, in fact, the Gudermannians [Section 33:14] of the quantities α and β cited in equations 31:7:1 and 31:7:2

Figure 35-5



35:7:3 $$A = \mathrm{gd}(\alpha) \qquad \text{and} \qquad B = \mathrm{gd}(\beta)$$

35:8 NUMERICAL VALUES

In its roster of inverse circular function routines, *Equator* offers a degree option in addition to the default radian output. A standard algorithm is used in the inverse tangent function routine. The inverse sine function and inverse cosecant function routines then exploit the relationships

35:8:1 $$\arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad |x| < 1$$

and

35:8:2 $$\mathrm{arccsc}(x) = \arctan\left(\frac{\mathrm{sgn}(x)}{\sqrt{x^2-1}}\right) \qquad |x| > 1$$

The routines that calculate the inverse cotangent function, inverse cosine function, and inverse secant function invoke equations 35:0:17. Special measures are taken when the argument is in the immediate vicinity of 0 or ±1. *Equator*'s keywords are simply the six-lettered symbols, for example **arcsin**.

35:9 LIMITS AND APPROXIMATIONS

As the argument tends towards infinity, the approach of four of the inverse circular functions to their ultimate value is as the reciprocal of the argument:

35:9:1 $$\mathrm{f}(x) \to \frac{\pi}{2} - \frac{1}{x} \qquad \mathrm{f} = \arctan \text{ or } \mathrm{arcsec}$$

35:9:2 $$\mathrm{f}(x) \to \frac{1}{x} \qquad \mathrm{f} = \mathrm{arccsc} \text{ or } \mathrm{arccot}$$

$$\Bigg\} \quad x \to \infty$$

As the argument approaches unity from one direction or the other, the behavior of four inverse circular functions is governed by the expressions

35:9:3 $$\arcsin(x) \to \frac{\pi}{2} - \sqrt{2(1-x)} \qquad x \to 1$$

35:9:4 $$\arccos(x) \to \sqrt{2(1-x)} \qquad x \to 1$$

35:9:5 $$\mathrm{arccsc}(x) \to \frac{\pi}{2} - \sqrt{2(x-1)} \qquad 1 \leftarrow x$$

35:9:6 $$\mathrm{arcsec}(x) \to \sqrt{2(x-1)} \qquad 1 \leftarrow x$$

Similar formulas describe the limiting behaviors as x approaches $-\infty$ or -1.

35:10 OPERATIONS OF THE CALCULUS

For positive x, derivatives and indefinite integrals of the six inverse circular functions are

	f = arccos	f = arcsin	f = arcsec	f = arccsc	f = arctan	f = arccot
$\frac{d}{dx}f(x) =$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{1}{1+x^2}$	$\frac{-1}{1+x^2}$
$\int_0^x f(t)\,dt =$	$x\arccos(x) + 1 - \sqrt{1-x^2}$	$x\arccos(x) - 1 + \sqrt{1-x^2}$	$x\,\mathrm{arcsec}(x) - \mathrm{arcosh}(x)$	$x\,\mathrm{arccsc}(x) + \mathrm{arcosh}(x)$	$x\arctan(x) - \ln\left(\sqrt{1+x^2}\right)$	$x\,\mathrm{arccot}(x) + \ln\left(\sqrt{1+x^2}\right)$

Notice that each of the indefinite integrals is of the form

35:10:1 $$\int_0^x f(t)\,dt = x\,f(x) + g(x) \qquad f = \text{arccos, arcsin, arcsec, arccsc, arctan, or arccot}$$

and thereby it is straightforward to demonstrate that integration by parts [Formula 0:10:11] results in

35:10:2 $$\int_0^x t\,f(t)\,dt = \frac{x}{2}\left[x\,f(x) + g(x)\right] - \frac{1}{2}\int_0^x g(t)\,dt$$

On the other hand, indefinite integrals of $f(t)/t$ are unknown other than as infinite series. See Spiegel [pages 82−84] for a long list of indefinite integrals of $t^{\pm n}f(t)$, where $n = 2,3,4,\cdots$ and f is an inverse circular function. Gradshteyn and Ryzhik [Section 2.8] list similar integrals, as well as some indefinite integrals of the powers $f^n(t)$.

Among definite integrals and Laplace transforms are

35:10:3 $$\int_0^1 t^{\nu}\arctan(t)\,dt = \frac{\pi - 2G(1+\frac{1}{2}\nu)}{4(1+\nu)} \qquad -1 \neq \nu > -2$$

35:10:4 $$\int_0^\infty \frac{\arctan(t)}{t^{3/2}}\,dt = \sqrt{2}\,\pi$$

35:10:5 $$\int_0^1 \frac{\arctan(t)}{t}\,dt = \int_1^\infty \frac{\mathrm{arccot}(t)}{t}\,dt = \int_0^\infty \frac{\arctan(t)}{t^2-1}\,dt = G$$

35:10:6 $$\int_0^1 \frac{\arcsin(t)}{t}\,dt = \frac{\pi}{2}\ln(2)$$

35:10:7 $$\int_0^1 \arcsin(t)\exp(-bt)\,dt = \frac{\pi}{2b}\left[I_0(b) - \mathbf{L}_0(b) - \exp(-b)\right]$$

35:10:8 $$\int_0^\infty \arctan(t)\exp(-st)\,dt = \mathfrak{L}\{\arctan(t)\} = \frac{\sin(s)\mathrm{Ci}(s) + \cos(s)\left[\frac{1}{2}\pi - \mathrm{Si}(s)\right]}{s}$$

35:10:9 $$\int_0^\infty \arctan\left(\sqrt{t}\right)\exp(-st)\,dt = \mathfrak{L}\left\{\arctan\left(\sqrt{t}\right)\right\} = \frac{\pi}{2s}\exp(s)\mathrm{erfc}\left(\sqrt{s}\right)$$

Bateman's G function from Section 44:14, Catalan's constant G from Section 1:7, the zero-order modified Bessel and modified Struve functions I_0 and $\mathtt{l}_0$ from Chapters 49 and 57, the cosine and sine integrals Ci and Si from Chapter 38, and the error function complement from Chapter 40 are included in the functions generated by these integrals. In their Section 4.5, Gradshteyn and Ryzhik list over 100 additional definite integrals involving the inverse circular functions.

35:11 COMPLEX ARGUMENT

Figures 35-6, 35-7 and 35-8 show the real and imaginary parts of the inverse circular cosine, sine and tangent functions. When the argument is purely imaginary, three of the inverse circular functions devolve into their inverse hyperbolic analogues [Chapter 31]:

35:11:1 $$\arcsin(iy) = i\,\mathrm{arsinh}(y)$$

35:11:2 $$\mathrm{arccot}(iy) = -i\,\mathrm{arcoth}(y)$$

35:11:3 $$\mathrm{arccsc}(iy) = -i\,\mathrm{arcsch}(y)$$

whereas the other three have a real component of π/2 and can be formulated by applying equation 35:0:17.

Inverse Laplace transforms include

35:11:4 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \mathrm{arccot}(bs)\frac{\exp(st)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\{\mathrm{arccot}(bs)\} = \frac{1}{t}\sin\left(\frac{t}{b}\right)$$

and

35:11:5 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s}\mathrm{arccot}(bs)\frac{\exp(st)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{\mathrm{arccot}(bs)}{s}\right\} = \mathrm{Si}\left(\frac{t}{b}\right)$$

the latter yielding a sine integral [Chapter 38].

Figure 35-6

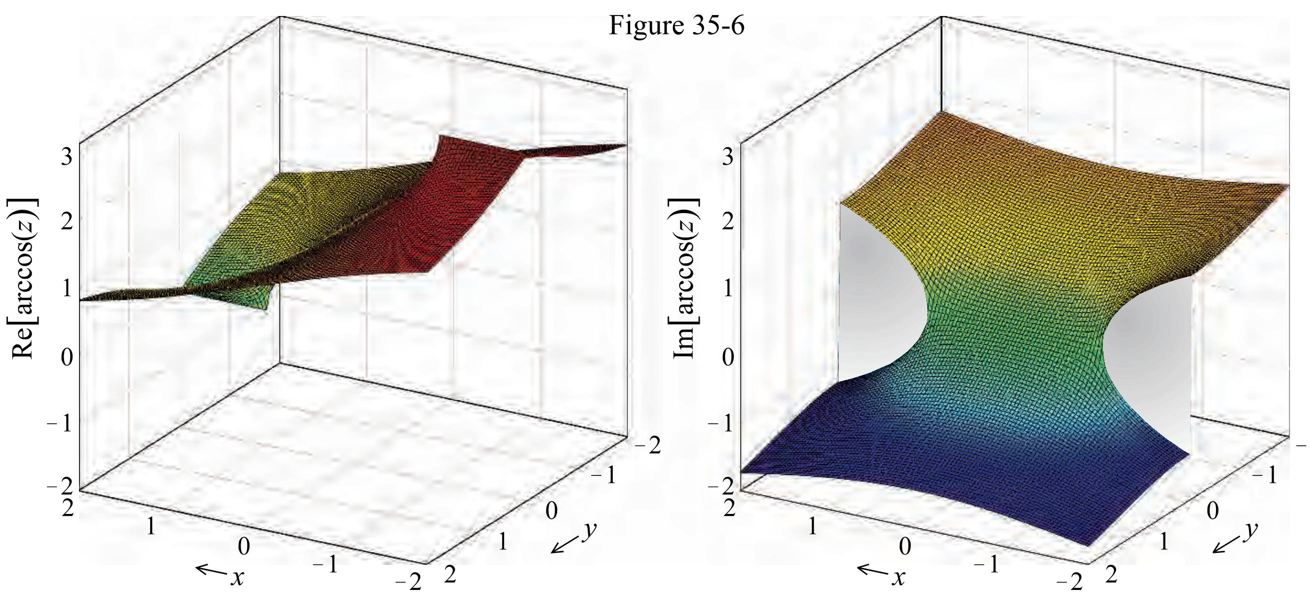


Figure 35-7

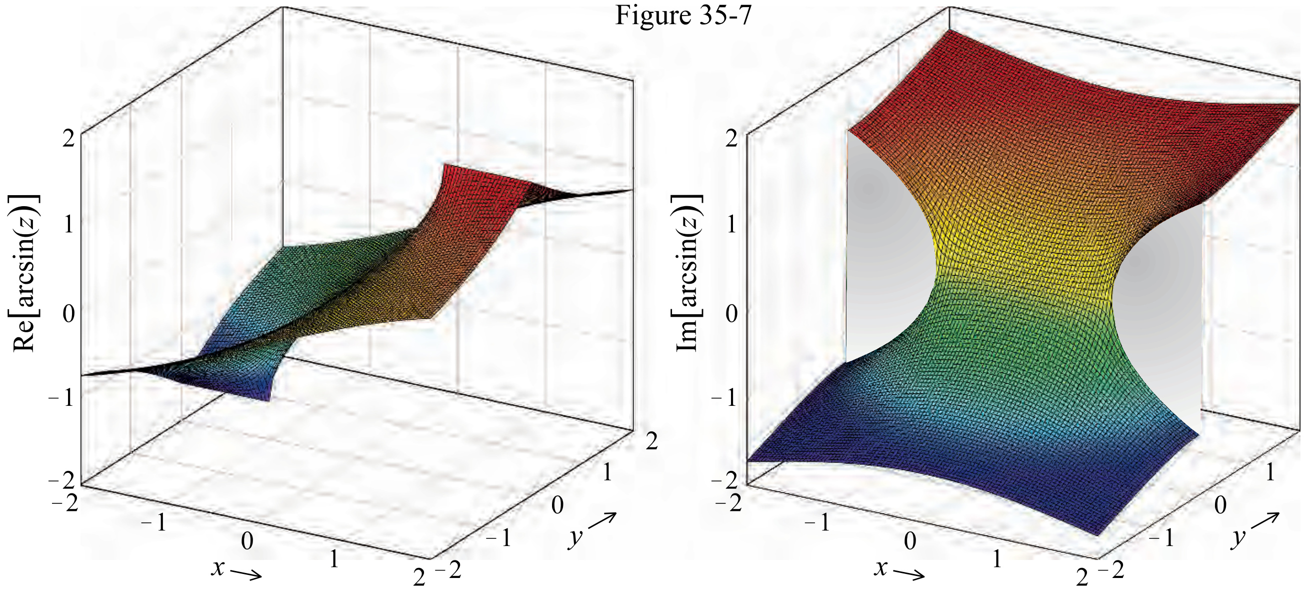
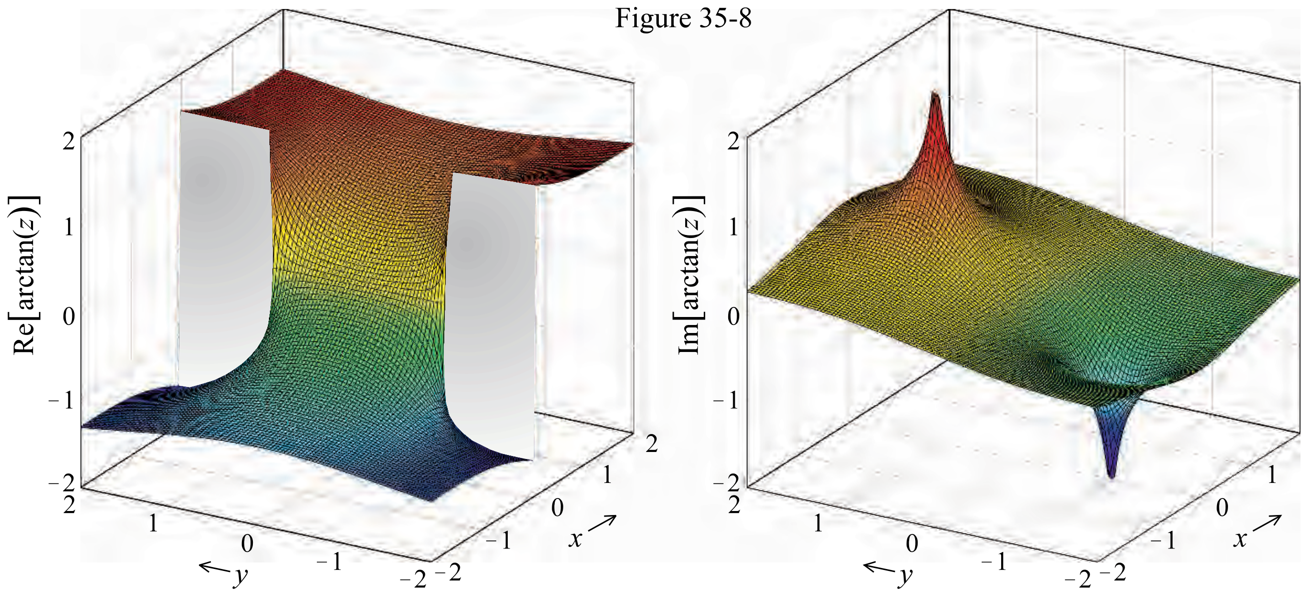


Figure 35-8



35:12 GENERALIZATIONS

If the restrictions are removed from equations 35:3:1–6, the *multivalued inverse circular functions* are thereby defined. These are distinguished by having a capitalized initial letter in their symbols. For example, the multivalued inverse cosine is defined by

35:12:1 $$x = \cos(\mathrm{f}) \qquad \text{implies} \qquad \mathrm{f} = \mathrm{Arccos}(x)$$

and similarly for the other five. The relationship to the single-valued function is

35:12:2 $$\mathrm{Arccos}(x) = \arccos(x) + k\pi \qquad k = 0, \pm 2, \pm 4, \cdots$$

and a similar relation holds for the Arcsin, Arcsec, and Arccsc functions; however, any integer value of *k*, even or odd, is applicable in the Arctan and Arccot analogs of 35:12:2.

The inverse cosine and inverse sine functions are special cases of the incomplete beta function [Chapter 58], while the inverse sine and tangent functions similarly specialize the Gauss hypergeometric function of Chapter 60:

35:12:3 $$\arccos(x) = \tfrac{1}{2}\mathrm{B}\left(\tfrac{1}{2},\tfrac{1}{2},1-x^2\right)$$

35:12:4 $$\arcsin(x) = \tfrac{1}{2}\mathrm{B}\left(\tfrac{1}{2},\tfrac{1}{2},x^2\right) = x\,\mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},x^2\right)$$

35:12:5 $$\arctan(x) = x\mathrm{F}\left(\tfrac{1}{2},1,\tfrac{3}{2},-x^2\right)$$

35:13 COGNATE FUNCTIONS

The inverse circular functions play a role in the algebra of the Jacobian elliptic functions [Chapter 63] and they are related through equations 35:11:1−3 to the inverse hyperbolic functions of Chapter 30.

35:14 RELATED TOPIC: two-dimensional coordinate systems

The location of any point P in a plane (the *cartesian* plane) may be specified by citing the values of two suitably chosen parameters, called *coordinates*. In addition, some means of "anchoring" the coordinates to the plane is needed. This latter requirement is usually satisfied by specifying a reference point (the *origin* O) and a reference direction (the *major axis*) from that point.

The simplest and most familiar coordinate system employs the *rectangular coordinates*, x and y. The x coordinate measures the shortest distance to point P from a line (the *minor axis*, the y-axis) that is perpendicular to the major axis (the x-axis) through the origin. See Figure 35-9. The y coordinate measures the perpendicular distance from the x-axis to the point P. Lines of constant x $(-\infty < x < \infty)$ are straight and parallel to the minor axis; lines of constant y $(-\infty < y < \infty)$ are straight and parallel to the major axis.

Also familiar is the *polar system*, with coordinates r and θ, illustrated in Figure 35-10. The r coordinate measures the distance from the origin to P. The θ coordinate is an angle, measured counterclockwise from the reference direction. Lines of constant r $(0 \le r < \infty)$ are concentric circles; lines of constant θ $(0 \le \theta < 2\pi)$ are straight and radiate from the origin. The interrelations

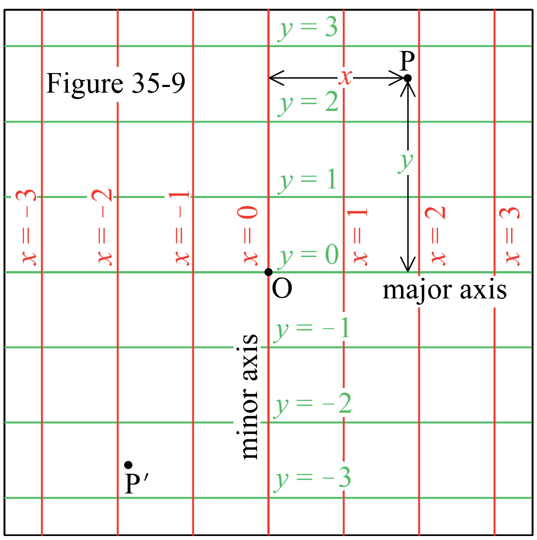


Figure 35-9

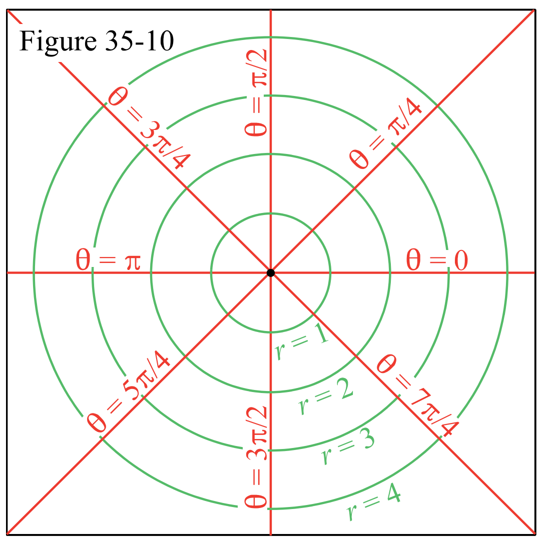


Figure 35-10

35:14:1 $$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

and

35:14:2 $$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

between the two systems are also in widespread use. However, whereas each system uniquely specifies the location of point P, blunders may occur in using these interrelation formulas. Ambiguities arise because the inverse tangent of a ratio takes no cognizance of the individual signs of the components of the ratio, so the points marked P and P′ in Figure 35-9 yield erroneously identical *polar coordinates* when the formulas in 35:14:1 are invoked. The second member of this formula should be modified to

35:14:3 $$\theta = \mathrm{sgn}(y)\mathrm{arccot}\left(\frac{x}{y}\right) \qquad -\pi < \theta < \pi \qquad y \neq 0$$

or

35:14:4 $$\theta = \arctan\left(\frac{y}{x}\right) + \frac{\pi}{2}[2 - \mathrm{sgn}(y) - \mathrm{sgn}(xy)] \qquad 0 \le \theta < 2\pi \qquad x, y \neq 0$$

to avoid the problem.

System	Primary coordinate α	Secondary coordinate β
rectangular	x (parallel straight lines)	y (parallel straight lines)
polar	r (concentric circles)	θ (radial straight lines)
parabolic	p (confocal parabolas)	q (confocal parabolas)
elliptical	η (horizontal ellipses)	ψ (horizontal hyperbolas)
bipolar	λ (circles centered on major axis)	ξ (circles centered on minor axis)

The rectangular and polar systems are in common use but three less familiar two-dimensional coordinate systems, included in the table above, are valuable alternatives in appropriate situations. The table names, and the figures illustrate, the shapes of the lines (the *grid lines*), which may be straight or curved, mapped out when either the primary or the secondary coordinate takes a constant value, the other being allowed to vary. Although each coordinate system may be defined and utilized without reference to any rival system, it is often more convenient to introduce new systems through their relationship to the rectangular system. Note that the symbols used for the coordinates are not standardized. Moreover, quite different forms may be encountered; for example, the coordinates adopted for the elliptical system are sometimes the equivalent of our $\sinh(\eta)$ and $\sin(\psi)$.

The *parabolic coordinate* (p,q) *system*, illustrated in Figure 35-11, is related to the rectangular system through the equations

35:14:5 $$x = \tfrac{1}{2}(p^2 - q^2) \qquad y = pq$$

Both sets of grid lines are parabolas [Section 11:14], their common focus being at the origin. In rectangular coordinates, the equations of the lines of constant p $(-\infty < p < \infty)$ and the lines of constant q $(q \ge 0)$ are

35:14:6 $$y = \sqrt{p^4 - 2p^2x} \qquad \text{and} \qquad y = \sqrt{q^4 + 2q^2x}$$

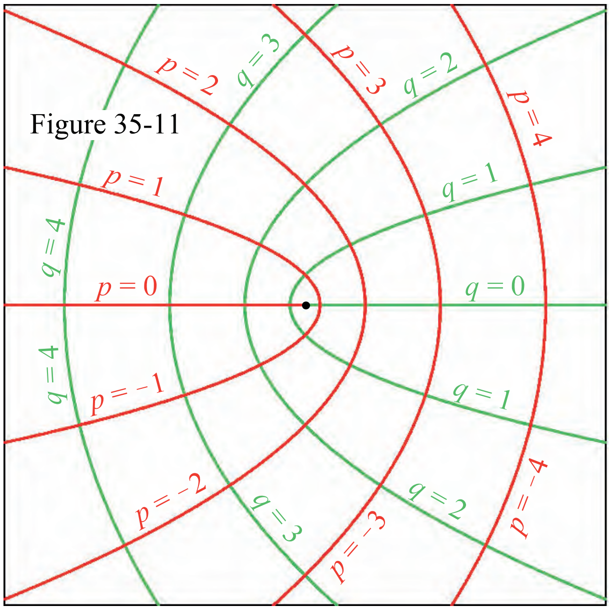


Figure 35-11

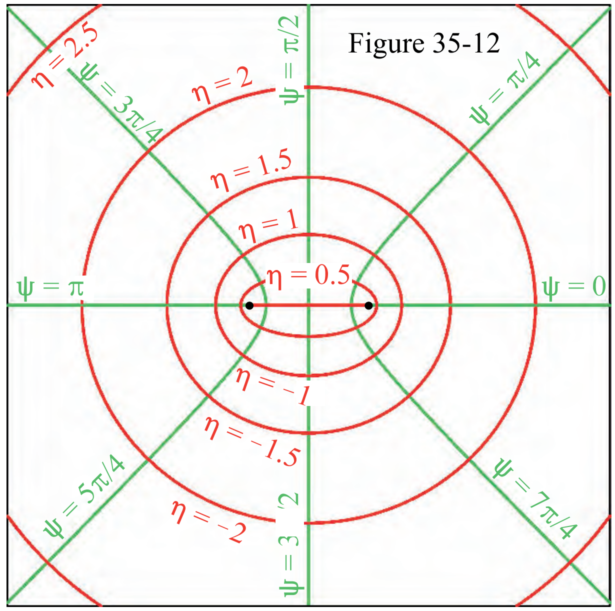


Figure 35-12

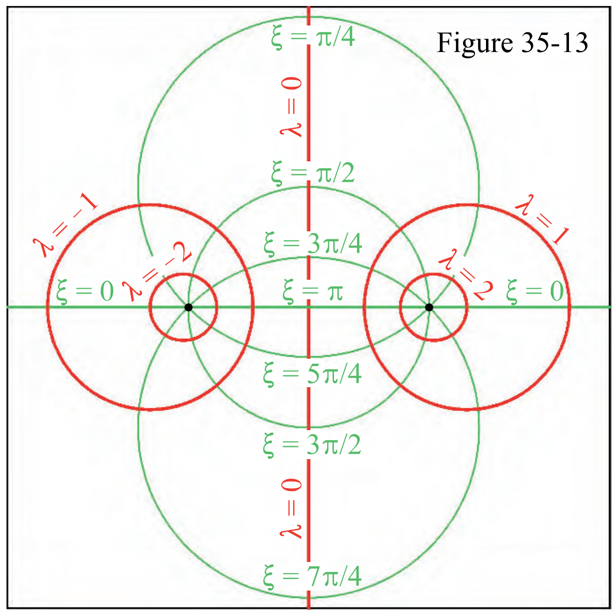


Figure 35-13

The reference elements of the elliptic and bipolar coordinate systems are two points, rather than the standard origin point and major axis direction. The two points are separated by a distance $2a$. For the purpose of comparison with the rectangular system, we place the origin of the latter midway between the paired points and place the major axis on the line joining them.

In the *elliptic coordinate* (η,ψ) *system*, illustrated in Figure 35-12, the two reference points are the foci of curves of the second degree [Section 15:15]. If a rectangular coordinate system is superimposed on the elliptic system, the equations

35:14:7 $$x = a\cosh(\eta)\cos(\psi) \qquad y = a\sinh(\eta)\sin(\psi)$$

interrelate the coordinates of the two systems. Lines of constant η $(-\infty < \eta < \infty)$ are horizontal ellipses [Section 13:14, but note that the a of Chapter 13 differs in meaning from that used here], whereas lines of constant ψ $(0 \le \psi < 2\pi)$ are horizontal hyperbolas [Section 14:14, again with the caveat that a has changed its meaning]. These lines of constant η and ψ obey the equations

35:14:8 $$y = \tanh(\eta)\sqrt{a^2\cosh^2(\eta) - x^2} \qquad \text{and} \qquad y = \tan(\psi)\sqrt{x^2 - a^2\cos^2(\psi)}$$

respectively, in rectangular coordinates.

In the *bipolar coordinate* (λ,ξ) *system*, all the grid lines pass through the two reference points, as in Figure 35-13, which are separated by $2a$. The equations interrelating the bipolar and rectangular coordinates are

35:14:9 $$x = \frac{a\sinh(\lambda)}{\cosh(\lambda) - \cos(\xi)} \qquad y = \frac{a\sin(\xi)}{\cosh(\lambda) - \cos(\xi)}$$

Lines of constant λ $(-\infty \le \lambda \le \infty)$ and lines of constant ξ $(0 \le \xi < 2\pi)$ are all circles. Their equations, in rectangular coordinates, are respectively

35:14:10 $$y=\sqrt{2ax\coth(\lambda)-a^2-x^2} \qquad \text{and} \qquad y=a\cot(\xi)\pm\sqrt{a^2\csc^2(\xi)-x^2}$$

It is evident from the figures that all five of these coordinate systems display reflection symmetry across the major and minor axes although, in the case of the parabolic system, reflection requires an interchange between the primary and secondary coordinates. These five systems are *orthogonal*; this means that the grid lines cross each other at right angles. Thus, if α and β are any pair of orthogonal coordinates, then the slope of the constant-α grid line is related to that of the constant-β grid line by their product being −1. This is true at any point in the plane. Taking the parabolic coordinate system as an example, the slope of the constant-p gridline is $-p/q$, while that of the constant-q gridline is q/p. Two functions which are interrelated in this way are said to be *conjugate harmonic functions*.

Consider two points in a plane, infinitesimally separated from each other. The distance between them, in rectangular coordinates, is

35:14:11 $$\mathrm{d}\ell=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$$

What is the corresponding distance in polar coordinates? One finds via equations 35:14:2 that it is

35:14:12 $$\mathrm{d}\ell=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}=\sqrt{\left[\cos(\theta)\mathrm{d}r-r\sin(\theta)\mathrm{d}\theta\right]^2+\left[\sin(\theta)\mathrm{d}r+r\cos(\theta)\mathrm{d}\theta\right]^2}=\sqrt{[1]^2\mathrm{d}r^2+[r]^2\mathrm{d}\theta^2}$$

The quantities that appear in square-brackets in the final expression of 35:14:12 are the so-called *scale factors* (or *metric coefficients*) of the corresponding coordinate. They are usually represented by the symbol h subscripted by the coordinate's symbol. Thus, for polar coordinates

35:14:13 $$h_r=1 \qquad \text{and} \qquad h_\theta=r$$

The corresponding scale factors for the parabolic, elliptical, and bipolar coordinate systems are

35:14:14 $$h_p=h_q=\sqrt{p^2+q^2}$$

35:14:15 $$h_\eta=h_\psi=a\sqrt{\sinh^2(\eta)+\sin^2(\psi)}$$

and

35:14:16 $$h_\lambda=\frac{a\sinh(\lambda)}{\cosh(\lambda)-\cos(\xi)} \qquad \text{and} \qquad h_\xi=\frac{a\sin(\xi)}{\cosh(\lambda)-\cos(\xi)}$$

Except for the rectangular system (wherein both scale factors are unity), the scale factors enter most problems in which the geometry or other property of a system is being examined in orthogonal coordinates. For example the area bounded by the four lines $\alpha=\alpha_0$, $\beta=\beta_0$, $\alpha=\alpha_1$, and $\beta=\beta_1$, in the (α,β) coordinate system is

35:14:17 $$\int_{\alpha_0}^{\alpha_1}\int_{\beta_0}^{\beta_1}h_\alpha h_\beta\,\mathrm{d}\beta\,\mathrm{d}\alpha$$

Scale factors may be defined for three-dimensional coordinate systems in a strictly analogous fashion; such scale factors are encountered in Section 46:14.

Conjugate harmonic functions, with their right-angular linkage are important in many realms of physics, because motion generally occurs, or fields of influence exist, perpendicularly to regions in which there is a uniform density of the agent causing the motion or field. Consider, for example, a hot elliptical disk inlaid into a planar medium of uniform thermal conductivity. Such a system is best described in elliptical coordinates. Heat flows away from the disk along routes shaped as hyperbolas, while isothermal lines (on which the temperature is uniform) are ellipses confocal with the disk and orthogonal to the hyperbolas. We have described a two-dimensional problem but, to match the space of our universe, systems of interest generally occupy three spatial dimensions. Much of the material in this section generalizes to three dimensions, as discussed in Section 46:14.

CHAPTER 36

PERIODIC FUNCTIONS

As sound and electromagnetic radiation of various frequencies, periodic functions, or nearly periodic functions, provide the medium by which most telecommunications take place. Series of *periodic function* also play an important role in solving many problems in applied mathematics that do not overtly have periodic properties.

36:1 NOTATION

We shall use per(x) to represent any periodic function, and occasionally qer(x) to represent a second periodic function.

Throughout the chapter, P will denote the *period* of per(x). The quantity $2\pi/P$ is known as the *frequency* or *angular frequency* of the function and is often denoted by ω.

36:2 BEHAVIOR

Apart from their repetitive characteristic, periodic functions share no common behavior. They may be simple or complicated, continuous or discontinuous. Examples of periodic functions are graphed later in the chapter.

36:3 DEFINITIONS

A function that satisfies the condition

36:3:1 $$f(x) = f(x + kP) \qquad k = \pm 1, \pm 2, \pm 3, \cdots$$

for all argument x is a periodic function. Its period is the smallest positive value of P that satisfies relation 36:3:1.

An aperiodic function may be converted, for example by the action of the fractional-value or modulo functions of Chapter 8, into a periodic function by having a segment replicated at regular intervals. Thus Figure 36-1 shows a graph of x^2 (mod 2), a periodic function of period 2.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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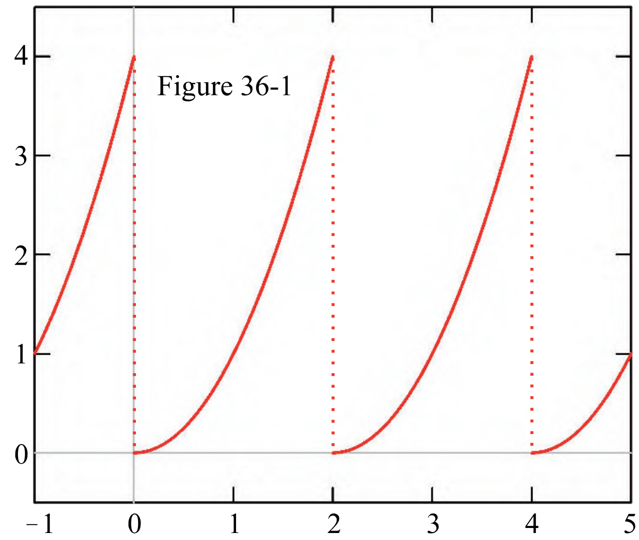
36:4 SPECIAL CASES

Several of the functions addressed elsewhere in this *Atlas* are periodic. The functions of Chapters 32 and 33 are periodic with periods of 2π, while the tangent and cotangent functions of Chapter 34 have periods of π. The comb function of Section 8:13, exponential theta functions of Section 27:13, and the multivalued inverse circular functions of Section 35:12, are also periodic. The Jacobian elliptic functions [Chapter 63] have periods determined by their modulus. Some common periodic waveforms are the subject of Section 34:14.



36:5 INTRARELATIONSHIPS

The fundamental property of periodicity

36:5:1
$$\mathrm{per}(x+P)=\mathrm{per}(x)$$

constitutes a recurrence formula for periodic functions. Periodic functions may be even, odd, or neither.

If two periodic functions, per(x) and qer(x) with periods P and Q, are added, subtracted, multiplied, or divided, the resultant function will be periodic only if the quotient P/Q is a rational number. Generally, per(x)±qer(x), per(x)qer(x) or per(x)/qer(x) will have a period equal to the least common multiple of P and Q. There are, however, exceptions to this rule. For example sin(x) and cos(x), each of period 2π, have products and quotients of period π, not 2π as predicted by the rule. The sum and difference, sin(x) ± cos(x) do, however, obey the rule.

If the argument x of the periodic function per(x) is replaced by a linear function of x, the resulting per($bx+c$) function will be periodic, but with a period altered to P/b. Replacement of the argument by any other aperiodic function of x destroys the periodicity; the functions per(ax^2+bx+c) or per($1/x$), for example, are not periodic. Multiplication or division by, or addition or subtraction of, any aperiodic function other than a constant likewise destroys the periodicity.

A function of a periodic function is itself periodic. It generally has the same period P as per(x) or occasionally a submultiple such as $P/2$.

36:6 EXPANSIONS

In a process known as *harmonic analysis*, any periodic function of period P may be represented, exactly or approximately, as the *Fourier series* (Jean Baptiste Joseph Fourier, 1768−1830, French mathematician)

36:6:1
$$\mathrm{per}(x)\left\{\begin{matrix}=\\ \approx\end{matrix}\right\}\frac{c_0}{2}+\sum_{j=1}^{\infty}c_j\cos\left(\frac{2j\pi x}{P}\right)+s_j\sin\left(\frac{2j\pi x}{P}\right)$$

Whether the = or the ≈ sign is appropriate in this equation depends on the convergence properties of Fourier series, the details of which are of some complexity [Hamming, Chapter 32] and will not be explored here. Suffice it to state that if the per(x) function is continuous, the series converges; if the function has discontinuities, the series converges except at the points of discontinuity. An example occurs later in this section.

The c_j and s_j coefficients in series 36:6:1 are known as *Fourier coefficients*; they may be calculated by the so-called *Euler formulas*

36:6:2 $$c_j = \frac{2}{P}\int_0^P \mathrm{per}(t)\cos\left(\frac{2j\pi t}{P}\right)\mathrm{d}t \qquad j = 0,1,2,\cdots$$

and

36:6:3 $$s_j = \frac{2}{P}\int_0^P \mathrm{per}(t)\sin\left(\frac{2j\pi t}{P}\right)\mathrm{d}t \qquad j = 1,2,3\cdots$$

The Fourier coefficients satisfy *Parseval's relation* (Marc-Antoine Parseval des Chênes, French nobleman, 1788−1829)

36:6:4 $$\frac{c_0^2}{2} + \sum_{j=1}^{\infty} c_j^2 + s_j^2 = \frac{2}{P}\int_{x_0}^{x_0+P} \mathrm{per}^2(t)\mathrm{d}t$$

provided that per(x) is finite everywhere.

The Fourier series 36:6:1 may be written in the alternative form

36:6:5 $$\mathrm{per}(x) = \frac{c_0}{2} + \sum_{j=1}^{\infty}\sqrt{c_j^2 + s_j^2}\ \cos\left\{\frac{2j\pi x}{P} - \arctan\left(\frac{s_j}{c_j}\right)\right\}$$

or in terms of exponential functions of imaginary argument

36:6:6 $$\mathrm{per}(x) = \sum_{j=-\infty}^{+\infty} a_j \exp\left(\frac{2ji\pi x}{P}\right)$$

In the latter representation, the Fourier coefficients a_j are complex numbers calculable by the unitary Euler formula

36:6:7 $$a_j = \frac{1}{P}\int_0^P \mathrm{per}(t)\exp\left\{\frac{-2ij\pi t}{P}\right\}\mathrm{d}t = \frac{c_{|j|} - \mathrm{sgn}(j)is_{|j|}}{2} \qquad j = 0,\pm1,\pm2,\cdots$$

All the formulas in this section condense somewhat when the period parameter P is replaced by $2\pi/\omega$, ω being the frequency. It is in this condensed format that harmonic analysis is commonly conducted, especially when the variable x represents time, as it often does.

Over a limited range of argument, Euler's formulas may be applied to functions that are not periodic. For example, choosing the square function $\mathrm{f}(x) = x^2$ over the interval $0 < x < 2$, one finds from equations 36:6:2 and 36:6:3 that

36:6:8 $$c_j = \int_0^2 t^2\cos(j\pi t)\mathrm{d}t = \begin{cases} 8/3 & j = 0 \\ 4/j^2\pi^2 & j = 1,2,3,\cdots \end{cases}$$

36:6:9 $$s_j = \int_0^2 t^2\sin(j\pi t)\mathrm{d}t = \frac{-4}{j\pi} \qquad j = 1,2,3,\cdots$$

and therefore

36:6:10 $$x^2 = \frac{4}{3} + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos(j\pi x)}{j^2\pi} - \frac{\sin(j\pi x)}{j} \qquad 0 < x < 2$$

If a similar replacement in $2 < x < 4$, $-2 < x < 0$, etc., the periodic function illustrated in Figure 36-1 is generated. This periodic function has one discontinuity per period and it is interesting to see how well formula 36:6:10, absent the restriction, handles this. One finds that at $x = 0$ (or $x = \pm2, \pm4,\cdots$), the formulas gives $(4/3)+4\zeta(2)/\pi^2 = 2$, the properties of a zeta number [Chapter 3] having assisted. Note that this answer is the mean of the values, 0 and 4, at each side of the discontinuity. This is generally true: at a discontinuity the Fourier series representation gives the "best answer that it can", the average of the values on either side of the discontinuity.

Though they are the usual, and normally the best, units into which periodic functions may be resolved, the cosine and sine are not the only candidates for this task. For example, as mentioned in Section 8:6, the sine function can be built from square-waves [Section 36:14], and so can the cosine. It follows that any periodic function may be resolved into a (usually infinite) set of square waves.

36:7 PARTICULAR VALUES

Of course the particular values of a periodic function depend on the identity of the function. In terms of the Fourier coefficients, however, one can cite the four special values

36:7:1 $$\mathrm{per}(0)=\mathrm{per}(P)=\tfrac{1}{2}c_0+c_1+c_2+c_3+c_4+c_5+c_6+\cdots$$

36:7:2 $$\mathrm{per}\left(\tfrac{1}{4}P\right)=\tfrac{1}{2}c_0+s_1-c_2-s_3+c_4+s_5-c_6-\cdots$$

36:7:3 $$\mathrm{per}\left(\tfrac{1}{2}P\right)=\tfrac{1}{2}c_0-c_1+c_2-c_3+c_4-c_5+c_6-\cdots$$

36:7:4 $$\mathrm{per}\left(\tfrac{3}{4}P\right)=\tfrac{1}{2}c_0-s_1-c_2+s_3+c_4-s_5-c_6+\cdots$$

Notice that at the end points of the period, and in its center, the Fourier sine coefficients play no role whatsoever.

36:8 NUMERICAL VALUES

If necessary, the numerical value of a periodic function may be calculated via its Fourier expansion.

36:9 LIMITS AND APPROXIMATIONS

A periodic function may be approximated by a truncated version of its Fourier expansion.

Moreover, a so-called *Fourier approximation* to an *aperiodic* function f(x) over a specified argument range $x_0 \le x \le x_1$, may be created for most functions. First change the variable to $x-x_0$ and replace P in Euler's formulas by x_1-x_0. The resulting definitions,

36:9:1 $$c_j=\frac{2}{x_1-x_0}\int_{x_0}^{x_1}\mathrm{f}(x-x_0)\cos\left(\frac{2j\pi(x-x_0)}{x_1-x_0}\right)\mathrm{d}x \qquad j=0,1,2,\cdots,J$$

and

36:9:2 $$s_j=\frac{2}{x_1-x_0}\int_{x_0}^{x_1}\mathrm{f}(x-x_0)\sin\left(\frac{2j\pi(x-x_0)}{x_1-x_0}\right)\mathrm{d}x \qquad j=1,2,3,\cdots,J$$

may be used to create a limited set of Fourier coefficients with $2J+1$ members. The Fourier approximation is then

36:9:3 $$\hat{\mathrm{f}}(x)\approx\frac{c_0}{2}+\sum_{j=1}^{J}c_j\cos\left(\frac{2j\pi(x-x_0)}{x_1-x_0}\right)+s_j\sin\left(\frac{2j\pi(x-x_0)}{x_1-x_0}\right) \qquad x_0\le x\le x_1$$

When any function, periodic or not, is approximated over a specified range by a truncated Fourier expansion, such an approximation is *best in the least-squares sense*. The meaning of the italicized phrase is that if $\hat{\mathrm{f}}(x)$ is an approximation to f(x) based on 36:9:3, then

36:9:4 $$\int_{x_0}^{x_1}[\hat{\mathrm{f}}(x)-\mathrm{f}(x)]^2\,\mathrm{d}x \le \int_{x_0}^{x_1}[\hat{\mathrm{f}}'(x)-\mathrm{f}(x)]^2\,\mathrm{d}x$$

where $\hat{\mathrm{f}}'(x)$ is any other $(2J+1)$-term approximation to $\mathrm{f}(x)$.

36:10 OPERATIONS OF THE CALCULUS

Differentiation of the periodic function 36:6:1 gives another periodic function of the same period. If $\mathrm{per}(x)$ is continuous, the derivative is

36:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{per}(x)=\frac{2\pi}{P}\sum_{j=1}^{\infty} js_j\cos\left(\frac{2j\pi x}{P}\right)-jc_j\sin\left(\frac{2j\pi x}{P}\right)$$

in terms of its Fourier coefficients. Discontinuities in $\mathrm{per}(x)$ will generate Dirac functions [Chapter 9] in $\mathrm{d}\{\mathrm{per}(x)\}/\mathrm{d}x$.

The $\frac{1}{2}c_0$ Fourier coefficient represents the average value of $\mathrm{per}(x)$ over the period:

36:10:2 $$\frac{1}{P}\int_{x_0}^{x_0+P}\mathrm{per}(t)\mathrm{d}t=\frac{c_0}{2} \qquad x_0 \text{ arbitrary}$$

Only if this coefficient is zero will indefinite integration of $\mathrm{per}(x)$ give rise to another periodic function, in which case the integral has the following Fourier expansion

36:10:3 $$\int_0^x \mathrm{per}(t)\,\mathrm{d}t=\frac{P}{2\pi}\sum_{j=1}^{\infty}\frac{c_j}{j}\sin\left(\frac{2j\pi x}{P}\right)+\frac{s_j}{j}\left[1-\cos\left(\frac{2j\pi x}{P}\right)\right] \qquad c_0=0$$

After subtraction of $\frac{1}{2}c_0$, differintegration [Section 12:14] with a lower limit of $-\infty$ converts $\mathrm{per}(x)$ into another periodic function of period P. In terms of the original Fourier coefficients, the differintegrated function is

36:10:4 $$\begin{aligned}\left.\frac{\mathrm{d}^v}{\mathrm{d}x^v}[\mathrm{per}(x)-\tfrac{1}{2}c_0]\right|_{-\infty} &= \sum_{j=1}^{\infty}\left(\frac{2j\pi}{P}\right)^v\left[c_j\cos\left(\frac{2j\pi x}{P}+\frac{v\pi}{2}\right)+s_j\sin\left(\frac{2j\pi x}{P}+\frac{v\pi}{2}\right)\right] \\ &= \sum_{j=1}^{\infty}\left(\frac{2j\pi}{P}\right)^v\left[\left\{c_j\cos\left(\tfrac{1}{2}v\pi\right)+s_j\sin\left(\tfrac{1}{2}v\pi\right)\right\}\cos\left(\frac{2j\pi x}{P}\right)+\left\{s_j\cos\left(\tfrac{1}{2}v\pi\right)-c_j\sin\left(\tfrac{1}{2}v\pi\right)\right\}\sin\left(\frac{2j\pi x}{P}\right)\right]\end{aligned}$$

See Section 64:14 for further discussion of this topic.

Laplace transforms of periodic functions obey the rule

36:10:5 $$\int_0^{\infty}\mathrm{per}(t)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{per}(t)\}=\frac{1}{1-\exp(-Ps)}\int_0^P\mathrm{per}(t)\exp(-st)\,\mathrm{d}t$$

Specific instances of this formula will be found in Section 36:14 for a variety of periodic functions.

36:11 COMPLEX ARGUMENT

For constant y, the periodic function $\mathrm{per}(x+iy)$ remains periodic in x. The Fourier coefficients of such functions are complex numbers that may be further resolved with the help of equations 32:11:1 and 32:11:2.

36:12 GENERALIZATIONS

A number of functions. such as the logarithmic function and the hyperbolic sine functions, ln and sinh, that are aperiodic when their arguments are real, are periodic functions in the complex plane.

The Jacobian elliptic functions [Chapter 63] exhibit *double periodicity* when their argument is complex. That is, they satisfy the recurrence relations

36:12:1 $$\mathrm{per}(x+iy)=\mathrm{per}(x+P+iy)=\mathrm{per}(x+iy+iQ)=\mathrm{per}(x+P+iy+iQ)$$

where P and Q are respectively the real and imaginary periods.

36:13 COGNATE FUNCTIONS

Periodic, and especially sinusoidal, functions are used as "carriers" of lower frequency signals. That is, the signal is imposed on the carrier, which is said to be *modulated* thereby. The ensemble is no longer periodic, but is described as *quasiperiodic*. Usually the signal is merely added to the carrier, so that the resulting quasiperiodic waveform resembles a periodic signal in its frequency attribute but lacks a constant amplitude: it is amplitude modulated (AM). Alternatively, the amplitude may be constant but the frequency may be perturbed slightly by the signal (frequency modulation, FM). Quasiperiodicity is a property used for conveying information in such transfers as light, sound, radio-waves and diagnostic signals of various kinds. Filtering or Fourier transformation [Section 32:15] is used to *demodulate* the wave and recover the signal.

36:14 RELATED TOPIC: waveforms

Certain commonly encountered periodic functions, often incorporating one or two discontinuities per period, are known as *waveforms*. The diagrams in Figure 36-2 give the names and display the shapes of some of the most important waveforms. Notice that these waveforms come in variants, as exemplified in diagrams (a) and (b), that differ only in *phase*.

The names of waveforms (g) and (h) have their origins in electrical technology. When alternating current (a.c.), which has a sinusoidal waveform (f), is converted to direct current (d.c.) the first step is to *rectify* it ("make it right") by converting it into one or other of waveforms (g) or (h).

Periodic functions may be Laplace transformed and often the result is quite simple. Listed below are the transforms of the eight functions depicted in Figure 36-2.

36:14:1 $$\mathfrak{L}\left\{\mathrm{per}_{(a)}(t)\right\}=\frac{1}{s}\tanh\left(\frac{Ps}{4}\right)$$

36:14:2 $$\mathfrak{L}\left\{\mathrm{per}_{(\mathrm{b})}(t)\right\}=\frac{1}{s}\left[1-\mathrm{sech}\left(\frac{Ps}{4}\right)\right]$$

36:14:3 $$\mathfrak{L}\left\{\mathrm{per}_{(\mathrm{c})}(t)\right\}=\frac{1}{s}\left[\frac{2}{Ps}-\mathrm{csch}\left(\frac{Ps}{2}\right)\right]$$

36:14:4 $$\mathfrak{L}\left\{\mathrm{per}_{(\mathrm{d})}(t)\right\}=\frac{4}{Ps^2}\tanh\left(\frac{Ps}{4}\right)$$

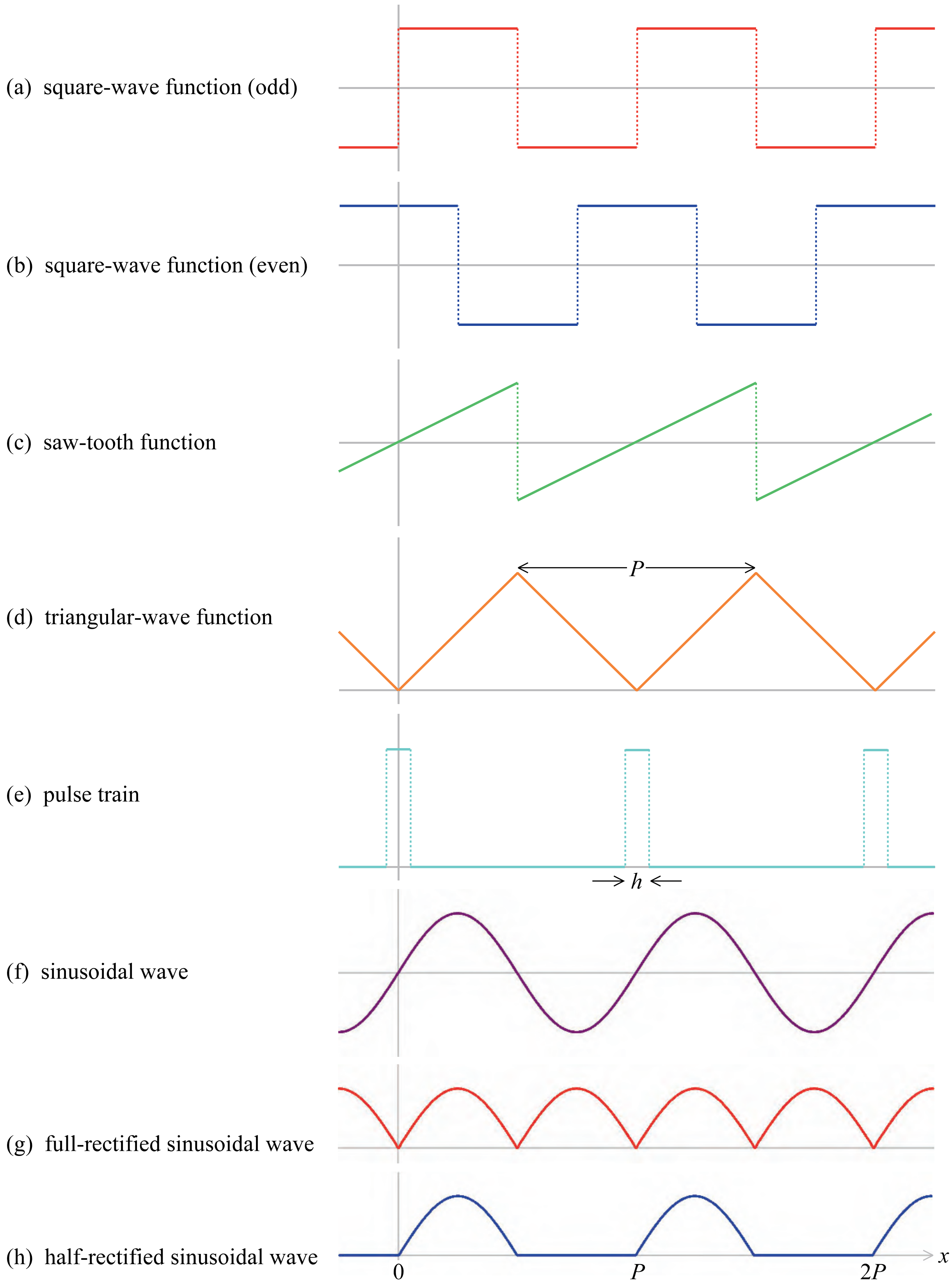

(a) square-wave function (odd)
(b) square-wave function (even)
(c) saw-tooth function
(d) triangular-wave function
P
(e) pulse train
h
(f) sinusoidal wave
(g) full-rectified sinusoidal wave
(h) half-rectified sinusoidal wave
0
P
2P
x

Figure 36-2

36:14:5 $$\mathcal{L}\{\mathrm{per}_{(\mathrm{e})}(t)\} = \frac{2}{s}\left[1 - \exp\left(\frac{hs}{2}\right) + \frac{2\sinh(hs/2)}{1-\exp(-Ps)}\right]$$

36:14:6 $$\mathcal{L}\{\mathrm{per}_{(\mathrm{f})}(t)\} = \frac{2\pi P}{4\pi^2 + P^2s^2}$$

36:14:7 $$\mathcal{L}\{\mathrm{per}_{(\mathrm{g})}(t)\} = \frac{2\pi P}{4\pi^2 + P^2s^2}\coth\left(\frac{Ps}{4}\right)$$

36:14:8 $$\mathcal{L}\{\mathrm{per}_{(\mathrm{h})}(t)\} = \frac{2\pi P}{\left(4\pi^2 + P^2s^2\right)\left[1-\exp(-Ps/2)\right]}$$

Of course, the subscript attaching to the per symbol in the above transforms relates to the items so identified in Figure 36-2 and in the following table, which lists each waveform's formula and its Fourier coefficients.

Fig	per(x)	$\frac{c_0}{2}$	c_j		s_1	s_j	
			$j=1,3,\cdots$	$j=2,4,\cdots$		$j=3,5,\cdots$	$j=2,4,\cdots$
(a)	$(-1)^{\mathrm{Int}(2x/P)}$	0	0	0	$\frac{4}{\pi}$	$\frac{4}{j\pi}$	0
(b)	$(-1)^{\mathrm{Int}(4x+P)/2P}$	0	$\frac{(-)^{(j-1)/2}4}{j\pi}$	0	0	0	0
(c)	$2\,\mathrm{frac}\left(\frac{1}{2}+\frac{x}{P}\right)-1$	0	0	0	$\frac{2}{\pi}$	$\frac{2}{j\pi}$	$\frac{-2}{j\pi}$
(d)	$(-)^{\mathrm{Int}(2x/P)}\left[4\,\mathrm{frac}\left(\frac{1}{2}+\frac{x}{P}\right)-2\right]$	1	$\frac{-8}{j^2\pi^2}$	0	0	0	0
(e)	$2\mathrm{u}\left\{\mathrm{frac}\left(\frac{2x-h}{2P}\right)-1+\frac{h}{P}\right\}$	$\frac{h}{P}$	$\frac{2}{j\pi}\sin\left(\frac{j\pi h}{P}\right)$	$\frac{2}{j\pi}\sin\left(\frac{j\pi h}{P}\right)$	0	0	0
(f)	$\sin\left(\frac{2\pi x}{P}\right)$	0	0	0	1	0	0
(g)	$\left\lvert\sin\left(\frac{2\pi x}{P}\right)\right\rvert$	$\frac{2}{\pi}$	0	$\frac{-4}{(j^2-1)\pi}$	0	0	0
(h)	$\frac{1}{2}\sin\left(\frac{2\pi x}{P}\right)+\frac{1}{2}\left\lvert\sin\left(\frac{2\pi x}{P}\right)\right\rvert$	$\frac{1}{\pi}$	0	$\frac{-2}{(j^2-1)\pi}$	$\frac{1}{2}$	0	0

Apart from the last two, the formulas above apply to a periodic function that ranges by 2 from minimum to maximum, thereby matching the figures in scale. The functions (a), (b), (c), and (f) are said to have an *amplitude* of unity: this term is not clearly defined for other periodic functions. Another measure of the intensity of a waveform is its *root-mean-square amplitude*,

36:14:9 $$per_{\mathrm{rms}} = \sqrt{\frac{1}{2}\int_{x_0}^{x_0+P}\mathrm{per}^2(x)\,\mathrm{d}x}$$

CHAPTER
37

THE EXPONENTIAL INTEGRALS Ei(x) AND Ein(x)

For $n = 1,2,3,\cdots$, indefinite integrals of the form $\int t^{-n}\exp(\pm t)\mathrm{d}t$ cannot be expressed as elementary functions. The *exponential integral* function Ei(x) fills this deficiency, but has an inconvenient discontinuity at $x = 0$. The *entire exponential integral* Ein(x) lacks discontinuities, being related by

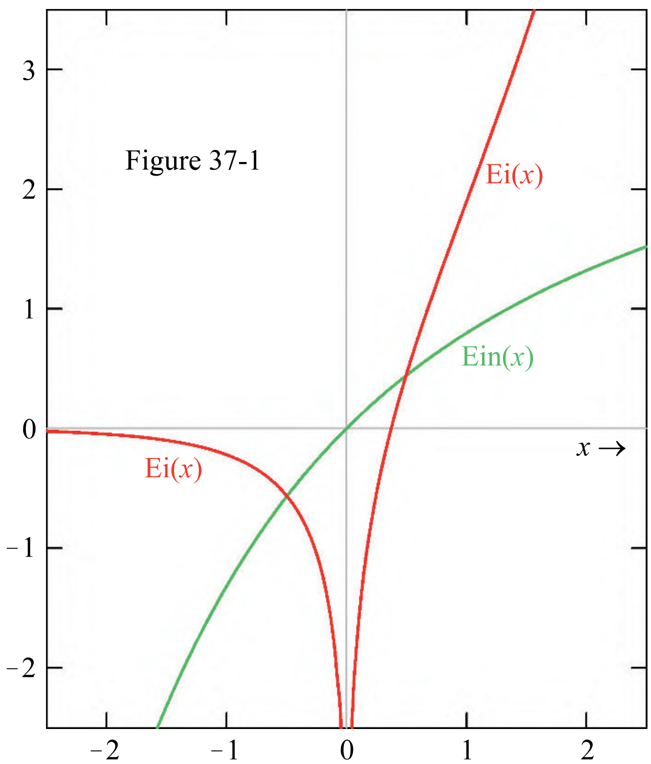
37:0:1 $$\mathrm{Ein}(x) = \gamma + \ln\left(|x|\right) - \mathrm{Ei}(-x) \qquad \gamma = 0.57721\,56649\,01533$$

to the exponential integral function. Here ln is the logarithmic function [Chapter 25] and γ is Euler's constant [Section 1:7].

37:1 NOTATION

Notations abound for the exponential integral function. Ei*(x), E*(x), $\bar{\mathrm{E}}\mathrm{i}(x)$, $\overline{\mathrm{Ei}}(x)$, $\mathrm{E}^{+}(x)$, and $\mathrm{E}^{-}(x)$ have all been used to denote Ei(x) or some closely related function. The definition of these symbols in other sources may not exactly match that used in this *Atlas*. Thompson [Chapter 5] recognizes three kinds of exponential integral, of which the "exponential integral of the second kind" is our Ei. His first and third kinds of "exponential integral" are the Schlömilch functions [Section 37:13] and the logarithmic integral [Section 25:13], respectively.

For negative argument, Ei(x) is often written $-\mathrm{E}_1(-x)$ for reasons that will be clear from Section 37:13. Adding to the confusion, ei(x) sometimes replaces $-\mathrm{Ei}(-x)$. This *Atlas* uses only Ei and Ein, defined as in Section 37:3 below.



K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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37:2 BEHAVIOR

Both functions accept any real argument and have unlimited ranges. As Figure 37-1 suggests, the Ei function tends towards $-\infty$ as its argument approaches zero from either direction, whereas Ein passes smoothly through zero. As their arguments increase, both functions approach ever larger values, the increase being dramatic in the case of Ei. The exponential integral function approaches zero at large negative arguments, whereas its entire congener acquires increasingly negative values indefinitely.

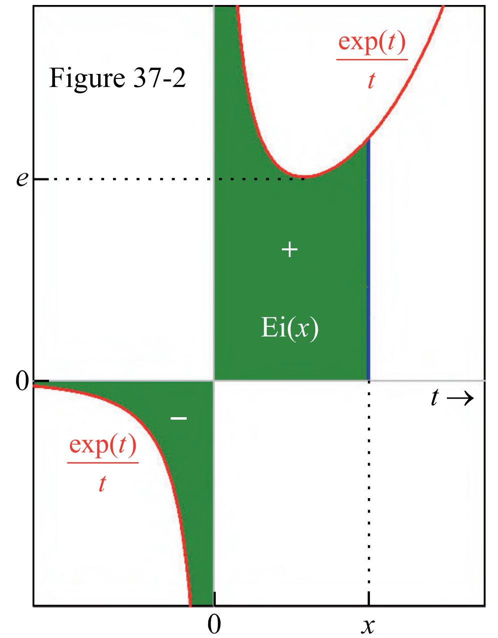
37:3 DEFINITIONS

The two functions are defined by the indefinite integrals:

37:3:1 $$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t}\,\mathrm{d}t$$

and

37:3:2 $$\mathrm{Ein}(x) = \int_{0}^{x} \frac{1-\exp(-t)}{t}\,\mathrm{d}t$$



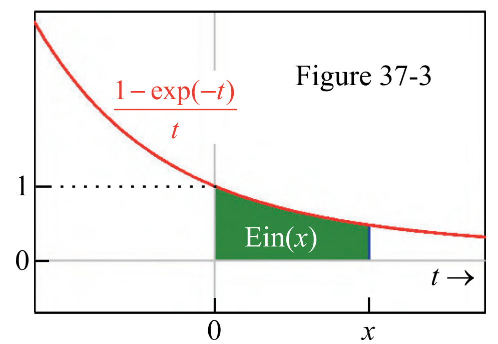
these definitions being illustrated in Figures 37-2 and 37-3. Because the integrand in 37:3:1 passes through a discontinuity, that integral is to be interpreted as a *Cauchy limit*:

37:3:3 $$\mathrm{Ei}(x) = \lim_{\varepsilon\to 0}\left\{\mathrm{Ei}(-\varepsilon) + \int_{\varepsilon}^{x} \frac{\exp(t)}{t}\,\mathrm{d}t\right\} \qquad x > \varepsilon > 0$$

whenever x is positive.

The exponential integral function may be defined as a definite integral in many ways. One such definition is

37:3:4 $$\mathrm{Ei}(-x) = \exp(-x)\int_{0}^{1} \frac{1}{\ln(t) - x}\,\mathrm{d}t \qquad x > 0$$



and several others are listed by Gradshteyn and Rhyzik [Section 8.212]. As well, the logarithmic integral [Section 25:13] can serve to define the exponential integral through the equivalence

37:3:5 $$\mathrm{Ei}(x) = \mathrm{li}\{\exp(x)\}$$

In addition to 37:3:2, equation 37:0:1 provides a definition of the entire exponential function. Being hypergeometric, this function may be synthesized [Section 43:14]; two steps are needed:

37:3:6 $$\exp(-x) \xrightarrow[2]{1} \frac{1-\exp(-x)}{x} \xrightarrow[2]{1} \frac{\mathrm{Ein}(x)}{x}$$

37:4 SPECIAL CASES

There are none.

37:5 INTRARELATIONSHIPS

We know of no simple reflection, recurrence, addition, or multiplication formulas, though the following apply:

37:5:1 $$\mathrm{Ei}(x)+\mathrm{Ein}(-x)=\mathrm{Ei}(-x)+\mathrm{Ein}(x)$$

37:5:2 $$\mathrm{Ei}(x)-\mathrm{Ein}(-x)=2x+\frac{x^3}{9}+\frac{x^5}{300}+\cdots=\sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!\left(j+\frac{1}{2}\right)}=2x\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j\left(\frac{3}{2}\right)_j}\left(4x^2\right)^j$$

37:5:3 $$\mathrm{Ei}(x+y)=\mathrm{Ei}(x)-\exp(x+y)\sum_{j=1}^{\infty}\frac{(-y)^j}{j!}\sum_{k=0}^{j-1}\frac{k!}{x^{k+1}}\qquad xy>0\qquad |x|\geq|y|$$

In the last of these formulas, interchange x and y if $|y|>|x|$.

37:6 EXPANSIONS

Two alternative power series:

37:6:1 $$\mathrm{Ein}(x)=x-\frac{x^2}{4}+\frac{x^3}{18}-\frac{x^4}{96}+\cdots=-\sum_{j=1}^{\infty}\frac{(-x)^j}{j!\,j}=x\sum_{j=0}^{\infty}\frac{(1)_j}{(2)_j(2)_j}(-x)^j$$

and

37:6:2 $$\exp(x)\,\mathrm{Ein}(x)=x+\frac{3x^2}{4}+\frac{11x^3}{36}+\cdots=\sum_{j=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j}\right)\frac{x^j}{j!}=\sum_{j=1}^{\infty}\left[\gamma+\psi(j+1)\right]\frac{x^j}{j!}$$

are available to express the entire exponential integral. The former demonstrates this function's hypergeometricity. In the latter, ψ is the digamma function [Chapter 44].

One may combine equation 37:0:1 and 37:6:1 to produce

37:6:3 $$\mathrm{Ei}(x)=\ln\left(|x|\right)+\gamma+x+\frac{x^2}{4}+\frac{x^3}{18}+\frac{x^4}{96}+\cdots=\ln\left(|x|\right)+\gamma+\sum_{j=1}^{\infty}\frac{x^j}{j!\,j}$$

The product of $\exp(-x)$ and the exponential integral is expansible as the continued fraction

37:6:4 $$\exp(-x)\,\mathrm{Ei}(x)=\frac{1}{x-}\,\frac{1}{1-}\,\frac{1}{x-}\,\frac{2}{1-}\,\frac{2}{x-}\,\frac{3}{1-}\,\frac{3}{x-}\,\frac{4}{1-\cdots}$$

or as the asymptotic series

37:6:5 $$\exp(-x)\,\mathrm{Ei}(x)\sim\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}+\frac{6}{x^4}+\frac{24}{x^5}+\cdots=\sum_{j=0}\frac{j!}{x^{j+1}}\qquad x\to\infty$$

the last being useful only for large, and preferably negative, argument.

37:7 PARTICULAR VALUES

The zeros of the two functions are

37:7:1 $$\mathrm{Ei}(0.37250\;74107\;81367)=0\qquad\text{and}\qquad\mathrm{Ein}(0)=0$$

The Ei(x) function displays a discontinuity of the $-\infty|-\infty$ variety at $x=0$; it inflects at $x=1$.

See 44:5:10 for the special significances of Ei(1) and Ei(−1).

37:8 NUMERICAL VALUES

With keywords **Ei** and **Ein**, *Equator* provides an exponential integral routine and a distinct entire exponential integral routine. For arguments in the domain $-2 \le x \le 75$, *Equator* uses many terms of expansion 37:6:1 to calculate Ein($-x$). Ei(x) is then computed via 37:0:1. For other arguments in the domain $|x| \le 705$, the continued fraction 37:6:5 is invoked, the entire exponential integral Ein(x) being then computed via 37:0:1. Though the relative accuracy in Ei(x) is compromised close to its zero [equation 37:7:1], both routines give numerical values that are precise to the number of digits reported.

37:9 LIMITS AND APPROXIMATIONS

Limiting expressions as the argument approaches infinity are

37:9:1 $$\mathrm{Ei}(x) \to \frac{\exp(x)}{x} \qquad x \to \pm\infty$$

37:9:2 $$\mathrm{Ein}(x) \to \ln(x) + \gamma \qquad x \to +\infty$$

and

37:9:3 $$\mathrm{Ein}(x) \to \frac{\exp(-x)}{x} \qquad x \to -\infty$$

while, close to $x = 0$, the following approximations hold

37:9:4 $$\mathrm{Ei}(x) \approx \ln\left(|x|\right) + x + \gamma \qquad |x| \text{ small}$$

and

37:9:5 $$\mathrm{Ein}(x) \approx x - \tfrac{1}{4}x^2 \qquad |x| \text{ small}$$

37:10 OPERATIONS OF THE CALCULUS

The following rules apply for differentiation and indefinite integration:

37:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ei}(bx) = \frac{\exp(bx)}{x}$$

37:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ein}(bx) = \frac{1-\exp(-bx)}{x}$$

37:10:3 $$\int_0^x \mathrm{Ei}(bt)\mathrm{d}t = x\,\mathrm{Ei}(bx) - \frac{\exp(bx)-1}{b}$$

37:10:4 $$\int_0^x \mathrm{Ein}(bt)\mathrm{d}t = x\,\mathrm{Ein}(bx) - x + \frac{1-\exp(-bx)}{b} = \frac{bx^2}{2}\sum_{j=0}^{\infty}\frac{(1)_j}{(2)_j(3)_j}(-bx)^j$$

37:10:5 $$\int_0^x t^n\,\mathrm{Ei}(bt)\mathrm{d}t = \frac{x^{n+1}\,\mathrm{Ei}(bx)}{n+1} + \frac{n!\left[\exp(bx)\mathrm{e}_n(-bx)-1\right]}{(n+1)(-b)^{n+1}} \qquad n = 0,1,2,\cdots$$

37:10:6
$$\int_0^x \exp(\beta t)\,\mathrm{Ei}(bt)\mathrm{d}t = \begin{cases} \dfrac{1}{\beta}\left[\exp(\beta x)\,\mathrm{Ei}(bx) - \mathrm{Ei}(\beta x + bx) + \ln\left(\left|\dfrac{\beta+b}{b}\right|\right)\right] & \beta + b \neq 0 \\ \dfrac{1}{b}\left[\ln(|bx|) + \gamma - \exp(-bx)\,\mathrm{Ei}(bx)\right] & \beta + b = 0 \end{cases}$$

The e_n function appearing in 37:10:5 above, and in 37:10:8 below, is the exponential polynomial [Section 26:12].

Among indefinite and definite integrals are

37:10:7
$$\int_{-c/b}^x \exp(\beta t)\,\mathrm{Ei}(bt+c)\mathrm{d}t = \begin{cases} \dfrac{1}{\beta}\exp\left(\dfrac{-\beta c}{b}\right)\left[\exp\left(\beta x + \dfrac{\beta c}{b}\right)\mathrm{Ei}(bx+c) - \mathrm{Ei}\left\{(\beta+b)\left(x+\dfrac{c}{b}\right)\right\} + \ln\left(\left|\dfrac{\beta+b}{b}\right|\right)\right] \\ \text{or, if } \beta = -b, \quad \dfrac{1}{b}\exp(c)\left[\ln\left(|bx+c|\right) + \gamma - \exp(-bx-c)\,\mathrm{Ei}(bx+c)\right] \end{cases}$$

37:10:8
$$\int_0^\infty \frac{\exp(t)\,\mathrm{Ei}(-t)}{t^\nu}\mathrm{d}t = -\pi\csc(\pi\nu)\Gamma(1-\nu) = -\Gamma(\nu)\Gamma^2(1-\nu) \qquad -1 < \nu < 0$$

and others may be found in Gradshteyn and Ryzhik [Sections 6.21−6.23]. The Γ function in 37:10:8 is the (complete) gamma function [Chapter 43].

The following Laplace transforms generate functions from Chapters 25, 31, 51, 43, and 60:

37:10:9
$$\int_0^\infty \mathrm{Ei}(\pm bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\{\mathrm{Ei}(\pm bt)\} = \frac{-1}{s}\ln\left(1 \mp \frac{s}{b}\right)$$

37:10:10
$$\int_0^\infty \frac{\mathrm{Ei}(-bt)}{\sqrt{t}}\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{\mathrm{Ei}(-bt)}{\sqrt{t}}\right\} = -2\sqrt{\frac{\pi}{s}}\mathrm{arsinh}\left(\sqrt{\frac{s}{b}}\right)$$

37:10:11
$$\int_0^\infty \mathrm{Ei}\left(\frac{-a}{t}\right)\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\mathrm{Ei}\left(\frac{-a}{t}\right)\right\} = \frac{-2\mathrm{K}_0\left(2\sqrt{as}\right)}{s} \qquad a > 0$$

37:10:12
$$\int_0^\infty t^{\nu-1}\,\mathrm{Ei}(-bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{t^{\nu-1}\,\mathrm{Ei}(-bt)\right\} = \frac{-\Gamma(\nu)\mathrm{F}\left(1,\nu,\nu+1,s/(s+b)\right)}{\nu(s+b)^\nu} \qquad \nu > 0$$

37:11 COMPLEX ARGUMENT

The real and imaginary parts of $\mathrm{Ei}(x + iy)$ are illustrated in Figure 37-4 overleaf. Apart from the negative pole at the origin, the real part has unexceptional behavior. The imaginary part is cut along the negative real axis, as it would otherwise be multivalued. Some authors cut the complex plane along the positive real axis.

For purely imaginary argument, one has

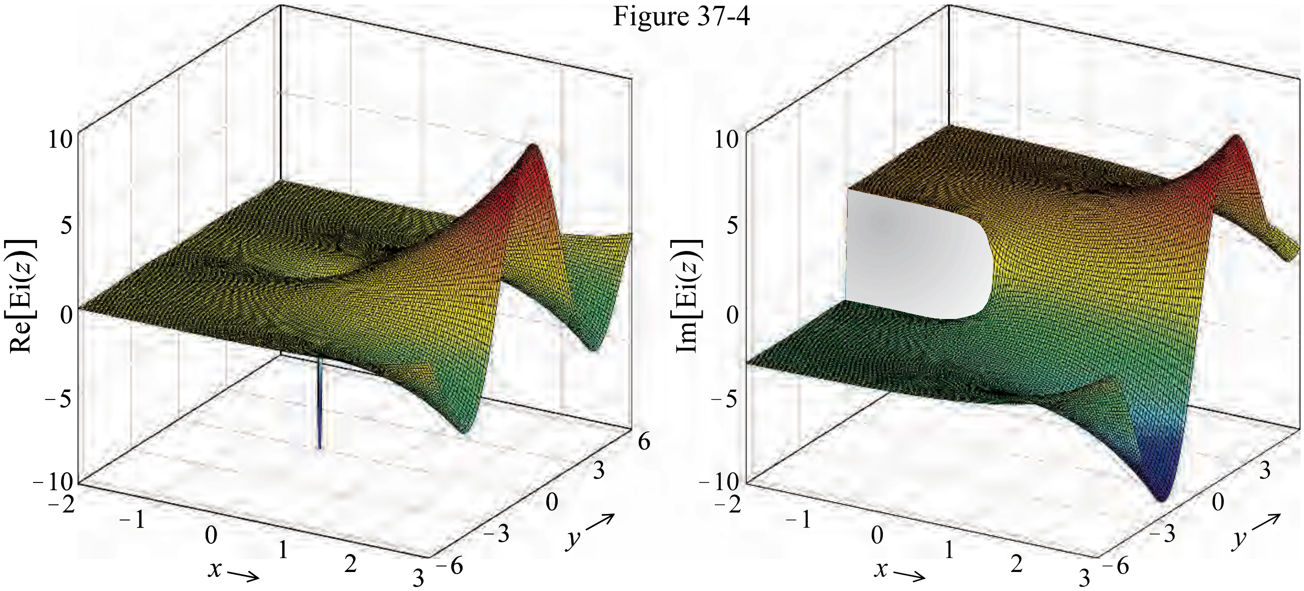
37:11:1
$$\mathrm{Ei}(iy) = \mathrm{Ci}(y) + i\left[\mathrm{Si}(y) - \frac{\pi}{2} + \mathrm{sgn}(y)\pi\right]$$

and

37:11:2
$$\mathrm{Ein}(iy) = \mathrm{Cin}(y) + i\,\mathrm{Si}(y)$$

The Si, Ci and Cin functions are discussed in Chapter 38.

Figure 37-4



37:12 GENERALIZATIONS

The exponential integral generalizes to two important functions: the upper incomplete gamma function [Chapter 45]

37:12:1 $$\mathrm{Ei}(x) = -\Gamma(0,-x)$$

and the Tricomi function [Chapter 48]

37:12:2 $$\mathrm{Ei}(x) = -\exp(x)\,\mathrm{U}(1,1,-x)$$

When x is positive, it is only the real part of $-\exp(x)\mathrm{U}(1,1,-x)$, that equals the exponential integral.

37:13 COGNATE FUNCTIONS

As established in equation 37:6:1, the entire exponential integral is an example of an $L = K+1 = 2$ hypergeometric function [Section 18:14]. As such, Ein has kinship with all the functions in Tables 18-3 and 18-4.

A family of functions, each defined by either a definite, or an equivalent indefinite, integral

37:13:1 $$\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t = x^{n-1}\int_x^\infty \frac{\exp(-t)}{t^n}\mathrm{d}t \qquad n = 0,1,2,\cdots$$

was investigated by Oscar Xavier Schlömilch (1823 − 1901, French-born, though his mathematical career was in Germany). These *Schlömilch functions* are generally denoted $\mathrm{E}_n(x)$ but the *Atlas* avoids this notation to prevent confusion with Euler polynomials [Chapter 20] that are symbolized identically. Early members of the family are

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t$	$\frac{\exp(-x)}{x}$	$-\mathrm{Ei}(-x)$	$\exp(-x) + x\,\mathrm{Ei}(x)$	$\frac{1-x}{2}\exp(-x) - \frac{x^2}{2}\mathrm{Ei}(x)$

and others may be found either from the simple recursion relationship

37:13:2 $$(n-1)\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t = \exp(-x) - x\int_1^\infty \frac{\exp(-xt)}{t^{n-1}}\mathrm{d}t \qquad n=2,3,4,\cdots$$

or by the general formulation

37:13:3 $$\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t = \frac{1}{(n-1)!}\left[(-)^n x^{n-1}\,\mathrm{Ei}(-x) + \exp(-x)\sum_{j=0}^{n-2} j!(-x)^{n-j-2}\right] \qquad n=2,3,4\cdots$$

The derivative of the nth Schlömilch function is the negative of its ($n-1$)th congener. Having the following asymptotic expansion, useful only for large x,

37:13:4 $$\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t \sim \frac{\exp(-x)}{x}\left[1-\frac{n}{x}+\frac{n(n+1)}{x^2}-\frac{n(n+1)(n+2)}{x^3}+\cdots\right] = \frac{\exp(-x)}{x}\sum_{j=0}(n)_j\left(\frac{-1}{x}\right)^j$$

the Schlömilch function is a special case of the upper incomplete gamma function,

37:13:5 $$\int_1^\infty \frac{\exp(-xt)}{t^n}\mathrm{d}t = x^{n-1}\Gamma(n-1,x)$$

Its numerical values are calculable by using the preceding formula in conjunction with *Equator*'s upper incomplete gamma function routine (keyword **gamupper**) [Section 45:8].

Two other relatives of the Ei function are the *alpha exponential integrals* and the *beta exponential integrals*, defined in the equations

37:13:6 $$\alpha_n(x) = \int_1^\infty t^n \exp(-xt)\mathrm{d}t = \frac{n!\exp(-x)\mathrm{e}_n(x)}{x^{n+1}} = \frac{\Gamma(n+1,x)}{x^{n+1}} \qquad n=0,1,2,\cdots$$

37:13:7 $$\beta_n(x) = \int_{-1}^1 t^n \exp(-xt)\mathrm{d}t = \frac{n!}{x^{n+1}}\left[\exp(x)\mathrm{e}_n(-x) - \exp(-x)\mathrm{e}_n(x)\right] \qquad n=0,1,2,\cdots$$

However, because these may be written explicitly – as shown – in terms of well-known functions, they are not further discussed in the *Atlas*. The $\mathrm{e}_n(x)$ function in these equations is the exponential polynomial of Section 26:13.

The asymptotic representation of the function

37:13:8 $$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right) \sim 1 - x + 2x^2 - 6x^3 + \cdots = \sum_{j=0}(1)_j(-x)^j \qquad \text{small } x$$

often known as *Euler's function*, together with its non-alternating counterpart

37:13:9 $$\frac{1}{x}\exp\left(\frac{-1}{x}\right)\mathrm{Ei}\left(\frac{1}{x}\right) \sim 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{j=0}(1)_j x^j \qquad \text{very small } x$$

are the $L = K-1 = 0$ prototypes or basis functions for a family of hypergeometric functions [Section 18:14]. From one or other of these prototypes, all the functions listed in Tables 18-7 and 18-8, as well as others, may be synthesized via the method explained in Section 43:14.

37:14 RELATED TOPIC: functions defined as an indefinite integral

Ein is one of a number of functions that are defined by an integral of the form

37:14:1 $$\mathrm{f}(x) = \int_0^x \mathrm{g}(t)\,\mathrm{d}t$$

where g is a more elementary function. The functions of Chapters 38, 39, 40, and 62 are also in this class. Such functions may be integrated indefinitely in a standard way, because

37:14:2 $$\int_0^x \mathrm{f}(t)\,\mathrm{d}t = x\,\mathrm{f}(x) - \int_0^x t\,\mathrm{g}(t)\,\mathrm{d}t$$

as is easily established by parts integration [Section 0:10]. Equation 37:10:4 is an example of this general rule. Moreover, it is not difficult to demonstrate that

37:14:3 $$\int_0^x t\,\mathrm{f}(t)\,\mathrm{d}t = \frac{1}{2}\left[x^2\,\mathrm{f}(x) - x\int_0^x t\,\mathrm{g}(t)\,\mathrm{d}t + \int_0^x\int_0^{t'} t\,\mathrm{g}(t)\,\mathrm{d}t\,\mathrm{d}t'\right]$$

and that

37:14:4 $$\int_0^x\int_0^{t'} \mathrm{f}(t)\,\mathrm{d}t\,\mathrm{d}t' = \frac{1}{2}\left[x^2\,\mathrm{f}(x) - x\int_0^x t\,\mathrm{g}(t)\,\mathrm{d}t - \int_0^x\int_0^{t'} t\,\mathrm{g}(t)\,\mathrm{d}t\,\mathrm{d}t'\right]$$

Hence, if the $x\mathrm{g}(x)$ function is susceptible to facile integration, various integrals involving the $\mathrm{f}(x)$ function may be derived easily.

If the lower limit in definition 37:14:1 is other than zero, the same principles hold, though the formulas are somewhat more elaborate.

37:15 RELATED TOPIC: popular integrals

Indefinite integrals of the general form

37:15:1 $$\int t^{\nu}\exp(\pm t)\,\mathrm{d}t$$

are incomplete gamma functions and are discussed in detail in Chapter 45. Nevertheless, certain special instances of integral 37:15:1 occur so frequently in scientific applications, that it is worthwhile to assemble them into the convenient table opposite. Though some apply more widely, the tabulated formulas are applicable generally to real positive x values. The erfc and ierfc functions occurring in this tabulation are discussed in Chapter 40. The daw function is Dawson's integral [Chapter 42] and the lower limits identified as x_{M} and x_{i} are the arguments at which $\mathrm{daw}\left(\sqrt{x}\right)$ acquires its maximum value and where it inflects [Figure 42-1]:

37:15:2 $$x_{\mathrm{M}} = 0.85403\,26565\,98197$$

37:15:3 $$x_{\mathrm{i}} = 1.8436\,50900\,13325$$

Specifically, the integrals tabulated are

37:15:4 $$\int_{x_0}^{x} t^{\nu}\exp(t)\,\mathrm{d}t \qquad \text{and} \qquad \int_{x}^{\infty} t^{\nu}\exp(-t)\,\mathrm{d}t \qquad \text{for} \qquad \nu = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \pm 2, \pm\tfrac{5}{2} \text{ and } \pm 3$$

where x_0 has been selected to nullify any "constant of integration". However, Table 37-1 may also be used to evaluate two other families of indefinite integrals, members of which arise frequently. Thus, one may use it to evaluate

37:15:5 $$\int \tau^{\mu}\exp\left(\frac{\pm 1}{\tau}\right)\mathrm{d}\tau \qquad \text{for} \qquad \mu = 0, \pm\tfrac{1}{2}, \pm 1, \tfrac{-3}{2}, -2, \tfrac{-5}{2}, -3, \tfrac{-7}{2}, -4, \tfrac{-9}{2} \text{ and } -5$$

by changing the integration variable from τ to $1/t$, and the power μ to $-(\nu/2)$. Similarly, changes from τ to $\sqrt{t}$ and from n to $2\nu-1$, permit use of the table to evaluate the integrals

37:15:6 $$\int \tau^n \exp\left(\pm\tau^2\right)\mathrm{d}\tau \qquad \text{for} \qquad n = 0, \pm1, \pm2, \pm3, \pm4, \pm5, -6 \text{ and } -7$$

Of course, the limits of integration change accordingly when the variable is altered. As with all indefinite integrals, it is a good idea to check the veracity of the result by differentiation.

Table 37-1

ν	x_0	$\int_{x_0}^{x} t^\nu \exp(t)\,\mathrm{d}t$	$\int_{x}^{\infty} t^\nu \exp(-t)\,\mathrm{d}t$
-3	$-\infty$	$\frac{1}{2}\mathrm{Ei}(x) - \frac{1+x}{2x^2}\exp(x)$	$\frac{1-x}{2x^2}\exp(-x) - \frac{1}{2}\mathrm{Ei}(-x)$
$-5/2$	x_i	$\frac{2}{3}\exp(x)\left[\mathrm{daw}\left(\sqrt{x}\right) - \frac{1+2x}{\sqrt{x^3}}\right]$	$\frac{2(1-2x)}{3\sqrt{x^3}}\exp(-x) + \frac{4\sqrt{\pi}}{3}\mathrm{erfc}\left(\sqrt{x}\right)$
-2	$-\infty$	$\mathrm{Ei}(x) - \frac{\exp(x)}{x}$	$\mathrm{Ei}(-x) + \frac{\exp(-x)}{x}$
$-3/2$	x_M	$2\exp(x)\left[2\,\mathrm{daw}\left(\sqrt{x}\right) - \frac{1}{\sqrt{x}}\right]$	$2\sqrt{\frac{\pi}{x}}\;\mathrm{ierfc}\left(\sqrt{x}\right)$
-1	$-\infty$	$\mathrm{Ei}(x)$	$-\mathrm{Ei}(-x)$
$-1/2$	0	$2\exp(x)\,\mathrm{daw}\left(\sqrt{x}\right)$	$\sqrt{\pi}\,\mathrm{erfc}\left(\sqrt{x}\right)$
0	$-\infty$	$\exp(x)$	$\exp(-x)$
$1/2$	0	$\exp(x)\left[\sqrt{x} - \mathrm{daw}\left(\sqrt{x}\right)\right]$	$\frac{\sqrt{\pi}}{2}\mathrm{erfc}\left(\sqrt{x}\right) + \sqrt{x}\exp(-x)$
1	$-\infty$	$[x-1]\exp(x)$	$[x+1]\exp(-x)$
$3/2$	0	$\sqrt{x}\exp(x)\left[x - \frac{3}{2}\left(1 - \frac{\mathrm{daw}\left(\sqrt{x}\right)}{\sqrt{x}}\right)\right]$	$\frac{3\sqrt{\pi}}{4}\mathrm{erfc}\left(\sqrt{x}\right) + \sqrt{x}\left(x + \tfrac{3}{2}\right)\exp(-x)$
2	$-\infty$	$\left[x^2 - 2x + 2\right]\exp(x)$	$\left[x^2 + 2x + 2\right]\exp(-x)$
$5/2$	0	$\sqrt{x}\exp(x)\left[x^2 - \frac{5x}{2} + \frac{15}{4}\left(1 - \frac{\mathrm{daw}\left(\sqrt{x}\right)}{\sqrt{x}}\right)\right]$	$\frac{15\sqrt{\pi}}{8}\mathrm{erfc}\left(\sqrt{x}\right) + \sqrt{x}\left(x^2 + \frac{5x}{2} + \frac{15}{4}\right)\exp(-x)$
3	$-\infty$	$\left[x^3 - 3x^2 + 6x - 6\right]\exp(x)$	$\left[x^3 + 3x^2 + 6x + 6\right]\exp(-x)$

CHAPTER 38

SINE AND COSINE INTEGRALS

This chapter addresses functions defined as indefinite integrals of $\sin(x)/x$, $\cos(x)/x$, and their hyperbolic counterparts. These functions are named *sine integral*, *cosine integral*, *hyperbolic sine integral*, and *hyperbolic cosine integral* and are given the symbols $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\mathrm{Shi}(x)$, and $\mathrm{Chi}(x)$. Because the $\mathrm{Ci}(x)$ and $\mathrm{Chi}(x)$ functions have discontinuities at $x = 0$, and are complex for negative argument, it is convenient to supplement this pair by an *entire cosine integral* defined by

38:0:1 $$\mathrm{Cin}(x) = \gamma + \ln\left(|x|\right) - \mathrm{Ci}(x) \qquad \gamma = 0.57721\,56649\,01533$$

and an *entire hyperbolic cosine integral*

38:0:2 $$\mathrm{Chin}(x) = \mathrm{Chi}(x) - \gamma - \ln\left(|x|\right)$$

these definitions involving the logarithmic function and Euler's constant. In addition, there are two useful auxiliary functions, introduced in Section 38:13.

Because their hyperbolic analogues find fewer applications, this chapter concentrates on the Si, Ci, and Cin functions. Moreover, the close relationship [equations 38:3:4−6] of the Shi, Chi, and Chin functions to the exponential integral functions of the previous chapter, makes the properties of the hyperbolic trio readily accessible by that route.

38:1 NOTATION

The initial letters of Shi and Chi are not always capitalized. The "h" that identifies the hyperbolic integrals may occur elsewhere in the function's symbol. Thus Sih sometimes replaces Shi and the anagram Cinh may substitute for Chin.

Some authors use $\mathrm{ci}(x)$ synonymously with $\mathrm{Ci}(x)$ but others employ it to denote the negative, $-\mathrm{Ci}(x)$. The notation $\mathrm{si}(x)$ is usually encountered with the meaning

38:1:1 $$\mathrm{si}(x) = \mathrm{Si}(x) - \tfrac{1}{2}\pi$$

Neither ci nor si finds use in this *Atlas*.

We adopt the symbolism $\mathrm{fi}(x)$ and $\mathrm{gi}(x)$ for the auxiliary sine and cosine integrals because there appears to be no standard notation. Abramowitz and Stegun [Section 5.2] use the generic $f(x)$ and $g(x)$, whereas Lebedev [page 37] employs $P(x)$ and $Q(x)$.

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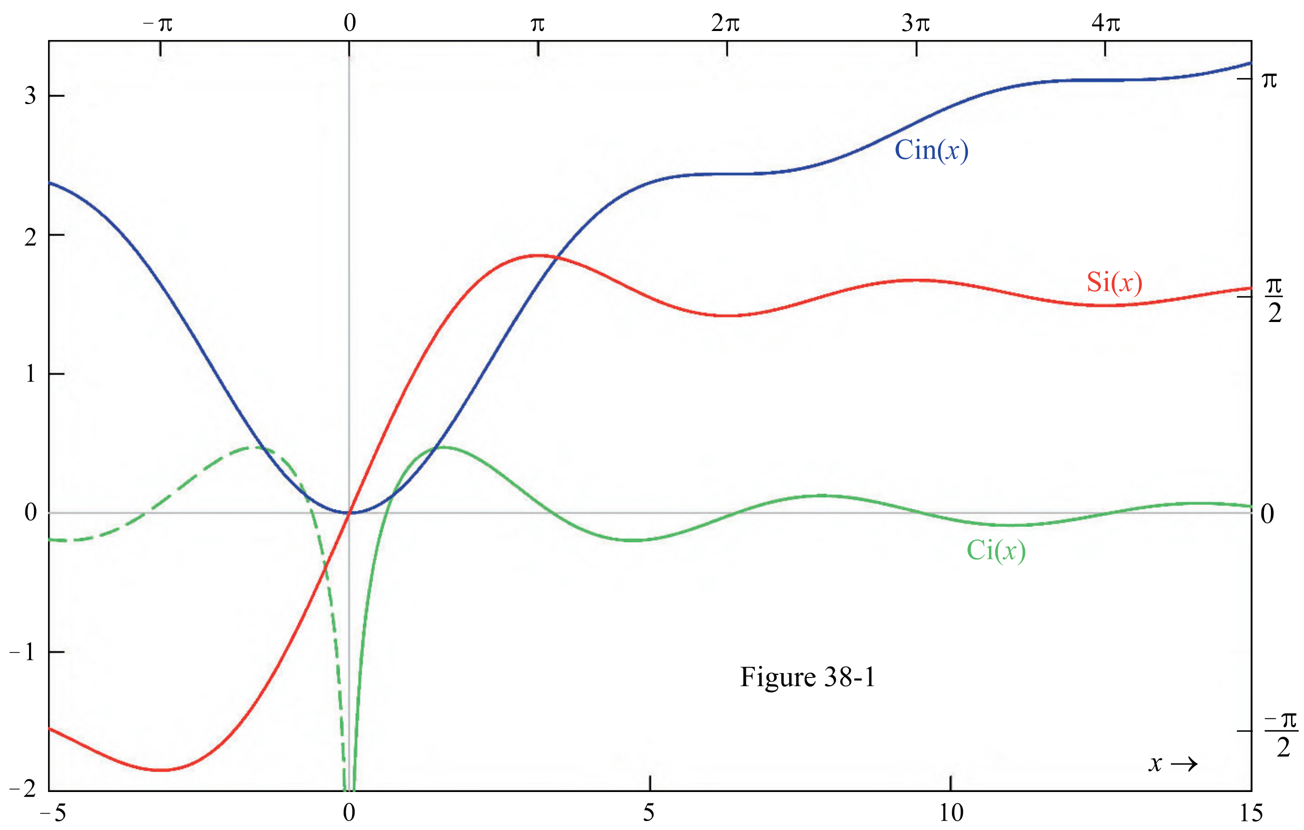


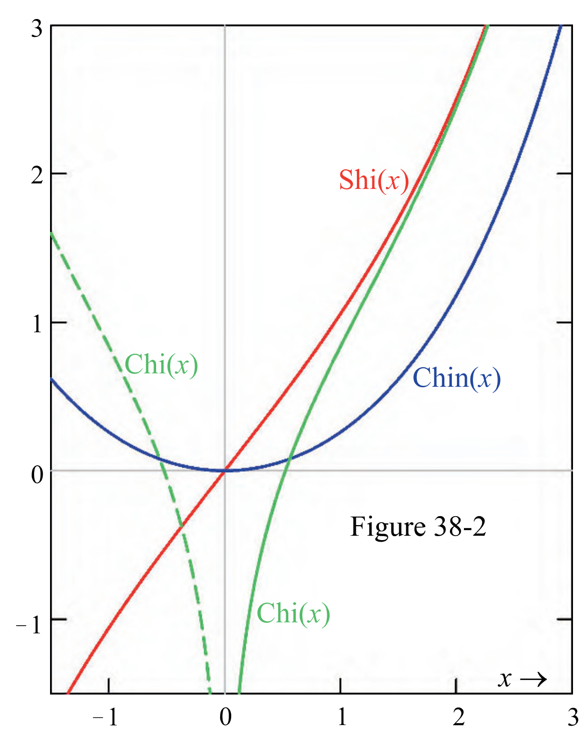
38:2 BEHAVIOR

The damped oscillatory behavior of the Ci and Si functions is evident in Figure 38-1. Respectively, these functions tend towards zero and $\pm\pi/2$ as their arguments approach $\pm\infty$. Ci suffers a discontinuity at an argument of zero. Having a value of zero at $x = 0$, Cin proceeds logarithmically towards infinity as the argument increases, through a series of plateaus.

Figure 38-2 maps the hyperbolic integrals Shi, Chi, and Chin. Both Shi(x) and Chi(x) converge to $\frac{1}{2}\mathrm{Ei}(x)$ as x increases [Ei is the exponential integral of Chapter 37]. Chi(x) resembles Ci(x) in having a discontinuity at $x = 0$, whereas Chin(x) displays a null minimum at this argument.

For $x < 0$, the Ci and Chi functions are complex, with an argument-independent imaginary component of πi. It is for this reason that the negative-x branches of these functions are shown as dashed lines in the figures. The dashed lines represent the *real* parts of the complex-valued functions.

For positive arguments, the difference $\frac{1}{2}\pi - \mathrm{Si}(x)$ crops up in relationships more frequently than does Si(x) itself. For example, when the square of this difference is added to $\mathrm{Ci}^2(x)$, the oscillatory nature of the two functions suffers mutual annihilation, as equation 38:13:9 demonstrates.



38:3 DEFINITIONS

Any of the following integrals define the sine integral function

38:3:1 $$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\mathrm{d}t = \frac{\pi}{2} - \int_x^\infty \frac{\sin(t)}{t}\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi x} \mathrm{sinc}(t)\mathrm{d}t$$

Note the negative sign in the following definition of the cosine integral function

38:3:2 $$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos(t)}{t}\mathrm{d}t$$

The entire cosine integral is defined by equation 38:0:1 or by the integral

38:3:3 $$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos(t)}{t}\mathrm{d}t$$

The condition $x \geq 0$ must be attached to definition 38:3:2 (and also to the integral definition in 38:3:5) to ensure that the function thereby defined is real.

In addition to integral definitions analogous to those of their circular counterparts, the hyperbolic Shi, Chi, and Chin functions may be defined through the exponential integral function of Chapter 37:

38:3:4 $$\mathrm{Shi}(x) = \int_0^x \frac{\sinh(t)}{t}\mathrm{d}t = \frac{\mathrm{Ei}(x) - \mathrm{Ei}(-x)}{2}$$

38:3:5 $$\mathrm{Chi}(x) = \int_{x_0}^x \frac{\cosh(t)}{t}\mathrm{d}t = \frac{\mathrm{Ei}(x) + \mathrm{Ei}(-x)}{2} \qquad x_0 = 0.52382\;25713\;89864$$

38:3:6 $$\mathrm{Chin}(x) = \int_0^x \frac{\cosh(t)-1}{t}\mathrm{d}t = \frac{-[\mathrm{Ein}(x) + \mathrm{Ein}(-x)]}{2}$$

The x_0 in integral 38:3:5 is the zero of the Chi function. Analogous to the final terms in equations 38:3:4−6, exponential integrals of imaginary argument serve to define the three circular functions through the formulas

38:3:7 $$\mathrm{Si}(x) = \frac{\mathrm{Ei}(ix) - \mathrm{Ei}(-ix)}{2i} - \mathrm{sgn}(x)\frac{\pi}{2} = -i\,\mathrm{Shi}(ix)$$

38:3:8 $$\mathrm{Ci}(x) = \frac{\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)}{2} = \mathrm{Chi}(ix) - \frac{\pi i}{2} \qquad x \geq 0$$

and

38:3:9 $$\mathrm{Cin}(x) = \frac{\mathrm{Ein}(ix) + \mathrm{Ein}(-ix)}{2} = \frac{\pi i}{2} - \mathrm{Chin}(ix) \qquad x \geq 0$$

The last three equations also provide representations of the circular functions in terms of their hyperbolic congeners of imaginary argument.

Being $L = K + 2 = 3$ hypergeometric functions [see equation 38:6:1,2], the Si, Shi, Cin, and Chin functions may be synthesized [Section 43:14] by the multistep processes

38:3:10 $$\mathrm{I}_0\left(2\sqrt{x}\right) \xrightarrow[3/2]{1} \frac{1}{2\sqrt{x}}\sinh\left(2\sqrt{x}\right) \xrightarrow[3/2]{1/2} \frac{1}{2\sqrt{x}}\mathrm{Shi}\left(2\sqrt{x}\right)$$

38:3:11 $$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[3/2]{1} \frac{1}{2\sqrt{x}}\sin\left(2\sqrt{x}\right) \xrightarrow[3/2]{1/2} \frac{1}{2\sqrt{x}}\mathrm{Si}\left(2\sqrt{x}\right)$$

38:3:12 $$\mathrm{I}_0\left(2\sqrt{x}\right) \xrightarrow[2]{1} \frac{1}{\sqrt{x}}\mathrm{I}_1\left(2\sqrt{x}\right) \xrightarrow[2]{1} \frac{1}{2\sqrt{x}}\left[\mathrm{I}_0\left(2\sqrt{x}\right)-1\right] \xrightarrow[3/2]{1/2} \frac{1}{x}\mathrm{Chin}\left(2\sqrt{x}\right)$$

38:3:13 $$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[2]{1} \frac{1}{\sqrt{x}}\mathrm{J}_1\left(2\sqrt{x}\right) \xrightarrow[2]{1} \frac{1}{2\sqrt{x}}\left[1-\mathrm{J}_0\left(2\sqrt{x}\right)\right] \xrightarrow[3/2]{1/2} \frac{1}{x}\mathrm{Chi}\left(2\sqrt{x}\right)$$

from the prototypical cylinder functions J_0 and I_0.

38:4 SPECIAL CASES

There are none.

38:5 INTRARELATIONSHIPS

Like the sine function itself, the sine integrals, circular and hyperbolic, are odd

38:5:1 $$\mathrm{f}(-x) = -\mathrm{f}(x) \qquad \mathrm{f} = \mathrm{Si} \text{ or } \mathrm{Shi}$$

whereas, emulating $\cos(x)$, the two entire cosine integrals are even

38:5:2 $$\mathrm{f}(-x) = \mathrm{f}(x) \qquad \mathrm{f} = \mathrm{Cin} \text{ or } \mathrm{Chin}$$

Expressions linking the Ci and Si functions to the auxiliary functions detailed in Section 38:13 are:

38:5:3 $$\mathrm{Ci}(x) = \sin(x)\,\mathrm{fi}(x) - \cos(x)\,\mathrm{gi}(x) \qquad x \geq 0$$

38:5:4 $$\mathrm{Si}(x) = \tfrac{1}{2}\pi - \cos(x)\,\mathrm{fi}(x) - \sin(x)\,\mathrm{gi}(x) \qquad x \geq 0$$

38:6 EXPANSIONS

The sine integrals, circular and hyperbolic, have the following power series expansions

38:6:1 $$\begin{matrix}\mathrm{Shi}\\ \mathrm{Si}\end{matrix}(x) = x \pm \frac{x^3}{18} + \frac{x^5}{600} \pm \frac{x^7}{35280} + \cdots = x\sum_{j=0}^{\infty}\frac{\left(\pm x^2\right)^j}{(2j+1)!(2j+1)} = x\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_j}{(1)_j\left(\tfrac{3}{2}\right)_j\left(\tfrac{3}{2}\right)_j}\left(\frac{\pm x^2}{4}\right)^j$$

Likewise the power series expansions for the two entire cosine integrals are

38:6:2 $$\begin{matrix}\mathrm{Chin}\\ \mathrm{Cin}\end{matrix}(x) = \frac{x^2}{4} \pm \frac{x^4}{96} + \frac{x^6}{4320} \pm \frac{x^8}{322560} + \cdots = -\sum_{j=1}^{\infty}\frac{(\pm x^2)^j}{(2j)!2j} = \frac{x^2}{4}\sum_{j=0}^{\infty}\frac{(1)_j}{\left(\tfrac{3}{2}\right)_j(2)_j(2)_j}\left(\frac{\pm x^2}{4}\right)^j$$

but series 38:8:1 and 38:8:2 will generally converge more rapidly.

The Ci and Chi functions do not have simple power series, but equations 38:0:1 and 38:0:2 may be combined with 38:6:2 to construct the expansions

38:6:3 $$\begin{matrix}\mathrm{Chi}\\ \mathrm{Ci}\end{matrix}(x) = \gamma + \ln(x) \pm \frac{x^2}{4} + \frac{x^4}{96} \pm \frac{x^6}{4320} + \frac{x^8}{322560} \pm \cdots \qquad x \geq 0$$

The sine integral function is also expansible in terms of spherical Bessel functions [Section 32:13]

38:6:4 $$\mathrm{Si}(x) = x\sum_{n=0}^{\infty}\mathrm{j}_n^2\left(\tfrac{1}{2}x\right) = \pi\sum_{n=0}^{\infty}\mathrm{J}_{n+\frac{1}{2}}^2\left(\tfrac{1}{2}x\right)$$

The asymptotic expansions

38:6:5 $$\mathrm{Ci}(x) \sim \sum_{j=0}\left[\frac{\sin(x)}{x} - (2j+1)\frac{\cos(x)}{x^2}\right]\frac{(2j)!}{(-x^2)^j} \qquad \text{large positive } x$$

and

38:6:6 $$\mathrm{Si}(x) \sim \frac{\pi}{2} - \sum_{j=0}\left[\frac{\cos(x)}{x} + (2j+1)\frac{\sin(x)}{x^2}\right]\frac{(2j)!}{(-x^2)^j} \qquad \text{large positive } x$$

arise from combining equations 38:5:3,4 with expansions 38:13:5,6 of the auxiliary sine and cosine integral functions.

38:7 PARTICULAR VALUES

The Si, Cin, Shi, and Chin functions are zero at $x = 0$, which is where, as x diminishes, Ci and Chi encounter discontinuities and become complex. The positive zero of the Chi(x) function lies at $x = 0.52382\ 25713\ 89864$. Ci($x$) has an infinite number of zeros, the earliest being at $x = 0.61650\ 54856\ 20716$; these are not evenly spaced but later instances do lie close to the inflections of this function, which are located as specified in equation 38:7: 3.

The sine and cosine integrals have extrema, as well as points of inflection, at the following argument values

38:7:1 $$\mathrm{Si}(x) \text{ has} \begin{cases} \text{local maxima at } x = +\pi, -2\pi, +3\pi, -4\pi, \cdots \\ \text{inflections at } x = 0, \pm r_1(1), \pm r_2(1), \pm r_3(1), \cdots \\ \text{local minima at } x = -\pi, +2\pi, -3\pi, +4\pi, \cdots \end{cases}$$

38:7:2 $$\mathrm{Ci}(x) \text{ has} \begin{cases} \text{local maxima at } x = \tfrac{1}{2}\pi, \tfrac{5}{2}\pi, \tfrac{9}{2}\pi, \tfrac{13}{2}\pi, \cdots \\ \text{inflections at } x = \rho_1(-1), \rho_2(-1), \rho_3(-1), \rho_4(-1), \cdots \\ \text{local minima at } x = \tfrac{3}{2}\pi, \tfrac{7}{2}\pi, \tfrac{11}{2}\pi, \tfrac{15}{2}\pi, \cdots \end{cases}$$

In these equations $r_j(1)$ is the jth positive root of the equation $\tan(x) = x$ and $\rho_j(-1)$ is the jth positive root of $\cot(x) = -x$ [Section 34:7].

In addition to its minimum at $x = 0$, the Cin(x) function has an infinite number of horizontal inflections. These are regularly spaced, thus:

38:7:3 $$\frac{d}{dx}\mathrm{Cin}(x) = \frac{d^2}{dx^2}\mathrm{Cin}(x) = 0 \qquad x = \pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots$$

38:8 NUMERICAL VALUES

Equator provides routines (keywords **Shi**, **Chi** and **Chin**) for the hyperbolic sine integral, hyperbolic cosine integral and entire hyperbolic cosine integral. The algorithms are based on equations 38:3:4−6, with 38:6:1 and 38:6:2 being used close to $x = 0$.

Equator's routine (keyword **Ci**) for calculating the cosine integral utilizes series 38:6:3 for arguments not exceeding 16. The entire cosine integral (keyword **Cin**) is based on 38:6:2 for small arguments, but a truncated version of the series

38:8:1 $$\mathrm{Cin}(x)=\frac{x-\sin(x)}{x}+x^2\sum_{j=0}^{\infty}\frac{(-x^2)^j}{(2j+3)!(2j+2)}$$

is used for $0.25 \le x \le 16$. Similarly, the routine (keyword **Si**) adopted by *Equator* for calculating the sine integral uses 38:6:1 for small arguments, and the series

38:8:2 $$\mathrm{Si}(x)=\frac{2-\cos(x)}{x}-\frac{\sin(x)}{x^2}+2x\sum_{j=0}^{\infty}\frac{(-x^2)^j}{(2j+3)!(2j+1)}$$

when $0.15 \le x \le 16$. For arguments in excess of 16, the Ci, Cin, and Si routines rely on formulas 38:5:3, 38:0:1, and 38:5:4, and on the algorithms described in Section 38:13.

38:9 LIMITS AND APPROXIMATIONS

As the argument approaches zero (from either direction in the first and third cases), similar limits are attained by both the circular and the hyperbolic functions. These lead to the approximations

38:9:1 $$\begin{matrix}\mathrm{Shi}\\\mathrm{Si}\end{matrix}(x)\approx x \qquad \text{small } x$$

38:9:2 $$\begin{matrix}\mathrm{Chi}\\\mathrm{Ci}\end{matrix}(x)\approx \ln(x)+\gamma \qquad \text{small positive } x$$

38:9:3 $$\begin{matrix}\mathrm{Chin}\\\mathrm{Cin}\end{matrix}(x)\approx \tfrac{1}{4}x^2 \qquad \text{small positive } x$$

For large arguments, the approximations

38:9:4 $$\mathrm{Si}(x)\approx\frac{\pi}{2}-\left(\frac{1}{x}-\frac{2}{x^3}\right)\cos(x)-\left(\frac{1}{x^2}-\frac{6}{x^4}\right)\sin(x) \qquad \text{large positive } x$$

and

38:9:5 $$\mathrm{Ci}(x)\approx\left(\frac{1}{x}-\frac{2}{x^3}\right)\sin(x)-\left(\frac{1}{x^2}-\frac{6}{x^4}\right)\cos(x) \qquad \text{large positive } x$$

are surprisingly accurate.

38:10 OPERATIONS OF THE CALCULUS

The derivatives and indefinite integrals of the six functions are listed below:

$f(x) =$	$\mathrm{Si}(x)$	$\mathrm{Ci}(x)$	$\mathrm{Cin}(x)$	$\mathrm{Shi}(x)$	$\mathrm{Chi}(x)$	$\mathrm{Chin}(x)$
$\frac{d}{dx}f(x) =$	$\frac{\sin(x)}{x}$	$\frac{\cos(x)}{x}$	$\frac{1-\cos(x)}{x}$	$\frac{\sinh(x)}{x}$	$\frac{\cosh(x)}{x}$	$\frac{\cosh(x)-1}{x}$
$\int_0^x f(t)\,dt =$	$x\mathrm{Si}(x)-1+\cos(x)$	$x\mathrm{Ci}(x)-\sin(x)$	$x\mathrm{Cin}(x)-x+\sin(x)$	$x\mathrm{Shi}(x)+1-\cosh(x)$	$x\mathrm{Chi}(x)-\sinh(x)$	$x\mathrm{Chin}(x)+x-\sinh(x)$

Entries in the final row are accessible through the standard formula 37:14:2. The six functions of this chapter are ideal candidates for the procedures exposed in Section 37:14, which may be used to generate expressions for

38:10:1 $$\int_0^x t\,\mathrm{f}(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^x\int_0^{t'} \mathrm{f}(t)\,\mathrm{d}t\,\mathrm{d}t' \qquad \mathrm{f}=\mathrm{Si},\mathrm{Ci},\mathrm{Cin},\mathrm{Shi},\mathrm{Chi},\text{or}\,\mathrm{Chin}$$

The following indefinite integrals,

38:10:2 $$\int_0^x \sin(t)\mathrm{Si}(bt)\,\mathrm{d}t = -\cos(x)\mathrm{Si}(bx) + \tfrac{1}{2}\mathrm{Si}\{(1+b)x\} - \tfrac{1}{2}\mathrm{Si}\{(1-b)x\}$$

38:10:3 $$\int_0^x \cos(t)\mathrm{Si}(bt)\,\mathrm{d}t = \sin(x)\mathrm{Si}(bx) + \tfrac{1}{2}\mathrm{Ci}\{|1+b|x\} - \tfrac{1}{2}\mathrm{Ci}\{|1-b|x\} - \tfrac{1}{2}\ln\left\{\left|\frac{b+1}{b-1}\right|\right\}$$

38:10:4 $$\int_x^\infty \sin(t)\mathrm{Ci}(bt)\,\mathrm{d}t = \cos(x)\mathrm{Ci}(bx) - \tfrac{1}{2}\mathrm{Ci}\{(1+b)x\} - \tfrac{1}{2}\mathrm{Ci}\{|1-b|x\} \qquad x>0, \quad b>0$$

and

38:10:5 $$\int_x^\infty \cos(t)\mathrm{Ci}(bt)\,\mathrm{d}t = -\sin(x)\mathrm{Ci}(bx) + \tfrac{1}{2}\mathrm{Si}\{(1+b)x\} + \tfrac{1}{2}\mathrm{Si}\{(1-b)x\} - \big(1+\mathrm{sgn}(1-b)\big)\frac{\pi}{4} \qquad x>0, \quad b>0$$

may be by supplemented by others listed by Gradsheyn and Ryzhik [Section 5.3]. As well, these authors present [Sections 6.26 and 6.27] many definite integrals of the Si and Ci functions, as well as their hyperbolic analogues.

An intriguing definite integral is

38:10:6 $$\int_0^\infty \mathrm{Ci}(bt)\mathrm{Ci}(\beta t)\,\mathrm{d}t = \begin{cases} \pi/2b & b \ge \beta \\ \pi/2\beta & \beta \ge b \end{cases}$$

The following Laplace transforms provide links to functions from Chapters 25 and 34:

38:10:7 $$\int_0^\infty \mathrm{Ci}(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{Ci}(bt)\} = \frac{-1}{2s}\ln\left(1+\frac{s^2}{b^2}\right)$$

38:10:8 $$\int_0^\infty \mathrm{Si}(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{Si}(bt)\} = \frac{1}{s}\mathrm{arccot}\left(\frac{s}{b}\right)$$

38:11 COMPLEX ARGUMENT

Figures 38-3 and 38-4 show the real and imaginary parts of the functions Si(x+iy) and Ci(x+iy). The most noteworthy aspects of these three-dimensional graphics are the pole evident in Re{Ci(x+iy)} and the cut along the negative real axis in Im{Ci(x+iy)}.

When the argument of the sine and cosine integrals is complex, these functions are described by the formulas

38:11:1 $$\mathrm{Si}(x+iy) = (x+iy)\sum_{j=0}^{\infty}\frac{\{-(x+iy)^2\}^j}{(2j+1)!(2j+1)}$$

38:11:2 $$\mathrm{Ci}(x+iy) = \gamma + \ln(x+iy) + \sum_{j=1}^{\infty}\frac{\{-(x+iy)^2\}^j}{(2j)!(2j)}$$

Figure 38-3

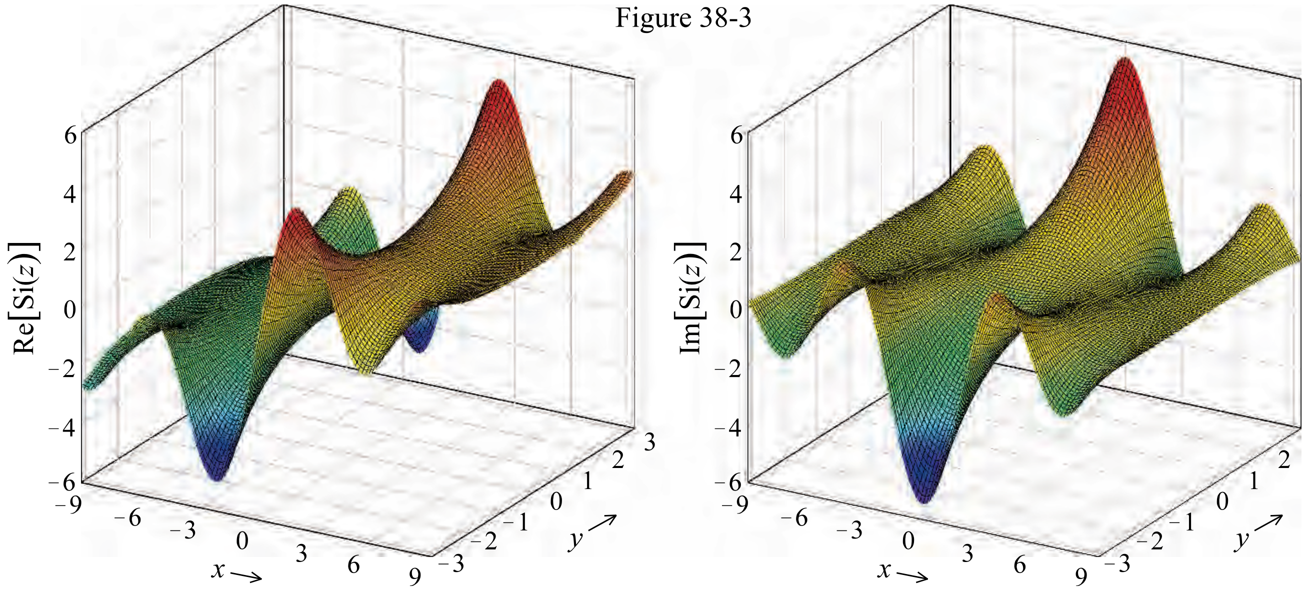
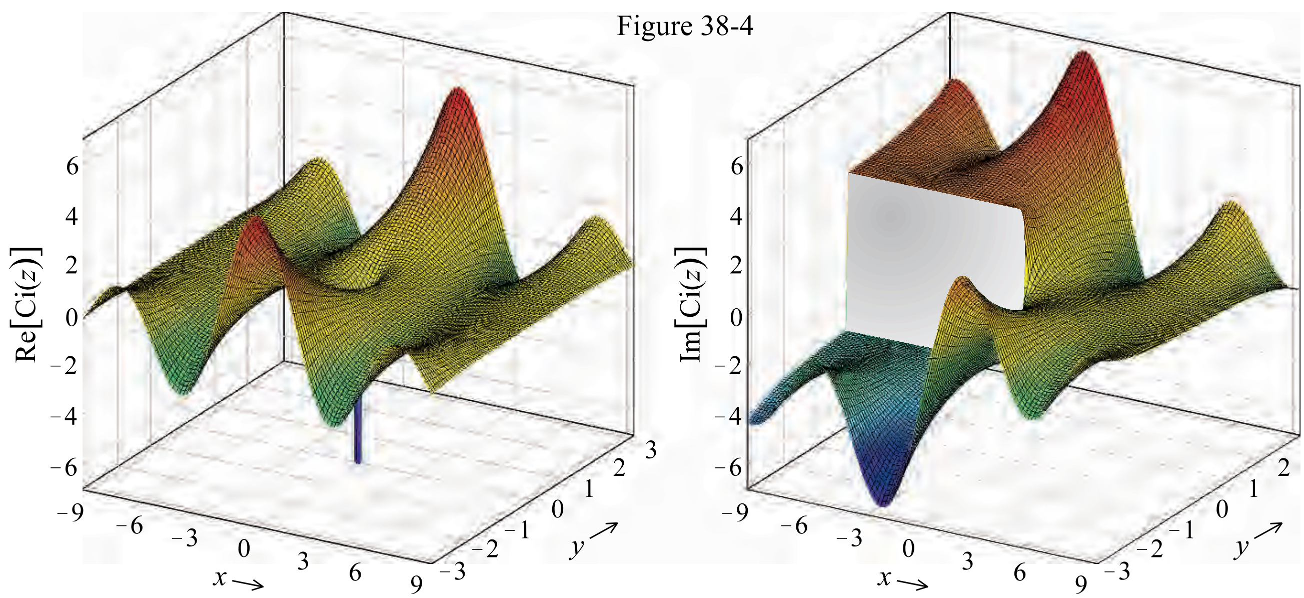


Figure 38-4



where $\ln(x+iy)$ is to be interpreted as $\frac{1}{2}\ln(x^2+y^2)+\mathrm{sgn}(y)i\,\mathrm{arccot}(x/|y|)$. The other four functions of this chapter are similarly constructed by replacing the x in the equations of Section 38:6 by $x+iy$. The expressions do not readily separate into real and imaginary parts.

Formulas for purely imaginary argument are:

f =	Si	Ci	Cin	Shi	Chi	Chin
$f(iy) =$	$i\,\mathrm{Shi}(y)$	$\mathrm{Chi}(-y)-\dfrac{\pi i}{2}$	$-\mathrm{Chin}(-y)-\dfrac{\pi i}{2}$	$i\,\mathrm{Si}(y)$	$\mathrm{Ci}(y)+\dfrac{\pi i}{2}$	$\dfrac{\pi i}{2}-\mathrm{Cin}(y)$

38:12 GENERALIZATIONS

Erdélyi et al. [*Higher Transcendental Functions*, Volume 2, page 147] cite the function

38:12:1 $$\int_0^x \frac{\sin\left(\sqrt{a^2+t^2}\right)}{\sqrt{a^2+t^2}}\mathrm{d}t = \int_a^{\sqrt{a^2+x^2}} \frac{\sin(t)}{\sqrt{t^2-a^2}}\mathrm{d}t$$

of which Si(x) is the $a = 0$ special case.

A generalization in a different direction is provided by *Böhmer integrals*. These bivariate functions, which are the subject of Section 39:12, devolve into sine and cosine integrals when their parameter is zero. One has

38:12:2 $$\mathrm{Si}(x) = \tfrac{1}{2}\pi - \mathrm{S}(0,x) \qquad \text{and} \qquad \mathrm{Ci}(x) = -\mathrm{C}(0,x)$$

38:13 COGNATE FUNCTIONS: the auxiliary sine and cosine integrals

The so-called *auxiliary sine integral* is defined, for nonnegative real argument x, by the alternative definite integrals

38:13:1 $$\mathrm{fi}(x) = \int_0^\infty \frac{\sin(t)}{t+x}\mathrm{d}t = \int_0^\infty \frac{\exp(-xt)}{t^2+1}\mathrm{d}t$$

of which the latter may be regarded as a Laplace transform. Similarly, the *auxiliary cosine integral* function has the definitions

38:13:2 $$\mathrm{gi}(x) = \int_0^\infty \frac{\cos(t)}{t+x}\mathrm{d}t = \int_0^\infty \frac{t\exp(-xt)}{t^2+1}\mathrm{d}t$$

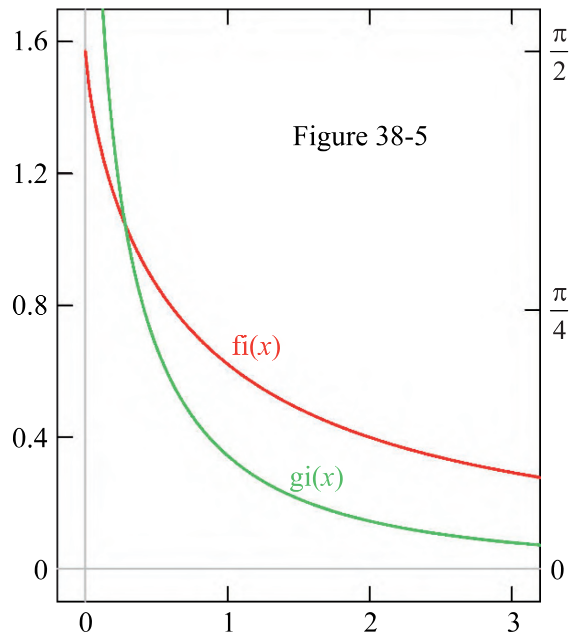


Figure 38-5 maps these auxiliary functions. Note that fi(x) takes a finite value, $\pi/2$, at $x = 0$, whereas gi(0) is infinite. Despite this, the integral

38:13:3 $$\int_0^\infty \mathrm{gi}(t)\mathrm{d}t = \frac{\pi}{2}$$

is finite, whereas the corresponding fi integral fails to converge.

The differential identities

38:13:4 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fi}(x) = -\mathrm{gi}(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{gi}(x) = \mathrm{fi}(x) - \frac{1}{x}$$

show that the two auxiliary functions provide the particular integrals [Section 24:14] in the solution of the following inhomogeneous differential equations:

38:13:5 $$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = \frac{1}{x} \qquad \mathrm{f}(x) = w_1\cos(x) + w_2\sin(x) + \mathrm{fi}(x)$$

38:13:6 $$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f}(x) + \mathrm{f}(x) = \frac{1}{x^2} \qquad \mathrm{f}(x) = w_1\cos(x) + w_2\sin(x) + \mathrm{gi}(x)$$

The auxiliary nature of these functions, and hence their names, arises from the relationships to the Si and Ci functions:

38:13:7 $$\mathrm{fi}(x)=\sin(x)\,\mathrm{Ci}(x)+\cos(x)\left[\tfrac{1}{2}\pi-\mathrm{Si}(x)\right]\qquad x\geq 0$$

38:13:8 $$\mathrm{gi}(x)=\sin(x)\left[\tfrac{1}{2}\pi-\mathrm{Si}(x)\right]-\cos(x)\,\mathrm{Ci}(x)\qquad x\geq 0$$

and thereby

38:13:9 $$\mathrm{fi}^2(x)+\mathrm{gi}^2(x)=\left[\frac{\pi}{2}-\mathrm{Si}(x)\right]^2+\mathrm{Ci}^2(x)$$

Relations 38:13:7 and 38:13:8, together with equations 32:6:1,2 and 38:6:1,3, enable convergent series to be constructed for the fi and gi functions. More important than these, however, are the asymptotic series

38:13:10 $$\mathrm{fi}(x)\sim\frac{1}{x}-\frac{2}{x^3}+\frac{24}{x^5}-\frac{720}{x^7}+\cdots+\frac{(2j)!}{x(-x^2)^j}+\cdots=\frac{1}{x}\sum_{j=0}(\tfrac{1}{2})_j(1)_j\left(\frac{-4}{x^2}\right)^j\qquad x\to\infty$$

and

38:13:11 $$\mathrm{gi}(x)\sim\frac{1}{x^2}-\frac{6}{x^4}+\frac{120}{x^6}-\frac{5040}{x^8}+\cdots+\frac{(2j+1)!}{x^2(-x^2)^j}+\cdots=\frac{1}{x^2}\sum_{j=0}(1)_j(\tfrac{3}{2})_j\left(\frac{-4}{x^2}\right)^j\qquad x\to\infty$$

into which the auxiliary sine and cosine functions may be developed.

Series 38:13:10 and 38:13:11 are asymptotic but, nevertheless, employment of the ε-transformation [Section 10:14] enables their summation to be used by *Equator* for $x > 16$. This is the approach used by the auxiliary sine integral and auxiliary cosine integral routines, under the keywords **fi** and **gi**. For $x \leq 16$ equations 38:13:7 and 38:13:8 are utilized.

CHAPTER 39

THE FRESNEL INTEGRALS C(x) AND S(x)

Named for the French physicist Augustin Jean Fresnel (1788 – 1829), the Fresnel cosine integral C(x) and the Fresnel sine integral S(x) play important roles in physical optics. The so-called auxiliary Fresnel integrals [Section 39:13] are also of some importance.

39:1 NOTATION

Though the names *Fresnel cosine integral* and *Fresnel sine integral* and the corresponding symbols C(x) and S(x) are general, there is a wide variation in the meaning to be associated with them. Thus, in addition to the definition, equation 39:3:1, given to C(x) in this *Atlas*, each of the following integrals is cited as the "Fresnel cosine integral" by at least one authority

39:1:1
$$\int_0^x \cos(t^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{2}}\,\mathrm{C}(x)$$

39:1:2
$$\int_0^x \frac{\cos(t)}{\sqrt{2\pi t}}\,\mathrm{d}t = \mathrm{C}\left(\sqrt{x}\right) \qquad x \ge 0$$

39:1:3
$$\int_0^x \cos\left(\frac{\pi t^2}{2}\right)\mathrm{d}t = \mathrm{C}\left(\sqrt{\frac{\pi}{2}}\,x\right)$$

39:1:4
$$\int_0^x \frac{\cos(t)}{2\sqrt{xt}}\,\mathrm{d}t = \sqrt{\frac{\pi}{2x}}\,\mathrm{C}\left(\sqrt{x}\right) \qquad x \ge 0$$

The left-hand side of each of the four equations above are the four alternative definitions of the "Fresnel cosine integral C(x)"; the right-hand side shows how this would be represented in our notation. In all cases, the definition adopted for the Fresnel sine integral involves straightforward replacement of cos in the integrand by sin.

Some authors adopt multiple definitions of these functions and use subscripts – as in $\mathrm{C}_1(x)$, $\mathrm{S}_2(x)$, etc. – to distinguish the options. There is, however, no greater unanimity in this secondary notation than in the primary.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_40, © Springer Science+Business Media, LLC 2009

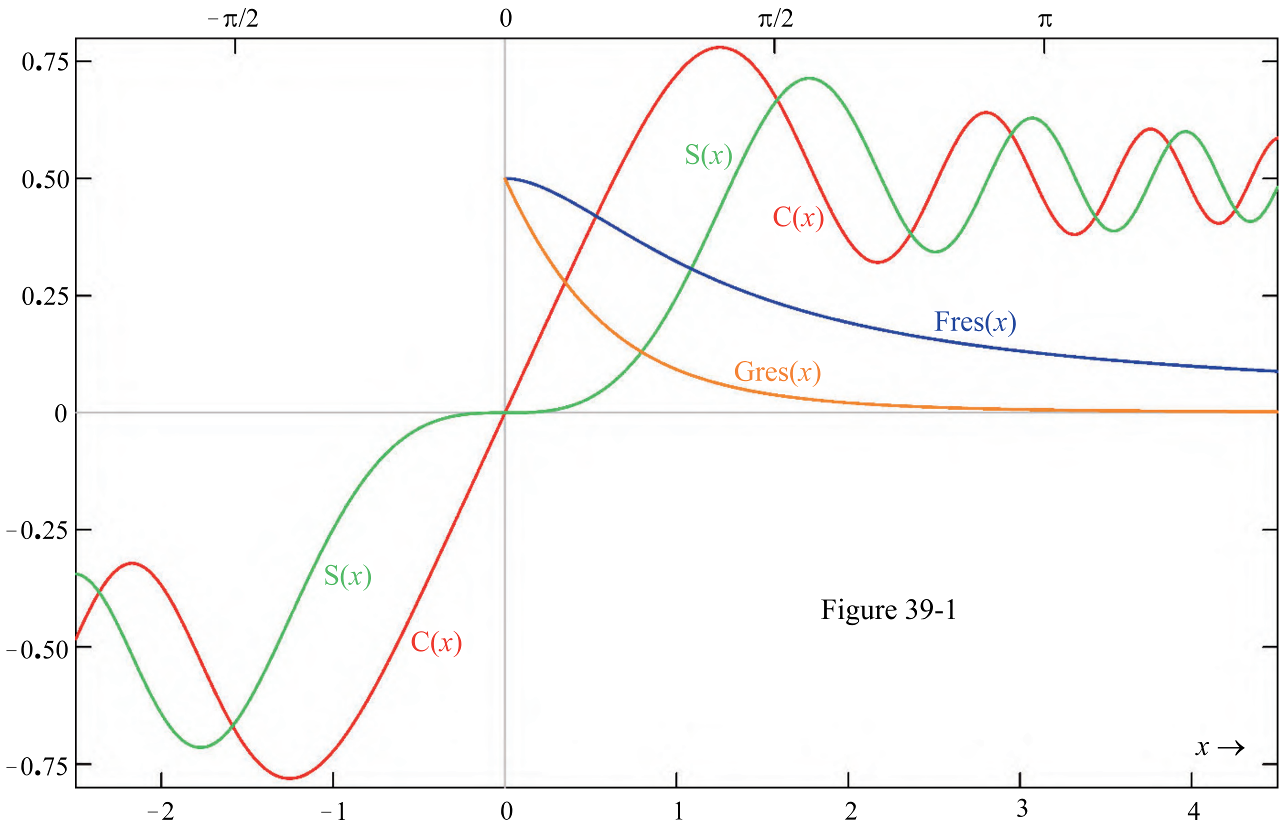


Figure 39-1

39:2 BEHAVIOR

Figure 39-1 maps the behaviors of the two functions. Outside a central zone, both C(x) and S(x) display damped oscillations that decrease in both period and amplitude as $x \to \pm\infty$, approaching the values ±½ in these limits.

Being odd functions, the Fresnel integrals lie solely within the first and third quadrants. In the first quadrant, the functions

39:2:1
$$\begin{matrix}\text{lower}\\ \text{bound}\end{matrix} = \frac{1}{2} - \mathrm{Fres}(x) - \mathrm{Gres}(x) \qquad \text{and} \qquad \begin{matrix}\text{upper}\\ \text{bound}\end{matrix} = \frac{1}{2} + \mathrm{Fres}(x) + \mathrm{Gres}(x)$$

delineate an envelope within which both the C(x) and S(x) functions exist. The oscillations of the Fresnel integrals are nullified by the operation

39:2:2
$$\sin^2\left(x^2\right)\left[\mathrm{C}(x) - \tfrac{1}{2}\right]^2 + \cos^2\left(x^2\right)\left[\mathrm{S}(x) - \tfrac{1}{2}\right]^2 = \mathrm{Fres}^2(x) + \mathrm{Gres}^2(x) \qquad x > 0$$

Fres and Gres are auxiliary functions portrayed in Figure 39-1 and discussed in Section 39:13.

39:3 DEFINITIONS

As their names imply, the Fresnel integrals are primarily defined as indefinite integrals:

39:3:1
$$\mathrm{C}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos\left(t^2\right)\mathrm{d}t = \frac{\mathrm{sgn}(x)}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos(t)}{\sqrt{t}}\mathrm{d}t$$

and

39:3:2 $$\mathrm{S}(x)=\sqrt{\frac{2}{\pi}}\int_0^x \sin\left(t^2\right)\mathrm{d}t=\frac{\mathrm{sgn}(x)}{\sqrt{2\pi}}\int_0^{x^2}\frac{\sin(t)}{\sqrt{t}}\mathrm{d}t$$

Equations 39:6:1 and 39:6:2 demonstrate that the Fresnel integrals may be represented hypergeometrically, which implies that they are amenable to synthesis. The synthetic routes are

39:3:3 $$\mathrm{J}_0\left(2\sqrt{x}\right)\xrightarrow[1/2]{1}\cos\left(2\sqrt{x}\right)\xrightarrow[5/4]{1/4}\sqrt{\frac{\pi}{4\sqrt{x}}}\,\mathrm{C}\left(\sqrt{2\sqrt{x}}\right)$$

39:3:4 $$\mathrm{J}_0\left(2\sqrt{x}\right)\xrightarrow[3/2]{1}\frac{1}{2\sqrt{x}}\sin\left(2\sqrt{x}\right)\xrightarrow[7/4]{3/4}\frac{3}{4}\sqrt{\frac{\pi}{\sqrt{x^3}}}\,\mathrm{S}\left(\sqrt{2\sqrt{x}}\right)$$

See Sections 39:11 and 42:14 for definitions based on an exponential function of complex argument.

39:4 SPECIAL CASES

There are none.

39:5 INTRARELATIONSHIPS

Both Fresnel integrals are odd functions

39:5:1 $$\mathrm{f}(-x)=-\mathrm{f}(x) \qquad \mathrm{f}=\mathrm{C}\text{ or }\mathrm{S}$$

Their relation to the auxiliary Fresnel integrals [Section 39:13] is given by:

39:5:2 $$\mathrm{C}(x)=\mathrm{sgn}(x)\left[\tfrac{1}{2}+\sin(x^2)\mathrm{Fres}(|x|)-\cos(x^2)\mathrm{Gres}(|x|)\right]$$

39:5:3 $$\mathrm{S}(x)=\mathrm{sgn}(x)\left[\tfrac{1}{2}-\cos(x^2)\mathrm{Fres}(|x|)-\sin(x^2)\mathrm{Gres}(|x|)\right]$$

39:6 EXPANSIONS

Power series expansions of the Fresnel integrals are

39:6:1 $$\mathrm{C}(x)=\sqrt{\frac{2}{\pi}}\left[x-\frac{x^5}{10}+\frac{x^9}{216}-\cdots\right]=\sqrt{\frac{2}{\pi}}\,x\sum_{j=0}^{\infty}\frac{(-x^4)^j}{(4j+1)(2j)!}=\sqrt{\frac{2}{\pi}}\,x\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)_j}{\left(\frac{1}{2}\right)_j(1)_j\left(\frac{5}{4}\right)_j}\left(\frac{-x^4}{4}\right)^j$$

39:6:2 $$\mathrm{S}(x)=\sqrt{\frac{2}{\pi}}\left[\frac{x^3}{3}-\frac{x^7}{42}+\frac{x^{11}}{1320}-\cdots\right]=\sqrt{\frac{2}{\pi}}\,x^3\sum_{j=0}^{\infty}\frac{(-x^4)^j}{(4j+3)(2j+1)!}=\sqrt{\frac{2}{\pi}}\frac{x^3}{3}\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j\left(\frac{7}{4}\right)_j}\left(\frac{-x^4}{4}\right)^j$$

Through relationships 39:5:2 and 39:5:3, the auxiliary functions from Section 39:13 permit the Fresnel integrals to be expanded as the power series

39:6:3 $$\mathrm{C}(x)=\sqrt{\frac{2}{\pi}}\left[x\cos(x^2)\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+1)!!}+2x^3\sin(x^2)\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+3)!!}\right]$$

39:6:4
$$\mathrm{S}(x)=\sqrt{\frac{2}{\pi}}\left[x\sin(x^2)\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+1)!!}-2x^3\cos(x^2)\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+3)!!}\right]$$

or alternatively as the asymptotic expansions

39:6:5
$$\mathrm{C}(x)\sim\frac{\mathrm{sgn}(x)}{2}+\frac{\sin(x^2)}{\sqrt{2\pi}\,x}\sum_{j=0}\frac{(4j-1)!!}{(-4x^4)^j}-\frac{\cos(x^2)}{\sqrt{8\pi}\,x^3}\sum_{j=0}\frac{(4j+1)!!}{(-4x^4)^j}\qquad \text{large } x$$

39:6:6
$$\mathrm{S}(x)\sim\frac{\mathrm{sgn}(x)}{2}-\frac{\cos(x^2)}{\sqrt{2\pi}\,x}\sum_{j=0}\frac{(4j-1)!!}{(-4x^4)^j}-\frac{\sin(x^2)}{\sqrt{8\pi}\,x^3}\sum_{j=0}\frac{(4j+1)!!}{(-4x^4)^j}\qquad \text{large } x$$

Fresnel integrals are expansible in terms of Bessel functions of half-odd order or, equivalently, spherical Bessel functions [Section 32:13]:

39:6:7
$$\mathrm{C}(x)=\mathrm{sgn}(x)\sum_{j=0}^{\infty}\mathrm{J}_{2j+\frac12}(x^2)\qquad\text{and}\qquad \mathrm{S}(x)=\mathrm{sgn}(x)\sum_{j=0}^{\infty}\mathrm{J}_{2j+\frac32}(x^2)$$

39:7 PARTICULAR VALUES

The Fresnel integrals have local maxima and minima, as well as points of inflection, at the argument values that follow

39:7:1
$$\mathrm{C}(x)\text{ has}\begin{cases}\text{maxima at } x=\sqrt{\tfrac12\pi},\,-\sqrt{\tfrac32\pi},\,\sqrt{\tfrac52\pi},\,-\sqrt{\tfrac72\pi},\,\sqrt{\tfrac92\pi},\cdots\\ \text{inflections at } x=0,\,\pm\sqrt{\pi},\,\pm\sqrt{2\pi},\,\pm\sqrt{3\pi},\,\pm\sqrt{4\pi},\cdots\\ \text{minima at } x=-\sqrt{\tfrac12 x},\,\sqrt{\tfrac32\pi},\,-\sqrt{\tfrac52\pi},\,\sqrt{\tfrac72\pi},\,-\sqrt{\tfrac92\pi},\cdots\end{cases}$$

39:7:2
$$\mathrm{S}(x)\text{ has}\begin{cases}\text{maxima at } x=\sqrt{\pi},\,-\sqrt{2\pi},\,\sqrt{3\pi},\,-\sqrt{4\pi},\,\sqrt{5\pi},\cdots\\ \text{inflections at } x=0,\,\pm\sqrt{\tfrac12\pi},\,\pm\sqrt{\tfrac32\pi},\,\pm\sqrt{\tfrac52\pi},\pm\sqrt{\tfrac72\pi},\cdots\\ \text{minima at } x=-\sqrt{\pi},\,\sqrt{2\pi},\,-\sqrt{3\pi},\,\sqrt{4\pi},\,-\sqrt{5\pi},\cdots\end{cases}$$

Some values of the extrema themselves are listed by Abramowitz and Stegun [Table 7.12, though recognize that these authors define Fresnel integrals differently than here]. As the argument increases towards $+\infty$, the values acquired by S(x) and C(x) at their extrema, and particularly at their inflections, move ever closer to ½.

39:8 NUMERICAL VALUES

With keywords **C** and **S**, *Equator* provides Fresnel cosine integral and Fresnel sine integral routines. For $|x|\le 0.5$, *Equator* uses equations 39:6:1 and 39:6:2. When the argument is in the range $0.5<|x|\le 4$, algorithms based on the equations

39:8:1
$$\mathrm{C}(x)=\frac{-1}{\sqrt{8\pi x^3}}\left[\cos\left(x^2\right)-2x^2\sin\left(x^2\right)+3\sum_{j=0}^{\infty}\frac{\left(-x^4\right)^j}{(2j)!(4j-3)}\right]$$

and

39:8:2 $$\mathrm{S}(x)=\frac{-1}{\sqrt{8\pi}x^3}\left[\sin\left(x^2\right)+2x^2\cos\left(x^2\right)+3x^2\sum_{j=0}^{\infty}\frac{\left(-x^4\right)^j}{(2j+1)!(4j-1)}\right]$$

are employed. Formulas 39:5:2 and 39:5:3 are used when $|x|>4$.

39:9 LIMITS AND APPROXIMATIONS

Close to $x=0$, the Fresnel integrals follow linear or cubic approximations:

39:9:1 $$\mathrm{C}(x)\approx\sqrt{\frac{2}{\pi}}\,x \qquad \text{small } x$$

39:9:2 $$\mathrm{S}(x)\approx\sqrt{\frac{2}{9\pi}}\,x^3 \qquad \text{small } x$$

Their limiting behaviors at large arguments of either sign are given by

39:9:3 $$\mathrm{C}(x)\to\frac{\mathrm{sgn}(x)}{2}+\frac{\sin(x^2)}{\sqrt{2\pi}\,x}-\frac{\cos(x^2)}{\sqrt{8\pi}\,x^3} \qquad x\to\pm\infty$$

39:9:4 $$\mathrm{S}(x)\to\frac{\mathrm{sgn}(x)}{2}-\frac{\cos(x^2)}{\sqrt{2\pi}\,x}-\frac{\sin(x^2)}{\sqrt{8\pi}\,x^3} \qquad x\to\pm\infty$$

The approximate relationship

39:9:5 $$\left[\mathrm{S}(x)-\tfrac{1}{2}\right]^2+\left[\mathrm{C}(x)-\tfrac{1}{2}\right]^2\approx\frac{1}{2\pi x^2} \qquad \text{large } x$$

is a consequence of equations 39:9:3 and 39:9:4 and has relevance to Cornu's spiral [Section 39:13].

39:10 OPERATIONS OF THE CALCULUS

Differentiation and indefinite integration of the Fresnel integrals lead to

39:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\,{\mathrm{C} \atop \mathrm{S}}(bx)=\sqrt{\frac{2}{\pi}}\,b\,{\cos \atop \sin}\left(b^2x^2\right)$$

39:10:2 $$\int_0^x \mathrm{C}(bt)\,\mathrm{d}t=x\mathrm{C}(bx)-\frac{\sin(b^2x^2)}{\sqrt{2\pi}\,b} \qquad \text{and} \qquad \int_0^x \mathrm{S}(bt)\,\mathrm{d}t=x\mathrm{S}(bx)-\frac{1-\cos(b^2x^2)}{\sqrt{2\pi}\,b}$$

The results in 39:10:2 above and 39:10:3 below stem from the procedures outlined in Section 37:14.

39:10:3 $$\int_0^x t\,\mathrm{C}(t)\,\mathrm{d}t=\frac{2x^2\mathrm{C}(x)+\mathrm{S}(x)}{4}-\frac{x\sin(x^2)}{\sqrt{8\pi}} \qquad \text{and} \qquad \int_0^x t\,\mathrm{S}(t)\,\mathrm{d}t=\frac{2x^2\,\mathrm{S}(x)-\mathrm{C}(x)}{4}+\frac{x\cos(x^2)}{\sqrt{8\pi}}$$

Among the definite integrals listed by Gradshteyn and Ryzhik [Section 6.32] are

39:10:4 $$\int_0^\infty t^{\nu}\left[\tfrac{1}{2}-\mathrm{S}(bt)\right]\mathrm{d}t=\frac{\Gamma(1+\frac{1}{2}\nu)\cos(\frac{1}{4}\nu\pi)}{\sqrt{2\pi}(1+\nu)b^{1+\nu}} \qquad b>0 \qquad -1<\nu<2$$

39:10:5 $$\int_0^\infty t^\nu\left[\mathrm{C}(bt)-\tfrac{1}{2}\right]\mathrm{d}t=\frac{\Gamma(1+\frac{1}{2}\nu)\sin(\frac{1}{4}\nu\pi)}{\sqrt{2\pi}(1+\nu)b^{1+\nu}} \qquad b>0 \qquad -1<\nu<2$$

and the surprising results

39:10:6 $$\int_0^\infty \mathrm{S}(bt)\sin\left(\beta t^2\right)\mathrm{d}t=\int_0^\infty \mathrm{C}(bt)\cos\left(\beta t^2\right)\mathrm{d}t=\begin{cases}\sqrt{\pi/(32\beta)} & 0<\beta<b^2\\ \sqrt{\pi/(128\beta)} & \beta=b^2\\ 0 & \beta>b^2\end{cases}$$

The following Laplace transforms apply

39:10:7 $$\int_0^\infty \mathrm{S}(bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{S}(bt)\}=\frac{1}{s}\left[\left\{\frac{1}{2}-\mathrm{C}\left(\frac{s}{2b}\right)\right\}\cos\left(\frac{s^2}{4b^2}\right)+\left\{\frac{1}{2}-\mathrm{S}\left(\frac{s}{2b}\right)\right\}\sin\left(\frac{s^2}{4b^2}\right)\right]$$

39:10:8 $$\int_0^\infty \mathrm{C}(bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{C}(bt)\}=\frac{1}{s}\left[\left\{\frac{1}{2}-\mathrm{S}\left(\frac{s}{2b}\right)\right\}\cos\left(\frac{s^2}{4b^2}\right)-\left\{\frac{1}{2}-\mathrm{C}\left(\frac{s}{2b}\right)\right\}\sin\left(\frac{s^2}{4b^2}\right)\right]$$

39:10:9 $$\int_0^\infty \mathrm{S}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\mathrm{S}\left(\sqrt{bt}\right)\right\}=\frac{\sqrt{b}}{2s}\sqrt{\frac{\sqrt{s^2+b^2}-s}{s^2+b^2}}$$

39:10:10 $$\int_0^\infty \mathrm{C}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\mathrm{C}\left(\sqrt{bt}\right)\right\}=\frac{\sqrt{b}}{2s}\sqrt{\frac{\sqrt{s^2+b^2}+s}{s^2+b^2}}$$

39:11 COMPLEX ARGUMENT

The Fresnel integrals may be regarded as arising from the integral of the Gaussian function $\exp(-z^2)$ of complex argument when the real and imaginary parts of the argument are of equal magnitude $t/\sqrt{2}$, with t real. For then

39:11:1 $$\exp\left\{-\left(\frac{t}{\sqrt{2}}+i\frac{t}{\sqrt{2}}\right)^2\right\}=\exp\left(-it^2\right)=\cos\left(t^2\right)-i\sin\left(t^2\right)$$

and accordingly

39:11:2 $$\int_0^z \exp\left\{-\left(\frac{t}{\sqrt{2}}+i\frac{t}{\sqrt{2}}\right)^2\right\}\mathrm{d}t=\int_0^z\left[\cos\left(t^2\right)-i\sin\left(t^2\right)\right]\mathrm{d}t=\sqrt{\frac{\pi}{2}}\left[\mathrm{C}(z)-i\mathrm{S}(z)\right]$$

No simple formulas express the real and imaginary parts of the Fresnel integrals when their arguments are complex. Each Fresnel integral has a zero at the origin of the complex plane and an infinite number of other complex zeros in the first and third quadrants [Abramowitz and Stegun, Table 7.11, but note the different definitions employed there].

39:12 GENERALIZATIONS: Böhmer integrals

Fresnel integrals generalize to *Böhmer* (or Boehmer) *integrals*, also known as *generalized Fresnel integrals* and defined, for $x \ge 0$ and $\nu < 1$, by

39:12:1 $$\mathrm{S}(\nu,x)=\int_x^\infty t^{\nu-1}\sin(t)\,\mathrm{d}t \qquad \mathrm{C}(\nu,x)=\int_x^\infty t^{\nu-1}\cos(t)\,\mathrm{d}t \qquad \nu<1$$

Alternative definitions are provided by the real and imaginary parts of the incomplete gamma function of complex argument as described in Section 45:11. Sometimes the order of citation of the variables in Böhmer integrals is reversed; thus you may encounter S(x,ν) and C(x,ν).

Böhmer integrals provide a bridge between Fresnel integrals, to which they reduce when the ν parameter is one-half:

39:12:2 $$\mathrm{S}\left(\tfrac{1}{2},x\right)=\sqrt{2\pi}\left[\tfrac{1}{2}-\mathrm{S}\left(\sqrt{x}\right)\right] \qquad \mathrm{C}\left(\tfrac{1}{2},x\right)=\sqrt{2\pi}\left[\tfrac{1}{2}-\mathrm{C}\left(\sqrt{x}\right)\right]$$

and the sine or cosine integrals [Chapter 38], to which reduction occurs when $\nu = 0$:

39:12:3 $$\mathrm{S}(0,x)=\frac{\pi}{2}-\mathrm{Si}(x) \qquad \mathrm{C}(0,x)=-\mathrm{Ci}(x)$$

The recurrence relations of Böhmer integrals

39:12:4 $$\nu\mathrm{S}(\nu,x)=-\mathrm{C}(\nu+1,x)-x^{\nu}\sin(x) \qquad \nu\mathrm{C}(\nu,x)=\mathrm{S}(\nu+1,x)-x^{\nu}\cos(x)$$

coupled with equations 39:12:1−3, permit any indefinite integral of the form

39:12:5 $$\int_x^\infty \frac{\sin(t)}{t^{n/2}}\mathrm{d}t \qquad \text{or} \qquad \int_x^\infty \frac{\cos(t)}{t^{n/2}}\mathrm{d}t \qquad n=1,2,3,\cdots$$

to be expressed in terms of Si(x), Ci(x), $\mathrm{S}\left(\sqrt{x}\right)$ or $\mathrm{C}\left(\sqrt{x}\right)$.

For zero argument, the Böhmer integrals adopt the values

39:12:6 $$\mathrm{S}(\nu,0)=\Gamma(\nu)\sin\left(\tfrac{1}{2}\nu\pi\right) \qquad \mathrm{C}(\nu,0)=\Gamma(\nu)\cos\left(\tfrac{1}{2}\nu\pi\right)$$

and these expressions, in which Γ is the (complete) gamma function [Chapter 43], provide the subtractive terms in the expansions

39:12:7 $$\mathrm{S}(\nu,x)-\mathrm{S}(\nu,0)=-x^{\nu+1}\sum_{j=0}^{\infty}\frac{\left(-x^2\right)^j}{(2j+1)!(2j+\nu+1)}=\frac{-x^{\nu+1}}{\nu+1}\sum_{j=0}^{\infty}\frac{\left(\frac{\nu+1}{2}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j\left(\frac{\nu+3}{2}\right)_j}\left(\frac{-x^2}{4}\right)^j$$

39:12:8 $$\mathrm{C}(\nu,x)-\mathrm{C}(\nu,0)=-x^{\nu}\sum_{j=0}^{\infty}\frac{\left(-x^2\right)^j}{(2j)!(2j+\nu)}=\frac{-x^{\nu}}{\nu}\sum_{j=0}^{\infty}\frac{\left(\frac{\nu}{2}\right)_j}{\left(\frac{1}{2}\right)_j(1)_j\left(\frac{\nu+2}{2}\right)_j}\left(\frac{-x^2}{4}\right)^j$$

These series are hypergeometric [Section 18:14], as are those that occur in the asymptotic series

39:12:9 $$\mathrm{S}(\nu,x)\sim\frac{\cos(x)}{x^{1-\nu}}\sum_{j=0}\left(\tfrac{1-\nu}{2}\right)_j\left(\tfrac{2-\nu}{2}\right)_j\left(\frac{-4}{x^2}\right)^j+(1-\nu)\frac{\sin(x)}{x^{2-\nu}}\sum_{j=0}\left(\tfrac{2-\nu}{2}\right)_j\left(\tfrac{3-\nu}{2}\right)_j\left(\frac{-4}{x^2}\right)^j \qquad \text{large } x$$

and

39:12:10 $$\mathrm{C}(\nu,x)\sim\frac{-\sin(x)}{x^{1-\nu}}\sum_{j=0}\left(\tfrac{1-\nu}{2}\right)_j\left(\tfrac{2-\nu}{2}\right)_j\left(\frac{-4}{x^2}\right)^j+(1-\nu)\frac{\cos(x)}{x^{2-\nu}}\sum_{j=0}\left(\tfrac{2-\nu}{2}\right)_j\left(\tfrac{3-\nu}{2}\right)_j\left(\frac{-4}{x^2}\right)^j \qquad \text{large } x$$

39:13 COGNATE FUNCTIONS: auxiliary Fresnel integrals

There appears to be no definitive symbolism for the *auxiliary Fresnel cosine integral* or the *auxiliary Fresnel sine integral*, to which this *Atlas* assigns the notations Fres(x) and Gres(x) respectively; their maps are included in

Figure 39-1. Abramowitz and Stegun [Chapter 7] use the symbols $f\left(\sqrt{\frac{2}{\pi}}\,x\right)$ and $g\left(\sqrt{\frac{2}{\pi}}\,x\right)$ for these functions.

The auxiliary Fresnel integrals may be defined as Laplace transforms:

39:13:1 $$\int_0^\infty \cos(t^2)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\cos(t^2)\} = \sqrt{\frac{\pi}{2}}\,\mathrm{Fres}\left(\tfrac{1}{2}s\right)$$

39:13:2 $$\int_0^\infty \sin(t^2)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\sin(t^2)\} = \sqrt{\frac{\pi}{2}}\,\mathrm{Gres}\left(\tfrac{1}{2}s\right)$$

or as the aperiodic components of the semiintegral and semiderivative [Section 12:14] of the sine function:

39:13:3 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\sin(x) = \sin\left(x-\tfrac{1}{4}\pi\right) + \sqrt{2}\,\mathrm{Fres}\left(\sqrt{x}\right)$$

39:13:4 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sin(x) = \sin\left(x+\tfrac{1}{4}\pi\right) - \sqrt{2}\,\mathrm{Gres}\left(\sqrt{x}\right)$$

Their relationships to the Fresnel integrals themselves

39:13:5 $$\mathrm{Fres}(x) = \left[\tfrac{1}{2}-\mathrm{S}(x)\right]\cos(x^2) - \left[\tfrac{1}{2}-\mathrm{C}(x)\right]\sin(x^2)$$

39:13:6 $$\mathrm{Gres}(x) = \left[\tfrac{1}{2}-\mathrm{S}(x)\right]\sin(x^2) + \left[\tfrac{1}{2}-\mathrm{C}(x)\right]\cos(x^2)$$

may also serve as definitions.

For large arguments, auxiliary Fresnel integrals may be expanded asymptotically

39:13:7 $$\mathrm{Fres}(x) \sim \frac{1}{\sqrt{2\pi}}\left[\frac{1}{x} - \frac{3}{4x^5} + \frac{105}{16x^9} - \cdots\right] = \frac{1}{\sqrt{2\pi}\,x}\sum_{j=0}\frac{(4j-1)!!}{(-4x^4)^j} = \frac{1}{\sqrt{2\pi}\,x}\sum_{j=0}\left(\tfrac{1}{4}\right)_j\left(\tfrac{3}{4}\right)_j\left(\frac{-4}{x^4}\right)^j$$

39:13:8 $$\mathrm{Gres}(x) \sim \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2x^3} - \frac{15}{8x^5} + \frac{945}{32x^9} - \cdots\right] = \frac{1}{\sqrt{8\pi}\,x^3}\sum_{j=0}\frac{(4j+1)!!}{(-4x^4)^j} = \frac{1}{\sqrt{8\pi}\,x^3}\sum_{j=0}\left(\tfrac{3}{4}\right)_j\left(\tfrac{5}{4}\right)_j\left(\frac{-4}{x^4}\right)^j$$

or, in combination with sinusoids, as the rapidly convergent power series

39:13:9 $$\mathrm{Fres}(x) - \frac{\cos\left(x^2+\frac{\pi}{4}\right)}{\sqrt{2}} = \sqrt{\frac{2}{\pi}}\left[\frac{2x^3}{3} - \frac{8x^7}{105} + \frac{32x^{11}}{10395} - \cdots\right] = \sqrt{\frac{8}{\pi}}x^3\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+3)!!} = \sqrt{\frac{8}{\pi}}x^3\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{4}x^4\right)^j}{\left(\frac{5}{4}\right)_j\left(\frac{7}{4}\right)_j}$$

and

39:13:10 $$\frac{\sin\left(x^2+\frac{\pi}{4}\right)}{\sqrt{2}} - \mathrm{Gres}(x) = \sqrt{\frac{2}{\pi}}\left[x - \frac{4x^5}{15} + \frac{16x^9}{945} - \cdots\right] = \sqrt{\frac{2}{\pi}}x\sum_{j=0}^{\infty}\frac{(-4x^4)^j}{(4j+1)!!} = \sqrt{\frac{2}{\pi}}x\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{4}x^4\right)^j}{\left(\frac{3}{4}\right)_j\left(\frac{5}{4}\right)_j}$$

All four of these series are hypergeometric [Section 18:14].

Equator's auxiliary Fresnel cosine integral and auxiliary Fresnel sine integral routines (keywords **Fres** and **Gres**) are based on equations 39:13:5 and 39:13:6 when $x \le 4$ and, with the ε-transformation [Section 10:14], on equations 39:13:7 and 39:13:8 otherwise.

39:14 RELATED TOPIC: the curvature and length of a plane curve

A *plane curve* lies in a two-dimensional space, the *cartesian plane*, and can be characterized by the equation $y = \mathrm{f}(x)$, where x and y are rectangular coordinates. Familiar properties of plane curves are their slopes, maxima, minima, inflections and zeros, as discussed in Section 0:7. Less familiar properties are the *curvature* of a plane curve and its *length* (or *arc length*) between two points.

The *curvature* of a curve obeying the function $y = \mathrm{f}(x)$ is given by the formula

39:14:1 $$\kappa(x) = \frac{\mathrm{d}^2\mathrm{f}/\mathrm{d}x^2}{\sqrt{\left[1+(\mathrm{d}\mathrm{f}/\mathrm{d}x)^2\right]^3}}$$

In general, the curvature varies from point to point. For example, the curvature of the sine function $y = \sin(x)$ is

39:14:2 $$\kappa(x) = \frac{-\sin(x)}{\sqrt{\left[1+\cos^2(x)\right]^3}}$$

at point x, being zero when $x = 0$ or π, and -1 when $x = \pi/2$. A negative κ implies that the curve is concave when viewed from below. The so-called *radius of curvature* is the reciprocal $1/|\kappa|$ of the curvature: it is the radius of the circle that osculates (kisses) the curve at the point in question. Figure 39-2 shows a segment of the sine function and its osculating circle at the point $x = \pi/2$.

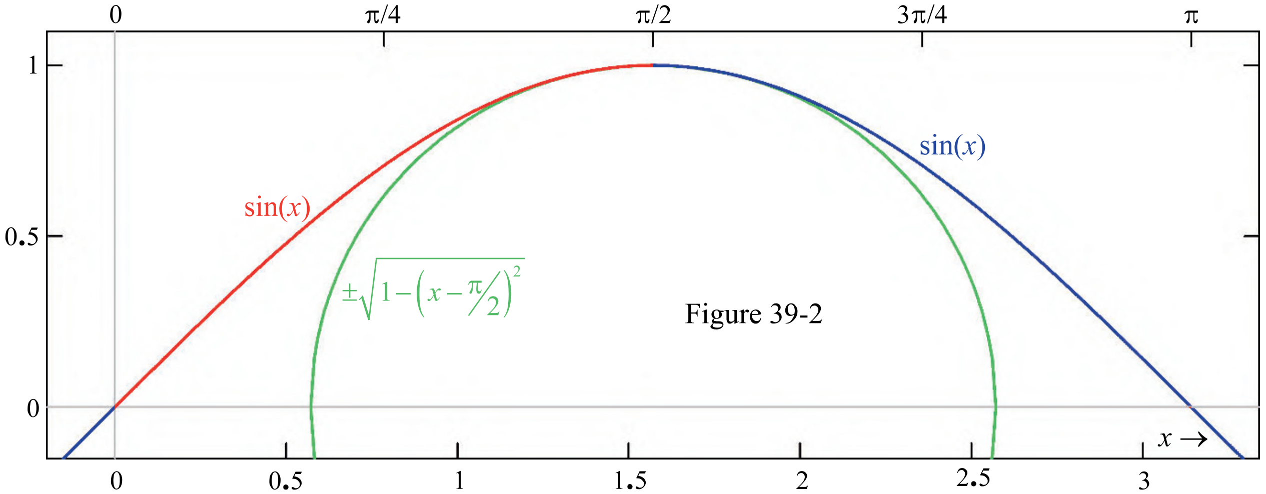


Figure 39-2

The *length* of the $y = \mathrm{f}(x)$ curve between the points x_0 and x_1 is given by the integral

39:14:3 $$\ell(x_0 \to x_1) = \int_{x_0}^{x_1} \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_{x_0}^{x_1} \sqrt{1+\left(\mathrm{d}\mathrm{f}/\mathrm{d}x\right)^2}\,\mathrm{d}x$$

For example, the length of the segment of the sine curve that is shown in red in Figure 39-2 can be found by recourse to equation 61:3:5 and Section 61:7 as

39:14:4 $$\ell\left(0 \to \tfrac{1}{2}\pi\right) = \int_0^{\pi/2} \sqrt{1+\cos^2(x)}\,\mathrm{d}x = \sqrt{2}\,\mathrm{E}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi g}{2} + \frac{1}{2g} = 1.9100\,98894\,51385$$

Here E is the complete elliptic integral of the second kind [Chapter 61] and g is Gauss's constant [Section 1:7]. See equation 28:14:11 for another application of formula 39:14:3.

The curve $\mathrm{f}(t)$ shown in Figure 39-3 has the property that its curvature is proportional to its length measured from the origin; specifically:

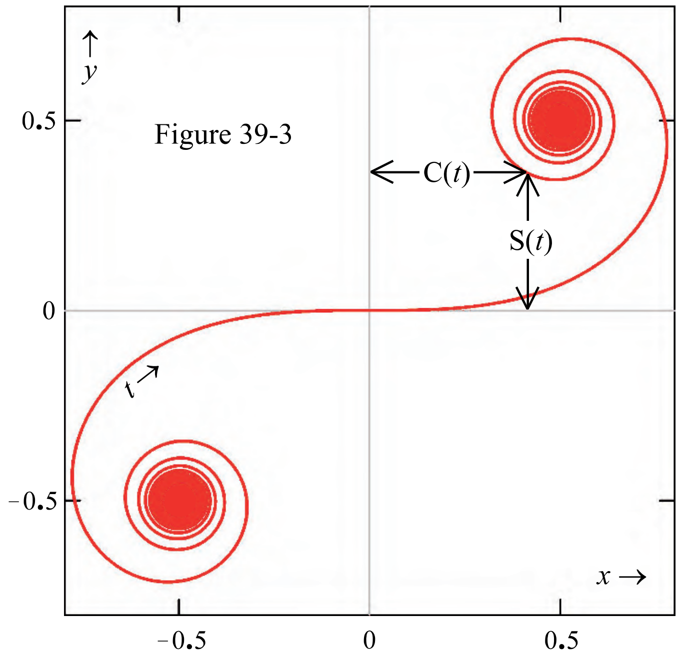
39:14:5 $$\kappa(t) = \pi\,\ell(0 \to t)$$

The $\mathrm{f}(t)$ curve is called a *clotoid curve* or *Cornu's spiral* (Marie Alfred Cornu, French physicist, 1841–1902) and plays an important role in the theory of diffraction. The ever-tightening spirals converge towards the points $x = \pm½$,

$y = \pm\frac{1}{2}$, in the cartesian plane. The ordinate and abscissa of any point on the curve are, in fact, the Fresnel integrals

39:14:6 $$x = \mathrm{C}(t) \qquad\qquad y = \mathrm{S}(t)$$

so that Cornu's spiral can be, and usually is, defined parametrically [Section 0:3] by this pair of equations as t journeys from $-\infty$ to $+\infty$.



CHAPTER 40

THE ERROR FUNCTION erf(x) AND ITS COMPLEMENT erfc(x)

The functions of this chapter are interrelated by

40:0:1 $$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$$

The inverse error function inverf(x) is briefly discussed, too. All three functions occur widely in statistics and in the solutions to problems in heat conduction and similar instances of the diffusion of matter or energy. The "error" in the name of erf(x) arises from its importance in probability theory, a subject touched on in Section 40:14.

40:1 NOTATION

The error function is sometimes given the symbol H(x) or $\Phi(x)$. The related notations $\Phi_1(x)$, $\Phi_2(x)$, etc. then refer to successive derivatives:

40:1:1 $$\Phi_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{erf}(x) \qquad n = 0,1,2,\cdots$$

These notations are avoided in the *Atlas*.

Sometimes the initial letter of the symbol is capitalized without change of meaning, but more often this implies multiplication by a $\sqrt{\pi}/2$ factor,

40:1:2 $$\mathrm{Erf}(x) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) \qquad \mathrm{Erfc}(x) = \frac{\sqrt{\pi}}{2}\mathrm{erfc}(x)$$

and the appropriate name *probability integral* is then given to the Erf function. Sadly, the same name is sometimes given to erf itself, though "error function" is more common. Beware of changed arguments: authors often attach the names *probability function* or *Gauss probability integral* to such functions as $\mathrm{erf}\left(x/\sqrt{2}\right)$ or $\left(1/\sqrt{2}\right)\mathrm{erf}\left(x/\sqrt{2}\right)$.

The notation "erf", defined as in Section 40:3, and the name "error function" are used exclusively in the *Atlas*. Alternative names for erfc(x) are *complementary error function* and *error function complement*, the latter being our choice.

See Section 42:1 for the significance of the erfi and Erfi symbols.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_41,

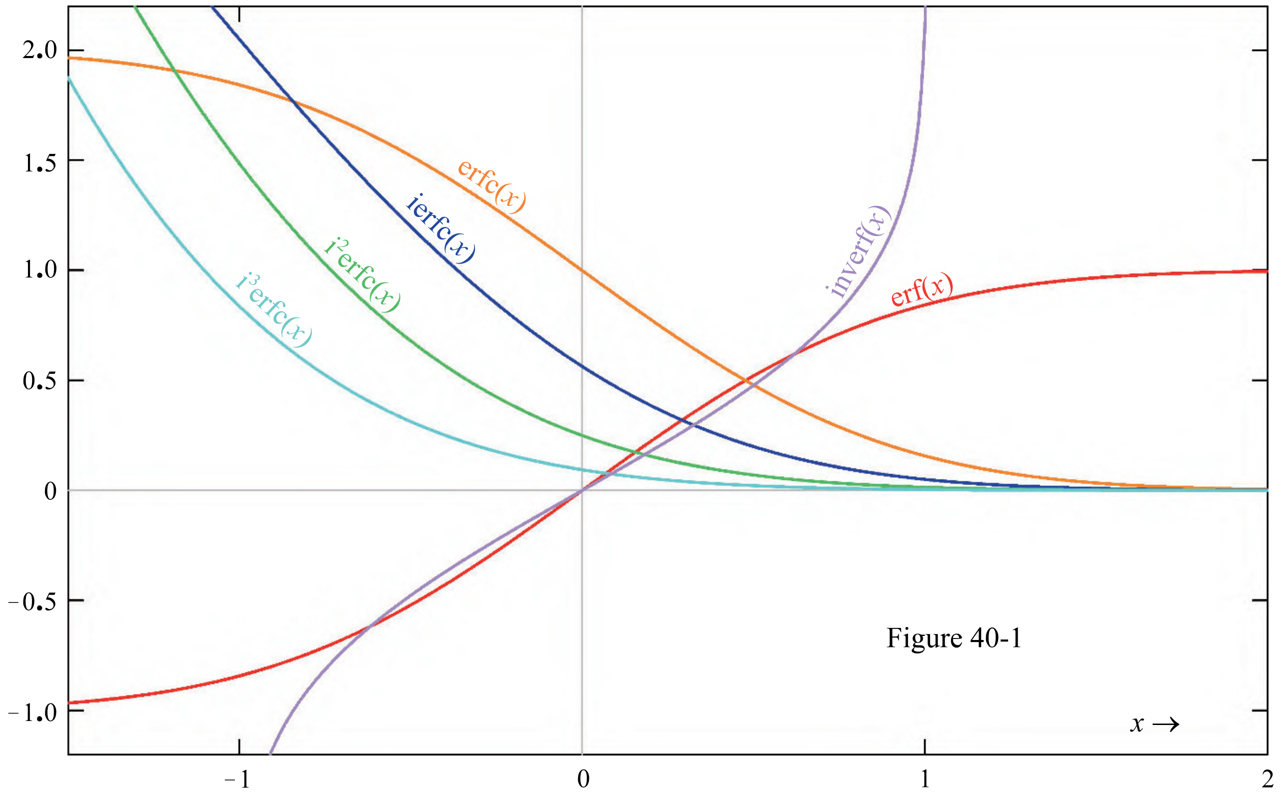


Figure 40-1

40:2 BEHAVIOR

Figure 40-1 includes maps of the error function and its complement. Both are sigmoidally shaped and rapidly approach horizontal limits as $x \to \pm\infty$. The ranges of the two functions are $-1 \le \mathrm{erf}(x) \le 1$ and $2 \ge \mathrm{erfc}(x) \ge 0$.

40:3 DEFINITIONS

The standard definitions,

40:3:1
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\mathrm{d}t$$

and

40:3:2
$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\mathrm{d}t$$

are illustrated in Figure 40-2, in which the colored zones, combined, have an area of unity. Other integral definitions of the error function include

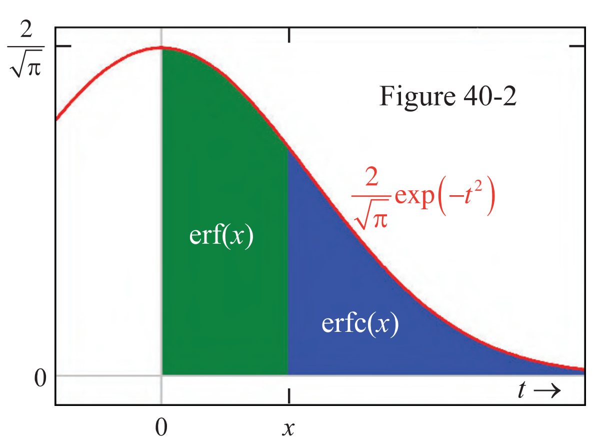


Figure 40-2

40:3:3
$$\mathrm{erf}(x) = \frac{\mathrm{sgn}(x)}{\sqrt{\pi}} \int_0^{x^2} \frac{\exp(-t)}{\sqrt{t}}\mathrm{d}t$$

40:3:4 $$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}}\int_0^1 \exp(-x^2t^2)\,\mathrm{d}t$$

and

40:3:5 $$\mathrm{erf}(x) = \frac{2}{\pi}\int_0^\infty \frac{\exp(-t^2)\sin(2xt)}{t}\,\mathrm{d}t$$

Equation 40:6:1 shows the error function to be an $L = K+1 = 2$ hypergeometric function. It may be synthesized [Section 43:14] from the exponential function by two alternative routes:

40:3:6 $$\exp(-x) \xrightarrow[\frac{3}{2}]{\frac{1}{2}} \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\mathrm{erf}\left(\sqrt{x}\right)$$

40:3:7 $$\exp(x) \xrightarrow[\frac{3}{2}]{1} \frac{1}{2}\sqrt{\frac{\pi}{x}}\exp(x)\,\mathrm{erf}\left(\sqrt{x}\right)$$

A synthetic route to the complementary error function is shown in equation 41:3:7.

The error function is generated by semiintegration or semidifferentiation [Section 12:14] of the exponential function

40:3:8 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\exp(x) = \exp(x)\,\mathrm{erf}\left(\sqrt{x}\right)$$

40:3:9 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\exp(x) = \exp(x)\,\mathrm{erf}\left(\sqrt{x}\right) + \frac{1}{\sqrt{\pi x}}$$

Other definitions based on semiintegration are inherent in equations 40:10:9 and 40:10:10.

The inverse error function is defined implicitly by

40:3:10 $$\mathrm{inverf}\{\mathrm{erf}(x)\} = x$$

and takes arguments only in the domain $-1 \le x \le 1$. It is illustrated in Figure 40-1.

40:4 SPECIAL CASES

There are none.

40:5 INTRARELATIONSHIPS

The error function and its inverse are odd functions

40:5:1 $$\mathrm{erf}(-x) = -\mathrm{erf}(x)$$

and

40:5:2 $$\mathrm{inverf}(-x) = -\mathrm{inverf}(x)$$

whereas its complement obeys the reflection formula

40:5:3 $$\mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x)$$

40:6 EXPANSIONS

The error function may be expanded as a power series in the following two ways

40:6:1
$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\left[x-\frac{x^3}{3}+\frac{x^5}{10}-\cdots\right]=\frac{x}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(-x^2)^j}{j!(j+\frac{1}{2})}=\frac{2x}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j}\left(-x^2\right)^j$$

40:6:2
$$\mathrm{erf}(x)=\frac{2\exp(-x^2)}{\sqrt{\pi}}\left[x+\frac{2x^3}{3}+\frac{4x^5}{15}+\cdots\right]=\exp(-x^2)\sum_{j=0}^{\infty}\frac{x^{2j+1}}{\Gamma\left(j+\frac{3}{2}\right)}=\frac{2x\exp(-x^2)}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{x^{2j}}{\left(\frac{3}{2}\right)_j}$$

while its complement has the asymptotic expansion, valid for large arguments

40:6:3
$$\mathrm{erfc}(x)\sim\frac{\exp(-x^2)}{\sqrt{\pi}\,x}\left[1-\frac{1}{2x^2}+\frac{3}{4x^4}-\cdots\right]=\frac{\exp(-x^2)}{\sqrt{\pi}\,x}\sum_{j=0}\frac{(2j-1)!!}{(-2x^2)^j}=\frac{\exp(-x^2)}{\sqrt{\pi}\,x}\sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\frac{-1}{x^2}\right)^j$$

A less familiar expansion of the error function is in terms of modified Bessel functions of half-integer order [Chapter 50], or equivalently the modified spherical Bessel functions i_j introduced in Section 28:13.

40:6:4
$$\mathrm{erf}(x)=\sqrt{2}\left[\mathrm{I}_{1/2}(x^2)-\mathrm{I}_{3/2}(x^2)-\mathrm{I}_{5/2}(x^2)+\mathrm{I}_{7/2}(x^2)+\cdots\right]=\frac{2x}{\sqrt{\pi}}\sum_{j=0}^{\infty}(-)^{\mathrm{Int}[(j+1)/2]}\mathrm{i}_j(x^2)$$

Note the unusual $+--++--+\cdots$ sign sequence.

The inverse error function has a power series in $\chi=\sqrt{\pi}\,x/2$, namely

40:6:5
$$\mathrm{inverf}(x)=\chi+\frac{\chi^3}{3}+\frac{7\chi^5}{30}+\cdots=\sum_{j=0}^{\infty}a_j\chi^{2j+1}\qquad\text{where } a_0=1,\text{ and } a_j=\frac{1}{2j+1}\sum_{k=1}^{j}\frac{a_{k-1}a_{j-k}}{k(2k-1)}$$

40:7 PARTICULAR VALUES

Included in the following tabulation are values with relevance to probability theory [Section 40:14].

	$x=-\infty$	$x=0$	$x=0.47693\,62762\,04470$	$x=1/\sqrt{2}$	$x=\sqrt{2}$	$x=\infty$
erf(x)	−1	0	0.5	0.68268 94921 37086	0.95449 97361 03642	1
erfc(x)	2	1	0.5	0.31731 05078 62914	0.045500 26389 63584	0

40:8 NUMERICAL VALUES

With keywords **erf** and **erfc**, *Equator* offers accurate values through its error function and error function complement routines. When $|x|\le 1.9$, the erf algorithm is based on formula 40:6:1. The erfc routine, for $|x|\le 1.66$, also uses 40:6:1 with 40:0:1. For larger x, algorithms are based on the continued fraction 41:6:5 of the next chapter. For x values exceeding about 5.5, erf(x) is not significantly different from unity, but erfc values significantly different from zero are calculable provided x does not exceed about 26.5.

Equator also provides an inverse error function routine (keyword **inverf**). The algorithm mostly employs expansion 40:6:5, but for $x>0.95$ relies on the recursion

40:8:1
$$\left[\mathrm{inverf}(x)\right]_{m+1}=\sqrt{\ln\left\{\frac{\exp\left\{\left[\mathrm{inverf}(x)\right]_m^2\right\}\mathrm{erfc}\left\{\left[\mathrm{inverf}(x)\right]_m\right\}}{1-x}\right\}}\qquad x>0$$

to improve the mth estimate of inverf(x) repeatedly. A starting estimate $[\mathrm{inverf}(x)]_0 = 5$ is employed with $\exp\{[\;]^2\}\mathrm{erfc}\{[\;]\}$ being calculated via the continued fraction 41:6:5.

These algorithms rely on reflection formula 40:5:1 or 40:5:3 when x is negative.

40:9 LIMITS AND APPROXIMATIONS

The approaches of the error function and its complement to their limits are governed by:

40:9:1
$$\mathrm{erf}(x)\to\mathrm{sgn}(x)\left[1-\frac{\exp(-x^2)}{\sqrt{\pi}\,x}\right]\qquad x\to\pm\infty$$

and

40:9:2
$$\mathrm{erfc}(x)\to\frac{\exp(-x^2)}{\sqrt{\pi}\,x}\qquad x\to\infty$$

40:9:3
$$\mathrm{erfc}(x)\to 2+\frac{\exp(-x^2)}{\sqrt{\pi}x}\qquad x\to-\infty$$

For small arguments of either sign the linear approximations

40:9:4
$$\mathrm{erf}(x)\approx\frac{2x}{\sqrt{\pi}}\qquad |x|\text{ small}$$

and

40:9:5
$$\mathrm{erfc}(x)\approx 1-\frac{2x}{\sqrt{\pi}}\qquad |x|\text{ small}$$

hold.

The approximation

40:9:6
$$\mathrm{inverf}(x)\approx\mathrm{sgn}(x)\sqrt{\ln\left\{\frac{\Upsilon}{\sqrt{\ln(\Upsilon^2)}}\right\}}\qquad \Upsilon=\frac{\sqrt{2/\pi}}{1-|x|}\qquad |x|\to 1$$

becomes useful as the magnitude of the argument approaches unity.

40:10 OPERATIONS OF THE CALCULUS

Because of the simplicity of relationship 40:0:1, it suffices to present formulas for either erf(x) or erfc(x), rather than for both.

Single and multiple differentiations give

40:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{erf}(bx)=\frac{2b}{\sqrt{\pi}}\exp\left(-b^2x^2\right)$$

and

40:10:2 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{erfc}(bx) = (-b)^n \frac{2\exp(-b^2x^2)}{\sqrt{\pi}}\mathrm{H}_{n-1}(bx) \qquad n = 1,2,3,\cdots$$

while single and multiple indefinite integrations yield

40:10:3 $$\int_0^x \mathrm{erf}(bt)\mathrm{d}t = x\,\mathrm{erf}(bx) - \frac{1-\exp(-b^2x^2)}{\sqrt{\pi}\,b}$$

and

40:10:4 $$\int_x^\infty\int_t^\infty\cdots\int_t^\infty \mathrm{erfc}(bt)(\mathrm{d}t)^n = \frac{\mathrm{i}^n\mathrm{erfc}(bx)}{b^n}$$

The H and i^nerfc functions are addressed in Chapter 24 and Section 40:13 respectively. Other useful indefinite integrals include

40:10:5 $$\int_0^x \mathrm{erf}\left(\sqrt{bt}\right)\mathrm{d}t = \sqrt{\frac{x}{\pi b}}\exp(-bx) + \left(x - \frac{1}{2b}\right)\mathrm{erf}\left(\sqrt{bx}\right)$$

and

40:10:6 $$\int_x^\infty t\,\mathrm{erfc}(bt)\,\mathrm{d}t = \left(\frac{1}{4b^2} - \frac{x^2}{2}\right)\mathrm{erfc}(bx) + \frac{x}{2\sqrt{\pi}\,b}\exp\left(-b^2x^2\right)$$

Among definite integrals are

40:10:7 $$\int_0^\infty t^\nu\,\mathrm{erfc}(bt)\,\mathrm{d}t = \frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{(1+\nu)\sqrt{\pi}\,b^{1+\nu}} \qquad b > 0 \qquad \nu > -1$$

and

40:10:8 $$\int_0^\infty \sin(t)\,\mathrm{erfc}(bt)\,\mathrm{d}t = 1 - \exp\left(\frac{-1}{4b^2}\right) \qquad b > 0$$

Section 41:10 lists many examples of integrals, indefinite and definite, in which the integrand is a product of an exponential function with the error function or its complement; others may be inferred from the Laplace transforms below.

Two noteworthy semiderivatives [Section 12:14] are

40:10:9 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\mathrm{erf}\left(\sqrt{x}\right) = \exp\left(\frac{-x}{2}\right)\mathrm{I}_0\left(\frac{x}{2}\right)$$

in which I_0 is a modified Bessel function [Chapter 49], and

40:10:10 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{\pi x}}\exp\left(\frac{-1}{x}\right)$$

Among Laplace transforms are the examples

40:10:11 $$\int_0^\infty \mathrm{erf}(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{erf}(bt)\} = \frac{1}{s}\exp\left(\frac{s^2}{4b^2}\right)\mathrm{erfc}\left(\frac{s}{2b}\right)$$

40:10:12 $$\int_0^\infty \mathrm{erf}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\mathrm{erf}\left(\sqrt{bt}\right)\right\} = \frac{1}{s}\sqrt{\frac{b}{s+b}}$$

and

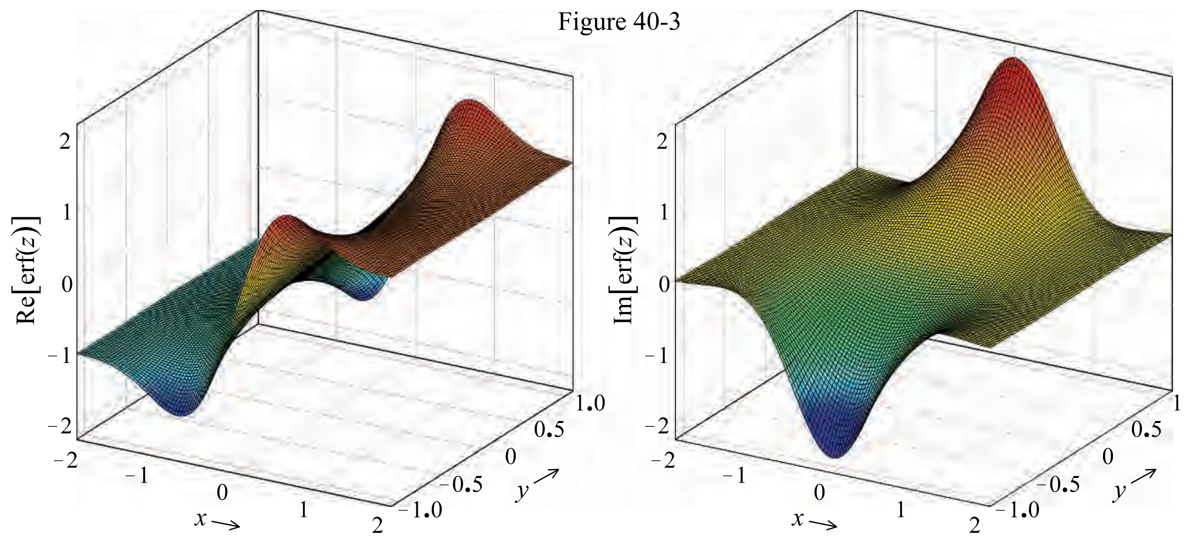
40:10:13 $$\int_0^\infty \mathrm{erfc}\left(\sqrt{\frac{a}{t}}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\mathrm{erfc}\left(\sqrt{\frac{a}{t}}\right)\right\} = \frac{1}{s}\exp\left(-2\sqrt{as}\right)$$

40:11 COMPLEX ARGUMENT

Confusingly, the function that has been called the "error function for complex argument" is *not* erf($x+iy$), as might be expected. Instead, it is the function discussed in Section 41:11.

Figure 40-3 shows the behaviors, close to the origin, of the real and imaginary parts of the erf($x+iy$) function. Farther from the origin, much more complicated behavior develops. Not apparent in the figure are the zeros, of which there is an infinite number; the coordinates of early examples are listed by Abramowitz and Stegun [page 327]. Though they are complicated, double-sum expansions for the real and imaginary parts of erf($x+iy$) can be constructed by combination of equations 40:6:1 and 10:11:1. An approximate expansion for erf($x+iy$) − erf(x) that has a relative error of only 10^{-16} has been reported [Abramowitz and Stegun, equation 7.1.29].

Figure 40-3



When $x = 0$, so that the argument is purely imaginary, the real part of erf($x+iy$) vanishes, leaving the purely imaginary result

40:11:1 $$\mathrm{erf}(iy) = \frac{2i}{\sqrt{\pi}}\int_0^y \exp(t^2)\,\mathrm{d}t = \frac{2i}{\sqrt{\pi}}\exp\left(y^2\right)\mathrm{daw}(y)$$

that involves Dawson's integral [Chapter 42]. Also see Section 42:14.

Inverse Laplace transforms of the error function and its complement include

40:11:2 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \mathrm{erfc}\left(\sqrt{bs}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\mathrm{erfc}\left(\sqrt{bs}\right)\right\} = \begin{cases} 0 & t < b \\ \dfrac{1}{\pi t}\sqrt{\dfrac{b}{t-b}} & t > b \end{cases}$$

40:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\operatorname{erf}(\sqrt{bs})}{\sqrt{s}}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{\operatorname{erf}(\sqrt{bs})}{\sqrt{s}}\right\} = \begin{cases} \dfrac{1}{\sqrt{\pi t}} & t < b \\ 0 & t > b \end{cases}$$

40:11:4
$$\int_{\alpha-i\infty}^{\alpha+i\infty} \operatorname{erf}\left(\frac{a}{\sqrt{s}}\right)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\operatorname{erf}\left(\frac{a}{\sqrt{s}}\right)\right\} = \frac{\sin\left(2a\sqrt{t}\right)}{\pi t}$$

40:12 GENERALIZATIONS

The error function and its complement generalize to the incomplete gamma function of Chapter 45, of which they are the $v = ½$ cases

40:12:1
$$\sqrt{\pi}\operatorname{erf}(x) = \operatorname{sgn}(x)\gamma\left(\tfrac{1}{2},x^2\right)$$

40:12:2
$$\sqrt{\pi}\operatorname{erfc}(x) = \Gamma\left(\tfrac{1}{2},x^2\right) \qquad x \ge 0$$

The complementary error function may also be regarded as a special case of the parabolic cylinder function of Chapter 46

40:12:3
$$\sqrt{\frac{\pi}{2}}\operatorname{erfc}(x) = \exp\left(\frac{-x^2}{2}\right)\mathrm{D}_{-1}\left(\sqrt{2}\,x\right)$$

Because, as detailed in Sections 45:12 and 46:12, the incomplete gamma and parabolic cylinder functions are themselves special cases of the Kummer or Tricomi functions [Chapters 47 and 48], we have the further generalizations

40:12:4
$$\frac{\sqrt{\pi}}{2x}\operatorname{erf}(x) = \mathrm{M}\left(\tfrac{1}{2},\tfrac{3}{2},-x^2\right) = \exp(-x^2)\mathrm{M}\left(1,\tfrac{3}{2},x^2\right)$$

40:12:5
$$\sqrt{\pi}\operatorname{erfc}(x) = \exp(-x^2)\,\mathrm{U}\left(\tfrac{1}{2},\tfrac{1}{2},x^2\right) = x\exp(-x^2)\,\mathrm{U}\left(1,\tfrac{3}{2},x^2\right) \qquad x > 0$$

40:13 COGNATE FUNCTIONS: repeated integrals of erfc

The *integral of the error function complement* ierfc(x) is defined by

40:13:1
$$\operatorname{ierfc}(x) = \int_x^\infty \operatorname{erfc}(t)\,\mathrm{d}t$$

The "i" in the symbol for this function should not be confused with the imaginary *i*; here it stands for "integral". A second integration

40:13:2
$$\mathrm{i}^2\operatorname{erfc}(x) = \int_x^\infty \operatorname{ierfc}(t)\,\mathrm{d}t$$

creates a *two-fold integral of the error function complement* and the process may be repeated indefinitely

40:13:3
$$\mathrm{i}^n\operatorname{erfc}(x) = \int_x^\infty \mathrm{i}^{n-1}\operatorname{erfc}(t)\,\mathrm{d}t$$

to produce a family of *n-fold integrals of the error function complement*. The notation is extended "backwards", so that one encounters

40:13:4 $$\mathrm{i}^1\mathrm{erfc}(x) = \mathrm{ierfc}(x)$$

40:13:5 $$\mathrm{i}^0\mathrm{erfc}(x) = \mathrm{erfc}(x)$$

and even

40:13:6 $$\mathrm{i}^{-1}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\exp\left(-x^2\right)$$

Apart from this last, all family members may be defined through the integral

40:13:7 $$\mathrm{i}^n\mathrm{erfc}(x) = \frac{2}{n!\sqrt{\pi}}\int_x^\infty (t-x)^n \exp(-t^2)\,\mathrm{d}t \qquad n = 0,1,2,\cdots$$

Their values at $x = 0$ are:

	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	all $n \geq -1$
$\mathrm{i}^n\mathrm{erfc}(0)$	$\frac{2}{\sqrt{\pi}}$	1	$\frac{1}{\sqrt{\pi}}$	$\frac{1}{4}$	$\frac{1}{6\sqrt{\pi}}$	$\frac{1}{32}$	$\frac{1}{60\sqrt{\pi}}$	$\frac{1}{384}$	$\frac{1}{840\sqrt{\pi}}$	$\frac{1}{6144}$	$\frac{1}{2^n\Gamma(1+\frac{1}{2}n)}$

The n = 1, 2, and 3 cases are illustrated in Figure 40-1; note that, unlike erfc itself, its repeated integrals are not sigmoidal.

The n-fold integral of the error function complement may be expanded as the power series

40:13:8 $$\mathrm{i}^n\mathrm{erfc}(x) = \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{(-2x)^j}{j!\,\Gamma\left(1+\frac{1}{2}(n-j)\right)}$$

or asymptotically

40:13:9 $$\mathrm{i}^n\mathrm{erfc}(x) \sim \frac{\exp(-x^2)}{2^n n!\sqrt{\pi}\,x^{n+1}}\sum_{j=0}\frac{(n+2j)!}{j!\left(-4x^2\right)^j} = \frac{\exp(-x^2)}{2^n\sqrt{\pi}x^{n+1}}\sum_{j=0}\frac{\left(\frac{n}{2}+\frac{1}{2}\right)_j\left(\frac{n}{2}+1\right)_j}{(1)_j}\left(\frac{-1}{x^2}\right)^j \qquad \text{large } x$$

The recurrence relation

40:13:10 $$\mathrm{i}^n\mathrm{erfc}(x) = \frac{-x}{n}\mathrm{i}^{n-1}\mathrm{erfc}(x) + \frac{1}{2n}\mathrm{i}^{n-2}\mathrm{erfc}(x) \qquad n = 1,2,3,\cdots$$

applies. By sufficient applications of this formula, one may express any repeated integral of the complementary error function in terms of erfc(x) and exp($-x^2$). Early examples are

$\mathrm{i}^1\mathrm{erfc}(x)$	$\mathrm{i}^2\mathrm{erfc}(x)$	$\mathrm{i}^3\mathrm{erfc}(x)$
$\frac{\exp(-x^2)}{\sqrt{\pi}} - x\,\mathrm{erfc}(x)$	$\frac{1+2x^2}{4}\mathrm{erfc}(x) - \frac{x}{2\sqrt{\pi}}\exp(-x^2)$	$\frac{1+x^2}{6\sqrt{\pi}}\exp(-x^2) - \frac{3x+2x^3}{12}\mathrm{erfc}(x)$

Equator's n-fold integral of the error function complement routine (keyword **inerfc**), adopts this principle.

Differentiation and indefinite integration obey the rules

40:13:11 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{i}^n\,\mathrm{erfc}(x) = -\mathrm{i}^{n-1}\mathrm{erfc}(x)$$

and

40:13:12 $$\int_x^\infty \mathrm{i}^n\mathrm{erfc}(t)\,\mathrm{d}t = \mathrm{i}^{n+1}\mathrm{erfc}(x)$$

The definite integral

40:13:13 $$\int_0^\infty \exp(bx)\,\mathrm{i}^n\,\mathrm{erfc}(x)\,\mathrm{d}x = \frac{b^n\exp(bx)}{\sqrt{\pi}} - \sum_{j=0}^{n-1}\frac{b^j}{2^{n-1-j}\,\Gamma\left(\dfrac{n+1-j}{2}\right)} \qquad n=0,1,2,\cdots$$

arises from applying parts integration [Section 0:10] a sufficient number of times.

An important differential equation and its solution are

40:13:14 $$a\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + x\frac{\mathrm{d}f}{\mathrm{d}x} - nf = 0 \qquad f = w_1\,\mathrm{i}^n\mathrm{erfc}\left(\frac{x}{\sqrt{2a}}\right) + w_2\,\mathrm{i}^n\mathrm{erfc}\left(\frac{-x}{\sqrt{2a}}\right) \qquad a>0$$

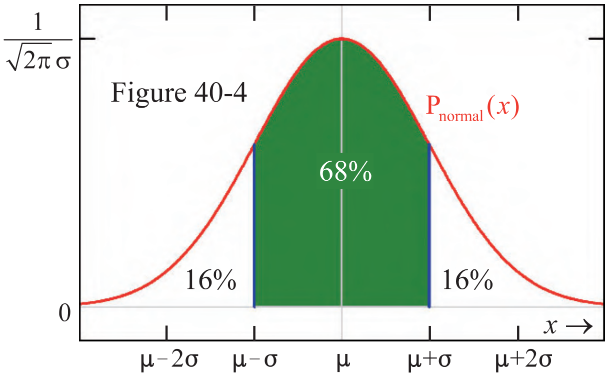
w_1 and w_2 being arbitrary constants.

The equation

40:13:15 $$\mathrm{i}^n\,\mathrm{erfc}(x) = \frac{\exp(-x^2)}{2^n\sqrt{\pi}}\,\mathrm{U}\left(\frac{1+n}{2},\frac{1}{2},x^2\right)$$

relates the n-fold integral of the error function complement to the Tricomi function [Chapter48].

40:14 RELATED TOPIC: normal probability



As introduced in Section 27:14, the so-called *normal* or *Gaussian distribution* has a probability function given by

40:14:1 $$P_{\mathrm{normal}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

in terms of the *mean* μ and *variance* σ^2 of the distribution. This equation fits the curve in Figure 40-4 that is often, though inaccurately, described as "bell shaped". The corresponding cumulative function is

40:14:2 $$F_{\mathrm{normal}}(x) = \int_{-\infty}^{x} P_{\mathrm{normal}}(t)\,\mathrm{d}t = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right) = 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)$$

Random events often obey, or are assumed to obey, this "normal" distribution. In some contexts, $x=\mu$ represents a "correct" outcome; the quantity $|x-\mu|$ is therefore termed the *error* of any outcome x, and σ is known as the *standard error* or *standard deviation*. The functions described by equations 40:14:1 and 40:14:2 are available from *Equator* under the keywords **Pnormal** and **Fnormal** and are discussed in Section 27:14..

According to equation 40:14:2, there is about a 68% probability of a single random event lying within one standard error of the mean; that is:

40:14:3 $$\int_{\mu-\sigma}^{\mu+\sigma} P_{\mathrm{normal}}(t)\,\mathrm{d}t = F_{\mathrm{normal}}\left(\frac{1}{\sqrt{2}}\right) - F_{\mathrm{normal}}\left(\frac{-1}{\sqrt{2}}\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) = 0.68268$$

Correspondingly, the green area in Figure 40-4 represents 68% of the total area enclosed between the curve and the x-axis. In a similar way, about 95% of random events lie within two standard errors of the correct value:

40:14:4
$$\int_{\mu-2\sigma}^{\mu+2\sigma} \mathrm{P}_{\text{normal}}(t)\,\mathrm{d}t = \mathrm{F}_{\text{normal}}\left(\sqrt{2}\right) - \mathrm{F}_{\text{normal}}\left(-\sqrt{2}\right) = \mathrm{erf}\left(\sqrt{2}\right) = 0.95450$$

This “nineteen times out of twenty” range of probability is often cited in presenting statistical data. The quantity 0.67449σ is known as the *probable error*; it is the error that leads to a probability of 50%:

40:14:5
$$\int_{\mu-0.67449\sigma}^{\mu+0.67449\sigma} \mathrm{P}_{\text{normal}}(t)\,\mathrm{d}t = \mathrm{erf}\left(\frac{0.67449}{\sqrt{2}}\right) = 0.50000$$

A frequent need in simulation studies by techniques known as *Monte Carlo methods* is to generate a set of numbers that mimic Gaussian-random occurrences. Before describing this topic, it is pertinent to discuss *standard random numbers* (more precisely standard *pseudo*random numbers) . A standard random number, v, is one that is equally likely to have any value in the range $0 \le v < 1$. *Equator* uses a modified linear congruential method based on the formula

40:14:6
$$v_j = \frac{[4561\,\mathrm{Int}(243000\,v'_j) + 51349]\left(\mathrm{mod}\ 243000\right)}{243000} \qquad j = 1,2,3,\cdots,J$$

to generate a standard random number, rounded to six digits. The large integers in this formula are among those recommended in Press et al. That book [Chapter 7] gives a very readable description of the hazards inherent in generating “random” numbers. The v' in formula 40:14:6 is a random number generated by a second, independent, random number generator. *Equator* treats its random number generator as the random(J,s) function. A user of the random number routine (keyword **random**) must supply a seed, s, as well as the number J of random numbers required. The seed can be any number, 1 being the default value. Reusing the seed, generates a set of identical numbers; different seeds produce sequences of different numbers.

Random numbers are uniformly distributed. Such numbers may be converted into others that have a normal or Gaussian distribution, with *mean* μ and *standard error* σ, by the *Box-Muller method*. The jth member of such a set of so-called *normally distributed random variates* is given by the formula

40:14:7
$$x_j^{\text{normal}} = \mu + \sqrt{2\sigma^2 \ln\left(\frac{1}{v_{2j}}\right)}\cos\left(2\pi v_{2j+1}\right) \qquad j = 1,2,3,\cdots,J$$

where each v is a standard random number. *Equator* uses this formula, and the v_j algorithm discussed above, in its normally distributed random variate routine (keyword **normal**). The user is asked to supply variables for the normal(μ,σ,J,s) function. A very large set of x_j^{normal} values accords almost perfectly with the distribution specified by equations 40:14:1 and 40:14:2.

CHAPTER 41

THE $\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)$ AND RELATED FUNCTIONS

Although it is merely a composite of the functions discussed in Chapters 11, 26, and 40, the function

41:0:1 $$\mathrm{f}(x) = \exp(x)\mathrm{erfc}\left(\sqrt{x}\right)$$

has sufficient importance to warrant separate treatment. This function is our prime concern in the present chapter, but several closely related functions, including

41:0:2 $$\exp(x)\mathrm{erfc}\left(-\sqrt{x}\right) = 2\exp(x) - \mathrm{f}(x)$$

41:0:3 $$\exp(x)\mathrm{erf}\left(\sqrt{x}\right) = \exp(x) - \mathrm{f}(x)$$

41:0:4 $$\sqrt{\pi x}\exp(x)\mathrm{erfc}\left(\sqrt{x}\right) = \sqrt{\pi x}\,\mathrm{f}(x)$$

41:0:5 $$\frac{1}{\sqrt{\pi x}} + \exp(x)\mathrm{erfc}\left(-\sqrt{x}\right) = \frac{1}{\sqrt{\pi x}} + 2\exp(x) - \mathrm{f}(x)$$

41:0:6 $$\exp\left(x^2\right)\mathrm{erfc}(x) = \mathrm{f}\left(x^2\right)$$

and

41:0:7 $$\exp\left(x^2\right)\mathrm{erf}(x) = \exp\left(x^2\right) - \mathrm{f}\left(x^2\right)$$

also receive mention. All these functions arise in solutions to practical problems in physics and engineering.

41:1 NOTATION

Symbol variants identified in Sections 1 of Chapters 11, 26, and 40 may lead to the replacement of our $\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)$ notation by such alternatives as $\left(2e^x/\pi^{1/2}\right)\mathrm{Erfc}\left(x^{1/2}\right)$ and $e^x[1-\Phi(\sqrt{x})]$. Unitary notations have frequently been employed in place of the $\exp(x^2)\mathrm{erfc}(x)$ product; these include $\mathrm{erc}(x)$, $\mathrm{eerfc}(x)$ and $\mathrm{experfc}(x)$.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_42, © Springer Science+Business Media, LLC 2009

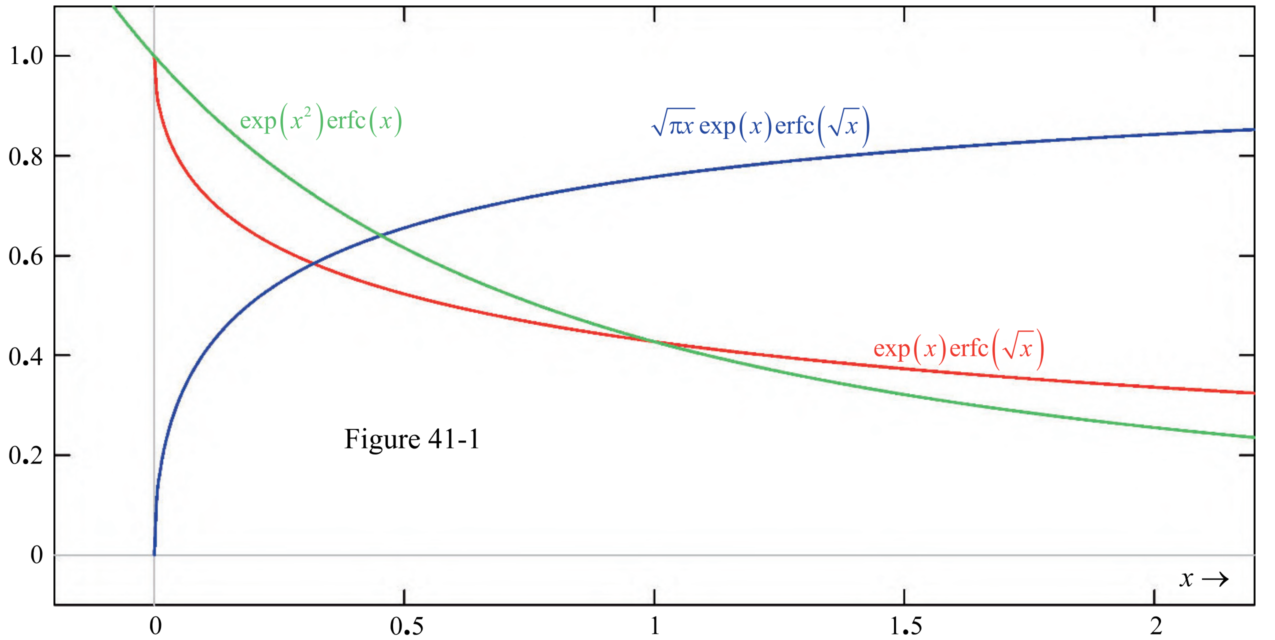


Figure 41-1

The *Atlas* avoids such abbreviations, though **experfc** serves as *Equator*'s keyword for its $\exp(x)\mathrm{erfc}(\sqrt{x})$ routine.

The $(\sqrt{\pi}/2)\exp(x^2)\mathrm{erf}(x)$ function has been represented by the symbol $D_+(x)$. Its twin, $D_-(x)$, is mentioned in Section 42:1.

41:2 BEHAVIOR

Apart from 41:0:6 and 41:0:6, all of the functions cited in the preamble to this chapter are defined as real quantities only for nonnegative real values of x. Moreover, these functions themselves are all positive and therefore Figures 41-1 and 41-2, which map the functions addressed in this chapter, portray the first quadrant predominantly.

As x increases, so does the exponential function $\exp(x)$, whereas the error function complement $\mathrm{erfc}(\sqrt{x})$ decreases. These conflicting effects mold the behavior of the three functions depicted in Figure 41-1. The $\sqrt{\pi x}\exp(x)\mathrm{erfc}(\sqrt{x})$ function gradually approaches unity as $x \to \infty$, whereas the other two functions decay slowly towards zero.

No similar compensation influences three of the four functions displayed in Figure 41-2, which increase dramatically as the argument becomes large. The $(1/\sqrt{\pi x}) + \exp(x)\mathrm{erfc}(-\sqrt{x})$ function initially decreases and displays a minimum at an argument close to 0.20405. This is the argument at which an inflection occurs in the $\exp(x)\mathrm{erfc}(-\sqrt{x})$ function.

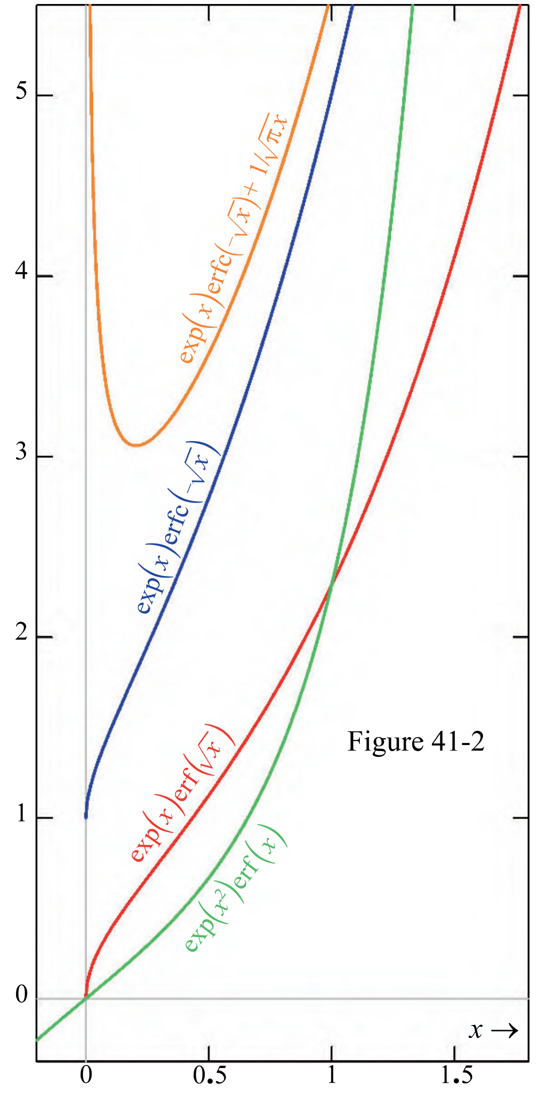


Figure 41-2

41:3 DEFINITIONS

Inasmuch as the functions of this chapter are composites of functions defined in earlier chapters, they need no definitions. Nevertheless, this chapter's primary function, 41:0:1, may be defined in two equivalent ways as an indefinite integral

41:3:1 $$\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)=\frac{1}{\sqrt{\pi}}\int_x^\infty\frac{\exp(x-t)}{\sqrt{t}}\mathrm{d}t=\frac{2}{\sqrt{\pi}}\int_{\sqrt{x}}^\infty\exp(x-t^2)\mathrm{d}t$$

and in three ways as a definite integral

41:3:2 $$\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{\exp(-t)}{\sqrt{t+x}}\mathrm{d}t=\frac{2\sqrt{x}}{\pi}\int_0^\infty\frac{\exp(-t^2)}{t^2+x}\mathrm{d}t=\frac{2}{\sqrt{\pi}}\int_0^\infty\exp\left(-t^2-2t\sqrt{x}\right)\mathrm{d}t$$

There are also two definitions as Laplace transforms

41:3:3 $$\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)=\sqrt{\frac{x}{\pi}}\int_0^\infty\frac{\exp(-xt)}{\sqrt{t+1}}\mathrm{d}t=\frac{1}{\pi}\int_0^\infty\frac{\exp(-xt)}{(t+1)\sqrt{t}}\mathrm{d}t$$

The $\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)$ function is the semiintegral [Section 12:14] of the exponential function:

41:3:4 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\exp(x)=\exp(x)\mathrm{erf}\left(\sqrt{x}\right)$$

Closely related is the syntheses [Section 43:14]

41:3:5 $$\exp(x)\xrightarrow[3/2]{1}\frac{1}{2}\sqrt{\frac{\pi}{x}}\exp(x)\mathrm{erf}\left(\sqrt{x}\right)$$

Another synthetic route is

41:3:6 $$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right)\xrightarrow[1]{1/2}\sqrt{\frac{\pi}{x}}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right)$$

A number of differential equations are satisfied by the functions of this chapter. These include the first-order examples

41:3:7 $$\frac{\mathrm{d}f}{\mathrm{d}x}-\pi f=\frac{\pm1}{\sqrt{x}}\qquad f=\exp(\pi x)\left[\mathrm{erfc}\left(\mp\sqrt{\pi x}\right)+\text{constant}\right]$$

and

41:3:8 $$\frac{\mathrm{d}f}{\mathrm{d}x}-\frac{\pi x}{2}f=\pm1\qquad f=\exp\left(\frac{\pi x^2}{4}\right)\left[\mathrm{erfc}\left(\mp\frac{\sqrt{\pi}\,x}{2}\right)+\text{constant}\right]$$

as well as the fractional differential equation

41:3:9 $$\frac{\mathrm{d}^{1/2}f}{\mathrm{d}x^{1/2}}=f-1\qquad f=\frac{1}{\sqrt{\pi x}}+\exp(x)\mathrm{erfc}\left(-\sqrt{x}\right)$$

41:4 SPECIAL CASES

There are none.

41:5 INTRARELATIONSHIPS

Equation 41:0:2 may be looked on as a reflection formula:

41:5:1 $$\exp(x)\operatorname{erfc}(-\sqrt{x}) - \exp(x) = -\exp(x)\operatorname{erfc}(\sqrt{x}) + \exp(x)$$

There exists the addition formula

41:5:2 $$\exp(x+y)\operatorname{erfc}(\sqrt{x+y}) = \exp(x+y)\operatorname{erfc}(\sqrt{x}) - \frac{\exp(y)}{\sqrt{\pi x}}\sum_{j=0}^{\infty}\frac{(2j-1)!!}{(2j)!!}\frac{\gamma(j+1,y)}{(-x)^j}$$

involving the lower incomplete gamma function [Chapter 45] and double factorials [Section 2:13].

41:6 EXPANSIONS

Power series are

41:6:1 $$\exp(x)\operatorname{erf}(\sqrt{x}) = 2\sqrt{\frac{x}{\pi}}\left[1 + \frac{2x}{3} + \frac{4x^2}{15} + \cdots\right] = 2\sqrt{\frac{x}{\pi}}\sum_{j=0}^{\infty}\frac{(2x)^j}{(2j+1)!!} = 2\sqrt{\frac{x}{\pi}}\sum_{j=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)_j}(x)^j$$

and

41:6:2 $$\exp(x)\operatorname{erfc}(\pm\sqrt{x}) = 1 \mp 2\sqrt{\frac{x}{\pi}} + x \mp \frac{4}{3}\sqrt{\frac{x^3}{\pi}} + \frac{x^2}{2} \mp \cdots = \sum_{j=0}^{\infty}\frac{\left(\mp\sqrt{x}\right)^j}{\Gamma\left(1+\frac{1}{2}j\right)}$$

The latter series, which involves the gamma function of Chapter 43, may have its alternate terms summed separately, thereby regenerating equation 41:0:3

41:6:3 $$\exp(x)\operatorname{erfc}(\pm\sqrt{x}) = \left[1 + x + \frac{x^2}{2} + \cdots\right] \mp 2\sqrt{\frac{x}{\pi}}\left[1 + \frac{2x}{3} + \frac{4x^2}{15} + \cdots\right] = \exp(x) \mp \exp(x)\operatorname{erf}(\sqrt{x})$$

Moreover, if the summation index in expansion 41:6:2 is allowed to adopt the additional value $j = -1$, then the function 41:0:5 is formed from the lower sign option:

41:6:4 $$\frac{1}{\sqrt{\pi x}} + \exp(x)\operatorname{erfc}(-\sqrt{x}) = \sum_{j=-1}^{\infty}\frac{x^{j/2}}{\Gamma\left(1+\frac{1}{2}j\right)} = \frac{1}{\sqrt{x}}\sum_{j=0}^{\infty}\frac{x^{j/2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}j\right)}$$

The $\exp(x)\operatorname{erfc}(\sqrt{x})$ function may be expanded as the continued fraction

41:6:5 $$\exp(x)\operatorname{erfc}(\sqrt{x}) = \frac{\sqrt{2/\pi}}{\sqrt{2x}+}\,\frac{1}{\sqrt{2x}+}\,\frac{2}{\sqrt{2x}+}\,\frac{3}{\sqrt{2x}+}\,\frac{4}{\sqrt{2x}+\cdots}$$

An important asymptotic series is

41:6:6 $$\sqrt{\pi x}\exp(x)\operatorname{erfc}(\sqrt{x}) \sim 1 - \frac{1}{2x} + \frac{3}{4x^2} - \cdots = \sum_{j=0}\frac{(2j-1)!!}{(-2x)^j} = \sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\frac{-1}{x}\right)^j$$

Though it is computationally useful only for large x, users of truncated versions of this summation have the advantage of knowing that the error is less than the magnitude of the first neglected term. Halving the last retained term [Section 10:13] is valuable numerically.

41:7 PARTICULAR VALUES

Apart from zero and ±infinity, the only noteworthy argument is that at which $(1/\sqrt{\pi x})+\exp(x)\mathrm{erfc}(-\sqrt{x})$ is minimal.

	$x=-\infty$	$x=0$	$x=0.20405\ 39388\ 07199$	$x=\infty$
$\exp(x)\mathrm{erfc}(\sqrt{x})$	undef	1		0
$\exp(x)\mathrm{erf}(\sqrt{x})$	undef	0		∞
$\exp(x)\mathrm{erfc}(-\sqrt{x})$	undef	1	inflection	∞
$\exp(x)\mathrm{erfc}(-\sqrt{x})$	undef	0		1
$(1/\sqrt{\pi x})+\exp(x)\mathrm{erfc}(-\sqrt{x})$	undef	∞	minimum	∞
$\exp(x^2)\mathrm{erfc}(x)$	∞	1		0
$\exp(x^2)\mathrm{erf}(x)$	$-\infty$	0		∞

41:8 NUMERICAL VALUES

Under the name exponential error function complement product, *Equator* provides a routine (keyword **experfc**) for calculating $\exp(x)\mathrm{erfc}(\sqrt{x})$. All the other functions of this chapter are then readily accessible through equations 41:0:2−6. For arguments smaller than 3, the output of the erfc algorithm from Section 40:8 is multiplied by exp(x). For larger arguments, the routine uses the continued fraction 41:6:5.

41:9 LIMITS AND APPROXIMATIONS

Equation 41:6:6, suitably rearranged where necessary, is useful in providing limiting $x\to\infty$ expressions for the various functions of this chapter. Equations 41:6:1 and 41:6:2 can perform a similar service for the $x\to 0$ limit.

Surprisingly, the empirical approximation

41:9:1 $$\exp(x)\mathrm{erfc}(\sqrt{x})\approx\frac{2}{\sqrt{\pi x}+\sqrt{\pi(x+2)-(2\pi-4)\exp(-\sqrt{5x/7})}}$$

is accurate to within 0.1% over the entire $0\le x\le\infty$ range. It is based on the inequality

41:9:2 $$\sqrt{x}+\sqrt{x+4/\pi}\le\frac{\sqrt{\pi}}{2\exp(x)\mathrm{erfc}(\sqrt{x})}\le\sqrt{x}+\sqrt{x+2}$$

41:10 OPERATIONS OF THE CALCULUS

Differentiation formulas include:

41:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\exp(bx)\mathrm{erfc}\left(\pm\sqrt{bx}\right)=b\exp(bx)\mathrm{erfc}\left(\pm\sqrt{bx}\right)\mp\sqrt{\frac{b}{\pi x}}$$

41:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\exp(bx)\mathrm{erf}\left(\pm\sqrt{bx}\right)=\pm b\exp(bx)\mathrm{erf}\left(\sqrt{bx}\right)\pm\sqrt{\frac{b}{\pi x}}$$

41:10:3 $$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\pi x}\exp(bx)\mathrm{erfc}\left(\sqrt{bx}\right)=\sqrt{\frac{\pi}{x}}\left(bx+\frac{1}{2}\right)\exp(bx)\mathrm{erfc}\left(\sqrt{bx}\right)-\sqrt{b}$$

41:10:4 $$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{\pi bx}}+\exp(bx)\mathrm{erfc}\left(-\sqrt{bx}\right)\right\}=b\exp(bx)\mathrm{erfc}\left(-\sqrt{bx}\right)+\sqrt{\frac{b}{\pi x}}\left[1-\frac{1}{2bx}\right]$$

and

41:10:5 $$\frac{\mathrm{d}}{\mathrm{d}x}\exp\left(\beta^2x^2\right)\mathrm{erfc}(bx)=2\exp\left(\beta^2x^2\right)\left[\beta^2x\,\mathrm{erfc}(bx)-\frac{b}{\sqrt{\pi}}\exp\left(-b^2x^2\right)\right]$$

Repeated differentiation of $\exp(x)\mathrm{erfc}\left(\pm\sqrt{x}\right)$ regenerates the original function together with an n-term series that is proportional to the first n terms in expansion 41:6:6, that is:

41:10:6 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(x)\mathrm{erfc}\left(\pm\sqrt{x}\right)=\exp(x)\mathrm{erfc}\left(\pm\sqrt{x}\right)\mp\frac{1}{\sqrt{\pi x}}\sum_{j=0}^{n-1}\frac{(2j-1)!!}{(-2x)^j}$$

Indefinite integrals include:

41:10:7 $$\int_0^x\exp(bt)\mathrm{erfc}\left(\sqrt{bt}\right)\mathrm{d}t=\frac{1}{b}\left[\exp(bx)\mathrm{erfc}\left(\sqrt{bx}\right)-1+2\sqrt{\frac{bx}{\pi}}\right]$$

41:10:8 $$\int_0^x\exp(t)\mathrm{erfc}\left(\sqrt{bt}\right)\mathrm{d}t=\exp(x)\mathrm{erfc}\left(\sqrt{bt}\right)-1+\sqrt{\frac{b}{b-1}}\mathrm{erfc}\left(\sqrt{bx-x}\right)\qquad b>1$$

41:10:9 $$\int_0^x\exp(\beta t)\mathrm{erfc}(bt)\mathrm{d}t=\frac{1}{\beta}\left[\exp(\beta x)\mathrm{erfc}(bx)-1\right]-\frac{1}{\beta}\exp\left(\frac{\beta^2}{4b^2}\right)\left[\mathrm{erfc}\left(bx-\frac{\beta}{2b}\right)-\mathrm{erfc}\left(\frac{-\beta}{2b}\right)\right]$$

41:10:10 $$\int_0^x\exp(-t)\,\mathrm{erfc}\left(\frac{a}{\sqrt{t}}\right)\mathrm{d}t=-\exp(-x)\,\mathrm{erfc}\left(\frac{a}{\sqrt{x}}\right)+\frac{1}{2}\sum_{\sigma=\pm}\exp(\sigma 2a)\,\mathrm{erfc}\left(\frac{a}{\sqrt{x}}+\sigma\sqrt{x}\right)$$

and

41:10:11 $$\int_x^\infty\exp(\beta t)\mathrm{erfc}(t)\mathrm{d}t=\frac{1}{\beta}\left[\exp\left(\frac{\beta^2}{4}\right)\mathrm{erfc}\left(x-\frac{\beta}{2}\right)-\exp(\beta x)\mathrm{erfc}(x)\right]$$

Representing the error function by Φ, Gradshteyn and Ryzhik [Sections 6.28−6.31] list over thirty definite integrals relevant to this chapter. Some of the more interesting, leading to functions from Chapters 60, 30, and 35, are

41:10:12 $$\int_0^\infty t^{\nu}\exp\left(at^2\right)\mathrm{erfc}(bt)\mathrm{d}t=\frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\sqrt{\pi}(1+\nu)b^{1+\nu}}\,\mathrm{F}\left(\frac{1+\nu}{2},\frac{2+\nu}{2},\frac{3+\nu}{2},\frac{a}{b^2}\right)\qquad b^2>a,\quad \nu>-1$$

41:10:13
$$\int_0^\infty \exp(at^2)\,\mathrm{erfc}(bt)\,\mathrm{d}t = \left\{\begin{array}{ll} \dfrac{1}{\sqrt{\pi a}}\,\mathrm{artanh}\left(\sqrt{a}/b\right) & a>0 \\ 1/(\sqrt{\pi}\,b) & a=0 \\ \dfrac{1}{\sqrt{-\pi a}}\,\arctan\left(\sqrt{-a}/b\right) & a<0 \end{array}\right\} \quad b>0$$

and

41:10:14
$$\int_0^\infty \exp(\beta t)\,\mathrm{erfc}\left(\sqrt{bt}\right)\mathrm{d}t = \begin{cases} \dfrac{1}{b-\beta+\sqrt{b^2-b\beta}} & \beta<0<b \\ \dfrac{\sqrt{b}-\sqrt{b-\beta}}{\beta\sqrt{b-\beta}} & 0<\beta<b \end{cases}$$

The operations of semidifferentiation and semiintegration give

41:10:15
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\exp(bx)\,\mathrm{erfc}\left(\pm\sqrt{bx}\right) = \frac{1}{\sqrt{\pi x}} \mp \sqrt{b}\exp(bx)\,\mathrm{erfc}\left(\pm\sqrt{bx}\right)$$

and

41:10:16
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\exp(bx)\,\mathrm{erfc}\left(\pm\sqrt{bx}\right) = \frac{\pm 1}{\sqrt{b}}\left[1-\exp(bx)\,\mathrm{erfc}\left(\pm\sqrt{bx}\right)\right]$$

The semiintegration formula

41:10:17
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\frac{1}{\sqrt{\pi x}}+\exp(x)\,\mathrm{erfc}\left(-\sqrt{x}\right)\right\} = \left\{\frac{1}{\sqrt{\pi x}}+\exp(x)\,\mathrm{erfc}\left(-\sqrt{x}\right)\right\}-1$$

is analogous to

41:10:18
$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\exp(x) = \int_0^x \exp(t)\,\mathrm{d}t = \exp(x)-1$$

illustrating the fact that the $(1/\sqrt{\pi x})+\exp(x)\mathrm{erfc}(-\sqrt{x})$ function plays the same role in the fractional calculus that the exponential function plays in the traditional calculus.

Examples of Laplace transforms involving the functions of this chapter are listed below

41:10:19
$$\int_0^\infty \exp(bt)\,\mathrm{erfc}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\exp(bt)\,\mathrm{erfc}\left(\sqrt{bt}\right)\right\} = \frac{1}{\sqrt{s}\left(\sqrt{s}+\sqrt{b}\right)}$$

41:10:20
$$\int_0^\infty \exp(bt)\,\mathrm{erf}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\exp(bt)\,\mathrm{erf}\left(\sqrt{bt}\right)\right\} = \frac{\sqrt{b}}{\sqrt{s}\,(s-b)}$$

41:10:21
$$\int_0^\infty \exp(-\beta t)\,\mathrm{erf}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\exp(-\beta t)\,\mathrm{erf}\left(\sqrt{bt}\right)\right\} = \frac{\sqrt{b}}{(s+\beta)\sqrt{s+\beta+b}}$$

41:10:22
$$\int_0^\infty \exp(bt)\,\mathrm{erfc}\left(\sqrt{bt}+\frac{a}{\sqrt{t}}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\exp(bt)\,\mathrm{erfc}\left(\sqrt{bt}+\frac{a}{\sqrt{t}}\right)\right\} = \frac{\exp\left\{-2a\left(\sqrt{s}+\sqrt{b}\right)\right\}}{\sqrt{s}\left(\sqrt{s}+\sqrt{b}\right)}$$

41:11 COMPLEX ARGUMENT

The function here symbolized W(x), and sometimes inappropriately called "the error function for complex argument" is $\exp(x)\mathrm{erfc}(-\sqrt{x})$ with x replaced by $-z^2$. That is,

41:11:1 $$\mathrm{W}(z) = \exp(-z^2)\mathrm{erfc}(-iz)$$

This is an important complex-valued function, the real and imaginary parts of which arise in practical problems. These parts, themselves real and bivariate, are described by the integrals

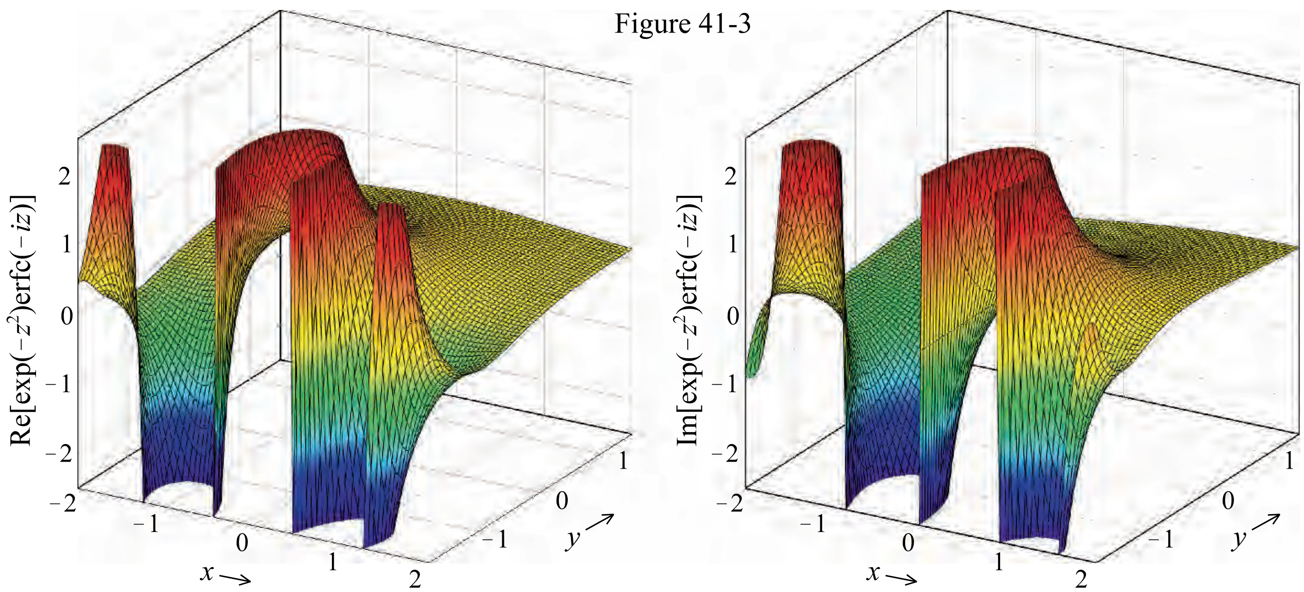
41:11:2 $$\mathrm{Re}\{\mathrm{W}(x+iy)\} = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{(x-t)^2+y^2}\mathrm{d}t$$

and

41:11:3 $$\mathrm{Im}\{\mathrm{W}(x+iy)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-t)\exp(-t^2)}{(x-t)^2+y^2}\mathrm{d}t$$

and are mapped in Figure 41-3.

Figure 41-3



Expression 41:11:2 is known as the *Voigt function* [Thompson, Section 19.7, who uses $V(-x,y)$ as equivalent to our Re{W(x + iy)} notation and discusses computational procedures]. When the argument of the W function is real ($y = 0$), the real and imaginary parts or the W function are respectively an exponential function and a Dawson's integral [next chapter]:

41:11:4 $$\mathrm{W}(x) = \exp(-x^2) + i\frac{2}{\sqrt{\pi}}\,\mathrm{daw}(x)$$

but when the argument is imaginary ($x = 0$), the function is purely real, without any imaginary component:

41:11:5 $$\mathrm{W}(iy) = \exp(y^2)\mathrm{erfc}(y)$$

Abramowitz and Stegun [Table 7.8] provide an extensive list of W(x + iy) values for x and y between 0 and 3.

The functions of this chapter occur frequently in problems that are tackled in Laplace space. Some valuable inverse Laplace transforms are

41:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \exp(bs)\mathrm{erfc}(\sqrt{bs})\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\exp(bs)\mathrm{erfc}(\sqrt{bs})\right\} = \frac{1}{\pi}\sqrt{\frac{b}{t}}\frac{1}{t+b}$$

41:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{\sqrt{s}}\exp(bs)\mathrm{erfc}(\sqrt{bs})\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{\sqrt{s}}\exp(bs)\mathrm{erfc}(\sqrt{bs})\right\} = \frac{1}{\sqrt{\pi(t+b)}}$$

41:11:8 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s}\exp(bs)\mathrm{erfc}(\sqrt{bs})\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{s}\exp(bs)\mathrm{erfc}(\sqrt{bs})\right\} = \frac{2}{\pi}\arctan\left(\sqrt{\frac{t}{b}}\right)$$

41:11:9 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \exp(b^2s^2)\mathrm{erfc}(bs)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\exp(b^2s^2)\mathrm{erfc}(bs)\right\} = \frac{1}{\sqrt{\pi}b}\exp\left(\frac{-t^2}{4b^2}\right)$$

41:11:10 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s}\exp(b^2s^2)\mathrm{erfc}(bs)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{s}\exp(b^2s^2)\mathrm{erfc}(bs)\right\} = \mathrm{erf}\left(\frac{t}{2b}\right)$$

41:11:11 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{\sqrt{s}}\exp\left(\frac{1}{bs}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{bs}}\right)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{\sqrt{s}}\exp\left(\frac{1}{bs}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{bs}}\right)\right\} = \frac{1}{\sqrt{\pi t}}\sinh\left(2\sqrt{\frac{t}{b}}\right)$$

and many more are given by Roberts and Kaufman [pages 317−320].

41:12 GENERALIZATIONS

Respectively, the $\exp(x)\mathrm{erf}(\sqrt{x})$ and $\exp(x)\mathrm{erfc}(\sqrt{x})$ functions are special cases of the Kummer function [Chapter 47]

41:12:1 $$\exp(x)\mathrm{erf}(\sqrt{x}) = 2\sqrt{\frac{x}{\pi}}\,\mathrm{M}\left(1,\tfrac{3}{2},x\right)$$

and the Tricomi function [Chapter 48]

41:12:2 $$\exp(x)\mathrm{erfc}(\sqrt{x}) = \frac{1}{\sqrt{\pi}}\mathrm{U}\left(\tfrac{1}{2},\tfrac{1}{2},x\right) = \sqrt{\frac{x}{\pi}}\,\mathrm{U}\left(1,\tfrac{3}{2},x\right)$$

The asymptotic representation 41:6:6 of $\sqrt{\pi x}\exp(x)\mathrm{erfc}(\sqrt{x})$ is an example of an $L=K-1=0$ hypergeometric function

41:12:3 $$\sqrt{\pi x}\exp(x)\mathrm{erfc}(\sqrt{x}) \sim \sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\frac{-1}{x}\right)^j \qquad \text{large } x$$

and therefore this function has kinship with those listed in Table 18-7.

41:13 COGNATE FUNCTIONS

Dawson's integral [Chapter 42] is closely allied to the functions of this chapter through the relationship

41:13:1 $$\mathrm{daw}(x) = \frac{-i\sqrt{\pi}}{2}\exp(-x^2)\mathrm{erf}(ix)$$

Products of exponential and complementary error functions crop up in discussions of parabolic cylinder functions [Chapter 46]. For example, the parabolic cylinder function of order -1 and argument $2\sqrt{x}$ is

41:13:2
$$\mathrm{D}_{-1}\left(2\sqrt{x}\right)=\sqrt{\frac{\pi}{2}}\exp(x)\mathrm{erfc}\left(\sqrt{2x}\right)$$

However, there is a crucial distinction between the functions of the present chapter and those of Chapter 46, which prevents simple connectivity between the two function families. Both involve functions of the form

41:13:3
$$\exp(ax)\mathrm{erfc}\left(\sqrt{bx}\right)$$

but differ in the magnitude of the a/b ratio. This ratio usually equals unity in the present chapter, but is $1/\sqrt{2}$ in the algebra of parabolic cylinder functions.

The $\exp(x^2)\mathrm{erfc}(-x)$ function is an instance of the Mittag-Leffler function [Section 45:14].

CHAPTER 42

DAWSON'S INTEGRAL daw(x)

Among other practical applications, Dawson's integral arises during studies of the propagation of electromagnetic radiation along the earth's surface. As well as daw(x) itself, $\mathrm{daw}\left(\sqrt{x}\right)$ and $x\,\mathrm{daw}\left(x\right)$ occur frequently and this is recognized in the text.

There are close connections between Dawson's integral and the functions of Chapter 41; thus the relationship

42:0:1 $$\frac{2}{\sqrt{\pi}}\mathrm{daw}\left(-i\sqrt{x}\right) = -i\exp\left(x\right)\mathrm{erf}\left(\sqrt{x}\right)$$

resembles that between a circular function and its hyperbolic counterpart.

42:1 NOTATION

Because no standard exists, the *Atlas* uses the symbol daw(x) to denote Dawson's integral of argument x. Abramowitz and Stegun [Chapter 7] use a generic $F(x)$, while Gradshteyn and Ryzhik adopt a symbolism equivalent to $\left(-i\sqrt{\pi}/2\right)\exp\left(-x^2\right)\mathrm{erf}\left(ix\right)$.

Unfortunately, the name "Dawson's integral" is also given to the product $\exp(x^2)\mathrm{daw}(x)$. Such products are generally defined via the error function of imaginary argument and the symbolism used reflects this approach. Thus the following notations

42:1:1 $$\mathrm{erfi}(x) = \frac{\mathrm{erf}(ix)}{i} = -i\Phi(ix) = \frac{2}{\sqrt{\pi}}\exp\left(x^2\right)\mathrm{daw}(x)$$

and

42:1:2 $$\mathrm{Erfi}(x) = \frac{\sqrt{\pi}}{2}\mathrm{erfi}(x) = \exp\left(x^2\right)\mathrm{daw}(x)$$

are commonly encountered, though Erfi(x) is sometimes regarded as synonymous with erfi(x). This *Atlas* employs daw(x) exclusively, though erfi is mentioned in Sections 42:11 and 42:14.

In certain applications, a rescaling of the argument leads to notational economies. The function $\sqrt{2x}\,\mathrm{daw}\left(\sqrt{x/2}\right)$ has been symbolized $D(x)$, but D is used with many other meanings, not all of which relate to Dawson's integral. $D_{-}(x)$ is a notation sometimes used for daw(x).

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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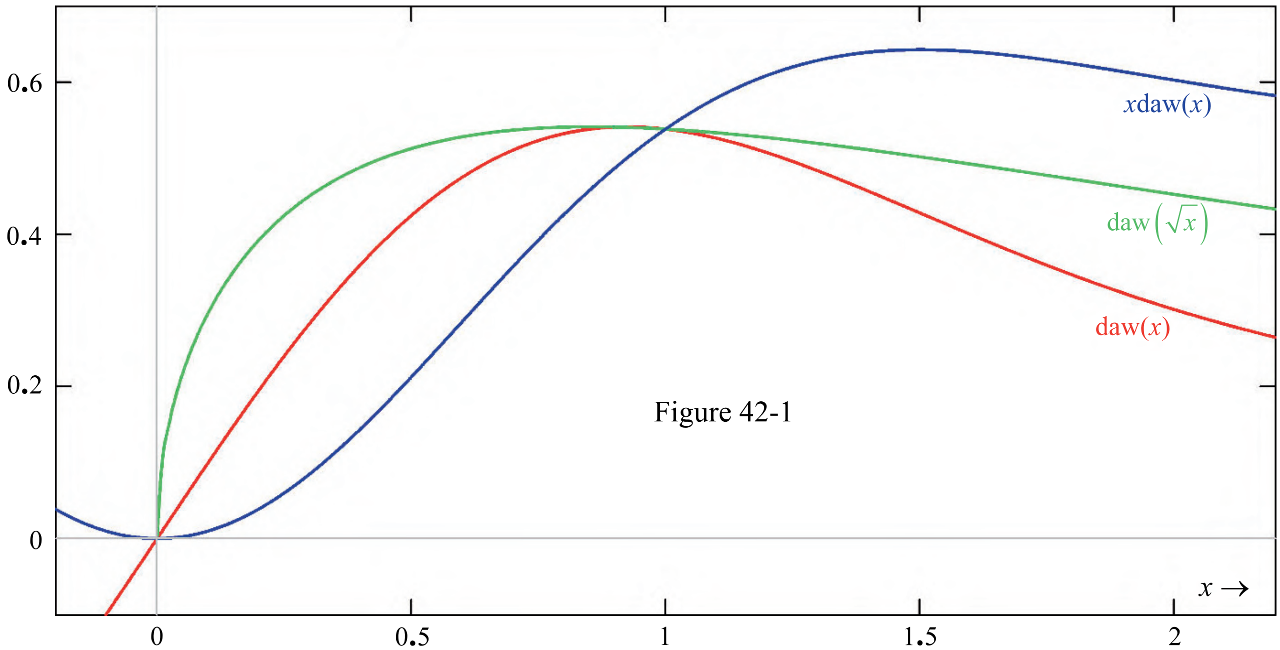


Figure 42-1

42:2 BEHAVIOR

Dawson's integral daw(x) and two of its modifications are mapped in Figure 42-1. Each of these functions is zero at $x = 0$, displays a maximum in the vicinity of $x = 1$, and then slowly declines, approaching ½ in the case of $x$daw(x) and zero in the other cases.

42:3 DEFINITIONS

The indefinite integrals,

42:3:1 $$\mathrm{daw}(x) = \int_0^x \exp\left(t^2 - x^2\right)\mathrm{d}t = \frac{\mathrm{sgn}(x)\exp\left(-x^2\right)}{2}\int_0^{x^2}\frac{\exp(t)}{\sqrt{t}}\mathrm{d}t$$

define Dawson's integral. It may also be defined, via 42:1:1, in terms of the error function of imaginary argument, or as the semiintegral of an exponential function

42:3:2 $$\mathrm{daw}\left(\sqrt{x}\right) = \frac{\sqrt{\pi}}{2}\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\exp(-x)$$

The hypergeometric formulations of Dawson's integral, given in expansions 42:6:2 and 42:6:4, open routes by which daw(x) may be synthesized [Section 43:14]

42:3:3 $$\exp(-x) \xrightarrow[3/2]{1} \frac{1}{\sqrt{x}}\mathrm{daw}\left(\sqrt{x}\right)$$

42:3:4 $$\frac{1}{x}\exp\left(\frac{-1}{x}\right)\mathrm{Ei}\left(\frac{1}{x}\right) \xrightarrow[1]{1/2} \frac{2}{\sqrt{x}}\mathrm{daw}\left(\frac{1}{\sqrt{x}}\right)$$

As the particular integral [Section 24:14] accompanying the exponential function, Dawson's integral appears as a solution to a simple differential equation:

42:3:5 $$\frac{\mathrm{d}\,\mathrm{f}}{\mathrm{d}x}+2x\,\mathrm{f}=1 \qquad \mathrm{f}=\mathrm{daw}(x)+(\text{constant})\exp\left(-x^2\right)$$

42:4 SPECIAL CASES

There are none.

42:5 INTRARELATIONSHIPS

Dawson's integral is an odd function

42:5:1 $$\mathrm{daw}(-x)=-\mathrm{daw}(x)$$

but $x\mathrm{daw}(x)$ is even.

42:6 EXPANSIONS

There are two power-series expansions, each sum of which may be written in several ways:

42:6:1 $$\mathrm{daw}(x)=x-\frac{2x^3}{3}+\frac{4x^5}{15}-\frac{8x^7}{105}+\cdots=x\sum_{j=0}^{\infty}\frac{\left(-2x^2\right)^j}{(2j+1)!!}=\frac{\sqrt{\pi}x}{2}\sum_{j=0}^{\infty}\frac{(-x^2)^j}{\Gamma\left(j+\frac{3}{2}\right)}=x\sum_{j=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)_j}\left(-x^2\right)^j$$

42:6:2 $$\exp(x^2)\,\mathrm{daw}(x)=x+\frac{x^3}{3}+\frac{x^5}{10}+\frac{x^7}{42}+\cdots=x\sum_{j=0}^{\infty}\frac{x^{2j}}{(2j+1)\,j!}=x\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j}\left(x^2\right)^j$$

The latter may be developed into

42:6:3 $$\mathrm{daw}(x)=\frac{1}{2x}\sum_{j=0}^{n-1}\frac{(2j-1)!!}{(2x^2)^j}+\frac{(2n-1)!!x}{\left(2x^2\right)^n\exp\left(x^2\right)}\sum_{j=0}^{\infty}\frac{x^{2j}}{(2j-2n+1)\,j!} \qquad n=1,2,3,\cdots$$

and, if x and n are large enough, the second term becomes negligible and there remains the asymptotic series

42:6:4 $$\mathrm{daw}(x)\sim\frac{1}{2x}+\frac{1}{4x^2}+\frac{3}{8x^3}+\frac{15}{16x^4}+\cdots=\frac{1}{2x}\sum_{j=0}\frac{(2j-1)!!}{\left(2x^2\right)^j}=\frac{1}{2x}\sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\frac{1}{x^2}\right)^j$$

A useful continued fraction expansion is

42:6:5 $$\sqrt{\frac{2}{x}}\,\mathrm{daw}\left(\sqrt{\frac{x}{2}}\right)=\frac{1}{1+}\;\frac{x}{3-x+}\;\frac{3x}{5-x+}\;\frac{5x}{7-x+}\;\frac{7x}{9-x+\cdots}$$

42:7 PARTICULAR VALUES

For unity argument, Dawson's integral evaluates to

42:7:1 $$\mathrm{daw}(1)=\int_0^1 \exp(t^2-1)\,\mathrm{d}t=\frac{1}{2e}\int_0^1 \frac{\exp(t)}{\sqrt{t}}\,\mathrm{d}t=\frac{1}{2e}\int_1^e \frac{\mathrm{d}t}{\ln^{1/2}(t)}=0.53807\,95069\,12768$$

and this value appears frequently below.

	$x=-\infty$	$x=-1$	$x=0$	$x=1$	$x=\infty$
daw(x)	0	−daw(1)	0	daw(1)	0
$\mathrm{daw}\left(\sqrt{x}\right)$	undef	undef	0	daw(1)	0
$x$daw(x)	½	daw(1)	0	daw(1)	½

Each of the curves shown in Figure 42-1 has a maximum and at least one inflection point. The maximum, at argument x_{M}, and the inflection point x_{i}, of $\mathrm{daw}\left(\sqrt{x}\right)$ are important because they appear in the table in Section 37:15. At these particular locations there are simple relationships, namely

42:7:2 $$\text{maximum}:\quad \mathrm{daw}\left(\sqrt{x_{\mathrm{M}}}\right)=\frac{1}{2\sqrt{x_{\mathrm{M}}}}\qquad x_{\mathrm{M}}=0.85403\,26565\,98197$$

42:7:3 $$\text{inflection}:\quad \mathrm{daw}\left(\sqrt{x_{\mathrm{i}}}\right)=\frac{2x_{\mathrm{i}}+1}{4\sqrt{x_{\mathrm{i}}^3}}\qquad x_{\mathrm{i}}=1.8436\,50900\,13325$$

between the values of the function and its argument.

42:8 NUMERICAL VALUES

With keyword **daw**, *Equator*'s Dawson's integral routine returns precise values of daw(x). The algorithm utilizes expansion 42:6:2 when $|x| < 6.5$, but for arguments of larger magnitude, series 42:6:4 is preferred, with implemention via the ε-transformation [Section 10:14]. For the very largest arguments, $\mathrm{daw}(x) \approx 1/(2x)$ suffices.

42:9 LIMITS AND APPROXIMATIONS

For arguments close to zero and infinity, the following limiting approximations apply

	daw(x)	$\mathrm{daw}\left(\sqrt{x}\right)$	$x$daw(x)
$x\to 0$	$x-\frac{2}{3}x^3$	$\sqrt{x}\left(1-\frac{2}{3}x\right)$	$x^2-\frac{2}{3}x^4$
$x\to\infty$	$\dfrac{2x^2+1}{4x^3}$	$\dfrac{2x+1}{4\sqrt{x^3}}$	$\dfrac{1}{2}+\dfrac{1}{4x^2}$

42:10 OPERATIONS OF THE CALCULUS

Differentiation of Dawson's integral yields

42:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{daw}(bx) = -b\left[2bx\,\mathrm{daw}(bx) - 1\right]$$

Repeated derivatives obey the following general formula

42:10:2
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{daw}(bx) = b^n[\mathrm{f}_n(bx) + \mathrm{g}_n(bx)\mathrm{daw}(bx)]$$

where f_n and g_n are polynomials in bx of which early members are

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\mathrm{f}_n(bx)$	1	$-2bx$	$-4+4b^2x^2$	$20bx-8b^3x^3$	$32-72b^2x^2+16b^4x^4$
$\mathrm{g}_n(bx)$	$-2bx$	$-2+4b^2x^2$	$12bx-8b^3x^3$	$12-48b^2x^2+16b^4x^4$	$-120bx+160b^3x^3-32b^5x^5$

The indefinite integral of daw(x) is not expressible in terms of named functions, but nonetheless it is quite a simple hypergeometric function [Section 18:14]:

42:10:3
$$\int_0^x \mathrm{daw}(t)\mathrm{d}t = \frac{x^2}{2}\sum_{j=0}^{\infty}\frac{(1)_j}{\left(\frac{3}{2}\right)_j(2)_j}\left(-x^2\right)^j$$

Formulas for the integrals and derivatives of $\mathrm{daw}\left(\sqrt{x}\right)$ are simpler than those of daw(x). Indeed, even the semiintegral, semiderivative and sesquiderivative expressions are surprisingly simple for this function. The following panel

$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\mathrm{daw}\left(\sqrt{x}\right)=$	$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\mathrm{daw}\left(\sqrt{x}\right)=$	$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\mathrm{daw}\left(\sqrt{x}\right)=$	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{daw}\left(\sqrt{x}\right)=$	$\frac{\mathrm{d}^{3/2}}{\mathrm{d}x^{3/2}}\mathrm{daw}\left(\sqrt{x}\right)=$
$\sqrt{x}-\mathrm{daw}\left(\sqrt{x}\right)$	$\frac{\sqrt{\pi}}{2}\left[1-\exp(-x)\right]$	$\frac{\sqrt{\pi}}{2}\exp(-x)$	$\frac{1}{2\sqrt{x}}-\mathrm{daw}\left(\sqrt{x}\right)$	$\frac{-\sqrt{\pi}}{2}\exp(-x)$

demonstrates how closely the function $\left(2/\sqrt{\pi}\right)\mathrm{daw}\left(\sqrt{x}\right)$ is associated with the exponential exp($-x$) function: differentiation converts $1-\exp(-x)$ to $\exp(-x)$, the halfway house along that route being $\left(2/\sqrt{\pi}\right)\mathrm{daw}\left(\sqrt{x}\right)$

42:10:4
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[1-\exp(-x)\right] = \frac{2}{\sqrt{\pi}}\mathrm{daw}\left(\sqrt{x}\right)$$

Some definite integrals and Laplace transforms include:

42:10:4
$$\int_0^\infty \exp(-at^2)\,\mathrm{daw}(bt)\mathrm{d}t = \frac{\mathrm{arcsch}\left(\sqrt{a}/b\right)}{2\sqrt{a+b^2}} \qquad a,b>0$$

42:10:5
$$\int_0^\infty \sin(\beta t)\,\mathrm{daw}(bt)\mathrm{d}t = \frac{\pi}{4b}\exp\left(\frac{-\beta^2}{4b^2}\right) \qquad \beta>0$$

42:10:6
$$\int_0^\infty \mathrm{daw}(bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\mathrm{daw}(bt)\right\} = \frac{1}{4b}\exp\left(\frac{s^2}{4b^2}\right)\mathrm{Ei}\left(\frac{-s^2}{4b^2}\right) \qquad b>0$$

42:10:7
$$\int_0^\infty t\,\mathrm{daw}(bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{t\,\mathrm{daw}(bt)\right\} = \frac{1}{2bs}+\frac{s}{8b^3}\mathrm{Ei}\left(\frac{-s^2}{4b^2}\right) \qquad b>0$$

42:10:8
$$\int_0^\infty \mathrm{daw}(\sqrt{bt})\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\mathrm{daw}\left(\sqrt{bt}\right)\right\} = \sqrt{\frac{\pi b}{s}}\frac{1}{2(s+b)} \qquad b>0$$

42:11 COMPLEX ARGUMENT

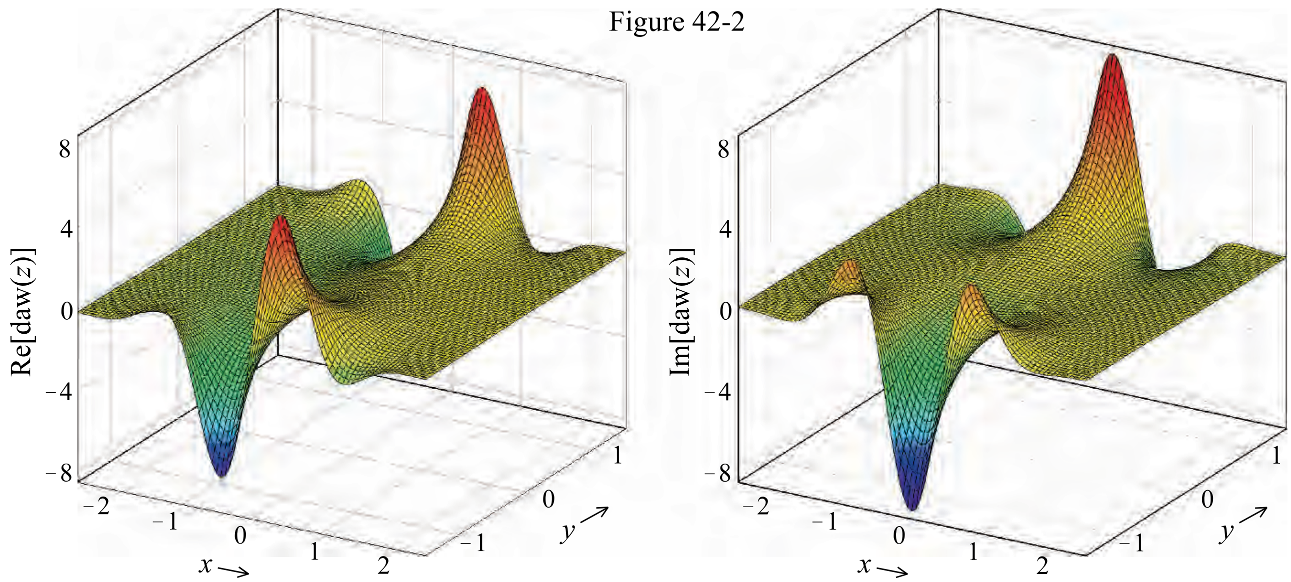
Figure 42-2 shows the real and imaginary parts of Dawson’s integral of complex variable, daw($x + iy$). It is related by

42:11:1 $$\mathrm{daw}(z)=\frac{i\sqrt{\pi}}{2}\left[\mathrm{W}(z)-\exp(-z^2)\right]$$

to the complex W function discussed in Section 41:11. For purely imaginary argument:

42:11:2 $$\mathrm{daw}(iy)=\frac{i\sqrt{\pi}}{2}\exp(y^2)\,\mathrm{erf}(y)$$

Figure 42-2



Inverse Laplace transformations involving Dawson’s integral include

42:11:3 $$\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt{s}}\,\mathrm{daw}\left(\frac{a}{\sqrt{s}}\right)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s=\mathfrak{I}\left\{\frac{\mathrm{daw}\left(a/\sqrt{s}\right)}{\sqrt{s}}\right\}=\frac{\sin\left(2a\sqrt{t}\right)}{2\sqrt{t}}$$

42:12 GENERALIZATIONS

Dawson’s integral is a special case of the entire incomplete gamma function [Chapter 45]

42:12:1 $$\mathrm{daw}(x)=\frac{\sqrt{\pi}}{2}x\exp\left(-x^2\right)\gamma\mathrm{n}\left(\tfrac{1}{2},-x^2\right)$$

and the Kummer function [Chapter 47]

42:12:2 $$\mathrm{daw}(x)=x\,\mathrm{M}\left(1,\tfrac{3}{2},-x^2\right)$$

42:13 COGNATE FUNCTIONS

Dawson's integral is the $\nu = 2$ instance of a set of peaked functions defined by

42:13:1 $$\int_0^x \exp\left(t^\nu - x^\nu\right)\mathrm{d}t \qquad \nu > 0$$

and expressible as $x\,\mathrm{M}\left(1,\frac{\nu+1}{\nu},-x^\nu\right)$ in terms of the Kummer function.

42:14 RELATED TOPIC: gaussian integrals of complex argument

The functions of Chapters 39, 40 and 42 have a kinship that is not immediately apparent, but which becomes evident on considering the integration of the Gaussian function of complex argument. This latter function may be written in rectangular or polar notation [Section 1:11] as

42:14:1 $$\frac{2}{\sqrt{\pi}}\exp\left(-z^2\right) = \frac{2}{\sqrt{\pi}}\exp\left(y^2 - x^2 - 2ixy\right) = \frac{2}{\sqrt{\pi}}\exp\left(-\rho^2[\cos(\theta) + i\sin(\theta)]^2\right)$$

The integral of this function with respect to z is simply the error function of complex argument of that is discussed in Section 40:10 and portrayed in Figure 40-3. For our present purpose, however, consider integrating the complex Gaussian function along a radial line in the complex plane, from the origin to some point (ρ,θ). With z constrained by the constancy of θ, then

42:14:2 $$\frac{2}{\sqrt{\pi}}\int_0^{\rho,\theta}\exp\left(-z^2\right)_{\text{constant}\,\theta}\mathrm{d}z = \frac{2\left[\cos(\theta) + i\sin(\theta)\right]}{\sqrt{\pi}}\int_0^{\rho}\exp\left(-z^2\left[\cos(\theta) + i\sin(\theta)\right]^2\right)\mathrm{d}z$$

For certain values of the angle θ, the integral is easily evaluated. Thus for $\theta = \pi/2$, then by equation 42:3:1

42:14:3 $$\frac{2}{\sqrt{\pi}}\int_0^{\rho,\theta}\exp\left(-z^2\right)_{\theta=\pi/2}\mathrm{d}z = \frac{2i}{\sqrt{\pi}}\int_0^{\rho}\exp\left(z^2\right)\mathrm{d}z = \frac{2i}{\sqrt{\pi}}\exp(\rho^2)\mathrm{daw}(\rho)$$

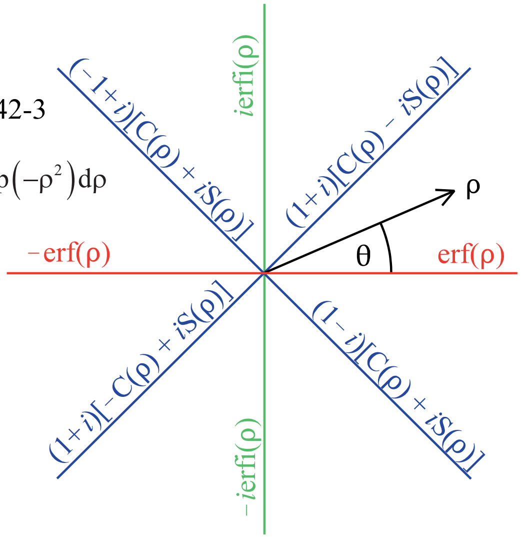
Likewise, when $\theta = \pi/4$, with help from formula 39:11:2, it can be shown that

42:14:4 $$\frac{2}{\sqrt{\pi}}\int_0^{\rho,\theta}\exp\left(-z^2\right)_{\theta=\pi/4}\mathrm{d}z = \sqrt{\frac{2}{\pi}}(1+i)\int_0^{\rho}\exp\left(-iz^2\right)\mathrm{d}z = (1+i)\left[\mathrm{C}(\rho) - i\mathrm{S}(\rho)\right]$$

There are eight angles for which similar calculations can be carried out and the results are displayed in Figure 42-3 as labels on the "spokes" that radiate from the origin at the appropriate angles. In this diagram, erfi(ρ) is used as a succinct alternative to $(2/\sqrt{\pi})\exp(\rho^2)\mathrm{daw}(\rho)$. The labels displayed are nothing but the complex expressions for erf(z) at the appropriate phase angles expressed in terms of the modulus ρ, but they do serve to demonstrate the sought linkage between the error function, Dawson's integral and the Fresnel integrals.

Figure 42-3

$$\frac{2}{\sqrt{\pi}}\int_0^{\rho}\exp\left(-\rho^2\right)\mathrm{d}\rho$$



CHAPTER
43

THE GAMMA FUNCTION $\Gamma(\nu)$

The *gamma function* is unusual in the simplicity of its recurrence properties. It is because of this that the gamma function (and its special case, the factorial) plays such an important role in the theory of other functions. The reciprocal $1/\Gamma(\nu)$ and the logarithm $\ln\{\Gamma(\nu)\}$ are also important and are discussed in this chapter, as is the related complete beta function $B(\nu,\mu)$, which is addressed in Section 43:13.

Formulas involving the gamma function often become simpler when written for argument $1+\nu$ rather than ν, and we have sometimes taken advantage of this fact. Because $\nu\Gamma(\nu) = \Gamma(1+\nu)$, such a change of argument is readily achieved.

43:1 NOTATION

The gamma function is also known as *Euler's integral of the second kind*. $\Gamma(1+\nu)$ is sometimes symbolized $\nu!$ or $\Pi(\nu)$ and termed the *factorial function* or *pi function*, respectively. To avoid possible confusion with the functions of Chapter 45, Γ is sometimes distinguished as the *complete gamma function*.

43:2 BEHAVIOR

The behavior of $\Gamma(\nu)$ for $-5 < \nu < 6$ is shown on the accompanying map, Figure 43-1; it is complicated. For positive argument, the gamma function passes through a shallow minimum between $\nu = 1$ and $\nu = 2$ and increases steeply as $\nu = 0$ or $\nu = \infty$ is approached. On the negative side, $\Gamma(\nu)$ is segmented: it has positive values for $-2 < \nu < -1$, $-4 < \nu < -3$, $-6 < \nu < -5$, $\cdots$ but negative values for $-1 < \nu < 0$, $-3 < \nu < -2$, $-5 < \nu < -4$, $\cdots$, with discontinuities at $\nu = 0, -1, -2, \cdots$. The gamma function never takes the value zero, but it comes very close between consecutive large negative integers.

The reciprocal $1/\Gamma(\nu)$ has no discontinuities. It rapidly approaches zero as $\nu \to \infty$ and equals zero at $\nu = 0, -1, -2, \cdots$. As shown on the map, its oscillations become increasingly violent as the argument becomes more negative.

The logarithm $\ln\{\Gamma(\nu)\}$ is usually only considered for $\nu > 0$. This is the convention adopted in drawing the map, which shows that $\ln\{\Gamma(\nu)\}$ is a positive function except between $\nu = 1$ and $\nu = 2$, where it is briefly and slightly negative.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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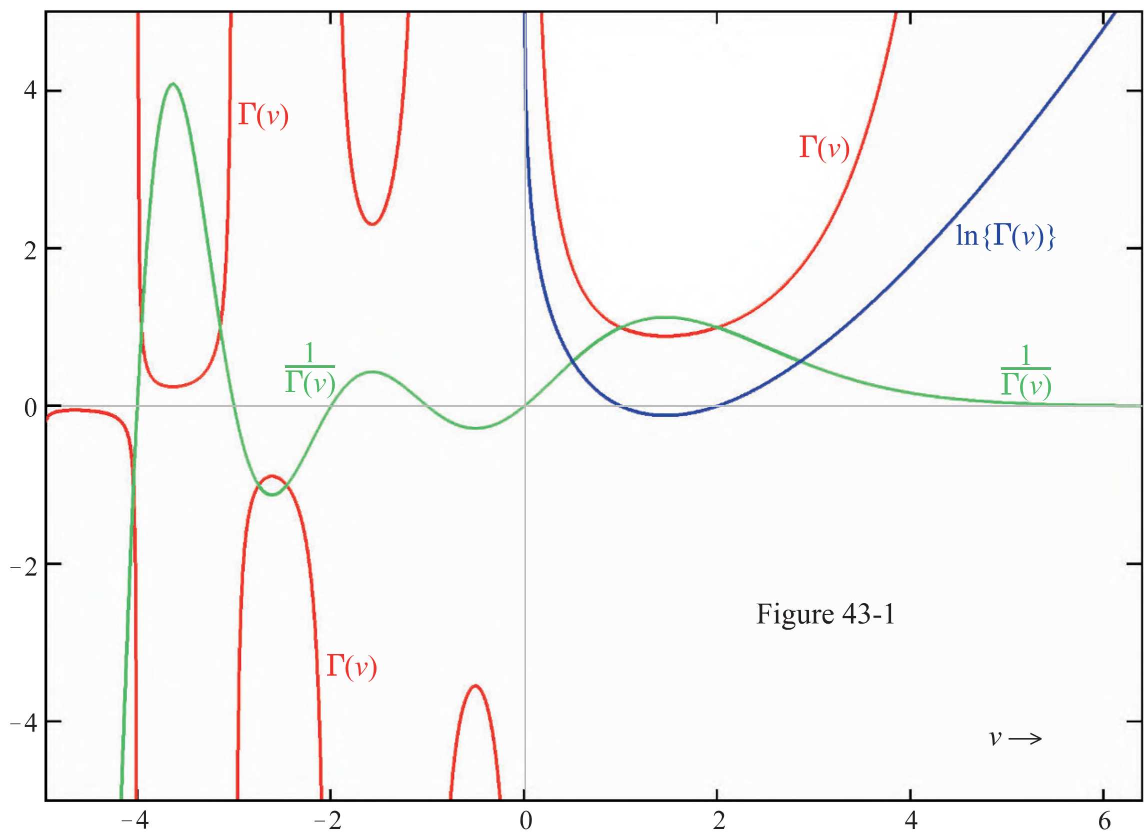


43:3 DEFINITIONS

Although it is restricted to positive arguments, the most useful definition of the (complete) gamma function is the *Euler integral*

43:3:1 $$\Gamma(v) = \int_0^\infty t^{v-1} \exp(-t)\,\mathrm{d}t \qquad v > 0$$

More comprehensive are the *Gauss limit definition*

43:3:2 $$\Gamma(v) = \lim_{n\to\infty}\left\{\frac{n^v}{v(1+v)\left(1+\frac{v}{2}\right)\cdots\left(1+\frac{v}{n-1}\right)\left(1+\frac{v}{n}\right)}\right\}$$

and the *infinite product definition of Weierstrass*

43:3:3 $$\frac{1}{\Gamma(v)} = v\exp(\gamma v)\prod_{j=1}^{\infty}\left(1+\frac{v}{j}\right)\exp\left(\frac{-v}{j}\right)$$

where γ is Euler's constant [Chapter 1].

The (complete) gamma function may be expressed as a definite integral in many ways apart from the one given above. Gradshteyn and Ryzhik [Section 8.31] give a long list of which the following are representative:

43:3:4 $$\Gamma(1+v) = \int_0^1 \ln^v\left(\frac{1}{t}\right)\mathrm{d}t \qquad v > -1$$

43:3:5 $$\Gamma(v) = s^v \int_0^\infty t^{v-1}\exp(-st)\,\mathrm{d}t \qquad v > 0 \qquad s > 0$$

43:3:6 $$\Gamma(v) = \omega^v \sec\left(\frac{\pi v}{2}\right)\int_0^\infty t^{v-1}\cos(\omega t)\,\mathrm{d}t \qquad 0 < v < 1 \qquad \omega > 0$$

The last two definitions may be regarded as Laplace and Fourier transforms, respectively.

Likewise, there are several ways of representing the logarithm of the gamma function by means of integrals. One is

43:3:7 $$\ln\{\Gamma(1+v)\} = \int_0^1 \left(\frac{t^v - 1}{t-1} - v\right)\frac{\mathrm{d}t}{\ln(t)} \qquad v > -1$$

and others may be found in Gradshteyn and Ryzhik [Section 8.34].

43:4 SPECIAL CASES

The (complete) gamma function reduces to the factorial function [Chapter 2] when its argument is a positive integer:

43:4:1 $$\Gamma(n) = (n-1)! \qquad n = 1,2,3,\cdots$$

The (complete) gamma function of odd multiples, positive or negative, of $\frac{1}{2}$ are all proportional to $\sqrt{\pi}$:

43:4:2 $$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1.7724\,53805\,99052$$

43:4:3 $$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \left(\tfrac{1}{2}\right)_n\sqrt{\pi} \qquad n = 0,1,2,\cdots$$

43:4:4 $$\Gamma\left(\tfrac{1}{2}-n\right) = \frac{(-2)^n}{(2n-1)!!}\sqrt{\pi} = \frac{(-4)^n}{(2n)!}\sqrt{\pi} = \frac{(-1)^n}{\left(\tfrac{1}{2}\right)_n}\sqrt{\pi} \qquad n = 0,1,2,\cdots$$

the constant of proportionality being a rational number (an integer or a fraction) involving double factorial [Section 2:13] or factorial [Chapter 2] functions or Pochhammer polynomials [Chapter 18].

In a similar fashion, the (complete) gamma functions of certain multiples of $\frac{1}{3}$ may be expressed in terms of $\Gamma(\frac{1}{3})$ or $\Gamma(\frac{2}{3})$:

43:4:5 $$\Gamma\left(\tfrac{1}{3}\right) = 2.6789\,38534\,70775$$

43:4:6 $$\Gamma\left(\tfrac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\,\Gamma\left(\tfrac{1}{3}\right)} = 1.3541\,17939\,42640$$

The proportionality constants in these cases involve triple factorials [Section 2:13] or Pochhammer polynomials [Chapter 18]:

43:4:7 $$\Gamma\left(n+\tfrac{1}{3}\right) = \frac{(3n-2)!!!\,\Gamma\left(\tfrac{1}{3}\right)}{3^n} = \left(\tfrac{1}{3}\right)_n\Gamma\left(\tfrac{1}{3}\right) \qquad n = 0,1,2,\cdots$$

43:4:8 $$\Gamma\left(n+\tfrac{2}{3}\right)=\frac{(3n-1)!!!}{3^n}\Gamma\left(\tfrac{2}{3}\right)=\left(\tfrac{2}{3}\right)_n\Gamma\left(\tfrac{2}{3}\right) \qquad n=0,1,2,\cdots$$

43:4:9 $$\Gamma\left(\tfrac{1}{3}-n\right)=\frac{(-3)^n}{(3n-1)!!!}\Gamma\left(\tfrac{1}{3}\right)=\frac{\Gamma\left(\tfrac{1}{3}\right)}{\left(\tfrac{1}{3}-n\right)_n} \qquad n=0,1,2,\cdots$$

43:4:10 $$\Gamma\left(\tfrac{2}{3}-n\right)=\frac{(-3)^n}{(3n-2)!!!}\Gamma\left(\tfrac{2}{3}\right)=\frac{\Gamma\left(\tfrac{2}{3}\right)}{\left(\tfrac{2}{3}-n\right)_n} \qquad n=0,1,2,\cdots$$

The (complete) gamma functions of one quarter and three quarters are related to *Gauss's constant*, g [Section 1:7]:

43:4:11 $$\Gamma\left(\tfrac{1}{4}\right)=\sqrt{(2\pi)^{3/2}g}=3.6256\ 09908\ 22191$$

43:4:12 $$\Gamma\left(\tfrac{3}{4}\right)=\frac{\sqrt{2}\,\pi}{\Gamma\left(\tfrac{1}{4}\right)}=\sqrt{\frac{1}{g}\sqrt{\frac{\pi}{2}}}=1.2254\ 16702\ 46518$$

Thence, using expressions 43:5:5 or 43:5:6, the (complete) gamma functions of such arguments as $n+\tfrac{1}{4}$ or $\tfrac{3}{4}-n$ may be formulated.

43:5 INTRARELATIONSHIPS

The (complete) gamma function obeys the reflection formulas

43:5:1 $$\Gamma(-v)=\frac{-\pi\csc(\pi v)}{v\Gamma(v)}=\frac{\pi\csc(-\pi v)}{\Gamma(1+v)}$$

and

43:5:2 $$\Gamma\left(\tfrac{1}{2}-v\right)=\frac{\pi\sec(\pi v)}{\Gamma(\tfrac{1}{2}+v)}$$

The recurrence formulas

43:5:3 $$\Gamma(1+v)=v\Gamma(v)$$

and

43:5:4 $$\Gamma(v-1)=\frac{\Gamma(v)}{v-1}$$

generalize to the formulas

43:5:5 $$\Gamma(n+v)=v(1+v)(2+v)\cdots(n-1+v)\Gamma(v)=(v)_n\Gamma(v) \qquad n=0,1,2,\cdots$$

and

43:5:6 $$\Gamma(v-n)=\frac{\Gamma(v)}{(v-1)(v-2)(v-3)\cdots(v-n)}=\frac{(-)^n\Gamma(v)}{(1-v)_n} \qquad n=0,1,2,\cdots$$

involving the Pochhammer polynomial [Chapter 18].

The duplication and triplication formulas

43:5:7
$$\Gamma(2v) = \frac{4^v}{2\sqrt{\pi}}\Gamma(v)\Gamma(\tfrac{1}{2}+v)$$

and

43:5:8
$$\Gamma(3v) = \frac{27^v}{2\pi\sqrt{3}}\Gamma(v)\Gamma(\tfrac{1}{3}+v)\Gamma(\tfrac{2}{3}+v)$$

are the $n = 2$ and 3 cases of the general *Gauss-Legendre formula*

43:5:9
$$\Gamma(nv) = \sqrt{\frac{2\pi}{n}}\,\frac{n^{nv}}{(2\pi)^{n/2}}\prod_{j=0}^{n-1}\Gamma\left(\frac{j}{n}+v\right) \qquad n = 2,3,4,\cdots$$

which applies for any positive integer multiplier n.

From 43:5:5 and 43:5:6, one may derive the expressions

43:5:10
$$\frac{\Gamma(n+v)}{\Gamma(v)} = (v)_n \qquad n = 1,2,3,\cdots$$

and

43:5:11
$$\frac{\Gamma(v-n)}{\Gamma(v)} = \frac{(-1)^n}{(1-v)_n} \qquad n = 1,2,3,\cdots$$

for the ratio of the (complete) gamma functions of two arguments that differ by an integer. These formulas may be used even when the individual gamma functions are infinite; for example, $\Gamma(v-3)/\Gamma(v) \to -\tfrac{1}{6}$ as $v \to 0$.

Because of the frequent occurrence of the reciprocal gamma function in power series expansions of transcendental functions, particular values of the latter functions often serve as sums of infinite series of reciprocal gamma functions. For example, on account of expansion 41:6:4, we have

43:5:12
$$\frac{1}{\Gamma(\tfrac{1}{2})} + \frac{1}{\Gamma(1)} + \frac{1}{\Gamma(\tfrac{3}{2})} + \cdots = \sum_{j=1}^{\infty}\frac{1}{\Gamma(\tfrac{j}{2})} = \frac{1}{\sqrt{\pi}} + e\,\mathrm{erfc}(-1) = 5.5731\ 69664\ 31004$$

while

43:5:13
$$\frac{1}{\Gamma(v)} - \frac{1}{\Gamma(v+1)} + \frac{1}{\Gamma(v+2)} - \cdots = \sum_{j=1}^{\infty}\frac{(-1)^j}{\Gamma(v+j)} = \frac{\gamma\mathrm{n}(v-1,-1)}{e}$$

follows from the expansion in Section 45:5, γn being the *entire incomplete gamma function* [Chapter 45].

43:6 EXPANSIONS

Early coefficients in the power series expansion of the (complete) gamma function

43:6:1
$$\Gamma(v) = \sum_{j=1}^{\infty} b_j v^j \qquad v \neq 0,-1,-2,\cdots$$

are $b_1 = 1$, $b_2 = -\gamma$, $b_3 = (6\gamma^2+\pi^2)/12$ and subsequent coefficients may be found from the recursion formula

43:6:2
$$b_{j+1} = \frac{-1}{j}\left[\gamma b_j + (-)^j\sum_{k=1}^{j-1}(-)^k\zeta(j+1-k)b_k\right] \qquad j = 1,2,3,\cdots$$

Here γ denotes Euler's constant [Section 1:7] and ζ(n) is the nth *zeta number* [Chapter 3]. A very similar expansion holds for the reciprocal of the gamma function

43:6:3
$$\frac{1}{\Gamma(v)}=\sum_{j=1}^{\infty}c_j\,v^j$$

and in this case $c_1 = 1$, $c_2 = \gamma$, and $c_3 = (6\gamma^2-\pi^2)/12$, with a recursion formula,

43:6:4
$$c_{j+1}=\frac{1}{j}\left[\gamma c_j+(-)^j\sum_{k=1}^{j-1}(-)^k\zeta(j+1-k)c_k\right] \qquad j=1,2,3,\cdots$$

that differs only in a sign from formula 43:6:2. Numerical values of c_j for $j = 1,2,3,\cdots,26$ are listed by Abramowitz and Stegun [page 256].

The power series expansion of the logarithm of the (complete) gamma function is less complicated:

43:6:5
$$\ln\{\Gamma(1+v)\}=-\gamma v+\frac{\pi^2}{12}v^2-\cdots=-\gamma v+\sum_{j=2}^{\infty}\frac{\zeta(j)}{j}(-v)^j \qquad -1<v\le 1$$

More rapidly convergent is the similar series

43:6:6
$$\ln\{\Gamma(1+v)\}=(1-\gamma)v+\frac{1}{2}\ln\left(\frac{\pi v(1-v)}{(1+v)\sin(\pi v)}\right)-\sum_{j}^{\infty}\frac{\zeta(j)-1}{j}v^j \qquad j=3,5,7,\cdots \qquad -1<v<1$$

The gamma function may also be expanded as the infinite product

43:6:7
$$\Gamma(1+v)=\prod_{j=1}^{\infty}\frac{(j+1)^v}{(j+v)j^{v-1}}$$

An asymptotic expansion of the gamma function is provided by *Stirling's formula* (James Stirling, 1692 – 1770, Scottish mathematician):

43:6:8
$$\Gamma(v)\sim\sqrt{\frac{2\pi}{v}}\,\exp(-v)v^v\left(1+\frac{1}{12v}+\frac{1}{288v^2}-\frac{139}{51840v^3}-\cdots\right) \qquad v\to\infty$$

The corresponding asymptotic expansion for the gamma function's logarithm is

43:6:9
$$\ln\{\Gamma(v)\}\sim\ln\left(\sqrt{2\pi}\right)-v+(v-\tfrac{1}{2})\ln(v)+\frac{1}{12v}-\frac{1}{360v^3}+\cdots+\frac{B_{2j}}{2j(2j-1)v^{2j-1}}+\cdots \qquad v\to\infty$$

where B_{2j} denotes a Bernoulli number [Chapter 4]. The series that is part of 43:6:9 is the asymptotic expansion of an integral that defines the *Binet function* (Jacques Phillipe Marie Binet, French physicist and astronomer, 1786 – 1856)

43:6:10
$$\frac{-v}{\pi}\int_0^{\infty}\frac{\ln\left(1-\exp(-2\pi t)\right)}{v^2+t^2}\,dt\sim\frac{1}{12v}-\frac{1}{360v^3}+\frac{1}{1260v^5}-\cdots=\sum_{j=1}\frac{B_{2j}}{2j(2j-1)v^{2j-1}}$$

Though exact only in the $j\to\infty$ or $v\to\infty$ limit, truncated versions of expansions 43:6:3 – 9 are remarkably accurate for modest values of the argument [see 43:9:1, for example]. Relationship 43:6:9 is a special case of *Barnes's asymptotic expansion* (Ernest William Barnes, English mathematician and bishop, 1874 – 1953)

43:6:11
$$\ln\{\Gamma(v+c)\}\sim(v+c)\ln(v)-v+\frac{1}{2}\ln\left(\frac{2\pi}{v}\right)+\frac{6c^2-6c+1}{12v}$$
$$-\frac{2c^3-3c^2+c}{12v^2}+\cdots-\frac{B_{j+1}(c)}{j(j+1)(-v)^j}+\cdots \qquad v\to\infty$$

where $B_{j+1}(c)$ denotes a *Bernoulli polynomial* [Chapter 20]. From this expansion one may also derive the useful expansion

43:6:12 $$\frac{\Gamma(v+c)}{\Gamma(v)} \sim v^c\left[1+\frac{c(c-1)}{2v}+\frac{c(c-1)(c-2)(3c-1)}{24v^2}+\frac{c^2(c-1)^2(c-2)(c-3)}{48v^3}+\cdots\right] \qquad v\to\infty$$

for the ratio of two gamma functions of large, but not very different, arguments.

43:7 PARTICULAR VALUES

In addition to those reported in 43:4:5,6 and 43:4:11,12, there are the following:

v	$-5/2$	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3	$7/2$	∞
$\Gamma(v)$	$\frac{-8\sqrt{\pi}}{15}$	$-\infty\|+\infty$	$\frac{4\sqrt{\pi}}{3}$	$+\infty\|-\infty$	$-2\sqrt{\pi}$	$-\infty\|+\infty$	$\sqrt{\pi}$	1	$\frac{\sqrt{\pi}}{2}$	1	$\frac{3\sqrt{\pi}}{4}$	2	$\frac{15\sqrt{\pi}}{8}$	∞

Local maxima or minima of $\Gamma(v)$ correspond to the zeros of the *digamma function* [Section 44:7].

43:8 NUMERICAL VALUES

Except at $v=0,-1,-2,\cdots$, (where discontinuities occur) and in the immediate vicinity of these values, *Equator*'s gamma function routine (keyword **Gamma**) returns values of $\Gamma(v)$ precise to 15 decimal digits for inputs in the range $-170 \le v \le 170$. Relation 43:4:1 is exploited if v is a positive integer. Otherwise, *Equator* calculates the integer value $I=\mathrm{Int}(v)$ and the fractional value $f=\mathrm{frac}(v)$ of the input argument, and then employs the algorithm

43:8:1 $$\Gamma(v)\begin{cases} \approx \dfrac{\pi P}{\sin(\pi f)}\displaystyle\sum_{j=1}^{28} c_j(1-f)^j & 0<f<1/2 \\ = P\sqrt{\pi} & f=1/2 \\ \approx P\Big/\displaystyle\sum_{j=1}^{28} c_j f^j & 1/2<f<1 \end{cases}$$

where c_j is the jth coefficient in equation 43:6:3. Pochhammer polynomials [Chapter 18] are used as follows

43:8:2 $$P=\begin{cases} 1/(v)_{|I|} & I<0 \\ 1 & I=0 \\ (f)_I & I\ge 1 \end{cases}$$

in the calculation of P. This strategy is based on equations 43:5:5 and 43:5:1.

Equator also has the facility, especially useful when v exceeds 170, to generate values of the logarithm of the gamma function via its ln- or log10-gamma function routines (keywords **lnGamma** and **log10Gamma**). For arguments in the range $0<v\le 170$, it suffices to take the logarithm of the value generated by the algorithm outlined above. For $v>170$, *Equator* uses the formula

43:8:3 $$\ln\{\Gamma(v)\} \approx v\ln(v)+\frac{\ln(2\pi/v)}{2}-v+\frac{1}{12v}-\frac{1}{360v^3}+\frac{1}{1260v^5}$$

which is a truncated version of 43:6:9, to produce ample precision. No values of $\ln\{\Gamma(v)\}$ are reported for negative v because $\Gamma(v)$ is itself frequently negative (and hence its logarithm is complex) in this range of argument.

43:9 LIMITS AND APPROXIMATIONS

In the limit of large argument,

43:9:1 $$\Gamma(v) \to \sqrt{\frac{2\pi}{v}}\exp\{-v\}v^{v} \qquad v \to \infty$$

this being the leading term in Stirling's approximation 43:6:8.

Close to its zeros, the reciprocal gamma function is approximated by

43:9:2 $$\frac{1}{\Gamma(v)} \approx (-)^{n} n!(v+n)\left[1-(v+n)\psi(n+1)\right] \qquad v \approx -n = 0,-1,-2,\cdots$$

where ψ denotes the digamma function [Chapter 44]. For example, close to the origin, $1/\Gamma(v)$ is well approximated by $v + \gamma v^2$.

The approximation

43:9:3 $$\frac{\Gamma(v+\frac{1}{2})}{\Gamma(v)} \approx \sqrt{v}\left[1-\frac{1}{8v}+\frac{1}{128v^{2}}+\frac{5}{1024v^{3}}\right] \qquad \text{large positive } v$$

is sometimes useful because of its rapid convergence. See the discussion surrounding equation 51:13:5 for another aspect of this ratio of gamma functions.

43:10 OPERATIONS OF THE CALCULUS

Differentiation of the (complete) gamma function and its logarithm yield

43:10:1 $$\frac{\mathrm{d}}{\mathrm{d}v}\Gamma(v) = \psi(v)\Gamma(v)$$

43:10:2 $$\frac{\mathrm{d}^{n}}{\mathrm{d}v^{n}}\ln\{\Gamma(v)\} = \psi^{(n-1)}(v) \qquad n = 1,2,3,\cdots$$

where ψ and $\psi^{(n)}$ are the digamma and polygamma functions [Chapter 44].

Few simple integrals involving the gamma function have been established, although there are numerous integrals of products and quotients of gamma functions [see Gradshteyn and Ryzhik, Sections 6.41 and 6.42]. A number of integrals involving the logarithm of the gamma function, and including

43:10:3 $$\int_{v}^{v+1}\ln\{\Gamma(t)\}\,\mathrm{d}t = \ln\left(\sqrt{2\pi}\right)+v\ln(v)-v \qquad v > 0$$

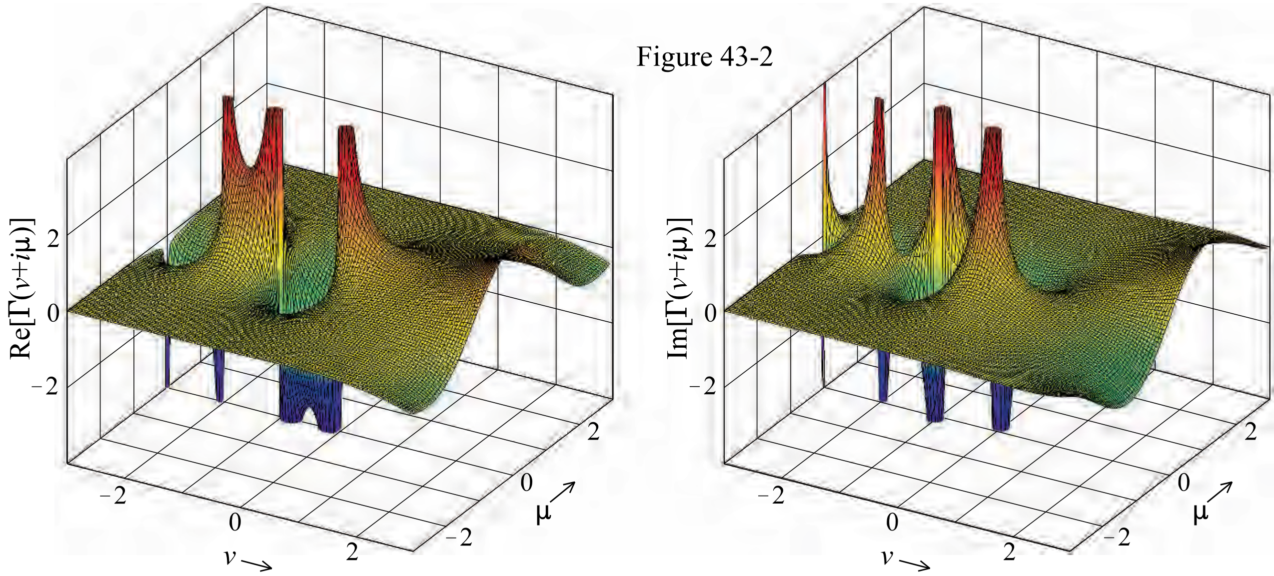
are also listed by Gradshteyn and Ryzhik [Section 6.44]. The latter formula leads to

43:10:4 $$\int_{1}^{n}\ln\{\Gamma(t)\}\,\mathrm{d}t = \frac{n-1}{2}\left[\ln(2\pi)-n\right]+\sum_{j=2}^{n-1} j\ln(j) \qquad n = 2,3,4,\cdots$$

43:11 COMPLEX ARGUMENT

Figure 43-2 portrays the (complete) gamma function of complex variable $v+i\mu$. Note that all the poles of $\Gamma(v+i\mu)$ lie along the nonpositive reaches of the real axis, $\mu = 0$. The real and imaginary components of the complex-valued function are contained in the formulation

Figure 43-2



43:11:1 $$\Gamma(v+i\mu)=\left[\cos(\theta)+i\sin(\theta)\right]|\Gamma(v)|\prod_{j=0}^{\infty}\frac{|j+v|}{\sqrt{\mu^2+(j+v)^2}}$$

where

43:11:2 $$\theta=\mu\,\psi(v)+\sum_{j=0}^{\infty}\frac{\mu}{j+v}-\arctan\left(\frac{\mu}{j+v}\right)$$

Here ψ is the digamma function [Chapter 44]. Tables from which $\Gamma(v+i\mu)$ may be evaluated are given by Abramowitz and Stegun [pages 277−287]. It follows from 43:11:1 that the product $\Gamma(v+i\mu)\Gamma(v-i\mu)$ is real; special cases are

43:11:3 $$\Gamma(1+i\mu)\Gamma(1-i\mu)=\pi\mu\,\mathrm{csch}(\pi\mu)$$

and

43:11:4 $$\Gamma(\tfrac{1}{2}+i\mu)\Gamma(\tfrac{1}{2}-i\mu)=\pi\,\mathrm{sech}(\pi\mu)$$

For purely imaginary argument

43:11:5 $$\Gamma(i\mu)=\sqrt{\frac{\pi}{\mu}\mathrm{csch}(\pi\mu)}\left[\sin(\theta)-i\cos(\theta)\right] \qquad \theta=-\gamma\mu+\sum_{j=1}^{\infty}\frac{\mu}{j}-\arctan\left(\frac{\mu}{j}\right)$$

where γ is Euler's constant [Section 1:7].

43:12 GENERALIZATIONS

The (complete) gamma function is a special case of the incomplete gamma function $\gamma(v\,,x)$ and of its complement $\Gamma(v,x)$

43:12:1 $$\Gamma(v)=\gamma(v,\infty) \qquad \text{and} \qquad \Gamma(v)=\Gamma(v,0)$$

both of which are discussed in Chapter 45. These two incomplete functions sum to the complete gamma function

43:12:2 $$\gamma(v,x)+\Gamma(v,x)=\Gamma(v)$$

for all values of x.

43:13 COGNATE FUNCTION: the complete beta function

Though it is also related to the functions in Chapters 2, 6, 18, 44 and 45, the closest relative of the (complete) gamma function is the *complete beta function* $\mathrm{B}(\nu,\mu)$, also known as *Euler's integral of the first kind* or simply as the *beta function*. Not to be confused with the beta numbers [Chapter 3], it is defined by the Euler integral

43:13:1 $$\mathrm{B}(\nu,\mu)=\int_0^1 t^{\nu-1}(1-t)^{\mu-1}\,\mathrm{d}t \qquad \nu>0 \qquad \mu>0$$

and is related to the gamma function through

43:13:2 $$\mathrm{B}(\nu,\mu)=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}$$

The interchangeability $\mathrm{B}(\mu,\nu)=\mathrm{B}(\nu,\mu)$ is implicit in both these definitions and is evident in the symmetry of Figure 43-3. This is a diagram illustrating the "shape" of the function in a region close to the origin.

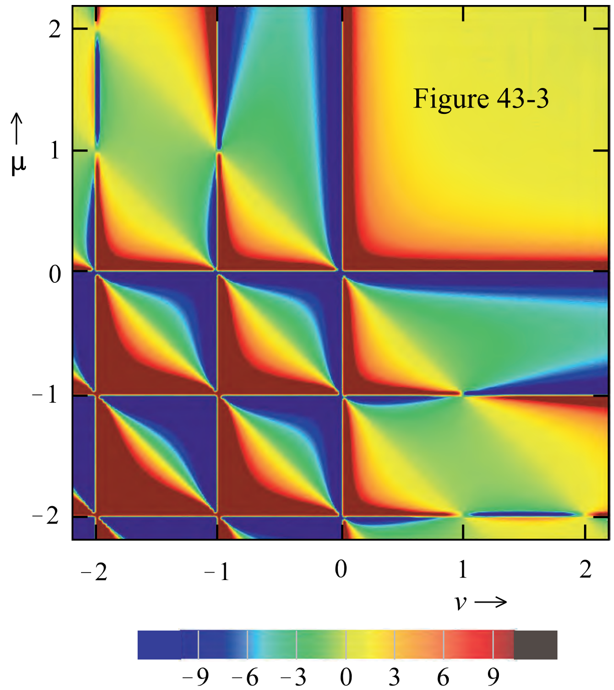


Figure 43-3

As with the gamma function, the complete beta function may be expressed as a definite integral in many ways other than 43:13:1. These include

43:13:3 $$\mathrm{B}(\nu,\mu)=\int_0^1 \frac{t^{\nu-1}+t^{\mu-1}}{(1+t)^{\nu+\mu}}\,\mathrm{d}t=\int_1^\infty \frac{t^{\nu-1}+t^{\mu-1}}{(1+t)^{\nu+\mu}}\,\mathrm{d}t \qquad \nu>0 \quad \mu>0$$

and

43:13:4 $$\mathrm{B}(\nu,\mu)=2\int_0^1 t^{2\nu-1}\left(1-t^2\right)^{\mu-1}\mathrm{d}t=2\int_0^{\pi/2}\sin^{2\nu-1}(t)\cos^{2\mu-1}(t)\,\mathrm{d}t \qquad \nu>0 \qquad \mu>0$$

and extensive lists may be found in Gradshteyn and Ryzhik [Section 8.38] and in Magnus, Oberhettinger and Soni [page 7]. The integral representations of $\mathrm{B}(\nu,\mu)$ apply only when both arguments are positive, but relationship 43:13:2 is valid for any pair of real arguments, as is the infinite-product expression

43:13:5 $$\mathrm{B}(\nu,\mu)=\prod_{j=0}^{\infty}\frac{(j+1)(\nu+\mu+j)}{(\nu+j)(\mu+j)}$$

There are numerous special cases of the beta function that apply when the two arguments have a particular relationship to each other. If ν and μ sum to zero, then

43:13:6 $$\mathrm{B}(\nu,-\nu)=\begin{cases}(-1)^\nu/|\nu| & \nu=\pm1,\pm2,\pm3,\cdots\\ \pm\infty & \nu=0\\ 0 & \text{otherwise}\end{cases}$$

or if they sum to a negative integer $-n$, then

43:13:7
$$\mathrm{B}(\nu,-\nu-n)=\left\{\begin{array}{ll}0 & \nu \text{ not an integer}\\ \dfrac{(-)^{n+\nu}(-\nu-n-1)!}{(n+1)_{-n-\nu}} & \nu=-n-1,-n-2,-n-3,\cdots\\ \pm\infty & \nu=0,-1,-2,\cdots,-n\\ \dfrac{(-)^{\nu}(\nu-1)!}{(n+1)_{\nu}} & \nu=1,2,3,\cdots\end{array}\right\} \quad n=1,2,3,\cdots$$

The uppermost diagonal mostly yellow/green line in Figure 43-3 illustrates the formulas in 47:13:6, while the lines parallel to this are where 47:13:7 hold sway. If one of the arguments is a moiety, if the arguments sum to moiety, or if the two arguments are identical, then the following set of equalities applies to the beta function

43:13:8
$$\frac{4^{\nu}}{2}\mathrm{B}(\nu,\nu)=\cos(\pi\nu)\,\mathrm{B}\left(\nu,\tfrac{1}{2}-\nu\right)=\mathrm{B}\left(\tfrac{1}{2}\,,\,\nu\right)=\sum_{j=0}^{\infty}\frac{(2j-1)!!}{(2j)!!(j+\nu)}$$

Some important particular values of the beta function are tabulated below

μ	$\mathrm{B}(\frac{-5}{2},\mu)$	$\mathrm{B}(-2,\mu)$	$\mathrm{B}(\frac{-3}{2},\mu)$	$\mathrm{B}(-1,\mu)$	$\mathrm{B}(\frac{-1}{2},\mu)$	$\mathrm{B}(0,\mu)$	$\mathrm{B}(\frac{1}{2},\mu)$	$\mathrm{B}(1,\mu)$	$\mathrm{B}(\frac{3}{2},\mu)$	$\mathrm{B}(2,\mu)$	$\mathrm{B}(\frac{5}{2},\mu)$
$5/2$	0	$\pm\infty$	π	$\pm\infty$	$-3\pi/2$	$\pm\infty$	$3\pi/8$	$2/15$	$3\pi/16$	$4/35$	$3\pi/128$
2	$4\pi/3$	$1/2$	$4\pi/3$	$\pm\infty$	-4	$\pm\infty$	$4/3$	$1/2$	$4/15$	$1/6$	$4/35$
$3/2$	0	$\pm\infty$	0	$\pm\infty$	$-\pi$	$\pm\infty$	$3\pi/4$	$3/8$	$\pi/8$	$4/15$	$3\pi/16$
1	-2	$-1/2$	$-2/3$	-1	-2	$\pm\infty$	2	1	$3/8$	$1/2$	$2/15$
$1/2$	0	$\pm\infty$	0	$\pm\infty$	0	$\pm\infty$	π	2	$3\pi/4$	$4/3$	$3\pi/8$
0	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$
$-1/2$	0	$\pm\infty$	0	$\pm\infty$	0	$\pm\infty$	0	-2	$-\pi$	-4	$-3\pi/2$

and some others are $\mathrm{B}(\tfrac{1}{4},\tfrac{1}{4})=\sqrt{8}\,\pi g$, $\mathrm{B}(\tfrac{1}{4},\tfrac{3}{4})=\sqrt{2}\,\pi$ and $\mathrm{B}(\tfrac{3}{4},\tfrac{3}{4})=\sqrt{2}/g$, where g is *Gauss's constant* [Section 1:7]. Yet other particular values of the beta function may be found by combining definition 43:13:2 with the findings of Section 43:4. In carrying out this exercise, recognize that the quotient of two gamma functions, both of which are infinite, is generally finite. For example, in view of formula 43:5:6, $\Gamma(-5)/\Gamma(0)=-1/120$.

Though there are exceptions to the rule, it is generally true that the beta function is infinite if one (or both) of its arguments is a nonpositive integer. Likewise, it is generally true that if one (or both) of the arguments is a positive integer n, then the beta function reduces to an expression involving a binomial coefficient

43:13:9
$$\mathrm{B}(\nu,\,n)=\frac{1}{\nu\binom{n+\nu-1}{n-1}}$$

though again, there are exceptions to this rule.

Intrarelationships of complete beta functions, such as

43:13:10
$$\mathrm{B}(\nu+1,\,\mu)=\frac{\nu}{\nu+\mu}\mathrm{B}(\nu,\,\mu)=\frac{\nu}{\mu}\mathrm{B}(\nu,\,\mu+1)$$

as well as expansions, such as

43:13:11 $$\mathrm{B}(v,\mu)=\frac{1}{v}+\frac{1-\mu}{1+v}+\frac{(1-\mu)(2-\mu)}{2(2+v)}+\cdots=\sum_{j=0}^{\infty}\frac{(1-\mu)_j}{j!(v+j)} \qquad \mu>0$$

and infinite sums, such as

43:13:12 $$\mathrm{B}(v,\mu)+\mathrm{B}(v+1,\mu)+\mathrm{B}(v+2,\mu)+\cdots=\sum_{j=0}^{\infty}\mathrm{B}(v+j,\mu)=\mathrm{B}(v,\mu-1)$$

may be established via the equation 43:13:2.

Equator's complete beta function routine (keyword **Beta**) first checks for the special cases outlined in equations 43:13:6 and 43:13:7 and sets the value of the function appropriately. If none of those conditions is met, and one or both of the arguments is a nonpositive integer, then *Equator* returns $\pm\infty$. For all other values of the arguments, *Equator* relies on the identity 43:13:2 to generate numerical values of the beta function.

43:14 RELATED TOPIC: function synthesis

The (complete) gamma function (or its special case, the factorial) is preeminent as a component of coefficients of power series expansions of the functions of this *Atlas*. The majority of these functions may be regarded as derived from one of four fundamental functions, namely those whose power-series expansions are

43:14:1 $$\sum_{j=0}^{\infty}\frac{x^j}{\Gamma^2(j+1)},\quad \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)},\quad \sum_{j=0}^{\infty}x^j,\ \text{or}\ \sum_{j=0}\Gamma(j+1)x^j$$

Because j here is a nonnegative integer, the gamma function $\Gamma(j+1)$ is equal to $j!$ or $(1)_j$ and it is the latter Pochhammer notation [Chapter 18] that is most convenient for our present purpose. The four fundamental functions, or prototypes, are:

43:14:2 $$\sum_{j=0}^{\infty}\frac{x^j}{\Gamma^2(j+1)}=\sum_{j=0}^{\infty}\frac{1}{(1)_j(1)_j}x^j=\begin{cases}\mathrm{I}_0\left(2\sqrt{x}\right) & x\ge 0\\ \mathrm{J}_0\left(2\sqrt{-x}\right) & x\le 0\end{cases}$$

43:14:3 $$\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)}=\sum_{j=0}^{\infty}\frac{1}{(1)_j}x^j=\exp(x)$$

43:14:4 $$\sum_{j=0}^{\infty}x^j=\frac{1}{1-x} \qquad |x|<1$$

and the asymptotic *Euler function*,

43:14:5 $$\sum_{j=0}\Gamma(j+1)x^j=\sum_{j=0}(1)_j x^j\sim\frac{1}{x}\exp\left(\frac{-1}{x}\right)\mathrm{Ei}\left(\frac{1}{x}\right) \qquad x\text{ small}$$

The zero-order Bessel and modified Bessel functions, J_0 and I_0, are treated in Chapters 52 and 49 respectively; the exponential integral function Ei is the subject of Chapter 37.

The four prototype functions specified in equations 43:14:2−5 have been termed *basis hypergeometric functions*, or *prototype hypergeometric functions*. This is because, in the terminology introduced in Section 18:14, they are the simplest members of four families characterized by the difference $L-K$. These latter numbers, L and K, are the integers that specify the numbers of Pochhammer polynomials present in the denominator and numerator, respectively, of the hypergeometric representation. Thus, for example, $\exp(x)$, with one denominatorial Pochhammer term and none in its numerator, is the simplest member of the $L=K+1$ family of hypergeometric functions. It is the task of the present section to demonstrate that any two hypergeometric that share the same value of $L-K$ can be interconverted, one to the other. In particular, any hypergeometric function may be formed from its own basis function. For example, any of the functions in Table 18-3 or 18-4 can be formed from the exponential function. This process, which is termed "*synthesis*", was first described in the literature of the fractional calculus [Oldham and Spanier, Chapter 9].

We represent the overall synthetic process by the symbol $\xrightarrow[\beta]{\alpha}$ where α and β are unequal real numbers each of which may be a positive integer or any fractional number, positive or negative. The net effect of this operation is to introduce the Pochhammer polynomial $(\alpha)_j$ into the function's list of numeratorial factors and $(\beta)_j$ into its denominatorial roster; for example:

43:14:6 $$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j}x^j \xrightarrow[\beta]{\alpha} \sum_{j=0}^{\infty}\frac{(a)_j(\alpha)_j}{(c_1)_j(c_2)_j(\beta)_j}x^j$$

This is accomplished by three operations applied sequentially to any hypergeometric function:

(i) multiply the input function by $x^{\alpha-1}/\Gamma(\alpha)$;

(ii) apply the differintegration [Section 12:14] operator $\mathrm{d}^{\alpha-\beta}/\mathrm{d}x^{\alpha-\beta}$, and;

(iii) divide by $x^{\beta-1}/\Gamma(\beta)$.

These three steps conspire to achieve the sought goal:

43:14:7 $$\sum_{j=0}^{\infty}\frac{(a)_j x^j}{(c_1)_j(c_2)_j} \xrightarrow{\text{(i)}} \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(a)_j x^{j+\alpha-1}}{(c_1)_j(c_2)_j} \xrightarrow{\text{(ii)}} \frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{(a)_j(\alpha)_j x^{j+\beta-1}}{(c_1)_j(c_2)_j(\beta)_j} \xrightarrow{\text{(iii)}} \sum_{j=0}^{\infty}\frac{(a)_j(\alpha)_j x^j}{(c_1)_j(c_2)_j(\beta)_j}$$

Equation 18:14:18 is the key to understanding step (ii). A straightforward example is the synthesis of Dawson's integral [Chapter 42] from the exponential function.

43:14:8 $$\exp(x)=\sum_{j=0}^{\infty}\frac{1}{(1)_j}x^j \xrightarrow[3/2]{1} \sum_{j=0}^{\infty}\frac{1}{(3/2)_j}x^j=\frac{\exp(x)}{\sqrt{x}}\mathrm{daw}\left(\sqrt{x}\right)$$

Notice that we refer to this process as a synthesis "of Dawson's integral" even though there is a multiplier, here $\exp(x)/\sqrt{x}$, also present and even though the product is $\mathrm{daw}(\sqrt{x})$, not daw(x).

At first sight, synthesis appears to increase the complement of denominatorial and numeratorial parameters, each by unity, but this is not invariable so because α and/or β may be chosen to match a preexisting parameter, resulting in cancellation. An example is provided by the synthesis

43:14:9 $$\frac{\sin\left(2\sqrt{x}\right)}{2\sqrt{x}}=\sum_{j=0}^{\infty}\frac{1}{(1)_j(3/2)_j}(-x)^j \xrightarrow[3/2]{1} \sum_{j=0}^{\infty}\frac{1}{(3/2)_j(3/2)_j}(-x)^j=\frac{\pi \mathrm{h}_0\left(2\sqrt{x}\right)}{4\sqrt{x}}$$

of a Struve function [Chapter 57]. The difference $L-K$ always remains constant in a synthetic procedure, though the individual L and K integers may increase, remain the same, or even decrease.

It is necessary to ensure that the hypergeometric variable matches that used in the synthetic process, which is x in all our examples throughout the *Atlas*, before invoking synthesis. Therefore function arguments may need adjustment. For example, in synthesizing the cosine function from the zero-order Bessel function, the synthetic operation acts on a function of argument $2\sqrt{x}$

43:14:10 $$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[1/2]{1} \cos\left(2\sqrt{x}\right)$$

Alternatively, of course, one can synthesize cos(x) from $\mathrm{J}_0(x)$ by adopting $x^2/4$ as the synthetic variable.

Sometimes more than one synthetic step is needed to create the sought function from the prototypical basis function, as in the example

43:14:11 $$\frac{1}{1-x} \xrightarrow[1]{-\frac{1}{2}} \sqrt{1-x} \xrightarrow[1]{\frac{1}{2}} \frac{2}{\pi}\mathrm{E}\left(\sqrt{x}\right)$$

which leads to a complete elliptic integral of the second kind [Chapter 61]. This example will serve to illustrate that two or more synthetic routes may lead to the same destination:

43:14:12 $$\frac{1}{1-x} \xrightarrow[\frac{3}{2}]{\frac{1}{2}} \frac{\mathrm{artanh}(\sqrt{x})}{\sqrt{x}} \xrightarrow[1]{\frac{3}{2}} \frac{1}{\sqrt{1-x}} \xrightarrow[1]{-\frac{1}{2}} \frac{2}{\pi}\mathrm{E}\left(\sqrt{x}\right)$$

Notice in equations 43:14:2−5 that a limitation sometimes exists on the magnitude of the variable. Thus, in the $L=K$ family of hypergeometric functions, x must have a magnitude of less than unity, and this restriction therefore transfers to the corresponding syntheses, such as those in the two previous equations. The restriction is most telling in the $L=K-1$ family for here the hypergeometric series are asymptotic and the corresponding syntheses, such as that generating the error function complement

43:14:13 $$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right) \xrightarrow[\frac{1}{2}]{1} \sqrt{\frac{\pi}{x}}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{\mathrm{x}}}\right)$$

are valid only in the limit of small x.

There exist several differintegration algorithms for zero lower limit [Oldham and Spanier, Chapter 8], though most are restricted to limited ranges of the differintegration order μ. One that is more versatile than most, and without undue complexity, is the *Grünwald algorithm*

43:14:14 $$\frac{d^{\mu}}{dx^{\mu}}f(x) \approx \left(\frac{J}{x}\right)^{\mu}\sum_{j=0}^{J-1}\frac{(-\mu)_j}{j!}f\left(\frac{J-j}{J}x\right)$$

that uses a large number, J, of evenly spaced data points in the range x/J to x. Such an algorithm permits differintegration to be carried out numerically. This algorithm may be incorporated into a scheme that implements function synthesis, permitting the synthetic operation

43:14:15 $$f(x) \xrightarrow[\beta]{\alpha} g(x)$$

to be carried out numerically, too, as well as algebraically When the three steps specified in 43:14:7 are combined into a single operation, with the differintegration step implemented through the 43:14:14 algorithm, one arrives at

43:14:16 $$g(x) \approx \frac{\Gamma(\beta)}{J^{\beta-1}\Gamma(\alpha)}\sum_{j=1}^{J}\frac{(\beta-\alpha)_{J-j}}{(J-j)!}j^{\alpha-1}f\left(\frac{jx}{J}\right)$$

This last expression is exact in the $J \to \infty$ limit. Though it often delivers a close approximation when J is a sufficiently large positive integer, the formula is unsuitable for calculating precise numerical values of $g(x)$. Moreover, this algorithm may occasionally fail totally. A case in point is provided in Section 61:3. Whereas synthesis 61:3:6 may be implemented through algorithm 43:14:16, synthesis 61:3:7 cannot be. The failure stems from the lack of differintegrability of an intermediate function.

CHAPTER 44

THE DIGAMMA FUNCTION $\psi(v)$

The *digamma function* arises on differentiation of the logarithm of the previous chapter's gamma function. Multiple differentiations generate the *polygamma functions*, addressed in Section 44:12. The related *Bateman's* G *function* is the subject of Section 44:13. All these functions are useful in summing certain algebraic and numerical series, an application discussed in Section 44:14.

44:1 NOTATION

The *digamma function* of argument v, sometimes known as the *psi function*, has the symbol $\psi(v)$. It occurs often as its sum $\psi(v)+\gamma$ with Euler's constant. This pair is frequently represented by $\Phi(v)$ that, especially when v is an integer n, is known as the nth *harmonic number*, being the sum of the reciprocals of the first n natural numbers:

44:1:1
$$\psi(n)+\gamma=\Phi(n)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Occasionally the name "harmonic number" is used, less appropriately, with non-integer arguments; in these cases it means $\psi(v)+\gamma$.

A few authors use a translated argument for the digamma function, with their $\psi(v-1)$ being our $\psi(v)$. Some authorities do not accord the function any name or special symbol, regarding it as merely the zeroth-order instance $\psi^{(0)}(v)$ of the polygamma function [Section 44:12].

In this *Atlas* the polygamma functions are denoted $\psi^{(1)}(v)$, $\psi^{(2)}(v)$, $\psi^{(3)}(v)$, etc., the Greek prefixes in their names being offset by 2 from the numerals in their symbols. Thus $\psi^{(1)}(v)$ is the *trigamma function* and $\psi^{(3)}(v)$ is the *pentagamma function*. Commonly, the symbols ψ' and ψ'' replace $\psi^{(1)}$ and $\psi^{(2)}$ in recognition of the role played by differentiation in their definitions.

44:2 BEHAVIOR

The red curve in Figure 44-1 shows the digamma function to be a monotonically increasing function for $v>0$. It is segmented for negative v, with discontinuities of the $+\infty|-\infty$ variety at each nonpositive integer value of the argument. The shapes within each of these segments are similar, but not identical.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_45, © Springer Science+Business Media, LLC 2009

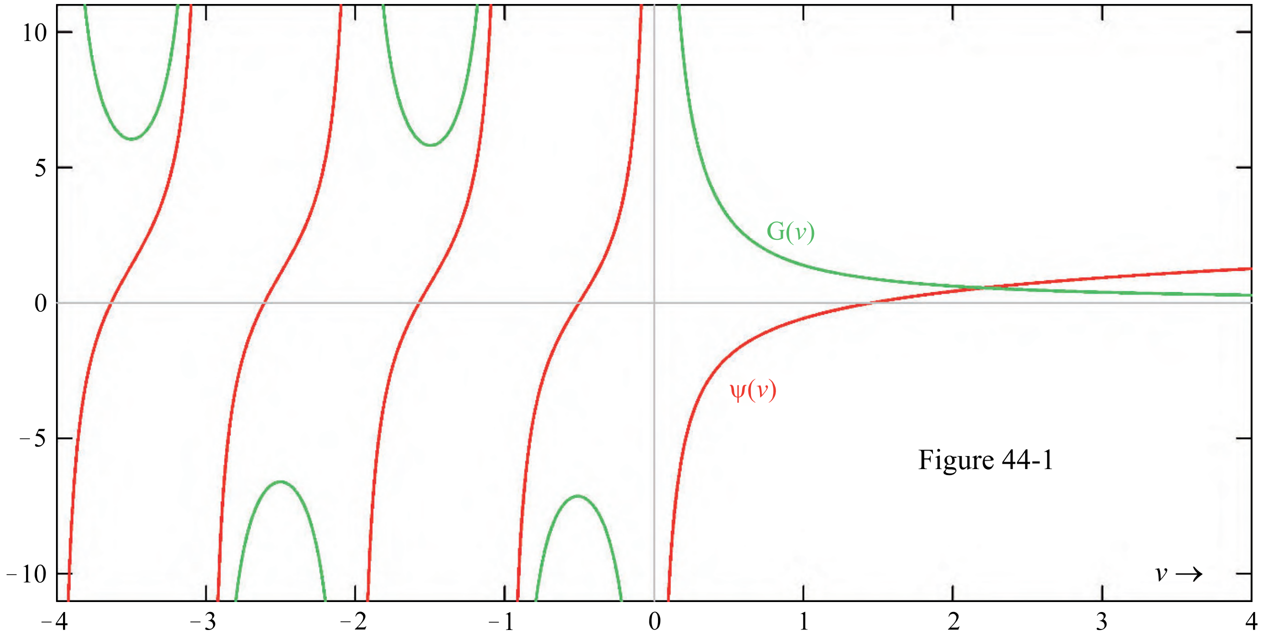


Figure 44-1

44:3 DEFINITIONS

In terms of the gamma function, the digamma function is defined by

44:3:1
$$\psi(\nu) = \frac{\mathrm{d}}{\mathrm{d}\nu}\ln\left(\Gamma(\nu)\right) = \frac{1}{\Gamma(\nu)}\frac{\mathrm{d}}{\mathrm{d}\nu}\Gamma(\nu)$$

It may also be defined as the limit

44:3:2
$$\psi(\nu) = \lim_{n\to\infty}\left\{\ln(n) - \sum_{j=0}^{n}\frac{1}{j+\nu}\right\}$$

Representations as definite integrals are numerous and include

44:3:3
$$\psi(\nu) - \psi(1) = \int_0^1 \frac{1-t^{\nu-1}}{1-t}\mathrm{d}t = \int_0^\infty \frac{\exp(-t)-\exp(-\nu t)}{1-\exp(-t)}\mathrm{d}t \qquad \nu > 0$$

The integral definition

44:3:4
$$\psi(\nu) = \int_0^\infty \left[\frac{\exp(-t)}{t} - \frac{\exp(-\nu t)}{1-\exp(-t)}\right]\mathrm{d}t \qquad \nu > 0$$

is due to Gauss, while

44:3:5
$$\psi(\nu) = \ln(\nu) - \frac{1}{2\nu} - 2\int_0^\infty \frac{t}{(t^2+\nu^2)[\exp(2\pi t)-1]}\mathrm{d}t \qquad \nu > 0$$

and

44:3:6
$$\psi(\nu) = \int_0^\infty \frac{\exp(-t)-(1+t)^{-\nu}}{t}\mathrm{d}t \qquad \nu > 0$$

are attributed respectively to Binet and Dirichlet (of Belgian extraction, Johann Peter Gustav Lejeune Dirichlet, 1805 − 1859, pursued mathematical studies in neighboring France and Germany). Other representations of the digamma function as a definite integral are given by Erdélyi et al. [*Higher Transcendental Functions*, Volume 1, Section 1.7.2].

44:4 SPECIAL CASES

In a sense, every rational value of the argument – that is, whenever v equals the ratio m/n of two integers – is a special case, for then the digamma function $\psi(v)$ is given by a formula that is inapplicable when v is irrational. This formula will be found in Section 44:7.

44:5 INTRARELATIONSHIPS

The digamma function satisfies the reflection formulas

44:5:1 $$\psi(1-v) = \psi(v) + \pi\cot(\pi v)$$

and

44:5:2 $$\psi\left(\tfrac{1}{2}-v\right) = \psi\left(\tfrac{1}{2}+v\right) - \pi\tan(\pi v)$$

The recurrence relationship

44:5:3 $$\psi(1+v) = \psi(v) + \frac{1}{v}$$

may be generalized to

44:5:4 $$\psi(n+v) = \psi(v) + \sum_{j=0}^{n-1}\frac{1}{j+v} \qquad n = 1,2,3,\cdots$$

or

44:5:5 $$\psi(v-n) = \psi(v) - \sum_{j=1}^{n}\frac{1}{v-j} \qquad n = 1,2,3,\cdots$$

though the latter may be invalid for certain values of n if v is an integer.

The duplication formula

44:5:6 $$\psi(2v) = \ln(2) + \tfrac{1}{2}\psi(v) + \tfrac{1}{2}\psi\left(v+\tfrac{1}{2}\right)$$

generalizes to

44:5:7 $$\psi(nv) = \ln(n) + \frac{1}{n}\sum_{j=0}^{n-1}\psi\left(v+\frac{j}{n}\right) \qquad n = 2,3,4,\cdots$$

The difference of two digamma functions may be expressed as a simple infinite sum, or as a hypergeometric function [Section 18:14] of unity argument:

44:5:8 $$\psi(v) - \psi(\mu) = (v-\mu)\sum_{j=0}^{\infty}\frac{1}{(j+v)(j+\mu)} = \frac{v-\mu}{v}\sum_{j=0}^{\infty}\frac{(1)_j(v-\mu+1)_j}{(2)_j(v+1)_j}$$

An alternating finite series of digamma functions with increasing positive integer arguments has a sum that depends on the parity of the argument of the final term:

44:5:9 $$\psi(1)-\psi(2)+\psi(3)-\cdots-(-)^n\psi(n)=\sum_{j=1}^{n}(-)^{j+1}\psi(j)=\begin{cases}\frac{1}{2}\psi\left(\frac{n+1}{2}\right)-\frac{1}{2}\gamma & n=1,3,5,\cdots\\ -\psi(n)+\frac{1}{2}\psi\left(\frac{n}{2}\right)-\frac{1}{2}\gamma & n=2,4,6,\cdots\end{cases}$$

Certain infinite series involving quotients of integer-argument digamma functions by the corresponding gamma function may be summed as follows

44:5:10 $$\pm\frac{\psi(1)}{\Gamma(1)}+\frac{\psi(2)}{\Gamma(2)}\pm\frac{\psi(3)}{\Gamma(3)}+\cdots=\sum_{j=1}^{\infty}\frac{(\pm)^j\psi(j)}{\Gamma(j)}=\begin{cases}-e\,\mathrm{Ei}(-1)=0.59634\;73623\;23194\\ \mathrm{Ei}(1)/e=0.69717\;48832\;35066\end{cases}$$

44:5:11 $$\pm\frac{\psi(1)}{\Gamma^2(1)}+\frac{\psi(2)}{\Gamma^2(2)}\pm\frac{\psi(3)}{\Gamma^2(3)}+\cdots=\sum_{j=1}^{\infty}\frac{(\pm)^j\psi(j)}{\Gamma^2(j)}=\begin{cases}\mathrm{K}_0(2)=0.11389\;38727\;49533\\ \frac{1}{2}\pi\mathrm{Y}_0(2)=0.80169\;62318\;83694\end{cases}$$

The Ei, K_0, and Y_0 functions will be found in Chapter 37, 51, and 54.

44:6 EXPANSIONS

In the Taylor series expansion

44:6:1 $$\psi(v)=\frac{-1}{v}-\gamma+\frac{\pi^2 v}{6}-Zv^2+\frac{\pi^4 v^3}{90}-\cdots=\frac{-1}{v}-\gamma+\frac{1}{v}\sum_{j=2}^{\infty}\zeta(j)(-v)^j \qquad 0\neq|v|<1$$

$\zeta(j)$ is the jth zeta number [Chapter 3] and Z is $\zeta(3)$, Apéry's constant [Section 3:7]. This expansion is valid only in two narrow domains of the argument but one of wider applicability can be written in the three equivalent forms:

44:6:2 $$\psi(v)+\gamma=\sum_{j=0}^{\infty}\frac{1}{j+1}-\frac{1}{j+v}=\sum_{j=0}^{\infty}\frac{v-1}{(j+1)(j+v)}=\frac{-1}{v}+v\sum_{j=1}^{\infty}\frac{1}{j(j+v)} \qquad v\neq 0,-1,-2,\cdots$$

Two other expansions are

44:6:3 $$\psi(v)=\ln(v)-\sum_{j=0}^{\infty}\frac{1}{j+v}-\ln\left(1+\frac{1}{j+v}\right) \qquad v>0$$

and

44:6:4 $$\psi(v)=-\gamma-\frac{v^2}{1-v^2}-\frac{\pi}{2}\cot(\pi v)-\frac{1}{v}\left[\frac{1}{2}+\sum_{k=3,5,7,\cdots}^{\infty}\{\zeta(k)-1\}v^k\right] \qquad 2>|v|\neq 0,1$$

An asymptotic expansion involving the Bernoulli numbers [Chapter 4]

44:6:5 $$\psi(v)\sim\ln(v)-\frac{1}{2v}-\frac{1}{12v^2}+\frac{1}{120v^4}-\frac{1}{252v^6}+\cdots=\ln(v)-\sum_{j=1}\frac{\mathrm{B}_j}{jv^j} \qquad \text{large } v$$

becomes exact as $v\to\infty$.

44:7 PARTICULAR VALUES

The negative of Euler's number

44:7:1 $$-\gamma=-\,0.57721\;56649\;01533$$

is an inescapable presence in almost all of the formulas for the digamma function. It appears in each particular value of the digamma function of positive integer argument

$\psi(0), \psi(-1), \psi(-2), \cdots$	$\psi(1)$	$\psi(2)$	$\psi(3)$	$\psi(4)$	$\psi(5)$	$\psi(6)$	$\psi(7)$	$\psi(8)$	$\psi(n)$
$+\infty\|-\infty$	$-\gamma$	$1-\gamma$	$\frac{3}{2}-\gamma$	$\frac{11}{6}-\gamma$	$\frac{25}{12}-\gamma$	$\frac{137}{60}-\gamma$	$\frac{49}{20}-\gamma$	$\frac{363}{140}-\gamma$	$-\gamma+\sum_{j=1}^{n-1}\frac{1}{j}$

and also, along with the logarithm of 4, when the argument is one-half of an odd integer of either sign:

$\psi\left(\frac{1}{2}\right)$	$\psi\left(\frac{-1}{2}\right)$ or $\psi\left(\frac{3}{2}\right)$	$\psi\left(\frac{-3}{2}\right)$ or $\psi\left(\frac{5}{2}\right)$	$\psi\left(\frac{-5}{2}\right)$ or $\psi\left(\frac{7}{2}\right)$	$\psi\left(\frac{-7}{2}\right)$ or $\psi\left(\frac{9}{2}\right)$	$\psi\left(\frac{1}{2}\pm n\right)$
$-\gamma-\ln(4)$	$2-\gamma-\ln(4)$	$\frac{8}{3}-\gamma-\ln(4)$	$\frac{46}{15}-\gamma-\ln(4)$	$\frac{352}{105}-\gamma-\ln(4)$	$-\gamma-\ln(4)+\sum_{j=1}^{n}\frac{2}{2j-1}$

where

44:7:2 $$-\gamma-\ln(4) = -1.9635\,10026\,02142$$

The general formulas are included in the above panels. Other general formulas include

44:7:3 $$\psi\left(\tfrac{1}{3}\pm n\right) = -\gamma-\frac{\pi}{\sqrt{12}}-\frac{\ln(27)}{2}+\sum_{j=1}^{n}\frac{6}{6j-3\mp1} \qquad n=0,1,2,\cdots$$

44:7:4 $$\psi\left(\tfrac{1}{4}\pm n\right) = -\gamma-\frac{\pi}{2}-\ln(8)+\sum_{j=1}^{n}\frac{4}{4j-2\mp1} \qquad n=0,1,2,\cdots$$

44:7:5 $$\psi\left(\tfrac{2}{3}\pm n\right) = -\gamma+\frac{\pi}{\sqrt{12}}-\frac{\ln(27)}{2}+\sum_{j=1}^{n}\frac{6}{6j-3\pm1} \qquad n=0,1,2,\cdots$$

and

44:7:6 $$\psi\left(\tfrac{3}{4}\pm n\right) = -\gamma+\frac{\pi}{2}-\ln(8)+\sum_{j=1}^{n}\frac{4}{4j-2\pm1} \qquad n=0,1,2,\cdots$$

In fact, though its formulation is too complicated to be generally useful, the *theorem of Gauss* states that the digamma function of any rational number is able to be expressed in terms of constants and simple functions (logarithms and circular functions). For any rational argument in the domain $0 < m/n < 1$, the Gauss formula is

44:7:7 $$\psi\left(\frac{m}{n}\right) = -\gamma-\ln(2n)-\frac{\pi}{2}\cot\left(\frac{m\pi}{2}\right)+\sum_{j=1}^{\mathrm{Int}[(n-1)/2]}\cos\left(\frac{2jm\pi}{n}\right)\ln\left\{\sin^2\left(\frac{j\pi}{n}\right)\right\} \qquad \begin{matrix} n=2,3,4,\cdots \\ m=1,2,3,\cdots,n-1 \end{matrix}$$

This relationship may be extended to encompass any fractional argument whatsoever by use of equation 44:5:4 or 44:5:5.

The digamma function has a single positive zero at $v=+1.4616\,32144\,96836$ and an infinite number of negative zeros, the first two being $r_{-1} = -0.50408\,30082\,64455$ and $r_{-2} = -1.5734\,98473\,16239$. The approximation

44:7:8 $$r_{-j} \approx -j+\frac{1}{\pi}\arctan\left(\frac{\pi}{\ln(j)}\right) \qquad j=1,2,3,\cdots$$

becomes increasingly accurate as j increases. This approximation may be used as the starting point of a Raphson rule procedure [Section 52:15] to progressively improve the estimate.

The digamma function has no extremum, reflecting the fact that the trigamma function lacks zeros, other than at infinity.

44:8 NUMERICAL VALUES

With keyword **digamma**, *Equator* provides a digamma function routine. For $x \geq 10$ the asymptotic expansion 44:6:5 is used. For positive $x < 10$, the recursion 44:5:4 is used to increase the argument until the asymptotic expansion is valid. When x is negative, the reflection formula 44:5:1 is employed.

44:9 LIMITS AND APPROXIMATIONS

The limiting approximation

44:9:1 $$\psi(v) \approx \ln\left(v - \tfrac{1}{2}\right) \qquad \text{large positive } v$$

can be improved with help from equation 44:6:5. The corresponding approximation for large negative argument is

44:9:2 $$\psi(v) \approx \ln\left(\tfrac{1}{2} - v\right) - \pi\cot(\pi v) \qquad \text{large negative } v$$

Close to the discontinuity at $v = -n$, the digamma function is well approximated by

44:9:3 $$\psi(v) \approx \psi(1+n) - \pi\cot\left\{\pi(v+n)\right\} \qquad v \approx -n = 0,1,2,\cdots$$

44:10 OPERATIONS OF THE CALCULUS

Single and double differentiation generates the trigamma and higher polygamma functions [Section 44:12]:

44:10:1 $$\frac{\mathrm{d}}{\mathrm{d}v}\psi(v) = \psi^{(1)}(v)$$

44:10:2 $$\frac{\mathrm{d}^n}{\mathrm{d}v^n}\psi(v) = \psi^{(n)}(v) \qquad n = 1,2,3,\cdots$$

Important indefinite integrals lead to logarithms

44:10:3 $$\int_1^v \psi(t)\,\mathrm{d}t = \ln\left\{\Gamma(v)\right\} \qquad v > 0$$

and

44:10:4 $$\int_v^{v+1} \psi(t)\,\mathrm{d}t = \ln(v) \qquad v > 0$$

either of which shows the integral of the digamma function between arguments 1 and 2 to be zero.

Many definite integrals, including

44:10:5 $$\int_0^1 \psi(t+\lambda)\mathrm{d}t = \ln(\lambda)$$

and

44:10:6 $$\int_0^1 \psi(t)\sin\left(n\pi t\right)\mathrm{d}t = -\pi/2 \qquad n = 2,4,6,\cdots$$

are listed by Gradshteyn and Ryzhik [Sections 6.46 and 6.47].

44:11 COMPLEX ARGUMENT

Notice the poles that occur along the real axis for $v = 0, -1, -2, \cdots$ in Figure 44-2, a three-dimensional graphic of the real and imaginary parts of the digamma function of $v + i\mu$. Formulas for these parts are contained in

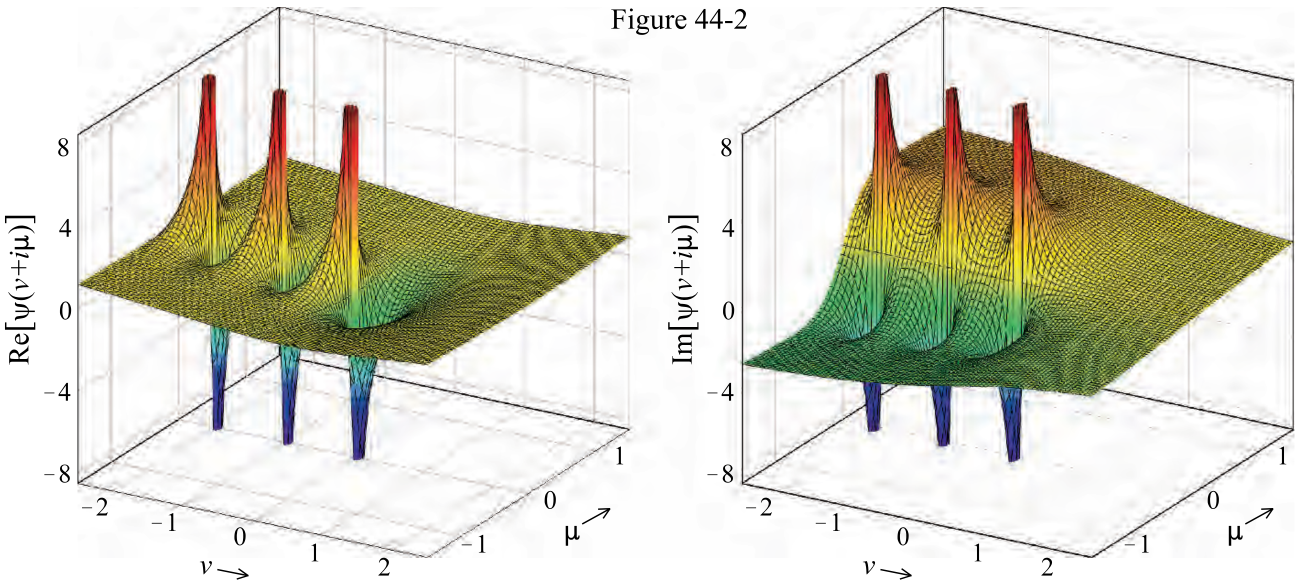
44:11:1 $$\psi(v+i\mu) = -\gamma - \frac{v}{v^2+\mu^2} + \sum_{j=1}^{\infty} \frac{jv+v^2+\mu^2}{j\left[(j+v)^2+\mu^2\right]} + i\sum_{j=0}^{\infty} \frac{\mu}{(j+v)^2+\mu^2}$$

Six pages [288−293] of numerical values are listed by Abramowitz and Stegun. The formula

44:11:2 $$\psi(i\mu) = -\gamma + \sum_{j=1}^{\infty} \frac{\mu^2}{j(j^2+\mu^2)} + i\left[\frac{1}{2\mu} + \frac{\pi}{2}\coth(\pi\mu)\right]$$

applies when the argument is purely imaginary.

Figure 44-2



Inverse Laplace transforms of the digamma function include

44:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \psi(bs)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\psi(bs)\} = \frac{-1}{b[1-\exp(-t/b)]}$$

44:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{-\psi(bs)}{s}\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{-\psi(bs)}{s}\right\} = \ln\left\{\exp\left(\frac{t}{b}\right)-1\right\} + \gamma - i\pi$$

and

44:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \left[\psi(bs+c)-\psi(bs+c')\right]\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\psi(bs+c)-\psi(bs+c')\} = \frac{\exp(-c't/b)-\exp(-ct/b)}{b\left[1-\exp(-t/b)\right]}$$

44:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} 2s\left[\psi(s+\tfrac{1}{2})-\psi(s)-\frac{1}{2s}\right]\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{2s\left[\psi(s+\tfrac{1}{2})-\psi(s)-\frac{1}{2s}\right]\right\} = \mathrm{sech}^2\left(\frac{t}{4}\right)$$

44:12 GENERALIZATIONS: polygamma functions

The trigamma function $\psi^{(1)}(v)$ is the derivative of the digamma function

44:12:1 $$\psi^{(1)}(v)=\frac{\mathrm{d}}{\mathrm{d}v}\psi(v)$$

Some particular values of this function are

$\psi^{(1)}(-n)$	$\psi^{(1)}(\frac{-3}{2})$	$\psi^{(1)}(\frac{-1}{2})$	$\psi^{(1)}(0)$	$\psi^{(1)}(\frac{1}{2})$	$\psi^{(1)}(1)$	$\psi^{(1)}(\frac{3}{2})$	$\psi^{(1)}(2)$	$\psi^{(1)}(n)$	$\psi^{(1)}(\infty)$
$+\infty\vert+\infty$	$\frac{\pi^2}{2}+\frac{40}{9}$	$\frac{\pi^2}{2}+4$	$+\infty\vert+\infty$	$\frac{\pi^2}{2}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{2}-4$	$\frac{\pi^2}{6}-1$	$\frac{\pi^2}{6}-\sum_{j=1}^{n-1}\frac{1}{j^2}$	0

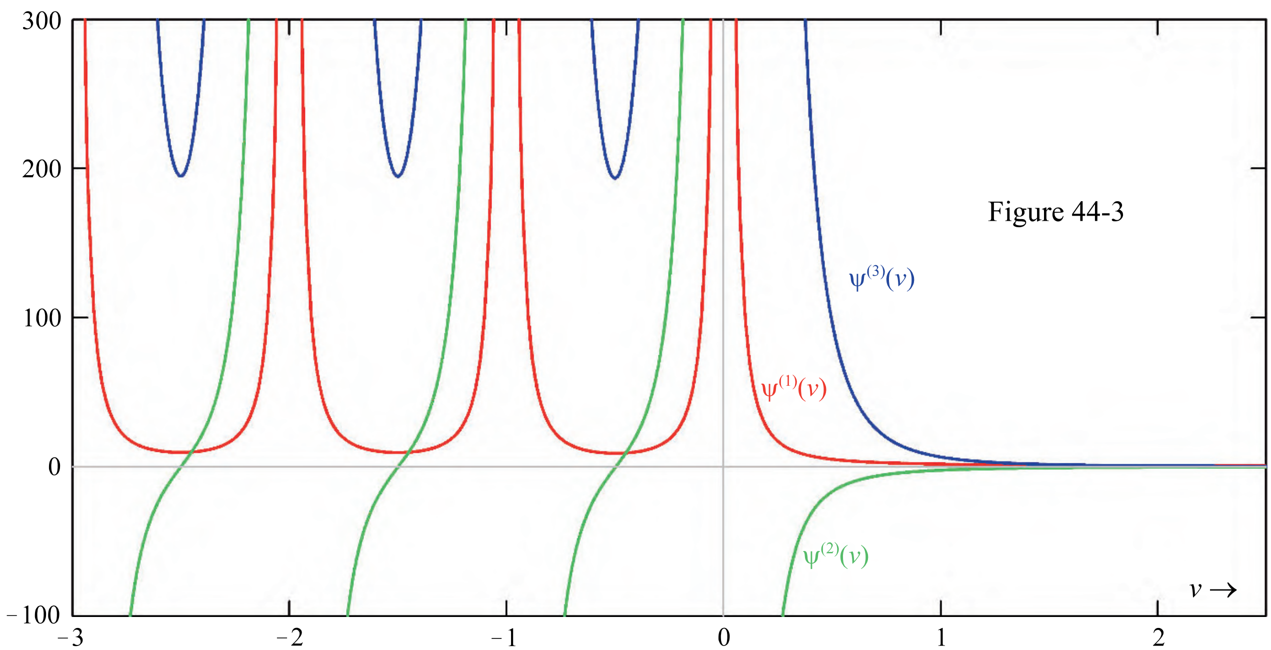
The tetragamma function $\psi^{(2)}(v)$ is the derivative of the trigamma function, the pentagamma function is the derivative of the tetragamma function, and so on; generally

44:12:2 $$\psi^{(n)}(v)=\frac{\mathrm{d}}{\mathrm{d}v}\psi^{(n-1)}(v)$$

The trigamma, tetragamma and pentagamma instances are illustrated in Figure 44-3. As do all polygamma functions, they resemble the digamma function in being monotonic for positive argument and segmented for negative v, with discontinuities at all nonpositive integer arguments. The cause of the repetitive behavior at negative argument can be understood by making an n-fold differentiation of the reflection formula 44:5:2 of the digamma function, to produce the reflection formula

44:12:3 $$(-)^n\psi^{(n)}(\tfrac{1}{2}-v)=\psi^{(n)}(\tfrac{1}{2}+v)-\pi\frac{\mathrm{d}^n}{\mathrm{d}v^n}\tan(\pi v)$$

for polygamma functions. Because of the periodic nature of the tangent function [Chapter 34], the second right-hand term, which generally has a large magnitude, responds only to the fractional value of v. In contrast, the aperiodic first right-hand term is of relatively small magnitude. So the second right-hand term in 44:12:4 swamps the first,



leading to the quasiperiodic behavior for $v < 0$.

The recursion formula

44:12:4 $$\psi^{(n)}(v+1) = \psi^{(n)}(v) + \frac{(-)^n n!}{v^{n+1}}$$

may be combined with the particular values

44:12:5 $$\psi^{(n)}(1) = (-)^{n+1} n! \zeta(n+1)$$

and

44:12:6 $$\psi^{(n)}\left(\tfrac{1}{2}\right) = (-2)^{n+1} n! \lambda(n+1)$$

to give the formulas

44:12:7 $$\psi^{(n)}(m) = (-)^{n+1} n! \left[\zeta(n+1) - \sum_{j=1}^{m-1} \frac{1}{j^{n+1}}\right] = (-)^{n+1} n! \sum_{j=m}^{\infty} \frac{1}{j^{n+1}} \qquad \begin{matrix} n = 1,2,3,\cdots \\ m = 1,2,3,\cdots \end{matrix}$$

and

44:12:8 $$\psi^{(n)}\left(\tfrac{1}{2} \pm m\right) = 2^{n+1} n! \left[(-)^{n+1} \lambda(n+1) + (\mp)^n \sum_{j=1}^{m} \frac{1}{(2j-1)^{n+1}}\right] \qquad \begin{matrix} n = 1,2,3,\cdots \\ m = 0,1,2,\cdots \end{matrix}$$

The ζ and λ functions are those from Chapter 3.

The behavior of $\psi^{(n)}(v)$ close to $v = 0$ is dominated by an $n!(-v)^{-n-1}$ term. After this dominant term has been subtracted, the residue may be expanded as a power series:

44:12:9 $$\psi^{(n)}(v) - \frac{n!}{(-v)^{n+1}} = (-)^{n+1} \sum_{j=0}^{\infty} \frac{(n+j)!\zeta(n+j+1)}{j!}(-v)^j \qquad -1 < v < 1$$

The poor convergence and limited domain of this series can be remedied by rewriting it as

44:12:10 $$\psi^{(n)}(v) = (-)^{n+1}\left[\frac{n!}{v^{n+1}} + \frac{n!}{(1+v)^{n+1}} + \sum_{j=0}^{\infty} \frac{(n+j)![\zeta(n+j+1)-1]}{j!}(-v)^j\right] \qquad -1 < v < 1$$

Another expansion shares its first two terms with those of 44:12:11

44:12:11 $$\psi^{(n)}(v) = (-)^{n+1} n! \left[\frac{1}{v^{n+1}} + \frac{1}{(1+v)^{n+1}} + \frac{1}{(2+v)^{n+1}} + \cdots\right] = n! \sum_{j=1}^{\infty} \left(\frac{-1}{j+v}\right)^{n+1}$$

The asymptotic expansion provided by

44:12:12 $$\psi^{(n)}(v) \sim \frac{-1}{(-v)^n}\left[(n-1)! + \frac{n!}{2v} + \frac{(n+1)!}{12v^2} - \frac{(n+3)!}{720v^4} + \cdots\right] = (-)^{n+1} \sum_{j=0} \frac{(j+n-1)!\mathrm{B}_j}{j! v^{n+j}}$$

establishes $-(n-1)!/(\frac{1}{2}-v)^n$ as a limiting expression for $\psi^{(n)}(v)$ as $v \to \infty$.

Be aware that, though in many respects the polygamma functions do generalize the digamma function, setting $n = 0$ in the formulas of this section does not necessarily give valid expressions for the digamma function. This caveat also applies to *Equator*'s polygamma function routine (keyword **polygamma**), which *cannot* be used to generate values of $\psi(v)$. This routine utilizes equations 44:12:12, with 44:12:3 if v is negative. *Equator* also has dedicated trigamma function and tetragamma function routines (keywords **trigamma** and **tetragamma**), which operate similarly.

The inverse Laplace transformation of the polygamma function is

44:12:13 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \psi^{(n)}(as) \frac{\exp(ts)}{2\pi i} \mathrm{d}s = \mathfrak{I}\left\{\psi^{(n)}(as)\right\} = \frac{t^n}{(-a)^{n+1}\left[1 - \exp(-t/a)\right]}$$

44:13 COGNATE FUNCTION: Bateman's G function

Harry Bateman (a prolific English mathematician, 1882 – 1946, who emigrated to the United States in 1910) made use of a function derived from the digamma function and defined by

44:13:1 $$G(v) = \psi\left(\frac{v+1}{2}\right) - \psi\left(\frac{v}{2}\right)$$

It is denoted $2\beta(v)$ by Gradshteyn and Ryzhik and is generalized by the bivariate eta function [Section 64:13] as

44:13:2 $$G(v) = 2\eta(1,v)$$

Bateman's G function, which is mapped in Figure 44-1, may be represented by the integral

44:13:3 $$G(v) = 2\int_0^\infty \frac{\exp(-vt)}{1+\exp(-t)}\,dt \qquad v>0$$

It is also an $L = K = 1$ hypergeometric function [Section 18:14] of argument ½:

44:13:4 $$G(v) = \frac{1}{v}\sum_{j=0}^{\infty} \frac{(1)_j}{(v+1)_j}\left(\frac{1}{2}\right)^j$$

A further definition is provided by formula 44:14:5.

The properties of Bateman's G function, including the reflection

44:13:5 $$G(1-v) = 2\pi\csc(\pi v) - G(v)$$

and recursion

44:13:6 $$G(1+v) = \frac{2}{v} - G(v)$$

formulas, may be deduced from those of the digamma function. The eta number of Chapter 3 is a component of two of the following expansions

44:13:7 $$\frac{G(v)}{2} = \frac{1}{v(1+v)} + \frac{1}{(2+v)(3+v)} + \frac{1}{(4+v)(5+v)} + \cdots = \sum_{j=0}^{\infty}\frac{1}{(2j+v)(2j+1+v)}$$

44:13:8 $$G(v) = \ln(4) + \frac{\pi^2}{6}(1-v) + \frac{3Z}{4}(1-v)^2 + \frac{7\pi^4}{360}(1-v)^3 + \cdots = 2\sum_{j=0}^{\infty}\eta(j+1)[1-v]^j$$

44:13:9 $$G(v) = \frac{1}{v} - \frac{2}{1-v^2} + \pi\csc(\pi v) + 2\sum_{j=0}^{\infty}\left[1-\eta(2j+1)\right]v^{2j}$$

each of which is limited in its domain of applicability.

Like the digamma function, Bateman's G function is unusually fecund in its particular values, a sampling of which is listed in the following panel

$G\left(\frac{-1}{2}\right)$	$G(0)$	$G\left(\frac{1}{4}\right)$	$G\left(\frac{1}{3}\right)$	$G\left(\frac{1}{2}\right)$	$G\left(\frac{2}{3}\right)$	$G\left(\frac{3}{4}\right)$	$G(1)$	$G\left(\frac{3}{2}\right)$	$G(2)$
$-4-\pi$	$-\infty\|+\infty$	$\sqrt{2}\pi + \sqrt{8}\ln\left(\sqrt{2}+1\right)$	$\frac{2\pi}{\sqrt{3}} + \ln(4)$	π	$\frac{2\pi}{\sqrt{3}} - \ln(4)$	$\sqrt{2}\pi - \sqrt{8}\ln\left(\sqrt{2}+1\right)$	$\ln(4)$	$4-\pi$	$2-\ln(4)$

Innumerable additions may be made via equations 44:7:6, 44:13:1, 44:13:4, and 44:13:5. *Equator*'s Bateman's G function routine (keyword **G**) utilizes definition 44:13:1.

44:14 RELATED TOPIC: summation of series of reciprocals

The digamma function, polygamma functions, and Bateman's G function are all tools useful in summing certain infinite (and finite) algebraic (and numerical) series.

Though the infinite series

44:14:1 $$\frac{1}{c}+\frac{1}{b+c}+\frac{1}{2b+c}+\frac{1}{3b+c}+\cdots=\sum_{j=0}^{\infty}\frac{1}{jb+c}=\frac{-1}{b}\psi\left(\frac{c}{b}\right) \qquad c\neq 0,-b,-2b,\cdots$$

does *not* converge, rendering the formula invalid, it is nevertheless useful in permitting the construction of the *finite* summation

44:14:2 $$\frac{1}{c}+\frac{1}{b+c}+\frac{1}{2b+c}+\cdots+\frac{1}{nb+c}=\sum_{j=0}^{n}\frac{1}{jb+c}=\frac{-1}{b}\left[\psi\left(\frac{c}{b}\right)-\psi\left(\frac{c}{b}+n+1\right)\right] \qquad c\neq 0,-b,-2b,\cdots$$

which does converge. The corresponding alternating series converges in both the infinite

44:14:3 $$\frac{1}{c}-\frac{1}{b+c}+\frac{1}{2b+c}-\frac{1}{3b+c}+\cdots=\sum_{j=0}^{\infty}\frac{(-1)^j}{jb+c}=\frac{1}{2b}\mathrm{G}\left(\frac{c}{b}\right) \qquad c\neq 0,-b,-2b,\cdots$$

and the finite

44:14:4 $$\frac{1}{c}-\frac{1}{b+c}+\frac{1}{2b+c}-\cdots+\frac{(-1)^n}{nb+c}=\sum_{j=0}^{n}\frac{(-1)^j}{jb+c}=\frac{1}{2b}\left[\mathrm{G}\left(\frac{c}{b}\right)+(-)^n\mathrm{G}\left(\frac{c}{b}+n+1\right)\right] \qquad c\neq 0,-b,-2b,\cdots$$

formats. Equations 1:14:5 and 1:14:10 are examples of finite numerical series summed in these ways. An example of an infinite algebraic series is

44:14:5 $$\frac{1}{x}-\frac{1}{1+x}+\frac{1}{2+x}-\frac{1}{3+x}+\cdots=\sum_{j=0}^{\infty}\frac{(-1)^j}{j+x}=\frac{\mathrm{G}(x)}{2} \qquad x\neq 0,-1,-2,\cdots$$

which can serve as an additional definition of Bateman's G function.

From equation 44:6:2 one may derive

44:14:6 $$\frac{1}{b+c}+\frac{1}{2(2b+c)}+\frac{1}{3(3b+c)}+\cdots=\sum_{j=1}^{\infty}\frac{1}{j(jb+c)}=\frac{1}{c}\left[\psi\left(\frac{b+c}{b}\right)+\gamma\right]$$

An example of a numerical series summed by this route is

44:14:7 $$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots=\sum_{j=1}^{\infty}\frac{1}{j(j+1)}=\psi(2)+\gamma=1$$

Other similar examples develop from setting $v = 1$ or ½ in equation 44:13:7; one thereby derives

44:14:8 $$\frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\cdots=\sum_{j=1}^{\infty}\frac{1}{2j(2j-1)}=\frac{\mathrm{G}(1)}{2}=\ln(2)$$

or

44:14:9 $$\frac{1}{1\times 3}+\frac{1}{5\times 7}+\frac{1}{9\times 11}+\cdots=\sum_{j=0}^{\infty}\frac{1}{(4j+1)(4j+3)}=\frac{\mathrm{G}(\frac{1}{2})}{8}=\frac{\pi}{8}$$

while combination of equations 44:14:6 and 44:14:7 yields

44:14:10 $$\frac{1}{1\times 2}-\frac{1}{2\times 3}+\frac{1}{3\times 4}-\cdots=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(j+1)}=\ln(4)-1$$

Similarly, sums of infinite series of reciprocal powers greater than unity can often be summed in terms of polygamma functions. Thus, for $n = 2,3,4,\cdots$

44:14:11 $$\frac{1}{c^n}+\frac{1}{(b+c)^n}+\frac{1}{(2b+c)^n}+\frac{1}{(3b+c)^n}+\cdots=\sum_{j=0}^{\infty}\frac{1}{(jb+c)^n}=\frac{(-1/b)^n}{(n-1)!}\psi^{(n-1)}\left(\frac{c}{b}\right)$$

which converges irrespective of the sign of b provided c/b is not a nonpositive integer. The corresponding alternating series may be constructed as follows:

44:14:12 $$\frac{1}{c^n}-\frac{1}{(b+c)^n}+\frac{1}{(2b+c)^n}-\cdots=\sum_{j=0}^{\infty}\frac{(-1)^j}{(jb+c)^n}=\frac{(-1/2b)^n}{(n-1)!}\left[\psi^{(n-1)}\left(\frac{c}{2b}\right)-\psi^{(n-1)}\left(\frac{b+c}{2b}\right)\right]$$

Again, finite series can be summed as a difference between two infinite series.

With the aid of partial fractionation [Sections 16:13 and 17:13], quite complicated algebraic series may be summed. For example, the partial fractionation

44:14:13 $$\mathrm{f}(j)=\frac{1+j}{4-4j-7j^2-2j^3}=\frac{3}{25(2+j)}-\frac{1}{5(2+j)^2}+\frac{6}{25(1-2j)}$$

permits the summation

44:14:14 $$\sum_{j=0}^{\infty}\mathrm{f}(j)=\frac{1}{25}\left[\sum_{j=0}^{\infty}\frac{3}{2+j}-\sum_{j=0}^{\infty}\frac{5}{(2+j)^2}+\sum_{j=0}^{\infty}\frac{6}{1-2j}\right]=\frac{-3\psi(2)}{25}-\frac{\psi^{(1)}(2)}{5}+\frac{3\psi(\frac{-1}{2})}{25}$$

which evaluates to $(8/25)-(\pi^2/30)-(3/25)\ln(4)$. The derivation of integral 35:10:3 provides yet another example.

CHAPTER 45

THE INCOMPLETE GAMMA FUNCTIONS

When one speaks of "the incomplete gamma function", without further qualification, one usually means $\gamma(v,x)$, but a less ambiguous name for this is the *lower incomplete gamma function*. Two other varieties of incomplete gamma functions, the *upper incomplete gamma function* $\Gamma(v,x)$, and the *entire incomplete gamma function* $\gamma\mathrm{n}(v,x)$, are also addressed in this chapter. The three are linked through the relationships

45:0:1 $$\Gamma(v)-\Gamma(v,x)=\gamma(v,x)=x^{v}\Gamma(v)\gamma\mathrm{n}(v,x)$$

where $\Gamma(v)$ is the (complete) gamma function of Chapter 43. The related Mittag-Leffler function is the subject of Section 45:14.

45:1 NOTATION

The v and x variables are respectively the parameter and the argument of these bivariate functions. We treat both as real.

The "incomplete" in the names of these functions reflects the restricted ranges of integration in definitions 45:3:1 and 45:3:2 contrasted with that in the integral, 43:3:1, that defines the (complete) gamma function:

45:1:2 $$\Gamma(v)=\int_0^\infty t^{v-1}\exp(-t)\mathrm{d}t=\int_0^x t^{v-1}\exp(-t)\mathrm{d}t+\int_x^\infty t^{v-1}\exp(-t)\mathrm{d}t=\gamma(v,x)+\Gamma(v,x)$$

Of course, the adjectives "lower" and "upper" refer to the ranges of the integration variable in the equations defining $\gamma(v,x)$ and $\Gamma(v,x)$. Incidentally, these adjectives also denote the case in which the Greek letter gamma is printed, a useful mnemonic. Note that the incompleteness of these functions relates to the v variable, not x: they are incomplete versions of $\Gamma(v)$, not $\Gamma(x)$.

The $\Gamma(v,x)$ function is complementary to $\gamma(v,x)$ in the sense that the addition of the two functions creates $\Gamma(v)$; it is for this reason that an alternative name for the upper incomplete gamma function $\Gamma(v,x)$ is the *complementary incomplete gamma function*.

Entire functions are ones that lack singularities (except perhaps at infinity) throughout the complex plane; the $\gamma\mathrm{n}(v,x)$ function possesses this property and hence has taken "entire" into its name.

Within this chapter, we shall often refer to $\gamma(v,x)$, $\Gamma(v,x)$, and $\gamma\mathrm{n}(v,x)$, for the sake of brevity, as "the lower function", "the upper function", and "the entire function".

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_46, © Springer Science+Business Media, LLC 2009

Alternative symbolisms abound. $\Gamma_x(\nu)$ has been used for $\gamma(\nu,x)$. The notations $P(\nu,x)$ and $Q(\nu,x)$ are often used, respectively for the quotients $\gamma(\nu,x)/\Gamma(\nu)$ and $\Gamma(\nu,x)/\gamma(\nu)$. These quotients, which find statistical applications, are said to be *regularized incomplete gamma functions*; they sum to unity. The symbol $\gamma^*(\nu,x)$ often replaces $\gamma\mathrm{n}(\nu,x)$ and did so in the first edition of the *Atlas*. You may encounter $E_\nu(x)$ or $K_\nu(x)$ as a symbol for $x^{\nu-1}\Gamma(1-\nu,x)$.

45:2 BEHAVIOR

For negative argument x, the lower and upper incomplete gamma functions are generally complex and this domain is therefore excluded from consideration within this chapter, except in discussions of the entire incomplete gamma function.

The lower function has no discontinuities when the parameter ν exceeds zero, but these are present when $\nu = 0, -1, -2, \cdots$. At any fixed value of ν, $\gamma(\nu,x)$ increases with x, as shown in Figure 45-1, ultimately approaching $\Gamma(\nu)$ as $x \to \infty$. The lower function is always positive for $\nu > 0$ and always negative when ν lies in one of the ranges $-1 < \nu < 0$, $-3 < \nu < -2$, etc. Otherwise, a single zero is displayed.

The upper incomplete gamma function has a simpler behavior that can be appreciated from Figure 45-2. It is invariably positive. At $x = 0$, $\Gamma(\nu,x) = \Gamma(\nu)$ when the ν parameter is positive, but it equals $+\infty$ when the ν is negative. As x increases from 0 to ∞, ν remaining fixed, the upper function invariably decreases monotonically, but at an ever-slower rate, to approach zero at $x = \infty$.

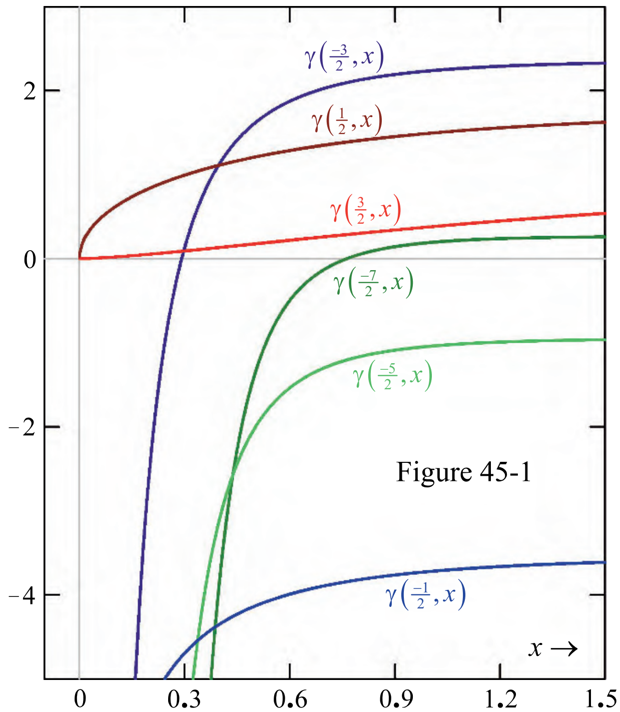


Figure 45-1

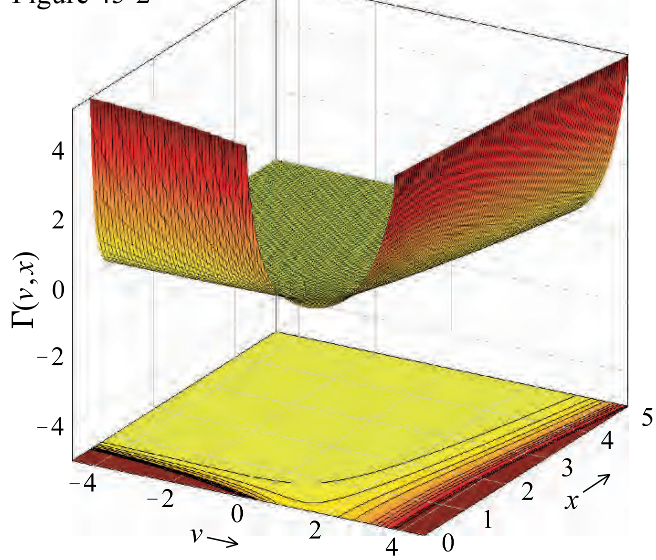
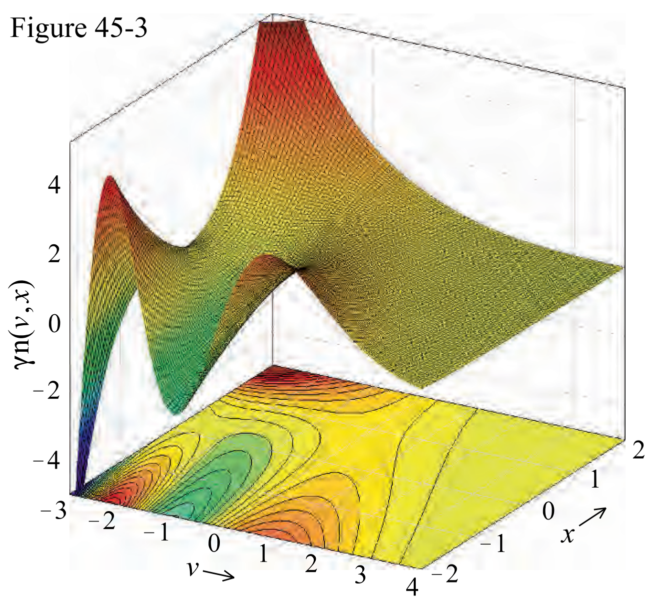
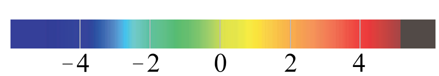
Figure 45-2


Figure 45-3




The discontinuities and infinities that complicate the behavior of the $\gamma(v,x)$ and $\Gamma(v,x)$ functions are absent from the $\gamma\mathrm{n}(v,x)$ function, which is real and finite whenever its variables are real and finite. Nonetheless, its behavior is far from simple, as Figure 45-3 will attest. The entire incomplete gamma function lacks zeros if $v>0$, has a single zero if $0>v>-1$, two zeros if $-1>v>-2$, three zeros if $-2>v>-3$, and so on.

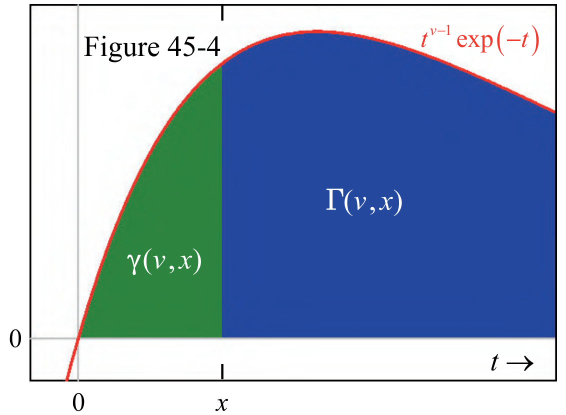
45:3 DEFINITIONS

Figure 45-4 illustrates the definition of the lower and upper incomplete gamma functions as the integrals

45:3:1 $$\gamma(v,x)=\int_0^x t^{v-1}\exp(-t)\mathrm{d}t$$

and

45:3:2 $$\Gamma(v,x)=\int_x^\infty t^{v-1}\exp(-t)\mathrm{d}t$$



The total area (both colored zones with the blue extended to infinity) in the diagram is $\Gamma(v)$. The first of these integrals diverges for nonpositive v but the recursion formula 45:5:1 can serve to extend the definition to encompass most negative parameters, though not for $v=0,-1,-2,\cdots$. The corresponding definition of the entire incomplete gamma function is

45:3:3 $$\gamma\mathrm{n}(v,x)=\frac{x^{-v}}{\Gamma(v)}\int_0^x t^{v-1}\exp(-t)\mathrm{d}t=\frac{\gamma(v,x)}{x^v\Gamma(v)} \qquad x>0$$

There are many definitions of the three functions as definite integrals. Some are:

45:3:4 $$\gamma(v,x)=x^{v/2}\int_0^\infty t^{(v-2)/2}\exp(-t)\mathrm{J}_v\left(2\sqrt{xt}\right)\mathrm{d}t \qquad v>0$$

45:3:5 $$\Gamma(v,x)=\frac{2x^{v/2}}{\Gamma(1-v)}\exp(-x)\int_0^\infty t^{-v/2}\exp(-t)\mathrm{K}_v\left(2\sqrt{xt}\right)\mathrm{d}t \qquad v<1$$

45:3:6 $$\Gamma(v,x)=\frac{x^v}{\Gamma(1-v)}\exp(-x)\int_0^\infty \frac{t^{-v}\exp(-t)}{t+x}\mathrm{d}t \qquad v<1$$

45:3:7 $$\Gamma(v,x)=\int_0^\infty \frac{\exp(-x-t)}{(x+t)^{1-v}}\mathrm{d}t \qquad v<1 \qquad x>0$$

45:3:8 $$\gamma\mathrm{n}(v,x)=\frac{1}{\Gamma(v)}\int_0^1 t^{v-1}\exp(-xt)\mathrm{d}t \qquad v>0$$

The J and K functions are the Bessel and Macdonald functions of Chapters 53 and 51.

There are several hypergeometric representations of the function trio, as identified in equations 45:6:1−4. It follows that synthesis [Section 43:14] can construct the functions from the corresponding basis function. For example, the entire incomplete gamma function is an $L=K+1=2$ hypergeometric function [Section 18:14] and, as such, it may be synthesized [Section 43:14] from the exponential function

45:3:9 $$\exp(-x) \xrightarrow[1+\nu]{\nu} \sum_{j=0}^{\infty} \frac{(\nu)_j}{(1)_j(1+\nu)_j}(-x)^j = \Gamma(1+\nu)\,\gamma\mathrm{n}(\nu,x)$$

Related to this is the observation that one of the *regularized incomplete gamma functions* arises by differintegration [Section 12:14] of the exponential function:

45:3:10 $$\frac{d^{-\nu}}{dx^{-\nu}}\exp(\pm x) = \exp(\pm x)\frac{\gamma(\nu,\pm x)}{\Gamma(\nu)}$$

45:4 SPECIAL CASES

When ν equals zero or ± 1, the three incomplete gamma functions take the following values

$\gamma(-1,x)$	$\gamma(0,x)$	$\gamma(1,x)$	$\Gamma(-1,x)$	$\Gamma(0,x)$	$\Gamma(1,x)$	$\gamma\mathrm{n}(-1,x)$	$\gamma\mathrm{n}(0,x)$	$\gamma\mathrm{n}(1,x)$
$+\infty\vert-\infty$	$-\infty\vert+\infty$	$1-\exp(-x)$	$\mathrm{Ei}(-x)+$ $\exp(-x)/x$	$-\mathrm{Ei}(-x)$	$\exp(-x)$	x	1	$\frac{1-\exp(-x)}{x}$

More generally, we have for positive integer parameters

45:4:1 $$\gamma(n,x) = (n-1)![1-e_{n-1}(x)\exp(-x)] \quad x \ge 0 \quad n=1,2,3,\cdots$$

45:4:2 $$\Gamma(n,x) = (n-1)!e_{n-1}(x)\exp(-x) \quad x \ge 0 \quad n=1,2,3,\cdots$$

45:4:3 $$\gamma\mathrm{n}(n,x) = x^{-n}[1-e_{n-1}(x)\exp(-x)] \quad n=1,2,3,\cdots$$

$e_{n-1}(x)$ being the exponential polynomial [Section 26:12], while for negative integer parameter

45:4:4 $$\gamma(-n,x) = (-)^n\infty \mid (-)^{n+1}\infty \quad n=0,1,2,\cdots$$

45:4:5 $$\Gamma(-n,x) = \frac{(-)^{n+1}}{n!}\left[\mathrm{Ei}(-x) + \frac{\exp(-x)}{x}\sum_{j=0}^{n-1}\frac{j!}{(-x)^j}\right] \quad n=0,1,2,\cdots$$

45:4:6 $$\gamma\mathrm{n}(-n,x) = x^n \quad n=0,1,2,\cdots$$

where Ei is the exponential integral of Chapter 37.

Incomplete gamma functions of moiety parameter reduce to the functions of Chapters 40 and 42:

$\gamma(\tfrac12,x)$	$\Gamma(\tfrac12,x)$	$\gamma\mathrm{n}(\tfrac12,x)$	$\gamma\mathrm{n}(\tfrac12,-x)$
$\sqrt{\pi}\,\mathrm{erf}(\sqrt{x})$	$\sqrt{\pi}\,\mathrm{erfc}(\sqrt{x})$	$\frac{\mathrm{erf}(\sqrt{x})}{\sqrt{x}}$	$\frac{2}{\sqrt{\pi x}}\exp(x)\,\mathrm{daw}(\sqrt{x})$

$x \ge 0$

and thereby the cases in which the parameter is an odd multiple of ½ are accessible through the recursion formulas 45:5:1−3.

A table in Section 37:15 lists the integrals

45:4:7 $$\int_x^{\infty} t^{\nu}\exp(-t)\,dt = \Gamma(\nu+1,x)$$

for 13 values of ν and hence provides access to 13 special cases of the upper incomplete gamma function.

45:5 INTRARELATIONSHIPS

Recursion formulas for the three varieties of the incomplete gamma function are

45:5:1 $$\gamma(\nu+1,x)=\nu\gamma(\nu,x)-x^{\nu}\exp(-x)$$

45:5:2 $$\Gamma(\nu+1,x)=\nu\Gamma(\nu,x)+x^{\nu}\exp(-x)$$

and

45:5:3 $$\gamma\mathrm{n}(\nu+1,x)=\frac{\gamma\mathrm{n}(\nu,x)}{x}-\frac{\exp(-x)}{x\Gamma(1+\nu)} \qquad x\neq 0$$

By iteration, the first of these may be generalized to

45:5:4 $$\gamma(\nu+n,x)=\left(\nu\right)_{n}\left[\gamma(\nu,x)-\exp(-x)\sum_{j=0}^{n-1}\frac{x^{j+\nu}}{(\nu)_{j+1}}\right]$$

and similar formulas may be derived for the other two.

Nielsen's expansion (Niels Nielsen, Danish mathematician, 1865 – 1931)

45:5:5 $$\gamma(\nu,x+y)=\gamma(\nu,x)+x^{\nu-1}\exp(-x)\sum_{j=0}^{\infty}\frac{(1-\nu)_{j}}{j!(-x)^{j}}\left[1-\exp(-y)\mathrm{e}_{j}(y)\right] \qquad |y|<|x|$$

provides an argument-addition formula for the lower incomplete gamma function.

45:6 EXPANSIONS

Two hypergeometric-style power-series expansions exist for the lower incomplete gamma function; one for the function itself:

45:6:1 $$\gamma(\nu,x)=\frac{x^{\nu}}{\nu}-\frac{x^{\nu+1}}{1+\nu}+\frac{x^{\nu+2}}{2(2+\nu)}-\frac{x^{\nu+3}}{6(3+\nu)}+\cdots=\sum_{j=0}^{\infty}\frac{(-)^{j}x^{\nu+j}}{j!(j+\nu)}=\frac{x^{\nu}}{\nu}\sum_{j=0}^{\infty}\frac{(\nu)_{j}}{(1)_{j}(\nu+1)_{j}}(-x)^{j}$$

and one for its product with the exponential function:

45:6:2 $$\exp(x)\gamma(\nu,x)=\frac{x^{\nu}}{\nu}+\frac{x^{\nu+1}}{\nu(1+\nu)}+\frac{x^{\nu+2}}{\nu(1+\nu)(2+\nu)}+\cdots=\sum_{j=0}^{\infty}\frac{x^{\nu+j}}{(\nu)_{j}}=\frac{x^{\nu}}{\nu}\sum_{j=0}^{\infty}\frac{1}{(\nu+1)_{j}}x^{j}$$

Corresponding to these, there are two expansions,

45:6:3 $$\gamma\mathrm{n}(\nu,x)=\frac{1}{\Gamma(\nu)}\left[\frac{1}{\nu}-\frac{x}{1!(1+\nu)}+\frac{x^{2}}{2!(2+\nu)}-\frac{x^{3}}{3!(3+\nu)}+\cdots\right]=\frac{1}{\Gamma(1+\nu)}\sum_{j=0}^{\infty}\frac{(\nu)_{j}}{(1)_{j}(1+\nu)_{j}}(-x)^{j}$$

and

45:6:4 $$\exp(x)\gamma\mathrm{n}(\nu,x)=\frac{1}{\Gamma(1+\nu)}+\frac{x}{\Gamma(2+\nu)}+\frac{x^{2}}{\Gamma(3+\nu)}+\cdots=\sum_{j=0}^{\infty}\frac{x^{j}}{\Gamma(j+1+\nu)}=\frac{1}{\Gamma(1+\nu)}\sum_{j=0}^{\infty}\frac{1}{(1+\nu)_{j}}(x)^{j}$$

for the entire incomplete gamma function. Another expansion is

45:6:5 $$\gamma\mathrm{n}(\nu,x)=\frac{\exp(-x)}{x^{\nu/2}}\sum_{j=0}^{\infty}x^{j/2}\mathrm{e}_{j}(-1)\,\mathrm{I}_{\nu+j}\left(2\sqrt{x}\right)$$

involving functions from Section 26:12 and Chapter 50.

The upper incomplete gamma function has the power-series expansion

45:6:6 $$\exp(x)\Gamma(v,x) \sim \frac{1}{x^{1-v}} - \frac{1-v}{x^{2-v}} + \frac{(1-v)(2-v)}{x^{3-v}} - \cdots = x^{v-1}\sum_{j=0}(1-v)_j\left(\frac{-1}{x}\right)^j$$

This is an asymptotic hypergeometric series, useful for large x. This upper function is expansible as the continued fraction

45:6:7 $$\Gamma(v,x) = \frac{x^v \exp(-x)}{x+}\,\frac{1-v}{1+}\,\frac{2-v}{x+}\,\frac{3-v}{1+}\,\frac{4-v}{x+\cdots}$$

which is surprisingly convergent, even for quite small values of the argument.

45:7 PARTICULAR VALUES

	$x=0$		$x=\infty$	
	$v<0$	$v>0$	$v<0$	$v>0$
$\gamma(v,x)$	$-\infty$	0	$\Gamma(v)$	$\Gamma(v)$
$\Gamma(v,x)$	$+\infty$	$\Gamma(v)$	0	0
$\gamma\text{n}(v,x)$	$1/\Gamma(1+v)$	$1/\Gamma(1+v)$	$+\infty$	0

45:8 NUMERICAL VALUES

Equator's lower incomplete gamma function routine (keyword **gamlower**) is based on formula 45:6:2. Of course, no useful value is returned if x is negative or v is a nonpositive integer.

Several different approaches are adopted by *Equator* in executing its upper incomplete gamma function routine, which uses the keyword **gamupper**. If v is a negative integer or zero, equation 45:4:5 is employed. If x is smaller than either v or 0.04, then the algorithm calculates $\Gamma(v,x)$ by subtracting $\gamma(v,x)$ from $\Gamma(v)$; if this leads to loss of precision, as it sometimes does, *Equator* returns only those digits that are significant. Otherwise, the continued fraction, formula 45:6:6 is used.

With keyword **gamentire**, *Equator*'s entire incomplete gamma function routine employs equation 45:6:3 when the argument is negative, but equation 45:6:4 when x is positive.

45:9 LIMITS AND APPROXIMATIONS

Some limiting expressions are:

45:9:1 $$\gamma(v,x\to 0) \to \frac{x^v}{v}$$

45:9:2 $$\gamma(v,x\to\infty) \to \Gamma(v) - \frac{\exp(-x)}{x^{1-v}}$$

45:9:3 $$\gamma(v\to 0,x) \to \frac{1}{v} - \gamma + \text{Ei}(-x) \qquad \gamma = 0.57721\,56649\,01533$$

45:9:4 $$\gamma(\nu\to -n,x)\to\frac{(-1)^n}{n!}\left[\frac{x^{n+\nu}}{n+\nu}-\mathrm{Ein}(x)+\sum_{k=1}^{n}\frac{1}{k}\right]\qquad n=1,2,3,\cdots$$

45:9:5 $$\Gamma(\nu,x\to\infty)\to\frac{\exp(-x)}{x^{1-\nu}}$$

45:9:6 $$\Gamma(\nu,x\to 0)\to\begin{cases} x-\gamma-\ln(x) & \nu=0\\ \Gamma(\nu)-x^{\nu}/\nu & \nu\neq 0,-1,-2,\cdots\end{cases}$$

45:9:7 $$\Gamma(\nu\to 0,x)\to -x^{\nu}\,\mathrm{Ei}(-x)$$

All seven of the formulas above hold for $x > 0$, but this restriction does not apply to

45:9:8 $$\gamma\mathrm{n}(\nu,x\to 0)\to\frac{1}{\Gamma(1+\nu)}\left[1-\frac{\nu x}{1+\nu}\right]$$

The formula

45:9:9 $$\gamma\mathrm{n}(\nu,x\to\infty)\to x^{-\nu}$$

provides a useful approximation when x is large and positive. It becomes exact when ν is a *nonpositive* integer and then it is applicable for x of any magnitude whatsoever, in accord with 45:4:6.

45:10 OPERATIONS OF THE CALCULUS

The derivative with respect to its argument of the lower and upper incomplete gamma functions may be expressed in alternative ways:

45:10:1 $$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\gamma(\nu,x)\\ &\frac{-\mathrm{d}}{\mathrm{d}x}\Gamma(\nu,x)\end{aligned}\right\}=\begin{cases}(\nu-1)\gamma(\nu-1,x)-\gamma(\nu,x)\\ x^{\nu-1}\exp(-x)\\ \Gamma(\nu,x)-(\nu-1)\Gamma(\nu-1,x)\end{cases}$$

while that of the entire function gives

45:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\gamma\mathrm{n}(\nu,x)=-\nu\,\gamma\mathrm{n}(\nu+1,x)$$

The interesting differintegration [Section 12:14] formula,

45:10:3 $$\frac{\mathrm{d}^{\mu}}{\mathrm{d}x^{\mu}}\left\{\exp(x)\gamma(\nu,x)\right\}=\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}\exp(x)\gamma(\nu-\mu,x)$$

is a consequence of equations 45:6:2 and 18:14:18; it holds provided neither ν nor $\nu-\mu$ is a nonpositive integer.

Examples of definite integrals and Laplace transforms are

45:10:4 $$\int_0^{\infty}t^{\mu-1}\Gamma(\nu,bt)\,\mathrm{d}t=\frac{\Gamma(\mu+\nu)}{\mu b^{\mu}}\qquad \mu+\nu>0,\quad \mu>0$$

45:10:5 $$\int_0^{\infty}\gamma(\nu,bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\gamma(\nu,bt)\right\}=\frac{\Gamma(\nu)}{s}\left(\frac{b}{b+s}\right)^{\nu}\qquad \nu>0\quad b>0$$

45:10:6 $$\int_0^\infty t^{\mu-1}\Gamma(\nu,bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{t^{\mu-1}\Gamma(\nu,bt)\right\} = \frac{b^\nu\Gamma(\mu+\nu)}{\mu(s+b)^{\mu+\nu}}\sum_{j=0}^{\infty}\frac{(\mu+\nu)_j}{(\mu+1)_j}\left(\frac{s}{s+b}\right)^j \qquad \mu+\nu>0<\mu$$

45:10:7 $$\int_0^\infty t^{\mu-1}\gamma(\nu,bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{t^{\mu-1}\gamma(\nu,bt)\right\} = \frac{b^\nu\Gamma(\mu+\nu)}{\nu(s+b)^{\mu+\nu}}\sum_{j=0}^{\infty}\frac{(\mu+\nu)_j}{(\nu+1)_j}\left(\frac{b}{s+b}\right)^j \qquad \mu+\nu>0<b$$

Notice that equation 45:10:4 is the special $s=0$ case of 45:10:6 and likewise 45:10:5 is a special instance of 45:10:7. The transform that appears in formula 45:10:7 may be written as $(\mu/\nu)b^{\nu-\mu}s^{-\nu}\Gamma(\mu+\nu)\mathrm{B}\big(\mu,\nu,(b/s+b)\big)$ in terms of the incomplete beta function of Chapter 58. A similar depiction is available for the right-hand member of 45:10:6.

45:11 COMPLEX ARGUMENT

The functions of this chapter are seldom encountered with complex argument and the *Atlas* excludes this possibility. With purely imaginary argument, the upper incomplete gamma function has real and imaginary parts that may be expressed as *Böhmer integrals* [Section 39:12]

45:11:1 $$\Gamma(\nu,iy) = \sin\left(\tfrac{1}{2}\pi\nu\right)\mathrm{S}(\nu,y) + \cos\left(\tfrac{1}{2}\pi\nu\right)\mathrm{C}(\nu,y) + i\left[\sin\left(\tfrac{1}{2}\pi\nu\right)\mathrm{C}(\nu,y) - \cos\left(\tfrac{1}{2}\pi\nu\right)\mathrm{S}(\nu,y)\right]$$

45:12 GENERALIZATIONS

The three incomplete gamma functions are special cases of either the Kummer function [Chapter 47] or the Tricomi function [Chapter 48]. In each case, there are two ways in which the incomplete gamma function is related to its parent:

45:12:1 $$\gamma(\nu,x) = \frac{x^\nu\exp(-x)}{\nu}\mathrm{M}(1,1+\nu,x) = \frac{x^\nu}{\nu}\mathrm{M}(\nu,1+\nu,-x)$$

45:12:2 $$\Gamma(\nu,x) = x^\nu\exp(-x)\mathrm{U}(1,1+\nu,x) = \exp(-x)\mathrm{U}(1-\nu,1-\nu,x)$$

45:12:3 $$\gamma\mathrm{n}(\nu,x) = \frac{\exp(-x)}{\Gamma(1+\nu)}\mathrm{M}(1,1+\nu,x) = \frac{1}{\Gamma(1+\nu)}\mathrm{M}(\nu,1+\nu,-x)$$

45:13 COGNATE FUNCTIONS

The lower incomplete gamma function $\gamma(\nu,x)$ derives from Euler's function of the second kind [the (complete) gamma function, $\Gamma(\nu)$, Chapter 43] by allowing the upper limit to become indefinite. In a strictly analogous fashion, the incomplete beta function $\mathrm{B}(\nu,\mu,x)$ [Chapter 58] derives from Euler's function of the *first* kind [the (complete) beta function, $\mathrm{B}(\nu,\mu)$, Section 43:13] by allowing the upper limit to become indefinite.

A trivariate function that might appropriately be called the *doubly incomplete gamma function* but has actually been named simply *gamma* is

45:13:1 $$\int_x^u t^{\nu-1}\exp(-t)\,\mathrm{d}t = \gamma(\nu,u) - \gamma(\nu,x) = \Gamma(\nu,x) - \Gamma(\nu,u)$$

It has statistical applications, but is not further addressed in this *Atlas*.

45:14 RELATED TOPIC: the Mittag-Leffler function

From equations 45:6:3 and 43:5:5, one may construct

45:14:1 $$\exp(x)\gamma\mathrm{n}(\nu-1,x)=\frac{1}{\Gamma(\nu)}\sum_{j=0}^{\infty}\frac{x^j}{(\nu)_j}=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+\nu)}$$

This is a bivariate function with an argument x and a single parameter ν. A generalization of this innominate function is the trivariate *Mittag-Leffler function*, sometimes called the *generalized exponential function*, for which the symbol $\mathrm{E}_{\mu,\nu}(x)$ is in use. It has two parameters, μ and ν, additional to its argument x, and is defined by

45:14:2 $$\mathrm{E}_{\mu,\nu}(x)=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j\mu+\nu)}=\frac{1}{\Gamma(\nu)}+\frac{x}{\Gamma(\mu+\nu)}+\frac{x^2}{\Gamma(2\mu+\nu)}+\frac{x^3}{\Gamma(3\mu+\nu)}+\cdots$$

The Mittag-Leffler function satisfies the recursion formula

45:14:3 $$\mathrm{E}_{\mu,\nu+\mu}(x)=\frac{1}{x}\left[\mathrm{E}_{\mu,\nu}(x)-\frac{1}{\Gamma(\mu)}\right]$$

and often arises in solving problems through the Laplace inversion [Section 26:15]

45:14:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{s^{\mu-\nu}}{s^{\mu}-a}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{s^{\mu-\nu}}{s^{\mu}-a}\right\}=t^{\nu-1}\,\mathrm{E}_{\mu,\nu}\left(at^{\mu}\right)$$

Early terms are absent from series 45:14:2 when the quotient ν/μ is a nonpositive integer. The presence of the μ multiplier prevents the Mittag-Leffler function from being a hypergeometric function, but it is closely related thereto. It becomes a true hypergeometric function [Section 18:14, with $L = K+m = m$] whenever μ is a positive integer m, for then the *Gauss-Legendre formula* 43:5:9 allows a rewriting of the definition as

45:14:5 $$\mathrm{E}_{m,\nu}(x)=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(jm+\nu)}=\frac{1}{\Gamma(\nu)}\sum_{j=0}^{\infty}\frac{1}{\left(\frac{\nu}{m}\right)_j\left(\frac{\nu+1}{m}\right)_j\cdots\left(\frac{\nu+m-1}{m}\right)_j}\left(\frac{x}{m^m}\right)^j$$

On the other hand, if μ is the reciprocal, say $1/n$, of a positive integer, the Mittag-Leffler function reduces to a sum of n hypergeometric functions of the $L = K+1 = 1$ variety:

45:14:6 $$\mathrm{E}_{\frac{1}{n},\nu}(x)=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\frac{j}{n}+\nu)}=\sum_{j=0}^{\infty}\frac{(x^n)^j}{\Gamma\left(j+\nu\right)}+x\sum_{j=0}^{\infty}\frac{(x^n)^j}{\Gamma\left(j+\nu+\frac{1}{n}\right)}+\cdots+x^{n-1}\sum_{j=0}^{\infty}\frac{(x^n)^j}{\Gamma\left(j+\nu+\frac{n-1}{n}\right)}$$

each of which may be seen via 45:14:1 to be an entire incomplete gamma function. The two techniques may be combined if μ equals the quotient m/n.

Equator's Mittag-Leffler function routine (keyword **Mittag**) calculates values of the function by simply summing sufficient terms in series 45:14:2. Discussion of the Mittag-Leffler function is often limited to cases in which all three variables, μ, ν, and x, are positive, and, in this circumstance, the algorithm returns accurate values of the function provided that neither x^j nor $\Gamma(j\mu+\nu)$ causes numerical overflow by exceeding 10^{308}. The largest term in this series occurs close to $j = (x^{1/\mu} - \nu)/\mu$ and many more terms than this must be summed before the partial sum approximates the infinite sum to our prescribed accuracy of 15 places. Accordingly, the routine is sometimes rather slow. If your patience becomes exhausted, press the "Esc" key. Negative values of any, or all, of the variables are accepted by *Equator*, but the detailed progress of the summation is then rather unpredictable. Nevertheless *Equator* strives never to return an inappropriate answer.

The Swedish mathematician Magnus Gösta Mittag-Leffler (1847 – 1927) originally introduced a bivariate version of the function 45:14:2, with ν fixed at unity, and this remains the most important subclass of the function that bears his name. Some of its special cases are

$E_{0,1}(x)$	$E_{\frac{1}{2},1}(x)$	$E_{1,1}(x)$	$E_{2,1}(x)$	$E_{3,1}(x)$	$E_{4,1}(x)$
$\frac{1}{1-x}$	$\exp(x^2)\operatorname{erfc}(-x)$	$\exp(x)$	$\cosh(\sqrt{x})$	$\frac{2}{3}\exp\left(\frac{-1}{2x^{1/3}}\right)\cos\left(\frac{\sqrt{3}\,x^{1/3}}{2}\right)+\frac{\exp(x^{1/3})}{3}$	$\frac{\cosh(x^{1/4})+\cos(x^{1/4})}{2}$

Two results for differentiation of this bivariate Mittag-Leffler function are

45:14:7 $$\frac{d}{dx}E_{\mu,1}(x)=\frac{1}{\mu}E_{\mu,\mu}(x)$$

and

45:14:8 $$\frac{d}{dx}E_{\mu,1}(x^{\mu})=\frac{x^{\mu-1}}{\mu}E_{\mu,\mu}(x^{\mu})$$

The fractional calculus [Section 12:14] also benefits from the Mittag-Leffler function. If

45:14:9 $$f(x)=x^{\mu-1}E_{\mu,\mu}(ax^{\mu})=\sum_{j=0}^{\infty}\frac{a^j x^{j\mu+\mu-1}}{\Gamma(j\mu+\mu)}$$

where a is a constant, then with the aid of 12:10:8, one may derive straightforwardly that

$$\frac{d^{\mu}}{dx^{\mu}}f(x)=\sum_{j=0}^{\infty}\frac{a^j x^{(j-1)\mu+\nu-1}}{\Gamma((j-1)\mu+\nu)}=\frac{x^{\nu-\mu-1}}{\Gamma(\nu-\mu)}+\sum_{j=0}^{\infty}\frac{a^{j+1}x^{j\mu+\nu-1}}{\Gamma(j\mu+\nu)}=\frac{x^{\nu-\mu-1}}{\Gamma(\nu-\mu)}+a\,f(x)$$

[see also Hilfer, Chapter II]. It follows that the solution to the important fractional differential equation

45:14:11 $$\frac{d^{\mu}}{dx^{\mu}}f(x)=a\,f(x) \qquad \text{is} \qquad f(x)=ax^{\mu-1}E_{\mu,\mu}(ax^{\mu})$$

or equivalently in the light of 45:14:8

45:14:12 $$f(x)=\mu\frac{d}{dx}E_{\mu,1}(ax^{\mu})$$

CHAPTER 46

THE PARABOLIC CYLINDER FUNCTION $D_\nu(x)$

This well-behaved bivariate function arises in the solution of many practical problems, including those that are conveniently formulated in parabolic cylindrical coordinates. The latter coordinate system, as well as others, is addressed in Section 46:14.

46:1 NOTATION

The *parabolic cylinder function* is also known as the *Weber function* (Heinrich Martin Weber, German mathematician, 1842–1913) or the *Weber-Hermite function*. The name *Whittaker function* is also encountered but confusion should be avoided with the identically named functions discussed in Section 48:14.

A parabolic cylinder is the three-dimensional shell formed by translating a two-dimensional parabola [Section 11:14] perpendicularly to the plane of that parabola. The function $D_\nu(x)$ has adopted the name of this body because it arises in the solution to physical problems dealing with spaces bounded by parabolic cylinders.

Apart from often being italicized, the symbol $D_\nu(x)$ is standard for the function defined in Section 46:3, the variables ν and x being known as the function's order and argument respectively. Expressions involving the parabolic cylinder function are often simpler when $\sqrt{2}\,x$ or $\sqrt{2x}$ rather than x itself, is regarded as the argument and the *Atlas* sometimes makes use of this simplifying property.

Some authors recognize two distinct parabolic cylinder functions. These are termed "the parabolic cylinder functions of the first and second kinds" and are functions of variables a and x that, in our symbolism, would be represented by

46:1:1 $$D_{-a-1/2}(x) \qquad \text{and} \qquad \frac{\Gamma(a+\frac{1}{2})}{\pi}\Big[D_{-a-1/2}(-x)+\sin(\pi a)D_{-a-1/2}(x)\Big]$$

respectively. The *Atlas* makes no reference to these functions, to which the notations $U(a,x)$ and $V(a,x)$ have been applied.

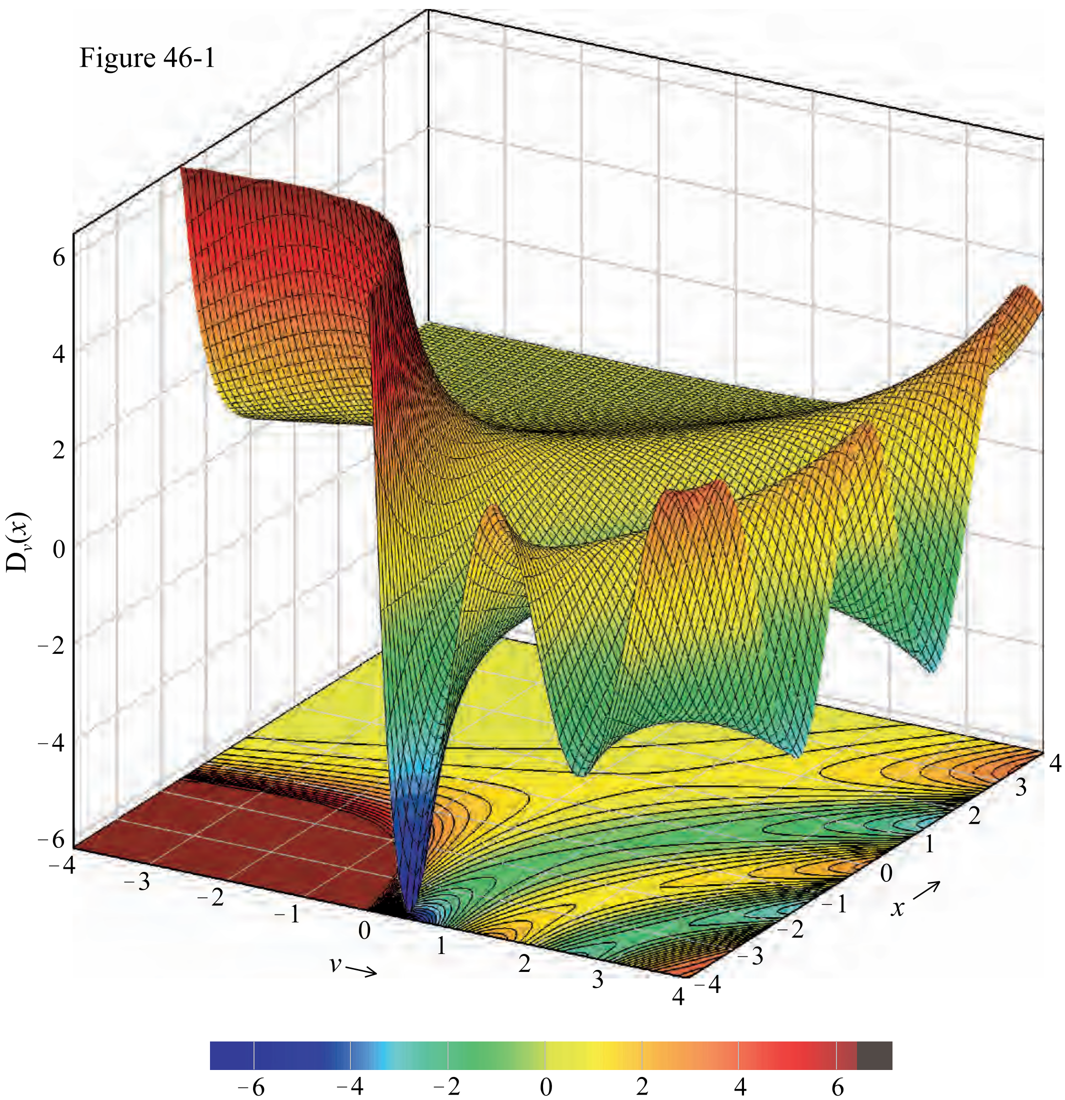
Other writers prefer to use $2^{\nu/2}\exp\left(x^2/2\right)D_\nu\left(\sqrt{2}x\right)$ as the canonical form and refer to this as the *Hermite function*, $H_\nu(x)$ [Section 24:13].

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_47,

46:2 BEHAVIOR

The $D_\nu(x)$ function is defined for all real values of ν and x, but there is a decided difference in properties between positive orders and negative orders. This dichotomy is evident in Figure 46-1. For negative orders less than about $\nu = -0.205$, the parabolic cylinder function is a positive, monotonically decreasing, function of x, lacking zeros or discontinuities. For positive orders, however, $D_\nu(x)$ develops zeros and extrema, the numbers of which are given in the accompanying table.

	Number of zeros	Number of maxima	Number of minima
$\nu \le -0.205$	0	0	0
$-0.205 < \nu \le 0$	0	1	0
$0 < \nu \le 1$	1	1	0
$1 < \nu \le 2$	2	1	1
$2 < \nu \le 3$	3	2	1
⋮	⋮	⋮	⋮
$n-1 < \nu \le n$	n	$n - \mathrm{Int}(\nu/2)$	$\mathrm{Int}(n/2)$



46:3 DEFINITIONS

Three definite integrals that may be used to define the parabolic cylinder function are

46:3:1 $$D_\nu(x) = \sqrt{\frac{2}{\pi}}\exp\left(\frac{x^2}{4}\right)\int_0^\infty t^\nu \exp\left(\frac{-t^2}{2}\right)\cos\left(xt - \frac{\nu\pi}{2}\right)dt \qquad \nu > -1$$

46:3:2 $$D_\nu(x) = \frac{1}{\Gamma(-\nu)}\exp\left(\frac{-x^2}{4}\right)\int_0^\infty t^{-\nu-1}\exp\left(\frac{-t^2}{2} - xt\right)dt \qquad \nu < 0$$

and

46:3:3 $$D_\nu(x) = \frac{1}{\Gamma(-\nu)}\exp\left(\frac{-x^2}{4}\right)\int_0^\infty \frac{\exp(-t)}{\sqrt{x^2+2t}\left[\sqrt{x^2+2t}-x\right]^{\nu+1}}dt \qquad \nu < 0 < x$$

many others being listed by Erdélyi et al. [*Higher Transcendental Functions*, Volume 2, Section 8.3].

A generating function for parabolic cylinder functions of integer order is

46:3:4 $$\exp\left(xt - \frac{x^2}{4} - \frac{t^2}{2}\right) = \sum_{n=0}^\infty \frac{t^n}{n!}D_n(x)$$

which closely resembles the generating function 24:3:1 for Hermite polynomials. Such parabolic cylinder functions are also defined by

46:3:5 $$\exp\left(\frac{x^2}{4}\right)\frac{d^n}{dx^n}\exp\left(\frac{-x^2}{2}\right) = (-)^n D_n(x)$$

The differential equation known as *Weber's equation* and its solution are

46:3:6 $$\frac{d^2 f}{dx^2} + \left(\nu + \frac{1}{2} - \frac{x^2}{4}\right)f = 0 \qquad f(x) = w_1 D_\nu(x) + w_2 D_\nu(-x)$$

unless ν is an integer n, in which case $w_1 D_n(x) + w_2 D_{-n-1}(ix)$ provides a solution. An application of Weber's equation will be found in Section 46:15.

Not being a hypergeometric function (other than asymptotically as in 46:6:5), the parabolic cylinder function is not open to straightforward synthesis [Section 43:14]. $D_\nu(x)$ may, however, be expressed as the difference of two $L = K+1 = 2$ hypergeometric functions,

46:3:7 $$D_\nu(x) = \sqrt{2^\nu \pi}\exp\left(\frac{-x^2}{4}\right)\left[\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)}\sum_{j=0}^\infty \frac{\left(\frac{-\nu}{2}\right)_j}{\left(\frac{1}{2}\right)_j (1)_j}\left(\frac{x^2}{2}\right)^j - \frac{\sqrt{2}x}{\Gamma\left(\frac{-\nu}{2}\right)}\sum_{j=0}^\infty \frac{\left(\frac{1-\nu}{2}\right)_j}{(1)_j\left(\frac{3}{2}\right)_j}\left(\frac{x^2}{2}\right)^j\right]$$

which is synthesizable. This formula applies for arguments of either sign.

46:4 SPECIAL CASES

When the order is the nonnegative integer n, one or other of the bracketed terms in 46:3:7 vanishes and the parabolic cylinder function reduces to the product of an exponential function and a *Hermite polynomial* [Chapter 24]:

46:4:1 $$D_n\left(\sqrt{2}x\right) = 2^{-n/2}\exp\left(-\tfrac{1}{2}x^2\right)H_n(x) \qquad n = 0,1,2,\cdots$$

The first few of these functions are:

$D_0(x)$	$D_1(x)$	$D_2(x)$	$D_3(x)$	$D_4(x)$
$\exp\left(\frac{-1}{4}x^2\right)$	$x\exp\left(\frac{-1}{4}x^2\right)$	$[x^2-1]\exp\left(\frac{-1}{4}x^2\right)$	$[x^3-3x]\exp\left(\frac{-1}{4}x^2\right)$	$[x^4-6x^2+3]\exp\left(\frac{-1}{4}x^2\right)$

A generating function for parabolic cylinder functions of nonnegative integer order is given in equation 46:3:4. Another rule applicable to parabolic cylinder functions only when the order is a nonnegative integer is the argument-reflection formula

46:4:2 $$D_n(-x) = (-)^n D_n(x) \qquad n = 0,1,2,\cdots$$

Parabolic cylinder functions of negative integer order reduce to products of an exponential function and a repeated integral of the error function complement [Section 40:13]

46:4:3 $$D_{-n-1}\left(\sqrt{2}x\right) = \sqrt{2^{n-1}\pi}\,\exp\left(\tfrac{1}{2}x^2\right)\mathrm{i}^n\mathrm{erfc}(x) \qquad n = 0,1,2,\cdots$$

but, as explained in Section 41:13, these products differ substantively from those encountered in Chapter 41.

When the order of $D_\nu(x)$ is an odd multiple, positive or negative, of ½, reduction occurs to a Macdonald function [Chapter 51] of an order that is an odd multiple of ¼. The simplest cases are

46:4:4 $$D_{-1/2}(x) = \sqrt{\frac{x}{2\pi}}\,K_{1/4}\left(\frac{x^2}{4}\right)$$

and

46:4:5 $$D_{1/2}(x) = \sqrt{\frac{x^3}{8\pi}}\left[K_{1/4}\left(\frac{x^2}{4}\right) + K_{3/4}\left(\frac{x^2}{4}\right)\right]$$

Starting from this pair, other instances are accessible through the recursion 46:5:1.

46:5 INTRARELATIONSHIPS

The parabolic cylinder function satisfies the recursion formula

46:5:1 $$D_{\nu+1}(x) = xD_\nu(x) - \nu D_{\nu-1}(x)$$

and an argument-addition formula that can be written in two alternative ways:

46:5:2 $$D_\nu(x+y) = \exp\left(\frac{2xy+y^2}{4}\right)\sum_{j=0}^{\infty}\frac{D_{j+\nu}(x)}{j!}(-y)^j = \exp\left(\frac{-2xy-y^2}{4}\right)\sum_{j=0}^{\infty}\binom{\nu}{j}y^j D_{\nu-j}(x)$$

The sum or difference $D_\nu(-x) \pm D_\nu(x)$ can be expressed in terms of the Kummer function [Chapter 47], each in two alternative ways:

46:5:3 $$D_\nu(-x) + D_\nu(x) = \frac{\sqrt{2^{\nu+2}\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\exp\left(\frac{-x^2}{4}\right)M\left(\tfrac{-\nu}{2},\tfrac{1}{2},\tfrac{1}{2}x^2\right) = \frac{\sqrt{2^{\nu+2}\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\exp\left(\frac{x^2}{4}\right)M\left(\tfrac{1+\nu}{2},\tfrac{1}{2},\tfrac{-1}{2}x^2\right)$$

46:5:4 $$D_\nu(-x) - D_\nu(x) = \frac{\sqrt{2^{\nu+3}\pi}}{\Gamma\left(\frac{-\nu}{2}\right)}x\exp\left(\frac{-x^2}{4}\right)M\left(\tfrac{1-\nu}{2},\tfrac{3}{2},\tfrac{1}{2}x^2\right) = \frac{\sqrt{2^{\nu+3}\pi}}{\Gamma\left(\frac{-\nu}{2}\right)}x\exp\left(\frac{x^2}{4}\right)M\left(\tfrac{2+\nu}{2},\tfrac{3}{2},\tfrac{-1}{2}x^2\right)$$

Subtraction of these two formulas leads to equation 46:3:6 when the Kummer functions are represented hypergeometrically.

An interesting property is that any parabolic cylinder function may be expressed as an infinite sum of its fellow functions of nonnegative integer order. In most cases there are two alternatives, involving sums of either even

46:5:5 $$D_\nu(x)=\frac{2^{(\nu+2)/2}}{\Gamma\left(\frac{-\nu}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{2}\right)^j D_{2j}(x)}{j!(2j-\nu)}\qquad \nu\neq 0,2,4,\cdots$$

or odd

46:5:6 $$D_\nu(x)=\frac{2^{(\nu+1)/2}}{\Gamma\left(\frac{1-\nu}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{2}\right)^j D_{2j+1}(x)}{j!(2j+1-\nu)}\qquad \nu\neq 1,3,5,\cdots$$

orders. Notice that, as ν approaches an even integer $2n$, the $j = n$ term in the 46:5:5 sum becomes dominant. Moreover, in this circumstance, equation 43:9:2 shows that $1/\Gamma(-\nu/2)\to(-)^n n!(n-\frac{1}{2}\nu)$. The net result is that, in the limit, 46:5:5 becomes the $D_{2n}(x)=D_{2n}(x)$ identity. In this sense, the prohibition accompanying equation 46:5:5 does not exist. The same is true for 46:5:6.

46:6 EXPANSIONS

The power series 46:3:7 may be written more concisely as

46:6:1 $$\exp\left(\frac{x^2}{2}\right)D_\nu\left(\sqrt{2}x\right)=\frac{2^{-\nu/2}}{2\Gamma(-\nu)}\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{j-\nu}{2}\right)}{j!}(-2x)^j$$

or in the computationally felicitous form

46:6:2 $$D_\nu\left(\sqrt{2}x\right)=\sqrt{2^\nu\pi}\exp\left(\tfrac{-1}{2}x^2\right)\sum_{j=0}^{\infty}c_jx^j\qquad c_0=\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)},\quad c_1=\frac{-2}{\Gamma\left(\frac{-\nu}{2}\right)},\quad c_j=\frac{2(j-\nu-2)}{j(j-1)}c_{j-2}$$

An expansion involving the cosine function is

46:6:3 $$D_\nu(x)=\sqrt{\frac{2^\nu}{\pi}}\exp\left(\frac{x^2}{4}\right)\sum_{j=0}^{\infty}\cos\left(\tfrac{1}{2}(j+\nu)\pi\right)\Gamma\left(\frac{j+\nu+1}{2}\right)\frac{\left(-\sqrt{2}x\right)^j}{j!}$$

while expansion as Hermite polynomials is possible in two distinct ways:

46:6:4 $$\exp\left(\frac{x^2}{2}\right)D_\nu\left(\sqrt{2}x\right)=\frac{2^{(\nu+2)/2}}{\Gamma\left(\frac{-\nu}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{4}\right)^j H_{2j}(x)}{j!(2j-\nu)}=\frac{2^{\nu/2}}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{4}\right)^j H_{2j+1}(x)}{j!(2j+1-\nu)}\qquad x>0$$

The similarity of these expansions to the equation pair 45:5:5 and 45:5:6 is evident.

Though limited to large positive values of the argument, the asymptotic expansion

46:6:5 $$\exp\left(\frac{x^2}{4}\right)\frac{D_\nu(x)}{x^\nu}\sim 1-\frac{\nu(\nu-1)}{2x^2}+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{8x^4}-\cdots=\sum_{j=0}\frac{(-\nu)_{2j}}{(1)_j}\left(\frac{-1}{2x^2}\right)^j$$

is a valuable property of the parabolic cylinder function.

46:7 PARTICULAR VALUES

The values acquired by the parabolic cylinder function at $x = 0$ and $\pm\infty$ are as follows

46:7:1 $$D_\nu(\infty)=0 \qquad D_\nu(0)=\frac{\sqrt{2^\nu}\pi}{\Gamma(\frac{1-\nu}{2})} \qquad D_\nu(-\infty)=\begin{cases}+\infty & \nu<0,\ 1<\nu<2,\ 3<\nu<4,\cdots \\ 0 & \nu=0,1,2,\cdots \\ -\infty & 0<\nu<1,\ 2<\nu<3,\cdots\end{cases}$$

For certain specific orders, the values at zero argument are

$D_{-3}(0)$	$D_{-2}(0)$	$D_{-3/2}(0)$	$D_{-1}(0)$	$D_{-1/2}(0)$	$D_0(0)$	$D_{1/2}(0)$	$D_1(0)$	$D_{3/2}(0)$	$D_2(0)$	$D_3(0)$
$\sqrt{\frac{\pi}{8}}$	1	$\sqrt{\frac{2}{g\sqrt{\pi}}}$	$\sqrt{\frac{\pi}{2}}$	$\sqrt{g\sqrt{\pi}}$	1	$\frac{1}{\sqrt{2g\sqrt{\pi}}}$	0	$\frac{-\sqrt{g\sqrt{\pi}}}{2}$	-1	0

Here g is Gauss's constant [Section 1.7] with $\sqrt{g\sqrt{\pi}}=1.2162\,80214\,25752$

As 46:7:1 requires, all parabolic cylinder functions of nonnegative integer order have a zero at $x=-\infty$. For this subclass, some other zeros occur at arguments of:

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
0	±1	$0,\ \pm\sqrt{3}$	$\pm\sqrt{3\pm\sqrt{6}}$	$0,\ \pm\sqrt{5\pm\sqrt{10}}$	$\pm\sqrt{5+\sqrt{40}\cos\left\{\frac{m\pi}{3}+\frac{1}{3}\arccos\left(\sqrt{\frac{2}{5}}\right)\right\}}$ $m=0,\pm2$

46:8 NUMERICAL VALUES

Exploiting the power series 46:3:6 as a means of calculating numerical values of the parabolic cylinder function is challenging because additive terms are sometimes of very similar magnitude and opposite sign. Nevertheless, by using special precision-conserving procedures, *Equator*'s parabolic cylinder function routine (keyword **D**) is mostly based on this equation, in its 46:6:2 guise. For certain positive values of x, however, *Equator* substitutes an ε-transformed [Section 10:14] implementation of equation 46:6:5, or a Gauss-Legendre [Section 24:15] numerical integration of equation 46:3:3. When ν is a nonnegative integer, equation 46:4:1 is used. By combining the four procedures, *Equator* is able to provide accurate values of $D_\nu(x)$ for the domains $-35\le\nu\le35$, $-35\le x\le35$.

46:9 LIMITS AND APPROXIMATIONS

When x is small and of either sign, the approach of the parabolic cylinder function to its zero-argument value, given in 46:7:1, follows the approximation

46:9:1 $$D_\nu(x)\approx\sqrt{2^\nu}\pi\left[\frac{4-(2\nu+1)x^2}{4\Gamma\left(\frac{1-\nu}{2}\right)}-\frac{\sqrt{2}x}{\Gamma\left(\frac{-\nu}{2}\right)}\right] \qquad \text{small } x$$

For large argument, an appropriate approximation is

46:9:2 $$D_\nu(x)\approx[x^\nu+\tfrac{1}{2}(\nu-\nu^2)x^{\nu-2}]\exp\left(-\tfrac{1}{4}x^2\right) \qquad \text{large positive } x$$

which arises by truncation of expansion 46:6:5.

The parabolic cylinder function becomes a decaying exponential function of its argument as the order becomes increasingly negative. When the argument is constant and modest in magnitude, this behavior is described by the limit

46:9:3 $$D_\nu(x) \to \sqrt{\frac{(-\nu)^\nu}{2}}\exp\left\{\frac{-\nu}{2} - x\sqrt{-\nu}\right\} \qquad \nu \to -\infty$$

46:10 OPERATIONS OF THE CALCULUS

Single and double differentiations obey the formulas

46:10:1 $$\frac{d}{dx}D_\nu(x) = \frac{x}{2}D_\nu(x) - D_{\nu+1}(x) = \nu D_{\nu-1}(x) - \frac{x}{2}D_\nu(x)$$

and

46:10:2 $$\frac{d^2}{dx^2}D_\nu(x) = \left[\frac{x^2}{4} - \frac{1}{2} - \nu\right]D_\nu(x)$$

The very general definite integral

46:10:3 $$\int_0^\infty t^p \exp(-at^2)D_\nu(t)dt = \sqrt{\frac{2^\nu \pi}{(4a+1)^{p+1}}}\frac{\Gamma(p+1)}{\Gamma\left(\frac{p-\nu}{2}+1\right)}F\left(\frac{p+1}{2}, \frac{-\nu}{2}, \frac{2+p-\nu}{2}, \frac{4a-1}{4a+1}\right)$$

in which F is a Gauss hypergeometric function [Chapter 60], has a plethora of special cases. Some of these, as well as many other definite integrals involving parabolic cylinder functions, are listed by Gradshteyn and Ryzhik [Section 7.7].

Most Laplace transform formulas for parabolic cylinder functions require that the function's argument be proportional to $\sqrt{t}$. Examples include

46:10:4 $$\int_0^\infty \frac{D_{2n}\left(a\sqrt{t}\right)}{\sqrt{t}}\exp(-st)\,dt = \mathfrak{L}\left\{\frac{D_{2n}\left(a\sqrt{t}\right)}{\sqrt{t}}\right\} = \frac{2^{n+1}\Gamma(n+\frac{1}{2})(a^2-4s)^n}{(a^2+4s)^{(2n+1)/2}} \qquad n = 0,1,2,\cdots$$

and

46:10:5 $$\int_0^\infty \frac{\exp(bt)D_\nu\left(a\sqrt{t}\right)}{\sqrt{t^{\nu+1}}}\exp(-st)\,dt = \mathfrak{L}\left\{\frac{\exp(bt)D_\nu\left(a\sqrt{t}\right)}{\sqrt{t^{\nu+1}}}\right\} = \frac{\sqrt{\pi}\left[a+\sqrt{2s+\frac{1}{2}a^2-2b}\right]^\nu}{\sqrt{s+\frac{1}{4}a^2-b}} \qquad \nu < 1$$

See Roberts and Kaufman for others.

46:11 COMPLEX ARGUMENT

The *Atlas* does not address the case of complex arguments. For purely imaginary argument, one has

46:11:1 $$D_\nu(iy) = \frac{\Gamma(1+\nu)}{\sqrt{2\pi}}\left\{\cos\left(\frac{\pi\nu}{2}\right)\left[D_{-\nu-1}(y) + D_{-\nu-1}(-y)\right] - i\sin\left(\frac{\pi\nu}{2}\right)\left[D_{-\nu-1}(y) - D_{-\nu-1}(-y)\right]\right\}$$

Note the change in the sign of the function's order that accompanies this imaginary transformation.

46:12 GENERALIZATIONS

The Kummer function [Chapter 47] and the Tricomi function [Chapter 48] may each be regarded as generalizing the parabolic cylinder function. In either case, there are two ways in which the parabolic cylinder function may be expressed as a special instance of the more general function:

46:12:1
$$D_\nu(x)=\sqrt{2^\nu\pi}\exp\left(\frac{-x^2}{4}\right)\left[\frac{M\left(\frac{-\nu}{2},\frac{1}{2},\frac{x^2}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}-x\frac{\sqrt{2}\,M\left(\frac{1-\nu}{2},\frac{3}{2},\frac{x^2}{2}\right)}{\Gamma\left(\frac{-\nu}{2}\right)}\right]\qquad x>0$$

46:12:2
$$D_\nu(x)=\sqrt{2^\nu\pi}\exp\left(\frac{x^2}{4}\right)\left[\frac{M\left(\frac{1+\nu}{2},\frac{1}{2},\frac{-x^2}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}-x\frac{\sqrt{2}\,M\left(\frac{2+\nu}{2},\frac{3}{2},\frac{-x^2}{2}\right)}{\Gamma\left(\frac{-\nu}{2}\right)}\right]\qquad x>0$$

46:12:3
$$D_\nu(x)=\sqrt{2^\nu}\exp\left(\frac{-x^2}{4}\right)U\left(\frac{-\nu}{2},\frac{1}{2},\frac{x^2}{2}\right)$$

and

46:12:4
$$D_\nu(x)=\sqrt{2^{\nu-1}}\,x\exp\left(\frac{-x^2}{4}\right)U\left(\frac{1-\nu}{2},\frac{3}{2},\frac{x^2}{2}\right)$$

46:13 COGNATE FUNCTIONS

Whittaker's notation [Section 48:13] may be adapted to express $D_\nu(x)$.

The so-called "parabolic cylinder functions of the first and second kinds" are mentioned in Section 46:1.

46:14 RELATED TOPIC: three-dimensional coordinate systems

Section 35:14 addresses two-dimensional *coordinate systems* and much of the current section represents a straightforward extension of the material discussed there.

In three-dimensional coordinate systems, the values of three suitably chosen parameters – the three *coordinates*, for which we use the generic symbols α, β, and γ – are used to locate any point in space uniquely. Specifying a value of one of these coordinates, but allowing the other two to adopt all values within their domains, generates a *coordinate surface*, of finite or infinite extent. Here we consider only *orthogonal coordinate* systems, in which the α-, β-, and γ- coordinate surfaces are mutually perpendicular at all points in space (except, perhaps, at one or two so-called *concentration points*, such as at $\eta=0$, $\psi=\pi$ in the elliptic cylinder case). Apart from the cartesian (x,y,z) system – in which all three coordinate surfaces are planes – the name *curvilinear coordinates* is applied to triplets of orthogonal coordinates. The figures in this section show examples of each of the three coordinate surfaces (the first in red, the second in green, the third in blue), to aid visualization of the shapes involved in the particular curvilinear coordinate system being illustrated.

Most of the important curvilinear coordinate systems arise by adding a third coordinate to one of the two-dimensional geometries addressed in Section 35:14. In the five cases listed in the first table, the three-dimensional system is generated simply by translating the two-dimensional counterpart in a direction perpendicular to the original plane. The distance moved (positive or negative) from that plane becomes the third coordinate, which is usually given the symbol z. In other cases, discussed below, the three-dimensional system is formed from a two-dimensional progenitor by rotation about an axis, the angle of rotation serving as the new coordinate.

Two-dimensional system	Three-dimensional system
rectangular (x,y)	cartesian (x,y,z)
polar (r,θ)	cylindrical (r,θ,z)
parabolic (p,q)	parabolic cylinder (p,q,z)
elliptical (η,ψ)	elliptic cylinder (η,ψ,z)
bipolar (λ,μ)	bipolar cylinder (λ,μ,z)

Of course, all three-dimensional coordinate systems require three *scale factors* [Section 35:14]. These, and other important properties, are detailed in the panels that accompany the present section. Included in these panels are equations expressing the relationship of the curvilinear coordinates to cartesian (x, y, z) coordinates. As well, the expression for the *Laplacian* [Section 46:15] is given.

Cylindrical coordinate system (r,θ,z)		
$0 \le r < \infty$	$h_r = 1$	$x = r\cos(\theta)$
$0 \le \theta < 2\pi$	$h_\theta = r$	$y = r\sin(\theta)$
$-\infty < z < \infty$	$h_z = 1$	$z = z$
$\nabla^2 F = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^2 F}{\partial z^2}$		

The case of parabolic cylinder coordinates is illustrated in Figure 46-2 which shows examples of constant-p, constant-q and constant-z coordinate surfaces. The first two are parabolic cylinders; the third is a plane.

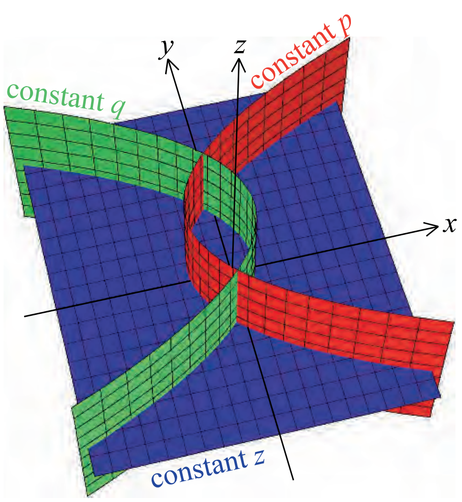


Figure 46-2

Parabolic cylinder coordinate system (p,q,z)		
$-\infty < p < \infty$	$\left.\begin{matrix} h_p \\ h_q \end{matrix}\right\} = \sqrt{p^2+q^2}$	$x = \frac{1}{2}\left[p^2 - q^2\right]$
$0 \le q < \infty$		$y = pq$
$-\infty < z < \infty$	$h_z = 1$	$z = z$
$\nabla^2 F = \frac{1}{h_p^2}\left[\frac{\partial^2 F}{\partial p^2} + \frac{\partial^2 F}{\partial q^2}\right] + \frac{\partial^2 F}{\partial z^2}$		

The choice of symbols for the cartesian, cylindrical and spherical coordinates has become standardized but, beyond this, there is no unanimity in the notation of other curvilinear systems. Moreover, recognize that a cartesian mesh may be overlaid onto another curvilinear system in several ways, so that the equations relating (α,β,γ) to (x,y,z) are not unique. Adding to the confusion is the fact that coordinates of different forms are in use; for example, elliptic cylinder coordinates equivalent to our $\sinh(\eta)$ and $\sin(\psi)$ may be encountered.

The *prolate spheroidal coordinate system* is obtained by rotating the two-dimensional elliptical system [Figure 35-12] about its major axis, whereas the *oblate spheroidal coordinate system* is generated by rotation about the minor axis. Figures 46-3 and 46-4 show typical coordinate surfaces for the prolate and oblate cases respectively.

Elliptic cylinder coordinate system (η,ψ,z)		
$-\infty<\eta<\infty$	h_η	$x = a\cosh(\eta)\cos(\psi)$
$0\le\psi<2\pi$	h_ψ $\Big\} = a\sqrt{\sinh^2(\eta)+\sin^2(\psi)}$	$y = a\sinh(\eta)\sin(\psi)$
$-\infty<z<\infty$	$h_z = 1$	$z = z$
$\nabla^2 F = \dfrac{1}{h_\eta^2}\left[\dfrac{\partial^2 F}{\partial\eta^2}+\dfrac{\partial^2 F}{\partial\psi^2}\right]+\dfrac{\partial^2 F}{\partial z^2}$		

Bipolar cylinder coordinate system (λ,ξ,z)		
$-\infty<\lambda<\infty$	h_λ	$x = h_\lambda\sinh(\lambda)$
$0\le\xi<2\pi$	h_ξ $\Big\} = \dfrac{a}{\cosh(\lambda)-\cos(\xi)}$	$y = h_\xi\sin(\xi)$
$-\infty<z<\infty$	$h_z = 1$	$z = z$
$\nabla^2 F = \dfrac{1}{h_\lambda^2}\left[\dfrac{\partial^2 F}{\partial\lambda^2}+\dfrac{\partial^2 F}{\partial\xi^2}\right]+\dfrac{\partial^2 F}{\partial z^2}$		

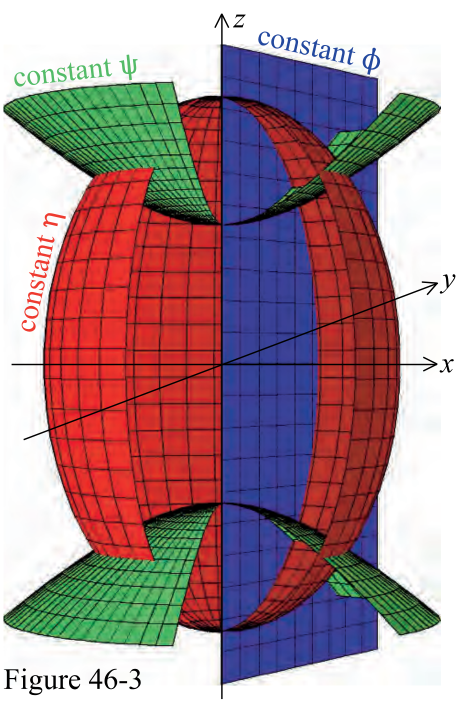


Figure 46-3

Prolate spheroidal coordinate system (η,ψ,ϕ)		
$0\le\eta<\infty$	h_η	$x = h_\phi\cos(\phi)$
$0\le\psi<\pi$	h_ψ $\Big\} = a\sqrt{\sinh^2(\eta)+\sin^2(\psi)}$	$y = h_\phi\sin(\phi)$
$0\le\phi<2\pi$	$h_\phi = a\sinh(\eta)\sin(\psi)$	$z = a\cosh(\eta)\cos(\psi)$
$\nabla^2 F = \dfrac{1}{h_\eta^2}\left[\dfrac{\partial^2 F}{\partial\eta^2}+\coth(\eta)\dfrac{\partial F}{\partial\eta}+\dfrac{\partial^2 F}{\partial\psi^2}+\cot(\psi)\dfrac{\partial F}{\partial\psi}\right]+\dfrac{1}{h_\phi^2}\dfrac{\partial^2 F}{\partial\phi^2}$		

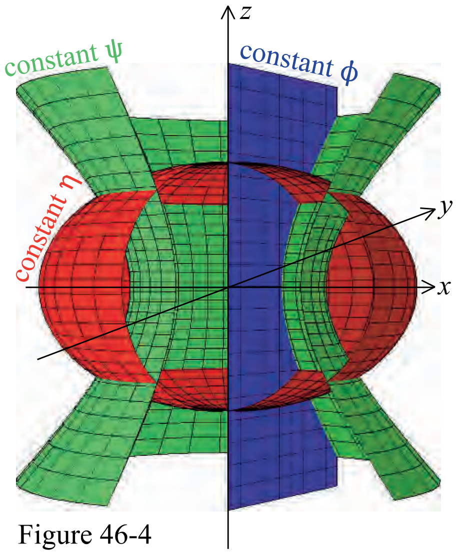


Figure 46-4

Oblate spheroidal coordinate system (η,ψ,ϕ)		
$0\le\eta<\infty$	$\left.\begin{matrix}h_\eta\\h_\psi\end{matrix}\right\}=a\sqrt{\sinh^2(\eta)+\sin^2(\psi)}$	$x=h_\phi\cos(\phi)$
$\frac{-1}{2}\pi<\psi<\frac{1}{2}\pi$		$y=h_\phi\sin(\phi)$
$0\le\phi<2\pi$	$h_\phi=a\cosh(\eta)\cos(\psi)$	$z=a\sinh(\eta)\sin(\psi)$
$\nabla^2\mathrm{F}=\frac{1}{h_\eta^2}\left[\frac{\partial^2\mathrm{F}}{\partial\eta^2}+\tanh(\eta)\frac{\partial\mathrm{F}}{\partial\eta}+\frac{\partial^2\mathrm{F}}{\partial\psi^2}+\tan(\psi)\frac{\partial\mathrm{F}}{\partial\psi}\right]+\frac{1}{h_\phi^2}\frac{\partial^2\mathrm{F}}{\partial\phi^2}$		

Two rotations are required in the creation of the *spherical coordinate system*. The two angles thereby introduced, ϕ and θ, are familiar in planetary cartography as *longitude* and *latitude*. The r, ϕ, and θ coordinate surfaces, exemplified in Figure 46-5, are respectively *spheres*, *half-planes* and *cones*. Note the inconsistent, but standard (though not universal) use of ϕ in spherical coordinate notation to denote the same angle that is conventionally denoted θ in cylindrical coordinates.

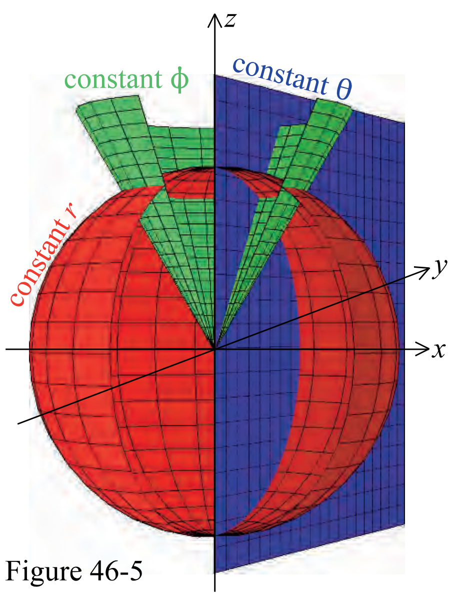


Figure 46-5

Spherical coordinate system (r,ϕ,θ)		
$0\le r<\infty$	$h_r=1$	$x=h_\theta\cos(\theta)$
$0\le\phi<\pi$	$h_\phi=r$	$y=h_\theta\sin(\theta)$
$\frac{-1}{2}\pi\le\theta\le\frac{1}{2}\pi$	$h_\theta=r\sin(\phi)$	$z=r\cos(\phi)$
$\nabla^2\mathrm{F}=\frac{\partial^2\mathrm{F}}{\partial r^2}+\frac{2}{r}\frac{\partial\mathrm{F}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\mathrm{F}}{\partial^2\phi}+\frac{\cot(\phi)}{r^2}\frac{\partial\mathrm{F}}{\partial\phi}+\frac{1}{h_\theta^2}\frac{\partial^2\mathrm{F}}{\partial\theta^2}$		

The three-dimensional *paraboloidal coordinate system* shown in Figure 46-6 arises by rotating the two-dimensional parabolic system about its major axis. The p and q coordinate surfaces are *paraboloids* of revolution; the ϕ surfaces are *half-planes*.

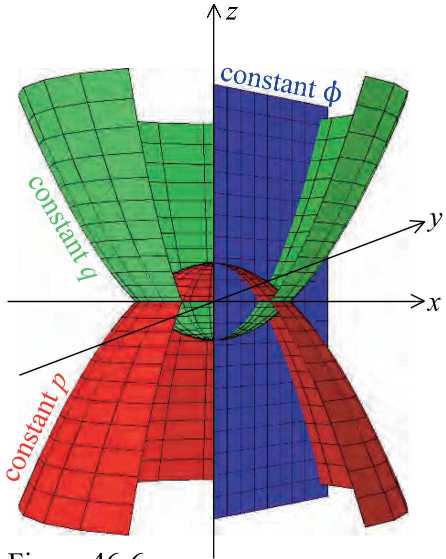


Figure 46-6

Paraboloidal coordinate system (p,q,ϕ)		
$0 \le p < \infty$	$\left.\begin{matrix} h_p \\ h_q \end{matrix}\right\} = \sqrt{p^2+q^2}$	$x = pq\cos(\phi)$
$0 \le q < \infty$		$y = pq\sin(\phi)$
$0 \le \phi < 2\pi$	$h_\phi = pq$	$z = \frac{1}{2}\left[p^2 - q^2\right]$
$\nabla^2 \mathrm{F} = \dfrac{1}{h_p^2}\left[\dfrac{\partial^2 \mathrm{F}}{\partial p^2} + \dfrac{1}{p}\dfrac{\partial \mathrm{F}}{\partial p} + \dfrac{\partial^2 \mathrm{F}}{\partial q^2} + \dfrac{1}{q}\dfrac{\partial \mathrm{F}}{\partial q}\right] + \dfrac{1}{h_\phi^2}\dfrac{\partial^2 \mathrm{F}}{\partial \phi^2}$		

The three-dimensional *toroidal coordinate system* arises by rotating the two-dimensional bipolar system about its minor axis. Figure 46-7 illustrates the shapes of the coordinate surfaces, which are doughnut-shaped *tori*, *spherical bowls*, and *half-planes*.

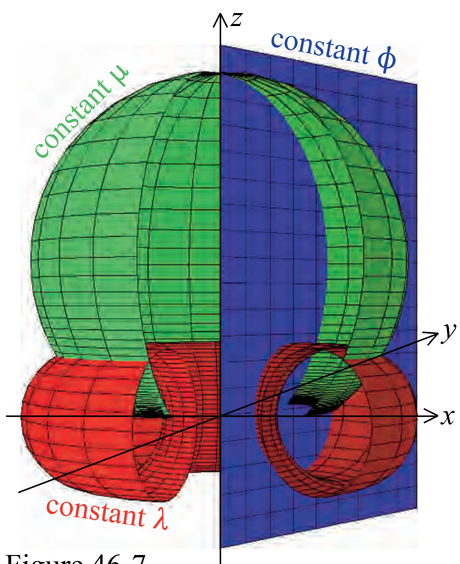


Figure 46-7

Toroidal coordinate system (λ,μ,ϕ)		
$0 \le \lambda < \infty$	$\left.\begin{matrix} h_\lambda \\ h_\mu \end{matrix}\right\} = \dfrac{a}{\cosh(\lambda) - \cos(\mu)}$	$x = h_\phi \cos(\phi)$
$0 \le \mu < 2\pi$		$y = h_\phi \sin(\phi)$
$0 \le \phi < 2\pi$	$h_\phi = h_\lambda \sinh(\lambda)$	$z = h_\lambda \sin(\mu)$
$\nabla^2 \mathrm{F} = \dfrac{1}{h_\lambda^3}\dfrac{\partial}{\partial \lambda}\left\{h_\lambda \dfrac{\partial \mathrm{F}}{\partial \lambda}\right\} + \dfrac{1}{h_\phi^3}\dfrac{\partial}{\partial \mu}\left\{h_\phi \dfrac{\partial \mathrm{F}}{\partial \mu}\right\} + \dfrac{1}{h_\phi^2}\dfrac{\partial^2 \mathrm{F}}{\partial \phi^2}$		

We have addressed nine curvilinear coordinate systems, but there are others. See Spiegel [pages 129,130] for the *conical coordinate system*, the *confocal paraboloidal coordinate* system and the *confocal ellipsoidal coordinate system*. The last-mentioned system is regarded by Morse and Feshbach [Chapter 5] as the most general orthogonal coordinate system, of which all others are degenerate instances.

46:15 RELATED TOPIC: the Laplacian

Among the most important equations in physics are *Laplace's equation*

46:15:1 $$\nabla^2 F = 0$$

describing static electric fields, the *diffusion equation* (*Fick's second law*)

46:15:2 $$\nabla^2 F = k\frac{\partial F}{\partial t}$$

(which, as *Fourier's equation*, also governs the conduction of heat), the *wave equation*

46:15:3 $$\nabla^2 F = k\frac{\partial^2 F}{\partial t^2}$$

for the propagation of vibrations, *Schrödinger's equation* of quantum mechanics

46:15:4 $$\nabla^2 F = k + V F$$

(this is the time-independent version), *Poisson's equation* of electrostatics

46:15:5 $$\nabla^2 F = kV$$

and *Helmholtz's equation* which describes the spatial aspects of harmonic motion

46:15:6 $$\nabla^2 F = k F$$

In these equations k represents a physical constant, V is a position-dependent quantity, and t is time. All six equations describe the spatial (and temporal in two cases) distribution of F, which is some *scalar property* (an "intensity" of something), such as pressure, electric potential, density, concentration, temperature, or probability.

The operator ∇^2, found in all these equations, is called the *Laplacian*; it performs double differentiation with respect to the spatial coordinates. In cartesian coordinates, the formulation of the Laplacian is simply

46:15:7 $$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

but in curvilinear coordinates [Section 46:14], the more complicated result is

46:15:8 $$\nabla^2 F = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial}{\partial\alpha}\left\{\frac{h_\beta h_\gamma}{h_\alpha}\frac{\partial F}{\partial\alpha}\right\} + \frac{\partial}{\partial\beta}\left\{\frac{h_\gamma h_\alpha}{h_\beta}\frac{\partial F}{\partial\beta}\right\} + \frac{\partial}{\partial\gamma}\left\{\frac{h_\alpha h_\beta}{h_\gamma}\frac{\partial F}{\partial\gamma}\right\}\right]$$

where α, β, and γ are the three coordinates and the h's are the corresponding scale factors [Section 35:14]. The forms adopted by the Laplacian for most of the three-dimensional orthogonal coordinate systems are listed in the panels of the preceding section, where the scale factors are also presented. The remainder of this section is an outline of how one might go about solving Helmholtz's equation in parabolic cylinder coordinates. This involves the important concept of *separability*, whereby the Laplacian is dismembered into three parts.

Helmholtz's equation in parabolic cylinder coordinates is

46:15:9 $$\nabla^2 F = \frac{1}{p^2+q^2}\left[\frac{\partial^2 F}{\partial p^2} + \frac{\partial^2 F}{\partial q^2}\right] + \frac{\partial^2 F}{\partial z^2} = k F$$

To proceed, the *separability assumption* is made. That is, it is assumed that F, a function of p, q, and z, can be expressed as a threefold product of functions, each of which depends on one coordinate only; that is

46:15:10 $$F(p,q,z) = P(p)Q(q)Z(z)$$

Such an assumption turns out to be widely, although not universally, valid. In the present example, substitution of 46:15:10 into 46:15:9 leads to

46:15:11
$$\frac{1}{p^2+q^2}\left[QZ\frac{d^2P}{dp^2}+PZ\frac{d^2Q}{dq^2}\right]+PQ\frac{d^2Z}{dz^2}=kPQZ$$

which may be rearranged into

46:15:12
$$\frac{1}{p^2+q^2}\left[\frac{1}{P}\frac{d^2P}{dp^2}+\frac{1}{Q}\frac{d^2Q}{dq^2}\right]=k-\frac{1}{Z}\frac{d^2Z}{dz^2}$$

Now, the left-hand side of equation 46:15:12 is independent of z, whereas the right-hand side depends only on z. The only way in which this is possible is for each side to equal the same constant, that we take to be positive and represent by $\frac{1}{4}\kappa^4$. Thus 46:15:12 devolves into two equations, namely

46:15:13
$$\frac{1}{P}\frac{d^2P}{dp^2}+\frac{1}{Q}\frac{d^2Q}{dq^2}=\frac{\kappa^4}{4}\left[p^2+q^2\right] \qquad \text{and} \qquad \frac{d^2Z}{dz^2}=\left(k-\frac{\kappa^4}{4}\right)Z$$

The second equation in 46:15:13 solves in terms of exponential functions [see 26:3:6] to

46:15:14
$$Z=w_1\exp\{Kz\}+w_2\exp\{-Kz\} \qquad \text{where} \qquad K=\sqrt{k-\tfrac{1}{4}\kappa^4}$$

if $4k>\kappa^4$, or according to 32:3:6 otherwise.

The first equation in 46:15:13 may be rearranged to

46:15:15
$$\frac{1}{P}\frac{d^2P}{dp^2}-\frac{\kappa^4p^2}{4}=\frac{\kappa^4q^2}{4}-\frac{1}{Q}\frac{d^2Q}{dq^2}$$

Once again it can be argued that each side of this equation must equal the same constant, because the left-hand is independent of q, whereas the right-hand is independent of p. Such constants are termed *separation constants*. In this case it is convenient to choose $-\kappa^2\,[\nu+\tfrac{1}{2}]$ to represent the separation constant, ν being yet another constant. Thereby one finds

46:15:16
$$\frac{1}{\kappa^2}\frac{d^2P}{dp^2}=\left(\frac{\kappa^2p^2}{4}-\nu-\frac{1}{2}\right)P \qquad \text{and} \qquad \frac{1}{\kappa^2}\frac{d^2Q}{dq^2}=\left(\frac{\kappa^2q^2}{4}+\nu+\frac{1}{2}\right)Q$$

These are instances of *Weber's equation* 46:3:4, and the solutions are in terms of parabolic cylinder functions:

46:15:17
$$P=w_3D_\nu(\kappa p)+w_4D_\nu(-\kappa p) \qquad \text{and} \qquad Q=w_5D_{-\nu-1}(\kappa q)+w_6D_{-\nu-1}(-\kappa q)$$

Having been broken asunder, the solution may now be reassembled into

46:15:18
$$F=\left[w_1\exp\{Kz\}+w_2\exp\{-Kz\}\right]\left[w_3D_\nu(\kappa p)+w_4D_\nu(-\kappa p)\right]\left[w_5D_{-\nu-1}(\kappa q)+w_6D_{-\nu-1}(-\kappa q)\right]$$

This result contains several arbitrary constants: the three separation constants and the weights $w_{1\text{ to }6}$. In principle at least, these constants (many are often zero and others coalesce) may be selected to match the boundary conditions of a wide range of problems in physics. Remember, however, that *any* values of the separation constants will satisfy the mathematics. It is often necessary to employ a weighted sum of solutions with different separation constants (or an integral in which the separation constant becomes the integration variable) in order to match certain boundary conditions and physical realities. A large number of the functions in the *Atlas* arose historically to satisfy fragments of equations 46:15:1 – 6 separated by procedures analogous to those just illustrated.

CHAPTER

47

THE KUMMER FUNCTION M(a, c, x)

Named for the short-lived German mathematician, Ernst Eduard Kummer (1860 − 1893), this trivariate function is one of the most important hypergeometric functions.

47:1 NOTATION

The symbol M(a,c,x) is often replaced by $\Phi(a,c,x)$ or ${}_1\mathrm{F}_1(a,c,x)$. The subscript 1's in the latter notation reflect the presence of one adjustable parameter in each of the numerator and denominator in definition 47:3:1. The location of the a and c in this definition is responsible for their names: “numeratorial parameter” and “denominatorial parameter”. As usual, x is the argument.

Collectively, the *Kummer function* and the *Tricomi function* [Chapter 48] are known as the *confluent hypergeometric functions* or *degenerate hypergeometric functions*. These puzzling names originate in the limiting operations described by definitions 47:3:4 and 48:3:5.

47:2 BEHAVIOR

Though the parameters and the argument may be complex, our interest is restricted here to real values of a, c, and x. Unless a is also a nonpositive integer of smaller magnitude [see Section 47:14], the Kummer function is undefined when $c = 0, -1, -2, \cdots$. Away from these forbidden values of the denominatorial parameter, M(a,c,x) suffers no discontinuities and is real.

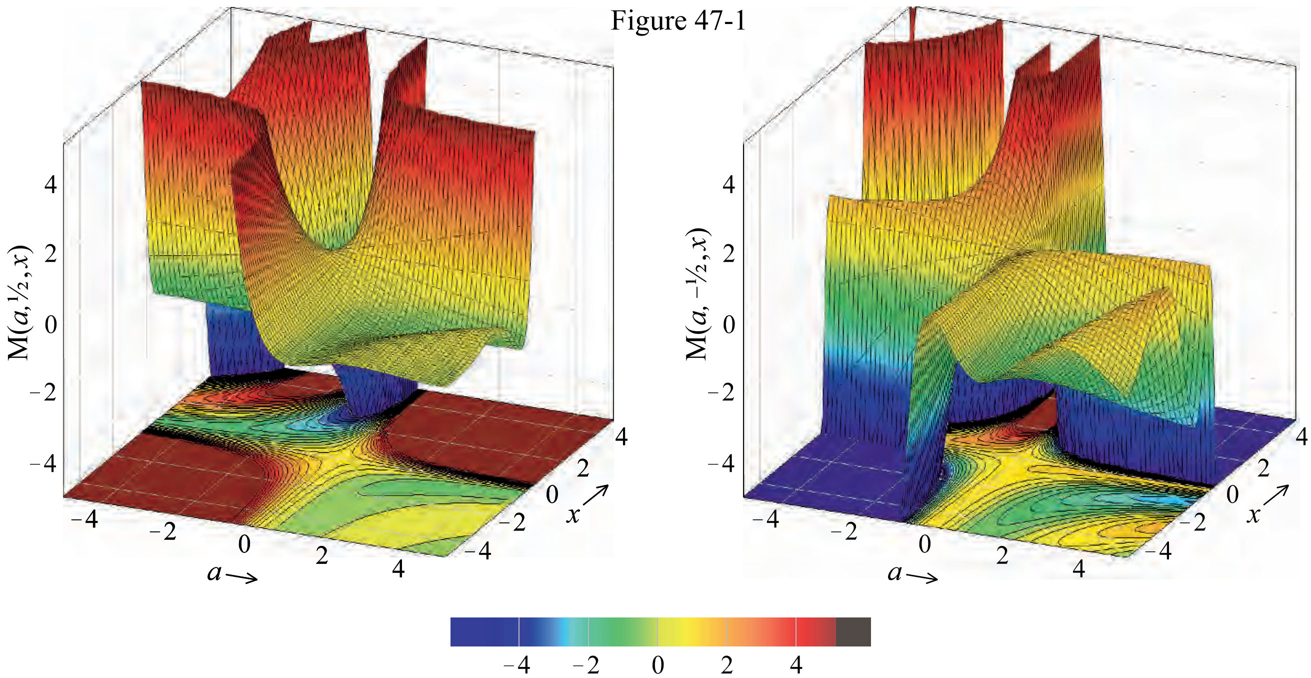
Because of its trivariate nature, it is unrewarding to attempt to depict the global behavior of the Kummer function graphically. Thus Figure 47-1 is designed to show only examples of the variety of the complicated behaviors that M(a,c,x) exhibits even when the denominatorial parameter is kept constant. The value $c = ½$ is used in the left-hand diagram, with $c = -½$ in the right.

For any pair of parameter values, a and c, a qualitative picture of how M(a,c,x) depends on x can be gleaned from knowledge of the properties of the Kummer function. The crucial information needed to construct such a picture may be acquired through the following considerations:

(a) M(a,c,x) is a single-valued function over the $-\infty < x < \infty$ domain and lacks discontinuities.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_48, © Springer Science+Business Media, LLC 2009

Figure 47-1



(b) The Kummer function invariably equals unity at $x = 0$ and its slope there is a/c.

(c) The graph of M versus x crosses the positive x-axis a number of times given in the left-hand diagram of Figure 47-2; the second diagram in the same figure gives the number of zero crossings in $-\infty < x < 0$.

(d) The total number of extrema exhibited by the Kummer function is revealed in Figure 47-3.

(e) As $x \to -\infty$, M usually approaches either $+\infty$ or $-\infty$, the sign of the limit being evident from the first diagram in Figure 47-4; the second diagram reveals which limiting value is acquired as $x \to +\infty$.

Of course, with the chosen a and c entered, *Equator* will provide quantitative information on ranging x through values between -500 and $+500$.

47:3 DEFINITIONS

Kummer's series is the prime definition of the eponymous function

47:3:1 $$\mathrm{M}(a,c,x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j (1)_j} x^j$$

Being hypergeometric, this series demonstrates the possibility of synthesis [Section 43:14] from the exponential function

47:3:2 $$\exp(x) \xrightarrow[c]{a} \mathrm{M}(a,c,x)$$

The Kummer function is equivalent to a *generalized Laguerre function* [Section 23:12]

47:3:3 $$\mathrm{M}(a,c,x) = \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)} \mathrm{L}_{-a}^{(c-1)}(x)$$

The Kummer function may also be defined by a limiting operation applied to the *Gauss hypergeometric function* [Chapter 60] in which the b parameter progresses towards infinity, its product with the argument remaining constant; that is

47:3:4 $$\mathrm{M}(a,c,x)=\lim_{b\to\infty}\mathrm{F}\left(a,b,c,\frac{x}{b}\right)$$

Provided that both parameters are positive, with c exceeding a, the Kummer function may be represented as the indefinite integral

47:3:5 $$\mathrm{M}(a,c,x)=\frac{\Gamma(c)x^{1-c}}{\Gamma(c-a)\Gamma(a)}\int_0^x\frac{t^{a-1}\exp(t)}{(x-t)^{1+a-c}}\mathrm{d}t \qquad 0<a<c$$

or as either of the following definite integrals:

47:3:6 $$\mathrm{M}(a,c,x)=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1\frac{t^{a-1}}{(1-t)^{1+a-c}}\exp(xt)\,\mathrm{d}t \qquad 0<a<c$$

47:3:7 $$\mathrm{M}(a,c,x)=\frac{2^{1-c}\Gamma(c)\exp\left(\frac{1}{2}x\right)}{\Gamma(c-a)\Gamma(a)}\int_{-1}^1\frac{(1+t)^{a-1}}{(1-t)^{1+a-c}}\exp\left(\frac{xt}{2}\right)\mathrm{d}t \qquad 0<a<c$$

The *confluent hypergeometric differential equation* and its solution are

47:3:8 $$x\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}-a\mathrm{f}=0 \qquad w_1\mathrm{M}(a,c,x)+w_2x^{1-c}\mathrm{M}(1+a-c,2-c,x)$$

provided that c is not an integer. The weights w_1 and w_2 are arbitrary. The restriction on c does not apply if the alternative solution $w_1\mathrm{M}(a,c,x)+w_2\mathrm{U}(a,c,x)$ is adopted. Here U is the *Tricomi function* [Chapter 48].

47:4 SPECIAL CASES

The entries in Table 18-4 are all instances of the Kummer function.

When the two parameters are equal to each other, the Kummer function reduces to an exponential function:

47:4:1 $$\mathrm{M}(a,a,x)=\exp(x)$$

Whenever the denominatorial parameter exceeds the numeratorial by unity, reduction occurs to an entire incomplete gamma function [Chapter 45]

47:4:2 $$\mathrm{M}(a,a+1,x)=\Gamma(1+a)\gamma\mathrm{n}(a,-x)$$

Setting $c=a+1$ in recursion 47:5:3 permits a weighted sum of the two equations above to serve as an expression for $\mathrm{M}(a,a+2,x)$; the procedure may be extended indefinitely to express $\mathrm{M}(a,a+n,x)$ for $n=3,4,5,\cdots$.

A Kummer function in which the denominatorial parameter is twice the numeratorial parameter is equivalent to an expression involving a modified Bessel function [Chapter 50]:

47:4:3 $$\mathrm{M}(a,2a,x)=\Gamma\left(a+\tfrac{1}{2}\right)\left[\tfrac{1}{4}x\right]^{(1-2a)/2}\exp\left(\tfrac{1}{2}x\right)\mathrm{I}_{(2a-1)/2}\left(\tfrac{1}{2}x\right)$$

The instance $\mathrm{M}(1,2,x)=[\exp(x)-1]/x$ is noteworthy.

When a is a negative integer, the Kummer function becomes a polynomial – in the most general case, a generalized Laguerre polynomial [Section 23:12] – divided by a binomial coefficient [Chapter 6]

47:4:4 $$\mathrm{M}(-n,c,x)=\frac{\mathrm{L}_n^{(c-1)}(x)}{\binom{n+c-1}{n}}$$

If its numeratorial parameter is zero, the Kummer function equals unity

47:4:5 $$\mathrm{M}(0,c,x)=1 \qquad c\neq 0,-1,-2,\cdots$$

and if a equals unity, then reduction occurs to an incomplete gamma function

47:4:6 $$\mathrm{M}(1,c,x)=1+x^{1-c}\exp(x)\gamma(c,x)$$

Use of the recursive weights listed in the first row of the table on the facing page leads to

47:4:7 $$\mathrm{M}(2,c,x)=\left[2-c+x\right]\left[1+x^{1-c}\exp(x)\gamma(c,x)\right]+c-1$$

and further applications of this recursion permit expressions for M(n,c,x) to be constructed, where n is any positive integer.

Except when a is a negative integer of equal or smaller magnitude [for which cases see Section 47:14], the Kummer function is undefined for $c = 0,-1,-2,\cdots$ [but see Section 47:12]. When $c = 1$, reduction occurs to a Laguerre function [Chapter 23]

47:4:8 $$\mathrm{M}(a,1,x)=\mathrm{L}_{-a}(x)=1+ax+\frac{a(a+1)}{(2!)^2}x^2+\frac{a(a+1)(a+2)}{(3!)^2}x^3+\cdots$$

and use of recursion 47:5:4 shows that M$(a,2,x) = [\mathrm{L}_{-a}(x)-\mathrm{L}_{1-a}(x)]/x$. Multiple applications of this recursion formula lead to expressions for M(a,m,x) for any positive integer m.

When $c=\frac{1}{2}$ or $\frac{3}{2}$ the Kummer function is expressible as the sum or difference of two parabolic cylinder function [Chapter 46]:

47:4:9 $$\mathrm{M}\left(a,\tfrac{1}{2},x\right)=\begin{cases}\dfrac{\Gamma(a+\frac{1}{2})}{2^{1-a}\sqrt{\pi}}\exp\left(\tfrac{1}{2}x\right)\left[\mathrm{D}_{-2a}\left(\sqrt{2x}\right)+\mathrm{D}_{-2a}\left(-\sqrt{2x}\right)\right] & x\ge 0\\[2ex] \dfrac{\Gamma(1-a)}{2^{a}\sqrt{2\pi}}\exp\left(\tfrac{1}{2}x\right)\left[\mathrm{D}_{2a-1}\left(\sqrt{-2x}\right)+\mathrm{D}_{2a-1}\left(-\sqrt{-2x}\right)\right] & x\le 0\end{cases}$$

47:4:10 $$\mathrm{M}\left(a,\tfrac{3}{2},x\right)=\begin{cases}\dfrac{\Gamma(a-\frac{1}{2})}{2^{2-a}\sqrt{2\pi x}}\exp\left(\tfrac{1}{2}x\right)\left[\mathrm{D}_{1-2a}\left(-\sqrt{2x}\right)-\mathrm{D}_{1-2a}\left(\sqrt{2x}\right)\right] & x>0\\[2ex] \dfrac{\Gamma(1-a)}{2^{a+1}\sqrt{-\pi x}}\exp\left(\tfrac{1}{2}x\right)\left[\mathrm{D}_{2a-2}\left(-\sqrt{-2x}\right)-\mathrm{D}_{2a-2}\left(\sqrt{-2x}\right)\right] & x<0\end{cases}$$

These same values of the denominatorial parameter lead to Hermite polynomials [Chapter 24]:

47:4:11 $$\mathrm{M}\left(-n,\tfrac{1}{2},x\right)=\frac{(-)^n n!}{(2n)!}\mathrm{H}_{2n}\left(\sqrt{x}\right)$$

47:4:12 $$\mathrm{M}\left(-n,\tfrac{3}{2},x\right)=\frac{(-)^n n!}{2(2n+1)!\sqrt{x}}\mathrm{H}_{2n+1}\left(\sqrt{x}\right) \qquad \Bigg\}\; n=0,1,2,\cdots$$

if the numeratorial parameter is a nonpositive integer. Recursion 47:5:3 may be used to increment or decrement the denominatorial parameter in any of formulas 47:4:8−12.

When *both* parameters are integers, rather dramatic changes occur to the Kummer function. These are addressed in Section 47:14.

47:5 INTRARELATIONSHIPS

Known as *Kummer's transformation*, the important identity

47:5:1 $$\mathrm{M}(a,c,-x)=\exp(-x)\mathrm{M}(c-a,c,x)$$

constitutes an argument-reflection formula for the Kummer function. One of a number of argument-addition formulas is

47:5:2
$$\mathrm{M}(a,c,x+y)=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j\,j!}y^j\,\mathrm{M}(a+j,c+j,x)$$

By setting $y=(v-1)x$, equation 47:5:2 becomes an argument-multiplication formula: others will be found in Erdélyi et al.[*Higher Transcendental Functions I*, Section 6.14].

There is no recursion formula applicable to the argument of a Kummer function, but six that apply to the parameters. An example is

47:5:3
$$\mathrm{M}(a,c,x)=\frac{(c-a)x}{c(c-1+x)}\mathrm{M}(a,c+1,x)+\frac{c-1}{c(c-1+x)}\mathrm{M}(a,c-1,x)$$

Each of the six enables $\mathrm{M}(a,c,x)$ to be expressed as a weighted sum of two members of its so-called *contiguous function* family, the members of which are $\mathrm{M}(a+1,c,x)$, $\mathrm{M}(a-1,c,x)$, $\mathrm{M}(a,c+1,x)$, and $\mathrm{M}(a,c-1,x)$. The corresponding weights are tabulated below.

$\mathrm{M}(a+1,c,x)$	$\mathrm{M}(a-1,c,x)$	$\mathrm{M}(a,c+1,x)$	$\mathrm{M}(a,c-1,x)$
$\frac{a}{2a-c+x}$	$\frac{a-c}{2a-c+x}$		
$\frac{a}{a+x}$		$\frac{(c-a)x}{c(a+x)}$	
$\frac{a}{a-c+1}$			$\frac{1-c}{a-c+1}$
	1	$\frac{x}{c}$	
	$\frac{a-c}{a-1+x}$		$\frac{c-1}{a-1+x}$
		$\frac{(c-a)x}{c(c-1+x)}$	$\frac{c-1}{c(c-1+x)}$

The final row corresponds to the example cited in 47:5:3, illustrating how the table is to be used. By employing such recursions more than once, very many relationships, such as

47:5:4
$$\mathrm{M}(a,c,x)=\frac{c-1}{x}\left[\mathrm{M}(a,c-1,x)-\mathrm{M}(a-1,c-1,x)\right]$$

linking $\mathrm{M}(a,c,x)$ to noncontiguous Kummer functions, may be constructed. One relationship derived by iterating the fourth entry J times is

47:5:5
$$\mathrm{M}(a,c,x)=\mathrm{M}(a-J,c,x)+\frac{x}{c}\sum_{j=0}^{J-1}\mathrm{M}(a-j,c,x)$$

Among summable infinite series of Kummer function is

47:5:6
$$\mathrm{M}(a,c,x)+\mathrm{M}(a-1,c,x)+\mathrm{M}(a-2,c,x)+\cdots=\sum_{j=0}^{\infty}\mathrm{M}(a-j,c,x)\quad=\frac{c-1}{x}\mathrm{M}(a,c-1,x)$$

though this formula is applicable only if $c>5/2$.

Synthetic operations [Section 43:14] can convert one Kummer function into another:

47:5:7 $$\mathrm{M}(a,c,x) \xrightarrow[c']{c} \mathrm{M}(a,c',x) \xrightarrow[a]{a'} \mathrm{M}(a',c',x)$$

47:6 EXPANSIONS

Kummer's series, equation 47:3:1, is convergent for all values of the three variables, provided that c is not a nonpositive integer. It may be written as the concatenation

47:6:1 $$\mathrm{M}(a,c,x) = 1 + \frac{ax}{c}\left(1 + \frac{(a+1)x}{2(c+1)}\left(1 + \frac{(a+2)x}{3(c+2)}\left(1 + \cdots \frac{(a+j)x}{(j+1)(c+j)}(1+\cdots)\right)\right)\right)$$

which is convenient computationally when the truncated series is being summed. The series terminates if a is a nonpositive integer. Series expansions involving cylinder functions [Section 49:14] are listed by Abramowitz and Stegun [Section 13.3].

Useful for large arguments of either sign are the asymptotic expansions

47:6:2 $$\mathrm{M}(a,c,x) \sim \begin{cases} \dfrac{\Gamma(c)x^{a-c}\exp(x)}{\Gamma(a)} \displaystyle\sum_{j=0} \frac{(c-a)_j(1-a)_j}{(1)_j}\left(\frac{1}{x}\right)^j & a \neq 0,-1,-2,\cdots \quad \text{large positive } x \\ \dfrac{\Gamma(c)}{\Gamma(c-a)[-x]^a} \displaystyle\sum_{j=0} \frac{(a)_j(1+a-c)_j}{(1)_j}\left(\frac{-1}{x}\right)^j & c-a \neq 0,-1,-2,\cdots \quad \text{large negative } x \end{cases}$$

The summations in these formulas are Tricomi functions [Chapter 48].

47:7 PARTICULAR VALUES

The Kummer function takes the value unity if either x or a is zero, irrespective of the values of the other variables.

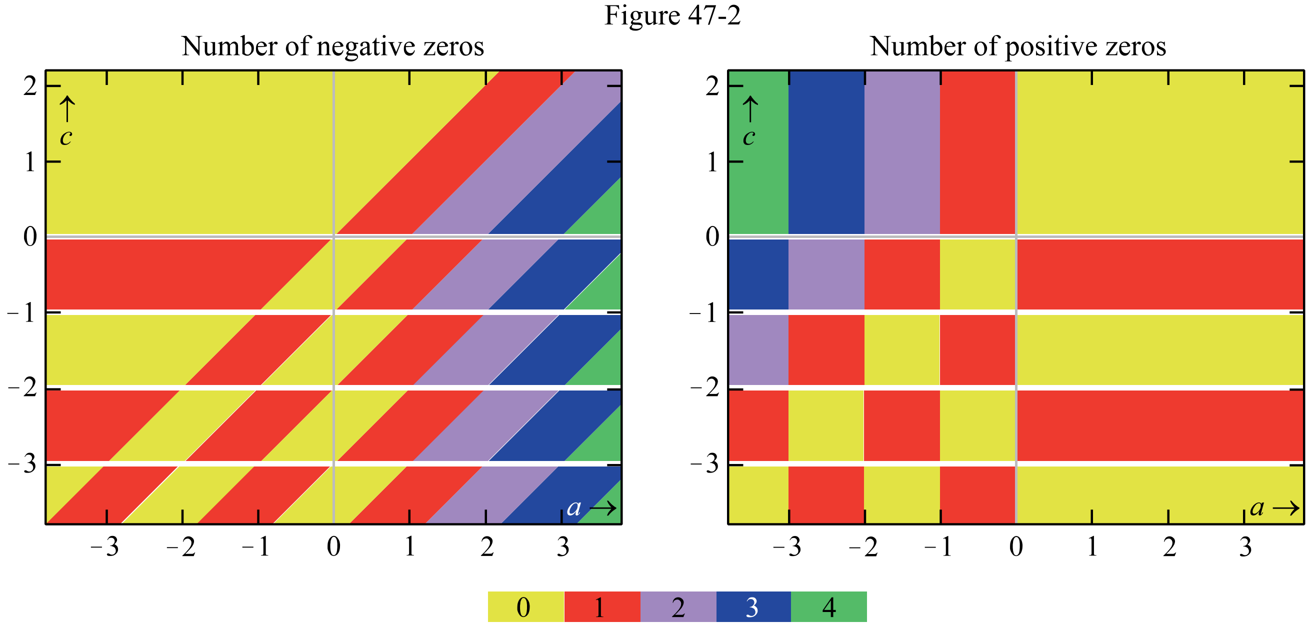
Depending on the signs and magnitudes of the parameters, the Kummer function, considered as a function of its argument, may or may not have zeros. If it has zeros, they may be "positive zeros" (that is, they represent crossings of the x-axis for positive x values), or "negative zeros" (when negative values of x causes M to be zero), or both. We know of no general formulas giving the *values* of these zeros, but there are rules giving their *number*. Figure 47-2 is color-coded to reveal the numbers of negative zeros (left-hand diagram) and positive zeros (right). Here our interest is confined to real zeros. The patterns established in these figures are readily extended to the entire a,c plane.

The number of extrema (local minima or maxima) of the Kummer function is calculable from Figure 47-2 because extrema and zeros are linked by the relationship

47:7:1 $$\begin{matrix}\text{location of}\\ \text{extremum of}\end{matrix}\ \mathrm{M}(a,c,x) = \begin{matrix}\text{location of}\\ \text{zero of}\end{matrix}\ \mathrm{M}(a+1,c+1,\,x)$$

which is a consequence of equation 47:10:1. It is this calculation that leads to Figure 47-3 which reports the total number of extrema possessed by the M(a,c,x) function. The white strips in these figures serve as a reminder of the absence of a definition for the Kummer function when the denominatorial c parameter is a nonpositive integer.

Figure 47-2

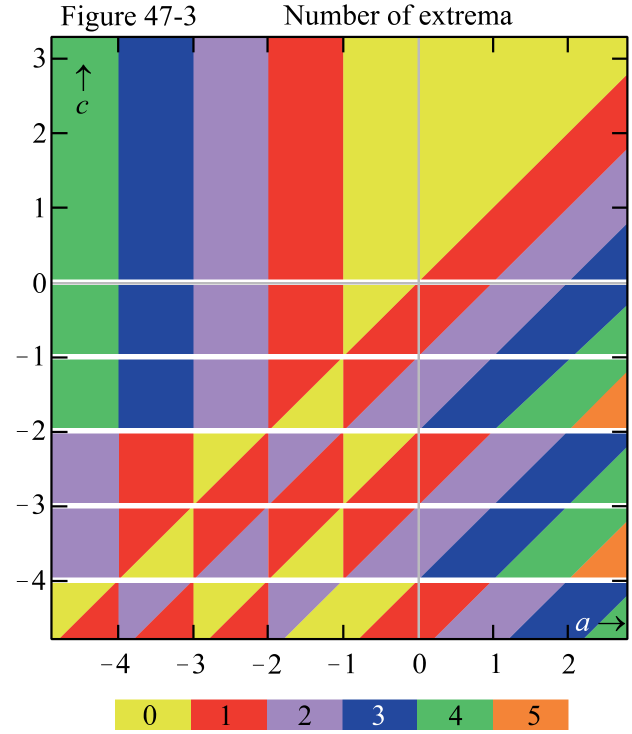


47:8 NUMERICAL VALUES

Abramowitz and Stegun [Chapter 13] devote twenty pages to tabulating numerical values of the Kummer function.

Equator's Kummer function routine (keyword **M**) is mostly based on equation 47:3:1 and can return values for the domains $|a| \le 15$, $|c| \le 15$ and $|x| \le 500$. The algorithm employs equation 47:5:1 when x is negative and adopts 47:4:1 for the special $a = c$ case.

Figure 47-3 Number of extrema



47:9 LIMITS AND APPROXIMATIONS

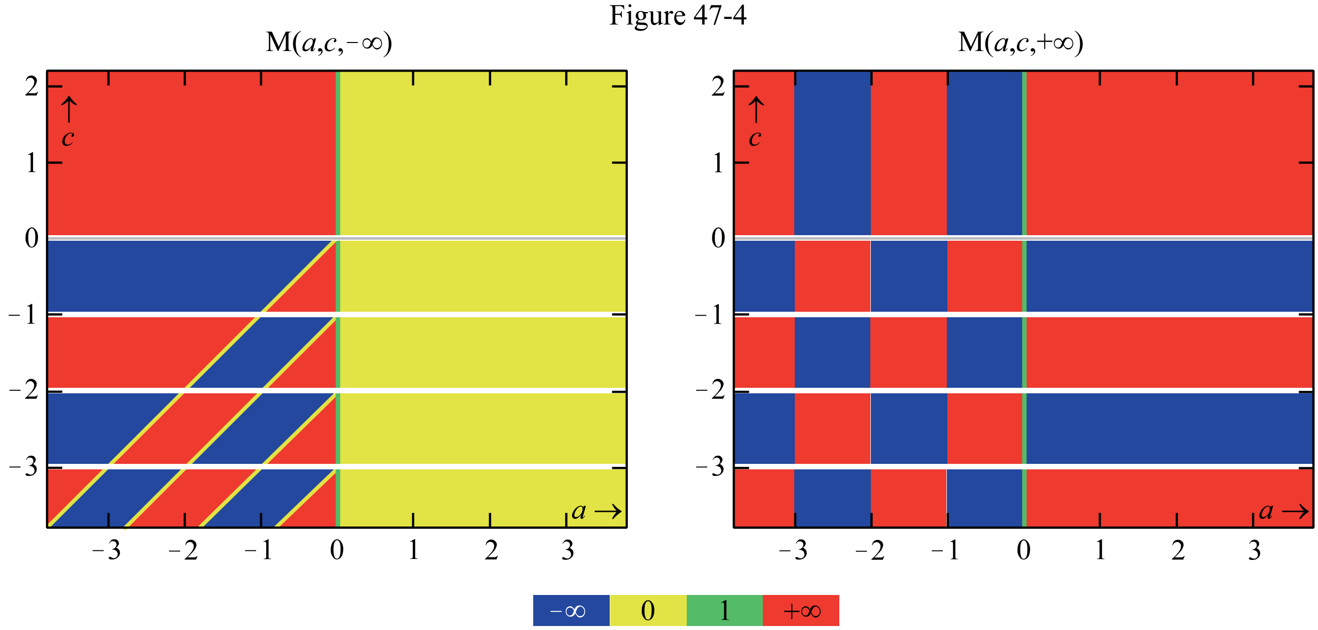
As the denominatorial parameter approaches one of its forbidden values $(0,-1,-2,\text{etc.})$ from either direction, the Kummer function becomes a "beheaded" Kummer series that is approximately proportional to another Kummer function:

$$\text{47:9:1} \qquad \mathrm{M}(a,c,x) \to \sum_{j=n+1}^{\infty} \frac{(a)_j}{(c)_j (1)_j} x^j \approx \frac{(-)^n (a)_{n+1} x^{n+1}}{(n+c) n! (n+1)!} \mathrm{M}(a+n+1, n+2, x) \qquad c \to -n = 0,-1,-2,\cdots$$

As the limit is attained, the denominatorial term $(n+c)$ vanishes, forcing $\mathrm{M}(a,c,x)$ towards either $+\infty$ or $-\infty$, and generating a $+\infty|-\infty$ or $-\infty|+\infty$ discontinuity.

The approach of the argument to $-\infty$ results in the Kummer function acquiring one of four values, and these are illustrated in the left-hand diagram of Figure 47-4. The unity value is at $a = 0$ only. The limit is $\mathrm{M}(a,c,x \to \infty) = 0$ in the yellow regions of figure, which includes the diagonal lines on which $a-c$ is a nonnegative integer. In the

Figure 47-4



red regions of the figure, $\mathrm{M}(a,c,x\to-\infty)=\infty$ while the limit $\mathrm{M}(a,c,x\to-\infty)=-\infty$ is characteristic of the blue regions.

With fixed values of a and c, $\mathrm{M}(a,c,x)$ approaches one of three values as $x\to\infty$. These values are $-\infty$, $+\infty$, and 1, the last being applicable only if $a=0$, as shown by the green strip in the right-hand diagram of Figure 47-4. For nonzero values of a, the limit is either $+\infty$ at the locations colored red in the diagram or $-\infty$ in the blue regions of the a,c plane. Note the exclusion from both diagrams of $c=0,-1,-2,\cdots$, where the Kummer function is undefined.

When one of a, c, or x is large, the other two remaining modest in magnitude, the following approximations hold:

47:9:2 $$\mathrm{M}(a,c,x)\approx\Gamma(c)\left[(2a-c)\frac{x}{2}\right]^{(1-c)/2}\exp\left(\frac{x}{2}\right)\mathrm{I}_{c-1}\left(\sqrt{(4a-2c)x}\right)\qquad a\text{ large and positive}$$

47:9:3 $$\mathrm{M}(a,c,x)\approx\frac{\Gamma(c)}{\sqrt{\pi}}\left[(c-2a)\frac{x}{2}\right]^{(1-2c)/4}\exp\left(\frac{x}{2}\right)\cos\left((1-2c)\frac{\pi}{4}+\sqrt{(2c-4a)x}\right)\qquad a\text{ large and negative}$$

47:9:4 $$\mathrm{M}(a,c,x)\approx\left(\frac{c}{c-x}\right)^{a}\qquad c\text{ large and of either sign}\qquad c\neq0,-1,-2,\cdots$$

47:9:5 $$\mathrm{M}(a,c,x)\approx\frac{\Gamma(c)}{\Gamma(a)}x^{a-c}\exp(x)\qquad x\text{ large and positive}\qquad a\neq0,-1,-2,\cdots$$

47:9:6 $$\mathrm{M}(a,c,x)\approx\frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}\qquad x\text{ large and negative}\qquad c-a\neq0,-1,-2,\cdots$$

47:10 OPERATIONS OF THE CALCULUS

The rules for differentiation and indefinite integration follow a pleasingly regular pattern:

47:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{M}(a,c,x)=\frac{a}{c}\mathrm{M}(a+1,c+1,x)$$

47:10:2
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{M}(a,c,x)=\frac{(a)_n}{(c)_n}\mathrm{M}(a+n,c+n,x)$$

47:10:3
$$\int_0^x \mathrm{M}(a,c,t)\,\mathrm{d}t=\frac{c-1}{a-1}\left[\mathrm{M}(a-1,c-1,x)-1\right] \qquad a\neq 1 \qquad c\neq 1$$

Other indefinite integrals, definite integrals and Laplace transforms include

47:10:4
$$\int_0^x \frac{t^{c-1}\,\mathrm{M}(a,c,t)}{(x-t)^{\mu+1}}\mathrm{d}t=\frac{\Gamma(c)\Gamma(-\mu)}{\Gamma(c-\mu)}\mathrm{M}(a,c-\mu,x) \qquad c>0 \qquad c-\mu>0$$

47:10:5
$$\int_0^\infty t^{v-1}\,\mathrm{M}(a,c,-t)\,\mathrm{d}t=\frac{\Gamma(v)\Gamma(c)\Gamma(a-v)}{\Gamma(a)\Gamma(c-v)} \qquad 0<v<a$$

47:10:6
$$\int_0^\infty \mathrm{M}(a,c,bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\{\mathrm{M}(a,c,bt)\}=\frac{1}{s}\mathrm{F}\left(a,1,c,\frac{b}{s}\right)$$

and

47:10:7
$$\int_0^\infty t^{c-1}\exp(vt)\,\mathrm{M}(a,c,bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{t^{c-1}\exp(vt)\,\mathrm{M}(a,c,bt)\right\}=\frac{\Gamma(c)(s-v)^{a-c}}{(s-v-b)^a}$$

Integration 47:10:4 is equivalent to the first step of synthesis 47:5:7 with $\mu=c-c'$. The function generated by the transformation in 47:10:6 is the Gauss hypergeometric function [Chapter 60].

In the language of the fractional calculus [Section 12:14], equation 47:10:4 may be rewritten

47:10:8
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\left\{x^{c-1}\,\mathrm{M}(a,c,x)\right\}=\frac{\Gamma(c)x^{c-1-\mu}}{\Gamma(c-\mu)}\mathrm{M}(a,c-\mu,x)$$

Another differintegration formula, generating a generic $L=K+1=2$ hypergeometric function, is

47:10:9
$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\mathrm{M}(a,c,x)=\frac{x^{-\mu}}{\Gamma(1-\mu)}\sum_{j=0}^\infty \frac{(a)_j}{(c)_j(1-\mu)_j}x^j$$

Equations 47:10:1−3 are all special cases of the previous equation.

47:11 COMPLEX ARGUMENT

We shall not pursue the properties of the quadrivariate function generated on replacement of the argument of the M(a,c,x) function by $x+iy$, though the formulas applicable to real argument generally carry over. When the argument is purely imaginary, the Kummer function has real and imaginary parts given by the $L=K+2=4$ hypergeometric functions

47:11:1
$$\mathrm{Re}\{\mathrm{M}(a,c,iy)\}=\sum_{j=0}^\infty \frac{\left(\frac{a}{2}\right)_j\left(\frac{a+1}{2}\right)_j}{\left(\frac{c}{2}\right)_j\left(\frac{c+1}{2}\right)_j\left(\frac{1}{2}\right)_j(1)_j}\left(\frac{-y^2}{4}\right)^j$$

and

47:11:2
$$\mathrm{Im}\{\mathrm{M}(a,c,iy)\} = \frac{ay}{c}\sum_{j=0}^{\infty}\frac{\left(\frac{a+1}{2}\right)_j\left(\frac{a+2}{2}\right)_j}{\left(\frac{c+1}{2}\right)_j\left(\frac{c+2}{2}\right)_j(1)_j\left(\frac{3}{2}\right)_j}\left(\frac{-y^2}{4}\right)^j$$

The first of the inverse Laplace transforms listed below generates a window function [Section 9:13], one that is nonzero only in the interval $0 < t < 1$

47:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\mathrm{M}(a,c,-s)\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\mathrm{M}(a,c,-s)\} = \frac{[\mathrm{u}(t)-\mathrm{u}(t-1)]\Gamma(c)t^{a-1}}{\Gamma(a)\Gamma(c-a)[1-t]^{a-c+1}} \qquad c > a > 0$$

47:11:4
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\mathrm{M}(a,c,-1/s)}{s^a}\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{\mathrm{M}(a,c,-1/s)}{s^a}\right\} = \frac{\Gamma(c)t^{(2a-c-1)/2}}{\Gamma(a)}\mathrm{J}_{c-1}\left(\sqrt{4t}\right)$$

A Bessel function [Chapter 53] is generated by the second.

47:12 GENERALIZATIONS

All hypergeometric functions [Section 18:14] with $L = K+1 \geq 2$ may be regarded as generalizations of the Kummer function, from which they may be synthesized [Section 43:14].

Modest generalization is provided by the *regularized Kummer function* or *entire Kummer function* defined by

47:12:1
$$\frac{\mathrm{M}(a,c,x)}{\Gamma(c)} = \sum_{j=0}^{\infty}\frac{(a)_j}{\Gamma(j+c)}\frac{x^j}{j!}$$

and symbolized $\varphi(a,c,x)$ by Lebedev [Section 9.9]. This function has the advantage of embracing those values $(0,-1,-2,\text{etc.})$ of the denominatorial parameter c that are excluded from the definition of the unmodified Kummer function because of the discontinuities that they cause. When c has one of these values, say $-n$, the regularized function remains finite and is given by

47:12:2
$$\frac{\mathrm{M}(a,-n,x)}{\Gamma(-n)} = \frac{(a)_{n+1}x^{n+1}}{(n+1)!}\mathrm{M}(a+n+1,n+2,x) \qquad n = 0,1,2,\cdots$$

It is unfortunate that the regularized Kummer function is not widely used because its properties are significantly simpler than those of $\mathrm{M}(a,c,x)$.

47:13 COGNATE FUNCTIONS

The Kummer function is closely linked to the Tricomi function [next chapter] through the definition

47:13:1
$$\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\mathrm{M}(a,c,x) + \frac{\Gamma(c-1)}{\Gamma(a)x^{c-1}}\mathrm{M}(1+a-c,2-c,x) = \mathrm{U}(a,c,x)$$

The first Whittaker function [Section 48:13] is an even closer relative.

47:14 RELATED TOPIC: the Kummer function with integer parameters

In terms of (complete) gamma functions, definition 47:3:1 may be rewritten as

47:14:1 $$\mathrm{M}(a,c,x)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{j=0}^{\infty}\frac{\Gamma(j+a)}{\Gamma(j+c)}\frac{x^j}{j!}$$

When both a and c are integers, say n and m respectively, each gamma function may be replaced by a corresponding factorial

47:14:2 $$\mathrm{M}(n,m,x)=\frac{(m-1)!}{(n-1)!}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{(j+m-1)!}\frac{x^j}{j!}$$

Cancellations will now commonly occur between the numerator and denominator of the summands, opening up the possibility of representing $\mathrm{M}(n,m,x)$ in different, and usually simpler, ways. However, authorities differ as to the appropriate interpretation of the Kummer function when a and c adopt certain integer values.

When both parameters are positive, with m not exceeding n, it is evident that all factors in $(m-1)!$ will be cancelled by those in $(n-1)!$, leaving only $(n-m)$ denominatorial factors. Similarly only $(n-m)$ factors remain in the numerator of the $(j+n-1)!/(j+m-1)!$ quotient after cancellations. One therefore finds

47:14:3 $$\mathrm{M}(n,m,x)=\frac{1}{(m)_{n-m}}\sum_{j=0}^{\infty}(j+m)_{n-m}\frac{x^j}{j!}\qquad 0<m\le n$$

A simple instance is when $n-m=1$, for then

47:14:4 $$\mathrm{M}(m{+}1,m,x)=\frac{1}{m}\sum_{j=0}^{\infty}(j+m)\frac{x^j}{j!}=\frac{1}{m}\sum_{j=1}^{\infty}\frac{x^j}{(j-1)!}+\sum_{j=0}^{\infty}\frac{x^j}{j!}=\left(\frac{x}{m}+1\right)\exp(x)\qquad n-1=m>0$$

and all other cases likewise reduce to the product of a polynomial with $\exp(x)$. Similar cancellations occur in the Kummer function when n does not exceed m, both being positive integers; in this event

47:14:5 $$\mathrm{M}(n,m,x)=(n)_{m-n}\sum_{j=0}^{\infty}\frac{x^j}{(j+n)_{m-n}\,j!}\qquad 0<n\le m$$

When m exceeds n by unity, reduction occurs to the expression $n!\gamma\mathrm{n}(n,-x)$ as predicted by 47:4:2 and when the difference is 2, a formula with a pair of entire incomplete gamma functions results:

47:14:6 $$\mathrm{M}(n,n+2,x)=(n+1)![\gamma\mathrm{n}(n,-x)-n\gamma\mathrm{n}(n+1,-x)]\qquad m-2=n>0$$

and so on. Note that, as expected from 47:4:1, both 47:14:3 and 47:14:5 reduce to $\mathrm{M}(n,n,x)=\exp(x)$ when the parameters are equal positive integers.

Next, cases in which the parameters are integers of opposite sign will be addressed. As an example of those instances in which n is negative and m positive, consider the case $\mathrm{M}(-4,3,x)$. Using the fundamental definition 47:3:1, we see that the leading terms in the expansion are

47:14:7 $$\mathrm{M}(-4,3,x)=1+\frac{-4}{3}\frac{x}{1!}+\frac{(-4)(-3)}{(3)(4)}\frac{x^2}{2!}+\frac{(-4)(-3)(-2)}{(3)(4)(5)}\frac{x^3}{3!}+\frac{(-4)(-3)(-2)(-1)}{(3)(4)(5)(6)}\frac{x^4}{4!}+\cdots$$

The x^5 and all subsequent terms will have a zero as a numeratorial factor. Hence the series terminates after the x^4, or in general after the x^{-n}, term. The early nonzero terms alternate in sign and their sum obeys the formula

47:14:8 $$\mathrm{M}(n,m,x)=\sum_{j=0}^{-n}\frac{(j-n)_j}{(m)_j}\frac{(-x)^j}{j!}\qquad n<0<m$$

However, when it is n that is positive and m negative, zeros invariably appear in most denominators and lead to the Kummer function being infinite; thus:

47:14:9 $$\mathrm{M}(n,m,x)=\pm\infty\qquad m<0<n$$

When both parameters are negative and n is the more negative of the pair, zeros again appear in denominators and cause the Kummer function to become infinite:

47:14:10 $$\mathrm{M}(n,m,x) = \pm\infty \qquad n < m < 0$$

Two schools of thought apply to the case of two negative parameters with m being the more negative. Early terms in the $n = -1$, $n = -2$ example are

47:14:11 $$\mathrm{M}(-1,-2,x) = 1 + \frac{-1}{-2}\frac{x}{1!} + \frac{(-1)(0)}{(-2)(-1)}\frac{x^2}{2!} + \frac{(-1)(0)(1)}{(-2)(-1)(0)}\frac{x^3}{3!} + \frac{(-1)(0)(1)(2)}{(-2)(-1)(0)(1)}\frac{x^4}{4!} + \cdots$$

If one considers that the two zeros in the x^3 and subsequent terms cancel, then the series terminates after the zeroth and first terms but, restarts with the x^3 term. In general, the missing terms are those that would have involved the $x^{1-n}, x^{2-n}, \cdots, x^{-m}$ powers. The formula

47:14:12 $$\mathrm{M}(n,m,x) = \sum_{j=0}^{-n} \frac{(n)_j}{(m)_j}\frac{x^j}{j!} + \frac{(-x)^{-m}}{(m)_{n-m}} \sum_{k=1}^{\infty} \frac{(k)_{n-m} x^k}{(k-m)!} \qquad m < n < 0$$

applies. However, an alternative interpretation is that the presence of a zero in the denominator of a term renders that term infinite, notwithstanding that a compensatory zero exists in the numerator. On this basis $\mathrm{M}(n,m,x)$ is infinite whenever $m < n < 0$. *Equator* sits on the fence and returns no definite answer. The present authors favor the interpretation embodied in 47:14:12 because it alone provides continuity with the Kummer functions of nearby noninteger parameters. When n and m are negative and equal, the "cancellation approach" equates the Kummer function to $\exp(x)$, whereas *Mathematica* returns the polynomial $\mathrm{e}_{-n}(x)$ [Section 26:12].

It remains to consider values of the Kummer function when one is, or both of the parameters are, zero. Definition 47:3:1 shows unequivocally that $\mathrm{M}(n,0,x) = \pm\infty$, unless n is also zero, in which case the cancellation approach finds $\mathrm{M}(0,0,x) = \exp(x)$. This approach applied to $n = 0$, $m < 0$ cases generates the series expansion

47:14:13 $$\mathrm{M}(0,m,x) = 1 + \frac{(-x)^{-m}}{(-m)!} \sum_{k=1}^{\infty} \frac{(k)_{-m} x^k}{(k-m)!} \qquad m < 0$$

which is simply 47:14:12 extended to embrace $n = 0$. If cancellation of denominatorial zeros is denied, the result is $\mathrm{M}(0,m,x) = 1$.

CHAPTER 48

THE TRICOMI FUNCTION $\mathrm{U}(a,c,x)$

Named for the Italian mathematician Francesco Giacomo Tricomi (1897 – 1978), this function, together with the Kummer function of the previous chapter, constitute the duo known as the *confluent hypergeometric functions*. The Whittaker functions and Bateman's confluent function, addressed in Section 48:13, are closely related.

48:1 NOTATION

Alternative names are *confluent hypergeometric function of the second kind*, *degenerate hypergeometric function of the second kind*, *Kummer function of the second kind*, and *Gordon function*. The notation $\Psi(a,c,x)$ is a common alternative to the $\mathrm{U}(a,c,x)$ that we use. Tricomi himself used the G symbol. On account of the asymptotic expansion 48:6:2, the equivalences

48:1:1 $$x^a\,\mathrm{U}(a,c,x) \sim \sum_{j=0} \frac{(a)_j(1+a-c)_j}{(1)_j}\left(\frac{-1}{x}\right)^j = {}_2\mathrm{F}_0\left(a,1+a-c,\frac{-1}{x}\right)$$

exist and therefore the hypergeometric function ${}_2\mathrm{F}_0$ plays a role in some writing equivalent to that played by the Tricomi function here.

The *Tricomi function* is trivariate, with an argument x and two parameters, a and c. In contrast to their status in the Kummer and Gauss hypergeometric functions, the parameters cannot be separately assigned to numeratorial and denominatorial roles; we call a and c the first and second parameters. The composite variable $1+a-c$ that appears in equation 48:1:1, and elsewhere in this chapter, is often more important than c itself in determining the properties of the Tricomi function; accordingly it is sometimes accorded its own symbol, b in this *Atlas*.

48:1:2 $$b = 1+a-c$$

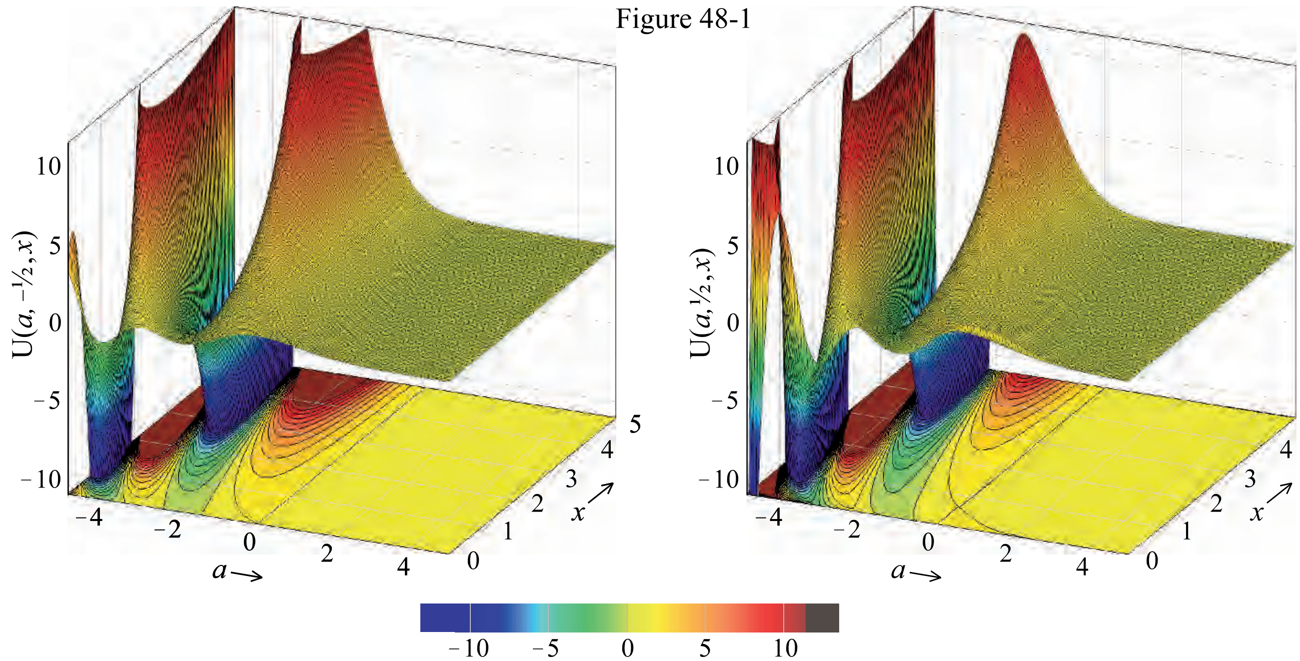
Here, you will sometimes find formulas written in terms of a and the *auxiliary parameter* b rather than a and c.

48:2 BEHAVIOR

The Tricomi function is defined for all real values of its three variables. We exclude complex variables here, though most of the formulas carry over into the complex plane. Also excluded from discussion in this *Atlas* is the $x<0$ domain, because $\mathrm{U}(a,c,x)$ is generally complex when its argument is negative.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_49,

Figure 48-1



Even three-dimensional graphs are inadequate to convey the global behavior of a trivariate function, such as $\mathrm{U}(a,c,x)$. However, some information can be gleaned from Figure 48-1, which shows instances of the function for two fixed values of the c parameter: $c = -\tfrac{1}{2}$, left; and $c = +\tfrac{1}{2}$, right. The behavior of the Tricomi function is not straightforward even at $x = 0$, as evidenced by Figure 48-1. Nevertheless, a qualitative picture of the behavior of $\mathrm{U}(a,c,x)$ for $0 < x < \infty$ can be assembled from the following considerations:

(a) Contrary to what a casual inspection of definition 48:3:1 might suggest, $\mathrm{U}(a,c,x)$ develops no discontinuities away from $x = 0$, being real and finite for all values of the parameters and all positive arguments.

(b) Figure 48-3, later in this chapter, may be used to assess the behavior of the Tricomi function at zero argument. As well, the table in Section 48:9 describes the Tricomi function's behavior close to $x = 0$.

(c) The number of positive zeros of the Tricomi function can be found from Figure 48-4 for any pair of a,c values.

(d) Originating in formula 48:10:1, the rule

$$\text{48:2:1} \qquad \begin{matrix}\text{number of}\\ \text{extrema of}\end{matrix}\ \mathrm{U}(a,c,x) = \begin{matrix}\text{number of}\\ \text{zeros of}\end{matrix}\ \mathrm{U}(a+1,c+1,x)$$

enables one to use the same figure to determine the number of local maxima or minima. Most often, there are equal numbers of zeros and extrema.

(e) Behavior at large arguments is primarily determined by the sign of the a parameter. If this is negative, the Tricomi function heads towards $+\infty$ as x increases. For positive a, the function asymptotically approaches zero with increasing argument.

To illustrate the behavioral diversity, Figure 48-2 shows a few graphs of $\mathrm{U}(a,c,x)$ for rather small arguments.

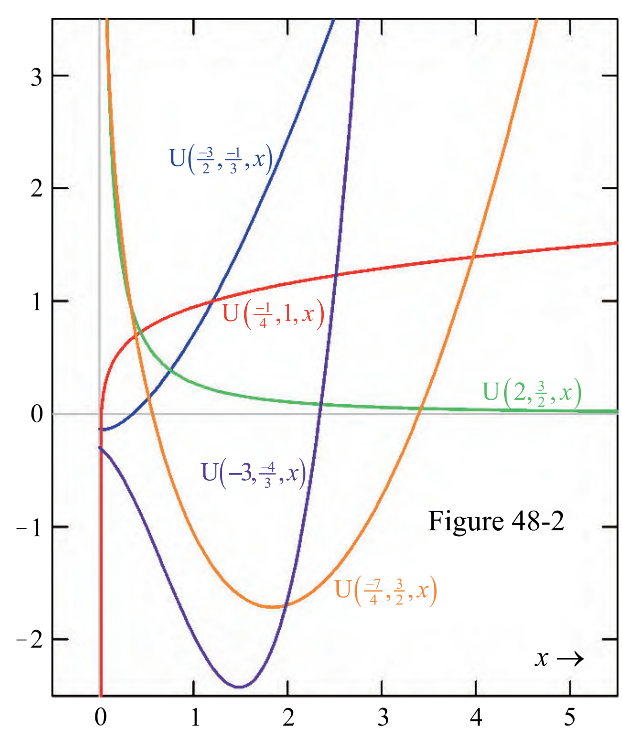


Figure 48-2

48:3 DEFINITIONS

The Tricomi function may be defined as a suitably weighted sum of two Kummer functions [previous chapter]

48:3:1
$$\mathrm{U}(a,c,x)=\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\mathrm{M}(a,c,x)+\frac{\Gamma(c-1)}{\Gamma(a)x^{c-1}}\mathrm{M}(1+a-c,2-c,x)$$

One or other of the weights becomes infinite if c is an integer. Nevertheless, as Lebedev [Section 9.10] shows, this definition may be developed into a finite-valued expression when c is an integer m of either sign. When this integer is nonpositive, the transformation 48:5:1 provides the replacement formula

48:3:2
$$\mathrm{U}(a,m,x)=x^{1-m}\,\mathrm{U}(1+a-m,2-m,x) \qquad c=m=0,-1,-2,\cdots$$

in which the second parameter is now positive. Positive integer c parameters can be accommodated through the complicated formula

48:3:3
$$\begin{aligned}\mathrm{U}(a,m,x)=&\frac{(m-2)!}{\Gamma(a)x^{m-1}}\sum_{k=0}^{m-2}\frac{(1+a-m)_k x^k}{(2-m)_k(1)_k} && c=m=1,2,3,\cdots\\ &+\frac{(-1)^m}{(m-1)!\Gamma(1+a-m)}\sum_{j=0}^{\infty}\left[\psi(j+a)-\psi(j+1)-\psi(j+m)+\ln(x)\right]\frac{(a)_j x^j}{(1)_j(m)_j} && a\neq 0,-1,-2,\cdots\end{aligned}$$

involving the digamma function [Chapter 44]. The second right-hand term vanishes when $m = 1$, and it contributes merely $1/[x\,\Gamma(a)]$ when $m = 2$. However, 48:3:3 is invalid when a is a nonpositive integer, in which case the definition becomes

48:3:4
$$\mathrm{U}(n,m,x)=\frac{(-)^n(m-n-1)!}{(m-1)!}\mathrm{M}(n,m,x)=(-)^n(m)_{-n}\sum_{j=0}^{-n}\frac{(n)_j}{(m)_j(1)_j}x^j \qquad \begin{matrix}a=n=0,-1,-2,\cdots\\ c=m=1,2,3,\cdots\end{matrix}$$

The Tricomi function may be defined by the limiting operation

48:3:5
$$\mathrm{U}(a,c,x)=\frac{1}{x^a}\lim_{\lambda\to\infty}\mathrm{F}\left(a,1+a-c,\lambda,\frac{x-\lambda}{x}\right)$$

applied to the Gauss hypergeometric function [Chapter 60] and it is this definition that is responsible for the name *confluent hypergeometric function* or *degenerate hypergeometric function* being applied to $\mathrm{U}(a,c,x)$.

The integral definition

48:3:6
$$\mathrm{U}(a,c,x)=\frac{1}{\Gamma(a)}\int_0^\infty\frac{t^{a-1}}{(1+t)^b}\exp(-xt)\,\mathrm{d}t \qquad b=1+a-c \qquad a>0 \qquad x>0$$

shows the Tricomi function to be the Laplace transform of a simple algebraic expression. Equivalent to this are

48:3:7
$$\mathrm{U}(a,c,x)=\frac{1}{\Gamma(a)}\int_0^1\frac{t^{a-1}}{(1-t)^c}\exp\left(\frac{-xt}{1-t}\right)\mathrm{d}t \qquad a>0 \qquad x>0$$

and the integrals shown in 48:8:1 and 48:8:2.

An arbitrarily weighted sum of two Tricomi functions, one of which has an exponential multiplier, solves the *confluent hypergeometric differential equation*

48:3:8
$$x\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}-a\mathrm{f}=0 \qquad \mathrm{f}=w_1\,\mathrm{U}(a,c,x)+w_2\exp(x)\,\mathrm{U}(c-a,c,-x)$$

Other solutions to this equation are mentioned in Section 47:3.

A two-step synthesis [Section 43:14] can generate the Tricomi function from the prototypical $L = K-1 = 0$

hypergeometric *Euler function*

48:3:9 $$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right) \xrightarrow[1]{a} \frac{1}{x^a}\exp\left(\frac{1}{x}\right)\Gamma\left(1-a,\frac{1}{x}\right) \xrightarrow[1]{b} \frac{1}{x^a}\mathrm{U}\left(a,c,\frac{1}{x}\right) \qquad \text{large } x$$

where, as elsewhere in this chapter, $b = 1+a-c$.

48:4 SPECIAL CASES

Certain relationships between the a and c parameters simplify the Tricomi function. When the two parameters are equal, reduction occurs to a function involving an upper incomplete gamma function [Chapter 45]:

48:4:1 $$\mathrm{U}(a,a,x) = \exp(x)\Gamma(1-a,x)$$

A simple power function results when the c parameter exceeds a by unity (that is, when $b = 0$):

48:4:2 $$\mathrm{U}(a,a+1,x) = x^{-a}$$

The two preceding formulas, coupled with one or more applications of recursion 48:5:3, permit expressions to be deduced for $\mathrm{U}(a, a\pm m, x)$. When the c parameter is twice the a parameter, the Tricomi function reduces to a Macdonald function [Chapter 51]

48:4:3 $$\mathrm{U}(a,2a,x) = \frac{x^{(1-2a)/2}}{\sqrt{\pi}}\exp\left(\frac{x}{2}\right)\mathrm{K}_{(2a-1)/2}\left(\frac{x}{2}\right)$$

and, if a is an integer, further simplification to exponentials is possible via the spherical Macdonald functions discussed in Section 26:13.

When the a parameter is a negative integer $-n$, the Tricomi function becomes a polynomial, specifically a generalized Laguerre polynomial [Section 23:12]:

48:4:4 $$\mathrm{U}(-n,c,x) = (-)^n n!\mathrm{L}_n^{(c-1)}(x) \qquad a=-n=-1,-2,-3,\cdots$$

This carries over to the case in which $a = 0$, whereby

48:4:5 $$\mathrm{U}(0,c,x) = 1$$

An expression similar to 48:4:1 results from setting a equal to unity:

48:4:6 $$\mathrm{U}(1,c,x) = x^{1-c}\exp(x)\Gamma(c-1,x) \sim \frac{1}{x}\sum_{j=0}(2-c)_j\left(\frac{-1}{x}\right)^j$$

The asymptotic hypergeometric representation of this result has been included to establish contact with Table 18-7, from which instances corresponding to specific c values can be found. Sufficient recursions based on the final row in the table on the facing page permit equations 48:4:5 and 48:4:6 to serve as building blocks from which expressions for $\mathrm{U}(n,c,x)$ may be constructed.

Cases in which the c parameter is an integer are discussed in the context of definitions 48:3:2 and 48:3:3. In particular, note that when $c = 1$, the latter equation may be written in the more concise form

48:4:7 $$\mathrm{U}(a,1,x) = \sum_{j=0}^{\infty}\left[2\psi(j+1)-\psi(j+a)-\ln(x)\right]\frac{\Gamma(j+a)x^j}{[\Gamma(a)j!]^2} \qquad a \neq 0,-1,-2,\cdots$$

A simple relationship, namely

48:4:8 $$\mathrm{U}(a,0,x) = x\,\mathrm{U}(a+1,2,x)$$

exists between Tricomi functions whose c parameters are 0 and 2; these are related to the *Bateman confluent function*, addressed in Section 48:13.

The following special cases, arising when the c parameter is half an odd integer, are noteworthy:

48:4:9 $$\mathrm{U}\left(a,\tfrac{1}{2},x\right)=\sqrt{x}\,\mathrm{U}\left(a+\tfrac{1}{2},\tfrac{3}{2},x\right)=2^{a}\exp\left(\tfrac{1}{2}x\right)\mathrm{D}_{-2a}\left(\sqrt{2x}\right)$$

48:4:10 $$\mathrm{U}\left(\tfrac{1}{2},\tfrac{1}{2},x\right)=\sqrt{x}\,\mathrm{U}\left(1,\tfrac{-3}{2},x\right)=\sqrt{\pi}\exp(x)\mathrm{erfc}\left(\sqrt{x}\right)$$

48:4:11 $$\mathrm{U}\left(\tfrac{-n}{2},\tfrac{1}{2},x\right)=\sqrt{x}\,\mathrm{U}\left(\tfrac{1-n}{2},\tfrac{3}{2},x\right)=2^{-n}\mathrm{H}_{n}\left(\sqrt{x}\right)$$

Functions from Chapters 46, 41, and 24 are involved.

48:5 INTRARELATIONSHIPS

The important transformation

48:5:1 $$\mathrm{U}(a,c,x)=x^{1-c}\,\mathrm{U}(1+a-c,2-c,x)$$

constitutes a reflection formula for the c parameter. This becomes an identity when $c=1$, while equation 48:4:8 is the $c=0$ example. The transformation exhibits a pleasing symmetry when written as

48:5:2 $$x^{a}\,\mathrm{U}(a,1+a-b,x)=x^{b}\,\mathrm{U}(b,1+b-a,x)\qquad b=1+a-c$$

in terms of the auxiliary parameter.

Recursion relations may be written expressing $\mathrm{U}(a,c,x)$ as the weighted sum of any two of its four *contiguous functions*; that is, in terms of any two of the functions in the heading of the table below. The body of the table lists the weights, which usually involve the argument as well as the parameters

$\mathrm{U}(a+1,c,x)$	$\mathrm{U}(a-1,c,x)$	$\mathrm{U}(a,c+1,x)$	$\mathrm{U}(a,c-1,x)$
$\dfrac{a(1+a-c)}{2a-c+x}$	$\dfrac{1}{2a-c+x}$		
$\dfrac{a(1+a-c)}{a+x}$		$\dfrac{x}{a+x}$	
a			1
	$\dfrac{1}{a-c}$	$\dfrac{x}{c-a}$	
	$\dfrac{1}{a-1+x}$		$\dfrac{c-a-1}{a-1+x}$
		$\dfrac{x}{c-1+x}$	$\dfrac{c-a-1}{c-1+x}$

Such relations, of which

48:5:3 $$(2a-c+x)\,\mathrm{U}(a,c,x)=a(1+a-c)\,\mathrm{U}(a+1,c,x)+\mathrm{U}(a-1,c,x)$$

is an example coming from the first row of the table, are known as *contiguity relationships*.

Numerically useful only for y not grossly exceeding unity, the argument-addition formula

48:5:4 $$\mathrm{U}(a,c,x+y)=\sum_{j=0}^{\infty}\frac{(a)_{j}(-y)^{j}}{(1)_{j}}\mathrm{U}(a+j,c+j,x)$$

converts to an argument-multiplication formula on replacement of y by $(v-1)x$.

48:6 EXPANSIONS

Utilizing the convergent power series expansion of the Kummer function, and the abbreviation $b = 1+a-c$, definition 48:3:1 leads to

48:6:1 $$\mathrm{U}(a,c,x)=\frac{\Gamma(1-c)}{\Gamma(b)}\left[1+\frac{ax}{c}+\frac{a(a+1)x^2}{c(c+1)2!}+\cdots\right]+\frac{\Gamma(c-1)}{\Gamma(a)x^{c-1}}\left[1+\frac{bx}{2-c}+\frac{b(b+1)x^2}{(2-c)(3-c)2!}+\cdots\right]$$

As its stands, this formula is inapplicable when the c parameter is an integer of either sign.

Again employing the $b = 1+a-c$ abbreviation, the important asymptotic series

48:6:2 $$\mathrm{U}(a,c,x)\sim x^{-a}\left[1-\frac{ab}{x}+\frac{a(a+1)b(b+1)}{2!x^2}-\cdots\right]=\frac{1}{x^a}\sum_{j=0}\frac{(a)_j(b)_j}{(1)_j}\left(\frac{-1}{x}\right)^j \qquad x \text{ large}$$

demonstrates the hypergeometric nature of the Tricomi function, establishes a connection with the functions in Table 18-8, and validates synthesis 48:3:9.

48:7 PARTICULAR VALUES

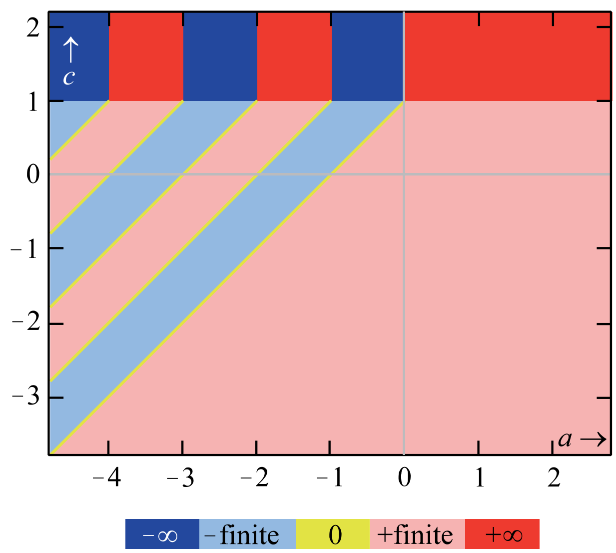
Whereas $\mathrm{U}(a,c,x)$ is everywhere finite for $0<x<\infty$, this is not always true at $x=0$ or $x=\infty$. See Section 48:9 for the latter circumstance.

Infinite values of $\mathrm{U}(a,c,0)$ are the rule when $c \geq 1$, unless a is a nonpositive integer. The situation is clarified in Figure 48-3. Red zones in the diagram correspond to positive values of $\mathrm{U}(a,c,0)$; the color is bright red where the value is $+\infty$, pale red where the positive value is finite. Likewise, blue signifies $\mathrm{U}(a,c,0)=-\infty$, whereas pale blue connotes a finite negative value. The Tricomi function of zero argument is zero on the yellow diagonal lines separating the red and blue zones:

48:7:1 $$\mathrm{U}(a,c,0)=0 \qquad a-c=-1,-2,-3,\cdots \qquad c<1$$

whereas values on the vertical separators are nonzero and various. A variety of values are acquired at zero argument:

Figure 48-3 Value of $\mathrm{U}(a,c,0)$

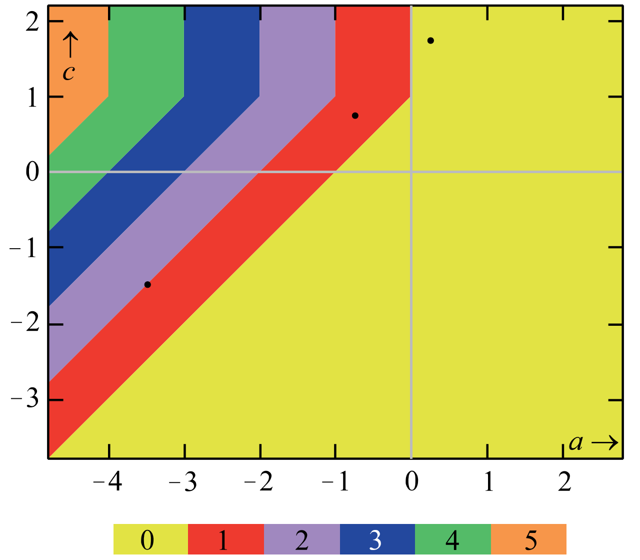


48:7:2 $$\mathrm{U}(a,c,0)=\begin{cases}1 & a=0 \quad \text{all } c\\ +\infty & a<0,\ -1<a\le-2,\ -3<a\le 4,\ \cdots \\ -\infty & 0<a\le-1,\ -2<a\le 3,\ \cdots \end{cases}\Bigg\}\ c>1 \\ \qquad\qquad \Gamma(1-c)/\Gamma(1+a-c) \qquad c\le 1$$

The frameworks of Figures 48-3 and 48-4 are similar. The purpose of the latter figure is to show the number of zeros exhibited by $\mathrm{U}(a,c,x)$ in the range $0<x<\infty$. This diagram is ambiguous if the point corresponding to the pair of a,c values lies on one of the diagonal lines, as at the point marked by the leftmost dot on Figure 48-4. This corresponds to $a=-7/2$, $c=-3/2$. Does $\mathrm{U}(-7/2,-3/2,x)$ have one zero? Or two? The answer is satisfying: there is one zero in $0<x<\infty$, but an extra zero on the domain boundary *at* $x=0$.

Figure 48-4 can be used in conjunction with rule 48:2:1 to predict the number of extrema displayed by U(a , c , x). It shows, for example, that $\mathrm{U}\left(-\tfrac{3}{4},\tfrac{3}{4},x\right)$ has none. This conclusion is reached by consideration of the two dots on Figure 48-4 which show the location in a,c space of the Tricomi functions linked by relation 48:2:1 when $a = -\tfrac{3}{4}$ and $c = \tfrac{3}{4}$.

Figure 48-4 Number of zeros of U(a,c,x)



48:8 NUMERICAL VALUES

The calculation of accurate values of the Tricomi function for wide domains of the three variables is a severe challenge. *Equator*'s Tricomi function routine (keyword **U**) treats the cases of integer a, c or b separately. Otherwise, depending on the values of a, c and x, one of four methods is used. Two of the quartet of methods are Gauss-Laguerre numerical integrations [Section 24:15] based on alternative definite integrals, namely

48:8:1 $$\mathrm{U}(a,c,x) = \frac{x^{1-c}}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}\exp(-t)}{(x+t)^b}\,\mathrm{d}t$$

and

48:8:2 $$\mathrm{U}(a,c,x) = \frac{1}{\Gamma(b)}\int_0^\infty \frac{t^{b-1}\exp(-t)}{(x+t)^a}\,\mathrm{d}t \qquad 1+a-c=b>0$$

The third method, suitable only for large arguments, is based on expansion 48:6:2 and employs the ε-transformation [Section10:14] to combat the asymptoticity. The fourth method utilizes the twin series formula 48:6:1. This formula is deceptively innocuous, from a computational standpoint, because the two moieties are frequently of comparable magnitude and opposite sign. To combat this problem, *Equator* adopts special precision optimization techniques in summing the twin series, reformulated as

48:8:3 $$\mathrm{U}(a,c,x) = \frac{\sqrt{|\Lambda|}\,\Gamma(1-c)}{\Gamma(b)}\sum_{j=0}^{\infty}\left\{\frac{(a)_j}{\sqrt{|\Lambda|}\,(c)_j} + \mathrm{sgn}(\Lambda)\frac{\sqrt{|\Lambda|}\,(b)_j}{(2-c)_j}\right\}\frac{x^j}{j!} \qquad \Lambda = \frac{\Gamma(b)\Gamma(c-1)}{\Gamma(a)\Gamma(1-c)}x^{1-c}$$

Equator's algorithm is designed to provide values, accurate to the number of digits displayed, over the domains $-15 \le a \le 15$, $-15 \le c \le 15$, and $10^{-6} \le x \le 500$ or the variables. It has been tested copiously but, because of the unusually refractory numerical properties of the Tricomi function, the user is cautioned to be on guard against algorithmic malfunction.

48:9 LIMITS AND APPROXIMATIONS

For small positive values of its argument, the value of U(a , c , x) is affected remarkably by the value of the c parameter. The formulas in the table below are composed of the two major terms contributing to the value of the Tricomi function when x is small.

$c<0$	$c=0$	$0<c<1$	$c=1$	$1<c<2$	$c=2$	$c>2$
$\dfrac{[c+ax]\Gamma(-c)}{-\Gamma(b)}$	$\dfrac{1+ax\ln(x)}{\Gamma(a+1)}$	$\dfrac{\Gamma(1-c)}{\Gamma(b)}+\dfrac{\Gamma(c-1)x^{1-c}}{\Gamma(a)}$	$\dfrac{2\gamma+\psi(a)+\ln(x)}{-\Gamma(a)}$	$\dfrac{\Gamma(c-1)}{\Gamma(a)x^{c-1}}+\dfrac{\Gamma(1-c)}{\Gamma(b)}$	$\dfrac{1}{\Gamma(a)x}+\dfrac{\ln(x)}{\Gamma(a-1)}$	$\dfrac{[c-2-bx]\Gamma(c-2)}{\Gamma(a)x^{c-1}}$

For large x, $\mathrm{U}(a,c,x)$ is approximated by x^{-a} and the $x\to\infty$ limit therefore takes one of three values:

48:9:1 $$\mathrm{U}(a,c,x)\approx x^{-a}\to\begin{Bmatrix}0 & a>0\\ 1 & a=0\\ \infty & a<0\end{Bmatrix} \qquad x\to\infty$$

48:10 OPERATIONS OF THE CALCULUS

Single or multiple differentiation of a particular Tricomi function yields another Tricomi function

48:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{U}(a,c,x)=-a\,\mathrm{U}(a+1,c+1,x)$$

48:10:2 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{U}(a,c,x)=(-)^n\left(a\right)_n\mathrm{U}(a+n,c+n,x)$$

as does indefinite integration

48:10:3 $$\int_0^x \mathrm{U}(a,c,t)\,\mathrm{d}t=\frac{1}{1-a}\left[\mathrm{U}(a-1,c-1,x)-\frac{\Gamma(c-2)}{\Gamma(1+a-c)}\right] \qquad c<2$$

Among the many formulas listed by Gradshteyn and Ryzhik [Section 7.6] for definite integrals and Laplace transforms of the Tricomi function (or the equivalent Whittaker W function) are:

48:10:4 $$\int_0^\infty t^{\nu-1}\,\mathrm{U}(a,c,t)\,\mathrm{d}t=\frac{\Gamma(\nu)\Gamma(a-\nu)\Gamma(1+\nu-c)}{\Gamma(a)\Gamma(1+a-c)} \qquad 0<\nu<a \qquad c<1+\nu$$

and

48:10:5 $$\int_0^\infty t^{\nu-1}\,\mathrm{U}(a,c,t)\exp(-st)\,\mathrm{d}t=\frac{\Gamma(\nu)\Gamma(\nu-c+1)}{\Gamma(\nu+a-c+1)}\mathrm{F}\left(\nu,\nu-c+1,\nu+a-c+1,1-s\right) \qquad 0<\nu>c-1 \quad 0<s\le 1$$

The F function in 48:10:5 is the Gauss hypergeometric function of Chapter 60.

48:11 COMPLEX ARGUMENT

The Tricomi function adopts complex values not only when its argument is complex, but also when the argument is real and negative. Neither situation is addressed in this *Atlas*.

Laplace inversion of the Tricomi function generates an algebraic function

48:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\mathrm{U}(a,c,s)\frac{\exp(st)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\mathrm{U}(a,c,s)\right\}=\frac{1}{\Gamma(a)}\frac{t^{a-1}}{\left(1+t\right)^{1+a-c}} \qquad a>0$$

hinting at a common way in which the Tricomi function enters practical problems.

48:12 GENERALIZATIONS

Inasmuch as it "degenerates" into $\mathrm{U}(a,c,x)$ via 48:3:5, the *Gauss hypergeometric function* [Chapter 60] may be regarded as a generalization of the Tricomi function.

48:13 COGNATE FUNCTIONS: Whittaker and Bateman confluent functions

The functions $\mathrm{M}_{\nu,\mu}(x)$ and $\mathrm{W}_{\nu,\mu}(x)$ are normalized versions of the Kummer and Tricomi functions, respectively. Named for Edmund Taylor Whittaker (English mathematician and astronomer, 1873 – 1956), these functions use parameters related to the familiar a and c (and b) through the equations

48:13:1
$$\nu = \tfrac{1}{2}c - a \qquad \mu = \tfrac{1}{2}c - \tfrac{1}{2} \qquad \text{or} \qquad a = \tfrac{1}{2} - \nu + \mu \qquad c = 2\mu + 1 \qquad \left(b = \tfrac{1}{2} - \nu - \mu\right)$$

Whittaker functions are also regarded as *confluent hypergeometric functions.* The relationships

48:13:2
$$\exp\left(\tfrac{1}{2}x\right)\mathrm{M}_{\nu,\mu}(x) = x^{c/2}\,\mathrm{M}\left(a,c,x\right) \qquad \exp\left(\tfrac{1}{2}x\right)\mathrm{W}_{\nu,\mu}(x) = x^{c/2}\,\mathrm{U}\left(a,c,x\right)$$

permit easy transition between the two notations. Certain relationships are more symmetrical in Whittaker's notation, for example the transformations 47:5:1 and 48:5:1 simplify to

48:13:3
$$\mathrm{M}_{\nu,\mu}(-x) = (-1)^{(2\mu+1)/2}\,\mathrm{M}_{-\nu,\mu}(x) \qquad (-1)^{(2\mu+1)/2} = -\sin(\mu\pi) + i\cos(\mu\pi)$$

and

48:13:4
$$\mathrm{W}_{\nu,-\mu}(x) = \mathrm{W}_{\nu,\mu}(x)$$

Moreover, the Whittaker functions display their kinship to cylinder functions [Section 49:14] more evidently than do the Kummer and Tricomi functions; thus when the ν parameter is zero, there is equivalence to modified Bessel I functions and Macdonald K functions:

48:13:5
$$\mathrm{M}_{0,\mu}(x) = 4^{\mu}\,\Gamma(1+\mu)\sqrt{x}\,\mathrm{I}_{\mu}\left(\tfrac{1}{2}x\right)$$

48:13:6
$$\mathrm{W}_{0,\mu}(x) = \sqrt{x/\pi}\,\mathrm{K}_{\mu}\left(\tfrac{1}{2}x\right)$$

On the other hand, some other formulas from Chapters 47 and 48 become more complicated in Whittaker's notation, so advantages and disadvantages are evenly balanced.

Arbitrarily weighted Whittaker functions are solutions to *Whittaker's differential equation*

48:13:7
$$x^2\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + \left[\tfrac{1}{4} - \mu^2 + \nu x - \tfrac{1}{4}x^2\right]\mathrm{f} = 0 \qquad \mathrm{f} = w_1\,\mathrm{M}_{\nu,\mu}(x) + w_2\,\mathrm{W}_{\nu,\mu}(x)$$

Equator's Whittaker M function and Whittaker W function routines (keywords **WhittakerM** and **WhittakerW**) calculate values through the equivalences contained in equations 48:13:1 and 48:13:2. Values are provided only for variables in the domains $|\nu| \le 7$, $|\mu| \le 7$, and $10^{-6} \le x \le 500$.

A function introduced by Bateman corresponds to the case of a Whittaker W function whose second parameter is −½, or to a Tricomi function for which $c = 0$. He defined

48:13:8
$$\kappa_{\nu}(x) = \frac{\mathrm{W}_{\frac{\nu}{2},\frac{-1}{2}}(2x)}{\Gamma\left(1+\frac{1}{2}\nu\right)} = \frac{\exp(-x)}{\Gamma\left(1+\frac{1}{2}\nu\right)}\,\mathrm{U}\left(\tfrac{-1}{2}\nu,0,2x\right)$$

To avoid confusion with the function discussed in Section 44:13, which is attributed to the same mathematician, the

Atlas adopts the name *Bateman confluent function*. We use the symbol $\kappa_v(x)$ to denote the Bateman confluent function of order v and argument x, though $k_v(x)$ is more usual. On account of transformation 48:5:1, the Tricomi functions with c parameters of 0 or 2 are both equivalent to Bateman confluent functions

48:13:9 $$\kappa_v(x) = \frac{\exp(-x)}{\Gamma(1+\frac{1}{2}v)}\mathrm{U}\left(\tfrac{-1}{2}v, 0, 2x\right) = \frac{2x\exp(-x)}{\Gamma(1+\frac{1}{2}v)}\mathrm{U}\left(1-\tfrac{1}{2}v, 2, 2x\right)$$

The integral representation

48:13:10 $$\kappa_v(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos\left\{x\tan(\theta) - v\theta\right\}\mathrm{d}\theta$$

also applies. Through its Bateman confluent function routine (keyword **kappa**) *Equator* provides accurate $\kappa_v(x)$ values. The routine is restricted to $|v| \le 28$ and $10^{-6} \le x \le 250$.

CHAPTER
49

THE MODIFIED BESSEL FUNCTIONS $I_n(x)$ OF INTEGER ORDER

These functions are the simplest of the *cylinder functions*, a large group of important functions, the tortured nomenclature of which is explained in Section 49:14. In the present chapter we address only those *modified Bessel functions* that have orders of 0,1,2,···, whereas the next chapter considers all orders, positive and negative, integer and noninteger. The present functions are, therefore, just special cases of the function to which Chapter 50 is devoted and they obey the same general rules. Nevertheless, it is appropriate to devote a separate chapter to $I_0(x)$, $I_1(x)$, $I_2(x)$, etc., because of their prime importance in applications. Moreover, this chapter will concentrate on $I_0(x)$ and $I_1(x)$ because, as detailed in Section 49:5, all other $I_n(x)$ functions can be expressed in terms of the first two family members.

49:1 NOTATION

The symbol $I_n(x)$ is standard for a modified Bessel function of order n and argument x. The name *hyperbolic Bessel function* is a common alternative, with *Basset function* occasionally encountered.

49:2 BEHAVIOR

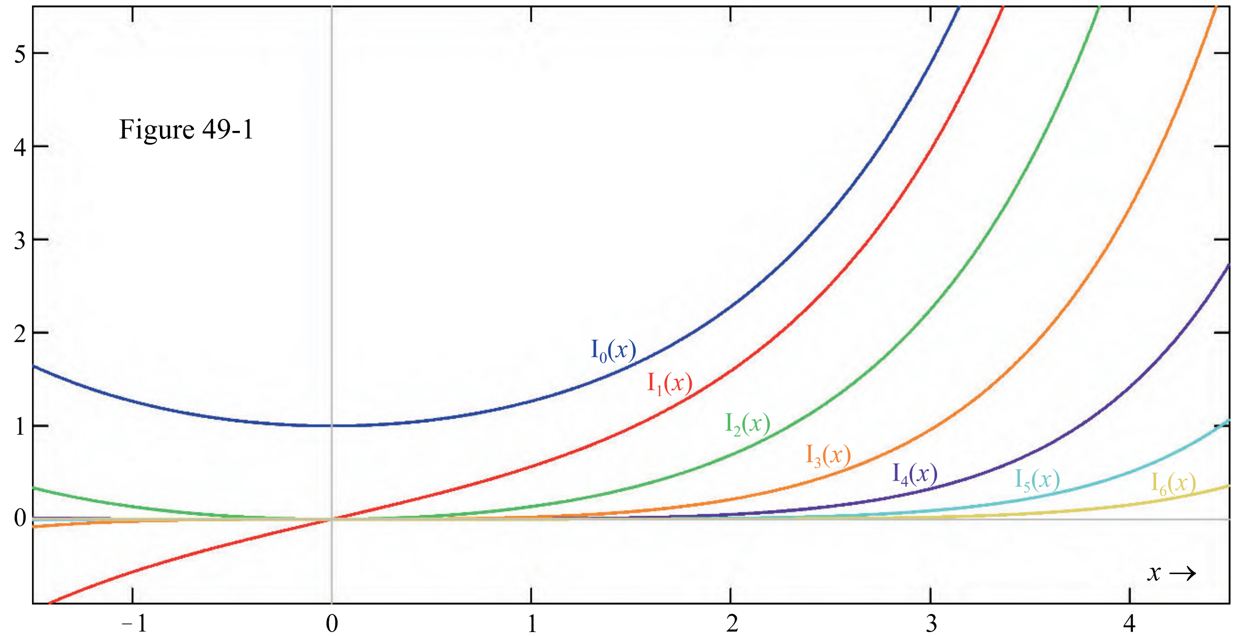
Figure 49-1 shows that modified Bessel functions of positive integer order increase rapidly with their argument x, but decrease steadily with order n.

In accord with 49:5:2, modified Bessel functions of negative integer order are identical with their positive-ordered counterparts and hence the former are not specifically mentioned in this chapter.

49:3 DEFINITIONS

All the definitions cited in Section 50:3 apply when v is a positive integer or zero. In addition, there are several definitions unique to integer orders of modified Bessel functions and especially to $I_0(x)$ and $I_1(x)$. Because of 49:5:2,

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_50,



the solutions given in 50:3:4−6 do not provide a *complete* solution to the differential equations reported there. For a complete solution when $\nu = n$, each $I_{-\nu}$ term in those solutions is to be replaced by the corresponding K_n, as in equations 51:3:7−9.

Modified Bessel functions of integer order may be defined through a number of generating functions, including

49:3:1 $$\exp\left(x\frac{1+t^2}{2t}\right) = \sum_{j=-\infty}^{\infty} I_j(x)t^j = I_0(x) + \sum_{j=1}^{\infty} I_j(x)\left[t^j + t^{-j}\right] = I_0(x) + I_1(x)\left[\frac{1+t^2}{t}\right] + I_2(x)\left[\frac{1+t^4}{t^2}\right] + \cdots$$

and

49:3:2 $$\exp\big(x\cos(\theta)\big) = \sum_{j=-\infty}^{\infty} I_j(x)\cos(j\theta) = I_0(x) + 2I_1(x)\cos(\theta) + 2I_2(x)\cos(2\theta) + \cdots$$

The integral definitions

49:3:3 $$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta)\exp\big(x\cos(\theta)\big)\mathrm{d}\theta = \frac{x^n}{(2n-1)!!\pi}\int_0^{\pi} \sin^{2n}(\theta)\exp\big(x\cos(\theta)\big)\mathrm{d}\theta$$

apply for all integer orders and, in addition, there are several, such as

49:3:4 $$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp\big(\pm x\sin(\theta)\big)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi} \cosh\big(x\sin(\theta)\big)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi} \cosh\big(x\cos(\theta)\big)\mathrm{d}\theta$$

and

49:3:5 $$I_1(x) = \frac{1}{\pi x}\int_0^{2x} \frac{(x-t)\exp(x-t)}{\sqrt{2xt-t^2}}\mathrm{d}t$$

that apply solely to the $n = 0$ or 1 cases. Definite integrals involving an angular integration variable, such as those in 49:3:4, may be adapted, as in

49:3:6 $$I_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cosh(xt)}{\sqrt{1-t^2}}\mathrm{d}t$$

and

49:3:7 $$\mathrm{I}_1(x) = \frac{2x}{\pi}\int_0^1 \sqrt{1-t^2}\cosh(xt)\,\mathrm{d}t$$

to a hyperbolic integrand.

Any modified Bessel function of positive integer order may be generated from its predecessor by a definite integration such as

49:3:8 $$\mathrm{I}_1(x) = x\int_0^1 t\,\mathrm{I}_0(xt)\,\mathrm{d}t, \qquad \mathrm{I}_2(x) = x\int_0^1 t^2\,\mathrm{I}_1(xt)\,\mathrm{d}t, \qquad \mathrm{I}_3(x) = x\int_0^1 t^3\,\mathrm{I}_2(xt)\,\mathrm{d}t, \qquad \text{etc.}$$

Modified Bessel functions of orders 0 and 1 are generated by the operations of semiintegration or semidifferentiation [Section 12:14] applied to functions involving exponentials [Chapter 26]

49:3:9 $$\mathrm{I}_0(x) = \frac{\exp(-x)}{\sqrt{\pi}}\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\frac{\exp(2x)}{\sqrt{x}}$$

49:3:10 $$\mathrm{I}_1(x) + \mathrm{I}_0(x) = \frac{\exp(-x)}{\sqrt{\pi}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\frac{\exp(2x)}{\sqrt{x}}$$

hyperbolic functions [Chapter 28]

49:3:11 $$\mathrm{I}_0\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sinh\left(\sqrt{x}\right) = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\frac{\cosh\left(\sqrt{x}\right)}{\sqrt{x}}$$

49:3:12 $$\mathrm{I}_1\left(\sqrt{x}\right) = \frac{1}{\sqrt{\pi x}}\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\sinh\left(\sqrt{x}\right) = 2\sqrt{\frac{x}{\pi}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\frac{\cosh\left(\sqrt{x}\right)}{\sqrt{x}}$$

or error functions [Chapter 40]

49:3:13 $$\mathrm{I}_0(x) = \frac{\exp(x)}{\sqrt{2}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\mathrm{erf}\left(\sqrt{2x}\right)$$

49:3:14 $$\mathrm{I}_1(x) = \frac{\exp(x)}{x}\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\frac{\mathrm{erf}\left(\sqrt{2x}\right)}{\sqrt{8}} - \frac{\sqrt{x}\exp(-2x)}{\sqrt{\pi}}\right\}$$

Modified Bessel functions of positive integer orders may be synthesized [Section 43:14] from the zero-order instance by

49:3:15 $$\mathrm{I}_0\left(2\sqrt{x}\right) \xrightarrow[n+1]{1} \frac{n!}{x^{n/2}}\mathrm{I}_n\left(2\sqrt{x}\right)$$

49:4 SPECIAL CASES

There are none. Be aware that, despite their integer subscripts, the modified spherical Bessel functions $\mathrm{i}_n(x)$ [Section 28:13] do *not* correspond to modified Bessel functions of integer order, but to those having an order equal to half an odd integer.

49:5 INTRARELATIONSHIPS

The argument-reflection formula

49:5:1 $$I_n(-x) = (-)^n I_n(x) \qquad n = 0, \pm1, \pm2, \cdots$$

shows that the modified Bessel functions are even or odd according to the parity of their integer orders, as is also evident in Figure 49-1. The order-reflection formula

49:5:2 $$I_{-n}(x) = I_n(x) \qquad n = 0, \pm1, \pm2, \cdots$$

asserts that modified Bessel functions of negative order merely duplicate their positive-ordered brethren, provided that the order is an integer.

Sufficient applications of the recursion formula

49:5:3 $$I_{n+1}(x) = \frac{-2n}{x} I_n(x) + I_{n-1}(x) \qquad n = 0, \pm1, \pm2, \cdots$$

permit any modified Bessel function of integer order to be expressed in terms of $I_0(x)$ and $I_1(x)$. The general formula may be written

49:5:4 $$I_n(x) = \mathrm{Wi}_n^{(0)}(x) I_0(x) + \mathrm{Wi}_n^{(1)}(x) I_1(x) \qquad n = 2, 3, 4, \cdots$$

where the Wi weighting functions are the polynomials are listed below that differ from the *Lommel polynomials*, discussed in Section 52:5 only by having different sign patterns.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\mathrm{Wi}_n^{(0)}(x)$	1	$\frac{-4}{x}$	$1+\frac{24}{x^2}$	$\frac{-12}{x}-\frac{192}{x^3}$	$1+\frac{144}{x^2}+\frac{1920}{x^4}$	$\frac{-24}{x}-\frac{1920}{x^3}-\frac{23040}{x^5}$
$\mathrm{Wi}_n^{(1)}(x)$	$\frac{-2}{x}$	$1+\frac{8}{x^2}$	$\frac{-8}{x}-\frac{48}{x^3}$	$1+\frac{72}{x^2}+\frac{384}{x^4}$	$\frac{-18}{x}-\frac{768}{x^3}-\frac{3840}{x^5}$	$1+\frac{288}{x^2}+\frac{9600}{x^4}+\frac{46080}{x^6}$

The right-hand side of the argument-multiplication formula

49:5:5 $$I_n(bx) = b^n \sum_{j=0}^{\infty} \left(\frac{b^2-1}{2} \right)^j \frac{x^j}{j!} I_{j+n}(x) \qquad n = 0, 1, 2, \cdots$$

is an infinite series. The multiplier b is unrestricted but setting $b = \sqrt{3}$ or $\sqrt{5}$ gives series that are particularly simple. Other sums of modified Bessel functions of integer orders are

49:5:6 $$\tfrac{1}{2} I_0(x) + I_2(x) + I_4(x) + I_6(x) + \cdots = \tfrac{1}{2}\cosh(x)$$

49:5:7 $$I_1(x) + I_3(x) + I_5(x) + \cdots = \tfrac{1}{2}\sinh(x)$$

and

49:5:8 $$\frac{I_2(x)}{2} + \frac{I_4(x)}{4} + \frac{I_6(x)}{6} + \cdots = \frac{K_0(x) + \left\{\gamma + \ln\left(\tfrac{1}{2}x\right)\right\} I_0(x)}{4}$$

and yet others arise by setting $t = 1$ or $t = -1$ in 49:3:1 or $\theta = \pi/2$ or π in 49:3:2. Euler's number γ [Chapter 1] occurs in formula 49:5:8, as does the zero-order case K_0 of the Macdonald function [Chapter 51].

49:6 EXPANSIONS

The power series, which may be written in terms of factorials or hypergeometrically, 49:6:1 $$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} \frac{\left(x^2/4\right)^j}{j!(j+n)!} = \frac{\left(\frac{1}{2}x\right)^n}{n!} \sum_{j=0}^{\infty} \frac{1}{(1)_j (n+1)_j} \left(\frac{x^2}{4}\right)^j \qquad n = 0,1,2,\cdots$$

converges with alacrity. For n = 0, 1, 2, and 3 the first few terms in the series are

$I_0(x)$	$I_1(x)$	$I_2(x)$	$I_3(x)$
$1+\frac{x^2}{4}+\frac{x^4}{64}+\frac{x^6}{2304}+\cdots$	$\frac{x}{2}+\frac{x^3}{16}+\frac{x^5}{384}+\frac{x^7}{18432}+\cdots$	$\frac{x^2}{8}+\frac{x^4}{96}+\frac{x^6}{3072}+\frac{x^8}{184320}+\cdots$	$\frac{x^3}{48}+\frac{x^5}{768}+\frac{x^7}{30720}+\cdots$

Asymptotic expansions for $I_n(x)$ follow by replacement of v in 50:6:2 by n. Each can be written as a hypergeometric function [Section 18:14] or in terms of factorials. The $n = 0$ and $n = 1$ cases are

49:6:2 $$I_0(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}\left[1+\frac{1}{8x}+\frac{9}{128x^2}+\cdots\right] = \frac{\exp(x)}{\sqrt{2\pi x}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{(1)_j} \left(\frac{1}{2x}\right)^j = \frac{\exp(x)}{\sqrt{2\pi x}} \sum_{j=0}^{\infty} \frac{\left[(2j)!\right]^2}{\left[j!\right]^3} \left(\frac{1}{32x}\right)^j$$

and

49:6:3 $$I_1(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}\left[1-\frac{3}{8x}-\frac{15}{128x^2}+\cdots\right] = \frac{\exp(x)}{\sqrt{2\pi x}} \sum_{j=0}^{\infty} \frac{\left(\frac{-1}{2}\right)_j \left(\frac{3}{2}\right)_j}{(1)_j} \left(\frac{1}{2x}\right)^j = \frac{\exp(x)}{\sqrt{2\pi x}} \sum_{j=0}^{\infty} \frac{\left[(2j+1)!!\right]^2}{j!(1-4j^2)} \left(\frac{1}{8x}\right)^j$$

being valid as $x \to \infty$.

49:7 PARTICULAR VALUES

All modified Bessel functions of integer order, except $I_0(x)$, take the value zero at $x = 0$ and have no other zeros. $I_0(x)$ equals unity at $x = 0$ and, when its argument is 2, equals the special number

49:7:1 $$I_0(2) = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right)^2 = 2.2795\ 85302\ 33607$$

49:8 NUMERICAL VALUES

An elegant method of calculating values of modified Bessel functions of integer order is based on the property that, for a given argument x, the ratio R_j of the values of two modified Bessel functions of successive orders and identical arguments obeys the very simple recursion

49:8:1 $$R_{j-1} = \frac{1}{R_j + (2j/x)} \qquad \text{where} \qquad R_j = \frac{I_{j+1}(x)}{I_j(x)}$$

This is a consequence of formula 49:5:3. Starting from the crude estimate $R_J \approx 2(J+1)/x$ of this ratio for a sufficiently large J, and employing 49:8:1 repeatedly, one may then develop the formula

49:8:2 $$I_n(x) = \frac{R_{n-1}R_{n-2}\cdots R_1 R_0 \exp(x)}{\left(\left(\left(\cdots\left((R_J+1)R_{J-1}+1\right)R_{J-2}+\cdots 1\right)R_2+1\right)R_1+1\right)2R_0+1}$$

to calculate $I_n(x)$. This formula, which is an adaptation of *Miller's method* [Section 52:8], has its basis in the $t = 1$ version of equation 49:3:1.

The Miller method is not employed by *Equator*. The modified Bessel function routine (keyword **I**) described in Section 50:8, which caters to both noninteger and integer orders, is used. The I_ν and I_n routines are identical.

49:9 LIMITS AND APPROXIMATIONS

All modified Bessel functions approach the same limit at large positive arguments

49:9:1
$$\mathrm{I}_n(x\to\infty)\to\frac{\exp(x)}{\sqrt{2\pi x}}$$

irrespective of their order, whereas they behave as power functions at small arguments

49:9:2
$$\mathrm{I}_n(x\to 0)\to\frac{1}{n!}\left(\frac{x}{2}\right)^n$$

49:10 OPERATIONS OF THE CALCULUS

Although all the formulas of Section 50:10 apply when $\nu = n$, the following differentiation and integration formulas are simpler than the general cases:

49:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{I}_0(x)=\mathrm{I}_1(x)$$

49:10:2
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{I}_1(x)=\mathrm{I}_0(x)-\frac{\mathrm{I}_1(x)}{x}$$

49:10:3
$$\int_0^x \mathrm{I}_0(t)\,\mathrm{d}t=\frac{\pi x}{2}\left[\mathrm{I}_0(x)\mathtt{l}_{-1}(x)-\mathrm{I}_1(x)\mathtt{l}_0(x)\right]$$

49:10:4
$$\int_0^x \mathrm{I}_1(t)\,\mathrm{d}t=\mathrm{I}_0(x)-1$$

49:10:5
$$\int_0^x t\,\mathrm{I}_0(t)\,\mathrm{d}t=x\,\mathrm{I}_1(x)$$

49:10:6
$$\int_0^x t\,\mathrm{I}_1(t)\,\mathrm{d}t=\frac{\pi x}{2}\left[\mathrm{I}_1(x)\mathtt{l}_0(x)-\mathrm{I}_0(x)\mathtt{l}_1(x)\right]$$

In these equations $\mathtt{l}_n$ represents a *modified Struve function* [Section 57:13].

An indefinite integral, specific to integer orders, is

49:10:7
$$\int_0^x \exp(-t)\,\mathrm{I}_n(t)\,\mathrm{d}t=\exp(-x)\left[(n+x)\mathrm{I}_0(x)+x\,\mathrm{I}_1(x)+2\sum_{j=1}^{n-1}(n-j)\,\mathrm{I}_j(x)\right]-n$$

Because the sum is empty if $n < 2$, the $n = 0$ and $n = 1$ cases of 49:10:7 are particularly simple.

The definite integrals listed in Section 50:10 apply when $\nu = n$, as do the differintegration formulas listed there. Special instances of the latter are the semioperations

49:10:8 $$\frac{d^{1/2}}{dx^{1/2}} I_0\left(\sqrt{x}\right) = \frac{\cosh\left(\sqrt{x}\right)}{\sqrt{\pi x}} \qquad \text{and} \qquad \frac{d^{-1/2}}{dx^{-1/2}} I_0\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}} \sinh\left(\sqrt{x}\right)$$

Among Laplace transforms of modified Bessel functions of integer order are:

49:10:9 $$\int_0^\infty I_0(bt)\exp(-st)\,dt = \mathfrak{L}\{I_0(bt)\} = \frac{1}{\sqrt{s^2-b^2}}$$

49:10:10 $$\int_0^\infty t^n I_n(bt)\exp(-st)\,dt = \mathfrak{L}\{t^n I_n(bt)\} = \frac{(2n-1)!!b^n}{\left(s^2-b^2\right)^{(2n+1)/2}}$$

49:11 COMPLEX ARGUMENT

Figure 49-2

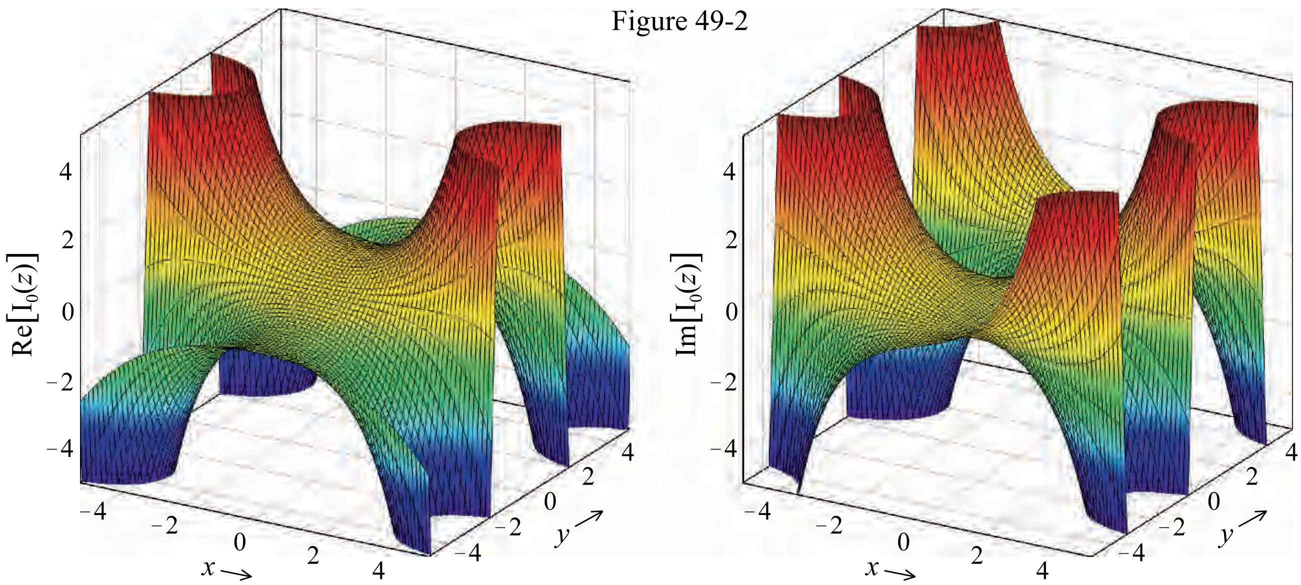


Figure 49-2 shows the real and imaginary parts of $I_0(z)$, where $z = x + iy$. Of course, taking sections through these three-dimensional graphs along the x-axis, where $y = 0$, gives the modified Bessel function of real argument. This implies

49:11:1 $$\mathrm{Re}\{I_0(x+0i)\} = I_0(x) \qquad \text{and} \qquad \mathrm{Im}\{I_0(x+0i)\} = 0$$

It is instructive, however, also to cut along the y-axis, where $x = 0$, because the resulting profile is that of the (unmodified) Bessel function [Chapter 52]

49:11:2 $$\mathrm{Re}\{I_0(0+iy)\} = J_0(y) \qquad \text{and} \qquad \mathrm{Im}\{I_0(0+iy)\} = 0$$

There is also interest in sectioning this figure diagonally, because this gives rise to Kelvin functions [Chapter 55]. Cutting along the $y = +x$ diagonal generates the following profiles

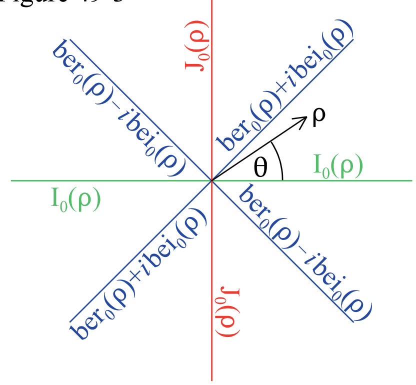
49:11:3 $$\mathrm{Re}\{I_0(x+ix)\} = \mathrm{ber}_0\left(\sqrt{2}x\right) \qquad \text{and} \qquad \mathrm{Im}\{I_0(x+ix)\} = \mathrm{bei}_0\left(\sqrt{2}x\right)$$

whereas along the other diagonal, the imaginary component has the opposite sign:

49:11:4 $$\mathrm{Re}\{I_0(x-ix)\} = \mathrm{ber}_0\left(\sqrt{2}x\right) \qquad \text{and} \qquad \mathrm{Im}\{I_0(x-ix)\} = -\mathrm{bei}_0\left(\sqrt{2}x\right)$$

The relationship between the various I, J, ber and bei cylinder functions, revealed for the zero order case in the equations above, may be further clarified by expressing the complex variable in polar $z = \rho\exp(i\theta)$, rather than in rectangular $z = x + iy$, coordinates. As illustrated diagrammatically in Figure 49-3, one then has

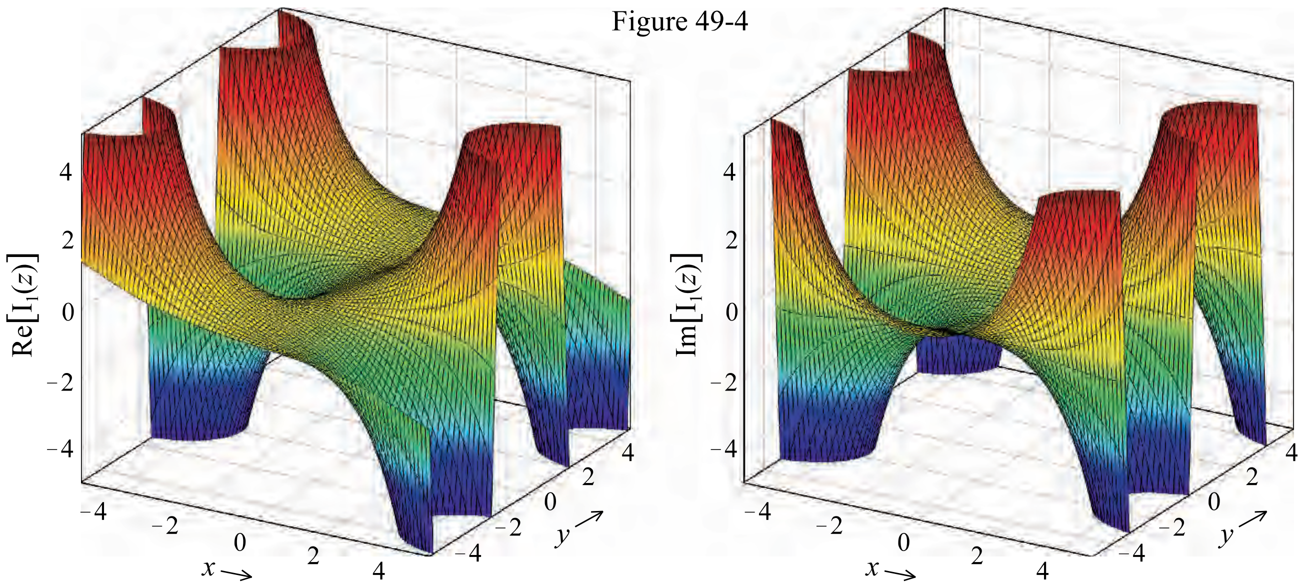
Figure 49-3



49:11:5 $$\mathrm{I}_0(z) = \mathrm{I}_0\big(\rho\exp(i\theta)\big) = \begin{cases} \mathrm{I}_0(\rho) & \theta = 0 \text{ or } \pi \\ \mathrm{ber}_0(\rho) + i\,\mathrm{bei}_0(\rho) & \theta = \frac{1}{4}\pi \text{ or } \frac{-3}{4}\pi \\ \mathrm{J}_0(\rho) & \theta = \pm\frac{1}{2}\pi \\ \mathrm{ber}_0(\rho) - i\,\mathrm{bei}_0(\rho) & \theta = \frac{3}{4}\pi \text{ or } \frac{-1}{4}\pi \end{cases}$$

Figure 49-4 shows the real and imaginary parts of $\mathrm{I}_1(x)$. Again, sectioning at right angles to the real axis gives rise to Bessel J functions, whereas scission at angles of 45° or 135° produces Kelvin ber_0 and bei_0 functions, though the relationships are somewhat different from the $n = 0$ case. The analogue of Figure 49-3 for the modified Bessel function of unity order is shown in the polar diagram Figure 49-5.

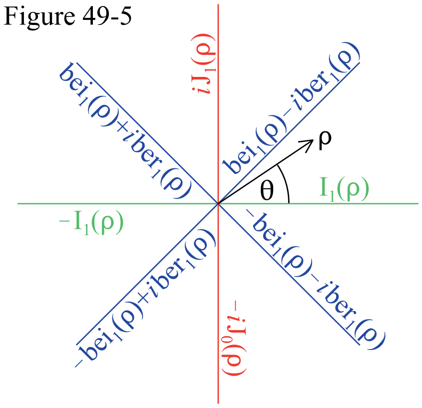
Figure 49-4



Properties of the general I_n function of complex argument may be deduced from equation 49:5:4 or from information in Section 50:11.

Interesting *window functions* [Section 9:13] arise from the inverse Laplace transformation of exponentially weighted modified Bessel functions of integer order. Two examples of these inverse transforms, which are nonzero for $0 < t < 2b$ but zero for $t > 2b$, are

Figure 49-5



49:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \exp(-bs)\mathrm{I}_0(bs)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\exp(-bs)\mathrm{I}_0(bs)\} = \frac{\mathrm{u}(t)-\mathrm{u}(t-2b)}{\pi\sqrt{t(2b-t)}}$$

and

49:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \exp(-bs)\,\mathrm{I}_1(bs)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\exp(-bs)\,\mathrm{I}_1(bs)\} = \frac{[\mathrm{u}(t)-\mathrm{u}(t-2b)](b-t)}{\pi b\sqrt{t(2b-t)}}$$

49:12 GENERALIZATIONS

The next chapter addresses modified Bessel functions $\mathrm{I}_\nu(x)$ of unrestricted order.

49:13 COGNATE FUNCTIONS

The modified Bessel functions of integer order are among the simplest of cylinder functions [Section 49:14], to all of which $\mathrm{I}_n(x)$ is closely related.

The modified Bessel function of zero order and argument $2\sqrt{x}$

49:13:1 $$\mathrm{I}_0\left(2\sqrt{x}\right) = \sum_{j=0}^{\infty}\frac{1}{(1)_j(1)_j}x^j$$

is one of the *hypergeometric basis functions* [Section 43:14]. As such, it is the progenitor of a large number of named, and an infinite number of unnamed, functions. The functions listed in Table 18-5 and 18-6 are examples of the functions that may be synthesized from $\mathrm{I}_0\left(2\sqrt{x}\right)$ or its counterpart $\mathrm{J}_0\left(2\sqrt{x}\right)$.

49:14 RELATED TOPIC: cylinder functions generally

Functions that satisfy either the differential equation known as *Bessel's equation*

49:14:1 $$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} - \left(\frac{\nu^2}{x^2} - 1\right)\mathrm{f} = 0$$

or that named *Bessel's modified equation*

49:14:2 $$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} - \left(\frac{\nu^2}{x^2} + 1\right)\mathrm{f} = 0$$

or certain very similar equations, are known collectively as *cylinder functions*, because they arise in the solution of problems involving cylindrical (and other) geometries. The name *Bessel functions* is also sometimes applied generically to this collection of functions, though not in this *Atlas*. All cylinder functions are bivariate, with an order ν and an argument x. Interest focuses on positive values of ν and x, though the definitions of some cylinder functions remain valid for negative arguments or orders. Rational orders, such as $\nu = 1$, $3/2$, or $1/3$ are of especial interest because of their frequent occurrence in practical applications.

The relationship between the functions that satisfy 49:14:1 and those that satisfy 49:14:2 is exactly like that between, say, $\cos(x)$ and $\cosh(x)$ and it would therefore be appropriate to use the adjectives "circular" and "hyperbolic" to distinguish the two groups of functions. In practice the term "circular" is seldom employed, whereas "hyperbolic" is in frequent use to designate solutions of 49:14:2 and was so used in the first edition of this *Atlas*. Another name is *Bessel functions of imaginary argument*. Most usually, however, the adjective "modified" is employed, and this is the preferred descriptor in this second edition of the *Atlas* and for *Equator*. Thus a certain solution of equation 49:14:1 is known generally as the *Bessel function*; whereas the *modified Bessel function* or *hyperbolic Bessel function* is the corresponding solution of 49:14:2.

Two solutions of the Bessel equation 49:14:1 are given the symbols $J_\nu(x)$ and $Y_\nu(x)$ and the names *Bessel function of the first kind* and *Bessel function of the second kind*. However, it is cumbersome to use such long names, so the unqualified name *Bessel function* is usually applied to the important $J_\nu(x)$ function, while $Y_\nu(x)$ is called the *Neumann function* (*Weber's function* is an alternative name and $N_\nu(x)$ is an alternative symbol). These two solutions have radically different properties near $x = 0$ and, on this basis, the Bessel function is described as "*regular*" and Neumann function as "*irregular*". For the most part, regular functions adopt the value zero when their argument is zero, whereas irregular functions often approach $\pm\infty$ in the $x \to 0$ limit.

As discussed in Section 24:14, second order differential equations can have only two "linearly independent" solutions. However, linear combinations of such solutions are also solutions of the differential equation and, in solving Bessel's equation, it is often convenient to use such combinations in preference to $J_\nu(x)$ and $Y_\nu(x)$. One such combination,

49:14:3 $$\cos(\nu\pi)J_\nu(x) - \sin(\nu\pi)Y_\nu(x) = J_{-\nu}(x)$$

provides a solution which corresponds to a J function of negative order, but which is inapplicable (because $J_{-\nu}(x)$ then merely duplicates $J_\nu(x)$ or its negative) when ν is an integer. Two other combinations, useful in certain applications, are the *Bessel function of the third kind* or the *Hankel function*

49:14:4 $$J_\nu(x) + iY_\nu(x) = H_\nu^{(1)}(x) \qquad \text{and} \qquad J_\nu(x) - iY_\nu(x) = H_\nu^{(2)}(x)$$

being known individually as *Hankel functions of the first and second kinds*.

The storyline is similar for the modified (or hyperbolic) family of functions. There is a regular solution $I_\nu(x)$, known as the *modified Bessel function of the first kind* or more simply as the *modified Bessel function*, as in earlier sections of this chapter. There is an irregular solution $K_\nu(x)$ described variously as the *modified Bessel function of the third kind*, *Bessel's function of the second kind of imaginary argument*, the *modified Hankel function*, the *Basset function* and the *Macdonald function*, the last being the name used exclusively in this edition of the *Atlas*. In the modified family, the combination that yields functions of negative order is

49:14:5 $$I_\nu(x) + \frac{2}{\pi}\sin(\nu\pi)K_\nu(x) = I_{-\nu}(x)$$

but is not the precise analogue of 49:14:3 because of the substantially different definitions of the Macdonald and Neumann functions.

In summary, the symbols and colloquial names adopted by this *Atlas* for the major cylinder functions are:

differential equation 49:14:1		differential equation 49:14:2	
regular solution	irregular solution	regular solution	irregular solution
$J_\nu(x)$ Bessel function [Chapters 52 and 53]	$Y_\nu(x)$ Neumann function [Chapter 54]	$I_\nu(x)$ modified Bessel function [Chapters 49 and 50]	$K_\nu(x)$ Macdonald function [Chapter 51]

Only when ν is half of an odd integer can solutions of equations 49:14:1 and 49:14:2 be expressed as functions simpler than cylinder functions. The names of such unusually simple functions incorporate the adjective "spherical" and use a lower case symbolism. Thus

49:14:6 $$j_n(x) = \sqrt{\frac{\pi}{2x}}\,J_{n+\frac{1}{2}}(x) \qquad \text{[see Section 32:13]}$$

49:14:7 $$y_n(x) = \sqrt{\frac{\pi}{2x}}\,Y_{n+\frac{1}{2}}(x) \qquad \text{[see Section 32:13]}$$

49:14:8 $$\mathrm{i}_n(x)=\sqrt{\frac{\pi}{2x}}\,\mathrm{I}_{n+\frac{1}{2}}(x) \qquad \text{[see Section 28:13]}$$

and

49:14:9 $$\mathrm{k}_n(x)=\sqrt{\frac{\pi}{2x}}\,\mathrm{K}_{n+\frac{1}{2}}(x) \qquad \text{[see Section 26:13]}$$

and these *spherical cylinder functions* are addressed in earlier chapters. As well, many of these spherical cylinder functions represent special cases of the Struve function [Chapter 57] and its modified counterpart. There are also *spherical Hankel functions*, $\mathrm{h}_n^{(1)}$ and $\mathrm{h}_n^{(2)}$, defined similarly but not discussed in this *Atlas*.

Cylinder functions of order $\frac{1}{3}$ are termed *Airy functions* and are addressed in Chapter 56. Besides providing solutions of instances of equation 49:14:1, these functions also solve the simpler differential equation

49:14:10 $$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}-x\,\mathrm{f}=0$$

known as *Airy's equation*.

Close cousins of the Bessel differential equations 49:14:1 and 49:14:2 are the two differential equations

49:14:11 $$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x}-\left(\frac{v^2}{x^2}\pm i\right)\mathrm{f}=0$$

which incorporate the imaginary i in place of unity. Solutions of these equations are known as *Kelvin functions* and are the subject of Chapter 55.

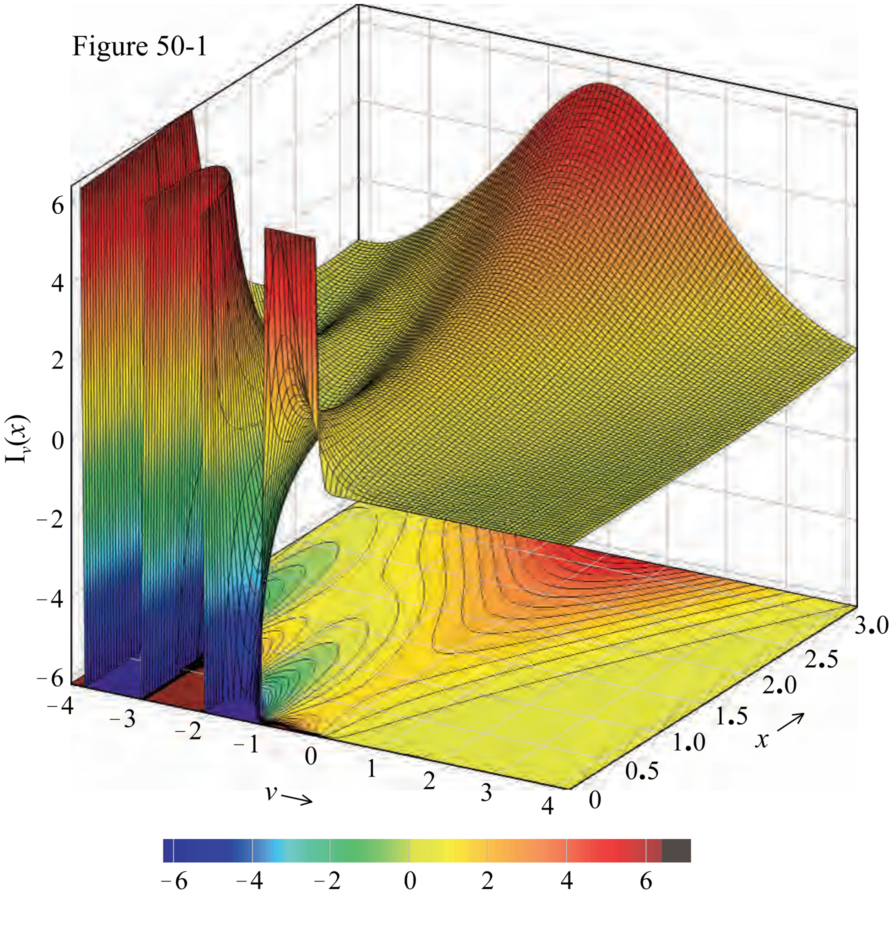
Not usually classified as cylinder functions, but closely related to them, are the *Struve functions* of Chapter 57. These participate in the solutions of inhomogeneous versions of equations 49:14:1 and 49:14:2; that is, analogues of these differential equations with nonzero right-hand sides.

CHAPTER 50

THE MODIFIED BESSEL FUNCTION $I_\nu(x)$ OF ARBITRARY ORDER

Because applications of the (unmodified) Bessel function predated and outnumber those of this chapter's function, the standard nomenclature treats the I function as subordinate to the J function, being designated as a "modified" version thereof. With equal, if not greater, validity one could regard the J function as a modification of the I function.

Figure 50-1



50:1 NOTATION

The names *hyperbolic Bessel function*, *Bessel function of imaginary argument*, and *Basset function* are alternative names. Sometimes the moniker *modified Bessel function of the first kind* is used.

The symbol $I_\nu(x)$ is standard, ν and x being termed the *order* and *argument* of the function. See Section 28:13 for the $i_\nu(x)$ symbol.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_51,

50:2 BEHAVIOR

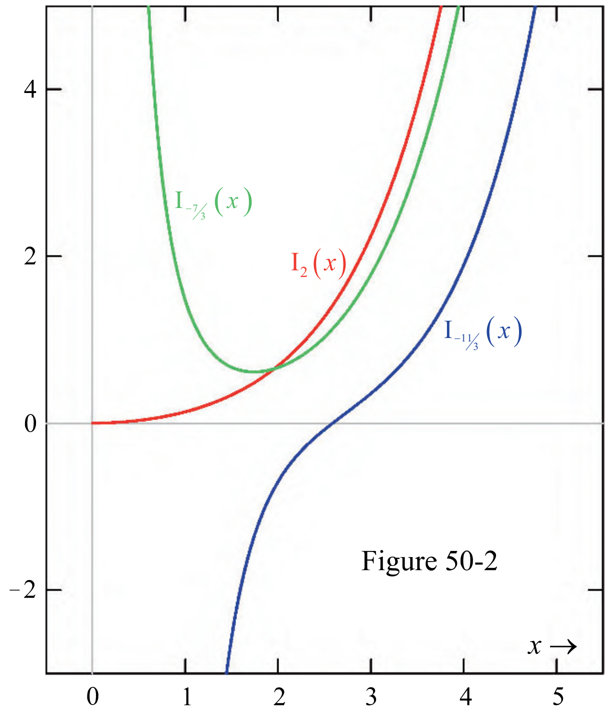


Figure 50-2

Unless v is an integer, the modified Bessel function $I_v(x)$ is complex when its argument is negative. Accordingly, x is treated as positive in most of this chapter and that is the only region depicted in Figures 50-1 and 50-2.

Irrespective of the order v, the modified Bessel function rises steeply as x increases through more positive values. Closer to $x = 0$, the behavior of $I_v(x)$ falls into several categories as detailed in Section 50:7. One example from each of the three main categories is shown in Figure 50-2. For $v > 0$, such as the $v = 2$ case graphed, the function increases monotonically with x from $I_v(0) = 0$. For orders in the ranges $-1 < v < 0$, $-3 < v < -2$, $-5 < v < -4$, etc., such as the $v = -7/3$ case graphed, $I_v(0) = +\infty$, which implies that the modified Bessel function must display a minimum when viewed as a function of x; such minimal values are positive and often occur not far from $x = 1$. For $-2 < v < -1$, $-4 < v < -3$, $-6 < v < -5$, etc., such as the $v = -11/3$ case graphed, $I_v(0) = -\infty$, and the modified Bessel function is once again a monotonic function of x, stretching in value from $-\infty$ to ∞, with a single zero but without an extremum.

Unless v is an integer, $I_v(x)$ is complex for negative x, whereas this is not the case for the composite function $I_v(x)/x^v$. This function equals $2^{-v}/\Gamma(1+v)$ at $x = 0$, and remains real for all real arguments. Being even, its properties for negative x are available through the reflection formula

50:2:1
$$\frac{I_v(-x)}{(-x)^v} = \frac{I_v(x)}{x^v}$$

50:3 DEFINITIONS

The following integrals define the modified Bessel function, though only for $v > -\tfrac{1}{2}$:

50:3:1
$$\frac{\sqrt{\pi}\,\Gamma(v+\tfrac{1}{2})}{(x/2)^v} I_v(x) = \int_{-1}^{1} (1-t^2)^{(2v-1)/2} \exp(\pm xt)\,\mathrm{d}t = \int_{-1}^{1} (1-t^2)^{(2v-1)/2} \cosh(xt)\,\mathrm{d}t$$

50:3:2
$$\frac{\sqrt{\pi}\,\Gamma(v+\tfrac{1}{2})}{(x/2)^v} I_v(x) = \int_0^{\pi} \sin^{2v}(\theta) \exp\{\pm x\cos(\theta)\}\,\mathrm{d}\theta = \int_0^{\pi} \sin^{2v}(\theta) \cosh\{x\cos(\theta)\}\,\mathrm{d}\theta$$

A Kummer function [Chapter 47] reduces to a modified Bessel function whenever the former function's parameters are in a one-to-two ratio:

50:3:3
$$\frac{(x/2)^v \exp(-x)}{\Gamma(1+v)} M\left(\tfrac{1}{2}+v, 1+2v, 2x\right) = I_v(x)$$

As well, a modified Bessel function may be defined through its complex algebraic relationship to an (unmodified) Bessel function as explained in Section 50:11, or via its series expansion 50:6:1.

Below are listed three differential equations and their solutions:

50:3:4
$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}-(x^2+\nu^2)f=0 \qquad f=w_1 I_\nu(x)+w_2 I_{-\nu}(x)$$

50:3:5
$$x\frac{d^2f}{dx^2}+(2\nu+1)\frac{df}{dx}-xf=0 \qquad f=\frac{w_1}{x^\nu}I_\nu(x)+\frac{w_2}{x^\nu}I_{-\nu}(x)$$

50:3:6
$$x\frac{d^2f}{dx^2}+(\nu+1)\frac{df}{dx}-f=0 \qquad f=\frac{w_1}{\left(2\sqrt{x}\right)^\nu}I_\nu\left(2\sqrt{x}\right)+\frac{w_2}{\left(2\sqrt{x}\right)^\nu}I_{-\nu}\left(2\sqrt{x}\right)$$

These differential equations, and several others, serve to define the modified Bessel function, but the listed solutions are incomplete when ν is an integer. See Section 49:3 for those cases.

The synthetic operations discussed in Section 43:14 can generate any modified Bessel function from the zero-order prototype:

50:3:7
$$I_0\left(2\sqrt{x}\right) \xrightarrow[\nu+1]{1} \frac{\Gamma(1+\nu)}{(x/2)^\nu}I_\nu\left(2\sqrt{x}\right)$$

50:4 SPECIAL CASES

Modified Bessel functions of integer order are the subject of Chapter 49.

When the order is an odd multiple of ±½, modified Bessel functions are termed *spherical* and they reduce to hyperbolic cosines or sines, as discussed in Section 28:13.

Modified Bessel functions of order $\pm\frac{1}{3}$ are closely related to the Airy functions of Chapter 56. The relationship

50:4:1
$$I_{\pm1/3}(x)=\frac{1}{2}\sqrt{\frac{3}{\hat{x}}}\left[\mathrm{Bi}(\hat{x})\mp\sqrt{3}\,\mathrm{Ai}(\hat{x})\right] \qquad \hat{x}=\left(\frac{3x}{2}\right)^{2/3}$$

is obeyed. When the order is $\pm\frac{2}{3}$, the corresponding formula involves the derivatives of the Airy functions

50:4:2
$$I_{\pm2/3}(x)=\frac{\sqrt{3}}{2\hat{x}}\left[\frac{d}{d\hat{x}}\mathrm{Bi}(\hat{x})\mp\sqrt{3}\frac{d}{d\hat{x}}\mathrm{Ai}(\hat{x})\right] \qquad \hat{x}=\left(\frac{3x}{2}\right)^{2/3}$$

These formulas involve a radical change in argument.

A similar change in argument is sometimes helpful in expressing a modified Bessel function as a weighted sum of two or more parabolic cylinder functions [Chapter 46]. Such expressions exist whenever the order is a positive or negative odd multiple of ¼. The simplest cases are

50:4:3
$$I_{\pm1/4}(x)=\frac{D_{-1/2}(-x')\mp D_{-1/2}(x')}{\sqrt{\pi x'}}$$

50:4:4
$$I_{\pm3/4}(x)=\frac{\mp2D_{1/2}(x')+2D_{1/2}(-x')\pm x'D_{-1/2}(x')+x'D_{-1/2}(-x')}{\sqrt{\pi(x')^3}}$$

$$x'=2\sqrt{x}$$

and others may be constructed via recursion 50:5:2.

50:5 INTRARELATIONSHIPS

A modified Bessel function of one order may be converted to another order by the synthetic process

50:5:1 $$\frac{\Gamma(v+1)}{(x/2)^v}\mathrm{I}_v\left(2\sqrt{x}\right) \xrightarrow[v']{v} \frac{\Gamma(v'+1)}{(x/2)^{v'}}\mathrm{I}_{v'}\left(2\sqrt{x}\right)$$

as prescribed in Section 43:14. As well, the recursion formula

50:5:2 $$\mathrm{I}_{v+1}(x) = \mathrm{I}_{v-1}(x) - \frac{2v}{x}\mathrm{I}_v(x)$$

may be applied to change the order of a modified Bessel function by an integer increment of decrement.

The two functions $\mathrm{I}_v(x)$ and $\mathrm{I}_{-v}(x)$ are identical if v is an integer; otherwise

50:5:3 $$\mathrm{I}_{-v}(x) = \mathrm{I}_v(x) + \frac{2\sin(v\pi)}{\pi}\mathrm{K}_v(x)$$

where $\mathrm{K}_v(x)$ is the Macdonald function of Chapter 51. This equation may be regarded as an order-reflection formula. There are similar order-reflection formulas, namely

50:5:4 $$\mathrm{I}_{v-\frac{1}{2}}(x)\mathrm{I}_{-v-\frac{1}{2}}(x) = \mathrm{I}_{\frac{1}{2}-v}(x)\mathrm{I}_{v+\frac{1}{2}}(x) + \frac{2\cos(v\pi)}{\pi x}$$

and

50:5:5 $$\mathrm{I}_{-v}(x)\mathrm{I}_{v\pm1}(x) = \mathrm{I}_v(x)\mathrm{I}_{-v\mp1}(x) + \frac{2\sin(v\pi)}{\pi x}$$

for the product of two modified Bessel functions whose orders sum to ±1.

On setting $b = i = \sqrt{-1}$, the argument-multiplication formula

50:5:6 $$\frac{\mathrm{I}_v(bx)}{b^v} = \sum_{j=0}^{\infty}\frac{\mathrm{I}_{j+v}(x)}{j!}\left(\frac{(b^2-1)x}{2}\right)^j$$

generates the summation

50:5:7 $$\mathrm{I}_v(x) - \frac{x}{1!}\mathrm{I}_{v+1}(x) + \frac{x^2}{2!}\mathrm{I}_{v+2}(x) - \cdots = \sum_{j=0}^{\infty}\frac{(-x)^j}{j!}\mathrm{I}_{j+v}(x) = \frac{\mathrm{I}_v(ix)}{i^v} = \mathrm{J}_v(x)$$

whereas setting $b = \sqrt{3}$ leads to a similar series, but lacking the alternating signs.

50:5:8 $$\mathrm{I}_v(x) + \frac{x}{1!}\mathrm{I}_{v+1}(x) + \frac{x^2}{2!}\mathrm{I}_{v+2}(x) + \cdots = \sum_{j=0}^{\infty}\frac{x^j}{j!}\mathrm{I}_{j+v}(x) = \frac{\mathrm{I}_v\left(\sqrt{3}x\right)}{3^{v/2}}$$

Replacing b in 50:5:6 by $1+(y/x)$ converts that equation into an argument-addition formula, though not a very useful one.

50:6 EXPANSIONS

The convergent power series expansion for the modified Bessel function

50:6:1 $$\mathrm{I}_v(x) = \frac{(x/2)^v}{\Gamma(1+v)} + \frac{(x/2)^{2+v}}{1!\Gamma(2+v)} + \frac{(x/2)^{4+v}}{2!\Gamma(3+v)} + \cdots = \sum_{j=0}^{\infty}\frac{(x/2)^{2j+v}}{j!\Gamma(j+1+v)} = \frac{(x/2)^v}{\Gamma(1+v)}\sum_{j=0}^{\infty}\frac{1}{(1+v)_j(1)_j}\left(\frac{x^2}{4}\right)^j$$

has a hypergeometric formulation. There is also the asymptotic series, valid for large x

50:6:2 $$\mathrm{I}_v(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}\left[1 + \frac{\frac{1}{4}-v^2}{2x} + \frac{\left(\frac{9}{4}-v^2\right)\left(\frac{1}{4}-v^2\right)}{8x^2} + \frac{\left(\frac{25}{4}-v^2\right)\left(\frac{9}{4}-v^2\right)\left(\frac{1}{4}-v^2\right)}{48x^3} + \cdots\right] = \frac{\exp(x)}{\sqrt{2\pi x}}\sum_{j=0}\frac{\left(\frac{1}{2}-v\right)_j\left(\frac{1}{2}+v\right)_j}{j!(2x)^j}$$

This series terminates when $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ but the expansion is not exact even then. See Section 54:14 for more information about this series and its relation to the *auxiliary cylinder functions* $\mathrm{fc}_\nu(x)$ and $\mathrm{gc}_\nu(x)$.

Equation 50:5:7 shows how a J Bessel function may be expressed as a series of I's; the converse is also possible:

50:6:3
$$I_\nu(x) = J_\nu(x) + \frac{x}{1!}J_{\nu+1} + \frac{x^2}{2!}J_{\nu+2} + \cdots = \sum_{j=0}^{\infty}\frac{x^j}{j!}J_{j+\nu}(x)$$

50:7 PARTICULAR VALUES

The panel below, in which $m = 1, 2, 3, \cdots$, shows the varied values adopted by $I_\nu(x)$ at $x = 0$:

	$\nu > 0$	$\nu = 0$	$-2m+1 < \nu < -2m+2$	$\nu = \pm m$	$-2m < \nu < -2m+1$
$I_\nu(0) =$	0	1	$+\infty$	0	$-\infty$

When $-2 < \nu < -1$, the modified Bessel function encounters a zero somewhere in the range $0 < x < 1.4$, but we know of no simple formula expressing the location of such a zero, except for the zero of $I_{-3/2}(x)$, which occurs at the value, close to 1.1997, of x such that $\coth(x) = x$. Similar zeros occur whenever ν lies between any two consecutive negative integers, the more negative being even.

On the other hand, when ν lies between two consecutive nonpositive integers, the more negative of which is odd, the modified Bessel function encounters no zero but rather a minimum. The minimum of $I_{-1/2}(x)$ occurs where $\coth(x) = 2x$.

50:8 NUMERICAL VALUES

With keyword **I**, *Equator*'s modified Bessel function routine relies exclusively on equation 50:6:1. Negative arguments are not accepted unless ν is an integer.

50:9 LIMITS AND APPROXIMATIONS

Irrespective of its order, the modified Bessel function approaches the same limit

50:9:1
$$I_\nu(x) \to \frac{\exp(x)}{\sqrt{2\pi x}} \qquad x \to \infty$$

for large positive argument. For small positive arguments, however, the order has a dominant influence on the function's magnitude, as the limiting operation

50:9:2
$$I_\nu(x) \to \frac{(x/2)^\nu}{\Gamma(1+\nu)} \qquad x \to 0$$

demonstrates.

50:10 OPERATIONS OF THE CALCULUS

The derivative of the modified Bessel function may be expressed in several ways:

50:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{I}_\nu(x) = \frac{\mathrm{I}_{\nu+1}(x)+\mathrm{I}_{\nu-1}(x)}{2} = \mathrm{I}_{\nu-1}(x) - \frac{\nu}{x}\mathrm{I}_\nu(x) = \mathrm{I}_{\nu+1}(x) + \frac{\nu}{x}\mathrm{I}_\nu(x)$$

Other differentiation formulas include the pleasingly symmetrical result

50:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{\pm\nu}\,\mathrm{I}_\nu(x)\right\} = x^{\pm\nu}\,\mathrm{I}_{\nu\mp1}(x)$$

and *Rayleigh's formula*

50:10:3 $$\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left\{x^{\pm\nu}\,\mathrm{I}_\nu(x)\right\} = x^{\pm\nu-n}\,\mathrm{I}_{\nu\mp n}(x) \qquad n=1,2,3,\cdots$$

Changing the argument x in Rayleigh's formula to $2\sqrt{x}$ leads to an expression for multiple differentiation of $x^{\nu/2}\,\mathrm{I}_\nu\left(2\sqrt{x}\right)$ that can be generalized to

50:10:4 $$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\left\{x^{\nu/2}\,\mathrm{I}_\mu\left(2\sqrt{x}\right)\right\} = x^{(\nu-\mu)/2}\,\mathrm{I}_{\nu-\mu}\left(2\sqrt{x}\right)$$

where μ is not necessarily a positive integer.

Formulas for indefinite integrals of the modified Bessel function include

50:10:5 $$\int_0^x \mathrm{I}_\nu(t)\,\mathrm{d}t = 2\left[\mathrm{I}_{\nu+1}(x) - \mathrm{I}_{\nu+3}(x) + \mathrm{I}_{\nu+5}(x) - \cdots\right] = 2\sum_{j=0}^{\infty}(-)^j\,\mathrm{I}_{\nu+1+2j}(x) \qquad \nu>-1$$

50:10:6 $$\int_0^x t^\nu\,\mathrm{I}_\nu(t)\mathrm{d}t = 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\tfrac{1}{2}+\nu\right)x\left[\mathrm{I}_\nu(x)\mathrm{L}_{\nu-1}(x) - \mathrm{I}_{\nu-1}(x)\mathrm{L}_\nu(x)\right] \qquad \nu>-\tfrac{1}{2}$$

50:10:7 $$\int_0^x t^{\nu+1}\,\mathrm{I}_\nu(t)\mathrm{d}t = x^{\nu+1}\,\mathrm{I}_{\nu+1}(x) \qquad \nu>-1$$

50:10:8 $$\int_0^x t^{1-\nu}\,\mathrm{I}_\nu(t)\,\mathrm{d}t = x^{1-\nu}\,\mathrm{I}_{\nu-1}(x) - \frac{2^{1-\nu}}{\Gamma(\nu)}$$

and

50:10:9 $$\int_0^x t^{-\nu}\exp(\pm t)\,\mathrm{I}_\nu(t)\,\mathrm{d}t = \frac{x^{1-\nu}\exp(\pm x)}{2\nu-1}\left[\pm\mathrm{I}_{\nu-1}(x) - \mathrm{I}_\nu(x)\right] \mp \frac{2^{1-\nu}}{(2\nu-1)\Gamma(\nu)} \qquad \nu\neq\tfrac{1}{2}$$

Formula 50:10:6 involves the modified Struve function from Chapter 57.

Among definite integrals and Laplace transforms are

50:10:10 $$\int_0^1 t^{1+\nu}\,\mathrm{I}_\nu(xt)\,\mathrm{d}t = \frac{\mathrm{I}_{1+\nu}(x)}{x} \qquad \nu>-1$$

50:10:11 $$\int_0^\infty \exp(-at^2)\,\mathrm{I}_\nu(bt)\,\mathrm{d}t = \sqrt{\frac{\pi}{4a}}\exp\left(\frac{b^2}{8a}\right)\mathrm{I}_{\nu/2}\left(\frac{b^2}{8a}\right)$$

50:10:12 $$\int_0^\infty \mathrm{I}_\nu(bt)\exp(-st)\,\mathrm{d}t = \mathcal{L}\left\{\mathrm{I}_\nu(bt)\right\} = \frac{\left(s-\sqrt{s^2-b^2}\right)^\nu}{b^\nu\sqrt{s^2-b^2}}$$

50:10:13 $$\int_0^\infty \exp(-at)\,I_\nu(bt)\exp(-st)\,dt = \mathfrak{L}\{\exp(-at)\,I_\nu(bt)\} = \frac{\left[\sqrt{s+a+b}-\sqrt{s+a-b}\right]^{2\nu}}{(2b)^\nu\sqrt{(s+a)^2-b^2}} \qquad \nu > -1$$

The derivative of the modified Bessel function $I_\nu(x)$ with respect to its order is a function of both ν and x:

50:10:14 $$\frac{\partial}{\partial\nu}I_\nu(x) = \ln\left(\frac{x}{2}\right)I_\nu(x) - \left(\frac{x}{2}\right)^\nu \sum_{j=0}^\infty \frac{\psi(\nu+j+1)}{j!\Gamma(\nu+j+1)}\left(\frac{x^2}{4}\right)^j = \sum_{j=0}^\infty \frac{\ln(x/2)-\psi(\nu+j+1)}{j!\Gamma(\nu+j+1)}\left(\frac{x}{2}\right)^{2j+\nu}$$

This formula plays an important role in the definition of the Macdonald function [Section 51:3]. The form adopted by this ν-derivative when $\nu = 0$, namely

50:10:15 $$\left[\frac{\partial}{\partial\nu}I_\nu(x)\right]_{\nu=0} = K_0(x)$$

is of special interest.

50:11 COMPLEX ARGUMENT

Not only is the modified Bessel function generally complex-valued when its argument is complex but also, unless the order is an integer, when it is negative. This is discussed in the context of equation 50:2:1, from which it follows that

50:11:1 $$I_\nu(-x) = (-)^\nu I_\nu(x) = [\cos(\nu\pi) + i\sin(\nu\pi)]I_\nu(x)$$

The second equality is a consequence of de Moivre's theorem, equation 12:11:1. The same theorem permits the modified Bessel function of complex argument to be defined by replacing the x in definition 50:6:1 by z and making the development [see Section 12:11]

50:11:2 $$z^{2j+\nu} = (x+iy)^{2j+\nu} = \rho^{2j+\nu}\exp\{i(2j+\nu)\theta\} = \rho^{2j+\nu}\begin{bmatrix}\cos\{(2j+\nu)\theta\} \\ +i\sin\{(2j+\nu)\theta\}\end{bmatrix} \qquad \begin{cases}\rho = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) + \frac{\pi}{2} - \frac{\pi}{2}\operatorname{sgn}(x)\end{cases}$$

whereby the definitions of the real and imaginary parts are found to be

50:11:3 $$\operatorname{Re}\{I_\nu(z)\} = \sum_{j=0}^\infty \frac{\cos\{(2j+\nu)\theta\}}{j!\Gamma(j+1+\nu)}\left(\frac{\rho}{2}\right)^{2j+\nu} \qquad \operatorname{Im}\{I_\nu(z)\} = \sum_{j=0}^\infty \frac{\sin\{(2j+\nu)\theta\}}{j!\Gamma(j+1+\nu)}\left(\frac{\rho}{2}\right)^{2j+\nu}$$

To prevent these parts having multiple values, the complex plane is cut along the negative branch of the real axis.

The name "Bessel function of imaginary argument" often given to $I_\nu(x)$ is a misnomer, being literally correct only when ν is a multiple of 4. For real arguments, the relationships between the I and J functions are

50:11:4 $$I_\nu(x) = i^{-\nu}J_\nu(ix) \qquad \text{or} \qquad J_\nu(x) = i^\nu I_\nu(-ix)$$

When their arguments are complex, the relationship between the two functions may be thought of as a rotation in the complex plane, as exemplified in Section 49:11 for the $\nu = 0$ and 1 cases. To ensure that the functions have single values only, restrictions need to be made that lead to the following relationships between the Bessel function and the modified Bessel function of complex argument

50:11:5 $$I_\nu(z) = \begin{cases}(-i)^\nu J_\nu(iz) & -\pi < \theta \le \frac{1}{2}\pi \\ i^\nu J_\nu(-iz) & \frac{1}{2}\pi < \theta \le \pi\end{cases}$$

where θ is the phase of z defined as in 50:11:2.

The real and imaginary parts of the modified Bessel function of a complex argument that has equal real and imaginary components serve to define two of the Kelvin functions [Chapter 55]

50:11:6 $$\mathrm{I}_\nu\left(\frac{x}{\sqrt{2}}+\frac{ix}{\sqrt{2}}\right)=\frac{\mathrm{ber}_\nu(x)+i\,\mathrm{bei}_\nu(x)}{i^\nu}$$

where i^ν is interpretable as $\cos(\nu\pi/2)+i\sin(\nu\pi/2)$.

An example of an inverse Laplace transform involving a modified Bessel function of arbitrary order is

50:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\exp(-b/s)\mathrm{I}_\nu(b/s)}{s}\frac{\exp(ts)}{2\pi i}\mathrm{d}s=\mathfrak{I}\left\{\frac{\exp(-b/s)\mathrm{I}_\nu(b/s)}{s}\right\}=\mathrm{J}_\nu^2\left(\sqrt{2bt}\right)$$

50:12 GENERALIZATIONS

Inasmuch as they solve a differential equation that differs from Bessel's modified equation 50:3:4 only in having a nonzero right-hand side, the modified Struve functions [Section 57:13] may be regarded as a generalization of the modified Bessel function.

50:13 COGNATE FUNCTIONS

There are familial relationships between $\mathrm{I}_\nu(x)$ and all the cylinder functions [Chapters 49 – 57]. Note in particular,

50:13:1 $$\mathrm{K}_\nu(x)=\frac{\pi[\mathrm{I}_{-\nu}(x)-\mathrm{I}_\nu(x)]}{2\sin(\nu\pi)}\qquad \nu\neq 0,\pm1,\pm2,\cdots$$

50:13:2 $$\mathrm{J}_\nu(x)=i^\nu\,\mathrm{I}_\nu(-ix)$$

and

50:13:3 $$\mathrm{Ai}(x)=\frac{\sqrt{x}}{3}\left[\mathrm{I}_{-1/3}\left(\frac{2}{3}x^{3/2}\right)-\mathrm{I}_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right]\qquad x>0$$

as well as 50:11:6.

CHAPTER 51

THE MACDONALD FUNCTION $K_\nu(x)$

The Macdonald function (Hector Munro Macdonald, Scottish mathematician, 1865 – 1935) provides a distinct solution to Bessel's modified differential equation, 51:3:7, applicable for all orders. Because a Macdonald function of noninteger order is related so simply, via the equation

51:0:1 $$\mathrm{K}_\nu(x) = \frac{\pi\left[\mathrm{I}_{-\nu}(x) - \mathrm{I}_\nu(x)\right]}{2\sin(\nu\pi)} \qquad \nu \neq 0, \pm1, \pm2, \cdots$$

to the functions of the preceding chapter, this chapter concentrates on the Macdonald functions of integer order, and particularly on the most important of these, $\mathrm{K}_0(x)$ and $\mathrm{K}_1(x)$.

51:1 NOTATION

In addition to the golden arches, the symbol associated with a Macdonald function is $\mathrm{K}_\nu(x)$, where ν is the order and x the argument. Alternative names include the *modified Bessel function of the second kind*, the *hyperbolic Bessel function of the third kind*, the *Bessel function of the second kind of imaginary argument*, the *Basset function*, and the *modified Hankel function*. The first edition of this *Atlas* used the "Basset function" name.

Avoid confusion with the unsubscripted K( ) symbol, which is used for the complete elliptic integral [Chapter 61]. See Section 51:13 for $\mathrm{Ki}_\nu(x)$. The $\mathrm{k}_n(x)$ notation for spherical Macdonald functions is explained in Section 51:4 below.

51:2 BEHAVIOR

The Macdonald function is infinite when $x = 0$, and generally complex for $x < 0$. Accordingly, we restrict attention here and throughout most of the chapter to $x > 0$.

For positive argument x and positive order ν, $\mathrm{K}_\nu(x)$ is a monotonically decreasing function of x and a monotonically increasing function of ν. The dependence on ν, however, slackens as x increases and disappears in the $x \to \infty$ limit. Some integer-ordered examples are portrayed in Figure 51-1; curves for noninteger orders interpolate smoothly between those shown. In accord with equation 51:5:1, Macdonald functions of negative order duplicate their positive-ordered counterparts exactly, and therefore will seldom be addressed specifically.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_52,

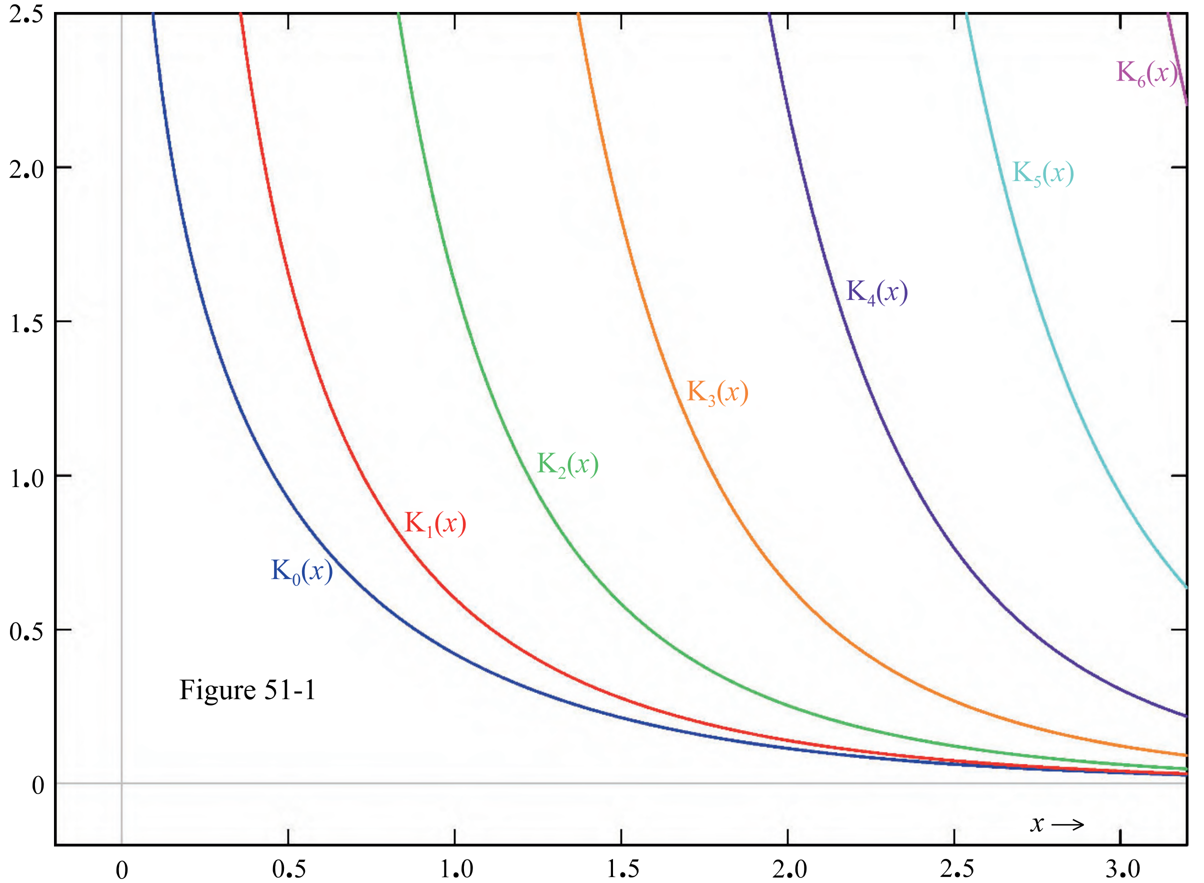


Figure 51-1

51:3 DEFINITIONS

A Tricomi function [Chapter 48] in which the *a* parameter is a moiety of the *c* parameter provides one definition of a Macdonald function

51:3:1 $$K_\nu(x) = \frac{\sqrt{\pi}\,(2x)^\nu}{\exp(x)} U\left(\tfrac{1}{2}+\nu, 1+2\nu, 2x\right) \qquad \nu \ge 0$$

and another is provided by the approach to infinity of the *a* parameter of a Tricomi function with a suitable argument:

51:3:2 $$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \lim_{a\to\infty}\left\{\Gamma(a-\nu)\,U\left(a, 1+\nu, \frac{x^2}{4a}\right)\right\} \qquad \nu \ge 0$$

Among the many integral representations of the Macdonald function assembled by Bateman and listed by Erdélyi et al. [*Higher Transcendental Functions*, Vol 2, pages 82−83] are

51:3:3 $$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\nu\right)}\left(\frac{x}{2}\right)^\nu \int_1^\infty \frac{\exp(-xt)}{\left(t^2-1\right)^{(1-2\nu)/2}}\,dt \qquad \nu > -\tfrac{1}{2}$$

and

51:3:4 $$K_\nu(x) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}(2x)^\nu \int_0^\infty \frac{\cos(t)}{\left(t^2+x^2\right)^{(2\nu+1)/2}}\,dt \qquad \nu > -\tfrac{1}{2}$$

Although each of definitions 51:3:1−4 imposes a lower limit on the order ν, this is of no practical consequence because of relation 51:5:1. In each definition ν could be replaced by $|\nu|$, effectively removing the restriction.

A Macdonald function is related to two modified Bessel functions through the equation

51:3:5 $$\mathrm{K}_\nu(x) = \frac{\pi}{2}\csc(\nu\pi)\left[\mathrm{I}_{-\nu}(x) - \mathrm{I}_\nu(x)\right] \qquad \nu \neq 0, \pm1, \pm2, \cdots$$

but because the right-hand member of this definition becomes indeterminate when ν is an integer, a limit must be introduced in those cases:

51:3:6 $$\mathrm{K}_n(x) = \frac{\pi}{2}\lim_{\nu\to n}\left\{\csc(\nu\pi)\left[\mathrm{I}_{-\nu}(x) - \mathrm{I}_\nu(x)\right]\right\} \qquad n = 0, \pm1, \pm2, \cdots$$

An arbitrarily weighted Macdonald function is part of the general solution of each of the differential equations listed below, the first of which is *Bessel's modified differential equation*.

51:3:7 $$x^2\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + x\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} - (x^2+\nu^2)\mathrm{f} = 0 \qquad \mathrm{f} = w_1\mathrm{I}_\nu(x) + w_2\mathrm{K}_\nu(x)$$

51:3:8 $$x\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + (2\nu+1)\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} - x\mathrm{f} = 0 \qquad \mathrm{f} = \frac{w_1}{x^\nu}\mathrm{I}_\nu(x) + \frac{w_2}{x^\nu}\mathrm{K}_\nu(x)$$

51:3:9 $$x\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + (\nu+1)\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} - \mathrm{f} = 0 \qquad \mathrm{f} = \frac{w_1}{\left(2\sqrt{x}\right)^\nu}\mathrm{I}_\nu\left(2\sqrt{x}\right) + \frac{w_2}{\left(2\sqrt{x}\right)^\nu}\mathrm{K}_\nu\left(2\sqrt{x}\right)$$

These solutions are valid for all values of ν.

Semiintegration [Section 12:14] is a powerful method of generating Macdonald functions from exponentials:

51:3:10 $$\mathrm{K}_0\left(\frac{1}{2x}\right) = \sqrt{\pi x}\exp\left(\frac{1}{2x}\right)\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\frac{1}{x}\exp\left(\frac{-1}{x}\right)\right\}$$

51:3:11 $$\mathrm{K}_1\left(\frac{1}{\sqrt{x}}\right) = \frac{\sqrt{\pi}}{2}x\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\frac{1}{\sqrt{x^3}}\exp\left(\frac{-1}{\sqrt{x}}\right)\right\}$$

51:3:12 $$\mathrm{K}_{1/4}\left(\frac{1}{2x^2}\right) = \sqrt{2\pi}x\exp\left(\frac{1}{2x^2}\right)\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\left\{\frac{1}{\sqrt{x^3}}\exp\left(\frac{-1}{x^2}\right)\right\}$$

51:4 SPECIAL CASES

Macdonald functions of orders that are odd multiples of ½ are related to the *spherical Macdonald functions*

51:4:1 $$\mathrm{K}_{n+\frac{1}{2}}(x) = \sqrt{\frac{2x}{\pi}}\,\mathrm{k}_n(x)$$

which are themselves proportional to $\exp(-x)$ in ways detailed in Section 26:13.

When its order ν is $1/3$ or ¼ respectively, the Macdonald function becomes an Airy function [Chapter 56] or a parabolic cylinder function [Chapter 46]:

51:4:2 $$\mathrm{K}_{1/3}(x) = \pi\sqrt{\frac{3}{\hat{x}}}\,\mathrm{Ai}(\hat{x}) \qquad \hat{x} = \sqrt[3]{\frac{9x^2}{4}}$$

51:4:3 $$K_{1/4}(x) = \sqrt{\frac{\pi}{\sqrt{x}}}\, D_{-1/2}\left(2\sqrt{x}\right)$$

51:5 INTRARELATIONSHIPS

With respect to its order, the Macdonald function is even, so the order-reflection formula is simply

51:5:1 $$K_{-\nu}(x) = K_\nu(x)$$

The corresponding argument-reflection formula is given in Section 51:11, because $K_\nu(-x)$ is generally complex.

The Macdonald function obeys the recursion formula

51:5:2 $$K_{\nu+1}(x) = \frac{2\nu}{x} K_\nu(x) + K_{\nu-1}(x)$$

Thereby it becomes possible to express any integer-ordered Macdonald function, of order 2 or more, via the formula

51:5:3 $$K_n(x) = \mathrm{Wk}_n^{(0)}(x) K_0(x) + \mathrm{Wk}_n^{(1)}(x) K_1(x) \qquad n = 2,3,4,\cdots$$

where the Wk weighting functions differ from the Lommel Wj polynomials discussed in Section 52:5 only in that all the polynomial terms in $\mathrm{Wk}_n^{(0)}(x)$ and $\mathrm{Wk}_n^{(1)}(x)$ have uniformly positive signs; for example

51:5:4 $$K_4(x) = \left[1 + \frac{24}{x^2}\right] K_0(x) + \left[\frac{8}{x} + \frac{48}{x^3}\right] K_1(x)$$

Formulas involving both the Macdonald and modified Bessel functions include

51:5:5 $$K_{\nu+1}(x) I_\nu(x) + K_\nu(x) I_{\nu+1}(x) = \frac{1}{x}$$

51:6 EXPANSIONS

Based on equations 51:3:5, 43:5:1 and 50:6:1, the average of two convergent power series, namely

51:6:1 $$K_\nu(x) = \frac{1}{2} \sum_{\sigma=\pm 1} \Gamma(-\sigma\nu) \left(\frac{x}{2}\right)^{\sigma\nu} \sum_{j=0}^{\infty} \frac{(x^2/4)^j}{j!(1+\sigma\nu)_j} \qquad \nu \neq 0, \pm 1, \pm 2, \cdots$$

represents the Macdonald function of noninteger order. The two series, which differ only in the sign of ν, may be coalesced only if ν is an odd multiple of ½. To obtain an expansion for integer orders, one must apply L'Hôpital's rule [Section 0:10] to definition 51:0:1. Digamma functions [Chapter 44] appear in the results which, for the zero-order and first-order cases, are

51:6:2 $$K_0(x) = \sum_{j=0}^{\infty} \left[\psi(j+1) - \ln\left(\frac{x}{2}\right)\right] \frac{(x/2)^{2j}}{j!\,j!}$$

and

51:6:3 $$K_1(x) = \frac{1}{x} - \sum_{j=0}^{\infty} \left[\frac{\psi(j+1) + \psi(j+2)}{2} - \ln\left(\frac{x}{2}\right)\right] \frac{(x/2)^{2j+1}}{j!(j+1)!}$$

The general formula is quite complicated; using Pochhammer polynomials [Chapter 18] it is

51:6:4
$$K_n(x) = \frac{(n-1)!}{2}\sum_{k=0}^{n-1}\frac{(x/2)^{2k-n}}{(1)_k(1-n)_k} + \frac{1}{n!}\left(\frac{-x}{2}\right)^n\sum_{j=0}^{\infty}\left[\frac{\psi(j+1)+\psi(n+j+1)}{2} - \ln\left(\frac{x}{2}\right)\right]\frac{(x^2/4)^j}{(1)_j(n+1)_j}$$

Alternatively, $K_n(x)$ for $n = 2, 3, 4, \cdots$ is accessible via equation 51:5:3.

Macdonald functions of integer order may also be expanded as the *Neumann series*:

51:6:5
$$K_0(x) = -\left[\gamma + \ln\left(\frac{x}{2}\right)\right]I_0(x) + 2\sum_{j=1}^{\infty}\frac{I_{2j}(x)}{j}$$

51:6:6
$$K_1(x) = \frac{1}{x}I_0(x) + \left[\gamma - 1 + \ln\left(\frac{x}{2}\right)\right]I_1(x) - \sum_{j=1}^{\infty}\frac{2j+1}{j^2+j}I_{2j+1}(x)$$

in which γ is Euler's constant [Section 1:7]. Higher-ordered functions may be accessed through equation 51:5:3.

The series

51:6:7
$$\sqrt{\frac{2x}{\pi}}\exp(x)K_\nu(x) \sim 1 - \frac{\frac{1}{4}-\nu^2}{2x} + \frac{(\frac{1}{4}-\nu^2)(\frac{9}{4}-\nu^2)}{8x^2} - \frac{(\frac{1}{4}-\nu^2)(\frac{9}{4}-\nu^2)(\frac{25}{4}-\nu^2)}{48x^3} + \cdots = \sum_{j=0}\frac{(\frac{1}{2}-\nu)_j(\frac{1}{2}+\nu)_j}{(1)_j}\left(\frac{-1}{2x}\right)^j$$

is generally asymptotic, but it terminates when $|\nu|$ is an odd multiple of ½ and under these circumstances it is exact. If x is positive and ν is an integer, series 51:6:7 alternates in sign once $j > n$, and under these conditions the true value lies between the last two partial sums of the truncated series. See Section 54:14 for further discussion of this series and for its relation to the auxiliary cylinder functions $fc_\nu(x)$ and $gc_\nu(x)$.

Series 51:6:7 establishes $\sqrt{1/\pi x}\,\exp(1/2x)K_\nu(1/2x)$ as an $L = K - 1 = 1$ hypergeometric function [Section 18:14], the simplest instance being

51:6:8
$$\frac{1}{\sqrt{\pi x}}\exp\left(\frac{1}{2x}\right)K_0\left(\frac{1}{2x}\right) = \sum_{j=0}\frac{(\frac{1}{2})_j(\frac{1}{2})_j}{(1)_j}(-x)^j$$

51:7 PARTICULAR VALUES

With real variables, the Macdonald function is supremely bland, lacking any zeros or extrema. For all orders, the function approaches $+\infty$ as x approaches zero from positive values, monotonically declines as the argument increases, and asymptotically approaches zero as $x \to \infty$. Thus there are no noteworthy particular values, though the special significance of $K_0(2)$ is evident in 44:5:11.

51:8 NUMERICAL VALUES

Equator's Macdonald function routine (keyword **K**) provides accurate values of $K_\nu(x)$ for wide domains of the variables x and ν. Series 51:6:7, aided by the ε-transformation, is the chosen procedure for larger arguments. Otherwise, expansion 51:6:1 provides the basis of the algorithm, replaced by 51:6:4 when ν is an integer. Of course 51:5:1 caters to negative orders.

51:9 LIMITS AND APPROXIMATIONS

The following two-term approximations apply to the Macdonald function when its argument is close to zero

	$\nu = 0$	$0 < \nu < 1$	$\nu = 1$	$\nu > 1$
$\mathrm{K}_\nu(x) \approx$ small x	$\ln\left(\dfrac{2}{x}\right) - \gamma$	$\dfrac{\frac{1}{2}\Gamma(\nu)}{(x/2)^\nu} + \dfrac{\frac{1}{2}\Gamma(-\nu)}{(2/x)^\nu}$	$\dfrac{1}{x} + \dfrac{x}{2}\ln\left(\dfrac{x}{2}\right)$	$\dfrac{\frac{1}{2}\Gamma(\nu)}{(x/2)^\nu}\left[1 - \dfrac{x^2}{4\nu-4}\right]$

As the argument approaches infinity, the Macdonald function approaches zero in an order-independent fashion such that

51:9:1 $$\lim_{x\to\infty} \mathrm{K}_\nu(x) = \sqrt{\frac{\pi}{2x}}\exp(-x)$$

51:10 OPERATIONS OF THE CALCULUS

The general differentiation formulas

51:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{K}_\nu(x) = -\frac{\mathrm{K}_{\nu-1}(x) + \mathrm{K}_{\nu+1}(x)}{2} = \frac{\nu}{x}\mathrm{K}_\nu(x) - \mathrm{K}_{\nu+1}(x) = -\mathrm{K}_{\nu-1}(x) - \frac{\nu}{x}\mathrm{K}_{\nu+1}(x)$$

have the special cases

51:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{K}_0(x) = -\mathrm{K}_1(x)$$

and

51:10:3 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{K}_1(x) = \frac{-\mathrm{K}_0(x) - \mathrm{K}_2(x)}{2} = -\mathrm{K}_0(x) - \frac{\mathrm{K}_1(x)}{x}$$

The formulas

51:10:4 $$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{\pm\nu}\,\mathrm{K}_\nu(x)\right\} = -x^{\pm\nu}\,\mathrm{K}_{\nu\mp 1}(x)$$

are easily derived.

Arising as special cases of formulas 51:10:8 and 51:10:6 respectively are the indefinite integrals

51:10:5 $$\int_0^x \mathrm{K}_0(t)\mathrm{d}t = \frac{\pi x}{2}\left[\mathrm{K}_0(x)\mathrm{L}_{-1}(x) + \mathrm{K}_1(x)\mathrm{L}_0(x)\right] \qquad \text{and} \qquad \int_x^\infty \mathrm{K}_1(t)\mathrm{d}t = \mathrm{K}_0(x)$$

in which $\mathrm{L}_n(x)$ is the modified Struve function of Section 57:13. Closed form expressions are available for the integrals of the following products of the Macdonald function and powers

51:10:6 $$\int_x^\infty t^{1-\nu}\,\mathrm{K}_\nu(t)\mathrm{d}t = x^{1-\nu}\,\mathrm{K}_{\nu-1}(x)$$

51:10:7 $$\int_0^x t^{\nu+1}\,\mathrm{K}_\nu(t)\,\mathrm{d}t = 2^\nu\Gamma(\nu+1) - x^{\nu+1}\,\mathrm{K}_{\nu+1}(x) \qquad \nu > -1$$

and

51:10:8 $$\int_0^x t^\nu\,\mathrm{K}_\nu(t)\,\mathrm{d}t = 2^{\nu-1}\sqrt{\pi}\Gamma(\nu+\tfrac{1}{2})x\left[\mathrm{K}_\nu(x)\mathrm{L}_{\nu-1}(x) + \mathrm{K}_{\nu-1}(x)\mathrm{L}_\nu(x)\right] \qquad \nu > -\tfrac{1}{2}$$

The identity $\mathrm{K}_{-\nu}(x) = \mathrm{K}_\nu(x)$ is useful in extending the above integrals, as well as those that follow. *King's integral*

51:10:9 $$\int_0^x \exp(t)\mathrm{K}_0(t)\,\mathrm{d}t = x\exp(x)\left[\mathrm{K}_0(x)+\mathrm{K}_1(x)\right]-1$$

is an important special case of the first formula below

51:10:10 $$\int_0^x t^\nu \exp(\pm t)\mathrm{K}_\nu(t)\,\mathrm{d}t = \frac{1}{2\nu+1}\left[x^{\nu+1}\exp(\pm x)[\mathrm{K}_\nu(x)\pm \mathrm{K}_{\nu-1}(x)]\mp 2^\nu\Gamma(\nu+1)\right] \qquad \nu>-\tfrac{1}{2}$$

51:10:11 $$\int_x^\infty t^{-\nu}\exp(t)\mathrm{K}_\nu(t)\,\mathrm{d}t = \frac{x^{1-\nu}\exp(x)}{2\nu-1}\left[\mathrm{K}_\nu(x)+\mathrm{K}_{\nu-1}(x)\right] \qquad \nu>\tfrac{1}{2}$$

The definite integral of the zero-order Macdonald function over $0\le t\le\infty$ is $\pi/2$, this being the $\nu=0$ instance of

51:10:12 $$\int_0^\infty \mathrm{K}_\nu(t)\,\mathrm{d}t = \frac{\pi}{2}\sec\left(\frac{\nu\pi}{2}\right) \qquad -1<\nu<1$$

but for $|\nu|\ge 1$ the integral diverges. Definite integrals and Laplace transforms include

51:10:13 $$\int_0^\infty t^\mu \mathrm{K}_\nu(t)\,\mathrm{d}t = 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right) \qquad 0\le\nu<\mu+1$$

51:10:14 $$\int_0^\infty \mathrm{K}_0(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{K}_0(bt)\} = \frac{\mathrm{arcosh}(s/b)}{\sqrt{s^2-b^2}}$$

51:10:15 $$\int_0^\infty \mathrm{K}_0\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\mathrm{K}_0\left(\sqrt{bt}\right)\right\} = \frac{-1}{2s}\exp\left(\frac{b}{4s}\right)\mathrm{Ei}\left(\frac{-b}{4s}\right)$$

51:10:16 $$\int_0^\infty t\,\mathrm{K}_1(bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{t\,\mathrm{K}_1(bt)\} = \frac{s}{b(s^2-b^2)} - \frac{b\,\mathrm{arcosh}(s/b)}{\sqrt{(s^2-b^2)^3}}$$

51:10:17 $$\int_0^\infty \frac{\mathrm{K}_\nu(\sqrt{bt})}{\sqrt{t}}\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\frac{\mathrm{K}_\nu(\sqrt{bt})}{\sqrt{t}}\right\} = \frac{\sec(\frac{1}{2}\nu\pi)}{2}\sqrt{\frac{\pi}{s}}\exp\left(\frac{b}{8s}\right)\mathrm{K}_{\nu/2}\left(\frac{b}{8s}\right) \qquad \nu<1$$

and others are accessible via definition 51:3:5 or 51:3:6. See Section 51:13 for multiple integrals of $\mathrm{K}_0(t)$ and for the indefinite integral of $\mathrm{K}_\nu(t)/t$.

51:11 COMPLEX ARGUMENT

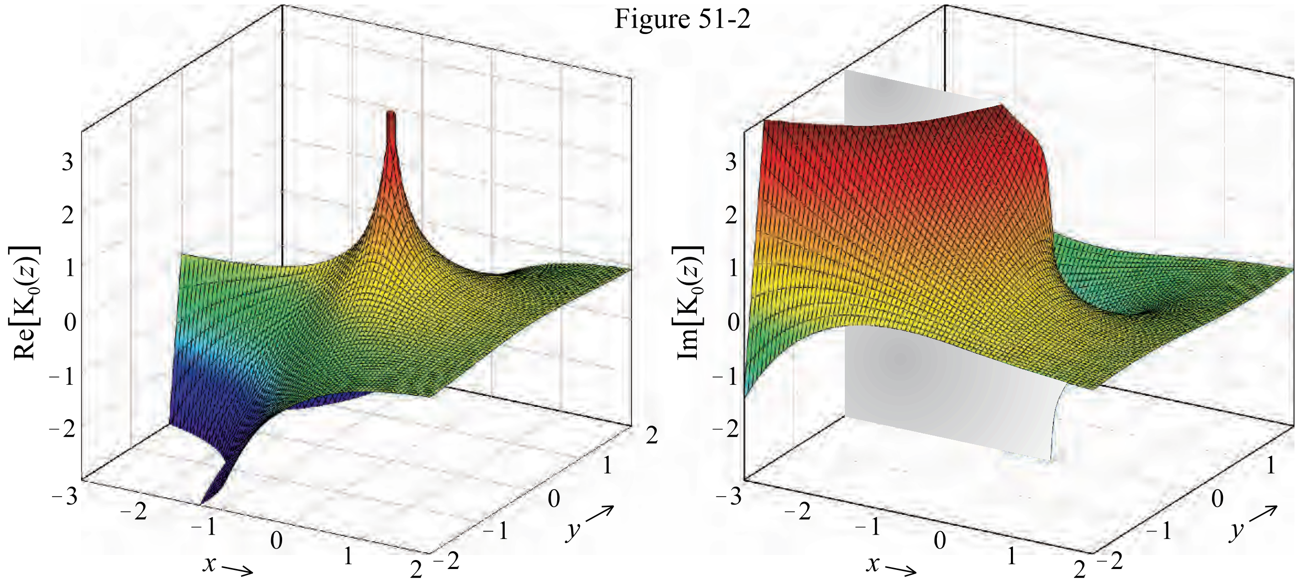
Generally, the Macdonald function adopts complex values not only when its argument is complex, but also when it is merely negative. Expansions 51:6:1 and 51:6:4 remain valid when x is replaced by $x+iy$. The complex-plane relationship of the Macdonald function to regular cylinder functions [Section 49:14] is summarized by

51:11:1 $$\mathrm{K}_\nu(z) = \tfrac{1}{2}\pi i^{\nu+1}\mathrm{H}_\nu^{(1)}(iz) = \tfrac{1}{2}\pi i^{\nu+1}\left[\mathrm{J}_\nu(iz)+i\,\mathrm{Y}_\nu(iz)\right]$$

It is this relation that explains the *modified Hankel function* name sometimes given to the K function. In this section, detailed discussion will be confined to the $\mathrm{K}_0(x+iy)$ and $\mathrm{K}_1(x+iy)$ cases.

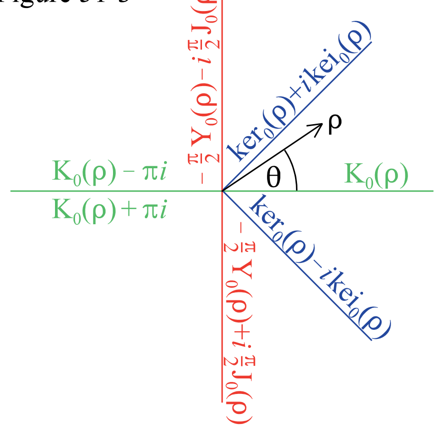
Figure 51-2 depicts the real and imaginary parts of $\mathrm{K}_0(x+iy)$. The singularity at the origin is the dominant

Figure 51-2



feature of the real part. To avoid multiple values in the imaginary part, it is necessary to cut the complex plane and, as with the logarithm, the scission is made conventionally along the negative real axis ($x < 0 = y$). Figure 51-3 is a polar diagram annotated by the formulas of the functions obtained by slicing the three-dimensional surface along the two axes and along certain diagonals. The alternative values listed for $K_0(-x+0i)$, reflect the values on either side of the cut.

Figure 51-3



This figure illustrates the fact that the functions of Chapters 53 and 54 represent the imaginary and real parts of the Macdonald function of imaginary argument or, equivalently, after it is rotated by 90° in the complex plane. Rotation by ±45° generates Kelvin functions [Chapter 55].

Figure 51-4 shows the real and imaginary parts of $K_1(x+iy)$ and

Figure 51-4

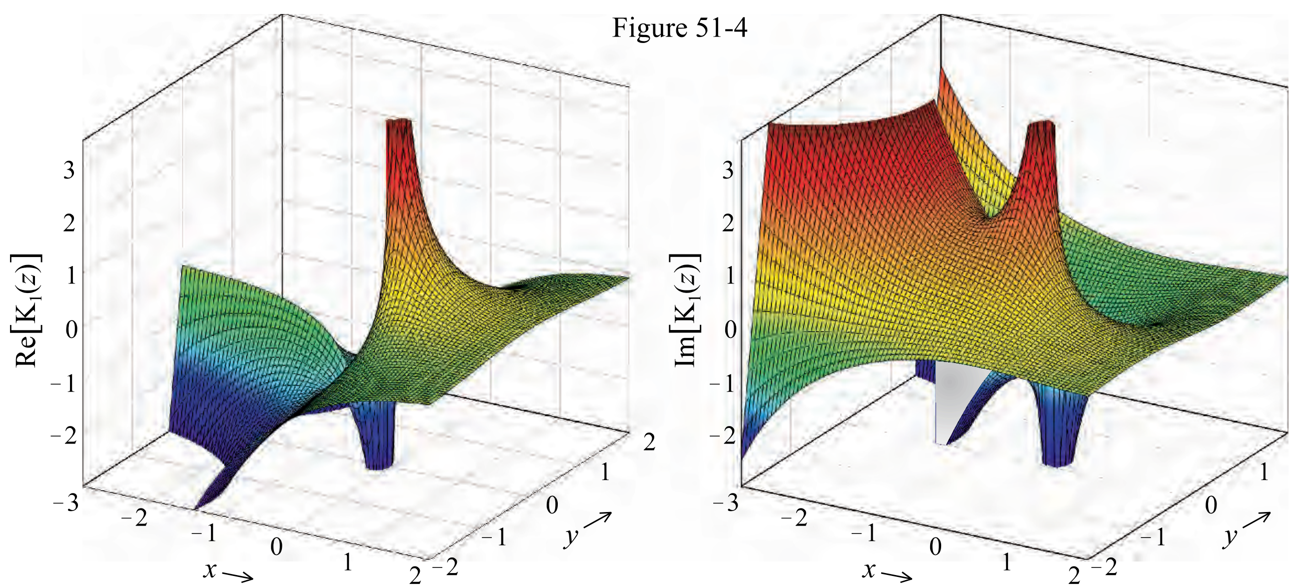
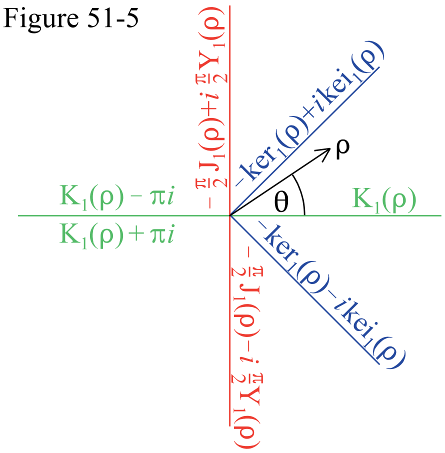


Figure 51-5 is a polar map showing the locations of other functions that share the same complex landscape. Notice, in contrast to the zero-order Macdonald function, that it is the *real* part of $K_1(iy)$ that corresponds to the J Bessel function.

Figure 51-5



There is a huge number of Laplace inversion formulas involving the Macdonald functions. Among those listed by Roberts and Kaufman (pages 303−312) are:

51:11:2 $$\int_{\alpha-i\infty}^{\alpha+i\infty} K_0(as)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{K_0(as)\} = \frac{u(t-a)}{\sqrt{t^2-a^2}}$$

51:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty} K_0\left(a\sqrt{s}\right)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{K_0\left(a\sqrt{s}\right)\right\} = \frac{1}{2t}\exp\left(\frac{-a^2}{4t}\right)$$

51:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty} K_n(as)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{K_n(as)\} = \frac{u(t-a)}{\sqrt{t^2-a^2}}T_n\left(\frac{t}{a}\right) \qquad n=0,1,2,\cdots$$

51:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty} K_\nu(as)\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{K_\nu(as)\} = \frac{u(t-a)}{\sqrt{t^2-a^2}}\cosh\left(\nu\,\mathrm{arcosh}(t/a)\right)$$

and

51:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{K_\nu\left(a\sqrt{s}\right)}{\sqrt{s}}\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{K_\nu\left(a\sqrt{s}\right)}{\sqrt{s}}\right\} = \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-a^2}{8t}\right)K_{\nu/2}\left(\frac{a^2}{8t}\right)$$

51:12 GENERALIZATIONS

The Macdonald function may be generalized to the Tricomi function or, more concisely, to the equivalent Whittaker function [Section 48:12]

51:12:1 $$K_\nu\left(\frac{x}{2}\right) = \sqrt{\pi}x^\nu\exp\left(\frac{-x}{2}\right)U\left(\nu+\tfrac{1}{2},2\nu+1,x\right) = \sqrt{\frac{\pi}{x}}W_{0,\nu}(x)$$

51:13 COGNATE FUNCTIONS

The symbol $\mathrm{Ki}_\nu(x)$ is used sometimes to indicate the indefinite integral of the $K_\nu(t)/t$ quotient:

51:13:1 $$\int_x^\infty \frac{K_\nu(t)}{t}\mathrm{d}t$$

which lacks a closed-form expression. Its zero-order version may be evaluated as the asymptotic series

51:13:2 $$\int_x^\infty \frac{K_0(t)}{t}\mathrm{d}t \sim \sqrt{\frac{\pi}{2x^3}}\exp(-x)\sum_{j=0} a_j\left(\frac{-1}{x}\right)^j \qquad \text{where} \qquad a_j = \frac{12j^2+1}{8j}a_{j-1} - \frac{(j-\frac{1}{2})^3}{2j}a_{j-2}$$

with $a_0 = 1$ and $a_1 = 13/8$. Moreover, it has the interesting Laplace transform

51:13:3 $$\mathcal{L}\left\{\int_x^\infty \frac{K_0(t)}{t}\,dt\right\} = \frac{\pi^2+4\operatorname{arcosh}^2(s)}{8s}$$

Confusingly, however, a similar symbol $\mathrm{Ki}_n(x)$ also finds use to denote the repeated integrals of the zero-order Macdonald function

51:13:4 $$\int_x^\infty\int_t^\infty\cdots\int_t^\infty K_0(t)[dt]^n$$

The latter construct may be generalized to $\mathrm{Ki}_\nu(x)$, where ν is positive, but not necessarily an integer, via the convolution formula

51:13:5 $$\frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}\,K_0(t)\,dt$$

The quantity defined by 51:13:5, or by 51:13:4 if ν is the positive integer n, is unrelated to that defined by 51:13:1. It takes the value $\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\nu\right)/2\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)$ when $x=0$.

CHAPTER 52

THE BESSEL FUNCTIONS $J_n(x)$ OF INTEGER ORDER

The great German astronomer and mathematician, Friedrich Wilhelm Bessel (1784 – 1846), encountered these functions in his studies of planetary motion and they have since found very widespread applications. This chapter addresses those Bessel functions that have integer orders, emphasizing the most important members, $J_0(x)$ and $J_1(x)$. Properties that are common to all Bessel functions, irrespective of order, are the subject of the next chapter.

52:1 NOTATION

The symbol $J_\nu(x)$ is universal for a Bessel function of order ν and argument x. This chapter is devoted to Bessel functions, J_n, of integer order n, a class to which the term *Bessel coefficient* is sometimes applied. The symbol $R_n(x)$ will be used to denote the ratio of two successive integer-order Bessel functions

52:1:1
$$R_n(x) = \frac{J_{n+1}(x)}{J_n(x)} \qquad n = 0,1,2,\cdots$$

Do not confuse the symbol $j_n(x)$ that we use for spherical Bessel functions [Section 32:13] with the symbols $j_n^{(k)}$ and $j_n'^{(k)}$ introduced in Section 52:7 of this chapter to denote the kth zero and the kth extremum, respectively, of the Bessel function $J_n(x)$. See that section too for the significance of J_n'.

52:2 BEHAVIOR

Defined for all real [and complex – see Section 52:11] argument x, Bessel functions of integer order are oscillatory, with oscillations that become increasingly damped as x approaches large values of either sign. Apart from the damping, $J_0(x)$ somewhat resembles the cosine function [Chapter 32] in taking unity value at $x = 0$ and acquiring values of zero not far from arguments of $\pm\pi/2$ and $\pm3\pi/2$. Likewise $J_1(x)$ bears some resemblance to $\sin(x)$ in having zeros at $x = 0$ and close to $\pm\pi$ and $\pm2\pi$. Similarities to sinusoids become less pronounced as the order increases. After the first few orders, $J_n(x)$ retains near-zero values over a range of arguments around $x = 0$ before

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_53,

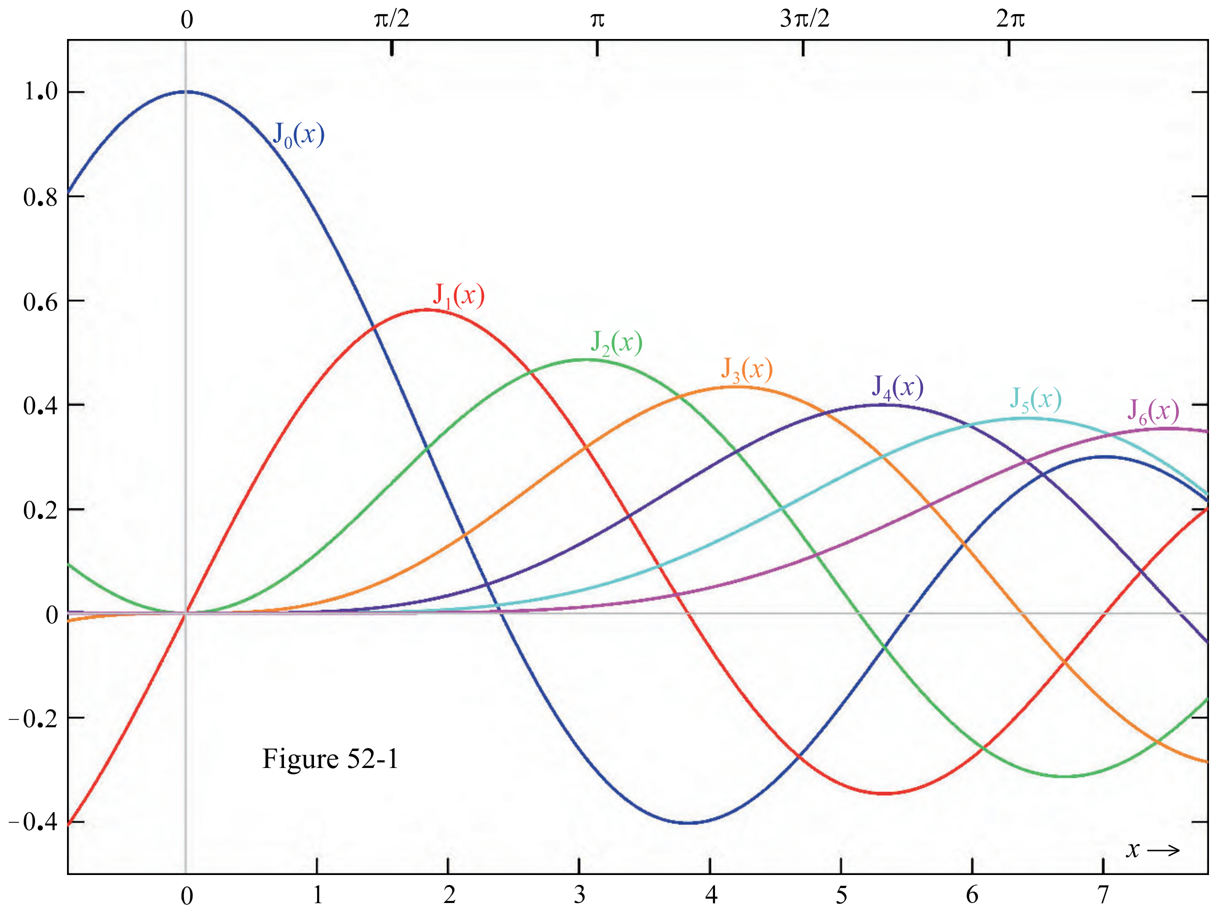


Figure 52-1

breaking into oscillation. This near-zero hiatus increases in width as n increases. For example, as Figure 52-1 illustrates, $J_6(x)$ never exceeds 0.01 in magnitude in the range $-2.9 < x < 2.9$.

Some regularities are apparent in Figure 52-1. Each local maximum or minimum of $J_0(x)$ corresponds to a zero of $J_1(x)$, though this rule has no parallel for higher orders. Notice that for $n \geq 1$, the argument of each extremum of $J_n(x)$ corresponds to a point of intersection of the $J_{n-1}(x)$ and $J_{n+1}(x)$ curves. Moreover observe that, at each zero of $J_n(x)$, its contiguous congeners $J_{n-1}(x)$ and $J_{n+1}(x)$ have equal magnitudes but opposite signs. Of course, these rules have their foundations in the systematic differentiation and recursion properties detailed later in this chapter.

For positive x, each Bessel function of integer order n encounters its first (and largest) local maximum at an argument that is close to $n\pi/2$. This rule breaks down for larger n, however, and for very large orders the first maximum of $J_n(x)$ occurs closer to $x = n$. Subsequent to the first maximum, there is a sequence of ever-smaller local maxima, with interspersed local minima; zeros occur almost midway between each minimum and its adjacent maxima.

52:3 DEFINITIONS

With f representing either the cosine or sine function, the functions $\mathrm{f}\{x\,\mathrm{f}(t)\}$ serve as generating functions for the Bessel functions of integer order. Thus, even-ordered Bessel functions are generated by

52:3:1 $$\cos\left\{x\begin{matrix}\cos\\\sin\end{matrix}(t)\right\} = J_0(x) \mp 2J_2(x)\cos(2t) + 2J_4(x)\cos(4t) \mp \cdots = J_0(x) + 2\sum_{j=1}^{\infty}(\mp)^j\cos(2jt)J_{2j}(x)$$

while generating functions for those of odd order comprise

52:3:2 $$\sin\left\{x\begin{matrix}\cos\\\sin\end{matrix}(t)\right\} = 2J_1(x)\begin{matrix}\cos\\\sin\end{matrix}(t) \mp 2J_3(x)\begin{matrix}\cos\\\sin\end{matrix}(3t) + \cdots = 2\sum_{j=0}^{\infty}(\mp)^j\begin{matrix}\cos\\\sin\end{matrix}\{(2j+1)t\}J_{2j+1}(x)$$

An exponential function generates integer-order Bessel functions of both parities:

52:3:3 $$\exp\left(\frac{t^2-1}{2t}x\right) = J_0(x) + \left[t-\frac{1}{t}\right]J_1(x) + \left[t^2+\frac{1}{t^2}\right]J_2(x) + \left[t^3-\frac{1}{t^3}\right]J_3(x) + \cdots = \sum_{j=-\infty}^{\infty}\left[t\,\mathrm{sgn}(j)\right]^j J_j(x)$$

Representations of integer-order Bessel functions as definite integrals include

52:3:4 $$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos\{x\sin(t) - nt\}\,dt \qquad n = 0,1,2,\cdots$$

52:3:5 $$J_n(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(nt)\cos\{x\sin(t)\}\,dt \qquad n = 0,2,4,\cdots$$

52:3:6 $$J_n(x) = \frac{2}{\pi}\int_0^{\pi/2}\sin(nt)\sin\{x\sin(t)\}\,dt \qquad n = 1,3,5,\cdots$$

and others arise from specializing those listed in Section 53:3.

One reason for the especial importance of Bessel functions of orders 0 and 1 is that they arise in many practical contexts as solutions of some of the simplest second-order differential equations, including the following:

52:3:7 $$x\frac{d^2f}{dx^2} + \frac{df}{dx} + xf = 0 \qquad f = w_1 J_0(x) + w_2 Y_0(x)$$

52:3:8 $$x\frac{d^2f}{dx^2} + \frac{df}{dx} + f = 0 \qquad f = w_1 J_0\left(2\sqrt{x}\right) + w_2 Y_0\left(2\sqrt{x}\right)$$

52:3:9 $$x\frac{d^2f}{dx^2} - \frac{df}{dx} + xf = 0 \qquad f = w_1 x J_1(x) + w_2 x Y_1(x)$$

52:3:10 $$x\frac{d^2f}{dx^2} + f = 0 \qquad f = w_1\sqrt{x}\,J_1\left(2\sqrt{x}\right) + w_2\sqrt{x}\,Y_1\left(2\sqrt{x}\right)$$

Here the w's are arbitrary weights and the Y's are Neumann functions [Chapter 54] of zero and unity orders.

The close relationship of the $J_0(x)$ and $J_1(x)$ functions to the sinusoids is borne out by the following formulas for semidifferentiation

52:3:11 $$J_0\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\sin\left(\sqrt{x}\right)$$

and semiintegration

52:3:12 $$J_1\left(\sqrt{x}\right) = \frac{1}{\sqrt{\pi x}}\frac{d^{-1/2}}{dx^{-1/2}}\sin\left(\sqrt{x}\right)$$

52:3:13 $$J_0(x) = \frac{\sec(x)}{\sqrt{\pi}}\frac{d^{-1/2}}{dx^{-1/2}}\frac{\cos(2x)}{\sqrt{x}} = \frac{\csc(x)}{\sqrt{\pi}}\frac{d^{-1/2}}{dx^{-1/2}}\frac{\sin(2x)}{\sqrt{x}}$$

The hypergeometric nature of the composite $x^{-n/2}J_n\left(2\sqrt{x}\right)$ function, evident in equation 52:6:1, allows the synthetic conversion [Chapter 43:14] of one Bessel function into another. To convert between integer orders, one uses

52:3:14 $$\frac{n!}{x^{n/2}}J_n\left(2\sqrt{x}\right) \xrightarrow[n'+1]{n+1} \frac{n'!}{x^{n'/2}}J_{n'}\left(2\sqrt{x}\right)$$

though the procedure applies equally to noninteger orders.

52:4 SPECIAL CASES

There are none. Despite their integer subscript, the functions $j_n(x)$ [spherical Bessel functions, Section 32:13] do *not* correspond to Bessel functions of integer order.

52:5 INTRARELATIONSHIPS

The formulas from Section 53:5 apply equally whether or not v is an integer. Here, emphasis is on those relationships that apply *only* when the order of the Bessel function is an integer of either sign.

The behavior implied by the argument-reflection formula

52:5:1 $$J_n(-x) = (-)^n J_n(x)$$

is evident in Figure 52-1. Thus a Bessel function of integer order is even or odd, according to the parity of its order. Moreover, this behavior is echoed by the order-reflection formula:

52:5:2 $$J_{-n}(x) = (-)^n J_n(x)$$

These reflection rules, which imply that $J_n(-x) = J_{-n}(x)$ and that $J_{-n}(-x) = J_n(x)$, apply only when n is an integer.

Sufficient applications of the recursion formula

52:5:3 $$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \qquad n = 0, \pm 1, \pm 2, \cdots$$

permit Bessel functions of any integer order to be expressed in terms of $J_0(x)$ and $J_1(x)$. The general formula, for Bessel functions of positive integer order, may be written

52:5:4 $$J_n(x) = \mathrm{Wj}_n^{(0)}(x)\,J_0(x) + \mathrm{Wj}_n^{(1)}(x)\,J_1(x) \qquad n = 2,3,4,\cdots$$

where the Wj multipliers are polynomials in $1/x$, early members of which are

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\mathrm{Wj}_n^{(0)}(x)$	-1	$\frac{-4}{x}$	$1-\frac{24}{x^2}$	$\frac{12}{x}-\frac{192}{x^3}$	$-1+\frac{144}{x^2}-\frac{1920}{x^4}$	$\frac{-24}{x}+\frac{1920}{x^3}-\frac{23040}{x^5}$
$\mathrm{Wj}_n^{(1)}(x)$	$\frac{2}{x}$	$-1+\frac{8}{x^2}$	$\frac{-8}{x}+\frac{48}{x^3}$	$1-\frac{72}{x^2}+\frac{384}{x^4}$	$\frac{18}{x}-\frac{768}{x^3}+\frac{3840}{x^5}$	$-1+\frac{288}{x^2}-\frac{9600}{x^4}+\frac{46080}{x^6}$

These weighting polynomials are given by the explicit formulation

52:5:5 $$\mathrm{Wj}_n^{(k)}(x) = (-)^{n-1}\left(\frac{-2}{x}\right)^{n+k-2}\sum_{j=0}^{\mathrm{Int}[(n+k-2)/2]}\frac{(n+k-2-j)!(n-j-1)!}{(n+k-2-2j)!\,j!(1-k+j)!}\left(\frac{-x^2}{4}\right)^j \qquad \begin{cases} k=0 \text{ or } 1 \\ n=2,3,4,\cdots \end{cases}$$

They are instances of *Lommel polynomials* [Erdélyi et al. *Higher Transcendental Functions*, Section 7.5.2. These authors use a notation to which ours is related through $Wj_n^{(0)}(x) = -R_{n-2,2}(x)$ and $Wj_n^{(1)}(x) = R_{n-1,1}(x)$].

The recursion formula 52:5:3 interrelates *three* Bessel functions, but when written for the quotient defined in 52:1:1, one finds a formula

52:5:6 $$R_n(x) = \frac{2n}{x} - \frac{1}{R_{n-1}(x)}$$

that relates only *two* Bessel function ratios, enhancing computational utility as explained in Section 52:8.

There is a plethora of summable series of integer-order Bessel functions. Some of these, such as the trio

52:5:7 $$\tfrac{1}{2}J_0(x) + J_2(x) + J_4(x) + J_6(x) + \cdots = \tfrac{1}{2}$$

52:5:8 $$\tfrac{1}{2}J_0(x) - J_2(x) + J_4(x) - J_6(x) + \cdots = \tfrac{1}{2}\cos(x)$$

and

52:5:9 $$J_1(x) - J_3(x) + J_5(x) - J_7(x) + \cdots = \tfrac{1}{2}\sin(x)$$

may be derived by recourse to the generating functions 52:3:1−3. Others, for example the infinite series

52:5:10 $$J_1(x) + 3J_3(x) + 5J_5(x) + 7J_7(x) + \cdots = \tfrac{1}{2}x$$

52:5:11 $$2J_2(x) - 4J_4(x) + 6J_6(x) - 8J_8(x) + \cdots = \tfrac{1}{2}xJ_1(x)$$

and

52:5:12 $$4J_2(x) + 16J_4(x) + 36J_6(x) + 64J_8(x) + \cdots = \tfrac{1}{2}x^2$$

follow from the properties of *Neumann series* [Section 53:14]. Still others, including

52:5:13 $$J_1^2(x) + J_2^2(2x) + J_3^2(3x) + J_4^2(4x) + \cdots = \tfrac{1}{2}\left[\left(1/\sqrt{1-x^2}\right) - 1\right] \qquad x < 1$$

are examples of *Kapteyn series* [[Erdélyi et al. *Higher Transcendental Functions, Volume 2*, pages 66−68] The unexpected sum of the series

52:5:14 $$J_n(x) - xJ_{n+1}(x) + \frac{x^2}{2!}J_{n+2}(x) - \frac{x^3}{3!}J_{n+3} + \cdots = \frac{J_n\left(\sqrt{3}x\right)}{3^{n/2}}$$

is a consequence of the *argument-multiplication formula*

52:5:15 $$J_n(bx) = b^n \sum_{j=0}^{\infty} \left(\frac{1-b^2}{2}\right)^j \frac{x^j}{j!} J_{j+n}(x)$$

A number of general formulas provide sums for infinite series of products of two Bessel functions. The simplest of these is *Neumann's addition formula*

52:5:16 $$J_n(x+y) = \sum_{j=-\infty}^{\infty} J_{n-j}(x)J_j(y)$$

and it is the source of the *duplication formulas*

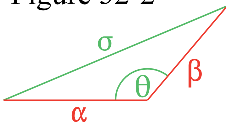
52:5:17 $$J_0(2x) = J_0^2(x) + 2\left[-J_1^2(x) + J_2^2(x) - J_3^2(x) + \cdots\right]$$

52:5:18 $$J_1(2x) = 2J_0(x)J_1(x) - 2J_1(x)J_2(x) + 2J_2(x)J_3(x) - 2J_3(x)J_4(x) + \cdots$$

as well as

52:5:19 $$\tfrac{1}{2}J_0^2(x) + J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots = \tfrac{1}{2}$$

Figure 52-2



There is another addition formula, going by the name of the *Gegenbauer's addition theorem*, but the addition in this instance is a vector addition. The length σ in Figure 52-2 is the vector sum of α and β. This addition theorem applied to the zero-order Bessel functions states that

52:5:20 $$J_0(\sigma) = J_0(\alpha)J_0(\beta) + 2\sum_{n=1}^{\infty} J_n(\alpha)J_n(\beta)\cos(n\theta)$$

where the figure identifies the angle θ, equal to $\arccos\{(\alpha^2+\beta^2-\sigma^2)/2\alpha\beta\}$ by the *law of cosines* [Section 34:15]. From the $\alpha = \beta = x$ version of this formula, one obtains 52:5:17 when $\theta = \pi$, 52:5:18 when $\theta = 0$, and

52:5:21 $$J_0\left(\sqrt{2}x\right) = J_0^2(x) - 2J_2^2(x) + 2J_4^2(x) - 2J_6^2(x) + 2J_8^2(x) - \cdots$$

when $\theta = \pi/2$. Lebedev [page 125] discusses generalizations of this theorem.

Series of Bessel coefficients whose arguments are multiples of the zeros, or the extrema, of Bessel functions arise from the procedure discussed in Section 52:14.

52:6 EXPANSIONS

The power series expansion of Bessel functions of integer order may be written in terms of either factorials or Pochhammer polynomials:

52:6:1 $$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{x^2/4}{(n+1)!} + \frac{\left(x^2/4\right)^2}{2!(n+2)!} - \cdots\right] = \sum_{j=0}^{\infty} \frac{(-)^j (x/2)^{n+2j}}{j!(n+j)!} = \frac{(x/2)^n}{n!}\sum_{j=0}^{\infty}\frac{1}{(1)_j (n+1)_j}\left(\frac{-x^2}{4}\right)^j$$

The final expression in the equation above exhibits the hypergeometric nature of the Bessel functions. The two most important instances are

52:6:2 $$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2306} + \frac{x^8}{147456} - \cdots = \sum_{j=0}^{\infty}\frac{(-)^j (x/2)^{2j}}{(j!)^2} = \sum_{j=0}^{\infty}\frac{1}{(1)_j (1)_j}\left(\frac{-x^2}{4}\right)^j$$

and

52:6:3 $$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \cdots = \sum_{j=0}^{\infty}\frac{(-)^j (x/2)^{2j+1}}{j!(j+1)!} = \frac{x}{2}\sum_{j=0}^{\infty}\frac{1}{(1)_j (2)_j}\left(\frac{-x^2}{4}\right)^j$$

The simplicity of the hypergeometric representation of the $J_0\left(2\sqrt{x}\right)$ function is responsible for its serving as a basis function of $L = K+2 = 2$ hypergeometric functions [Sections 43:14 and 18:14].

With $j_n^{(k)}$ denoting the kth positive zero of $J_n(x)$, as discussed in the next section, expansions as infinite products are as follows

52:6:4 $$J_0(x) = \left[1 - \left(\frac{x}{j_0^{(1)}}\right)^2\right]\left[1 - \left(\frac{x}{j_0^{(2)}}\right)^2\right]\left[1 - \left(\frac{x}{j_0^{(3)}}\right)^2\right]\cdots = \prod_{k=1}^{\infty}\left\{1 - \left(\frac{x}{j_0^{(k)}}\right)^2\right\}$$

52:6:5 $$J_1(x) = \frac{x}{2}\left[1 - \left(\frac{x}{j_1^{(1)}}\right)^2\right]\left[1 - \left(\frac{x}{j_1^{(2)}}\right)^2\right]\left[1 - \left(\frac{x}{j_1^{(3)}}\right)^2\right]\cdots = \frac{x}{2}\prod_{k=1}^{\infty}\left\{1 - \left(\frac{x}{j_1^{(k)}}\right)^2\right\}$$

for the first two integer-ordered Bessel functions.

The Bessel ratio defined in equation 52:1:1 may be expanded in partial fractions

52:6:6 $$\frac{J_{n+1}(x)}{J_n(x)} = R_n(x) = \frac{2x}{\left(j_n^{(1)}\right)^2 - x^2} + \frac{2x}{\left(j_n^{(2)}\right)^2 - x^2} + \frac{2x}{\left(j_n^{(3)}\right)^2 - x^2} + \cdots$$

or as a continued fraction of either infinite extent

52:6:7 $$R_n(x) = \frac{\frac{1}{2}x}{1+n-}\,\frac{\frac{1}{4}x^2}{2+n-}\,\frac{\frac{1}{4}x^2}{3+n-}\,\frac{\frac{1}{4}x^2}{4+n-}\,\frac{\frac{1}{4}x^2}{5+n-\cdots}$$

or finite extent

52:6:8 $$R_n(x) = \frac{\frac{1}{2}x}{1+n-}\,\frac{\frac{1}{4}x^2}{2+n-}\,\frac{\frac{1}{4}x^2}{3+n-}\cdots\frac{\frac{1}{4}x^2}{m-1+n-}\,\frac{\frac{1}{4}x^2}{m+n-\frac{1}{2}x\,R_{m+n}(x)}$$

Valid for large argument, the asymptotic expansions of integer-ordered Bessel functions are exemplified by

52:6:9 $$J_0(x) \sim \frac{\cos(x)}{\sqrt{\pi x}}\left[1-\frac{1}{8x}-\frac{9}{128x^2}+\frac{225}{3072x^3}+\cdots\right]+\frac{\sin(x)}{\sqrt{\pi x}}\left[1+\frac{1}{8x}-\frac{9}{128x^2}-\frac{225}{3072x^3}+\cdots\right]$$

and

52:6:10 $$J_1(x) \sim \frac{\sin(x)}{\sqrt{\pi x}}\left[1+\frac{3}{8x}+\frac{15}{128x^2}-\frac{315}{3072x^3}-\cdots\right]-\frac{\cos(x)}{\sqrt{\pi x}}\left[1-\frac{3}{8x}+\frac{15}{128x^2}+\frac{315}{3072x^3}-\cdots\right]$$

Note the unusual sequencing of signs. See Sections 53:6 and 54:14 for the general formulations.

52:7 PARTICULAR VALUES

With the sole exception of $J_0(x)$, which adopts the value unity at $x = 0$, all Bessel functions of integer order are zero at $x = 0$, $x = +\infty$, and $x = -\infty$. The zero at $x = 0$ has a multiplicity [Section 0:7] of n. That is,

52:7:1 $$J_n(0) = \frac{dJ_n}{dx}(0) = \frac{d^2 J_n}{dx^2}(0) = \cdots = \frac{d^{n-1} J_n}{dx^{n-1}}(0) = 0 \qquad \text{whereas} \qquad \frac{d^n J_n}{dx^n}(0) = 2^{-n}$$

$j_0^{(k)}$	$J_0'\left(j_0^{(k)}\right)$	$j_1^{(k)}$	$J_1'\left(j_1^{(k)}\right)$		$j_0'^{(k)}$	$J_0\left(j_0'^{(k)}\right)$	$j_1'^{(k)}$	$J_1\left(j_1'^{(k)}\right)$
0.77π	−0.5191	1.22π	−0.4028	$k = 1$	0.00π	+1.0000	0.59π	+0.5819
1.76π	+0.3403	2.23π	+0.3001	$k = 2$	1.22π	−0.4028	1.70π	−0.3461
2.75π	−0.2715	3.24π	−0.2497	$k = 3$	2.23π	+0.3001	2.72π	+0.2733
3.75π	+0.2325	4.24π	+0.2184	$k = 4$	3.24π	−0.2497	3.73π	−0.2333
4.75π	−0.2065	5.24π	−0.1965	$k = 5$	4.24π	+0.2184	4.73π	+0.2070
5.75π	+0.1877	6.24π	+0.1801	$k = 6$	5.24π	−0.1965	5.73π	−0.1880
6.73π	−0.1733	7.24π	−0.1672	$k = 7$	6.24π	+0.1801	6.74π	+0.1735

In addition to any zero at $x = 0$, each Bessel function has an infinite number of other zeros. The distribution of early positive zeros in a few integer-ordered Bessel functions can be examined in Figure 52-1. Some approximate numerical values for the zeros of J_0 and J_1 will be found in the first and third columns of the table above. Notice that the spacing between consecutive zeros is very close to π. We denote the location of the kth *positive zero of the Bessel function* of order n by $j_n^{(k)}$ (the notation $j_{n,k}$ is a common alternative). Of course $-j_n^{(k)}$ is also a zero, all zeros being defined implicitly by

52:7:2 $$J_n\left(\pm j_n^{(k)}\right) = 0 \qquad k = 1,2,3,\cdots$$

As Figure 52-1 shows, the occurrence of the first zero is increasingly delayed at n increases. For small values of n and k the crude approximation

52:7:3 $$j_n^{(k)} \approx \sqrt{2}\left(n-\tfrac{1}{2}\right)+k\pi \qquad n \text{ and } k \text{ small}$$

holds. To refine this approximation, the Newton-Raphson technique [Section 52:15] may be used.

Equator provides a routine for finding the zeros of Bessel functions of integer order. This procedure makes repeated use of the Newton-Raphson root-finding method [Section 52:15], in the convenient formulation

52:7:4 $$\left(j_n^{(k)}\right)_{m+1} = \left(j_n^{(k)}\right)_m - \frac{J_n\left(j_n^{(k)}\right)_m}{\dfrac{d}{dx}J_n\left(j_n^{(k)}\right)_m} = \left(j_n^{(k)}\right)_m\left[1-\frac{1}{n-\left(j_n^{(k)}\right)_m R_n\left(j_n^{(k)}\right)_m}\right]$$

to improve the mth estimate $\left(j_n^{(k)}\right)_m$ of the zero in creating the $(m+1)$th. The elaborate notation $\left(j_n^{(k)}\right)_m$ represents the mth estimate of the kth zero of the nth order Bessel function and, of course, $J_n\left(j_n^{(k)}\right)_m$ means the Bessel function of order n at that argument. The development of the final expression in 52:7:4 employs equations 52:15:2 and 52:10:1, with definition 52:1:1.

Values of the Bessel function zeros, especially those of J_0 and J_1, are needed in a number of practical problems. Often the application requires, in addition to the zero itself, values of a so-called *associated value of a zero of the Bessel function* for specified values of n and k. This is the value of the *derivative* of the nth Bessel function at its kth zero. The usual notation is $J'_n\left(j_n^{(k)}\right)$, though formula 52:10:1 leads to two equivalent representations:

52:7:5 $$\frac{d}{dx}J_n\left(j_n^{(k)}\right) = J'_n\left(j_n^{(k)}\right) = J_{n-1}\left(j_n^{(k)}\right) = -J_{n+1}\left(j_n^{(k)}\right)$$

The intricate notation in 52:7:5 can be confusing. Figure 52-3 may help. It shows features of a typical integer-order Bessel function J_n in the vicinity of its kth zero, together with its two contiguous congeners J_{n-1} and J_{n+1}.

With the single keyword **JZeros**, *Equator*'s zeros and their associated values of the Bessel function routine sequentially generates the first 9 positive zeros of $J_n(x)$ and their associated values, but the user may replace the "9"

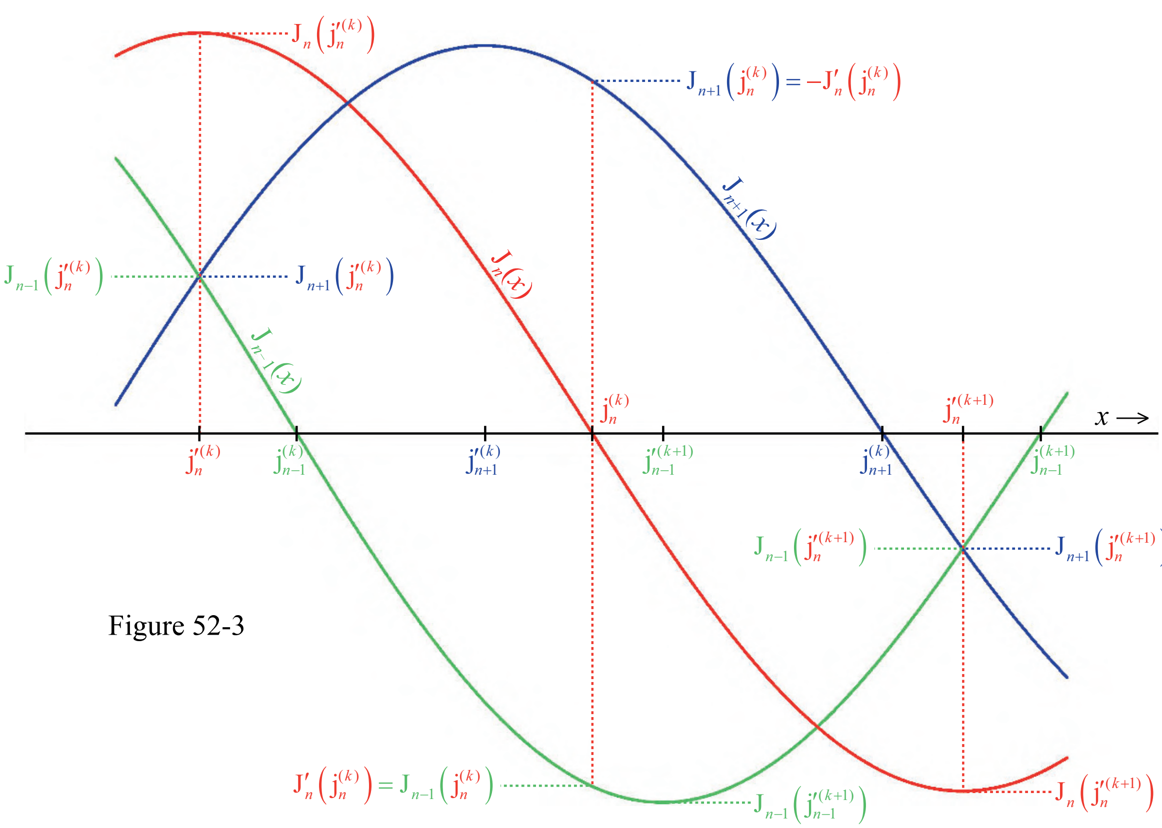


Figure 52-3

by any number up to 250. Any positive integer up to $n = 980$ may be input, but no j_n exceeding 998 is available. This routine uses formulas 52:9:3 to provide $\left(j_n^{(1)}\right)_0$, the initial estimate of the first positive zero of the Bessel function of interest. This estimate of $j_n^{(1)}$ is then refined repeatedly via recursion 52:7:4 until no change occurs. The R ratio required by each recursion is computed by Miller's method, as explained in the next section. Miller's method is used also to calculate the associated value of the first zero as $-J_{n+1}\left(j_n^{(1)}\right)$, in consequence of equation 52:7:5. After the first zero and its associated value have been output, the routine moves on to the second zero and its associated value, then to the third, and so on. The initial estimate of each new zero is provided by the empirical equation

52:7:6 $$\left(j_n^{(k+1)}\right)_0 = j_n^{(k)} + \pi\left[1 + 0.114 k^p \sqrt{n}\right] \qquad p = -3/\left(2n^{0.191}\right)$$

The zeros of the Bessel functions are important in some applications but, in certain others, it is the nonnegative arguments that lead to local maxima or minima of the integer-order Bessel functions that are important. The right-hand columns of the table shown earlier in this section detail the approximate locations and values of early extrema of J_0 and J_1. The argument that generates the kth *extremum of the nth Bessel function* is denoted here by $j_n'^{(k)}$. Thus

52:7:7 $$\frac{d}{dx} J_n\left(j_n'^{(k)}\right) = 0 \qquad k = 1, 2, 3, \cdots$$

When n is positive, odd-k extrema are maxima, even-k extrema are minima. Several examples of a Bessel function extremum, either a local maximum or a local minimum, can be identified in Figure 52-3. The value of the function itself at a Bessel extremum is commonly known as its "associated value", though the adjective "associated" in this name is redundant. Each extremum's value, for example the kth, $J_n(j_n'^{(k)})$, is related through the following relations to the values of its contiguous Bessel functions of the same argument. The general relationship

52:7:8 $$J_n\left(j_n'^{(k)}\right) = \frac{j_n'^{(k)}}{n} J_{n-1}\left(j_n'^{(k)}\right) = \frac{j_n'^{(k)}}{n} J_{n+1}\left(j_n'^{(k)}\right) \qquad \begin{cases} n = 1, 2, 3, \cdots \\ k = 1, 2, 3, \cdots \end{cases}$$

is replaced by

52:7:9 $$J_0\left(j_0'^{(k)}\right) = J_1'\left(j_0'^{(k)}\right) = J_1'\left(j_0'^{(k-1)}\right) \qquad k = 2, 3, 4 \cdots \qquad \text{and} \qquad J_0\left(j_0'^{(1)}\right) = 1$$

when $n = 0$. Again, Figure 52-3 may help you to decipher the elaborate notation.

Equator provides an extrema and their associated values of the Bessel function routine with the keyword **Jextrema**. The principles on which this operates are analogous to those for the zero-finding routine described earlier in this section. The same options and restrictions apply. The salient differences are that equation 52:9:4 is used to find the first extremum and that the equation

52:7:10 $$\left(j_n'^{(k)}\right)_{m+1} = \left(j_n'^{(k)}\right)_m \left[1 - \frac{n - \left(j_n'^{(k)}\right)_m R_n\left(j_n'^{(k)}\right)_m}{n^2 - n + \left(j_n'^{(k)}\right)_m \left\{R_n\left(j_n'^{(k)}\right)_m - \left(j_n'^{(k)}\right)_m\right\}}\right]$$

replace 52:7:4 as the means of estimate improvement. The extrema of the Bessel function of zero order are the zeros of the first-order Bessel function

52:7:11 $$j_0'^{(k)} = j_1^{(k-1)}$$

and that is how they are calculated by *Equator*.

52:8 NUMERICAL VALUES

In view of definition 52:1:1, expansion 52:5:7 may be reformulated as the infinite concatenation

52:8:1
$$\frac{1}{J_0(x)}=1+2\sum_{k=1}^{\infty}\frac{J_{2k}(x)}{J_0(x)}=1+2R_0(x)R_1(x)\Big[1+R_2(x)R_3(x)\Big[1+R_4(x)R_5(x)\Big[1+R_6(x)R_7(x)\big[1+\cdots\big]\Big]\Big]\Big]$$

Now, recursion 52:5:5 may be rewritten as

52:8:2
$$R_{n-1}(x)=\frac{1}{(2n/x)-R_n(x)}$$

and this formula has the useful property that, provided n exceeds x, an error in R_n induces a smaller error in R_{n-1}. This means that including sufficient terms in a curtailed version of concatenation 52:8:1, by setting $R_K(x)=1$, where K is a large enough even number, provides a formula for calculating $J_0(x)$ to any desired accuracy. This, then, leads to the formula

52:8:3
$$J_n(x)=\frac{R_{n-1}(x)R_{n-2}(x)R_{n-3}(x)\cdots R_2(x)R_1(x)R_0(x)}{\Big[\Big[\big[R_{K-1}(x)R_{K-2}(x)+1\big]R_{K-3}(x)R_{K-4}(x)+\cdots+1\Big]R_3(x)R_2(x)+1\Big]2R_1(x)R_0(x)+1}$$

for *any* Bessel function of integer order. The algorithm is known as *Miller's method* [Jeffery Charles Percy Miller, English mathematician, 1906 − 1981] and is the procedure adopted by *Equator*'s Bessel function routine (keyword **J**) whenever v is an integer n lying within $-999\le n\le 999$. The argument x can take any value in the domain $|x|\le 999$. Relationships 52:5:1 and 52:5:2 are exploited when x and/or n is negative.

Miller's method is used by *Equator* also for refining initial estimates of $j_n^{(k)}$ and $j_n'^{(k)}$ via equations 52:7:4 and 52:7:10.

52:9 LIMITS AND APPROXIMATIONS

Apart from the first few members of the family, $J_n(x)$ adopts increasingly positive values, as the argument increases from zero, as long as x remains sufficiently less than n. In this region, the Bessel function behaves as a power function, pursuing the approximation

52:9:1
$$J_n(x)\approx\frac{1}{n!}\left(\frac{x}{2}\right)^n \qquad x<<2\sqrt{n+1} \qquad n\ge 5$$

Based on *McMahon's expansion* (Percy Alexander McMahon, 1854 − 1929, English mathematician born in Malta), the approximation

52:9:2
$$j_n^{(k)}\approx\frac{\Upsilon}{4}\left(1-\frac{8n^2-2}{\Upsilon^2}\left(1-\frac{28n^2-31}{3\Upsilon^2}\right)\right) \qquad \Upsilon=\pi(4k+2n-1) \qquad \text{large } k$$

to the Bessel zeros is excellent when $k>>n$, but even for $k=1$, $n=10$, it gives an answer that is adequate to serve as the initial estimate for a sequence of *Raphson rule* [Section 52:15] improvements. For large orders, the location of the first zero is well approximated by the empirical formula

52:9:3
$$j_n^{(1)}\approx n+1.8558n^{1/3}+1.055n^{-1/3} \qquad \text{large positive } n$$

Clearly, this approximation fails for $n=0$, but even for $n=1$ it is good enough to initiate an improvement sequence. Though approximation 52:9:2 is designed to find *zeros*, it provides good estimates of the kth Bessel *extremum* if the definition of Υ is changed to $\pi(4k+2n-3)$. Likewise, modification of 52:9:3 to

52:9:4
$$j_n'^{(1)}\approx n+0.8086n^{1/3}+0.072n^{-1/3} \qquad \text{large positive } n$$

provides an excellent approximation of the first Bessel extremum (always a maximum). Abramowitz and Stegun [Section 9.5] list additional terms for formulas 52:9:2−4.

For very large arguments the oscillations of the Bessel functions become sinusoidal. For example

52:9:5 $$\left.\begin{aligned} J_0(x) &\to \sqrt{2/\pi x}\,\sin(x+\tfrac{1}{4}\pi) \\ J_1(x) &\to \sqrt{2/\pi x}\,\sin(x-\tfrac{1}{4}\pi) \end{aligned}\right\} \quad n\to\infty$$
52:9:6

52:10 OPERATIONS OF THE CALCULUS

Beyond those listed here, other formulas for differentiation and integration come from specializing results from Section 53:10 to integer orders.

The formulas

52:10:1 $$\frac{d}{dx}J_0(x) = -J_1(x)$$

and

52:10:2 $$\frac{d}{dx}J_1(x) = J_0(x) - \frac{J_1(x)}{x} = \frac{J_0(x)-J_2(x)}{2}$$

are important special cases of the differentiation formula

52:10:3 $$\frac{d}{dx}J_n(x) = \frac{J_{n-1}(x)-J_{n+1}(x)}{2} = J_{n-1}(x) - \frac{n}{x}J_n(x) = \frac{n}{x}J_n - J_{n+1}(x)$$

The coefficients occurring in the numerators of the formulas

52:10:4 $$\frac{d^2}{dx^2}J_n(x) = \frac{J_{n-2}(x)-2J_n(x)+J_{n+2}(x)}{2^2}$$

and

52:10:5 $$\frac{d^3}{dx^3}J_n(x) = \frac{J_{n-3}(x)-3J_{n-1}(x)+3J_{n+1}-J_{n+3}(x)}{2^3}$$

for multiple differentiation will be recognized as binomial coefficients [Chapter 6] of alternating sign, and this property generalizes.

Struve functions [Chapter 57] occur in the formula for integration of even-ordered Bessel functions

52:10:6 $$\int_0^x J_n(t)\,dt = \frac{\pi x}{2}\left[J_1(x)h_0(x)-J_0(x)h_1(x)\right] + xJ_0(x) - 2\left\{J_1(x)+J_3(x)+\cdots+J_{n-1}(x)\right\} \qquad n=0,2,4,\cdots$$

but not in those for odd integer orders

52:10:7 $$\int_0^x J_n(t)\,dt = 1 - J_0(x) - 2\left\{J_2(x)+J_4(x)+\cdots+J_{n-1}(x)\right\} \qquad n=1,3,5,\cdots$$

These integration formulas are simpler when $n=0$ or 1, for then the terms in braces are zero. Other useful indefinite integrals include

52:10:8 $$\int_x^\infty \frac{J_0(t)}{t}\,dt = \ln\left(\frac{2}{x}\right) - \gamma - \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{(j!)^2 j}\left(\frac{-x^2}{4}\right)^j$$

52:10:9 $$\int_x^\infty \frac{J_1(t)}{t}\,dt = 1 - xJ_0(x) + J_1(x) - \frac{\pi x}{2}\left[J_1(x)h_0(x)-J_0(x)h_1(x)\right]$$

and

52:10:10 $$\int_0^x J_0(t)\,J_1(x-t)\,dt = J_0(x) - \cos(x)$$

The operations of semidifferentiation and semiintegration [Section 12:14] generate sinusoids when applied to Bessel functions of appropriate argument, as in the examples

52:10:11 $$\frac{d^{1/2}}{dx^{1/2}} J_0\left(\sqrt{x}\right) = \frac{\cos\left(\sqrt{x}\right)}{\sqrt{\pi x}}$$

and

52:10:12 $$\frac{1}{2}\frac{d^{-1/2}}{dx^{-1/2}} J_0\left(\sqrt{x}\right) = \frac{d^{1/2}}{dx^{1/2}}\left\{\sqrt{x}\,J_1\left(\sqrt{x}\right)\right\} = \frac{\sin\left(\sqrt{x}\right)}{\sqrt{\pi x}}$$

Definite integrals and Laplace transforms include

52:10:13 $$\int_0^\infty J_n(x)\,dx = 1 \qquad n = 0,1,2,\cdots$$

52:10:14 $$\int_0^\infty \frac{J_n(bt)}{t}\,dt = \frac{1}{n} \qquad n = 1,2,3,\cdots$$

52:10:15 $$\int_0^\infty \cos(\beta t)\,J_0(bt)\,dt = \begin{cases} 0 & \beta < b \\ (\beta^2 - b^2)^{-1/2} & \beta > b \end{cases}$$

52:10:16 $$\int_0^\infty \frac{[1 - J_0(\beta t)]\,J_0(bt)}{t}\,dt = \begin{cases} 0 & 0 < \beta \le b \\ \ln(\beta/b) & 0 < b \le \beta \end{cases}$$

52:10:17 $$\int_0^\infty \frac{t\,J_0(bt)}{\sqrt{t^2 + a^2}}\,dt = \frac{\exp(-ab)}{b}$$

52:10:18 $$\int_0^\infty t^n\,J_n(bt)\exp(-st)\,dt = \mathcal{L}\left\{t^n\,J_n(bt)\right\} = \frac{(2n-1)!!b^n}{\left(s^2 + b^2\right)^{(2n+1)/2}}$$

and

52:10:19 $$\int_0^\infty \frac{J_1\left(\sqrt{bt}\right)}{\sqrt{t}}\exp(-st)\,dt = \mathcal{L}\left\{\frac{J_1\left(\sqrt{bt}\right)}{\sqrt{t}}\right\} = \frac{2}{\sqrt{b}}\left[1 - \exp\left(\frac{-b}{4s}\right)\right]$$

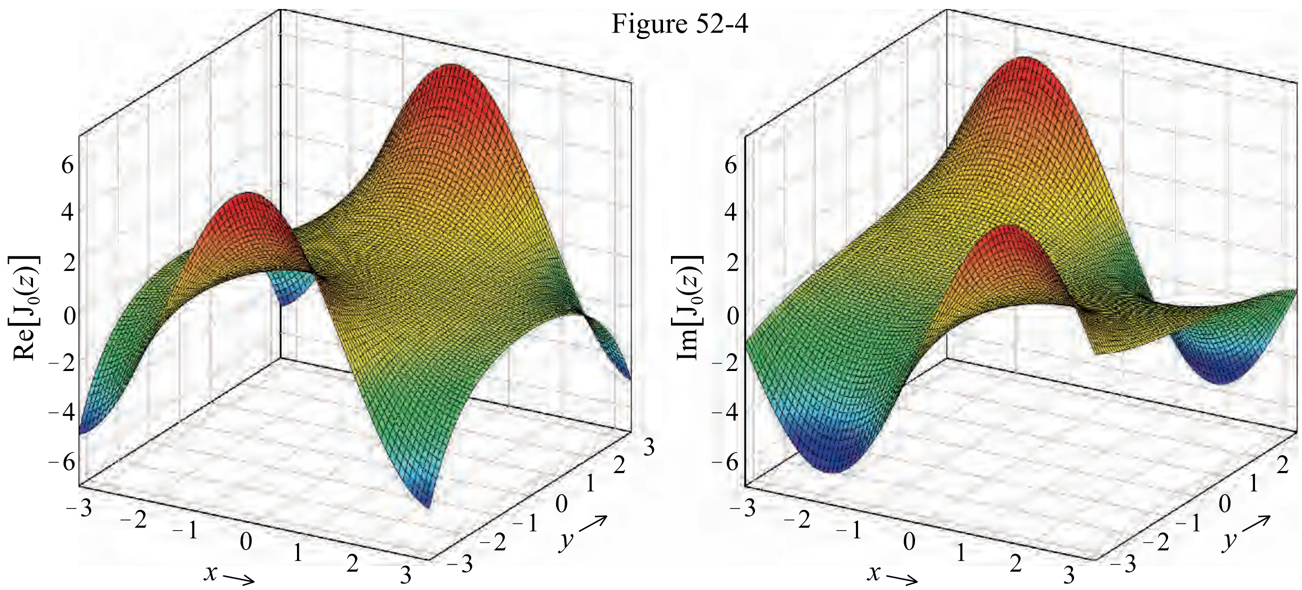
With $n = 0$ or 1, Erdélyi et al. [*Tables of Integral Transforms*, Volume 2, Pages 9−21] list 79 transforms of the form

52:10:20 $$\int_0^\infty \sqrt{yt}\,J_n(yt)\,f(t)\,dt$$

These (or sometimes the corresponding integrals without the $\sqrt{yt}$ factor) are known as *Hankel transforms* of f(t). Some examples in which the order is arbitrary will be found in Section 53:10.

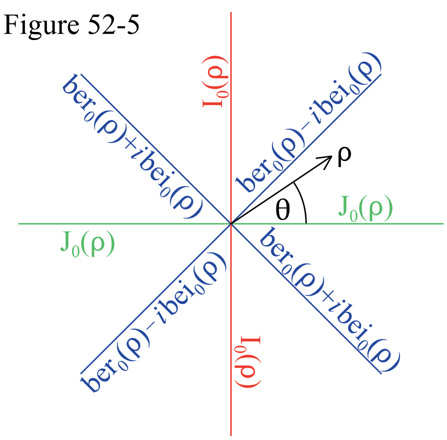
52:11 COMPLEX ARGUMENT

Figure 52-4



There are no discontinuities or singularities in the complex plane for any integer-ordered Bessel function, which means that they are classified as *entire functions*. Figure 52-4 shows the magnitudes of the real and imaginary parts of the $J_0(x+iy)$ function for values of the real and complex variables in the domain $-\pi$ to $+\pi$. A careful comparison of Figure 52-4 with Figure 49-2, shows that, for both the real and the imaginary parts, the Bessel and modified Bessel functions, each of order zero, may be interconverted merely by rotation through an angle of 90°. The reason is clear on study of the polar diagrams Figures 52-5 and 49-3: they are identical apart from orientation. Figure 52-5 shows how several functions arise as specializations of $J_0(z)$ in the complex plane because

Figure 52-5



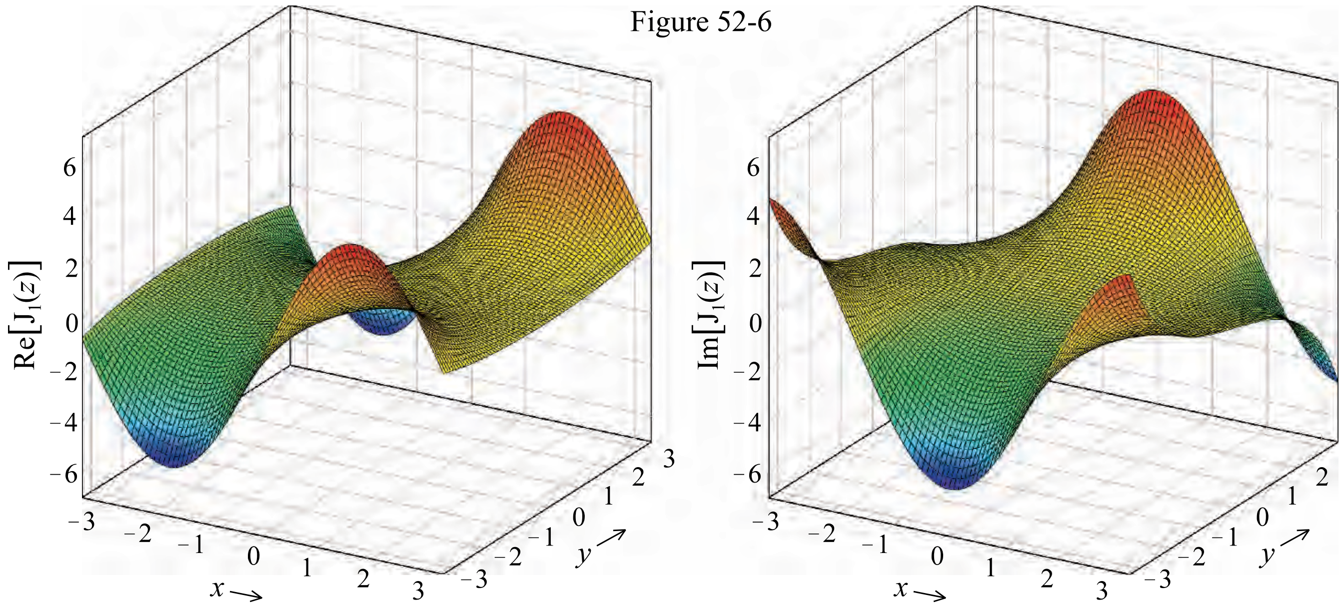
$$\text{52:11:1} \quad J_0(z) = J_0\big(\rho\exp(i\theta)\big) = \begin{cases} J_0(\rho) & \theta = 0 \text{ or } \pm\pi \\ \mathrm{ber}_0(\rho) - i\,\mathrm{bei}_0(\rho) & \theta = \tfrac{1}{4}\pi \text{ or } -\tfrac{3}{4}\pi \\ I_0(\rho) & \theta = \pm\tfrac{1}{2}\pi \\ \mathrm{ber}_0(\rho) + i\,\mathrm{bei}_0(\rho) & \theta = \tfrac{3}{4}\pi \text{ or } -\tfrac{1}{4}\pi \end{cases}$$

The ber and bei functions are addressed in Chapter 55.

Figures 52-6 and 52-7 are the analogues of Figures 52-4 and 52-5 for the first-order Bessel function. Again there is rotational correspondence between the three-dimensional diagrams and those of Chapter 49 but now a 90° rotation brings coincidence between the *real* moiety of Figure 52-6 and the *imaginary* moiety of Figure 49-4, and vice versa. The functions sharing a complex plane with $J_1(\rho)$ are indicated in the polar Figure 52-7. Again, a single three-dimensional landscape incorporates the two-dimensional terrain of several functions.

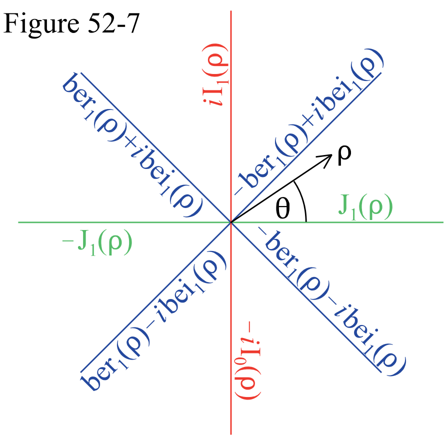
For general integer order, Bessel functions of complex argument and integer order are related to other cylinder functions through the equations

Figure 52-6



52:11:2 $$J_n(z) = \begin{cases} i^n \, \mathrm{I}_n(-iz) \\ \mathrm{H}_n^{(1)}(z) - i\,\mathrm{Y}_n(z) \\ \mathrm{H}_n^{(2)}(z) + i\,\mathrm{Y}_n(z) \\ \pm\mathrm{ber}_n\left(\mp\frac{1+i}{\sqrt{2}}z\right) \pm i\,\mathrm{bei}_n\left(\mp\frac{1+i}{\sqrt{2}}z\right) \end{cases}$$

Figure 52-7



The $\mathrm{H}_n^{()}$ and Y_n are Hankel [Section 49:14] and Neumann [Chapter 54] functions, respectively.

The three-dimensional diagrams in this section focus on the region near the origin. For a more global viewpoint, see Section 53:11. Likewise, see Section 53:11 for inverse Laplace transforms, which apply equally to integer and non-integer values of the order v.

52:12 GENERALIZATIONS

The next chapter addresses the generalization of the order of Bessel functions to arbitrary real values. The general $J_v(x)$ specializes to $J_n(x)$ when v becomes an integer, but another function also specializes to $J_n(x)$ when its order v becomes an integer. This is the *Anger function*. For arbitrary order, this function is defined by the integral

52:12:1 $$\frac{1}{\pi}\int_0^\pi \cos\{x\sin(t) - vt\}\,\mathrm{d}t$$

which becomes identical with definition 52:3:4 when v is an integer. Closely related is the *Weber function*

52:12:2 $$\frac{-1}{\pi}\int_0^\pi \sin\{x\sin(t) - vt\}\,\mathrm{d}t$$

See Erdélyi et al. [*Higher Transcendental Functions*, Section 7.5.3] or Thompson [Section 15.2] for further information on the Anger and Weber functions, neither of which is addressed further in this *Atlas*.

The function $J_0\left(2\sqrt{x}\right)$ is the prototype, or basis function [Section 43:14] of the $L = K+2 = 2$ family of hypergeometric functions and, from this viewpoint, all the functions in the right-hand column of Table 18-5 represent generalizations of the zero-order Bessel function.

52:13 COGNATE FUNCTIONS

The Struve functions [Chapter 57] have much in common with Bessel functions.

In common with many other functions in this *Atlas*, Bessel functions are complemented by a pair of *auxiliary functions*. These describe the properties of Bessel functions at large arguments more efficiently than do the formulas for the J functions themselves. They are symbolized $\mathrm{fc}_\nu(x)$ and $\mathrm{gc}_\nu(x)$ and are named *auxiliary cylinder functions*, because they serve *all* the functions in Chapters 49–56, and not solely the Bessel functions. Read about these functions in Section 54:14.

52:14 RELATED TOPIC: the orthogonality of Bessel functions

An important definite integral involving the product of two Bessel functions is

52:14:1
$$\int_0^1 t\,J_n(\alpha t)J_n(\beta t)\,dt = \begin{cases} \dfrac{1}{\beta^2-\alpha^2}\left[\alpha J_n(\beta)\dfrac{dJ_n}{dt}(\alpha) - \beta J_n(\alpha)\dfrac{dJ_n}{dt}(\beta)\right] & \beta \neq \alpha \\ \dfrac{1}{2\alpha^2}\left[\left\{\alpha\dfrac{dJ_n}{dt}(\alpha)\right\}^2 - (n^2-\alpha^2)\left\{J_n(\alpha)\right\}^2\right] & \beta = \alpha \end{cases}$$

On choosing α and β to be any two of the zeros of the nth-order Bessel function, say $j_n^{(k)}$ and $j_n^{(l)}$, the upper and lower right-hand options of 52:14:1 become, respectively, zero and half the square of the associated value of the zero. Thus

52:14:2
$$\int_0^1 t\,J_n\left(j_n^{(k)}t\right)J_n\left(j_n^{(\ell)}t\right)dt = \begin{cases} 0 & \ell \neq k \\ \frac{1}{2}\left\{J'\left(j_n^{(k)}\right)\right\}^2 & \ell = k \end{cases}$$

In the terminology discussed in Section 21:14, the functions $J_n\left(j_n^{(1)}t\right), J_n\left(j_n^{(2)}t\right), J_n\left(j_n^{(3)}t\right), \cdots$, which we now abbreviate to $\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$, ⋯ , are seen to be orthogonal on the interval $0 \le t \le 1$ with a weight function of $w(t) = t$.

Hence, by a development that parallels the one in Section 21:15, many functions $f(x)$ defined in the $0 \le x \le 1$ domain may be expanded in the orthogonal series

52:14:3
$$f(x) = c_1\Psi_1(x) + c_2\Psi_2(x) + c_3\Psi_3(x) + \cdots = \sum_{k=1}^{\infty} c_k\Psi_k(x) = \sum_{k=1}^{\infty} c_k J_n\left(j_n^{(k)}x\right)$$

where

52:14:4
$$c_k = \frac{2}{\left\{J'\left(j_n^{(k)}\right)\right\}^2}\int_0^1 t\,f(t)\,J_n\left(j_n^{(k)}t\right)dt$$

While this orthogonality expansion may be based on any integer-order Bessel function, J_0 is often the most convenient.

Instead of choosing the α and β in 52:14:1to be the *zeros* of a Bessel function, one may opt to make them *extrema*. Then the analogue of 52:14:2 becomes

52:14:5
$$\int_0^1 t\,J_n\left(j_n'^{(k)}t\right)J_n\left(j_n'^{(\ell)}t\right)dt = \begin{cases} 0 & \ell \neq k \\ \tfrac{1}{2}\left[1-\left(n/j_n'^{(k)}\right)^2\right]\left\{J'\left(j_n^{(k)}\right)\right\}^2 & \ell = k \end{cases}$$

Accordingly, orthogonal expansions may be developed from this basis too. In fact, as Spiegel [pages 144−145] shows, the orthogonality of Bessel functions may be generalized more widely still.

52:15 RELATED TOPIC: root-finding by the Newton-Raphson method

Frequently a need exists for *numerical inversion*; that is, to find the numerical value of a function's argument at which the function equals a known number, c. The sought value, denoted r in this section, is called a *root* of the equation $g(x) = c$ or, if $c = 0$, a *zero* of the function f. When no analytical method of finding r exists, iterative methods must be used and there follows a description of one of the commonest numerical methods, attributed to Sir Isaac Newton (1643 −1727) and Joseph Raphson (1648 −1715), contemporary English mathematicians who discovered the method independently.

The root of $g(x) = c$ is, of course, the zero of the difference function $f(x) = g(x)-c$, so it suffices to couch our description as one of finding a number r such that $f(r) \approx 0$, to whatever degree of accuracy is desired. To carry out the Newton-Raphson method requires three items:

(a) a method of calculating f accurately,
(b) a method of calculating the derivative df/dx accurately, and
(c) a preliminary estimate r_0 of the zero.

How close to r the estimate r_0 needs to be depends on the shape of the function f. The estimate should certainly be closer than it is to any other zero, or to any extremum, that $f(x)$ may possess.

The method is iterative: one first uses r_0 to find a better approximation r_1, then improves this to r_2, and so on. The Newton-Raphson method is based on the reasonable premise that the third and subsequent right-hand terms in the *Taylor expansion* [Section 0:5]

52:15:1
$$f(r) = f(r_m) + (r-r_m)\frac{df}{dx}(r_m) + \frac{(r-r_m)^2}{2}\frac{d^2f}{dx^2}(r_m) + \cdots \qquad m = 0,1,2,\cdots$$

will be small if r_m is sufficiently close to r. Moreover, by definition, the left-hand term is zero. It follows that

52:15:2
$$r_m - \frac{f(r_m)}{(df/dx)(r_m)} \approx r$$

This approximation to r is assigned to be the next estimate r_{m+1}:

52:15:3
$$r_{m+1} = r_m - \frac{f(r_m)}{(df/dx)(r_m)} \qquad m = 0,1,2,\cdots$$

and the procedure is repeated. Convergence is generally very rapid. Formula 52:15:3 is named the *Raphson rule* and it finds widespread application in the location of such zeros as those of the Bessel function.

There are pitfalls in blindly using the Newton-Raphson method. Press et al. [Chapter 9] give a very readable description of these and of alternative root-finding procedures.

CHAPTER

53

THE BESSEL FUNCTION $J_\nu(x)$ OF ARBITRARY ORDER

The Bessel family of functions are among the most fructuous. The classic assemblage of information on the many valuable properties of these functions, as well as other cylinder functions, is contained in the copious treatise of the English mathematician George Neville Watson (1886 – 1965). The preceding chapter deals with the special properties of those Bessel functions for which the order ν is an integer; here ν is any real number.

53:1 NOTATION

The symbol $J_\nu(x)$ is universal for a Bessel function of order ν and argument x. As explained in Section 49:14, the name "Bessel function" is often taken to apply generally to all or many of the cylinder functions [those of Chapters 49 – 56]. When that broadened nomenclature is espoused, the J function is said to be a *Bessel function of the first kind*. In this *Atlas*, however, "of the first kind" is redundant because, in the absence of a "modified" qualifier, Bessel's name is attached here only to those cylinder functions that are denoted by the letter J; that is, to the functions addressed in this chapter and the preceding one.

A change of argument to $2\sqrt{x}$ often simplifies formulas. Accordingly, the adoption of *Clifford's notation*

53:1:1 $$C_\nu(x) = x^{-\nu/2} J_\nu\left(2\sqrt{x}\right)$$

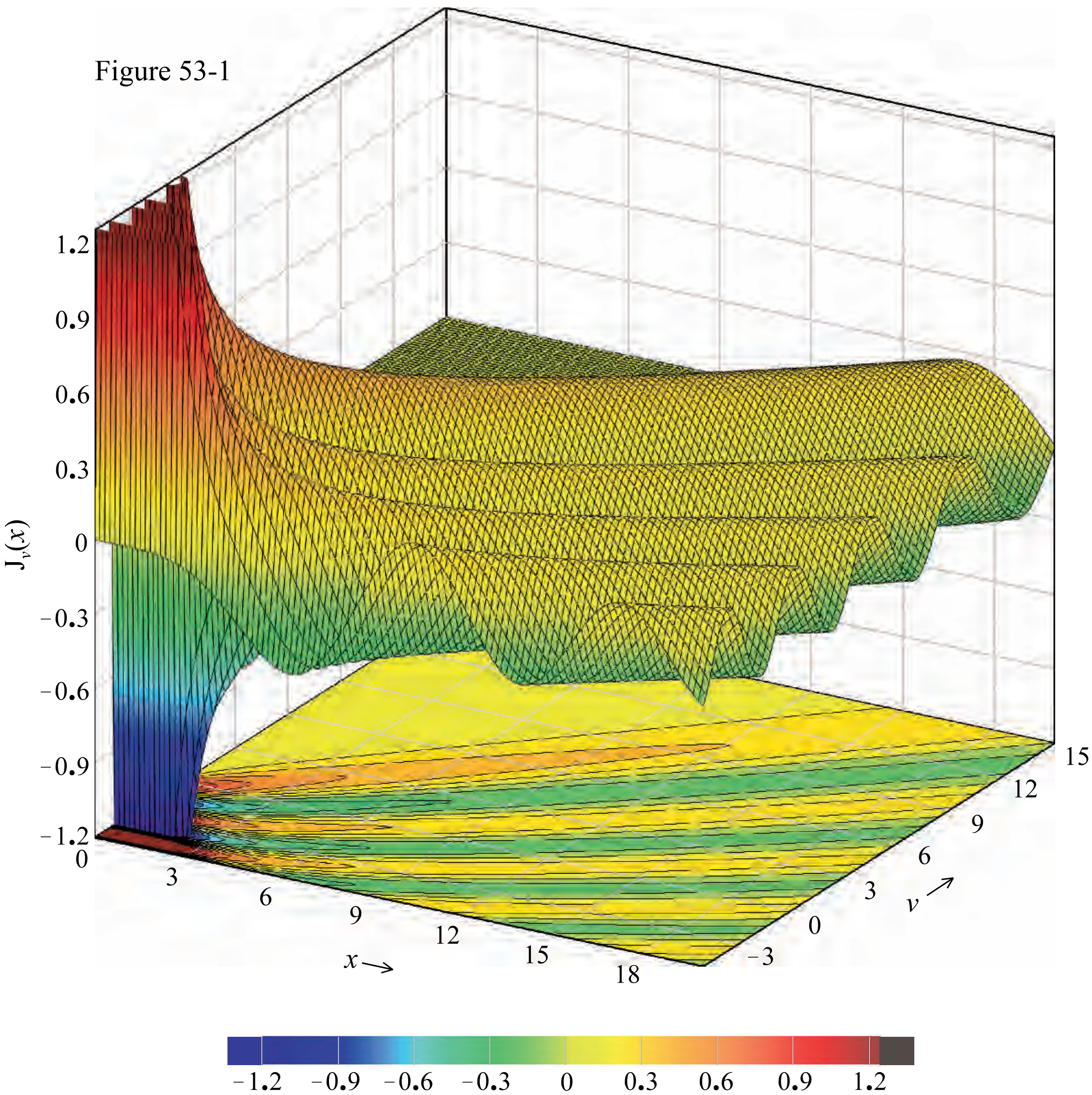
(William Kingdon Clifford, 1845 – 1879, English philosopher, mathematician, and writer of fairytales) frequently abbreviates the algebra. One advantage of Clifford's notation is that, whereas $J_\nu(x)$ is generally defined as a real function only for $x \ge 0$, $C_\nu(x)$ is real for all values of its argument. Nevertheless this notation is not used in the *Atlas*, because J is traditional.

53:2 BEHAVIOR

Except when ν is an integer, the Bessel function $J_\nu(x)$ is complex for negative argument. Therefore, other than in Section 11, $x \ge 0$ is the only argument range considered in this chapter, and the only range depicted in Figure 53-1.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_54, © Springer Science+Business Media, LLC 2009

Figure 53-1



As this diagram suggests, there is an infinite number of alternating zones of positive and negative values crossing the x,ν plane diagonally. This implies that $J_\nu(x)$, viewed as a function of either x or ν, displays an infinite number of zeros, separated by local maxima and minima. The only discontinuities occur on the $x = 0$ boundary and even there, only when ν encounters a nonpositive integer.

At zero argument, the Bessel function is zero when $\nu > 0$. In this range of orders, and with x positive, there is no conspicuous distinction between integer-ordered and noninteger-ordered behavior. Thus the general pattern discussed in Section 52:2 carries over to noninteger ν's. The Bessel function is confined in value to the narrow range $-0.403 < J_\nu(x) \le 1$ whenever ν is positive. The same range of magnitude applies when the order has one of the values $0, -1, -2, \cdots$. Bessel functions of these nonpositive integer orders merely duplicate those of the corresponding positive order, though with a sign change if ν is odd. In contrast, however, when ν is a negative *non*integer, $J_\nu(x)$ comes to acquire a limitlessly large magnitude, positive or negative, as $x = 0$ is approached.

At constant small ν, $J_\nu(x)$ is oscillatory throughout the whole $x > 0$ domain. As ν takes larger values, of either sign, there develops a zone, extending approximately over $0 < x < |\nu|$, in which the Bessel function behaves monotonically, prior to breaking into oscillations. Once they are established, the oscillations steadily attenuate, remaining centered about a magnitude of zero.

53:3 DEFINITIONS

The function $(\frac{1}{2}x)^\nu/\Gamma(1+\nu)$ is a generating function for the $J_\nu(x)$, $J_{\nu+1}(x)$, $J_{\nu+2}(x)$, $\cdots$ set of Bessel functions:

53:3:1
$$\frac{\left(\frac{1}{2}x\right)^\nu}{\Gamma(1+\nu)} = \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^j}{j!} J_{\nu+j}(x)$$

The related expansion 53:6:1 often serves as a definition.

Bessel functions may be defined by a number of definite integrals, including

53:3:2
$$J_\nu(x) = \frac{\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\pi \cos\{x\cos(t)\}\sin^{2\nu}(t)\,dt$$

53:3:3
$$J_\nu(x) = \frac{2\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 [1-t^2]^{\nu-\frac{1}{2}} \cos(xt)\,dt$$

and

53:3:4
$$J_\nu(x) = \frac{2(2/x)^\nu}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{\sin(xt)}{[t^2-1]^{\nu+\frac{1}{2}}}\,dt$$

These do not necessarily converge for all orders. Over a dozen additional integral representations are given by Gradshteyn and Ryzhik [Section 8.41].

The solution to an important differential equation, called *Bessel's equation*:

53:3:5
$$x\frac{d}{dx}\left(x\frac{df}{dx}\right) = x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} = (\nu^2-x^2)f \qquad f = w_1x^{-\nu/2}J_\nu(x) + \begin{cases} w_2x^{-\nu/2}\,Y_\nu(x) \\ w_2x^{-\nu/2}\,J_{-\nu}(x) \end{cases}$$

is another way of defining the Bessel function and this is how the function often arises in practice. A second differential equation, the *Bessel-Clifford equation*, has a similar solution

53:3:6
$$x\frac{d^2f}{dx^2} + (1+\nu)\frac{df}{dx} + f = 0 \qquad f = w_1x^{-\nu/2}J_\nu(x) + \begin{cases} w_2x^{-\nu/2}\,Y_\nu\left(2\sqrt{x}\right) \\ w_2x^{-\nu/2}\,J_{-\nu}\left(2\sqrt{x}\right) \end{cases}$$

In these solutions the w's are arbitrary weights. Very many other second-order differential equations are solved by the Bessel function. In fact *any* differential equation that can be manipulated into the form

53:3:7
$$x^2\frac{d^2f}{dx^2} + (1-2a)x\frac{df}{dx} + \left[a^2-\nu^2p^2+\left(bpx^p\right)^2\right]f = 0 \qquad f = w_1x^a J_\nu(bx^p) + \begin{cases} w_2x^a\,Y_\nu(bx^p) \\ w_2x^a\,J_{-\nu}(bx^p) \end{cases}$$

for any real values of a, b, ν and p, is solved by the arbitrarily weighted combination of cylinder functions shown. A Bessel function also solves another very general differential equation

53:3:8
$$\frac{d^2f}{dx^2} + a^2x^{(1/\nu)-2}f = 0 \qquad f = w_1\sqrt{x}\,J_\nu\left(2\nu ax^{1/2\nu}\right) + \begin{cases} w_2\sqrt{x}\,Y_\nu\left(2\nu ax^{1/2\nu}\right) \\ w_2\sqrt{x}\,J_{-\nu}\left(2\nu ax^{1/2\nu}\right) \end{cases}$$

Two alternatives are shown for the second solution in the four previous equations. The first, incorporating a Neumann function [Chapter 54], is valid unreservedly. The second alternative applies only if ν is *not* an integer.

The differintegration [Section 12:14] formula

53:3:9 $$\frac{d^{\frac{1}{2}-\nu}}{dx^{\frac{1}{2}-\nu}}\sin\left(\sqrt{x}\right)=\frac{\sqrt{\pi x^{\nu}}}{2^{1-\nu}}J_\nu\left(\sqrt{x}\right)$$

which is valid for all ν, illustrates the close relationship of the Bessel function to the sine function. Any Bessel function may be synthesized [Section 43:14] from the zero-order case:

53:3:10 $$J_0\left(2\sqrt{x}\right)\xrightarrow[\nu+1]{1}\frac{\Gamma(1+\nu)}{x^{\nu/2}}J_\nu\left(2\sqrt{x}\right)$$

Both these formulas benefit from being redrafted in Clifford's notation [Section 53:1].

53:4 SPECIAL CASES

The special properties of Bessel functions of integer order are addressed in Chapter 52.

Bessel functions of an order that is one-half of an odd integer, positive or negative, are known as *spherical Bessel functions*. They are represented by a lower-case symbol

53:4:1 $$\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)=j_n(x)\qquad n=0,\pm1,\pm2,\cdots$$

and are the subject of Section 32:13. As well, such Bessel functions are coincident in magnitude with *Neumann functions* [Chapter 54] of the same argument but of an order of opposite sign

53:4:2 $$J_{n+\frac{1}{2}}(x)=(-)^n\,Y_{-n-\frac{1}{2}}(x)\qquad n=0,\pm1,\pm2,\cdots$$

When the order is $\frac{1}{3}$ or $-\frac{1}{3}$, the Bessel function is related, somewhat tortuously, to the *Airy functions* Ai and Bi of Chapter 56:

53:4:3 $$J_{\pm\frac{1}{3}}(x)=\frac{1}{2}\sqrt{\frac{3}{\hat{x}}}\left[\sqrt{3}\mathrm{Ai}(-\hat{x})\mp\mathrm{Bi}(-\hat{x})\right]\qquad \hat{x}=\sqrt[3]{\frac{9x^2}{4}}$$

Similarly, the Bessel functions of order $\frac{2}{3}$ or $-\frac{2}{3}$ are expressible through derivatives of the Airy functions:

53:4:4 $$J_{\pm\frac{2}{3}}(x)=\frac{\sqrt{3}}{2\hat{x}}\left[\frac{d\,\mathrm{Bi}}{d\hat{x}}(-\hat{x})\pm\sqrt{3}\frac{d\,\mathrm{Ai}}{d\hat{x}}(-\hat{x})\right]$$

The recursion formula 53:5:3 allows expressions for such Bessel functions as $J_{-\frac{4}{3}}(x)$ and $J_{\frac{5}{3}}(x)$ to be constructed from 53:4:3 and 53:4:4.

53:5 INTRARELATIONSHIPS

The reflection-about-zero property, applicable to both argument and order, that is such a simplifying feature when the order of a Bessel function is an integer, does not extend to noninteger orders. Order-reflection generalizes to the relationship

53:5:1 $$J_{-\nu}(x)=\cos(\nu\pi)J_\nu(x)-\sin(\nu\pi)Y_\nu(x)$$

that applies for all ν, reducing to $J_{-\nu}(x)=(-)^\nu J_\nu(x)$ when ν is an integer. The similar formula

53:5:2 $$J_{-1-\nu}(x)J_\nu(x)=-J_{1+\nu}(x)J_{-\nu}(x)-\frac{2\sin(\nu\pi)}{\pi x}$$

likewise simplifies when ν is an integer.

Bessel functions of all orders obey the recursion formula

53:5:3 $$\mathrm{J}_{\nu+1}(x) = \frac{2\nu}{x}\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x)$$

and *Neumann's addition theorem*

53:5:4 $$\mathrm{J}_\nu(x \pm y) = \sum_{n=-\infty}^{\infty} \mathrm{J}_{\nu\mp n}(x)\mathrm{J}_n(y) \qquad |y|<|x|$$

The argument-multiplication formula

53:5:5 $$\mathrm{J}_\nu(bx) = b^\nu \sum_{j=0}^{\infty}\left[\tfrac{1}{2}(1-b^2)x\right]^j \frac{\mathrm{J}_{\nu+j}(x)}{j!}$$

may be reformulated as

53:5:6 $$\sum_{j=0}^{\infty}\frac{x^j}{j!}\mathrm{J}_{\nu+j}(ax) = \left(\frac{a}{a-2}\right)^{\nu/2}\mathrm{J}_\nu\left(\sqrt{a^2-2a}\,x\right)$$

and setting $a = 1$ reveals that

53:5:7 $$\mathrm{J}_\nu(x) + x\mathrm{J}_{\nu+1}(x) + \frac{x^2}{2!}\mathrm{J}_{\nu+2}(x) + \frac{x^3}{3!}\mathrm{J}_{\nu+3}(x) + \cdots = (-i)^\nu \mathrm{J}_\nu(ix) = \mathrm{I}_\nu(x)$$

A summation formula is

53:5:8 $$\sum_{j=0}^{\infty}\frac{\nu+2j}{j!}\Gamma(\nu+j)\mathrm{J}_{\nu+2j}(x) = \left(\frac{x}{2}\right)^\nu$$

When $\nu = 0$ or 1, this reduces to 52:5:7 or 52:5:10.

53:6 EXPANSIONS

Bessel functions are expansible as the power series

53:6:1 $$\mathrm{J}_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)} - \frac{(x/2)^{\nu+2}}{1!\Gamma(2+\nu)} + \frac{(x/2)^{\nu+4}}{2!\Gamma(3+\nu)} - \cdots = \sum_{j=0}^{\infty}\frac{(-)^j(x/2)^{2j+\nu}}{j!\Gamma(1+\nu+j)}$$

that is convergent for all orders, though some leading terms vanish if ν is a negative integer. That this is an $L = K+2 = 2$ hypergeometric series [Section 18:14] is evident by rewriting the sum as in

53:6:2 $$\mathrm{J}_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)}\sum_{j=0}^{\infty}\frac{1}{(1)_j(1+\nu)_j}\left(\frac{-x^2}{4}\right)^j$$

As is often the case, this last expression may be written more economically in Clifford's notation [equation 53:1:1]. The product of two Bessel functions may also be expanded hypergeometrically

53:6:3 $$\mathrm{J}_\nu(x)\mathrm{J}_\mu(x) = \frac{(x/2)^{\nu+\mu}}{\Gamma(1+\nu)\Gamma(1+\mu)}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)_j\left(1+\frac{1}{2}\nu+\frac{1}{2}\mu\right)_j}{(1)_j(1+\nu)_j(1+\mu)_j(1+\nu+\mu)_j}\left(-x^2\right)^j$$

in this case with $L = K+2 = 4$.

Any Bessel function of positive order may be expanded as an infinite product

53:6:4 $$\mathrm{J}_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)}\left[1-\left(\frac{x}{\mathrm{j}_\nu^{(1)}}\right)^2\right]\left[1-\left(\frac{x}{\mathrm{j}_\nu^{(2)}}\right)^2\right]\left[1-\left(\frac{x}{\mathrm{j}_\nu^{(3)}}\right)^2\right]\cdots = \frac{(x/2)^\nu}{\Gamma(1+\nu)}\prod_{k=1}^{\infty}1-\left(\frac{x}{\mathrm{j}_\nu^{(k)}}\right)^2$$

where $j_\nu^{(k)}$ is the kth positive zero of the function.

Though they are more useful for integer orders, equations 52:6:6−8 remain valid when n is replaced by ν.

When x, but not $|\nu|$, is large, *Hankel's asymptotic expansion*

53:6:5 $$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\left[\mathrm{fc}_\nu(x)\cos\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)-\mathrm{gc}_\nu(x)\sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right]$$

is useful, where the *auxiliary cylinder functions* fc_ν and gc_ν are defined and discussed in Section 54:14.

53:7 PARTICULAR VALUES

At zero argument, the Bessel function acquires one of four values

53:7:1 $$J_\nu(0)=\begin{cases}-\infty & -2<\nu<-1,\ -4<\nu<-3,\ -6<\nu<-5,\cdots\\ +\infty & -1<\nu<0,\ -3<\nu<-2,\ -5<\nu<-4,\cdots\\ 1 & \nu=0\\ 0 & \begin{cases}\nu=-1,-2,-3,\cdots\\ \nu>0\end{cases}\end{cases}$$

with discontinuities at each nonpositive integer. All Bessel functions approach zero as $x\to\infty$.

The argument values that cause $J_\nu(x)$ to adopt a magnitude of zero are known as the *zeros of the Bessel function*. The kth such value is denoted $j_\nu^{(k)}$ in this *Atlas*.

53:7:2 $$J_\nu(j_\nu^{(k)})=0 \qquad k=1,2,3,\cdots$$

There is an infinite number of such zeros. The allocation of the number k may easily mislead. As 53:7:1 shows, the Bessel function is often zero at $x=0$, but this is not counted in numbering the zeros (or you could consider it the $j_\nu^{(0)}$ member). Thus $j_\nu^{(1)}$ is the "first" positive value of x for which $J_\nu(x)=0$. This is an appropriate designation for $\nu\geq -1$ but may be confusing for $\nu<-1$ because not all zeros exist (as real values) in this range of orders. As an example, for $-4<\nu<-3$ the $j_\nu^{(4)}$, $j_\nu^{(5)}$, $j_\nu^{(6)}$, $\cdots$ zeros exist, but there is no real $j_\nu^{(1)}$, $j_\nu^{(2)}$, $j_\nu^{(3)}$, $\cdots$. For nonnegative integer order the numbering of zeros is straightforward; such zeros and their associated values are discussed at some length in Section 52:7. The same principles apply to noninteger orders, but the *Equator* routines are not applicable unless ν is a positive integer. Explicit formulas exist for all zeros when the order has one of the four values listed below. See the table in Section 32:13 for the origin of these formulas.

	$\nu=-3/2$	$\nu=-1/2$	$\nu=1/2$	$\nu=3/2$
$j_\nu^{(k)}$	$\rho_{k-1}(-1)$ [Section 34:7]	$\left(k-\frac{1}{2}\right)\pi$	$k\pi$	$r_k(1)$ [Section 34:7]

The symbol $j_\nu'^{(k)}$ denotes the kth extremum of $J_\nu(x)$. Such extrema are discussed in Section 52:7 for the cases of nonnegative integer ν and the same principles carry over to noninteger orders, with the caveat that small-k extrema may not exist when ν is a negative noninteger.

53:8 NUMERICAL VALUES

Equator provides a Bessel function routine (keyword **J**) that caters to both integer and noninteger values of the order ν. See Section 52:8 for the algorithm adopted in the former case.

Equator uses three distinct procedures when the order is not an integer. For negative noninteger orders, either series 53:6:2 is summed over a sufficient number of terms, or equation 53:6:5 is used to calculate $J_\nu(x)$ from the auxiliary cylinder functions $fc_\nu(x)$ and $gc_\nu(x)$, aided by an ε-transformation [Section 10:14].

Analogous to Miller's method [Section 52:8], but based on equation 53:5:8, the formula

53:8:1
$$J_\nu(x) = \frac{\left(\tfrac{1}{2}x\right)^\nu / \Gamma(\nu+1)}{[[[R_{\nu+K-1}(x)R_{\nu+K-2}(x)+\lambda_{(K-2)/2}]R_{\nu+K-3}(x)R_{\nu+K-4}(x)+\cdots+\lambda_2]R_{\nu+3}(x)R_{\nu+2}(x)+\lambda_1]R_{\nu+1}(x)R_\nu(x)+\lambda_0}$$

is used by *Equator* for positive noninteger orders. Here $\lambda_k = (\nu+2k)(\nu)_k/(k!\nu)$ and, starting with $R_{\nu+K}(x) = 1$, each $R_{\nu+k}(x)$ is calculated from $R_{\nu+k+1}(x)$ by recursion 52:8:2 which applies equally to integer and noninteger orders.

53:9 LIMITS AND APPROXIMATIONS

As its argument approaches zero, the value of the Bessel function of order ν is dominated generally by the first term in expansion 52:6:1

53:9:1
$$J_\nu(x) \to \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu \qquad 0 \leftarrow x \qquad \nu \neq -1,-3,-5,\cdots$$

However, when ν is a negative integer, the first $-\nu$ terms are nullified by the presence of the denominatorial gamma function. The $(1-\nu)$th term is therefore dominant and this leads to the limit

53:9:2
$$J_\nu(x) \to \frac{1}{(-\nu)!}\left(\frac{-x}{2}\right)^{-\nu} \qquad 0 \leftarrow x \qquad \nu = -1,-2,-3,\cdots$$

which duplicates 53:9:1 if $-\nu$ is even, but is its negative if ν is odd.

For very large arguments the Bessel functions become sinusoidal with a period of 2π

53:9:3
$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\left\{x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right\} \qquad x \to \infty$$

This formula applies as the argument x becomes large, the order remaining modest. Conversely, if the order ν increases, with the argument remaining modest

53:9:4
$$J_\nu(x) \to \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^\nu \qquad \nu \to \infty$$

where e is the base of natural logarithms.

53:10 OPERATIONS OF THE CALCULUS

The derivative of the Bessel function may be expressed in three equivalent ways:

53:10:1
$$\frac{d}{dx}J_\nu(x) = \frac{J_{\nu-1}(x) - J_{\nu+1}(x)}{2} = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

Moreover

53:10:2 $$\frac{d}{dx}\left\{x^{\pm\nu}\,J_\nu(x)\right\} = \pm x^{\pm\nu}\,J_{\nu\mp1}(x)$$

Apart from integer order cases [for which see 52:10:5 and 52:10:6] indefinite integrals of $J_\nu(x)$ cannot be expressed in terms of a finite number of named functions. One has, however

53:10:3 $$\int_0^x J_\nu(t)\,dt = 2\sum_{j=0}^{\infty} J_{1+\nu+2j}(x) \qquad \nu > -1$$

53:10:4 $$\int_0^x t^{1+\nu}\,J_\nu(t)\,dt = x^{1+\nu}\,J_{1+\nu}(x) \qquad \nu > -1$$

and

53:10:5 $$\int_0^x t^{1-\nu}\,J_\nu(t)\,dt = \frac{2^{1-\nu}}{\Gamma(\nu)} - x^{1-\nu}\,J_{\nu-1}(x)$$

Over 100 pages of Gradshteyn and Ryzhik's classic compendium are devoted to definite integrals of Bessel and other cylinder functions. A few representative Bessel function entries are

53:10:6 $$\int_0^\infty J_\nu(bx)\,dx = \frac{1}{b} \qquad \nu > -1$$

53:10:7 $$\int_0^\infty \frac{J_\nu(t)}{t^\mu}\,dt = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}{2^\mu\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}$$

53:10:8 $$\int_0^{\pi/2} J_\nu\{a\cos(x)\}\,dx = \int_0^{\pi/2} J_\nu\{a\sin(x)\}\,dx = \frac{\pi}{2}J^2_{\nu/2}\left(\tfrac{1}{2}a\right)$$

and

53:10:9 $$\int_0^\infty \frac{1}{x}\exp\left(\frac{-1}{x}\right)J_\nu(bx)\,dx = 2\,J_\nu\left(\sqrt{2b}\right)K_\nu\left(\sqrt{2b}\right) \qquad b > 0$$

Definite integrals of the form

53:10:10 $$\int_0^\infty f(t)\,J_\nu(yt)\,dt \qquad \text{or} \qquad \int_0^\infty f(t)\,J_\nu(yt)\sqrt{yt}\,dt$$

are known as *Hankel transforms* and copious tables are given by Erdélyi et al. [*Tables of Integral Transforms*, Volume 2, Chapter 8]. Two illustrative examples are

53:10:11 $$\int_0^\infty \frac{t^{1+\nu}}{t^2+a^2}\,J_\nu(yt)\,dt = a^\nu\,K_\nu(ay) \qquad -1 < \nu < 3/2$$

and

53:10:12 $$\int_0^\infty t^{\nu+\frac{1}{2}}\cos(at^2)\,J_\nu(yt)\sqrt{yt}\,dt = \frac{(y/2a)^{\nu+\frac{1}{2}}}{\sqrt{2a}}\sin\left(\frac{y^2}{4a}-\frac{\nu\pi}{2}\right) \qquad -1 < \nu < 1/2$$

Laplace transforms involving Bessel function are also plentiful and include

53:10:13 $$\int_0^\infty J_\nu(bt)\exp(-st)\,dt = \mathfrak{L}\{J_\nu(bt)\} = \frac{\left(\sqrt{s^2+b^2}-s\right)^\nu}{b^\nu\sqrt{s^2+b^2}}$$

53:10:14 $$\int_0^\infty t^\nu \mathrm{J}_\nu(bt)\exp(-st)\,\mathrm{d}t = \mathcal{L}\left\{t^\nu \mathrm{J}_\nu(bt)\right\} = \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}\frac{(2b)^\nu}{(s^2+b^2)^{\nu+\frac{1}{2}}} \qquad \nu > -1$$

53:10:15 $$\int_0^\infty \frac{\mathrm{J}_\nu(bt)}{t}\exp(-st)\,\mathrm{d}t = \mathcal{L}\left\{\frac{\mathrm{J}_\nu(bt)}{t}\right\} = \frac{\left(\sqrt{s^2+b^2}-s\right)^\nu}{\nu b^\nu}$$

and

53:10:16 $$\int_0^\infty t^{\nu/2}\mathrm{J}_\nu\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathcal{L}\left\{t^{\nu/2}\mathrm{J}_\nu\left(\sqrt{bt}\right)\right\} = \left(\frac{b}{4}\right)^{\nu/2}\frac{\exp(-b/4s)}{s^{1+\nu}}$$

The definite integral

53:10:17 $$\int_0^\infty \frac{\mathrm{J}_0(yt)\sin(\omega t)\exp(-st)}{t}\,\mathrm{d}t = \arcsin\left(\frac{2\omega}{\sqrt{s^2+(\omega+y)^2}+\sqrt{s^2+(\omega-y)^2}}\right)$$

may be regarded as a Hankel transform, a Fourier transform, a Laplace transform, or merely as a definite integral.

The formula describing the operation of differintegration [Section 12:14]

53:10:18 $$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}x^{\nu/2}\mathrm{J}_\nu\left(2\sqrt{x}\right) = x^{(\nu-\mu)/2}\mathrm{J}_{\nu-\mu}\left(2\sqrt{x}\right)$$

is another that benefits from being expressed in Clifford's notation, 53:1:1.

53:11 COMPLEX ARGUMENT

The $\mathrm{J}_\nu(x+iy)$ function has an infinity of zeros and they are all real if $\nu \geq -1$. This reality extends to all negative *integer* orders. However, a few of the zeros, in fact $2\mathrm{Int}(-\nu)$ of them, are complex when ν is negative and noninteger. These zeros are *conjugate*; that is, if one lies at the point $a + bi$ in the complex plane, there is another zero at $z = a - bi$. Two of those zeros are purely imaginary if $\mathrm{Int}(-\nu)$ is odd.

Unless ν is an integer, the $\mathrm{J}_\nu(x+iy)$ surface in the complex plane must be cut to avoid multiple values. The cut is customarily made along the negative reach of the real axis. The depth of the cut is not constant, being given by

53:11:1 $$\mathrm{J}_\nu(-x+i0)-\mathrm{J}_\nu(-x-i0) = 2i\sin(\nu\pi)\mathrm{J}_\nu(x) \qquad x>0$$

Two inverse Laplace transforms are:

53:11:2 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\mathrm{J}_\nu(b/s)}{s}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathcal{I}\left\{\frac{\mathrm{J}_\nu(b/s)}{s}\right\} = \mathrm{I}_\nu\left(\sqrt{2bt}\right)\mathrm{J}_\nu\left(\sqrt{2bt}\right) \qquad \nu>1$$

53:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\mathrm{J}_{2\nu}(\sqrt{a/s})}{s^\mu}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathcal{I}\left\{\frac{\mathrm{J}_{2\nu}\left(\sqrt{a/s}\right)}{s^\mu}\right\} = t^{\mu-1}\sum_{j=0}^{\infty}\frac{(-)^j(at/4)^{j+\nu}}{j!\Gamma(j+\mu+\nu)\Gamma(j+2\nu+1)} \qquad \mu+\nu>0$$

53:12 GENERALIZATIONS

Bessel functions are the simplest $L = K+2 = 2$ hypergeometric functions. Each of the functions in the right-hand column of Table 18-5 could therefore claim to be a generalized Bessel function.

53:13 COGNATE FUNCTIONS

Other cylinder functions are related to Bessel functions through the complex-argument formulas that follow. Some of these relationships require that *ν not* be an integer

53:13:1 $$\mathrm{I}_\nu(z) = (-i)^\nu \, \mathrm{J}_\nu(iz)$$

53:13:2 $$\mathrm{K}_\nu(x) = \frac{\pi\csc(\nu\pi)}{2}\left[\frac{\mathrm{J}_{-\nu}(iz)}{(-i)^\nu} - (-i)^\nu \, \mathrm{J}_\nu(iz)\right]$$

53:13:3 $$\mathrm{Y}_\nu(z) = \cot(\nu\pi)\,\mathrm{J}_\nu(z) - \csc(\nu\pi)\,\mathrm{J}_{-\nu}(z)$$

53:13:4 $$\mathrm{H}_\nu^{(1)}(z) = [1 + i\cot(\nu\pi)]\,\mathrm{J}_\nu(z) - i\csc(\nu\pi)\,\mathrm{J}_{-\nu}(z)$$

53:13:5 $$\mathrm{H}_\nu^{(2)}(z) = [1 - i\cot(\nu\pi)]\,\mathrm{J}_\nu(z) + i\csc(\nu\pi)\,\mathrm{J}_{-\nu}(z)$$

53:13:6 $$\mathrm{ber}_\nu(z) = \frac{1}{2}\left[\mathrm{J}_\nu\left(\frac{-1+i}{\sqrt{2}}z\right) + \mathrm{J}_\nu\left(\frac{-1-i}{\sqrt{2}}z\right)\right] \qquad \mathrm{Re}(z) \ge \mathrm{Im}(z)$$

53:13:7 $$\mathrm{bei}_\nu(z) = \frac{1}{2i}\left[\mathrm{J}_\nu\left(\frac{-1+i}{\sqrt{2}}z\right) - \mathrm{J}_\nu\left(\frac{-1-i}{\sqrt{2}}z\right)\right] \qquad \mathrm{Re}(z) \ge \mathrm{Im}(z)$$

When *ν is* an integer, the subtractions in equations 53:13:2−4, and the addition in 53:13:5, must be replaced by limiting operations; for example

53:13:8 $$\mathrm{K}_n(x) = (-i)^n \frac{n\pi}{2} \lim_{\nu \to n} \frac{\mathrm{J}_{-\nu}(iz) - \mathrm{J}_\nu(iz)}{\nu}$$

The auxiliary cylinder functions $\mathrm{fc}_\nu(x)$ and $\mathrm{gc}_\nu(x)$, discussed in Section 54:14, are useful cognate functions.

53:14 RELATED TOPIC: Neumann series

If, for some restricted or unrestricted range of its argument *x*, a function f(*x*) is expansible as a *Maclaurin series*

53:14:1 $$\mathrm{f}(x) = \sum_{j=0}^{\infty} a_j x^j \qquad \text{where} \qquad a_j = \frac{1}{j!}\frac{\mathrm{d}^j \mathrm{f}}{\mathrm{d}x^j}(0)$$

then it may also be expanded as the so-called *Neumann series*

53:14:2 $$\mathrm{f}(x) = \left(\frac{2}{x}\right)^\nu \sum_{k=0}^{\infty} b_k \, \mathrm{J}_{\nu+k}(x)$$

The choice of *ν* is arbitrary, except that negative integers are forbidden. The relationship between the *b* coefficients in the Neumann series and the *a* coefficients in the Maclaurin series is

53:14:3 $$b_k = (\nu + k)\sum_j \frac{2^j \Gamma\left(\nu + \frac{k+j}{2}\right)}{\left(\frac{k-j}{2}\right)!} a_j \qquad \begin{cases} j = 0, 2, 4, \cdots, k & k \text{ even} \\ j = 1, 3, 5, \cdots, k & k \text{ odd} \end{cases}$$

Equations 32:6:9, 32:6:10, 39:6:7, and 52:5:10 are examples of Neumann series.

Setting f(*x*) to unity in 53:14:2, leads to the formula

53:14:4
$$x^\nu = 2^\nu \sum_{j=0}^{\infty} (2j+\nu)\frac{\Gamma(j+\nu)}{j!} J_{2j+\nu}(x) \qquad \nu \neq -1,-2,-3,\cdots$$

for the expansion of an arbitrary power. Equations 52:5:7, 52:5:10 and 52:5:12 are simple examples.

An alternative means of converting a power series into a series of Bessel functions employs the *modified Neumann series*. The replacement series is

53:14:5
$$\mathrm{f}(x) = \sum_{j=0}^{k} c_k \left(\tfrac{1}{2}\sqrt{x}\right)^{k-\nu} J_{k+\nu}\left(\sqrt{x}\right) \qquad \nu \neq -1,-2,-3,\cdots$$

The c coefficients are related to those of the power series through

53:14:6
$$c_k = \sum_{j=0}^{k} \frac{4^j \Gamma(1+j+\nu)}{(k-j)!} a_j$$

Formulas 53:5:5 and 53:5:7, as well as 52:5:15, originate in this way.

In mathematical physics it is often useful to expand pertinent functions in terms of cylinder functions. The expansions discussed in this section provide ways of doing this. Another avenue relies on the orthogonality of Bessel functions, as outlined in Section 52:14.

53:15 RELATED TOPIC: discontinuous Bessel integrals

By and large, integration is a "smoothing" operation. Thus, if a function has discontinuities, these are often ameliorated when the function is integrated. It is, therefore, surprising to find that certain integrals may actually generate discontinuities. Such is the case for the definite integral on the left-hand side of equation 53:15:1, which evaluates to two quite different Gauss hypergeometric functions [Chapter 60] according to the value of the parameter x.

53:15:1
$$\int_0^\infty \frac{J_\nu(t)J_\mu(xt)}{t^\lambda}\,\mathrm{d}t = \begin{cases} \dfrac{\Gamma(\sigma-\lambda)x^\mu}{2^\lambda\Gamma(1+\mu)\Gamma(\sigma-\mu)} \mathrm{F}\left(\sigma-\lambda\,,\ \sigma-\lambda-\nu, 1+\mu, x^2\right) & \begin{cases} 0 \le x < 1 \\ \lambda > -1 \end{cases} \\[2ex] \dfrac{\Gamma(\lambda)\Gamma(\sigma-\lambda)}{2^\lambda\Gamma(\sigma)\Gamma(\sigma-\nu)\Gamma(\sigma-\mu)} & \begin{cases} x = 1 \\ \lambda > 0 \end{cases} \\[2ex] \dfrac{\Gamma(\sigma-\lambda)x^{\lambda-\nu-1}}{2^\lambda\Gamma(1+\nu)\Gamma(\sigma-\nu)} \mathrm{F}\left(\sigma-\lambda\,,\sigma-\lambda-\mu, 1+\nu, \dfrac{1}{x^2}\right) & \begin{cases} x > 1 \\ \lambda > -1 \end{cases} \end{cases}$$

In this formula, which involves gamma functions galore, we are using the convenient abbreviation

53:15:2
$$\sigma = \frac{1+\nu+\mu+\lambda}{2}$$

Inasmuch as $\cos(t) = \sqrt{\pi t/2}\,J_{-1/2}(t)$, equation 52:10:13 is a special case of 53:15:1, as are the formulas

53:15:3
$$\int_0^\infty \cos(bt)J_\nu(t)\,\mathrm{d}t = \left.\begin{cases} \cos\{\nu\arcsin(b)\}/\sqrt{1-b^2} & 0 \le b < 1 \\ -\sin(\nu\pi/2)/\left[\sqrt{b^2-1}\left(b+\sqrt{b^2-1}\right)^\nu\right] & b > 1 \end{cases}\right\} \quad \nu > -1$$

and

53:15:4
$$\int_0^\infty \sin(bt)\mathrm{J}_\nu(t)\,\mathrm{d}t = \begin{cases} \sin\{\nu\arcsin(b)\}/\sqrt{1-b^2} & 0 \le b < 1 \\ \cos(\nu\pi/2)/\left[\sqrt{b^2-1}\left(b+\sqrt{b^2-1}\right)^\nu\right] & b > 1 \end{cases} \quad \nu > -2$$

Figure 53-2 is a cross-sectional diagram of a system that will furnish an example of the application of such discontinuous integrals to a physical problem. Consider the steady state (the ultimate condition when changes no longer occur at finite distances) for a medium of thermal conductivity k in contact with the circular end of a heated conductive rod (of radius a) and with the coplanar end of the insulator that encases the cylindrical rod. The whole system was originally at a lower temperature T_0, but the end of the rod has maintained a constant temperature T_1 for so long that the temperature distribution in the vicinity of the rod is virtually unchanging.

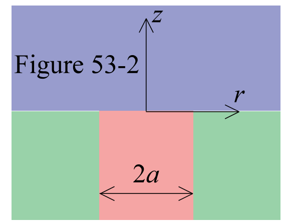


Figure 53-2

The thermal conditions within the conducting medium obey the steady-state version of *Fourier's law* [Section 46:15] which, in cylindrical coordinates [Section 46:14], is the differential equation

53:15:5
$$\nabla^2\mathrm{T} = \frac{\partial^2\mathrm{T}}{\partial r^2} + \frac{1}{r}\frac{\partial\mathrm{T}}{\partial r} + \frac{\partial^2\mathrm{T}}{\partial z^2} = k\frac{\partial\mathrm{T}}{\partial t} = 0$$

because symmetry allows the θ-coordinate to be ignored. $\mathrm{T}(r,z)$ is the temperature of the medium, a function of the r and z coordinates. Of the three *boundary conditions*, two reflect the fact that, remote from the rod, the medium will have retained its original temperature

53:15:6
$$\left.\begin{matrix}\mathrm{T}(\infty,z) \\ \mathrm{T}(r,\infty)\end{matrix}\right\} = T_0$$

while the third asserts that the medium in contact with the hot rod will acquire its temperature

53:15:7
$$\mathrm{T}(r<a,0) = T_1$$

It is straightforward to demonstrate that the differential equation and the boundary conditions in 53:15:6 are satisfied by the formula

53:15:8
$$\mathrm{T}(r,z) = T_0 + \int_0^\infty \mathrm{w}(u)\mathrm{J}_0\left(\frac{r}{a}u\right)\exp\left(\frac{-z}{a}u\right)\mathrm{d}u$$

w being an arbitrary weighting function of the dummy variable u. This weighting function needs to be elucidated to discover the temperature distribution.

A discontinuity in the expression describing temperature is expected at $r = a$ on the $z = 0$ surface. With formula 53:15:1 in mind, one can fathom from the properties of Gauss hypergeometric functions [Chapter 60] that both ν and λ need to equal ½ if T is to be constant in $0 \le r < a$. Furthermore, the choice $\mathrm{w}(u) = (T_1 - T_0)\sqrt{2/\pi u}\;\mathrm{J}_{1/2}(u)$ provides satisfaction for the third boundary condition and converts the $z = 0$ version of equation 53:15:8 to

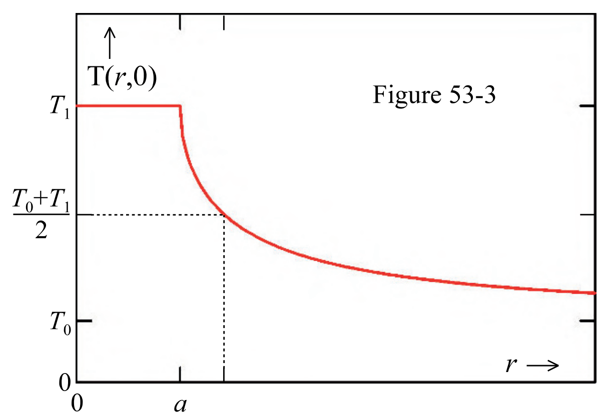


Figure 53-3

53:15:9
$$\mathrm{T}(r,0) = \begin{cases} T_0 + (T_1 - T_0)\mathrm{F}\left(\tfrac{1}{2},0,1,r^2/a^2\right) = T_1 & r \le a \\ T_0 + (T_1 - T_0)\dfrac{2a}{\pi r}\mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},a^2/r^2\right) = T_0 + \dfrac{2}{\pi}(T_1 - T_0)\arcsin\left(\dfrac{a}{r}\right) & r \ge a \end{cases}$$

This surface result is diagrammed in Figure 53-3. With w(u) now known, the integral in 53:15:8 may be evaluated leading, via 53:10:15, to the complete solution

53:15:10 $$\mathrm{T}(r,z)=T_0+\frac{2}{\pi}(T_1-T_0)\arcsin\left(\frac{2a}{\sqrt{z^2+(a+r)^2}+\sqrt{z^2+(a-r)^2}}\right)$$

This equation describes the three-dimensional temperature distribution, illustrated colorfully in Figure 53-4.

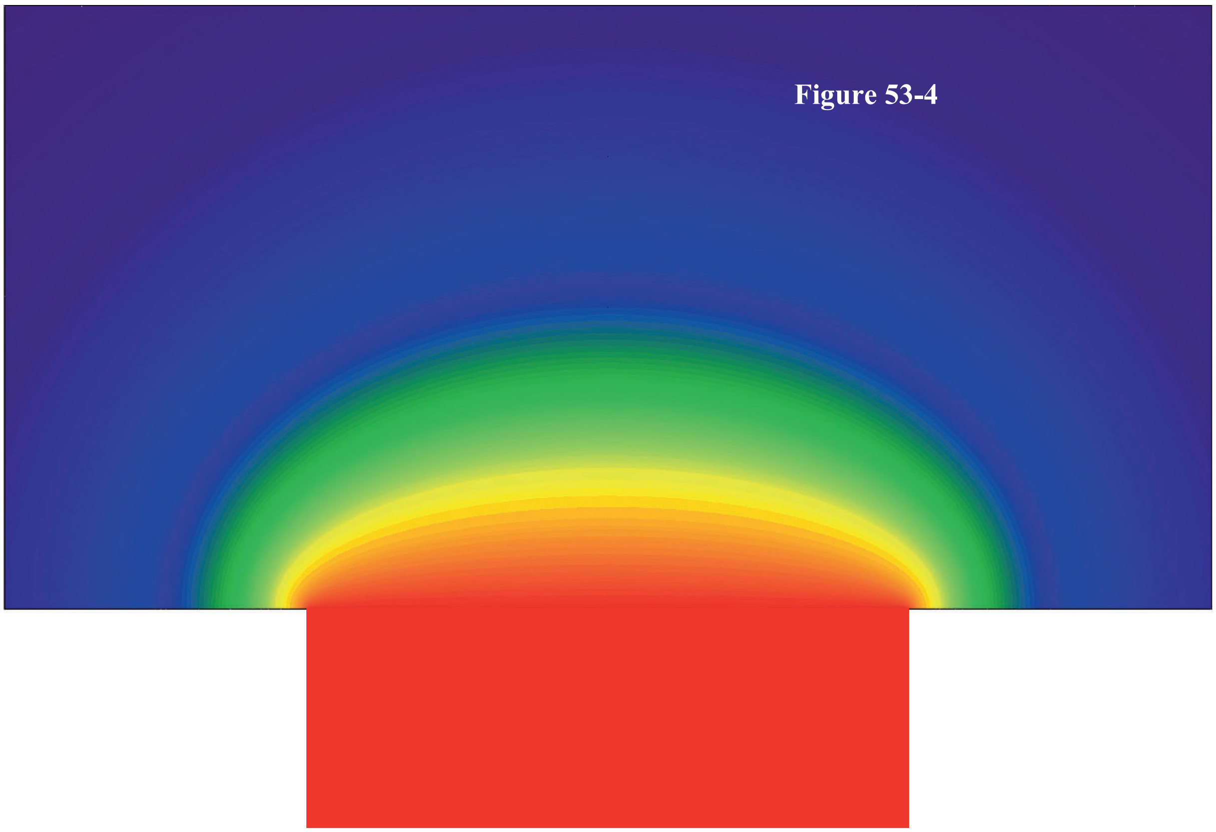


Figure 53-4

The temperature distribution along the z axis, $r=0$, is seen to be the simple

53:15:11 $$\mathrm{T}(0,z)=T_0+\frac{2}{\pi}(T_1-T_0)\arctan\left(\frac{z}{a}\right)$$

as depicted in Figure 53-5. Notice that at a distance equal to the radius of the rod, the temperature has adopted a value midway between the rod's temperature and that at infinity.

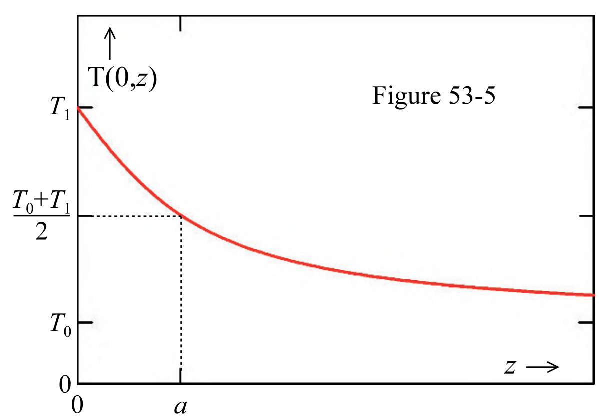


Figure 53-5

CHAPTER 54

THE NEUMANN FUNCTION $Y_\nu(x)$

Named for the redoubtable German mathematician Franz Ernst Neumann (1798 – 1895), this function appears in the solution to many physical problems, especially those involving cylindrical symmetry. In the terminology introduced in Section 49:14, the Neumann function is the "irregular" counterpart of the Bessel function. Through formula 54:3:1, the properties of Neumann functions of *noninteger* order can be deduced easily from those of the Bessel functions, and therefore this chapter concentrates on Neumann functions $Y_n(x)$ of *integer* order and particularly on the $n = 0$ and $n = 1$ cases.

54:1 NOTATION

The symbolism $N_\nu(x)$ is a common replacement for $Y_\nu(x)$ in denoting the Neumann function of order ν and argument x. The alternative names *Bessel function of the second kind* and *Weber's function* are encountered.

54:2 BEHAVIOR

Because the Neumann function is complex when its argument is negative, our attention in most of this chapter is restricted to the range $x \geq 0$. In this range $Y_\nu(x)$ is real and well defined for all real orders, though there are discontinuities at $x = 0$.

Superficially, the three-dimensional map of $Y_\nu(x)$ shown in Figure 54-1 closely resembles that of $J_\nu(x)$ in Figure 53-1. The most noteworthy distinction lies in the $0 < x < \nu$ region where the Neumann function acquires values which are negative and large, in contrast to the small and positive values of Bessel function in the same zone. Away from this region, bands of alternating positive and negative sign are seen to cross the x,ν plane diagonally, indicative of an infinite number of zeros and extrema. In this behavior the Neumann and Bessel functions are similar, though the ripples are out of phase by about $\pi/2$.

Figure 54-2 shows graphs of several Neumann functions of small integer order. Away from small arguments, the close similarity to Bessel functions is again evident (compare to Figure 52-1). Notice that Neumann functions of order $-2, -4, -6, \cdots$ are exact duplicates of $Y_2(x)$, $Y_4(x)$, $Y_6(x)$, $\cdots$, whereas those of odd negative integer order differ from their positive counterparts only in sign, as reiterated in 54:5:2.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_55, © Springer Science+Business Media, LLC 2009

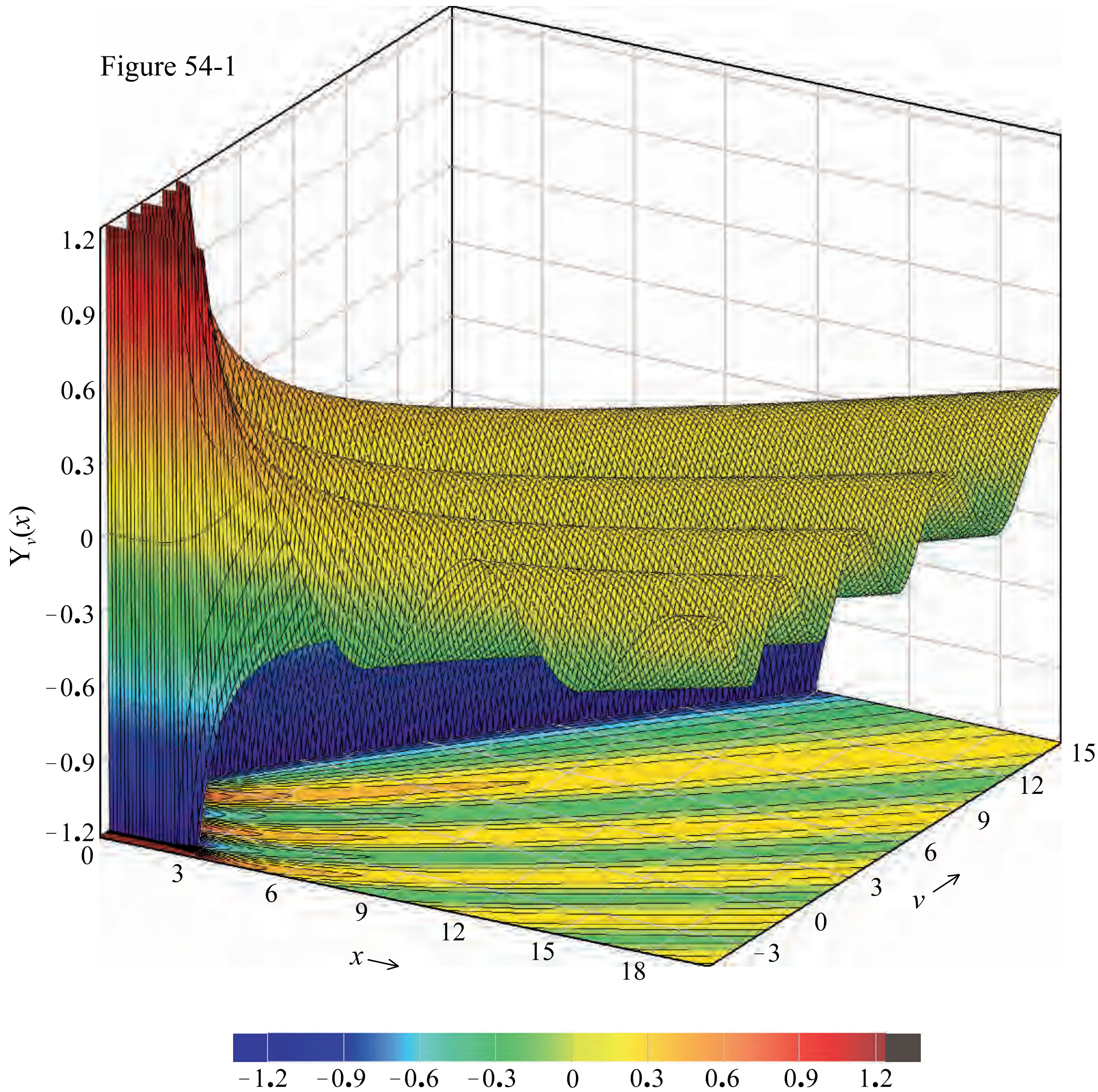


Figure 54-1

54:3 DEFINITIONS

For noninteger order, the Neumann function is defined in terms of Bessel functions:

54:3:1 $$Y_\nu(x) = \cot(\nu\pi)\,J_\nu(x) - \csc(\nu\pi)\,J_{-\nu}(x) \qquad \nu \neq 0, \pm1, \pm2, \cdots$$

For integer orders, this definition must be replaced by an equivalent limiting operation:

54:3:2 $$Y_n(x) = \lim_{\nu\to n}\left\{\frac{\cos(\nu\pi)\,J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}\right\} \qquad n = 0, \pm1, \pm2, \cdots$$

Two integral definitions ascribed to the Neumann function of zero order are

54:3:3 $$Y_0(x) = \frac{-2}{\pi}\int_0^\infty \cos\{x\cosh(t)\}\,dt = \frac{-2}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{t^2-1}}\,dt$$

but the former does not converge acceptably. The latter generalizes to

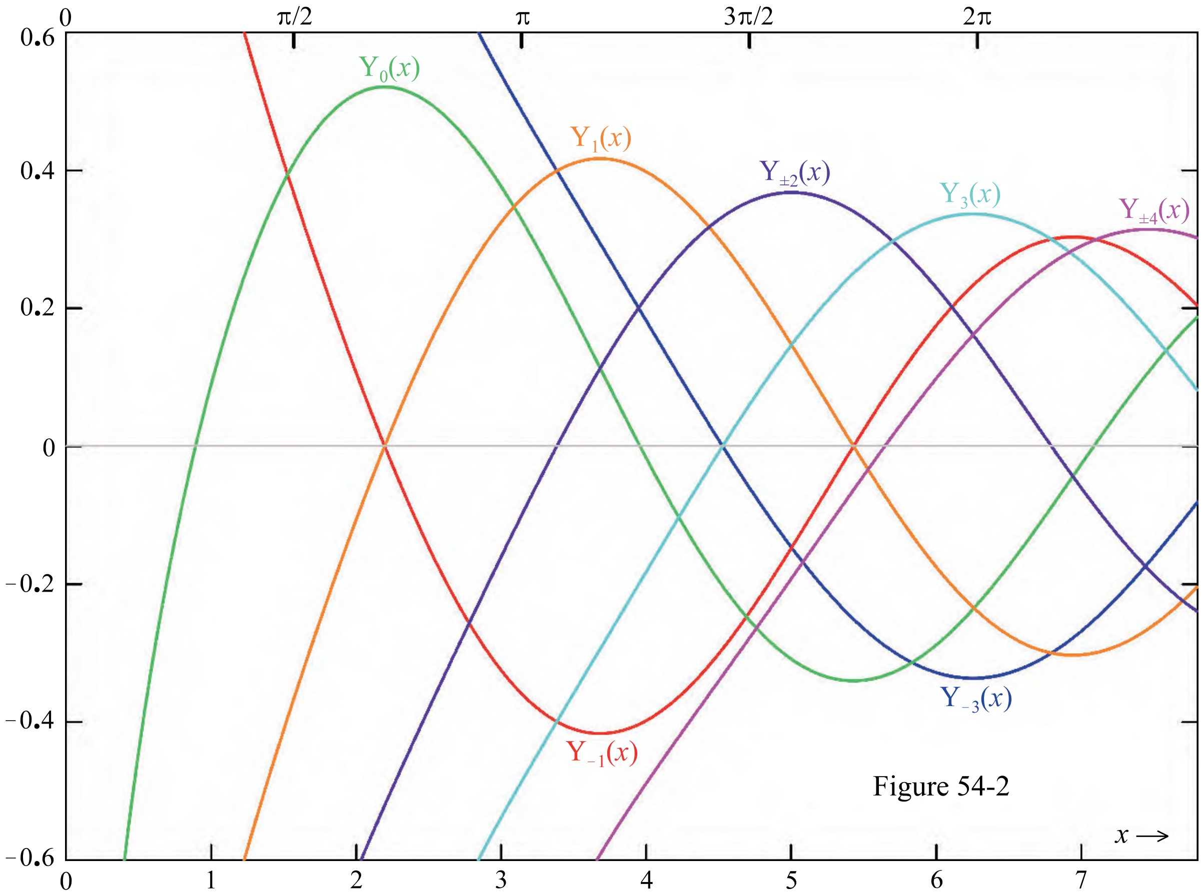


Figure 54-2

54:3:4
$$Y_\nu(x) = \frac{-2(2/x)^\nu}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{\cos(xt)}{(t^2-1)^{(1+2\nu)/2}}\,dt$$

but this is valid only for $|\nu| < 1$. Other integral representations of the Neumann function will be found in Gradshteyn and Ryzhik [Section 8.41].

The $Y_\nu(x)$ function is one of the solutions to each of the differential equations 53:3:5−8. It is in this way that the Neumann function most often arises in practice.

The semiintegration [Section12:14] formula

54:3:5
$$\frac{d^{-1/2}}{dx^{-1/2}}\left\{\frac{1}{x}\cos\left(\frac{1}{\sqrt{x}}\right)\right\} = \sqrt{\frac{\pi}{x}}\,Y_0\left(\frac{1}{\sqrt{x}}\right)$$

indicates a close relationship between the Neumann function of zero order and the cosine function.

54:4 SPECIAL CASES

In a sense, all Neumann functions of noninteger order are "special", inasmuch as they alone are linearly related, through equation 54:3:1, to Bessel functions.

A Neumann function of an order that is one-half of an odd integer, positive or negative, is known as a *spherical Neumann function*. It is represented by a lower-case symbol

54:4:1 $$\sqrt{\frac{\pi}{2x}}\,Y_{n+\frac{1}{2}}(x)=y_n(x)\qquad n=0,\pm1,\pm2,\cdots$$

and is the subject of Section 32:13. Such Neumann functions are coincident in magnitude with Bessel functions [Chapter 54] of the same argument but of an order of opposite sign:

54:4:2 $$Y_{n+\frac{1}{2}}(x)=-(-)^n\,J_{-n-\frac{1}{2}}(x)\qquad n=0,\pm1,\pm2,\cdots$$

A Neumann function of order $\frac{1}{3}$ or $-\frac{1}{3}$ is related, as is the corresponding Bessel function, to the Airy functions Ai and Bi of Chapter 56:

54:4:3 $$Y_{\pm 1/3}(x)=\frac{-1}{2}\sqrt{\frac{3}{\hat{x}}}\left[\sqrt{3}\mathrm{Bi}(-\hat{x})\pm \mathrm{Ai}(-\hat{x})\right]\qquad \hat{x}=\sqrt[3]{\frac{9x^2}{4}}$$

A Neumann function of order $\frac{2}{3}$ or $-\frac{2}{3}$ is similarly related to the derivatives of the Airy functions.

54:5 INTRARELATIONSHIPS

The general order-reflection property of the Neumann function

54:5:1 $$Y_{-\nu}(x)=\cos(\nu\pi)\,Y_\nu(x)+\sin(\nu\pi)\,J_\nu(x)$$

reduces to

54:5:2 $$Y_{-n}(x)=(-)^n\,Y_n(x)\qquad n=1,2,3,\cdots$$

when the order is an integer.

A Neumann function of any order obeys the recursion formula

54:5:3 $$Y_{\nu+1}(x)=\frac{2\nu}{x}Y_\nu(x)-Y_{\nu-1}(x)$$

and accordingly, any Neumann function of integer (only!) order may be expressed in terms of $Y_0(x)$ and $Y_1(x)$:

54:5:4 $$Y_n(x)=\mathrm{Wj}_n^{(0)}(x)\,Y_0(x)+\mathrm{Wj}_n^{(1)}(x)\,Y_1(x)\qquad n=2,3,4,\cdots$$

Because the form of recursion 54:5:3 is identical to that for Bessel functions, the weighting multipliers are likewise identical to the *Lommel polynomials* given in the table in Section 52:5.

In addition to 54:3:1 and 54:5:1, several other relationships, including

54:5:5 $$Y_\nu(x)\pm Y_{-\nu}(x)=\begin{matrix}\cot\\ \tan\end{matrix}\left(\frac{\nu\pi}{2}\right)\left[\pm J_\nu(x)-J_{-\nu}(x)\right]$$

54:5:6 $$Y_\nu^2(x)+J_\nu^2(x)=Y_{-\nu}^2(x)+J_{-\nu}^2(x)$$

and

54:5:7 $$Y_\nu(x)\,J_{\nu+1}(x)-Y_{\nu+1}(x)\,J_\nu(x)=\frac{2}{\pi x}$$

interconnect Neumann and Bessel functions.

54:6 EXPANSIONS

Simple power series do not exist for a Neumann function of positive integer order. However, an expansion incorporating logarithmic and power components can be developed through *L'Hôpital's rule* [Section 0:10] from definition 54:3:2. For nonnegative integer *n*, the expansion is

54:6:1 $$Y_n(x)=\frac{2}{\pi}\ln\left(\frac{x}{2}\right)J_n(x)-\frac{1}{\pi}\left(\frac{x}{2}\right)^n\sum_{j=0}^{\infty}\frac{\psi(j+1)+\psi(j+n+1)}{j!(j+n)!}\left(\frac{-x^2}{4}\right)^j-\frac{1}{\pi}\left(\frac{2}{x}\right)^n\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x^2}{4}\right)^j$$

where ψ is the digamma function of Chapter 44. The final sum makes no contribution to the $n=0$ version of this expansion. For the zeroth- and first-order cases, the leading terms of this complicated series are

54:6:2 $$Y_0(x)=\frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma-\left\{\ln\left(\frac{x}{2}\right)-1+\gamma\right\}\frac{x^2}{4}+\left\{\ln\left(\frac{x}{2}\right)-\frac{3}{2}+\gamma\right\}\frac{x^4}{64}-\cdots\right]$$

and

54:6:3 $$Y_1(x)=\frac{2}{\pi}\left[\frac{-1}{x}+\left\{\ln\left(\frac{x}{2}\right)-\frac{1}{2}+\gamma\right\}\frac{x}{2}-\left\{\ln\left(\frac{x}{2}\right)-\frac{3}{4}+\gamma\right\}\frac{x^3}{64}+\left\{\ln\left(\frac{x}{2}\right)-\frac{19}{12}+\gamma\right\}\frac{x^5}{384}-\cdots\right]$$

The γ term is Euler's constant [Section 1:7].

A Neumann function of positive integer order may also be expressed largely in terms of Bessel functions:

54:6:4 $$Y_n(x)=\frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)-\psi(n+1)\right\}J_n(x)-\sum_{j=1}^{\infty}\frac{(-)^j(n+2j)}{j(n+j)}J_{n+2j}(x)\right]-\frac{n!}{\pi}\left(\frac{2}{x}\right)^n\sum_{j=0}^{n-1}\frac{(x/2)^j}{j!(n-j)}J_j(x)$$

The final term equals zero when $n=0$ and $-2/\pi x$ when $n=1$.

The expression in terms of auxiliary cylinder functions [Section 54:14]

54:6:5 $$Y_\nu(x)=\sqrt{\frac{2}{\pi x}}\left[\mathrm{fc}_\nu(x)\sin\left\{x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right\}+\mathrm{gc}_\nu(x)\cos\left\{x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right\}\right]$$

holds irrespective of whether ν is integer or not. Useful when x is large, asymptotic expansions for the auxiliary functions fc_ν and gc_ν are reported in the cited section.

54:7 PARTICULAR VALUES

At zero argument, the Neumann function acquires one of three values:

54:7:1 $$Y_\nu(0)=\begin{cases}0 & \nu=\frac{-1}{2},\frac{-3}{2},\frac{-5}{2},\cdots\\ -\infty & \begin{cases}\nu>\frac{-1}{2}\\ \frac{-5}{2}<\nu<\frac{-3}{2},\ \frac{-9}{2}<\nu<\frac{-7}{2},\ \frac{-13}{2}<\nu<\frac{-11}{2},\cdots\end{cases}\\ +\infty & \frac{-3}{2}<\nu<\frac{-1}{2},\ \frac{-7}{2}<\nu<\frac{-5}{2},\ \frac{-11}{2}<\nu<\frac{-9}{2},\cdots\end{cases}$$

All Neumann functions are zero when $x=\infty$. Equation 44:5:11 gives a special significance to $Y_0(2)$.

For each order, integer or noninteger, positive or negative, the Neumann function has an infinite number of zeros, separated by local maxima and minima. The disposition of these particular values is very reminiscent of those of the corresponding Bessel function [Sections 52:7 and 53:7], but their locations, values, and associated values are distinct. If $y_\nu^{(k)}$ denotes the *k*th *zero of the Neumann function* of nonnegative order ν, then

54:7:2 $$\nu<y_\nu^{(1)}<j_\nu^{(1)}<y_\nu^{(2)}<j_\nu^{(2)}<y_\nu^{(3)}<\cdots$$

where $j_\nu^{(k)}$ is the corresponding Bessel zero. Similar nested sequencing is followed by the extrema of the Neumann and Bessel functions. Zeros and extrema of the Neumann function are less important than those of the Bessel function and are not further addressed in the *Atlas*: nor does *Equator* provide their values.

54:8 NUMERICAL VALUES

Designed for all orders and arguments in the domains $|\nu| \le 22$ and $0 \le 300$, *Equator* provides a Neumann function routine (keyword **Y**). In fact, the routine often performs far beyond those bounds, and the reported digits are always significant. Formula 54:3:1 is used exclusively for noninteger orders, but equations 54:6:2, 54:6:5, 54:5:2, and 54:8:1 are all called upon, in different regions, when ν is an integer. The last mentioned formula is the quintic interpolation

54:8:1 $$Y_n(x) \approx \tfrac{150}{256}\left[Y_{n+h}(x) + Y_{n-h}(x)\right] - \tfrac{25}{256}\left[Y_{n+2h}(x) + Y_{n-2h}(x)\right] + \tfrac{3}{256}\left[Y_{n+3h}(x) + Y_{n-3h}(x)\right]$$

which, with $h = 1/500$, enables an integer-ordered Neumann function to be calculated from values of six nearby noninteger-ordered Neumann functions.

54:9 LIMITS AND APPROXIMATIONS

As its argument approaches zero from positive values, the limiting expression for the $Y_\nu(x)$ function is determined by one or other of the following formulas:

54:9:1 $$\lim_{0 \leftarrow x} Y_\nu(x) \to \begin{cases} \dfrac{-\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu & \begin{cases} \nu > 0 \\ \nu = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots \end{cases} \\ (2/\pi)\left[\ln(x/2) + \gamma\right] & \nu = 0 \\ \dfrac{-\Gamma(-\nu)}{\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu)}\left(\dfrac{x}{2}\right)^\nu & \text{otherwise} \end{cases}$$

Reduction to simpler equivalents occurs if ν is an integer or half an odd integer – see formulas 43:4:1−4.

As Figure 54-1 suggests, at any constant value of its argument greater than its order, the Neumann function $Y_\nu(x)$ is an oscillatory function of its order, as well as of x. As ν becomes large and negative, this oscillation becomes sinusoidal and obedient to the formula

54:9:2 $$Y_\nu(x) \to -\sqrt{\frac{-2}{\pi\nu}}\left(\frac{-ex}{2\nu}\right)^\nu \cos(\nu\pi) \qquad \nu \to -\infty$$

where e is the base of natural logarithms.

For very large arguments the Neumann functions become sinusoidal in x with a period of 2π:

54:9:3 $$Y_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) \qquad x \to \infty$$

This formula applies as x becomes large, the order remaining modest. Conversely, the relationship

54:9:4 $$Y_\nu(x) \to -\sqrt{\frac{2}{\nu\pi}}\left(\frac{2\nu}{ex}\right)^\nu \qquad \nu \to \infty$$

comes to be obeyed as the order ν increases, with the argument remaining modest.

54:10 OPERATIONS OF THE CALCULUS

The derivative of the Neumann function may be expressed in three equivalent ways:

54:10:1
$$\frac{d}{dx}Y_\nu(x)=\frac{Y_{\nu-1}(x)-Y_{\nu+1}(x)}{2}=Y_{\nu-1}(x)-\frac{\nu}{x}Y_\nu(x)=\frac{\nu}{x}Y_\nu(x)-Y_{\nu+1}(x)$$

Notice that these formulas are exact analogues of those in 53:10:1 for the Bessel function. Likewise the expressions for the derivatives of $x^{\pm\nu}Y_\nu(x)$ strictly resemble 53:10:2.

Integrals include

54:10:2
$$\int_0^x t^\nu Y_{\nu-1}(t)\,dt = x^\nu Y_\nu(x)+\frac{2^\nu\Gamma(\nu)}{\pi} \qquad \nu>0$$

54:10:3
$$\int_0^\infty Y_\nu(t)\,dt=-\tan\left(\tfrac{1}{2}\nu\pi\right) \qquad -1<\nu<1$$

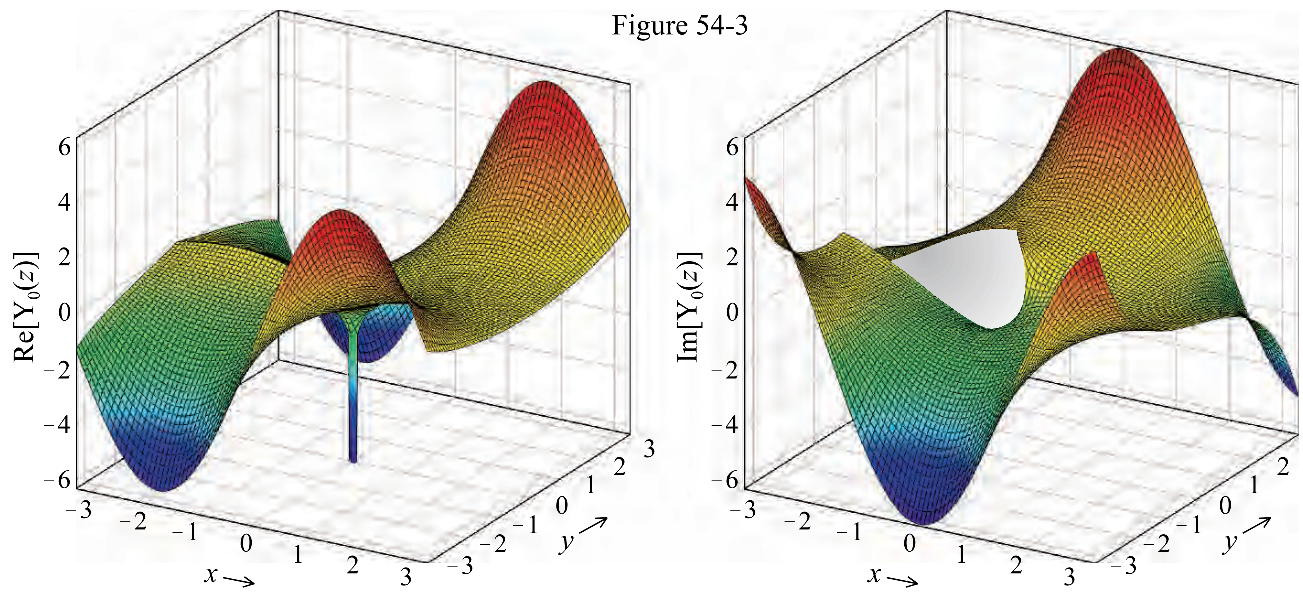
and

54:10:4
$$\int_0^\infty t^\mu Y_\nu(t)\,dt=\frac{2^\mu}{\pi}\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right)\sin\left\{\tfrac{1}{2}\pi(\mu-\nu)\right\} \qquad \mu<\tfrac{1}{2} \qquad 1+\mu>|\nu|$$

Other definite integrals will be found in Sections 6.5–6.7 or Gradshteyn and Ryzhik.

54:11 COMPLEX ARGUMENT

Figure 54-3



With $z=x+iy$, Figures 54-3 and 54-4 show the real and imaginary parts of the most important of the Neumann functions, $Y_0(z)$ and $Y_1(z)$, in the vicinity of the origin. In each case, the Riemann surface is cut along the negative real axis. The complex values on each side of the cut are given by

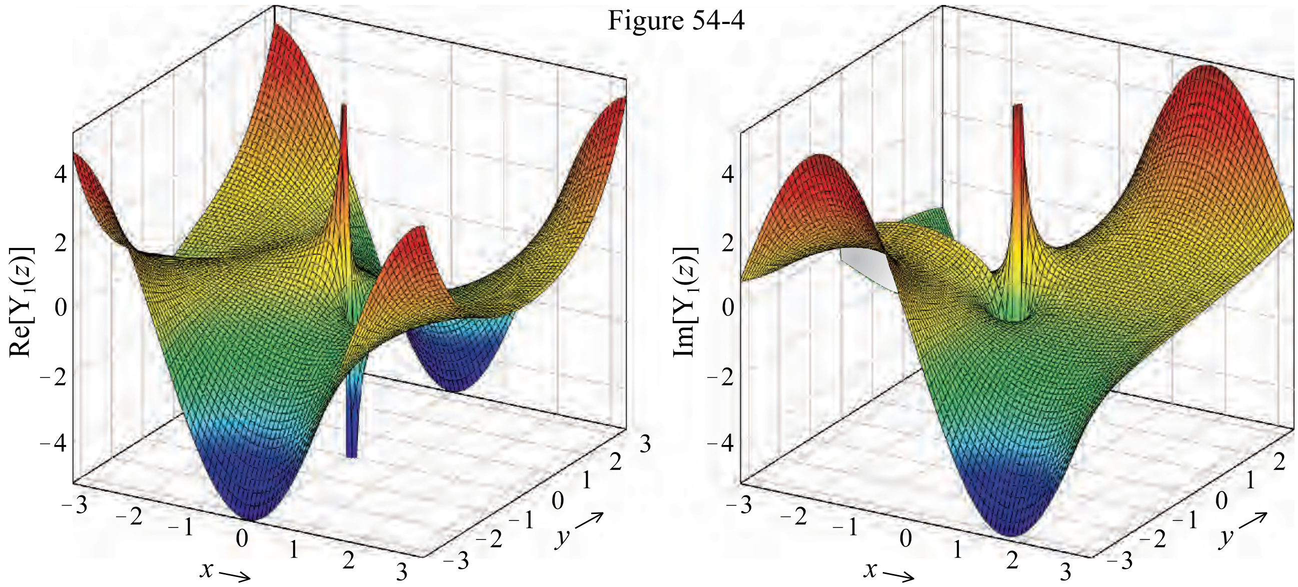
54:11:1
$$Y_0(-x\pm 0i)=Y_0(x)\pm 2i\,J_0(x) \qquad x>0$$

and

54:11:2
$$Y_1(-x\pm 0i)=-Y_1(x)\mp 2i\,J_1(x) \qquad x>0$$

illustrating (a) an argument-reflection principle for the real part of integer-ordered Neuman functions and (b) that the average value of the imaginary parts on either side of the cut is zero. The latter rule holds for complex-valued

Figure 54-4



Neumann functions of all orders.

For purely imaginary argument

54:11:3 $$\mathrm{Y}_0(iy) = i\,\mathrm{I}_0(y) - \frac{2}{\pi}\mathrm{K}_0(y)$$

and

54:11:4 $$\mathrm{Y}_1(iy) = -\mathrm{I}_1(y) + \frac{2i}{\pi}\mathrm{K}_1(y)$$

54:12 GENERALIZATIONS

By virtue of the limiting process

54:12:1 $$\lim_{\lambda\to 0}\left\{\lambda^\nu \mathrm{Q}_{1/2}^{(\nu)}\big(\cos(\lambda x)\big)\right\} = \frac{-\pi}{2}\mathrm{Y}_{-\nu}(x)$$

the *second Legendre function* [Chapter 59] might be considered as a generalization of the Neumann function.

54:13 COGNATE FUNCTIONS: Hankel functions

In his short life, the German mathematician Hermann Hankel (1839 − 1873) made several noteworthy contributions. Named in his honor are the two *Hankel functions* defined by

54:13:1 $$\mathrm{H}_\nu^{(1)}(z) = \mathrm{J}_\nu(z) + i\,\mathrm{Y}_\nu(z)$$

and

54:13:2 $$\mathrm{H}_\nu^{(2)}(z) = \mathrm{J}_\nu(z) - i\,\mathrm{Y}_\nu(z)$$

They are also known as *Bessel functions of the third kind*. In some engineering applications, this pair of complex solutions to Bessel's equation are more felicitous than are the real J, Y pair. Asymptotic expansions of the Hankel functions are given in the next section.

54:14 RELATED TOPIC: asymptotic representation of cylinder functions

Two auxiliary functions $\mathrm{fc}_\nu(x)$ and $\mathrm{gc}_\nu(x)$ may be defined in terms of the Neumann and Bessel functions:

54:14:1
$$\mathrm{fc}_\nu(x) = \sqrt{\tfrac{1}{2}\pi x}\left[\mathrm{Y}_\nu(x)\sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)+\mathrm{J}_\nu(x)\cos\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right]$$

and

54:14:2
$$\mathrm{gc}_\nu(x) = \sqrt{\tfrac{1}{2}\pi x}\left[\mathrm{Y}_\nu(x)\cos\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)-\mathrm{J}_\nu(x)\sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right]$$

Figure 54-5 shows the behavior of the functions for a typical order, $\nu = 3$. Notice that, though definitions 54:14:1 and 54:14:2 show the auxiliary functions to be composed of four functions, all of which are oscillatory, the combination is such that no oscillations remain. The argument of the cosine and sine in the two equations above occurs so frequently in this section that the adoption of the abbreviation

54:14:3
$$\chi = x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi$$

is a worthwhile convenience.

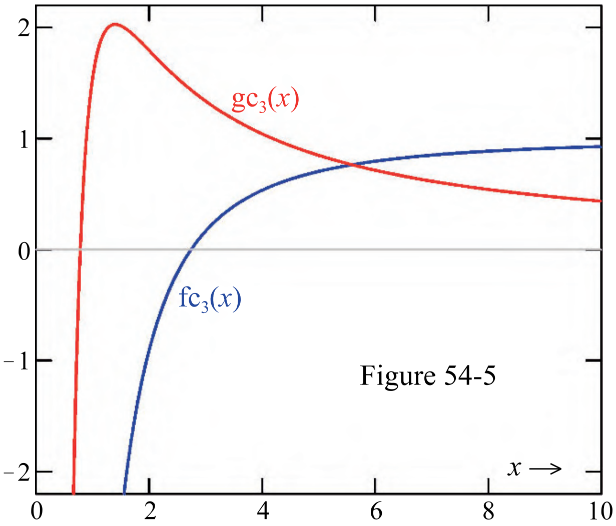


Figure 54-5

Under the names auxiliary cylinder fc function and auxiliary cylinder gc function (keywords **fc** and **gc**). *Equator* provides routines for the functions defined in equation 54:14:1 and 54:14:2. Because applications of fc and gc are limited to large arguments, the algorithms are operative only in the domain $40 \le |x| \le 450$ for $|\nu| \le 24$, though arguments as small as 8 are served for more restricted orders. Aided by ε-transformation [Section 10:14], the routines utilize the following two asymptotic representations

54:14:4
$$\mathrm{fc}_\nu(x) \sim \sum_{j=0}\frac{\left(\tfrac{1}{2}-\nu\right)_{2j}\left(\tfrac{1}{2}+\nu\right)_{2j}}{(1)_{2j}}\left(\frac{-1}{4x^2}\right)^j = 1-\frac{9-40\nu^2+16\nu^4}{128x^2}+\frac{11025-51664\nu^2+31584\nu^4-5376\nu^6+256\nu^8}{98304x^4}-\cdots$$

and

54:14:5
$$\mathrm{gc}_\nu(x) \sim \frac{-1}{2x}\sum_{j=0}\frac{\left(\tfrac{1}{2}-\nu\right)_{2j+1}\left(\tfrac{1}{2}+\nu\right)_{2j+1}}{(1)_{2j+1}}\left(\frac{-1}{4x^2}\right)^j = -\frac{1-4\nu^2}{8x}+\frac{225-1036\nu^2+560\nu^4-64\nu^6}{3072x^3}-\cdots$$

Although the utility of these series is generally limited to sufficiently large values of x, they terminate for orders that are odd multiples of one-half and in these circumstances the formulas are exact for all arguments.

Four important cylinder functions can be expressed as weighted sums of the auxiliary functions:

54:14:6
$$\begin{matrix}\text{cylinder}\\\text{function}\end{matrix} = \sqrt{\frac{2}{\pi x}}\left[\mathrm{Wf}(\chi)\mathrm{fc}_\nu(x)+\mathrm{Wg}(\chi)\mathrm{gc}_\nu(x)\right]$$

where the weighting functions are given in the accompanying table, providing a useful large-argument representation of these cylinder functions.

cylinder function	$\mathrm{Wf}(\chi)$	$\mathrm{Wg}(\chi)$
$\mathrm{J}_\nu(x)$	$\cos(\chi)$	$-\sin(\chi)$
$\mathrm{Y}_\nu(x)$	$\sin(\chi)$	$\cos(\chi)$
$\mathrm{H}_\nu^{(1)}(x)$	$\exp(i\chi)$	$i\exp(i\chi)$
$\mathrm{H}_\nu^{(2)}(x)$	$\exp(-i\chi)$	$-i\exp(-i\chi)$

If a set of terms, dependent on the cylinder function variables ν and x, are defined by

54:14:7 $$\Upsilon_j = \frac{\left(\tfrac{1}{2}-\nu\right)_j \left(\tfrac{1}{2}+\nu\right)_j}{j!\sqrt{2\pi x}\,(2x)^j} \qquad j = 0,1,2,\cdots$$

they are easily calculated through the recursion

54:14:8 $$\Upsilon_j = \frac{(j-\tfrac{1}{2})^2-\nu^2}{2jx}\Upsilon_{j-1} \qquad \Upsilon_0 = \frac{1}{\sqrt{2\pi x}}$$

Series of these terms provide asymptotic expansions for as many as twelve cylinder functions through the formulas:

54:14:9 $$\mathrm{fc}_\nu(x)/\sqrt{2\pi x} \sim \Upsilon_0 - \Upsilon_2 + \Upsilon_4 - \Upsilon_6 + \cdots$$

54:14:10 $$\mathrm{gc}_\nu(x)/\sqrt{2\pi x} \sim -\Upsilon_1 + \Upsilon_3 - \Upsilon_5 + \Upsilon_7 - \cdots$$

54:14:11 $$\exp(-x)\,\mathrm{I}_\nu(x) \sim \Upsilon_0 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \cdots$$

54:14:12 $$\exp(x)\mathrm{K}_\nu(x)/\pi \sim \Upsilon_0 - \Upsilon_1 + \Upsilon_2 - \Upsilon_3 + \Upsilon_4 - \cdots$$

54:14:13 $$\tfrac{1}{2}\mathrm{J}_\nu(x) \sim \cos(\chi)[\Upsilon_0 - \Upsilon_2 + \Upsilon_4 - \cdots] + \sin(\chi)[\Upsilon_1 - \Upsilon_3 + \Upsilon_5 - \cdots]$$

54:14:14 $$\tfrac{1}{2}\mathrm{Y}_\nu(x) \sim \sin(\chi)[\Upsilon_0 - \Upsilon_2 + \Upsilon_4 - \cdots] - \cos(\chi)[\Upsilon_1 - \Upsilon_3 + \Upsilon_5 - \cdots]$$

54:14:15 $$\tfrac{1}{2}\mathrm{H}_\nu^{(1)}(x) \sim \exp(i\chi)[\Upsilon_0 - i\Upsilon_1 - \Upsilon_2 + i\Upsilon_3 + \Upsilon_4 - \cdots]$$

54:14:16 $$\tfrac{1}{2}\mathrm{H}_\nu^{(2)}(x) \sim \exp(-i\chi)[\Upsilon_0 + i\Upsilon_1 - \Upsilon_2 - i\Upsilon_3 + \Upsilon_4 + \cdots]$$

54:14:17 $$\exp\left(\frac{-x}{\sqrt{2}}\right)\mathrm{ber}_\nu(x) \sim \sum_{j=0}\Upsilon_j \cos\left(\frac{x}{\sqrt{2}} + \frac{\nu\pi}{2} - \frac{j\pi}{4} - \frac{\pi}{8}\right)$$

54:14:18 $$\exp\left(\frac{-x}{\sqrt{2}}\right)\mathrm{bei}_\nu(x) \sim \sum_{j=0}\Upsilon_j \sin\left(\frac{x}{\sqrt{2}} + \frac{\nu\pi}{2} - \frac{j\pi}{4} - \frac{\pi}{8}\right)$$

54:14:19 $$\exp\left(\frac{x}{\sqrt{2}}\right)\frac{\mathrm{ker}_\nu(x)}{\pi} \sim \sum_{j=0}(-)^j\Upsilon_j \cos\left(\frac{x}{\sqrt{2}} + \frac{\nu\pi}{2} + \frac{j\pi}{4} + \frac{\pi}{8}\right)$$

54:14:20 $$\exp\left(\frac{x}{\sqrt{2}}\right)\frac{\mathrm{kei}_\nu(x)}{\pi} \sim -\sum_{j=0}(-)^j\Upsilon_j \sin\left(\frac{x}{\sqrt{2}} + \frac{\nu\pi}{2} + \frac{j\pi}{4} + \frac{\pi}{8}\right)$$

The ber, bei, ker, and kei functions in the last four equations are *Kelvin functions* from Chapter 55.

CHAPTER 55

THE KELVIN FUNCTIONS

Four families of bivariate functions, $\mathrm{ber}_\nu(x)$, $\mathrm{bei}_\nu(x)$, $\mathrm{ker}_\nu(x)$, and $\mathrm{kei}_\nu(x)$, are grouped under the heading of Kelvin functions. Those of zero order, $\nu = 0$, are of prime importance and are emphasized in this chapter. They arise in such problems as the electrical "skin effect" in cylindrical conductors.

55:1 NOTATION

These functions are named for the Scottish physicist and applied mathematician William Thomson (1824 – 1907), who took the title Baron Kelvin of Largs on his elevation to the peerage in 1866. The alternative name *Thomson functions* is preferred by some with antipathy to the aristocracy.

The initial letter of the symbol is often capitalized. The symbols originate from the names of **Be**ssel or **Ke**lvin coupled with the qualifiers **r**eal and **i**maginary. In the terminology introduced in Section 49:14, ber and bei are regular functions; ker and kei are irregular. When the symbols are encountered without a subscript, an order of zero is implied:

55:1:1 $$\mathrm{ber}(x) = \mathrm{ber}_0(x), \quad \mathrm{kei}(x) = \mathrm{kei}_0(x), \quad \text{etc.}$$

55:2 BEHAVIOR

Only Kelvin functions of real positive argument are addressed in this *Atlas*. All four functions are oscillatory at sufficiently large positive arguments but, like Bessel functions, the onset of the oscillations is increasingly delayed as ν increases. Figure 55-1 shows the overall behaviors of the four functions.

Once established, the oscillations of the ber and bei functions increase in amplitude exponentially, while those of the ker and kei functions decrease exponentially. Stated more precisely, the $\mathrm{ber}_\nu(x)$ and $\mathrm{bei}_\nu(x)$ functions increase with x as $(1/\sqrt{2\pi x})\exp(x/\sqrt{2})$ whereas $\mathrm{ker}_\nu(x)$ and $\mathrm{kei}_\nu(x)$ decrease as $\sqrt{\pi/2x}\exp(-x/\sqrt{2})$. If one corrects for these amplitude-perturbing factors, as was done in constructing Figure 55-2, the four Kelvin functions soon become quasiperiodic, and eventually sinusoidal with a period of $\sqrt{8}\,\pi$.

When the order is zero, the amplitude-adjusted Kelvin functions settle into a repetitive pattern very rapidly, as Figure 55-2 attests. The color-coded curves in this diagram relate to the composite functions shown beside the

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Figure 55-1

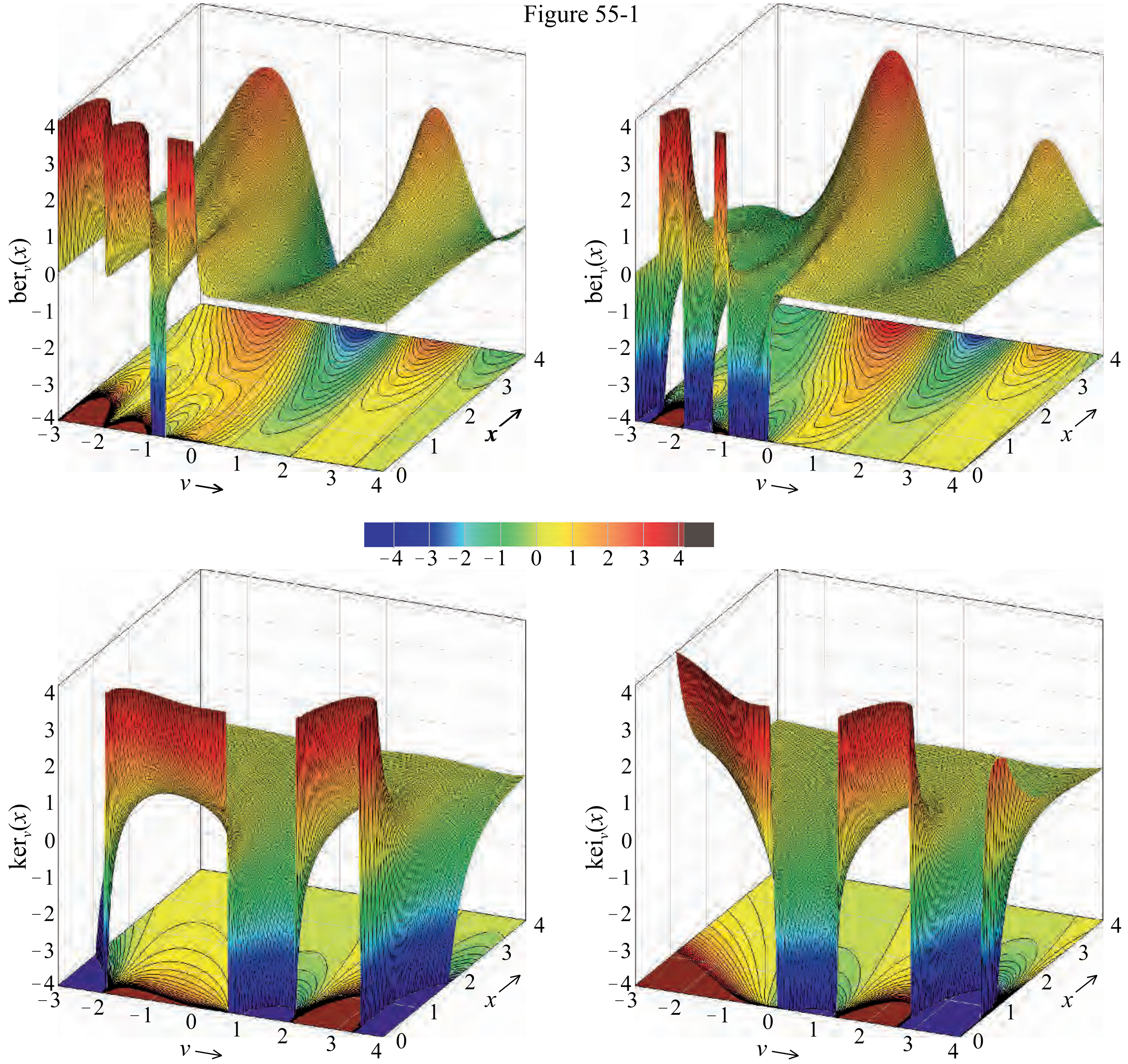


figure. It is evident that, as x increases, the four amplitude-adjusted Kelvin functions come to differ among themselves, and from $\sin\left(x/\sqrt{2}\right)$, only in the phase of their oscillations. All five curves rapidly evolve to satisfy

55:2:1
$$\left(\begin{matrix}\text{amplitude}\\ \text{adjusted}\\ \text{zero order}\\ \text{Kelvin function}\end{matrix}\right)_{x\to\infty} \to \sin\left(\frac{x}{\sqrt{2}}+\phi\right) \qquad \begin{cases}\phi = 3\pi/8 & \text{for ber}\\ \phi = -\pi/8 & \text{for bei}\\ \phi = 5\pi/8 & \text{for ker}\\ \phi = -7\pi/8 & \text{for kei}\end{cases}$$

The same general behavior is exhibited by amplitude-adjusted Kelvin functions of all orders. The amplitude-adjustment factors are independent of the order ν and, after adjustment, each function is an approximate sinusoid which approaches a perfect sinusoid with increasing argument. The order ν of the function affects only the phase of the oscillations and the alacrity with which the aperiodic component of the function dies away.

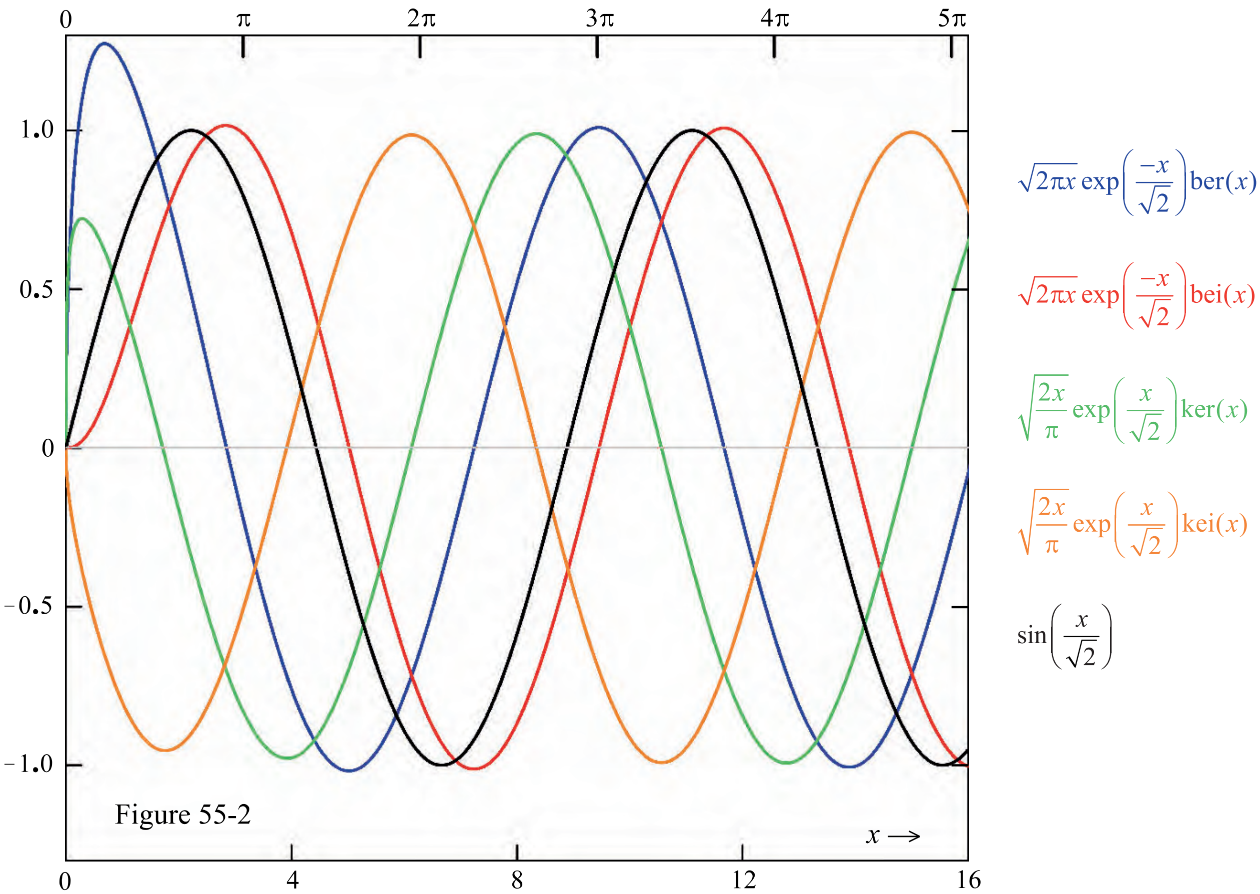


Figure 55-2

55:3 DEFINITIONS

Kelvin functions of real order and positive real argument are themselves invariably real. Despite this, the most common definitions of ber, bei, ker, and kei involve a complex variable. The complex variable in question is one that has real and imaginary parts of equal magnitude, namely one of the four values $z = \pm(x \pm ix)/\sqrt{2}$. These values are represented in Figure 55-3 by the points labeled A, B, C, and D, all lying on a circle of radius x, in the complex plane illustrated.

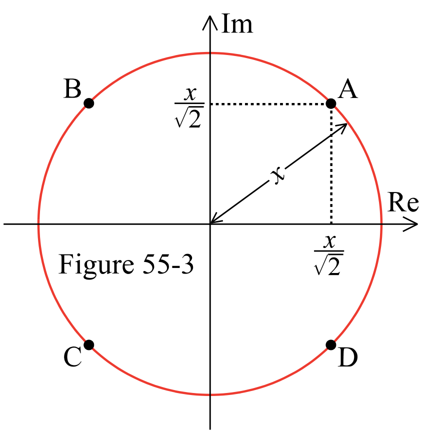


Figure 55-3

The regular Kelvin functions ber and bei are most simply defined through the corresponding Bessel functions. One pair of definitions, based on two of the points in Figure 55-3, is

55:3:1
$$\mathrm{ber}_\nu(x) = \frac{\mathrm{J}_\nu(z_\mathrm{B}) + \mathrm{J}_\nu(z_\mathrm{C})}{2} = \frac{1}{2}\mathrm{J}_\nu\left(\frac{-x}{\sqrt{2}} + \frac{ix}{\sqrt{2}}\right) + \frac{1}{2}\mathrm{J}_\nu\left(\frac{-x}{\sqrt{2}} - \frac{ix}{\sqrt{2}}\right)$$

55:3:2
$$\mathrm{bei}_\nu(x) = \frac{\mathrm{J}_\nu(z_\mathrm{B}) - \mathrm{J}_\nu(z_\mathrm{C})}{2i} = \frac{i}{2}\mathrm{J}_\nu\left(\frac{-x}{\sqrt{2}} - \frac{ix}{\sqrt{2}}\right) - \frac{i}{2}\mathrm{J}_\nu\left(\frac{-x}{\sqrt{2}} + \frac{ix}{\sqrt{2}}\right)$$

Equivalently the ber and bei functions may be considered the real and imaginary parts of the Bessel function of point B

55:3:3 $$J_\nu(z_B)=\mathrm{Re}\left\{J_\nu\left(\frac{-x}{\sqrt{2}}+\frac{ix}{\sqrt{2}}\right)\right\}+i\,\mathrm{Im}\left\{J_\nu\left(\frac{-x}{\sqrt{2}}+\frac{ix}{\sqrt{2}}\right)\right\}=\mathrm{ber}_\nu(x)+i\,\mathrm{bei}_\nu(x)$$

and it is this definition that is responsible for the be**r** and be**i** notation. See Section 49:11 and equation 50:11:6 for analogous definitions based on the *modified* Bessel function.

In a fashion similar to 55:3:3, the irregular Kelvin functions may be defined with reference to the Macdonald function of a complex variable at point A but, unless $\nu = 0$, with a complicating multiplicative factor equal to $\cos(\nu\pi/2) - i\sin(\nu\pi/2)$.

55:3:4 $$\begin{matrix}\mathrm{ker}_\nu\\ \mathrm{kei}_\nu\end{matrix}(x)=\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left\{\exp\left(-\tfrac{\nu}{2}\pi i\right)K_\nu(z_A)\right\}=\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left\{\exp\left(-\tfrac{\nu}{2}\pi i\right)K_\nu\left(x/\sqrt{2}+ix/\sqrt{2}\right)\right\}$$

The irregular Kelvin functions of zero order may be represented by a number of integrals, including:

55:3:5 $$\mathrm{ker}(x)=\frac{-x^2}{8}\int_0^\infty \mathrm{Ci}\left(\frac{1}{t}\right)\exp\left(\frac{-x^2t}{4}\right)dt=\int_0^\infty\frac{t^3J_0(xt)}{1+t^4}dt=\frac{x}{4}\int_0^\infty\ln(1+t^4)J_1(xt)\,dt$$

55:3:6 $$\mathrm{kei}(x)=\frac{x^2}{8}\int_0^\infty\left[\mathrm{Si}\left(\frac{1}{t}\right)-\frac{\pi}{2}\right]\exp\left(\frac{-x^2t}{4}\right)dt=-\int_0^\infty\frac{t\,J_0(xt)}{1+t^4}dt=\frac{-x}{2}\int_0^\infty\arctan(t^2)J_1(xt)\,dt$$

These integrals, which involve functions from Chapters 35, 38, and 52, provide alternative definitions.

The solution to the following differential equation involves arbitrarily weighted pairs of Kelvin functions

55:3:7 $$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}-(\nu^2+ix^2)f=0\qquad f=w_1\left[\mathrm{ber}_\nu(x)+i\,\mathrm{bei}_\nu(x)\right]+w_2\left[\mathrm{ker}_\nu(x)+i\,\mathrm{kei}_\nu(x)\right]$$

55:4 SPECIAL CASES

In general, simple power series do not exist for Kelvin functions. Special exceptions are the regular $\mathrm{ber}_\nu(x)$ and $\mathrm{bei}_\nu(x)$ functions for which the order is a multiple of ⅓, including $\nu = 0$. The zero-order series are reported in equations 55:6:1 and 55:6:2; the others may be deduced easily from equation 55:6:9.

Kelvin functions of order ½ reduce to products of exponential and sinusoidal functions:

55:4:1 $$\begin{matrix}\mathrm{ber}_{1/2}\\ \mathrm{bei}_{1/2}\end{matrix}(x)=\frac{1}{\sqrt{2\pi x}}\left[\exp\left(\frac{x}{\sqrt{2}}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{x}{\sqrt{2}}+\frac{\pi}{8}\right)\mp\exp\left(\frac{-x}{\sqrt{2}}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)\right]$$

55:4:2 $$\begin{matrix}\mathrm{ker}_{1/2}\\ \mathrm{kei}_{1/2}\end{matrix}(x)=-\sqrt{\frac{\pi}{2x}}\exp\left(\frac{-x}{\sqrt{2}}\right)\begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)$$

55:5 INTRARELATIONSHIPS

The order-reflection formulas

55:5:1 $$\begin{matrix}\mathrm{ber}_{-\nu}\\ \mathrm{bei}_{-\nu}\end{matrix}(x)=\cos(\nu\pi)\begin{matrix}\mathrm{ber}_\nu\\ \mathrm{bei}_\nu\end{matrix}(x)\pm\sin(\nu\pi)\begin{matrix}\mathrm{bei}_\nu\\ \mathrm{ber}_\nu\end{matrix}(x)+\frac{2}{\pi}\sin(\nu\pi)\begin{matrix}\mathrm{ker}_\nu\\ \mathrm{kei}_\nu\end{matrix}(x)$$

55:5:2 $$\begin{matrix}\mathrm{ker}_{-\nu}\\ \mathrm{kei}_{-\nu}\end{matrix}(x)=\cos(\nu\pi)\begin{matrix}\mathrm{ker}_\nu\\ \mathrm{kei}_\nu\end{matrix}(x)\mp\sin(\nu\pi)\begin{matrix}\mathrm{kei}_\nu\\ \mathrm{ker}_\nu\end{matrix}(x)$$

all reduce to

55:5:3 $$\mathrm{f}_{-n}(x) = (-)^n \mathrm{f}_n(x) \qquad \mathrm{f} = \text{ber, bei, ker, or kei} \qquad n = 0, \pm1, \pm2, \cdots$$

when the order is an integer. Negative arguments generally make the Kelvin functions complex. Exceptions are the integer-ordered regular functions, for which the following argument-reflection formula applies:

55:5:4 $$\mathrm{f}_n(-x) = (-)^n \mathrm{f}_n(x) \qquad \mathrm{f} = \text{ber or bei} \qquad n = 0, \pm1, \pm2, \cdots$$

The recurrence relations of the Kelvin functions are

55:5:5 $$\mathrm{fer}_{\nu+1}(x) = -\nu\sqrt{2}\,\frac{\mathrm{fer}_\nu(x) - \mathrm{fei}_\nu(x)}{x} - \mathrm{fer}_{\nu-1}(x)$$

55:5:6 $$\mathrm{fei}_{\nu+1}(x) = -\nu\sqrt{2}\,\frac{\mathrm{fer}_\nu(x) + \mathrm{fei}_\nu(x)}{x} - \mathrm{fei}_{\nu-1}(x)$$

$$\left.\right\} \quad \mathrm{fer} = \text{ber or ker} \qquad \mathrm{fei} = \text{bei or kei}$$

55:6 EXPANSIONS

Because of their salient importance, we first report the convergent expansions for the functions of zero order. The regular pair are $L = K+4 = 4$ hypergeometric functions:

55:6:1 $$\mathrm{ber}(x) = 1 - \frac{x^4}{64} + \frac{x^8}{147546} - \cdots = \sum_{j=0}^{\infty} \frac{\left(-x^4/16\right)^j}{\left[(2j)!\right]^2} = \sum_{j=0}^{\infty} \frac{1}{\left(\tfrac{1}{2}\right)_j \left(\tfrac{1}{2}\right)_j (1)_j (1)_j} \left(\frac{-x^4}{256}\right)^j$$

55:6:2 $$\mathrm{bei}(x) = \frac{x^2}{4} - \frac{x^6}{2304} + \frac{x^{10}}{14754600} - \cdots = \sum_{j=0}^{\infty} \frac{(-)^j \left(x^4/16\right)^{j+\frac{1}{2}}}{\left[(2j+1)!\right]^2} = \frac{x^2}{4} \sum_{j=0}^{\infty} \frac{1}{(1)_j (1)_j \left(\tfrac{3}{2}\right)_j \left(\tfrac{3}{2}\right)_j} \left(\frac{-x^4}{256}\right)^j$$

These regular functions appear in the following series describing the irregular functions of zero order:

55:6:3 $$\mathrm{ker}(x) = \left[\ln\left(\frac{2}{x}\right) - \gamma\right] \mathrm{ber}(x) + \frac{\pi}{4} \mathrm{bei}(x) + \sum_{j=0}^{\infty} \frac{(-x^4/16)^j}{[(2j)!]^2} \sum_{k=1}^{2j} \frac{1}{k}$$

55:6:4 $$\mathrm{kei}(x) = \left[\ln\left(\frac{2}{x}\right) - \gamma\right] \mathrm{bei}(x) - \frac{\pi}{4} \mathrm{ber}(x) + \frac{x^2}{4} \sum_{j=0}^{\infty} \frac{(-x^4/16)^j}{[(2j+1)!]^2} \sum_{k=1}^{2j+1} \frac{1}{k}$$

Asymptotic series, valid for large x, are

55:6:5 $$\begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}(x) \sim \frac{\exp\left(x/\sqrt{2}\right)}{\sqrt{2\pi x}} \sum_{j=0} \frac{[(2j)!]^2}{[j!]^3 [32x]^j} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{x}{\sqrt{2}} - \frac{j\pi}{4} - \frac{\pi}{8}\right) \mp \frac{1}{\pi} \begin{matrix}\mathrm{kei}\\ \mathrm{ker}\end{matrix}(x)$$

55:6:6 $$\begin{matrix}\mathrm{ker}\\ \mathrm{kei}\end{matrix}(x) \sim \sqrt{\frac{\pi}{2x}} \exp\left(\frac{-x}{\sqrt{2}}\right) \sum_{j=0} \frac{[(2j)!]^2}{[j!]^3 [32x]^j} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{-x}{\sqrt{2}} + \frac{3j\pi}{4} - \frac{\pi}{8}\right)$$

For arbitrary order the regular Kelvin functions may be expanded as the series

55:6:7 $$\begin{matrix}\mathrm{ber}_\nu\\ \mathrm{bei}_\nu\end{matrix}(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+\nu}}{j!\,\Gamma(1+j+\nu)} \begin{matrix}\cos\\ \sin\end{matrix}\left(\tfrac{1}{2} j\pi + \tfrac{3}{4}\nu\pi\right)$$

but the irregular functions are most easily calculated via their regular cohorts, through the equations

55:6:8 $$\begin{matrix}\mathrm{ker}_\nu\\ \mathrm{kei}_\nu\end{matrix}(x) = \frac{\pi}{2}\left[\csc(\nu\pi) \begin{matrix}\mathrm{ber}_{-\nu}\\ \mathrm{bei}_{-\nu}\end{matrix}(x) - \cot(\nu\pi) \begin{matrix}\mathrm{ber}_\nu\\ \mathrm{bei}_\nu\end{matrix}(x) \mp \begin{matrix}\mathrm{bei}_\nu\\ \mathrm{ber}_\nu\end{matrix}(x)\right] \qquad \nu \neq 0, \pm1, \pm2, \cdots$$

which involve orders of $-\nu$ as well as ν. However, this last formula fails for integer orders and must be replaced by the lengthy alternative

55:6:9
$$\begin{aligned}\begin{matrix}\mathrm{ker}_n\\ \mathrm{kei}_n\end{matrix}(x) = \ln\left(\frac{2}{x}\right)\begin{matrix}\mathrm{ber}_n\\ \mathrm{bei}_n\end{matrix}(x) \pm \frac{\pi}{4}\begin{matrix}\mathrm{bei}_n\\ \mathrm{ber}_n\end{matrix}(x) \pm \frac{1}{2}\left(\frac{2}{x}\right)^n \sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x^2}{4}\right)^j \begin{matrix}\cos\\ \sin\end{matrix}\left(\tfrac{3}{4}n\pi + \tfrac{1}{2}j\pi\right) \\ + \frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty}\frac{\psi(1+j)+\psi(1+n+j)}{j!(n+j)!}\left(\frac{x^2}{4}\right)^j \begin{matrix}\cos\\ \sin\end{matrix}\left(\tfrac{3}{4}n\pi + \tfrac{1}{2}j\pi\right)\end{aligned} \qquad n = 0,1,2,\cdots$$

To handle negative integers, first apply the reflection 55:5:3.

Valid for large values of the argument are the asymptotic expansions

55:6:10
$$\begin{matrix}\mathrm{ker}_\nu\\ \mathrm{kei}_\nu\end{matrix}(x) \sim \pm\sqrt{\frac{\pi}{2x}}\exp\left(\frac{-x}{\sqrt{2}}\right)\sum_{j=0}\frac{(\frac{1}{2}-\nu)_j(\frac{1}{2}+\nu)_j}{j!(-2x)^j}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{x}{\sqrt{2}}+\frac{\nu\pi}{2}+\frac{j\pi}{4}+\frac{\pi}{8}\right)$$

55:6:11
$$\begin{matrix}\mathrm{ber}_\nu\\ \mathrm{bei}_\nu\end{matrix}(x) \sim \frac{\exp\left(x/\sqrt{2}\right)}{\sqrt{2\pi x}}\sum_{j=0}\frac{(\frac{1}{2}-\nu)_j(\frac{1}{2}+\nu)_j}{j!(2x)^j}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{x}{\sqrt{2}}+\frac{\nu\pi}{2}-\frac{j\pi}{4}-\frac{\pi}{8}\right) \mp \frac{\mathrm{ker}_\nu(x)}{\pi}\begin{matrix}\sin\\ \cos\end{matrix}(2\nu\pi) - \frac{\mathrm{kei}_\nu(x)}{\pi}\begin{matrix}\cos\\ \sin\end{matrix}(2\nu\pi)$$

The last two terms in expansion 55:6:11 are optional; the series converges to $\mathrm{ber}_\nu(x)$ or $\mathrm{bei}_\nu(x)$ without them, but convergence is accelerated by their presence.

55:7 PARTICULAR VALUES

Kelvin functions display diverse behavior at $x = 0$ but are mostly zero or infinite there. Noteworthy values are

ber(0) = 1	bei(0) = 0	ker(0) = $+\infty$	kei(0) = $-\pi/4$	$\mathrm{ker}_{\pm 2}(0) = \frac{1}{2}$

The damped oscillations of the $\mathrm{ker}_\nu(x)$ and $\mathrm{kei}_\nu(x)$ functions lead rapidly towards a value of zero as $x \to \infty$, whereas $\mathrm{ber}_\nu(x)$ and $\mathrm{bei}_\nu(x)$ oscillate around zero with ever increasing amplitude as this limit is approached.

All Kelvin functions have an infinite number of positive real zeros. Some of these, for the zero-order cases, are listed below. The values are listed in a format that demonstrates the almost-uniform spacing of the zeros, with the kth and $(k-1)$th members separated by a gap very close to $\sqrt{2}\,\pi$.

k	zeros of ber	zeros of bei	zeros of ker	zeros of kei
1	$0.6412\sqrt{2}\,\pi$	$1.1313\sqrt{2}\,\pi$	$0.3868\sqrt{2}\,\pi$	$0.8811\sqrt{2}\,\pi$
2	$1.6293\sqrt{2}\,\pi$	$2.1282\sqrt{2}\,\pi$	$1.3791\sqrt{2}\,\pi$	$1.8781\sqrt{2}\,\pi$
3	$2.6276\sqrt{2}\,\pi$	$3.1271\sqrt{2}\,\pi$	$2.3775\sqrt{2}\,\pi$	$2.8771\sqrt{2}\,\pi$
4	$3.6268\sqrt{2}\,\pi$	$4.1266\sqrt{2}\,\pi$	$3.3768\sqrt{2}\,\pi$	$3.8766\sqrt{2}\,\pi$
99	$98.625\sqrt{2}\,\pi$	$99.125\sqrt{2}\,\pi$	$98.375\sqrt{2}\,\pi$	$98.875\sqrt{2}\,\pi$
100	$99.625\sqrt{2}\,\pi$	$100.125\sqrt{2}\,\pi$	$99.375\sqrt{2}\,\pi$	$99.875\sqrt{2}\,\pi$

This regularity carries over to Kelvin functions of other orders. If $r_\nu^{(k)}$ denotes the kth zero of a Kelvin function of order ν then, especially when k is large, a good approximation is

55:7:1 $$\frac{r_\nu^{(k)}}{\sqrt{2\pi}} \approx k - \frac{\nu}{2} + \frac{\lambda}{8}$$

where $\lambda = -3$ for ber_ν, $\lambda = +1$ for bei_ν, $\lambda = -5$ for ker_ν, and $\lambda = -1$ for kei_ν.

55:8 NUMERICAL VALUES

For zero order only, *Equator* provides a routine for each of the four Kelvin functions, each keyword being identical with the symbol. For example, the Kelvin ber function routine has the keyword **ber**.

The ber and bei routines use equations 55:6:1 and 55:6:2 for arguments up to 26. For $26 \le x \le 1000$, the formulas in 55:6:5 are employed without, however, the contributions from the irregular functions, these being insignificant for this range of arguments.

For the ker and kei functions, *Equator* uses equations 55:6:3 and 55:6:4 for arguments less than 4. In the domain $4 \le x \le 990$, the asymptotic formulation 55:6:6 is utilized, with assistance from the ε-transformation [Section 10:16] for arguments up to 20.

55:9 LIMITS AND APPROXIMATIONS

The panel below shows the limiting forms generally adopted by the Kelvin functions for small arguments

$\mathrm{ber}_\nu(0 \leftarrow x)$	$\mathrm{bei}_\nu(0 \leftarrow x)$	$\mathrm{ker}_\nu(0 \leftarrow x)$	$\mathrm{kei}_\nu(0 \leftarrow x)$
$\dfrac{(x/2)^\nu \cos\left(\frac{3}{4}\nu\pi\right)}{\Gamma(1+\nu)}$	$\dfrac{(x/2)^\nu \sin\left(\frac{3}{4}\nu\pi\right)}{\Gamma(1+\nu)}$	$\dfrac{\Gamma(\nu)\cos\left(\frac{3}{4}\nu\pi\right)}{2(x/2)^\nu}$	$\dfrac{-\Gamma(\nu)\sin\left(\frac{3}{4}\nu\pi\right)}{2(x/2)^\nu}$

These limits may be inapplicable when ν is a multiple of 1/3 and for the irregular functions when ν is a small integer.

With λ abbreviating $x/\sqrt{2} + \nu\pi/2$, the corresponding limits for large arguments are

$\mathrm{ber}_\nu(x \to \infty)$	$\mathrm{bei}_\nu(x \to \infty)$	$\mathrm{ker}_\nu(x \to \infty)$	$\mathrm{kei}_\nu(x \to \infty)$
$\dfrac{\exp\left(x/\sqrt{2}\right)}{\sqrt{2\pi x}}\cos\left(\lambda - \dfrac{\pi}{8}\right)$	$\dfrac{\exp\left(x/\sqrt{2}\right)}{\sqrt{2\pi x}}\sin\left(\lambda - \dfrac{\pi}{8}\right)$	$\dfrac{\exp\left(-x/\sqrt{2}\right)}{\sqrt{2x/\pi}}\cos\left(\lambda + \dfrac{\pi}{8}\right)$	$\dfrac{-\exp\left(-x/\sqrt{2}\right)}{\sqrt{2x/\pi}}\sin\left(\lambda + \dfrac{\pi}{8}\right)$

From these, the simpler limits acquired by such combinations as $\mathrm{ber}_\nu(x)\mathrm{bei}_\nu(x)$, $\mathrm{ker}_\nu^2(x) + \mathrm{kei}_\nu^2(x)$, and $\mathrm{ber}_\nu(x)\mathrm{kei}_\nu(x) - \mathrm{bei}_\nu(x)\mathrm{ker}_\nu(x)$ are readily derived.

55:10 OPERATIONS OF THE CALCULUS

The differentiation formulas

$$\left.\begin{aligned} &\text{55:10:1} \quad \frac{d}{dx}\mathrm{fer}_\nu(x) = \left[\mathrm{fer}_{\nu+1}(x) + \mathrm{fei}_{\nu+1}(x) - \mathrm{fer}_{\nu-1}(x) - \mathrm{fei}_{\nu-1}(x)\right]/\sqrt{8} \\ &\text{55:10:2} \quad \frac{d}{dx}\mathrm{fei}_\nu(x) = \left[\mathrm{fei}_{\nu+1}(x) - \mathrm{fer}_{\nu+1}(x) - \mathrm{fei}_{\nu-1}(x) + \mathrm{fer}_{\nu-1}(x)\right]\sqrt{8} \end{aligned}\right\} \begin{aligned} &\mathrm{fer} = \mathrm{ber} \text{ or } \mathrm{ker} \\ &\mathrm{fei} = \mathrm{bei} \text{ or } \mathrm{kei} \end{aligned}$$

may be combined with recursions 55:5:5 and 55:5:6 to produce a number of alternative versions. An example of an indefinite integral is

55:10:3 $$\int_0^x t^{1\pm\nu}\mathrm{ber}_\nu(t)\,\mathrm{d}t = \frac{\mp x^{1\pm\nu}}{\sqrt{2}}\left[\mathrm{ber}_{1\pm\nu}(x) - \mathrm{bei}_{1\pm\nu}(x)\right]$$

Section 9.9 of Abramowitz and Stegun contains several other indefinite integrals of Kelvin functions.

Definite integrals and Laplace transforms of zero-order Kelvin functions include

55:10:4 $$\int_0^\infty \begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}\left(2\sqrt{t}\right)\exp\left(\frac{-t}{x}\right)\mathrm{d}t = x\begin{matrix}\cos\\ \sin\end{matrix}(x)$$

55:10:5 $$\int_0^\infty \begin{matrix}\mathrm{ker}\\ \mathrm{kei}\end{matrix}\left(2\sqrt{t}\right)\exp\left(\frac{-t}{x}\right)\mathrm{d}t = \frac{-x}{2}\left[\begin{matrix}\cos\\ \sin\end{matrix}(x)\mathrm{Ci}(x) \pm \begin{matrix}\sin\\ \cos\end{matrix}(x)\left[\mathrm{Si}(x) - \tfrac{1}{2}\pi\right]\right]$$

55:10:6 $$\int_0^\infty \begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}(bt)\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}(bt)\right\} = \sqrt{\frac{1}{2\sqrt{s^4+b^4}} \pm \frac{s^2}{2(s^4+b^4)}}$$

55:10:7 $$\int_0^\infty \begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}\left(\sqrt{bt}\right)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\begin{matrix}\mathrm{ber}\\ \mathrm{bei}\end{matrix}\left(\sqrt{bt}\right)\right\} = \frac{1}{s}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{b}{4s}\right)$$

Other definite integrals of Kelvin functions will be found in Section 6.8 of Gradshteyn and Ryzhik.

55:11 COMPLEX ARGUMENT

Kelvin functions of complex argument are seldom encountered and the *Atlas* ignores this possibility.

55:12 GENERALIZATIONS

The function

55:12:1 $$\mathrm{f}_0(\theta,x) = \sum_{j=0}^\infty\left[\frac{\{\tfrac{1}{2}x[\cos(\theta)+i\sin(\theta)]\}^j}{j!}\right]^2 = \sum_{j=0}^\infty \frac{\exp(2ji\theta)}{(j!)^2}\left(\frac{x^2}{4}\right)^j$$

may be considered the progenitor of all regular zero-order cylinder functions because it has the special cases

$\mathrm{f}_0(0,x)$	$\mathrm{f}_0\left(\frac{\pi}{4},x\right)$	$\mathrm{f}_0\left(\frac{\pi}{2},x\right)$	$\mathrm{f}_0\left(\frac{3\pi}{4},x\right)$	$\mathrm{f}_0(\pi,x)$
$\mathrm{I}_0(x)$	$\mathrm{ber}_0(x)+i\,\mathrm{bei}_0(x)$	$\mathrm{J}_0(x)$	$\mathrm{ber}_0(x)-i\,\mathrm{bei}_0(x)$	$\mathrm{I}_0(x)$

Kelvin functions are those instances of this function that have equal magnitudes of "realness" and "imaginaryness".

55:13 COGNATE FUNCTIONS

Just as equation 55:3:3 may be said to define the $\mathrm{ber}_\nu(x)$ and $\mathrm{bei}_\nu(x)$ functions in terms of the Bessel function $\mathrm{J}_\nu\left\{(-x+ix)/\sqrt{2}\right\}$, so may the functions $\mathrm{her}_\nu(x)$ and $\mathrm{hei}_\nu(x)$ be defined as the real and imaginary parts of the Hankel function $\mathrm{H}_\nu^{(1)}\left\{(-x+ix)/\sqrt{2}\right\}$ [Section 54:13]. These are sometimes described as *Kelvin functions of the third kind*.

CHAPTER 56

THE AIRY FUNCTIONS Ai(x) AND Bi(x)

George Biddell Airy, 1801–1892, was an English astronomer and mathematician. The functions that carry his name find use in electromagnetic theory and elsewhere. The derivatives of the two Airy functions and a pair of auxiliary functions also receive brief mention in this chapter.

56:1 NOTATION

The symbols Ai(x) and Bi(x) are used universally for the Airy functions. The primed notations Ai′(x) and Bi′(x) denote the derivatives. Sometimes the designations "of the first kind" and "of the second kind" are applied, respectively, to distinguish Ai and Bi.

This *Atlas* use fai(x) and gai(x) to symbolize the functions that Abramowitz and Stegun [Section 10.4] denote by $f(x)$ and $g(x)$. These are named *auxiliary Airy functions*, but they are not "auxiliary" in the sense usually attributed to that designation; they are simply weighted sums of Ai(x) and Bi(x). Other functions, related but innominate, are hai(x), Gi(x) and Hi(x).

In formulas involving Airy functions, it is often more convenient to use the *auxiliary argument*

56:1:1 $$\hat{x} = \tfrac{2}{3}\left(\sqrt{|\,x\,|}\right)^3 \qquad x = \sqrt[3]{\tfrac{9}{4}\hat{x}^2}$$

We avoid writing this definition as $2x^{3/2}/3$ to emphasize that $\hat{x}$ is invariably real and positive. When $x = 0$ or $x = 9/4$, the two arguments coincide.

56:2 BEHAVIOR

In the study of mathematical functions, it is common to find function pairs, such as sin(x) and sinh(x), that are distinguished as "circular" and "hyperbolic" and that have distinctive properties. The most striking distinction is that circular functions are oscillatory, whereas hyperbolic functions are generally monotonic. Airy functions possess both these properties at once: they are "hyperbolic" for positive argument and "circular" when $x < 0$. This

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_57,

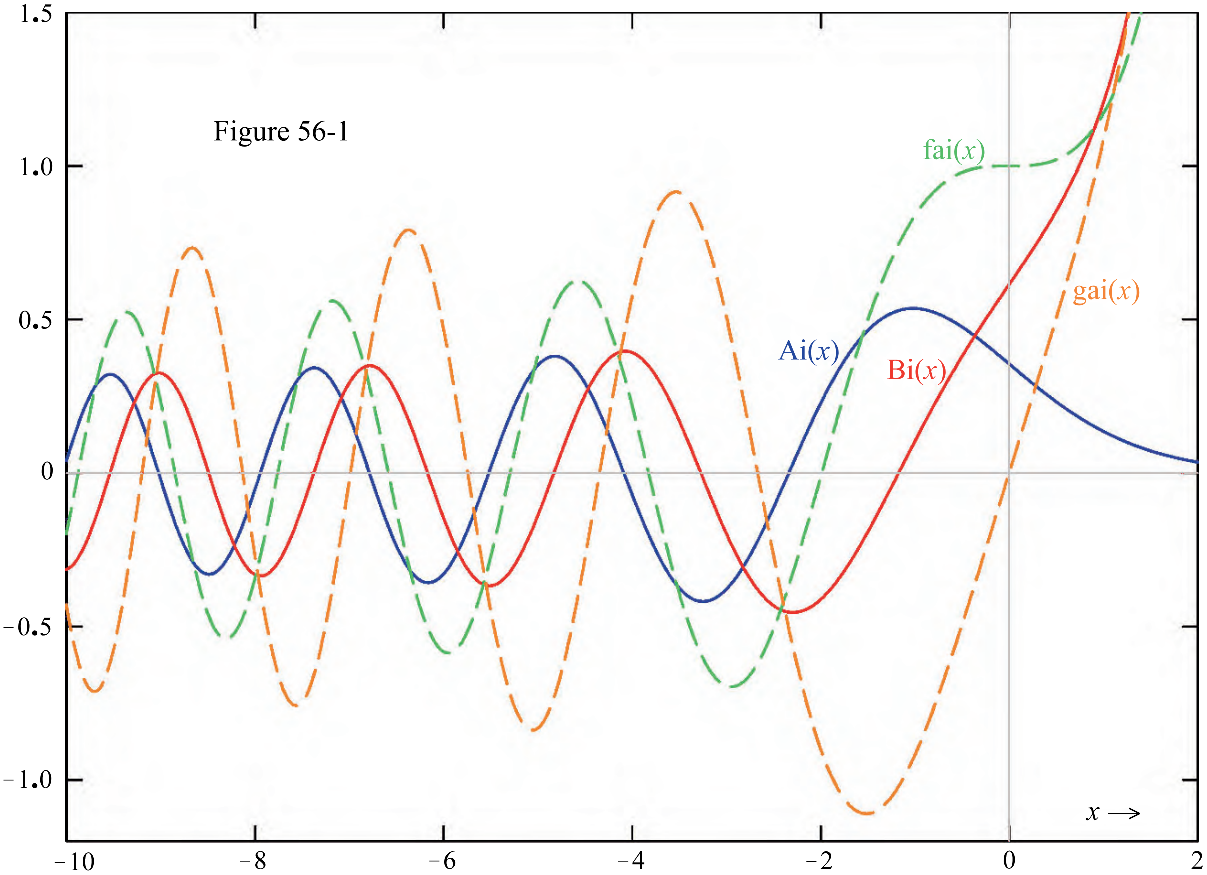


Figure 56-1

hermaphroditic behavior is evident in Figure 56-1. Notice that in their negative realm, both the amplitude and the period of the Ai(x) and Bi(x) functions shrink as x becomes more negative.

The behavior of the auxiliary Airy functions is generally similar to that of Ai(x) and Bi(x). These too are plotted on Figure 56-1. Like Bi(x), fai(x) and gai(x) increase exponentially on the positive side, whereas Ai(x) declines exponentially.

56:3 DEFINITIONS

Airy functions and their auxiliaries are cylinder functions of order ⅓. They may be defined through the major cylinder functions as follows:

56:3:1
$$\mathrm{Ai}(x) = \begin{cases} \left.\begin{array}{l} \frac{1}{3}\sqrt{x}\left[\mathrm{I}_{-1/3}(\hat{x}) - \mathrm{I}_{1/3}(\hat{x})\right] \\ (1/\pi)\sqrt{x/3}\,\mathrm{K}_{1/3}(\hat{x}) \end{array}\right\} & x > 0 \\ \left.\begin{array}{l} \frac{1}{3}\sqrt{-x}\left[\mathrm{J}_{-1/3}(\hat{x}) + \mathrm{J}_{1/3}(\hat{x})\right] \\ \sqrt{-x/3}\left[\mathrm{Y}_{-1/3}(\hat{x}) - \mathrm{Y}_{1/3}(\hat{x})\right] \end{array}\right\} & x < 0 \end{cases}$$

56:3:2
$$\mathrm{Bi}(x)=\begin{cases}\sqrt{x/3}\left[\mathrm{I}_{-1/3}(\hat{x})+\mathrm{I}_{1/3}(\hat{x})\right] & x>0\\ \sqrt{-x/3}\left[\mathrm{J}_{-1/3}(\hat{x})-\mathrm{J}_{1/3}(\hat{x})\right] & \\ \frac{-1}{3}\sqrt{-x}\left[\mathrm{Y}_{-1/3}(\hat{x})+\mathrm{Y}_{1/3}(\hat{x})\right] & \end{cases}\Bigg\}\quad x<0$$

56:3:3
$$\mathrm{fai}(x)=\frac{\sqrt[3]{9}\,\Gamma(2/3)}{2}\left[\frac{\mathrm{Bi}(x)}{\sqrt{3}}+\mathrm{Ai}(x)\right]=\begin{cases}\left[\Gamma(2/3)\sqrt{x}/\sqrt[3]{3}\right]\mathrm{I}_{-1/3}(x) & x>0\\ \left[\Gamma(2/3)\sqrt{-x}/\sqrt[3]{3}\right]\mathrm{J}_{-1/3}(x) & x<0\end{cases}$$

56:3:4
$$\mathrm{gai}(x)=\frac{\sqrt[3]{3}\,\Gamma(1/3)}{2}\left[\frac{\mathrm{Bi}(x)}{\sqrt{3}}-\mathrm{Ai}(x)\right]=\begin{cases}\left[\Gamma(1/3)\sqrt{x}/\sqrt[3]{9}\right]\mathrm{I}_{1/3}(x) & x>0\\ \left[\Gamma(1/3)\sqrt{-x}/\sqrt[3]{9}\right]\mathrm{J}_{1/3}(x) & x<0\end{cases}$$

The *Airy integral* serves to define the Ai function

56:3:5
$$\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^\infty \cos\left(xt+\frac{t^3}{3}\right)\mathrm{d}t$$

but the corresponding representation of the Bi function is more circuitous. First one defines two other functions

56:3:6
$$\mathrm{Gi}(x)=\frac{1}{\pi}\int_0^\infty \sin\left(xt+\frac{t^3}{3}\right)\mathrm{d}t \qquad \text{and} \qquad \mathrm{Hi}(x)=\frac{1}{\pi}\int_0^\infty \exp\left(xt-\frac{t^3}{3}\right)\mathrm{d}t$$

that are encountered again in Section 56:10; then

56:3:7
$$\mathrm{Bi}(x)=\mathrm{Gi}(x)+\mathrm{Hi}(x)$$

The auxiliary Airy functions are hypergeometric [Section 18:14] and may be synthesized [Section 43:14] from the zero-order modified Bessel function:

56:3:8
$$\mathrm{fai}\left(\sqrt[3]{9x}\right)\xleftarrow[2/3]{1}\ \mathrm{I}_0\left(2\sqrt{x}\right)\xrightarrow[4/3]{1}\frac{\mathrm{gai}\left(\sqrt[3]{9x}\right)}{\sqrt[3]{9x}} \qquad x>0$$

Similar synthetic routes from the J_0 function create auxiliary Airy functions of negative argument. *Airy's differential equation* and its solutions are

56:3:9
$$\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}-x\mathrm{f}=0 \qquad \mathrm{f}=w_1\mathrm{Ai}(x)+w_2\mathrm{Bi}(x) \qquad \text{or} \qquad \mathrm{f}=w_3\mathrm{fai}(x)+w_4\mathrm{gai}(x)$$

It is in solving this differential equation that Airy functions generally arise in applications.

56:4 SPECIAL CASES

There are none.

56:5 INTRARELATIONSHIPS

The relationship between the auxiliary and the definitive Airy functions is reported in definitions 56:3:3 and 56:3:4. These may be rephrased to show that Ai is a weighted sum of fai and gai, as is Bi:

56:5:1 $$\mathrm{Ai}(x)=\mathrm{Ai}(0)\mathrm{fai}(x)+\mathrm{Ai}'(0)\mathrm{gai}(x)\approx 0.35503\ \mathrm{fai}(x)-0.25882\ \mathrm{gai}(x)$$

and

56:5:2 $$\mathrm{Bi}(x)=\mathrm{Bi}(0)\mathrm{fai}(x)+\mathrm{Bi}'(0)\mathrm{gai}(x)\approx 0.61493\ \mathrm{fai}(x)+0.44829\ \mathrm{gai}(x)$$

More precise numerical values of the weighting constants are reported in Section 56:7.

56:6 EXPANSIONS

The power series expansion of the auxiliary Airy functions may be expressed with coefficients in terms of either triple factorials [Section 2:13] or Pochhammer polynomials [Chapter 19]:

56:6:1 $$\mathrm{fai}(x)=1+\frac{x^3}{6}+\frac{x^6}{180}+\frac{x^9}{12960}+\cdots=\sum_{j=0}^{\infty}\frac{(3j+1)!!!x^{3j}}{(3j+1)!}=\sum_{j=0}^{\infty}\frac{1}{\left(\frac{2}{3}\right)_j(1)_j}\left(\frac{x^3}{9}\right)^j$$

56:6:2 $$\mathrm{gai}(x)=x+\frac{x^4}{12}+\frac{x^7}{504}+\frac{x^{10}}{45360}+\cdots=\sum_{j=0}^{\infty}\frac{(3j+2)!!!x^{3j+1}}{(3j+2)!}=x\sum_{j=0}^{\infty}\frac{1}{(1)_j\left(\frac{4}{3}\right)_j}\left(\frac{x^3}{9}\right)^j$$

The powers x^2, x^5, x^8, $\cdots$, are absent from both these series but these powers alone are present in a third series

56:6:3 $$\mathrm{hai}(x)=\frac{x^2}{2}+\frac{x^5}{40}+\frac{x^8}{2240}+\frac{x^{11}}{246400}+\cdots=\sum_{j=0}^{\infty}\frac{(3j+3)!!!x^{3j+2}}{(3j+3)!}=\frac{x^2}{2}\sum_{j=0}^{\infty}\frac{1}{\left(\frac{4}{3}\right)_j\left(\frac{5}{3}\right)_j}\left(\frac{x^3}{9}\right)^j$$

which complements the other two. A weighted sum of the first two of these equations can generate power series for each of Ai and Bi, the weights being given by the equations in Section 56:5. The sum of the three functions expanded in formulas 56:6:1−3 is proportional to the Hi function, defined in 56:3:6:

56:6:4 $$\frac{3^{7/6}}{2}\Gamma\left(2/3\right)\mathrm{Hi}(x)=\mathrm{fai}(x)+\mathrm{gai}(x)+\mathrm{hai}(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+\cdots=\sum_{j=0}^{\infty}\frac{(j-2)!!!}{j!}x^j$$

Coefficients defined by

56:6:5 $$a_j=\frac{(6j-1)!!}{j!(2j-1)!!(216)^j}\qquad j=0,1,2,\cdots$$

and obeying the recursion formula

56:6:6 $$a_j=\left(\frac{j-1}{2}+\frac{5}{72j}\right)a_{j-1}\qquad j=1,2,3,\cdots\qquad \text{with}\qquad a_0=1$$

appear in asymptotic series for Ai(x) and Bi(x) for both positive and negative arguments of large magnitude. These series are

56:6:7 $$\begin{matrix}\mathrm{Ai}\\ \mathrm{Bi}\end{matrix}(x)\sim\left(\tfrac{3}{4}\mp\tfrac{1}{4}\right)\frac{\exp(\mp\hat{x})}{\sqrt{\pi\sqrt{x}}}\left[a_0\mp\frac{a_1}{\hat{x}}+\frac{a_2}{\hat{x}^2}\mp\frac{a_3}{\hat{x}^3}+\cdots\right]\qquad \begin{matrix}x\text{ large}\\ \text{and}\\ \text{positive}\end{matrix}$$

and

56:6:8 $$\begin{matrix}\mathrm{Ai}\\ \mathrm{Bi}\end{matrix}(x)\sim\frac{1}{\sqrt{\pi\sqrt{-x}}}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\left(\hat{x}+\tfrac{1}{4}\pi\right)\left[a_0-\frac{a_2}{\hat{x}^2}+\frac{a_4}{\hat{x}^4}-\cdots\right]\mp\begin{matrix}\cos\\ \sin\end{matrix}\left(\hat{x}+\tfrac{1}{4}\pi\right)\left[\frac{a_1}{\hat{x}}-\frac{a_3}{\hat{x}^3}+\frac{a_5}{\hat{x}^5}-\cdots\right]\right\}\qquad \begin{matrix}x\text{ large}\\ \text{and}\\ \text{negative}\end{matrix}$$

Note that both the original argument x and the hatted auxiliary argument [56:1:1] $\hat{x}$ appear in these formulas.

56:7 PARTICULAR VALUES

The values attained by the various Airy functions at zero argument are

Ai(0)	Bi(0)	fai(0)	gai(0)	Ai′(0)	Bi′(0)
$\dfrac{3^{-2/3}}{\Gamma(2/3)} = 0.35502\ 80538\ 87817$	$\sqrt{3}\,\mathrm{Ai}(0)$	1	0	$\dfrac{-\mathrm{Bi}'(0)}{\sqrt{3}}$	$\dfrac{3^{1/6}}{\Gamma(1/3)} = 0.44828\ 83573\ 53826$

Note that $\mathrm{Ai}(0)\mathrm{Bi}'(0) = -\mathrm{Ai}'(0)\mathrm{Bi}(0) = 1/(2\pi)$ and that this is in concurrence with 56:13:1.

Each of the Airy functions, as well as their auxiliaries, has an infinite number of zeros, all of which are at negative arguments. No simple formulas describe these zeros.

56:8 NUMERICAL VALUES

Equator has Airy Ai function, Airy Bi function, auxiliary Airy fai function, and auxiliary Airy gai function routines, with keywords **Ai**, **Bi**, **fai**, and **gai**. Equations 56:6:8, 56:3:3, and 56:3:4 are used for arguments in the domain $-100 \le x \le -4.85$; equations 56:5:1, 56:6:1, and 56:6:2 cover the domain $-4.65 < x < 3$; and equations 56:5:7, 56:5:1, 56:6:1, and 56:6:2 are called upon for the domain $3 \le x \le 100$.

56:9 LIMITS AND APPROXIMATIONS

For small arguments of either sign, the approximations

56:9:1 $$\begin{matrix}\mathrm{Ai}\\ \mathrm{Bi}\end{matrix}(x) \approx \left[1+\frac{x^3}{6}\right]\left[\begin{matrix}\mathrm{Ai}\\ \mathrm{Bi}\end{matrix}(0) + x\begin{matrix}\mathrm{Ai}'\\ \mathrm{Bi}'\end{matrix}(0)\right] \qquad \text{small } x$$

are valid.

For large positive and large negative arguments, the following limits are approached

56:9:2 $$\mathrm{Ai}(x) \to \frac{\exp\left(-\sqrt{4x^3/9}\right)}{2\sqrt{\pi\sqrt{x}}} \qquad \mathrm{Bi}(x) \to \frac{\exp\left(\sqrt{4x^3/9}\right)}{\sqrt{\pi\sqrt{x}}} \qquad x \to \infty$$

56:9:3 $$\mathrm{Ai}(x) \to \frac{\cos\left(\sqrt{-4x^3/9} - \frac{1}{4}\pi\right)}{\sqrt{\pi\sqrt{-x}}} \qquad \mathrm{Bi}(x) \to \frac{-\sin\left(\sqrt{-4x^3/9} - \frac{1}{4}\pi\right)}{\sqrt{\pi\sqrt{-x}}} \qquad x \to -\infty$$

56:10 OPERATIONS OF THE CALCULUS

The derivatives of the definitive and auxiliary variables are linked by

56:10:1 $$\frac{d}{dx}\hat{x} = \sqrt{x} = \sqrt[3]{\sqrt{3/2}\,\hat{x}} \qquad \frac{d}{d\hat{x}}x = \sqrt[3]{\frac{9}{4\hat{x}}} = \frac{\sqrt[3]{2/3}}{\sqrt{x}}$$

The form adopted by the derivative of the Airy function depends on the sign of the argument:

56:10:2
$$\mathrm{Ai}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ai}(x) = \begin{cases} \left.\begin{array}{l} \tfrac{1}{3}x\left[\mathrm{I}_{2/3}(\hat{x}) - \mathrm{I}_{-2/3}(\hat{x})\right] \\ (-x/\sqrt{3}\pi)\mathrm{K}_{2/3}(\hat{x}) \end{array}\right\} & x > 0 \\ \tfrac{1}{3}x\left[\mathrm{J}_{-2/3}(\hat{x}) - \mathrm{J}_{2/3}(\hat{x})\right] & x < 0 \end{cases}$$

56:10:3
$$\mathrm{Bi}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Bi}(x) = \begin{cases} \left(x/\sqrt{3}\right)\left[\mathrm{I}_{-2/3}(\hat{x}) + \mathrm{I}_{2/3}(\hat{x})\right] & x > 0 \\ \left(-x/\sqrt{3}\right)\left[\mathrm{J}_{-2/3}(\hat{x}) + \mathrm{J}_{2/3}(\hat{x})\right] & x < 0 \end{cases}$$

These formulas serve as the definitions of the primed quantities Ai′ and Bi′. Two differentiations lead to

56:10:4
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{f} = x\mathrm{f} \qquad \mathrm{f} = \mathrm{Ai}, \mathrm{Bi}, \mathrm{fai}, \text{ or gai}$$

which is a consequence of Airy's differential equation 56:3:10.

The definite integrals

56:10:5
$$\int_{-\infty}^{0}\mathrm{Ai}(t)\mathrm{d}t = \frac{2}{3} \qquad \text{and} \qquad \int_{0}^{\infty}\mathrm{Ai}(t)\mathrm{d}t = \frac{1}{3}$$

may be combined with each other and with the indefinite integral

56:10:6
$$\int_{-\infty}^{x}\mathrm{Ai}(t)\mathrm{d}t = \pi\left[\mathrm{Ai}(x)\mathrm{Hi}'(x) - \mathrm{Hi}(x)\mathrm{Ai}'(x)\right]$$

to produce expressions for integrals over the ranges $\int_{-\infty}^{\infty}$, $\int_{0}^{x}$, and $\int_{x}^{\infty}$. Integrals with an upper limit of +∞ do not converge for Bi, but

56:10:7
$$\int_{-\infty}^{x}\mathrm{Bi}(t)\mathrm{d}t = \int_{0}^{x}\mathrm{Bi}(t)\mathrm{d}t = \pi\left[\mathrm{Bi}(x)\mathrm{Hi}'(x) - \mathrm{Hi}(x)\mathrm{Bi}'(x)\right] = \pi\left[\mathrm{Gi}(x)\mathrm{Hi}'(x) - \mathrm{Hi}(x)\mathrm{Gi}'(x)\right]$$

Hi and Gi are the functions defined in 56:3:6 and 56:3:7; Hi′ and Gi′ are their derivatives, namely

56:10:8
$$\mathrm{Hi}'(x) = \frac{1}{\pi}\int_{0}^{\infty} t\exp\left(xt - \frac{t^3}{3}\right)\mathrm{d}t$$

56:10:9
$$\mathrm{Gi}'(x) = \frac{1}{\pi}\int_{0}^{\infty} t\cos\left(xt + \frac{t^3}{3}\right)\mathrm{d}t$$

56:11 COMPLEX ARGUMENT

The Airy functions are well defined for complex argument $z = x+iy$. Figures 56-2 and 56-3 show the real and imaginary parts of Ai and Bi close to the origin.

56:12 GENERALIZATIONS

In as much as equations 56:3:1 and 56:3:2 show Ai and Bi to be special cases of the I, K, J and Y cylinder functions, these latter may be said to generalize the Airy functions.

Figure 56-2

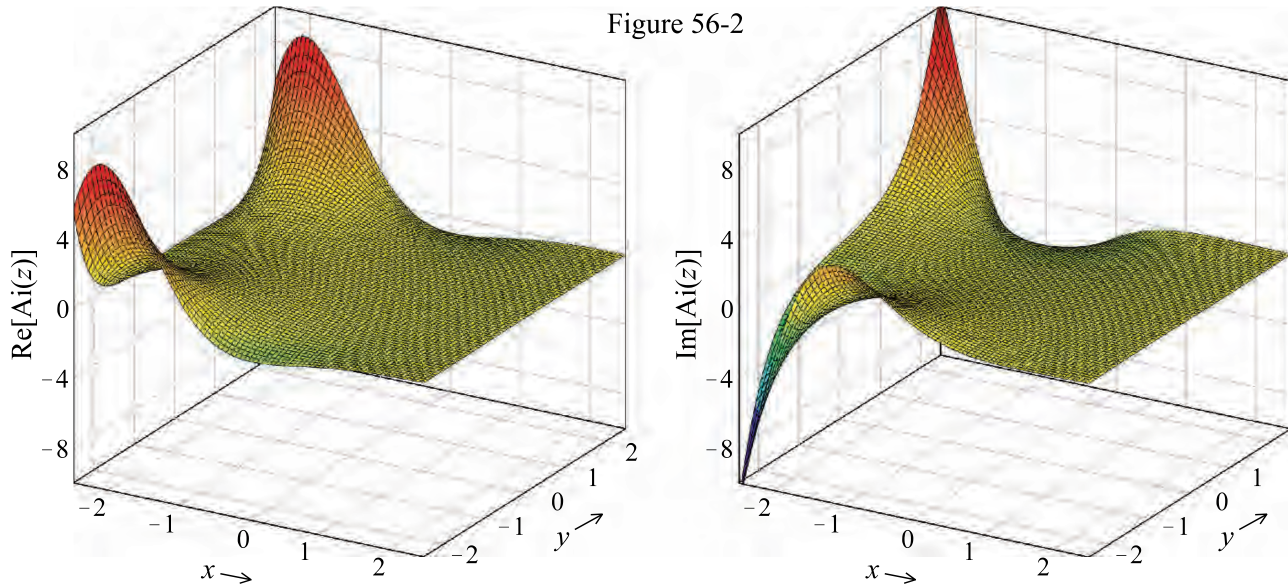
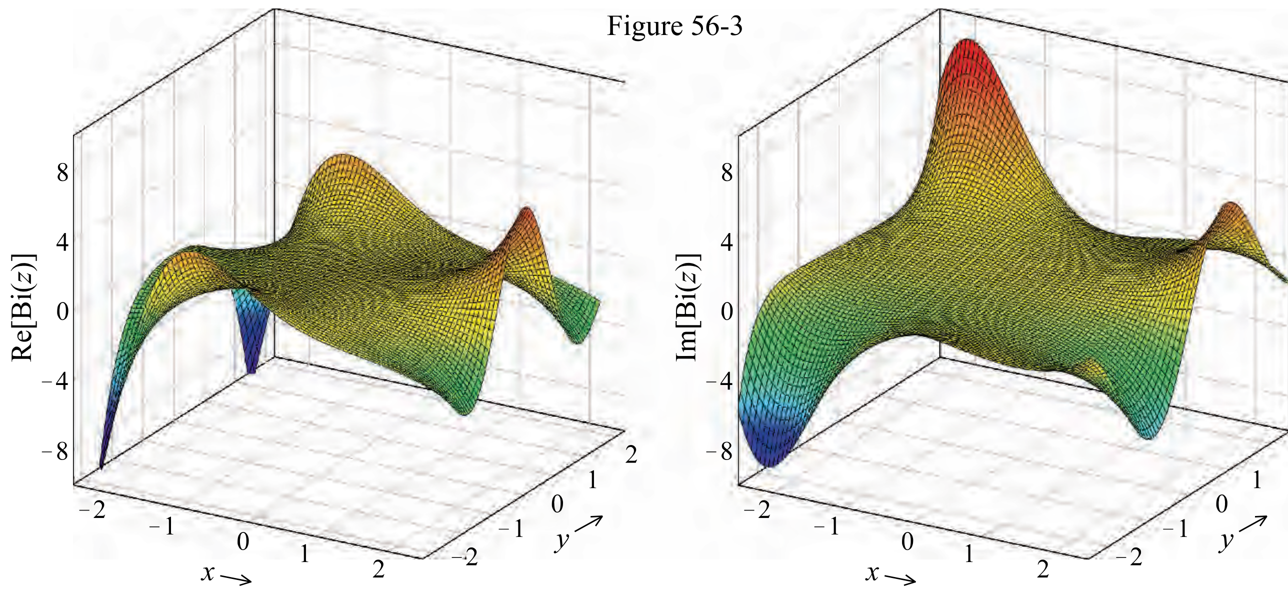


Figure 56-3



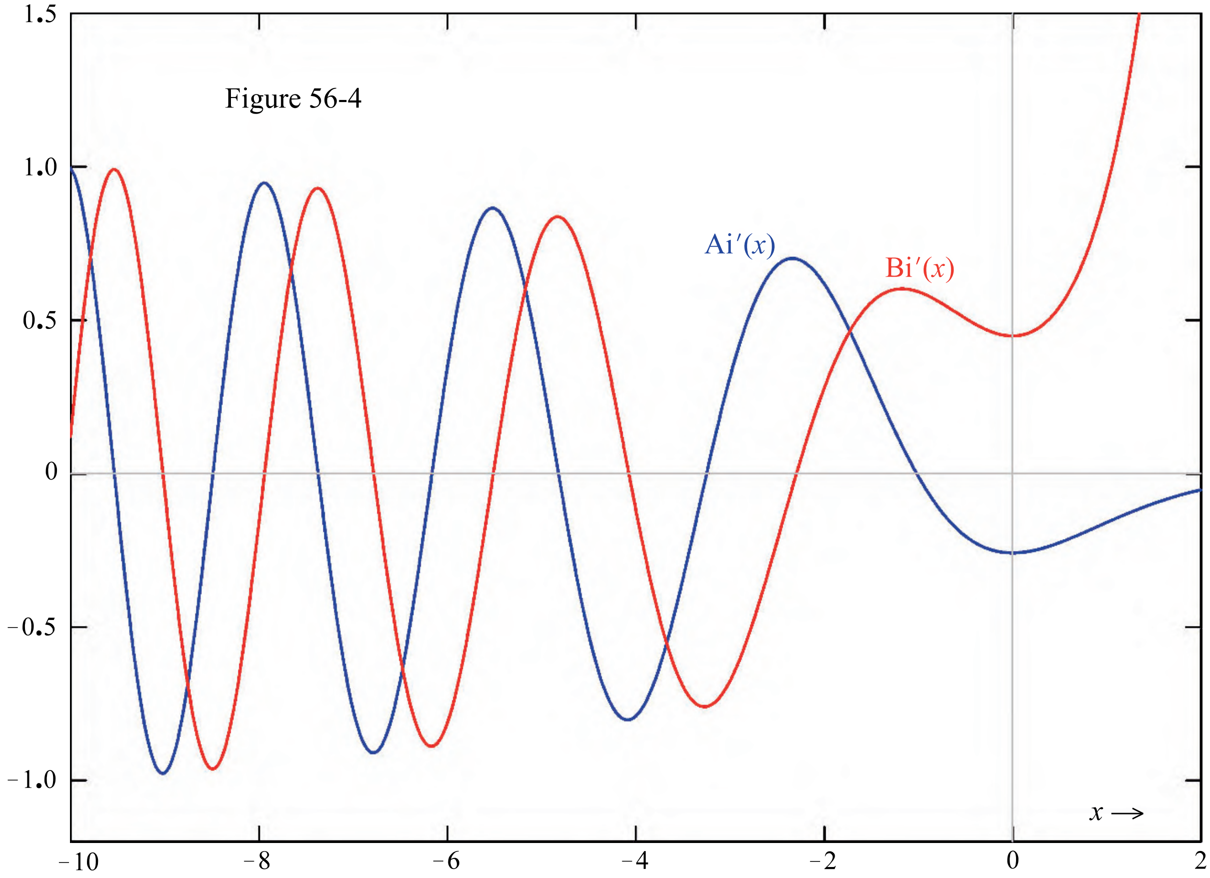
56:13 COGNATE FUNCTIONS: Airy derivatives

Some authorities treat the derivatives Ai′ and Bi′ as Airy functions in their own right, on a par with Ai and Bi. These functions are mapped in Figure 56-4 and are seen to have rather similar behaviors to their parents. One distinction, however, is that whereas the amplitude of the oscillations of Ai and Bi decrease as the argument becomes more negative, those of Ai′ and Bi′ increase. Another is that Ai′ is negative for $x \geq 0$.

Equations 56:10:2 and 56:10:3 constitute definitions of the Airy derivatives and show them to be cylinder functions of order $2/3$. The interrelation

56:13:1
$$\mathrm{Ai}(x)\,\mathrm{Bi}'(x) - \mathrm{Bi}(x)\,\mathrm{Ai}'(x) = \frac{1}{\pi}$$

which is the Wronskian [Section 24:14] of Ai and Bi, links all four functions.



The Airy derivatives are straightforwardly related to the Bessel, and modified Bessel, functions of two-thirds order by the expressions

56:13:2
$$\mathrm{Ai}'(x)=\begin{cases}(x/3)\left[\mathrm{I}_{-2/3}(\hat{x})-\mathrm{I}_{2/3}(\hat{x})\right] & x>0\\ (x/3)\left[\mathrm{J}_{-2/3}(\hat{x})-\mathrm{J}_{2/3}(\hat{x})\right] & x<0\end{cases}$$

and

56:13:3
$$\mathrm{Bi}'(x)=\begin{cases}\left(x/\sqrt{3}\right)\left[\mathrm{I}_{-2/3}(\hat{x})+\mathrm{I}_{2/3}(\hat{x})\right] & x>0\\ \left(-x/\sqrt{3}\right)\left[\mathrm{J}_{-2/3}(\hat{x})+\mathrm{J}_{2/3}(\hat{x})\right] & x<0\end{cases}$$

CHAPTER 57

THE STRUVE FUNCTION $h_\nu(x)$

This bivariate function is closely related to cylinder functions, and especially to the Neumann function. The Struve function has a variety of applications, such as describing the vibrations of thin disks and in the theory of electromagnetism. There is also a hyperbolic analogue that, having fewer applications, has been relegated to Section 57:13.

57:1 NOTATION

The standard notation for the Struve function is $H_\nu(x)$, the "H" often being rendered in bold type. To avoid confusion with Hermite polynomials and functions, this *Atlas* adopts the $h_\nu(x)$ symbol. Likewise $\mathtt{l}_\nu(x)$ is used here, as an unambiguous alternative to the usual $L_\nu(x)$, for representing the modified Struve function.

In preference to h_ν, we employ h_n as the symbol for the Struve function whenever the order is restricted to integer values.

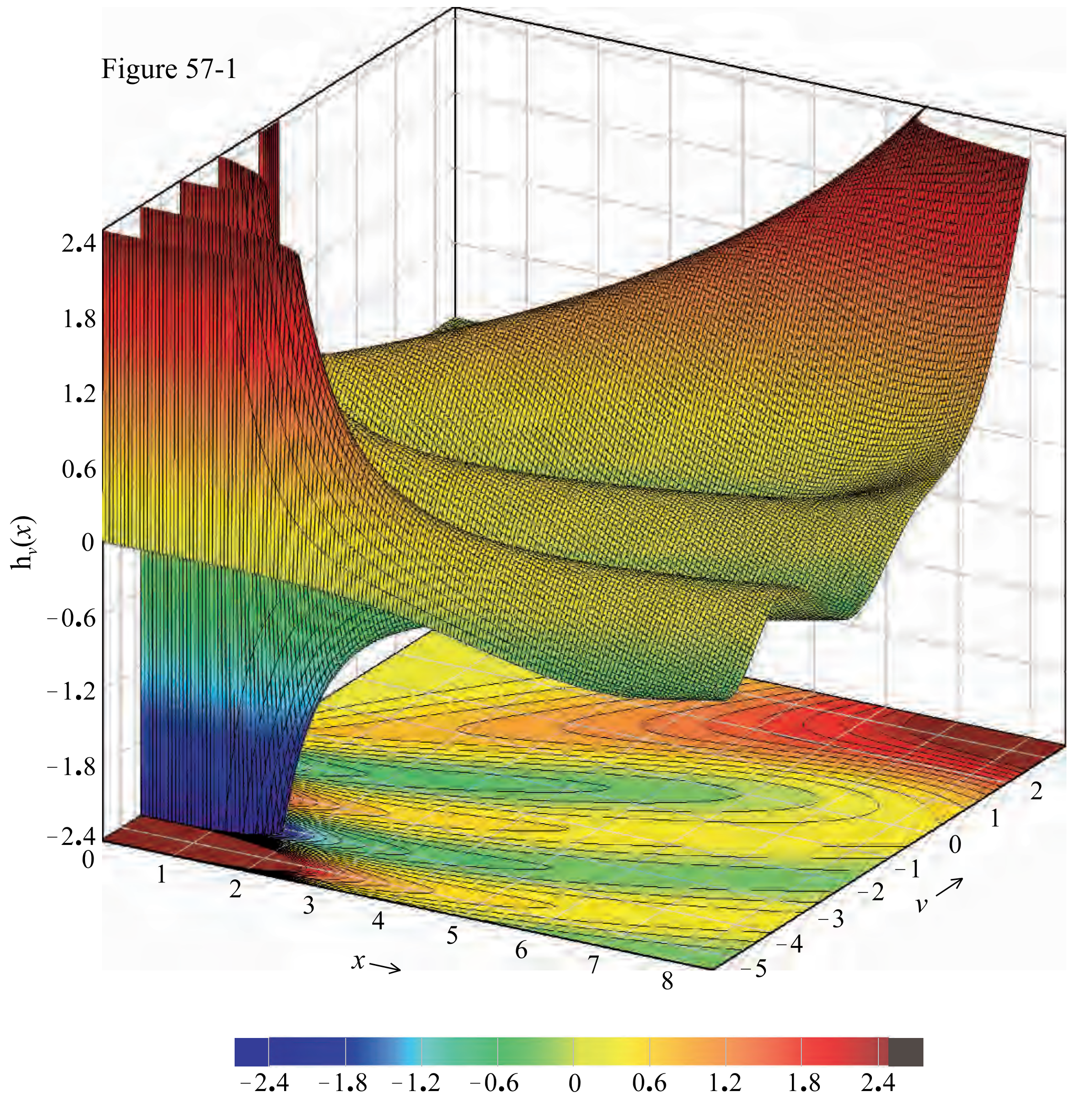
57:2 BEHAVIOR

For negative argument x, the Struve function is defined as a real function only when the order is an integer. Accordingly, Figure 57-1 is restricted to $x \ge 0$. It shows $h_\nu(x)$ to be an oscillatory function of both its argument and its order, except when $\nu > x > 0$. For any given order, the oscillations are of a period close to 2π and have a declining amplitude as x increases.

For negative orders less than -1 and other than $-3/2, -5/2, -7/2, \cdots$, the Struve function approaches either $-\infty$ or $+\infty$ at small arguments, but soon becomes oscillatory as x increases. These oscillations are centered on zero. However, the corresponding oscillations for positive order are not centered on zero. Figure 57-2 illustrates this asymmetry clearly and demonstrates that for $\nu > 1/2$, $x > 0$, the Struve function is invariably positive.

Away from $x = 0$, the Struve function $h_\nu(x)$ has no zeros if the order exceeds one-half, but an infinite number of zeros if $\nu \le 1/2$. The interesting case of $\nu = 1/2$ is among those graphed in Figure 57-2; this function has both a zero and a minimum at each nonnegative multiple of 2π.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_58, © Springer Science+Business Media, LLC 2009



57:3 DEFINITIONS

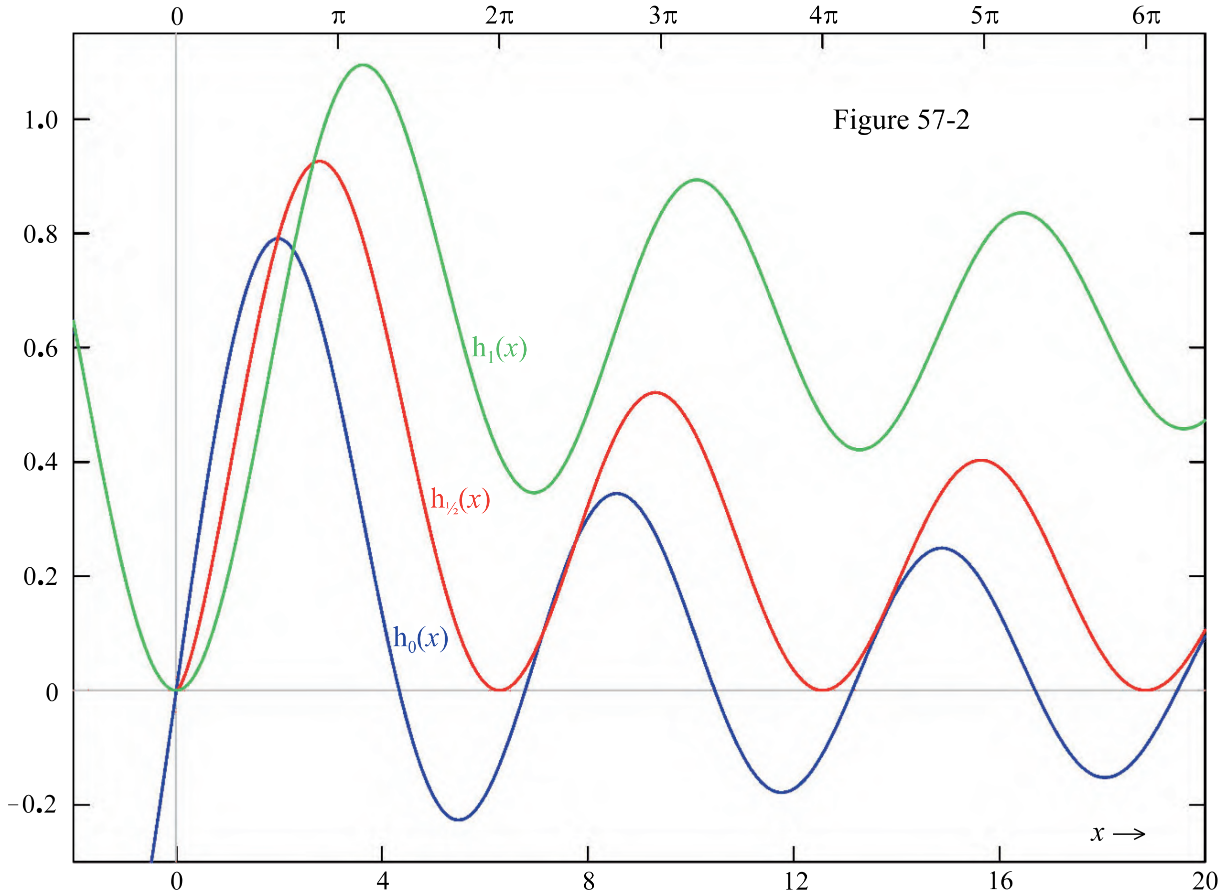
Differing from definition 53:3:3 of the Bessel function only in the replacement of cos by sin, the definite integral

57:3:1 $$h_\nu(x) = \frac{2\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \sin(xt)\,dt \qquad \nu > -\tfrac{1}{2}$$

represents the Struve function for a wide domain of orders. For $\nu \le -\frac{1}{2}$, this definition may be supplemented by recursion 57:5:1. Alternative integral representations result from replacement of t in 57:3:1 by $\sin(\theta)$ or $\cos(\theta)$. A similar integral

57:3:2 $$h_\nu(x) - Y_\nu(x) = \frac{2\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty \left(1+t^2\right)^{\nu-\frac{1}{2}} \exp(-xt)\,dt \qquad \nu > -\tfrac{1}{2}$$

represents the difference between the Struve and *Neumann functions* of common order and argument. In this case, it is the replacement of t by $\tan(\theta)$ or $\sinh(\alpha)$ that provides an equivalent integral representation.



Because $\mathrm{h}_\nu(x)$ is an $L = K+2 = 2$ hypergeometric function, it may be synthesized [Section 43:14] from the zero-order Bessel function, two synthetic steps being needed:

57:3:3
$$\mathrm{J}_0\left(2\sqrt{x}\right) \xrightarrow[3/2]{1} \frac{\sin\left(2\sqrt{x}\right)}{2\sqrt{x}} \xrightarrow[\nu+3/2]{1} \frac{\sqrt{\pi}\,\Gamma(\nu+3/2)}{2x^{(\nu+1)/2}}\mathrm{h}_\nu\left(2\sqrt{x}\right)$$

The Struve function is the particular integral [Section 24:14] in the solution of the inhomogeneous *Bessel's differential equation*

57:3:4
$$x^2\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}x^2} + x\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} + \left(x^2-\nu^2\right)\mathrm{f} = \frac{4\left(\frac{1}{2}x\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \qquad \mathrm{f} = w_1\,\mathrm{J}_\nu(x) + w_2\,\mathrm{Y}_\nu(x) + \mathrm{h}_\nu(x)$$

It is because $\mathrm{h}_\nu(x)-\mathrm{Y}_\nu(x)$ also solves his equation that this function difference crops up frequently in the mathematics of the Struve function.

57:4 SPECIAL CASES

The integer order instances of the Struve function are special inasmuch as they alone exist as real functions in the $x < 0$ domain. These functions are even if their order is odd, and vice versa

57:4:1
$$\mathrm{h}_n(-x) = -(-)^n\,\mathrm{h}_n(x) \qquad n = 0, \pm1, \pm2, \cdots$$

There is also an order-reflection formula, specific to integer orders

57:4:2 $$\mathrm{h}_{-n}(x) = (-)^n \left[\mathrm{h}_n(x) - \frac{2}{\pi x^{n-1}} \sum_{j=0}^{n-1} \frac{(2n-2j-3)!!}{(2j+1)!!} x^{2j} \right] \qquad n = 1,2,3,\cdots$$

for example $\mathrm{h}_{-1}(x) = (2/\pi) - \mathrm{h}_1(x)$. Integer orders are also "special" as regards their importance in applications.

When the order ν is half of an odd negative integer, the Struve and Neumann functions coalesce, being each then expressible as a spherical Bessel function [Section 32:13]

57:4:3 $$\mathrm{h}_{-n-\frac{1}{2}}(x) = \mathrm{Y}_{-n-\frac{1}{2}}(x) = (-)^n \sqrt{\frac{2x}{\pi}}\, \mathrm{j}_n(x) \qquad n = 0,1,2,\cdots$$

and thence in terms of sinusoids, for example

57:4:4 $$\mathrm{h}_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

A similar reduction to elementary functions occurs when the order is a moiety of an odd *positive* integer. The simplest case is depicted in Figure 57-2

57:4:5 $$\mathrm{h}_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[1 - \cos(x)\right]$$

and another is

57:4:6 $$\mathrm{h}_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{1-\cos(x)}{x} + \frac{x}{2} - \sin(x) \right]$$

Others may be constructed through the recursion

57:4:7 $$\mathrm{h}_{n+\frac{1}{2}}(x) = \frac{2n-1}{x} \mathrm{h}_{n-\frac{1}{2}}(x) - \mathrm{h}_{n-\frac{3}{2}}(x) + \frac{1}{n!} \sqrt{\frac{2}{\pi x}} \left(\frac{x}{2} \right)^n \qquad n = 2,3,4,\cdots$$

this being a special case of equation 57:5:1.

57:5 INTRARELATIONSHIPS

The recursion formula for the Struve function

57:5:1 $$\mathrm{h}_{\nu+1}(x) = \frac{2\nu}{x} \mathrm{h}_\nu(x) - \mathrm{h}_{\nu-1}(x) + \frac{\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}$$

is more complicated than that for many other functions, because of the third right-hand term. This term equals $2x^n/[(2n+1)!!\pi]$ when n is zero or a positive integer, and $-2(-2n-3)!!(-x)^n/\pi$ when n is a negative integer.

57:6 EXPANSIONS

The power series expansion of the Struve function is

57:6:1 $$\mathrm{h}_\nu(x) = \sum_{j=0}^{\infty} \frac{(-)^j \left(\frac{1}{2}x\right)^{2j+\nu+1}}{\Gamma(j+\frac{3}{2})\Gamma(j+\nu+\frac{3}{2})} = \frac{2(\frac{1}{2}x)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{(\frac{3}{2})_j (\nu+\frac{3}{2})_j} \left(\frac{-x^2}{4} \right)^j$$

of which the most important cases are

$\frac{\pi}{2}h_{-1}(x)=$	$\frac{\pi}{2}h_0(x)=$	$\frac{\pi}{2}h_1(x)=$
$1-\frac{x^2}{3}+\frac{x^4}{45}-\frac{x^6}{1575}+\cdots$	$x-\frac{x^3}{9}+\frac{x^5}{225}-\frac{x^7}{11025}+\cdots$	$\frac{x^2}{3}-\frac{x^4}{45}+\frac{x^6}{1575}-\frac{x^8}{99225}+\cdots$

The most important asymptotic series generates the difference between the Neumann and Struve functions

57:6:2 $$h_\nu(x)-Y_\nu(x)\sim\frac{1}{\pi}\sum_{j=0}\frac{\Gamma\left(j+\tfrac{1}{2}\right)}{\Gamma\left(\nu-j+\tfrac{1}{2}\right)\left(\tfrac{1}{2}x\right)^{2j-\nu+1}}=\frac{\left(\tfrac{1}{2}x\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{1}{2}\right)}\sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\tfrac{1}{2}-\nu\right)_j\left(\frac{-4}{x^2}\right)^j$$

with the examples shown below

$\frac{\pi}{2}\left[h_{-1}(x)-Y_{-1}(x)\right]\sim$	$\frac{\pi}{2}\left[h_0(x)-Y_0(x)\right]\sim$	$\frac{\pi}{2}\left[h_1(x)-Y_1(x)\right]\sim$
$\frac{-1}{x^2}+\frac{3}{x^4}-\frac{45}{x^6}+\frac{1575}{x^8}-\cdots$	$\frac{1}{x}-\frac{1}{x^3}+\frac{9}{x^5}-\frac{225}{x^7}+\cdots$	$1+\frac{1}{x^2}-\frac{3}{x^4}+\frac{45}{x^6}-\cdots$

Integer-order Struve functions expand simply in terms of Bessel functions, for example

57:6:3 $$\frac{\pi}{4}h_0(x)=J_1(x)+\frac{J_3(x)}{3}+\frac{J_5(x)}{5}+\frac{J_7(x)}{7}+\frac{J_9(x)}{9}+\cdots+\frac{J_{2j+1}(x)}{2j+1}+\cdots$$

and

57:6:4 $$\frac{\pi}{4}h_1(x)=\frac{1-J_0(x)}{2}+\frac{J_2(x)}{3}+\frac{J_4(x)}{15}+\frac{J_6(x)}{35}+\frac{J_8(x)}{63}+\cdots+\frac{J_{2j}(x)}{4j^2-1}+\cdots$$

57:7 PARTICULAR VALUES

The value adopted by the Struve function at $x=0$ (or, more precisely, in the limit as x approaches zero from real positive values) depends on the order:

57:7:1 $$h_\nu(0)=\begin{cases}0 & \begin{cases}\nu>-1\\ \nu=-\tfrac{3}{2},-\tfrac{5}{2},-\tfrac{7}{2},\cdots\end{cases}\\ 2/\pi & \nu=-1\\ +\infty & -\tfrac{3}{2}<\nu<-1,-\tfrac{7}{2}<\nu<-\tfrac{5}{2},-\tfrac{11}{2}<\nu<-\tfrac{9}{2},\cdots\\ -\infty & -\tfrac{5}{2}<\nu<-\tfrac{3}{2},-\tfrac{9}{2}<\nu<-\tfrac{7}{2},-\tfrac{13}{2}<\nu<-\tfrac{11}{2},\cdots\end{cases}$$

Similarly, there is an order-dependence at infinity:

57:7:2 $$h_\nu(\infty)=\begin{cases}+\infty & \nu>1\\ 2/\pi & \nu=1\\ 0 & \nu<1\end{cases}$$

Apart from a zero value that might exist at $x=0$, the Struve function has no zeros for $\nu>\tfrac{1}{2}$. There are, however, an infinite number of zeros for $\nu\le\tfrac{1}{2}$. For large arguments, these occur at the roots of the equation

57:7:3 $$\sin\left(r-\tfrac{1}{2}\nu\pi+\tfrac{3}{4}\pi\right)\approx\frac{\left(\tfrac{1}{2}r\right)^{\nu-1/2}}{\Gamma\left(\nu+1/2\right)}\qquad \mathrm{h}_\nu(r)=0,\quad \text{large } r,\quad \nu\le 1/2$$

and therefore, for very large arguments, the zeros are found at $\left(k+\tfrac{1}{2}\nu+\tfrac{1}{4}\right)\pi$, where k is a large integer.

57:8 NUMERICAL VALUES

Equator's Struve function routine (keyword **h**) evaluates $\mathrm{h}_\nu(x)$ throughout the domains $|\nu|\le 168$ and $|x|\le 300$. Equation 57:6:1 is employed for arguments up to a magnitude of 20, but beyond this *Equator* utilizes formula 57:6:2 with an ε-transformation [Section 10:16].

57:9 LIMITS AND APPROXIMATIONS

Close to $x=0$, the Struve function behaves as a simple power:

57:9:1 $$\mathrm{h}_\nu(0\leftarrow x)\to\frac{x^{\nu+1}}{2^\nu\sqrt{\pi}\,\Gamma\left(\nu+3/2\right)}$$

As the argument approaches infinity, the Struve function generally behaves either as a power or as a damped sinusoid

57:9:2 $$\mathrm{h}_\nu(x\to\infty)\to\begin{cases}\left(\tfrac{1}{2}x\right)^{\nu-1}/\left[\sqrt{\pi}\,\Gamma\left(\nu+1/2\right)\right] & \nu>1/2\\ \sqrt{2/\pi x}\left[1+\sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right] & \nu=1/2\\ \sqrt{2/\pi x}\,\sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right) & \nu<1/2\end{cases}$$

57:10 OPERATIONS OF THE CALCULUS

The differentiation formula

57:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{h}_\nu(x)=\frac{\mathrm{h}_{\nu-1}(x)-\mathrm{h}_{\nu+1}(x)}{2}+\frac{\left(\tfrac{1}{2}x\right)^\nu}{2\sqrt{\pi}\,\Gamma\left(\nu+3/2\right)}$$

has the simple special case $(\mathrm{d}/\mathrm{d}x)\mathrm{h}_0(x)=\mathrm{h}_{-1}(x)$. The formula

57:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}x^\nu\,\mathrm{h}_\nu(x)=x^\nu\,\mathrm{h}_{\nu-1}(x)$$

is also noteworthy.

Indefinite integrals include

$\int_0^x t\,\mathrm{h}_0(t)\,\mathrm{d}t=x\,\mathrm{h}_1(x)$	$\int_0^x \mathrm{h}_1(t)\,\mathrm{d}t=\frac{2x}{\pi}-\mathrm{h}_0(x)$	$\int_0^x \mathrm{h}_{-1}(t)\,\mathrm{d}t=\mathrm{h}_0(x)$	$\int_0^x t^{1-\nu}\,\mathrm{h}_\nu(t)\,\mathrm{d}t=\frac{2^{1-\nu}x}{\sqrt{\pi}\,\Gamma(\nu+1/2)}-\frac{\mathrm{h}_{\nu-1}(x)}{x^{\nu-1}}$

and a particularly simple definite integral is

57:10:3
$$\int_0^\infty \frac{\mathrm{h}_0(t)}{t}\,\mathrm{d}t = \frac{\pi}{2}$$

There are many Laplace transforms [Section 26:15] involving the Struve function, of which some are

$\mathcal{L}\left\{\mathrm{h}_{-n-\frac{1}{2}}(bt)\right\}$	$\mathcal{L}\left\{t^{-\frac{1}{2}}\,\mathrm{h}_{-\frac{1}{2}}(bt)\right\}$	$\mathcal{L}\left\{\sqrt{t}\,\mathrm{h}_{-\frac{1}{2}}(bt)\right\}$	$\mathcal{L}\left\{\mathrm{h}_0(bt)\right\}$	$\mathcal{L}\left\{t^{-\frac{1}{2}}\,\mathrm{h}_{\frac{1}{2}}(bt)\right\}$	$\mathcal{L}\left\{\sqrt{t}\,\mathrm{h}_{\frac{1}{2}}(bt)\right\}$
$\dfrac{(-)^n\left[\sqrt{s^2+b^2}-s\right]^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}\sqrt{s^2+b^2}}$	$\sqrt{\dfrac{2}{\pi b}}\arctan\left(\dfrac{b}{s}\right)$	$\dfrac{\sqrt{2b/\pi}}{s^2+b^2}$	$\dfrac{2\operatorname{arsinh}(b/s)}{\pi\sqrt{b^2+s^2}}$	$\dfrac{\ln\left(1+\dfrac{b^2}{s^2}\right)}{\sqrt{2\pi b}}$	$\dfrac{\sqrt{2b^3/\pi}}{s\left(s^2+b^2\right)}$

57:11 COMPLEX ARGUMENT

This *Atlas* does not address Struve functions of complex arguments. For purely imaginary argument, one finds

57:11:1
$$\mathrm{h}_\nu(iy) = i^{1-\nu}\mathtt{l}_\nu(y) = \left[\sin\left(\tfrac{1}{2}\nu\pi\right) + i\cos\left(\tfrac{1}{2}\nu\pi\right)\right]\mathtt{l}_\nu(y)$$

where $\mathtt{l}_\nu$ is the modified, or hyperbolic, Struve function addressed in Section 57:13.

Simple functions often result from inverse Laplace transformation of $\mathrm{h}_\nu - \mathrm{Y}_\nu$ differences; the simplest case is

57:11:2
$$\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\mathrm{h}_0(s)-\mathrm{Y}_0(s)\right]\exp(ts)\frac{\mathrm{d}s}{2\pi i} = \mathcal{I}\left\{\mathrm{h}_0(s)-\mathrm{Y}_0(s)\right\} = \frac{2}{\pi\sqrt{t^2+1}}$$

57:12 GENERALIZATIONS

Inasmuch as its integer-order cases differ from the corresponding Struve function only by sign and an additive polynomial [Abramowitz and Stegun, formulas 12.3.6,7], *Weber's function* may be regarded as a generalization of the $\mathrm{h}_n(x)$ function. Weber's function is not addressed in this *Atlas*, though it is mentioned briefly in Section 52:12.

57:13 COGNATE FUNCTIONS: the modified Struve function

Just as the modified Bessel function, $\mathrm{I}_\nu(x)$ is the hyperbolic counterpart of the Bessel function $\mathrm{J}_\nu(x)$, so the modified Struve function $\mathtt{l}_\nu(x)$ is the hyperbolic counterpart of the Struve function $\mathrm{h}_\nu(x)$. The relationship between the two functions

57:13:1
$$\mathtt{l}_\nu(x) = i^{-1-\nu}\,\mathrm{h}_\nu(ix)$$

involves operations in the complex plane. The modified Struve function is real whenever its argument x is real and positive but, for negative arguments, $\mathtt{l}_\nu(x)$ is complex unless the order ν is an integer. In common with other cylinder functions in the "modified" category, the modified Struve function is not oscillatory,

As a comparison of Figures 57-3 and 49-1 will confirm, there are strong qualitative similarities between the modified Struve functions $\mathtt{l}_0(x)$, $\mathtt{l}_1(x)$, $\mathtt{l}_2(x)$, $\cdots$ and the corresponding modified Bessel functions $\mathrm{I}_1(x)$, $\mathrm{I}_2(x)$, $\mathrm{I}_3(x)$, … . Figure 57-3 also includes examples of modified Struve functions of negative integer order. These have less similarity to their Bessel counterparts, but there is an *identity* between an $\mathtt{l}_\nu(x)$ function of negative half-odd-integer order and the corresponding $\mathrm{I}_\nu(x)$ function of *positive* half-odd-integer order

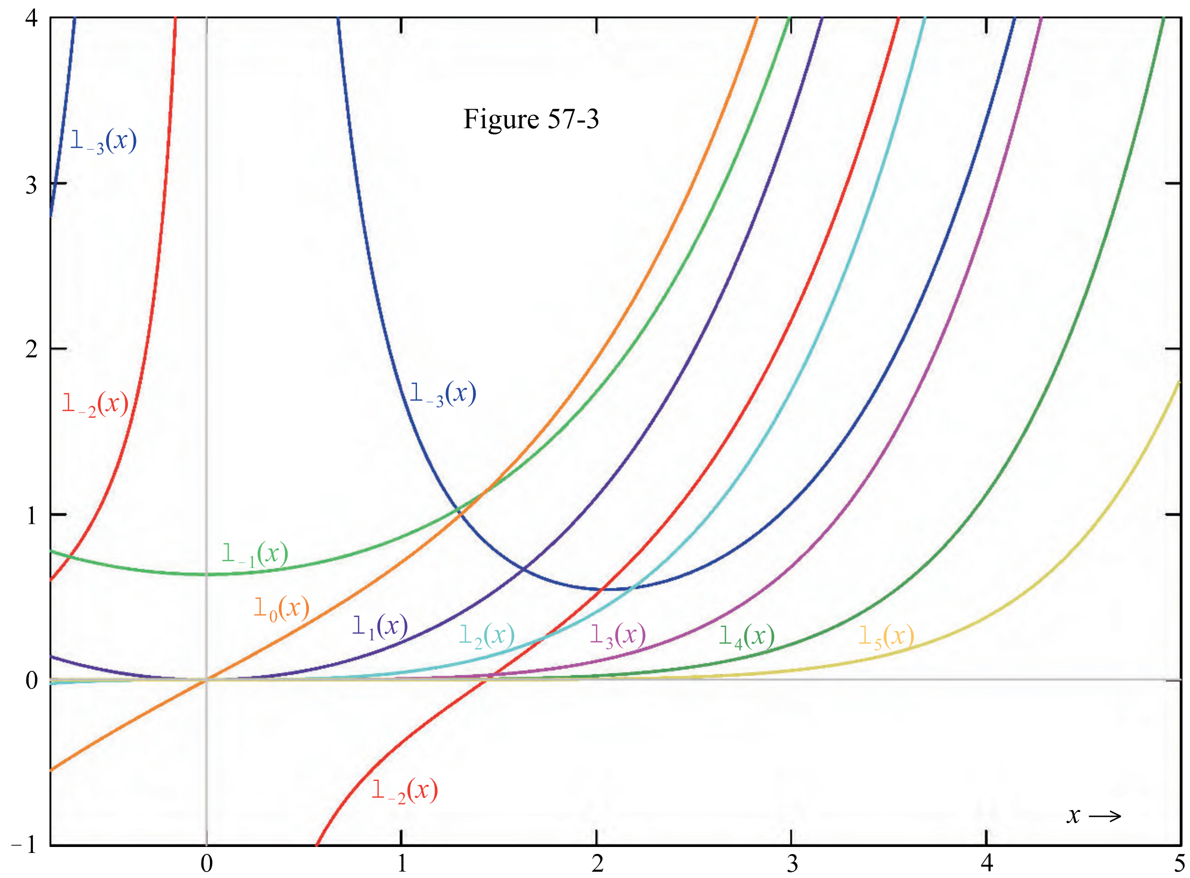


Figure 57-3

57:13:2 $$\mathrm{l}_{-n-1/2}(x) = \mathrm{I}_{n+1/2}(x) \qquad n = 0,1,2,\cdots$$

[see also Section 28:12].

Many definitions of $\mathrm{l}_\nu(x)$ are analogous to those of $\mathrm{h}_\nu(x)$. Thus the defining equation

57:13:3 $$\mathrm{l}_\nu(x) = \frac{2\left(\tfrac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})}\int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \sinh(xt)\,\mathrm{d}t \qquad \nu > -\tfrac{1}{2}$$

is a close parallel of the definition 57:3:1 of the Struve function. There are other integral representations, but some of those integrals have problematic convergence properties. Some formulas, of which

57:13:4 $$\mathrm{l}_1(x) = \mathrm{I}_1(x) - \frac{2}{\pi}\int_0^\infty \frac{\mathrm{h}_1(xt)}{t^2+1}\mathrm{d}t$$

is a particularly simple instance, relate modified Struve functions to other cylinder functions of the same order.

Likewise the properties of modified Struve functions mostly mirror those of their unmodified counterparts. The order-reflection formula

57:13:5 $$\mathrm{l}_{-n}(x) = \mathrm{l}_n(x) - \frac{2x}{\pi(-x)^n}\sum_{j=0}^{n-1}\frac{(2n-2j-3)!!}{(2j+1)!!}(-x^2)^j \qquad n = 1,2,3,\cdots$$

is similar to 57:4:2, while the order-recursion formula

57:13:6 $$\mathrm{l}_{\nu+1}(x) = \mathrm{l}_{\nu-1}(x) - \frac{2\nu}{x}\mathrm{l}_\nu(x) - \frac{\left(\tfrac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\tfrac{3}{2})}$$

fails to identify with 57:5:1 only by a sign change. The expansion of $\mathtt{l}_\nu(x)$ in powers of x,

57:13:7 $$\mathtt{l}_\nu(x)=\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}x\right)^{2j+\nu+1}}{\Gamma\left(j+\tfrac{3}{2}\right)\Gamma\left(j+\nu+\tfrac{3}{2}\right)}=\frac{2\left(\tfrac{1}{2}x\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{3}{2}\right)}\sum_{j=0}^{\infty}\frac{1}{\left(\tfrac{3}{2}\right)_j\left(\nu+\tfrac{3}{2}\right)_j}\left(\frac{x^2}{4}\right)^j$$

matches 57:6:1 except that the alternating signs are now absent. It is a similar story with asymptotic expansions 57:6:2, from which the right-hand side of

57:13:8 $$\mathrm{I}_{-\nu}(x)-\mathtt{l}_\nu(x)\sim\frac{1}{\pi}\sum_{j=0}\frac{(-)^j\,\Gamma\left(j+\tfrac{1}{2}\right)}{\Gamma\left(\nu-j+\tfrac{1}{2}\right)\left(\tfrac{1}{2}x\right)^{2j-\nu+1}}=\frac{\left(\tfrac{1}{2}x\right)^{\nu-1}}{\pi\,\Gamma\left(\nu+\tfrac{1}{2}\right)}\sum_{j=0}\left(\tfrac{1}{2}\right)_j\left(\tfrac{1}{2}-\nu\right)_j\left(\frac{4}{x^2}\right)^j \qquad \text{large } x$$

differs only in lacking alternating signs. Accordingly, the panels in Section 57:6 are easily adapted to provide expansions for special cases of the modified Struve function. Expansion 57:13:7 forms the basis of *Equator*'s modified Struve function routine (keyword **l**), which delivers exact values of $\mathtt{l}_\nu(x)$ for all $-125 \le \nu \le 125$ and $0 \le x \le 500$ and also for negative x if ν is an integer.

As the argument x increases, the increasing value of $\mathtt{l}_\nu(x)$ soon ceases to be significantly dependent on the sign of ν, and ultimately even becomes independent of the magnitude of ν, approaching $\exp(x)$ asymptotically.

The derivative of the modified Struve function is given by

57:13:9 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathtt{l}_\nu(x)=\frac{\mathtt{l}_{\nu-1}(x)+\mathtt{l}_{\nu+1}(x)}{2}+\frac{\left(\tfrac{1}{2}x\right)^{\nu}}{2\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{1}{2}\right)}$$

a formula that may be combined with equation 57:13:6 to produce alternatives. Interesting links to other functions are encountered on Laplace transformation of the modified Struve function; for example, the transforms

57:13:10 $$\int_0^\infty \mathtt{l}_{-n-\frac{1}{2}}(bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\mathtt{l}_{-n-\frac{1}{2}}(bt)\right\}=\frac{\left(s-\sqrt{s^2-b^2}\right)^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}\sqrt{s^2-b^2}} \qquad s>b,\quad n=0,1,2,\cdots$$

57:13:11 $$\int_0^\infty \mathtt{l}_0(bt)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{\mathtt{l}_0(bt)\right\}=\frac{2\arcsin(b/s)}{\sqrt{s^2+b^2}} \qquad s>b$$

57:13:12 $$\int_0^\infty t^{\nu/2}\mathtt{l}_\nu\left(2\sqrt{t}\right)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{t^{\nu/2}\mathtt{l}_\nu\left(2\sqrt{t}\right)\right\}=\frac{1}{s^{\nu+1}}\exp\left(\frac{1}{s}\right)\mathrm{erf}\left(\frac{1}{\sqrt{s}}\right)$$

57:13:13 $$\int_0^\infty t^{\nu/2}\mathtt{l}_{-\nu}\left(2\sqrt{t}\right)\exp(-st)\,\mathrm{d}t=\mathfrak{L}\left\{t^{\nu/2}\mathtt{l}_{-\nu}\left(2\sqrt{t}\right)\right\}=\frac{\exp(1/s)}{\Gamma\left(\tfrac{1}{2}-\nu\right)s^{\nu+1}}\gamma\left(\tfrac{1}{2}-\nu\,,\,\frac{1}{s}\right)$$

generate functions from Chapter 35, Chapter 40 and 45.

CHAPTER 58

THE INCOMPLETE BETA FUNCTION $\mathrm{B}(\nu,\mu,x)$

This trivariate function subsumes a number of simpler functions. As demonstrated in Section 58:14, the indefinite integral of any circular or hyperbolic function, raised to an arbitrary power, is one-half of an incomplete beta function.

58:1 NOTATION

The adjective "incomplete" reflects the less-than-unity upper limit in the definition 58:3:1 of the incomplete beta function $\mathrm{B}(\nu,\mu,x)$ contrasted with that in the definition

58:1:1 $$\mathrm{B}(\nu,\mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\,\mathrm{d}t = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}$$

of the (complete) beta function of Section 43:13. The incompleteness destroys the symmetry of this beta function: the ν and μ parameters are not interchangeable in this chapter.

You may encounter the variables ordered in the sequence $\mathrm{B}(x,\nu,\mu)$. An alternative notation is $\mathrm{B}_x(\nu,\mu)$. Also in common use is the *regularized incomplete beta function*, defined by

58:1:2 $$\mathrm{I}_x(\nu,\mu) = \frac{\mathrm{B}(\nu,\mu,x)}{\mathrm{B}(\nu,\mu)}$$

that is important in statistics, but is not further addressed here.

58:2 BEHAVIOR

The incomplete beta function is generally defined as a real-valued function only for arguments in the range $0 \le x \le 1$, being zero at $x = 0$ and becoming the (complete) beta function at $x = 1$. However, extension to negative arguments of magnitude exceeding unity is possible through the formula

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_59, © Springer Science+Business Media, LLC 2009

58:2:1 $$\mathrm{B}(n,\mu,x)=(-)^n\,\mathrm{B}\left(n,1-\mu-n,\frac{-x}{1-x}\right)\qquad x<-1,\qquad \nu=n=1,2,3,\cdots$$

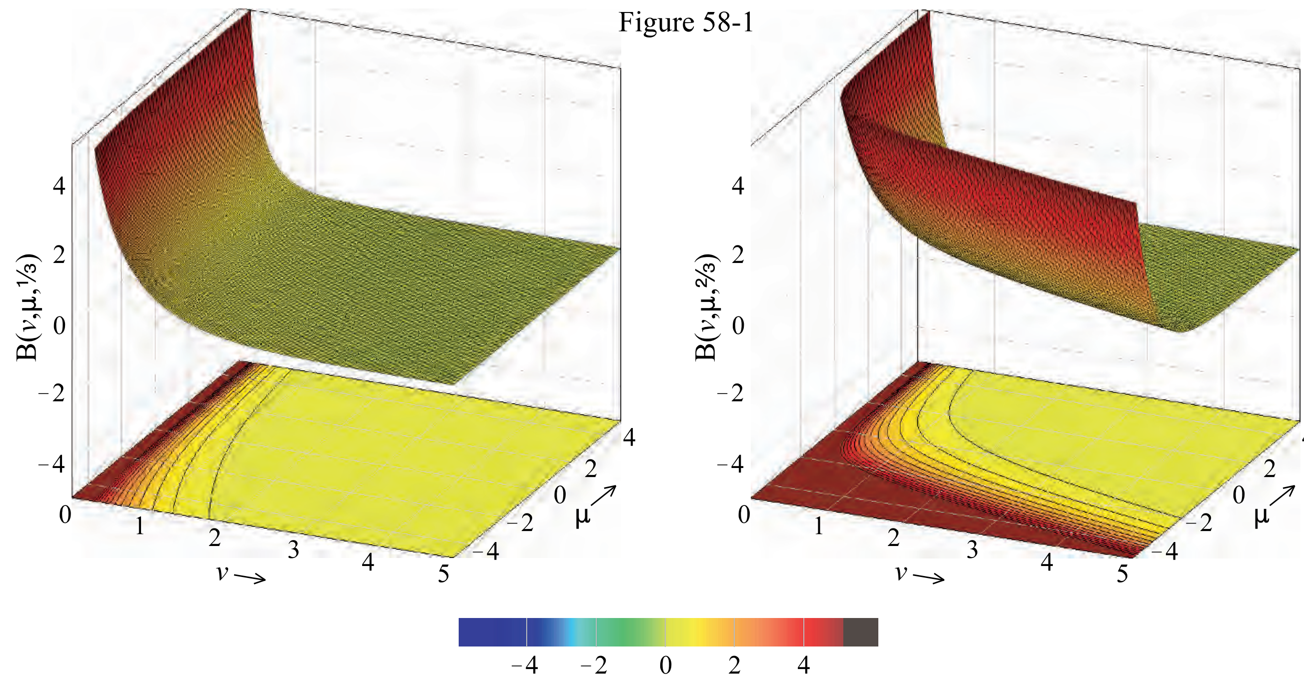
when the ν parameter is a positive integer. Likewise, extension to arguments exceeding unity is enabled whenever the μ parameter is a positive integer, via the formula

58:2:2 $$\mathrm{B}(\nu,m,x)=\frac{(m-1)!}{(\nu)_m}-(-)^m\,\mathrm{B}\left(m,1-\nu-m,\frac{x-1}{x}\right)\qquad x\ge 1,\qquad \mu=m=1,2,3,\cdots$$

in which $(\nu)_m$ is a Pochhammer polynomial [Chapter 18]. Beyond noting these extensions, the *Atlas* does not discuss the behavior of the incomplete beta function outside the $0<x<1$ domain, other than in some special cases discussed in Section 58:4. We treat all three variables as real but, apart from this, the μ parameter is unrestricted, whereas ν must be positive.

The behavior of the incomplete beta function is quite bland: when the other two variables are held constant, the function decreases monotonically with ν and with μ, but increases monotonically with x. These properties are illustrated in Figure 58-1, in which the argument is fixed at either $\frac{1}{3}$ (left) or $\frac{2}{3}$ (right).

Figure 58-1



58:3 DEFINITIONS

The fundamental definition as an indefinite integral

58:3:1 $$\mathrm{B}(\nu,\mu,x)=\int_0^x t^{\nu-1}(1-t)^{\mu-1}\,\mathrm{d}t\qquad 0\le x\le 1$$

may be recast in several useful ways, including

58:3:2 $$\mathrm{B}\left(\nu,\mu,\frac{x}{1+x}\right)=\int_0^x \frac{t^{\nu-1}}{(1+t)^{\nu+\mu}}\,\mathrm{d}t\qquad 0\le x\le\infty$$

58:3:3 $$\mathrm{B}\left(\nu,\mu,\sin^2(x)\right)=2\int_0^x \sin^{2\nu-1}(t)\cos^{2\mu-1}(t)\,\mathrm{d}t \qquad 0\le x\le \tfrac{1}{2}\pi$$

and

58:3:4 $$\mathrm{B}\left(\nu,\mu,\tanh^2(x)\right)=2\int_0^x \tanh^{2\nu-1}(t)\,\mathrm{sech}^{2\mu}(t)\,\mathrm{d}t \qquad 0\le x\le \infty$$

As well, the incomplete beta function may be expressed as a number of definite integrals, the simplest being

58:3:5 $$\mathrm{B}(\nu,\mu,x)=x^{\nu}\int_0^1 t^{\nu-1}(1-xt)^{\mu-1}\,\mathrm{d}t \qquad 0\le x\le 1$$

Using the notation introduced in Section 43:14, the incomplete beta function may be synthesized by the route

58:3:6 $$\frac{1}{1-x}\ \xrightarrow[\nu+1]{\nu+\mu}\ \frac{\nu}{x^{\nu}(1-x)^{\mu}}\mathrm{B}(\nu,\mu,x)$$

This illustrates the hypergeometricity of the function, which leads to its being definable as the infinite series 58:6:1.

58:4 SPECIAL CASES

To ensure that the incomplete beta function is not complex, we require that its argument lie between zero and unity in other sections of this chapter. This requirement can often be relaxed in the special cases treated in this section, being replaced by a stipulation that x^{ν} or $(1-x)^{\mu}$ be real. Section 12:2 addresses the conditions under which a noninteger power is real.

When ν is the positive integer n, but μ is not an integer, the incomplete beta function reduces to an algebraic function:

58:4:1 $$\mathrm{B}(n,\mu,x)=(n-1)!\left[\frac{1}{(\mu)_n}-\frac{x^n}{(1-x)^{1-\mu}}\sum_{j=1}^{n}\frac{1}{(n-j)!(\mu)_j}\left(\frac{1-x}{x}\right)^j\right] \qquad n=1,2,3,\cdots$$

The simplest case is

58:4:2 $$\mathrm{B}(1,\mu,x)=\frac{1-(1-x)^{\mu}}{\mu}$$

Formula 58:2:1 may also be useful in these cases.

When μ is the positive integer m, the incomplete beta function is the product of x^{ν} and a polynomial of degree $m-1$in x. There is no restriction on the values of ν and x in these cases, provided that x^{ν} is real and well defined [Section 12:2]. The formula

58:4:3 $$\mathrm{B}(\nu,m,x)=x^{\nu}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-x)^j}{j+\nu} \qquad m=1,2,3,\cdots \qquad x>0$$

applies, the simplest instance being $\mathrm{B}(\nu,1,x)=x^{\nu}/\nu$. Formula 58:2:2 also applies in this case. When μ is zero, the incomplete beta function becomes a simple infinite series

58:4:4 $$\mathrm{B}(\nu,0,x)=\sum_{j=0}^{\infty}\frac{x^{j+\nu}}{j+\nu}=\ln_{\nu}\left(\frac{1}{1-x}\right)=x^{\nu}\Phi(x,1,\nu) \qquad \nu>0 \qquad 0<x<1$$

equivalent to a *generalized logarithm* [Section 25:12] or a *Lerch function* [Section 64:12]. Negative integer values of μ lead to

58:4:5
$$\mathrm{B}(\nu,m,x)=\frac{x^\nu}{(-m)!}\sum_{j=0}^{\infty}\frac{(j-m)!x^j}{j!(j+\nu)} \qquad \mu=m=-1,-2,-3,\cdots$$

of which 58:4:4 is the $m=0$ version. Convergence of this series requires $-1<x<1$, but nonetheless may be very slow. Moreover x^ν must be well defined and ν cannot be a nonpositive integer.

When both parameters are positive integers, the incomplete beta function reduces to a polynomial:

58:4:6
$$\mathrm{B}(n,m,x)=x^n\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-x)^j}{j+n} \qquad n,m=1,2,3,\cdots \qquad 0<x<1$$

Other finite series result when one or other of the parameters is an odd positive multiple of ½, the other being a positive integer:

58:4:7
$$\mathrm{B}(n+\tfrac12,m+1,x)=x^{n+\frac12}\sum_{j=0}^{m}\binom{m}{j}\frac{(-x)^j}{j+n+\frac12}$$

58:4:8
$$\mathrm{B}(n+1,m+\tfrac12,x)=\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{1-(1-x)^{j+m+\frac12}}{j+m+\frac12}$$

$$n,m=0,1,2,\cdots \qquad 0<x<1$$

Other results in which each parameter is a fraction or zero include

$\mathrm{B}(\tfrac12,\tfrac12,x)$	$\mathrm{B}(\tfrac32,\tfrac32,x)$	$\mathrm{B}(\tfrac12,0,x)$	$\mathrm{B}(\tfrac{2\pm1}{4},0,x)$
$2\arcsin(\sqrt{x})$	$\dfrac{\arccos(1-2x)}{8}-\dfrac{1-2x}{4}\sqrt{x-x^2}$	$2\operatorname{artanh}(\sqrt{x})$	$2\operatorname{artanh}(x^{\frac14})\mp 2\arctan(x^{\frac14})$

When the two parameters differ only in sign, there is a simple result

58:4:9
$$\mathrm{B}(\nu,-\nu,x)=\frac{1}{\nu}\left(\frac{x}{1-x}\right)^\nu \qquad \nu>0 \qquad 0<x<1$$

but we know of no comparable general formula for $\mathrm{B}(\nu,\nu,x)$.

58:5 INTRARELATIONSHIPS

The important formula

58:5:1
$$\mathrm{B}(\nu,\mu,x)=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}-\mathrm{B}(\mu,\nu,1-x)$$

constitutes a reflection formula for the argument of the incomplete beta function, but also shows the effect of interchanging the parameters.

The reflection formula

58:5:2
$$\mu\,\mathrm{B}(\nu+1,\mu,x)=\nu\,\mathrm{B}(\nu,\mu+1,x)-x^\nu(1-x)^\mu$$

increases one parameter at the expense of the other, so that the sum of the parameters remains constant. There is a large number of intrarelationships connecting three incomplete beta functions, of which

58:5:3
$$\mathrm{B}(\nu,\mu,x)=\mathrm{B}(\nu+1,\mu,x)+\mathrm{B}(\nu,\,\mu+1,x)$$

is the simplest.

58:6 EXPANSIONS

The power series

58:6:1 $$\mathrm{B}(\nu,\mu,x)=\frac{x^\nu(1-x)^\mu}{\nu}\left[1+\frac{\nu+\mu}{\nu+1}x+\frac{(\nu+\mu)(\nu+\mu+1)}{(\nu+1)(\nu+2)}x^2+\cdots\right]=\frac{(1-x)^\mu}{\nu}\sum_{j=0}^{\infty}\frac{(\nu+\mu)_j}{(\nu+1)_j}x^{j+\nu}\qquad 0<x<1$$

shows the incomplete beta function to be an $L=K=1$ hypergeometric function [Section 18:14]. Though technically convergent for all parameter values, this series often converges at glacial speed if x is close to unity. In these circumstances the alternative series

58:6:2 $$\mathrm{B}(\nu,\mu,x)=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}-\frac{x^\nu(1-x)^\mu}{\mu}\sum_{j=0}^{\infty}\frac{(\nu+\mu)_j}{(\mu+1)_j}(1-x)^j\qquad 0<x<1$$

will converge more rapidly.

58:7 PARTICULAR VALUES

An argument of one-half is the most fructuous for particular values, some of which are:

$\mathrm{B}(\nu,\nu,\tfrac12)$	$\mathrm{B}(\nu,-\nu,\tfrac12)$	$\mathrm{B}(\nu,1-\nu,\tfrac12)$	$\mathrm{B}(\tfrac14,\tfrac14,\tfrac12)$	$\mathrm{B}(\tfrac12,\tfrac12,\tfrac12)$	$\mathrm{B}(\tfrac34,\tfrac34,\tfrac12)$	$\mathrm{B}(\tfrac32,\tfrac32,\tfrac12)$
$\dfrac{\sqrt{\pi}\,\Gamma(\nu)}{4^\nu\Gamma(\nu+\tfrac12)}$	$\dfrac{1}{\nu}$	$\dfrac{\mathrm{G}(\nu)}{2}$	$\sqrt{2}\pi g$	$\dfrac{\pi}{2}$	$\dfrac{1}{\sqrt{2}\,g}$	$\dfrac{\pi}{16}$

Here g is Gauss's constant [Section 1:8] and G(ν) is the Bateman G function [Section 44:13].

58:8 NUMERICAL VALUES

In its incomplete beta function routine (keyword **incompBeta**), *Equator* generally relies upon series 58:4:5, 58:6:1, and 58:6:2. Values of the argument outside the $0\le x\le 1$ range are not accepted.

58:9 LIMITS AND APPROXIMATIONS

The incomplete beta function approaches the limit zero as its argument declines towards zero. The approximation

58:9:1 $$\mathrm{B}(\nu,\mu,x)\approx x^\nu\left[\frac{1}{\nu}+\frac{1-\mu}{1+\nu}x\right]\qquad \text{small } x$$

holds excellently during this approach. The corresponding approximation

58:9:2 $$\mathrm{B}(\nu,\mu,x)\approx \mathrm{B}(\nu,\mu)-(1-x)^\mu\left[\frac{1}{\mu}+\frac{1-\nu}{1+\mu}(1-x)\right]\qquad x\to 1$$

covers the approach of the argument to unity, in which limit the incomplete beta function becomes complete.

58:10 OPERATIONS OF THE CALCULUS

Differentiation and indefinite integration of the incomplete beta function with respect to its argument give

58:10:1
$$\frac{d}{dx}B(\nu,\mu,x) = x^{\nu-1}(1-x)^{\mu-1}$$

and

58:10:2
$$\int_0^x B(\nu,\mu,t)\,dt = x\,B(\nu,\mu,x) - B(\nu+1,\mu,x)$$

Equation 58:10:2 is an instance of one of the general rules elaborated in Section 37:14.

58:11 COMPLEX ARGUMENT

The incomplete beta function is seldom encountered with complex argument and that possibility is not addressed in this *Atlas*.

58:12 GENERALIZATIONS

The Gauss hypergeometric function of Chapter 60 represents a generalization of the incomplete beta functions. Two ways in which a connection may be made between a Gauss function and the incomplete beta function are

58:12:1
$$\nu x^{-\nu}\,B(\nu,\mu,x) = F(\nu,1-\mu,1+\nu,x)$$

and

58:12:2
$$\frac{\nu\,B(\nu,\mu,x)}{x^{\nu}(1-x)^{\mu}} = F(1,\nu+\mu,1+\nu,x)$$

58:13 COGNATE FUNCTIONS

The incomplete beta function has some similarities to the incomplete gamma function [Chapter 45] and to the Legendre functions of Chapter 59. As an $L=K=1$ hypergeometric function, it also has liaison with all the functions in Table 18-1. In fact, the incomplete beta function may be regarded as the prototype $L=K=1$ hypergeometric function

58:13:1
$$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j = \frac{c-1}{x^a}\left(\frac{x}{1-x}\right)^{1+a-c} B\left(c-1,1+a-c,x\right) \qquad c-1 \neq 1,2,3,\cdots$$

58:14 RELATED TOPIC: integrals of powered circular and hyperbolic functions

If f is any one of the *circular functions* of Chapters 33, 34 or 35, then the indefinite integral of that function, raised to an arbitrary power λ, is expressible as one-half of an incomplete beta function:

58:14:1 $$\int_0^x \mathrm{f}^{\lambda}(t)\,\mathrm{d}t = \tfrac{1}{2}\mathrm{B}\left(\nu,\mu,\sin^2(x)\right) \qquad 0<x<\tfrac{\pi}{2} \qquad \mathrm{f} = \cos, \sin, \sec, \csc, \tan, \cot$$

The relationships between the power λ and the parameters of the incomplete beta function are:

f =	cos	sin	sec	csc	tan	cot
$\nu =$	$\frac{1}{2}$	$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}$	$\frac{1}{2}(1-\lambda)$	$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}(1-\lambda)$
$\mu =$	$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}$	$\frac{1}{2}(1-\lambda)$	$\frac{1}{2}$	$\frac{1}{2}(1-\lambda)$	$\frac{1}{2}(1+\lambda)$

For example:

58:14:2 $$\int_0^{\pi/4} \tan^5(t)\,\mathrm{d}t = \tfrac{1}{2}\mathrm{B}(3,-2,\tfrac{1}{2})$$

In some cases, a limit is imposed on the magnitude of the power by the requirement that ν be positive; thus, unless λ is less than unity, the integral of $\csc^{\lambda}(t)$ diverges. The same assignment of parameters also allows the complementary integrals to be expressed as incomplete beta functions:

58:14:3 $$\int_x^{\pi/2} \mathrm{f}^{\lambda}(t)\,\mathrm{d}t = \mathrm{B}\left(\mu,\nu,\cos^2(x)\right) \qquad \mu>0$$

Note the inversion of the parameters on passing from formula 58:14:1 to 58:14:3; of course any prohibition on nonpositive values now transfers to μ.

Similar formulas exists for the *hyperbolic functions* of Chapters 28−30:

58:14:4 $$\int_0^x \mathrm{g}^{\lambda}(t)\,\mathrm{d}t = \tfrac{1}{2}\mathrm{B}\left(\nu,\mu,\tanh^2(x)\right) \quad \nu>0$$

58:14:5 $$\int_x^{\infty} \mathrm{g}^{\lambda}(t)\,\mathrm{d}t = \tfrac{1}{2}\mathrm{B}\left(\mu,\nu,\mathrm{sech}^2(x)\right) \quad \mu>0$$

$$\left.\begin{matrix} \\ \end{matrix}\right\} \begin{cases} 0<x<\infty \\ \mathrm{g} = \cosh, \sinh, \mathrm{sech}, \mathrm{csch}, \tanh, \coth \end{cases}$$

The relationships between the power λ and the parameters of the incomplete beta function are

g =	cosh	sinh	sech	csch	tanh	coth
$\nu =$	$\frac{1}{2}$	$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}$	$\frac{1}{2}(1-\lambda)$	$\frac{1}{2}(1+\lambda)$	$\frac{1}{2}(1-\lambda)$
$\mu =$	$\frac{-1}{2}\lambda$	$\frac{-1}{2}\lambda$	$\frac{1}{2}\lambda$	$\frac{1}{2}\lambda$	0	0

for these hyperbolic functions. For example,

58:14:6 $$\int_1^{\infty} \mathrm{sech}^{3/2}(t)\mathrm{d}t = \tfrac{1}{2}\mathrm{B}\left(\tfrac{3}{4},\tfrac{1}{2},\mathrm{sech}^2(1)\right)$$

As before, the range of the power may be restricted by the condition placed on the parameter. Thus, the condition attaching to 58:14:5 cannot be met for the tanh and coth functions, but 58:14:4 *can* be applied, leading to

58:14:7 $$\int_0^x \tanh^{\lambda}(t)\mathrm{d}t = \tfrac{1}{2}\mathrm{B}\left(\frac{1+\lambda}{2},0,\tanh^2(x)\right) = \tfrac{1}{2}\ln_{(1+\lambda)/2}\left(\cosh^2(x)\right)$$

for the hyperbolic tangent case.

CHAPTER

59

THE LEGENDRE FUNCTIONS $\mathrm{P}_\nu(x)$ AND $\mathrm{Q}_\nu(x)$

The two bivariate functions of this chapter arise in several physical contexts, most notably in describing phenomena within, or on the surface of, a sphere. Applications of these functions concentrate on real arguments of magnitudes less than unity and accordingly the domain $-1 < x < 1$ receives emphasis in this chapter. *Associated Legendre functions*, which are trivariate extensions of $\mathrm{P}_\nu(x)$ and $\mathrm{Q}_\nu(x)$, are discussed briefly in Section 59:12. The final section of this chapter illustrates how these functions arise in physical applications.

59:1 NOTATION

$\mathrm{P}_\nu(x)$ and $\mathrm{Q}_\nu(x)$ are known respectively as the *Legendre function of the first kind* and the *Legendre function of the second kind*, of degree ν. Not infrequently the argument x is replaced by a hyperbolic or circular cosine; the *Atlas* uses notations such as $\mathrm{P}_\nu(\cosh(\alpha))$ and $\mathrm{Q}_\nu(\cos(\theta))$ when these replacements are made.

The term *spherical harmonic* is used collectively to embrace both the Legendre functions and the associated Legendre functions [Section 59:12].

Regrettably, the definitions of $\mathrm{P}_\nu(x)$ and $\mathrm{Q}_\nu(x)$ are by no means uniform from one author to the next, and varying symbolism are in use. Abramowitz and Stegun [Page 332] list several of these alternatives. Sometimes different symbols are adopted according as the argument is real or complex. As elsewhere in this *Atlas*, our emphasis here is on real variables. Accordingly, an equation in this chapter will sometimes imply only that the real parts of each side are equal.

59:2 BEHAVIOR

Apart from the isolated instances addressed in Sections 59:4 and 59:5, both Legendre functions adopt complex values when $x \le -1$. This domain receives no further attention here. The Q Legendre function is mostly complex also when its argument exceeds -1, but the definition discussed in Section 59:11 accords it the real values discussed in most of this chapter. $\mathrm{Q}_\nu(x)$ is unequivocally real only for $-1 < x < 1$.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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Each Legendre function exhibits such different properties on either side of $x = 1$ that effectively it behaves as a distinct function in each region. Accordingly, we display their landscapes as the four graphs shown opposite.

The values acquired by the two Legendre functions at arguments of -1, 0, and $+1$ are reported in detail in Section 59:7 and their general behavior within $-1 < x < 1$ is evident from Figures 59-1 and 59-2. Because of the degree-reflection formulas 59:5:3 and 59:5:4, it suffices to consider only degrees of $\nu \geq -\frac{1}{2}$, and the two figures are so restricted. The landscape is rippled for both functions, with multiple zeros and local extrema.

Figures 59-3 and 59-4 are three-dimensional graphs of the Legendre functions in the $x > 1$ domain. $P_\nu(x)$ exhibits a rather bland landscape; notice the reflection symmetry that exists across the plane $\nu = -\frac{1}{2}$, in consequence of formula 59:5:3. The repetitive $-\infty|+\infty$ discontinuities that occur in Figure 59-4 whenever the degree of $Q_\nu(x)$ encounters a negative integer value may be attributed to the cotangent term in formula 59:5:4.

59:3 DEFINITIONS

Several definite integrals, including the following, represent the Legendre functions:

59:3:1
$$P_\nu(x) = \frac{1}{\pi}\int_0^\pi \left[x + \sqrt{x^2-1}\cos(\phi)\right]^\nu d\phi \qquad x > 1 \qquad \text{all } \nu$$

59:3:2
$$Q_\nu(x) = \int_0^\infty \left[x + \sqrt{x^2-1}\cosh(t)\right]^{-\nu-1} dt \qquad x > 1 \qquad \nu > -1$$

There are also several definitions of the Legendre functions as indefinite integrals:

59:3:3
$$P_\nu(\cos(\theta)) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos\left((\nu+\frac{1}{2})\phi\right)}{\sqrt{\cos(\phi)-\cos(\theta)}} d\phi \qquad 0 < \theta < \pi$$

59:3:4
$$P_\nu(\cosh(\alpha)) = \frac{\sqrt{2}}{\pi}\cot\left((\nu+\tfrac{1}{2})\pi\right)\int_\alpha^\infty \frac{\sinh\left((\nu+\frac{1}{2})t\right)}{\sqrt{\cosh(t)-\cosh(\alpha)}} dt \qquad \alpha > 0 \qquad -1 < \nu < 0$$

59:3:5
$$Q_\nu(\cosh(\alpha)) = \frac{1}{\sqrt{2}}\int_\alpha^\infty \frac{\exp\left(-(\nu+\frac{1}{2})t\right)}{\sqrt{\cosh(t)-\cosh(\alpha)}} dt \qquad \alpha > 0 \qquad \nu > -1$$

Despite appearances to the contrary, the integrands in the definitions

59:3:6
$$P_\nu(x) = \frac{1}{\pi}\int_1^\infty \left\{ i\left[x + it\sqrt{1-x^2}\right]^{-1-\nu} - \left[x - it\sqrt{1-x^2}\right]^{-1-\nu}\right\} \frac{dt}{\sqrt{t^2-1}} \qquad -1 < x < 1 \qquad \nu > -1$$

and

59:3:7
$$Q_\nu(x) = \frac{1}{2}\int_1^\infty \left\{ \left[x + it\sqrt{1-x^2}\right]^{-1-\nu} + \left[x - it\sqrt{1-x^2}\right]^{-1-\nu}\right\} \frac{dt}{\sqrt{t^2-1}} \qquad -1 < x < 1 \qquad \nu > -1$$

are wholly real because all imaginary terms cancel after binomial expansions.

On account of formula 59:5:3, the degree ν in any definition of $P_\nu(x)$ may be replaced by $-\nu - 1$. A similar replacement for $Q_\nu(x)$ is valid only if ν is an odd multiple of ½.

The traditional definition of the Legendre functions is as solutions of *Legendre's differential equation*

59:3:8
$$(1-x^2)\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + \nu(\nu+1)f = 0 \qquad f = w_1 P_\nu(x) + w_2 Q_\nu(x)$$

the w's being arbitrary constants.

Figure 59-1 Figure 59-2

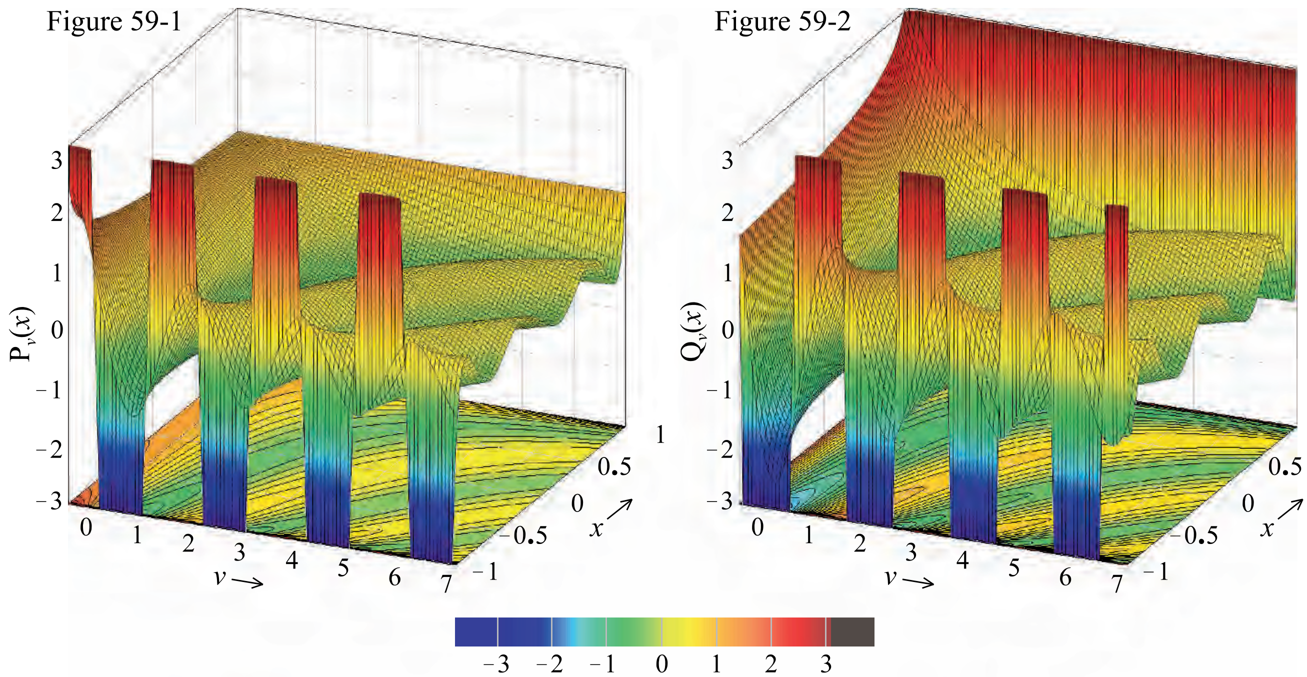
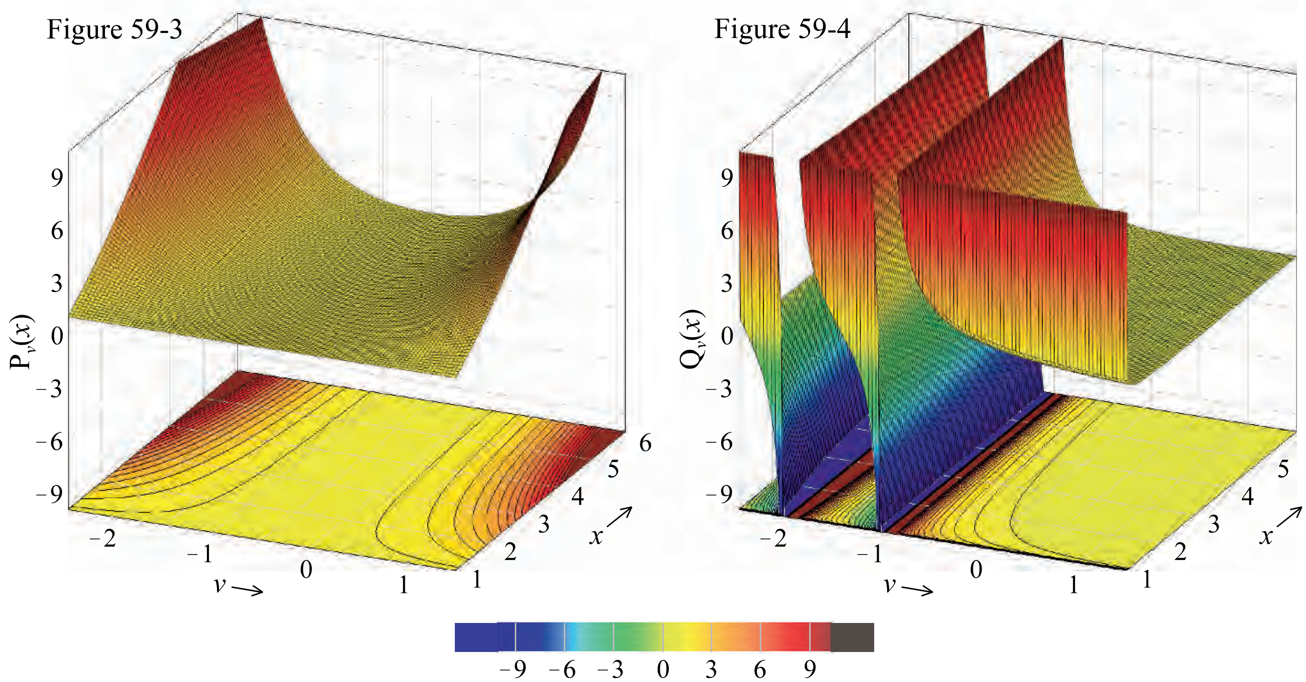


Figure 59-3 Figure 59-4



The hypergeometric formulation, equation 59:6:1, of the Legendre function of the first kind opens the way to its synthesis [Section 43:14] from a power function:

59:3:9 $$(1-x)^\nu \xrightarrow[1]{1+\nu} P_\nu(1-2x)$$

Synthesis of the second Legendre function is more elaborate:

59:3:10 $$(1-x)^{-\nu/2} \xrightarrow[\nu+\frac{1}{2}]{(\nu+1)/2} \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)x^{\nu/2}}Q_{\nu-1}\left(\frac{1}{\sqrt{x}}\right)$$

The Legendre function of the first kind can be expressed in many ways as special cases of the Gauss hypergeometric function [Section 59:13] and occasionally $P_\nu(x)$ is defined by this route.

59:4 SPECIAL CASES

When ν is an integer of either sign, $P_\nu(x)$ reduces to one of the Legendre polynomials discussed in Chapter 21

59:4:1 $$\left.\begin{matrix} P_\nu(x) \\ P_{-\nu-1}(x) \end{matrix}\right\} = P_n(x) = \text{polynomial} \qquad \begin{cases} \nu = n = 0,1,2,\cdots \\ \nu = -n-1 = -1,-2,-3,\cdots \end{cases}$$

Legendre functions of the second kind are not defined for negative integer degree, but when ν is a *non*negative integer, $Q_\nu(x)$ reduces to one of the functions $Q_n(x)$ described in Section 21:13, many of which are graphed in Figure 21-3. Some examples are

	$\nu=-2$	$\nu=-1$	$\nu=0$	$\nu=1$	$\nu=2$	$\nu=3$
$P_\nu(x)$	x	1	1	x	$\frac{3}{2}x^2-\frac{1}{2}$	$\frac{5}{2}x^3-\frac{3}{2}x$
$Q_\nu(x)$	undef	undef	$Q_0(x)$	$xQ_0(x)-1$	$P_2(x)Q_0(x)-\frac{3}{2}x$	$P_3(x)Q_0(x)-\frac{5}{2}x^2+\frac{2}{3}$

where the zero-degree instance of the Legendre function of the second kind is an inverse hyperbolic function [Chapter 31]

59:4:2 $$Q_0(x) = \begin{cases} \operatorname{artanh}(x) & -1<x<1 \\ \operatorname{arcoth}(x) & |x|>1 \end{cases}$$

When the degree ν is an odd multiple of ½, each of the Legendre functions reduces to an expression involving complete elliptic integrals [Chapter 61] with square-root moduli. The prime examples are for $\nu = -\frac{1}{2}$:

59:4:3 $$P_{-1/2}(x) = \frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

59:4:4 $$Q_{-1/2}(x) = K\left(\sqrt{\frac{1+x}{2}}\right)$$

The modulus of the elliptic integral in these formulas is imaginary for $x > 1$ in the case of P, or $x < -1$ for Q. Nevertheless, these formulas define real Legendre functions in these x domains because [see 61:11:4] complete elliptic integrals are real when their moduli are imaginary. However, for $x < -1$ in the case of P, or $x > 1$ for Q, the modulus of the elliptic integral exceeds unity whereupon its value, and hence that of the Legendre function, becomes complex. Because the reflection formula $f_{-\nu-1}(x) = f_\nu(x)$ applies to *both* Legendre functions in these special cases, each expression for a Legendre function of a degree that is an odd multiple of ½ applies to two degrees that average $-\frac{1}{2}$. For example

59:4:5 $$P_{-3/2}(x) = P_{1/2}(x) = \frac{4}{\pi}E\left(\sqrt{\frac{1-x}{2}}\right) - \frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

and

59:4:6 $$Q_{-3/2}(x) = Q_{1/2}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) - 2E\left(\sqrt{\frac{1+x}{2}}\right)$$

By employing two equations from 59:4:3-6 as the starting point, recursion formula 59:5:5 may be used to develop formulas expressing the Legendre function of any other degree that is an odd multiple of ½.

59:5 INTRARELATIONSHIPS

The argument-reflection formulas

59:5:1 $$P_n(-x) = (-)^{|n+1/2|-1/2} P_n(x) \qquad n = 0, \pm1, \pm2, \cdots$$

59:5:2 $$Q_n(-x) = -(-)^n Q_n(x) \qquad n = 0, 1, 2, \cdots$$

apply only when the degree is an integer, whereas the degree-reflection formulas

59:5:3 $$P_{-\nu-1}(x) = P_\nu(x)$$

59:5:4 $$Q_{-\nu-1}(x) = Q_\nu(x) - \pi\cot(\nu\pi) P_\nu(x)$$

are of general applicability.

The same degree-recursion formula

59:5:5 $$(\nu+1)f_{\nu+1}(x) = (2\nu+1)x f_\nu(x) - \nu f_{\nu-1}(x) \qquad f = P \text{ or } Q$$

applies to both kinds of Legendre function.

The following equations link the two kinds of Legendre function but apply only when the degree is not an integer and when $-1 < x < 1$

59:5:6 $$Q_\nu(\pm x) = \frac{\pi}{2}\left[\cot(\nu\pi) P_\nu(\pm x) - \csc(\nu\pi) P_\nu(\mp x)\right]$$

59:5:7 $$P_\nu(\pm x) = -\frac{2}{\pi}\left[\cot(\nu\pi) Q_\nu(\pm x) + \csc(\nu\pi) Q_\nu(\mp x)\right]$$

59:6 EXPANSIONS

As detailed in Section 59:13, Legendre functions may be expressed as Gauss hypergeometric functions [Chapter 60] in a multitude of ways. Because each of these latter functions may be expanded as the (usually infinite) power series 60:6:1, the number of expansions of $P_\nu(x)$ and $Q_\nu(x)$ is huge. The listing presented here is not exhaustive. In all cases the strategy is to express the Legendre function as a Gauss hypergeometric function, or as a weighted sum of two such functions, and then expand the latter function(s) as power series.

For reasons explained in Section 59:13, the expansion

59:6:1 $$P_\nu(x) = F\left(-\nu, 1+\nu, 1, \frac{1-x}{2}\right) = \sum_{j=0}^{\infty} \frac{(-\nu)_j (1+\nu)_j}{(1)_j (1)_j}\left(\frac{1-x}{2}\right)^j = 1 - \frac{\nu(1+\nu)}{2}(1-x) + \cdots$$

valid in the range $-1 < x < 3$ range of argument, and the more complicated expansion

59:6:2 $$Q_\nu(x) = \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{1+\nu}} F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, \frac{1}{x^2}\right) = \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{1+\nu}}\left[1 + \frac{\nu^2+3\nu+2}{(8\nu+12)x^2} + \cdots\right]$$

which has validity for $x > 1$, are seen as fundamental. Should ν equal one of the values $-3/2, -5/2, -7/2, \cdots$, the first $(-\nu-\tfrac{1}{2})$ terms in 59:6:2 are zero. If you wish to avoid this, replace each ν on the right-hand side by $-\nu-1$.

To illustrate the twin-Gauss approach to expanding Legendre functions, first consider the two expansions

59:6:3 $$\mathrm{F}\left(-\tfrac{1}{2}\nu,\tfrac{1}{2}+\tfrac{1}{2}\nu,\tfrac{1}{2},x^2\right)=\sum_{j=0}^{\infty}\frac{\left(\tfrac{-1}{2}\nu\right)_j\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)_j}{\left(\tfrac{1}{2}\right)_j(1)_j}x^{2j}=1-\frac{\nu^2+\nu}{2}x^2+\frac{\nu^4+2\nu^3-5\nu^2-6\nu}{24}x^4-\cdots$$

and

59:6:4 $$\mathrm{F}\left(\tfrac{1}{2}-\tfrac{1}{2}\nu,1+\tfrac{1}{2}\nu,\tfrac{3}{2},x^2\right)=\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}-\tfrac{1}{2}\nu\right)_j\left(1+\tfrac{1}{2}\nu\right)_j}{\left(\tfrac{3}{2}\right)_j(1)_j}x^{2j}=1-\frac{\nu^2+\nu-1}{6}x^2+\frac{\nu^4+2\nu^3-12\nu^2-13\nu+12}{120}x^4-\cdots$$

both applicable for $-1 < x < 1$. The Legendre functions are weighted sums of these two Gauss hypergeometric functions of argument x^2:

59:6:5 $$\mathrm{P}_\nu(x)=\frac{\sqrt{\pi}}{\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}\nu\right)\Gamma\left(1+\tfrac{1}{2}\nu\right)}\mathrm{F}\left(-\tfrac{1}{2}\nu,\tfrac{1}{2}+\tfrac{1}{2}\nu,\tfrac{1}{2},x^2\right)\pm\frac{2\sqrt{\pi}}{\Gamma\left(\tfrac{\nu}{2}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)}x\mathrm{F}\left(\tfrac{1}{2}-\tfrac{1}{2}\nu,1+\tfrac{1}{2}\nu,\tfrac{3}{2},x^2\right)$$

59:6:6 $$\mathrm{Q}_\nu(x)=\frac{\sqrt{\pi^3}\,\Gamma\left(1+\tfrac{1}{2}\nu\right)}{\Gamma^2\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}\nu\right)}x\mathrm{F}\left(\tfrac{1}{2}-\tfrac{1}{2}\nu,1+\tfrac{1}{2}\nu,\tfrac{3}{2},x^2\right)+\frac{\sqrt{\pi^3}\,\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)}{2\Gamma^2\left(1+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{-1}{2}\nu\right)}\mathrm{F}\left(-\tfrac{1}{2}\nu,\tfrac{1}{2}+\tfrac{1}{2}\nu,\tfrac{1}{2},x^2\right)$$

Notice that, in each of these formulas, there is an x multiplier in one or other of the weights, ensuring that all nonnegative integer powers of x appear in the ultimate expansions.

There are also expansions as trigonometric functions:

59:6:7 $$\mathrm{P}_\nu\left(\cos(\theta)\right)=\frac{2\Gamma(1+\nu)}{\sqrt{\pi}\,\Gamma\left(\tfrac{3}{2}+\nu\right)}\sum_{j=0}^{\infty}\frac{(1+\nu)_j\left(\tfrac{1}{2}\right)_j}{\left(\tfrac{3}{2}+\nu\right)_j(1)_j}\sin\left((1+\nu+2j)\theta\right)\qquad 0<\theta<\pi$$

and

59:6:8 $$\mathrm{Q}_\nu\left(\cos(\theta)\right)=\frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)}\sum_{j=0}^{\infty}\frac{(1+\nu)_j\left(\tfrac{1}{2}\right)_j}{\left(\tfrac{3}{2}+\nu\right)_j(1)_j}\cos\left((1+\nu+2j)\theta\right)\qquad 0<\theta<\pi$$

though these sums are often slow to converge.

59:7 PARTICULAR VALUES

In this section we look into the values acquired by $\mathrm{P}_\nu(x)$ and $\mathrm{Q}_\nu(x)$ at the arguments $x = -1, 0, 1$ and ∞. The first and second tables below apply to Legendre function of the first and second kinds respectively. Observe that a plethora of formulas is needed to cover particular values at zero argument; mostly, these are special cases of the formulas

59:7:1 $$\mathrm{P}_\nu(0)=\frac{\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\cos\left(\tfrac{1}{2}\nu\pi\right)}{\sqrt{\pi}\Gamma\left(1+\tfrac{1}{2}\nu\right)}\qquad\text{and}\qquad \mathrm{Q}_\nu(0)=\frac{-\sqrt{\pi}\,\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\sin\left(\tfrac{1}{2}\nu\pi\right)}{2\Gamma\left(1+\tfrac{1}{2}\nu\right)}$$

Gauss's constant g [Section 1:7] occurs frequently in these tables, as do the double and quadruple factorial functions that are addressed in Section 2:3. Note the example $(15)!!!! = 15\times11\times7\times3$, and that quadruple factorials of $1, -1$ and -3 all equal unity.

ν	$P_\nu(-1)$	$P_\nu(0)$	$P_\nu(1)$	$P_\nu(\infty)$
$-3,-5,-7,-9,\cdots$	1	$\dfrac{(-)^{(\nu+1)/2}(-\nu-2)!!}{(-\nu-1)!!}$	1	$+\infty$
$\frac{-5}{2},\frac{-9}{2},\frac{-13}{2},\frac{-17}{2},\cdots$	$+\infty$	$(-)^{(1+2\nu)/4}\dfrac{\sqrt{2}(-2\nu-4)!!!!g}{(-2\nu-2)!!!!}$	1	$+\infty$
$1,-2,3,-4,\cdots$	-1	0	1	$+\infty$
$\frac{-3}{2},\frac{-7}{2},\frac{-11}{2},\frac{-15}{2},\cdots$	$-\infty$	$(-)^{(3+2\nu)/4}\dfrac{\sqrt{2}(-2\nu-4)!!!!}{(-2\nu-2)!!!!\pi g}$	1	$+\infty$
$-1,0$	1	1	1	1
$\frac{-1}{2},\frac{3}{2},\frac{7}{2},\frac{11}{2},\cdots$	$+\infty$	$(-)^{(2\nu+1)/4}\dfrac{\sqrt{2}(2\nu-2)!!!!g}{(2\nu)!!!!}$	1	$+\infty$
$\frac{1}{2},\frac{5}{2},\frac{9}{2},\frac{13}{2},\cdots$	$-\infty$	$(-)^{(2\nu-1)/4}\dfrac{\sqrt{2}(2\nu-2)!!!!}{(2\nu)!!!!\pi g}$	1	$+\infty$
$2,4,6,8,\cdots$	1	$\dfrac{(-)^{\nu/2}(\nu-1)!!}{\nu!!}$	1	$+\infty$

ν	$Q_\nu(-1)$	$Q_\nu(0)$	$Q_\nu(1)$	$Q_\nu(\infty)$
$\frac{-5}{2},\frac{-9}{2},\frac{-13}{2},\frac{-17}{2},\cdots$	$\dfrac{\pi}{2}$	$(-)^{(1+2\nu)/4}\dfrac{(-2\nu-4)!!!!\pi g}{\sqrt{2}(-2\nu-2)!!!!}$	$+\infty\vert+\infty$	0
$\frac{-3}{2},\frac{-7}{2},\frac{-11}{2},\frac{-15}{2},\cdots$	$\dfrac{-\pi}{2}$	$(-)^{(1-2\nu)/4}\dfrac{(-2\nu-4)!!!!}{\sqrt{2}(-2\nu-2)!!!!g}$	$+\infty\vert+\infty$	0
$-1,-2,-3,-4,\cdots$	undef	undef	undef	undef
$\frac{-1}{2},\frac{3}{2},\frac{7}{2},\frac{11}{2},\cdots$	$\dfrac{\pi}{2}$	$(-)^{(2\nu+1)/4}\dfrac{(2\nu-2)!!!!\pi g}{\sqrt{2}(2\nu)!!!!}$	$+\infty\vert+\infty$	0
$0,2,4,6,\cdots$	$-\infty$	0	$+\infty\vert+\infty$	0
$\frac{1}{2},\frac{5}{2},\frac{9}{2},\frac{13}{2},\cdots$	$\dfrac{-\pi}{2}$	$(-)^{(2\nu+3)/4}\dfrac{(2\nu-2)!!!!}{\sqrt{2}(2\nu)!!!!g}$	$+\infty\vert+\infty$	0
$1,3,5,7,\cdots$	$+\infty$	$(-)^{(\nu-1)/2}\dfrac{(\nu-1)!!}{\nu!!}$	$+\infty\vert+\infty$	0

59:8 NUMERICAL VALUES

Equator's routines for the Legendre function of the first kind and Legendre function of the second kind (keywords **P** and **Q**) employ equations 59:6:1 or 59:6:5 and 59:6:2 or 59:6:6 to evaluate $P_\nu(x)$ and $Q_\nu(x)$ in the domains $|\nu| \le 100$ and $x \ge -1$. Where functions are complex, only the real part is returned. For integer ν values of the Legendre function of the first kind are also provided for the $x < -1$ region.

59:9 LIMITS AND APPROXIMATIONS

Close to $x = 1$, the Legendre function of the first kind becomes linear

59:9:1 $$P_\nu(x) \approx 1 + \frac{\nu+\nu^2}{2}(x-1) \qquad |1-x| \text{ small}$$

whereas that of the second kind approaches infinity in accord with the relationship

59:9:2 $$Q_\nu(x) \approx \ln\left(\sqrt{\frac{2}{|1-x|}}\right) - \gamma - \psi(1+\nu) \qquad |1-x| \text{ small}$$

The corresponding limiting behaviors as $-1 \leftarrow x$ are described by

59:9:3 $$P_\nu(x) \approx \cos(\nu\pi) + \frac{\sin(\nu\pi)}{\pi}\left[\gamma + 2\psi(1+\nu) + \ln\left(\frac{1+x}{2}\right)\right] \qquad \begin{matrix}(1+x)\text{ small}\\ \text{and positive}\end{matrix}$$

and

59:9:4 $$Q_\nu(x) \approx \frac{-\pi\sin(\nu\pi)}{2} + \frac{\cos(\nu\pi)}{2}\left[\gamma + 2\psi(1+\nu) + \ln\left(\frac{1+x}{2}\right)\right] \qquad \begin{matrix}(1+x)\text{ small}\\ \text{and positive}\end{matrix}$$

The γ and $\psi(\;)$ terms in these formulas are Euler's constant [Section 1:7] and the digamma function [Chapter 44].

When the argument is large and positive, the limiting expressions are

	$\nu < -\frac{1}{2}$	$\nu = -\frac{1}{2}$	$\nu \ge 0$
$P_\nu(x \to \infty)$	$\dfrac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu)}(2x)^{-\nu-1}$	$\dfrac{\sqrt{2}}{\pi}\dfrac{\ln(8x)}{\sqrt{x}}$	$\dfrac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}(2x)^{\nu}$

	$\nu < -\frac{1}{2}$ but $\ne -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$	$\nu = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$	$\nu \le 3,\ \nu \ne -1, -2, -3, \cdots$
$Q_\nu(x \to \infty)$	$\dfrac{\sqrt{\pi}\,\Gamma\left(-\nu-\frac{1}{2}\right)\cot(\nu\pi)}{\Gamma(-\nu)}(2x)^{-\nu-1}$	$\dfrac{(-2\nu-2)!!\pi\sqrt{2}}{\left(-\frac{1}{2}-\nu\right)!}(4x)^{\nu}$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{\nu+1}}$

59:10 OPERATIONS OF THE CALCULUS

Either differentiation or indefinite integration of a Legendre function yields an associated Legendre function [Section 59:12]

59:10:1 $$\frac{d}{dx}f_\nu(x) = \left\{\begin{matrix} -f_\nu^{(1)}(x)/\sqrt{1-x^2} & -1 < x < 1 \\ f_\nu^{(1)}(x)/\sqrt{x^2-1} & x > 1 \end{matrix}\right\} \qquad f = P \text{ or } Q$$

59:10:2 $$\int_x^1 P_\nu(t)\,dt = \sqrt{1-x^2}\,P_\nu^{(-1)}(x) \qquad -1<x<1$$

59:10:3 $$\int_1^x P_\nu(t)\,dt = \sqrt{x^2-1}\,P_\nu^{(-1)}(x) \qquad x>1$$

As an alternative to 59:10:1, the derivatives of Legendre functions may be expressed, without recourse to associated functions, as

59:10:4 $$\frac{d}{dx}f_\nu(x) = \frac{\nu+1}{1-x^2}\left[x\,f_\nu(x) - f_{\nu+1}(x)\right] = \frac{\nu}{1-x^2}\left[f_{\nu-1}(x) - x\,f_\nu(x)\right] \qquad x>-1 \qquad f = P \text{ or } Q$$

or by many other formulas discovered by incorporating the recursion 59:5:5.

Noteworthy integrals and Laplace transforms of the first kind of Legendre function include

59:10:5 $$\int_x^1 t\,P_\nu(t)\,dt = \frac{\left(1-x^2\right)P_\nu(x) + x\sqrt{1-x^2}\,P_\nu^{(1)}(x)}{(1-\nu)(2+\nu)} \qquad -1<x<1$$

59:10:6 $$\int_0^1 t^\lambda P_\nu(t)\,dt = \frac{\sqrt{\pi}\,\Gamma(1+\lambda)}{2^{1+\lambda}\Gamma\left(1+\frac{1}{2}\lambda-\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\lambda+\frac{1}{2}\nu\right)} \qquad \lambda>-1$$

59:10:7 $$\int_{-1}^1 (1+t)^\lambda P_\nu(t)\,dt = \frac{2^{1+\lambda}\Gamma^2(1+\lambda)}{\Gamma(2+\lambda+\nu)\Gamma(1+\lambda-\nu)} \qquad \lambda>-1$$

59:10:8 $$\int_0^\infty P_\nu(1+bt)\exp(-st)\,dt = \mathscr{L}\{P_\nu(1+bt)\} = \sqrt{\frac{2}{\pi bs}}\exp\left(\frac{s}{b}\right)K_{\nu+1/2}\left(\frac{s}{b}\right)$$

and many others are given by Gradshteyn and Ryzhik [Section 7.1].

Formulas for integrals of products of Legendre functions include

59:10:9 $$\int_{-1}^1 P_\nu(t)P_\omega(t)\,dt = \frac{2\pi\sin\{(\nu-\omega)\pi\} - 4\sin(\nu\pi)\sin(\omega\pi)[\psi(1+\nu)-\psi(1+\omega)]}{\pi^2(\nu-\omega)(1+\nu+\omega)} \qquad \nu+\omega\neq-1$$

59:10:10 $$\int_0^1 P_\nu(t)P_\omega(t)\,dt = \frac{2\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\omega\right)\sin\left(\frac{1}{2}\nu\pi\right)\cos\left(\frac{1}{2}\omega\pi\right)}{\pi\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)\Gamma\left(1+\frac{1}{2}\omega\right)(\nu-\omega)(1+\nu+\omega)} + \frac{2\Gamma\left(1+\frac{1}{2}\omega\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)\sin\left(\frac{1}{2}\omega\pi\right)\cos\left(\frac{1}{2}\nu\pi\right)}{\pi\Gamma\left(\frac{1}{2}+\frac{1}{2}\omega\right)\Gamma(1+\frac{1}{2}\nu)(\omega-\nu)(1+\omega+\nu)}$$

59:10:11 $$\int_{-1}^1 P_\nu(t)Q_\omega(t)\,dt = \frac{\pi[1-\cos\{(\nu-\omega)\pi\}] - 2\sin(\nu\pi)\cos(\omega\pi)[\psi(1+\nu)-\psi(1+\omega)]}{\pi(\nu-\omega)(1+\nu+\omega)} \qquad \nu>0<\omega$$

and

59:10:12 $$\int_{-1}^1 Q_\nu(t)Q_\omega(t)\,dt = \frac{[\psi(\nu+1)-\psi(\omega+1)][1+\cos(\nu\pi)\cos(\omega\pi)] - \frac{1}{2}\pi\sin(\nu\pi-\omega\pi)}{(\omega-\nu)(1+\nu+\omega)}$$

$$\omega\neq-\nu-1,-1,-2,-3,\cdots$$

where $\Gamma(\ )$ and $\psi(\ )$ are the gamma and digamma functions [Chapters 43 and 44]. Other integrals of this sort are listed in Section 3.12 of Erdélyi et al. [*Higher Transcendental Functions*, Volume 1], though not all of them appear to be correct. One cannot set $\omega=\nu$ in any of these formulas without carefully investigating the behavior as $\omega\to\nu$. Some consequences are as follows; they mostly involve the trigamma function [Section 44:12],

$\int_{-1}^{1} P_\nu^2(t)\,dt =$	$\int_{-1}^{1} P_\nu(t)Q_\nu(t)\,dt =$	$\int_{-1}^{1} Q_\nu^2(t)\,dt =$	$\int_{1}^{\infty} P_\nu(t)Q_\nu(t)\,dt =$
$\dfrac{2\pi^2 - 4\sin^2(\nu\pi)\psi^{(1)}(1+\nu)}{\pi^2(1+2\nu)}$	$\dfrac{-\sin(2\nu\pi)\psi^{(1)}(1+\nu)}{\pi(1+2\nu)}$	$\dfrac{\frac{1}{2}\pi^2 - [1+\cos^2(\nu\pi)]\psi^{(1)}(1+\nu)}{1+2\nu}$	∞

The formulas 59:10:9 − 12 are useful, not only in their own right, but also as the source of other definite integrals by taking advantage of such identities as $P_0(t) = 1$, $P_1(t) = t$, and $\frac{2}{3}P_2(t) - \frac{1}{3}P_0(t) = t^2$. The orthogonality properties of Legendre functions of the first kind are revealed by setting $m = 0$ in equation 59:12:11.

59:11 COMPLEX ARGUMENT

To avoid the Legendre functions of complex argument being multivalued, the complex plane must be cut. For the $P_\nu(z)$ function, it is conventional to make the cut along the line $-\infty \le x \le -1$, $y = 0$, whereas a longer cut $-\infty \le x \le +1$, $y = 0$ is needed for $Q_\nu(z)$. Resulting from the cut are ambiguities in the values of Legendre functions when the argument is real. To resolve the ambiguity, it is conventional, but not universal, to assign the average of the values on either side of the cut. That is

59:11:1 $$f_\nu(x) = \lim_{\delta\to 0}\left\{\frac{f_\nu(x+i\delta)+f_\nu(x-i\delta)}{2}\right\} \qquad f = P \text{ or } Q$$

When their arguments or degrees are complex, the Legendre functions are generally complex, too, but the *Atlas* does not pursue this topic except for some comments on *conical functions*. These are Legendre functions of complex degree, of which the real part is −½. They arise in solving differential equations such as those listed in Section 46:15 for a space shaped as a cone or possessing certain other geometries [Lebedev, Sections 8.5, 8.9 and 8.12]. Notwithstanding its complex degree, the conical function of the first kind is real

59:11:2 $$P_{\frac{-1}{2}+i\lambda}(x) = F\left(\tfrac{1}{2}+i\lambda, \tfrac{1}{2}-i\lambda, 1, \frac{1-x}{2}\right) = \sum_{j=0}^{\infty}\frac{[(1-x)/2]^j}{(j!)^2}\prod_{k=0}^{j-1}\lambda^2+\left(k+\tfrac{1}{2}\right)^2$$

Inverse Laplace transformation of Legendre functions generates a spherical instance of the Macdonald [Section 26:13] or the modified Bessel [Section 28:13] functions

59:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty} P_\nu(bs)\frac{\exp(ts)}{2\pi i}\,ds = \mathfrak{I}\{P_\nu(bs)\} = \frac{-2}{\pi^2 b}\sin(\nu\pi)k_\nu\left(\frac{t}{b}\right)$$

59:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty} Q_\nu(bs)\frac{\exp(ts)}{2\pi i}\,ds = \mathfrak{I}\{Q_\nu(bs)\} = \frac{1}{b}i_\nu\left(\frac{t}{b}\right)$$

59:12 GENERALIZATIONS: the associated Legendre functions

The trivariate *associated Legendre functions* $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ are generalizations of the functions that are the main subject of this chapter, inasmuch as

59:12:1 $$P_\nu^{(0)}(x) = P_\nu(x) \qquad \text{and} \qquad Q_\nu^{(0)}(x) = Q_\nu(x)$$

The third variable μ is the *order* of the function. Some authors describe $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ as "Legendre functions" and consider $P_\nu(x)$ and $Q_\nu(x)$ as merely the zero-order instances. Associated Legendre functions having a degree that is an odd multiple of ½ are sometimes known as *toroidal functions*.

Linear combinations of $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ satisfy the *associated Legendre differential equation*

59:12:2 $$(1-x^2)\frac{d^2f}{dx^2} - 2x\frac{df}{dx} + \left[\nu(1+\nu) - \frac{\mu^2}{1-x^2}\right]f = 0 \qquad f = w_1 P_\nu^{(\mu)}(x) + w_2 Q_\nu^{(\mu)}(x)$$

This commonly arises as the trigonometric equivalent

59:12:3 $$\frac{d^2f}{d\theta^2} + \cot(\theta)\frac{df}{d\theta} + \left[\nu(1+\nu) - \mu^2\csc^2(\theta)\right]f = 0 \qquad f = w_1 P_\nu^{(\mu)}\left(\cos(\theta)\right) + w_2 Q_\nu^{(\mu)}\left(\cos(\theta)\right)$$

The latter is valid only for $|\theta| < \pi$, which is the primary domain of interest for associated Legendre functions, corresponding to $-1 < x < 1$. For reasons made evident in Section 59:14, applications almost invariably require that the order be a nonnegative integer and henceforth we mostly impose this restriction and replace μ by m.

The differential equation 59:12:2 might suggest that associated Legendre functions of orders m and $-m$ would be defined identically, but this is not so. Instead one has the order-reflection formulas

59:12:4 $$f_\nu^{(-m)}(x) = \frac{f_\nu^{(m)}(x)}{(-\nu)_m(1+\nu)_m} \qquad f = \text{ P or Q} \qquad m = 0,1,2,\cdots$$

For example $f_\nu^{(-1)}(x) = -f_\nu^{(1)}(x)/(\nu^2+\nu)$. The degree-reflection formulas are

59:12:5 $$P_{-\nu}^{(m)}(x) = P_{\nu-1}^{(m)}(x) \qquad \text{and} \qquad Q_{-\nu}^{(m)}(x) = Q_{\nu-1}^{(m)}(x) - \pi\cot(\nu\pi)P_{\nu-1}^{(m)}(x) \qquad m = 0,1,2,\cdots$$

Integral representations of the associated Legendre functions that are extensions of those listed in Section 59:3 exist, but are not reported here. The multiple differentiation formulas

59:12:6 $$P_\nu^{(m)}(x) = (-)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_\nu(x)$$

and

59:12:7 $$Q_\nu^{(m)}(x) = (-)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}Q_\nu(x) \qquad \nu \neq -1,-2,-3,\cdots$$

provide the simplest definitions for integer orders and lead to most of the formulas reported here. Be aware that the $(-)^m$ multiplier in these formulas is omitted by some authorities, so that odd-ordered associated Legendre functions may be encountered with signs that differ from those here. The full story around associated Legendre functions is quite complicated, especially when the argument is complex; see Gradshteyn and Ryzhik [Section 8.70] for enlightenment.

The associated Legendre function $P_\nu^{(\mu)}(x)$ may be expressed as a weighted sum of any two of its *contiguous functions*, that is, of any two of $P_{\nu-1}^{(\mu)}(x)$, $P_{\nu+1}^{(\mu)}(x)$, $P_\nu^{(\mu-1)}(x)$, and $P_\nu^{(\mu+1)}(x)$. The appropriate weights are contained in the table overleaf, which applies equally to associated Legendre functions of either kind. To illustrate the use of this table, note that the final row implies

59:12:8 $$f_\nu^{(\mu)}(x) = \frac{-\sqrt{1-x^2}}{2\mu x}\left[f_\nu^{(\mu+1)} + (\nu-\mu+1)(\nu+\mu)f_\nu^{(\mu-1)}\right] \qquad f = \text{P or Q}$$

a result that may be reorganized into an order-recursion relationship. Likewise the top row of the table provides a degree-recursion formula, generalizing equation 59:5:5.

There are many formulas that connect associated Legendre functions with Gauss hypergeometric functions and thus facilitate expansion. In particular,

59:12:9 $$P_\nu^{(m)}(x) = \frac{(\nu-m+1)_{2m}}{(-2)^m m!}\left(1-x^2\right)^{m/2} F\left(m-\nu, \nu+m+1, m+1, \frac{1-x}{2}\right)$$

$\mathrm{f}_{\nu-1}^{(\mu)}(x) \quad \mathrm{f}=\mathrm{P}$ or Q	$\mathrm{f}_{\nu+1}^{(\mu)}(x) \quad \mathrm{f}=\mathrm{P}$ or Q	$\mathrm{f}_{\nu}^{(\mu-1)}(x) \quad \mathrm{f}=\mathrm{P}$ or Q	$\mathrm{f}_{\nu}^{(\mu+1)}(x) \quad \mathrm{f}=\mathrm{P}$ or Q
$\dfrac{\nu+\mu}{(2\nu+1)x}$	$\dfrac{\nu-\mu+1}{(2\nu+1)x}$		
$\dfrac{1}{x}$		$\dfrac{-(\nu-\mu+1)\sqrt{1-x^2}}{x}$	
$\dfrac{\nu+\mu}{(\nu-\mu)x}$			$\dfrac{\sqrt{1-x^2}}{(\nu-\mu)x}$
	$\dfrac{1}{x}$	$\dfrac{-(\nu+\mu)\sqrt{1-x^2}}{x}$	
	$\dfrac{\nu-\mu+1}{(\nu+\mu+1)x}$		$\dfrac{-\sqrt{1-x^2}}{(\nu+\mu+1)x}$
		$\dfrac{-(\nu-\mu+1)(\nu+\mu)\sqrt{1-x^2}}{2\mu x}$	$\dfrac{-\sqrt{1-x^2}}{2\mu x}$

and

59:12:10
$$\mathrm{Q}_\nu^{(m)}(x)=\frac{(-)^m\sqrt{\pi}\,\Gamma(1+\nu+m)}{2^{\nu+1}\Gamma\left(\frac{3}{2}+\nu\right)}\frac{\left(x^2-1\right)^{m/2}}{x^{1+\nu+m}}\mathrm{F}\left(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m,1+\tfrac{1}{2}\nu+\tfrac{1}{2}m,\tfrac{3}{2}-\tfrac{1}{2}m,\frac{1}{x^2}\right)$$

are the analogues of equations 59:6:1 and 59:6:2, to which they reduce when $m=0$.

Some examples of associated Legendre functions in which both the degree and the order are positive integers are

$\mathrm{P}_1^{(1)}(x)=\mathrm{P}_{-2}^{(1)}(x)=-\sqrt{1-x^2}$	$\mathrm{Q}_1^{(1)}(x)=\dfrac{-x}{\sqrt{1-x^2}}-\sqrt{1-x^2}\,\mathrm{artanh}(x)$
$\mathrm{P}_2^{(1)}(x)=\mathrm{P}_{-3}^{(1)}(x)=-3x\sqrt{1-x^2}$	$\mathrm{Q}_2^{(1)}(x)=\dfrac{2-3x^2}{\sqrt{1-x^2}}-3x\sqrt{1-x^2}\,\mathrm{artanh}(x)$
$\mathrm{P}_2^{(2)}(x)=\mathrm{P}_{-3}^{(2)}(x)=3\left(1-x^2\right)$	$\mathrm{Q}_2^{(2)}(x)=\dfrac{5x-3x^3}{1-x^2}+3\left(1-x^2\right)\mathrm{artanh}(x)$

Note that these are rarely polynomials, though they sometimes go by the misnomer "associated Legendre polynomials". $\mathrm{P}_n^{(m)}(x)=0$ whenever $|m|$ exceeds the larger of n and $-n-1$, a result that has consequence in Section 59:14.

The associated Legendre function of the first kind satisfies the orthogonality relationship [Section 21:14]

59:12:11
$$\int_{-1}^{1}\mathrm{P}_n^{(m)}(t)\mathrm{P}_N^{(m)}(t)\,\mathrm{d}t=\begin{cases}0 & N\neq n\\[4pt] \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & N=n\end{cases}$$

Operative for arguments between −1 and 1, for $|\nu| \le 150$, and for integer m not exceeding 150 in magnitude, *Equator* provides an associated Legrendre function of the first kind routine and an associated Legendre function of the second kind routine (keywords **assocP** and **assocQ**). For ν not less than −½ and positive m, these algorithms generally utilize the following double series

59:12:12
$$P_\nu^{(m)}(x) = \frac{2^m}{(1-x^2)^{m/2}\sqrt{\pi}}\left[\begin{array}{l}\cos\left(\dfrac{\nu+m}{2}\pi\right)\dfrac{\Gamma(\frac{1+\nu+m}{2})}{\Gamma\left(\frac{2+\nu-m}{2}\right)}F\left(\dfrac{-\nu-m}{2},\dfrac{1+\nu-m}{2},\dfrac{1}{2},x^2\right)\\ \qquad +2x\sin\left(\dfrac{\nu+m}{2}\pi\right)\dfrac{\Gamma(\frac{2+\nu+m}{2})}{\Gamma\left(\frac{1+\nu-m}{2}\right)}F\left(\dfrac{1-\nu-m}{2},\dfrac{2+\nu-m}{2},\dfrac{3}{2},x^2\right)\end{array}\right]$$

and

59:12:13
$$Q_\nu^{(m)}(x) = \frac{2^{m-1}\sqrt{\pi}}{(1-x^2)^{m/2}}\left[\begin{array}{l}-\sin\left(\dfrac{\nu+m}{2}\pi\right)\dfrac{\Gamma(\frac{1+\nu+m}{2})}{\Gamma\left(\frac{2+\nu-m}{2}\right)}F\left(\dfrac{-\nu-m}{2},\dfrac{1+\nu-m}{2},\dfrac{1}{2},x^2\right)\\ \qquad +2x\cos\left(\dfrac{\nu+m}{2}\pi\right)\dfrac{\Gamma(\frac{2+\nu+m}{2})}{\Gamma\left(\frac{1+\nu-m}{2}\right)}F\left(\dfrac{1-\nu-m}{2},\dfrac{2+\nu-m}{2},\dfrac{3}{2},x^2\right)\end{array}\right]$$

but, whenever equation 59:12:9 proves to provide answers with more precision, it is substituted. Formula 59:12:4 or 59:12:5 is employed, where necessary, if the order or degree is negative.

59:13 COGNATE FUNCTIONS: certain Gauss hypergeometric functions

When the variable x in Legendre's differential equation 59:3:18 is replaced by $X = (1-x)/2$, the result may be written as

59:13:1
$$X(1-X)\frac{d^2f}{dX^2} + [1-\{(-\nu)+(1+\nu)+1\}X]\frac{df}{dX} - (-\nu)(1+\nu)f = 0$$

Compare this with Gauss's differential equation 60:3:7: the two are identical if $a = -\nu$, $b = 1+\nu$, and $c = 1$. It follows that a solution to Legendre's differential equation is

59:13:2
$$f = F\left(-\nu,1+\nu,1,\frac{1-x}{2}\right)$$

This is the result that is identified with $P_\nu(x)$ in 59:6:1. Evidently the first kind of Legendre function is a particular Gauss hypergeometric function. This latter function is addressed in some detail in the next chapter; here the focus is on how the Legendre functions are expressible as Gauss functions.

Now set $g = x^{1+\nu}f$, where f is the dependent variable in Legendre's differential equation 59:3:8. Thereby that equation becomes

59:13:3
$$\left(x^4-x^2\right)\frac{d^2g}{dx^2} + 2x\left(1+\nu-\nu x^2\right)\frac{dg}{dx} + (1+\nu)(2+\nu)g = 0$$

and if the independent variable is now replaced by $\chi = x^{-2}$, the differential equation adopts a form that can be rewritten

59:13:4
$$\chi(1-\chi)\frac{d^2g}{d\chi^2} + \left[\left(\tfrac{3}{2}+\nu\right) - \left\{\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)+\left(1+\tfrac{1}{2}\nu\right)+1\right\}\chi\right]\frac{dg}{d\chi} - \left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\left(1+\tfrac{1}{2}\nu\right)g = 0$$

This equation again identifies with the Gauss hypergeometric differential equation 60:3:7, this time if the choices

$a = (1+\nu)/2$, $b = (2+\nu)/2$, and $c = (3+2\nu)/2$ are made. We therefore have

59:13:5 $$f = x^{-1-\nu}g = x^{-1-\nu}\,F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$$

as a second solution to the Legendre differential equation. This is the expression ascribed to $Q_\nu(x)$ in equation 59:6:2, apart from a constant factor of $\sqrt{\pi}\,\Gamma(1+\nu)/[2^{1+\nu}\Gamma(\tfrac{3}{2}+\nu)]$.

In summary, particular Gauss hypergeometric functions solve Legendre's differential equation:

59:13:6 $$\left(1-x^2\right)\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + \nu(\nu+1)f = 0 \qquad f = w_1 F\left(-\nu, 1+\nu, 1, \tfrac{1}{2}-\tfrac{1}{2}x\right) + \frac{w_2}{x^{1+\nu}}F\left(1+\tfrac{1}{2}\nu, \tfrac{1}{2}+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$$

Legendre functions are nothing but these Gauss functions "in disguise", with suitable weighting factors, as prescribed in equations 59:6:1 and 59:6:2. Gauss hypergeometric functions are unusually flexible, with the great number of intrarelationships described in Section 60:5. This flexibility passes on to Legendre functions, as the tabulation below amply demonstrates. This lists examples of hypergeometric representations of each kind of Legendre function.

$P_\nu(x)$	$Q_\nu(x)$
$F\left(-\nu, 1+\nu, 1, \tfrac{1}{2}-\tfrac{1}{2}x\right)$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)[2x]^{1+\nu}}F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$
$\left[\dfrac{x+1}{2}\right]^\nu F\left(-\nu, -\nu, 1, \dfrac{x-1}{x+1}\right)$	$\dfrac{2^{-1-\nu}\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)[x+1]^{1+\nu}}F\left(1+\nu, 1+\nu, 2+2\nu, \dfrac{2}{1+x}\right)$
$F\left(\tfrac{-1}{2}\nu, \tfrac{1}{2}+\tfrac{1}{2}\nu, 1, 1-x^2\right)$	$\dfrac{-\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)}\sqrt{\dfrac{\pi\left(x-\sqrt{x^2-1}\right)^{1+2\nu}}{2\sqrt{x^2-1}}}\,F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}+\nu, \dfrac{\sqrt{x^2-1}-x}{2\sqrt{x^2-1}}\right)$
$\left[x+\sqrt{x^2-1}\right]^\nu F\left(-\nu, \tfrac{1}{2}, 1, \dfrac{2\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)\left[x+\sqrt{x^2-1}\right]^{1+\nu}}F\left(\tfrac{1}{2}, 1+\nu, \tfrac{3}{2}+\nu, \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$
$x^\nu F\left(\tfrac{-1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\nu, 1, \dfrac{x^2-1}{x^2}\right)$	$\dfrac{2^{-1-\nu}\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)\left[x^2-1\right]^{(1+\nu)/2}}F\left(1+\nu, 1+\nu, \tfrac{3}{2}+\nu, \dfrac{\sqrt{x^2-1}-x}{2\sqrt{x^2-1}}\right)$

In consequence of the reflection formula 59:5:5, additional entries may be added to the first (only!) column by replacing each ν by $-\nu-1$. The domains of the tabulated Gauss hypergeometric functions are restricted, because the argument of an $F(a,b,c,x)$ function is limited to $-\infty < x < 1$ (sometimes $x = 1$ is admissible). Note, however, that though the Gauss hypergeometric function is defined for $-\infty < x < 1$, the corresponding series, equation 60:3:1, requires $-1 < x < 1$ for convergence.

The table above shows ways in which a Legendre function may be expressed as a *single* Gauss hypergeometric function. There are very many other representations as sums of two or more such functions; Section 60:5 provides routes for creating these.

59:14 RELATED TOPIC: solving Laplace's equation in spherical coordinates

Functions that satisfy Laplace's equation [Section 46:15] are known as *harmonic functions*. Legendre functions satisfy this equation when spherical coordinates [Section 46:14] are employed, explaining why "spherical harmonic" is often considered synonymous with "Legendre function".

In Section 46:14, the Laplacian operator was presented in several coordinate systems. Of course, the spherical coordinates are the preferred means of indexing three-dimensional space for studies involving the interior or exterior of spheres, or portions thereof. Laplace's equation in these coordinates is

59:14:1 $$\frac{\partial^2 F}{\partial r^2}+\frac{2}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2}+\frac{\cot(\phi)}{r^2}\frac{\partial F}{\partial \phi}+\frac{\csc^2(\phi)}{r^2}\frac{\partial^2 F}{\partial \theta^2}=0$$

where F is the pertinent scalar quantity (temperature, potential, or the like). If coordinate separability is asserted, so that $F(r,\phi,\theta)=R(r)\Phi(\phi)\Theta(\theta)$, then we may trisect equation 59:14:1 in a manner strictly analogous to that used in Section 46:15, arriving at three separated equations. The first of these, addressing the dependence of F on the radial coordinate, is

59:14:2 $$\frac{d^2 R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\nu(\nu+1)}{r^2}R=0$$

where $\nu(\nu+1)$ is a conveniently formulated separation constant. The solution of this equation is mentioned in Section 32:13 but will not concern us here. The second separated equation, expressing how F is affected by the longitude is

59:14:3 $$\frac{d^2\Theta}{d\theta^2}=-\mu^2\Theta$$

with μ^2 being the new separation constant. The third separated equation, which incorporates both separation constants, conveys the dependence of F on the latitudinal angle ϕ and is

59:14:4 $$\frac{d^2\Phi}{d\phi^2}+\cot(\phi)\frac{d\Phi}{d\phi}+\left[\nu(\nu+1)-\mu^2\csc^2(\phi)\right]\Phi=0$$

Each of the three separated equations is a typical second-order ordinary differential equation, solutions of which are available through the procedures described in Section 24:14. Our prime interest here is in the solution of 59:14:4. Because this equation is identical, apart from notation, with 59:12:3, its solution is a weighted sum of associated Legendre functions:

59:14:5 $$\Phi(\phi)=w_1 P_\nu^{(\mu)}\left(\cos(\phi)\right)+w_2 Q_\nu^{(\mu)}\left(\cos(\phi)\right)$$

When the region of interest is the *surface* of a sphere, equation 59:14:2 is not pertinent and the overall solution is a composite of solutions of the latitudinal and longitudinal separated equations, 59:14:4 and 59:14:3, with arbitrary weighting factors

59:14:6 $$F(\phi,\theta)=\Phi(\phi)\Theta(\theta)=\left[w_5 P_\nu^{(\mu)}\left(\cos(\phi)\right)+w_6 Q_\nu^{(\mu)}\left(\cos(\phi)\right)\right]\left[w_3\sin(\mu\theta)+w_4\cos(\mu\theta)\right]$$

One can argue from the geometry of a sphere that μ must be an integer (as otherwise $\sin\{\mu(\theta+2\pi)\}$ would not equal $\sin\{\mu\theta\}$, as it must). In most physical situations, w_6 must be zero (as otherwise, F would be infinite at $\phi=0$, the "north pole"). Because $\theta=\pi$ corresponds to another point on the sphere's surface (the "south pole", in fact) and $P_\nu^{(m)}(-1)$ is infinite for all noninteger values of ν, we are forced to conclude that ν must also be an integer (which we can treat as nonnegative because P_{-n} merely duplicates P_{n-1}). As noted in the preceding section, the associated Legendre functions $P_n^{(m)}$ vanish for $|m|>n$. These various considerations permit us to simplify 59:14:6 to

59:14:7 $$F(\phi,\theta)=P_n^{(m)}\left(\cos(\phi)\right)\left[w_1\sin(m\theta)+w_2\cos(m\theta)\right]\qquad n=0,1,2,\cdots\qquad m=-n,-n+1,\cdots,n$$

Depending on the application, the n and m numbers may be referred to as *eigenvalues* or *quantum numbers*. The remaining weighting factors, w_1 and w_2, are independent of ϕ and θ but will generally depend on n and m, so that $w_1(n,m)$ and $w_2(n,m)$ is a more informative symbolism. A large, possibly an infinite, number of weights might be needed in a complete solution that matches the boundary conditions of a physical problem, but we shall not pursue any particular application.

To summarize, the equation

59:14:8
$$F(\phi,\theta) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^{(m)}(\cos(\phi))\left[w_1(n,m)\sin(m\theta) + w_2(n,m)\cos(m\theta)\right]$$

provides a general solution to Laplace's equation on the surface of a sphere. The components of this solution, that is the terms $\sin(m\theta)\,P_n^{(m)}(\cos(\phi))$ and $\cos(m\theta)\,P_n^{(m)}(\cos(\phi))$, possibly with normalizing multipliers, are known as *surface harmonics* or sometimes "spherical harmonics".

CHAPTER 60

THE GAUSS HYPERGEOMETRIC FUNCTION F(a, b, c, x)

In Section 18:14 a hypergeometric function is defined as any function that is expansible as the (usually infinite) series

60:0:1 $$1+\frac{a_1a_2a_3\cdots a_K}{c_1c_2c_3\cdots c_L}x+\frac{(a_1)_2(a_2)_2(a_3)_2\cdots(a_K)_2}{(c_1)_2(c_2)_2(c_3)_2\cdots(c_L)_2}x^2+\cdots+\frac{(a_1)_j(a_2)_j(a_3)_j\cdots(a_K)_j}{(c_1)_j(c_2)_j(c_3)_j\cdots(c_L)_j}x^j+\cdots$$

where K and L are nonnegative integers. The jth term of this series incorporates $(K+L)$ Pochhammer polynomials [Chapter 18] of degree j. Name for the renowned German physicist and mathematician Carl Friedrich Gauss (1777 − 1855), the Gauss hypergeometric function is the special case of series 60:0:1 in which K and L each equal 2, with one of the c parameters being constrained to be unity. Thus the Gauss hypergeometric function is quadrivariate. Though it shares the properties of the general hypergeometric function [Section 18:14], the Gauss version has additional unique features.

60:1 NOTATION

Although a_1 and a_2 occasionally replace them, symbols a and b are general for the *numeratorial parameters* of the Gauss hypergeometric function, with c being used to denote the single unconstrained *denominatorial parameter*; x is the argument. The immutable denominatorial parameter is understood to be unity and does not appear explicitly in the usual F(a,b,c,x) notation.

The doubly subscripted symbolism ${}_2F_1(a,b,c,x)$ is often encountered, the "2" and "1" serving as reminders of the numbers of adjustable parameters appearing as Pochhammer polynomials in the numerator and denominator of each term in the power series expansion. A varied punctuation, F(a,b;c;x) or F(a;b|c|x), may be employed as a means of emphasizing the interchangeability only of the a and b variables. Other notations for the Gauss function are

60:1:1 $${}_2F_1\left[\begin{matrix}a,b;x\\c\end{matrix}\right] \qquad \text{and} \qquad \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\left[x\,\frac{a-1,b-1}{0,c-1}\right]$$

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60:2 BEHAVIOR

Generally the Gauss hypergeometric function is defined only for those real values of the argument that lie in the $-\infty < x < 1$ domain ; $x = 1$ is included if $c > (a+b)$. However, if one of the four quantities a, b, $c-a$, or $c-b$ is a nonpositive integer, $F(a,b,c,x)$ may be reduced to a polynomial, in which event one of equations 60:4:10 or 60:4:11 (or their analogues with a and b interchanged) applies and x is unrestricted. Nonpositive integer values of the c parameter are generally forbidden [though see Section 60:4], but otherwise these variables may adopt any real value.

Apart from equaling unity when its argument is zero, the behavior of the Gauss hypergeometric function is so dependent on its three parameters that no useful general description can be given. Nor is any graphical depiction feasible for a quadrivariate function.

60:3 DEFINITIONS

The most commonly encountered definition is as a summation that may be written in terms of the gamma function or, more economically, through Pochhammer polynomials

60:3:1 $$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(j+a)\Gamma(j+b)}{\Gamma(c)}\frac{x^j}{j!} = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j(1)_j}x^j \qquad |x| \begin{cases} <1 & c \le a+b \\ \le 1 & c > a+b \end{cases}$$

The restriction on the magnitude of x may be discarded if the series terminates, as it does if a or b is a nonpositive integer. Transformation 60:5:3 permits a more extensive domain of definition

60:3:2 $$F(a,b,c,x) = (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(c-b)_j}{(c)_j(1)_j}\left(\frac{x}{x-1}\right)^j \qquad -\infty < x < \tfrac{1}{2}$$

As always, the roles of a and b in 60:3:2 may be interchanged.

Several definite integrals can serve as definitions of the Gauss hypergeometric function. These include

60:3:3 $$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}}{(1-t)^{1+b-c}(1-xt)^a}\,dt \qquad c > b > 0 \qquad x < 1$$

60:3:4 $$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^\infty \frac{(t-1)^{c-b-1}t^{a-c}}{(t-x)^a}\,dt \qquad (1+a) > c > b \qquad x < 1$$

and

60:3:5 $$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty \frac{t^{b-1}}{(1+t)^{c-a}(1+t-xt)^a}\,dt \qquad c > b > 0 \qquad x < 1$$

A Gauss hypergeometric function may be synthesized from a binomial function [Section 6:14]:

60:3:6 $$(1-x)^{-a} \xrightarrow[c]{b} F(a,b,c,x) \qquad -1 < x < 1$$

The notation is described in Section 43:14. An arbitrarily weighted sum of two Gauss hypergeometric functions solves the *hypergeometric differential equation*

60:3:7 $$x(1-x)\frac{d^2 f}{dx^2} + \left[c-(1+a+b)x\right]\frac{df}{dx} - ab f = 0 \qquad f = w_1 F(a,b,c,x) + w_2 F(a+c-1,b+c-1,2-c,x)$$

The a and b parameters may be interchanged in all the formulas in this section and, indeed, throughout the chapter.

60:4 SPECIAL CASES

A substantial fraction of the functions treated in this *Atlas* are instances of the Gauss hypergeometric function; a random selection of these, including functions from Chapters 22, 32, 35, 61, and 58, is:

$F(-n,n,\tfrac{1}{2},x)$	$F(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},x)$	$F(\tfrac{1}{2},\tfrac{1}{2},1,k^2)$	$F(\mu,1,\nu,x)$	$F(-\nu,\tfrac{1}{2}-\nu,\tfrac{1}{2},-x)$
$T_n(1-2x)$	$\dfrac{1}{\sqrt{x}}\arcsin\left(\sqrt{x}\right)$	$\dfrac{2}{\pi}K(k)$	$\dfrac{[\nu-1]B(\nu-1,\mu-\nu+1,x)}{(1-x)^{\mu-\nu+1}\,x^{\nu-1}}$	$(1+x)^{\nu}\cos\left\{2\nu\arctan\left(\sqrt{x}\right)\right\}$

There are many circumstances in which a Gauss hypergeometric function simplifies to an algebraic function. A case in point is

60:4:1 $$F\left(\frac{\nu+1}{2},\frac{\nu}{2}+1,\nu+1,-x\right)=\left(\frac{2}{x}\right)^{\nu}\frac{\left(\sqrt{1+x}-1\right)^{\nu}}{\sqrt{1+x}}$$

and many others will be found in Table 18-2. Because

60:4:2 $$F(a,1,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j$$

all the functions in Table 18-1 are also Gauss hypergeometric functions.

Certain relationships between the numeratorial parameters, namely

60:4:3 $$b=a-\tfrac{1}{2}\quad\text{or}\quad a+\tfrac{1}{2}\quad\text{or}\quad 1-a$$

or among the denominatorial parameter and the numeratorial parameters:

60:4:4 $$c=2a\quad\text{or}\quad 2b\quad\text{or}\quad a+b+\tfrac{1}{2}\quad\text{or}\quad a+b-\tfrac{1}{2}\quad\text{or}\quad \frac{a+b+1}{2}\quad\text{or}\quad 1+a-b\quad\text{or}\quad 1+b-a$$

cause the quadrivariate Gauss hypergeometric function to reduce to the trivariate *associated Legendre function* [Section 59:13]. The equivalences are all of the form

60:4:5 $$F(a,b,c,x)=f(x)P_{\nu}^{(\mu)}(X)$$

where ν and μ reflect two of the three parameters a, b and c, while X depends only on the argument x. The function $f(x)$ incorporates parameters as well as powers of x and $(1-x)$. Of the ten possibilities specified in 60:4:3 and 60:4:4, seven are disclosed in Table 60-1 as the first seven entries; the three omissions are readily derived by interchanging a and b in the asterisked entries. Further reduction of the associated Legendre function may occur, as detailed in Section 59:12.

The final two entries in Table 60-1 provide details of formulas that apply when the denominatorial parameter equals $\tfrac{1}{2}$ or $\tfrac{3}{2}$, in which event

60:4:6 $$F\left(a,b,1\mp\tfrac{1}{2},x\right)=f(x)\left[P_{\nu}^{(\mu)}(X)\pm P_{\nu}^{(\mu)}(-X)\right]$$

Of course, the validity of the formulas embodied in Table 60-1 requires that both the Gauss hypergeometric function and the associated Legendre function(s) be well defined. This generally means $x<1$ and $X>-1$.

The *incomplete beta function* [Chapter 58] is another trivariate function that is a common special case of the Gauss hypergeometric function. If either of the numeratorial parameters, say b, equals unity, then there is cancellation with the phantom denominatorial $(1)_j$, leading to

60:4:7 $$F(a,1,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j(1)_j}{(c)_j(1)_{j)}}x^j=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j=\frac{c-1}{x^{c-1}(1-x)^{1+a-c}}B(c-1,1+a-c,x)$$

Table 60-1

Relation	f(x)	v	μ	X
$b=a-\tfrac{1}{2}$ *	$\dfrac{\Gamma(c)}{2^{1-c}}\sqrt{\lvert x\rvert^{1-c}(1-x)^{c-2a}}$	$2a-c-1$	$1-c$	$\dfrac{1}{\sqrt{1-x}}$
$b=1-a$	$\Gamma(c)\sqrt{\left[\dfrac{\lvert x\rvert}{1-x}\right]^{1-c}}$	$-a$	$1-c$	$1-2x$
$c=2a$ *	$2^{2a-1}\Gamma\left(a+\tfrac{1}{2}\right)\sqrt{\dfrac{x^{1-2a}}{(1-x)^{b-a+\frac{1}{2}}}}$	$b-a-\tfrac{1}{2}$	$\tfrac{1}{2}-a$	$\dfrac{2-x}{2\sqrt{1-x}}$
$c=a+b-\tfrac{1}{2}$	$2^{a+b-\frac{3}{2}}\Gamma\left(a+b-\tfrac{1}{2}\right)\sqrt{\dfrac{\lvert x\rvert^{\frac{3}{2}-a-b}}{1-x}}$	$b-a-\tfrac{1}{2}$	$\tfrac{3}{2}-a-b$	$\sqrt{1-x}$
$c=a+b+\tfrac{1}{2}$	$2^{a+b-\frac{1}{2}}\Gamma\left(a+b+\tfrac{1}{2}\right)\sqrt{\lvert x\rvert^{\frac{1}{2}-a-b}}$	$a-b-\tfrac{1}{2}$	$\tfrac{1}{2}-a-b$	$\sqrt{1-x}$
$c=\dfrac{a+b+1}{2}$	$\Gamma\left(\dfrac{a+b+1}{2}\right)\sqrt{\lvert x-x^2\rvert^{(1-a-b)/2}}$	$\dfrac{a-b-1}{2}$	$\dfrac{1-a-b}{2}$	$1-2x$
$c=a-b+1$ *	$\Gamma(a-b+1)\sqrt{\dfrac{\lvert x\rvert^{b-a}}{(1-x)^{2b}}}$	$-b$	$b-a$	$\dfrac{1+x}{1-x}$
$c=\tfrac{1}{2}$	$\dfrac{\Gamma\left(a+\tfrac{1}{2}\right)\Gamma\left(b+\tfrac{1}{2}\right)}{2^{\frac{3}{2}-a-b}\sqrt{\pi}}\sqrt{\dfrac{1}{\lvert 1-x\rvert^{a+b-\frac{1}{2}}}}$	$a-b-\tfrac{1}{2}$	$\tfrac{1}{2}-a-b$	$\sqrt{\lvert x\rvert}$
$c=\tfrac{3}{2}$	$\dfrac{-\Gamma\left(a-\tfrac{1}{2}\right)\Gamma\left(b-\tfrac{1}{2}\right)}{2^{\frac{7}{2}-a-b}\sqrt{\pi x}}\sqrt{\dfrac{1}{\lvert 1-x\rvert^{a+b-\frac{3}{2}}}}$	$a-b-\tfrac{1}{2}$	$\tfrac{3}{2}-a-b$	$\sqrt{x}$

If one of the numeratorial parameters is less by unity than the denominatorial c parameter, then, for example

60:4:8 $$\mathrm{F}(a,c-1,c,x)=\frac{c-1}{x^{c-1}}\mathrm{B}\left(c-1,1-a,\,x\right)$$

Powers, polynomials, inverse hyperbolic functions or inverse trigonometric functions may arise by further specialization of the incomplete beta functions as detailed in Chapter 58.

If either of the numeratorial parameters, say b, equals the denominatorial parameter, then there is cancellation resulting in

60:4:9 $$\mathrm{F}(a,c,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j(c)_j}{(c)_j(1)_{j)}}x^j=\sum_{j=0}^{\infty}\frac{(a)_j}{(1)_j}x^j=\frac{1}{(1-x)^a}$$

If either of the numeratorial parameters exceeds c by a positive integer, for example if $b-c=n=1,2,3,\cdots$, then the Gauss hypergeometric function reduces to a binomial function multiplied by a polynomial, specifically a *Jacobi polynomial* [Section 22:12] of degree n:

60:4:10 $$F(a,c+n,c,x)=\frac{1}{(1-x)^{a+n}}\sum_{j=0}^{n}\binom{n}{j}\frac{(c-a)_j}{(c)_j}(-x)^j=\frac{n!P_n^{(c-1,-a-n)}(1-2x)}{(c)_n(1-x)^{a+n}}$$

A similar reduction occurs if either of the numeratorial parameters is a nonpositive integer. If, for example, $b=-n$ where $n=0,1,2,3,\cdots$, then

60:4:11 $$F(a,-n,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j(-n)_j}{(c)_j(1)_j}x^j=\sum_{j=0}^{n}\binom{n}{j}\frac{(a)_j}{(c)_j}(-x)^j=\frac{n!P_n^{(c-1,a-n-c)}(1-2x)}{(c)_n}$$

The specialization to an nth degree polynomial that is manifested in equation 60:4:11 can be attributed to the property

60:4:12 $$(-n)_j=\begin{cases}1 & j=0\\ (-n)(-n+1)(-n+2)\cdots(-n+j-1) & 1\le j\le n=2,3,4,\cdots\\ 0 & j\ge n+1\end{cases}$$

of the numeratorial Pochhammer polynomial. This introduces a zero multiplier into all terms in the expansion of the Gauss hypergeometric function for which j exceeds n. If c were to equal a nonpositive integer, a similarly arising zero would eventually occur as a divisor in the $F(a,b,c,x)$ series and it is this consideration that leads to the exclusion of nonpositive integer c values from the domain of the Gauss hypergeometric function prescribed in Section 60:2. Nevertheless, negative integer values are open to c, provided that either (or both) a or b is zero or a less negative integer, in which cases equation 60:4:10 holds.

60:5 INTRARELATIONSHIPS

The Gauss hypergeometric function is symmetrical with respect to the interchange of its two numeratorial parameters:

60:5:1 $$F(b,a,c,x)=F(a,b,c,x)$$

There are two important transformations. The first

60:5:2 $$F(a,b,c,x)=(1-x)^{c-a-b}F(c-a,c-b,c,x)$$

relates two Gauss hypergeometric functions of common argument, whereas the second

60:5:3 $$F(a,b,c,x)=(1-x)^{-a}F\left(a,c-b,c,\frac{x}{x-1}\right)=(1-x)^{-b}F\left(c-a,b,c,\frac{x}{x-1}\right)$$

links a Gauss function with argument in the domain $-1\le x<1$ to one with an argument between ½ and $-\infty$.

Each Gauss hypergeometric functions has six so-called *contiguous functions*; the contiguous functions of $F(a,b,c,x)$ are those shown in the header line of Table 60-2. $F(a,b,c,x)$ may be expressed as the weighted sum of any two of its contiguous functions: thus there are fifteen *contiguity relationships*. The weighting functions appropriate for nine of these fifteen are listed in the body of Table 60-2. The absent six would appear in the pair of columns marked by asterisks and they can be found by interchanging a and b in the entries that share the row with those asterisks. An example of how to use this table is provided by the fifth row, after an $a\rightleftarrows b$ interchange; thus

60:5:4 $$F(a,b,c,x)=\frac{b}{1+b-c}F(a,b+1,c,x)+\frac{1-c}{1+b-c}F(a,b,c-1,x)$$

Three of the contiguity relations in the table engender recursion formulas for the individual parameters. For

Table 60-2

$\mathrm{F}(a+1,b,c,x)$	$\mathrm{F}(a-1,b,c,x)$	$\mathrm{F}(a,b+1,c,x)$	$\mathrm{F}(a,b-1,c,x)$	$\mathrm{F}(a,b,c+1,x)$	$\mathrm{F}(a,b,c-1,x)$
$\dfrac{a(1-x)}{2a-c+(b-a)x}$	$\dfrac{a-c}{2a-c+(b-a)x}$	*	*		
$\dfrac{a}{a-b}$		$\dfrac{-b}{a-b}$			
$\dfrac{a(1-x)}{a+b-c}$	*	*	$\dfrac{b-c}{a+b-c}$		
$\dfrac{a(1-x)}{a-(c-b)x}$		*		$\dfrac{(c-a)(c-b)x}{c[(c-b)x-a]}$ *	
$\dfrac{a}{1+a-c}$		*			$\dfrac{1-c}{1+a-c}$ *
	$\dfrac{c-a}{(b-a)(1-x)}$		$\dfrac{b-c}{(b-a)(1-x)}$		
	$\dfrac{1}{1-x}$		*	$\dfrac{(b-c)x}{c(1-x)}$ *	
	$\dfrac{a-c}{a-1+(1+b-c)x}$		*		$\dfrac{(c-1)(1-x)}{a-1+(1+b-c)x}$ *
				$\dfrac{(c-a)(c-b)x}{c-c^2-(1+a+b-2c)cx}$	$\dfrac{(1-c)(1-x)}{1-c-(1+a+b-2c)x}$

example, the first tabular entry leads to

60:5:5 $$\mathrm{F}(a+1,b,c,x)=\frac{2a-c+(b-a)x}{a(1-x)}\mathrm{F}(a,b,c,x)-\frac{a-c}{a(1-x)}\mathrm{F}(a-1,b,c,x)$$

The fifteen contiguity relations provide relationships between three Gauss hypergeometric functions of common argument. In addition, a great number of relationships exist that link trios of these functions of dissimilar arguments. Three typical formulas of this sort are:

60:5:6 $$\frac{\mathrm{F}(a,b,c,x)}{\Gamma(c)}=\sum_{\lambda=a,}^{b}\frac{\Gamma(a+b-2\lambda)(-x)^{-\lambda}}{\Gamma(a+b-\lambda)\Gamma(c-\lambda)}\mathrm{F}\left(\lambda,\,\lambda-c+1,\,2\lambda-a-b+1,\frac{1}{x}\right)\qquad x<0$$

60:5:7 $$\frac{\mathrm{F}(a,b,c,x)}{\Gamma(c)}=\sum_{\lambda=a,}^{b}\frac{\Gamma(a+b-2\lambda)(1-x)^{-\lambda}}{\Gamma(a+b-\lambda)\Gamma(c-\lambda)}\mathrm{F}\left(\lambda,\,\lambda+c-a-b,\,2\lambda-a-b+1,\frac{1}{1-x}\right)\qquad x<1$$

60:5:8 $$\frac{\mathrm{F}(a,b,c,x)}{\Gamma(c)}=\sum_{\lambda=c,}^{a+b}\frac{\Gamma(a+b+c-2\lambda)(1-x)^{\lambda-a-b}}{\Gamma(a+c-\lambda)\Gamma(b+c-\lambda)}\mathrm{F}\left(\lambda-a,\,\lambda-b,\,2\lambda-a-b-c+1,1-x\right)\qquad x<1$$

The Σ notation in these three equations indicates that their right-hand members consist of two terms that differ only by λ adopting either of the two indicated values. These formulas may fail if the argument of one of the numeratorial gamma functions equals a nonpositive integer; see Abramowitz and Stegun [equations 15.3.10 – 14] for expansions that apply in these special circumstances.

With rare exceptions, such as those just noted, the formulas exhibited in this section are valid for all values of the a, b and c parameters. They are said to be *linear transformations*. There also exist *quadratic transformations*, which are applicable only when the parameters satisfy 60:4:3 or 60:4:4. For these, the interested reader is referred to the Bateman manuscript [Erdélyi et al., *Higher Transcendental Functions*, Section 2.9 and 2.11], where a comprehensive listing will be found and where *cubic transformations* are also addressed.

60:6 EXPANSIONS

The *Gauss series*

60:6:1 $$F(a,b,c,x)=1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{2c(c+1)}x^2+\frac{a(a+1)(a+3)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3+\cdots$$

is the prototype expansion and follow directly from definition 60:3:1. It may be rewritten as a concatenation

60:6:2 $$F(a,b,c,x)=1+\frac{abx}{c}\left(1+\frac{(a+1)(b+1)x}{2(c+1)}\left(1+\frac{(a+2)(b+2)x}{3(c+2)}\left(1+\cdots\right)\right)\right)$$

Moreover, each of the transformations detailed in Section 60:5, except for the first, leads to a different series expansion, often with a different domain of convergence. Thus, for example, whereas expansion 60:6:1 generally requires $|x|<1$, the series

60:6:3 $$\frac{F(a,b,c,x)}{(1-x)^{-a}}=1+\frac{a(c-b)}{c}\left(\frac{x}{x-1}\right)+\frac{a(a+1)(c-b)(c-b+1)}{2c(c+1)}\left(\frac{x}{x-1}\right)^2+\cdots$$

converges for any argument less than ½. Likewise, the double series developed from transformation 60:5:7 is convergent for all $x<1$.

60:7 PARTICULAR VALUES

$F(a,b,c,-\infty)$		$F(a,b,c,0)$	$F(a,b,c,1)$	
0	$a<0$ and $b<0$	undefined if $c=0$ and $ab\neq 0$	$\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$	$c>a+b$
1	a or $b=0$, the other <0	otherwise 1	otherwise undefined	
∞	a or $b>0$			

In addition to those noted in the panel above, which apply generally, the following particular values apply when the parameters are interrelated in specific ways:

60:7:1 $$F\left(a,b,\frac{1+a+b}{2},\tfrac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}\qquad a+b\neq -1,-3,-5,\cdots$$

60:7:2 $$\mathrm{F}\left(a,1-a,c,\tfrac{1}{2}\right)=\frac{2^{1-c}\sqrt{\pi}\,\Gamma(c)}{\Gamma\left(\dfrac{a+c}{2}\right)\Gamma\left(\dfrac{1-a+c}{2}\right)}\qquad c\neq 0,-1,-2,\cdots$$

60:7:3 $$\mathrm{F}\left(a,b,1+a-b,-1\right)=\frac{2^{-a}\sqrt{\pi}\,\Gamma(1+a-b)}{\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)}\qquad a-b\neq -1,-2,-3,\cdots$$

60:8 NUMERICAL VALUES

Equator's Gauss hypergeometric function routine (keyword **F**) returns values for wide domains of the parameters a, b, and c, and for most values of x not exceeding unity. Truncated versions of four series or double series are used according to the magnitude of the argument: 60:5:8 for $0.8\le x<1$, 63:3:1 for $-0.25\le x\le 0.8$, 60:3:2 for $-5<x<-0.25$, and 60:5:6 for $x\le -5$. Where appropriate, advantage is taken of the truncation corrections 60:9:5 and 60:9:6. In all cases, a and b are first interchanged if $a>b$.

60:9 LIMITS AND APPROXIMATIONS

As any one of its parameters approaches infinity, the Gauss hypergeometric function becomes one of the confluent hypergeometric functions discussed in Chapters 47 and 48:

60:9:1 $$\lim_{a\to\infty}\left\{\mathrm{F}(a,b,c,x\right\}=\mathrm{M}(b,c,ax)$$

60:9:2 $$\lim_{b\to\infty}\left\{\mathrm{F}(a,b,c,x\right\}=\mathrm{M}(a,c,bx)$$

60:9:3 $$\lim_{c\to\infty}\left\{\mathrm{F}(a,b,c,x\right\}=\left(\frac{c}{1-x}\right)^{a}\mathrm{U}\left(a,1+a-b,\frac{c}{1-x}\right)=\left(\frac{c}{1-x}\right)^{b}\mathrm{U}\left(1-a+b,b,\frac{c}{1-x}\right)$$

Limits 60:9:1 and 60:9:2 are quantitatively useful only if x is negative, while 60:9:3 is of dubious quantitative utility.

When the argument x is close to unity, the summation in 60:3:1 is notoriously slow to converge. In this circumstance, the approximation

60:9:4 $$\frac{(a)_j(b)_j}{(c)_j(1)_j}\approx 1+\frac{a+b-c-1}{j}\qquad \text{large } j$$

is useful in providing an estimate of the truncation error. It lead from definition 60:3:1 to the formula

60:9:5 $$\mathrm{F}(a,b,c,x)\approx\sum_{j=0}^{J-1}\frac{(a)_j(b)_j}{(c)_j(1)_j}x^j+\frac{(a)_J(b)_J x^J}{(c)_J(1)_J(1-x)}\left[1+\frac{(a+b-c-1)x}{J(1-x)}\right]\qquad J>\left|\frac{(a+b-c-1)x}{1-x}\right|$$

A similar truncation approximation, applied to series 60:3:2 leads to

60:9:6 $$\frac{\mathrm{F}(a,b,c,x)}{(1-x)^{-a}}\approx\sum_{j=0}^{J-1}\frac{(a)_j(b-c)_j}{(c)_j(1)_j}\left(\frac{-x}{1-x}\right)^j+\frac{(a)_J(b-c)_J(-x)^J}{(c)_J(1)_J(1-x)^{J-1}}\left[1-\frac{a-b-1}{J}x\right]\qquad J>\left|(a-b-1)x\right|$$

60:10 OPERATIONS OF THE CALCULUS

The operations of differentiation

60:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(a,b,c,x)=\frac{ab}{c}\mathrm{F}(a+1,b+1,c+1,x)$$

multiple differentiation

60:10:2 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{F}(a,b,c,x)=\frac{(a)_j(b)_j}{(c)_j}\mathrm{F}(a+n,b+n,c+n,x)$$

and integration

60:10:3 $$\int_0^x \mathrm{F}(a,b,c,t)\,\mathrm{d}t=\frac{c-1}{(a-1)(b-1)}\big[\mathrm{F}(a-1,b-1,c-1,x)-1\big] \qquad a,b,c\neq 1$$

modify the parameters equally. Other operations, exemplified in the four equations that follow, allow selective modification of single parameters of the Gauss hypergeometric function.

60:10:4 $$x^{1-b}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^{n+b-1}\mathrm{F}(a,b,c,x)\right\}=(b)_n\mathrm{F}(a,b+n,c,x)$$

60:10:5 $$\frac{x^{1-c+b}}{(1-x)^{a+b-c-n}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{\frac{x^{n-1-b+c}}{(1-x)^{c-a-b}}\mathrm{F}(a,b,c,x)\right\}=(c-b)_n\mathrm{F}(a,b-n,c,x)$$

60:10:6 $$\frac{1}{(1-x)^{a+b-c-n}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{\frac{1}{(1-x)^{c-a-b}}\mathrm{F}(a,b,c,x)\right\}=\frac{(c-a)_n(c-b)_n}{(c)_n}\mathrm{F}(a,b,c+n,x)$$

60:10:7 $$x^{1+n-c}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^{c-1}\mathrm{F}(a,b,c,x)\right\}=(c-n)_n\mathrm{F}(a,b,c-n,x)$$

Some of the operations in this paragraph may be disallowed, such as when they create a nonpositive integer denominatorial parameter. The need to avoid complex algebra may restrict validity to $0<x<1$, but extension to $-\infty<x<1$ is often possible.

Differintegration [Section 12:14] of the product of a Gauss hypergeometric function and a power obeys the rule

60:10:8 $$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left\{x^\mu\mathrm{F}(a,b,c,x)\right\}=\frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\nu)}x^{\mu-\nu}\sum_{j=0}^{\infty}\frac{(a)_j(b)_j(1+\mu)_j}{(c)_j(1)_j(1+\mu-\nu)_j}x^j$$

and generally yields a $L=K=3$ hypergeometric function [Section 18:14]. Certain choices of ν and μ will, however, lead to parameter cancellation, resulting in a hypergeometric function with $L=K=2$, or even $L=K=1$, as exemplified in equations 60:10:11 and 60:10:12 below. In the notation introduced in Section 43:14, the process described by equation 60:10:8 would be written

60:10:9 $$\mathrm{F}(a,b,c,x)\ \xrightarrow[1+\mu-\nu]{1+\mu}\ \sum_{j=0}^{\infty}\frac{(a)_j(b)_j(1+\mu)_j}{(c)_j(1)_j(1+\mu-\nu)_j}x^j$$

and described as a "synthesis". Examples of syntheses starting from a Gauss hypergeometric function are

60:10:10 $$\mathrm{F}(a,2c-a+\tfrac{1}{2},c,x)\ \xrightarrow[2c-1]{c-\frac{1}{2}}\ \sum_{j=0}^{\infty}\frac{(a)_j(2c-a+\frac{1}{2})_j(c-\frac{1}{2})_j}{(c)_j(2c-1)_j(1)_j}x^j=\left[\mathrm{F}(\tfrac{1}{2}a,c-\tfrac{1}{2}a-\tfrac{1}{2},c,x)\right]^2$$

60:10:11 $$F(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},x)\ \xrightarrow[2]{1}\ \sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j(\frac{1}{2})_j}{(\frac{3}{2})_j(2)_j}x^j=\frac{2}{x}\left[\sqrt{x}\arcsin\left(\sqrt{x}\right)-1+\sqrt{1-x}\right]$$

and

60:10:12 $$F(a,b,c,x)\ \xrightarrow[a]{c}\ \sum_{j=0}^{\infty}\frac{(b)_j}{(1)_j}x^j=\frac{1}{(1-x)^b}$$

The Laplace transform of a Gauss hypergeometric function is an $L=K-1=1$ hypergeometric function:

60:10:13 $$\int_0^{\infty}F(a,b,c,t)\exp(-st)\,dt=\mathfrak{L}\{F(a,b,c,t)\}=\frac{1}{s}\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\left(\frac{1}{s}\right)^j$$

60:11 COMPLEX ARGUMENT

Generally $F(a,b,c,x+iy)$ is complex valued, but the *Atlas* does not pursue this topic.

The inverse Laplace transformation formula

60:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty}F(a,b,c,-s)\frac{\exp(st)}{2\pi i}\,ds=\mathfrak{I}\{F(a,b,c,-s)\}=t^{a-1}\exp(-t)\,U(c-b,a-b+1,t)$$

which is generally valid for for $s>-\frac{1}{2}$, yields a *Tricomi function* [Chapter 48]. Other formulas may be derived from 48:10:5.

60:12 GENERALIZATIONS

The (generalized) hypergeometric function detailed in Section 18:14 may be regarded as a generalization of the Gauss version.

There are several more profound generalizations of the Gauss hypergeometric function. These are discussed in Chapter 4 of *Higher Transcendental Functions* [Erdélyi et al.].

60:13 COGNATE FUNCTIONS

The *rationalized Gauss hypergeometric function* or *regularized Gauss hypergeometric function* is defined as

60:13:1 $$\frac{F(a,b,c,x)}{\Gamma(c)}=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!\,\Gamma(c+j)}x^j$$

It has the advantage of displaying no discontinuities as the denominatorial parameter c moves through negative values.

CHAPTER 61

THE COMPLETE ELLIPTIC INTEGRALS K(k) AND E(k)

The complete elliptic integrals of the first and second kinds are rather simple univariate functions. These functions, together with their incomplete analogues addressed in Chapter 62 and the complete elliptic integrals of the third kind [Section 61:12], comprise the main members of the elliptic integral family. The family members were devised, primarily by Legendre, to allow the canonical representation of certain important integrals, a topic explored in Section 62:12. The intriguing and useful "elliptic nome" is addressed in Section 61:15, as are the four Neville theta functions.

61:1 NOTATION

With few exceptions, this *Atlas* uses "argument" to describe the sole variable of a univariate function. To achieve unity with the functions of Chapters 62 and 63, we violate this rule here and use the name *modulus* (*module* is also encountered), and the symbol k (p is also in use) to denote the variable of K(k), the *complete elliptic integral of the first kind*, and E(k), the *complete elliptic integral of the second kind*. Similarly, the bivariate *complete elliptic integral of the third kind* [Section 61:12] is denoted $\Pi(v,k)$.

As their names imply, the functions discussed in Chapters 61, 62 and 63 are ultimately related to an ellipse. The modulus k is, in fact, the eccentricity of this ellipse, equal to $\sqrt{1-(b/a)^2}$, where b and a are the lengths of the minor and major semiaxes [Section 13:14]. When the eccentricity is zero, $k = 0$ and the ellipse becomes a circle. As $k \to 1$, the ellipse degenerates into a straight-line segment.

Not all authors use the modulus as the variable of K and E. There is a surprisingly large number of alternative related variables that are encountered in discussions of complete elliptic integrals and elliptic functions. Replacements for k include the *complementary modulus*

61:1:1 $$k' = \sqrt{1-k^2} = \frac{b}{a}$$

(the symbol q has sometimes been used, as in the first edition of this *Atlas*), the *modular angle*,

61:1:2 $$\alpha = \arcsin(k)$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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the *complementary modular angle*,

61:1:3 $$\alpha' = \arccos(k)$$

the *parameter*,

61:1:4 $$m = k^2$$

the *complementary parameter*,

61:1:5 $$m_1 = 1 - m = (k')^2 = 1 - k^2$$

and the *elliptic nome q* [Section 61:14]. Figure 61-1 illustrates geometric relations between k, k', and α in terms of an ellipse and its circumscribed circle. See the table in Section 61:7 for special values acquired by many of these various "modulus substitutes". Beware of confusion because other authors often use such substitutions as K(k^2) and E(arcsin(k)) where we would use K(k) and E(k).

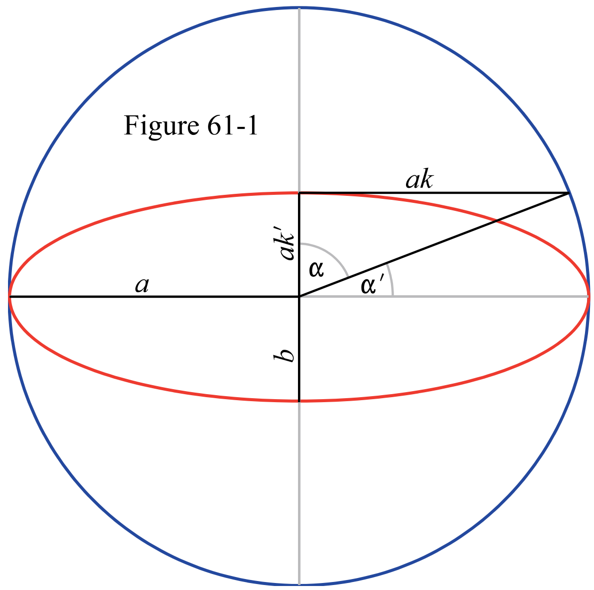


Figure 61-1

Another source of confusion is that E is the conventional symbol not only for the complete elliptic integral of the second kind, but also for the *incomplete* elliptic integral of the second kind, as discussed in Chapter 62.

In addressing bivariate and trivariate elliptic functions, it is sometimes convenient to use K(k) or K(k') as a variable, replacing or supplementing k itself. In this circumstance the abbreviations

61:1:6 $$K = \mathrm{K}(k)$$

and

61:1:7 $$K' = \mathrm{K}(k') = \mathrm{K}\left(\sqrt{1-k^2}\right)$$

are often adopted. In a similar way, E and E' are frequently used to abbreviate E(k) and E(k'). Notice our adherence to the *Atlas* rule that roman characters from the Latin alphabet denote functions, whereas *italic* characters signify numbers.

61:2 BEHAVIOR

Though they may be defined outside this domain, as in Section 61:11, by far the most important values of the modulus lie within $0 \le k \le 1$. This is the range covered by Figure 61-2, which maps the behavior, not only of K(k) and E(k), but also k', K(k'), E(k'), and q(k). Note that the ranges of the first and second kinds of complete elliptic integral are $\frac{1}{2}\pi \le \mathrm{K}(k) \le \infty$ and $1 \le \mathrm{E}(k) \le \frac{1}{2}\pi$, provided that $0 \le k \le 1$.

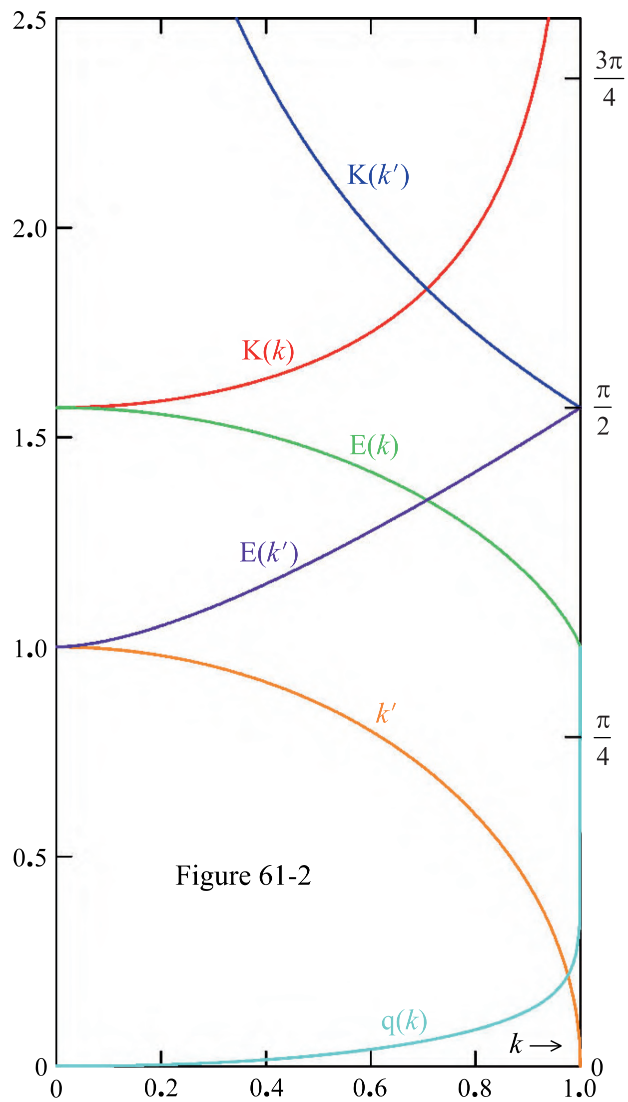


Figure 61-2

61:3 DEFINITIONS

The geometry of an ellipse of semiaxes a and b can serve to define the two complete elliptic integrals, though in very different ways. Thus, the perimeter of this ellipse [Section 13:14] equals $4a\mathrm{E}\left(\sqrt{1-(b/a)^2}\right)$ or $4a\mathrm{E}(k)$, while the *common mean* [mc, Section 61:14] of the two semiaxes provides a definition of K(k) through the relationship.

61:3:1 $$\mathrm{mc}(b,a)=\frac{\pi a}{2\mathrm{K}\left(\sqrt{1-(b/a)^2}\right)}=\frac{\pi a}{2\mathrm{K}(k)}$$

The complete elliptic integrals are generated from elementary functions by the operations of semidifferentiation or semiintegration with respect to the parameter k^2:

61:3:2 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}(k^2)^{1/2}}\arcsin(k)=\frac{1}{\sqrt{\pi}}\mathrm{K}(k)$$

61:3:3 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}(k^2)^{-1/2}}\frac{\sqrt{1-k^2}}{k}=\frac{2}{\sqrt{\pi}}\mathrm{E}(k)$$

Each of the two complete elliptic integrals may be defined by any one of three definite integrals:

61:3:4 $$\mathrm{K}(k)=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\mathrm{d}\theta=\int_0^1\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\mathrm{d}t=\int_0^\infty\frac{1}{\sqrt{(1+t^2)[1+(1-k^2)t^2]}}\mathrm{d}t$$

and

61:3:5 $$\mathrm{E}(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2(\theta)}\,\mathrm{d}\theta=\int_0^1\sqrt{\frac{1-k^2t^2}{1-t^2}}\,\mathrm{d}t=\int_0^\infty\sqrt{\frac{1+(1-k^2)t^2}{(1+t^2)^3}}\,\mathrm{d}t$$

The two definitions that involve the angle θ are illustrated in Figures 62-2 and 62-3 of the *next* chapter. Section 61:13 lists other integrals that may be expressed in terms of K(k) and E(k).

In the notation described in Section 43:14, the two complete elliptic integrals may be synthesized as follows:

61:3:6 $$\frac{1}{\sqrt{1-k}}\ \xrightarrow[1]{1/2}\ \frac{2}{\pi}\mathrm{K}\left(\sqrt{k}\right)$$

61:3:7 $$\frac{1}{\sqrt{1-k}}\ \xrightarrow[1]{-1/2}\ \frac{2}{\pi}\mathrm{E}\left(\sqrt{k}\right)$$

Related to these syntheses are the representations of the K and E functions as Gauss hypergeometric functions [Chapter 60]:

61:3:8 $$\mathrm{K}(k)=\frac{\pi}{2}\mathrm{F}\left(1/2,1/2,1,k^2\right)$$

61:3:9 $$\mathrm{E}(k)=\frac{\pi}{2}\mathrm{F}\left(-1/2,1/2,1,k^2\right)$$

61:4 SPECIAL CASES

There are none.

61:5 INTRARELATIONSHIPS

Both elliptic functions are even:

61:5:1 $$f(-k) = f(k) \qquad f = \mathrm{K} \text{ or } \mathrm{E}$$

The four elliptic integrals K(k), E(k), K(k'), and E(k') are linked by the remarkable *Legendre relation*

61:5:2 $$\mathrm{K}(k)\mathrm{E}(k') + \mathrm{K}(k')\mathrm{E}(k) = \tfrac{1}{2}\pi + \mathrm{K}(k)\mathrm{K}(k')$$

Complete elliptic integrals of modulus $2\sqrt{k}\,/(1+k)$ are related to those of modulus k through the transformations

61:5:3 $$\mathrm{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathrm{K}(k)$$

61:5:4 $$\mathrm{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2\mathrm{E}(k)}{1+k} - (1-k)\mathrm{K}(k)$$

A similar transformation links complete elliptic integrals of modulus $(1-k')/(1+k')$ to those of modulus k:

61:5:5 $$\mathrm{K}\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}\mathrm{K}(k) \qquad k' = \sqrt{1-k^2}$$

61:5:6 $$\mathrm{E}\left(\frac{1-k'}{1+k'}\right) = \frac{\mathrm{E}(k) + k'\mathrm{K}(k)}{1+k'}$$

The domain of definition, from 0 to 1, is preserved by these transformations.

61:6 EXPANSIONS

Alternative ways of formulating the power series

61:6:1 $$\frac{2}{\pi}\mathrm{K}(k) = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\times3}{2\times4}\right)^2 k^4 + \left(\frac{1\times3\times5}{2\times4\times6}\right)^2 k^6 + \cdots = \sum_{j=0}^{\infty}\left[\frac{(2j-1)!!}{(2j)!!}k^j\right]^2 = \sum_{j=0}^{\infty}\frac{(\tfrac{1}{2})_j(\tfrac{1}{2})_j}{(1)_j(1)_j}k^{2j}$$

61:6:2 $$\frac{2}{\pi}\mathrm{E}(k) = 1 - \left(\frac{1}{2}\right)^2\frac{k^2}{1} - \left(\frac{1\times3}{2\times4}\right)^2\frac{k^4}{3} - \left(\frac{1\times3\times5}{2\times4\times6}\right)^2\frac{k^6}{5} - \cdots = \sum_{j=0}^{\infty}\frac{1}{1-2j}\left[\frac{(2j-1)!!}{(2j)!!}k^j\right]^2 = \sum_{j=0}^{\infty}\frac{(-\tfrac{1}{2})_j(\tfrac{1}{2})_j}{(1)_j(1)_j}k^{2j}$$

are available via equations 61:3:8 and 61:3:9 coupled with the expansions reported in Section 60:6. The final expressions in 61:6:1 and 61:6:2 demonstrate that the complete elliptic integrals are hypergeometric functions. More useful than 61:6:1 when k is close to unity is the elaborate expansion

61:6:3 $$\mathrm{K}(k) = \frac{2K'}{\pi}\ln\left(\frac{4}{k'}\right) + 2\sum_{j=1}^{\infty}\left[\frac{(2j-1)!!}{(2j)!!}(k')^j\right]^2\sum_{n=1}^{2j}\frac{(-1)^n}{n}$$

the analogue of which is

61:6:4 $$\mathrm{E}(k) = 1 + \frac{2(K'-E')}{\pi}\ln\left(\frac{4}{k'}\right) + 2\sum_{j=1}^{\infty}\left[\frac{(2j-3)!!(2j-1)!!}{(2j-2)!!(2j)!!}(k')^{2j}\right]\left[\frac{1}{4j-8j^2} + \sum_{n=1}^{2j-2}\frac{(-1)^n}{n}\right]$$

The infinite product expansion

61:6:5 $$\mathrm{K}(k)=\frac{\pi}{2}\prod_{j=1}^{\infty}\left[1+k_j\right] \qquad k_1=\frac{1-k'}{1+k'} \qquad k_{j+1}=\frac{1-\sqrt{1-k_j^2}}{1+\sqrt{1-k_j^2}}$$

converges to the first kind of complete elliptic integral.

An expansion in terms of sines of multiples of the modular angle is provided by

61:6:6 $$\mathrm{K}(k)=\pi\left[\sin(\alpha)+\tfrac{1}{4}\sin(5\alpha)+\tfrac{9}{64}\sin(9\alpha)+\cdots\right]=\pi\sum_{j=0}^{\infty}\frac{[(2j)!]^2}{\left[2^j j!\right]^4}\sin\{(4j+1)\alpha\} \qquad \alpha=\arcsin(k)$$

See Section 61:15 for expansions involving the nome.

61:7 PARTICULAR VALUES

The table shows that many variables and functions coalesce with their complements when the modulus equals $1/\sqrt{2}$.

	$k=0$	$k=1/\sqrt{2}$	k = 1
k'	1	$1/\sqrt{2}$	0
m	0	½	1
m'	1	½	0
α	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$
α'	1	$\frac{1}{4}\pi$	0
$\mathrm{q}(k)=q$	0	$\exp(-\pi)$	1
$\mathrm{K}(k)=K$	$\frac{1}{2}\pi$	$\pi g/\sqrt{2}$	∞
$\mathrm{E}(k)=E$	$\frac{1}{2}\pi$	$\left(\pi g^2+1\right)/\left(\sqrt{8}\,g\right)$	1
$\mathrm{K}(k')=K'$	∞	$\pi g/\sqrt{2}$	$\frac{1}{2}\pi$
$\mathrm{E}(k')=E'$	1	$\left(\pi g^2+1\right)/\left(\sqrt{8}\,g\right)$	$\frac{1}{2}\pi$

In this table, g is Gauss's constant [Section 1:8]. The transformations in Section 61:5 may be used to develop other particular values.

61:8 NUMERICAL VALUES

A popular method of computing values of complete elliptic integrals of the first and second kinds relies on the common-mean procedure described in Section 61:14. In the terminology developed in that section

61:8:1 $$\mathrm{K}(k)=\frac{\pi}{2\mathrm{mc}(k',1)} \qquad k'=\sqrt{1-k^2}$$

and

61:8:2 $$\mathrm{E}(k)=\mathrm{K}(k)\left[1-\frac{k^2}{2}-\sum_{j=1}^{\infty}\mathrm{ma}(j,k',1)-\mathrm{mg}(j,k',1)\right]$$

and this is the route followed by *Equator* in its complete elliptic integral of the first kind routine (keyword **EllipticE**). However, for its complete elliptic integral of the second kind routine (keyword **EllipticK**), *Equator* relies on equation 61:3:9 and the algorithm described in Section 60:8, unless the modulus lies in the domain $0.99 \le k < 1$ in which case the procedure is replaced by series 61:6:4.

61:9 LIMITS AND APPROXIMATIONS

The earliest terms in expansions 61:6:1−4 provide limiting expressions for the complete elliptic integrals of the first and second kinds:

61:9:1 $$\mathrm{K}(k\to 0)=\frac{\pi}{2}\left[1+\frac{k^2}{4}\right]$$

61:9:2 $$\mathrm{K}(k\to 1)=\ln\left(4/k'\right)\qquad k'=\sqrt{1-k^2}$$

61:9:3 $$\mathrm{E}(k\to 0)=\frac{\pi}{2}\left[1-\frac{k^2}{4}\right]$$

61:9:4 $$\mathrm{E}(k\to 1)=1+\frac{(k')^2}{2}\left[\ln\left(\frac{4}{k'}\right)-\frac{1}{2}\right]\qquad k'=\sqrt{1-k^2}$$

61:10 OPERATIONS OF THE CALCULUS

Formulas for differentiation are:

61:10:1 $$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{K}(k)=\frac{\mathrm{E}(k)}{k(1-k^2)}-\frac{\mathrm{K}(k)}{k}$$

61:10:2 $$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{E}(k)=\frac{\mathrm{E}(k)-\mathrm{K}(k)}{k}$$

The indefinite integrals of the complete elliptic integrals cannot be written as established functions, though they are simple $L=K=3$ hypergeometric functions:

61:10:3 $$\int_0^k \mathrm{K}(t)\,\mathrm{d}t=\frac{\pi k}{2}\sum_{j=0}^{\infty}\frac{(\tfrac12)_j(\tfrac12)_j(\tfrac12)_j}{(1)_j(1)_j(\tfrac32)_j}\left(k^2\right)^j$$

61:10:4 $$\int_0^k \mathrm{E}(t)\,\mathrm{d}t=\frac{\pi k}{2}\sum_{j=0}^{\infty}\frac{(-\tfrac12)_j(\tfrac12)_j(\tfrac12)_j}{(1)_j(1)_j(\tfrac32)_j}\left(k^2\right)^j$$

However, the related indefinite integrals

61:10:5 $$\int_0^k t\,\mathrm{K}(t)\,\mathrm{d}t=\mathrm{E}(k)-(1-k^2)\mathrm{K}(k)$$

and

61:10:6
$$\int_0^k t\,\mathrm{E}(t)\,\mathrm{d}t = \frac{(1+k^2)\,\mathrm{E}(k)-(1-k^2)\,\mathrm{K}(k)}{3}$$

do have straightforward representations.

Catalan's and Gauss's constants , G and g, [Section 1:7] appear in the formulas for certain definite integrals of complete elliptic integrals:

	$\int_0^1 \mathrm{f}(k)\,\mathrm{d}k$	$\int_0^1 \mathrm{f}(k')\,\mathrm{d}k$	$\int_0^1 \frac{\mathrm{f}(k)}{k'}\mathrm{d}k$	$\int_0^1 \frac{\mathrm{f}(k)}{1+k}\mathrm{d}k$	$\int_0^1 \frac{\mathrm{f}(k)-\frac{1}{2}\pi}{k}\mathrm{d}k$	$\int_0^1 \frac{\mathrm{f}(k')-1}{k}\mathrm{d}k$
f = K	$2G$	$\frac{1}{4}\pi^2$	$\frac{1}{2}\pi^2 g^2$	$\frac{1}{8}\pi^2$	$\pi\ln(2)-2G$	
f = E	$G+½$	$\frac{1}{8}\pi^2$	$\frac{\pi^2 g^2}{4}+\frac{1}{4g^2}$		$\frac{\pi}{4}[1-\ln(4)]$	$\ln(4)-1$

61:11 COMPLEX ARGUMENT

Figures 61-3 and 61-4 respectively show the behavior of the first and second kinds of complete elliptic integral when the modulus equals $x+iy$. Each left-hand diagram shows the real part, while the right-hand diagram depicts the imaginary part.

The complete elliptic integrals of the first and second kinds are complex, even for real moduli, if k exceeds unity. In that case, the formulas that apply are

61:11:1
$$\mathrm{K}(k) = \frac{1}{k}\mathrm{K}\left(\frac{1}{k}\right)+\frac{i}{k}\mathrm{K}\left(\frac{\sqrt{k^2-1}}{k}\right) \qquad k>1$$

and

Figure 61-3

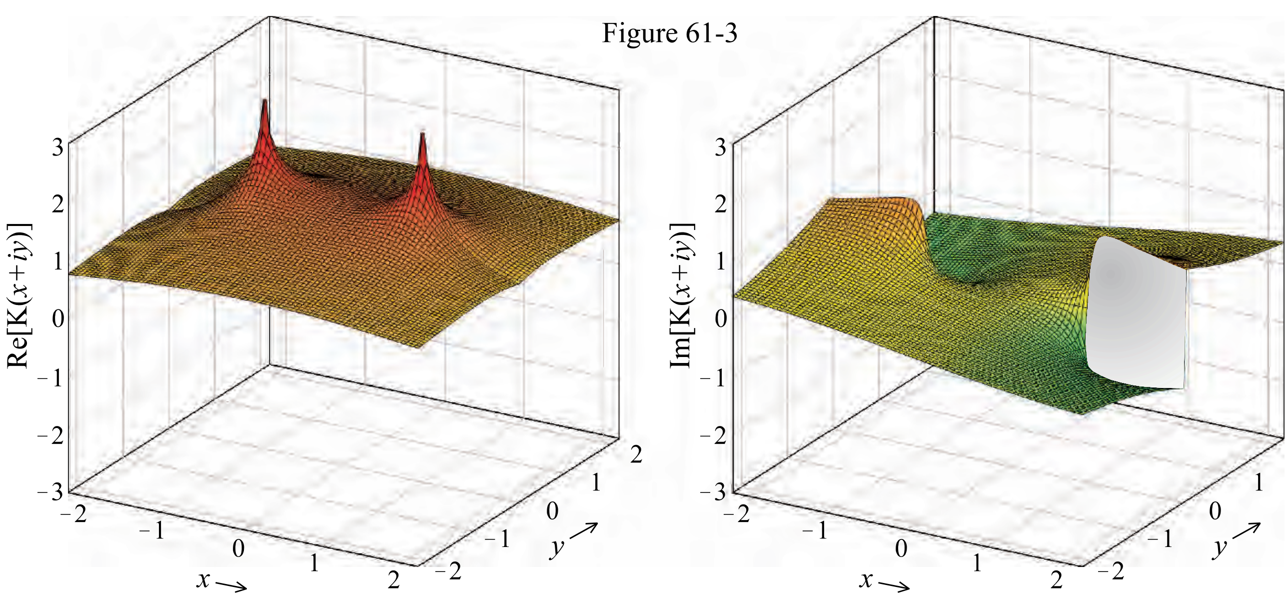
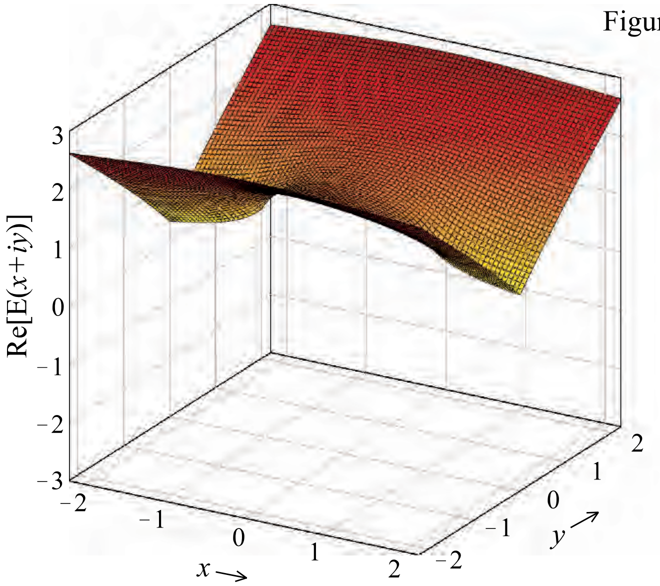
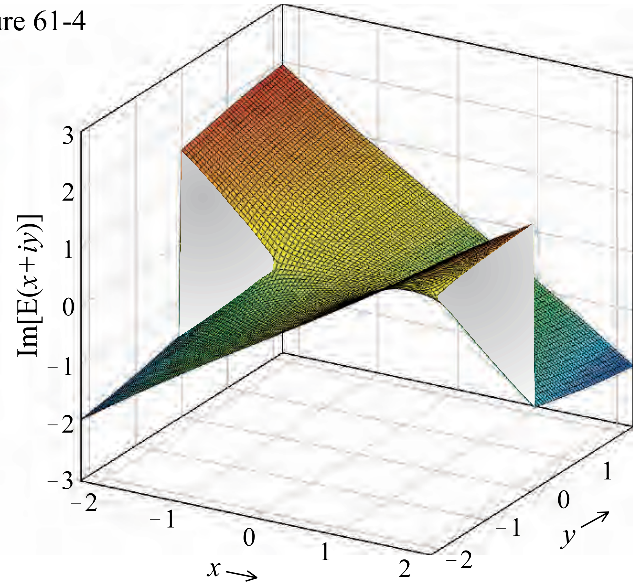


Figure 61-4





61:11:2 $$\mathrm{E}(k) = k\,\mathrm{E}\left(\frac{1}{k}\right) - \frac{k^2-1}{k}\mathrm{K}\left(\frac{1}{k}\right) + i\left[k\,\mathrm{E}\left(\frac{\sqrt{k^2-1}}{k}\right) - \frac{1}{k}\mathrm{K}\left(\frac{\sqrt{k^2-1}}{k}\right)\right] \qquad k>1$$

The complete elliptic integrals are generally complex when their modulus is complex. For purely imaginary argument, these functions are real:

61:11:3 $$\mathrm{K}(ik) = \frac{1}{\sqrt{1+k^2}}\mathrm{K}\left(\frac{k}{\sqrt{1+k^2}}\right) \qquad \text{and} \qquad \mathrm{E}(ik) = \sqrt{1+k^2}\,\mathrm{E}\left(\frac{k}{\sqrt{1+k^2}}\right)$$

For example $\mathrm{K}(i) = \frac{1}{2}\pi g$ and $\mathrm{E}(i) = (\pi g^2+1)/2g$, g being Gauss's constant [Section 1:7].

Modified Bessel functions [Chapter 49] arise from the Laplace inversion of complete elliptic integrals:

61:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\mathrm{K}\left(\frac{b}{s}\right) - \frac{\pi}{2}\right]\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\mathrm{K}\left(\frac{b}{s}\right) - \frac{\pi}{2}\right\} = \frac{\pi b}{2}\mathrm{I}_0\left(\frac{bt}{2}\right)\mathrm{I}_1\left(\frac{bt}{2}\right)$$

61:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\left[\frac{\pi}{2} - s\mathrm{E}\left(\frac{b}{s}\right)\right]\frac{\exp(ts)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\frac{\pi}{2} - s\mathrm{E}\left(\frac{b}{s}\right)\right\} = \frac{\pi b}{2t}\mathrm{I}_0\left(\frac{bt}{2}\right)\mathrm{I}_1\left(\frac{bt}{2}\right)$$

Other examples will be found in Roberts and Kaufman (page 312).

61:12 GENERALIZATIONS: mostly the complete elliptic integral of the third kind

The most important generalizations of the complete elliptic integrals arise by allowing the upper limit to become a variable in definitions 61:3:1 and 61:3:2. These are the incomplete elliptic integrals and are the subject of the next chapter. The remainder of this section is concerned with the *complete elliptic integral of the third kind* $\Pi(\nu,k)$, which is a generalization of that of the first kind inasmuch as $\Pi(0,k) = \mathrm{K}(k)$.

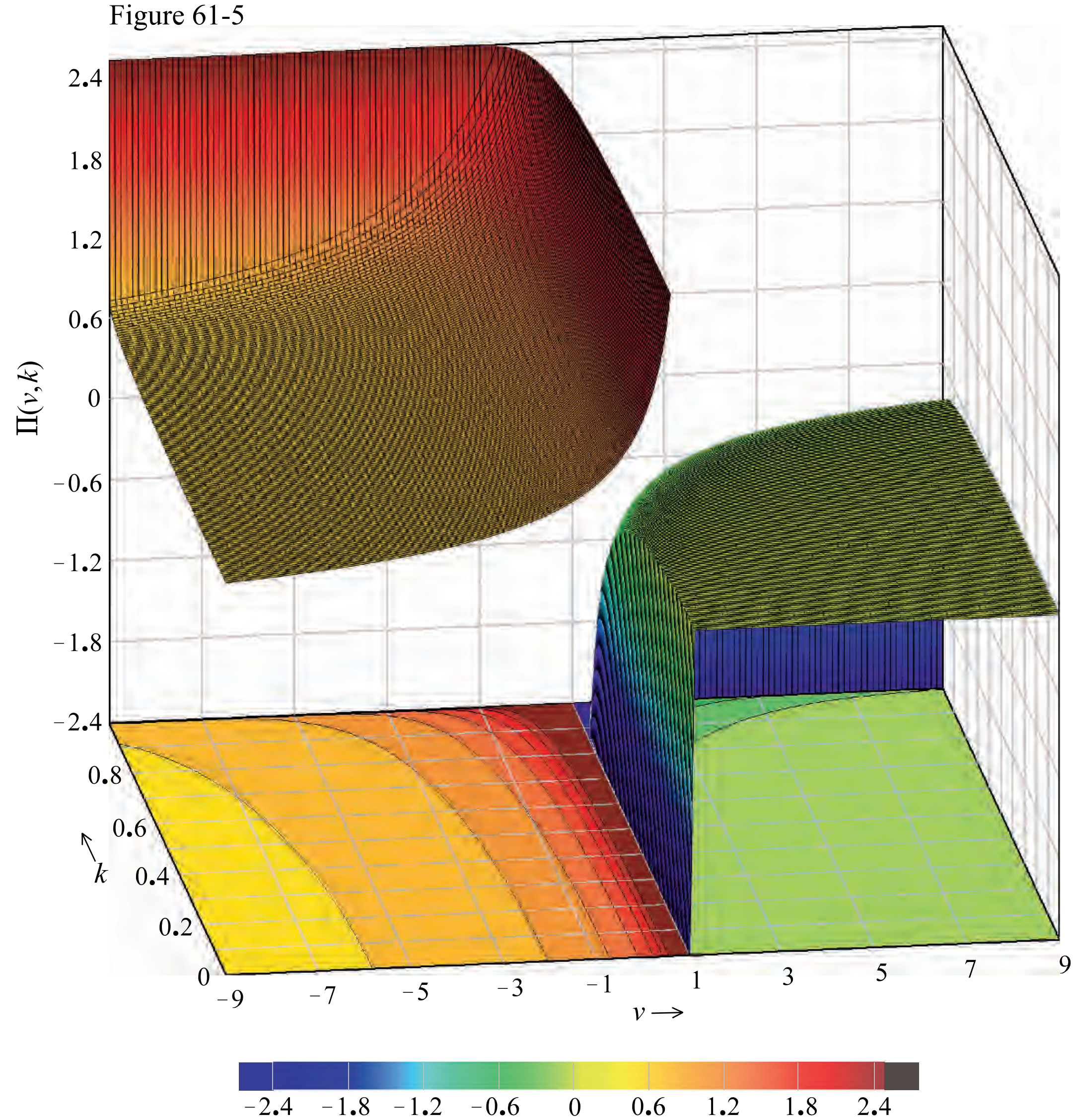
The complete elliptic integral of the third kind is defined by

61:12:1 $$\Pi(\nu,k)=\int_0^1 \frac{1}{(1-\nu t^2)\sqrt{(1-t^2)(1-k^2t^2)}}\,dt=\int_0^{\pi/2}\frac{1}{\left[1-\nu\sin^2(\theta)\right]\sqrt{1-k^2\sin^2(\theta)}}\,d\theta$$

where ν is a number, treated as real here, known as the *characteristic*. Be aware of the wide variety of definitions and notations. Thus the integrals in 61:12:1 would be denoted $\Pi_1(\nu,k)$, $\Pi(-\nu\setminus\arcsin(k))$, $\Pi(\sqrt{\nu},k)$, $\Pi(k,\nu,\frac{1}{2}\pi)$, $\Pi(\frac{1}{2}\pi,-\nu,k)$, or even $\frac{1}{\nu}\Pi(\frac{-1}{\nu},k,\frac{1}{2}\pi)$ by different authors. In the first edition of this *Atlas*, the sign of ν differed from that adopted here.

Interest concentrates on the domain $0\le k<1$ of the *modulus*. Figure 61-5 shows the behavior of $\Pi(\nu,k)$ in this modular domain. When all the variables in the defining integral in 16:12:1 are real, the elliptic integral of the third kind is also real, though some authorities regard $\Pi(\nu,k)$ as complex when $\nu>1$.

Figure 61-5



The complete elliptic integral of the third kind suffers a discontinuity at $\nu=1$ and adopts special values when the characteristic equals zero or k^2, as tabulated overleaf.

$\nu = 0$	$\nu = k^2$	$\nu = 1$
$\Pi(0,k) = \mathrm{K}(k)$	$\Pi(k^2,k) = \dfrac{\mathrm{E}(k)}{1-k^2}$	$\Pi(1,k) = +\infty \mid -\infty$

For other values of ν, the third kind of elliptic integral can always be expressed in terms of *incomplete* elliptic integrals of the first and second kinds [Chapter 62], but the formula allowing this is quadripartite, depending on the magnitude of the characteristic ν. In the formulation below, we use K and E to represent the complete integrals K(k) and E(k):

61:12:2
$$\Pi(\nu,k) = \begin{cases} \dfrac{K}{1-\nu} + \sqrt{\dfrac{-\nu}{(1-\nu)(k^2-\nu)}}\left[\dfrac{\pi}{2} + (K-E)\mathrm{F}(k',\phi) - K\,\mathrm{E}(k',\phi)\right] & \nu < 0 \\ K - \sqrt{\dfrac{\nu}{(1-\nu)(k^2-\nu)}}\left[E\,\mathrm{F}(k,\phi') - K\,\mathrm{E}(k,\phi')\right] & 0 < \nu < k^2 \\ K + \sqrt{\dfrac{\nu}{(1-\nu)(\nu-k^2)}}\left[\dfrac{\pi}{2} + (K-E)\mathrm{F}(k',\phi'') - K\mathrm{E}(k',\phi'')\right] & k^2 < \nu < 1 \\ \sqrt{\dfrac{\nu}{(\nu-1)(\nu-k^2)}}\left[E\,\mathrm{F}(k,\phi''') - K\,\mathrm{E}(k,\phi''')\right] & \nu > 1 \end{cases}$$

The angles that appear in these formulas are defined by

61:12:3
$$\phi = \arcsin\left(\frac{1}{\sqrt{1-\nu}}\right) \qquad \phi' = \arcsin\left(\frac{\sqrt{\nu}}{k}\right) \qquad \phi'' = \arcsin\left(\frac{\sqrt{1-\nu}}{k'}\right) \qquad \phi''' = \arcsin\left(\frac{1}{\sqrt{\nu}}\right)$$

The formulas in expression 61:12:2 are used by *Equator* to calculate values of the complete elliptic integral of the third kind, with keyword **EllipticPi**. The values required for the F(k,ϕ), E(k,ϕ), F(k',ϕ), and E(k',ϕ) functions are obtained by the methods explained in Section 62:8.

61:13 COGNATE FUNCTIONS

In addition to those of the first, second, and third kinds, the following complete elliptic integrals

61:13:1
$$\mathrm{D}(k) = \int_0^{\pi/2} \frac{\sin^2(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\,d\theta = \frac{\mathrm{K}(k)-\mathrm{E}(k)}{k^2}$$

61:13:2
$$\mathrm{B}(k) = \int_0^{\pi/2} \frac{\cos^2(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\,d\theta = \frac{\mathrm{E}(k)-(1-k^2)\mathrm{K}(k)}{k^2}$$

and

61:13:3
$$\mathrm{C}(k) = \mathrm{G}(k) = \int_0^{\pi/2} \frac{\sin^2(\theta)\cos^2(\theta)}{\left[1-k^2\sin^2(\theta)\right]^{3/2}}\,d\theta = \frac{2-k^2}{k^4}\mathrm{K}(k) - \frac{2}{k^4}\mathrm{E}(k)$$

may be encountered in some works, though not in the *Atlas*.

61:14 RELATED TOPIC: means

A *mean* of two positive numbers, x and y, is a third positive number that represents some sort of compromise between the two. It invariably has a value intermediate between x and y.

Familiar means are the *arithmetic mean*

61:14:1 $$\mathrm{ma}(x,y) = \frac{x+y}{2}$$

also known as the *average*, the *geometric mean*

61:14:2 $$\mathrm{mg}(x,y) = \sqrt{xy}$$

the *harmonic mean*

61:14:3 $$\mathrm{mh}(x,y) = \frac{2}{(1/x)+(1/y)} = \frac{2xy}{x+y} = \frac{\mathrm{mg}^2(x,y)}{\mathrm{ma}(x,y)}$$

and the *root-mean-square*

61:14:4 $$\mathrm{mr}(x,y) = \sqrt{\frac{x^2+y^2}{2}}$$

If y exceeds x, the sequence of inequalities

61:14:5 $$x < \mathrm{mh}(x,y) < \mathrm{mg}(x,y) < \mathrm{ma}(x,y) < \mathrm{mr}(x,y) < y$$

applies. These four means are special cases of a *generalized mean* defined by

61:14:6 $$\mathrm{m}(n,x,y) = \left(\frac{x^n+y^n}{2}\right)^{1/n}$$

The harmonic mean, arithmetic mean, and root-mean-square are the $n = -1$, $n = 1$ and $n = 2$ cases of this general formula, while the geometric mean corresponds to the $n \to 0$ limit. *Equator*'s generalized mean routine (keyword **m**) delivers $\mathrm{m}(n,x,y)$ for most combinations of positive values of x and y and any real value of n.

The means discussed above can be broadened in obvious ways to encompass more than two arguments, but this is not true of the *common mean*, the topic to which attention now turns. Also called the *arithmeticogeometric mean*, this mean, $\mathrm{mc}(x,y)$, lies between $\mathrm{mg}(x,y)$ and $\mathrm{ma}(x,y)$ in the hierarchy cited in 61:14:5.

A procedure may be implemented in which, starting from two positive numbers x and y, geometric and arithmetic means are formed repeatedly. The symbols $\mathrm{mg}(j,x,y)$ and $\mathrm{ma}(j,x,y)$, defined by the iterations

61:14:7 $$\mathrm{mg}(j,x,y) = \mathrm{mg}\big(\mathrm{mg}(j-1,x,y),\mathrm{ma}(j-1,x,y)\big)$$

61:14:8 $$\mathrm{ma}(j,x,y) = \mathrm{ma}\big(\mathrm{mg}(j-1,x,y),\mathrm{ma}(j-1,x,y)\big)$$

$$\left\{\begin{array}{l}\mathrm{mg}(0,x,y) = x \\ \mathrm{ma}(0,x,y) = y\end{array}\right. \qquad j = 1,2,3,\cdots$$

establish this *arithmeticogeometric procedure*. As j increases, $\mathrm{mg}(j,x,y)$ and $\mathrm{ma}(j,x,y)$ rapidly approach a common value known as the common mean of x and y and here denoted $\mathrm{mc}(x,y)$:

61:14:9 $$\mathrm{mc}(x,y) = \lim_{j\to\infty}\{\mathrm{mg}(j,x,y)\} = \lim_{j\to\infty}\{\mathrm{ma}(j,x,y)\}$$

Based on this procedure, *Equator* generates values of $\mathrm{mc}(x,y)$ through its common mean routine (keyword **mc**), fewer than 5 iterations generally being adequate.

The common mean provides definitions of Gauss's constant g [Section 1:7], the first kind of complete elliptic integral $\mathrm{K}(k)$ [Section 61:3], and the elliptic nome $\mathrm{q}(k)$ [Section 61:15]. Arithmeticogeometric procedures are also used, in the context of Landen's transformation, in computing values of the incomplete elliptic integrals [Section 62:8].

61:15 RELATED TOPIC: the elliptic nome

Associated with the functions of Chapters 61–63 is a univariate function known as the *nome* or the *elliptic nome*. When considered as a function of the modulus k, we use the symbol $\mathrm{q}(k)$ [the alternative $\mathrm{N}(p)$ was adopted in the first edition of the *Atlas*], but when other functions employ the nome as a variable, the more usual symbol q will be employed. This function receives modular values in the domain $0 \le k \le 1$ and its range is likewise $0 \le \mathrm{q}(k) \le 1$. As Figure 61-1 illustrates, its approach to the upper limit is at first gentle but ultimately it becomes incredibly steep, as confirmed by a comparison of the values $\mathrm{q}(0.999995) \approx \frac{1}{2}$ and $\mathrm{q}(1) = 1$.

The definition of the nome is in terms of the complete elliptic integral of the first kind and its complement:

61:15:1 $$\mathrm{q}(k) = \exp\left\{\frac{-\pi \mathrm{K}(k')}{\mathrm{K}(k)}\right\} = \exp\left\{\frac{-\pi K'}{K}\right\}$$

This definition, combined with equation 61:8:1 into

61:15:2 $$\mathrm{q}(k) = \exp\left\{\frac{-\pi\,\mathrm{mc}(k,1)}{\mathrm{mc}(k',1)}\right\} \qquad k' = \sqrt{1-k^2}$$

provides a direct method of relating the nome to k. See the preceding section for the significance of mc(,). The reflection formula

61:15:3 $$\mathrm{q}(k) = \exp\left(\frac{\pi^2}{\ln\{\mathrm{q}(k')\}}\right)$$

relates a nome of modulus less than $1/\sqrt{2}$ value to one with a modulus greater than this value. Two expansions of the nome, in terms of the modulus are

61:15:4 $$\mathrm{q}(k) = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \cdots \qquad \lambda = \frac{1-\sqrt{k'}}{2+2\sqrt{k'}}$$

and

61:15:5 $$\mathrm{q}(k) = \frac{1}{2\kappa} + \frac{2}{\kappa^2} + \frac{21}{2\kappa^3} + \frac{62}{\kappa^4} + \frac{6257}{16\kappa^5} + \frac{10293}{4\kappa^6} + \frac{279025}{16\kappa^7} + \frac{483127}{4\kappa^8} + \frac{435506703}{512\kappa^9} + \cdots \qquad \kappa = \frac{8}{k^2}$$

This last formula is used by *Equator*'s elliptic nome routine (keyword **q**) when $0 \le k \le 0.2$; otherwise equation 61:15:1 is used to compute $\mathrm{q}(k)$.

The utility of the nome arises from the large number of elliptic functions that can be expressed as rapidly convergent series or products of terms involving q. These functions include the modulus itself

61:15:6 $$k = \left[\frac{q^{1/4} + q^{9/4} + q^{25/4} + \cdots}{\frac{1}{2} + q + q^4 + q^9 + \cdots}\right]^2 = 4\sqrt{q}\left[\frac{1+q^2}{1+q}\right]^4\left[\frac{1+q^4}{1+q^3}\right]^4\left[\frac{1+q^6}{1+q^5}\right]^4 \cdots$$

the complementary modulus

61:15:7 $$k' = \left[\frac{\frac{1}{2} - q + q^4 - q^9 + \cdots}{\frac{1}{2} + q + q^4 + q^9 + \cdots}\right]^2 = \left[\frac{1-q}{1+q}\right]^4\left[\frac{1-q^3}{1+q^3}\right]^4\left[\frac{1-q^5}{1+q^5}\right]^4 \cdots$$

the complete elliptic integrals of the first

61:15:8 $$\frac{\mathrm{K}(k)}{2\pi} = \left[\tfrac{1}{2} + q + q^4 + q^9 + \cdots\right]^2 = \sqrt{\frac{\sqrt{q}}{4k}}\left[\frac{1-q^2}{1-q}\right]^2\left[\frac{1-q^4}{1-q^3}\right]^2\left[\frac{1-q^6}{1-q^5}\right]^2 \cdots$$

and second

61:15:9 $$\mathrm{E}(k)=\mathrm{K}(k)-\frac{\pi^2}{\mathrm{K}(k)}\left[\frac{q-4q^4+9q^9-\cdots}{\tfrac{1}{2}-q+q^4-q^9+\cdots}\right]$$

kinds, the incomplete elliptic function E(k,φ) of the second kind [Chapter 62] expressed in terms of its cohort F(k,φ) of the first kind

61:15:10 $$\mathrm{E}(k,\varphi)=\frac{\mathrm{E}(k)\mathrm{F}(k,\varphi)}{K}+\frac{2\pi}{K}\left[\frac{q\sin(\pi\mathrm{F}(k,\varphi)/K)}{1-q^2}+\frac{q^2\sin(2\pi\mathrm{F}(k,\varphi)/K)}{1-q^4}+\frac{q^3\sin(3\pi\mathrm{F}(k,\varphi)/K)}{1-q^6}+\cdots\right]$$

the elliptic amplitude [Section 62:3]

61:15:11 $$\varphi=2\left[\frac{\pi\mathrm{F}(k,\varphi)}{4K}+\frac{q\sin(\pi\mathrm{F}(k,\varphi)/K)}{1+q^2}+\frac{q^2\sin(2\pi\mathrm{F}(k,\varphi)/K)}{2[1+q^4]}+\frac{q^3\sin(3\pi\mathrm{F}(k,\varphi)/K)}{3[1+q^6]}+\cdots\right]$$

and all twelve of the Jacobian elliptic functions. Some of the latter expansions will be found in Section 63:6.

An important role for the nome is to serve as one of the variables in theta functions. A first encounter with theta functions can be bewildering! Be alert to the existence of many different kinds of theta function in the literature, in addition to such modified versions as that discussed in Section 27:13. Three kinds are described in this *Atlas*. Not only are these three kinds named and sometimes defined differently by different authorities, but the symbols used to represent their variables may be permuted mercilessly. All three kinds of theta function are bivariate with one of those variables being a *periodic variable*, by which is meant that incrementing this variable by a constant (the "period" of the function is the smallest such increment) leaves the theta function's value unchanged. The other variable is aperiodic but there is no unanimity on whether the periodic or the aperiodic symbol is cited first in the function's notation. Indeed, the aperiodic variable may not even appear in the notations adopted by some authors.

In Section 27:13 a four-member family of so-called *exponential theta functions* is encountered. Each of the four is defined there in two ways, one of which is exemplified by

61:15:12 $$\theta_1(v,t)=2\sum_{j=0}^{\infty}(-)^j\exp\left\{-(j+\tfrac{1}{2})^2\pi^2t\right\}\sin\left\{(2j+1)\pi v\right\}$$

This is the definition of the exponential theta-one function given in equation 27:13:7; the three other exponential theta functions, $\theta_2(v,t)$, $\theta_3(v,t)$, and $\theta_4(v,t)$, are defined similarly. Notice in 61:15:12 that v is the periodic variable and that its period is 2. This is the case also for the exponential theta-two function, but $\theta_3(v,t)$, and $\theta_4(v,t)$ have periods of unity.

Members of a second quartet of theta functions are called *elliptic theta functions* or *Jacobi theta functions*. Their notations and series definitions are

61:15:13 $$\vartheta_1(q,x)=2\left[q^{1/4}\sin(x)-q^{9/4}\sin(3x)+q^{25/4}\sin(5x)-\cdots\right]$$

61:15:14 $$\vartheta_2(q,x)=2\left[q^{1/4}\cos(x)+q^{9/4}\cos(3x)+q^{25/4}\cos(5x)+\cdots\right]$$

61:15:15 $$\vartheta_3(q,x)=1+2\left[q\cos(2x)+q^4\cos(4x)+q^9\cos(6x)+\cdots\right]$$

61:15:16 $$\vartheta_4(q,x)=1-2\left[q\cos(2x)-q^4\cos(4x)+q^9\cos(6x)-\cdots\right]$$

Alternative symbols for ϑ_1 and ϑ_4 are H and Θ, these being Jacobi's choices, but the variables serving these symbols are not always q and x. Compare the four ϑ expansions with equations 27:13:7-10 and observe that each elliptic theta function is identical to the corresponding exponential theta function, apart from a radical change in the variables, the order of which is switched. The connections are

61:15:17 $$\vartheta_j(q,x)=\theta_j\left(\frac{x}{\pi},\frac{-\ln(q)}{\pi^2}\right)\qquad\text{or}\qquad\theta_j(v,t)=\vartheta_j\left(\exp(-\pi^2t),\pi v\right)\qquad j=1,2,3,4$$

so that the nome q replaces $\exp(-\pi^2 t)$ as the aperiodic variable, while x replaces πv as the periodic variable. Thereby the period becomes 2π for ϑ_1 and ϑ_2 but π for ϑ_3 and ϑ_4. Particular values of the elliptic theta functions are as tabulated here:

	$\vartheta_1(q,x)$	$\vartheta_2(q,x)$	$\vartheta_3(q,x)$	$\vartheta_4(q,x)$
$x=0$	0	$\sqrt{2kK/\pi}$	$\sqrt{2K/\pi}$	$\sqrt{2k'K/\pi}$
$x=\pi/2$	$\sqrt{2kK/\pi}$	0	$\sqrt{2k'K/\pi}$	$\sqrt{2K/\pi}$

These identities permit a number of power series to be summed through equations 61:15:13−16:

61:15:18
$$1+q^2+q^6+q^{12}+q^{20}+\cdots=\sum_{j=0}^{\infty}q^{j(j+1)}=\frac{\vartheta_1\left(q,\frac{1}{2}\pi\right)}{2q^{1/4}}=\frac{\vartheta_2(q,0)}{2q^{1/4}}=\sqrt{\frac{kK}{2\pi\sqrt{q}}}$$

61:15:19
$$1+q+q^4+q^9+q^{16}+\cdots=\sum_{j=0}^{\infty}q^{j^2}=\frac{1+\vartheta_3(q,0)}{2}=\frac{1}{2}+\sqrt{\frac{K}{2\pi}}$$

and

61:15:20
$$1-q+q^4-q^9+q^{16}-\cdots=\sum_{j=0}^{\infty}(-)^j q^{j^2}=\frac{1+\vartheta_4(q,0)}{2}=\frac{1}{2}+\sqrt{\frac{k'K}{2\pi}}$$

Equator does not provide a dedicated means of calculating values of the elliptic theta functions because these are so easily found from the exponential theta function routines described in Section 27:13, via the variables changes specified in the equivalence 61:15:17.

To add more elaboration, a third quartet of theta functions will now be broached. These were introduced by the English mathematician Eric Harold Neville (1889−1961) and bear his name. Again, one of the two variables is periodic, but the periods are $4K$ or $2K$, in contrast to the 2π or π of the elliptic theta functions. There are several other ways in which *Neville's theta functions*, which play an important role in Chapter 63, differ from elliptic theta functions. The aperiodic variable is taken as the nome q for elliptic theta functions but the modulus k fills that role here for the Neville theta functions. Whereas the distinguishing subscripts are the numbers 1, 2, 3, and 4, for elliptic theta functions, these are replaced by the letters s, c, d, and n in Neville's foursome. And, finally, a normalizing multiplier is introduced. Notwithstanding these differences, there remains a close liaison between the two kinds of theta function, the connections being

61:15:21
$$\vartheta_{\mathrm{s}}(k,x)=\sqrt{\frac{\pi}{2kk'K}}\,\vartheta_1\left(q,\frac{\pi x}{2K}\right)$$

61:15:22
$$\vartheta_{\mathrm{c}}(k,x)=\sqrt{\frac{\pi}{2kK}}\,\vartheta_2\left(q,\frac{\pi x}{2K}\right)$$

61:15:23
$$\vartheta_{\mathrm{d}}(k,x)=\sqrt{\frac{\pi}{2K}}\,\vartheta_3\left(q,\frac{\pi x}{2K}\right)$$

61:15:24
$$\vartheta_{\mathrm{n}}(k,x)=\sqrt{\frac{\pi}{2k'K}}\,\vartheta_4\left(q,\frac{\pi x}{2K}\right)$$

$$\left.\right\} \quad K=\mathrm{K}(k) \qquad k'=\sqrt{1-k^2} \qquad q=\exp\left(\frac{-\pi\mathrm{K}(k')}{\mathrm{K}(k)}\right)$$

The normalizing factors in these formulas serve to ensure that $\vartheta_{\mathrm{c}}(k,x)$, $\vartheta_{\mathrm{n}}(k,x)$, and $\vartheta_{\mathrm{d}}(k,x)$ all equal unity at

$x = 0$, whereas $\vartheta_{\mathrm{s}}(k,x)$ – which equals zero when its argument is zero – is normalized to equate $\vartheta_{\mathrm{s}}(k,K)$ and $\vartheta_{\mathrm{n}}(k,K)$.

The Neville theta functions are defined as real-valued functions for $0 \le k \le 1$, though ϑ_{s} and ϑ_{n} become unbounded at $x = \pm K$ as $k \to 1$. The x domain is unlimited because all four functions are periodic in x, the periods of the s- and c-varieties being $4K$, while the d- and n-versions have periods of $2K$. The theta-s function is odd with respect to the periodic argument x; the other three are even. Values of ϑ_{c} range between -1 and $+1$; those of ϑ_{s} occupy a wider range, symmetrical around zero, with the extrema depending on k. Values of ϑ_{d} range between 1 and a smaller positive k-dependent minimum, while those of ϑ_{n} range upwards from 1 to a k-dependent maximum. These behaviors conform to the special cases to which the Neville theta functions reduce at extreme values of k:

	$\vartheta_{\mathrm{s}}(q,x)$	$\vartheta_{\mathrm{c}}(q,x)$	$\vartheta_{\mathrm{d}}(q,x)$	$\vartheta_{\mathrm{e}}(q,x)$
$k = 0$	sin(x)	cos(x)	1	1
$k = 1$	sinh(x)	1	1	1

and to the following values that these functions acquire at particular values of x:

	$\vartheta_{\mathrm{s}}(q,x)$	$\vartheta_{\mathrm{c}}(q,x)$	$\vartheta_{\mathrm{d}}(q,x)$	$\vartheta_{\mathrm{e}}(q,x)$
$x = 0, \pm 4K, \pm 8K, \cdots$	0	1	1	1
$x = K, -3K, 5K, -7K, \cdots$	$1/\sqrt{k'}$	0	$\sqrt{k'}$	$1/\sqrt{k'}$
$x = \pm 2K, \pm 6K, \pm 10K, \cdots$	0	-1	1	1
$x = -K, 3K, -5K, 7K, \cdots$	$-1/\sqrt{k'}$	0	$\sqrt{k'}$	$1/\sqrt{k'}$

The four Neville functions are interrelated through

61:15:25 $$\vartheta_{\mathrm{n}}^2(k,x) - \vartheta_{\mathrm{s}}^2(k,x) = \vartheta_{\mathrm{c}}^2(k,x) = \vartheta_{\mathrm{d}}^2(k,x) - [k'\vartheta_{\mathrm{s}}(k,x)]^2$$

61:15:26 $$\sqrt{k'}\,\vartheta_{\mathrm{s}}(k,x) = \vartheta_{\mathrm{c}}(k,K-x)$$

61:15:27 $$\sqrt{k'}\,\vartheta_{\mathrm{n}}(k,x) = \vartheta_{\mathrm{d}}(k,K-x)$$

and by other relationships that can be deduced from information in Chapter 63.

Equations 61:15:21–24, the equivalences 61:15:17, and ultimately the routines from Chapter 27, allow *Equator* to compute Neville's theta-s, theta-c, theta-d, and theta-n functions (keywords **theta-s**, **theta-c**, **theta-d**, and **theta-n**). However, formula 61:15:26 is invoked in calculating $\vartheta_{\mathrm{s}}(k,x)$ for moduli $k \le 0.16$. Moreover, when $|x| \le 0.001$ and $k \ge 0.9$, *Equator* resorts to the formula

61:15:28 $$\vartheta_{\mathrm{s}}(k,x) \approx \vartheta_{\mathrm{n}}(k,x)\left[\tanh(x) + \frac{1-k^2}{8}\{\sinh(2x) - 2x\}\mathrm{sech}^2(x)\right]$$

as a means of calculating theta-s. Though the formulas of this section apply for all values of the periodic variable x, computational precision suffers if x strays too far from zero. This is because of the subtractive loss of significance inherent in such operations as x(mod $4K$). Accordingly, the argument is restricted to the domain $-8K \le x \le 8K$ when Neville's theta functions are calculated by *Equator*.

Note that, because $\vartheta_n(k,x)$ approaches unity as $k \to 1$, expression 61:15:28 correctly predicts that $\vartheta_{\mathrm{s}}(k,x)$ approaches tanh(x) as $k \to 1$. Yet, paradoxically, $\vartheta_{\mathrm{s}}(k,x)$ equals sinh(x) *at* $k = 1$. A similar discontinuity afflicts $\vartheta_{\mathrm{c}}(k,x)$, which approaches $k = 1$ as sech (x) but equals unity *at* $k = 1$.

CHAPTER

62

THE INCOMPLETE ELLIPTIC INTEGRALS F(k,φ) AND E(k,φ)

These two bivariate functions – and a third, mentioned briefly in Section 62:12 – arise in many practical problems, including the motions of particles, pendulums and planets. Analysis of these motions often leads to the indefinite integrals cited in Section 62:14 and it was in formalizing these integrals that the incomplete elliptic integrals arose historically.

Numerical quadrature may be used to calculate values of these integrals and an efficient method of performing this task accurately is described in Section 62:15.

There is a strong kinship between the functions of Chapters 61, 62, and 63. We refer to them collectively as members of the "elliptic family of functions".

62:1 NOTATION

The incomplete elliptic integrals are sometimes named *Legendre's elliptic integrals*. The F and E varieties are distinguished as being of the "first kind" and "second kind" respectively. The adjective "incomplete" is appropriate because the upper integration bound in definitions 62:3:1 and 62:3:2 is generally less than the value $\varphi = \pi/2$ required to "complete" the integrals that define the functions of Chapter 61.

It is unfortunate that the character "E" is the symbol for the complete, as well as the incomplete, elliptic integral of the second kind. Both usages are deeply entrenched.

The variable φ is termed the *amplitude* of the incomplete elliptic integral, k being its *modulus*. However, as in Chapter 61, there are several "modulus substitutes". When these are used, the comma in F(k,φ) and E(k,φ) is frequently replaced by some other separator, and the amplitude then receives first mention. Thus, for example, F(φ|m) and E(φ\α) are commonly employed when the parameter or the modular angle [equation 61:1:4 or 61:1:2] replaces the modulus. One may even find the modulus to be merely implied, as in F(φ). It is usual for the amplitude to be treated as an angle and denoted φ but you may encounter F(k,x) where x is sin(φ). The *Atlas* invariably indicates the presence of the modulus k, and avoids modulus substitutes. In this chapter we never adopt K or E as an abbreviation for K(k) or E(k), though they are often so used elsewhere.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_63, © Springer Science+Business Media, LLC 2009

Figure 62-1

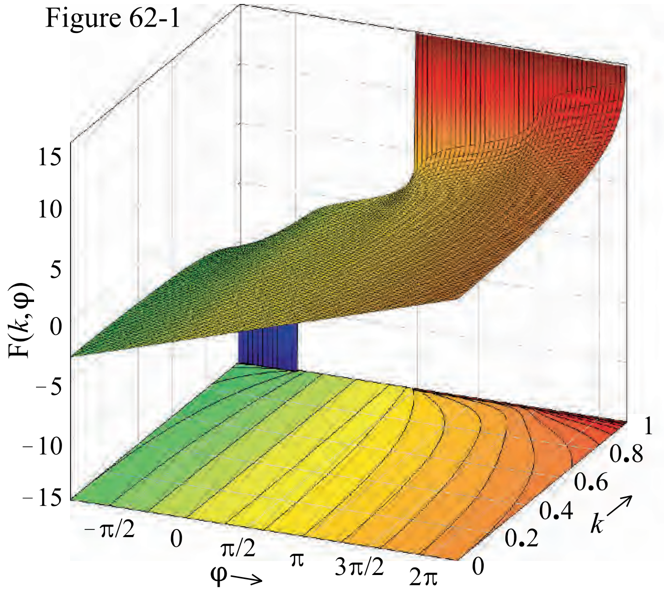
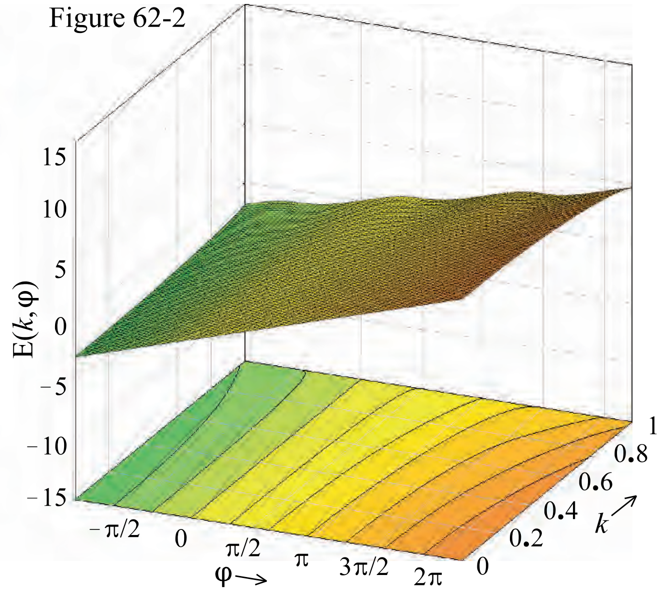


Figure 62-2



62:2 BEHAVIOR

By far the most important modular domain for the incomplete elliptic integrals is $0 \le k \le 1$. This is the only domain addressed in this section and the one illustrated in Figures 62-1 and 62-2. Observe that the rather bland terrain of these functions is dominated by a ramp-like steady increase in value with increasing amplitude φ. As k increases, the ramp becomes mildly rippled and, in the F(k,φ) case, the oscillations become prominent as the modulus approaches unity. The figures suggest the possibility of resolving the incomplete elliptic integrals into two components and, indeed, each incomplete elliptic function may be decomposed into a linear function of its amplitude and a periodic function:

62:2:1
$$\mathrm{f}(k,\varphi) = \frac{2\varphi}{\pi}\,\mathrm{f}\left(k,\frac{\pi}{2}\right) + \left(\begin{array}{c}\text{a periodic}\\ \text{function of}\\ \text{period } \pi\end{array}\right) \qquad \begin{array}{c}\mathrm{f} = \mathrm{F} \text{ or } \mathrm{E}\\ \mathrm{f}\left(k,\dfrac{\pi}{2}\right) = \mathrm{K}(k) \text{ or } \mathrm{E}(k)\end{array}$$

Each component is an odd function of the amplitude φ. Except for E(1,φ) which obeys equation 62:4:3, the periodic component is not sinusoidal. The periodicity property is not shared by F(1,φ), which is undefined outside $-\pi/2 \le \varphi \le \pi/2$.

One consequence of formula 62:2:1, and the properties of the functions vis-à-vis the modulus, is that it suffices to examine the behavior of F(k,φ) and E(k,φ) only in the intervals $0 \le k \le 1$ and $0 \le \varphi < \pi$. Felicitous reflection, recursion, and transformation properties then permit extension to other values of the variables.

62:3 DEFINITIONS

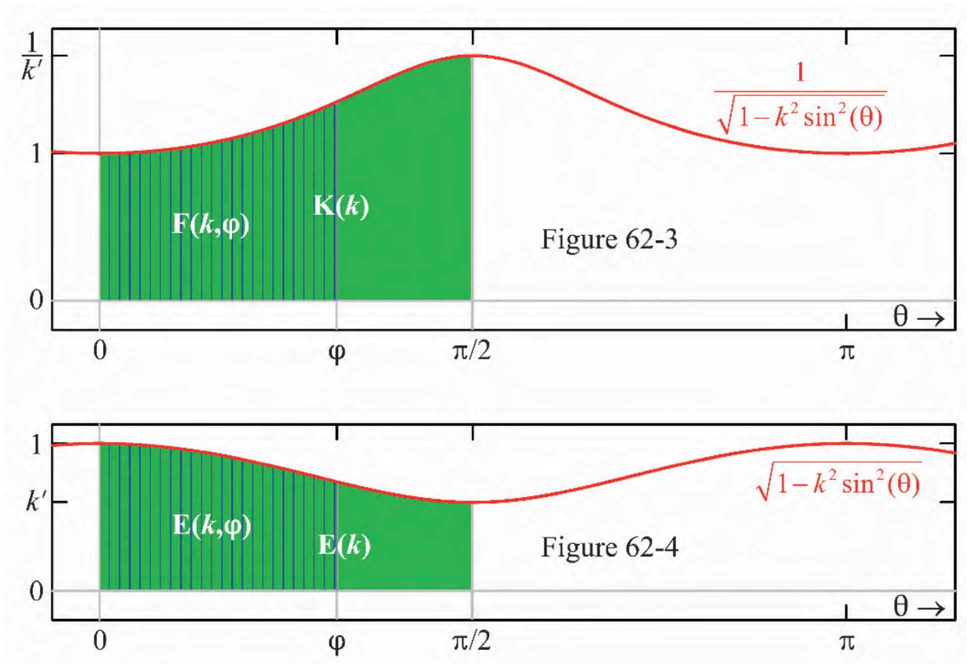
As their names imply, the most common definitions of the incomplete elliptic integrals are as integrals, typically the indefinite integrals in equations 62:3:1 and 62:3:2. The restriction $-1 \le k \le 1$ is general. Moreover, of these six integrals, all but the primary definitions (those that have φ as their upper bound) are limited in domain to $|\varphi| < \pi/2$.

$$\text{62:3:1}\qquad \mathrm{F}(k,\varphi)=\int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\,\mathrm{d}\theta=\int_0^{\sin(\varphi)}\frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,\mathrm{d}t=\int_0^{\tan(\varphi)}\frac{1}{\sqrt{1+t^2}\sqrt{1+(k')^2t^2}}\,\mathrm{d}t$$

$$\text{62:3:2}\qquad \mathrm{E}(k,\varphi)=\int_0^{\varphi}\sqrt{1-k^2\sin^2(\theta)}\,\mathrm{d}\theta=\int_0^{\sin(\varphi)}\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t=\int_0^{\tan(\varphi)}\frac{\sqrt{1+(k')^2t^2}}{\sqrt{\left(1+t^2\right)^3}}\,\mathrm{d}t$$

$$\left.\right\}\quad k'=\sqrt{1-k^2}$$

Figures 62-3 and 62-4 illustrate the primary definitions and the corresponding definitions of the *complete* elliptic integrals [Chapter 61].

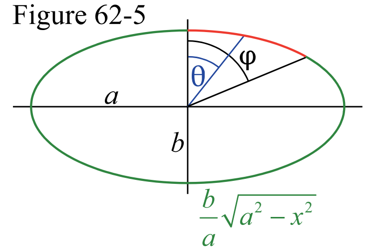


As suggested by their names, the incomplete elliptic integrals are related to the geometry of the ellipse and geometric definitions of F(k,φ) and E(k,φ) exist. Inspect the ellipse shown in Figure 62-5. By exploiting the formula in Section 39:14, the red segment of its perimeter is found to have a length which is related to the angle φ through

$$\text{62:3:2}\qquad \begin{matrix}\text{length of the}\\ \text{red segment}\end{matrix}=\int_0^{\varphi}\sqrt{a^2-(a^2-b^2)\sin^2(\theta)}\,\mathrm{d}\theta=a\,\mathrm{E}(k,\varphi)\qquad k=\frac{\sqrt{a^2-b^2}}{a}$$

Thus the second kind of incomplete elliptic integral is defined geometrically. The endpoint of that red segment can be considered to have reached its present position by traveling along the red elliptic path from the zenith of the ellipse. During this journey, the length of the blue line joining the origin to the moving point will have steadily increased as the angle θ opened from zero to φ. Then, because the instantaneous length of the line is $ab/\sqrt{a^2-(a^2-b^2)\sin^2(\theta)}$, the

Figure 62-5



$$\text{62:3:3}\qquad \begin{matrix}\text{average length}\\ \text{of the blue line}\end{matrix}=\frac{1}{\varphi}\int_0^{\varphi}\frac{ab}{\sqrt{a^2-(a^2-b^2)\sin^2(\theta)}}\,\mathrm{d}\theta=\frac{b}{\varphi}\mathrm{F}(k,\varphi)$$

This provides a geometric definition of the first kind of incomplete elliptic integral.

The incomplete elliptic functions are closely related to the integrals of reciprocal square-roots of the cubic function, in ways explained in Section 62:14. This is how the F(k,φ), E(k,φ) and Π(v,k,φ) functions were discovered and these integrals can serve as definitions.

62:4 SPECIAL CASES

The incomplete elliptic integrals become equal to their amplitudes when their moduli are zero,

62:4:1 $$F(0,\varphi) = E(0,\varphi) = \varphi$$

to the *inverse gudermannian* [Section 33:14] or *sine* [Chapter 32] functions when their moduli are unity

62:4:2 $$F(1,\varphi) = \mathrm{invgd}(\varphi) = \ln\left(\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi)\right) \qquad |\varphi| < \tfrac{1}{2}\pi$$

62:4:3 $$E(1,\varphi) = 2n + (-)^n \sin(\varphi) \qquad \text{all } \varphi, \quad n = \mathrm{Int}\left(\frac{\varphi}{\pi} + \frac{1}{2}\right)$$

and to various instances of the *incomplete beta function* [Chapter 58] when k equals $2^{\pm 1/2}$

62:4:4 $$F\left(\sqrt{2},\varphi\right) = \tfrac{1}{4}B\left(\tfrac{1}{2},\tfrac{1}{4},\sin^2(2\varphi)\right) \qquad 0 \le \varphi \le \tfrac{1}{4}\pi$$

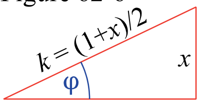
62:4:5 $$E\left(\sqrt{2},\varphi\right) = \tfrac{1}{4}B\left(\tfrac{1}{2},\tfrac{3}{4},\sin^2(2\varphi)\right) \qquad 0 \le \varphi \le \tfrac{1}{4}\pi$$

62:4:6 $$F\left(\frac{1}{\sqrt{2}},\varphi\right) = \frac{1}{\sqrt{8}}B\left(\tfrac{1}{2},\tfrac{1}{4},1-\cos^4(\varphi)\right) \qquad 0 \le \varphi \le \tfrac{1}{2}\pi$$

62:4:7 $$E\left(\frac{1}{\sqrt{2}},\varphi\right) = \frac{1}{2}F\left(\frac{1}{\sqrt{2}},\varphi\right) + \frac{1}{\sqrt{32}}B\left(\tfrac{1}{2},\tfrac{3}{4},1-\cos^4(\varphi)\right) \qquad 0 \le \varphi < \tfrac{1}{2}\pi$$

When the modulus and amplitude are interrelated such that $k = (1+x)/2$ where $x = k\sin(\varphi)$, the first kind of incomplete elliptic integral is a simple semiintegral [Section 12:14]:

Figure 62-6



62:4:8 $$\sqrt{\frac{2}{\pi}}F(k,\varphi) = \frac{d^{-1/2}}{dx^{-1/2}}\frac{1}{\sqrt{1-x^2}}$$

62:5 INTRARELATIONSHIPS

The reflection formulas

62:5:1 $$f(-k,\varphi) = f(k,\varphi) \qquad f = F \text{ or } E$$

and

62:5:2 $$f(k,-\varphi) = -f(k,\varphi) \qquad f = F \text{ or } E$$

show the incomplete elliptic integrals to be even with respect to their modulus but odd with respect to amplitude. Equation 62:5:2 is subsumed in the more general reflection formulas

62:5:3 $$F\left(k,\tfrac{1}{2}n\pi - \varphi\right) = 2nK(k) - F\left(k,\tfrac{1}{2}n\pi + \varphi\right) \qquad n = 0,\pm1,\pm2,\cdots$$

and

62:5:4 $$E\left(k,\tfrac{1}{2}n\pi - \varphi\right) = 2nE(k) - E\left(k,\tfrac{1}{2}n\pi + \varphi\right) \qquad n = 0,\pm1,\pm2,\cdots$$

Figures 62-3 and 62-4 help to illustrate the recursion formulas

62:5:5 $$F\left(k,\varphi + n\pi\right) = 2nK(k) + F\left(k,\varphi\right) \qquad n = 0,\pm1,\pm2,\cdots$$

and

62:5:6 $$E\left(k,\varphi + n\pi\right) = 2nE(k) + E\left(k,\varphi\right) \quad n = 0,\pm1,\pm2,\cdots$$

which are responsible for the periodicity noted in equation 62:2:1. Valuable formulas express the difference between a complete and an incomplete elliptic integral in terms of functions of an auxiliary amplitude ψ:

62:5:7 $$\mathrm{K}(k)-\mathrm{F}(k,\varphi)=\mathrm{F}(k,\psi)$$

62:5:8 $$\mathrm{E}(k)-\mathrm{E}(k,\varphi)=\mathrm{E}(k,\psi)-k^2\sin(\varphi)\sin(\psi)$$

$$\sin(\psi)=\frac{\cos(\varphi)}{\sqrt{1-k^2\sin^2(\varphi)}} \qquad |\varphi|<\frac{\pi}{2}$$

Two important transformations are attributed to Englishman John Landen (a surveyor and amateur mathematician, 1719–1790). The idea behind these transformations is that two incomplete elliptic integrals will be related to each other if their moduli and amplitudes are themselves suitably interrelated. Here we let two colors distinguish between the members of an interrelated pair. The required link between the two moduli is

62:5:9 $$\frac{1-k'}{1+k'}=k \qquad \text{or equivalently} \qquad \frac{2\sqrt{k}}{1+k}=k$$

where, as always, k' means $\sqrt{1-k^2}$. The paired amplitudes need to obey

62:5:10 $$\varphi+\arctan\left(k'\tan(\varphi)\right)=\varphi \qquad \text{or equivalently} \qquad \frac{\varphi+\arcsin\left(k\sin(\varphi)\right)}{2}=\varphi$$

For example, amplitude $k=1/4$ is paired with $k=4/5$ and $\varphi=1.0000$ with $\varphi=0.60598$. If these relationships exist, then it can be demonstrated that the incomplete elliptic integrals of the first kind are interrelated by

62:5:11 $$(1+k')\mathrm{F}(k,\varphi)=\mathrm{F}(k,\varphi) \qquad \text{or equivalently} \qquad \frac{1+k}{2}\mathrm{F}(k,\varphi)=\mathrm{F}(k,\varphi)$$

Continuing with our example, one finds F(k,φ) = 1.0087 and F(k,φ) = 0.63045. There is a similar, but more elaborate, relation between E(k,φ) and E(k,φ), namely

62:5:12 $$\frac{2[\mathrm{E}(k,\varphi)+k'\mathrm{F}(k,\varphi)]}{1+k'}-\frac{(1-k')\sin(\varphi)\cos(\varphi)}{\sqrt{1-k^2\sin^2(\varphi)}}=\mathrm{E}(k,\varphi)$$

and the equivalent inverse transformation is more complicated still. Geometrically, the different colors correspond to a pair of ellipses of differing eccentricity. Calculating red functions from blue corresponds to decreasing the eccentricity of the ellipse and is called *descending Landen transformation*. Conversely going from red to blue constitutes an *ascending Landen transformation*.

Of course, sequences of transformations are possible and such recurrent Landen transformation is computationally useful. As one proceeds through a sequence, the modulus rapidly approaches either zero (for the descending option, corresponding to the ellipse having progressed towards a circle) or unity (for the ascending option, corresponding to the ellipse approaching a straight line segment). Eventually, one of formulas 62:9:5–8 may be used to provide an accurate value for the final, and thence the original, incomplete elliptic integral.

Thus far in this section, as in most of the chapter and in most applications, it has been implicitly assumed that $0\le k\le 1$. However, the properties of incomplete elliptic integrals of moduli greater than unity are accessible through the formulas

62:5:13 $$\mathrm{F}\left(\frac{1}{k},\varphi\right)=k\,\mathrm{F}(k,\chi)$$

62:5:14 $$\mathrm{E}\left(\frac{1}{k},\varphi\right)=\frac{1}{k}\mathrm{E}(k,\chi)-\frac{1-k^2}{k}\mathrm{F}(k,\chi)$$

$$0<k\le 1 \qquad \chi=\arcsin\left\{\frac{\sin(\varphi)}{k}\right\}$$

Some authorities regard the incomplete elliptic functions as complex-valued when their moduli exceed unity, with an imaginary component in addition to the real component given by 62:5:13 and 62:5:14.

62:6 EXPANSIONS

With coefficients defined by the recursions

62:6:1 $$a_j = a_{j-1} - \left[\frac{(2j-1)!!}{(2j)!!}k^j\right]^2 \qquad a_0 = \frac{2\mathrm{K}(k)}{\pi} - 1$$

and

62:6:2 $$b_j = b_{j-1} - \frac{1}{2j-1}\left[\frac{(2j-1)!!}{(2j)!!}k^j\right]^2 \qquad b_0 = 1 - \frac{2\mathrm{E}(k)}{\pi}$$

the expansions

62:6:3 $$\mathrm{F}(k,\varphi) = (1+a_0)\varphi - \sin(\varphi)\cos(\varphi)\sum_{j=0}^{\infty} a_j \frac{(2j)!!}{(2j+1)!!}\sin^{2j}(\varphi)$$

and

62:6:4 $$\mathrm{E}(k,\varphi) = (1-b_0)\varphi + \sin(\varphi)\cos(\varphi)\sum_{j=0}^{\infty} b_j \frac{(2j)!!}{(2j+1)!!}\sin^{2j}(\varphi)$$

are particularly useful for small moduli, for which convergence is rapid.

Conversely, the expansions

62:6:5 $$\mathrm{F}(k,\varphi) = (1+\alpha_0)\mathrm{invgd}(\varphi) - \frac{\tan(\varphi)}{\cos(\varphi)}\sum_{j=0}^{\infty}\alpha_j \frac{(2j)!!}{(2j+1)!!}[-\tan^2(\varphi)]^j$$

and

62:6:6 $$\mathrm{E}(k,\varphi) = \frac{1-\cos(\varphi)\sqrt{1-k^2\sin^2(\varphi)}}{\sin(\varphi)} + \beta_0\mathrm{invgd}(\varphi) + \frac{\tan(\varphi)}{\cos(\varphi)}\sum_{j=0}^{\infty}\beta_{j+1}\frac{(2j)!!}{(2j+1)!!}[-\tan^2(\varphi)]^j$$

are useful when the modulus is close to unity and the amplitude does not exceed π/4. Here the coefficients obey the recursions:

62:6:7 $$\alpha_j = \alpha_{j-1} - \left[\frac{(2j-1)!!}{(2j)!!}\right]^2 (1-k^2)^j \qquad \alpha_0 = \frac{2}{\pi}\mathrm{K}\left(\sqrt{1-k^2}\right) - 1$$

62:6:8 $$\beta_j = \beta_{j-1} - \frac{2j-1}{2j}\left[\frac{(2j-3)!!}{(2j-2)!!}\right]^2 (1-k^2)^j \qquad \beta_0 = \frac{2}{\pi}\left[\mathrm{K}\left(\sqrt{1-k^2}\right) - \mathrm{E}\left(\sqrt{1-k^2}\right)\right]$$

The formulas in this section involve the *double factorial* and *inverse gudermannian functions* [Sections 2:13 and 33:15].

62:7 PARTICULAR VALUES

When their amplitudes are zero, so also are the incomplete elliptic integrals.

62:7:1 $$\mathrm{F}(k,0) = \mathrm{E}(k,0) = 0$$

These functions acquire their "complete" status when the amplitude φ equals π/2:

62:7:2 $$\mathrm{F}(k,\tfrac{1}{2}\pi) = \mathrm{K}(k) \qquad \text{and} \qquad \mathrm{E}(k,\tfrac{1}{2}\pi) = \mathrm{E}(k)$$

and these relationships may be generalized to

62:7:3 $$F\left(k,\tfrac{1}{2}n\pi\right) = n\mathrm{K}(k) \qquad -1<k<1$$

62:7:4 $$E\left(k,\tfrac{1}{2}n\pi\right) = n\mathrm{E}(k) \qquad -1\le k\le 1$$

$$n = 0,\pm1,\pm2,\cdots$$

62:8 NUMERICAL VALUES

A popular numerical procedure for calculating values of incomplete elliptic integrals relies on *Landen's descending transformation* [Section 62:5] implemented through procedures described in Section 61:14]. In that latter section, we define the means mg(, ,) and ma(, ,), values of both of which are used to compute a succession of φ_j values through the formula

62:8:1 $$\varphi_{j+1} = \varphi_j + \frac{1}{2^{j+1}}\arctan\left(\frac{\left[\mathrm{ma}(j,k',1)-\mathrm{mg}(j,k',1)\right]\tan(2^j\varphi_j)}{\mathrm{ma}(j,k',1)+\mathrm{mg}(j,k',1)\tan^2(2^j\varphi_j)}\right) \qquad \begin{array}{c}\mathrm{ma}(0,k',1)=1\\ \mathrm{mg}(0,k',1)=k'\\ \varphi_0=\varphi\end{array}$$

This recursion generates values that rapidly approach a limit from which the first kind of incomplete integral is calculable

62:8:2 $$F(k,\varphi) = \frac{2\mathrm{K}(k)}{\pi}\lim_{j\to\infty}\varphi_j$$

To compute the incomplete elliptic integral of the second kind, the formula

62:8:3 $$E(k,\varphi) = \frac{2\mathrm{E}(k)}{\pi}\lim_{j\to\infty}\left\{\varphi_j\right\} + \frac{1}{2}\sum_{j=0}^{\infty}\left[\mathrm{ma}(j,k',1)-\mathrm{mg}(j,k',1)\right]\sin\left(2^{j+1}\varphi_{j+1}\right)$$

is used.

This Landen approach is adopted, when $0.9\le k<1$, by *Equator*'s routines for the incomplete elliptic integral functions of the first and second kinds. The keywords **ellipF** and **ellipE** were designed to provide the trite mnemonic that, because the integrals are incomplete, so are the keywords. In contrast, the keywords for the complete elliptic integrals include the complete word "Elliptic". For $k<0.9$, straightforward numerical integration of the leftmost integrals in formulas 62:3:1 and 62:3:2 is employed, the Romberg procedure described in Section 62:15 being implemented. However, in computing F(k,φ) for $\varphi<0.0004$, the two-term expansion 62:9:1 replaces the Romberg method. For $|\varphi|>\pi/2$, recursion 62:5:5 or 62:5:6 is adopted.

62:9 LIMITS AND APPROXIMATIONS

For values of the amplitude close to 0 or $\tfrac{1}{2}\pi$, the following limiting approximations are applicable

62:9:1 $$F(k,\varphi)\approx\varphi+\tfrac{1}{6}k^2\varphi^3\pm\cdots$$

62:9:2 $$E(k,\phi)\approx\varphi-\tfrac{1}{6}k^2\varphi^3+\cdots$$

φ small

62:9:3 $$F(k,\varphi)\approx \mathrm{K}(k)-\{1/k'\}\left[\tfrac{1}{2}\pi-\varphi\right]+\left\{k^2/6(k')^3\right\}\left[\tfrac{1}{2}\pi-\varphi\right]^3-\cdots$$

62:9:4 $$E(k,\phi)\approx \mathrm{E}(k)-k'\left[\tfrac{1}{2}\pi-\varphi\right]-\left\{k^2/6k'\right\}\left[\tfrac{1}{2}\pi-\varphi\right]^3-\cdots$$

$(\tfrac{1}{2}\pi-\varphi)$ small, $k'=\sqrt{1-k^2}$

As the modulus of the incomplete elliptic integrals approaches zero, the limiting approximations

62:9:5 $$F(k,\varphi) \approx \varphi + \tfrac{1}{4}\{\varphi - \sin(\varphi)\cos(\varphi)\}k^2 + \cdots$$

62:9:6 $$E(k,\phi) \approx \varphi - \tfrac{1}{4}\{\varphi - \sin(\varphi)\cos(\varphi)\}k^2 - \cdots \qquad k \text{ small}$$

hold. The approaches of the incomplete elliptic integrals to their limiting values as the modulus approaches unity are governed by

62:9:7 $$F(k \to 1,\varphi) = \mathrm{invgd}(\varphi) + \tfrac{1}{2}[\mathrm{invgd}(\varphi) - \sec(\varphi)\tan(\varphi)](1-k)$$

and

62:9:8 $$E(k \to 1,\varphi) = \sin(\varphi) + \left[\mathrm{invgd}(\varphi) - \sin(\varphi)\right](1-k)$$

62:10 OPERATIONS OF THE CALCULUS

The formulas

62:10:1 $$\frac{d}{d\varphi}F(k,\varphi) = \frac{1}{\sqrt{1-k^2\sin^2(\varphi)}}$$

and

62:10:2 $$\frac{d}{d\varphi}E(k,\varphi) = \sqrt{1-k^2\sin^2(\varphi)}$$

describe the differentiation of the incomplete elliptic integrals with respect to their amplitudes. With respect to the modulus, the derivatives are

62:10:3 $$\frac{\partial}{\partial k}F(k,\varphi) = \frac{E(k,\varphi)}{k-k^3} - \frac{F(k,\varphi)}{k} - \frac{k\sin(\varphi)\cos(\varphi)}{\left(1-k^2\right)\sqrt{1-k^2\sin^2(\varphi)}}$$

and

62:10:4 $$\frac{\partial}{\partial k}E(k,\varphi) = \frac{E(k,\varphi) - F(k,\varphi)}{k}$$

Formulas for indefinite integration of F(k,φ) or E(k,φ) themselves do not exist, but related indefinite integrals include

62:10:5 $$\int_0^{\varphi} \sin(\theta)F(k,\theta)\,d\theta = \frac{\arcsin\left(k\sin(\varphi)\right)}{k} - \cos(\varphi)F(k,\varphi)$$

62:10:6 $$\int_0^{\varphi} \sin(\theta)E(k,\theta)\,d\theta = \frac{\arcsin\left(k\sin(\varphi)\right)}{2k} - \cos(\varphi)E(k,\varphi) + \frac{\sin(\varphi)}{2}\sqrt{1-k^2\sin^2(\varphi)}$$

62:10:7 $$\int_0^{\varphi} \frac{F(k,\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\,d\theta = \frac{1}{2}F^2(k,\varphi)$$

The integrals above are with respect to amplitude; those following are with respect to modulus,

62:10:8 $$\int_0^{k} t\,F(t,\varphi)\,dt = E(k,\varphi) - (1-k^2)F(k,\varphi) - \left[1-\sqrt{1-k^2\sin^2(\varphi)}\right]\cot(\varphi)$$

and

62:10:9
$$\int_0^k t\left[3\mathrm{E}(t,\varphi)-\mathrm{F}(t,\varphi)\right]\mathrm{d}t = k^2\,\mathrm{E}(k,\varphi)$$

See Gradshteyn & Ryzhik [Sections 5.12 and 6.111−113] for several other indefinite integrals and many definite integrals.

62:11 COMPLEX ARGUMENT

Incomplete elliptic integrals of complex argument are rather complicated quadrivariate functions. Here we confine attention to cases of imaginary amplitudes or imaginary moduli.

For imaginary amplitude, the incomplete elliptic functions are themselves imaginary:

62:11:1
$$\mathrm{F}(k,i\varphi) = i\,\mathrm{F}(k',\mathrm{gd}(\varphi)) \qquad k' = \sqrt{1-k^2}$$

62:11:2
$$\mathrm{E}(k,i\varphi) = i\left[\mathrm{F}\left(k',\mathrm{gd}(\varphi)\right) - \mathrm{E}\left(k',\mathrm{gd}(\varphi)\right) + \tanh(\varphi)\sqrt{1+k^2\sinh^2(\varphi)}\right]$$

Hyperbolic functions and the gudermannian function [Section 33:15] appear in 62:11:2.

When it is the modulus that is imaginary, the incomplete elliptic functions are real:

62:11:3
$$k''\,\mathrm{F}(ik,\varphi) = \mathrm{K}\left(\frac{k}{k''}\right) - \mathrm{F}\left(\frac{k}{k''},\frac{\pi}{2}-\varphi\right) = \mathrm{F}\left(\frac{k}{k''},\Upsilon\right) \qquad k'' = \sqrt{1+k^2}$$

62:11:4
$$\frac{\mathrm{E}(ik,\varphi)}{k''} = \mathrm{E}\left(\frac{k}{k''}\right) - \mathrm{E}\left(\frac{k}{k''},\frac{\pi}{2}-\varphi\right) = \mathrm{E}\left(\frac{k}{k''},\Upsilon\right) - \frac{k^2\sin(\Upsilon)\cos(\Upsilon)}{k''\sqrt{1+k^2\cos^2(\Upsilon)}}$$

The auxiliary amplitude Υ that provides an alternative formulation in these equations is $\arctan\{k''\tan(\varphi)\}$.

There are several other imaginary transformations. See Gradshteyn & Ryzhik [Table 8.127] for these.

62:12 GENERALIZATIONS: the incomplete elliptic integral of the third kind

The trivariate function defined by

62:12:1
$$\Pi(\nu,k,\varphi) = \int_0^{\varphi} \frac{1}{[1-\nu\sin^2(\theta)]\sqrt{1-k^2\sin^2(\theta)}}\mathrm{d}\theta = \int_0^{\sin(\varphi)} \frac{1}{\left(1-\nu t^2\right)\sqrt{1-t^2}\sqrt{1-k^2t^2}}\mathrm{d}t$$

is named the *incomplete elliptic integral of the third kind*. The second equality requires $|\varphi| < \pi/2$. Inasmuch as

62:12:2
$$\Pi(0,k,\varphi) = \mathrm{F}\left(k,\varphi\right) \qquad \text{and} \qquad \Pi\left(\nu,k,\tfrac{1}{2}\pi\right) = \Pi(\nu,k)$$

this function may be regarded as a generalization of either the first kind of incomplete elliptic function or the third kind of complete elliptic integral [Section 61:12]. Beware of notational inconsistency among different authors. In particular, note that the sign attributed to the *characteristic* ν may be the opposite of that used here. Moreover, it may be the reciprocal of our ν, or its square root, that is regarded as the variable of record.

For $\nu > 1$, the integral in 62:12:1 converges only if $\varphi < \arcsin\left(1/\sqrt{\nu}\right)$. In this section of the *Atlas*, and in *Equator*, attention is confined to the following domains

62:12:3
$$0 \le k \le 1, \begin{cases} \nu \le 1, & |\varphi| < \frac{1}{2}\pi \\ \nu > 1, & |\varphi| < \arcsin\left(1/\sqrt{\nu}\right) \end{cases}$$

within which the third kind of incomplete elliptic integral is invariably real. When the variables acquire certain special values (other than those already cited in 62:12:2) the third kind of incomplete elliptic integral is expressible as in the panels below, though unstated restrictions may apply.

$\Pi(k^2,k,\varphi)$	$\Pi(1,k,\varphi)$
$\dfrac{\mathrm{E}(k,\varphi)}{1-k^2}-\dfrac{k^2\sin(\varphi)\cos(\varphi)}{(1-k^2)\sqrt{1-k^2\sin^2(\varphi)}}$	$\mathrm{F}(k,\varphi)-\dfrac{\mathrm{E}(k,\varphi)}{1-k^2}+\dfrac{\tan(\varphi)}{1-k^2}\sqrt{1-k^2\sin^2(\varphi)}$

$\Pi(v,0,\varphi)$		$\Pi(v,1,\varphi),\ v\neq 1$	
$\dfrac{\operatorname{arctan}\left\{\sqrt{1-v}\tan(\varphi)\right\}}{\sqrt{1-v}}$	$v<1$	$\dfrac{\operatorname{invgd}(\varphi)}{1-v}+\dfrac{\sqrt{-v}}{1-v}\operatorname{arctan}\left\{\sqrt{-v}\sin(\varphi)\right\}$	$v<0$
$\dfrac{\operatorname{artanh}\left\{\sqrt{v-1}\tan(\varphi)\right\}}{\sqrt{v-1}}$	$v>1$	$\dfrac{\operatorname{invgd}(\varphi)}{1-v}-\dfrac{\sqrt{v}}{1-v}\operatorname{artanh}\left\{\sqrt{v}\sin(\varphi)\right\}$	$v>0$

Involving the *complete* elliptic integral of the third kind and the incomplete elliptic integral of the first kind, the formula

62:12:4 $$\Pi(v,k,\varphi)=\Pi(v,k)-\frac{k^2}{k^2-v}\mathrm{F}(k,\psi)+\frac{v(1-k^2)}{(1-v)(k^2-v)}\Pi\left(\frac{k^2-v}{1-v},k,\psi\right) \qquad \sin(\psi)=\frac{\cos(\varphi)}{\sqrt{1-k^2\sin^2(\varphi)}}$$

interrelates two incomplete elliptic integrals that differ in characteristic and amplitude. Abramowitz and Stegun [pages 599 and 600] present formulas by which the incomplete elliptic integrals of the third kind may be expressed via the elliptic nome [Section 61:15].

Though it is complicated, the series

62:12:5 $$\Pi(v,k,\varphi)=\sum_{j=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_j}{(1)_j}\left(k^2\right)^j\sum_{n=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)_{j+n}}{(1)_{j+n}}v^n\left(\varphi-\frac{\sin(2\varphi)}{2}\sum_{m=1}^{j+n-1}\frac{(1)_m}{\left(\tfrac{3}{2}\right)_m}\left[\sin^2(\varphi)\right]^m\right)$$

has computational utility.

Numerical values are accessible via *Equator*'s incomplete elliptic integral of the third kind routine (keyword **ellipPi**). The algorithm generally employs *segmented Romberg integration* [Section 62:15] of the first integral in formula 62:12:1 at an approximation order of 8. *Equator* accepts inputs that conform to the restriction in 62:12:3.

62:13 COGNATE FUNCTIONS

The inverse Jacobian elliptic functions [Section 63:13] provide examples of incomplete elliptic integrals of the first kind.

Two functions that are closely related to the incomplete elliptic integrals are *Jacobi's zeta function*

62:13:1 $$\mathrm{E}(k,\varphi)-\frac{\mathrm{E}(k)\,\mathrm{F}(k,\varphi)}{\mathrm{K}(k)}$$

and *Heumann's lambda function*

62:13:2
$$\frac{2}{\pi}\left[\mathrm{K}(k)\,\mathrm{E}(k',\varphi)-\left\{\mathrm{K}(k)-\mathrm{E}(k)\right\}\mathrm{F}(k',\varphi)\right] \qquad k'=\sqrt{1-k^2}$$

Respectively, these are customarily symbolized Z(k,φ) and $\Lambda_0(k,\varphi)$ but neither finds use in this *Atlas*. Nor does *Equator* cater to either function. However, their values are easily calculated "by hand" from the elliptic integrals that are components of formulas 62:13:1 and 62:13:2.

62:14 RELATED TOPIC: codifying Abelian integrals

Historically, incomplete elliptic integrals arose as the outcome of a successful attempt by mathematicians to codify the indefinite integrals of rational functions [Section 17:12] of two variables, t and u, where u^2 is either a cubic or quartic function of t. Such integrals are now known as Abelian integrals (Niels Henrik Abel, Norwegian mathematician, 1802 − 1829). A typical Abelian integral might be

62:14:1
$$\int^{x}\frac{\mathrm{p}(t)+u\,\mathrm{p}'(t)}{\mathrm{p}''(t)+u\,\mathrm{p}'''(t)}\,\mathrm{d}t$$

where the p's are arbitrary polynomials. In a taxing piece of algebra [summarized in Erdélyi et al., *Higher Transcendental Functions*, Chapter 13], Legendre proved that any Abelian integral could be reduced to elementary functions together with one or more standard integrals of the following three forms

62:14:2
$$\int\frac{\mathrm{d}t}{u},\quad \int\frac{t\,\mathrm{d}t}{u},\quad \int\frac{\mathrm{d}t}{(t-\mathrm{Constant})u} \qquad \text{where}\quad \pm u^2=\begin{cases}\mathrm{p}_3(t)=t^3+at^2+bt+c & \text{or}\\ \mathrm{p}_4(t)=t^4+a_3t^3+a_2t^2+a_1t+a_0\end{cases}$$

Addressing only the cubic option, here we shall illustrate how these three standard forms are expressible as incomplete elliptic integrals of the first, second, and third kinds. The treatment is not exhaustive, though possibly exhausting.

Consider the first standard integral, $\int \mathrm{d}t/\sqrt{\pm[t^3+at^2+bt+c]}$. If, as we assume, the coefficients a, b, and c are real, then the cubic function in the integrand's denominator has five special cases as enumerated in Section 16:4. We ignore these simpler possibilities and concentrate on the general cases. The properties of cubic functions [Chapter 16] then ensure that the $1/\sqrt{\pm\mathrm{p}_3(t)}$ function develops either one or three singularities. The triple singularity possibility will be addressed first.

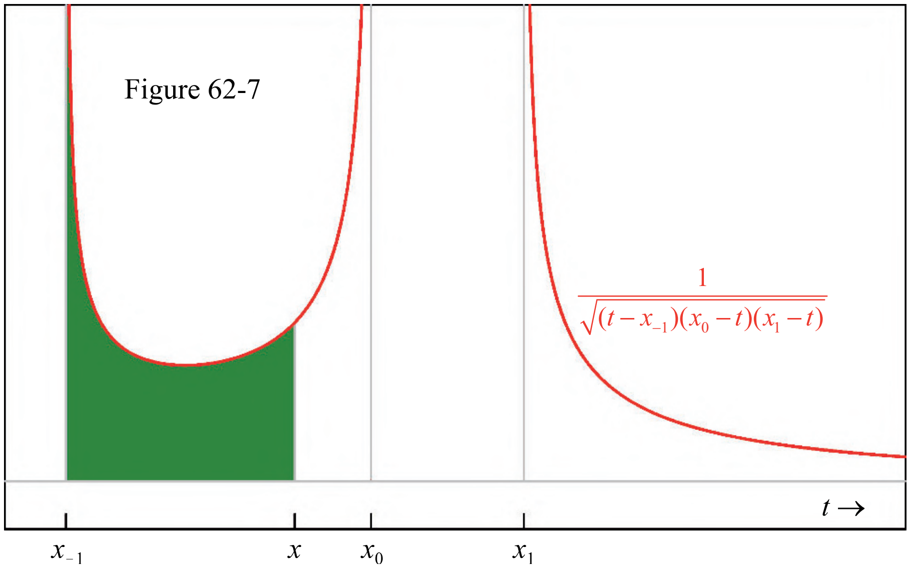
The locations of these singularities will be symbolized $t=x_{-1}$, x_0, and x_1, where $x_{-1}<x_0<x_1$; they are the zeros of the cubic function and are calculable via the formulas in Section 16:7. The three singularities appear in Figure 62-7 overleaf, which is a typical graph of the function $1/\sqrt{t^3+at^2+bt+c}$. The function is imaginary in the figure's vacant zones, and the three singularities are therefore of the real|imaginary variety. It is convenient to choose bounds for the indefinite integrals $\int \mathrm{d}t/\sqrt{\mathrm{p}_3(t)}$ at the singularities. There are then four appropriate integrals

62:14:3
$$\mathrm{I}_3=\int_{x_{-1}}^{x}\frac{\mathrm{d}t}{\sqrt{\mathrm{p}_3(t)}},\qquad \mathrm{I}_4=\int_{x}^{x_0}\frac{\mathrm{d}t}{\sqrt{\mathrm{p}_3(t)}},\qquad \mathrm{I}_7=\int_{x_1}^{x}\frac{\mathrm{d}t}{\sqrt{\mathrm{p}_3(t)}},\qquad \text{and}\qquad \mathrm{I}_8=\int_{x}^{\infty}\frac{\mathrm{d}t}{\sqrt{\mathrm{p}_3(t)}}$$

In the first two of these integrals $x_{-1}<x<x_0$; in the second pair, $x_1<x<\infty$. Had we addressed the integrand $1/\sqrt{-[t^3+at^2+bt+c]}$, a different set of indefinite integrals, namely

62:14:4
$$\mathrm{I}_1=\int_{-\infty}^{x}\frac{\mathrm{d}t}{\sqrt{-\mathrm{p}_3(t)}},\qquad \mathrm{I}_2=\int_{x}^{x_{-1}}\frac{\mathrm{d}t}{\sqrt{-\mathrm{p}_3(t)}},\qquad \mathrm{I}_5=\int_{x_0}^{x}\frac{\mathrm{d}t}{\sqrt{-\mathrm{p}_3(t)}},\qquad \text{and}\qquad \mathrm{I}_6=\int_{x}^{x_1}\frac{\mathrm{d}t}{\sqrt{-\mathrm{p}_3(t)}}$$

would have eventuated and the corresponding curve would have occupied the zones presently vacant in Figure 62-7.



The colored area in this diagram represents the I_3 integral, which will serve as our exemplar. The final equality in

62:14:5
$$I_3 = \int_{x_{-1}}^{x} \frac{dt}{\sqrt{p_3(t)}} = \int_{x_{-1}}^{x} \frac{dt}{\sqrt{t^3+at^2+bt+c}} = \int_{x_{-1}}^{x} \frac{dt}{\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}}$$

follows from a fundamental property [equation 17:3:3] of polynomials. Now define θ, φ, k, and w such that

62:14:6
$$\frac{t-x_{-1}}{x_1-x_{-1}} = \sin^2(\theta), \qquad \frac{x-x_{-1}}{x_1-x_{-1}} = \sin^2(\varphi), \qquad \frac{x_0-x_{-1}}{x_1-x_{-1}} = k^2, \qquad \text{and} \qquad \frac{4}{x_1-x_{-1}} = w^2$$

Then it is straightforward to demonstrate that

62:14:7
$$\int_{x_{-1}}^{x} \frac{dt}{\sqrt{p_3(t)}} = \int_{x_{-1}}^{x} \frac{dt}{\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}} = w\int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} = w\mathrm{F}(k,\varphi)$$

By these changes in variable, it has been established that, apart from the multiplier w, I_3 is simply an elliptic integral of the first kind. Similarly all the other seven indefinite integrals may be expressed as elliptic integrals of the first kind, with the same multiplier w, but different amplitudes. In all cases, the modulus is either k or its complement $k' = \sqrt{(x_1-x_0)/(x_1-x_{-1})}$, thus

62:14:8
$$I_j = w\mathrm{F}(k \text{ or } k', \varphi_j) \qquad j = 1,2,\cdots,7,8$$

The first eight rows of the table opposite summarize the final outcome of the integrations in 62:14:3 and 62:14:4.

Next, we need to consider the standard form $\int dt/\sqrt{\pm p_3(t)}$ when the cubic $p_3(t)$ has a single singularity, say at $t = x_0$. In this case, there are only four pertinent indefinite integrals

62:14:9
$$I_9 = \int_{-\infty}^{x} \frac{dt}{\sqrt{-p_3(t)}}, \qquad I_{10} = \int_{x}^{x_0} \frac{dt}{\sqrt{-p_3(t)}}, \qquad I_{11} = \int_{x_0}^{x} \frac{dt}{\sqrt{p_3(t)}}, \qquad \text{and} \qquad I_{12} = \int_{x}^{\infty} \frac{dt}{\sqrt{p_3(t)}}$$

Taking integral I_{10} as our example, the factoring this time is

62:14:10
$$I_{10} = \int_{x_0}^{x} \frac{dt}{\sqrt{p_3(t)}} = \int_{x_0}^{x} \frac{dt}{\sqrt{t^3+at^2+bt+c}} = \int_{x_0}^{x} \frac{dt}{\sqrt{t-x_0}\sqrt{(t-\rho)^2+\iota^2}}$$

where ρ and ±ι are the real and imaginary parts of the complex zero [Section 17:3] of $p_3(t)$, again calculable from the formulas in Section 16:7. Now if one defines

Integral I_j	Modulus	Amplitude φ_j
I_1	k'	$\arcsin\left(\sqrt{(x_1-x_{-1})/(x_1-x)}\right)$
I_2	k'	$\arccos\left(\sqrt{(x_0-x_{-1})/(x_0-x)}\right)$
I_3	k	$\arcsin\left(\sqrt{(x-x_{-1})/(x_0-x_{-1})}\right)$
I_4	k	$\arccos\left(\sqrt{k'(x-x_{-1})/k(x_1-x)}\right)$
I_5	k'	$\arcsin\left(\sqrt{(x-x_0)/k'(x-x_{-1})}\right)$
I_6	k'	$\arccos\left(\sqrt{(x-x_0)/(x_1-x_0)}\right)$
I_7	k	$\arcsin\left(\sqrt{(x-x_1)/(x-x_0)}\right)$
I_8	k	$\arccos\left(\sqrt{(x-x_1)/k(x-x_{-1})}\right)$
I_9	k'	$2\operatorname{arccot}\left(\sqrt{(x_0-x)/\sqrt{(x_0-\rho)^2+\iota^2}}\right)$
I_{10}	k'	$2\arctan\left(\sqrt{(x_0-x)/\sqrt{(x_0-\rho)^2+\iota^2}}\right)$
I_{11}	k	$2\arctan\left(\sqrt{(x-x_0)/\sqrt{(x_0-\rho)^2+\iota^2}}\right)$
I_{12}	k	$2\operatorname{arccot}\left(\sqrt{(x-x_0)/\sqrt{(x_0-\rho)^2+\iota^2}}\right)$

62:14:11 $$\frac{1}{\sqrt{(x_0-\rho)^2+\iota^2}}=w^2, \quad t-x_0=\frac{\tan^2\left(\frac{1}{2}\theta\right)}{w^2}, \quad x-x_0=\frac{\tan^2\left(\frac{1}{2}\varphi\right)}{w^2}, \quad \text{and} \quad k^2=\frac{1+(x_0-\rho)w^2}{2}$$

then it may be demonstrated that

62:14:12 $$\mathrm{I}_{10}=\int_{x_0}^{x}\frac{\mathrm{d}t}{\sqrt{\mathrm{p}_3(t)}}=\int_{x_0}^{x}\frac{\mathrm{d}t}{\sqrt{t-x_0}\sqrt{(t-\rho)^2+\iota^2}}=w\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2(\theta)}}=w\mathrm{F}(k,\varphi)$$

The I_{10} integral is seen to be expressible, as before, as an incomplete elliptic integral of the first kind, though the multiplier, modulus and amplitude are formulated somewhat differently than for the triple-singularity case. The I_9, I_{11}, and I_{12} integrals may be expressed as analogues of 62:14:12, as included in the table. A general equation analogous to 62:14:8 applies in these four cases too, but w, k, and k' require different interpretation.

To this point, only the first of the standard integral forms in 62:14:2 has been addressed. A less thorough description will be given of the reduction of the other two forms. Rather than attempting to be comprehensive, a single example of each will suffice.

To illustrate an integral of the form $\int t\,\mathrm{d}t/\sqrt{\mathrm{p}_3(t)}$, we choose an example that differs from I_3 only by the presence of a numeratorial t. This integral may first be split into two

62:14:13 $$\int_{x_{-1}}^{x} \frac{t\,\mathrm{d}t}{\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}} = x_1 \int_{x_{-1}}^{x} \frac{\mathrm{d}t}{\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}} - \int_{x_{-1}}^{x} \frac{\sqrt{x_1-t}\;\mathrm{d}t}{\sqrt{(t-x_{-1})(x_0-t)}}$$

The first right-hand moiety will now be recognized as an incomplete elliptic integral of the first kind, in fact $x_1\mathrm{I}_3$. The substitutions

62:14:14 $$\frac{t-x_{-1}}{x_0-x_{-1}} = \sin^2(\theta), \qquad \frac{x-x_{-1}}{x_0-x_{-1}} = \sin^2(\varphi), \qquad \frac{x_0-x_{-1}}{x_1-x_{-1}} = k^2, \qquad \text{and} \qquad 4\left(x_1-x_{-1}\right) = w^2$$

then transform the second moiety as follows

62:14:15 $$\int_{x_{-1}}^{x} \frac{\sqrt{x_1-t}\;\mathrm{d}t}{\sqrt{(t-x_{-1})(x_0-t)}} = w\int_0^{\varphi} \sqrt{1-k^2\sin^2(\theta)}\,\mathrm{d}\theta = w\mathrm{E}(k,\varphi)$$

generating an incomplete elliptic integral of the second kind. Thus the original integral turns out to be a weighted sum of incomplete elliptic integrals of the first and second kinds.

Again with a close resemblance to I_3, we shall choose

62:14:16 $$\int_{x_{-1}}^{x} \frac{\mathrm{d}t}{(t-C)\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}}$$

as an example of an integral that conforms to the third of the standard forms in 62:14:2. The substitutions

62:14:17 $$\frac{t-x_{-1}}{x_0-x_{-1}} = \sin^2(\theta), \quad \frac{r-x_{-1}}{x_0-x_{-1}} = \sin^2(\varphi), \quad \frac{x_0-x_{-1}}{x_1-x_{-1}}{}' = k^2, \quad \frac{x_0-x_{-1}}{x_{-1}-C} = v, \quad \frac{-2vk}{\left(x_1-x_{-1}\right)^{3/2}} = w$$

followed by considerable algebra, lead to

62:14:18 $$\int_{x_{-1}}^{x} \frac{\mathrm{d}t}{(t-C)\sqrt{(t-x_{-1})(x_0-t)(x_1-t)}} = \int_0^{\varphi} \frac{w\,\mathrm{d}\theta}{[1-v\sin^2(\theta)]\sqrt{1-k^2\sin^2(\theta)}} = w\Pi(v,k,\varphi)$$

and so convert the exemplary integral into the canonical form of the third kind of incomplete elliptic integral, with a multiplier.

62:15 RELATED TOPIC: Simpson's approximation and Romberg integration

Functions defined as integrals, such as the complete and incomplete elliptic integrals, can be evaluated by *numerical integration*, an approximation in which the integral is replaced by a weighted sum of function values. In rather special circumstances, the Gauss quadrature technique [Section 24:15] is useful. Otherwise, one or other of the two general approaches mentioned in Section 4:14 may be adopted. Each of these latter relies on summing values of the function at equally-spaced instances of the integration variable. Here we discuss refinements to that approach in which the function values, again at equally spaced nodes, are judiciously weighted prior to summation, so that

62:15:1 $$\int_0^1 \mathrm{f}(t)\,\mathrm{d}t \approx \sum_{j=0}^{2^k} w_j^{(k)}\,\mathrm{f}\left(\frac{j}{2^k}\right) \qquad \text{where} \qquad \sum_{j=0}^{2^k} w_j^{(k)} = 1$$

For simplicity, we have chosen the integration limits to be zero and unity and made the number of segments into which the integration domain is subdivided equal to an integer power of 2, namely 2^k. Here k is named the *approximation order*. In 62:15:1, the function is sampled at the junctions of the segments but, alternatively, the sampling may be made at the center of each segment. The number of sampled function values, or "points" as they are often called, is then either $2^k + 1$ or 2^k.

The *trapezoidal approximation* is described in Section 4:14 and its inherent error can be assessed from the *Euler-Maclaurin formula*, equation 4:14:1. For $x_0 = 0$ and $x_J = 1$, this formula involves $(\mathrm{B}_n/n!)\mathrm{d}^{n-1}\mathrm{f}/\mathrm{d}x^{n-1}|_0^1$ terms, the origins of which are explained in the cited section. However, the form of these terms has little significance here and the nth such term will be abbreviated to ε_n throughout this section. Moreover, the replacement $h = 2^{-k}$ is adopted to cater to the present needs, whereby rearrangement of equation 4:14:1 leads to

62:15:2
$$\Delta_{\mathrm{t}}^{(k,0)} = \int_0^1 \mathrm{f}(t)\,\mathrm{d}t - \frac{\mathrm{f}(0)+\mathrm{f}(1)}{2^{k+1}} - \sum_{j=1}^{2^k-1}\frac{\mathrm{f}(2^{-k}j)}{2^k} \sim \sum_{n=2,4}\frac{-\varepsilon_n}{2^{kn}} = -\frac{\varepsilon_2}{4^k} - \frac{\varepsilon_4}{16^k} - \frac{\varepsilon_6}{64^k} - \cdots$$

The leftmost symbol represents the error made in replacing the integral by a trapezoidal approximation of order k. Notice that the weight $w_j^{(k)}$ asigned to each point is 2^{-k} except for the end points, which have halved weights.

Section 4:14 also reports the *second Euler-Maclaurin formula*, which enables the error in the *midpoint approximation* to be expressed as

62:15:3
$$\Delta_{\mathrm{m}}^{(k,0)} = \int_0^1 \mathrm{f}(t)\,\mathrm{d}t - \sum_{j=0}^{2^k-1}\frac{\mathrm{f}\left((j+\tfrac{1}{2})2^{-k}\right)}{2^k} \sim \sum_{n=2,4}\left[\frac{1-2^{1-n}}{2^{kn}}\right]\varepsilon_n = \frac{\tfrac{1}{2}\varepsilon_2}{4^k} + \frac{\tfrac{7}{8}\varepsilon_4}{16^k} + \frac{\tfrac{31}{32}\varepsilon_6}{64^k} + \cdots$$

Generally, the magnitudes of the ε terms are in the sequence $|\varepsilon_2| > |\varepsilon_4| > |\varepsilon_6| > \cdots$ and so it follows that, despite employing one less point, the midpoint approximation appears as about twice as good as the trapezoidal approximation of the same order, the error being of opposite sign.

We can do better still, however, with a mixture of the two options. Add one-third of equation 62:15:2 to two-thirds of equation 62:15:3, after adjusting k. The motive for this mixing is that thereby the ε_2 term (usually the largest of the ε's) disappears:

62:15:4
$$\Delta_{\mathrm{s}}^{(k,1)} = \frac{1}{3}\Delta_{\mathrm{t}}^{(k-1)} + \frac{2}{3}\Delta_{\mathrm{m}}^{(k-1)} = \int_0^1 \mathrm{f}(t)\,\mathrm{d}t - \sum_{j=0}^{2^k} w_j^{(k)}\,\mathrm{f}\left(\frac{j}{2^k}\right) \sim \sum_{n=2,4}\left[\frac{2^n-4}{3\times 2^{kn}}\right]\varepsilon_n = \frac{4\varepsilon_4}{16^k} + \frac{20\varepsilon_6}{64^k} + \cdots$$

This is *Simpson's approximation* (for Thomas Simpson, English mathematician, 1710 – 1761, although he was not the first to apply it) and, for order k, it carries the following weights

62:15:5
$$w_j^{(k)} = \begin{cases} 2^{-k}/3 & j = 0, 2^k \\ 2^{2-k}/3 & j = 1,3,5,\cdots,2^k-1 \\ 2^{1-k}/3 & j = 2,4,6,\cdots,2^k-2 \end{cases}$$

Examples of the weights for the $k = 2$ cases of the trapezoidal, midpoint, and Simpson approximations are listed in the following table.

	$x=0$	$x=\frac{1}{8}$	$x=\frac{1}{4}$	$x=\frac{3}{8}$	$x=\frac{1}{2}$	$x=\frac{5}{8}$	$x=\frac{3}{4}$	$x=\frac{7}{8}$	$x=1$
trapezoidal	$\frac{1}{8}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{8}$
midpoint		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$	
Simpson's	$\frac{1}{12}$		$\frac{1}{3}$		$\frac{1}{6}$		$\frac{1}{3}$		$\frac{1}{12}$

Simpson's approximation is generally more accurate than the midpoint scheme, which is a marginal improvement over the trapezoidal. The midpoint method alone is useable if the integrand is infinite at either or both of the integral's limits.

The same procedure that serves to create the superior Simpson approximation from the prototypal trapezoidal and midpoint methods can be used to further improve any one of these three still further. When fully developed, the technique is called *Romberg integration* [Werner Romberg, Norwegian mathematician, 1909−2003] and, though it may also be applied to the midpoint family, we shall address only its origin from the trapezoidal method. The $k = 1$, and $k = 2$ orders of trapezoidal approximation, and their corresponding errors, are respectively

62:15:6 $$\int_0^1 \mathrm{f}(t)\,\mathrm{d}t \approx \frac{\mathrm{f}(0)+\mathrm{f}(1)}{4} + \frac{\mathrm{f}(\frac{1}{2})}{2} \qquad \Delta_\mathrm{t}^{(1,0)} = -\frac{\varepsilon_2}{4} - \frac{\varepsilon_4}{16} - \frac{\varepsilon_6}{64} - \frac{\varepsilon_8}{256} - \cdots$$

and

62:15:7 $$\int_0^1 \mathrm{f}(t)\,\mathrm{d}t \approx \frac{\mathrm{f}(0)+\mathrm{f}(1)}{8} + \frac{\mathrm{f}(\frac{1}{4})+\mathrm{f}(\frac{1}{2})+\mathrm{f}\left(\frac{3}{4}\right)}{4} \qquad \Delta_\mathrm{t}^{(2,0)} = -\frac{\varepsilon_2}{16} - \frac{\varepsilon_4}{256} - \frac{\varepsilon_6}{4096} - \cdots$$

A valuable tactic, termed a *Richardson extrapolation*, is to weight these equations and add them so that the leading ε term vanishes. Here, we must multiply 62:15:7 by $\frac{4}{3}$ and 62:15:6 by $-\frac{1}{3}$. The subsequent addition of the weighted equations leads to

62:15:8 $$\int_0^1 \mathrm{f}(t)\,\mathrm{d}t \approx \frac{\mathrm{f}(0)+\mathrm{f}(1)}{12} + \frac{\mathrm{f}(\frac{1}{2})}{6} + \frac{\mathrm{f}(\frac{1}{4})+\mathrm{f}\left(\frac{3}{4}\right)}{3} \qquad \Delta_\mathrm{t}^{(2,1)} = \frac{\varepsilon_4}{64} + \frac{5\varepsilon_6}{1024} + \frac{21\varepsilon_8}{16384} + \cdots$$

This exactly matches the $k = 2$ version of Simpson's approximation. Generally, in fact, the kth-order Simpson approximation arises from applying the Richardson extrapolation to the kth-order and ($k-1$)th-order trapezoidal approximations. The first digit of the two-digit superscript attaching to Δ_t in expressions 62:15:2, and 62:15:6−8 is k, the approximation order; the second signifies r the number of Richardson extrapolations that have been performed. The way in which the procedure subsequently develops is illustrated in Figure 62-8. Each box in the leftmost column of that figure represents a simple k-order trapezoidal approximation; the second column contains k-order Simpson approximations. The boxes outlined in red represent the ultimate Romberg approximations, the errors in which contain no ε_n term for $n < 2k$.

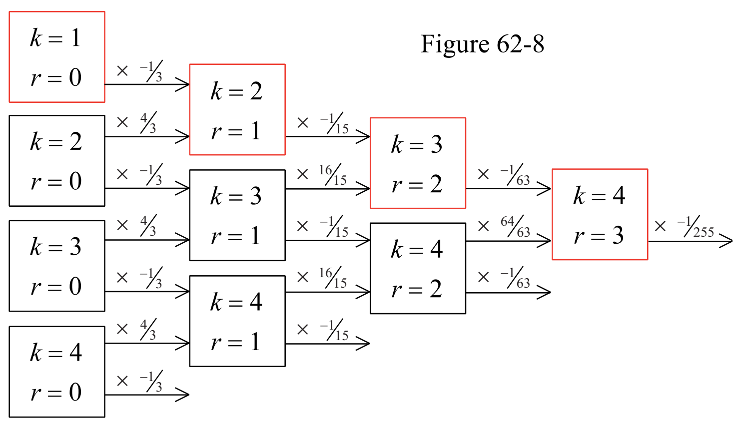


Figure 62-8

Notice that each stage in the propagation scheme for the Richardson extrapolation shown in Figure 62-8 obeys the rule

62:15:9 $$\frac{4^r}{4^r-1}\boxed{\begin{matrix}k\\ r-1\end{matrix}} - \frac{1}{4^r-1}\boxed{\begin{matrix}k-1\\ r-1\end{matrix}} = \boxed{\begin{matrix}k\\ r\end{matrix}}$$

Of course, one need not pursue the entire scheme of Figure 62-8 to implement Romberg integration; you may go directly to the ultimate approximation for any k of your choice. The following table lists the weights appropriate to those ultimate approximations (the red boxes) for approximation orders k of 1 through 8. The header row lists values of x to which the weights in the column below it apply. To conserve tabular space and to assist in preserving computational precision, the numerators have sometimes been factored, and the factors A = 24613875 and B = 1143068355 have been withdrawn from some denominators.

1	$\frac{1}{2}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$\frac{1,3,5,\cdots,15}{16}$	$\frac{1,3,5,\cdots,31}{32}$	$\frac{1,3,5,\cdots,63}{64}$	$\frac{1,3,5,\cdots,127}{128}$	$\frac{1.3,5.\cdots,255}{256}$
$\frac{1}{4}$	$\frac{1}{2}$							
$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$						
$\frac{7}{180}$	$\frac{7}{90}$	$\frac{1}{15}$	$\frac{8}{45}$					
$\frac{31}{1620}$	$\frac{31}{810}$	$\frac{109}{2835}$	$\frac{88}{2835}$	$\frac{256}{2835}$				
$\frac{3937}{413100}$	$\frac{3937}{206550}$	$\frac{1531}{80325}$	$\frac{13864}{722925}$	$\frac{11008}{722925}$	$\frac{32768}{722925}$			
$\frac{64897}{13632300}$	$\frac{64897}{6816150}$	$\frac{281653}{29582091}$	$\frac{7040984}{739552275}$	$\frac{202496}{21130065}$	$\frac{622592}{82172475}$	$\frac{16777216}{739552275}$		
$\frac{18977737}{324\,\mathrm{A}}$	$\frac{18977737}{162\,\mathrm{A}}$	$\frac{177945571}{1519\,\mathrm{A}}$	$\frac{4804530872}{41013\,\mathrm{A}}$	$\frac{2^{8}\times18766729}{41013\,\mathrm{A}}$	$\frac{14509309952}{123039\,\mathrm{A}}$	$\frac{11458838528}{123039\,\mathrm{A}}$	$\frac{2^{35}}{123039\,\mathrm{A}}$	
$\frac{8191\times149431}{900\,\mathrm{B}}$	$\frac{8191\times149431}{450\,\mathrm{B}}$	$\frac{34651\times 3407166487}{43405425\,\mathrm{B}}$	$\frac{1336\times 29456518447}{14468475\,\mathrm{B}}$	$\frac{597053\times 878848}{192913\,\mathrm{B}}$	$\frac{32768\times 3602769179}{43405425\,\mathrm{B}}$	$\frac{2^{24}\times7083941}{43405425\,\mathrm{B}}$	$\frac{2^{35}\times2731}{43405425\,\mathrm{B}}$	$\frac{2^{48}}{43405425\,\mathrm{B}}$

If, rather than 0 to 1, the integration bounds are from x_0 to x, the kth order Romberg approximation becomes

62:15:10 $$\int_{x_0}^{x} \mathrm{f}(t)\,\mathrm{d}t \approx (x-x_0)\sum_{j=0}^{2^k} w_j^{(k)}\,\mathrm{f}\left(x_0+\frac{x-x_0}{2^k}j\right)$$

where the weights $w_j^{(k)}$ are drawn from the kth row of the table above, and the column corresponding to the value of $j/2^k$ in the header row. The Romberg method generates excellent numerical estimates of most integrals, provided that the integrand is free of discontinuities. In calculating complete and incomplete elliptic integrals, *Equator* adopts the Romberg procedure at an approximation order of 8.

If, between x_0 and x, the integrand has one or more particularly "difficult" regions, such as where f has an unusually large magnitude, steepness, or curvature, it may be prudent to increase the accuracy of numerical quadrature by partitioning the integration range into several unequal segments. The choice of the intersegmental points, x_1, x_2, x_3, $\cdots$, is made such that narrower segments encompass the more difficult regions, thereby packing points more closely together. For a three-segment Romberg integration, the quadrature formula would be

62:15:11 $$\sum_{j=0}^{2^k} w_j^{(k)}\left[(x_1-x_0)\,\mathrm{f}\left(x_0+\frac{x_1-x_0}{2^k}j\right)+(x_2-x_1)\,\mathrm{f}\left(x_1+\frac{x_2-x_1}{2^k}j\right)+(x-x_2)\,\mathrm{f}\left(x_2+\frac{x-x_2}{2^k}j\right)\right]$$

For example, to counter the possibly steep rise as the argument approaches φ, *Equator*'s incomplete elliptic integral of the third kind routine applies a six-segment Romberg quadrature to integral 62:12:1, with the five intersegmental points being $\frac{3}{4}\varphi$, $\frac{15}{16}\varphi$, $\frac{63}{64}\varphi$, $\frac{255}{256}\varphi$, and $\frac{1023}{1024}\varphi$.

CHAPTER 63

THE JACOBIAN ELLIPTIC FUNCTIONS

The bivariate functions of this chapter have several interesting properties. One is their ability to bridge the gap between circular functions and hyperbolic functions. Another, discussed in Section 63:11, is their double periodicity. Three of these twelve functions – $\mathrm{cn}(k,x)$, $\mathrm{sn}(k,x)$, and $\mathrm{dn}(k,x)$ – were described by the prolific Prussian mathematician Karl Gustav Jacob Jacobi (1804 – 1851), and these receive emphasis here. The other nine, introduced by the Englishman James Whitbread Lee Glaisher (1848 – 1928), are often regarded as subordinate, because they can be constructed so easily from Jacobi's trio.

Unlike most other functions, the symbols of the Jacobian elliptic function have, in themselves, mathematical significance. These symbols consist of two letters. The first is drawn from the set (c,s,d,n); the second is a different letter from the same set, for a total of 4×3 = 12. It is useful to think of an elliptic function as a quotient of two functions, for example

63:0:1 $$\mathrm{cs}(k,x) = \frac{a \text{ “c” } function}{an \text{ “s” } function}$$

As described in Section 63:8, these single-letter "c", "s", "n" and "d" functions are, in fact, Neville theta functions, but this is unimportant here. The value of representation 63:0:1 is that the important rules for multiplying or dividing two or more Jacobian elliptic functions, exemplified by the following:

63:0:2 $$\mathrm{sc}(k,x)\,\mathrm{cs}(k,x) = 1$$

63:0:3 $$\mathrm{cn}(k,x)\,\mathrm{nd}(k,x) = \mathrm{cd}(k,x)$$

63:0:4 $$\frac{\mathrm{dn}(k,x)}{\mathrm{sn}(k,x)} = \mathrm{ds}(k,x)$$

63:0:5 $$\mathrm{ns}(k,x)\,\mathrm{dc}(k,x) = \mathrm{ds}(k,x)\,\mathrm{nc}(k,x)$$

become self-evident on the basis of such partitioning. Other interrelations among elliptic functions, not evident from the symbolism, are:

63:0:6 $$1 = \mathrm{cn}^2(k,x) + \mathrm{sn}^2(k,x) = \mathrm{dn}^2(k,x) + k^2\,\mathrm{sn}^2(k,x)$$

63:0:7 $$\mathrm{nd}^2(k,x) = \mathrm{cd}^2(k,x) + \mathrm{sd}^2(k,x) = 1 + k^2\,\mathrm{sd}^2(k,x)$$

63:0:8 $$\mathrm{ns}^2(k,x) = \mathrm{cs}^2(k,x) + 1 = \mathrm{ds}^2(k,x) - k^2$$

63:0:9 $$\mathrm{nc}^2(k,x) = 1 + \mathrm{sc}^2(k,x) = \mathrm{dc}^2(k,x) - k^2\,\mathrm{sc}^2(k,x)$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_64, © Springer Science+Business Media, LLC 2009

Having a pythagorean flavor, these latter formulas, and others, follow easily from their geometric interpretation, as explored in Section 63:3.

63:1 NOTATION

The "Jacobian" (sometimes "Jacobi") adjective is not always attached to these elliptic functions. Alternatively, only the sn, cn, and dn functions may be associated with Jacobi, the others being called *Glaisher functions*. The name *cosine-amplitude* is given to cn, *sine-amplitude* to sn, and *delta-amplitude* to dn; the others have not been individually named.

The Jacobian elliptic functions are bivariate, with *modulus* k and *argument* x as the standard variables. You may encounter notations such as cn(x), suggesting a single variable only; the second variable is then implied, being treated as a constant unworthy of mention. The symbol p has been used for the modulus and u commonly replaces x. As well, the order of citation of the variables may be reversed, as in sn(u,k). Rarely, tn is used for sc, because of its tangent-like properties. In common with the functions of Chapters 61 and 62, Jacobian elliptic functions are often symbolized with "modulus substitutes"; their use may be signaled by replacement of the comma by some other separator, as in dn($x|m$) or cs($x\backslash\alpha$) where $m = k^2$ and $\alpha =$ arcsin(k).

The twelve Jacobian elliptic functions form four groups, according to the second letter of the function's name. Thus sc, dc, and nc are said to be *copolar*: they all possess poles of type c.

Several supplementary univariate and bivariate functions arise in discussions of Jacobian elliptic functions. These are the *complementary modulus* $k' = \sqrt{1-k^2}$, the *complete elliptic integrals* [Chapter 61] K(k) or K, and E(k) or E, the incomplete elliptic integrals [Chapter 62] F(k,φ) and E(k,φ), and the *amplitude*. The symbol φ is appropriate for the last, but am(k,x) often replaces it, to emphasize that it shares variables with the elliptic functions, to which it is related through the identities

63:1:1 $$\mathrm{am}(k,x) = \arcsin\{\mathrm{sn}(k,x)\} = \arccos\{\mathrm{cn}(k,x)\} = \arcsin\left\{\sqrt{1-\mathrm{dn}^2(k,x)}\,/\,k\right\} = \varphi \qquad x \le K$$

The function that is denoted dn(k,x) in the *Atlas* may be symbolized Δ(k,φ) elsewhere.

Capitalizing the initial letter of the symbol for a Jacobian elliptic function has been used to indicate the indefinite integral of the square of the function [Section 63:10]

63:1:2 $$\mathrm{Ef}(k,x) = \int_0^x \mathrm{ef}^2(k,t)\,\mathrm{d}t \qquad \mathrm{ef} = \mathrm{cn, sn, dn, cd, sd, nd, cs, ds, ns, sc, dc, nc}$$

but this convention is not adopted here. Note our usage, in 63:1:2 and elsewhere, of "ef" as a stand-in for certain – or, as here, all – elliptic functions.

63:2 BEHAVIOR

The Jacobian elliptic functions display interesting properties when the modulus and/or the argument are imaginary or complex. However, except in Sections 63:11, this chapter treats k and x as real. Moreover, we generally assume $0 \le k \le 1$, which covers the most important values of the modulus, though equations 63:5:1 and 63:5:16−19 show how this domain may be extended to all real values.

Figure 63-1, is a three dimensional representation of the cn(k,x), sn(k,x) and dn(k,x) functions; that is, the three Jabobian elliptic functions that belong to the copolar group n. Likewise, Figures 63-2, 63-3 and 63-4 each depict a trio of functions belonging to the other copolar groups. Many of the properties of the twelve functions are evident

Figure 63-1

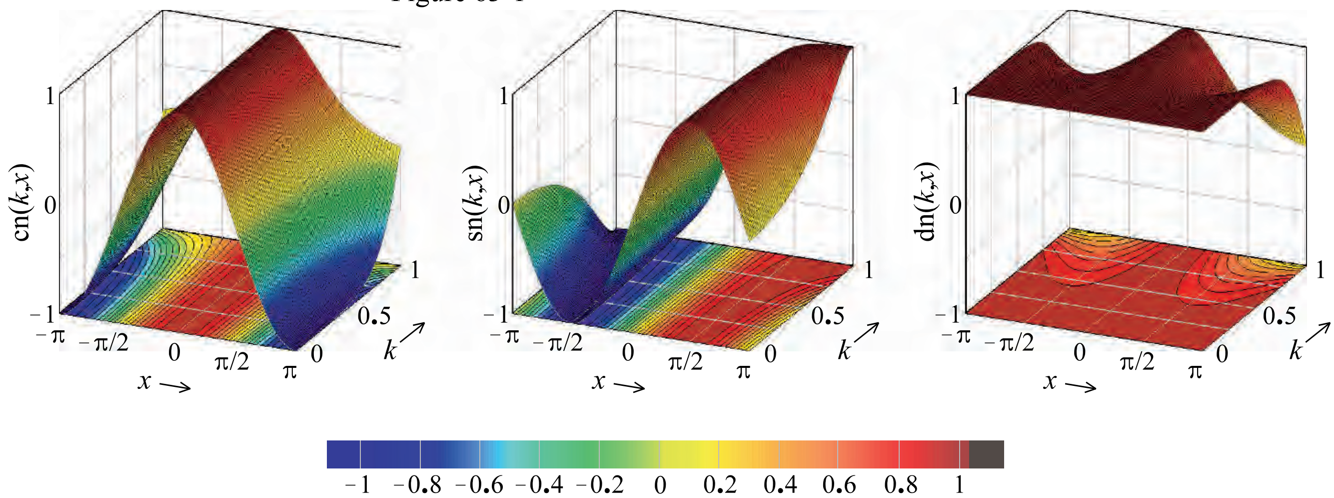


Figure 63-2

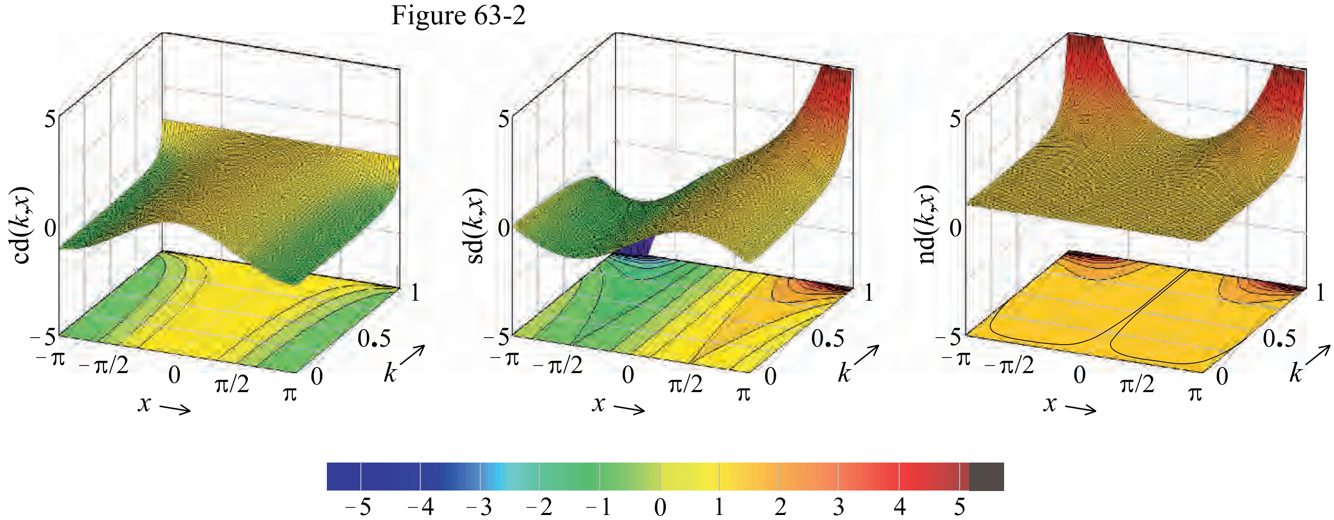


Figure 63-3

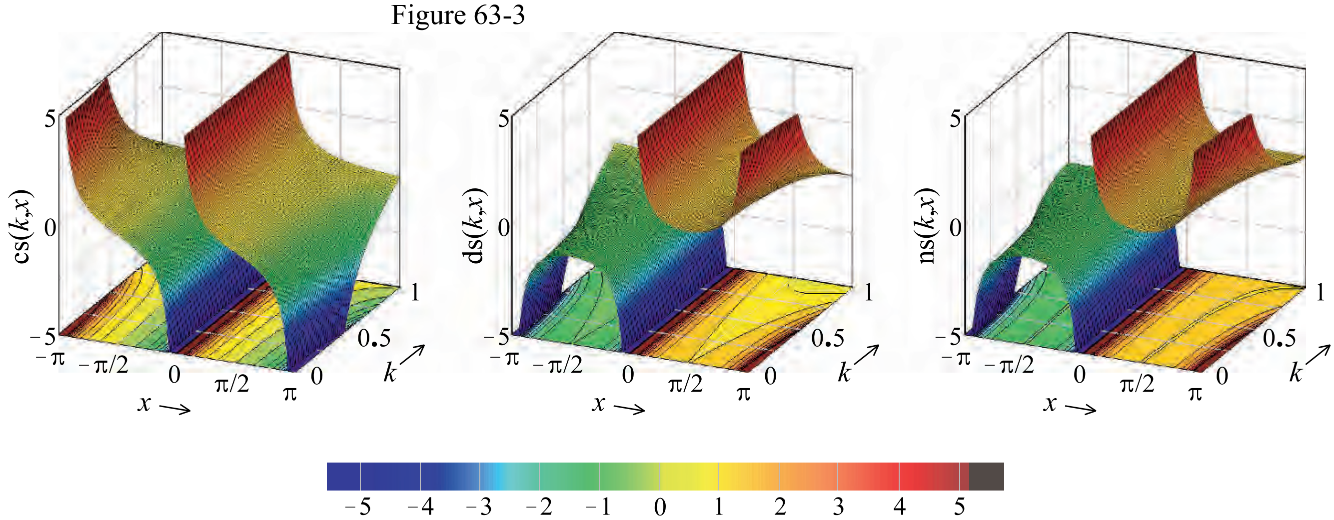
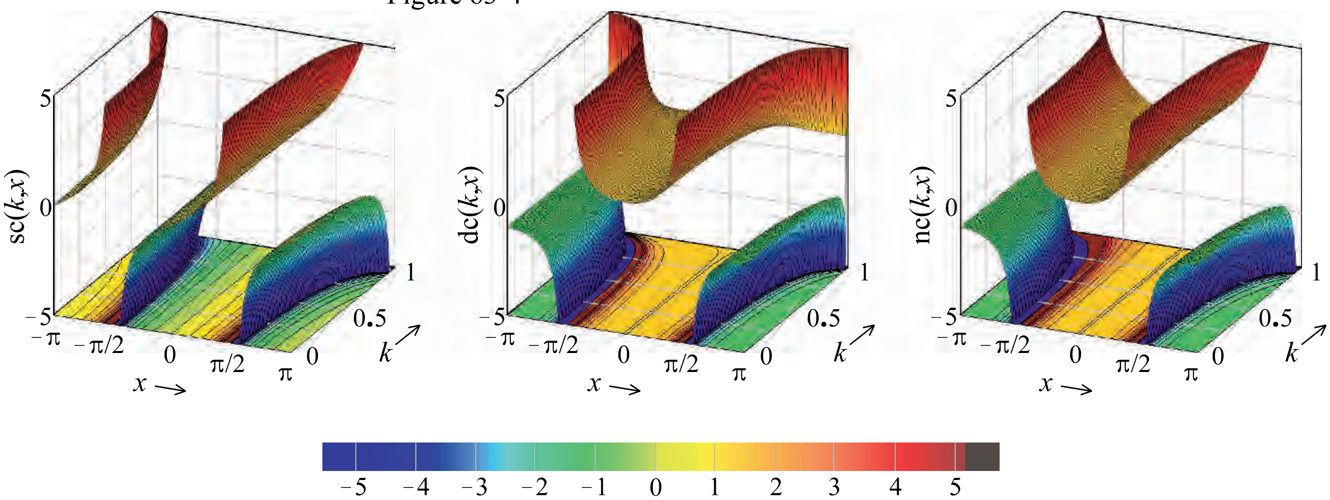


Figure 63-4



from these graphs. For the most part, the functions coincide with circular functions [those of Chapter 32, 33, and 34] when $k = 0$ but evolve as k increases to become hyperbolic functions [from Chapters 28−30] when $k = 1$. For example, $\mathrm{sc}(0,x) = \tan(x)$, whereas $\mathrm{sc}(1,x) = \sinh(x)$. See Section 63:4 for a complete listing.

Except when $k = 1$, all elliptic functions $\mathrm{ef}(k,x)$ are periodic in their argument x, with a period of either $4K$ or $2K$; specifically:

63:2:1
$$\left.\begin{aligned} &\mathrm{ef}(k,x+4nK)=\mathrm{ef}(k,x) \qquad \mathrm{ef} = \mathrm{sn, cn, ds, ns, dc, nc, sd, cd} \\ &\mathrm{ef}(k,x+2nK)=\mathrm{ef}(k,x) \qquad \mathrm{ef} = \mathrm{dn, cs, sc, nd} \end{aligned}\right\} n=\pm1,\pm2,\cdots$$

In this behavior, the elliptic functions are analogous to the circular functions, which have periods of 2π or π. This reflects the fact that, in elliptic algebra, the complete elliptic integral K plays the role that the right-angle, $\pi/2$, fills in circular trigonometry. Likewise, the concept of quadrants, familiar in the context of circular functions, can usefully be extended to elliptic functions. The table opposite reports the ranges of values adopted by the twelve elliptic functions in each of the four "quadrants". Information about the ranges, zeros, extrema and discontinuities of the functions can also be gleaned from a careful inspection of this tabulation.

The lengthening period as k increases is brought out particularly clearly in Figure 63-4. These three diagrams illustrate that when k reaches unity, the period becomes infinite, as appropriate for the hyperbolic functions that most elliptic functions become in that limit. The argument x is used as one of the variables in Figures 63-1 through 63-4, but it is the ratio x/K, where K denotes the complete elliptic integral $K(k)$, that is more significant in many respects and that was chosen in drawing Figures 63-5, 63-6 and 63-7. These diagrams show how the behavior of Jacobi's three functions – $\mathrm{cn}(k,x)$, $\mathrm{sn}(k,x)$, and $\mathrm{dn}(k,x)$ – is affected by the value of the modulus. Notice that the dependence on k is weak when k is small, but becomes dramatic as k approaches unity.

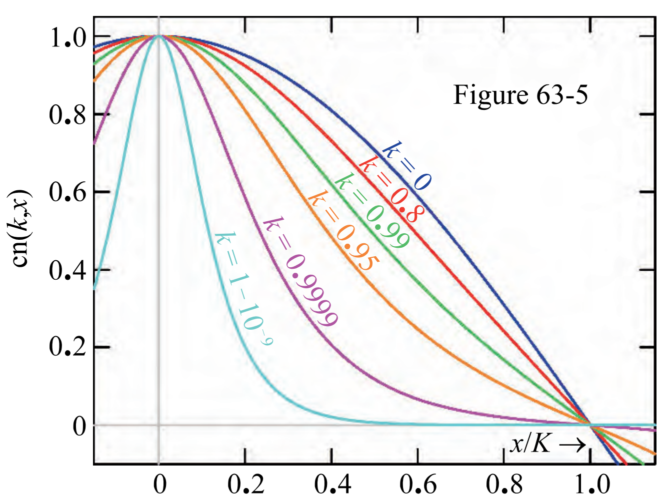


Figure 63-5

	$0 \le x \le K$	$K \le x \le 2K$	$2K \le x \le 3K$	$3K \le x \le 4K$
cn	$1 \ge \mathrm{cn}(k,x) \ge 0$	$0 \ge \mathrm{cn}(k,x) \ge -1$	$-1 \le \mathrm{cn}(k,x) \le 0$	$0 \le \mathrm{cn}(k,x) \le 1$
sn	$0 \le \mathrm{sn}(k,x) \le 1$	$1 \ge \mathrm{sn}(k,x) \ge 0$	$0 \ge \mathrm{sn}(k,x) \ge -1$	$-1 \le \mathrm{sn}(k,x) \le 0$
dn	$1 \ge \mathrm{dn}(k,x) \ge k'$	$k' \le \mathrm{dn}(k,x) \le 1$	$1 \ge \mathrm{dn}(k,x) \ge k'$	$k' \le \mathrm{dn}(k,x) \le 1$
cd	$1 \ge \mathrm{cd}(k,x) \ge 0$	$0 \ge \mathrm{cd}(k,x) \ge -1$	$-1 \le \mathrm{cd}(k,x) \le 0$	$0 \le \mathrm{cd}(k,x) \le 1$
sd	$0 \le \mathrm{sd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{sd}(k,x) \ge 0$	$0 \ge \mathrm{sd}(k,x) \ge -1/k'$	$-1/k' \le \mathrm{sn}(k,x) \le 0$
nd	$1 \le \mathrm{nd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{nd}(k,x) \ge 1$	$1 \le \mathrm{nd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{nd}(k,x) \ge 1$
cs	$+\infty \ge \mathrm{cs}(k,x) \ge 0$	$0 \ge \mathrm{cs}(k,x) \ge -\infty$	$+\infty \ge \mathrm{cs}(k,x) \ge 0$	$0 \ge \mathrm{cs}(k,x) \ge -\infty$
ds	$+\infty \ge \mathrm{ds}(k,x) \ge k'$	$k' \le \mathrm{ds}(k,x) \le +\infty$	$-\infty \le \mathrm{ds}(k,x) \le -k'$	$-k' \ge \mathrm{ds}(k,x) \ge -\infty$
ns	$+\infty \ge \mathrm{ns}(k,x) \ge 1$	$1 \le \mathrm{ns}(k,x) \le +\infty$	$-\infty \le \mathrm{ns}(k,x) \le -1$	$-1 \ge \mathrm{ns}(k,x) \ge -\infty$
sc	$0 \le \mathrm{sc}(k,x) \le +\infty$	$-\infty \le \mathrm{sc}(k,x) \le 0$	$0 \le \mathrm{sc}(k,x) \le +\infty$	$-\infty \le \mathrm{sc}(k,x) \le 0$
dc	$1 \le \mathrm{dc}(k,x) \le +\infty$	$-\infty \le \mathrm{dc}(k,x) \le -1$	$-1 \ge \mathrm{dc}(k,x) \ge -\infty$	$+\infty \ge \mathrm{dc}(k,x) \ge 1$
nc	$1 \le \mathrm{nc}(k,x) \le +\infty$	$-\infty \le \mathrm{nc}(k,x) \le -1$	$-1 \ge \mathrm{nc}(k,x) \ge -\infty$	$+\infty \ge \mathrm{nc}(k,x) \ge 1$
am	$0 \le \mathrm{am}(k,x) \le \pi/2$	$\pi/2 \le \mathrm{am}(k,x) \le \pi$	$\pi \le \mathrm{am}(k,x) \le 3\pi/2$	$3\pi/2 \le \mathrm{am}(k,x) \le 2\pi$

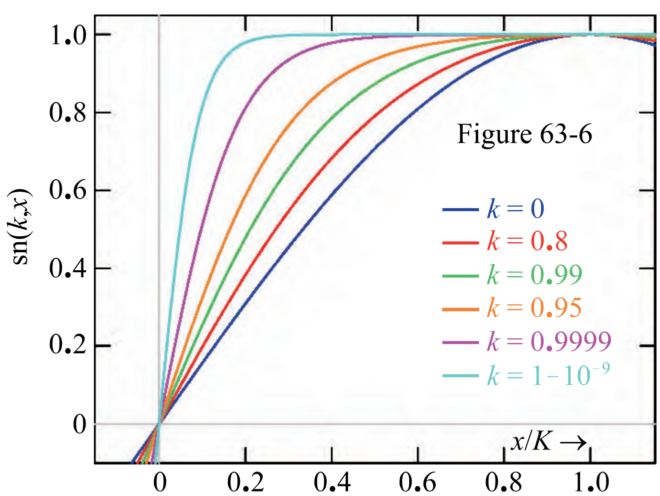


Figure 63-6

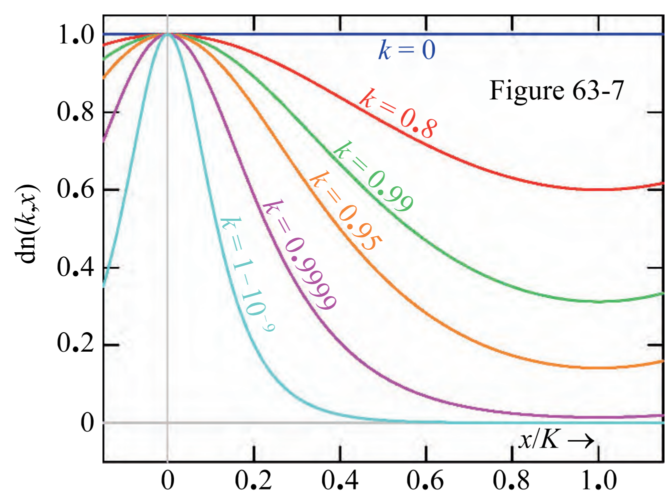


Figure 63-7

63:3 DEFINITIONS

Let the symbol x be assigned to the incomplete elliptic integral of the first kind, of modulus k and amplitude φ. Then the expression

63:3:1
$$x = \mathrm{F}(k,\varphi) = \int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\,d\theta$$

gives x as a function of k and φ. Equally well, one may regard the amplitude φ as a function of k and x. The six circular functions of φ then serve to define six of the Jacobian elliptic functions, as follows:

cos(φ) =	sin(φ) =	sec(φ) =	csc(φ) =	tan(φ) =	cot(φ) =
cn(k,x)	sn(k,x)	nc(k,x)	ns(k,x)	sc(k,x)	cs(k,x)

These definitions raise the pertinent question of how φ is to be found from known values of k and x. There is no straightforward way of doing this, but that does not undermine the validity of the definition. One may think of elliptic functions as being inverse functions [Section 0:3] of incomplete elliptic integrals; for example, treating k as a constant, cn(k,x) is the inverse function of F(k,arccos(x)), as detailed in Section 63:13.

The delta-amplitude is defined by

63:3:2 $$\mathrm{dn}(k,x) = \frac{\partial\varphi}{\partial x} = \sqrt{1-k^2\sin^2(\varphi)}$$

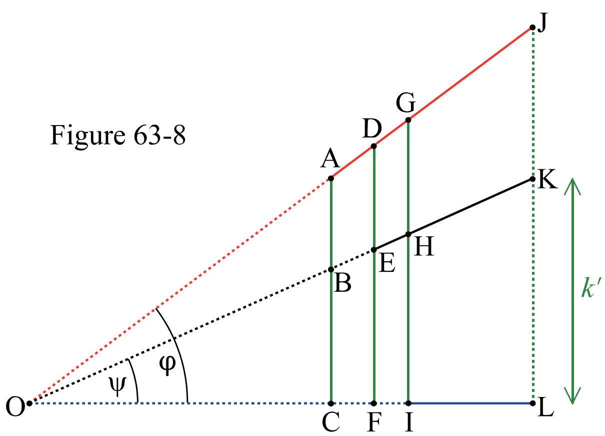
and the remaining five Jacobian elliptic functions may be defined from this, either by the definitions shown in the panel below

nd(k,x) =	cd(k,x) =	sd(k,x) =	dc(k,x) =	ds(k,x) =
1/dn(k,x)	cn(k,x)/dn(k,x)	sn(k,x)/dn(k,x)	dn(k,x)nc(k,x)	dn(k,x)ns(k,x)

or in numerous other ways allowed by the partitioning rules exemplified in equations 63:0:1−4.

In Section 33:3, a set of three similar triangles is described by means of which the six circular functions may be defined. A similar exercise is undertaken in Section 29:3 of the *Atlas* for the six hyperbolic functions. In much the same way, there exists a trigonometric construct that permits the defining of the twelve elliptic functions in an appealing way.

Figure 63-8



First construct the triangle OAC, right-angled at C, with the hypotenuse OA of unit length and with the angle AOC equal to the elliptic amplitude φ. Then cn(k,x) and sn(k,x) are defined as the lengths OC and CA. The OAC triangle, and the ensuing construction, are illustrated in Figure 63-8. In this diagram, dotted lines are all of unity length. Now extend the line OC to point I, such that OI is of unity length, and erect a perpendicular at that point to meet an extension of line OA at point G. Then OG and GI have lengths equal to nc(k,x) and sc(k,x) respectively. The next construction is to further extend lines OI and OG, to points L and J respectively, until the perpendicular distance JL between them becomes equal to unity. Then the lengths OL and OJ equal cs(k,x) and ns(k,x). Six of the elliptic functions have now been defined; the other six require further construction. Construct a line at an angle

63:3:3 $$\psi = \arctan\{k'\tan(\varphi)\} \qquad k' = \sqrt{1-k^2}$$

to line OL, as shown in the figure. This line will cut lines AC, GI and JL at points B, H and K. The length KL is thereby equal to the complementary modulus k'. The lengths of lines OB, OH and OK now define the elliptic functions dn(k,x), dc(k,x), and ds(k,x). The final construction is to measure unity length along line OK to a point E and erect the vertical line FED through that point. Lengths OD, OF and DF then define the remaining elliptic functions nd(k,x), cd(k,x), and sd(k,x).

Figure 63-9 is an exploded view of the Figure 63-8, marked with the length elements assigned to the various functions. Note that each of the four diagrams in this figure corresponds to a copolar group, and that the first

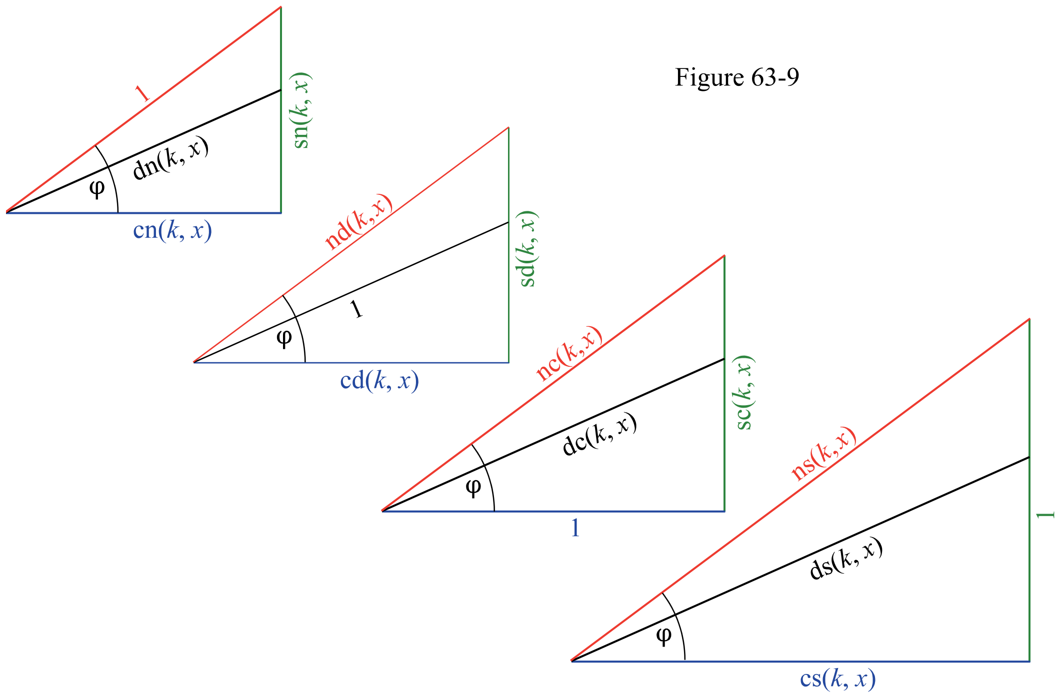


Figure 63-9

vindicates the early equalities in equation 63:1:1. Pythagorean arguments applied to various triangles lead directly to formulas 63:0:4−7. The four diagrams are "similar" in the geometric sense; that is, they differ in size but not in shape. This similarity enables one to assert, for example, that the ratio of the lengths of two red lines must equal the ratio of the lengths of two blue lines and therefore

63:3:4 $$\frac{\mathrm{nd}(k,x)}{1}=\frac{\mathrm{cd}(k,x)}{\mathrm{cn}(k,x)}$$

This relation leads directly to 63:0:3. In fact, all the "partitioning" rules discussed in Section 63:0 are validated by similarity arguments arising from Figure 63-9.

63:4 SPECIAL CASES

As reported in Section 63:2, each Jacobian elliptic function ef(k,x) reduces, when $k = 0$, to a circular function of x, or to unity, whereas it reduces to a hyperbolic function or unity, when $k = 1$.

	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc	am
$k = 0$	cos	sin	1	cos	sin	1	cot	csc	csc	tan	sec	sec	x
$k = 1$	sech	tanh	sech	1	sinh	cosh	csch	csch	coth	sinh	1	cosh	gd

The table identifies the particular circular or hyperbolic function for each ef(0,x) and ef(1,x). The tabulation includes the elliptic amplitude am(k,x), which is seen to equal its argument when $k = 0$ and the gudermannian function [Section 33:14] of x when $k = 1$.

63:5 INTRARELATIONSHIPS

Every elliptic function is even with respect to its modulus

63:5:1 $$\mathrm{ef}(-k,x)=\mathrm{ef}(k,x) \qquad \text{all ef}$$

and either even or odd with respect to its argument

63:5:2 $$\mathrm{ef}(k,-x)=\begin{cases}\mathrm{ef}(k,x) & \text{ef} = \text{dn, cn, nc, dc, nd, cd}\\ -\mathrm{ef}(k,x) & \text{ef} = \text{sn, ns, cs, ds, sc, sd}\end{cases}$$

Note that the presence of an "s" in the symbol of the elliptic function ensures its oddness with respect to its argument.

Jacobi's trio of elliptic functions satisfies the following addition formulas:

63:5:3 $$\mathrm{cn}(k,x\pm y)=\frac{\mathrm{cn}(k,x)\mathrm{cn}(k,y)\mp\mathrm{sn}(k,x)\mathrm{sn}(k,y)\mathrm{dn}(k,x)\mathrm{dn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

63:5:4 $$\mathrm{sn}(k,x\pm y)=\frac{\mathrm{sn}(k,x)\mathrm{cn}(k,y)\mathrm{dn}(k,y)\pm\mathrm{cn}(k,x)\mathrm{dn}(k,x)\mathrm{sn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

63:5:5 $$\mathrm{dn}(k,x\pm y)=\frac{\mathrm{dn}(k,x)\mathrm{dn}(k,y)\mp k^2\mathrm{sn}(k,x)\mathrm{cn}(k,x)\mathrm{sn}(k,y)\mathrm{cn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

These equations are easily converted to argument-duplication formulas that give values of ef(k,2x). Other important special cases are listed in the following table, which also includes expressions for the three prime elliptic functions of half-argument.

	ef = cn	ef = sn	ef = dn
ef(k,x±K)	$\mp k'\mathrm{sd}(k,x)$	±cd(k,x)	$k'\mathrm{nd}(k,x)$
ef(k,x±2K)	−cn(k,x)	−sn(k,x)	dn(k,x)
$\mathrm{ef}\left(k,\tfrac{1}{2}x\right)$	$\pm\sqrt{\dfrac{\mathrm{cn}(k,x)+\mathrm{dn}(k,x)}{1+\mathrm{dn}(k,x)}}$	$\pm\sqrt{\dfrac{1-\mathrm{cn}(k,x)}{1+\mathrm{dn}(k,x)}}$	$\sqrt{\dfrac{\mathrm{cn}(k,x)+\mathrm{dn}(k,x)}{1+\mathrm{cn}(k,x)}}$

The principle of *Landen transformation* is explained in Section 62:5. When employed to transform the delta-amplitude in the ascending mode, the procedure is

63:5:6 $$\frac{2\sqrt{k}}{1+k}=k \qquad \frac{(1+k)}{2}x=x \qquad \frac{\sqrt{\mathrm{dn}^2(k,x)-1+k^2}+\mathrm{dn}(k,x)}{1+k}=\mathrm{dn}(k,x)$$

and serves to increase the modulus at the expense of a decrease in the argument. The corresponding formulation for the descending Landen transformation is

63:5:7 $$\frac{1-k'}{1+k'}=k \qquad (1+k')x=x \qquad \frac{\mathrm{dn}^2(k,x)+k'}{(1+k')\mathrm{dn}(k,x)}=\mathrm{dn}(k,x)$$

and similar – though generally more complicated – formulas apply to other elliptic functions.

What is called the *Jacobi real transformation* establishes a relationship between a Jacobian elliptic functions with modulus in the domain $1\le k\le\infty$ and one in the standard domain $0\le k\le 1$. All these transformations are of the form

63:5:8 $$\mathrm{ef}\left(\frac{1}{k},x\right) = w\,\mathrm{ef}'\left(k,\frac{x}{k}\right)$$

where the multiplier w is either 1, k, or $1/k$, and ef′ is another – or sometimes the same – elliptic function. The panel below lists the correspondences between ef and $w$ef′

ef =	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc
$w$ef′ =	dn	$k$sn	cn	dc	$k$sc	nc	$\frac{1}{k}$ds	$\frac{1}{k}$cs	$\frac{1}{k}$ns	$k$sd	cd	nd

63:6 EXPANSIONS

The first five terms in the power series for the sine-amplitude, cosine-amplitude and delta-amplitude functions, as well as for the elliptic amplitude itself, are

63:6:1 $$\mathrm{cn}(k,x) = 1 - \frac{1}{2!}x^2 + \frac{1+4k^2}{4!}x^4 - \frac{1+44k^2+16k^4}{6!}x^6 + \frac{1+408k^2+912k^4+64k^6}{8!}x^8 - \cdots$$

63:6:2 $$\mathrm{sn}(k,x) = x - \frac{1+k^2}{3!}x^3 + \frac{1+14k^2+k^4}{5!}x^5 - \frac{1+135(k^2+k^4)+k^6}{7!}x^7 + \frac{1+1228(k^2+k^6)+5478k^4+k^8}{9!}x^9 - \cdots$$

63:6:3 $$\mathrm{dn}(k,x) = 1 - \frac{k^2}{2!}x^2 + \frac{4k^2+k^4}{4!}x^4 - \frac{16k^2+44k^4+k^6}{6!}x^6 + \frac{64k^2+912k^4+408k^6+k^8}{8!}x^8 - \cdots$$

63:6:4 $$\mathrm{am}(k,x) = x - \frac{k^2}{3!}x^3 + \frac{4k^2+k^4}{5!}x^5 - \frac{16k^2+44k^4+k^6}{7!}x^7 + \frac{64k^2+912k^4+408k^6+k^8}{9!}x^9 - \cdots$$

General formulas for the coefficients in these series are unknown. These four expansions are computationally useful whenever x is close to zero.

Gradshteyn and Ryzhik [Section 8.146] give a comprehensive listing of expansions of the Jacobian elliptic functions, as well as some of their logarithms and squares, in terms of the nome q [Section 61:15]. The most important are:

63:6:5 $$\mathrm{sn}(k,x) = \frac{2\pi}{kK}\left[\frac{q^{1/2}}{1-q}\sin\left(\frac{\pi x}{2K}\right) + \frac{q^{3/2}}{1-q^3}\sin\left(\frac{3\pi x}{2K}\right) + \frac{q^{5/2}}{1-q^5}\sin\left(\frac{5\pi x}{2K}\right) + \cdots\right]$$

63:6:6 $$\mathrm{cn}(k,x) = \frac{2\pi}{kK}\left[\frac{q^{1/2}}{1+q}\cos\left(\frac{\pi x}{2K}\right) + \frac{q^{3/2}}{1+q^3}\cos\left(\frac{3\pi x}{2K}\right) + \frac{q^{5/2}}{1+q^5}\cos\left(\frac{5\pi x}{2K}\right) + \cdots\right]$$

63:6:7 $$\mathrm{dn}(k,x) = \frac{2\pi}{K}\left[\frac{1}{4} + \frac{q}{1+q^2}\cos\left(\frac{\pi x}{K}\right) + \frac{q^2}{1+q^4}\cos\left(\frac{2\pi x}{K}\right) + \frac{q^3}{1+q^6}\cos\left(\frac{3\pi x}{K}\right) + \cdots\right]$$

63:6:8 $$\mathrm{am}(k,x) = \frac{\pi x}{2K} + 2\left[\frac{q}{1+q^2}\sin\left(\frac{\pi x}{K}\right) + \frac{q^2/2}{1+q^4}\sin\left(\frac{2\pi x}{K}\right) + \frac{q^3/3}{1+q^6}\sin\left(\frac{3\pi x}{K}\right) + \cdots\right]$$

In these equations we are using K to represent K(k).

63:7 PARTICULAR VALUES

Simple expressions arise for each Jacobian elliptic function when the argument is $K/2$, corresponding to an argument that bisects the first "quadrant". Moreover, each elliptic function often coincides in value there with another of its eleven congeners. The pairings, and the values acquired, are listed below. Note that $k'=\sqrt{1-k^2}$.

$\mathrm{ds}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{cn}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sn}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{cd}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{dn}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{cs}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{nc}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{ns}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{dc}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sc}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{nd}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sd}\left(k,\tfrac{1}{2}K\right)$
$\sqrt{k'(1+k')}$	$\sqrt{\dfrac{k'}{1+k'}}$	$\dfrac{1}{\sqrt{1+k'}}$	$\sqrt{k'}$	$\sqrt{\dfrac{1+k'}{k'}}$	$\sqrt{1+k'}$	$\dfrac{1}{\sqrt{k'}}$	$\dfrac{1}{\sqrt{k'(1+k')}}$

The same values reoccur, possibly with a change of sign, at the midpoints of *all* quadrants.

The particular values when x is a multiple of K (that is, where adjacent quadrants meet) is evident from the table in Section 63:2. All such values are drawn from the nine-member set 0, $\pm k'$, ±1, $\pm1/k'$, and $\pm\infty|\mp\infty$.

63:8 NUMERICAL VALUES

Using $\mathrm{dn}(k_0,x_0)$ as illustrative, one popular technique for evaluating Jacobian elliptic functions is to use the Landen transformation in either its descending mode [equation 63:5:7] or ascending mode [equation 63:5:6], to progressively decrease or increase the modulus of the delta-amplitude until it has reached (after, say, n transformations) a value so close to unity or zero that the approximation

63:8:1 $$\mathrm{dn}(k,x)\approx 1-\tfrac{1}{2}k^2\sin^2(x) \qquad k \text{ small}$$

or

63:8:2 $$\mathrm{dn}(k,x)\approx \mathrm{sech}(x)\left[1+\tfrac{1}{4}\left(1-k^2\right)\left(\sinh^2(x)+x\tanh(x)\right)\right] \qquad (1-k)\text{ small}$$

may be applied validly. These approximations arise from limits 63:9:3.

Another route to calculating Jacobian elliptic functions exploits the partitioning principle described in Section 63:0. The single-letter functions are, in fact, the Neville's theta functions and accordingly any of the twelve elliptic can be calculated as

63:8:3 $$\mathrm{ef}(k,x)=\frac{\vartheta_{\mathrm{e}}(k,x)}{\vartheta_{\mathrm{f}}(k,x)} \qquad \text{all ef functions}$$

Applying the routines described in Section 61:15, this is the procedure used by *Equator*'s Jacobian elliptic cn function routine and eleven other similarly named functions (keywords **cn**, **sn**, **dn**, **sd**, **cd**, **nd**, **sc**, **dc**, **nc**, **cs**, **ds**, and **ns**). Values of all twelve elliptic functions are available for variables in the domains $0 \le k \le 1$ and $-8\mathrm{K}(k) \le x \le 8\mathrm{K}(k)$.

Equator also has a routine (keyword **am**) which calculates the elliptic amplitude by the algorithm

63:8:4 $$\mathrm{am}(k,x)=\pi\,\mathrm{Int}\left(\frac{x}{2K}\right)+\left\{\begin{array}{ll}\arcsin\{\mathrm{sn}(k,2yK)\} & 0\le y\le 0.1\\ \arccos\{\mathrm{cn}(k,2yK)\} & 0.1<y<0.9\\ \pi-\arcsin\{\mathrm{sn}(k,2yK)\} & 0.9\le y<1\end{array}\right\} \qquad \begin{array}{l} y=\mathrm{frac}(x/2K)\\ K=\mathrm{K}(k)\end{array}$$

63:9 LIMITS AND APPROXIMATIONS

The following limiting approximations apply as the modulus of certain elliptic functions approaches the value zero or unity:

63:9:1
$$\mathrm{sn}(k,x) \approx \begin{cases} \sin(x) - \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right]\cos(x) & k \to 0 \\ \tanh(x) + \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right]\mathrm{sech}(x) & k \to 1 \end{cases}$$

63:9:2
$$\mathrm{cn}(k,x) \approx \begin{cases} \cos(x) + \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right]\sin(x) & k \to 0 \\ \mathrm{sech}(x) - \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right]\tanh(x) & k \to 1 \end{cases}$$

63:9:3
$$\mathrm{dn}(k,x) \approx \begin{cases} 1 - \frac{1}{2}k^2\sin^2(x) & k \to 0 \\ \mathrm{sech}(x) + \frac{1}{4}(k')^2\left[\sinh(x) + x\,\mathrm{sech}(x)\right]\tanh(x) & k \to 1 \end{cases}$$

63:9:4
$$\mathrm{am}(k,x) \approx \begin{cases} x - \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right] & k \to 0 \\ \mathrm{gd}(x) + \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right] & k \to 1 \end{cases}$$

Limiting expressions as $x \to 0$ are available by curtailing expansions 63:6:1−4.

63:10 OPERATIONS OF THE CALCULUS

The derivative of an arbitrary elliptic function ef(k,x) with respect to its argument is proportional to the product of the two other elliptic functions (e′f and e″f) that, with ef, constitute a copolar group. The constant α of proportionality may, or may not, depend on k, as follows

63:10:1
$$\frac{\partial}{\partial x}\mathrm{ef}(k,x) = \alpha\,\mathrm{e'f}(k,x)\,\mathrm{e''f}(k,x) \qquad \begin{cases} \alpha=1 \text{ for ef = sc, sn, sd, nc} \\ \alpha=-1 \text{ for ef = cn, cs, ds, ns} \\ \alpha=k^2 \text{ for ef = nd or } -k^2 \text{ for ef = dn} \\ \alpha=k'^2 \text{ for ef = dc or } -k'^2 \text{ for ef = cd} \end{cases}$$

The derivatives with respect to the modulus again involve the proportionality constant α but another term β, reflecting the individuality of the ef elliptic function, is also involved

63:10:2
$$\frac{\partial}{\partial k}\mathrm{ef}(k,x) = \alpha\,\mathrm{e'f}(k,x)\,\mathrm{e''f}(k,x)\left[\frac{x}{k} + \beta\frac{\mathrm{sn}(k,x)}{k'^2} - \frac{E(k,\varphi)}{k\,k'^2}\right] \qquad \begin{cases} \beta=0 \text{ for ef = cd,dc} \\ \beta=k\,\mathrm{cd}(k,x) \text{ for ef = cn,nc,sc,sn,cs,ns} \\ \beta=k\,\mathrm{dc}(k,x) \text{ for ef = sd,ds} \\ \beta=[\mathrm{dc}(k,x)]/k \text{ for ef = dn,nd} \end{cases}$$

Expressions for indefinite integrals of the forms

63:10:3
$$I_1 = \int_0^x \mathrm{ef}(k,t)\,\mathrm{d}t \qquad \text{and} \qquad I_2 = \int_0^x \mathrm{ef}^2(k,t)\,\mathrm{d}t \qquad \mathrm{f} \neq \mathrm{s}$$

exist for nine of the elliptic functions as listed below. Although those of pole type s diverge, their complements

63:10:4 $$I_1^* = \int_x^K \mathrm{es}(k,t)\,\mathrm{d}t \qquad \text{and} \qquad I_2^* = \int_x^K \mathrm{es}^2(k,t)\,\mathrm{d}t \qquad \mathrm{e} = \mathrm{n, c, d}$$

remain finite and are tabulated instead. For brevity in this table, each elliptic function ef(k,x) is denoted simply by the symbol ef.

	I_1 or I_1^*	I_2 or I_2^*
sn	$[\ln\{(\mathrm{dn}-k\mathrm{cn})/(1-k)\}]/k$	$[x-\mathrm{E}(k,\varphi)]/k^2$
cn	$[\arccos(\mathrm{dn})]/k$	$[\mathrm{E}(k,\varphi)-(k')^2x]/k^2$
dn	φ	$\mathrm{E}(k,\varphi)$
nc	$[\ln(\mathrm{dc}+k'\mathrm{sc})]/k'$	$x+[\mathrm{sn\,dc}-\mathrm{E}(k,\varphi)]/k'^2$
sc	$[\ln\{(\mathrm{dc}+k'\mathrm{nc})/(1+k')\}]/k'$	$[\mathrm{sn\,dc}-\mathrm{E}(k,\varphi)]/k'^2$
dc	$\ln(\mathrm{nc}+\mathrm{sc})$	$x+\mathrm{sn\,dc}-\mathrm{E}(k,\varphi)$
nd	$[\arccos(\mathrm{cd})]/k'$	$[\mathrm{E}(k,\varphi)-k^2\mathrm{sn\,cd}]/k'^2$
sd	$[\arcsin\{kk'(\mathrm{nd}-\mathrm{cd})\}]kk'$	$[\mathrm{E}(k,\varphi)-(k')^2x-k^2\mathrm{sn\,cd}]/(k\,k')^2$
cd	$[\ln(\mathrm{nd}+k\mathrm{sd})]/k$	$[x+k^2\mathrm{sn\,cd}-\mathrm{E}(k,\varphi)]/k^2$
ns	$\ln\{k'/(\mathrm{ds}-\mathrm{cs})\}$	$\mathrm{E}(k,\varphi)+\mathrm{cn\,ds}-\mathrm{E}(k)+K-x$
cs	$\ln\{(1-k')/(\mathrm{ns}-\mathrm{ds})\}$	$\mathrm{E}(k,\varphi)+\mathrm{cn\,ds}-\mathrm{E}(k)$
ds	$-\ln(\mathrm{ns}-\mathrm{cs})$	$\mathrm{E}(k,\varphi)+\mathrm{cn\,ds}-\mathrm{E}(k)+k'^2(K-x)$

Equation 63:10:1 is helpful in evaluating the integrals of many products and quotients of elliptic functions; some of these are listed by Gradshteyn and Ryzhik [Sections 5.131–139].

63:11 COMPLEX ARGUMENT

The real and imaginary parts of Jacobi's three functions are:

63:11:1 $$\mathrm{cn}(k,x+iy)=\frac{\mathrm{cs}(k,x)\mathrm{ns}(k',y)-i\mathrm{dn}(k,x)\mathrm{dc}(k',y)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y)+k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

63:11:2 $$\mathrm{sn}(k,x+iy)=\frac{\mathrm{ds}(k',y)\mathrm{nc}(k',y)+i\mathrm{ds}(k,x)\mathrm{cn}(k,x)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y)+k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

63:11:3 $$\mathrm{dn}(k,x+iy)=\frac{\mathrm{ds}(k,x)\mathrm{ds}(k',y)-ik^2\mathrm{cn}(k,x)\mathrm{nc}(k',y)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y)+k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

When the argument is imaginary, these formulas, and their nine other cohorts, reduce to Jacobi imaginary transformations, each of which establishes a relationship between a Jacobian elliptic functions of imaginary argument and one with real argument. All these transformations are of the form

63:11:4 $$\mathrm{ef}(k,iy) = w\,\overline{\mathrm{ef}}\,(k',y) \qquad k' = \sqrt{1-k^2}$$

where the multiplier w is either 1, i, or $-i$, and $\overline{\mathrm{ef}}$ is another – or sometimes the same – elliptic function. The panel below lists the correspondences between ef and $w\overline{\mathrm{ef}}$

ef =	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc
$w\overline{\mathrm{ef}}$ =	nc	i sc	dc	nd	i sd	cd	$-i$ ns	$-i$ ds	$-i$ cs	i sn	dn	cn

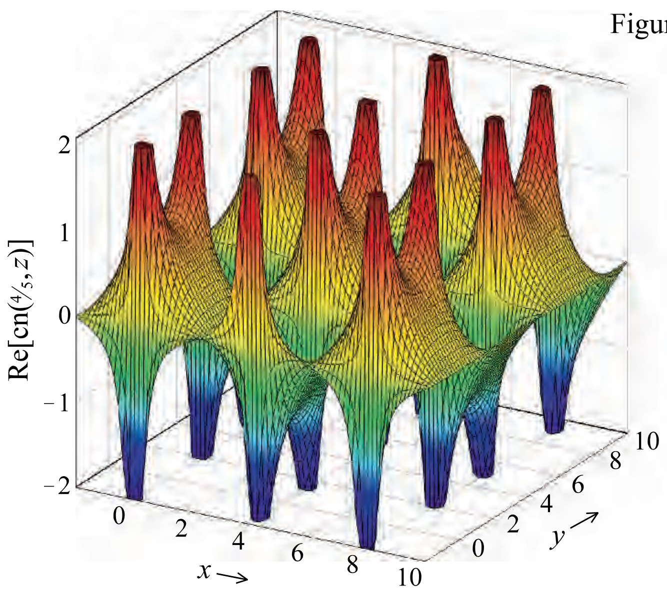
Notice that whether this transformation produces an imaginary or a real result depends on whether or not an "s" appears in the symbol for the elliptic function.

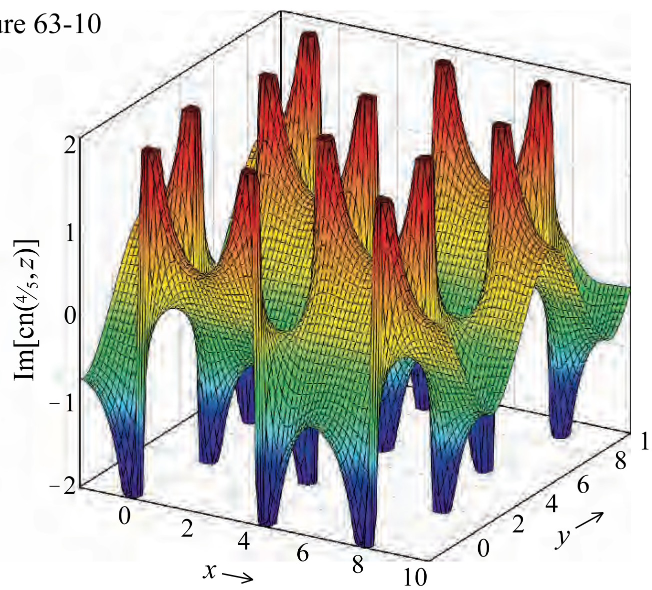
As elaborated in Section 63:2, an elliptic function of real argument is periodic in x with a period that is either 4K(k) or 2K(k). This periodicity is retained when the argument becomes complex but, as the previous paragraph demonstrates, an elliptic function of complex argument is periodic along the imaginary axis, too, with a period that is either 4iK(k') or 2iK(k'). The assignment of real and imaginary periods is made in the table to the right.

	Period	
	real	imaginary
cn,ds,nc,sd	4K(k)	4iK(k')
dn,cs,sc,nd	2K(k)	4iK(k')
sn,ns,dc,cd	4K(k)	2iK(k')

This double periodicity is clearly exemplified in Figure 63-10, which depicts the real and imaginary parts of cn(k,z) = cn(k,x+iy) with $k = 4/5$. Note that, because there are two poles per period, the spacing of poles is $\tfrac{1}{2}[4\mathrm{K}(4/5)] = 3.9906$ along the real dimension and $\tfrac{1}{2}[4\mathrm{K}(3/5)] = 3.5015$ along the imaginary axis.

Figure 63-10





63:12 GENERALIZATIONS

We are aware of no direct generalizations of the Jacobian elliptic functions having been made.

63:13 COGNATE FUNCTIONS: inverse elliptic functions

Among a number of functions that are related to Jacobian elliptic functions, we here mention only their inverses. As with other inverses of periodic functions, ambiguities arise in assigning the principal values of the inverse elliptic functions. Beware of differing conventions.

With k invariant, the inverse elliptic functions are various incomplete elliptic integrals of the first kind [Chapter 62]. From the prototype

63:13:1 $$\mathrm{invam}(k,y)=\int_0^y \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\,\mathrm{d}\theta=\mathrm{F}(k,y)$$

it follows that

63:13:2 $$\mathrm{invcn}(k,y)=\int_y^1 \frac{1}{\sqrt{1-t^2}\sqrt{k^2t^2+1-k^2}}\,\mathrm{d}t=\mathrm{F}(k,\arccos(y))$$

63:13:3 $$\mathrm{invsn}(k,y)=\int_0^y \frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,\mathrm{d}t=\mathrm{F}(k,\arcsin(y))$$

63:13:4 $$\mathrm{invdn}(k,y)=\int_y^1 \frac{1}{\sqrt{1-t^2}\sqrt{t^2-1+k^2}}\,\mathrm{d}t=\mathrm{F}\left(k,\arcsin\left\{\frac{\sqrt{1-y^2}}{k}\right\}\right)\qquad k'<y<1$$

with similar results for the other nine inverse Jacobian elliptic functions. Of course, when k equals zero or unity, they generally reduce to the functions of Chapters 35 or 31. Jeffrey [Chapter 12] lists a few other properties of these inverse functions.

63:14 RELATED TOPIC: Weierstrassian elliptic functions

The elliptic family of functions addressed in Chapters 61−63 are largely the creation of Legendre and Jacobi, but there is a parallel formalism due to the German Karl Theodor Wilhelm Weierstrass (teacher and mathematical innovator, 1815 −1897). Of course, the two rival systems are related. Here we shall point out some of those relationships, but stop well short of a comprehensive description of the Weierstrass system.

There are three interrelated parameters in the Weierstrass system that play a role equivalent to that played by Legendre's modulus k and complementary modulus k'. The equivalences are

63:14:1 $$k=\sqrt{\frac{e_2-e_1}{e_1-e_3}}\qquad k'=\sqrt{\frac{e_1-e_2}{e_1-e_3}}\qquad \text{where}\qquad e_1+e_2+e_3=0$$

Likewise, the role of determining the real and imaginary periods of the Weierstrass's elliptic functions, played in the Legendre system by $\mathrm{K}(k)$ and $\mathrm{K}(k')$, is taken by two new variables

63:14:2 $$\omega_1=\frac{2\,\mathrm{K}(k)}{\sqrt{e_1-e_2}}\qquad\text{and}\qquad \omega_2=\frac{2i\,\mathrm{K}(k')}{\sqrt{e_1-e_2}}$$

The principal Weierstrassian elliptic function, usually symbolized $\mathrm{P}(z)$ with some fancy typographic rendering of the "P", is expressible in terms of particular Jacobian elliptic functions as the alternatives

63:14:3 $$e_1+(e_1-e_3)\,\mathrm{cs}^2\left(k,z\sqrt{e_1-e_3}\right)=e_2+(e_1-e_3)\,\mathrm{ds}^2\left(k,z\sqrt{e_1-e_3}\right)=e_3+(e_1-e_3)\,\mathrm{sn}^2\left(k,z\sqrt{e_1-e_3}\right)$$

CHAPTER 64

THE HURWITZ FUNCTION $\zeta(v,u)$

Together with the bivariate eta function and the Lerch function, both of which are also addressed in this chapter, the Hurwitz function provides a means of summing the interesting series listed in formulas 64:3:3, 64:13:1, and 64:12:3. These functions can all be characterized as "logarithm-like". The Hurwitz function plays an invaluable role in the Weyl differintegration of periodic functions, a topic discussed in Section 64:14.

64:1 NOTATION

The symbol $\zeta(\,,\,)$ is standard for this function, but a variety of names − *generalized zeta function*, *Riemann's zeta function*, *Riemann's function*, *generalized Riemann zeta function*, *bivariate zeta function*, *Hurwitz zeta function*, and *Hurwitz function* − are commonly applied to it. We adopt the last of these names to avoid confusion with the function of Chapter 3 and to recognize the contributions of the German mathematician Adolf Hurwitz (1859 − 1919).

The variables v and u will be respectively termed the order and parameter of the Hurwitz function $\zeta(v,u)$. It is to achieve unity with the notation for the Lerch function [Section 64:12] that we resist the temptation to call u the argument of the function.

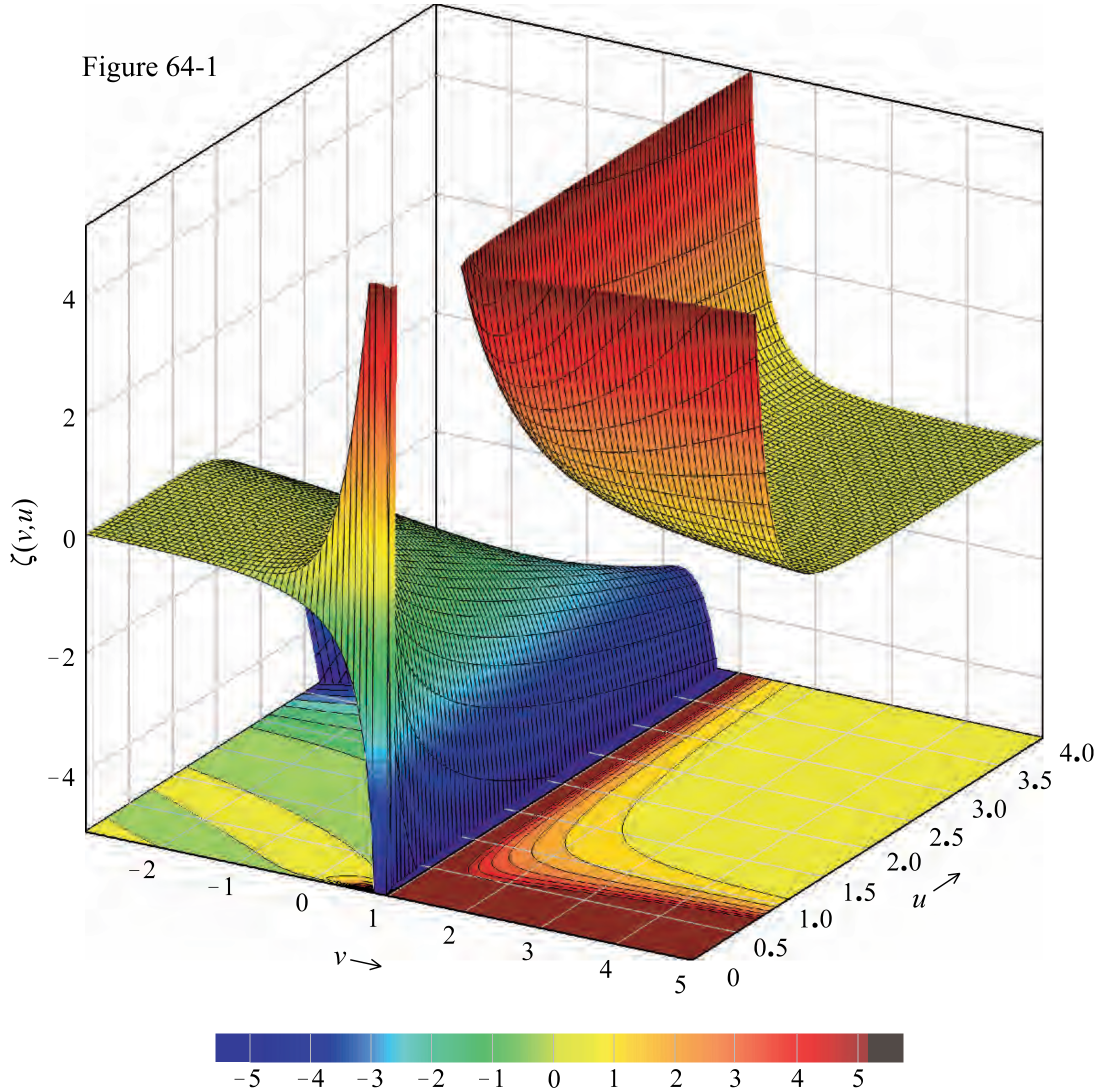
64:2 BEHAVIOR

We generally treat only real values of v and u, and exclude $v = 1$, where a $-\infty|+\infty$ discontinuity occurs. There is no unanimity in the definition of the Hurwitz function for negative parameter and accordingly the $u < 0$ domain is generally omitted from consideration in this *Atlas*, as it is in Figure 64-1. The status of $\zeta(v,0)$ is also questionable; here we regard it as infinite when v is greater then zero, but elsewhere it is considered to equal $\zeta(v,1)$.

The discontinuity in the Hurwitz function at $v = 1$ is the dominant feature in the landscape of the Hurwitz function shown in the figure. The function is invariably positive for $v > 1$, but it may have either sign for $v < 1$. In the latter domain, the Hurwitz function has a number of zeros, concentrated in the region of small u.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_65,

Figure 64-1



64:3 DEFINITIONS

The difference between two digamma functions [Chapter 44] provides a generating function for Hurwitz functions of integer orders of 2 and greater

64:3:1 $$t\left[\psi(u)-\psi(u-t)\right]=t^2\zeta(2,u)+t^3\zeta(3,u)+t^4\zeta(4,u)+\cdots=\sum_{n=2}^{\infty}\zeta(n,u)t^n$$

The Hurwitz function may be defined through the integral

64:3:2 $$\Gamma(v)\zeta(v,u)=\int_0^{\infty}\frac{t^{v-1}\exp(-ut)}{1-\exp(-t)}\mathrm{d}t \qquad v>1,\ u>0$$

which may be regarded as a Laplace transform [Section 26:15]. More complicated is *Hermite's integral*

64:3:3 $$\zeta(v,u)=\frac{u^{-v}}{2}+\frac{u^{1-v}}{v-1}+2\int_0^{\infty}\frac{\sin\left(v\arctan(t/u)\right)}{\left(u^2+t^2\right)^{v/2}[\exp(2\pi t)-1]}\mathrm{d}t$$

and

64:3:4 $$\zeta(v,u)=\frac{1}{\Gamma(v)}\int_0^1\frac{t^{u-1}\left[-\ln(t)\right]^{v-1}}{(1-t)}\mathrm{d}t \qquad v>1,\ u>0$$

The most transparent definition is as the series

64:3:5 $$\zeta(v,u)=\frac{1}{u^v}+\frac{1}{(1+u)^v}+\frac{1}{(2+u)^v}+\frac{1}{(3+u)^v}+\cdots=\sum_{j=0}^{\infty}(j+u)^{-v} \qquad v>1$$

but this converges only when the order exceeds unity, and then often very slowly. To extend this definition to cover nonpositive integer u, some authorities, but not this *Atlas*, exclude from the defining series any term that generates an infinity. With a similar objective, other authors replace the summand in 64:3:3 by $[(j+u)^2]^{-v/2}$, but this modification is not adopted here either. A definition, valid only in the narrow domain of the parameter, is provided by *Hurwitz's formula*

64:3:6 $$\frac{\zeta(v,u)}{\Gamma(1-v)}=2\sum_{k=1}^{\infty}\frac{\sin(2k\pi u+\frac{1}{2}\pi v)}{(2k\pi)^{1-v}} \qquad v<0,\quad 0\le u\le 1$$

Ways to mitigate the restrictions imposed by these series definitions are discussed in Section 64:6.

The Hurwitz function may be defined also by Weyl differintegration [Section 64:14] of a simple algebraic function with respect to the logarithm of the differintegration variable:

64:3:7 $$\zeta(v,u)=\frac{\mathrm{d}^{-v}}{[\mathrm{d}\ln(t)]^{-v}}\frac{t^u}{1-t}\bigg|_{-\infty}^{0}$$

Some of the definitions in this section may be extended to noninteger negative parameters, $u<0$, but neither the *Atlas* nor *Equator* caters to this domain.

64:4 SPECIAL CASES

When the order is unity, the Hurwitz function suffers a discontinuity but, for any other positive integer order n, $\zeta(v,u)$ is equivalent to a polygamma function [Section 44:12]

64:4:1 $$\zeta(n,u)=\frac{(-)^n\psi^{(n-1)}(u)}{(n-1)!} \qquad n=2,3,4,\cdots$$

When the order is a nonpositive integer $-n$, the Hurwitz function can be expressed as a Bernoulli polynomial [Chapter 19]

64:4:2 $$\zeta(-n,u)=\frac{-\mathrm{B}_{n+1}(u)}{n+1} \qquad n=0,1,2,\cdots$$

The first few cases are

$\zeta(0,u)$	$\zeta(-1,u)$	$\zeta(-2,u)$	$\zeta(-3,u)$	$\zeta(-4,u)$
$\frac{1}{2}-u$	$\frac{-1}{12}+\frac{1}{2}u-\frac{1}{2}u^2$	$\frac{-1}{6}u+\frac{1}{2}u^2-\frac{1}{3}u^3$	$\frac{1}{120}-\frac{1}{4}u^2+\frac{1}{2}u^3-\frac{1}{4}u^4$	$\frac{1}{30}u-\frac{1}{3}u^3+\frac{1}{2}u^4-\frac{1}{5}u^5$

64:5 INTRARELATIONSHIPS

An obvious consequence of definition 64:3:5 is the recursion formula

64:5:1 $$\zeta(\nu,u+1)=\zeta(\nu,u)-u^{-\nu}$$

and this may be iterated to

64:5:2 $$\zeta(\nu,J+u)=\zeta(\nu,u)-\sum_{j=0}^{J-1}\frac{1}{(j+u)^{\nu}}$$

The duplication formula

64:5:3 $$\zeta(\nu,2u)=2^{-\nu}\left[\zeta(\nu,u)+\zeta(\nu,u+\tfrac{1}{2})\right]$$

may be generalized to

64:5:4 $$\zeta(\nu,mu)=m^{-\nu}\sum_{j=0}^{m-1}\zeta\left(\nu,u+\frac{j}{m}\right)\qquad m=2,3,4,\cdots$$

which leads to such relationships as

64:5:5 $$\zeta(\nu,u+\tfrac{1}{4})+\zeta(\nu,u+\tfrac{3}{4})=4^{\nu}\zeta(\nu,4u)-2^{\nu}\zeta(\nu,2u)$$

Certain series of Hurwitz functions of positive integer order, with monotone or alternating signs, may be summed:

64:5:6 $$\zeta(2,u)\pm\zeta(3,u)+\zeta(4,u)\pm\zeta(5,u)+\cdots=\begin{cases}1/(u-1) & u>1\\ 1/u & u>0\end{cases}$$

64:5:7 $$\frac{\zeta(2,u)}{2}\pm\frac{\zeta(3,u)}{3}+\frac{\zeta(4,u)}{4}\pm\frac{\zeta(5,u)}{5}+\cdots=\begin{cases}\psi(u)-\ln(u-1) & u>1\\ \ln(u)-\psi(u) & u>0\end{cases}$$

The latter sums involve the digamma function and logarithms [Chapters 44 and 25].

64:6 EXPANSIONS

The seminal expansion of the Hurwitz function is series 64:3:3. When this is incorporated into the Euler-Maclaurin formula 4:14:1, with h set to unity and the discrete j variable treated as a continuous variable t, the result

64:6:1 $$\zeta(\nu,u)=\sum_{j=0}^{\infty}(j+u)^{-\nu}\sim\int_0^{\infty}(t+u)^{-\nu}\mathrm{d}t+\sum_{n=0}\frac{\mathrm{B}_n}{n!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}(t+u)^{-\nu}\Big|_{t=0}^{t=\infty}$$

emerges. After carrying out the indicated integration and differentiations, one discovers the formula

64:6:2 $$\zeta(\nu,u)\sim\frac{1}{2u^{\nu}}+\frac{u^{1-\nu}}{\nu-1}\left[2-\sum_{j=0}\frac{(\nu-1)_{2j}\left|\mathrm{B}_{2j}\right|}{(2j)!}\left(\frac{-1}{u^2}\right)^{j}\right]$$

in which a Pochhammer polynomial [Chapter 18] occurs. Though technically asymptotic, this series converges well.

For computational purposes, it may sometimes be preferable to apply the Euler-Maclaurin transformation, not to the seminal series 64:3:3 itself, but to formula 64:5:2. This leads to

64:6:3 $$\zeta(\nu,u)\sim\sum_{j=0}^{J-1}\frac{1}{(j+u)^{\nu}}+\frac{(J+u)^{1-\nu}}{\nu-1}+\frac{1}{2(J+u)^{\nu}}+\frac{\nu}{12(J+u)^{1+\nu}}-\frac{\nu(\nu+1)(\nu+2)}{720(J+u)^{3+\nu}}+\cdots$$

where J is an arbitrary positive integer. The right-hand terms in this formula, other than the first, may be regarded as an expression for the remainder when the seminal series is truncated after the Jth term. Another expression for the same remainder is

64:6:4
$$\frac{1}{v-1}\left[\frac{J+u+v-1}{(J+u)^{v}}+\sum_{j=J}^{\infty}\frac{j+u+v}{(j+u+1)^{v}}-\frac{1}{(j+u)^{v-1}}\right]$$

Hurwitz's formula, equation 6:3:5, is valid over a narrow domain, but by invoking recursion 63:5:2 to derive

64:6:5
$$\zeta(v,u)=2\Gamma(1-v)\sum_{k=1}^{\infty}\frac{\sin(2k\pi u+\frac{1}{2}\pi v)}{(2k\pi)^{1-v}}-\sum_{j=0}^{\mathrm{Int}(u)-1}\left[j+\mathrm{frac}(u)\right]^{-v}\qquad v<1,\quad u\geq 0$$

its validity may be broadened.

64:7 PARTICULAR VALUES

Although the Hurwitz function is often known as the "generalized zeta function", there is an important mismatch between the definition $\sum(j+u)^{-v}$ of the Hurwitz function $\zeta(v,u)$ and the definition $\sum j^{-v}$ of the *zeta number* $\zeta(v)$ [Chapter 3] in that the former definition starts at $j=0$, whereas the latter starts at $j=1$. Hence when $u=0$, the Hurwitz function reduces to the zeta number only when the order is nonpositive. Particular values of the Hurwitz function for instances of positive integer parameters, and in the general case, are

$\zeta(v,0)$	$\zeta(v,1)$	$\zeta(v,2)$	$\zeta(v,3)$	$\zeta(v,4)$	$\zeta(v,m)$
$\begin{cases}\infty & v>0\\ \zeta(v) & v\leq 0\end{cases}$	$\zeta(v)$	$\zeta(v)-1$	$\zeta(v)-1-2^{-v}$	$\zeta(v)-1-2^{-v}-3^{-v}$	$\zeta(v)-\sum_{j=1}^{m-1}\frac{1}{j^{v}}$

For parameters that equal an odd multiple of ½, the Hurwitz function may be expressed in terms of the *lambda number* of Chapter 3

$\zeta(v,\frac{1}{2})$	$\zeta(v,\frac{3}{2})$	$\zeta(v,\frac{5}{2})$	$\zeta(v,m+\frac{1}{2})$
$2^{v}\lambda(v)$	$2^{v}\left[\lambda(v)-1\right]$	$2^{v}\left[\lambda(v)-1-3^{-v}\right]$	$2^{v}\left[\lambda(v)-\sum_{j=1}^{m}(2j-1)^{-v}\right]$

When the parameter is an odd multiple of ¼, the Hurwitz function involves also the *beta number* from the same chapter. The prototypes are

64:7:1
$$\zeta(v,\tfrac{1}{4})=4^{v}\left[\frac{\lambda(v)+\beta(v)}{2}\right]\qquad\text{and}\qquad\zeta(v,\tfrac{3}{4})=4^{v}\left[\frac{\lambda(v)-\beta(v)}{2}\right]$$

Notice that these formulas concur with the general rule 64:5:4, when 64:5:3 is taken into account.

64:8 NUMERICAL VALUES

Equator provides accurate values of $\zeta(v,u)$ for all $|v|\leq 100$ and $0\leq u\leq 100$. With keyword **Hurwitz**, the Hurwitz function routine uses formula 64:4:2 for negative integer orders and equation 64:6:5 for negative noninteger v not greater than -3.5. For all other orders, the expansion 64:6:2 is exploited via the ε-transformation [Section 10:14].

64:9 LIMITS AND APPROXIMATIONS

When the order is unity, the Hurwitz function suffers a discontinuity, but certain modifications of the Hurwitz function remain finite as $v = 1$ is approached. Thus

64:9:1 $$\lim_{v\to 1}\left\{\frac{\zeta(v,u)}{\Gamma(1-v)}\right\} = -1$$

and

64:9:2 $$\lim_{v\to 1}\left\{\zeta(v,u) - \frac{1}{v-1}\right\} = -\psi(u)$$

These results apply irrespective of whether unity is approached from smaller or larger values.

When u is small, the approximation

64:9:3 $$\zeta(v,u) \approx u^{-v} + \zeta(v) - vu\zeta(v+1) \qquad \text{small } u$$

involving the *digamma function* [Chapter 44], holds. Adding terms $\binom{-v}{j}u^j\zeta(v+j)$ with $j = 2,3,\cdots$ progressively improves the approximation.

64:10 OPERATIONS OF THE CALCULUS

Differentiation with respect to the *order* yields

64:10:1 $$\frac{\partial}{\partial v}\zeta(v.u) = -\sum_{j=0}^{\infty}\frac{\ln(j+u)}{(j+u)^v} \qquad v > 0$$

of which the special case

64:10:2 $$\frac{\partial\zeta}{\partial v}(0,u) = \ln\left(\frac{\Gamma(u)}{\sqrt{2\pi}}\right)$$

is noteworthy. Single and multiple differentiations with respect to the *parameter* yield

64:10:3 $$\frac{\partial}{\partial u}\zeta(v,u) = -v\zeta(v+1,u) \qquad \text{and} \qquad \frac{\partial^n}{\partial u^n}\zeta(v,u) = (-)^n(v)_n\zeta(v+n,u)$$

and these formulas may be generalized to the differintegration [Section 12:14] result

64:10:4 $$\frac{\partial^\mu}{\partial u^\mu}\zeta(v,u) = \frac{\Gamma(1-v)}{\Gamma(1-v-\mu)}\zeta(v+\mu,u)$$

where μ is not necessarily an integer. In these formulas $(v)_n$ denotes a Pochhammer polynomial [Chapter 18] and Γ symbolizes the gamma function [Chapter 43]. Note that both formulas in 64:10:3 accord with 64:10:4, as does 64:10:5.

Formulas for indefinite integration with respect to the parameter include:

64:10:5 $$\int_u^\infty \zeta(v,t)\mathrm{d}t = \frac{\zeta(v-1,u)}{v-1} \qquad v > 2$$

64:10:6 $$\int_1^u \zeta(v,t)\mathrm{d}t = \frac{\zeta(v-1,u)-\zeta(v-1)}{1-v} \qquad v < 2$$

64:10:7 $$\int_0^u \zeta(v,t)\mathrm{d}t = \frac{\zeta(v-1,u)-\zeta(v-1)}{1-v} \qquad v<1$$

and lead to the following interesting definite integrals

64:10:8 $$\int_0^1 \zeta(v,t)\mathrm{d}t = 0 \qquad v<1$$

64:10:9 $$\int_1^2 \zeta(v,t)\mathrm{d}t = \frac{1}{v-1}$$

The parts-integration procedure [Section 0:10] that produces the result

64:10:10 $$\int_1^u t\,\zeta(v,t)\mathrm{d}t = \frac{u\zeta(v-1,u)-\zeta(v-1)}{1-v} - \frac{\zeta(v-2,u)-\zeta(v-2)}{(1-v)(2-v)}$$

may be iterated to generate expressions for integrals of $t^n\zeta(v,t)$ where $n=2,3,4,\cdots$.

The *Böhmer integrals* that are the subject of Section 39:12 appear in the integration formulas

64:10:11 $$\int_x^{1+x} \begin{matrix}\sin\\ \cos\end{matrix}\left(2\pi t\right)\zeta(v,t)\mathrm{d}t = \left(2\pi\right)^{v-1}\begin{matrix}\mathrm{S}\\ \mathrm{C}\end{matrix}\left(2\pi x,1-v\right) \qquad v>1, \quad x\ge 0$$

There is a close connection between these results and the discussion in Section 64:14.

64:11 COMPLEX ARGUMENT

There is interest in the function $\zeta\left(\tfrac{1}{2}+iy,u\right)$ in the context of *Riemann's hypothesis* [Section 3:11] but this topic will not be pursued here.

Inverse Laplace transformation leads to hyperbolic functions [Chapters 29 and 30]:

64:11:1 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \zeta(v,s)\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\zeta(v,s)\right\} = \frac{t^{v-1}}{2\Gamma(v)}\left[\coth\left(\frac{t}{2}\right)+1\right]$$

64:11:2 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \zeta(v,s+\tfrac{1}{2})\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\left\{\zeta(v,s+\tfrac{1}{2})\right\} = \frac{t^{v-1}\operatorname{csch}(t/2)}{2\Gamma(v)}$$

64:12 GENERALIZATION: the Lerch function

Named for the Czech mathematician Mathias Lerch (1860 – 1922), the trivariate function $\Phi(x,v,u)$ generalizes the Hurwitz function because

64:12:1 $$\Phi(1,v,u) = \zeta(v,u)$$

However, inasmuch as, for appropriate of the nonunity variables,

64:12:2 $$\Phi(x,1,u) = \frac{1}{x^u}\ln_u\left(\frac{1}{1-x}\right) \qquad \text{and} \qquad \Phi(x,v,1) = \frac{-1}{x}\operatorname{polyln}_v(1-x)$$

it may equally well be regarded as a generalization of either of the functions mentioned in Section 25:12 – the generalized logarithmic function, or the polylogarithm.

Note that, though the *Atlas* prefers that the terminal character in the symbol of a multivariate function be the argument, we defer to the convention that the three variables of the Lerch function are cited in the order: argument x, order v, and parameter u. Though extensions may be possible, we generally impose the restrictions $|x| \le 1$, $v \ne 1$, and $u > 0$ throughout this section. Analogous to equations 64:3:2 and 64:3:5, are the definitions of the Lerch function as an integral

64:12:3
$$\Phi(x,v,u) = \frac{1}{\Gamma(v)}\int_0^\infty \frac{t^{v-1}\exp(-ut)}{1-x\exp(-t)}\mathrm{d}t$$

or as an infinite series

64:12:4
$$\Phi(x,v,u) = \frac{1}{u^v} + \frac{x}{(1+u)^v} + \frac{x^2}{(2+u)^v} + \frac{x^3}{(3+u)^v} + \cdots = \sum_{j=0}^{\infty}\frac{x^j}{(j+u)^v}$$

Yet another definition of the Lerch function is as a Weyl differintegral [Section 64:14] with respect to the logarithm of the argument:

64:12:5
$$\Phi(x,v,u) = x^{-u}\frac{\mathrm{d}^{-v}}{[\mathrm{d}\ln(t)]^{-v}}\frac{t^u}{1-t}\bigg|_{-\infty}^{\ln(x)} \qquad x>0$$

Many of the properties of the Lerch function, such as its recursion

64:12:6
$$\Phi(x,v,1+u) = \frac{1}{x}\left[\Phi(x,v,u) - \frac{1}{u^v}\right]$$

echo those of the Hurwitz function. The following limit governs the approach of the argument to unity

64:12:7
$$\lim_{x\to1}\left\{\frac{\Phi(x,v,u)}{(1-x)^{v-1}}\right\} = \Gamma(1-v) \qquad v<1$$

A surprisingly large number of familiar functions arise by specializing one or more of the variables of the Lerch function. Specializations of the order to integer values lead to the following special cases:

$\Phi(x,1,u)$	$\Phi(x,0,u)$	$\Phi(x,-1,u)$	$\Phi(x,-n,u)$	$\Phi(-x,1,\tfrac12)$
$\dfrac{\mathrm{B}(u,0,x)}{x^u}$	$\dfrac{1}{1-x}$	$\dfrac{u+x-xu}{(1-x)^2}$	$u\Phi(x,1-n,u) + x\dfrac{\partial}{\partial x}\Phi(x,1-n,u)$	$\dfrac{2}{\sqrt{x}}\arctan\left(\sqrt{x}\right)$

in which an *incomplete beta function* [Chapter 58] is found. When the parameter is specialized, *polylogarithms* [Section 25:12] often appear:

$\Phi(x,1,1)$	$\Phi(x,2,1)$	$\Phi(x,1,\tfrac12)$	$\Phi(x,v,0)$	$\Phi(x,v,\tfrac12)$
$\dfrac{-\ln(1-x)}{x}$	$\dfrac{-\mathrm{diln}(1-x)}{x}$	$\dfrac{2}{\sqrt{x}}\mathrm{arctanh}\left(\sqrt{x}\right)$	$-\mathrm{polyln}_v(1-x)$	$\dfrac{\mathrm{polyln}_v(1-x)}{\sqrt{x}} - \dfrac{\mathrm{polyln}_v\left(1-\sqrt{x}\right)}{2^{-v}\sqrt{x}}$

Other special cases include

$\Phi(1,v,u)$	$\Phi(0,v,u)$	$\Phi(-1,v,u)$	$\Phi(1,v,2u)$	$\Phi(-x,1,\tfrac12)$	$\Phi(-\tfrac12,1,\tfrac12)$
$\zeta(v,u)$	u^{-v}	$\eta(v,u)$	$\dfrac{\zeta(v,u)+\zeta(v,u+\tfrac12)}{2^v}$	$\dfrac{2}{\sqrt{x}}\arctan\left(\sqrt{x}\right)$	$\dfrac{\pi}{\sqrt{2}}$

Equator's Lerch function routine (keyword **Lerch**) relies on expansion 64:12:4. Because, when x approaches unity, this series is slow to converge, a more convergent series representing the remainder is appended following curtailment of the original expansion. The formula used by *Equator* is

64:12:8
$$\Phi(x,u,\nu) \approx \sum_{j=0}^{J-1} \frac{x^j}{(j+u)^\nu} - x^J \sum_{m=0}^{M} \binom{-\nu}{m} \frac{\mathrm{polyln}_{-m}(x)}{(J+u)^{m+\nu}}$$

This expression for the remainder has its origin in equation 25:12:5. J is chosen large enough that only a few terms of the m-series are needed.

64:13 COGNATE FUNCTION: the bivariate eta function

The definition of this function as a series

64:13:1
$$\eta(\nu,u) = \frac{1}{u^\nu} - \frac{1}{(1+u)^\nu} + \frac{1}{(2+u)^\nu} - \frac{1}{(3+u)^\nu} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+u)^\nu}$$

differs from the corresponding definition of the Hurwitz function only by the presence of alternating signs. Other similarities to $\zeta(\nu,u)$ are its recurrence and duplication formulas:

64:13:2
$$\eta(\nu,u+1) = u^{-\nu} - \eta(\nu,u)$$

64:13:3
$$\eta(\nu,2u) = 2^{-\nu}\left[\zeta(\nu,u) - \zeta(\nu,u+\tfrac{1}{2})\right]$$

which likewise differ from their Hurwitz analogues only by signs. Equation 64:13:3 provides a route to calculate the bivariate eta function from the Hurwitz function; another way is

64:13:4
$$\eta(\nu,u) = 2^{1-\nu}\zeta(\nu,\tfrac{1}{2}u) - \zeta(\nu,u)$$

This latter equation is the one used by *Equator*'s bivariate eta function routine, which uses the keyword **eta**.

Some special cases of the bivariate eta function are

$\eta(\nu,0)$	$\eta(\nu,1)$	$\eta(\nu,2)$	$\eta(\nu,\tfrac{1}{2})$	$\eta(1,u)$	$\eta(1,1)$	$\eta(1,\tfrac{1}{2})$
$\begin{cases} \infty & \nu>0 \\ -\eta(\nu) & \nu<0 \end{cases}$	$\eta(\nu)$	$1-\eta(\nu)$	$2^\nu\beta(\nu)$	$\frac{\mathrm{G}(u)}{2}$	$\ln(2)$	$\frac{\pi}{2}$

Here $\eta(\nu)$ and $\beta(\nu)$ are *eta* and *beta numbers* from Chapter 3; $\mathrm{G}(u)$ is *Bateman's G function* [Section 44:13].

64:14 RELATED TOPIC: Weyl differintegration

The Hurwitz and Lerch functions of this chapter have strong connections with the fractional calculus [Section 12:14], as does the bivariate eta function. For example, equations 64:3:6 and 64:12:4 show how the first two of these functions can be generated from simple algebraic expressions by the operations of the fractional calculus.

Differintegration is the operation that unifies differentiation and integration and extends the concept to fractional orders. Except when the order μ of differintegration is a nonnegative integer, a lower limit must be specified for the differintegral of a function $f(x)$ to be fully characterized. Any number will serve as this lower limit but the most common are 0 and $-\infty$.

Differintegration with a lower limit of $-\infty$ is called *Weyl differintegration* (Hermann Klaus Hugo Weyl, German mathematician, 1885–1955). Notations vary greatly, but our symbolism for a Weyl differintegral, and a definition applicable when $\mu<0$, is through the *Riemann-Liouville integral transform*

64:14:1 $$\left.\frac{d^\mu}{dt^\mu}f(t)\right|_{-\infty}^{x}=\frac{1}{\Gamma(-\mu)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{1+\mu}}dt \qquad \mu<0$$

When μ is positive, the definition still relies on this transform but subsequently differentiates it a sufficient number of times

64:14:2 $$\left.\frac{d^\mu}{dt^\mu}f(t)\right|_{-\infty}^{x}=\frac{d^n}{dx^n}\left\{\frac{1}{\Gamma(n-\mu)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{1+\mu-n}}dt\right\} \qquad n>\mu\geq 0$$

Consider, for example, the function $f(x)=\exp(-bx)$ where b is a positive constant. Then a change of the integration variable in 64:14:1 to $b(x-t)$ easily establishes that

64:14:3 $$\left.\frac{d^\mu}{dt^\mu}\exp(-bt)\right|_{-\infty}^{x}=b^\mu\exp(-bt) \qquad b>0$$

and the same result is also given by 64:14:2. As a second example, the *Randles-Sevčik function*, important in electrochemistry, may be defined, for negative x, as the Weyl semiderivative of the function $1/[1+\exp(-x)]$

64:14:4 $$\left.\frac{d^{1/2}}{dt^{1/2}}\frac{1}{1+\exp(-t)}\right|_{-\infty}^{x}=\left.\frac{d^{1/2}}{dt^{1/2}}\sum_{j=1}^{\infty}(-)^{j+1}\exp(jt)\right|_{-\infty}^{x}=\sum_{j=1}^{\infty}(-)^{j+1}\sqrt{j}\exp(jx) \qquad x<0$$

Other representations of this functions are

64:14:5 $$\exp(x)\Phi\left(-\exp(x),\tfrac{1}{2},1\right) \qquad \text{and} \qquad \sqrt{\frac{\pi}{2}}\sum_{j=1}^{\infty}\frac{\sqrt{X_j-x}\left(X_j+2x\right)}{X_j^3} \qquad X_j=\sqrt{(2j-1)^2\pi^2+x^2}$$

the latter not being restricted to negative x.

Of course, as with regular differentiation and integration, one may differintegrate with respect to a function, instead of with respect to a variable. For example, replacing t in equation 64:14:3 by a logarithm leads to

64:14:6 $$\left.\frac{d^\mu}{[d\ln(t)]^\mu}t^{-b}\right|_{0}^{\ln(x)}=b^\mu x^{-b} \qquad b>0$$

This formula lies at the heart of definitions 64:3:6 and 64:12:4.

The Hurwitz and bivariate eta functions play vital roles in the *Weyl differintegration of periodic functions*. Let $\mathrm{per}(x)$ be such a function and its period be P. With definition 64:14:1 applied to this periodic function [Chapter 36],

64:14:7 $$\Gamma(-\mu)\left.\frac{d^\mu}{dt^\mu}\mathrm{per}(t)\right|_{-\infty}^{x}=\int_{-\infty}^{x}\frac{\mathrm{per}(t)}{(x-t)^{1+\mu}}dt=\sum_{j=0}^{\infty}\int_{x-P-jP}^{x-jP}\frac{\mathrm{per}(t)}{(x-t)^{1+\mu}}dt=\frac{1}{P^{1+\mu}}\sum_{j=0}^{\infty}\int_{0}^{P}\frac{\mathrm{per}(x-\lambda-jP)}{[j+(\lambda/P)]^{1+\mu}}d\lambda$$

where, in the final step, the integration variable was changed to $\lambda=x-t-jP$. The final integrand is seen to involve the Hurwitz summands from equation 64:3:3, whence

64:14:8 $$\left.\frac{d^\mu}{dt^\mu}\mathrm{per}(t)\right|_{-\infty}^{x}=\frac{P^{-1-\mu}}{\Gamma(-\mu)}\int_{0}^{P}\mathrm{per}(x-\lambda)\zeta\left(1+\mu,\frac{\lambda}{P}\right)d\lambda \qquad \mu<0$$

For $-1<\mu<0$, this result requires that the integrals in 64:14:7 converge which, in turn, requires that the mean value of the periodic function be zero over its period. This requirement can be discarded when μ is positive, in which case the formula for Weyl differintegration of a periodic function, derived from 64:14:2, is

64:14:9
$$\left.\frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}}\mathrm{per}(t)\right|_{-\infty}^{x}=\frac{P^{-1-\mu}}{\Gamma(-\mu)}\int_{0}^{P}[\mathrm{per}(x-\lambda)-\mathrm{per}(x)]\zeta\left(1+\mu,\frac{\lambda}{P}\right)\mathrm{d}\lambda \qquad \mu>0$$

Because increasing x by P leaves the right-hand members of 64:14:8 and 64:14:9 unchanged, it is evident that the Weyl differintegral of a periodic function is itself periodic and of unchanged period.

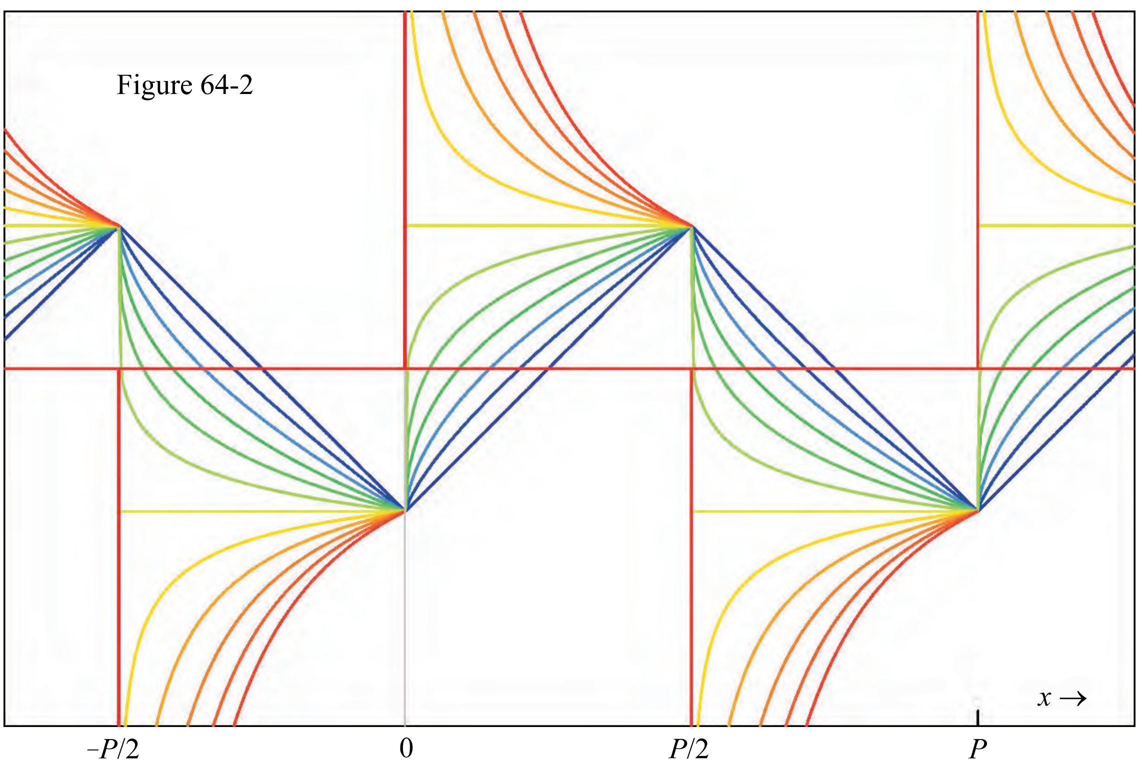
Though the rather complicated formulas of the previous paragraph apply to *all* periodic functions, Weyl differintegration of the cosine function merely scales the function and shifts its phase

64:14:10
$$\left.\frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}}\cos(\omega t)\right|_{-\infty}^{x}=\omega^{\mu}\cos\left(\omega x+\frac{\mu\pi}{2}\right)$$

and a similar result holds for the sine. As a final example, consider the *square-wave function* $(-1)^{\mathrm{Int}(2x/P)}$ [Section 36:14]. The result of differintegrating this periodic function to order μ can be expressed succinctly in terms of the bivariate eta function:

64:14:11
$$\left.\frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}}(-1)^{\mathrm{Int}(2x/P}\right|_{-\infty}^{x}=\frac{(2/P)^{\mu}}{\Gamma(1-\mu)}\eta\left(\mu,\frac{2x}{P}\right)$$

The waveforms produced for the cases of orders $\mu=1, 5/6, 2/3, 1/2, 1/3, 1/6, 0, -1/6, -1/3, -1/2, -2/3, 5/6$, and -1 are illustrated in Figure 64-2. They demonstrate a transition from square-wave to triangular-wave behavior as the order moves from 0 to −1, corresponding to increasingly robust integration. Increasing the order of differentiation, as μ transitions from 0 to 1, leads ultimately to a set of Dirac functions [Chapter 9], spiking alternately in the positive and negative directions. In the figure, the curves have been normalized to accentuate the familial pattern.



APPENDIX
A

USEFUL DATA

Information generally useful to scientists and engineers of all stripes is collected in this appendix.

Since 1960 *Le Système International d'Unités* has been the only system of measurement recognized globally. The *SI*, as it is usually abbreviated, is used universally in science and increasingly in engineering, architecture and medicine. The system specifies primary units, derived units, and prefixes.

The values (in primary *SI* units) of selected constants, conversion factors, and standards are available in *Equator*. This facility is indicated by the presence of keywords in the tables of this appendix.

A:1 NOTATION

There are seven base units, precisely defined by international agreement. Also there are two supplementary units, geometrically based. These *primary units* are:

Property	Name of unit	Symbol of unit
length	metre or meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd
plane angle	radian	rad
solid angle	steradian	sr

All other units in *SI* are derived from these nine.

A:2 DERIVED *SI* UNITS

Other than the primary units themselves, all units in *SI* are constructed by multiplying and/or dividing primary units. The symbols of these *derived units* use superscripts to indicate the exponent of each primary unit. The number of derived units is legion and only a few representative examples are included in the following listing:

Property	Name of unit	Symbol of unit
volume	cubic meter	m^3
acceleration	meter per square second	$m\ s^{-2}$
temperature gradient	kelvin per meter	$K\ m^{-1}$
luminance	candela per square meter	$cd\ m^{-2}$
reaction rate	mole per cubic meter per second	$mol\ m^{-3}\ s^{-1}$
	etc.	

Some derived units are given special names, which mostly honor a founding scientist. These named units are also given symbols, which usefully abbreviate the full list of primary unit symbols. A comprehensive listing of the *named units* that are part of *SI* is

Property	Named unit	Symbol	Equivalent primary-unit symbol
frequency	hertz	Hz	s^{-1}
force, weight	newton	N	$kg\ m\ s^{-2}$
pressure, stress	pascal	Pa	$kg\ m^{-1}\ s^{-2}$
energy, work	joule	J	$kg\ m^2\ s^{-2}$
power	watt	W	$kg\ m^2\ s^{-3}$
electric charge	coulomb	C	$s\ A$
electric potential	volt	V	$kg\ m^2\ s^{-1}\ A^{-1}$
electric resistance	ohm	Ω	$kg\ m^2\ s^{-1}\ A^{-2}$
electric conductance	siemens	S	$s^3\ A^2\ kg^{-1}\ m^{-2}$
electric capacitance	farad	F	$s^4\ A^2\ kg^{-1}\ m^{-2}$
electric inductance	henry	H	$kg\ m^2\ s^{-2}\ A^{-2}$
magnetic flux density	tesla	T	$kg\ s^{-2}\ A^{-1}$
magnetic flux	weber	Wb	$kg\ s^{-2}\ A^{-2}$
luminous flux	lumen	lm	$cd\ sr^{-1}$
illuminance	lux	lx	$cd\ m^{-2}\ sr^{-1}$
radioactivity	becquerel	Bq	s^{-1}
absorbed dose	gray	Gy	$m^2\ s^{-2}$
dose equivalent	sievert	Sv	$m^2\ s^{-2}$
catalytic activity	katal	kat	$mol\ s^{-1}$

Of course, derived units may combine named and primary units. A few examples of such composite *SI* units are

Property	Unit	Symbol	Equivalent primary-unit symbol
entropy, heat capacity	joule per kelvin	$J\,K^{-1}$	$kg\,m^2\,s^{-2}\,K^{-1}$
surface tension	newton per meter	$N\,m^{-1}$	$kg\,s^{-2}$
dynamic viscosity	pascal second	Pa s	$kg\,m^{-1}\,s^{-1}$
permittivity	farad per meter	$F\,m^{-1}$	$s^4\,A^2\,kg^{-1}\,m^{-3}$
thermal conductivity	watt per meter per kelvin	$W\,m^{-1}\,K^{-1}$	$kg\,m^{-1}\,s^{-3}\,K^{-1}$

Instead of negative superscripts, the solidus may be used; for example the unit of thermal conductivity may be symbolized $kg/(m\,s^3\,K)$. Unit symbols are not pluralized or punctuated.

A:3 NON-*SI* UNITS

Though not part of *SI*, there are several units that are officially condoned, or that are used frequently in conjunction with *SI* units. These include

Property	Non-*SI* Unit	Symbol	Equivalent *SI* quantity	Keyword
length	ångström	Å	10^{-10} m	
area	hectare	ha	$10^4\ m^2$	
volume	litre or liter	L	$10^{-3}\ m^3$	
mass	tonne or metric ton	t	10^3 kg	
mass	atomic mass unit	u	$1.6605\ 3878 \times 10^{-27}$ kg	**amu**
mass	dalton	Da	$1.6605\ 3878 \times 10^{-27}$ kg	**amu**
temperature	degree Celsius	°C	T °C = (T + 273.15) K	
pressure	bar	bar	10^5 Pa	
energy	electron volt	eV	$1.6021\ 7649 \times 10^{-19}$ J	**eV**
	etc.			

Here and elsewhere, uncertain digits are shown in a smaller font.

A major advantage of *SI* is that it eliminates the *conversion factors* that so often arise when such traditional units as inches, pounds and gallons are used. The first edition of this *Atlas* listed some conversion factors, but several websites offer much more comprehensive unit-interconversion facilities. Three that are free of advertisements [Philips, Public literature, and Wikipedia] are listed in Appendix B.

A:4 *SI* PREFIXES

Twenty *prefixes* are employed optionally in *SI* to form decimal submultiples

Factor:	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}	10^{-15}	10^{-18}	10^{-21}	10^{-24}
Prefix:	deci	centi	milli	micro	nano	pico	femto	atto	zepto	yocto
Symbol:	d	c	m	μ	n	p	f	a	z	y

or multiples

Factor:	10^1	10^2	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}	10^{24}
Prefix:	deka	hecto	kilo	mega	giga	tera	peta	exa	zetta	yotta
Symbol:	da	h	k	M	G	T	P	E	Z	Y

of *SI* units. These prefixes may be applied to any primary or named unit, except to the kilogram. For mass units, the prefix is applied to the gram (10^{-3} kg) instead. Thus

A:4:1 $$174\times10^{-7}\,\text{kg} = 17.4\,\text{mg} \qquad \textit{not} \qquad 17.4\,\mu\text{kg}$$

A superscript applied to a prefixed unit exponentiates the unit *and its prefix*, not just to the unit. For example:

A:4:2 $$0.013534\,\text{kg cm}^{-3} = 0.013534\,\text{kg}\,(10^{-2}\text{m})^{-3} = 13534\,\text{kg m}^{-3}$$

A:5 UNIVERSAL CONSTANTS

Among the *fundamental constants* that the U.S. National Institute for Standards and Technology [see "National Institute for Standards and Technology" in Appendix B] recognizes as in frequent use are

Name and symbol	Value	Keyword
velocity of light, c	299 792 458 m s^{-1}	**lightc**
Planck's constant, h	6.6260 690 × 10^{-34} J s	**Planckh**
Boltzmann's constant, k	1.3806 50 × 10^{-23} J K^{-1}	**Boltzmannk**
Avogadro's (Löschmidt's) constant, L	6.0221 418 × 10^{23} mol^{-1}	**AvogadroL**
Faraday's constant, F	96485.340 C mol^{-1}	**FaradayF**
gas constant, R	8.31447 J K^{-1} mol^{-1}	**gasR**
permittivity of space (electric constant), ε_0	8.8541 87817 62039 × 10^{-12} F m^{-1}	**epsilon0**
permeability of space (magnetic constant), μ_0	1.2566 37061 43592 × 10^{-6} N A^{-2}	**mu0**
electron charge, q_e	−1.6021 7649 × 10^{-19} C	**electronq**
electron mass, m_e	9.1093 822 × 10^{-31} kg	**electronm**
proton mass, m_p	1.6726 2164 × 10^{-27} kg	**protonm**
neutron mass, m_n	1.6749 2721 × 10^{-27} kg	**neutronm**
gravitational constant, G	6.6743 × 10^{-11} N m^2 kg^{-2}	**gravityG**
Rydberg constant, R_∞	1.0973 73156 853 × 10^7 m^{-1}	**RydbergR**
fine-structure constant, α	7.2973 52538 × 10^{-3}	**alpha**

The most recent values [Mohr, Taylor and Newell] of *physical constants* are listed here.

A:6 TERRESTRIAL CONSTANTS AND STANDARDS

Quantity	Value	Keyword
earth's mean radius	6.3710×10^{6} m	
earth's mass	5.9736×10^{24} kg	
earth's angular velocity	$7.2921\,159 \times 10^{-5}$ rad s^{-1}	
siderial year	$3.1558\,1500 \times 10^{7}$ s	
standard gravitational acceleration (sea level, 45° latitude), g	9.8066 5 m s^{-2}	**earthg**
escape velocity	11 186 m s^{-1}	
standard laboratory temperature, T_{std}	298.15 K	**standardT**
standard atmospheric pressure (older), P_{std}	101 325 Pa	**standardP**
standard atmospheric pressure (newer), 1 bar	10^{5} Pa	

A:7 THE GREEK ALPHABET

alpha	beta	gamma	delta	epsilon	zeta	eta	theta	iota	kappa	lambda	mu
Α, α	Β, β	Γ, γ	Δ, δ	Ε, ε	Ζ, ς, ζ	Η, η	Θ, θ, ϑ	Ι, ι	Κ, κ	Λ, λ	Μ, μ

nu	xi	omicron	pi	rho	sigma	tau	upsilon	phi	chi	psi	omega
Ν, ν	Ξ, ξ	Ο, ο	Π, π	Ρ, ρ	Σ, σ	Τ, τ	ϒ, υ	Φ, ϕ, φ	Χ, χ	Ψ, ψ	Ω, ω

Some Greek prefixes:

Quantitative: hemi- half, mono- one, bi- or di- two, tri- three, tetra- four, penta- five, hexa- six, hepta-, seven, octa- or octo- eight, nona- nine, deca- ten, dodeca- twelve, icosa- twenty, oligo- several, poly- many.

Qualitative: a- or an- without, anti- against, hetero- different, homo- same, hyper- less or below, hypo- more or beyond, iso- equal, ortho- straight, macro- long, mega- great or large, meta- between or among, micro- small, para- alongside, pseudo- false, quasi- almost.

APPENDIX B

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APPENDIX C

EQUATOR, THE ATLAS FUNCTION CALCULATOR

Equator, designed to be an integral part of the *Atlas*, is a software package that generates numerical values of more than 200 mathematical functions, including most of those that are the subjects of the preceding chapters. It resides on the CD that will be found within the back cover of the print version of the book. Users of the electronic version may purchase the *Equator* CD from booksellers. Consult Section C:12 for a comprehensive listing of the functions to which *Equator* caters, together with the corresponding keywords recognized by *Equator*.

In addition to its primary goal of providing function values, *Equator* will perform a number of subsidiary tasks, as described in Sections C:6, C:9, and C:10.

C:1 GETTING STARTED

The software must be first be installed from the CD onto the hard drive of a personal computer running at least Windows XP. The present version is not designed to operate on Macintosh computers or on Unix/Linux systems.

You may need "administrator privileges" in order to install *Equator*, *the Atlas function calculator*. Memory and processor requirements adequate for running Windows XP will satisfy *Equator*'s needs. During its installation, *Equator* will check if Microsoft's *.Net Framework 2.0* is already installed on your computer. If not, *Equator* will automatically download and install *.Net Framework 2.0* for you.

The following simple steps will enable you to install Equator:

- Insert the CD into your computer's disk drive. If the installation program does not immediately start, the "autorun" feature may be disabled on your computer. If so, double-click the CD drive icon under My Computer; then double-click the setup.exe file.
- If it is absent from your computer, you will be asked to install *.Net Framework 2.0* and you will need to be connected to the internet to do so. Click Accept when the license agreement appears. The required files will be downloaded and installed. This may take several minutes.
- An Application Install - Security Warning screen may appear. Just click the Install button.
- Follow the prompts to complete the installation of *Equator*. You will be required to accept a licensing agreement.
- A shortcut to *Equator* will have been placed in the Start menu and an *Equator* icon will have appeared on your desktop.

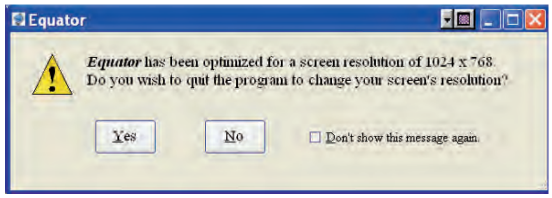
● The first time you run *Equator* you will be prompted to input your name, address, and licensing code. The code will be found on the CD pouch in the printed book or with the CD that you purchased.

Should you ever need to uninstall *Equator*, double-click Add or Remove Programs on the Windows Control Panel. Select Equator from the list and click Remove.

C:2 BASIC EQUATOR OPERATION

Double-click the Equator icon on your desktop. If your computer monitor is not set to 1024 × 768, you will see the following screen

C:2:1



You may, or may not, wish to adjust the resolution. Click the small ☐ checkbox to avoid being reminded of this next time.

Every function calculable by *Equator* has a name and a **keyword** [Section C:12]. *Equator*'s opening screen,

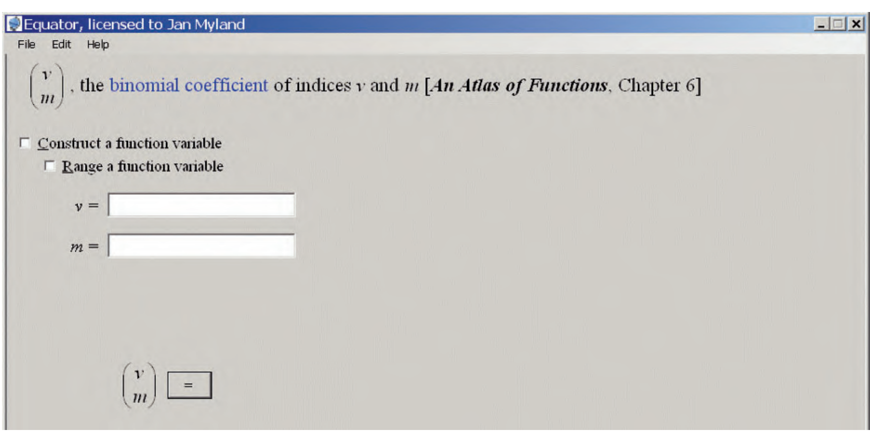
C:2:2

invites you to type either the name or the **keyword** into the header box. Alternatively, you may scroll down through the comprehensive alphabetized list to locate the sought name or **keyword**, then select and click it. The experienced user will use the typing option and will type the keyword, once familiar with it. As you type, *Equator* will try to anticipate your choice and will also display the corresponding mathematical symbol. For example, if wanting a binomial coefficient you type the keyword "bincoef", the screen will show:

C:2:3

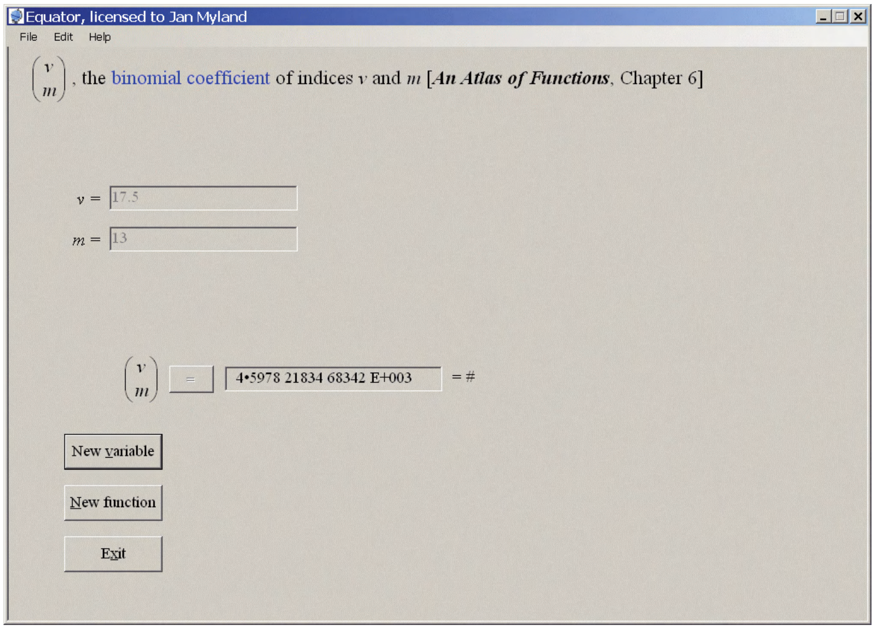
Once the sought function's name or **keyword** appears, click the Confirm button. In the binomial coefficient example, this brings up the following starting screen:

C:2:4



Because the binomial coefficient is bivariate, there are two input boxes but, depending on the function, there can be as many as four or as few as zero input boxes. For now, ignore the small square checkboxes. Recognize that you are being asked to type numbers into the $v =$ and $m =$ boxes. You might, for example, enter "17.5" and "13" for these variables. Then, on clicking the = button, you will find that *Equator* responds immediately with the answer screen:

C:2:5



At any time prior to clicking =, you can return to the start screen by pressing the Esc key.

With the calculation complete, there are now three buttons that you might click. One of these allows you to proceed to calculate the value of a different binomial coefficient. Another permits the choice of a new function. The third exits *Equator*.

The basics of *Equator* have now been covered, but there are several additional optional features.

C:3 MEMORIES AND CONSTANTS

Notice the " = #" message following the answer in screenshot C:2:5. This indicates that the answer has been stored by *Equator* in a #-memory, in case you might need to use it in some subsequent "chain" calculation. If, to continue with the previous example, you wished to find the logarithm of the binomial coefficient $\binom{17.5}{13}$, you need only summon the logarithm routine and type "#" into the argument box. Clicking = then generates the logarithm of the binomial coefficient.

The #-memory is ephemeral. Numbers so stored are retained only until the next time = is clicked. Sometimes, however, you may want to use a calculated answer more than once. To preserve an answer, type ctrl-M. This stores the answer into a more permanent M-memory. To reuse it, simply type "M" or "m" wherever the stored value is needed. It will remain stored until you overwrite it, or end the *Equator* session.

The values of seven mathematical constants (exact to 15 digits) and many of the most widely used physical constants are encoded in *Equator* and you may use them freely by typing in the keyword instead of a numerical value. See Appendix A for a listing of available physical constants, with their keywords. These constants are in primary *SI* units [Section A:3] and are the 2007 internationally recognized standard values.

C:4 VARIABLE CONSTRUCTION

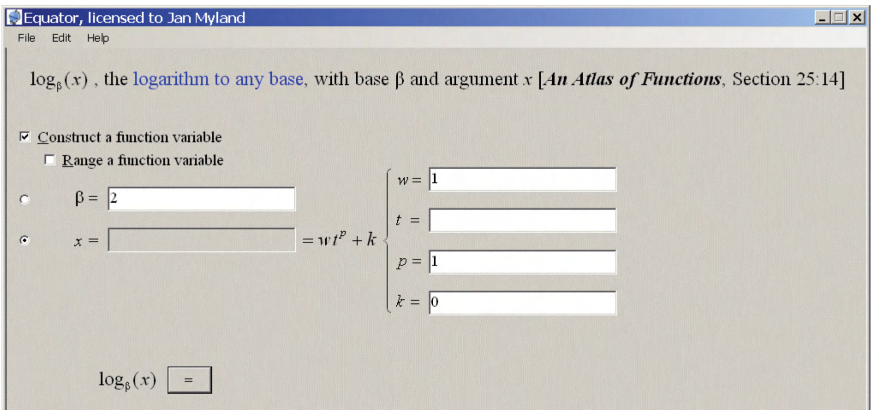
Knowing the value of some input quantity t, one often needs to find the value, not of the function f(t) of t itself, but of some modified variable, such as f(πt) or f($1+t^2$). *Equator* caters to this need by providing a facility to construct a desired input variable in accordance with the formula

C:4:1 $$x = wt^p + k$$

The default values of the multiplier w, power p, and the constant k are 1, 1, and 0, but these parameters may be altered at will by the user.

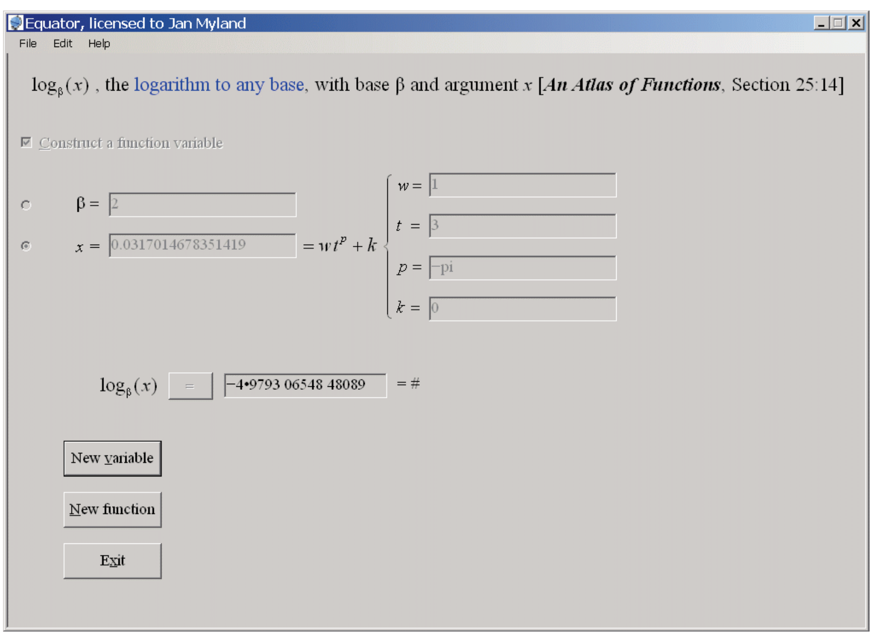
For example, you might want to compute $\log_2(3^{-\pi})$. Click New function and then choose the function logarithm to any base (or the keyword **loganybase**), which computes $\log_\beta(x)$. Enter 2 into the β box. Next click the checkbox ☐ Construct a function variable and also, to signify that it is argument x that is being constructed rather than the base β, click the ○ button to the left of $x =$ ▭. The screen now appears as:

C:4:2



Enter “3” into the t box and “$-$pi” into the p box, leaving k and w with their default values. Then, on clicking =, the screen appears as:

C:4:3



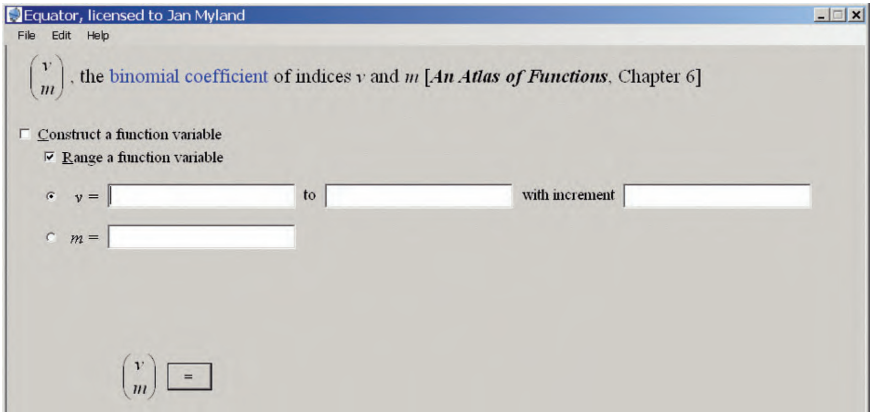
Notice that, as well as the answer being generated, the calculated value of the argument x is shown, “grayed out” in the appropriate variable box.

No more than one variable may be constructed in this way.

C:5 VARIABLE RANGING

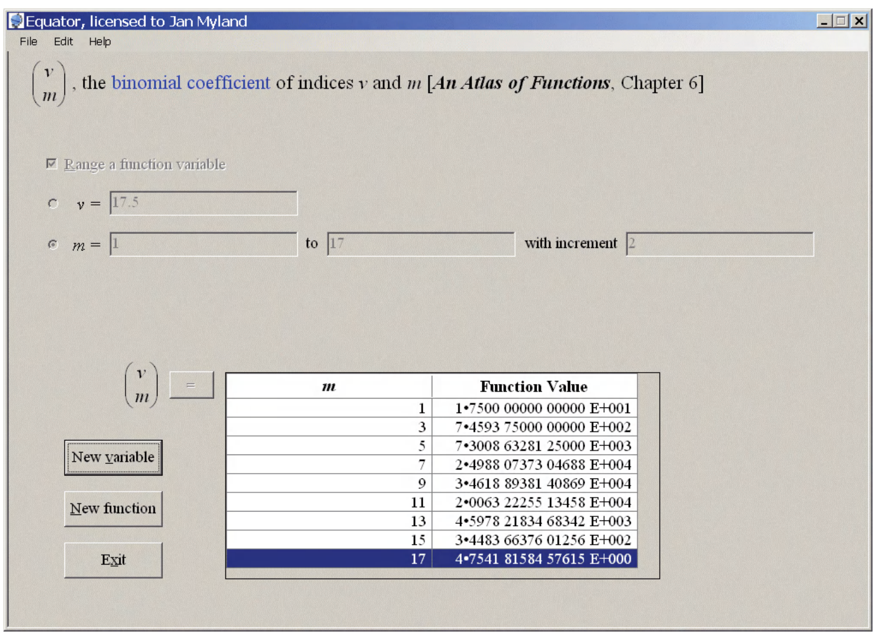
You might wish to determine, not just a single value of, say, a binomial coefficient, but a whole range of values. *Equator* provides for this need with its "variable ranging" feature. Test this out by again choosing the binomial coefficient function (keyword **bincoef**) and then clicking the checkbox ☐ Range a function variable. The screen

C:5:1



appears. Notice that the ○ button opposite the first variable, v, is already selected, on the assumption that it is this variable, rather than m, that you wish to range. If this is not your wish, click the button opposite m, converting it from ■ to ■ and allowing you to insert values into the appropriate boxes. If you choose $v = 17.5$ and decide to range m from 1 to 17 in steps of 2 then, after clicking =, the screen will be

C:5:2

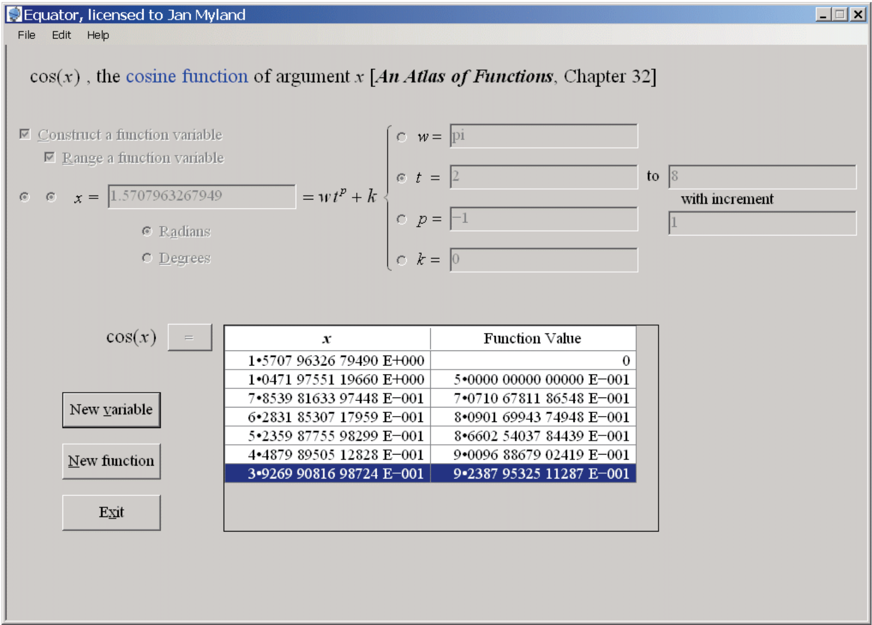


Equator will always evaluate the function at the value of the variable shown in the "to" box, even if this value is not in the chosen sequence. For example, if you ask for output from 1 to 10 in steps of 2, you will be given f(1), f(3), f(5), f(9) and f(10) values.

You can combine the construction and ranging features of *Equator* by ranging on any one of the w, t, p, or k construction tools. Simply click both ☐ checkboxes and the ○ button alongside whichever one of the tools you choose to range. Imagine, for example, that you wish to find the cosine of several submultiples of π; perhaps cos(π/2), cos(π/3), cos(π/4), ···, cos(π/8). Screenshot C:5:3 shows how this may be accomplished and what the answers are.

You cannot construct one variable and range another.

C:5:3



C:6 OTHER USES OF RANGING

Equator has no plotting capability, but should you wish to graph the output of a ranged calculation, you can easily transfer the data to graphing software, such as *Excel* or *SigmaPlot*®. *Equator*'s Edit menu provides a Copy results to clipboard facility which is useful for data transfer, not only for graphing, but also for creating hard copy or for another purpose.

Every time you use the ranging feature, *Equator* maintains a running sum of all the values it enters into the rightmost data column. Normally no use is made of this sequential addition and the user is unaware of its occurrence. However, if you wish to see the sum, type ctrl-S and the sum will appear as an additional final entry in the rightmost column. This sum is simply

C:6:1
$$\text{sum} = \mathrm{f}(x_1) + \mathrm{f}(x_2) + \mathrm{f}(x_3) + \cdots + \mathrm{f}(x_N)$$

where x_1 is the first argument, and $\mathrm{f}(x_N)$ is the last. This facility can be used, for instance, as a means of summing certain finite series. Alternatively, if you type ctrl-A, the average, sum/N, appears. *Equator* computes N as $2+\mathrm{Int}\{(x_N - x_1)/h\}$ where x_1, x_N, and h are the contents of the three ranging boxes.

Likewise, *Equator* automatically compiles another kind of sum each time a ranging calculation is performed. The sum in this case is called the trapezoidal area and it is

C:6:2
$$\begin{matrix}\text{trapezoidal}\\ \text{area}\end{matrix} = h\left[\tfrac{1}{2}\mathrm{f}(x_1) + \mathrm{f}(x_2) + \mathrm{f}(x_3) + \cdots + \tfrac{1}{2}\mathrm{f}(x_{N-1})\right] + \frac{x_N - x_{N-1}}{2}\left[\mathrm{f}(x_{N-1}) + \mathrm{f}(x_N)\right]$$

where h is the interval. To access this value, type ctrl-T. The trapezoidal area is output at the bottom of the rightmost column. It provides an approximation [Section 4:14] to the definite integral

C:6:3
$$\int_{x_1}^{x_N} \mathrm{f}(t)\,\mathrm{d}t \approx \begin{matrix}\text{trapezoidal}\\ \text{area}\end{matrix}$$

How good this approximation is depends on the properties of the function, as well as on N, the number of data you have specified.

These three facilities are inoperative if the ranged data consists of more than one column, or if any of the data are unavailable or out of range. The value of the sum, average or trapezoidal area is computed by unsophisticated summation; no steps are taken to ensure that all reported digits are significant.

C:7 DATA INPUT

You may enter a numerical value into *Equator* as an integer or a decimal number in either fixed or floating point (scientific) notation. Thus −765, 0.00573, and 12.987E−78 are all acceptable inputs, but not E−78 without a preceding number. E may be replaced by e, but not by 10. The minus sign, −, must precede a negative input, but + is optional for a positive number. Up to 15 digits may be input; any extra will be ignored. With decimal notation, *Equator* assumes that all digits beyond those input are zeros; thus 17.666 is treated as 17.666 00000 00000. The magnitude of input numbers may range between E−308 and E307.

As alternatives to numbers, the following "number substitutes" may be input:

- the # symbol, representing the value last calculated by *Equator*. If the calculation involved ranging, it is the result of the *final* calculation that is stored in #.
- the M or m symbol, representing a number previously placed in the M-memory.
- any one of the seven keywords listed here, representing a mathematical constant.
- any one of the twenty keywords listed in Appendix A, representing a fundamental physical constant, a conversion factor or an accepted standard. For the most part, these constants are known to less than 15-digit precision, but *Equator* takes no cognizance of this limited precision in calculations involving these constants.

Symbol and name	Keyword	Value
π, Archimedes's constant	**pi**	3.1415 92653 58979
G, Catalan's constant	**catalan**	0.91596 55941 77219
e, base of natural logarithms	**ebase**	2.7182 81828 45905
γ, Euler's constant	**euler**	0.57721 56649 01533
g, Gauss's constant	**gauss**	0.83462 68416 74073
Z, Apéry's constant	**apery**	1.2020 56903 15959
υ, golden section	**golden**	1.6180 33988 74990

- fractions, such as 17/369 or −pi/8.4E−7. The format must be two numbers (or number substitutes) separated by a solidus, "/". *Equator* converts the fraction to a decimal number before utilizing it in calculations, but it does remember that a fraction was input, because this is important in some computations.

Equator makes no general provision for the input of complex or imaginary numbers, but the complex number raised to a real power, the square-root function, and the exponential function of complex argument routines do accept real and/or imaginary input.

As elsewhere in the *Atlas*, this appendix uses the period as the decimal separator. This is the standard in many geographical regions. If, however, you are working in a country where the comma customarily fills the separator role, then (unless that setting in your computer has been changed), you must use a comma when inputting decimal numbers and *Equator* will respond in that system. Otherwise, *Equator* will use the period as the decimal separator and you should too. If you use commas *for any purpose*, when the decimal separator is the period, they are ignored. Conversely, periods will be ignored whenever the decimal separator is the comma. For visual convenience, you may wish to insert spaces within a number; that's okay: *Equator* will ignore them.

C:8 DATA OUTPUT

Answers that are integers smaller in magnitude than 1E16 are reported as integers. Function values *f* that lie in the range $0.1 \leq |f| < 1000$ are output in fixed-point decimal format. Otherwise a standard floating point notation, exemplified by

C:8:1 $$-1.2345\ 67890\ 12345\text{E}+123$$

is adopted. *Equator* can rarely output nonzero numbers outside the ranges $-1\text{E}307 \leq f \leq -1\text{E}-308$ and $1\text{E}-308 \leq f \leq 1\text{E}307$.

A few of *Equator*'s routines (the power function, cubic zeros, etc.) provide dual outputs: the real and imaginary parts of the complex answer. Some other routines (for example, the zeros, and associated values, of the Bessel functions) provide two real answers. In these cases, it is the penultimate output (often the real part) that is retained in the #-memory.

Equator strives to generate answers with "15-digit precision", by which we mean 14 exact digits, with the fifteenth having some uncertainty. This is not always possible. Generally, however, *Equator* is able to detect when precision is likely to have been lost during its calculations and it then curtails the output, such that all the reported digits (of which there may be as few as 3) are significant; that is, only the final reported digit is ever uncertain. If, for any reason, you wish to know what the original 15-digit answer was, just type "#": the uncurtailed answer will usually be displayed.

If *Equator* cannot confidently provide a function value, even one with only three significant digits, it will report "unavailable" or give some explanation of its failure. Such failures may occasionally occur even in regions where the *Atlas* text suggests that function values are accessible.

C:9 OTHER *EQUATOR* FEATURES

As do many computer programs, *Equator* allows mouse-less operation. Thus, instead of clicking on the Confirm button, you can type alt-C. Likewise, you may navigate between the various input boxes by use of the keyboard Tab key. The Confirm or = buttons may be similarly replaced by the Enter key.

Equator allows you to input arguments to the trigonometric and Gudermannian functions in radians (the default) or degree measure. Simply click the appropriate ○ button. Likewise you have the choice of radian or degree output from the inverse trigonometric and inverse Gudermannian functions and the elliptic amplitude routines.

Read about the rational approximation routine (keyword **rational**) in Section 8:13 of the *Atlas*. This provides a means of approximating a decimal number by a quotient of two integers. Thus it finds the numerator and denominator in such relationships as

C:9:1 $$\pi \approx \frac{355}{113}$$

The nearest binary approximant routine (keyword **bin**), described in the following section is another approximation feature of *Equator*.

Equator usually gives an answer almost immediately. Rarely, however, the operation may be slow. The 

message at the top of the screen will confirm that calculation is proceeding. If you wish to abort a computation, simply press the Esc key.

C:10 ARITHMETIC PROCESSING

So that you will have little need to supplement your *Equator* calculations by using a regular calculator, *Equator* has routines that carry out simple arithmetic operations. Routines for addition, subtraction, multiplication, division, square root and exponentiation exist and have the keywords **+**, **−**, *****, **/**, **sqrt**, and **power**. As well, the construction feature may be used for these purposes. However, *Equator* has another powerful tool to supplement function computation.

The "construction" feature, described in Section C:4 is a valuable way of tailoring your data prior to using them in evaluating the function of interest. There is often a requirement to tailor *Equator*'s output similarly, to suit your particular need. For example, you might have just calculated arctan($2x/\pi$), when what you actually need is $\sqrt{x}\,\arctan(2x/\pi)-\pi$. To aid in such a supplementary task, *Equator* provides an arithmetic function (keyword **arith**). This is a quadrivariate function defined by

C:10:1 $$\mathrm{arith}(W,T,P,K) = WT^{P} + K$$

where each of W, T, P, and K may be any number or number substitute. To use the arithmetic function in the example cited, one may choose the four parameters as follows

C:10:2 $$\mathrm{arith}(\#, x, \tfrac{1}{2}, -\mathrm{pi}) = \sqrt{x}\,\arctan(2x/\pi) - \pi$$

However the arithmetic function is more powerful than that example suggests because any one of the four parameters may itself be constructed, opening the possibility of using as many as seven parameters in formulating the output. Any one of those seven may be ranged in the normal way.

Though the arithmetic function is used most often as a sequel to a prior calculation (via the # or M memories), that is not its only application. For instance the quantity $(\pi g+1)/2g$, which occurs in Section 61:11, may be calculated as arith(pi/2,1,1,0.5/gauss), or in other ways. It may also be used to construct more elaborate input variables than the unaided "construction feature" can handle. As you become familiar with *Equator, the Atlas function calculator*, you will find the arithmetic function invaluable.

C:11 ACCURACY

A whole book could be, and perhaps one day will be, written about the measures that *Equator* adopts to preserve accuracy in calculating function values. Here we merely mention three sources of inaccuracy and methods by which some of these hazards may be mitigated. First note that there are two ways in which inaccuracies may be characterized. One is in terms of the absolute error

C:11:1 $$\begin{matrix}\text{absolute}\\ \text{error}\end{matrix} = \hat{\mathrm{f}}(x) - \mathrm{f}(x)$$

where $\mathrm{f}(x)$ is the true value of the function at the argument x and $\hat{\mathrm{f}}(x)$ is the calculated value. The second is the relative error

C:11:2 $$\begin{matrix}\text{relative}\\ \text{error}\end{matrix} = \frac{\hat{\mathrm{f}}(x) - \mathrm{f}(x)}{\mathrm{f}(x)} \approx \frac{\hat{\mathrm{f}}(x) - \mathrm{f}(x)}{\hat{\mathrm{f}}(x)}$$

A "good" algorithm is one that generates a small absolute error. However, it is the latter measure, the relative error, that determines how many digits can legitimately be reported as a function value. To report 15 digits requires that the relative error be on the order of 1E−15. It is the relative error that is usually thought of as reflecting "how accurate" an answer is.

Unless some special procedure can be exploited, *there is inevitably a large inaccuracy associated with* *computing the value of a function at an argument close to one of its zeros*. The truth of that statement is clear on inspection of equation C:11:2. Whenever x is close to a zero [Section 0:7 of the *Atlas*] of the function, the denominator in C:11:2 will be very small, which places extremely severe demands on the accuracy of computation of $\hat{f}(x)$ if the relative error is to be acceptably small. Of course, you may not even know that your input variable(s) correspond to a function value close to a zero. A small numerical value of the answer will provide a clue that this is the case and explain why *Equator* returns an answer with severely curtailed precision. An "inadequate significance" message may be returned if your argument is extremely close to a zero.

A second issue relates to number systems [Section 8:14]. Like most computer programs, *Equator* operates with binary numbers. On the other hand, the user inputs a decimal number. Hence, an early step in the operation of *Equator* is the conversion of the user's decimal number into a binary number. Now, some decimal numbers, such as 73.244 16503 90625, are also exact binary numbers (in this example 1010110.110000000001), whereas another decimal number, such as 73.244 39024 39025, which is superficially similar, is not. The first number converts exactly to a binary number, whereas the second number will be converted by *Equator* into the nearest binary number. This obligatory approximation that *Equator* makes may or may not engender a serious error in the output. It depends on how "steep" the function is at the argument in question; that is, the inaccuracy depends on the magnitude of $\partial f/\partial x$. Sometimes this derivative can be very large indeed (especially close to a discontinuity). *If the input variable is not an exactly binary number, inevitable inaccuracies may be present in the output, especially if the function is steep.* These inaccuracies will arise, of course, even if *Equator* returns an exact answer – it is the exact answer for a *different* argument. If you suspect that your answers may risk contamination from this cause, and if your particular problem allows, it will generally help if you use only inputs that are exactly binary. For this purpose, *Equator* has a nearest binary approximant routine (keyword **bin**), described in Section 8:14. This algorithm outputs, in decimal, the nearest exactly binary number, of no more than 15 decimal digits, to the input number. Of course, an alternative is to restrict yourself to input numbers, such as 73, 73.75, or 73.03125, that you recognize on sight as exactly binary.

Precision is lost whenever two numbers of like sign are subtracted, and *Equator* takes cognizance of this loss by outputting only digits that are significant. Numerous such subtractions are especially destructive of precision and hence *there is the danger of severe loss of precision whenever a computation incorporates the modulo operation* [Section 8:12], *or its special case, extraction of the fractional-part*, as a necessary part of the routine. A case in point that arises frequently is the calculation of such functions as $\sin(\pi x)$ when x is large. To evade this problem, *Equator* provides a reperiodized sine function routine and a reperiodized cosine function (keywords **sinpi** and **cospi**) that internally multiply the x by π and provide accurate $\sin(\pi x)$ and $\cos(\pi x)$ values, no matter how large x might be.

C:12 EQUATOR KEYWORDS

In addition to recognizing the full names of functions and sometimes their synonyms, *Equator* recognizes the following **keywords**, which are sequences of up to ten characters, mnemonically mirroring either the symbol or the name of the function. There follows a comprehensive listing of all *Equator*'s routines and the corresponding keywords. The final column lists the relevant chapter (or appendix) and often the section too.

Name and symbol	Keyword	Chap/Secn
addition, +	**+**	C:10
Airy Ai function, $\mathrm{Ai}(x)$	**Ai**	56
Airy Bi function, $\mathrm{Bi}(x)$	**Bi**	56
Apery's constant, Z	**apery**	1:7

Name and symbol	Keyword	Chap/Secn
Archimedes's constant, π	**pi**	1:7
arithmetic function, arith(W,T, P, K)	**arith**	C:10
associated Laguerre polynomial, $\mathrm{L}_n^{(m)}(x)$	**assocLpoly**	23:12
associated Legendre function of the first kind, $\mathrm{P}_\nu^{(m)}(x)$	**assocP**	59:12
associated Legendre function of the second kind, $\mathrm{Q}_\nu^{(m)}(x)$	**assocQ**	59:12
associated value of extremum of Bessel function, $\mathrm{J}\left(j_n'^{(k)}\right)$	**extremeJ**	52:7
atomic mass unit, u	**amu**	A:3
auxiliary Airy fai function, fai(x)	**fai**	56:6
auxiliary Airy gai function, gai(x)	**gai**	56:6
auxiliary cosine integral, gi(x)	**gi**	38:13
auxiliary cylinder fc function, $\mathrm{fc}_\nu(x)$	**fc**	54:14
auxiliary cylinder gc function, $\mathrm{gc}_\nu(x)$	**gc**	54:14
auxiliary Fresnel cosine integral, Fres(x)	**Fres**	39:13
auxiliary Fresnel sine integral, Gres(x)	**Gres**	39:13
auxiliary sine integral, fi(x)	**fi**	38:13
Avogadro's (Löschmidt's) constant, L	**AvogadroL**	A:5
base of natural logarithms, e	**ebase**	1:7
Bateman G function, G(ν)	**G**	44:13
Bateman's confluent function, $\kappa_\nu(x)$	**kappa**	48:13
Bernoulli number, B_n	**Bnum**	4
Bernoulli polynomial, $\mathrm{B}_n(x)$	**Bpoly**	19
Bessel function, $\mathrm{J}_n(x)$ or $\mathrm{J}_\nu(x)$	**J**	52 or 53
(complete) beta function, B(ν,μ)	**Beta**	43:13
beta number, $\beta(\nu)$	**betanum**	3
binomial coefficient, $\binom{\nu}{m}$	**bincoef**	6
bivariate eta function, $\eta(\nu,u)$	**eta**	64:13
Boltzmann's constant, k	**Boltzmannk**	A:5
Catalan's constant, G	**catalan**	1:7
Chebyshev gamma coefficient, $\gamma_j^{(n)}$	**Chebygamma**	22:5
Chebyshev polynomial of the first kind, $\mathrm{T}_n(x)$	**Tpoly**	22
Chebyshev polynomial of the second kind, $\mathrm{U}_n(x)$	**Upoly**	22
Chebyshev tau coefficient, $\tau_k^{(n)}$	**Chebytau**	22:6
Clausen's integral, Clausen(x)	**Clausen**	32:14
common mean, mc(x,y)	**mc**	61:14

Name and symbol	Keyword	Chap/Secn
complete beta function, $B(\nu,\mu)$	**Beta**	43:13
complete elliptic integral of the first kind, $K(k)$	**EllipticK**	61
complete elliptic integral of the second kind, $E(k)$	**EllipticE**	61
complete elliptic integral of the third kind, $\Pi(\nu,k)$	**EllipticPi**	61:12
complete gamma function, $\Gamma(\nu)$	**Gamma**	43
complex number raised to a real power, $(x+iy)^n$ or $(x+iy)^\nu$	**compower**	10:11 or 12:8
cosecant function, $\csc(x)$	**csc**	33
cosine function, $\cos(x)$	**cos**	32
cosine integral, $\mathrm{Ci}(x)$	**Ci**	38
cotangent function, $\cot(x)$	**cot**	34
cotangent root, $\rho_n(b)$	**rho**	34:7
cubic function, x^3+ax^2+bx+c	**cubic**	16
cubic zeros, $r_3(a,b,c,n)$	**r3**	16:7
cumulative function for a Boltzmann distribution, $F_{\mathrm{Boltzmann}}(\mu,x)$	**FBoltzmann**	27:14
cumulative function for a Laplace distribution, $F_{\mathrm{Laplace}}(\mu,\sigma,x)$	**FLaplace**	27:14
cumulative function for a logistic distribution, $F_{\mathrm{logistic}}(\mu,\sigma,x)$	**Flogistic**	27:14
cumulative function for a lognormal distribution, $F_{\mathrm{lognormal}}(\mu,\sigma,x)$	**Flognormal**	27:14
cumulative function for a Lorentz distribution, $F_{\mathrm{Lorentz}}(\mu,\sigma,x)$	**FLorentz**	27:14
cumulative function for a Maxwell distribution, $F_{\mathrm{Maxwell}}(\mu,x)$	**FMaxwell**	27:14
cumulative function for a normal distribution, $F_{\mathrm{normal}}(\mu,\sigma,x)$	**Fnormal**	27:14
cumulative function for a Rayleigh distribution, $F_{\mathrm{Rayleigh}}(\mu,x)$	**FRayleigh**	27:14
Dawson's integral, $\mathrm{daw}(x)$	**daw**	42
Debye function, $\int_0^x \frac{t^n \mathrm{d}t}{\exp(t)-1}$	**Debye**	3:15
decadic logarithm, $\log_{10}(x)$	**log10**	25:14
digamma function, $\psi(\nu)$	**digamma**	44
dilogarithm, $\mathrm{diln}(x)$	**diln**	25:12
discrete Chebyshev polynomial, $t_n^{(J)}(x)$	**discCheby**	22:13
division, ÷	**/**	C:10
double factorial function, $n!!$	**!!**	2:13
electron charge, q_e	**electronq**	A:5
electron mass, m_e	**electronm**	A:5
electron volt, eV	**eV**	A:3
elliptic amplitude, $\mathrm{am}(k,x)$	**am**	63

Name and symbol	Keyword	Chap/Secn
elliptic nome, q(k)	**q**	61:15
entire cosine integral, Cin(x)	**Cin**	38
entire exponential integral, Ein(x)	**Ein**	37
entire hyperbolic cosine integral, Chin(x)	**Chin**	38
entire incomplete gamma function, γn(v,x)	**gamentire**	45
error function, erf(x)	**erf**	40
error function complement, erfc(x)	**erfc**	40
eta number, $\eta(v)$	**etanum**	3
Euler number, E_n	**Enum**	5
Euler polynomial, $\mathrm{E}_n(x)$	**Epoly**	20
Euler's constant, γ	**euler**	1:7
exponential error function complement product, $\exp(x)\mathrm{erfc}(\sqrt{x})$	**experfc**	41
exponential function, exp(x)	**exp**	26
exponential function of complex argument, exp(x+iy)	**complexp**	26:11
exponential integral, Ei(x)	**Ei**	37
exponential polynomial, e(x)	**epoly**	26:12
exponential theta-four function, $\theta_4(v,t)$	**theta4**	27:13
exponential theta-one function, $\theta_1(v,t)$	**theta1**	27:13
exponential theta-three function, $\theta_3(v,t)$	**theta3**	27:13
exponential theta-two function, $\theta_2(v,t)$	**theta2**	27:13
extrema, and their (associated) values, of the Bessel function, $\mathrm{j}_n'^{(k)}$ and $\mathrm{J}(j_n'^{(k)})$	**Jextrema**	52:7
factorial function, $n!$	**!**	2
Faraday's constant, F	**FaradayF**	A:5
Fibonacci number, Fib(n)	**Fibnum**	23:14
Fibonacci polynomial, $\mathrm{Fib}_n(x)$	**Fibpoly**	23:14
fine structure constant, α	**alpha**	A:5
fractional-part function, Fp(x)	**Fp**	8
fractional-value function, frac(x)	**frac**	8
Fresnel cosine integral, C(x)	**C**	39
Fresnel sine integral, S(x)	**S**	39
(complete) gamma function, $\Gamma(v)$	**Gamma**	43
gas constant, R	**gasR**	A:5
Gauss hypergeometric function, F(a,b,c,x)	**F**	60
Gauss's constant, g	**gauss**	1:7

Name and symbol	Keyword	Chap/Secn
Gegenbauer polynomial, $C_n^{(\lambda)}(x)$	**Cpoly**	22:12
generalized mean, m(x,y,n)	**m**	61:14
golden section, υ	**golden**	23:14
gravitational constant, G	**gravityG**	A:5
Gudermannian function, gd(x)	**gd**	33:15
Hermite polynomial, $H_n(x)$	**Hpoly**	24
Hurwitz function, $\zeta(v,u)$	**Hurwitz**	64
hyperbolic cosecant function, csch(x)	**csch**	29
hyperbolic cosine function, cosh(x)	**cosh**	28
hyperbolic cosine integral, Chi(x)	**Chi**	38
hyperbolic cotangent function, coth(x)	**coth**	30
hyperbolic secant function, sech(x)	**sech**	29
hyperbolic sine function, sinh(x)	**sinh**	28
hyperbolic sine integral, Shi(x)	**Shi**	38
hyperbolic tangent function, tanh(x)	**tanh**	30
incomplete elliptic integral of the first kind, F(k,φ)	**ellipF**	62
incomplete elliptic integral of the second kind, E(k,φ)	**ellipE**	62
incomplete elliptic integral of the third kind, $\Pi(v,k,\varphi)$	**ellipPi**	62:12
incomplete beta function, $B(v,\mu,x)$	**incompBeta**	58
integer-part function, Ip(x)	**Ip**	8
integer-value function, Int(x)	**Int**	8
inverse cosecant function, arccsc(x)	**arccsc**	35
inverse cosine function, arccos(x)	**arccos**	35
inverse cotangent function, arccot(x)	**arccot**	35
inverse error function, inverf(x)	**inverf**	40
inverse Gudermannian function, invgd(x)	**invgd**	33:15
inverse hyperbolic cosecant function, arcsch(x)	**arcsch**	31
inverse hyperbolic cosine function, arcosh(x)	**arcosh**	31
inverse hyperbolic cotangent function, arcoth(x)	**arcoth**	31
inverse hyperbolic secant function, arsech(x)	**arsech**	31
inverse hyperbolic sine function, arsinh(x)	**arsinh**	31
inverse hyperbolic tangent function, artanh(x)	**artanh**	31
inverse secant function, arcsec(x)	**arcsec**	35
inverse sine function, arcsin(x)	**arcsin**	35

Name and symbol	Keyword	Chap/Secn
inverse tangent function, $\arctan(x)$	**arctan**	35
Jacobi polynomial, $P_n^{(v,\mu)}(x)$	**Jacobipoly**	22:12
Jacobian elliptic cd function, $\mathrm{cd}(k,x)$	**cd**	63
Jacobian elliptic cn function, $\mathrm{cn}(k,x)$	**cn**	63
Jacobian elliptic cs function, $\mathrm{cs}(k,x)$	**cs**	63
Jacobian elliptic dc function, $\mathrm{dc}(k,x)$	**dc**	63
Jacobian elliptic dn function, $\mathrm{dn}(k,x)$	**dn**	63
Jacobian elliptic ds function, $\mathrm{ds}(k,x)$	**ds**	63
Jacobian elliptic nc function, $\mathrm{nc}(k,x)$	**nc**	63
Jacobian elliptic nd function, $\mathrm{nd}(k,x)$	**nd**	63
Jacobian elliptic ns function, $\mathrm{ns}(k,x)$	**ns**	63
Jacobian elliptic sc function, $\mathrm{sc}(k,x)$	**sc**	63
Jacobian elliptic sd function, $\mathrm{sd}(k,x)$	**sd**	63
Jacobian elliptic sn function, $\mathrm{sn}(k,x)$	**sn**	63
Kelvin bei function, $\mathrm{bei}(x)$	**bei**	55
Kelvin ber function, $\mathrm{ber}(x)$	**ber**	55
Kelvin kei function, $\mathrm{kei}(x)$	**kei**	55
Kelvin ker function, $\mathrm{ker}(x)$	**ker**	55
Kummer function, $M(a,c,x)$	**M**	47
Laguerre polynomial, $L_n(x)$	**Lpoly**	23
lambda number, $\lambda(v)$	**lambdanum**	3
Langevin function, $\coth(x)-(1/x)$	**Langevin**	30:14
Legendre function of the first kind, $P_v(x)$	**P**	59
Legendre function of the second kind, $Q_v(x)$	**Q**	59
Legendre polynomial, $P_n(x)$	**Ppoly**	21
Lerch function, $\Phi(x,v,u)$	**Lerch**	64:12
logarithm to any base, $\log_\beta(x)$	**loganybase**	25:14
logarithm to base 10 of the factorial function, $\log_{10}(n!)$	**log10!**	2:8
logarithm to base 10 of the gamma function, $\log_{10}\{\Gamma(v)\}$	**log10Gamma**	43:8
logarithmic factorial function, $\ln(n!)$	**ln!**	2:8
logarithmic function, $\ln(x)$	**ln**	25
logarithmic gamma function, $\ln\{\Gamma(v)\}$	**lnGamma**	43:8
logarithmic integral, $\mathrm{li}(x)$	**li**	25:13
lower incomplete gamma function, $\gamma(v,x)$	**gamlower**	45

Name and symbol	Keyword	Chap/Secn
Macdonald function, $K_\nu(x)$	**K**	51
Mittag-Leffler function, $E_{\nu,\mu}(x)$	**Mittag**	45:13
modified Struve function, $\mathtt{l}_\nu(x)$	**l**	57
modified (hyperbolic) Bessel function, $I_n(x)$ or $I_\nu(x)$	**I**	49 or 50
modified spherical Bessel function, $i_n(x)$	**i**	28:13
modulo function, $v(\mathrm{mod}\ m)$	**mod**	8:12
multiplication, $\times$	*****	C:10
n-fold integral of the error function complement, $i^n\mathrm{erfc}(x)$	**inerfc**	40:13
nearest binary approximant, $\mathrm{bin}(x)$	**bin**	8:14
Neumann function, $Y_\nu(x)$	**Y**	54
neutron mass, m_n	**neutronm**	A:5
Neville's c theta function, $\vartheta_\mathrm{c}(k,x)$	**theta-c**	61:15
Neville's d theta function, $\vartheta_\mathrm{d}(k,x)$	**theta-d**	61:15
Neville's n theta function, $\vartheta_\mathrm{n}(k,x)$	**theta-n**	61:15
Neville's s theta function, $\vartheta_\mathrm{s}(k,x)$	**theta-s**	61:15
normally distributed random variates, $\mathrm{normal}(\mu,\sigma,J,s)$	**normal**	40:14
parabolic cylinder function, $D_\nu(x)$	**D**	46
permeability of free space, m_0	**mu0**	A:5
permittivity of free space, ε_0	**epsilon0**	A:5
Planck's constant, h	**Planckh**	A:5
Pochhammer polynomial, $(x)_n$	**Poch**	18
polygamma function, $\psi^{(n)}(\nu)$	**polygamma**	44
power function, x^n or x^ν	**power**	10 or 12
probability function for a Boltzmann distribution, $P_{\mathrm{Boltzmann}}(\mu,x)$	**PBoltzmann**	27:14
probability function for a Laplace distribution, $P_{\mathrm{Laplace}}(\mu,\sigma,x)$	**PLaplace**	27:14
probability function for a logistic distribution, $P_{\mathrm{logistic}}(\mu,\sigma,x)$	**Plogistic**	27:14
probability function for a lognormal distribution, $P_{\mathrm{lognormal}}(\mu,\sigma,x)$	**Plognormal**	27:14
probability function for a Lorentz distribution, $P_{\mathrm{Lorentz}}(\mu,\sigma,x)$	**PLorentz**	27:14
probability function for a Maxwell distribution, $P_{\mathrm{Maxwell}}(\mu,x)$	**PMaxwell**	27:14
probability function for a normal distribution, $P_{\mathrm{normal}}(\mu,\sigma,x)$	**Pnormal**	27:14
probability function for a Rayleigh distribution, $P_{\mathrm{Rayleigh}}(\mu,x)$	**PRayleigh**	27:14
proton mass, m_p	**protonm**	A:5
quadratic function, ax^2+bx+c	**quadratic**	15
quadratic zeros, $r_2(a,b,c,n)$	**r2**	15:7

Name and symbol	Keyword	Chap/Secn
quartic zeros, $r_4(a_3,a_2,a_1,a_0)$	**r4**	16:12
rational approximants, n/d	**rational**	8:13
reperiodized cosine function, $\cos(\pi x)$	**cospi**	32:8
reperiodized sine function, $\sin(\pi x)$	**sinpi**	32:8
Rydberg constant, R_∞	**RydbergR**	A:5
sampling function, $\mathrm{sinc}(x)$	**sinc**	32:13
secant function, $\sec(x)$	**sec**	33
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SYMBOL INDEX

Red numbers in brackets identify the section in which the symbol is encountered, or the equation by which it is defined. Numbers colored in teal imply brief mention in the cited section of a symbol that is not otherwise discussed or used in the *Atlas*. Temporary abbreviations, used only locally, are not always indexed. Symbols used in the appendices are not indexed either.

UPPERCASE LATIN

LOWERCASE LATIN

$\mathrm{i}_n(x)$	modified spherical Bessel function of order n and argument x [28:13]
i	$=\sqrt{-1}$, imaginary operator [1:11]
$\mathrm{ierfc}(x)$	complementary error function integral [40:13]
$\mathrm{i}^{\#}\mathrm{ierfc}(x)$	$\{\#=-1,0,1,2,\cdots,n\}$, #-fold integral of the error function complement [40:13]
$\mathrm{i}^{-1}\mathrm{erfc}(x)$	$=\left(2/\sqrt{\pi}\right)\exp(-x^2)$ [40:13:6]
$\mathrm{inverf}(x)$	inverse error function [40:8]
$\mathrm{invgd}(\theta)$	inverse Gudermannian function of (angular) argument θ [33:15]
$\mathrm{inv\#}(k,y)$	$\{\#=\mathrm{am},\mathrm{cn},\mathrm{sn},\mathrm{dn}\}$ inverse elliptic functions [63:13]
$\mathrm{j}_\nu^{(k)}$	kth positive zero of the Bessel function $\mathrm{J}_\nu(x)$ [52:7, 53:7]
$\left(\mathrm{j}_n^{(k)}\right)_m$	mth estimate of kth positive zero of the Bessel function $\mathrm{J}_n(x)$ [52:7]
$\mathrm{j}'^{(k)}_\nu$	kth positive extremum of the Bessel function $\mathrm{J}_\nu(x)$ [52:7, 53:7]
$j_{n,k}$	Bessel zero [52:7]
j,k	summation indices [0:6]
$\mathrm{j}_n(x)$	spherical Bessel function of order n and argument x [32:13:7]
k,k'	eccentricity (ellipticity), complementary eccentricity of an ellipse [13:1]
k	modulus of the elliptic family of functions [62:1]
k'	$=\sqrt{1-k^2}$, complementary elliptic modulus [62:1]
k	physical constant [46:15], thermal conductivity [53:15]
k	eccentricity of a conic section [15:15]
$\mathrm{ker}(x),\mathrm{kei}(x)$	zero-order Kelvin functions argument x [55:3]
$\mathrm{ker}_\nu(x),\mathrm{kei}_\nu(x)$	Kelvin functions of order ν and argument x [55:3]
$k_\nu(x)$	Bateman confluent function [48:13]
$\mathrm{k}_n(x)$	spherical Macdonald function of order n and argument x [26:13]
ℓ,s	sides of a rectangle [23:14]
$\ell(x_0\to x_1)$	length of a plane curve between points x_0 and x_1 [39:14]
$\mathtt{l}_\nu(x)$	modified Struve function [57:13]
$\mathrm{li}(x)$	logarithmic integral of x [25:13]
$\ln(x)$	logarithmic function of argument x [25:1]
$\ln^{-1}(x)$	natural antilogarithm (that is, the exponential) of x [26:1]
$\ln_\nu(x)$	generalized logarithmic function of argument x and order ν [25:12]
$\log_\beta(x)$	logarithm of x to base β [25:14]
$\log_{10}(x)$	decadic logarithm [25:14]
m	integer variable [0:1]
m	$=k^2$, elliptic parameter [61:1]
m_1	complementary elliptic parameter [61:1]
$\mathrm{m\#}(x,y)$	$\{\#=\mathrm{a},\mathrm{g},\mathrm{h},\mathrm{r},\mathrm{c}\}$ arithmetic, geometric, harmonic, root-mean-square, common mean of x and y [61:14]
$\mathrm{m}(n,x,y)$	generalized mean of x and y [61:14]
$\mathrm{m\#}(j,x,y)$	$\{\#=\mathrm{g},\mathrm{a}\}$ iterated mean [61:14:7, 61:14:8]
n	integer variable [0:1]
$\mathrm{n\#}(k,x)$	$\{\#=\mathrm{d},\mathrm{s},\mathrm{c}\}$ Jacobian elliptic functions of modulus k and argument x [63:3]
p	percentile [27:14]
$\mathrm{per}(x),\mathrm{qer}(x)$	periodic functions of x [36:1]
per_{rms}	root-mean-square amplitude of the periodic function $\mathrm{per}(x)$ [36:14]
p	chemists' cologarithm [25:14]

$\mathrm{polyln}_v(x)$	polylogarithm of order v and argument x [25:12]
$\mathrm{p}_n(x)$	polynomial function of degree n and argument x [17:1:1]
p, q	parabolic coordinates [35:14]
p, q, z	parabolic cylinder coordinates [46:14]
$\mathrm{q}(k)$	elliptic nome of modulus k [61:15]
q	$=\mathrm{q}(k)$, elliptic nome of modulus k, serving as a variable [61:15]
r	zero of a polynomial, or other, function [0:7, 17:3, 52:15]
r	correlation coefficient [7:14]
r, θ	polar coordinates [35:14]
r, θ, z	cylindrical coordinates [46:14]
r_2, r_3, r_4	zero of a quadratic, cubic, quartic function [15:7, 16:7, 16:12]
$\mathrm{r}_n(b)$	nth positive root of the equation $\tan(x) = bx$ [34:7]
$r_v^{(k)}$	kth positive root of a Kelvin function [55:7]
s	Laplace variable [0:10]
s	$=(a+b+c)/2$, semiperimeter of a triangle [34:15]
s	seed [40:14]
s_j	jth sine Fourier coefficient [36:6]
$\sec^{-1}(x)$	inverse secant [35.1]
$\sec(x), \sec(\theta)$	secant function of argument x or θ [33:3]
$\mathrm{sech}(x)$	hyperbolic secant of argument x [29:1]
$\mathrm{sech}^{-1}(x)$	inverse hyperbolic secant of argument x [31:1]
$\mathrm{s\#}(k, x)$	$\{\# = \mathrm{n}, \mathrm{d}, \mathrm{c}\}$ Jacobian elliptic functions of modulus k and argument x [63:3]
$\mathrm{sgn}(x)$	signum function of x, the sign of x [8:1]
$\mathrm{si}(x)$	sine integral [38:1]
$\mathrm{sign}(x), \mathrm{sg}(x)$	signum function [8:1]
$\sin(x), \sin(\theta)$	sine function of argument x or θ [32:1]
$\mathrm{sinc}(x)$	sampling function [32:13:5]
$\sinh(x)$	hyperbolic sine of argument x [28:1]
$\mathrm{sh}(x)$	hyperbolic sine [28:1]
$\tan(x), \tan(\theta)$	tangent function of argument x or θ [34:3]
$\mathrm{tg}(x)$	tangent [34:1]
$\mathrm{th}(x)$	hyperbolic tangent [30:1]
$\tanh(x)$	hyperbolic tangent of argument x [30:1]
$\tanh^{-1}(x)$	inverse hyperbolic tangent of argument x [31:1]
$\mathrm{triln}(x)$	trilogarithm of x [25:12]
$t, t_{1/2}$	time, doubling or halving interval [26:0]
t	integration variable, dummy variable [0:10]
$\mathrm{t}_j^{(J)}(y)$	jth of a $J+1$ member set of discrete Chebyshev polynomials [22:13]
$\mathrm{u}(x)$	Heaviside function located at $x=0$ [9:1]
$\mathrm{u}(x-a)$	Heaviside function located at $x=a$ [9:1]
$\mathrm{vers}(x)$	versine function [32:13:4]
$w, w_\#$	$\{\# = 1, 2, \cdots\}$ arbitrary weighting factor [24:14]
$\mathrm{w}(t)$	a weighting function of the variable t [21:14]
x_i	argument at which a function $\mathrm{f}(x)$ inflects [0:7]
x_j^{normal}	jth normally distributed random variate [40:14]

UPPERCASE GREEK

LOWERCASE GREEK

$\alpha_n(x)$	alpha exponential integral [37:13]
β	base of a logarithm [25:14]
$\beta(\nu)$	beta number of order ν [3:3]
$\beta(\nu)$	$= \mathrm{G}(\nu)/2$ [44:13]
$\beta_n(x)$	beta exponential integral [37:13]
γ	Euler's constant, ≈ 0.57722 [1:7:7]
$\gamma(\nu,x)$	lower incomplete gamma function of argument x and parameter ν [45:1]
$\gamma\mathrm{n}(\nu,x)$	entire incomplete gamma function of argument x and parameter ν [45:1]
$\gamma^*(\nu,x)$	$= \gamma\mathrm{n}(\nu,x)$, entire incomplete gamma function [45:1]
$\gamma_j^{(n)}$	jth Chebyshev gamma coefficient of degree n [22:5]
$\delta(x)$	Dirac function located at $x=0$ [9:1]
$\delta(x-a)$	Dirac function located at $x=a$ [9:1]
$\delta^{(1)}(x-a)$	unit-moment function located at $x=a$ [9:12]
$\delta_{n,m}$	Kronecker function (nonzero only when $n=m$) [9:13]
$\zeta(\nu)$	zeta number of order ν [3:3]
$\zeta(\nu,u)$	Hurwitz function of order ν and parameter u [64:3]
$\eta(\nu)$	eta number of order ν [3:3]
$\eta(\nu,u)$	bivariate eta function of order ν and parameter u [64:13]
η,ψ	elliptical coordinates [35:14]
η,ψ,z	elliptic cylinder coordinates [46:14]
η,ψ,ϕ	spheroidal coordinates, prolate or oblate [46:14]
θ,ϕ	variables that may be regarded as angles [0:1]
θ	angular polar coordinate, latitude [35:14]
$\theta(x-a)$	Heaviside function [9:1]
$\theta_\#(\nu,x)$	$\{\#=1,2,3,4\}$ exponential theta functions of (periodic) parameter ν and argument x [27:13]
$\hat{\theta}_\#(\nu,x)$	$\{\#=1,2,3,4\}$ modified exponential theta functions [27:13]
$\vartheta_\#(q,x)$	$\{\#=1,2,3,4\}$ elliptic theta functions of periodic parameter q and periodic variable x [61:15]
$\vartheta_\#(k,x)$	$\{\#=\mathrm{s},\mathrm{c},\mathrm{d},\mathrm{n}\}$ Neville's theta functions of modulus k and periodic variable x [61:15]
$\kappa(x)$	curvature of a plane curve at a point x [39:14]
$\kappa_\nu(x)$	Bateman's confluent function of argument x and order ν [48:13]
$\lambda(\nu)$	lambda number of order ν [3:3]
λ,ξ	bipolar coordinates [35:14]
λ,ξ,z	bipolar cylinder coordinates [46:14]
λ,μ,ϕ	toroidal coordinates [46:14]
μ	mean of a distribution [27:14]
ν,μ	variables that are often, but not necessarily, integers [0:1]
ν	standard random number [40:14]
π	Archimedes's constant, ≈ 3.1416 [1:7]
ρ,ι	real or imaginary part of a complex zero of a polynomial function [17:3]
$\rho_n(b)$	nth root of the equation $\cot(x)=bx$ [34:7]
σ	standard deviation of a distribution [27:14]
σ	sign of a specified variable [31:0]
$\sigma_{1\#}$	$\{\#=2,3,4\} = +1$ in the first and #th quadrants but -1 in the others [32:13:4]
$\sigma_n^{(m)}$	Stirling number of the second kind of indices n and m [2:14]
τ	$= \tan(x/2)$ [34:14]

NONLITERAL SYMBOLS & MODIFIERS

SUBJECT INDEX

Page references are in numerical order, the major citation being in **bold type**. Such a citation as "Secns 5" means that the information sought may often be found in Section 5 of the chapter devoted to the function of interest.

CUSTOMER NOTE: Please read the following before opening the CD pouch which contains *Equator, the Atlas function calculator*.

Opening this pouch may preclude your returning the book to the bookseller.

This disk is designed to operate under at least Windows XP. Simply load the CD and follow the prompts to install the software or see Appendix C for more detailed installation instructions.

The *Equator* program is fully described in Appendix C of this *Atlas*.