The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of chapter two and the Newton--Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg} x - 1)^{p-1}}{\sin^2 x} \ln \operatorname{tg} x\,dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \qquad (1)$$

By making the natural substitution $\operatorname{ctg} x - 1 = u$, we obtain

$$\int_0^\infty u^{p-1} \ln(1+u)\,du = \frac{\pi}{p} \operatorname{cosec} p\pi. \qquad (2)$$

Integrals similar to formula (1) are omitted in this new edition. Instead, we have formula (2) and the formula obtained from the integral (1) by making the substitution $\operatorname{ctg} x = \upsilon$.

As a second example, let us take

$$I = \int_0^{\frac{\pi}{2}} \ln \operatorname{tg}^p x + \operatorname{ctg}^p x) \ln \operatorname{tg} x \, dx = 0.$$

The substitution $\operatorname{tg} x = u$ yields

$$I = \int_0^{\infty} \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} \, du.$$

If we now set $\upsilon = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{\upsilon e^{\upsilon}}{1 + e^{2\upsilon}} \ln(e^{p\upsilon} + e^{-p\upsilon}) \, d\upsilon = \int_{-\infty}^{\infty} \upsilon \, \frac{\ln 2 \operatorname{ch} p\upsilon}{2 \operatorname{ch} \upsilon} \, d\upsilon.$$

The integrand is odd and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows:

First we have the elementary functions:

1.The function $f(x) = x$.

2.The exponential function.

3.The hyperbolic functions.

4.The trigonometric functions.

5.The logarithmic function.

6.The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)

7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.

9. Elliptic functions.

10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.

11. Probability integrals and Fresnel's integrals.

12. The gamma function and related functions.

13. Bessel functions.

14. Mathieu functions.

15. Legendre functions.

16. Orthogonal polynomials.

17. Hypergeometric functions.

18. Confluent hypergeometric functions.

19. Parabolic cylinder functions.

20. Meijer's and MacRobert's functions.

21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* For any natural number n, the involution $(a+bx)^n$ of the binomial $a+bx$ is a polynomial. If n is a negative integer, $(a+bx)^n$ is a rational function. If n is irrational, the function $(a+bx)^n$ is not even an algebraic function.

We shall distinguish between all these functions and those listed above and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x+\cos x}{\sin x>-\cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1.Polynomials (listed in order of their degree).

2.Rational operators.

3.Algebraic Operators (Expressions Of The Form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).

4.Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions (whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$) are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \operatorname{tg} x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \, \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear (in the order determined by the position of $\varphi_2(x)$ in the list) after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions

are the trigonometric and power functions (that is, $\varphi_2(x) = x$). Then come combinations of trigonometric and exponential functions,

combinations of trigonometric functions, exponential functions, and powers, etc.,

combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function e^1 comes after e^x as regards complexity, but $\ln x$ and $\ln\frac{1}{x}$ are equally complex since $\ln\frac{1}{x} = -\ln x$. In the section on "powers and algebraic functions", polynomials, rational functions, and powers of powers are formed from power functions of the form $(a+bx)^n$ and $(\alpha+\beta x)^\nu$.

Use of the Tables

Prepared by Alan Jeffrey for the English language edition.

For the effective use of the tables contained in this book it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled The Order of Presentation of the Formulas and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln\sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Ryzhik and Gradshteyn's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are

α, β, γ, δ,t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second and third kinds.

An abbreviated notation is used for the trigonometric and hyperbolic functions $\tan x$, $\cot x$, $\sinh x$, $\cosh x$, $\tanh x$ and $\coth x$ which are denoted, respectively, by $\operatorname{tg} x$, $\operatorname{ctg} x$, $\operatorname{sh} x$, $\operatorname{ch} x$, $\operatorname{th} x$ and $\operatorname{cth} x$. Also the four inverse hyperbolic functions $\operatorname{Arsh} z$, $\operatorname{Arch} z$, $\operatorname{Arth} z$ and $\operatorname{Arcth} z$ are introduced through the definitions

$$\begin{aligned}
\arcsin z &= \frac{1}{i} \operatorname{Arsh}(iz) \\
\arccos z &= \frac{1}{i} \operatorname{Arch}(z) \\
\operatorname{arctg} z &= \frac{1}{i} \operatorname{Arth}(iz) \\
\operatorname{arcctg} z &= i \operatorname{Arcth}(iz)
\end{aligned}$$

or,

$$\begin{aligned}
\operatorname{Arsh} z &= \frac{1}{i} \arcsin(iz) \\
\operatorname{Arch} z &= i \arccos z \\
\operatorname{Arth} z &= \frac{1}{i} \operatorname{arctg}(iz) \\
\operatorname{Arcth} z &= \frac{1}{i} \operatorname{arcctg}(-iz)
\end{aligned}$$

The numerical constants **C** and **G** which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant **C** is defined by the limit

$$\mathbf{C} = \lim_{s\to\infty} \left(\sum_{m=1}^{s} \frac{1}{m} - \ln s \right) = 0.577215\ldots .$$

On occasions other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

Catalan's constant **G** is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2} \frac{da}{\sqrt{1-k^2\sin^2 a}}$$

by the expression

$$\mathbf{G} = \frac{1}{2}\int_0^1 \mathbf{K}\,dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots .$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the Index of Special Functions and Notations that appears at the front of the book, and by then referring to the defining formula or section number listed there. We now present a brief

discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for} \qquad |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for} \qquad |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for} \qquad n = 0,\ 1, \ldots ,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots .$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \qquad \text{for} \qquad n = 0,\ 1, \ldots .$$

The E_n are all integral and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$.

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \ldots .$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$\begin{gathered} B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66, \\ B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \ldots . \end{gathered}$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$\begin{gathered} B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x) \qquad n = 0, 1, \ldots \\ E_{n-1}(x) = \frac{2^n}{n} \left\{ B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right\} \end{gathered}$$

or

$$E_{n-1}(x) = \frac{2}{n}\left\{ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right\} \qquad n = 1,\ 2, \ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{k}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}(2^{n-k}-1)B_{n-k}B_k(x) \qquad n = 2,\ 3, \ldots\ .$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right)\left\{B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right\}$$

the Euler polynomial of order n.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\begin{aligned}
\operatorname{ns} u &= \frac{1}{\operatorname{sn} u} & \operatorname{nc} u &= \frac{1}{\operatorname{cn} u} & \operatorname{nd} u &= \frac{1}{\operatorname{dn} u}\\
\operatorname{sc} u &= \frac{\operatorname{sn} u}{\operatorname{cn} u} & \operatorname{cs} u &= \frac{\operatorname{cn} u}{\operatorname{sn} u} & \operatorname{ds} u &= \frac{\operatorname{dn} u}{\operatorname{sn} u}\\
\operatorname{sd} u &= \frac{\operatorname{sn} u}{\operatorname{dn} u} & \operatorname{cd} u &= \frac{\operatorname{cn} u}{\operatorname{dn} u} & \operatorname{dc} u &= \frac{\operatorname{dn} u}{\operatorname{cn} u}
\end{aligned}$$

The following elliptic integral of the third kind is defined by Ryzhik and Gradshteyn to be

$$\begin{aligned}
\Pi(\varphi,\ n^2,\ k) &= \int_0^{\varphi} \frac{da}{(1-n^2\ \sin^2 a)\sqrt{1-k^2\ \sin^2 a}}\\
&= \int_0^{\sin\varphi} \frac{dx}{(1-n^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \qquad (-\infty < n^2 < \infty).
\end{aligned}$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\mathrm{zn}(u,k)$, frequently written $Z(u)$, is defined by the relation

$$\mathrm{zn}(u,k) = Z(u) = \int_0^u \left\{ \mathrm{dn}^2\, v - \frac{\mathbf{E}}{\mathbf{K}} \right\} dv = \mathbf{E}(u) - \frac{\mathbf{E}}{\mathbf{K}}\, u.$$

This is related to the theta functions by the relationship

$$\mathrm{zn}(u,k) = \frac{\partial}{\partial u}\, \ln \Theta(u)$$

giving

$$\text{(i)} \qquad \mathrm{zn}\,(u,\,k) = \frac{\pi}{2\mathbf{K}} \frac{\vartheta_1'\left(\dfrac{\pi u}{2\mathbf{K}}\right)}{\vartheta_1\left(\dfrac{\pi u}{2\mathbf{K}}\right)} - \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}$$

$$\text{(ii)} \qquad \mathrm{zn}\,(u,\,k) = \frac{\pi}{2\mathbf{K}} \frac{\vartheta_2'\left(\dfrac{\pi u}{2\mathbf{K}}\right)}{\vartheta_2\left(\dfrac{\pi u}{2\mathbf{K}}\right)} + \frac{\mathrm{dn}\, u\, \mathrm{sn}\, u}{\mathrm{cn}\, u}$$

$$\text{(iii)} \qquad \mathrm{zn}\,(u,\,k) = \frac{\pi}{2\mathbf{K}} \frac{\vartheta_3'\left(\dfrac{\pi u}{2\mathbf{K}}\right)}{\vartheta_3\left(\dfrac{\pi u}{2\mathbf{K}}\right)} - k^2\, \frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\text{(iv)} \qquad \mathrm{zn}\,(u,\,k) = \frac{\pi}{2\mathbf{K}} \frac{\vartheta_4'\left(\dfrac{\pi u}{2\mathbf{K}}\right)}{\vartheta_4\left(\dfrac{\pi u}{2\mathbf{K}}\right)}.$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

$\Gamma(z)$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -\mathbf{C} + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$-\Psi'(z-1) = -\Psi'(z) + \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^\infty e^{-zt} t^{-n}\, dt \qquad (n = 0,\, 1, \ldots, \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $Ei(z)$ and to the logarithmic integral $li(z)$ through the expressions

$$E_1(z) = -\mathrm{Ei}\,(-z) = \int_z^{\infty} e^{-t} t^{-1}\,dt$$

and

$$\mathrm{li}\,(z) = \int_0^z \frac{dt}{\ln t} = \mathrm{Ei}\,(\ln z) \qquad (z > 1).$$

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The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1}\{e^{-z} - zE_{n-1}(z)\}, \qquad (n > 1)$$

and

$$E_n'(z) = -E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z}\left\{1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots\right\} \qquad [\,|\arg z| < 3\pi/2],$$

while for large n,

$$E_n(x) = \frac{e^{-x}}{x+n}\left\{1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n,\,x)\right\}\,,$$

where

$$-0.36n^{-4} \le R(n,\,x) \le \left(1 + \frac{1}{x+n-1}\right) n^{-4} \qquad (x > 0).$$

The sine and cosine integrals si(x) and ci(x) are related to the functions Si(x) and Ci(x) by the integrals

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt = \mathrm{si}\,(x) + \pi/2$$

and

$$\mathrm{Ci}\,(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t}\,dt.$$

The hyperbolic sine and cosine integrals Shi(x) and Chi(x) are defined by the relations

$$\mathrm{Shi}\,(x) = \int_0^x \frac{\sinh t}{t}\,dt$$

and

$$\mathrm{Chi}\,(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t}\,dt.$$

Some authors write

$$\mathrm{Cin}\,(x) = \int_0^x \frac{(1 - \cos t)}{t}\,dt$$

when

$$\mathrm{Cin}\,(x) = -\mathrm{Ci}\,(x) + \ln x + \mathbf{C}.$$

The error function erf(x) is defined by the relation

$$\mathrm{erf}\,(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

and the complementary error function erfc(x) is related to the error function erf(x) and to Φ(x) by the expression

$$\mathrm{erfc}\,(x) = 1 - \mathrm{erf}\,(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Ryzhik and Gradshteyn as

$S(x)$ $C(x)$

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 \, dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 \, dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2} \, dt, \qquad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2} \, dt$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} \, dt, \qquad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} \, dt$$

These are related by the expressions

$$S(x) = S_1\left(x\sqrt{\frac{2}{\pi}}\right) = S_2(x^2)$$

and

$$C(x) = C_1\left(x\sqrt{\frac{2}{\pi}}\right) = C_2(x^2)$$

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $\mathrm{He}_n(x)$ by the relations

$$\mathrm{He}_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right)$$

and

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n = 0$$

and

$$\frac{d^2\,\mathrm{He}_n}{dx^2} - x\frac{d\,\mathrm{He}_n}{dx} + n\,\mathrm{He}_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2xH_n - 2nH_{n-1}$$

and

$$\mathrm{He}_{n+1} = x\,\mathrm{He}_n - n\,\mathrm{He}_{n-1}\ .$$

The first six orthogonal polynomials He_n are

$$\mathrm{He}_0 = 1, \quad \mathrm{He}_1 = x, \quad \mathrm{He}_2 = x^2 - 1, \quad \mathrm{He}_3 = x^3 - 3x, \quad \mathrm{He}_4 = x^4 - 6x^2 + 3, \quad \mathrm{He}_5 = x^5 - 10x^3 + 15x.$$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involves the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n}\Gamma(n+1)J_n(z).$$

In the book by Gray, Mathews and MacRobert the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi Y_n(z) + (\ln 2 - \mathbf{C})J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $Hs_\nu(z)$ and $Hi_\nu(z)$ and some authors write $G_\nu(z) = \left(\frac{1}{2}\right)\pi i H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n+1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z)O_0(t) + 2\sum_{k=1}^{\infty} J_k(z)O_k(t),$$

giving

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{[n/2]} \frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \qquad \text{for} \quad n = 1,\, 2, \ldots\,,$$

where $\left[\frac{1}{2}n\right]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1)O_{n+1}(t) + (n+1)O_{n-1}(t) - \frac{2(n^2-1)}{t}O_n(t) = \frac{2n}{t}\sin^2\frac{n\pi}{2}.$$

The Airy functions Ai(z) and Bi(z) are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{3}\sqrt{z}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} = \frac{1}{\pi}\sqrt{\frac{z}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right), \\
\mathrm{Ai}(-z) &= \frac{1}{3}\sqrt{z}\left\{J_{1/3}\left(\frac{2}{3}z^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}
\end{aligned}$$

and by

$$\begin{aligned}
\mathrm{Bi}(z) &= \sqrt{\frac{z}{3}}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}, \\
\mathrm{Bi}(-z) &= \sqrt{\frac{z}{3}}\left\{J_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - J_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}.
\end{aligned}$$

The differential equation

$$\frac{d^2y}{dz^2} + (az^2 + bz + c)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left(\frac{1}{4}z^2 + a\right) y = 0 \qquad \text{and} \qquad \frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right) y = 0$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right) y = 0$$

are sometimes written $U(a, z)$ and $V(a, z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a,\ z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a,\ z) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + a\right)\{D_{-a-\frac{1}{2}}(-z) + (\sin \pi a) D_{-a-\frac{1}{2}}(z)\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Ryzhik and Gradshteyn is

$$\frac{d^2y}{dz^2} + (a - 2k^2 \cos 2z)y = 0 \qquad \text{with} \quad k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ, λ and h^2 and b and $c = 2\sqrt{q}$, respectively. The

a q h θ λ h^2 b $c = 2\sqrt{q}$

periodic solutions $\mathrm{se}_n(z,q)$ and $\mathrm{ce}_n(z,q)$ and the modified periodic solutions $\mathrm{Se}_n(z,q)$ and $\mathrm{Ce}_n(z,q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: Tables relating to Mathieu functions. National Bureau of Standards, Columbia University Press, New York, 1951.

Index of Special Functions and Notations

Notation	Name of the function	formula no.
$\mathrm{am}\,(u,k)$	Amplitude (of an elliptic function)	8.141
B_n	Bernoulli numbers	9.61, 9.71
$\mathrm{B}_n(x)$	Bernoulli polynomials	9.620
$\mathrm{B}(x,y)$	Beta functions	8.38
$\mathrm{B}_x(p,q)$	Incomplete beta functions	8.39
$\beta(x)$		8.37
$\mathrm{bei}_\nu\,(z)$, $\mathrm{ber}_\nu\,(z)$	Thomson's functions	8.56
$\mathbf{C}$	Euler's constant	9.73, 8.367
$C(x)$	Fresnel's cosine-integral	8.25
$C_\nu(a)$	Young's function	3.76
$C_n^\lambda(t)$	Gegenbauer's polynomials	8.93
$C_\nu^\lambda(t)$	Gegenbauer's function	8.932 1
$\mathrm{ce}_{2n}(z,q)$, $\mathrm{ce}_{2n+1}(z,q)$	Periodic Mathieu functions (Mathieu functions of the first kind)	8.61
$\mathrm{Ce}_{2n}(z,q)$, $\mathrm{Ce}_{2n+1}(z,q)$	Associated (modified) Mathieu functions of the first kind	8.63
$\mathrm{chi}(x)$	Hyperbolic-cosine-integral function	8.22
$\mathrm{ci}(x)$	Cosine-integral	8.23
$\mathrm{cn}(u)$	Cosine-amplitude	8.14
$D(k) \equiv D$		8.112
$D(\varphi,k)$		8.111
$D_n(z), D_p(z)$	Parabolic cylinder functions	9.24–9.25
$\mathrm{dn}\,u$	Delta amplitude	8.14
e_1, e_2, e_3		8.162
E_n	Euler numbers	9.63, 9.72
$E(\varphi,k)$	Elliptic integral of the second kind	8.11–8.12
$\mathbf{E}(k) = \mathbf{E}$, $\mathbf{E}(k') = \mathbf{E}$	Complete elliptic integral of the second kind	8.11–8.12

Notation	Name of the function	formula no.
$E(p; a_r : q; \varrho_s : x)$	MacRobert's function	9.4
$\mathbf{E}_\nu(z)$	Weber's function	8.58
$Ei(z)$	Exponential-integral function	8.21
$\overline{E}i(z)$	Related exponential integral	8.21
$\Phi(x)$	Probability integral	8.25
$\mathrm{erf}\,(x)=\Phi(x)$	Error function	8.25

Notation	Name of the function	formula no.
$H_n(z)$	Hermite polynomials	8.95
$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Modified Bessel functions	8.406, 8.43
$I_x(p,q)$	Incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger's function	8.58
$k_\nu(x)$	Bateman's function	9.21
$\boldsymbol{K}(k) = \boldsymbol{K}, \quad \boldsymbol{K}(k') = \boldsymbol{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Modified Bessel functions	8.407, 8.43
$\operatorname{kei}(z), \quad \operatorname{ker}(z)$	Thomson's functions	8.56
$\xi(s)$		9.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\operatorname{li}(x)$	Logarithm-integral	8.24
$\lambda(x,y)$		9.640
$M_{\lambda,\,\mu}(z)$	Whittaker's functions	9.22, 9.23
$\mu(x,\,\beta), \quad \mu(x,\,\beta,\,\alpha)$		9.640
$N_\nu(z)$	Bessel functions of the second kind (Neumann functions)	8.403, 8.41
$\nu(x)$		9.640
$\nu(x), \nu(x,\alpha)$		9.640
$O_n(x)$	Neumann's polynomials	8.59
$\wp(u)$	Weierstrass elliptic function	8.16
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_n(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$	Riemann's differential equation (diagram)	9.160
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	8.96
$\Pi(x)$	Lobachevskiy's angle of parallelism	1.48
$\Pi(\varphi,n,k)$	Elliptic integral of the third kind	8.11
$\Phi(x)$	Probability integral	8.25
$\Phi(z,s,v)$	Lerch function	9.55

Notation	Name of the function	formula no.
$\Phi(\alpha,\,\gamma;\,x) = {}_1\mathrm{F}_1(\alpha;\,\gamma;\,x)$		9.21
$\Phi_1(\alpha,\beta,\gamma,x,y), \quad \Phi_2(\beta,\beta',\gamma,x,y),$ $\Phi_3(\beta,\gamma,x,y)$	Confluent hypergeometric series in two variables	9.26
$\psi(x)$	Euler's psi function	8.36
$\Psi(a,c;x)$	Confluent hypergeometric function	9.21
$Q_\nu^\mu(z), Q_\nu^\mu(x)$	Ass. Legendre func. of the 2nd kind	8.7, 8.8

The letter k (when not used as an index of summation) denotes a number in the interval [0, 1]. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1-k^2}$ is denoted by k'.

$R(x)$	A rational function
$\operatorname{Re} z \equiv x, \quad \operatorname{Im} z \equiv y$	The real and imaginary parts of the complex number $z = x + iy$.
$\overline{z} = x - iy$	The complex conjugate of $z = x + iy$.
$\arg z$	The argument of the complex number $z = x + iy$.
$\operatorname{sign} x$	The sign (signum) of the real number x; $\operatorname{sign} x = +1$ for $x > 0$; $\operatorname{sign} x = -1$ for $x < 0$.
$[x]$	The integral part of the real number x.
$\int_a^{(b+)} \quad \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a, proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve C.
$n!$	$= 1 \cdot 2 \cdot 3 \ldots n, \qquad 0! = 1.$
$(2n+1)!!$	$= 1 \cdot 3 \ldots (2n+1).$
$(2n)!!$	$= 2 \cdot 4 \ldots (2n).$
$\binom{p}{n}$	$= \frac{p(p-1)\ldots(p-n+1)}{1\cdot 2 \ldots n} = \frac{p!}{n!(p-n)!}, \qquad \binom{p}{0} = 1. \qquad [n, p = 0, 1, \ldots; p \geq n]$
$x(x-1)\ldots(x-n+1)/n!$	$[n = 0, 1, \ldots]$
$(a)_n$	$= a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$
$\sum_{k=m}^{n} u_k$	$= u_m + u_{m+1} + \ldots + u_n$. If $n < m$, we define $\sum_{k=m}^{n} u_k = 0$.
$\sum_n{}', \quad \sum_{m,n}{}'$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0. If there exists an $M > 0$ such that $\lvert g(z)\rvert \leq M\lvert f(z)\rvert$ in some sufficiently small neighborhood of the point z_0, we write $g(z) = O(f(z))$.
$0!! = 1$	
$(-1)!! = 1$	(cf. 3.372 for $n = 0$.)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$.)
$\sum_{k \in \phi} (\cdots) = 0$	An empty $\sum$ has the value 0
$\prod_{k \in \phi} (\cdots) = 1$	An empty $\prod$ has the value 1
$\delta_{ij} = \begin{cases} 1, & i{\neq}j \\ 0, & i{\neq}j \end{cases}$	Kronecker
$\delta_{ij} = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$	Heaviside step function

$\sum_{k=1} \qquad (1-x)^3$

$$\sum_{k=1}^{n} k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2}\binom{q}{1} B_2 n^{q-1} + \frac{1}{4}\binom{q}{3} B_4 n^{q-3} + \frac{1}{6}\binom{q}{5} B_6 n^{q-5} + \cdots =$$
$$= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720} n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240} n^{q-5} - \cdots$$

[last term contains either n or n^2].

CE 332

2

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

CE 333

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

CE333

3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

CE 333

4. $\sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$.

CE 333

5. $\sum_{k=1}^{n} k^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1)$.

CE 333

6. $\sum_{k=1}^{n} k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4+6n^3-3n+1)$.

CE 333

CE 333

0.122

$$\sum_{k=1}^{n}(2k-1)^q = \frac{2^q}{q+1}n^{q+1} - \frac{1}{2}\binom{q}{1}2^{q-1}B_2n^{q-1} - \frac{1}{4}\binom{q}{3}2^{q-3}(2^3-1)B_4n^{q-3} - \cdots$$

[last term contains either n or n^2.]

1. $\sum_{k=1}^{n}(2k-1) = n^2.$

2. $\sum_{k=1}^{n}(2k-1)^2 = \frac{1}{3}n(4n^2-1).$

JO (32a)

3. $\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1).$

JO (32b)

4*. $\sum_{k=1}^{n}(mk-1) = \frac{1}{2}[m(n+1)-2].$

5*. $\sum_{k=1}^{n}(mk-1)^2 = \frac{1}{6}n[m^2(n+1)(2n+1)-6m(n+1)+6].$

6*. $\sum_{k=1}^{n}(mk-1)^3 = \frac{1}{4}n[m^3n(n+1)^2-2m^2(n+1)(2n+1)+6m(n+1)-4].$

0.123

$$\sum_{k=1}^{n}k(k+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5).$$

0.124

1. $\sum_{k=1}^{q}k(n^2-k^2) = \frac{1}{4}q(q+1)(2n^2-q^2-q) \qquad [q=1,\,2,\ldots].$

$2^*.\quad \sum_{k=1}^{n} k(k+1)^3 = \frac{1}{60}n(n+1)(12n^3+63n^2+107n+58).$

3

0.125

$$\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1.$$

AD (188.1)

0.126

$$\sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}} K_{n+\frac{1}{2}}\left(\frac{1}{2}\right).$$

WA 94

0.13 Sums of reciprocals of natural numbers

0.131

$$\sum_{k=1}^{n} \frac{1}{k} = C + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1)\dots(n+k-1)},$$

where

$$A_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)(3-x)\dots(k-1-x)\,dx.$$

$$A_2 = \frac{1}{12}, \qquad A_3 = \frac{1}{12},$$

$$A_4 = \frac{19}{120}, \qquad A_5 = \frac{9}{20},$$

0.132

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \frac{1}{2}(\boldsymbol{C} + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{(2^3-1)B_4}{64n^4} + \dots \qquad [n \to \infty]$$

JO (71a)a

0.133

$$\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.$$

JO (184f)

0.14 Sums of products of reciprocals of natural numbers

0.141

1. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)} = \frac{n}{p(p+nq)}.$$

GI III (64)a

2. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)[p+(k+1)q]} = \frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]}.$$

GI III (65)a

3. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]} = \\ = \frac{1}{(l+1)q}\left\{\frac{1}{p(p+q)\dots(p+lq)} - \frac{1}{(p+nq)[p+(n+1)q]\dots[p+(n+l)q]}\right\}.$$

AD (1856)a

4. $$\sum_{k=1}^{n} \frac{1}{[1+(k-1)q][1+(k-l)q+p]} = \frac{1}{p}\left[\sum_{k=1}^{n} \frac{1}{1+(k-1)q} - \sum_{k=1}^{n} \frac{1}{1+(k-1)q+p}\right].$$

GI III (66)a

0.142

$$\sum_{k=1}^{n} \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!}.$$

0.15 Sums of the binomial coefficients

0.151

1. $\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}.$

KR 64 (70.1)

2. $1 + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}.$

KR 62 (58.1)

3. $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}.$

KR 62 (58.1)

4. $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \qquad [n \geq 1].$

KR 64 (70.2)

0.152

1. $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right).$

KR 62 (59.1)

2. $\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right).$

KR 62 (59.2)

3. $\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right).$

KR 62 (59.3)

0.153

0.153

1. $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4}\right).$

KR 63 (60.1)

2. $\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4}\right).$

KR 63 (60.2)

3. $\binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4}\right).$

KR 63 (60.3)

4. $\binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4}\right).$

KR 63 (60.4)

0.154

1. $\sum_{k=0}^{n} (k+1)\binom{n}{k} = 2^{n-1}(n+2) \qquad [n \geq 0].$

KR 63 (66.1)

2. $\sum_{k=1}^{n} (-1)^{k+1} k \binom{n}{k} = 0 \qquad [n \geq 2].$

KR 63 (66.2)

3. $\sum_{k=0}^{N} (-1)^{k} \binom{N}{k} k^{n-1} = 0 \qquad [N \geq n \geq 1; \quad 0^0 \equiv 1].$

4. $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} k^{n} = (-1)^n n! \qquad [n \geq 0; \quad 0^0 \equiv 1].$

5. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (\alpha + k)^n = (-1)^n n! \qquad [n \geq 0; \quad 0^0 \equiv 1].$

6. $\sum_{k=0}^{N} (-1)^k \binom{N}{k} (\alpha + k)^{n-1} = 0 \qquad [N \geq n \geq 1, \quad 0^0 \equiv 1 \quad N,\, n \in N^+].$

0.155

1. $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1}.$

KR 63 (67)

2. $\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}.$

KR 63 (68.1)

3. $\sum_{k=0}^{n} \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1} - 1}{n+1}.$

KR 63 (68.2)

4. $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^{n} \frac{1}{m}.$

KR 64 (69)

0.156

1. $\sum_{k=0}^{p} \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p} \qquad [m \text{ is a natural number}].$

KR 64 (71.1)

2. $\sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}.$

KR 64 (71.2)

0.157

1. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$

KR 64 (72.1)

2. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}.$

KR 64 (72.2)

3. $\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0.$

KR 64 (72.3)

4. $\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}.$

KR 64 (72.4)

0.158*

1. $\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k = 4^n - \binom{2n}{n}.$

2. $\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^2 = 4^n - \binom{2n}{n} 3.4^n.$

6

3. $\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^3 = (6n+13)4^n - 18n \binom{2n}{n}.$

4. $\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^4 = (32n^2+104n) \binom{2n}{n} - (60n+75)4^n.$

0.159*

1. $\sum_{k=0}^{n} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k = \frac{1}{2} \left[4^n - \binom{2n}{n} \right].$

2. $\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^2=\frac{1}{2}\left[(2n+1)\binom{2n}{n}-4^n\right].$

3. $\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^3=\frac{(3n+2)}{4}\cdot 4^n-\frac{1}{2}\binom{2n}{n}(3n+1).$

0.160*

1. $\sum_{k=n+1}^{2n}\binom{2n}{k}\alpha^k+\frac{1}{2}\binom{2n}{n}\alpha^n+\frac{(1+\alpha)^{2n-1}(1-\alpha)}{2}\sum_{k=0}^{n-1}\binom{2k}{k}\left[\frac{\alpha}{(1+\alpha)^2}\right]^k=\frac{1}{2}(1+\alpha)^{2n}.$

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

The series

0.211

$$\sum_{k=1}^{\infty}u_k=u_1+u_2+u_3+\dots$$

is said to *converge absolutely* if the series

0.212

$$\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\cdots,$$

composed of the absolute values of its terms converges. If the series 0.211 converges and the series 0.212 diverges, the series 0.211 is said to *converge conditionally*. Every absolutely convergent series converges.

0.22 Convergence tests

0.221

Suppose that

$$\lim_{k\to\infty} |u_k|^{\frac{1}{k}} = q.$$

If $q < 1$, the series 0.211 converges absolutely. On the other hand, if $q > 1$, the series 0.211 diverges. (Cauchy)

7

0.222

Suppose that

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = q.$$

Here, if $q < 1$, the series 0.211 converges absolutely. If q>1, the series 0.211 diverges. If $\left|\frac{u_{k+1}}{u_k}\right|$ approaches 1 but remains greater than unity, the series 0.211 diverges. (d'Alembert)

0.223

Suppose that

$$\lim_{k\to\infty} k\left\{\left|\frac{u_k}{u_{k+1}}\right| - 1\right\} = q.$$

Here, if $q > 1$, the series 0.211 converges absolutely. If $q < 1$, the series 0.211 diverges. (Raabe)

0.224

Suppose that $f(x)$ is a positive decreasing function and that

$$\lim_{k\to\infty} \frac{e^k f(e^k)}{f(k)} = q.$$

for natural k. If $q < 1$, the series $\sum_{k=1}^{\infty} f(k)$

$$q > 1$$

0.225

Suppose that

$$\left|\frac{u_k}{u_{k+1}}\right| = 1 + \frac{q}{k} + \frac{|\upsilon_k|}{k^p},$$

where $p > 1$ and the $|\upsilon_k|$ are bounded, that is, the $|\upsilon_k|$ are all less than some M, which is independent of k. Here, if $q > 1$, the series 0.211 converges absolutely. If $q \leq 1$, this series diverges. (Gauss)

0.226

Suppose that a function $f(x)$ defined for $x \geq q \geq 1$ is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges according as the integral

$$\int_q^{\infty} f(x)\, dx.$$

converges or diverges (the Cauchy integral test.)

8

0.227

Suppose that all terms of a sequence $u_1,\ u_2, \ldots\ , u_n$ are positive. In such a case, the series

1. $$\sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \ldots$$

is called an *alternating series.*

If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

2. $$u_{k+1} < u_k \qquad \text{and} \qquad \lim_{k\to\infty} u_k = 0,$$

the series 0.227 1. converges. Here, the remainder of the series is

3. $$\sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left| \sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^{n} (-1)^{k+1} u_k \right| < u_{n+1}. \qquad \text{(Leibnitz)}$$

0.228

If the series

1. $$\sum_{k=1}^{\infty} v_k = v_1 + v_2 + \ldots + v_k + \ldots$$

converges and the numbers u_k form a monotonic bounded sequence, that is, if $|u_k| < M$ for some number M and for all k, the series

2. $$\sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \ldots + u_k v_k + \ldots$$

converges. (Abel)

0.229

If the partial sums of the series 0.228 1. are bounded and if the numbers u_k constitute a monotonic sequence that approaches zero, that is, if

$$\left| \sum_{k=1}^{n} v_k \right| < M \qquad [n = 1, 2, \ldots] \qquad \text{and} \qquad \lim_{k\to\infty} u_k = 0,$$

then the series 0.228 2. converges. (Dirichlet)

0.23-0.24 Examples of numerical series

0.231

Progressions

1. $\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} \qquad [|q| < 1].$

2. $\sum_{k=0}^{\infty} (a+kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} \qquad [|q| < 1]$

0.113

9

0.232

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2$

1.511

2. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - 2\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4}$

1.643

0.233

1. $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots = \zeta(p) \qquad [\operatorname{Re} p > 1]$

WH

2. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^p} = (1 - 2^{1-p})\zeta(p) \qquad [\operatorname{Re} p > 0]$

3*. $\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$

FI II 721

4. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2n}} = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}|.$

JO (165)

5. $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2\cdot(2n)!}|B_{2n}|.$

JO (184b)

6. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}|E_{2n}|.$

JO (184d)

0.234

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12}.$

EU

2. $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$

EU

3. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \boldsymbol{G}.$

FI II 482

4. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$

5. $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$

EU

10

6. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^5} = \frac{5\pi^5}{1536}.$

EU

7. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2.$

8^6. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2\ln 2.$

0.235

$$S_n = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^n},$$
$$S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2 - 8}{16}, \quad S_3 = \frac{32 - 3\pi^2}{64}, \quad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$$

JO (186)

0.236

1. $\sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)} = 2\ln 2 - 1.$

BR 51a

2. $\sum_{k=1}^{\infty} \frac{1}{k(9k^2-1)} = \frac{3}{2}(\ln 3 - 1).$

BR 51a

3. $\sum_{k=1}^{\infty} \frac{1}{k(36k^2-1)} = -3 + \frac{3}{2}\ln 3 + 2\ln 2.$

4 $\sum_{k=1}^{\infty} \frac{k}{(4k^2-1)^2} = \frac{1}{8}.$

BR 52

5. $\sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)^2} = \frac{3}{2} - 2\ln 2.$

BR 52

6. $\sum_{k=1}^{\infty} \frac{12k^2-1}{k(4k^2-1)^2} = 2\ln 2.$

AD (6917.3), BR 52

7[6]. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2\ln 2.$

0.237

1. $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}.$

AD (6917.2), BR 52

2. $\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}.$

3. $\sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4}$

0.133

11

4. $\sum_{k=1,\, k\neq m}^{\infty}{}' \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2} \qquad [m \text{ is an integer}].$

AD (6916.1)

AD (6916.2)

0.238

1. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2}.$$

GI III (93)

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2}(1 - \ln 2).$$

GI III (94)a

3. $$\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4}\ln 3 + \frac{\pi}{12\sqrt{3}}.$$

GI III (95)

0.239

1. $$\sum_{k=1}^{\infty} (-1)^{n+1} \frac{1}{3k-2} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2\right)$$

GI III (85), BR* 161 (1)

2. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3}\left(\frac{n}{\sqrt{3}} - \ln 2\right).$$

BR* 161 (1)}

3. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}}[\pi + 2\ln(\sqrt{2}+1)].$$

BR* 161 (1)

4. $$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2}\ln 2.$$

GI III (87)

5. $$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{2k-1} = \frac{\pi}{2\sqrt{2}}.$$

6. $\sum_{k=1}^{\infty}(-1)^{\left[\frac{k+5}{3}\right]}\frac{1}{2k-1}=\frac{5\pi}{12}.$

GI III (88)

7. $\sum_{k=1}^{\infty}\frac{1}{(8k-1)(8k+1)}=\frac{1}{2}-\frac{\pi}{16}(\sqrt{2}+1).$

0.241

1. $\sum_{k=1}^{\infty}\frac{1}{2^k k}=\ln 2.$

JO (172g)

2. $\sum_{k=1}^{\infty}\frac{1}{2^k k^2}=\frac{\pi^2}{12}-\frac{1}{2}(\ln 2)^2.$

JO (174)

3*. $\sum_{n=0}^{\infty}\binom{2n}{n}p^n=\frac{1-\sqrt{1-4p}}{\sqrt{1-4p}-(1-4p)} \qquad \left[0\le p<\frac{1}{4}\right].$

4*. $\sum_{n=1}^{\infty}\frac{p^n}{n^2}=\frac{\pi^2}{6}-\int_1^p\frac{\ln(1-x)}{x}\,dx \qquad [0\le p\le 1].$

5*. $\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j=4^i-\binom{2i}{i} \qquad \left[\binom{n}{m}=0,\quad m<0\right].$

6*. $\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^2=4i\binom{2i}{i}-3\cdot 4^i \qquad \left[\binom{n}{m}=0,\quad m<0\right].$

7*. $\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^3=(6i+13)4^i-18i\binom{2i}{i} \qquad \left[\binom{n}{m}=0,\quad m<0\right].$

$$8^*.\ \sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^4=(32i^2+104i)\binom{2i}{i}-(60i+75)4^i$$

$$\left[\binom{n}{m}=0,\quad m<0\right].$$

$$9^*.\ \sum_{j=n+1}^{2n}\binom{2n}{j}k^j+\frac{1}{2}\binom{2n}{n}k^n+\frac{(1+k)^{2n-1}(1-k)}{2}\sum_{i=0}^{n-1}\binom{2i}{i}\left[\frac{k}{(1+k)^2}\right]^i=\frac{1}{2}(1+k)^{2n}.$$

$$10^*.\ \sum_{k=0}^{i}\binom{i+k}{k}2^{i-k}=4^i.$$

$$11^*.\ \sum_{k=0}^{i}\binom{i+k}{k}^{i-k}k=(i+1)4^i-(2i+1)\binom{2i}{i}.$$

$$12^*.\ \sum_{k=0}^{i}\binom{2i}{k}=\frac{1}{2}\left(4^i+\binom{2i}{i}\right).$$

$$13^*.\ \sum_{k=0}^{i}\binom{2i}{k}k=\frac{i}{2}4^i.$$

$$14^*.\ \sum_{k=0}^{i}\binom{2i}{k}k^2=(2i+1)i4^{i-1}-\frac{i^2}{2}\binom{2i}{i}.$$

0.242

$$\sum_{k=0}^{\infty}(-1)^k\frac{1}{n^{2k}}=\frac{n^2}{n^2+1}\quad(n>1).$$

0.243

$$1.\quad\sum_{k=1}^{\infty}\frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]}=\frac{1}{(l+1)q}\frac{1}{p(p+q)\dots(p+lq)}$$

2.[7] $$\sum_{k=1}^{\infty} \frac{x^{k-1}}{[p+(k-1)q][p+(k-1)q+1][p+(k-1)q+2]\dots[p+(k-1)q+l]} =$$
$$= \frac{1}{l!}\int_0^1 \frac{t^{p-1}(1-t)^t}{1-xt^q}\,dt \qquad [q>0, \quad x^2 \le 1].$$

BR* 161 (2), AD (6.704)

3. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\left[\frac{1}{x}\ \mathrm{th}\left[\frac{(2k+1)\pi x}{2}\right] + x\mathrm{t\ h}\left[\frac{(2k+1)\pi}{2x}\right]\right] = \frac{\pi^3}{16}.$$

0.244

1. $$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p}\int_0^1 \frac{x^p - x^q}{1-x}\,dx \qquad [p>-1, \quad q>-1, \quad p \neq q].$$

GI III (90)

2. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{p+(k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q}\,dt \qquad [p>0, \quad q>0].$$

3*. $$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p}\sum_{m=p+1}^{q}\frac{1}{m} \qquad [q>p>-1, \quad p \text{ and } q \text{ integers}].$$

BR* 161 (1)

Summations of reciprocals of factorials

0.245

1. $$\sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828\dots$$

2. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} = 0.36787\dots$$

3. $$2\sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787\dots$$

4. $\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$

5. $\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2}\left(e + \frac{1}{e}\right) = 1.54308\ldots$

6. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2}\left(e - \frac{1}{e}\right) = 1.17520\ldots$

7. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030\ldots$

8. $\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147\ldots$

14
0.246

1. $\sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530\ldots$

2. $\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855\ldots$

3. $\sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2).$

4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078\ldots$

0.247

$$\sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2)\cdot(n-1)!}.$$

0.248

$$\sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$\begin{array}{llll} S_1 = e, & S_2 = 2e, & S_3 = 5e, & S_4 = 15e, \\ S_5 = 52e, & S_6 = 203e, & S_7 = 877e, & S_8 = 4140e. \end{array}$$

0.249[8]

$$\sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e - 1.$$

0.25 Infinite products

0.250

Suppose that a sequence of numbers $a_1, a_2, \ldots, a_k, \ldots$ is given. If the limit $\lim_{n\to\infty} \prod_{k=1}^{n} (1+a_k)$ exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product* $\prod_{k=1}^{\infty} (1+a_k)$ and we write

1. $$\lim_{n\to\infty} \prod_{k=1}^{n} (1+a_k) = \prod_{k=1}^{\infty} (1+a_k).$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no a_k is equal to -1.

FI II 400

0.251

For the infinite product 0.250 1. to converge, it is necessary that $\lim_{k\to\infty} a_k = 0$.

0.252

If $a_k > 0$ or $a_k < 0$ for all values of the index k starting with some particular value, then, for the product 0.250 1. to converge, it is necessary and sufficient that the series $\sum_{k=1}^{\infty} a_k$ converges.

0.253

The product $\prod_{k=1}^{\infty} (1 + a_k)$ is said to converge absolutely if the product $\prod_{k=1}^{\infty} (1 + |a_k|)$ converges.

FI II 403

0.254

Absolute convergence of an infinite product implies its convergence.

0.255

The product $\prod_{k=1}^{\infty} (1 + a_k)$ converges absolutely if, and only if, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

FI II 406

0.26 Examples of infinite products

0.261

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}.$$

EU
FI II 403

0.262

1. $$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}.$$

FI II 401

2. $$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi}.$$

FI II 401

FI II 401

0.263

$$\frac{2}{1} \cdot \left(\frac{4}{3}\right)^{\frac{1}{2}} \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{\frac{1}{4}} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{1/8} \ldots = e.$$

0.264

$$\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}} = e^C.$$

FI II 402

0.265

$$\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \ldots = \frac{2}{\pi}.$$

FI II 402

0.266

$$\prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} \qquad [|x| < 1].$$

FI II 401

0.3 Functional Series

0.30 Definitions and theorems

0.301

The series

1. $$\sum_{k=1}^{\infty} f_k(x),$$

x

0.302

A series that converges for all values of x in a region M is said to *converge uniformly* in that region if, for every $\varepsilon \geq 0$, there exists a number N such that, for $n > N$, the inequality

$$\left|\sum_{k=n+1}^{\infty} f_k(x)\right| < \varepsilon$$

holds for *all x* in M.

0.303

If the terms of the functional series 0.301 1. satisfy the inequalities

$$|f_k(x)| < u_k \quad (k = 1,\ 2,\ 3, \ldots),$$

throughout the region M, where the u_k are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_k + \ldots\ ,$$

the series 0.301 1. converges uniformly in M. (Weierstrass)

0.304

Suppose that the series 0.301 1. converges uniformly in a region M and that a set of functions $g_k(x)$ constitutes (for each x) a monotonic sequence, and that these functions are uniformly bounded, that is, suppose that a number L exists such that the inequalities

1. $$|g_n(x)| \leq L$$

hold for all n and x. Then, the series

2. $$\sum_{k=1}^{\infty} f_k(x) g_k(x)$$

converges uniformly in the region M. (Abel)

0.305

Suppose that the partial sums of the series 0.301 1. are uniformly bounded; that is, suppose that, for some L and for all n and x in M, the inequalities

$$\left|\sum_{k=1}^{n} f_k(x)\right| < L$$

FI II 451

hold. Suppose also that for each x the functions $g_n(x)$ constitute a monotonic sequence that approaches zero uniformly in the region M. Then, the series 0.304 2. converges uniformly in the region M. (Dirichlet)

0.306[6]

If the functions $f_k(x)$ ($k = 1, 2, 3, \dots$) are integrable on the interval $[a, b]$ and if the series 0.301 1. made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x)\right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\, dx \qquad [a \le x \le b].$$

FI II 459
FI II 451

0.307

Suppose that the functions $f_k(r)$ (for ($k = 1, 2, 3, \dots$) have continuous derivatives $f_k'(x)$ on the interval $[a, b]$. If the series 0.301 1. converges on this interval and if the series $\sum_{k=1}^{\infty} f_k'(x)$ of these derivatives converges uniformly, the series 0.301 1. may be differentiated termwise; that is,

$$\left\{\sum_{k=1}^{\infty} f_k(x)\right\}' = \sum_{k=1}^{\infty} f_k'(x).$$

FI II 460

0.31 Power series

0.311

A functional series of the form

1. $$\sum_{k=0}^{\infty} a_k (x-\xi)^k = a_0 + a_1(x-\xi) + a_2(x-\xi)^2 + \dots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point ξ and a radius equal to R; at every interior point of this circle, the power series 0.311 1. converges absolutely and outside this circle, it diverges. This circle is called the *circle of convergence* and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ($R = +\infty$).

0.312

Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^{x} \left\{ \sum_{k=0}^{\infty} a_k (x-\xi)^k \right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-\xi)^{k+1},$$
$$\frac{d}{dx} \left\{ \sum_{k=0}^{\infty} a_k (x-\xi)^k \right\} = \sum_{k=1}^{\infty} k a_k (x-\xi)^{k-1}.$$

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

Operations on power series

0.313

Division of power series.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0} \sum_{k=1}^{n} c_{n-k} a_k - b_n = 0,$$

or

$$c_n = \frac{(-1)^n}{a_0^n} \begin{vmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & \cdots & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & \cdots & 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}.$$

AD (6360)

0.314

Power series raised to powers.

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0} \sum_{k=1}^{m} (kn - m + k) a_k c_{m-k} \quad \text{for } m \geq 1 \qquad [n \text{ is a natural number}].$$

AD (6361)

0.315

The substitution of one series into another.

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k \qquad y = \sum_{k=1}^{\infty} a_k x^k;$$
$$c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2 a_1 a_2 b_2 + a_1^3 b_3,$$
$$c_4 = a_4 b_1 + a_2^2 b_2 + 2 a_1 a_3 b_2 + 3 a_1^2 a_2 b_3 + a_1^4 b_4, \quad \ldots$$

AD (6362)

0.316

Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k; \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

FI II 372

Taylor series

0.317

If a function $f(x)$ has derivatives of all orders throughout a neighborhood of a point ξ, then we may write the series

$$1. \qquad f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots ,$$

which is known as the Taylor series of the function $f(x)$.

The Taylor series converges to the function $f(x)$ if the remainder

$$2. \qquad R_n(x) = f(x) - f(\xi) - \sum_{k=1}^{n} \frac{(x-\xi)^h}{k!} f^{(k)}(\xi)$$

approaches zero as $n \to \infty$.

The following are different forms for the remainder of a Taylor series.

$$3. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < \theta < 1]. \qquad \text{(Lagrange)}$$

$$4. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi+\theta(x-\xi)) \qquad [0 < \theta < 1]. \qquad \text{(Cauchy)}$$

$$5. \quad R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi+\theta(x-\xi)) \qquad [0 < \theta < 1], \text{(Schlömilch)}$$

where $\psi(x)$ is an arbitrary function satisfying the following two conditions: 1) It and its derivative $\psi'(x)$ are continuous in the interval $(0, x-\xi)$; 2) the derivative $\psi'(x)$ does not change sign in that interval. If we set $\psi(x) = x^{p+1}$, we obtain the following form for the remainder:

$$R_n(x) = \frac{(x-\xi)^{n+1}(1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi+\theta(x-\xi)) \qquad [0 < p \le n; \quad 0 < \theta < 1]. \qquad \text{(Rouché)}$$

6. $R_n(x) = \frac{1}{n!}\int_{\xi}^{x} f^{(n+1)}(t)(x-t)^n\,dt.$

20

0.318

Other forms in which a Taylor series may be written:

1. $f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!}f'(a) + \frac{x^2}{2!}f''(a) + \dots$

2. $\sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots \qquad$ [Maclaurin series]

0.319

The Taylor series of functions of several variables:

$$f(x,y) = f(\xi,\eta) + (x-\xi)\frac{\partial f(\xi,\eta)}{\partial x} + (y-\eta)\frac{\partial f(\xi,\eta)}{\partial y} + \\ + \frac{1}{2!}\left\{(x-\xi)^2\frac{\partial^2 f(\xi,\,\eta)}{\partial x^2} + 2(x-\xi)(y-\eta)\frac{\partial^2 f(\xi,\,\eta)}{\partial x\,\partial y} + (y-\eta)^2\frac{\partial^2 f(\xi,\,\eta)}{\partial y^2}\right\} + \dots$$

0.32 Fourier series

0.320

Suppose that $f(x)$ is a *periodic* function of period $2l$ and that it is absolutely integrable (possibly improperly) over the interval $(-l,l)$. The following trigonometric series is called the *Fourier series* of $f(x)$:

1. $\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\} \frac{k\pi x}{l} + b_k \sin\} \frac{k\pi x}{l}\right),$

2. $$a_k = \frac{1}{l}\int_{-l}^{l} f(t)\,\cos\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\,\cos\frac{k\pi t}{l}\,dt \quad (k = 0,\,1,\,2,\ldots),$$

3.[7] $$b_k = \frac{1}{l}\int_{-l}^{l} f(t)\,\sin\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\,\sin\frac{k\pi t}{l}\,dt \quad (k = 1,\,2,\ldots).$$

Convergence tests

0.321

The Fourier series of a function $f(x)$ at a point x_0 converges to the number

$$\frac{f(x_0+0)+f(x_0-0)}{2},$$

if, for some $h>0$, the integral

$$\int_0^h \frac{|f(x_0+t)+f(x_0-t)-f(x_0+0)-f(x_0-0)|}{t}\,dt$$

21

exists. Here, it is assumed that the function $f(x)$ either is continuous at the point x_0 or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits $f(x_0+0)$ and $f(x_0-0)$ exist. (Dini)

0.322

The Fourier series of a periodic function $f(x)$ that satisfies the Dirichlet conditions on the interval $[a,\,b]$ converges at every point x_0 to the value $\frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$. (Dirichlet)

We say that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a,\,b]$ if it is bounded on that interval and if the interval $[a,\,b]$ can be partitioned into a finite number of subintervals inside each of which the function $f(x)$ is continuous and monotonic.

0.323

The Fourier series of a function $f(x)$ at a point x_0 converges to $\frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$ if $f(x)$ is of bounded variation in some interval (x_0-h, x_0+h) with center at x_0. (Jordan-Dirichlet) FI III 528

The definition of a function of bounded variation. Suppose that a function $f(x)$ is defined on some interval $[a\,b]$, where $a<b$. Let us

$<$

partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

FI III 524

and let us form the sum

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \, .$$

Different partitions of the interval $[a, b]$ (that is, different choices of points of division x_i) yield, generally speaking, different sums. If the set of these sums is bounded above, we say that the function $f(x)$ is *of bounded variation* on the interval $[a, b]$. The least upper bound of these sums is called the *total variation* of the function $f(x)$ on the interval $[a, b]$.

0.324

Suppose that a function $f(x)$ is piecewise-continuous on the interval $[a,\ b]$ and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point x_0 of the interval $[a,\ b]$, the Fourier series of the function $f(x)$ converges to $\frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$.

0.325

A function *f*(*x*) defined in the interval (0, *l*) can be expanded in a cosine series of the form

1. $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

where

2. $$a_k = \frac{2}{l}\int_0^l f(t) \cos \frac{k\pi t}{l} \, dt.$$

0.326

A function *f*(*x*) defined in the interval (0, *l*) can be expanded in a sine series of the form

1. $$\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

where

2. $$b_k = \frac{2}{l}\int_0^l f(t)\,\sin\frac{k\pi t}{l}\,dt.$$

The convergence tests for the series 0.325 1. and 0.326 1. are analogous to the convergence tests for the series 0.320 1. (see 0.321-0.324).

0.327

The Fourier coefficients a_k and b_k (given by formulas 0.320 2. and 0.320 3.) of an absolutely integrable function approach zero as $k \to \infty$.

If a function *f*(*x*) is square-integrable on the interval $(-l, l)$, the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{l}\int_{-l}^{l} f^2(x)\,dx. \qquad \text{(A.M. Lyapunov)}$$

FI III 705

0.328

Suppose that $f(x)$ and $\varphi(x)$ are two functions that are square-integrable on the interval $(-l,\ l)$ and that $a_k,\ b_k$ and $\alpha_k,\ \beta_k$ are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k) = \frac{1}{l}\int_{-l}^{l} f(x)\varphi(x)\,dx.$$

FI III 709

For examples of Fourier series, see 1.44-1.45

0.33 Asymptotic series

0.330

Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. 0.227 3.).

Asymptotic series have many properties that are analogous to the properties of convergent series and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}.$$

23

The definition of an asymptotic expansion. The divergent series $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ is called the *asymptotic expansion* of a function $f(z)$ in a given region of values of arg z if the expression $R_n(z) = z^n[f(z) - S_n(z)]$, where $S_n(z) = \sum_{k=0}^{n} \frac{A_k}{2^k}$, satisfies the condition $\lim_{|z|\to\infty} R_n(z) = 0$ for fixed n.

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series*.

FI II 820

0.331

Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.

2. One asymptotic series can be divided by another provided that the first term A_0 of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic.

FI II 823-825

3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible.

FI II 824

4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an

asymptotic series in only one manner.

WH

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

0.410

$$\frac{d}{da}\int_{\psi(a)}^{\varphi(a)} f(x,a)\,dx = f(\varphi(a),a)\frac{d\varphi(a)}{da} - f(\psi(a),a)\frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da}f(x,a)\,dx.$$

FI II 680
FI II 820

0.411

In particular,

1. $\frac{d}{da}\int_{b}^{a} f(x)\,dx = f(a).$

2. $\frac{d}{db}\int_{b}^{a} f(x)\,dx = -f(b).$

0.42 The nth derivative of a product

(Leibniz rule)

Suppose that u and v are n-times-differentiable functions of x. Then,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^n v}{dx^n} + \binom{n}{1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \binom{n}{2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \binom{n}{3}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \ldots + v\frac{d^n u}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u+v)^{(n)}.$$

0.43 The nth derivative of a composite function

0.430

If $f(x)=F(y)$ and $y=\varphi(x)$, then

$$1.\quad \frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} F'(y) + \frac{U_2}{2!} F''(y) + \frac{U_3}{3!} F'''(y) + \ldots + \frac{U_n}{n!} F^{(n)}(y),$$

where

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} y \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} y^2 \frac{d^n}{dx^n} y^{k-2} - \ldots + (-1)^{k-1} k y^{k-1} \frac{d^n y}{dx^n}.$$

AD (7361) GO

$$2.\quad \frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!j!h!\ldots k!} \frac{d^m F}{dy^m} \left(\frac{y'}{1!}\right)^i \left(\frac{y''}{2!}\right)^j \left(\frac{y'''}{3!}\right)^h \cdots \left(\frac{y^{(l)}}{l!}\right)^k,$$

Here, the symbol $\sum$ indicates summation over all solutions in non negative integers of the equation $i + 2j + 3h + \ldots + lk = n$ and $m = i + j + h + \ldots + k$.

0.431

$$1.\quad (-1)^n \frac{d^n}{dx^n} F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}} F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}} \frac{n}{1!} F^{(n-1)}\left(\frac{1}{x}\right) + \\ + \frac{(n-1)(n-2)}{x^{2n-2}} \frac{n(n-1)}{2!} F^{(n-2)}\left(\frac{1}{x}\right) + \ldots$$

AD (7362.1)

$$2.\quad (-1)^n \frac{d^n}{dx^n} e^{\frac{a}{x}} = \frac{1}{x^n} e^{\frac{a}{x}} \left\{ \left(\frac{a}{x}\right)^n + (n-1)\binom{n}{1}\left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2)\binom{n}{2}\left(\frac{a}{x}\right)^{n-2} + \right. \\ \left. + (n-1)(n-2)(n-3)\binom{n}{3}\left(\frac{a}{x}\right)^{n-3} + \ldots \right\}.$$

AD (7362.2)

0.432

1. $$\frac{d^n}{dx^n}F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!}(2x)^{n-2}F^{(n-1)}(x^2) + \\ + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}F^{(n-2)}(x^2) + \\ + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}(2x)^{n-6}F^{(n-3)}(x^2) + \dots$$

AD (7363.1)

2. $$\frac{d^n}{dx^n}e^{ax^2} = (2ax)^n e^{ax^2}\left\{1 + \frac{n(n-1)}{1!(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!(4ax^2)^2} + \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!(4ax^2)^3} + \cdots\right\}.$$

AD (7363.2)

3. $$\frac{d^n}{dx^n}(1+ax^2)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times \\ \times\left\{1 + \frac{n(n-1)}{1!(p-n+1)}\frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)}\left(\frac{1+ax^2}{4ax^2}\right)^2 + \dots\right\},$$

AD (7363.3)

4. $$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1}\frac{(2m-1)!!}{m}\sin(m\arccos x).$$

AD (7363.4)

0.433

1. $$\frac{d^n}{dx^n}F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!}\frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!}\frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots$$

AD (7364.1)

2. $$\frac{d^n}{dx^n}(1+a\sqrt{x})^{2n-1} = \frac{(2n-1)!!}{2^n}\frac{a}{\sqrt{x}}\left(a^2 - \frac{1}{x}\right)^{n-1}.$$

AD (7364.2)

0.434

$$\frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{ -\binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n y^2}{dx^n} - \dots \right\}.$$

AD (737.1)

0.435

$$\frac{d^n}{dx^n} \ln y = \left\{ \binom{n}{1} \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n y^2}{dx^n} + \binom{n}{3} \frac{1}{3 \cdot y^3} \frac{d^n y^3}{dx^n} - \dots \right\}.$$

AD (737.2)

0.44 Integration by substitution

0.440

Let $f(x)$ and $g(x)$ be continuous in $[a, b]$. Further, let $g'(x)$ exist and be continuous there. Then

$$\int_a^b f[g(x)] g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

1. Elementary Functions

1.1 Power of Binomials

1.11 Power series

1.110

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!} x^2 + \dots + \frac{q(q-1)\dots(q-k+1)}{k!} x^k + \dots .$$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series converges for $q > -1$ and diverges for $q \le -1$. For $x = 1$, the series converges absolutely for $q > 0$. For $x = -1$, it converges absolutely for $q > 0$ and diverges for $q < 0$. If $q = n$ is a natural number, the series 1.110 is reduced to the finite sum 1.111.

FI II 425

1.111

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}.$$

1.112

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$

1.121

2. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}.$

3. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \ldots .$

4. $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \ldots .$

1.113

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k \qquad [x^2 < 1].$$

1.114

1. $(1+\sqrt{1+x})^q = 2^q \left\{ 1 + \frac{q}{1!}\left(\frac{x}{4}\right) + \frac{q\,(q-3)}{2!}\left(\frac{x}{4}\right)^2 + \right.$ $\left. + \frac{q(q-4)(q-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots \right\} \qquad [x^2 < 1,\ q \text{ is a real number}].$

AD (6351.1)

2. $(x+\sqrt{1+x^2})^q = 1 + \sum_{k=0}^{\infty} \frac{q^2(q^2-2^2)(q^2-4^2)\ldots[q^2-(2k)^2]x^{2k+2}}{(2k+2)!} +$ $+ qx + q\sum_{k=1}^{\infty} \frac{(q^2-1^2)(q^2-3^2)\ldots[q^2-(2k-1)^2]}{(2k+1)!}x^{2k+1}$ $[x^2 < 1,\ q \text{ is a real number}].$

1.12 Series of rational fractions

1.121

1. $$\frac{x}{1-x} = \sum_{k=1}^{\infty} \frac{2^{k-1}x^{2^{k-1}}}{1+x^{2^{k-1}}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1-x^{2^k}} \qquad [x^2 < 1].$$

AD (6350.3)

2. $$\frac{1}{x-1} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{x^{2^{k-1}}+1} \qquad [x^2 > 1].$$

AD (6350.3)

1.2 The Exponential Function

1.21 Series representations

1.211

1.[7] $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

2. $$a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!}.$$

3. $$e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}.$$

1.212

$$e^x(1+x) = \sum_{k=0}^{\infty} \frac{x^k(k+1)}{k!}.$$

1.213

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}x^{2k}}{(2k)!} \qquad [x < 2\pi].$$

1.214

$$e^{e^x} = e\left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots\right).$$

AD (6460.3)

1.215

1. $$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots$$

AD (6460.4)

2. $$e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots\right).$$

AD (6460.5)

3. $$e^{\operatorname{tg} x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots$$

AD (6460.6)

1.216

1. $$e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots$$

AD (6460.7)

2. $$e^{\operatorname{arctg} x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots$$

AD (6460.8)

1.217

1. $$\pi\frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x\sum_{k=1}^{\infty}\frac{1}{x^2 + k^2} \text{(cf. } \mathbf{1.421}\ 3.\text{)}.$$

AD (6707.1)

1.421

2. $$\frac{2\pi}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x \sum_{k=1}^{\infty} (-1)^k \frac{1}{x^2 + k^2} \qquad (\text{cf. } \mathbf{1.422}\ 3.).$$

AD (6707.2)

1.422

1.22 Functional relations

1.221

1. $a^x = e^{x \ln a}$.

2. $a^{\log_a x} = a^{\frac{1}{\log_x a}} = x$.

1.222

1. $e^x = \operatorname{ch} x + \operatorname{sh} x$.

2. $e^{ix} = \cos x + i \sin x$.

1.223

$$e^{ax} - e^{bx} = (a-b)x \exp\left[\frac{1}{2}(a+b)x\right] \prod_{k=1}^{\infty}\left[1 + \frac{(a-b)^2 x^2}{2k^2\pi^2}\right].$$

MO 216

1.23 Series of exponentials

1.231

$$\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x} \qquad [a > 1 \quad \text{and} \quad x < 0 \quad \text{or} \quad 0 < a < 1 \quad \text{and} \quad x > 0].$$

1. $\operatorname{th} x = 1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-2kx} \qquad [x > 0].$

2. $\operatorname{sech} x = 2\sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x} \qquad [x > 0].$

3. $\operatorname{cosech} x = 2\sum_{k=0}^{\infty} e^{-(2k+1)x} \qquad [x > 0].$

1.3-1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\operatorname{sh} x = \frac{1}{i}\sin(ix), \qquad \sin x = \frac{1}{i}\operatorname{sh}(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\operatorname{ch} x = \cos(ix), \qquad \cos x = \operatorname{ch}(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice-versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful ``pairs" are included in the list.

1.31 The basic functional relations

1.311

1. $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}); \quad = -i\operatorname{sh}(ix).$

2. $\operatorname{sh} x = \frac{1}{2}(e^{x} - e^{-x}); \quad = -i\sin(ix).$

3. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}); \quad = \operatorname{ch}(ix).$

4. $\operatorname{ch} x = \frac{1}{2}(e^{x} + e^{-x}); \quad = \cos(ix).$

5. $\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{1}{i}\operatorname{th}(ix)$

6. $\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{1}{i}\operatorname{tg}(ix)$

1.312

1. $\cos^2 x + \sin^2 x = 1.$

2. $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1.$

1.313

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x.$

2. $\operatorname{sh}(x \pm y) = \operatorname{sh} x \operatorname{ch} y \pm \operatorname{sh} y \operatorname{ch} x.$

3. $\sin(x \pm iy) = \sin x \operatorname{ch} y \pm i \operatorname{sh} y \cos x.$

4. $\operatorname{sh}(x \pm iy) = \operatorname{sh} x \cos y \pm i \sin y \operatorname{ch} x.$

5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$

6. $\operatorname{ch}(x \pm y) = \operatorname{ch} x \operatorname{ch} y \pm \operatorname{sh} x \operatorname{sh} y.$

7. $\cos(x \pm iy) = \cos x \operatorname{ch} y \mp i \sin x \operatorname{sh} y.$

8. $\operatorname{ch}(x \pm iy) = \operatorname{ch} x \cos y \pm i \operatorname{sh} x \sin y.$

9. $\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}.$

10. $\operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x \operatorname{th} y}.$

1.314

1. $\sin x \pm \sin y = 2\sin\frac{1}{2}(x \pm y)\cos\frac{1}{2}(x \mp y).$

2. $\operatorname{sh} x \pm \operatorname{sh} y = 2\operatorname{sh}\frac{1}{2}(x \pm y)\operatorname{ch}\frac{1}{2}(x \mp y).$

3. $\cos x + \cos y = 2\cos\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y).$

4. $\operatorname{ch} x + \operatorname{ch} y = 2\operatorname{ch}\frac{1}{2}(x + y)\operatorname{ch}\frac{1}{2}(x - y).$

5. $\cos x - \cos y = 2\sin\frac{1}{2}(x + y)\sin\frac{1}{2}(y - x).$

6. $\operatorname{ch} x - \operatorname{ch} y = 2\operatorname{sh}\frac{1}{2}(x + y)\operatorname{sh}\frac{1}{2}(x - y).$

7. $\operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}.$

8. $\operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x \operatorname{ch} y}.$

1.315

1. $\sin^2 x - \sin^2 y = \sin(x + y)\sin(x - y) = \cos^2 y - \cos^2 x.$

2. $\operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{sh}(x + y)\operatorname{sh}(x - y) = \operatorname{ch}^2 x - \operatorname{ch}^2 y.$

3. $\cos^2 x - \sin^2 y = \cos(x + y)\cos(x - y) = \cos^2 y - \sin^2 x.$

4. $\operatorname{sh}^2 x + \operatorname{ch}^2 y = \operatorname{ch}(x+y)\operatorname{ch}(x-y) = \operatorname{ch}^2 x + \operatorname{sh}^2 y.$

1.316

1. $(\cos x + i \sin x)^n = \cos nx + i \sin nx.$ 2. $(\operatorname{ch} x + \operatorname{sh} x)^n = \operatorname{sh} nx + \operatorname{ch} nx$

[n is an integer].

1.317

1. $\sin \frac{x}{2} = \pm\sqrt{\frac{1}{2}(1-\cos x)}.$

2. $\operatorname{sh} \frac{x}{2} = \pm\sqrt{\frac{1}{2}(\operatorname{ch} x - 1)}.$

3. $\cos \frac{x}{2} = \pm\sqrt{\frac{1}{2}(1+\cos x)}.$

4. $\operatorname{ch} \frac{x}{2} = \sqrt{\frac{1}{2}(\operatorname{ch} x + 1)}.$

5. $\operatorname{tg} \frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}.$

6. $\operatorname{th} \frac{x}{2} = \frac{\operatorname{ch} x - 1}{\operatorname{sh} x} = \frac{\operatorname{sh} x}{\operatorname{ch} x + 1}.$

The signs in front of the radical in formulas 1.317 1., 1.317 2., and 1.317 3. are taken so as to agree with the signs of the left hand members. The sign of the left hand members depends in turn on the value of *x*.

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.320

1. $$\sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}.$$

KR 56 (10, 2)

2. $$\operatorname{sh}^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \operatorname{ch} 2(n-k)x + \binom{2n}{n} \right\}.$$

3. $$\sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x$$

4. $\operatorname{sh}^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \operatorname{sh}(2n-2k-1)x.$

5. $\cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2\binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}.$

KR 56 (10, 1)

6. $\operatorname{ch}^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2\binom{2n}{k} \operatorname{ch} 2(n-k)x + \binom{2n}{n} \right\}.$

7. $\cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos(2n-2k-1)x.$

KR 56 (10, 3)

8. $\operatorname{ch}^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \operatorname{ch}(2n-2k-1)x.$

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Special cases

1.321

1. $\sin^2 x = \frac{1}{2}(-\cos 2x + 1).$

2. $\sin^3 x = \frac{1}{4}(-\sin 3x + 3\sin x).$

3. $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3).$

4. $\sin^5 x = \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x).$

5. $\sin^6 x = \frac{1}{32}(-\cos 6x + 6\cos 4x - 15\cos 2x + 10).$

6. $\sin^7 x = \frac{1}{64}(-\sin 7x + 7\sin 5x - 21\sin 3x + 35\sin x)$.

1.322

1. $\operatorname{sh}^2 x = \frac{1}{2}(\operatorname{ch} 2x - 1)$.

2. $\operatorname{sh}^3 x = \frac{1}{4}(\operatorname{sh} 3x - 3\operatorname{sh} x)$.

3. $\operatorname{sh}^4 x = \frac{1}{8}(\operatorname{ch} 4x - 4\operatorname{ch} 2x + 3)$.

4. $\operatorname{sh}^5 x = \frac{1}{16}(\operatorname{sh} 5x - 5\operatorname{sh} 3x + 10\operatorname{sh} x)$.

5. $\operatorname{sh}^6 x = \frac{1}{32}(\operatorname{ch} 6x - 6\operatorname{ch} 4x + 15\operatorname{ch} 2x + 10)$.

6. $\operatorname{sh}^7 x = \frac{1}{64}(\operatorname{sh} 7x - 7\operatorname{sh} 5x + 21\operatorname{sh} 3x + 35\operatorname{sh} x)$.

1.323

1. $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$.

2. $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

3. $\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$.

4. $\cos^5 x = \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x).$

5. $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10).$

6. $\cos^7 x = \frac{1}{64}(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x).$

1.324

1. $\operatorname{ch}^2 x = \frac{1}{2}(\operatorname{ch} 2x + 1).$

2. $\operatorname{ch}^3 x = \frac{1}{4}(\operatorname{ch} 3x + 3\operatorname{ch} x).$

3. $\operatorname{ch}^4 x = \frac{1}{8}(\operatorname{ch} 4x + 4\operatorname{ch} 2x + 3).$

4. $\operatorname{ch}^5 x = \frac{1}{16}(\operatorname{ch} 5x + 5\operatorname{ch} 3x + 10\operatorname{ch} x).$

5. $\operatorname{ch}^6 x = \frac{1}{32}(\operatorname{ch} 6x + 6\operatorname{ch} 4x + 15\operatorname{ch} 2x + 10).$

6. $\operatorname{ch}^7 x = \frac{1}{64}(\operatorname{ch} 7x + 7\operatorname{ch} 5x + 21\operatorname{ch} 3x + 35\operatorname{ch} x).$

1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

1.331

$$1^3.\ \sin nx = n\cos^{n-1}x\sin x + \binom{n}{3}\cos^{n-3}x\sin^3 x +$$
$$\binom{n}{5}\cos^{n-5}x\sin^5 x - \dots;$$
$$= \sin x\left\{2^{n-1}\cos^{n-1}x - \binom{n-2}{1}2^{n-3}\cos^{n-3}x + \right.$$
$$\left. + \binom{n-3}{2}2^{n-5}\cos^{n-5}x - \binom{n-4}{3}2^{n-7}\cos^{n-7}x + \dots\right\}.$$

AD (3.175)

$$2.\quad \operatorname{sh} nx = x\sum_{k=1}^{[(n+1)/2]}\binom{n}{2k-1}\operatorname{sh}^{2k-2}x\operatorname{ch}^{n-2k+1}x;$$
$$= \operatorname{sh} x\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n-k-1}{k}2^{n-2k-1}\operatorname{ch}^{n-2k-1}x.$$

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$$3.\quad \cos nx = \cos^n x - \binom{n}{2}\cos^{n-2}x\sin^2 x + \binom{n}{4}\cos^{n-4}x\sin^4 x - \dots;$$
$$= 2^{n-1}\cos^n x - \frac{n}{1}2^{n-3}\cos^{n-2}x + \frac{n}{2}\binom{n-3}{1}2^{n-5}\cos^{n-4}x -$$
$$- \frac{n}{3}\binom{n-4}{2}2^{n-7}\cos^{n-6}x + \dots.$$

AD (3.175)

$$4^3.\ \operatorname{ch} nx = \sum_{k=0}^{[n/2]}\binom{n}{2k}\operatorname{sh}^{2k}x\operatorname{ch}^{n-2k}x = 2^{n-1}\operatorname{ch}^n x + n\sum_{k=1}^{[n/2]}(-1)^k\frac{1}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\operatorname{ch}^{n-2k}x.$$

1.332

$$1.\quad \sin 2nx = 2n\cos x\left\{\sin x - \frac{4n^2-2^2}{3!}\sin^3 x + \frac{(4n^2-2^2)(4n^2-4^2)}{5!}\sin^5 x - \dots\right\};$$
$$= (-1)^{n-1}\cos x\left\{2^{2n-1}\sin^{2n-1}x - \frac{2n-2}{1!}2^{2n-3}\sin^{2n-3}x + \right.$$
$$+ \frac{(2n-3)(2n-4)}{2!}2^{2n-5}\sin^{2n-5}x -$$
$$\left. - \frac{(2n-4)(2n-5)(2n-6)}{3!}2^{2n-7}\sin^{2n-7}x + \dots\right\}.$$

2. $$\sin(2n-1)x = (2n-1)\left\{\sin x - \frac{(2n-1)^2-1^2}{3!}\sin^3 x + \right.$$
$$\left. + \frac{[(2n-1)^2-1^2][(2n-1)^2-3^2]}{5!}\sin^5 x - \dots\right\};$$
$$= (-1)^{n-1}\left\{2^{2n-2}\sin^{2n-1} x - \frac{2n-1}{1!}2^{2n-4}\sin^{2n-3} x + \right.$$
$$+ \frac{(2n-1)(2n-4)}{2!}2^{2n-6}\sin^{2n-5} x -$$
$$\left. - \frac{(2n-1)(2n-5)(2n-6)}{3!}2^{2n-8}\sin^{2n-7} x + \dots\right\}.$$

AD (3.174)
AD (3.172)

3. $$\cos 2nx = 1 - \frac{4n^2}{2!}\sin^2 x + \frac{4n^2(4n^2-2^2)}{4!}\sin^4 x - \frac{4n^2(4n^2-2)(4n^2-4^2)}{6!}\sin^6 x + \dots;$$
$$= (-1)^n\left\{2^{2n-1}\sin^{2n} x - \frac{2n}{1!}2^{2n-3}\sin^{2n-2} x + \right.$$
$$\left. + \frac{2n(2n-3)}{2!}2^{2n-5}\sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!}2^{2n-7}\sin^{2n-6} x + \dots\right\}.$$

AD (3.173)a
AD (3.171)

4. $$\cos(2n-1)x = \cos x\left\{1 - \frac{(2n-1)^2-1^2}{2!}\sin^2 x + \right.$$
$$\left. + \frac{[(2n-1)^2-1^2][(2n-1)^2-3^2]}{4!}\sin^4 x - \dots\right\};$$
$$= (-1)^{n-1}\cos x\left\{2^{2n-2}\sin^{2n-2} x - \frac{2n-3}{1!}2^{2n-4}\sin^{2n-4} x + \right.$$
$$+ \frac{(2n-4)(2n-5)}{2!}2^{2n-6}\sin^{2n-6} x -$$
$$\left. - \frac{(2n-5)(2n-6)(2n-7)}{3!}2^{2n-8}\sin^{2n-8} x + \dots\right\}.$$

AD (3.174)
AD (3.172)

By using the formulas and values of 1.30, we can write formulas for $\operatorname{sh} 2nx$, $\operatorname{sh}(2n-1)x$, $\operatorname{ch} 2nx$, and $\operatorname{ch}(2n-1)x$ that are analogous to those of 1.332, just as was done in the formulas in 1.331.

Special cases

1.333

1. $\sin 2x = 2\sin x \cos x.$

2. $\sin 3x = 3\sin x - 4\sin^3 x.$

3. $\sin 4x = \cos x(4\sin x - 8 \sin^3 x).$

4. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x.$

5. $\sin 6x = \cos x(6\sin x - 32 \sin^3 x + 32\sin^5 x).$

6. $\sin 7x = 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x.$

1.334

1. $\operatorname{sh} 2x = 2\operatorname{sh} x \operatorname{ch} x.$

2. $\operatorname{sh} 3x = 3\operatorname{sh} x + 4\operatorname{sh}^3 x.$

3. $\operatorname{sh} 4x = \operatorname{ch} x(4\operatorname{sh} x + 8\operatorname{sh}^3 x).$

4. $\operatorname{sh} 5x = 5\operatorname{sh} x + 20\operatorname{sh}^3 x + 16\sin^5 x.$

5. $\operatorname{sh} 6x = \operatorname{ch} x(6\operatorname{sh} x + 32\operatorname{sh}^3 x + 32\operatorname{sh}^5 x).$

6. $\operatorname{sh} 7x = 7\operatorname{sh} x + 56 \operatorname{sh}^3 x + 112\operatorname{sh}^5 x + 64\operatorname{sh}^7 x.$

1.335

1. $\cos 2x = 2\cos^2 x - 1.$

2. $\cos 3x = 4\cos^3 x{-}3\cos x.$

3. $\cos 4x = 8\cos^4 x{-}8\cos^2 x{+}1.$

4. $\cos 5x = 16\cos^5 x{-}20\cos^3 x{+}5\cos x.$

5. $\cos 6x = 32\cos^6 x{-}48\cos^4 x{+}18\cos^2 x{-}1.$

6. $\cos 7x = 64\cos^7 x{-}112\cos^5 x{+}56\cos^3 x{-}7\cos x.$

1.336

1. $\operatorname{ch} 2x = 2\operatorname{ch}^2 x{-}1.$

2. $\operatorname{ch} 3x = 4\operatorname{ch}^3 x{-}3\operatorname{ch} x.$

3. $\operatorname{ch} 4x = 8\operatorname{ch}^4 x{-}8\operatorname{ch}^2 x{+}1.$

4. $\operatorname{ch} 5x = 16\operatorname{ch}^5 x{-}20\operatorname{ch}^3 x{+}5\operatorname{ch} x.$

5. $\operatorname{ch} 6x = 32\operatorname{ch}^6 x{-}48\operatorname{ch}^4 x{+}18\operatorname{ch}^2 x{-}1.$

6. $\operatorname{ch} 7x = 64\operatorname{ch}^7 x{-}112\operatorname{ch}^5 x{+}56\operatorname{ch}^3 x{-}7\operatorname{ch} x.$

1.34 Certain sums of trigonometric and hyperbolic functions

1.341

1. $$\sum_{k=0}^{n-1} \sin(x+ky) = \sin\left(x+\frac{n-1}{2}y\right)\sin\frac{ny}{2}\operatorname{cosec}\frac{y}{2}.$$

AD (361.8)

2. $$\sum_{k=0}^{n-1} \operatorname{sh}(x+ky) = \operatorname{sh}\left(x+\frac{n-1}{2}y\right)\operatorname{sh}\frac{ny}{2}\frac{1}{\operatorname{sh}\frac{y}{2}}.$$

3. $$\sum_{k=0}^{n-1} \cos(x+ky) = \cos\left(x+\frac{n-1}{2}y\right)\sin\frac{ny}{2}\operatorname{cosec}\frac{y}{2}.$$

AD (361.9)

4. $$\sum_{k=0}^{n-1} \operatorname{ch}(x+ky) = \operatorname{ch}\left(x+\frac{n-1}{2}y\right)\operatorname{sh}\frac{ny}{2}\frac{1}{\operatorname{sh}\frac{y}{2}}.$$

5. $$\sum_{k=0}^{2n-1} (-1)^k\cos(x+ky) = \sin\left(x+\frac{2n-1}{2}y\right)\sin ny\sec\frac{y}{2}.$$

JO (202)

6. $$\sum_{k=0}^{n-1} (-1)^k\sin(x+ky) = \sin\left\{x+\frac{n-1}{2}(y+\pi)\right\}\sin\frac{n(y+\pi)}{2}\sec\frac{y}{2}.$$

AD (202a)

Special cases

1.342

1. $$\sum_{k=1}^{n} \sin kx = \sin\frac{n+1}{2}x\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}.$$

AD (361.1)

2*. $$\sum_{k=0}^{n} \cos kx = \cos\frac{n+1}{2}x\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}+1 = \cos\frac{nx}{2}\sin\frac{n+1}{2}x\operatorname{cosec}\frac{x}{2}$$
$$= \frac{1}{2}\left[1+\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\right].$$

3. $\sum_{k=1}^{n} \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x.$

AD (361.7)

4. $\sum_{k=1}^{n} \cos(2k-1)x = \frac{1}{2}\sin 2nx \operatorname{cosec} x.$

JO (207)

1.343

1. $$\sum_{k=1}^{n} (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos\left(\frac{2n+1}{2}x\right)}{2\cos\frac{x}{2}}.$$

AD (361.11)

2. $$\sum_{k=1}^{n} (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1}\frac{\sin 2nx}{2\cos x}.$$

AD (361.10)

3. $$\sum_{k=1}^{n} \cos(4k-3)x + \sum_{k=1}^{n} \sin(4k-1)x = \sin 2nx(\cos 2nx + \sin 2nx)(\cos x + \sin x)\operatorname{cosec} 2x.$$

JO (208)

1.344

1. $$\sum_{k=1}^{n-1} \sin\frac{\pi k}{n} = \operatorname{ctg}\frac{\pi}{2n}.$$

AD (361.19)

2. $$\sum_{k=1}^{n-1} \sin\frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right).$$

AD (361.18)

AD (361.17)

1.35 Sums of powers of trigonometric functions of multiple angles

1.351

1. $$\sum_{k=1}^{n} \sin^2 kx = \frac{1}{4}[(2n+1)\sin x - \sin(2n+1)x]\operatorname{cosec} x;$$
$$= \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2\sin x}.$$

AD (361.3)

2. $$\sum_{k=1}^{n} \cos^2 kx = \frac{n-1}{2} + \frac{1}{2}\cos nx \sin(n+1)x \operatorname{cosec} x;$$
$$= \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2\sin x}.$$

AD (361.4)a

3. $$\sum_{k=1}^{n} \sin^3 kx = \frac{3}{4}\sin\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2} - \frac{1}{4}\sin\frac{3(n+1)x}{2}\sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}.$$

JO (210)

4. $$\sum_{k=1}^{n} \cos^3 kx = \frac{3}{4}\cos\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2} + \frac{1}{4}\cos\frac{3(n+1)}{2}x \sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}.$$

JO (211)a

5. $$\sum_{k=1}^{n} \sin^4 kx = \frac{1}{8}[3n - 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x].$$

JO (212)

6. $$\sum_{k=1}^{n} \cos^4 kx = \frac{1}{8}[3n + 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x].$$

JO (213)

1.352

AD (361.5)

2. $\sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \frac{2n-1}{2} x}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}.$

AD (361.6)

1.353

1. $\sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin (n-1) x}{1 - 2p \cos x + p^2}.$

AD (361.12)a

2. $\sum_{k=1}^{n-1} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x - p^n \operatorname{sh} nx + p^{n+1} \operatorname{sh} (n-1) x}{1 - 2p \operatorname{ch} x + p^2}.$

3. $\sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos nx + p^{n+1} \cos (n-1) x}{1 - 2p \cos x + p^2}.$

AD (361.13)a

4. $\sum_{k=0}^{n-1} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x - p^n \operatorname{ch} nx + p^{n+1} \operatorname{ch} (n-1) x}{1 - 2p \operatorname{ch} x + p^2}.$

JO (396)

1.36 Sums of products of trigonometric functions of multiple angles

1.361

1. $\sum_{k=1}^{n} \sin kx \sin(k+1)x = \frac{1}{4}[(n+1) \sin 2x - \sin 2(n+1)x] \operatorname{cosec} x.$

JO (214)

2. $\sum_{k=1}^{n} \sin kx \sin(k+2)x = \frac{n}{2} \cos 2x - \frac{1}{2} \cos (n+3)x \sin nx \operatorname{cosec} x.$

3. $$2\sum_{k=1}^{n} \sin kx \cos(2k-1)y = \sin\left\{ny + \frac{n+1}{2}x\right\} \sin\frac{n(x+2y)}{2} \operatorname{cosec}\frac{x+2y}{2} - \\ -\sin\left\{ny - \frac{n+1}{2}x\right\} \sin\frac{n(2y-x)}{2} \operatorname{cosec}\frac{2y-x}{2}.$$

JO (217)

1.362

1. $$\sum_{k=1}^{n} \left(2^k \sin^2 \frac{x}{2^k}\right)^2 = \left(2^n \sin\frac{x}{2^n}\right)^2 - \sin^2 x.$$

AD (361.15)

2. $$\sum_{k=1}^{n} \left(\frac{1}{2^k} \sec\frac{x}{2^k}\right)^2 = \operatorname{cosec}^2 x - \left(\frac{1}{2^n} \operatorname{cosec}\frac{x}{2^n}\right)^2.$$

AD (361.14)

1.37 Sums of tangents of multiple angles

1.371

1. $$\sum_{k=0}^{n} \frac{1}{2^k} \operatorname{tg}\frac{x}{2^k} = \frac{1}{2^n} \operatorname{ctg}\frac{x}{2^n} - 2\operatorname{ctg} 2x.$$

AD (361.16)

2. $$\sum_{k=0}^{n} \frac{1}{2^{2k}} \operatorname{tg}^2\frac{x}{2^k} = \frac{2^{2n+2}-1}{3\cdot 2^{2n-1}} + 4\operatorname{ctg}^2 2x - \frac{1}{2^{2n}} \operatorname{ctg}^2\frac{x}{2^n}.$$

AD (361.20)

1.38 Sums leading to hyperbolic tangents and cotangents

1.381

1. $$\sum_{k=0}^{n-1} \frac{\operatorname{th} x \dfrac{1}{n\sin^2\dfrac{2k+1}{4n}\pi}}{1 + \dfrac{\operatorname{th}^2 x}{\operatorname{tg}^2\dfrac{2k+1}{4n}\pi}} = \operatorname{th} 2nx.$$

2. $$\sum_{k=1}^{n-1} \frac{\operatorname{th} x \dfrac{1}{n \sin^2 \dfrac{k\pi}{2n}}}{1 + \dfrac{\operatorname{th}^2 x}{\operatorname{tg}^2 \dfrac{k\pi}{2n}}} = \operatorname{cth} 2nx - \frac{1}{2n}(\operatorname{th} x + \operatorname{cth} x).$$

JO (403)

3. $$\sum_{k=0}^{n-1} \frac{\operatorname{th} x \dfrac{2}{(2n+1) \sin^2 \dfrac{2k+1}{2(2n+1)}\pi}}{1 + \dfrac{\operatorname{th}^2 x}{\operatorname{tg}^2 \dfrac{2k+1}{2(2n+1)}\pi}} = \operatorname{th}(2n+1)x - \frac{\operatorname{th} x}{2n+1}.$$

JO (404)

4. $$\sum_{k=1}^{n} \frac{\operatorname{th} x \dfrac{2}{(2n+1) \sin^2 \dfrac{k\pi}{2(2n+1)}}}{1 + \dfrac{\operatorname{th}^2 x}{\operatorname{tg}^2 \dfrac{k\pi}{(2n+1)}}} = \operatorname{cth}(2n+1)x - \frac{\operatorname{cth} x}{2n+1}.$$

JO (405)

1.382

1. $$\sum_{k=0}^{n-1} \frac{1}{\dfrac{\sin^2 \dfrac{2k+1}{4n}\pi}{\operatorname{sh} x} + \dfrac{1}{2} \operatorname{th} \dfrac{x}{2}} = 2n \operatorname{th} nx.$$

JO (406)

2. $$\sum_{k=1}^{n-1} \frac{1}{\dfrac{\sin^2 \dfrac{k\pi}{2n}}{\operatorname{sh} x} + \dfrac{1}{2} \operatorname{th} \dfrac{x}{2}} = 2n \operatorname{cth} nx - 2 \operatorname{cth} x.$$

JO (407)

3. $$\sum_{k=0}^{n-1} \frac{1}{\dfrac{\sin^2 \dfrac{2k+1}{2(2n+1)}\pi}{\operatorname{sh} x} + \dfrac{1}{2} \operatorname{th} \dfrac{x}{2}} = (2n+1) \operatorname{th} \frac{(2n+1)x}{2} - \operatorname{th} \frac{x}{2}.$$

4. $$\sum_{k=1}^{n} \frac{1}{\dfrac{\sin^2 \dfrac{k\pi}{2n+1}}{\operatorname{sh} x} + \dfrac{1}{2} \operatorname{th} \dfrac{x}{2}} = (2n+1) \operatorname{cth} \frac{(2n+1)x}{2} - \operatorname{cth} \frac{x}{2}.$$

JO (409)

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.391

1. $$\sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \dfrac{k\pi}{n}}\right) \qquad [n - \text{even}].$$

JO (568)

2. $$\cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \dfrac{(2k-1)\pi}{2n}}\right) \qquad [n - \text{even}].$$

JO (569)

3. $$\sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \dfrac{k\pi}{n}}\right) \qquad [n - \text{odd}].$$

JO (570)

4. $$\cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \dfrac{(2k-1)\pi}{2n}}\right) \qquad [n - \text{odd}].$$

JO (571)a

1.392

1. $$\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(x + \frac{k\pi}{n}\right).$$

JO (548)

JO (549)

1.393

1. $$\prod_{k=0}^{n-1} \cos\left(x + \frac{2k}{n}\pi\right) = \frac{1}{2^{n-1}} \cos nx \qquad [n - \text{odd}].$$
$$= \frac{1}{2^{n-1}} \left[(-1)^{\frac{n}{2}} - \cos nx\right] \qquad [n - \text{even}].$$

JO (543)

2. $$\prod_{k=0}^{n-1} \sin\left(x + \frac{2k}{n}\pi\right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \qquad [n - \text{odd}].$$
$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \qquad [n - \text{even}].$$

JO (544)

1.394

$$\prod_{k=0}^{n-1} \left\{x^2 - 2xy\cos\left(\alpha + \frac{2k\pi}{n}\right) + y^2\right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}.$$

JO (573)

1.395

1. $$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{\cos x - \cos\left(y + \frac{2k\pi}{n}\right)\right\}.$$

JO (573)

2. $$\operatorname{ch} nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{\operatorname{ch} x - \cos\left(y + \frac{2k\pi}{n}\right)\right\}.$$

JO (538)

1.396

1. $$\prod_{k=1}^{n-1} \left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right) = \frac{x^{2n} - 1}{x^2 - 1}.$$

2. $\prod_{k=1}^{n} \left(x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x - 1}.$

KR 58 (28.2)

3. $\prod_{k=1}^{n} \left(x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x + 1}.$

KR 58 (28.3)

4. $\prod_{k=0}^{n-1} \left(x^2 - 2x \cos \frac{(2k+1)\pi}{2n} + 1 \right) = x^{2n} + 1.$

KR 58 (28.4)

1.41 The expansion of trigonometric and hyperbolic functions in power series

1.411

1. $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$

2. $\operatorname{sh} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$

3. $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$

4. $\operatorname{ch} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$

5. $\operatorname{tg} x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} |B_{2k}| x^{2k-1} \qquad \left[x^2 < \frac{\pi^2}{4} \right].$

FI II 523

6. $\operatorname{th} x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17}{315} x^7 + \ldots = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k} x^{2k-1} \qquad \left[x^2 < \frac{\pi^2}{4} \right].$

7. $\operatorname{ctg} x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \qquad [x^2 < \pi^2].$

FI II 523a

8. $\operatorname{cth} x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \ldots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} \qquad [x^2 < \pi^2].$

9. $\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \left[x^2 < \frac{\pi^2}{4}\right].$

CE 330a

10. $\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \ldots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \qquad \left[x^2 < \frac{\pi^2}{4}\right]$

CE 330

11. $\operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)|B_{2k}|x^{2k-1}}{(2k)!} \qquad [x^2 < \pi^2]$

CE 329a

12. $\operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \ldots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)B_{2k}}{(2k)!} x^{2k-1} \qquad [x^2 < \pi^2].$

JO (418)

43

1.412

1. $\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}x^{2k}}{(2k)!}.$

JO (452)a

2. $\cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}x^{2k}}{(2k)!}.$

JO (443)

3. $\sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1}-3}{(2k+1)!} x^{2k+1}.$

JO (452a)a

4. $\cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(3^{2k}+3)x^{2k}}{(2k)!}.$

1.413

1. $\operatorname{sh} x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!}.$

JO (508)

2. $\operatorname{ch} x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^{k} \frac{2^{2k} x^{4k}}{(4k)!}.$

JO (507)

3. $\operatorname{sh} x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}.$

JO (510)

4. $\operatorname{ch} x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}.$

JO (509)

1.414

1. $\cos\left[n \ln(x+\sqrt{1+x^2})\right] = 1 - \sum_{k=0}^{\infty} (-1)^{k} \frac{(n^2+0^2)(n^2+2^2)\dots[n^2+(2k)^2]}{(2k+2)!} x^{2k+2} \qquad [x^2 < 1].$

AD (6456.1)

2. $\sin\left[n \ln(x+\sqrt{1+x^2})\right] = nx - n \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n^2+1^2)(n^2+3^2)\dots[n^2+(2k-1)^2] x^{2k+1}}{(2k+1)!} \qquad [x^2 < 1].$

AD (6456.2)

Power series for $\ln \sin x$, $\ln \cos x$ and $\ln \operatorname{tg} x$ see 1.518.

1.42 Expansion in series of simple fractions

1.421

1. $\operatorname{tg} \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2}.$

2. $\operatorname{th}\frac{\pi x}{2}=\frac{4\pi}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2+x^2}.$

3. $\operatorname{ctg}\pi x=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2-k^2}=\frac{1}{\pi x}+\frac{x}{\pi}\sum_{\substack{h=-\infty\\k\neq 0}}^{\infty}\frac{1}{k(x-k)}.$

AD (6495.2), JO (450a)

4. $\operatorname{cth}\pi x=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2+k^2}\qquad (cf. 1.2171.).$

1.217

5. $\operatorname{tg}^2\frac{\pi x}{2}=x^2\sum_{k=1}^{\infty}\frac{2(2k-1)^2-x^2}{(1^2-x^2)^2(3^2-x^2)^2\ldots[(2k-1)^2-x^2]^2}.$

JO (450)

1.422

1. $\sec\frac{\pi x}{2}=\frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k-1}{(2k-1)^2-x^2}.$

AD (6495.3)a

2. $\sec^2\frac{\pi x}{2}=\frac{4}{\pi^2}\sum_{k=1}^{\infty}\left\{\frac{1}{(2k-1-x)^2}+\frac{1}{(2k-1+x)^2}\right\}.$

JO (451)a

3. $\operatorname{cosec}\pi x=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{x^2-k^2}\qquad$ (see also **1.217** 2.).

AD (6495.4)a

4. $\operatorname{cosec}^2 \pi x = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2} = \frac{1}{\pi^2 x^2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{x^2+k^2}{(x^2-k^2)^2}.$

JO (446)

5. $\frac{1 + x \operatorname{cosec} x}{2x^2} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x^2 - k^2\pi^2)}.$

JO (449)

6. $\operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{1}{x-k} + \frac{1}{k} \right).$

JO (450b)

1.423

$$\frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \operatorname{ctg} \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1 - k^2 m^2)^2}.$$

JO (477)

1.43 Representation in the form of an infinite product

1.431

1. $\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2} \right).$

EU

2. $\operatorname{sh} x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2} \right).$

EU

3. $\cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2\pi^2} \right).$

4. $\operatorname{ch} x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2\pi^2}\right).$

EU

1.432

1. $\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(2k\pi + y^2)}\right)\left(1 - \frac{x^2}{(2k\pi - y)^2}\right).$

AD (653.2)

2. $\operatorname{ch} x - \cos y = 2\left(1 + \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(2k\pi + y)^2}\right)\left(1 + \frac{x^2}{(2k\pi - y)^2}\right).$

AD (653.1)

1.433

$$\cos\frac{\pi x}{4} - \sin\frac{\pi x}{4} = \prod_{k=1}^{\infty}\left[1 + \frac{(-1)^k x}{2k-1}\right].$$

BR* 189

1.434

$$\cos^2 x = \frac{1}{4}(\pi + 2x)^2 \prod_{k=1}^{\infty}\left[1 - \left(\frac{\pi + 2x}{2k\pi}\right)^2\right]^2.$$

MO 216

1.435

$$\frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty}\left(1 - \frac{x}{k-a}\right)\left(1 + \frac{x}{k+a}\right).$$

MO 216

1.436

$$1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty}\left[1 - \left(\frac{x}{k-a}\right)^2\right].$$

MO 216

1.437

1.437

$$\frac{\sin 3x}{\sin x} = -\prod_{k=-\infty}^{\infty}\left[1-\left(\frac{2x}{x+k\pi}\right)^2\right].$$

MO 216

1.438

$$\frac{\operatorname{ch} x - \cos a}{1-\cos a} = \prod_{k=-\infty}^{\infty}\left[1+\left(\frac{x}{2k\pi+a}\right)^2\right].$$

MO 216

1.439

1. $$\sin x = x\prod_{k=1}^{\infty}\cos\frac{x}{2^k} \qquad [|x|<1].$$

AD (615), MO 216

2. $$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left[1-\frac{4}{3}\sin^2\left(\frac{x}{3^k}\right)\right].$$

MO 216

1.44-1.45 Trigonometric (Fourier) series

1.441

1. $$\sum_{k=1}^{\infty}\frac{\sin kx}{k} = \frac{\pi-x}{2} \qquad [0<x<2\pi].$$

FI III 539

2. $$\sum_{k=1}^{\infty}\frac{\cos kx}{k} = \frac{1}{2}\ln\frac{1}{2(1-\cos x)} \qquad [0<x<2\pi].$$

FI III 530a, AD (6814)

3. $$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sin kx}{k} = \frac{x}{2} \qquad [-\pi<x<\pi].$$

4. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln\left(2\cos\frac{x}{2}\right) \qquad [-\pi < x < \pi].$

FI III 550

1.442

1. $\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{x} \qquad [0 < x < 2\pi].$

FI III 541

2. $\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2}\ln\operatorname{ctg}\frac{x}{2} \qquad [0 < x < \pi].$

BR* 168, JO (266), GI III(195)

3. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right) \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right].$

BR* 168, JO (268)a

4*. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \begin{cases} \frac{\pi}{4}\left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right] \\ -\frac{\pi}{4}\left[\frac{\pi}{2} < x < \frac{3\pi}{2}\right] \end{cases}.$

BR* 168, JO (269)

1.443

1. $$\sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k}\varrho^k;$$
$$= (-1)^{n-1}\frac{1}{2}\frac{(2\pi)^{2n}}{(2n)!} B_{2n}\left(\frac{x}{2}\right) \qquad \left[0 < x < 1, \quad \varrho = \frac{x}{2} - [x/2]\right].$$

CE 340, GE 71

2. $$\sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1}\rho^k;$$
$$= (-1)^{n-1}\frac{1}{2}\frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{x}{2}\right) \qquad \left[0 < x < 1; \quad \rho = \frac{x}{2} - [x/2]\right].$$

3. $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \qquad [0 \le x \le 2\pi].$

FI III 547

4. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \qquad [-\pi \le x \le \pi].$

FI III 544

47

$$\left.\begin{aligned}
&5.\quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}\\
&6.\quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48}\\
&7.\quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240}
\end{aligned}\right\} \qquad [0 \le x \le 2\pi]$$

AD (6818)
AD (6617)

1.444

1. $\sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x)\sin^2 x - \sin x \cos x \ln\left(4\sin^2 x\right) \qquad [0 \le x \le \pi].$

BR* 168, GI III (190)

2. $\sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln\left(4\sin^2 x\right) \qquad [0 \le x \le \pi].$

BR* 168

3. $\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2}(1 + \cos x) - \sin x \ln\left|2\cos\frac{x}{2}\right|.$

MO 213

4. $\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2}\sin x - (1 + \cos x)\ln\left|2\cos\frac{x}{2}\right|$

MO 213

MO 213

$$6^6.\ \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right) \qquad [-\pi \le x \le \pi].$$

FI III 546

$$7.\ \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4}\sin x \qquad \left[0 \le x \le \frac{\pi}{2}\right].$$

JO (591)

1.445

$$1.\ \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2+\alpha^2} = \frac{\pi}{2}\frac{\operatorname{sh}\alpha(\pi-x)}{\operatorname{sh}\alpha\pi} \qquad [0 < x < 2\pi].$$

BR* 157, JO (411)

$$2.\ \sum_{k=1}^{\infty} \frac{\cos kx}{k^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\operatorname{ch}\alpha(\pi-x)}{\operatorname{sh}\alpha\pi} - \frac{1}{2\alpha^2} \qquad [0 \le x \le 2\pi].$$

BR* 257, JO (410)

$$3.\ \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\operatorname{ch}\alpha x}{\operatorname{sh}\alpha\pi} - \frac{1}{2\alpha^2} \qquad [-\pi \le x \le \pi].$$

FI III 546

$$4.\ \sum_{k=1}^{\infty} (-1)^{k-1}\frac{k \sin kx}{k^2+\alpha^2} = \frac{\pi}{2}\frac{\operatorname{sh}\alpha x}{\operatorname{sh}\alpha\pi} \qquad [-\pi < x < \pi].$$

FI III, 546

$$5.\ \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2-\alpha^2} = \pi\frac{\sin\{\alpha[(2m+1)\pi - x]\}}{2\sin\alpha\pi} \qquad \left[\text{when } x = 2m\pi, \quad \sum \ldots = 0\right].$$
$$[2m\pi < x < (2m+2)\pi,\ \alpha\text{—not an integer}].$$

MO 213

MO 213

7. $$\sum_{k=1}^{\infty}(-1)^k \frac{k\sin kx}{k^2-\alpha^2} = \pi\frac{\sin[\alpha(2m\pi - x)]}{2\sin\alpha\pi} \qquad \left[\text{when } x=(2m+1)\pi, \quad \sum\ldots=0\right]$$
$$[(2m-1)\pi < x < (2m+1)\pi, \ \alpha\text{—not an integer}].$$

FI III 545a

8. $$\sum_{k=1}^{\infty}(-1)^k \frac{\cos kx}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2}\frac{\cos[\alpha(2m\pi - x)]}{\alpha\sin\alpha\pi}$$
$$[(2m-1)\pi \le x \le (2m+1)\pi, \ \alpha\text{—not an integer}].$$

FI III 545a

1.446

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8}\cos^2 x - \frac{1}{3}\cos x \qquad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right].$$

BR* 256, GI III (189)

1.447

$$\left.\begin{array}{ll} 1. & \displaystyle\sum_{k=1}^{\infty} p^k \sin kx = \frac{p\sin x}{1-2p\cos x+p^2} \\ 2. & \displaystyle\sum_{k=0}^{\infty} p^k \cos kx = \frac{1-p\cos x}{1-2p\cos x+p^2} \\ 3. & \displaystyle 1+2\sum_{k=1}^{\infty} p^k \cos kx = \frac{1-p^2}{1-2p\cos x+p^2} \end{array}\right\} \qquad [|p|<1].$$

FI II 559a, MO 213
FI II 559
FI II 559

1.448

$$\left.\begin{array}{ll} 1. & \displaystyle\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \operatorname{arctg}\frac{p\sin x}{1-p\cos x} \\ 2. & \displaystyle\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = \ln\frac{1}{\sqrt{1-2p\cos x+p^2}} \\ 3. & \displaystyle\sum_{k=1}^{\infty} \frac{p^{2k-1}\sin(2k-1)x}{2k-1} = \frac{1}{2}\operatorname{arctg}\frac{2p\sin x}{1-p^2} \\ 4. & \displaystyle\sum_{k=1}^{\infty} \frac{p^{2k-1}\cos(2k-1)x}{2k-1} = \frac{1}{4}\ln\frac{1+2p\cos x+p^2}{1-2p\cos x+p^2} \end{array}\right\} \qquad [0<x<2\pi, \quad p^2 \le 1].$$

$$5.\quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1+2p\sin x + p^2}{1-2p\sin x + p^2}$$

$$6.\quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \operatorname{arctg} \frac{2p\cos x}{1-p^2}$$

$$[0 < x < \pi, \quad p^2 \le 1].$$

JO (597)
JO (261)

1.449

$$1.\quad \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p\cos x} \sin(p\sin x)$$

$$2.\quad \sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p\cos x} \cos(p\sin x)$$

$$[p^2 \le 1].$$

JO (485)
JO (486)

Fourier expansions of hyperbolic functions

1.451

$$1.\quad \operatorname{sh} x = \cos x \sum_{k=0}^{\infty} \frac{(1^2+0^2)(1^2+2^2)\dots[1^2+(2k)^2]}{(2k+1)!} \sin^{2k+1} x.$$

JO (504)

$$2.\quad \operatorname{ch} x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2+1^2)(1^2+3^2)\dots[1^2+(2k-1)^2]}{(2k)!} \sin^{2k} x.$$

JO (503)

1.452

$$1.\quad \operatorname{sh}(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\cos(2k+1)\theta}{(2k+1)!}$$

$$2.\quad \operatorname{ch}(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k}\cos 2k\theta}{(2k)!}$$

$$[x^2 < 1].$$

JO (390)

JO (390)
JO (391)

$$\left.\begin{aligned} &3.\quad \operatorname{sh}(x\cos\theta) = \operatorname{cosec}(x\sin\theta)\sum_{k=1}^{\infty}\frac{x^{2k}\sin 2k\theta}{(2k)!} \\ &4.\quad \operatorname{ch}(x\cos\theta) = \operatorname{cosec}(x\sin\theta)\sum_{k=0}^{\infty}\frac{x^{2k+1}\sin(2k+1)\theta}{(2k+1)!}\end{aligned}\right\}\qquad [x^2<1],\quad [x\sin\theta\neq 0].$$

JO (392)
JO (393)

1.46 Series of products of exponential and trigonometric functions

1.461

1. $$\sum_{k=0}^{\infty} e^{-kt}\sin kx = \frac{1}{2}\frac{\sin x}{\operatorname{ch} t - \cos x}\qquad [t>0].$$

MO 213

2. $$1 + 2\sum_{k=1}^{\infty} e^{-kt}\cos kx = \frac{\operatorname{sh} t}{\operatorname{ch} t - \cos x}\qquad [t>0].$$

MO 213

1.462

$$\sum_{k=1}^{\infty}\frac{\sin kx \sin ky}{k}e^{-2k|t|} = \frac{1}{4}\ln\left[\frac{\sin\dfrac{x+y}{2} + \operatorname{sh}^2 t}{\sin^2\dfrac{x-y}{2} + \operatorname{sh}^2 t}\right].$$

MO 214

1.463

1. $$e^{x\ \cos\varphi}\cos(x\sin\varphi) = \sum_{n=0}^{\infty}\frac{x^n\cos n\varphi}{n!}\qquad [x^2<1].$$

AD (6476.1)

2. $$e^{x\ \cos\varphi}\sin(x\sin\varphi) = \sum_{n=1}^{\infty}\frac{x^n\sin n\varphi}{n!}\qquad [x^2<1].$$

1.47 Series of hyperbolic functions

1.471

1. $$\sum_{k=1}^{\infty} \frac{\operatorname{sh} kx}{k!} = e^{\operatorname{ch} x} \operatorname{sh}(\operatorname{sh} x).$$

JO (395)

2. $$\sum_{k=0}^{\infty} \frac{\operatorname{ch} kx}{k!} = e^{\operatorname{ch} x} \operatorname{ch}(\operatorname{sh} x).$$

JO (394)

3.* $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \operatorname{th} \frac{(2m+1)\pi x}{2} + x \operatorname{th} \frac{(2m+1)\pi}{2x}\right] = \frac{\pi^3}{16}.$$

1.472

1. $$\sum_{k=1}^{\infty} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x}{1 - 2p \operatorname{ch} x + p^2} \qquad [p^2 < 1].$$

JO (396)

2. $$\sum_{k=0}^{\infty} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x}{1 - 2p \operatorname{ch} x + p^2} \qquad [p^2 < 1].$$

JO (397)a

1.48 Lobachevskiy's "Angle of parallelism" $\Pi(x)$

1.480

Definition.

1. $\Pi(x) = 2 \operatorname{arcctg} e^x = 2 \operatorname{arctg} e^{-x} \qquad [x \geq 0].$
2. $\Pi(x) = \pi - \Pi(-x) \qquad [x < 0].$

LO III 183, LOI 193
LO III 297, LOI 120

1.481

Functional relations

1. $\sin \Pi(x) = \frac{1}{\operatorname{ch} x}.$

LO III 297

2. $\cos \Pi(x) = \operatorname{th} x.$

LO III 297

3. $\operatorname{tg} \Pi(x) = \frac{1}{\operatorname{sh} x}.$

LO III 297

4. $\operatorname{ctg} \Pi(x) = \operatorname{sh} x.$

LO III 297

5. $\sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}.$

LO III 297

6. $\cos \Pi(x+y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}.$

LO III 183

1.482

Connection with the gudermannian.

$$\operatorname{gd}(-x) = \Pi(x) - \frac{\pi}{2}.$$

(Definite) integral of the angle of parallelism; cf. 4.581 and 4.561.

1.49 The hyperbolic amplitude (the Gudermannian) gd x

1.490

Definition.

1. $\operatorname{gd} x = \int_0^x \frac{dt}{\operatorname{ch} t} = 2 \operatorname{arctg} e^x - \frac{\pi}{2}.$

JA

2. $x = \int_0^{\operatorname{gd} x} \frac{dt}{\cos t} = \ln \operatorname{tg} \left(\frac{\operatorname{gd} x}{2} + \frac{\pi}{4} \right).$

JA

1.491

Functional relations.

1. $\operatorname{ch} x = \sec (\operatorname{gd} x).$

AD (343.1), JA

2. $\operatorname{sh} x = \operatorname{tg} (\operatorname{gd} x).$

AD (343.2), JA

3. $e^x = \sec (\operatorname{gd} x) + \operatorname{tg} (\operatorname{gd} x) = \operatorname{tg} \left(\frac{\pi}{4} + \frac{\operatorname{gd} x}{2} \right) = \frac{1 + \sin (\operatorname{gd} x)}{\cos (\operatorname{gd} x)}.$

AD (343.5), JA

4. $\operatorname{th} x = \sin (\operatorname{gd} x).$

AD (343.3), JA

5. $\operatorname{th} \frac{x}{2} = \operatorname{tg} \left(\frac{1}{2} \operatorname{gd} x \right).$

6. $\operatorname{arctg}(\operatorname{th} x) = \frac{1}{2} \operatorname{gd} 2x.$

AD (343.6a)

1.492

If $\gamma = \operatorname{gd} x$, then $ix = \operatorname{gd} i\gamma$.

JA

1.493

Series expansion.

1. $\frac{\operatorname{gd} x}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{th}^{2k+1} \frac{x}{2}.$

JA

2. $\frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \operatorname{tg}^{2k+1} \left(\frac{1}{2} \operatorname{gd} x\right).$

JA

3. $\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \cdots$

JA

4. $x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \ldots \qquad \left[\operatorname{gd} x < \frac{\pi}{2}\right].$

JA

1.5 The Logarithm

1.51 Series representation

1.511

1.512

1. $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} \qquad [0 < x \le 2].$

2. $\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right] = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \qquad [0 < x].$

3. $\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x-1}{x}\right)^k \qquad \left[x \ge \frac{1}{2}\right].$

AD (644.6)

1.513

1. $\ln \frac{1+x}{1-x} = 2\sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1} \qquad [x^2 < 1].$

FI II 421

2. $\ln \frac{x+1}{x-1} = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)x^{2k-1}} \qquad [x^2 > 1].$

AD (644.9)

3. $\ln \frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{kx^k} \qquad [x \le -1 \quad \text{or} \quad x > 1].$

JO (88a)

4. $\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} \qquad [-1 \le x < 1].$

JO (88b)

53

5. $\frac{1-x}{x} \ln \frac{1}{1-x} = 1 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} \qquad [-1 \le x < 1].$

6. $$\frac{1}{1-x}\ln\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^{k} \frac{1}{n} \qquad [x^2 < 1].$$

JO (88e)

7. $$\frac{(1-x)^2}{2x^3}\ln\frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \qquad [-1 \le x < 1].$$

AD (6445.1)

1.514

$$\ln(1-2x\cos\varphi+x^2) = -2\sum_{k=1}^{\infty}\frac{\cos k\varphi}{k}x^k; \quad \ln(x+\sqrt{1+x^2}) = \operatorname{Arsh} x.$$

see **1.631, 1.641, 1.642, 1.646** $[x^2 \le 1, \quad x\cos\varphi \neq 1].$

MO 98, FI II 485

1.631

1.641

1.642

1.646

1.515

1. $$\ln(1+\sqrt{1+x^2}) = \ln 2 + \frac{1\cdot 1}{2\cdot 2}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}x^4 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6 - \dots ;$$
$$= \ln 2 - \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!}{2^{2k}(k!)^2}x^{2k} \qquad [x^2 \le 1].$$

JO (91)

2. $$\ln(1+\sqrt{1+x^2}) = \ln x + \frac{1}{x} - \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \dots ;$$
$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!}{2^{2k-1}\cdot k!(k-1)!(2k+1)x^{2k+1}} \qquad [x^2 \ge 1].$$

AD (644.4)

3. $$\sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) = x - \sum_{k=1}^{\infty}(-1)^k\frac{2^{2k-1}(k-1)!k!}{(2k+1)!}x^{2k+1} \qquad [x^2 \le 1].$$

JO (93)

4. $$\frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}(k!)^2}{(2k+1)!} x^{2k+1} \qquad [x^2 \le 1].$$

JO (94)

1.516

1. $$\frac{1}{2}\{\ln(1\pm x)\}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1} x^{k+1}}{k+1} \sum_{n=1}^{k} \frac{1}{n} \qquad [x^2 < 1].$$

JO (86), JO (85)

2. $$\frac{1}{6}\{\ln(1+x)\}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+2}}{k+2} \sum_{n=1}^{k} \frac{1}{n+1} \sum_{m=1}^{n} \frac{1}{m} \qquad [x^2 < 1].$$

AD (644.14)

3. $$-\ln(1+x)\cdot\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n} \qquad [x^2 < 1].$$

JO (87)

4. $$\frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}} \ln\frac{1+\sqrt{x}}{1-\sqrt{x}} + 2\ln(1-x)\right\} = \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(2k-1)2k(2k+1)} \qquad [0 < x < 1].$$

AD (6445.2)

1.517

1.[6] $$\frac{1}{2x}\left\{1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \operatorname{arctg}\sqrt{x}\right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1)2k(2k+1)} \qquad [0 < x \le 1].$$

AD (6445.3)

2. $$\frac{1}{2}\operatorname{arctg} x \ln\frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{4k-2}}{2k-1} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{2n-1} \qquad [x^2 < 1].$$

BR* 163

AD (6455.3)

1.518

1. $$\ln\sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots : \\ = \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!} \qquad [0 < x < \pi].$$

AD (643.1)a

2.[3] $$\ln\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots , \\ = -\sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!} x^{2k} = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

FI II 524

3. $$\ln\operatorname{tg} x = \ln x + \frac{x^2}{3} + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18,900}x^8 + \dots ; \\ = \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1}-1)2^{2k} B_{2k} x^{2k}}{k(2k)!} \qquad \left[0 < x < \frac{\pi}{2}\right].$$

AD (643.3)a

Power series for $\frac{\cos}{\sin}\left\{n \ln(x + \sqrt{1-x^2})\right\}$ cf. 1.414.

1.52 Series of logarithms

1.431

1.521

1. $$\sum_{k=1}^{\infty} \ln\left(1 - \frac{4x^2}{(2k-1)^2\pi^2}\right) = \ln\cos x \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right].$$

2. $$\sum_{k=1}^{\infty} \ln\left(1 - \frac{x^2}{k^2\pi^2}\right) = \ln\sin x - \ln x \qquad [0 < x < \pi].$$

1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

$$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}; \quad 0 \leq \arccos x \leq \pi \qquad [-1 \leq x \leq 1].$$

FI II 553

$$-\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}; \quad 0 < \operatorname{arcctg} x < \pi \qquad [-\infty < x < +\infty].$$

FI II 552

1.62 - 1.63 Functional relations

1.621

The relationship between the inverse and the direct trigonometric functions.

$$1. \quad \arcsin(\sin x) = x - 2n\pi \qquad \left[2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2}\right];$$
$$= -x+(2n+1)\pi \qquad \left[(2n+1)\pi - \frac{\pi}{2} \leq x \leq (2n+1)\pi + \frac{\pi}{2}\right].$$

$$2. \quad \arccos(\cos x) = x - 2n\pi \qquad [2n\pi \leq x \leq (2n+1)\pi];$$
$$= -x + 2(n+1)\pi \qquad [(2n+1)\pi \leq x \leq 2(n+1)\pi].$$

$$3. \quad \operatorname{arctg}(\operatorname{tg} x) = x - n\pi \qquad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right].$$

$$4. \quad \operatorname{arcctg}(\operatorname{ctg} x) = x - n\pi \qquad [n\pi < x < (n+1)\pi].$$

1.622

The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

1. $\arcsin z = \frac{1}{i} \ln (iz + \sqrt{1 - z^2}) = \frac{1}{i} \operatorname{Arsh} (iz)$

2. $\arccos z = \frac{1}{i} \ln (z + \sqrt{z^2 - 1}) = \frac{1}{i} \operatorname{Arch} z.$

3. $\operatorname{arctg} z = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz} = \frac{1}{i} \operatorname{Arth} (iz).$

4. $\operatorname{arcctg} z = \frac{1}{2i} \ln \frac{iz - 1}{iz + 1} = i \operatorname{Arcth} (iz).$

5. $\operatorname{Arsh} z = \ln (z + \sqrt{z^2 + 1}) = \frac{1}{i} \arcsin (iz).$

6. $\operatorname{Arch} z = \ln (z + \sqrt{z^2 - 1}) = i \arccos z.$

7. $\operatorname{Arth} z = \frac{1}{2} \ln \frac{1 + z}{1 - z} = \frac{1}{i} \operatorname{arctg} (iz).$

8. $\operatorname{Arcth} z = \frac{1}{2} \ln \frac{z + 1}{z - 1} = \frac{1}{i} \operatorname{arcctg} (-iz).$

Relations between different inverse trigonometric functions

1.623

1. $\arcsin x + \arccos x = \frac{\pi}{2}.$

2. $\operatorname{arctg} x + \operatorname{arcctg} x = \frac{\pi}{2}.$

NV 43

1.624

1. $$\arcsin x = \arccos\sqrt{1-x^2} \qquad [0 \le x \le 1];$$
$$= -\arccos\sqrt{1-x^2} \qquad [-1 \le x \le 0].$$

NV 47 (5)

2. $$\arcsin x = \operatorname{arctg}\frac{x}{\sqrt{1-x^2}} \qquad [x^2 < 1].$$

NV 46 (2)

3. $$\arcsin x = \operatorname{arcctg}\frac{\sqrt{1-x^2}}{x} \qquad [0 < x \le 1];$$
$$= \operatorname{arcctg}\frac{\sqrt{1-x^2}}{x} - \pi \qquad [-1 \le x < 0].$$

NV 49 (10)

4. $$\arccos x = \arcsin\sqrt{1-x^2} \qquad [0 \le x \le 1];$$
$$= \pi - \arcsin\sqrt{1-x^2} \qquad [-1 \le x \le 0].$$

NV 48 (6)

5. $$\arccos x = \operatorname{arctg}\frac{\sqrt{1-x^2}}{x} \qquad [0 < x \le 1];$$
$$= \pi + \operatorname{arctg}\frac{\sqrt{1-x^2}}{x} \qquad [-1 \le x < 0].$$

NV 48 (8)

6. $$\arccos x = \operatorname{arcctg}\frac{x}{\sqrt{1-x^2}} \qquad [-1 \le x < 1].$$

NV 46 (4)

7. $$\operatorname{arctg} x = \arcsin\frac{x}{\sqrt{1+x^2}}.$$

8. $\operatorname{arctg} x = \arccos \frac{1}{\sqrt{1+x^2}} \qquad [x \geq 0];$

$= -\arccos \frac{1}{\sqrt{1+x^2}} \qquad [x \leq 0].$

NV 48 (7)

57

9. $\operatorname{arctg} x = \operatorname{arcctg} \frac{1}{x} \qquad [x > 0],$

$= -\operatorname{arcctg} \frac{1}{x} - \pi \qquad [x < 0].$

NV 49 (9)

10. $\operatorname{arctg} x = \arcsin \frac{1}{\sqrt{1+x^2}} \qquad [x > 0];$

$= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} \qquad [x < 0].$

NV 49 (11)

11. $\operatorname{arcctg} x = \arccos \frac{x}{\sqrt{1+x^2}}.$

NV 46 (4)

12. $\operatorname{arcctg} x = \operatorname{arctg} \frac{1}{x} \qquad [x > 0];$

$= \pi + \operatorname{arctg} \frac{1}{x} \qquad [x < 0].$

NV 49 (12)

1.625

1. $\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad [xy \leq 0 \quad \text{or} \quad x^2+y^2 \leq 1];$

$= \pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad [x > 0, \quad y > 0 \quad \text{and} \quad x^2+y^2 > 1];$

$= -\pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad [x < 0, \quad y < 0, \quad \text{and} \quad x^2+y^2 > 1].$

NV 54(1), GI I (880)

2. $\arcsin x + \arcsin y = \arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) \qquad [x \geq 0, \quad y \geq 0];$

$= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) \qquad [x < 0, \quad y < 0].$

3. $\arcsin x + \arcsin y = \operatorname{arctg} \dfrac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} \qquad [xy \le 0 \quad \text{or} \quad x^2+y^2 < 1];$

$$= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi \qquad [x > 0, \quad y > 0 \quad \text{and} \quad x^2+y^2 > 1];$$

$$= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi \qquad [x < 0, \quad y < 0 \quad \text{and} \quad x^2+y^2 > 1].$$

NV 56

4. $\arcsin x - \arcsin y = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad [xy \ge 0 \quad \text{or} \quad x^2+y^2 \le 1];$

$$= \pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad [x > 0, \quad y < 0 \quad \text{and} \quad x^2+y^2 > 1];$$

$$= -\pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad [x < 0, \quad y > 0 \quad \text{and} \quad x^2+y^2 > 1].$$

NV 55(2)

58

5. $\arcsin x - \arcsin y = \arccos\left(x\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) \qquad [xy > y];$

$$= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) \qquad [x < y].$$

NV 56

6. $\arccos x + \arccos y = \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x + y \ge 0];$

$$= 2\pi - \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x + y < 0].$$

NV 57 (3)

7. $\arccos x - \arccos y = \arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x \ge y];$

$$= \arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x < y].$$

NV 57 (4)

8. $\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \dfrac{x+y}{1-xy} \qquad [xy < 1];$

$$= \pi + \operatorname{arctg} \frac{x+y}{1-xy} \qquad [x > 0, \quad xy > 1];$$

$$= -\pi + \operatorname{arctg} \frac{x+y}{1-xy} \qquad [x < 0, \quad xy > 1].$$

9. $\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \frac{x-y}{1+xy} \qquad [xy > -1]$

$$= \pi + \operatorname{arctg} \frac{x-y}{1+xy} \qquad [x > 0, \quad xy < -1];$$

$$= -\pi + \operatorname{arctg} \frac{x-y}{1+xy} \qquad [x < 0, \quad xy < -1].$$

NV 59(6)

1.626

1. $2 \arcsin x = \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[|x| \le \frac{1}{\sqrt{2}}\right];$

$$= \pi - \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[\frac{1}{\sqrt{2}} < x \le 1\right];$$

$$= -\pi - \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[-1 \le x < -\frac{1}{\sqrt{2}}\right].$$

NV 61 (7)

2. $2 \arccos x = \arccos \left(2x^2 - 1\right) \qquad [0 \le x \le 1];$

$$= 2\pi - \arccos \left(2x^2 - 1\right) \qquad [-1 \le x < 0].$$

NV 61 (8)

3. $2 \operatorname{arctg} x = \operatorname{arctg} \frac{2x}{1-x^2} \qquad [|x| < 1];$

$$= \operatorname{arctg} \frac{2x}{1-x^2} + \pi \qquad [x > 1];$$

$$= \operatorname{arctg} \frac{2x}{1-x^2} - \pi \qquad [x < -1].$$

NV 61 (9)

1.627

1. $\operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \frac{\pi}{2} \qquad [x > 0];$

$$= -\frac{\pi}{2} \qquad [x < 0].$$

GI I (878)

2. $\operatorname{arctg} x + \operatorname{arctg} \frac{1-x}{1+x} = \frac{\pi}{4} \qquad [x > -1];$

$$= -\frac{3}{4}\pi \qquad [x < -1].$$

1. $\arcsin \frac{2x}{1+x^2} = -\pi - 2 \operatorname{arctg} x \qquad [x \leq -1];$

$= 2 \operatorname{arctg} x \qquad [-1 \leq x \leq 1];$

$= \pi - 2 \operatorname{arctg} x \qquad [x \geq 1].$

NV 65

2. $\arccos \frac{1-x^2}{1+x^2} = 2 \operatorname{arctg} x \qquad [x \geq 0];$

$= -2 \operatorname{arctg} x \qquad [x \leq 0].$

NV 66

1.629

$$\frac{2x-1}{2} - \frac{1}{\pi} \operatorname{arctg} \left(\operatorname{tg} \frac{2x-1}{2} \pi \right) = E(x).$$

GI (886)

1.631

Relations between the inverse hyperbolic functions.

1. $\operatorname{Arsh} x = \operatorname{Arch} \sqrt{x^2+1} = \operatorname{Arth} \frac{x}{\sqrt{x^2+1}}.$

JA

2. $\operatorname{Arch} x = \operatorname{Arsh} \sqrt{x^2-1} = \operatorname{Arth} \frac{\sqrt{x^2-1}}{x}.$

JA

3. $\operatorname{Arth} x = \operatorname{Arsh} \frac{x}{\sqrt{1-x^2}} = \operatorname{Arch} \frac{1}{\sqrt{1-x^2}} = \operatorname{Arcth} \frac{1}{x}.$

JA

4. $\operatorname{Arsh} x \pm \operatorname{Arsh} y = \operatorname{Arsh} \left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right).$

JA

5. $\operatorname{Arch} x \pm \operatorname{Arch} y = \operatorname{Arch} \left(xy \pm \sqrt{(x^2-1)(y^2-1)}\right).$

6. $\operatorname{Arth} x \pm \operatorname{Arth} y = \operatorname{Arth} \frac{x \pm y}{1 \pm xy}.$

JA

1.64 Series representations

1.641

1. $$\arcsin x = \frac{\pi}{2} - \arccos x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots ;$$
$$= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) \qquad [x^2 \leq 1].$$

FI II 479

2. $$\operatorname{Arsh} x = x - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots ;$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1};$$
$$= xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2\right) \qquad [x^2 \leq 1].$$

FI II 480

1.642

1. $$\operatorname{Arsh} x = \ln 2x + \frac{1}{2} \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4x^4} + \dots ;$$
$$= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \qquad [x \geq 1].$$

AD (6480.2)a

2. $$\operatorname{Arch} x = \ln 2x - \sum_{k=1}^{\infty} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \qquad [x \geq 1].$$

AD (6480.3)a

1.643

1. $$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots ;$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad [x^2 \leq 1].$$

2. $\operatorname{Arth} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \qquad [x^2 < 1].$

AD (6480.4)

1.644

1. $$\begin{aligned} \operatorname{arctg} x &= \frac{x}{\sqrt{1+x^2}} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{x^2}{1+x^2}\right)^k \\ &= \frac{x}{\sqrt{1+x^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{1+x^2}\right) \qquad [x^2 < \infty] \end{aligned}$$

AD (641.3)

2. $$\begin{aligned} \operatorname{arctg} x &= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \\ &= \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)x^{2k+1}} \qquad [x^2 \geq 1] \qquad (\text{see also } \mathbf{1.643}). \end{aligned}$$

AD (641.4)

1.643

1.645

1. $$\begin{aligned} \operatorname{arcsec} x &= \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2 \cdot 3x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \cdots = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)! x^{-(2k+1)}}{(k!)^2 2^{2k}(2k+1)}; \\ &= \frac{\pi}{2} - \frac{1}{x} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{x^2}\right) \qquad [x^2 > 1]. \end{aligned}$$

AD (641.5)

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2. $(\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 x^{2k+2}}{(2k+1)!(k+1)} \qquad [x^2 \leq 1].$

AD (642.2), GI III (152)a

3. $(\arcsin x)^3 = x^3 + \frac{3!}{5!} 3^2 \left(1 + \frac{1}{3^2}\right) x^5 + \frac{3!}{7!} 3^2 \cdot 5^2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right) x^7 + \ldots \qquad [x^2 \leq 1].$

BR* 188, AD (642.2), GI III (153)a

1.646

1. $$\operatorname{Arsh}\frac{1}{x} = \operatorname{Arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}(k!)^2(2k+1)} x^{-2k-1} \qquad [x^2 \geq 1].$$

AD (6480.5)

2. $$\operatorname{Arch}\frac{1}{x} = \operatorname{Arsech} x = \ln\frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2 2k} x^{2k} \qquad [0 < x \leq 1].$$

AD (6480.6)

3. $$\operatorname{Arsh}\frac{1}{x} = \operatorname{Arcosech} x = \ln\frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k)!}{2^{2k}(k!)^2 2k} x^{2k} \qquad [0 < x \leq 1].$$

AD (6480.7)a

4. $$\operatorname{Arth}\frac{1}{x} = \operatorname{Arcth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1} \qquad [x^2 > 1].$$

AD (6480.8)

1.647* Due to T. Harrett.

1. $$T_n = \sum_{k=1}^{\infty} \frac{\tanh\,(2k-1)(\pi/2)}{(2k-1)]^{4n+3}} = \\ = \frac{\pi^{4n+3}}{2}\left(2\sum_{j=1}^{n} \frac{(-1)^{j-1}(2^{2j}-1)(2^{4n-2j+4}-1)B^*_{2j-1}B^*_{4n-2j+3}}{(2j)!(4n-2j+4)!} + \right. \\ \left. + \frac{(-1)^n(2^{2n+2}-1)^2 B^{*2}_{2n+1}}{[(2n+2)!]^2}\right), \quad n = 0, 1, 2, \ldots,$$

2. $$S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\operatorname{sech}\,(2k-1)(\pi/2)}{(2k-1)^{4n+1}} = \\ = \frac{\pi^{4n+1}}{2^{4n+3}}\left(2\sum_{j=1}^{n-1} \frac{(-1)^j B^*_{2j}B^*_{4n-2j}}{(2j)!(4n-2j)!} + \frac{2B^*_{4n}}{(4n)!} + \frac{(-1)^n B^{*2}_{2n}}{[(2n)!]^2}\right), \qquad n = 1, 2, \ldots.$$

(Summation term on right to be omitted for n=1.) (See page xxix for definition of B^*_r.)

2. Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign $(=)$ means that the functions on the left and right of this symbol differ by a constant. For example (see 2.01 15.), we write

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arcctg} x,$$

although

$$\operatorname{arctg} x = -\operatorname{arcctg} x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example, $\int \frac{dx}{\sqrt{1+x^2}} = \ln|x+\sqrt{1+x^2}|$). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula 2.02 8. for $n=-1$ or formula 2.02 15. for $a=b$). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters $x,\ y,\ t,\dots$ denote independent variables; $f,\ g,\ \varphi,\dots$ denote functions of $x,\ y,\ t,\dots$;
$f',\ g',\ \varphi',\dots,\ f'',\ g'',\ \varphi'',\dots$ denote their first, second, etc., derivatives; $a,\ b,\ m,\ p,\dots$ denote constants, by which we

$x,\, y,\, t, \dots$ $f,\, g,\, \varphi, \dots$ $x,\, y,\, t, \dots$

$f',\, g',\, \varphi', \dots ,\, f'',\, g'',\, \varphi'', \dots$ $a,\, b,\, m,\, p, \dots$

generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is

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made provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas 2.148 4. and 2.424 6., we make no remark since it is clear from the form of these formulas themselves that n must be a natural number (that is, a positive integer).

2.01 The basic integrals

1. $$\int x^n\, dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1).$$

For $n = -1$

2. $$\int \frac{dx}{x} = \ln x.$$

3. $$\int e^x\, dx = e^x.$$

4. $$\int a^x\, dx = \frac{a^x}{\ln a}.$$

5. $$\int \sin x\, dx = -\cos x.$$

6. $$\int \cos x\, dx = \sin x.$$

7. $$\int \frac{dx}{\sin^2 x} = -\text{ctg}\, x.$$

8. $$\int \frac{dx}{\cos^2 x} = \text{tg}\, x.$$

9. $$\int \frac{\sin x}{\cos^2 x}\, dx = \sec x.$$

10. $$\int \frac{\cos x}{\sin^2 x}\, dx = -\text{cosec}\, x.$$

11. $$\int \text{tg}\, x\, dx = -\ln \cos x.$$

12. $$\int \text{ctg}\, x\, dx = \ln \sin x.$$

20. $\int \operatorname{sh} x\, dx = \operatorname{ch} x.$

21. $\int \operatorname{ch} x\, dx = \operatorname{sh} x.$

22. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$

23. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$

24. $\int \operatorname{th} x\, dx = \ln \operatorname{ch} x.$

25. $\int \operatorname{cth} x\, dx = \ln \operatorname{sh} x.$

26. $\int \frac{dx}{\operatorname{sh} x} = \ln \operatorname{th} \frac{x}{2}.$

2.02 General formulas

1. $\int af\, dx = a\int f\, dx.$

2. $\int [af \pm b\varphi \pm c\psi \pm \dots]\, dx = a\int f\, dx \pm b\int \varphi\, dx \pm c\int \psi\, dx \pm \dots$

3. $\frac{d}{dx}\int f\, dx = f.$

4. $\int f'\, dx = f.$

5. $\int f'\varphi\, dx = f\varphi - \int f\varphi'\, dx$ [integration by parts].

6. $\int f^{(n+1)}\varphi\, dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \dots + (-1)^n \varphi^{(n)} f + (-1)^{n+1}\int \varphi^{(n+1)} f\, dx.$

7. $\displaystyle\int f(x)\,dx = \int f[\varphi(y)]\varphi'(y)\,dy \qquad [x = \varphi(y)] \qquad \text{[change of variable]}.$

8. $\displaystyle\int (f)^n f'\,dx = \frac{(f)^{n+1}}{n+1}.$

For $n = -1$

$$\int \frac{f'\,dx}{f} = \ln f.$$

9. $\displaystyle\int (af+b)^n f'\,dx = \frac{(af+b)^{n+1}}{a(n+1)}.$

10. $\displaystyle\int \frac{f'\,dx}{\sqrt{af+b}} = \frac{2\sqrt{af+b}}{a}.$

11. $\displaystyle\int \frac{f'\varphi - \varphi' f}{\varphi^2}\,dx = \frac{f}{\varphi}.$

12. $\displaystyle\int \frac{f'\varphi - \varphi' f}{f\varphi}\,dx = \ln\frac{f}{\varphi}.$

13. $\displaystyle\int \frac{dx}{f(f\pm\varphi)} = \pm\int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi(f\pm\varphi)}.$

14. $\displaystyle\int \frac{f'\,dx}{\sqrt{f^2+a}} = \ln(f + \sqrt{f^2+a}).$

For $a = b$

$$\int \frac{f\,dx}{(f+a)^2} = \int \frac{dx}{f+a} - a\int \frac{dx}{(f+a)^2}.$$

16. $$\int \frac{f\,dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi\,dx}{(f+\varphi)^n}.$$

17. $$\int \frac{f'\,dx}{p^2+q^2f^2} = \frac{1}{pq}\operatorname{arctg}\frac{qf}{p}.$$

18. $$\int \frac{f'\,dx}{q^2f^2-p^2} = \frac{1}{2pq}\ \ln\frac{qf-p}{qf+p}.$$

19. $$\int \frac{f\,dx}{1-f} = -x + \int \frac{dx}{1-f}.$$

20. $$\int \frac{f^2\,dx}{f^2-a^2} = \frac{1}{2}\int \frac{f\,dx}{f-a} + \frac{1}{2}\int \frac{f\,dx}{f+a}.$$

21. $$\int \frac{f'\,dx}{\sqrt{a^2-f^2}} = \arcsin\frac{f}{a}.$$

22. $$\int \frac{f'\,dx}{af^2+bf} = \frac{1}{b}\ \ln\frac{f}{af+b}.$$

23. $$\int \frac{f'\,dx}{f\sqrt{f^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{f}{a}.$$

24. $$\int \frac{(f'\varphi - f\varphi')\,dx}{f^2+\varphi^2} = \operatorname{arctg}\frac{f}{\varphi}.$$

25. $\displaystyle\int \frac{(f'\varphi - f\varphi')\,dx}{f^2 - \varphi^2} = \frac{1}{2}\,\ln\frac{f-\varphi}{f+\varphi}.$

2.1 Rational Functions

2.10 General integration rules

2.101

To integrate an arbitrary rational function $\frac{F(x)}{f(x)}$, where $F(x)$ and $f(x)$ are polynomials with no common factors, we first need to separate out the integral part $E(x)$ (where $E(x)$ is a polynomial), if there is an integral part, and then to integrate separately the integral part and the remainder, thus:

$$\int \frac{F(x)\,dx}{f(x)} = \int E(x)\,dx + \int \frac{\varphi(x)}{f(x)}\,dx.$$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions*

2.102

If $a,\ b,\ c, \dots, m$ are roots of the equation $f(x) = 0$ and if $\alpha, \beta, \gamma, \dots, \mu$ are their corresponding multiplicities, so that $f(x) = (x-a)^\alpha (x-b)^\beta \dots (x-m)^\mu$ then, $\frac{\varphi(x)}{f(x)}$ can be decomposed into the following partial fractions:

$$\frac{\varphi(x)}{f(x)} = \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \dots + \frac{A_1}{x-a} + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \dots + \frac{B_1}{x-b} + \dots + \\ + \frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \dots + \frac{M_1}{x-m},$$

where the numerators of the individual fractions are determined by the following formulas:

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \qquad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \dots, M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \qquad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \ldots, M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$\psi_1(x) = \frac{\varphi(x)(x-a)^\alpha}{f(x)}, \qquad \psi_2(x) = \frac{\varphi(x)(x-b)^\beta}{f(x)}, \ldots, \psi_m(x) = \frac{\varphi(x)(x-m)^\mu}{f(x)}.$$

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If $a, b, \ldots, m$ are simple roots, that is, if $\alpha = \beta = \ldots = \mu$=1, then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \qquad B = \frac{\varphi(b)}{f'(b)}, \qquad \ldots, \qquad M = \frac{\varphi(m)}{f'(m)}.$$

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If some of the roots of the equation $f(x) = 0$ are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1x + N_1}{x^2 + 2Bx + C} + \frac{M_2x + N_2}{(x^2 + 2Bx + C)^2} + \ldots + \frac{M_px + N_p}{(x^2 + 2Bx + C)^p}.$$

2.103

Thus, the integration of a proper rational fraction $\dfrac{\varphi(x)}{f(x)}$ reduces to integrals of the form $\int \frac{g\,dx}{(x-a)^\alpha}$ or $\int \frac{Mx+N}{(A+2Bx+Cx^2)^p}\,dx$. Fractions of the first form yield rational functions for $\alpha > 1$ and logarithms for $\alpha = 1$. Fractions of the second form yield rational functions and logarithms or arctangents:

1. $$\int \frac{g\,dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}.$$

2. $$\int \frac{g\,dx}{x-a} = g \int \frac{d(x-a)}{x-a} = g\,\ln|x-a|.$$

3. $$\int \frac{Mx+N}{(A+2Bx+Cx^2)^p}\,dx = \frac{NB - MA + (NC - MB)x}{2(p-1)(AC-B^2)(A+2Bx+Cx^2)^{p-1}} + \frac{(2p-3)(NC-MB)}{2(p-1)(AC-B^2)} \int \frac{dx}{(A+2Bx+Cx^2)^{p-1}}.$$

4. $$\int \frac{dx}{A+2Bx+Cx^2} = \frac{1}{\sqrt{AC-B^2}} \operatorname{arctg} \frac{Cx+B}{\sqrt{Ac-B^2}} \qquad [AC > B^2];$$
$$= \frac{1}{2\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| \qquad [AC < B^2].$$

5. $$\int \frac{(Mx+N)\,dx}{A+2Bx+Cx^2} = \frac{M}{2C} \ln|A+2Bx+Cx^2| + \frac{NC-MB}{C\sqrt{AC-B^2}} \operatorname{arctg} \frac{Cx+B}{\sqrt{AC-B^2}} \qquad [AC > B^2];$$
$$= \frac{M}{2C} \ln|A+2Bx+Cx^2| + \frac{NC-MB}{2C\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| \qquad [AC < B^2].$$

The ostrogradskiy-hermite method

2.104

By means of the Ostrogradskiy-Hermite method, we can find the rational part of $\int \frac{\varphi(x)}{f(x)}\,dx$ without finding the roots of the equation $f(x) = 0$ and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)}\,dx = \frac{M}{D} + \int \frac{N\,dx}{Q}.$$

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Here, M, N, D, and Q are rational functions of x. Specifically, D is the greatest common divisor of the function $f(x)$ and its derivative $f'(x)$; $Q = \frac{f(x)}{D}$; M is a polynomial of degree no higher than $m-1$, where m is the degree of the polynomial D; N is a poynomial of degree no higher than $n-1$, where n is the degree of the polynomial Q. The coefficients of the polynomials M and N are determined by equating the coefficients of like powers of x in the following identity:

$$\varphi(x) = M'Q - M(T-Q') + ND$$

where $T = \frac{f'(x)}{D}$ and M' and Q' are the derivatives of the polynomials M and Q.

2.11-2.13 Forms containing the binomial a + bx^k

2.110

Reduction formulas for $z = a + bx^k$

LA 126(4)

2. $$\int x^n z_k^m \, dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} \, dx.$$

LA 126 (6)

3. $$\int x^n z_k^m \, dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} \, dx.$$

LA 125 (1)

4. $$\int x^n z_k^m \, dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} \, dx.$$

LA 125 (2)

5. $$\int x^n z_k^m \, dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m \, dx.$$

LA 126 (3)

6. $$\int x^n z_k^m \, dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m \, dx.$$

LA 126 (5)

Forms containing the binomial $z_1 = a + bx$

2.111

1. $$\int z_1^m \, dx = \frac{z_1^{m+1}}{b(m+1)}.$$

For $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1.$$

2. $$\int \frac{x^n\,dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b}\int \frac{x^{n-1}\,dx}{z_1^m}.$$

For $n = m - 1$, we may use the formula

3.[8] $$\int \frac{x^{m-1}\,dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b}\int \frac{x^{m-2}\,dx}{z_1^{m-1}}.$$

For $m = 1$

$$\int \frac{x^n\,dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2x^{n-2}}{(n-2)b^3} - \ldots + (-1)^{n-1}\frac{a^{n-1}x}{1\cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}}\ \ln z_1.$$

4. $$\int \frac{x^n\,dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1}\frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1}\frac{a^n}{b^{n+1}z_1} + (-1)^{n+1}\frac{na^{n-1}}{b^{n+1}}\ \ln z_1.$$

2.112

1. $$\int \frac{x\,dx}{z_1} = \frac{x}{b} - \frac{a}{b^2}\ \ln z_1.$$

2. $$\int \frac{x^2\,dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3}\ \ln z_1.$$

2.113

1. $$\int \frac{dx}{z_1^2} = -\frac{1}{bz_1}.$$

2. $\displaystyle\int \frac{x\,dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2}\ \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2}\ \ln z_1.$

3. $\displaystyle\int \frac{x^2\,dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3}\ \ln z_1.$

2.114

1. $\displaystyle\int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}.$

2. $\displaystyle\int \frac{x\,dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2}\right] \frac{1}{z_1^2}.$

3. $\displaystyle\int \frac{x^2\,dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right] \frac{1}{z_1^2} + \frac{1}{b^3}\ \ln z_1.$

4[6]. $\displaystyle\int \frac{x^3\,dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2}x^2 - 2\frac{a^2}{b^3}x - \frac{5}{2}\frac{a^3}{b^4}\right] \frac{1}{z_1^2} - 3\frac{a}{b^4}\ \ln z_1.$

70

2.115

1. $\displaystyle\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}.$

2. $\displaystyle\int \frac{x\,dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right] \frac{1}{z_1^3}.$

3. $\displaystyle\int \frac{x^2\,dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right] \frac{1}{z_1^3}.$

4. $\displaystyle\int \frac{x^3\,dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right] \frac{1}{z_1^3} + \frac{1}{b^4}\ \ln z_1.$

1. $$\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}.$$

2. $$\int \frac{x\,dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right]\frac{1}{z_1^4}.$$

3. $$\int \frac{x^2\,dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right]\frac{1}{z_1^4}.$$

4. $$\int \frac{x^3\,dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right]\frac{1}{z_1^4}.$$

2.117

1. $$\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)}\int \frac{dx}{x^{n-1}z_1^m}.$$

2. $$\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}.$$

3. $$\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a}\int \frac{dx}{xz_1^{m-1}}.$$

4. $$\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n}\ \ln \frac{z_1}{x}.$$

2.118

1. $$\int \frac{dx}{xz_1} = -\frac{1}{a}\ \ln \frac{z_1}{x},$$

2.119

1. $\int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2} \ln \frac{z_1}{x}.$

2. $\int \frac{dx}{x^2z_1^2} = -\left[\frac{1}{ax} + \frac{2b}{a^2}\right] \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{z_1}{x}.$

3. $\int \frac{dx}{x^3z_1^2} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3}\right] \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{z_1}{x}.$

2.121

1. $\int \frac{dx}{xz_1^3} = \left[\frac{3}{2a} + \frac{bx}{a^2}\right] \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{z_1}{x}.$

2. $\int \frac{dx}{x^2z_1^3} = -\left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3}\right] \frac{1}{z_1^2} + \frac{3b}{a^4} \ln \frac{z_1}{x}.$

3. $\int \frac{dx}{x^3z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4}\right] \frac{1}{z_1^2} - \frac{6b^2}{a^5} \ln \frac{z_1}{x}.$

2.122

1. $\int \frac{dx}{xz_1^4} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3}\right] \frac{1}{z_1^3} - \frac{1}{a^4} \ln \frac{z_1}{x}.$

2. $\int \frac{dx}{x^2z_1^4} = -\left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4}\right] \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{z_1}{x}.$

3. $\int \frac{dx}{x^3z_1^4} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5}\right] \frac{1}{z_1^3} - \frac{10b^2}{a^6} \ln \frac{z_1}{x}.$

1. $\int \frac{dx}{xz^5} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4}\right]\frac{1}{z_1^4} - \frac{1}{a^5}\ln\frac{z_1}{x}.$

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2. $\int \frac{dx}{x^2z_1^5} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2x}{3a^3} - \frac{35b^3x^2}{2a^4} - \frac{5b^4x^3}{a^5}\right]\frac{1}{z_1^4} + \frac{5b}{a^6}\ln\frac{z_1}{x}.$

3. $\int \frac{dx}{x^3z_1^5} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2x} + \frac{125b^2}{4a^3} + \frac{65b^3x}{a^4} + \frac{105b^4x^2}{2a^5} + \frac{15b^5x^3}{a^6}\right]\frac{1}{z_1^4} - \frac{15b^2}{a^7}\ln\frac{z_1}{x}.$

2.124

Forms containing the binomial $z_2 = a + bx^2$

1. $\int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}}\operatorname{arctg} x\sqrt{\frac{b}{a}} \qquad [ab > 0] \qquad$ (see also **2.141** 2.);

$= \frac{1}{2i\sqrt{ab}}\ln\frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \qquad [ab < 0] \qquad$ (see also **2.143** 2. and **2.143** 3.).

2.143

2.141

2. $\int \frac{x\,dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \qquad$ (see also **2.145** 2., **2.145** 6. and **2.18**).

2.18

2.145

Forms containing the binomial $z_3 = a + bx^3$

Notation: $\alpha = \sqrt[3]{\frac{a}{b}}$

2.125

1. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3}\,dx}{z_3^m}.$$

2. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)} \int \frac{x^n\,dx}{z_3^{m-1}}.$$

LA 133 (1)

2.126

1. $$\int \frac{dx}{z_3} = \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\operatorname{arctg}\frac{x\sqrt{3}}{2\alpha - x}\right\};$$
$$= \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\operatorname{arctg}\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$ (see also **2.141** 3. and **2.143** 4.).

2.143

2.141

2. $$\int \frac{x\,dx}{z_3} = -\frac{1}{3b\alpha}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} - \sqrt{3}\operatorname{arctg}\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$ (see also **2.145** 3. and **2.145** 7 .).

2.145

2.145

3. $$\int \frac{x^2\,dx}{z_3} = \frac{1}{3b}\ln(1+x^3\alpha^{-3}) = \frac{1}{3b}\ln z_3.$$

4. $$\int \frac{x^3\,dx}{z_3} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{z_3}$$ (see **2.126** 1.).

2.126

5. $$\int \frac{x^4\,dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b}\int \frac{x\,dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 2.\text{)}.$$

2.126

2.127

1. $$\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a}\int \frac{dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 1.\text{)}.$$

2.126

2. $$\int \frac{x\,dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a}\int \frac{x\,dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 2.\text{)}.$$

2.126

3. $$\int \frac{x^2\,dx}{z_3^2} = -\frac{1}{3bz_3}.$$

4. $$\int \frac{x^3\,dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b}\int \frac{dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 1.\text{)}.$$

2.126

2.128

1. $$\int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1} z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)}\int \frac{dx}{x^{n-3} z_3^m}.$$

2. $$\int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1} z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)}\int \frac{dx}{x^n z_3^{m-1}}.$$

2.129

1. $$\int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}.$$

2. $$\int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x\,dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 2.\text{)}.$$

2.126

3. $$\int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 1.\text{)}.$$

2.126

2.131

1. $$\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}$$

2. $$\int \frac{dx}{x^2 z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2}\right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x\,dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 2.\text{)}.$$

2.126

3. $$\int \frac{dx}{x^3 z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2}\right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3} \qquad \text{(see } \mathbf{2.126}\ 1.\text{)}.$$

2.126

Forms containing the binomial $z_4 = A + Bx^4$

Notations: $\alpha = \sqrt[4]{\frac{a}{b}} \quad \alpha' = \sqrt[4]{\frac{-a}{b}}$

1.[8] $$\int \frac{dx}{z_4} = \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x\sqrt{2} + \alpha^2}{x^2 - \alpha x\sqrt{2} + \alpha^2} + 2 \operatorname{arctg} \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\} \qquad [ab > 0] \qquad \text{(see also } \mathbf{2.141}\ 4.).$$
$$= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2 \operatorname{arctg} \frac{x}{\alpha'} \right\} \qquad [ab < 0] \qquad \text{(see also } \mathbf{2.143}\ 5.).$$

2.143

2.141

2. $$\int \frac{x\,dx}{z} = \frac{1}{2\sqrt{ab}} \operatorname{arctg} x^2 \sqrt{\frac{b}{a}} \qquad [ab > 0] \qquad \text{(see also } \mathbf{2.145}\ 4.).$$
$$= \frac{1}{4i\sqrt{ab}} \ln \frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}} \qquad [ab < 0] \qquad \text{(see also } \mathbf{2.145}\ 8.).$$

2.145

3. $$\int \frac{x^2\,dx}{z_4} = \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x\sqrt{2} + \alpha^2}{x^2 + \alpha x\sqrt{2} + \alpha^2} + 2 \operatorname{arctg} \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\} \qquad [ab > 0];$$
$$= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2 \operatorname{arctg} \frac{x}{\alpha'} \right\} \qquad [ab < 0].$$

4. $$\int \frac{x^3\,dx}{z_4} = \frac{1}{4b} \ln z_4$$

2.133

1. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m - n - 5}{4a(m-1)} \int \frac{x^n\,dx}{z_4^{m-1}}$$

LA 134 (1)

2. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n + 1 - 4m)b} - \frac{(n-3)a}{b(n + 1 - 4m)} \int \frac{x^{n-4}\,dx}{z_4^m}.$$

2.134

2. $$\int \frac{x\,dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a}\int \frac{x\,dx}{z_4} \qquad (\text{see } \mathbf{2.132}\ 2\,.).$$

2.132

3. $$\int \frac{x^2\,dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a}\int \frac{x^2\,dx}{z_4} \qquad (\text{see } \mathbf{2.132}\ 3\,.).$$

2.132

4. $$\int \frac{x^3\,dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}.$$

2.135

$$\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1} z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a}\int \frac{dx}{x^{n-4} z_4^m}.$$

For $n = 1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a}\int \frac{dx}{xz_4^{m-1}} - \frac{b}{a}\int \frac{dx}{x^{-3} z_4^m}.$$

2.136

1. $$\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a}\ln\frac{x^4}{z_4}.$$

2. $$\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x^2\,dx}{z_4} \qquad (\text{see } \mathbf{2.132}\ 3\,.).$$

2.132

2.14 Forms containing the binomial 1 ± x^n

2.141

1. $\int \frac{dx}{1+x} = \ln(1+x).$

2. $\int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arcctg} x$ (see also **2.124** 1.).

2.124

3. $\int \frac{dx}{1+x^3} = \frac{1}{3}\ln\frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x\sqrt{3}}{2-x}$ (see also **2.126** 1.)

2.126

4. $\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{1-x^2}$ (see also **2.132** 1.).

2.132

2.142

$$\int \frac{dx}{1+x^n} = -\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} P_k \cos\frac{2k+1}{n}\pi + \frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} Q_k \sin\frac{2k+1}{n}\pi \qquad [n\text{—a positive even number}];$$

$$= \frac{1}{n}\ln(1+x) - \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k \cos\frac{2k+1}{n}\pi + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k \sin\frac{2k+1}{n}\pi$$

[n—a positive odd number].

TI (45)
TI (43)a

2.143

1. $\int \frac{dx}{1-x} = -\ln(1-x)$.

2. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} = \operatorname{Arth} x \qquad [-1 < x < 1] \qquad$ (see also **2.141** 1.).

2.141

3. $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\frac{x-1}{x+1} = -\operatorname{Arcth} x \qquad [x > 1, \quad x < -1]$.

4. $\int \frac{dx}{1-x^3} = \frac{1}{3}\ln\frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x\sqrt{3}}{2+x} \qquad$ (see also **2.126** 1.).

2.126

5. $\int \frac{dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x}{1-x} + \frac{1}{2}\operatorname{arctg} x = \frac{1}{2}(\operatorname{Arth} x + \operatorname{arctg} x) \qquad$ (see also **2.132** 1.)

2.132

2.144

1. $\int \frac{dx}{1-x^n} = \frac{1}{n}\ln\frac{1+x}{1-x} - \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} P_k \cos\frac{2k}{n}\pi + \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} Q_k \sin\frac{2k}{n}\pi$ [n—a positive even number].

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\frac{2k}{n}\pi + 1\right), \qquad Q_k = \operatorname{arctg}\frac{x - \cos\frac{2k}{n}\pi}{\sin\frac{2k}{n}\pi}.$$

2. $\int \frac{dx}{1-x^n} = -\frac{1}{n}\ln(1-x) + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k \cos\frac{2k+1}{n}\pi + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k \sin\frac{2k+1}{n}\pi$
[n—a positive odd number].

$$P_k = \frac{1}{2}\ln\left(x^2 + 2x\cos\frac{2k+1}{n}\pi + 1\right), \qquad Q_k = \operatorname{arctg}\frac{x + \cos\frac{2k+1}{n}\pi}{\sin\frac{2k+1}{n}\pi}.$$

1. $\displaystyle\int \frac{x\,dx}{1+x} = x - \ln(1+x).$

2. $\displaystyle\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2).$

3. $\displaystyle\int \frac{x\,dx}{1+x^3} = -\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$ (see also **2.126** 2.).

2.126

4. $\displaystyle\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\operatorname{arctg} x^2.$

5. $\displaystyle\int \frac{x\,dx}{1-x} = -\ln(1-x) - x.$

6. $\displaystyle\int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\ln(1-x^2).$

7. $\displaystyle\int \frac{x\,dx}{1-x^3} = -\frac{1}{6}\ln\frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$ (see also **2.126** 2.).

2.126

8. $\displaystyle\int \frac{x\,dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x^2}{1-x^2}$ (see also **2.132** 2.).

2.132

2.146

For *m* and *n*—natural numbers.

1. $$\int \frac{x^{m-1}\,dx}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n}\ln\left\{1 - 2x\cos\frac{2k-1}{2n}\pi + x^2\right\} + \\ + \frac{1}{n}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n}\operatorname{arctg}\frac{x-\cos\frac{2k-1}{2n}\pi}{\sin\frac{2k-1}{2n}\pi} \qquad [m < 2n].$$

2. $$\int \frac{x^{m-1}\,dx}{1+x^{2n+1}} = (-1)^{m+1}\frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left\{1-2x\cos\frac{2k-1}{2n+1}\pi + x^2\right\} + \\ + \frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\operatorname{arctg}\frac{x-\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi} \qquad [m \le 2n].$$

TI (46)a

3. $$\int \frac{x^{m-1}\,dx}{1-x^{2n}} = \frac{1}{2n}\{(-1)^{m+1}\ln(1+x) - \ln(1-x)\} - \frac{1}{2n}\sum_{k=1}^{n-1}\cos\frac{km\pi}{n}\ln\left(1-2x\cos\frac{k\pi}{n}+x^2\right) + \\ + \frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{km\pi}{n}\operatorname{arctg}\frac{x-\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}} \qquad [m < 2n].$$

TI (48)

4. $$\int \frac{x^{m-1}\,dx}{1-x^{2n+1}} = -\frac{1}{2n+1}\ln(1-x) + \\ + (-1)^{m+1}\frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left(1+2x\cos\frac{2k-1}{2n+1}\pi + x^2\right) + \\ + (-1)^{m+1}\frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\operatorname{arctg}\frac{x+\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi} \qquad [m \le 2n].$$

TI (50)

2.147

1. $$\int \frac{x^m\,dx}{1-x^{2n}} = \frac{1}{2}\int\frac{x^m\,dx}{1-x^n} + \frac{1}{2}\int\frac{x^m\,dx}{1+x^n}.$$

2. $$\int \frac{x^m\,dx}{(1+x^2)^n} = -\frac{1}{2n-m-1}\cdot\frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1}\int\frac{x^{m-2}\,dx}{(1+x^2)^n}.$$

LA 139 (28)

3. $$\int \frac{x^m}{1+x^2}\,dx = \frac{x^{m-1}}{m-1} - \int\frac{x^{m-2}}{1+x^2}\,dx.$$

LA 139 (33)

5. $$\int \frac{x^m\,dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2}\,dx}{1-x^2}.$$

2.148

1. $$\int \frac{dx}{x^m(1+x^2)^n} = -\frac{1}{m-1}\frac{1}{x^{m-1}(1+x^2)^{n-1}} - \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}(1+x^2)^n}.$$

For $m=1$

$$\int \frac{dx}{x(1+x^2)^n} = \frac{1}{2n-2}\frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x(1+x^2)^{n-1}}.$$

For $m=1$ and $n=1$

$$\int \frac{dx}{x(1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}}.$$

LA 139 (31)

LA 139 (29)

2. $$\int \frac{dx}{x^m(1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2}(1+x^2)}.$$

3. $$\int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2}\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)^{n-1}}.$$

FI II 40

4. $$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!}\operatorname{arctg} x.$$

TI (91)

2.149

1. $$\int \frac{dx}{x^m(1-x^2)^n} = -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1}} + \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}(1-x^2)^n}.$$

For $m=1$

$$\int \frac{dx}{x(1-x^2)^n} = \frac{1}{2(n-1)(1-x^2)^{n-1}} + \int \frac{dx}{x(1-x^2)^{n-1}}.$$

For $m=1$ and $n=1$

$$\int \frac{dx}{x(1-x^2)} = \ln \frac{x}{\sqrt{1-x^2}}.$$

2. $$\int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2}\frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1-x^2)^{n-1}}.$$

LA 139 (35)

3. $$\int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n\cdot(n-1)!}\ln\frac{1+x}{1-x}.$$

TI (91)

2.15 Forms containing pairs of binomials: a + bx and α + βx

Notations: $z = a + bx$; $t = \alpha + \beta x$; $\Delta = a\beta - \alpha b$

2.151

$$\int z^n t^m\,dx = \frac{z^{n+1}t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b}\int z^n t^{m-1}\,dx.$$

2.152

1. $$\int \frac{z}{t}\,dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2}\ln t.$$

2. $$\int \frac{t}{z}\,dx = \frac{\beta x}{b} - \frac{\Delta}{b^2}\ln z.$$

2.153

$$\int \frac{t^m\,dx}{z^n} = \frac{1}{(m-n+1)b}\frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b}\int \frac{t^{m-1}\,dx}{z^n};$$
$$= \frac{1}{(n-1)\Delta}\frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta}\int \frac{t^m\,dx}{z^{n-1}};$$
$$= -\frac{1}{(n-1)b}\frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b}\int \frac{t^{m-1}}{z^{n-1}}\,dx.$$

2.154

$$\int \frac{dx}{zt} = \frac{1}{\Delta}\ln\frac{t}{z}.$$

$$\int \frac{dx}{z^n t^m} = -\frac{1}{(m-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{t^{m-1}z^n};$$
$$= \frac{1}{(n-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta}\int \frac{dx}{t^m z^{n-1}}.$$

2.156

$$\int \frac{x\,dx}{zt} = \frac{1}{\Delta}\left(\frac{a}{b}\ln z - \frac{\alpha}{\beta}\ln t\right).$$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160

Reduction formulas for $R_k a + bx^k + cx^{2k}$

1. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma}\int x^{m+k-1}R_k^n\,dx - \frac{(m+2k+2kn)c}{ma}\int x^{m+2k-1}R_k^n\,dx.$$

2. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m}\int x^{m+k-1}R_k^{n-1}\,dx - \frac{2ckn}{m}\int x^{m+2k-1}R_k^{n-1}\,dx.$$

3. $$\int x^{m-1}R_k^n\,dx = \frac{x^{m-2k}R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c}\int x^{m-2k-1}R_k^n\,dx - \frac{(m-k+kn)b}{(m+2kn)c}\int x^{m-k-1}R_k^n\,dx;$$
$$= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn}\int x^{m-1}R_k^{n-1}\,dx + \frac{bkn}{m+2kn}\int x^{m+k-1}R_k^{n-1}\,dx.$$

2.161

Forms containing the trinomial $R_2 = a + bx^2 + cx^4$

$$\text{Notations:}\quad f = \frac{b}{2} - \frac{1}{2}\sqrt{b^2-4ac},\quad g = \frac{b}{2} + \frac{1}{2}\sqrt{b^2-4ac},$$
$$h = \sqrt{b^2-4ac},\quad q = \sqrt[4]{\frac{a}{c}},\quad l = 2a(n-1)(b^2-4ac),\quad \cos\alpha = -\frac{b}{2\sqrt{ac}}.$$

1. $$\int \frac{dx}{R_2} = \frac{c}{h}\left\{\int \frac{dx}{cx^2+f} - \int \frac{dx}{cx^2+g}\right\}\qquad [h^2>0];$$
$$= \frac{1}{4cq^3\sin\alpha}\left\{\sin\frac{\alpha}{2}\ln\frac{x^2+2qx\cos\frac{\alpha}{2}+q^2}{x^2-2qx\cos\frac{\alpha}{2}+q^2} + 2\cos\frac{\alpha}{2}\operatorname{arctg}\frac{x^2-q^2}{2qx\sin\frac{\alpha}{2}}\right\}\qquad [h^2<0].$$

2. $$\int \frac{x\,dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2+f}{cx^2+g} \qquad [h^2 > 0];$$
$$= \frac{1}{2cq^2 \sin\alpha} \operatorname{arctg} \frac{x^2 - q^2 \cos\alpha}{q^2 \sin\alpha} \qquad [h^2 < 0].$$

LA 146 (9)a
LA 146 (6)

3. $$\int \frac{x^2\,dx}{R_2} = \frac{g}{h} \int \frac{dx}{cx^2+g} - \frac{f}{h} \int \frac{dx}{cx^2+f} \qquad [h^2 > 0].$$

LA 146 (7)

4. $$\int \frac{dx}{R_2^2} = \frac{bcx^3 + (b^2-2ac)x}{lR_2} + \frac{b^2-6ac}{l} \int \frac{dx}{R_2} + \frac{bc}{l} \int \frac{x^2\,dx}{R_2}.$$

5. $$\int \frac{dx}{R_2^n} = \frac{bcx^3 + (b^2-2ac)x}{lR_2^{n-1}} + \frac{(4n-7)bc}{l} \int \frac{x^2\,dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l} \int \frac{dx}{R_2^{n-1}}$$
$$[n > 1].$$

LA 146

6. $$\int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a} \int \frac{dx}{x^{m-2}R^n} - \frac{(m+4n-5)b}{(m-1)a} \int \frac{dx}{x^{m-4}R_2^n}.$$

LA 147 (12)a

2.17 Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of x

Notations: $R = a + bx + cx^2$; $\Delta = 4ac - b^2$

2.171

1. $$\int x^{m+1} R^n\,dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)} \int x^{m-1} R^n\,dx - \frac{b(m+n+1)}{c(m+2n+2)} \int x^m R^n\,dx.$$

TI (97)

LA 142(3), TI (96)a

3. $$\int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta}\int \frac{dx}{R^n}$$

TI (94)a

4. $$\int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1}\sum_{k=0}^{n-1}\frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\dots(n-k)\Delta^{k+1}R^{n-k}} + \\ + 2^n\frac{(2n-1)!!c^n}{n!\Delta^n}\int\frac{dx}{R}.$$

TI (96)a

2.172[3]

$$\int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}}\ln\frac{\sqrt{-\Delta}-(b+2cx)}{(b+2cx)+\sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}}\operatorname{Arth}\frac{b+2cx}{\sqrt{-\Delta}} \qquad [\Delta < 0];$$
$$= \frac{-2}{b+2cx} \qquad [\Delta = 0];$$
$$= \frac{2}{\sqrt{\Delta}}\operatorname{arctg}\frac{b+2cx}{\sqrt{\Delta}} \qquad [\Delta > 0].$$

82

2.173

1. $$\int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta}\int\frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

2. $$\int \frac{dx}{R^3} = \frac{b+2cx}{\Delta}\left\{\frac{1}{2R^2}+\frac{3c}{\Delta R}\right\} + \frac{6c^2}{\Delta^2}\int\frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

2.174

1. $$\int \frac{x^m\,dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c}\int\frac{x^{m-1}\,dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c}\int\frac{x^{m-2}\,dx}{R^n}.$$

$m = 2n - 1$

2. $$\int \frac{x^{2n-1}\,dx}{R^n} = \frac{1}{c}\int \frac{x^{2n-3}\,dx}{R^{n-1}} - \frac{a}{c}\int \frac{x^{2n-3}\,dx}{R^n} - \frac{b}{c}\int \frac{x^{2n-2}\,dx}{R^n}.$$

2.175

1. $$\int \frac{x\,dx}{R} = \frac{1}{2c}\ln\}\,R - \frac{b}{2c}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

2. $$\int \frac{x\,dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

3. $$\int \frac{x\,dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

4. $$\int \frac{x^2\,dx}{R} = \frac{x}{c} - \frac{b}{2c^2}\ln R + \frac{b^2-2ac}{2c^2}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

2.172

5. $$\int \frac{x^2\,dx}{R^2} = \frac{ab+(b^2-2ac)x}{c\Delta R} + \frac{2a}{\Delta}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172})$$

2.172

6. $$\int \frac{x^2\,dx}{R^3} = \frac{ab+(b^2-2ac)x}{2c\,\Delta R^2} + \frac{(2ac+b^2)(b+2cx)}{2c\Delta^2 R} + \frac{2ac+b^2}{\Delta^2}\int \frac{dx}{R} \qquad (\text{see } \mathbf{2.172}).$$

7. $$\int \frac{x^3\,dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^3}\ln R - \frac{b(b^2-3ac)}{2c^3}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

8. $$\int \frac{x^3\,dx}{R^2} = \frac{1}{2c^2}\ln R + \frac{a(2ac-b^2)+b(3ac-b^2)x}{c^2\Delta R} - \frac{b(6ac-b^2)}{2c^2\Delta}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

9. $$\int \frac{x^3\,dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right)\frac{1}{2R^2} - \frac{3ab}{2c\Delta}\int \frac{dx}{R^2} \qquad \text{(see } \mathbf{2.173}\ 1.\text{)}.$$

2.173

2.176

$$\int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)}\int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)}\int \frac{dx}{x^{m-2}R^n}.$$

83

2.177

1. $$\int \frac{dx}{xR} = \frac{1}{2a}\ln\frac{x^2}{R} - \frac{b}{2a}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

2. $$\int \frac{dx}{xR^2} = \frac{1}{2a^2}\ln\frac{x^2}{R} + \frac{1}{2aR}\left\{1 - \frac{b(b+2cx)}{\Delta}\right\} - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right)\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

2.173

2.172

4. $$\int \frac{dx}{x^2 R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

5. $$\int \frac{dx}{x^2 R^2} = -\frac{b}{a^3} \ln \frac{x^2}{R} - \frac{a + bx}{a^2 x R} + \frac{(b^2 - 3ac)(b + 2cx)}{a^2 \Delta R} - \frac{1}{\Delta} \left(\frac{b^4}{a^3} - \frac{6b^2 c}{a^2} + \frac{6c^2}{a} \right) \int \frac{dx}{R}$$
(see **2.172**).

2.172

6. $$\int \frac{dx}{x^2 R^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3} \qquad \text{(see } \mathbf{2.173} \text{ and } \mathbf{2.177} \text{ 3.)}.$$

2.177

2.173

7. $$\int \frac{dx}{x^3 R} = -\frac{ac - b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2 x} - \frac{1}{2ax^2} + \frac{b(3ac - b^2)}{2a^3} \int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}.$$

2.172

8. $$\int \frac{dx}{x^3 R^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2 x} \right) \frac{1}{R} + \left(\frac{3b^2}{a^2} - \frac{2c}{a} \right) \int \frac{dx}{xR^2} + \frac{9bc}{2a^2} \int \frac{dx}{R^2}$$
(see **2.173** 1. and **2.177** 2.).

2.177

2.173

9. $\int \frac{dx}{x^3R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2x}\right)\frac{1}{R^2} + \left(\frac{6b^2}{a^2} - \frac{3c}{a}\right)\int \frac{dx}{xR^3} + \frac{10bc}{a^2}\int \frac{dx}{R^3}$ (see **2.173** 2., **2.177** 3.).

2.177

2.173

2.18 Forms containing the quadratic trinomial $a + bx + cx^2$ and the binomial $\alpha + \beta x$

Notations:

$R = a + bx + cx^2;\quad z = \alpha + \beta x;\quad A = \alpha\beta^2 - ab\beta + c\alpha^2;$

$B = b\beta - 2c\alpha;\quad \Delta = 4ac - b^2.$

1. $$\int z^m R^n\,dx = \frac{\beta z^{m-1}R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c}\int z^{m-1}R^n\,dx - \frac{(m-1)A}{(m+2n+1)c}\int z^{m-2}R^n\,dx.$$

2. $$\begin{aligned}\int \frac{R^n\,dx}{z^m} &= -\frac{1}{(m-2n-1)\beta}\frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2}\int \frac{R^{n-1}\,dx}{z^m} - \\ &\quad - \frac{nB}{(m-2n-1)\beta^2}\int \frac{R^{n-1}\,dx}{z^{m-1}};\\ &= \frac{-\beta}{(m-1)A}\frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A}\int \frac{R^n\,dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A}\int \frac{R^n\,dx}{z^{m-2}};\\ &= -\frac{1}{(m-1)\beta}\frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2}\int \frac{R^{n-1}\,dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2}\int \frac{R^{n-1}\,dx}{z^{m-2}}\end{aligned}$$

LA 418 (6)
LA 148 (5)
LA 184 (4)a

3. $$\begin{aligned}\int \frac{z^m\,dx}{R^n} &= \frac{\beta}{(m-2n+1)c}\frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c}\int \frac{z^{m-1}\,dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c}\int \frac{z^{m-2}\,dx}{R^n};\\ &= \frac{b+2cx}{(n-1)\Delta}\frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta}\int \frac{z^m\,dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta}\int \frac{z^{m-1}\,dx}{R^{n-1}}\end{aligned}$$

$$4^3. \int \frac{dx}{z^m R^n} = -\frac{\beta}{(m-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{(m+n-2)B}{(m-1)A} \int \frac{dx}{z^{m-1} R^n} - \frac{(m+2n-3)c}{(m-1)A} \int \frac{dx}{z^{m-2} R^n};$$

$$= \frac{\beta}{2(n-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{B}{2A} \int \frac{dx}{z^{m-1} R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A} \int \frac{dx}{z^m R^{n-1}}.$$

LA 148 (8)
LA 148 (7)

For $m = 1$ and $n = 1$

$$\int \frac{dx}{zR} = \frac{\beta}{2A} \ln \frac{z^2}{R} - \frac{B}{2A} \int \frac{dx}{R}.$$

For $A = 0$

$$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m+n-1)B} \frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B} \int \frac{dx}{z^{m-1} R^n}.$$

LA 148 (9)

2.2 Algebraic Functions

2.20 Introduction

2.201

The integrals $\int R\left(x, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^r, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^s, \dots\right) dx$, where $r, s, \dots$ are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x+\beta}{\gamma x+\delta} = t^m,$$

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where m is the common denominator of the fractions $r, s, \dots$.

2.202

Integrals of the form $\int x^m (a+bx^n)^p \, dx$,* Transl. The authors term such integrals "integrals of binomial differentials".

$\int x^m(a+bx^n)^p\,dx$

where m, n, and p are rational numbers, can be expressed in terms of elementary functions only in the following

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(a) When p is an integer; then, this integral takes the form of the sum of the integrals shown in 2.201;

(b) When $\frac{m+1}{n}$ is an integer: by means of the substitution $x^n = z$, this integral can be transformed to the form $\frac{1}{n}\int (a+bz)^p z^{\frac{m+1}{n}-1}\,dz$, which we considered in 2.201;

(c) When $\frac{m+1}{n}+p$ is an integer; by means of the same substitution $x^n = z$, this integral can be reduced to an integral of the form $\frac{1}{n}\int \left(\frac{a+bz}{z}\right)^p z^{\frac{m+1}{n}+p-1}\,dz$, considered in 2.201;

For reduction formulas for integrals of binomial differentials, see 2.110.

2.21 Forms containing the binomial $a+bx^k and \sqrt{x}$

Notation: $z_1 = A + Bx$.

2.211

$$\int \frac{dx}{z_1\sqrt{x}} = \frac{2}{\sqrt{ab}}\operatorname{arctg}\sqrt{\frac{bx}{a}} \qquad [ab>0];$$
$$= \frac{1}{i\sqrt{ab}}\ln\frac{a-bx+2i\sqrt{xab}}{z_1} \qquad [ab<0].$$

2.212

$$\int \frac{x^m\sqrt{x}}{z_1}\,dx = 2\sqrt{x}\sum_{k=0}^{m}\frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1}\frac{a^{m+1}}{b^{m+1}}\int\frac{dx}{z_1\sqrt{x}} \qquad (\text{see } \mathbf{2.211}).$$

2.211

2.213

1. $$\int\frac{\sqrt{x}\,dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int\frac{dx}{z_1\sqrt{x}} \qquad (\text{see } \mathbf{2.211}).$$

2.211

2. $$\int \frac{x\sqrt{x}\,dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right) 2\sqrt{x} + \frac{a^2}{b^2}\int \frac{dx}{z_1\sqrt{x}} \qquad \text{(see } \mathbf{2.211}\text{)}.$$

2.211

3. $$\int \frac{x^2\sqrt{x}\,dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3}\right) 2\sqrt{x} - \frac{a^3}{b^3}\int \frac{dx}{z_1\sqrt{x}} \qquad \text{(see } \mathbf{2.211}\text{)}.$$

2.211

4. $$\int \frac{dx}{z_1^2\sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a}\int \frac{dx}{z_1\sqrt{x}} \qquad \text{(see } \mathbf{2.211}\text{)}.$$

2.211

5. $$\int \frac{\sqrt{x}\,dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b}\int \frac{dx}{z_1\sqrt{x}} \qquad \text{(see } \mathbf{2.211}\text{)}.$$

2.211

6. $$\int \frac{x\sqrt{x}\,dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b}\int \frac{\sqrt{x}\,dx}{z_1^2} \qquad \text{(see } \mathbf{2.213}\ 5.\text{)}.$$

2.213

7. $$\int \frac{x^2\sqrt{x}\,dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right) \frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2}\int \frac{\sqrt{x}\,dx}{z_1^2} \qquad \text{(see } \mathbf{2.213}\ 5.\text{)}.$$

2.213

8. $$\int \frac{dx}{z_1^3\sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2 z_1}\right)\sqrt{x} + \frac{3}{8a^2}\int \frac{dx}{z_1\sqrt{x}} \qquad (\text{see } \mathbf{2.211}).$$

2.211

9. $$\int \frac{\sqrt{x}\,dx}{z_1^3} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1}\right)\sqrt{x} + \frac{1}{8ab}\int \frac{dx}{z_1\sqrt{x}} \qquad (\text{see } \mathbf{2.211}).$$

2.211

10. $$\int \frac{x\sqrt{x}\,dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b}\int \frac{\sqrt{x}\,dx}{z_1^3} \qquad (\text{see } \mathbf{2.213}\ 9.).$$

2.213

11. $$\int \frac{x^2\sqrt{x}\,dx}{z_1^3} = \left(\frac{x^2}{b} + \frac{5ax}{b^2}\right)\frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2}\int \frac{\sqrt{x}\,dx}{z_1^3} \qquad (\text{see } \mathbf{2.213}\ 9.).$$

2.213

Notations: $z_2 = a + bx^2, \quad \alpha = \sqrt[4]{\frac{a}{b}}, \quad alpha' = \sqrt[4]{-\frac{a}{b}}.$

2.214

$$\int \frac{dx}{z_2\sqrt{x}} = \frac{1}{b\alpha^3\sqrt{2}}\left[\ln\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{z_2}} + \operatorname{arctg}\frac{\alpha\sqrt{2x}}{\alpha^2 - x}\right] \qquad \left[\frac{a}{b} > 0\right];$$
$$= \frac{1}{2b\alpha'^3}\left(\ln\frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2\operatorname{arctg}\frac{\sqrt{x}}{\alpha'}\right) \qquad \left[\frac{a}{b} < 0\right].$$

2.215

$$\int \frac{\sqrt{x}\,dx}{z_2} = \frac{1}{b\alpha\sqrt{2}}\left[-\ln\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{z_2}} + \operatorname{arctg}\frac{\alpha\sqrt{2x}}{\alpha^2 - x}\right] \qquad \left[\frac{a}{b} > 0\right];$$
$$= \frac{1}{2b\alpha'}\left[\ln\frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} + 2\operatorname{arctg}\frac{\sqrt{x}}{\alpha'}\right] \qquad \left[\frac{a}{b} < 0\right].$$

1. $\int \frac{x\sqrt{x}\,dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{dx}{z_2\sqrt{x}}$ (see **2.214**).

2.214

2. $\int \frac{x^2\sqrt{x}\,dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b}\int \frac{\sqrt{x}\,dx}{z_2}$ (see **2.215**).

2.215

3. $\int \frac{x}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a}\int \frac{dx}{z_2\sqrt{x}}$ (see **2.214**).

2.214

4. $\int \frac{\sqrt{x}\,dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a}\int \frac{\sqrt{x}\,dx}{z_2}$ (see **2.215**).

2.215

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5. $\int \frac{x\sqrt{x}\,dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b}\int \frac{dx}{z_2\sqrt{x}}$ (see **2.214**).

2.214

6. $\int \frac{x^2\sqrt{x}\,dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b}\int \frac{\sqrt{x}\,dx}{z_2}$ (see **2.215**).

2.215

2.214

8. $\int \frac{\sqrt{x}\,dx}{z_2^3} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2z_2}\right) x\sqrt{x} + \frac{5}{32a^2}\int \frac{\sqrt{x}\,dx}{z_2}$ (see **2.215**).

2.215

9. $\int \frac{x\sqrt{x}\,dx}{z_2^3} = \frac{(bx^2 - 3a)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab}\int \frac{dx}{z_2\sqrt{x}}$ (see **2.214**).

2.214

10. $\int \frac{x^2\sqrt{x}\,dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b}\int \frac{\sqrt{x}\,dx}{z_2^3}$ (see **2.216** 8.).

2.216

2.22-2.23 Forms containing $\sqrt[n]{(a + bx)^k}$

Notation: $z = A + Bx$.

2.220

$$\int x^n \sqrt[l]{z^{lm+f}}\,dx = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{ln - lk + l\,(m+1) + f} \right\} \frac{l\sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}.$$

The square root

2.221

$$\int x^n \sqrt{z^{2m-1}}\,dx = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{2n - 2k + 2m + 1} \right\} \frac{2\sqrt{z^{2m+1}}}{b^{n+1}}.$$

1. $$\int \frac{dx}{\sqrt{z}} = \frac{2}{b}\sqrt{z}.$$

2. $$\int \frac{x\,dx}{\sqrt{z}} = \left(\frac{1}{3}z - a\right)\frac{2\sqrt{z}}{b^2}.$$

3. $$\int \frac{x^2\,dx}{\sqrt{z}} = \left(\frac{1}{5}z^2 - \frac{2}{3}az + a^2\right)\frac{2\sqrt{z}}{b^3}.$$

2.223

1. $$\int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}.$$

2. $$\int \frac{x\,dx}{\sqrt{z^3}} = (z + a)\frac{2}{b^2\sqrt{z}}.$$

3. $$\int \frac{x^2\,dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2\right)\frac{2}{b^3\sqrt{z}}.$$

2.224

1. $$\int \frac{z^m\,dx}{x^n\sqrt{z}} = -\frac{z^m\sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)}\,\frac{b}{a}\int \frac{z^m\,dx}{x^{n-1}\sqrt{z}}.$$

2. $$\begin{aligned}\int \frac{z^m\,dx}{x^n\sqrt{z}} = {} & -z^m\sqrt{z}\left\{\frac{1}{(n-1)ax^{n-1}} + \right.\\ & \left. + \sum_{k=1}^{n-2} \frac{(2m-2n+3)(2m-2n+5)\dots(2m-2n+2k+1)}{2^k(n-1)(n-2)\dots(n-k-1)x^{n-k-1}}\,\frac{b^k}{a^{k+1}}\right\} + \\ & + \frac{(2m-2n+3)(2m-2n+5)\dots(2m-3)(2m-1)}{2^{n-1}(n-1)!x}\,\frac{b^{n-1}}{a^{n-1}}\int \frac{z^m\,dx}{x\sqrt{z}}.\end{aligned}$$

For n=1

2.225

1. $\int \frac{\sqrt{z}\,dx}{x} = 2\sqrt{z} + a\int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

2. $\int \frac{\sqrt{z}\,dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

3. $\int \frac{\sqrt{z}\,dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a}\int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

2.226

1. $\int \frac{\sqrt{z^3}\,dx}{x} = \left(\frac{z}{3} + a\right) 2\sqrt{z} + a^2\int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

2. $\int \frac{\sqrt{z^3}\,dx}{x^2} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a}\int \frac{\sqrt{z^3}\,dx}{x}$ (see **2.226** 1.).

2.226

3. $\int \frac{\sqrt{z^3}\,dx}{x^3} = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right)\sqrt{z^5} + \frac{3b^2}{8a^2}\int \frac{\sqrt{z^3}\,dx}{x}$ (see **2.226** 1.).

2.227

$$\int \frac{dx}{xz^m\sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1)a^{m-k}z^k\sqrt{z}} + \frac{1}{a^m}\int \frac{dx}{x\sqrt{z}}.$$ (see **2.224** 4.).

2.224

2.228

1. $$\int \frac{dx}{x^2\sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4.).

2.224

2. $$\int \frac{dx}{x^3\sqrt{z}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right)\sqrt{z} + \frac{3b^2}{8a^2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4.).

2.224

2.229

1. $$\int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}} + \frac{1}{a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4.).

2.224

2. $$\int \frac{dx}{x^2\sqrt{z^3}} = \left(-\frac{1}{ax} - \frac{3b}{a^2}\right)\frac{1}{\sqrt{z}} - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4.).

2.224

3. $$\int \frac{dx}{x^3\sqrt{z^3}} = \left(-\frac{1}{2ax^2} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3}\right)\frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4.).

Cube root

2.231

1. $$\int \sqrt[3]{z^{3m+1}} x^n \, dx = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3\,(m+1) + 1} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}.$$

2. $$\int \frac{x^n \, dx}{\sqrt[3]{z^{3m+2}}} = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3\,(m-1) - 2} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+2}}}.$$

3. $$\int \sqrt[3]{z^{3m+2}} x^n \, dx = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3\,(m+1) + 2} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}.$$

4. $$\int \frac{x^n \, dx}{\sqrt[3]{z^{3m+1}}} = \left\{ \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3\,(m-1) - 1} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+1}}}.$$

5. $$\int \frac{z^n \, dx}{x^m \sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1) a x^{m-1}} + \frac{3n - 3m + 4}{3(m-1)} \frac{b}{a} \int \frac{z^n \, dx}{x^{m-1} \sqrt[3]{z^2}}.$$

For $m = 1$

$$\int \frac{z^n \, dx}{x \sqrt[3]{z^2}} = \frac{3z^n}{(3n-2) \sqrt[3]{z^2}} + a \int \frac{z^{n-1} \, dx}{x \sqrt[3]{z^2}}.$$

2.232

$$\int \frac{dx}{x\sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3}\operatorname{arctg} \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}.$$

2.233

1. $$\int \frac{\sqrt[3]{z}\,dx}{x} = 3\sqrt[3]{z} + a\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**).

2.232

2. $$\int \frac{\sqrt[3]{z}\,dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**).

2.232

3. $$\int \frac{\sqrt[3]{z}\,dx}{x^3} = \left(-\frac{1}{2ax^2} + \frac{b}{3a^2x}\right) z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**).

2.232

4. $$\int \frac{dx}{x^2\sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**).

2.232

5. $$\int \frac{dx}{x^3\sqrt[3]{z^2}} = \left[-\frac{1}{2ax^2} + \frac{5b}{6a^2x}\right]\sqrt[3]{z} + \frac{5b^2}{9a^2}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**).

2.232

2.234

1. $$\int \frac{z^n\,dx}{x^m \sqrt[3]{z^2}} = -\frac{z^n \sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)}\,\frac{b}{a}\int \frac{z^n\,dx}{x^{m-1}\sqrt[3]{z}}.$$

For $m=1$:

2. $$\int \frac{z^n\,dx}{x\sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a\int \frac{z^{n-1}\,dx}{x\sqrt[3]{z}}.$$

3. $$\int \frac{dx}{xz^n\sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a}\int \frac{\sqrt[3]{z^2}\,dx}{xz^n}.$$

2.235

$$\int \frac{dx}{x\sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}}\left\{\frac{3}{2}\ln\frac{\sqrt[3]{z}-\sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3}\operatorname{arctg}\frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z}+2\sqrt[3]{a}}\right\}.$$

2.236

1. $$\int \frac{\sqrt[3]{z^2}\,dx}{x} = \frac{3}{2}\sqrt[3]{z^2} + a\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**).

2.235

2. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a}\sqrt[3]{z^2} + \frac{2b}{3}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**).

2.235

3. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^3} = \left[-\frac{1}{2ax^2} + \frac{b}{6a^2x}\right] z^{5/3} - \frac{b^2}{6a^2}\sqrt[3]{z^2} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**).

4. $$\int \frac{dx}{x^2 \sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a} \int \frac{dx}{x \sqrt[3]{z}}$$ (see **2.235**).

2.235

5. $$\int \frac{dx}{x^3 \sqrt[3]{z}} = \left[-\frac{1}{2ax^2} + \frac{2b}{3a^2 x} \right] \sqrt[3]{z} + \frac{2b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z}}$$ (see **2.235**).

2.235

2.24 Forms containing $\sqrt{a + bx}$ and the binomial $\alpha + \beta x$

Notation: $z = a + bx, \quad t = \alpha + \beta x, \quad \Delta = a\beta - b\alpha.$

2.241

1. $$\int \frac{z^m t^n \, dx}{\sqrt{z}} = \frac{2}{(2n + 2m + 1)\beta} t^{n+1} z^{m-1} \sqrt{z} + \frac{(2m - 1)\Delta}{(2n + 2m + 1)\beta} \int \frac{z^{m-1} t^n \, dx}{\sqrt{z}}.$$

LA 176 (1)

2. $$\int \frac{t^n z^m \, dx}{\sqrt{z}} = 2\sqrt{z^{2m+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k - 2p + 2m + 1}.$$

2.242

1. $$\int \frac{t \, dx}{\sqrt{z}} = \frac{2a\sqrt{z}}{b} + \beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2}.$$

2. $$\int \frac{t^2 \, dx}{\sqrt{z}} = \frac{2\alpha^2 \sqrt{z}}{b} + 2\alpha\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + \beta^2 \left(\frac{z^2}{5} - \frac{2}{3} za + a^2 \right) \frac{2\sqrt{z}}{b^3}.$$

12. $$\int \frac{t^3 z^3\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^7}}{7b} + 3\alpha^2\beta\left(\frac{z}{9} - \frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7}\right)\frac{2\sqrt{z^7}}{b^3} + \\ + \beta^3\left(\frac{z^3}{13} - \frac{3z^2a}{11} + \frac{3za^2}{9} - \frac{a^3}{7}\right)\frac{2\sqrt{z^7}}{b^4}.$$

2.243

1. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{t^{n+1}}{z^m}\sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta}\int \frac{t^n\,dx}{z^{m-1}\sqrt{z}}; \\ = -\frac{2}{(2m-1)b}\frac{t^n}{z^m}\sqrt{z} + \frac{2n\beta}{(2m-1)b}\int \frac{t^{n-1}\,dx}{z^{m-1}\sqrt{z}}.$$

LA 176 (2)

2. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}}\sum_{k=0}^{n}\binom{n}{k}\frac{a^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p-2m+1}.$$

2.244

1. $$\int \frac{t\,dx}{z\sqrt{z}} = -\frac{2a}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2\sqrt{z}}.$$

2. $$\int \frac{t^2\,dx}{z\sqrt{z}} = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2\sqrt{z}} + \frac{2\beta^2\left(\frac{z^2}{3} - 2za - a^2\right)}{b^3\sqrt{z}}.$$

3. $$\int \frac{t^3\,dx}{z\sqrt{z}} = -\frac{2\alpha^3}{b\sqrt{z}} + \frac{6\alpha^2\beta(z+a)}{b^2\sqrt{z}} + \frac{6\alpha\beta^2\left(\frac{z^2}{3} - 2za - a^2\right)}{b^3\sqrt{z}} + \frac{2\beta^3\left(\frac{z^3}{5} - z^2a + 3za^2 + a^3\right)}{b^4\sqrt{z}}.$$

4. $$\int \frac{t\,dx}{z^2\sqrt{z}} = -\frac{2a}{3b\sqrt{z^3}} - \frac{2\beta\left(z - \frac{a}{3}\right)}{b^2\sqrt{z^3}}.$$

5. $$\int \frac{t^2\,dx}{z^2\sqrt{z}} = -\frac{2a^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta\left(z - \frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{2\beta^2\left(z^2 + 2az - \frac{a^2}{3}\right)}{b^3\sqrt{z^3}}.$$

6. $$\int \frac{t^3\,dx}{z^2\sqrt{z}} = -\frac{2\alpha^3}{3b\sqrt{z^3}} - \frac{6\alpha^2\beta\left(z - \frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{6\alpha\beta^2\left(z^2 + 2za - \frac{a^2}{3}\right)}{b^3\sqrt{z^3}} + \\ + \frac{2\beta^3\left(\frac{z^3}{3} - 3z^2a - 3za^2 + \frac{a^3}{3}\right)}{b^4\sqrt{z^3}}.$$

7. $$\int \frac{t\,dx}{z^3\sqrt{z}} = -\frac{2\alpha}{5b\sqrt{z^5}} - \frac{2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}}.$$

8. $$\int \frac{t^2\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{2\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}}.$$

9. $$\int \frac{t^3\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{6\alpha\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}} + \\ + \frac{2\beta^3\left(z^3 + 3z^2a - za^2 + \frac{a^3}{5}\right)}{b^4\sqrt{z^5}}.$$

2.245

1. $$\int \frac{z^m\,dx}{t^n\sqrt{z}} = -\frac{2}{(2n-2m-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta}\int \frac{z^{m-1}\,dx}{t^n\sqrt{z}};$$
$$= -\frac{1}{(n-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta}\int \frac{z^{m-1}}{t^{n-1}\sqrt{z}}\,dx;$$
$$= -\frac{1}{(n-1)\Delta}\frac{z^m}{t^{n-1}}\sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta}\int \frac{z^m\,dx}{t^{n-1}\sqrt{z}}.$$

2. $$\int \frac{z^m\,dz}{t^n\sqrt{z}} = -z^m\sqrt{z}\Bigg\{\frac{1}{(n-1)\Delta}\frac{1}{t^{n-1}} + \\ + \sum_{k=2}^{n-1}\frac{(2n-2m-3)(2n-2m-5)\dots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k}\frac{1}{t^{n-k}}\Bigg\} - \\ -\frac{(2n-2m-3)(2n-2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}\cdot(n-1)!\Delta^n}\int\frac{z^m\,dx}{t\sqrt{z}}.$$

For $n = 1$

3. $$\int \frac{z^m\,dx}{t\sqrt{z}} = \frac{2}{(2m-1)\beta}\frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta}\int\frac{z^{m-1}\,dx}{t\sqrt{z}}.$$

4. $$\int \frac{z^m\,dx}{t\sqrt{z}} = 2\sum_{k=0}^{m-1}\frac{\Delta^k}{(2m-2k-1)\beta^{k+1}}\frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m}\int\frac{dx}{t\sqrt{z}}.$$

2.246

$$\int \frac{dx}{t\sqrt{z}} = \frac{1}{\sqrt{\beta\Delta}}\ln\frac{\beta\sqrt{z}-\sqrt{\beta\Delta}}{\beta\sqrt{z}+\sqrt{\beta\Delta}} \qquad [\beta\Delta > 0];$$
$$= \frac{2}{\sqrt{-\beta\Delta}}\operatorname{arctg}\frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} \qquad [\beta\Delta < 0];$$
$$= -\frac{2\sqrt{z}}{bt} \qquad [\Delta = 0].$$

2.247

$$\int \frac{dx}{tz^m\sqrt{z}} = \frac{2}{z^{m-1}\sqrt{z}} + \sum_{k=1}^{m}\frac{\beta^{k-1}z^k}{\Delta^k(2m-2k+1)} + \frac{\beta^m}{\Delta^m}\int\frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

2.248

1. $$\int \frac{dx}{tz\sqrt{z}} = \frac{2}{\Delta\sqrt{z}} + \frac{\beta}{\Delta}\int\frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

2.246

3. $$\int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

4. $$\int \frac{dx}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

5. $$\int \frac{dx}{t^2 z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

6. $$\int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

7. $$\int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

8. $$\int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2}\int \frac{dx}{t\sqrt{z}} \qquad \text{(see } \mathbf{2.246}\text{)}.$$

2.246

2.246

10. $\int \frac{dx}{t^3 z^2 \sqrt{z}} = -\frac{1}{2\Delta t^2 z \sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 t z \sqrt{z}} + \frac{35b^2}{12\Delta^2 z \sqrt{z}} + \frac{35b^2\beta}{4\Delta^4 \sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

11. $\int \frac{dx}{t^3 z^3 \sqrt{z}} = -\frac{1}{2\Delta t^2 z^2 \sqrt{z}} + \frac{9b}{4\Delta^2 t z^2 \sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2 \sqrt{z}} + \frac{21b^2\beta}{4\Delta^4 z \sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5 \sqrt{z}} +$
$+ \frac{63b^2\beta^3}{8\Delta^5} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

12. $\int \frac{z\,dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

13. $\int \frac{z^2\,dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

14. $\int \frac{z^3\,dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

15. $\int \frac{z\,dx}{t^2\sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta} \int \frac{dx}{t\sqrt{z}}$ (see **2.246**).

2.246

16. $$\int \frac{z^2\,dx}{t^2\sqrt{z}} = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2}\int \frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

17. $$\int \frac{z^3\,dx}{t^2\sqrt{z}} = -\frac{z^3\sqrt{z}}{\Delta t} + \frac{bz^2\sqrt{z}}{\beta\Delta} + \frac{5bz\sqrt{z}}{3\beta^2} + \frac{5b\Delta\sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3}\int \frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

18[3]. $$\int \frac{z\,dx}{t^3\sqrt{z}} = -\frac{z\sqrt{z}}{2\Delta t^2} + \frac{bz\sqrt{z}}{4\Delta^2 t} - \frac{b^2\sqrt{z}}{4\beta\Delta^2} + \frac{b^2}{8\beta\Delta}\int \frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

19. $$\int \frac{z^2\,dx}{t^3\sqrt{z}} = -\frac{z^2\sqrt{z}}{2\Delta t^2} + \frac{bz^2\sqrt{z}}{4\Delta^2 t} + \frac{b^2 z\sqrt{z}}{4\beta\Delta^2} + \frac{3b^2\sqrt{z}}{4\beta^2\Delta} + \frac{3b^2}{8\beta^2}\int \frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

20. $$\int \frac{z^3\,dx}{t^3\sqrt{z}} = -\frac{z^3\sqrt{z}}{2\Delta t^2} + \frac{3bz^3\sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2\sqrt{z}}{4\beta\Delta^2} + \frac{5b^2 z\sqrt{z}}{4\beta^2\Delta} + \frac{15b^2\sqrt{z}}{4\beta^3} + \\ + \frac{15b^2\Delta}{8\beta^3}\int \frac{dx}{t\sqrt{z}} \qquad (\text{see } \mathbf{2.246}).$$

2.246

2.249

1. $$\int \frac{dx}{z^m t^n\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{\sqrt{z}}{t^{n-1}z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta}\int \frac{dx}{t^n z^{m-1}\sqrt{z}};$$
$$= -\frac{1}{(n-1)\Delta}\frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta}\int \frac{dx}{t^{n-1}z^m\sqrt{z}}.$$

$$2. \quad \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{\sqrt{z}}{z^m} \left\{ \frac{-1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \right.$$
$$+ \sum_{k=2}^{n-1} (-1)^k \frac{(2n+2m-3)(2n+2m-5)\dots(2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k} \cdot \left. \frac{1}{t^{n-k}} \right\} +$$
$$+(-1)^{n-1} \frac{(2n+2m-3)(2n+2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}} \int \frac{dx}{tz^m\sqrt{z}}.$$

For $n = 1$

$$\int \frac{dx}{z^m t \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{1}{z^{m-1}\sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{tz^{m-1}\sqrt{z}}.$$

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2.25 Forms containing $\sqrt{a + bx + cx^2}$

Integration techniques

2.251

It is possible to rationalize the integrand in integrals of the form $\int R(x, \sqrt{a+bx+cx^2})\, dx$ by using one or more of the following three substitutions, known as the ``Euler substitutions".

1) $\sqrt{a+bx+cx^2} = xt \pm \sqrt{a}$ for $a > 0$;

2) $\sqrt{a+bx+cx^2} = t \pm x\sqrt{c}$ for $c > 0$;

3) $\sqrt{c(x-x_1)(x-x_2)} = t(x-x_1)$ when x_1 and x_2 are real roots of the equation $a + bx + cx^2 = 0$.

2.252

Besides the Euler substitutions, there is also the following method of calculating integrals of the form $\int R(x, \sqrt{a+bx+cx^2})\, dx$. By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of x and an expression of the form $\frac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, where $P_1(x)$ and $P_2(x)$ are both polynomials. By separating the integral portion of the rational function $\frac{P_1(x)}{P_2(x)}$ from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals each of which is in one of the following three forms:

I. $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$, where $P(x)$ is a polynomial of some degree r;

II. $\int \frac{dx}{(x+p)^k\sqrt{a+bx+cx^2}}$;

III. $\int \frac{(Mx+N)\,dx}{(a+\beta x+x^2)^m\sqrt{c(a_1+b_1x+x^2)}}, \qquad \left(a_1=\frac{a}{c}, \quad b_1=\frac{b}{c}\right).$

I. $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}} = Q(x)\sqrt{a+bx+cx^2} + \lambda \int \frac{dx}{\sqrt{a+bx+cx^2}}$, where $Q(x)$ is a polynomial of degree $(r-1)$. Its coefficients, and also the number λ, can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)(a+bx+cx^2) + \frac{1}{2}Q(x)(b+2cx) + \lambda.$$

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Integrals of the form $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$ (where $r \leq 3$) can also be calculated by use of formulas 2.26.

II. Integrals of the form $\int \frac{P(x)\,dx}{(x+p)^k\sqrt{a+bx+cx^2}}$, where the degree n of the polynomial $P(x)$ is

lower than k can, by means of the substitution $t=\frac{1}{x+p}$, be reduced to an integral of the form $\int \frac{P(t)\,dt}{\sqrt{a+\beta t+\gamma t^2}}$. (See also 2.281).

III. Integrals of the form $\int \frac{(Mx+N)\,dx}{(\alpha+\beta x+x^2)^m\sqrt{c(a_1+b_1x+x^2)}}$ can be calculated by the following procedure.

If $b_1 \neq \beta$, by using the substitution

$$x = \frac{a_1-\alpha}{\beta-b_1} + \frac{t-1}{t+1}\frac{\sqrt{(a_1-\alpha)^2-(\alpha b_1-a_1\beta)(\beta-b_1)}}{\beta-b_1}$$

we can reduce this integral to an integral of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c(t^2+q)}}$, where $P(t)$ is a polynomial of degree no higher than $2m-1$. The integral $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{t^2+q}}$ can be reduced to the sum of integrals of the forms $\int \frac{t\,dt}{(t^2+p)^k\sqrt{t^2+q}}$ and $\int \frac{dt}{(t^2+p)^k\sqrt{t^2+q}}$.

If $b_1=\beta$, we can reduce it to integrals of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c(t^2+q)}}$ by means of the substitution $t=x+\frac{b_1}{2}$.

The integral $\int \frac{t\,dt}{(t^2+p)^k\sqrt{c(t^2+q)}}$ can be evaluated by means of the substitution $t^2+q=u^2$.

The integral $\int \frac{dt}{(t^2+p)^k\sqrt{c(t^2+q)}}$ can be evaluated by means of the substitution $\frac{t}{\sqrt{t^2+q}}=v$ (see also 2.283).

2.26 Forms containing $\sqrt{\mathrm{a} + \mathrm{bx} + \mathrm{cx}^2}$ *and integral powers of x*

Notation: $R=a+bx+cx^2, \quad \Delta=4ac-b^2$

Simplified formulas for the case $b=0$. See 2.27.

2.260

1. $$\int x^m\sqrt{R^{2n+1}}\,dx = \frac{x^{m-1}\sqrt{R^{2n+3}}}{(m+2n+2)c} - \frac{(2m+2n+1)b}{2(m+2n+2)c}\int x^{m-1}\sqrt{R^{2n+1}}\,dx - \\ - \frac{(m-1)a}{(m+2n+2)c}\int x^{m-2}\sqrt{R^{2n+1}}\,dx.$$

TI (192)a
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2. $$\int \sqrt{R^{2n+1}}\,dx = \frac{2cx+b}{4(n+1)c}\sqrt{R^{2n+1}} + \frac{2n+1}{8(n+1)}\frac{\Delta}{c}\int\sqrt{R^{2n-1}}\,dx.$$

TI (188)

3. $$\int \sqrt{R^{2n+1}}\,dx = \frac{(2cx+b)\sqrt{R}}{4(n+1)c}\left\{R^n + \sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)}\left(\frac{\Delta}{c}\right)^{k+1}R^{n-k-1}\right\} + \\ + \frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{\Delta}{c}\right)^{n+1}\int\frac{dx}{\sqrt{R}}.$$

TI (190)

2.261

[6] For n =-1

$$\begin{aligned}\int\frac{dx}{\sqrt{R}} &= \frac{1}{\sqrt{c}}\ln(2\sqrt{cR}+2cx+b) \qquad [c>0];\\ &= \frac{1}{\sqrt{c}}\operatorname{Arsh}\frac{2cx+b}{\sqrt{\Delta}} \qquad [c>0, \quad \Delta>0];\\ &= \frac{-1}{\sqrt{-c}}\arcsin\frac{2cx+b}{\sqrt{-\Delta}} \qquad [c<0, \quad \Delta<0];\\ &= \frac{1}{\sqrt{c}}\ln(2cx+b) \qquad [c>0, \quad \Delta=0].\end{aligned}$$

2.262

1. $$\int \sqrt{R}\,dx = \frac{(2cx+b)\sqrt{R}}{4c} + \frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

2. $$\int x\sqrt{R}\,dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx+b)b}{8c^2}\sqrt{R} - \frac{b\Delta}{16c^2}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

3. $$\int x^2\sqrt{R}\,dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right)\sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{(2cx+b)\sqrt{R}}{4c} + \\ + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

4. $$\int x^3\sqrt{R}\,dx = \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right)\sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{(2cx+b)\sqrt{R}}{4c} - \\ - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

5. $$\int \sqrt{R^3}\,dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right)(2cx+b)\sqrt{R} + \frac{3\Delta^2}{128c^2}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

6. $$\int x\sqrt{R^3}\,dx = \frac{\sqrt{R^5}}{5c} - (2cx+b)\left(\frac{b}{16c^2}\sqrt{R^3} + \frac{3\Delta b}{128c^3}\sqrt{R}\right) - \frac{3\Delta^2 b}{256c^3}\int \frac{dx}{\sqrt{R}} \qquad (\text{see } \mathbf{2.261}).$$

2.261

7. $$\int x^2\sqrt{R^3}\,dx = \left(\frac{x}{6c} - \frac{7b}{60c^2}\right)\sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) + \\ + \left(\frac{7b^2}{4c} - a\right)\frac{\Delta^2}{256c^3}\int\frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261}\text{).}$$

2.261

8. $$\int x^3\sqrt{R^3}\,dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2}\right)\sqrt{R^5} - \\ - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) - \\ - \left(\frac{3b^2}{4c} - a\right)\frac{3\Delta^2 b}{512c^4}\int\frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261}\text{).}$$

2.261

2.263

1. $$\int\frac{x^m\,dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c}\int\frac{x^{m-1}\,dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c}\int\frac{x^{m-2}\,dx}{\sqrt{R^{2n+1}}}.$$

TI (193)a

For $m = 2n$

2. $$\int\frac{x^{2n}\,dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c}\int\frac{x^{2n-1}}{\sqrt{R^{2n+1}}}\,dx + \frac{1}{c}\int\frac{x^{2n-2}}{\sqrt{R^{2n-1}}}\,dx.$$

TI (194)a

3. $$\int\frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta}\int\frac{dx}{\sqrt{R^{2n-1}}}.$$

TI (189)

4. $$\int\frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}}\left\{1 + \sum_{k=1}^{n-1}\frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\frac{c^k}{\Delta^k}R^k\right\} \\ [n \geq 1].$$

2.264

1. $\int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

2. $$\int \frac{x\,dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**).

2.261

3. $$\int \frac{x^2\,dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{R} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right)\int \frac{dx}{\sqrt{R}}$$ (see **2.261**).

2.261

4. $$\int \frac{x^3\,dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^2}\right)\int \frac{dx}{\sqrt{R}}$$ (see **2.261**).

2.261

5. $$\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}.$$

6. $$\int \frac{x\,dx}{\sqrt{R^3}} = -\frac{2(2a+bx)}{\Delta\sqrt{R}}.$$

7. $$\int \frac{x^2\,dx}{\sqrt{R^3}} = -\frac{(\Delta - b^2)x - 2ab}{c\Delta\sqrt{R}} + \frac{1}{c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**).

8. $\displaystyle\int \frac{x^3\,dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b(10ac - 3b^2)x + a(8ac - 3b^2)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2}\int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

2.265

$$\int \frac{\sqrt{R^{2n+1}}}{x^m}\,dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-1}}\,dx + {} + \frac{(2n-m+4)c}{(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-2}}\,dx.$$

TI (195)

For $m = 1$

$$\int \frac{\sqrt{R^{2n+1}}}{x}\,dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2}\int \sqrt{R^{2n-1}}\,dx + a\int \frac{\sqrt{R^{2n-1}}}{x}\,dx.$$

TI (198)

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m}\,dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b}\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}}.$$

LA 169 (3)

For $m = 0$ see 2.260 2. and 2.260 3.

For $n = -1$ and $m = 1$:

2.266[8]

$$\begin{aligned}
\int \frac{dx}{x\sqrt{R}} &= -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} && [a > 0];\\
&= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} && [a < 0, \quad \Delta < 0];\\
&= \frac{1}{\sqrt{-a}} \operatorname{arctg} \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} && [a < 0];\\
&= -\frac{1}{\sqrt{a}} \operatorname{Arsh} \frac{2a+bx}{x\sqrt{\Delta}} && [a > 0, \quad \Delta > 0];\\
&= -\frac{1}{\sqrt{a}} \operatorname{Arth} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} && [a > 0];\\
&= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} && [a > 0, \quad \Delta = 0];\\
&= -\frac{2\sqrt{bx+cx^2}}{bx} && [a = 0, \quad b \neq 0].\\
&= -\frac{1}{\sqrt{a}} \operatorname{Arch}\left(\frac{2a+bx}{x\sqrt{-\Delta}}\right) && [a > 0, \quad \Delta < 0]
\end{aligned}$$

2.267

1. $$\int \frac{\sqrt{R}\,dx}{x} = \sqrt{R} + a\int \frac{dx}{x\sqrt{R}} + \frac{b}{2}\int \frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261} \text{ and } \mathbf{2.266}\text{).}$$

2.266

2.261

2. $$\int \frac{\sqrt{R}\,dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{R}} + c\int \frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261} \text{ and } 2.266\text{).}$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{bx+cx^2}}{x^2}\,dx = -\frac{2\sqrt{bx+cx^2}}{x} + c\int \frac{dx}{\sqrt{bx+cx^2}} \qquad \text{(see } \mathbf{2.261}\text{).}$$

2.261

3. $$\int \frac{\sqrt{R}\,dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right)\sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right)\int \frac{dx}{x\sqrt{R}} \qquad \text{(see2.266).}$$

2.266

For $a = 0$

$$\int \frac{\sqrt{bx+cx^2}}{x^3}\,dx = -\frac{2\sqrt{(bx+cx^2)^3}}{3bx^3}$$

2.266

2.261

5. $$\int \frac{\sqrt{R^3}}{x^2}\,dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx+b}{a}\sqrt{R^3} + \frac{3}{4}(2cx+3b)\sqrt{R} + \\ +\frac{3}{2}ab\int \frac{dx}{x\sqrt{R}} + \frac{3(4ac+b^2)}{8}\int \frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261} \text{ and } \mathbf{2.266}\text{).}$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^2} = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b}{4}\sqrt{bx+cx^2} + \frac{3b^2}{8}\int \frac{dx}{\sqrt{bx+cx^2}} \qquad \text{(see } \mathbf{2.261}\text{).}$$

2.261

6. $$\int \frac{\sqrt{R^3}}{x^3}\,dx = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right)\sqrt{R^5} + \frac{bcx+2ac+b^2}{4a^2}\sqrt{R^3} + \frac{3(bcx+2ac+b^2)}{4a}\sqrt{R} + \\ +\frac{3}{8}(4ac+b^2)\int \frac{dx}{x\sqrt{R}} + \frac{3}{2}bc\int \frac{dx}{\sqrt{R}} \qquad \text{(see } \mathbf{2.261} \text{ and } \mathbf{2.266}\text{).}$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^3}\,dx = \left(c - \frac{2b}{x}\right)\sqrt{bx+cx^2} + \frac{3bc}{2}\int \frac{dx}{\sqrt{bx+cx^2}} \qquad \text{(see } \mathbf{2.261}\text{).}$$

$$\int \frac{dx}{x^m\sqrt{R^{2n+1}}} = -\frac{1}{(m-1)ax^{m-1}\sqrt{R^{2n-1}}} - \\ -\frac{(2n+2m-3)b}{2(m-1)a}\int\frac{dx}{x^{m-1}\sqrt{R^{2n+1}}} - \frac{(2n+m-2)c}{(m-1)a}\int\frac{dx}{x^{m-2}\sqrt{R^{2n+1}}}.$$

2.268
TI (196)

For $m = 1$

$$\int \frac{dx}{x\sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a\sqrt{R^{2n-1}}} - \frac{b}{2a}\int\frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a}\int\frac{dx}{x\sqrt{R^{2n-1}}}.$$

TI (199)

For $a = 0$

$$\int \frac{dx}{x^m\sqrt{(bx+cx^2)^{2n+1}}} = -\frac{2}{(2n+2m-1)bx^m\sqrt{(bx+cx^2)^{2n-1}}} - \\ -\frac{(4n+2m-2)c}{(2n+2m-1)b}\int\frac{dx}{x^{m-1}\sqrt{(bx+cx^2)^{2n+1}}} \qquad \text{(cf. } \mathbf{2.265}\text{)}$$

2.265

2.269

1. $\int \frac{dx}{x\sqrt{R}}$ (see **2.266**).

2.266

2. $\int \frac{dx}{x^2\sqrt{R}} = -\frac{\sqrt{R}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{R}}$ (see **2.266**)

2.266

For $a = 0$

$$\int \frac{dx}{x^2\sqrt{bx+cx^2}} = \frac{2}{3}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x}\right)\sqrt{bx+cx^2}.$$

3. $$\int \frac{dx}{x^3\sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right)\sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right)\int \frac{dx}{x\sqrt{R}}$$ (see **2.266**)

2.266

For $a = 0$

$$\int \frac{dx}{x^3\sqrt{bx+cx^2}} = \frac{2}{5}\left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x}\right)\sqrt{bx+cx^2}.$$

4. $$\int \frac{dx}{x\sqrt{R^3}} = -\frac{2(bcx - 2ac + b^2)}{a\Delta\sqrt{R}} + \frac{1}{a}\int \frac{dx}{x\sqrt{R}}$$ (see **2.266**)

2.266

For $a = 0$

$$\int \frac{dx}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3}\left(-\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3}\right)\frac{1}{\sqrt{bx+cx^2}}.$$

5. $$\int \frac{dx}{x^2\sqrt{R^3}} = -\frac{A}{\sqrt{R}} + \frac{1}{a^2\Delta\sqrt{R}}[(3b^2 - 8ac)cx + (3b^2 - 10ac)b] - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{R}}$$
where $A = \left(-\frac{1}{ax} - \frac{b(10ac - 3b^2)}{a^2\Delta} - \frac{c(8ac - 3b^2)x}{a^2\Delta}\right)$. (see **2.266**).

2.266

For $a = 0$

$$\int \frac{dx}{x^2\sqrt{(bx+cx^2)^3}} = \frac{2}{5}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4}\right)\frac{1}{\sqrt{bx+cx^2}}.$$

6. $$\int \frac{dx}{x^3\sqrt{R^3}} = \left(-\frac{1}{ax^2} + \frac{5b}{2a^2x} - \frac{15b^4 - 62acb^2 + 24a^2c^2}{2a^3\Delta} - \right.$$
$$\left. - \frac{bc(15b^2 - 52ac)x}{2a^3\Delta}\right)\frac{1}{2\sqrt{R}} + \frac{15b^2 - 12ac}{8a^3}\int \frac{dx}{x\sqrt{R}} \qquad \text{(see } \mathbf{2.266}\text{)}.$$

2.266

For $a = 0$

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{7}\left(-\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5}\right)\frac{1}{\sqrt{bx+cx^2}}.$$

2.27 Forms containing $\sqrt{a + cx^2}$ and integral powers of x

Notations: $u = A + Cx^2$.

$$I_1 = \frac{1}{\sqrt{c}}\ln(x\sqrt{c} + u) \qquad [c > 0];$$
$$= \frac{1}{\sqrt{-c}}\arcsin x\sqrt{-\frac{c}{a}} \qquad [c < 0 \quad \text{and} \quad a > 0].$$
$$I_2 = \frac{1}{2\sqrt{a}}\ln\frac{u - \sqrt{a}}{u + \sqrt{a}} \qquad [a > 0 \quad \text{and} \quad c > 0];$$
$$= \frac{1}{2\sqrt{a}}\ln\frac{\sqrt{a} - u}{\sqrt{a} + u} \qquad [a > 0 \quad \text{and} \quad c < 0];$$
$$= \frac{1}{\sqrt{-a}}\operatorname{arcsec} x\sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}}\arccos\frac{1}{x}\sqrt{-\frac{a}{c}} \qquad [a < 0 \quad \text{and} \quad c > 0].$$

2.271

1. $$\int u^5\,dx = \frac{1}{6}xu^5 + \frac{5}{24}axu^3 + \frac{5}{16}a^2xu + \frac{5}{16}a^3I_1.$$

DW

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2. $\int u^3\,dx = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2 I_1.$

DW

3. $\int u\,dx = \frac{1}{2}xu + \frac{1}{2}aI_1.$

DW

4. $\int \frac{dx}{u} = I_1.$

DW

5. $\int \frac{dx}{u^3} = \frac{1}{a}\frac{x}{u}$

DW

6. $\int \frac{dx}{u^{2n+1}} = \frac{1}{a^n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}\binom{n-1}{k}\frac{c^k x^{2k+1}}{u^{2k+1}}.$

7. $\int \frac{x\,dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}}.$

DW

2.272

1. $\int x^2u^3\,dx = \frac{1}{6}\frac{xu^5}{c} - \frac{1}{24}\frac{axu^3}{c} - \frac{1}{16}\frac{a^2xu}{c} - \frac{1}{16}\frac{a^3}{c}I_1.$

DW

2. $\int x^2u\,dx = \frac{1}{4}\frac{xu^3}{c} - \frac{1}{8}\frac{axu}{c} - \frac{1}{8}\frac{a^2}{c}I_1.$

3. $\int \frac{x^2}{u}\,dx = \frac{1}{2}\frac{xu}{c} - \frac{1}{2}\frac{a}{c}I_1.$

DW

4. $\int \frac{x^2}{u^3}\,dx = -\frac{x}{cu} + \frac{1}{c}I_1.$

DW

5. $\int \frac{x^2}{u^5}\,dx = \frac{1}{3}\frac{x^3}{au^3}.$

DW

6. $\int \frac{x^2\,dx}{u^{2n+1}} = \frac{1}{a^{n-1}}\sum_{k=0}^{n-2}\frac{(-1)^k}{2k+3}\binom{n-2}{k}\frac{c^k x^{2k+3}}{u^{2k+3}}.$

7. $\int \frac{x^3\,dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2u^{2n-3}} + \frac{a}{(2n-1)c^2u^{2n-1}}.$

DW

2.273

1. $\int x^4u^3\,dx = \frac{1}{8}\frac{x^3u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2xu^3}{64c^2} + \frac{3a^3xu}{128c^2} + \frac{3a^4}{128c^2}I_1.$

DW

2. $\int x^4u\,dx = \frac{1}{6}\frac{x^3u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2xu}{16c^2} + \frac{a^3}{16c^2}I_1.$

DW

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3. $\int \frac{x^4}{u}\,dx = \frac{1}{4}\frac{x^3u}{c} - \frac{3}{8}\frac{axu}{c^2} + \frac{3}{8}\frac{a^2}{c^2}I_1.$

DW

DW

5. $$\int \frac{x^4}{u^5}\,dx = -\frac{x}{c^2u} - \frac{1}{3}\frac{x^3}{cu^3} + \frac{1}{c^2}I_1.$$

DW

6. $$\int \frac{x^4}{u^7}\,dx = \frac{1}{5}\frac{x^5}{au^5}.$$

DW

7. $$\int \frac{x^4\,dx}{u^{2n+1}} = \frac{1}{a^{n-2}}\sum_{k=0}^{n-3}\frac{(-1)^k}{2k+5}\binom{n-3}{k}\frac{c^kx^{2k+5}}{u^{2k+5}}.$$

8. $$\int \frac{x^5\,dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3u^{2n-5}} + \frac{2a}{(2n-3)c^{u2n-3}} - \frac{a^2}{(2n-1)c^3u^{2n-1}}.$$

DW

2.274

1. $$\int x^6u^3\,dx = \frac{1}{10}\frac{x^5u^5}{c} - \frac{ax^3u^5}{16c^2} + \frac{a^2xu^5}{32c^3} - \frac{a^3xu^3}{128c^3} - \frac{3a^4xu}{256c^3} - \frac{3}{256}\frac{a^5}{c^3}I_1.$$

2. $$\int x^6u\,dx = \frac{1}{8}\frac{x^5u^3}{c} - \frac{5}{48}\frac{ax^3u^3}{c^2} + \frac{5a^2xu^3}{64c^3} - \frac{5a^3xu}{128c^3} - \frac{5}{128}\frac{a^4}{c^3}I_1.$$

3. $$\int \frac{x^6}{u}\,dx = \frac{1}{6}\frac{x^5u}{c} - \frac{5}{24}\frac{ax^3u}{c^2} + \frac{5}{16}\frac{a^2xu}{c^3} - \frac{5}{16}\frac{a^3}{c^3}I_1.$$

DW

4. $$\int \frac{x^6}{u^3}\,dx = \frac{1}{4}\frac{x^5}{cu} - \frac{5}{8}\frac{ax^3}{c^2u} - \frac{15}{8}\frac{a^2x}{c^3u} + \frac{15}{8}\frac{a^2}{c^3}I_1.$$

DW

DW

6. $\int \frac{x^6}{u^7}\,dx = -\frac{23}{15}\frac{x^5}{cu^5} - \frac{7}{3}\frac{ax^3}{c^2u^5} - \frac{a^2x}{c^3u^5} + \frac{1}{c^3}I_1.$

DW

7. $\int \frac{x^6}{u^9}\,dx = \frac{1}{7}\frac{x^7}{au^7}.$

DW

8. $\int \frac{x^6\,dx}{u^{2n+1}} = \frac{1}{a^{n-3}}\sum_{k=0}^{n-4}\frac{(-1)^k}{2k+7}\binom{n-4}{k}\frac{c^kx^{2k+7}}{u^{2k+7}}.$

9. $\int \frac{x^7\,dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4u^{2n-7}} + \frac{3a}{(2n-5)c^4u^{2n-5}} - \frac{3a^2}{(2n-3)c^4u^{2n-3}} + \frac{a^3}{(2n-1)c^4u^{2n-1}}.$

DW

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2.275

1. $\int \frac{u^5}{x}\,dx = \frac{u^5}{5} + \frac{1}{3}au^3 + a^2u + a^3I_2$

DW

2. $\int \frac{u^3}{x}\,dx = \frac{u^3}{3} + au + a^2I_2.$

DW

3. $\int \frac{u}{x}\,dx = u + aI_2.$

DW

4. $\int \frac{dx}{xu} = I_2.$

5. $\displaystyle\int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1)a^{n-k}u^{2k+1}}.$

6. $\displaystyle\int \frac{u^5}{x^2}\,dx = -\frac{u^5}{x} + \frac{5}{4}cxu^3 + \frac{15}{8}acxu + \frac{15}{8}a^2 I_1.$

DW

7. $\displaystyle\int \frac{u^3}{x^2}\,dx = -\frac{u^3}{x} + \frac{3}{2}cxu + \frac{3}{2}aI_1.$

DW

8. $\displaystyle\int \frac{u}{x^2}\,dx = -\frac{u}{x} + cI_1.$

DW

9. $\displaystyle\int \frac{dx}{x^2u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u}\right)^{2k-1} \right\}.$

2.276

1. $\displaystyle\int \frac{u^5}{x^3}\,dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2 cI_2.$

DW

2. $\displaystyle\int \frac{u^3}{x^3}\,dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2.$

DW

3. $\displaystyle\int \frac{u}{x^3}\,dx = -\frac{u}{2x^2} + \frac{c}{2}I_2.$

DW

4. $\displaystyle\int \frac{dx}{x^3u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2.$

5. $\int \frac{dx}{x^3u^3} = -\frac{1}{2ax^2u} - \frac{3c}{2a^2u} - \frac{3c}{2a^2}I_2.$

DW

6. $\int \frac{dx}{x^3u^5} = -\frac{1}{2ax^2u^3} - \frac{5}{6}\frac{c}{a^2u^3} - \frac{5}{2}\frac{c}{a^3u} - \frac{5}{2}\frac{c}{a^3}I_2.$

DW

7. $\int \frac{u^5}{x^4}\,dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2xu}{2} + \frac{5}{2}acI_1.$

DW

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8. $\int \frac{u^3}{x^4}\,dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + cI_1.$

DW

9. $\int \frac{u}{x^4}\,dx = -\frac{u^3}{3ax^3}.$

DW

10. $\int \frac{dx}{x^4u^{2n+1}} = \frac{1}{a^{n+2}}\left\{-\frac{u^3}{3x^3} + (n+1)\frac{cu}{x} + \sum_{k=2}^{n+1}\frac{(-1)^k}{2k-3}\binom{n+1}{k}c^k\left(\frac{x}{u}\right)^{2k-3}\right\}.$

2.277

1. $\int \frac{u^3}{x^5}\,dx = -\frac{u^3}{4x^4} - \frac{3}{8}\frac{cu^3}{ax^2} + \frac{3}{8}\frac{c^2u}{a} + \frac{3}{8}c^2I_2.$

DW

2. $\int \frac{u}{x^5}\,dx = -\frac{u}{4x^4} - \frac{1}{8}\frac{cu}{ax^2} - \frac{1}{8}\frac{c^2}{a}I_2.$

3. $\int \frac{dx}{x^5 u} = -\frac{u}{4ax^4} + \frac{3}{8}\frac{cu}{a^2x^2} + \frac{3}{8}\frac{c^2}{a^2} I_2.$

DW

4. $\int \frac{dx}{x^5 u^3} = -\frac{1}{4ax^4 u} + \frac{5}{8}\frac{c}{a^2x^2u} + \frac{15}{8}\frac{c^2}{a^3u} + \frac{15}{8}\frac{c^2}{a^3} I_2.$

DW

2.278

1. $\int \frac{u^3}{x^6}\,dx = -\frac{u^5}{5ax^5}.$

DW

2. $\int \frac{u}{x^6}\,dx = -\frac{u^3}{5ax^5} + \frac{2}{15}\frac{cu^3}{a^2x^3}.$

DW

3. $\int \frac{dx}{x^6 u} = \frac{1}{a^3}\left(-\frac{u^5}{5x^5} + \frac{2}{3}\frac{cu^3}{x^3} - \frac{c^2u}{x}\right).$

DW

4. $\int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}}\left\{-\frac{u^5}{5x^5} + \frac{1}{3}\binom{n+2}{1}\frac{cu^3}{x^3} - \binom{n+2}{2}\frac{c^2u}{x} + \sum_{k=3}^{n+2}\frac{(-1)^k}{2k-5}\binom{n+2}{k}c^k\left(\frac{x}{u}\right)^{2k-5}\right\}.$

2.28 Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials

Notation: $R = A + Bx + Cx^2$

See also 2.252.

2.281[3]

$$\int \frac{dx}{(x+p)^n\sqrt{R}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c + (b-2pc)t + (a - bp + cp^2)t^2}} \qquad \left[t = \frac{1}{x+p} > 0\right].$$

$$1.^3\quad \int \frac{\sqrt{R}\,dx}{x+p} = c\int \frac{x\,dx}{\sqrt{R}} + (b-cp)\int \frac{dx}{\sqrt{R}} + (a-bp+cp^2)\int \frac{dx}{(x+p)\sqrt{R}} \qquad [x+p>0].$$

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$$2.\quad \int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}.$$

$$3.\quad \int \frac{\sqrt{R}\,dx}{(x+p)(x+q)} = \frac{1}{q-p}\int \frac{\sqrt{R}\,dx}{x+p} + \frac{1}{p-q}\int \frac{\sqrt{R}\,dx}{x+q}.$$

$$4.\quad \int \frac{(x+p)\sqrt{R}\,dx}{x+q} = \int \sqrt{R}\,dx + (p-q)\int \frac{\sqrt{R}\,dx}{x+q}.$$

$$5.\quad \int \frac{(rx+s)\,dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}.$$

2.283

$$\int \frac{(Ax+B)\,dx}{(p+R)^n\sqrt{R}} = \frac{A}{c}\int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c}\int \frac{(1-c\upsilon^2)^{n-1}\,d\upsilon}{\left[p+a-\frac{b^2}{4c}-cp\upsilon^2\right]^n},$$

where $u=\sqrt{R}$ and $\upsilon = \frac{b+2cx}{2c\sqrt{R}}$.

2.284

$$\int \frac{Ax+B}{(p+R)\sqrt{R}}\,dx = \frac{A}{c}I_1 + \frac{2Bc-Ab}{\sqrt{c^2p[b^2-4(a+p)c]}}I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}}\operatorname{arctg}\sqrt{\frac{R}{p}} \qquad [p>0]; \qquad = \frac{1}{2\sqrt{-p}}\ln\frac{\sqrt{-p}-\sqrt{R}}{\sqrt{-p}+\sqrt{R}} \qquad [p<0].$$

$$I_2 = \operatorname{arctg}\sqrt{\frac{p}{b^2-4(a+p)c}}\frac{b+2cx}{\sqrt{R}} \qquad [p\{b^2-4(a+p)c\}>0, \quad p<0];$$
$$= -\operatorname{arctg}\sqrt{\frac{p}{b^2-4(a+p)c}}\frac{b+2cx}{\sqrt{R}} \qquad [p\{b^2-4(a+p)c\}>0, \quad p>0];$$
$$= \frac{1}{2i}\ln\frac{\sqrt{4(a+p)c-b^2}\sqrt{R}+\sqrt{p}(b+2cx)}{\sqrt{4(a+p)c-b^2}\sqrt{R}-\sqrt{p}(b+2cx)} \qquad [p\{b^2-4(a+p)c\}<0, \quad p>0];$$
$$= \frac{1}{2i}\ln\frac{\sqrt{b^2-4(a+p)c}\sqrt{R}-\sqrt{-p}(b+2cx)}{\sqrt{b^2-4(a+p)c}\sqrt{R}+\sqrt{-p}(b+2cx)} \qquad [p\{b^2-4(a+p)c\}<0, \quad p<0].$$

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.290

Integrals of the form $\int R(x, \sqrt{P(x)})\,dx$, where $P(x)$ is a third- or fourth-degree polynomial can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see 8.11). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see 3.13-3.17).

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2.291

Certain integrals of the form $\int R(x, \sqrt{P(x)})\,dx$, where $k \geq 2$ and $P_n(x)$ is a polynomial of not more than fourth degree, can be reduced to integrals of the form $\int R(x, \sqrt[k]{P_n(x)})\,dx$. Below are examples of this procedure.

1. $$\int\frac{dx}{\sqrt{1-x^6}} = -\int\frac{dz}{\sqrt{3+3z^2+z^4}} \qquad \left[x^2 = \frac{1}{1+z^2}\right].$$

2. $$\int\frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2}\int\frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}} \qquad [x^2 = z].$$

3. $$\int(a+2bx+cx^2+gx^3)^{\pm 1/3}\,dx = \frac{3}{2}\int\frac{z^2A^{\pm\frac{1}{3}}\,dz}{B}$$

$$\left[a+2bx+cx^2 = z^3, \quad A = g\left(\frac{-b+\sqrt{b^2+(z^3-a)c}}{c}\right)^3 + z^3, \quad B = \sqrt{b^2+(z^3-a)\,c}\right]$$

$$4. \quad \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}} =$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad [x = z+\sqrt{z^2-1}];$$

$$= -\frac{1}{\sqrt{2}}\int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad [x = z-\sqrt{z^2-1}],$$

where

$$p = 2a(4z^3-3z)+2b(2z^2-1)+2cz+d.$$

$$5. \quad \int \frac{dx}{\sqrt{a+bx^2+cx^4+bx^6+ax^8}} = \frac{1}{2}\int \frac{dy}{\sqrt{y}\sqrt{a+by+cy^2+by^3+ay^4}} \qquad [x=\sqrt{y}];$$

$$= -\frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad [y = z+\sqrt{z^2-1}];$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad [y = z-\sqrt{z^2-1}],$$

where $p = 2a(2z^2-1)+2bz+c$.

$$6. \quad \int \frac{dx}{\sqrt{a+bx^4+cx^8}} = \frac{1}{2}\sqrt[8]{\frac{a}{c}}\int \frac{dt}{\sqrt{t}\sqrt{a+b_1t^2+at^4}} \qquad \left[x = \sqrt[8]{\frac{a}{c}}\sqrt{t}\right];$$

$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad [t = z+\sqrt{z^2-1}];$$

$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad [t = z-\sqrt{z^2-1}],$$

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where $p = 2a(2z^2-1)+b_1\,;\, b_1 = b\sqrt{\tfrac{a}{c}}$.

$$7. \quad \int \frac{x\,dx}{\sqrt[4]{a+bx^2+cx^4}} = 2\int \frac{z^2\,dz}{\sqrt{A+Bz^4}} \qquad [a+bx^2+cx^4 = z^4, \quad A = b^2-4ac, \quad B = 4c].$$

$$8. \quad \int \frac{dx}{\sqrt[4]{a+2bx^2+cx^4}} = \int \frac{\sqrt{b^2-a(c-z^4)}+b}{(c-z^4)\sqrt{b^2-a(c-z^4)}}z^2\,dz = \int R_1(z^4)z^2\,dz + \int \frac{R_2(z^4)z^2\,dz}{\sqrt{b^2-a(c-z^4)}},$$

where $R_1(z^4)$ and $R_2(z^4)$ are rational functions of z^4; $a + 2bx^2 + cx^4 = x^4 z^4$.

2.292

In certain cases, integrals of the form $\int R(x,\ \sqrt{P(x)})\,dx,$ where $P(x)$ is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = -f_1\left(\frac{1}{k^2 x}\right), \qquad f_2(x) = -f_2\left(\frac{1-k^2x}{k^2(1-x)}\right), \qquad f_3(x) = -f_3\left(\frac{1-x}{1-k^2x}\right),$$

hold, then

1. $$\int \frac{f_1(x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_1(z)\,dz \qquad \left[zx = \sqrt{x(1-x)(1-k^2x)}\right];$$

2. $$\int \frac{f_2(x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_2(z)\,dz \qquad \left[z = \frac{\sqrt{x(1-k^2x)}}{\sqrt{1-x}}\right];$$

3. $$\int \frac{f_3(x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_3(z)\,dz \qquad \left[z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}}\right],$$

where $R_1(z)$, $R_2(z)$, and $R_3(z)$ are rational functions of z.

2.3 The Exponential Function

2.31 Forms containing e^{ax}

2.311

2.312

a^x in the integrands should be replaced with $e^{x \ln a} = a^x$.

112
2.313

1. $$\int \frac{dx}{a + be^{mx}} = \frac{1}{am}[mx - \ln(a + be^{mx})].$$

PE (410)

2. $$\int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x).$$

PE (409)

2.314

$$\int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \operatorname{arctg}\left(e^{mx}\sqrt{\frac{a}{b}}\right) \qquad [ab > 0];$$
$$= \frac{1}{2m\sqrt{-ab}} \ln \frac{b + e^{mx}\sqrt{-ab}}{b - e^{mx}\sqrt{-ab}} \qquad [ab < 0].$$

PE (411)

2.315

$$\int \frac{dx}{\sqrt{a + be^{mx}}} = \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} \qquad [a > 0];$$
$$= \frac{2}{m\sqrt{-a}} \operatorname{arctg} \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} \qquad [a < 0].$$

2.32 The exponential combined with rational functions of x

2.321

1. $$\int x^m e^{ax}\, dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1} e^{ax}\, dx.$$

2. $$\int x^n e^{ax}\, dx = e^{ax}\left(\frac{x^n}{a} + \sum_{k=1}^{n} (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k}\right).$$

2.322

1. $$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$$

2. $$\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right).$$

3. $$\int x^3e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right).$$

4*. $$\int x^4e^{ax}\,dx = e^{ax}\left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5}\right).$$

2.323

$$\int P_m(x)e^{ax}\,dx = \frac{e^{ax}}{a}\sum_{k=0}^{m}(-1)^k\frac{P^{(k)}(x)}{a^k},$$

where $P_m(x)$ is a polynomial in x of degree m and $P^{(k)}(x)$ is the k-th derivative of $P_m(x)$ with respect to x.

2.324

1. $$\int \frac{e^{ax}\,dx}{x^m} = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int\frac{e^{ax}\,dx}{x^{m-1}}\right].$$

2. $$\int \frac{e^{ax}}{x^n}\,dx = -e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!}\mathrm{Ei}(ax).$$

2.325

1. $$\int \frac{e^{ax}}{x}\,dx = \mathrm{Ei}(ax).$$

2.326

$$\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$$

2.33[8]

$$\int e^{-(ax^2+2bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\operatorname{erf}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right) \qquad [a\neq 0]$$

2.4 Hyperbolic Functions

2.41-2.43 Powers of sh x, ch x, th x and cth x

2.411

$$\begin{aligned}
\int \operatorname{sh}^p x \operatorname{ch}^q x\,dx &= \\
&= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \operatorname{sh}^p x \operatorname{ch}^{q-2} x\,dx; \\
&= \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{p+q} - \frac{p-1}{p+q}\int \operatorname{sh}^{p-2} x \operatorname{ch}^{q} x\,dx; \\
&= \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{q+1} - \frac{p-1}{q+1}\int \operatorname{sh}^{p-2} x \operatorname{ch}^{q+2} x\,dx; \\
&= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+1} - \frac{q-1}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^{q-2} x\,dx; \\
&= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{p+1} - \frac{p+q+2}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^{q} x\,dx; \\
&= -\frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \operatorname{sh}^{p} x \operatorname{ch}^{q+2} x\,dx.
\end{aligned}$$

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2.412

1. $$\begin{aligned}
\int \operatorname{sh}^p x \operatorname{ch}^{2n} x\,dx = \frac{\operatorname{sh}^{p+1} x}{2n+p}\Bigg\{&\operatorname{ch}^{2n-1} x + \\
&+ \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\operatorname{ch}^{2n-2k-1} x\Bigg\} + \\
&+ \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \operatorname{sh}^p x\,dx.
\end{aligned}$$

p -2 -4 $\ldots$ $-2n$ p

number and $n = 0$, we have

$$2.\quad \int \operatorname{sh}^{2m} x\,dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\operatorname{sh}(2m-2k)x}{2m-2k}.$$

TI (543)

$$3.\quad \int \operatorname{sh}^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \frac{\operatorname{ch}(2m-2k+1)x}{2m-2k+1};$$
$$= (-1)^n \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\operatorname{ch}^{2k+1} x}{2k+1}.$$

GU ((351)) (5)
TI (544)

$$4.\quad \int \operatorname{sh}^p x \operatorname{ch}^{2n+1} x\,dx = \frac{\operatorname{sh}^{p+1} x}{2n+p+1} \left\{ \operatorname{ch}^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)\operatorname{ch}^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: -1, -3, $\ldots$, $-(2n+1)$.

2.413

$$1.\quad \int \operatorname{ch}^p x \operatorname{sh}^{2n} x\,dx = \frac{\operatorname{ch}^{p+1} x}{2n+p} \Bigg\{ \operatorname{sh}^{2n-1} x + $$
$$+ \sum_{k=1}^{n-1} (-1)^k \frac{(2n-1)(2n-3)\ldots(2n-2k+1)\operatorname{sh}^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)} \Bigg\} +$$
$$+ (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \operatorname{ch}^p x\,dx.$$

This formula is applicable for arbitrary real p except for the following negative even integers: -2, -4, $\ldots$, $-2n$. If p is a natural number and $n = 0$, we have

$$2.\quad \int \operatorname{ch}^{2m} x\,dx = \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \frac{\operatorname{sh}(2m-2k)x}{2m-2k}.$$

TI (541)

$$3.\quad \int \operatorname{ch}^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \frac{\operatorname{sh}(2m-2k+1)x}{2m-2k+1};$$
$$= \sum_{k=0}^{m} \binom{m}{k} \frac{\operatorname{sh}^{2k+1} x}{2k+1}.$$

4. $$\int \operatorname{ch}^p x \operatorname{sh}^{2n+1} x\,dx = \frac{\operatorname{ch}^{p+1} x}{2n+p+1}\left\{\operatorname{sh}^{2n} x + \right.$$
$$\left. + \sum_{k=1}^{n} (-1)^k \frac{2^k n(n-1)\dots(n-k+1)\operatorname{sh}^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: -1, -3, $\dots$, $-(2n+1)$.

2.414

1. $$\int \operatorname{sh} ax\,dx = \frac{1}{a}\operatorname{ch} ax.$$

2. $$\int \operatorname{sh}^2 ax\,dx = \frac{1}{4a}\operatorname{sh} 2ax - \frac{x}{2}.$$

3. $$\int \operatorname{sh}^3 x\,dx = -\frac{3}{4}\operatorname{ch} x + \frac{1}{12}\operatorname{ch} 3x = \frac{1}{3}\operatorname{ch}^3 x - \operatorname{ch} x.$$

4. $$\int \operatorname{sh}^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\operatorname{sh} 2x + \frac{1}{32}\operatorname{sh} 4x = \frac{3}{8}x - \frac{3}{8}\operatorname{sh} x \operatorname{ch} x + \frac{1}{4}\operatorname{sh}^3 x \operatorname{ch} x.$$

5. $$\int \operatorname{sh}^5 x\,dx = \frac{5}{8}\operatorname{ch} x - \frac{5}{48}\operatorname{ch} 3x + \frac{1}{80}\operatorname{ch} 5x;$$
$$= \frac{4}{5}\operatorname{ch} x + \frac{1}{5}\operatorname{sh}^4 x \operatorname{ch} x - \frac{4}{15}\operatorname{ch}^3 x.$$

6. $$\int \operatorname{sh}^6 x\,dx = -\frac{5}{16}x + \frac{15}{64}\operatorname{sh} 2x - \frac{3}{64}\operatorname{sh} 4x + \frac{1}{192}\operatorname{sh} 6x;$$
$$= -\frac{5}{16}x + \frac{1}{6}\operatorname{sh}^5 x \operatorname{ch} x - \frac{5}{24}\operatorname{sh}^3 x \operatorname{ch} x + \frac{5}{16}\operatorname{sh} x \operatorname{ch} x.$$

7. $$\int \operatorname{sh}^7 x\,dx = -\frac{35}{64}\operatorname{ch} x + \frac{7}{64}\operatorname{ch} 3x - \frac{7}{320}\operatorname{ch} 5x + \frac{1}{448}\operatorname{ch} 7x;$$
$$= -\frac{24}{35}\operatorname{ch} x + \frac{8}{35}\operatorname{ch}^3 x - \frac{6}{35}\operatorname{ch} x \operatorname{sh}^4 x + \frac{1}{7}\operatorname{ch} x \operatorname{sh}^6 x.$$

8. $\int \operatorname{ch} ax\, dx = \frac{1}{a}\operatorname{sh} ax.$

9. $\int \operatorname{ch}^2 ax\, dx = \frac{x}{2} + \frac{1}{4a}\operatorname{sh} 2ax.$

10. $\int \operatorname{ch}^3 x\, dx = \frac{3}{4}\operatorname{sh} x + \frac{1}{12}\operatorname{sh} 3x = \operatorname{sh} x + \frac{1}{3}\operatorname{sh}^3 x.$

11. $\int \operatorname{ch}^4 x\, dx = \frac{3}{8}x + \frac{1}{4}\operatorname{sh} 2x + \frac{1}{32}\operatorname{sh} 4x = \frac{3}{8}x + \frac{3}{8}\operatorname{sh} x \operatorname{ch} x + \frac{1}{4}\operatorname{sh} x \operatorname{ch}^3 x.$

12. $$\int \operatorname{ch}^5 x\, dx = \frac{5}{8}\operatorname{sh} x + \frac{5}{48}\operatorname{sh} 3x + \frac{1}{80}\operatorname{sh} 5x; \\ = \frac{4}{5}\operatorname{sh} x + \frac{1}{5}\operatorname{ch}^4 x \operatorname{sh} x + \frac{4}{15}\operatorname{sh}^3 x.$$

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13. $$\int \operatorname{ch}^6 x\, dx = \frac{5}{16}x + \frac{15}{64}\operatorname{sh} 2x + \frac{3}{64}\operatorname{sh} 4x + \frac{1}{192}\operatorname{sh} 6x; \\ = \frac{5}{16}x + \frac{5}{16}\operatorname{sh} x \operatorname{ch} x + \frac{5}{24}\operatorname{sh} x \operatorname{ch}^3 x + \frac{1}{6}\operatorname{sh} x \operatorname{ch}^5 x.$$

14. $$\int \operatorname{ch}^7 x\, dx = \frac{35}{64}\operatorname{sh} x + \frac{7}{64}\operatorname{sh} 3x + \frac{7}{320}\operatorname{sh} 5x + \frac{1}{448}\operatorname{sh} 7x; \\ = \frac{24}{35}\operatorname{sh} x + \frac{8}{35}\operatorname{sh}^3 x + \frac{6}{35}\operatorname{sh} x \operatorname{ch}^4 x + \frac{1}{7}\operatorname{sh} x \operatorname{ch}^6 x.$$

2.415

1. $\int \operatorname{sh} ax \operatorname{ch} bx\, dx = \frac{\operatorname{ch}(a+b)x}{2(a+b)} + \frac{\operatorname{ch}(a-b)x}{2(a-b)}.$

2. $\int \operatorname{sh} ax \operatorname{ch} ax\, dx = \frac{1}{4a}\operatorname{ch} 2ax.$

3. $\int \operatorname{sh}^2 x \operatorname{ch} x \, dx = \frac{1}{3} \operatorname{sh}^3 x.$

4. $\int \operatorname{sh}^3 x \operatorname{ch} x \, dx = \frac{1}{4} \operatorname{sh}^4 x.$

5. $\int \operatorname{sh}^4 x \operatorname{ch} x \, dx = \frac{1}{5} \operatorname{sh}^5 x.$

6. $\int \operatorname{sh} x \operatorname{ch}^2 x \, dx = \frac{1}{3} \operatorname{ch}^3 x.$

7. $\int \operatorname{sh}^2 x \operatorname{ch}^2 x \, dx = -\frac{x}{8} + \frac{1}{32} \operatorname{sh} 4x.$

8. $\int \operatorname{sh}^3 x \operatorname{ch}^2 x \, dx = \frac{1}{5} \left(\operatorname{sh}^2 x - \frac{2}{3} \right) \operatorname{ch}^3 x.$

9. $\int \operatorname{sh}^4 x \operatorname{ch}^2 x \, dx = \frac{x}{16} - \frac{1}{64} \operatorname{sh} 2x - \frac{1}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x.$

10. $\int \operatorname{sh} x \operatorname{ch}^3 x \, dx = \frac{1}{4} \operatorname{ch}^4 x.$

11. $\int \operatorname{sh}^2 x \operatorname{ch}^3 x \, dx = \frac{1}{5} \left(\operatorname{ch}^2 x + \frac{2}{3} \right) \operatorname{sh}^3 x.$

12. $$\int \operatorname{sh}^3 x \operatorname{ch}^3 x \, dx = -\frac{3}{64} \operatorname{ch} 2x + \frac{1}{192} \operatorname{ch} 6x = \frac{1}{48} \operatorname{ch}^3 2x - \frac{1}{16} \operatorname{ch} 2x;$$
$$= \frac{\operatorname{sh}^6 x}{6} + \frac{\operatorname{sh}^4 x}{4} = \frac{\operatorname{ch}^6 x}{6} - \frac{\operatorname{ch}^4 x}{4}.$$

14. $$\int \operatorname{sh} x \operatorname{ch}^4 x\, dx = \frac{1}{5}\operatorname{ch}^5 x.$$

15. $$\int \operatorname{sh}^2 x \operatorname{ch}^4 x\, dx = -\frac{x}{16} - \frac{1}{64}\operatorname{sh} 2x + \frac{1}{64}\operatorname{sh} 4x + \frac{1}{192}\operatorname{sh} 6x.$$

16. $$\int \operatorname{sh}^3 x \operatorname{ch}^4 x\, dx = \frac{1}{7}\operatorname{ch}^3 x \left(\operatorname{sh}^4 x + \frac{3}{5}\operatorname{sh}^2 x - \frac{2}{5}\right) = \frac{1}{7}\left(\operatorname{sh}^2 x - \frac{2}{5}\right)\operatorname{ch}^5 x.$$

17. $$\int \operatorname{sh}^4 x \operatorname{ch}^4 x\, dx = \frac{3x}{128} - \frac{1}{128}\operatorname{sh} 4x + \frac{1}{1024}\operatorname{sh} 8x.$$

2.416

1. $$\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{2n} x}\, dx = \frac{\operatorname{sh}^{p+1}}{2n-1}\left\{\operatorname{sech}^{2n-1} x + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\operatorname{sech}^{2n-2k-1} x\right\} + \\ + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \operatorname{sh}^p x\, dx.$$

This formula is applicable for arbitrary real p. For $\int \operatorname{sh}^p x\, dx$, where p is a natural number, see 2.412 2. and 2.412 3. For $n = 0$ and p a negative integer, we have for this integral:

2. $$\int \frac{dx}{\operatorname{sh}^{2m} x} = \frac{\operatorname{ch} x}{2m-1}\left\{-\operatorname{cosech}^{2m-1} x + \right. \\ \left. + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)}\operatorname{cosech}^{2m-2k-1} x\right\}.$$

3. $$\int \frac{dx}{\operatorname{sh}^{2m+1} x} = \frac{\operatorname{ch} x}{2m}\left\{-\operatorname{cosech}^{2m} x + \right. \\ \left. + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)}\operatorname{cosech}^{2m-2k} x\right\} + \\ + (-1)^m \frac{(2m-1)!!}{(2m)!!} \ln \operatorname{th} \frac{x}{2}.$$

$$1.\quad \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{2n+1} x}\,dx = \frac{\operatorname{sh}^{p+1} x}{2n}\Bigg\{\operatorname{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)}\operatorname{sech}^{2n-2k} x\Bigg\} + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\operatorname{sh}^p x}{\operatorname{ch} x}\,dx.$$

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This formula is applicable for arbitrary real p. For $n=0$ and p integral, we have

$$2.\quad \int \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch} x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\operatorname{sh}^{2k} x + (-1)^m \ln\,\operatorname{ch} x; = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\binom{m}{k}\operatorname{ch}^{2k} x + (-1)^m \ln\,\operatorname{ch} x \qquad [m \geq 1].$$

$$3.\quad \int \frac{\operatorname{sh}^{2m} x}{\operatorname{ch} x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k-1}\operatorname{sh}^{2k-1} x + (-1)^m \operatorname{arctg}(\operatorname{sh} x) \qquad [m \geq 1].$$

$$4.\quad \int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch} x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln \operatorname{th} x.$$

$$5.\quad \int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch} x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \operatorname{arctg} \operatorname{sh} x.$$

2.418

$$1.\quad \int \frac{\operatorname{ch}^p x}{\operatorname{sh}^{2n} x}\,dx = -\frac{\operatorname{ch}^{p+1} x}{2n-1}\Bigg\{\operatorname{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\operatorname{cosech}^{2n-2k-1} x\Bigg\} + \frac{(-1)^n(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \operatorname{ch}^p x\,dx.$$

2. $$\int \frac{dx}{\operatorname{ch}^{2m} x} = \frac{\operatorname{sh} x}{2m-1}\left\{\operatorname{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{sech}^{2m-2k-1} x\right\}.$$

3. $$\int \frac{dx}{\operatorname{ch}^{2m+1} x} = \frac{\operatorname{sh} x}{2m}\left\{\operatorname{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \operatorname{sech}^{2m-2k} x\right\} + \\ + \frac{(2m-1)!!}{(2m)!!} \operatorname{arctg} \operatorname{sh} x.$$

2.419

1. $$\int \frac{\operatorname{ch}^p x}{\operatorname{sh}^{2n+1} x}\,dx = -\frac{\operatorname{ch}^{p+1} x}{2n}\left\{\operatorname{cosech}^{2n} x + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \operatorname{cosech}^{2n-2k} x\right\} + \\ + \frac{(-1)^n (2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\operatorname{ch}^p x}{\operatorname{sh} x}\,dx.$$

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This formula is applicable for arbitrary real p. For $n = 0$ and p an integer

2. $$\int \frac{\operatorname{ch}^{2m} x}{\operatorname{sh} x}\,dx = \sum_{k=1}^{m} \frac{\operatorname{ch}^{2k-1} x}{2k-1} + \ln \operatorname{th} \frac{x}{2}.$$

3. $$\int \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh} x}\,dx = \sum_{k=1}^{m} \frac{\operatorname{ch}^{2k} x}{2k} + \ln \operatorname{sh} x; \\ = \sum_{k=1}^{m} \binom{m}{k} \frac{\operatorname{sh}^{2k} x}{2k} + \ln \operatorname{sh} x.$$

4. $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^{2m} x} = \sum_{k=1}^{m} \frac{\operatorname{sech}^{2m-2k+1} x}{2m-2k+1} + \ln \operatorname{th} \frac{x}{2}.$$

2.421

1. $$\int \frac{\operatorname{sh}^{2n+1} x}{\operatorname{ch}^m x}\,dx = \sum_{\substack{k=0\\k\neq\frac{m-1}{2}}}^{n} (-1)^{n+k}\binom{n}{k}\frac{\operatorname{ch}^{2k-m+1} x}{2k-1+1} + s(-1)^{n+\frac{m-1}{2}}\binom{n}{\frac{m-1}{2}}\ln\operatorname{ch} x.$$

2. $$\int \frac{\operatorname{ch}^{2n+1} x}{\operatorname{sh}^m x}\,dx = \sum_{\substack{k=0\\k\neq\frac{m-1}{2}}}^{n} \binom{n}{k}\frac{\operatorname{sh}^{2k-m+1} x}{2k-m+1} + s\binom{n}{\frac{m-1}{2}}\ln\operatorname{sh} x.$$

[In formulas 2.421 1. and 2.421 2., $s=1$ for m odd and $m<2n+1$; in all other cases, $s=0$.]

2.422

1. $$\int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1}\binom{m+n-1}{k}\operatorname{th}^{2k-2m+1} x.$$

GI ((351))(11, 13)

2. $$\int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} = \sum_{\substack{k=0\\k\neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k}\binom{m+n}{k}\operatorname{th}^{2k-2m} x + (-1)^m\binom{m+n}{m}\ln\operatorname{th} x.$$

GI ((351))(15)

2.423

1. $$\int \frac{dx}{\operatorname{sh} x} = \ln\operatorname{th}\frac{x}{2} = \frac{1}{2}\ln\frac{\operatorname{ch} x-1}{\operatorname{ch} x+1}.$$

2. $$\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$$

3. $$\int \frac{dx}{\operatorname{sh}^3 x} = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} - \frac{1}{2}\ln\operatorname{th}\frac{x}{2}.$$

4. $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x} = -\frac{\operatorname{ch} x}{3\operatorname{sh}^3 x} + \frac{2}{3}\operatorname{cth} x = -\frac{1}{3}\operatorname{cth}^3 x + \operatorname{cth} x.$

5. $\displaystyle\int \frac{dx}{\operatorname{sh}^5 x} = -\frac{\operatorname{ch} x}{4\operatorname{sh}^4 x} + \frac{3}{8}\frac{\operatorname{ch} x}{\operatorname{sh}^2 x} + \frac{3}{8}\ln \operatorname{th}\frac{x}{2}.$

6. $$\begin{aligned}\int \frac{dx}{\operatorname{sh}^6 x} &= -\frac{\operatorname{ch} x}{5\operatorname{sh}^5 x} + \frac{4}{15}\operatorname{cth}^3 x - \frac{4}{5}\operatorname{cth} x;\\ &= -\frac{1}{5}\operatorname{cth}^5 x + \frac{2}{3}\operatorname{cth}^3 x - \operatorname{cth} x.\end{aligned}$$

7. $\displaystyle\int \frac{dx}{\operatorname{sh}^7 x} = -\frac{\operatorname{ch} x}{6\operatorname{sh}^2 x}\left(\frac{1}{\operatorname{sh}^4 x} - \frac{5}{4\operatorname{sh}^2 x} + \frac{15}{8}\right) - \frac{5}{16}\ln\ \operatorname{th}\frac{x}{2}.$

8. $\displaystyle\int \frac{dx}{\operatorname{sh}^8 x} = \operatorname{cth} x - \operatorname{cth}^3 x + \frac{3}{5}\operatorname{cth}^5 x - \frac{1}{7}\operatorname{cth}^7 x.$

9. $$\begin{aligned}\int \frac{dx}{\operatorname{ch} x} &= \operatorname{arctg}(\operatorname{sh} x) = 2\operatorname{arctg}(e^x);\\ &= \arcsin(\operatorname{th} x);\\ &= \operatorname{gd} x.\end{aligned}$$

10. $\displaystyle\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$

11. $\displaystyle\int \frac{dx}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} + \frac{1}{2}\operatorname{arctg}(\operatorname{sh} x).$

12. $$\begin{aligned}\int \frac{dx}{\operatorname{ch}^4 x} &= \frac{\operatorname{sh} x}{3\operatorname{ch}^3 x} + \frac{2}{3}\operatorname{th} x;\\ &= -\frac{1}{3}\operatorname{th}^3 x + \operatorname{th} x.\end{aligned}$$

13. $\displaystyle\int \frac{dx}{\operatorname{ch}^5 x} = \frac{\operatorname{sh} x}{4\operatorname{ch}^4 x} + \frac{3}{8}\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + \frac{3}{8}\operatorname{arctg}(\operatorname{sh} x).$

14. $$\int \frac{dx}{\operatorname{ch}^6 x} = \frac{\operatorname{sh} x}{5\operatorname{ch}^5 x} - \frac{4}{15}\operatorname{th}^3 x + \frac{4}{5}\operatorname{th} x;$$
$$= \frac{1}{5}\operatorname{th}^5 x - \frac{2}{3}\operatorname{th}^3 x + \operatorname{th} x.$$

15.[6] $$\int \frac{dx}{\operatorname{ch}^7 x} = \frac{\operatorname{sh} x}{6\operatorname{ch}^2 x}\left(\frac{1}{\operatorname{ch}^4 x} + \frac{5}{4\operatorname{ch}^2 x} + \frac{15}{8}\right) + \frac{5}{16}\operatorname{arctg}(\operatorname{sh} x).$$

16. $$\int \frac{dx}{\operatorname{ch}^8 x} = -\frac{1}{7}\operatorname{th}^7 x + \frac{3}{5}\operatorname{th}^5 x - \operatorname{th}^3 x + \operatorname{th} x.$$

17. $$\int \frac{\operatorname{sh} x}{\operatorname{ch} x}\, dx = \ln\, \operatorname{ch} x.$$

18. $$\int \frac{\operatorname{sh}^2 x}{\operatorname{ch} x}\, dx = \operatorname{sh} x - \operatorname{arctg}(\operatorname{sh} x).$$

19. $$\int \frac{\operatorname{sh}^3 x}{\operatorname{ch} x}\, dx = \frac{1}{2}\operatorname{sh}^2 x - \ln \operatorname{ch} x;$$
$$= \frac{1}{2}\operatorname{ch}^2 x - \ln \operatorname{ch} x.$$

20. $$\int \frac{\operatorname{sh}^4 x}{\operatorname{ch} x}\, dx = \frac{1}{3}\operatorname{sh}^3 x - \operatorname{sh} x + \operatorname{arctg}(\operatorname{sh} x).$$

21. $$\int \frac{\operatorname{sh} x}{\operatorname{ch}^2 x}\, dx = -\frac{1}{\operatorname{ch} x}.$$

22. $$\int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\, dx = x - \operatorname{th} x.$$

33. $$\int \frac{\operatorname{ch} x}{\operatorname{sh} x}\, dx = \ln \operatorname{sh} x.$$

34. $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh} x}\, dx = \operatorname{ch} x + \ln \operatorname{th} \frac{x}{2}.$$

35. $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh} x}\, dx = \frac{1}{2}\operatorname{ch}^2 x + \ln \operatorname{sh} x.$$

36. $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh} x}\, dx = \frac{1}{3}\operatorname{ch}^3 x + \operatorname{ch} x + \ln \operatorname{th} \frac{x}{2}.$$

37. $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^2 x}\, dx = -\frac{1}{\operatorname{sh} x}.$$

38. $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x}\, dx = x - \operatorname{cth} x.$$

39. $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^2 x}\, dx = \operatorname{sh} x - \frac{1}{\operatorname{sh} x}.$$

40. $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^2 x}\, dx = \frac{3}{2}x + \frac{1}{4}\operatorname{sh} 2x - \operatorname{cth} x.$$

41. $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^3 x}\, dx = -\frac{1}{2\operatorname{sh}^2 x}; \\ = -\frac{1}{2}\operatorname{cth}^2 x.$$

42. $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^3 x}\, dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \ln\, \operatorname{th}\frac{x}{2}.$$

43. $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^3 x}\, dx = -\frac{1}{2\operatorname{sh}^2 x} + \ln \operatorname{sh} x; \\ = -\frac{1}{2}\operatorname{cth}^2 x + \ln\, \operatorname{sh} x.$$

44. $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^3 x}\, dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \operatorname{ch} x + \frac{3}{2}\ln \operatorname{th}\frac{x}{2}.$$

45. $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^4 x}\, dx = -\frac{1}{3\operatorname{sh}^3 x}.$$

46. $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^4 x}\, dx = -\frac{1}{3}\operatorname{cth}^3 x.$$

47. $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^4 x}\, dx = -\frac{1}{\operatorname{sh} x} - \frac{1}{3\operatorname{sh}^3 x}.$$

48. $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^4 x}\, dx = -\frac{1}{3}\operatorname{cth}^3 x - \operatorname{cth} x + x.$$

49. $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln \operatorname{th} x.$$

50. $\displaystyle\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^2 x} = \frac{1}{\operatorname{ch} x} + \ln \operatorname{th} \frac{x}{2}.$

51. $\displaystyle\int \frac{dx}{\operatorname{sh} x \ \operatorname{ch}^3 x} = \frac{1}{2 \operatorname{ch}^2 x} + \ln \ \operatorname{th} x;$

$$= -\frac{1}{2} \operatorname{th}^2 x + \ln \operatorname{th} x.$$

52. $\displaystyle\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^4 x} = \frac{1}{\operatorname{ch} x} + \frac{1}{3 \operatorname{ch}^3 x} + \ln \operatorname{th} \frac{x}{2}.$

53. $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch} x} = -\frac{1}{\operatorname{sh} x} - \operatorname{arctg} \operatorname{sh} x.$

54. $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2 \operatorname{cth} 2x.$

55. $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^3 x} = -\frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} - \frac{1}{\operatorname{sh} x} - \frac{3}{2} \operatorname{arctg} \operatorname{sh} x.$

56. $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^4 x} = \frac{1}{3 \operatorname{sh} x \operatorname{ch}^3 x} - \frac{8}{3} \operatorname{cth} 2x.$

57. $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \ \operatorname{ch} x} = -\frac{1}{2 \operatorname{sh}^2 x} - \ln \operatorname{th} x;$

$$= -\frac{1}{2} \operatorname{cth}^2 x + \ln \operatorname{cth} x.$$

58. $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^2 x} = -\frac{1}{\operatorname{ch} x} - \frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{3}{2} \ln \operatorname{th} \frac{x}{2}.$

59. $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \ \operatorname{ch}^3 x} = -\frac{2 \operatorname{ch} 2x}{\operatorname{sh}^2 2x} - 2 \ \ln \operatorname{th} x;$

$$= \frac{1}{2} \operatorname{th}^2 x - \frac{1}{2} \operatorname{cth}^2 x - 2 \ \ln \operatorname{th} x.$$

60. $$\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^4 x} = -\frac{2}{\operatorname{ch} x} - \frac{1}{3 \operatorname{ch}^2 x} - \frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{5}{2} \ln \operatorname{th} \frac{x}{2}.$$

61. $$\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch} x} = \frac{1}{\operatorname{sh} x} - \frac{1}{3 \operatorname{sh}^3 x} + \operatorname{arctg} \operatorname{sh} x.$$

62. $$\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^2 x} = -\frac{1}{3 \operatorname{ch} x \operatorname{sh}^3 x} + \frac{8}{3} \operatorname{cth} 2x.$$

63. $$\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^3 x} = \frac{2}{\operatorname{sh} x} - \frac{1}{3 \operatorname{sh}^3 x} + \frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{5}{2} \operatorname{arctg} \operatorname{sh} x.$$

64. $$\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^4 x} = 8 \operatorname{cth} 2x - \frac{8}{3} \operatorname{cth}^3 2x.$$

2.424

1. $$\int \operatorname{th}^p x \, dx = -\frac{\operatorname{th}^{p-1} x}{p-1} + \int \operatorname{th}^{p-2} x \, dx \qquad [p \neq 1].$$

2. $$\int \operatorname{th}^{2n+1} x \, dx = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k} \binom{n}{k} \frac{1}{\operatorname{ch}^{2k} x} + \ln \operatorname{ch} x; \\ = -\sum_{k=1}^{n} \frac{\operatorname{th}^{2n-2k+2} x}{2n-2k+2} + \ln \operatorname{ch} x.$$

3. $$\int \operatorname{th}^{2n} x \, dx = -\sum_{k=1}^{n} \frac{\operatorname{th}^{2n-2k+1} x}{2n-2k+1} + x.$$

GU ((351))(12)

4. $$\int \operatorname{cth}^p x \, dx = -\frac{\operatorname{cth}^{p-1} x}{p-1} + \int \operatorname{cth}^{p-2} x \, dx \qquad [p \neq 1].$$

5. $$\int \operatorname{cth}^{2n+1} x \, dx = -\sum_{k=1}^{n} \frac{1}{2n} \binom{n}{k} \frac{1}{\operatorname{sh}^{2k} x} + \ln \operatorname{sh} x; \\ = -\sum_{k=1}^{n} \frac{\operatorname{cth}^{2n-2k+2} x}{2n-2k+2} + \ln \operatorname{sh} x.$$

6. $$\int \operatorname{cth}^{2n} x\, dx = -\sum_{k=1}^{n} \frac{\operatorname{cth}^{2n-2k+1} x}{2n-2k+1} + x.$$

GU ((351))(14)

For formulas containing powers of $\operatorname{th} x$ and $\operatorname{cth} x$ equal to $n = 1, 2, 3, 4$, see 2.423 17., 2.423 22., 2.423 27., 2.423 32., 2.423 33., 2.423 38., 2.423 43., 2.423 48.

Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

2.425

1. $$\int \operatorname{sh}(ax+b)\operatorname{sh}(cx+d)\, dx = \frac{1}{2(a+c)}\operatorname{sh}[(a+c)x+b+d] - \frac{1}{2(a-c)}\operatorname{sh}[(a-c)x+b-d] \qquad [a^2 \neq c^2].$$

GU ((352))(2a)

2. $$\int \operatorname{sh}(ax+b)\operatorname{ch}(cx+d)\, dx = \frac{1}{2(a+c)}\operatorname{ch}[(a+c)x+b+d] + \frac{1}{2(a-c)}\operatorname{ch}[(a-c)x+b-d] \qquad [a^2 \neq c^2].$$

GU ((352))(2c)

3. $$\int \operatorname{ch}(ax+b)\operatorname{ch}(cx+d)\, dx = \frac{1}{2(a+c)}\operatorname{sh}[(a+c)x+b+d] + \frac{1}{2(a-c)}\operatorname{sh}[(a-c)x+b-d] \qquad [a^2 \neq c^2].$$

GU ((352))(2b)

When $a = c$:

4. $$\int \operatorname{sh}(ax+b)\operatorname{sh}(ax+d)\, dx = -\frac{x}{2}\operatorname{ch}(b-d) + \frac{1}{4a}\operatorname{sh}(2ax+b+d).$$

GU ((352))(3a)

GU ((352))(3c)

6. $$\int \operatorname{ch}(ax+b)\operatorname{ch}(ax+d)\,dx = \frac{x}{2}\operatorname{ch}(b-d) + \frac{1}{4a}\operatorname{sh}(2ax+b+d).$$

GU ((352))(3b)

2.426

1. $$\int \operatorname{sh} ax\ \operatorname{sh} bx\ \operatorname{sh} cx\ dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} - \\ - \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} - \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

GU ((352))(4a)

2. $$\int \operatorname{sh} ax\ \operatorname{sh} bx\ \operatorname{ch} cx\,dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} - \\ - \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

GU ((352))(4b)

3. $$\int \operatorname{sh} ax\ \operatorname{ch} bx\ \operatorname{ch} cx\,dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

GU ((352))(4c)

4. $$\int \operatorname{ch} ax\ \operatorname{ch} bx\ \operatorname{ch} cx\,dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} + \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

GU ((352))(4d)

2.427

1. $$\int \operatorname{sh}^p x \operatorname{sh} ax\,dx = \frac{1}{p+a}\left\{\operatorname{sh}^p x \operatorname{ch} ax - p\int \operatorname{sh}^{p-1} x \operatorname{ch}(a-1)x\,dx\right\}.$$

$$2.\quad \int \operatorname{sh}^p x \operatorname{sh}(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times$$
$$\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma\left(p-2k+1\right)} \operatorname{sh}^{p-2k} x \operatorname{ch}(2n-2k+1)x - \right.\right.$$
$$\left. - \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma\left(p-2k\right)} \operatorname{sh}^{p-2k-1} x \operatorname{sh}(2n-2k)x \right] +$$
$$\left. + \frac{\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma\left(p+1-2n\right)} \int \operatorname{sh}^{p-2n} x \operatorname{sh} x\,dx \right\} \quad [p \text{ is not a negative integer }].$$

$$3.\quad \int \operatorname{sh}^p x \operatorname{sh} 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times$$
$$\times \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma\left(p-2k+1\right)} \operatorname{sh}^{p-2k} x \operatorname{ch}(2n-2k)x - \right.$$
$$\left. - \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma\left(p-2k\right)} \operatorname{sh}^{p-2k-1} x \operatorname{sh}(2n-2k-1)x \right]$$

[p is not a negative integer].

GU ((352))(5)a

2.428

$$1.\quad \int \operatorname{sh}^p x \operatorname{ch} ax\,dx = \frac{1}{p+a} \left\{ \operatorname{sh}^p x \operatorname{sh} ax - p \int \operatorname{sh}^{p-1} x \operatorname{sh}(a-1)x\,dx \right\}.$$

$$2.\quad \int \operatorname{sh}^p x \operatorname{ch}(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times$$
$$\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma\left(p-2k+1\right)} \operatorname{sh}^{p-2k} x \operatorname{sh}(2n-2k+1)x - \right.\right.$$
$$\Gamma\left(\frac{p-1}{\ }+n-2k\right)$$

3. $$\int \operatorname{sh}^p x \operatorname{ch} 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times \times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma\left(p-2k+1\right)} \operatorname{sh}^{p-2k} x \operatorname{sh}(2n-2k)x - -\frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma\left(p-2k\right)} \operatorname{sh}^{p-2k-1} x \operatorname{ch}(2n-2k-1)x \right] + +\frac{\Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\Gamma\left(p+1-2n\right)} \int \operatorname{sh}^{p-2n} x\, dx \right\} \quad [p \text{ is not a negative integer }].$$

GU ((352))(6)a

2.429

1. $$\int \operatorname{ch}^p x \operatorname{sh} ax\, dx = \frac{1}{p+a} \left\{ \operatorname{ch}^p x \operatorname{ch} ax + p \int \operatorname{ch}^{p-1} x \operatorname{sh}(a-1)x\, dx \right\}.$$

2. $$\int \operatorname{ch}^p x \operatorname{sh}(2n+1)x\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\Gamma\left(p-k+1\right)} \operatorname{ch}^{p-k} x \operatorname{ch}(2n-k+1)x + + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma\left(p-n+1\right)} \int \operatorname{ch}^{p-n} x \operatorname{sh}(n+1)x\, dx \right\}$$

$[p$ is not a negative integer$]$.

3. $$\int \operatorname{ch}^p x \operatorname{sh} 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\Gamma\left(p-k+1\right)} \operatorname{ch}^{p-k} x \operatorname{ch}(2n-k)x + +\frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\Gamma\left(p-n+1\right)} \int \operatorname{ch}^{p-n} x \operatorname{sh} nx\, dx \right\} \quad [p \text{ is not a negative integer }].$$

GU ((352))(7)a

2.431

1. $$\int \operatorname{ch}^p x \operatorname{ch} ax\, dx = \frac{1}{p+a}\left\{\operatorname{ch}^p x \operatorname{sh} ax + p\int \operatorname{ch}^{p-1} x \operatorname{ch}(a-1)x\, dx\right\}.$$

2. $$\int \operatorname{ch}^p x \operatorname{ch}(2n+1)x\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)}\operatorname{ch}^{p-k} x \operatorname{sh}(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma(p-n+1)}\int \operatorname{ch}^{p-n} x \operatorname{ch}(n+1)x\, dx\right\}$$

[p is not a negative integer].

3. $$\int \operatorname{ch}^p x \operatorname{ch} 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)}\operatorname{ch}^{p-k} x \operatorname{sh}(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\Gamma(p-n+1)}\operatorname{ch}^{p-n} x \operatorname{ch} nx\, dx\right\}$$

[p is not a negative integer].

GU ((352))(8)a

2.432

1. $$\int \operatorname{sh}(n+1)x \operatorname{sh}^{n-1} x\, dx = \frac{1}{n}\operatorname{sh}^n x \operatorname{sh} nx.$$

2. $$\int \operatorname{sh}(n+1)x \operatorname{ch}^{n-1} x\, dx = \frac{1}{n}\operatorname{ch}^n x \operatorname{ch} nx.$$

3. $$\int \operatorname{ch}(n+1)x \operatorname{sh}^{n-1} x\, dx = \frac{1}{n}\operatorname{sh}^n x \operatorname{ch} nx.$$

4. $$\int \operatorname{ch}(n+1)x \operatorname{ch}^{n-1} x\, dx = \frac{1}{n}\operatorname{ch}^n x \operatorname{sh} nx.$$

1. $$\int \frac{\operatorname{sh}(2n+1)x}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{sh}(2n-2k)x}{2n-2k} + x.$$

2. $$\int \frac{\operatorname{sh} 2nx}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{sh}(2n-2k-1)x}{2n-2k-1}.$$

GU ((352))(5d)

3. $$\int \frac{\operatorname{ch}(2n+1)x}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{ch}(2n-2k)x}{2n-2k} + \ln \operatorname{sh} x.$$

4. $$\int \frac{\operatorname{ch} 2nx}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{ch}(2n-2k-1)x}{2n-2k-1} + \ln \operatorname{th}\frac{x}{2}.$$

GU ((352))(6d)

5. $$\int \frac{\operatorname{sh}(2n+1)x}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1} (-1)^k \frac{\operatorname{ch}(2n-2k)x}{2n-2k} + (-1)^n \ln \operatorname{ch} x.$$

6. $$\int \frac{\operatorname{sh} 2nx}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1} (-1)^k \frac{\operatorname{ch}(2n-2k-1)x}{2n-2k-1}.$$

GU ((352))(7d)

7. $$\int \frac{\operatorname{ch}(2n+1)x}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1} (-1)^k \frac{\operatorname{sh}(2n-2k)x}{2n-2k} + (-1)^n x.$$

8. $$\int \frac{\operatorname{ch} 2nx}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1} (-1)^k \frac{\operatorname{sh}(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\operatorname{th} x).$$

GU ((352))(8d)

9. $$\int \frac{\operatorname{sh} 2x}{\operatorname{sh}^n x}\,dx = -\frac{2}{(n-2)\operatorname{sh}^{n-2} x}.$$

$n = 2$

10. $$\int \frac{\operatorname{sh} 2x}{\operatorname{sh}^2 x}\, dx = 2\, \ln \operatorname{sh} x.$$

11. $$\int \frac{\operatorname{sh} 2x\, dx}{\operatorname{ch}^n x} = \frac{2}{(2-n)\operatorname{ch}^{n-2} x}.$$

For $n = 2$:

12. $$\int \frac{\operatorname{sh} 2x}{\operatorname{ch}^2 x}\, dx = 2\, \ln \operatorname{ch} x.$$

13. $$\int \frac{\operatorname{ch} 2x}{\operatorname{sh} x}\, dx = 2 \operatorname{ch} x + \ln \operatorname{th} \frac{x}{2}.$$

14. $$\int \frac{\operatorname{ch} 2x}{\operatorname{sh}^2 x}\, dx = -\operatorname{cth} x + 2x.$$

15. $$\int \frac{\operatorname{ch} 2x}{\operatorname{sh}^3 x}\, dx = -\frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} + \frac{3}{2} \ln \operatorname{th} \frac{x}{2}.$$

16. $$\int \frac{\operatorname{ch} 2x}{\operatorname{ch} x}\, dx = 2 \operatorname{sh} x - \operatorname{arcsin}(\operatorname{th} x).$$

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17. $$\int \frac{\operatorname{ch} 2x}{\operatorname{ch}^2 x}\, dx = -\operatorname{th} x + 2x.$$

18. $$\int \frac{\operatorname{ch} 2x}{\operatorname{ch}^3 x}\, dx = -\frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{3}{2} \operatorname{arcsin}(\operatorname{th} x).$$

19. $$\int \frac{\operatorname{sh} 3x}{\operatorname{sh} x}\, dx = x + \operatorname{sh} 2x.$$

20. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{sh}^2 x}\, dx = 3\ \ln \operatorname{th} \frac{x}{2} + 4 \operatorname{ch} x.$

21. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{sh}^3 x}\, dx = -3 \operatorname{cth} x + 4x.$

22. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch}^n x}\, dx = \frac{4}{(3-n)\operatorname{ch}^{n-3} x} - \frac{1}{(1-n)\operatorname{ch}^{n-1} x}.$

For $n = 1$ and $n = 3$:

23. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch} x}\, dx = 2 \operatorname{sh}^2 x - \ln \operatorname{ch} x.$

24. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch}^3 x}\, dx = \frac{1}{2 \operatorname{ch}^2 x} + 4 \ln \operatorname{ch} x.$

25. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh}^n x}\, dx = \frac{4}{(3-n)\operatorname{sh}^{n-3} x} + \frac{1}{(1-n)\operatorname{sh}^{n-1} x}.$

For $n = 1$ and $n = 3$:

26. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh} x}\, dx = 2 \operatorname{sh}^2 x + \ln \operatorname{sh} x.$

27. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh}^3 x}\, dx = -\frac{1}{2 \operatorname{sh}^2 x} + 4 \ln \operatorname{sh} x.$

28. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{ch} x}\, dx = \operatorname{sh} 2x - x.$

2.44-2.45 Rational functions of hyperbolic functions

2.441

$$1.\quad \int \frac{A+B\ \operatorname{sh} x}{(a+b\ \operatorname{sh} x)^n}\,dx = \frac{aB-bA}{(n-1)(a^2+b^2)}\cdot\frac{\operatorname{ch} x}{(a+b\operatorname{sh} x)^{n-1}} + \\ +\frac{1}{(n-1)(a^2+b^2)}\int\frac{(n-1)(aA+bB)+(n-2)(aB-bA)\operatorname{sh} x}{(a+b\operatorname{sh} x)^{n-1}}\,dx.$$

For $n=1$:

$$2.\quad \int \frac{A+B\operatorname{sh} x}{a+b\operatorname{sh} x}\,dx = \frac{B}{b}x - \frac{aB-bA}{b}\int\frac{dx}{a+b\operatorname{sh} x} \qquad \text{(see } \mathbf{2.441}\ 3.\text{)}.$$

$$3.\quad \int\frac{dx}{a+b\ \operatorname{sh} x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a\operatorname{th}\frac{x}{2}-b+\sqrt{a^2+b^2}}{a\operatorname{th}\frac{x}{2}-b-\sqrt{a^2+b^2}}; \\ = \frac{2}{\sqrt{a^2+b^2}}\operatorname{Arth}\frac{a\operatorname{th}\frac{x}{2}-b}{\sqrt{a^2+b^2}}.$$

2.442

$$1.\quad \int \frac{A+B\operatorname{ch} x}{(a+b\operatorname{sh} x)^n}\,dx = -\frac{B}{(n-1)b(a+b\operatorname{sh} x)^{n-1}} + A\int\frac{dx}{(a+b\operatorname{sh} x)^n}.$$

For $n=1$:

$$2.\quad \int \frac{A+B\operatorname{ch} x}{a+b\operatorname{sh} x}\,dx = \frac{B}{b}\ln(a+b\operatorname{sh} x) + A\int\frac{dx}{a+b\operatorname{sh} x} \qquad \text{(see } \mathbf{2.441}\ 3.\text{)}$$

2.441

2.443

1. $$\int \frac{A + B \operatorname{ch} x}{(a + b \operatorname{ch} x)^n}\, dx = \frac{aB - bA}{(n-1)(a^2 - b^2)} \cdot \frac{\operatorname{sh} x}{(a + b \operatorname{ch} x)^{n-1}} + \\ + \frac{1}{(n-1)(a^2 - b^2)} \int \frac{(n-1)(aA - bB) + (n-2)(aB - bA) \operatorname{ch} x}{(a + b \operatorname{ch} x)^{n-1}}\, dx.$$

For $n = 1$:

2. $$\int \frac{A + B \operatorname{ch} x}{a + b \operatorname{ch} x}\, dx = \frac{B}{b} x - \frac{aB - bA}{b} \int \frac{dx}{a + b \operatorname{ch} x} \qquad (\text{see } \mathbf{2.443}\ 3.).$$

2.443

3. $$\int \frac{dx}{a + b \operatorname{ch} x} = \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \qquad [b^2 > a^2, \quad x < 0]; \\ - \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \qquad [b^2 > a^2, \quad x > 0]; \\ = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}}{a + b - \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}} \qquad [a^2 > b^2].$$

2.444

1. $$\int \frac{dx}{\operatorname{ch} a + \operatorname{ch} x} = \operatorname{cosech} a \left[\ln \operatorname{ch} \frac{x + a}{2} - \ln \operatorname{ch} \frac{x - a}{2} \right]; \\ = 2 \operatorname{cosech} a \operatorname{Arth} \left(\operatorname{th} \frac{x}{2} \operatorname{th} \frac{a}{2} \right).$$

2. $$\int \frac{dx}{\cos a + \operatorname{ch} x} = 2 \cos a \operatorname{arctg} \left(\operatorname{th} \frac{x}{2} \operatorname{tg} \frac{a}{2} \right).$$

2.445

1. $$\int \frac{A + B \operatorname{sh} x}{(a + b \operatorname{ch} x)^n}\, dx = -\frac{B}{(n-1) b (a + b \operatorname{ch} x)^{n-1}} + A \int \frac{dx}{(a + b \operatorname{ch} x)^n}.$$

For $n = 1$:

$$2. \quad \int \frac{A + B \operatorname{sh} x}{a + b \operatorname{ch} x}\, dx = \frac{B}{b} \ln(a + b \operatorname{ch} x) + A \int \frac{dx}{a + b \operatorname{ch} x}$$

2.443

In evaluating definite integrals by use of formulas 2.441 2.442 2.443 and 2.445, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{Arsh}\left(-\frac{a}{b}\right)$$

in formulas 2.441 or 2.442 or over the points

$$x = \operatorname{Arch}\left(-\frac{a}{b}\right)$$

in formulas 2.443 or 2.445. Formulas 2.443 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

2.446

$$1. \quad \int \frac{A + B \operatorname{ch} x}{(\varepsilon + \operatorname{ch} x)^n}\, dx = \frac{B \operatorname{sh} x}{(1-n)(\varepsilon + \operatorname{ch} x)^n} + \left(\varepsilon A + \frac{n}{n-1} B\right) \frac{(n-1)!}{(2n-1)!!} \operatorname{sh} x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \cdot \cdot \frac{\varepsilon^h}{(\varepsilon + \operatorname{ch} x)^{n-k}} \qquad [\varepsilon = \pm 1, \quad n > 1].$$

For $n = 1$:

$$2. \quad \int \frac{A + B \operatorname{ch} x}{\varepsilon + \operatorname{ch} x}\, dx = Bx + (\varepsilon A - B) \frac{\operatorname{ch} x - \varepsilon}{\operatorname{sh} x} \qquad [\varepsilon = \pm 1].$$

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2.447

$$1. \quad \int \frac{\operatorname{sh} x\, dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{a \ln \operatorname{ch}\left(x + \operatorname{Arth} \frac{b}{a}\right) - bx}{a^2 - b^2} \qquad [a > |b|];$$

$$= \frac{bx - a \ln \operatorname{sh}\left(x + \operatorname{Arth} \frac{a}{b}\right)}{b^2 - a^2} \qquad [b > |a|].$$

For $a = b = 1$:

$$2. \quad \int \frac{\operatorname{sh} x\, dx}{\operatorname{ch} x + \operatorname{sh} x} = \frac{x}{2} + \frac{1}{4} e^{-2x}.$$

For $a = -b = 1$:

$$3. \quad \int \frac{\operatorname{sh} x\, dx}{\operatorname{ch} x - \operatorname{sh} x} = -\frac{x}{2} + \frac{1}{4} e^{2x}.$$

MZ 215

2.448

$$1. \quad \int \frac{\operatorname{ch} x\, dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{ax - b \ln \operatorname{ch}\left(x + \operatorname{Arth}\frac{b}{a}\right)}{a^2 - b^2} \qquad [a > |b|];$$

$$= \frac{-ax + b \ln \operatorname{sh}\left(x + \operatorname{Arth}\frac{a}{b}\right)}{b^2 - a^2} \qquad [b > |a|].$$

For $a = b = 1$:

$$2. \quad \int \frac{\operatorname{ch} x\, dx}{\operatorname{ch} x + \operatorname{sh} x} = \frac{x}{2} - \frac{1}{4} e^{-2x}.$$

For $a = -b = 1$:

$$3. \quad \int \frac{\operatorname{ch} x\, dx}{\operatorname{ch} x - \operatorname{sh} x} = \frac{x}{2} + \frac{1}{4} e^{2x}.$$

MZ 214, 215

2.449

$$1.^{6} \quad \int \frac{dx}{(a \operatorname{ch} x + b \operatorname{sh} x)^n} = \frac{1}{\sqrt{(a^2 - b^2)^n}} \int \frac{dx}{\operatorname{sh}^n \left(x + \operatorname{Arth}\frac{b}{a}\right)} \qquad [a > |b|];$$

$$= \frac{1}{\sqrt{(b^2 - a^2)^n}} \int \frac{dx}{\operatorname{ch}^n \left(x + \operatorname{Arth}\frac{a}{b}\right)} \qquad [b > |a|].$$

$n = 1$

$$2.\quad \int \frac{dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 - b^2}} \operatorname{arctg} \left| \operatorname{sh} \left(x + \operatorname{Arth} \frac{b}{a} \right) \right| \qquad [a > |b|];$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \operatorname{th} \frac{x + \operatorname{Arth} \dfrac{a}{b}}{2} \right| \qquad [b > |a|].$$

For $a = b = 1$:

$$3.\quad \int \frac{ax}{\operatorname{ch} x + \operatorname{sh} x} = -e^{-x} = \operatorname{sh} x - \operatorname{ch} x.$$

For $a = -b = 1$:

$$4.\quad \int \frac{dx}{\operatorname{ch} x - \operatorname{sh} x} = e^{x} = \operatorname{sh} x + \operatorname{ch} x.$$

MZ 214

2.451

$$1.\quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} \, dx$$
$$= \frac{Bc - Cb + (Ac - Ca) \operatorname{ch} x + (Ab - Ba) \operatorname{sh} x}{(1 - n)(a^2 - b^2 + c^2)(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \times$$
$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \operatorname{ch} x - (n-2)(Ac - Ca) \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} \, dx$$
$$[a^2 + c^2 \neq b^2];$$
$$= \frac{Bc - Cb - Ca \operatorname{ch} x - Ba \operatorname{sh} x}{(n-1)a(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} + \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2} \right] (c \operatorname{ch} x + b \operatorname{sh} x) \frac{(n-1)!}{(2n-1)!!} \times$$
$$\times \sum_{k=0}^{n-1} \frac{(2n - 2k - 3)!!}{(n - k - 1)! a^k} \frac{1}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-k}} \qquad [a^2 + c^2 = b^2].$$

$$2.\quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x + c \operatorname{sh} x} \, dx$$
$$= \frac{Cb - Bc}{b^2 - c^2} \ln(a + b \operatorname{ch} x + c \operatorname{sh} x) +$$
$$+ \frac{Bb - Cc}{b^2 - c^2} x + \left(A - a \frac{Bb - Cc}{b^2 - c^2} \right) \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} \qquad [b^2 \neq c^2] \qquad \text{(see 2.4514)}.$$

3. $$\int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x \pm b \operatorname{sh} x}\, dx = \frac{C \mp B}{2a}(\operatorname{ch} x \mp \operatorname{sh} x) + \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2}\right] x + \\ + \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2}\right] \ln(a + b \operatorname{ch} x \pm b \operatorname{sh} x) \qquad [ab \neq 0].$$

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4. $$\int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x}$$
$$= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \operatorname{arctg} \frac{(b-a) \operatorname{th} \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} \qquad [b^2 > a^2 + c^2 \quad \text{and} \quad a \neq b]:$$
$$= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \operatorname{th} \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \operatorname{th} \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} \qquad [b^2 < a^2 + c^2 \quad \text{and} \quad a \neq b];$$
$$= \frac{1}{c} \ln\left(a + c \operatorname{th} \frac{x}{2}\right) \qquad [a = b, \quad c \neq 0];$$
$$= \frac{2}{(a-b) \operatorname{th} \frac{x}{2} + c} \qquad [b^2 = a^2 + c^2].$$

GU ((351))(18)

2.452

1. $$\int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x)(a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x)}\, dx$$
$$= A_0 \ln \frac{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x} + A_1 \int \frac{dx}{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x} + A_2 \int \frac{dx}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x}$$

where

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} b_1 & c_1 \\ B & C \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ C & A \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2\right].$$

GU ((351))(19)

2. $$\int \frac{A\operatorname{ch}^2 x + 2B\ \operatorname{sh} x \operatorname{ch} x + C\operatorname{sh}^2 x}{a\operatorname{ch}^2 x + 2b\operatorname{sh} x\operatorname{ch} x + c\operatorname{sh}^2 x}\,dx$$
$$= \frac{1}{4b^2-(a+c)^2}\{[4Bb-(A+C)(a+c)]x +$$
$$+ [(A+C)b - B(a+c)]\ln(a\operatorname{ch}^2 x + 2b\operatorname{sh} x\operatorname{ch} x + c\operatorname{sh}^2 x) +$$
$$+ [2(A-C)b^2 - 2Bb(a-c) + (Ca-Ac)(a+c)]f(x)\},$$

where

$$f(x) = \frac{1}{2\sqrt{b^2-ac}}\ln\frac{c\ \operatorname{th} x + b - \sqrt{b^2-ac}}{c\ \operatorname{th} x + b + \sqrt{b^2-ac}} \qquad [b^2 > ac];$$
$$= \frac{1}{\sqrt{ac-b^2}}\operatorname{arctg}\frac{c\ \operatorname{th} x + b}{\sqrt{ac-b^2}} \qquad [b^2 < ac];$$
$$= -\frac{1}{c\ \operatorname{th} x + b} \qquad [b^2 = ac].$$

GU ((351))(24)

2.453

1. $$\int \frac{(A+B\operatorname{sh} x)\,dx}{\operatorname{sh} x(a+b\operatorname{sh} x)} = \frac{1}{a}\left[A\ \ln\left|\operatorname{th}\frac{x}{2}\right| + (aB-bA)\int\frac{dx}{a+b\operatorname{sh} x}\right] \qquad (\text{see } \mathbf{2.441}\ 3.).$$

2.441

2. $$\int \frac{(A+B\operatorname{sh} x)\,dx}{\operatorname{sh} x(a+b\operatorname{ch} x)} = \frac{A}{a^2-b^2}\left(a\ \ln\left|\operatorname{th}\frac{x}{2}\right| + b\ \ln\left|\frac{a+b\operatorname{ch} x}{\operatorname{sh} x}\right|\right) + B\int\frac{dx}{a+b\operatorname{ch} x} \qquad (\text{see } \mathbf{2.443}\ 3.).$$

2.443

For $a^2 = b^2\ (= 1)$:

3. $$\int \frac{(A+B\operatorname{sh} x)\,dx}{\operatorname{sh} x(1+\operatorname{ch} x)} = \frac{A}{2}\left(\ln\left|\operatorname{th}\frac{x}{2}\right| - \frac{1}{2}\operatorname{th}^2\frac{x}{2}\right) + B\operatorname{th}\frac{x}{2}.$$

4. $$\int \frac{(A+B\operatorname{sh} x)\,dx}{\operatorname{sh} x(1-\operatorname{ch} x)} = \frac{A}{2}\left(-\ln\left|\operatorname{cth}\frac{x}{2}\right| + \frac{1}{2}\operatorname{cth}^2\frac{x}{2}\right) + B\operatorname{cth}\frac{x}{2}.$$

2.454

1. $$\int \frac{(A + B\,\mathrm{sh}\,x)\,dx}{\mathrm{ch}\,x(a + b\,\mathrm{sh}\,x)} = \frac{1}{a^2+b^2}\left[(Aa+Bb)\mathrm{arctg}(\mathrm{sh}\,x) + (Ab - Ba)\ln\left|\frac{a + b\ \mathrm{sh}\,x}{\mathrm{ch}\,x}\right|\right].$$

2. $$\int \frac{(A + B\,\mathrm{ch}\,x)\,dx}{\mathrm{sh}\,x(a + b\,\mathrm{sh}\,x)} = \frac{1}{a}\left(A\ \ln\left|\mathrm{th}\frac{x}{2}\right| + B\ \ln\left|\frac{\mathrm{sh}\,x}{a + b\,\mathrm{sh}\,x}\right| - Ab\int\frac{dx}{a + b\,\mathrm{sh}\,x}\right) \qquad (\text{see } \mathbf{2.441}\ 3.).$$

2.441

2.455

1. $$\int \frac{(A + B\,\mathrm{ch}\,x)\,dx}{\mathrm{sh}\,x(a + b\,\mathrm{ch}\,x)} = \frac{1}{a^2-b^2}\left[(Aa+Bb)\ln\left|\mathrm{th}\frac{x}{2}\right| + (Ab - Ba)\ln\left|\frac{a + b\,\mathrm{ch}\,x}{\mathrm{sh}\,x}\right|\right].$$

For $a^2 = b^2 (= 1)$:

2. $$\int \frac{(A + B\,\mathrm{ch}\,x)\,dx}{\mathrm{sh}\,x(1 + \mathrm{ch}\,x)} = \frac{A+B}{2}\ln\left|\mathrm{th}\frac{x}{2}\right| - \frac{A-B}{4}\mathrm{th}^2\frac{x}{2}.$$

3. $$\int \frac{(A + B\,\mathrm{ch}\,x)\,dx}{\mathrm{sh}\,x(1 - \mathrm{ch}\,x)} = \frac{A+B}{4}\mathrm{cth}^2\frac{x}{2} - \frac{A-B}{2}\ln\mathrm{cth}\frac{x}{2}.$$

2.456

$$\int \frac{(A + B\ \mathrm{ch}\,x)\,dx}{\mathrm{ch}\,x(a + b\ \mathrm{sh}\,x)} = \frac{A}{a^2+b^2}\left[a\,\mathrm{arctg}(\mathrm{sh}\,x) + b\ \ln\left|\frac{a + b\,\mathrm{sh}\,x}{\mathrm{ch}\,x}\right|\right] + + B\int\frac{dx}{a + b\,\mathrm{sh}\,x} \qquad (\text{see } \mathbf{2.441}\ 3.).$$

2.441

2.457

$$\int \frac{(A + B\,\mathrm{ch}\,x)\,dx}{\mathrm{ch}\,x(a + b\,\mathrm{ch}\,x)} = \frac{1}{a}\left[A\,\mathrm{arctg}\,\mathrm{sh}\,x - (Ab - Ba)\int\frac{dx}{a + b\,\mathrm{ch}\,x}\right] \qquad (\text{see } \mathbf{2.443}\ 3.).$$

2.458

1. $$\int \frac{dx}{a + b\,\mathrm{sh}^2 x} = \frac{1}{\sqrt{a(b-a)}}\mathrm{arctg}\left(\sqrt{\frac{b}{a} - 1}\,\mathrm{th}\,x\right) \qquad \left[\frac{b}{a} > 1\right]$$
$$= \frac{1}{\sqrt{a(a-b)}}\mathrm{Arth}\left(\sqrt{1 - \frac{b}{a}}\ \mathrm{th}\,x\right) \qquad \left[0 < \frac{b}{a} < 1 \ \text{ or } \ \frac{b}{a} < 0 \ \text{ and } \ \mathrm{sh}^2 x < -\frac{a}{b}\right];$$
$$= \frac{1}{\sqrt{a(a-b)}}\mathrm{Arcth}\left(\sqrt{1 - \frac{b}{a}}\mathrm{th}\,x\right) \qquad \left[\frac{b}{a} < 0 \quad \text{and} \quad \mathrm{sh}^2 x > -\frac{a}{b}\right].$$
MZ 195

2. $$\int \frac{dx}{a + b\,\mathrm{ch}^2 x} = \frac{1}{\sqrt{-a(a+b)}}\mathrm{arctg}\left(\sqrt{-\left(1 + \frac{b}{a}\right)}\mathrm{cth}\,x\right) \qquad \left[\frac{b}{a} < -1\right];$$
$$= \frac{1}{\sqrt{a(a+b)}}\mathrm{Arth}\left(\sqrt{1 + \frac{b}{a}}\mathrm{cth}\,x\right) \qquad \left[-1 < \frac{b}{a} < 0 \quad \text{and} \quad \mathrm{ch}^2 x > -\frac{a}{b}\right];$$
$$= \frac{1}{\sqrt{a(a+b)}}\ \mathrm{Arcth}\left(\sqrt{1 + \frac{b}{a}}\ \mathrm{cth}\,x\right)$$
$$\left[\frac{b}{a} > 0 \quad \text{or} \quad -1 < \frac{b}{a} < 0 \quad \text{and} \quad \mathrm{ch}^2 x < -\frac{a}{b}\right].$$
MZ 202

For $a^2 = b^2 = 1$:

3. $$\int \frac{dx}{1 + \mathrm{sh}^2 x} = \mathrm{th}\,x.$$

4. $$\int \frac{dx}{1 - \mathrm{sh}^2 x} = \frac{1}{\sqrt{2}}\mathrm{Arth}(\sqrt{2}\,\mathrm{th}\,x) \qquad [\mathrm{sh}^2 x < 1];$$
$$= \frac{1}{\sqrt{2}}\mathrm{Arcth}(\sqrt{2}\mathrm{th}\,x) \qquad [\mathrm{sh}^2 x > 1].$$

5. $$\int \frac{dx}{1 + \mathrm{ch}^2 x} = \frac{1}{\sqrt{2}}\mathrm{Arcth}(\sqrt{2}\,\mathrm{cth}\,x).$$

6. $\displaystyle\int \frac{dx}{1-\operatorname{ch}^2 x} = \operatorname{cth} x.$

2.459

1. $$\int \frac{dx}{(a+b\operatorname{sh}^2 x)^2} = \frac{1}{2a(b-a)}\left[\frac{b\operatorname{sh} x\operatorname{ch} x}{a+b\operatorname{sh}^2 x} + (b-2a)\int\frac{dx}{a+b\operatorname{sh}^2 x}\right] \qquad (\text{see } \mathbf{2.458}\ 1.).$$

2.458
MZ 196

2. $$\int \frac{dx}{(a+b\operatorname{ch}^2 x)^2} = \frac{1}{2a(a+b)}\left[-\frac{b\operatorname{sh} x\operatorname{ch} x}{a+b\operatorname{ch}^2 x} + (2a+b)\int\frac{dx}{a+b\operatorname{ch}^2 x}\right] \qquad (\text{see } \mathbf{2.458}\ 2.).$$

2.458
MZ 203

3. $$\int \frac{dx}{(a+b\operatorname{sh}^2 x)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{th} x) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{th} x}{1+p^2\operatorname{th}^2 x} + \right.$$
$$\left. + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\operatorname{th}^2 x\right)\frac{2p\operatorname{th} x}{\left(1+p^2\operatorname{th}^2 x\right)^2}\right] \qquad \left[p^2 = \frac{b}{a}-1>0\right];$$
$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q\operatorname{th} x) + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\operatorname{th} x}{1-q^2\operatorname{th}^2 x} + \right.$$
$$\left. + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\operatorname{th}^2 x\right)\frac{2q\operatorname{th} x}{\left(1-q^2\operatorname{th}^2 x\right)^2}\right] \qquad \left[q^2 = 1-\frac{b}{a}>0\right].$$

MZ 196

* If $\frac{b}{a}<0$ and $\operatorname{ch}^2 x > -\frac{a}{b}$, then $\varphi(x) = \operatorname{Arth}(q\operatorname{cth} x)$. If $\frac{b}{a}<0$, but $\operatorname{ch}^2 x < -\frac{a}{b}$, or if $\frac{b}{a}>0$, then $\varphi(x) = \operatorname{Arcth}(q\operatorname{cth} x)$.

4. $$\int \frac{dx}{(a+b\operatorname{ch}^2 x)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{cth} x) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{cth} x}{1+p^2\operatorname{cth}^2 x} + \right.$$
$$\left. + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\operatorname{cth}^2 x\right)\frac{2p\operatorname{cth} x}{\left(1+p^2\operatorname{cth}^2 x\right)^2}\right] \qquad \left[p^2 = -1-\frac{b}{a}>0\right];$$
$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\varphi(x)^* + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\operatorname{cth} x}{1-q^2\operatorname{cth}^2 x} + \right.$$
$$\left. + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\operatorname{cth}^2 x\right)\frac{2q\operatorname{cth} x}{\left(1-q^2\operatorname{cth}^2 x\right)^2}\right] \qquad \left[q^2 = 1+\frac{b}{a}>0\right].$$

1. $\int \sqrt{\operatorname{th} x}\, dx = \operatorname{Arth}\sqrt{\operatorname{th} x} - \operatorname{arctg}\sqrt{\operatorname{th} x}.$

MZ 221

2. $\int \sqrt{\operatorname{cth} x}\, dx = \operatorname{Arcth}\sqrt{\operatorname{cth} x} - \operatorname{arctg}\sqrt{\operatorname{cth} x}.$

MZ 222

2.462

1. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{a^2 + \operatorname{sh}^2 x}} = \operatorname{Arsh}\frac{\operatorname{ch} x}{\sqrt{a^2-1}} = \ln(\operatorname{ch} x + \sqrt{a^2 + \operatorname{sh}^2 x}) \qquad [a^2 > 1];$$
$$= \operatorname{Arch}\frac{\operatorname{ch} x}{\sqrt{1-a^2}} = \ln(\operatorname{ch} x + \sqrt{a^2 + \operatorname{sh}^2 x}) \qquad [a^2 < 1];$$
$$= \ln \operatorname{ch} x \qquad [a^2 = 1].$$

2. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{a^2 - \operatorname{sh}^2 x}} = \arcsin\frac{\operatorname{ch} x}{\sqrt{a^2+1}} \qquad [\operatorname{sh}^2 x < a^2].$$

3. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{\operatorname{sh}^2 x - a^2}} = \operatorname{Arch}\frac{\operatorname{ch} x}{\sqrt{a^2+1}} = \ln(\operatorname{ch} x + \sqrt{\operatorname{sh}^2 x - a^2}) \qquad [\operatorname{sh}^2 x > a^2].$$

MZ 199

4. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a^2 + \operatorname{sh}^2 x}} = \operatorname{Arsh}\frac{\operatorname{sh} x}{a} = \ln(\operatorname{sh} x + \sqrt{a^2 + \operatorname{sh}^2 x}).$$

5. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a^2 - \operatorname{sh}^2 x}} = \arcsin\frac{\operatorname{sh} x}{a} \qquad [\operatorname{sh}^2 x < a^2].$$

6. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{\operatorname{sh}^2 x - a^2}} = \operatorname{Arch}\frac{\operatorname{sh} x}{a} = \ln(\operatorname{sh} x + \sqrt{\operatorname{sh}^2 x - a^2}) \qquad [\operatorname{sh}^2 x > a^2].$$

7. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{a^2 + \operatorname{ch}^2 x}} = \operatorname{Arsh}\frac{\operatorname{ch} x}{a} = \ln(\operatorname{ch} x + \sqrt{a^2 + \operatorname{ch}^2 x}).$$

8. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{a^2 - \operatorname{ch}^2 x}} = \arcsin \frac{\operatorname{ch} x}{a} \qquad [\operatorname{ch}^2 x < a^2].$$

9. $$\int \frac{\operatorname{sh} x\, dx}{\sqrt{\operatorname{ch}^2 x - a^2}} = \operatorname{Arch} \frac{\operatorname{ch} x}{a} = \ln(\operatorname{ch} x + \sqrt{\operatorname{ch}^2 x - a^2}) \qquad [\operatorname{ch}^2 x > a^2].$$

MZ 215, 216

10. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a^2 + \operatorname{ch}^2 x}} = \operatorname{Arsh} \frac{\operatorname{sh} x}{\sqrt{a^2 + 1}} = \ln(\operatorname{sh} x + \sqrt{a^2 + \operatorname{ch}^2 x}).$$

11. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a^2 - \operatorname{ch}^2 x}} = \arcsin \frac{\operatorname{sh} x}{\sqrt{a^2 - 1}} \qquad [\operatorname{ch}^2 x < a^2].$$

12. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{\operatorname{ch}^2 x - a^2}} = \operatorname{Arch} \frac{\operatorname{sh} x}{\sqrt{a^2 - 1}} \qquad [a^2 > 1];$$
$$= \ln \operatorname{sh} x \qquad [a^2 = 1].$$

MZ 206

13. $$\int \frac{\operatorname{cth} x\, dx}{\sqrt{a + b \operatorname{sh} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \qquad [b \operatorname{sh} x > 0, \quad a > 0];$$
$$= 2\sqrt{a} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \qquad [b \operatorname{sh} x < 0, \quad a > 0];$$
$$= 2\sqrt{-a} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{sh} x\right)} \qquad a < 0.$$

14. $$\int \frac{\operatorname{th} x\, dx}{\sqrt{a + b \operatorname{ch} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x} \qquad [b \operatorname{ch} x > 0, \quad a > 0];$$
$$= 2\sqrt{a} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x} \qquad [b \operatorname{ch} x < 0, \quad a > 0];$$
$$= 2\sqrt{-a} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{ch} x\right)} \qquad [a < 0].$$

MZ 220, 221

2.463

1. $$\int \frac{\operatorname{sh} x\sqrt{a + b\operatorname{ch} x}}{p + q\operatorname{ch} x}\,dx = 2\sqrt{\frac{aq - bp}{q}}\operatorname{Arcth}\sqrt{\frac{q(a + b\operatorname{ch} x)}{aq - bp}} \qquad \left[b\operatorname{ch} x > 0, \quad \frac{aq - bp}{q} > 0\right];$$
$$= 2\sqrt{\frac{aq - bp}{q}}\operatorname{Arth}\sqrt{\frac{q(a + b\operatorname{ch} x)}{aq - bp}} \qquad \left[b\operatorname{ch} x < 0, \quad \frac{aq - bp}{q} > 0\right];$$
$$= 2\sqrt{\frac{bp - aq}{q}}\operatorname{Arth}\sqrt{\frac{q(a + b\operatorname{ch} x)}{bp - aq}} \qquad \left[\frac{aq - bp}{q} < 0\right].$$

MZ 220

2. $$\int \frac{\operatorname{ch} x\sqrt{a + b\operatorname{sh} x}}{p + q\operatorname{sh} x}\,dx = 2\sqrt{\frac{aq - bp}{q}}\operatorname{Arcth}\sqrt{\frac{q(a + b\operatorname{sh} x)}{aq - bp}} \qquad \left[b\operatorname{sh} x > 0, \quad \frac{aq - bp}{q} > 0\right];$$
$$= 2\sqrt{\frac{aq - bp}{q}}\operatorname{Arth}\sqrt{\frac{q(a + b\operatorname{sh} x)}{aq - bp}} \qquad \left[b\operatorname{sh} x < 0, \quad \frac{aq - bp}{q} > 0\right];$$
$$= 2\sqrt{\frac{bp - aq}{q}}\operatorname{Arth}\sqrt{\frac{q(a + b\operatorname{sh} x)}{bp - aq}} \qquad \left[\frac{aq - bp}{q} < 0\right].$$

MZ 221

2.464

1. $$\int \frac{dx}{\sqrt{k^2 + k'^2\operatorname{ch}^2 x}} = \int \frac{dx}{\sqrt{1 + k'^2\operatorname{sh}^2 x}} = F(\arcsin(\operatorname{th} x), k) \qquad [x > 0].$$

BY (295.00)(295.10)

2. $$\int \frac{dx}{\sqrt{\operatorname{ch}^2 x - k^2}} = \int \frac{dx}{\sqrt{\operatorname{sh}^2 x + k'^2}} = F\left(\arcsin\left(\frac{1}{\operatorname{ch} x}\right), k\right) \qquad [x > 0].$$

BY (295.40)(295.30)

3. $$\int \frac{dx}{\sqrt{1 - k'^2\operatorname{ch}^2 x}} = F\left(\arcsin\left(\frac{\operatorname{th} x}{k}\right), k\right) \qquad \left[0 < x < \operatorname{Arch}\frac{1}{k'}\right].$$

BY (295.20)

In 2.464 4.-2.464 8., we set $\alpha = \arccos\frac{1-\operatorname{sh} 2ax}{1+\operatorname{sh} 2ax}$, $r = \frac{1}{\sqrt{2}}$ $\quad [ax > 0]$:

BY (296.50)

5. $$\int \sqrt{\operatorname{sh} 2ax}\, dx = \frac{1}{2a}[F(\alpha,\, r) - 2E(\alpha,\, r)] + \frac{1}{a}\frac{\sqrt{\operatorname{sh} 2ax(1+\operatorname{sh}^2 2ax)}}{1+\operatorname{sh} 2ax}.$$

BY (296.53)

6. $$\int \frac{\operatorname{ch}^2 2ax\, dx}{(1+\operatorname{sh} 2ax)^2\sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a}E(\alpha,\, r).$$

BY (296.51)

7. $$\int \frac{(1-\operatorname{sh} 2ax)^2\, dx}{(1+\operatorname{sh} 2ax)^2\sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a}[2E(\alpha,\, r) - F(\alpha,\, r)].$$

BY (296.55)

8. $$\int \frac{\sqrt{\operatorname{sh} 2ax}\, dx}{(1+\operatorname{sh} 2ax)^2} = \frac{1}{4a}[F(\alpha,\, r) - E(\alpha,\, r)].$$

BY (296.54)

In 2.464 9.-2.464 15., we set $\alpha = \arcsin\sqrt{\frac{\operatorname{ch} 2ax - 1}{\operatorname{ch} 2ax}}$, $r = \frac{1}{\sqrt{2}}$ $[x \neq 0]$:

9. $$\int \frac{dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{1}{a\sqrt{2}}F(\alpha, r).$$

BY (296.00)

10. $$\int \sqrt{\operatorname{ch} 2ax}\, dx = \frac{1}{a\sqrt{2}}[F(\alpha,\, r) - 2E(\alpha,\, r)] + \frac{\operatorname{sh} 2ax}{a\sqrt{\operatorname{ch} 2ax}}.$$

BY (296.03)

11. $$\int \frac{dx}{\sqrt{\operatorname{ch}^3 2ax}} = \frac{1}{a\sqrt{2}}[2E(\alpha,\, r) - F(\alpha,\, r)].$$

BY (296.04)

BY (296.04)

13. $\int \frac{\operatorname{sh}^2 2ax\,dx}{\sqrt{\operatorname{ch} 2ax}} = -\frac{\sqrt{2}}{3a}F(\alpha,\,r) + \frac{1}{3a}\ \operatorname{sh} 2ax\sqrt{\operatorname{ch} 2ax}.$

BY (296.07)

14. $\int \frac{\operatorname{th}^2 2ax\,dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{\sqrt{2}}{3a}F(\alpha,\,r) - \frac{\operatorname{th} 2ax}{3a\sqrt{\operatorname{ch} 2ax}}.$

BY (296.05)

15. $\int \frac{\sqrt{\operatorname{ch} 2ax}\,dx}{sp^2 + (1-p^2)\operatorname{ch} 2ax} = \frac{1}{a\sqrt{2}}\Pi(\alpha,\,p^2,\,r).$

BY (296.02)

In 2.464 16.-2.464 20., we set

$$\alpha = \arccos\frac{\sqrt{a^2+b^2} - a - b\operatorname{sh} x}{\sqrt{a^2+b^2} + a + b\operatorname{sh} x},$$

$$r = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}} \qquad \left[a > 0, \quad b > 0, \quad x > -\operatorname{Arsh}\frac{a}{b}\right]:$$

16. $\int \frac{dx}{\sqrt{a + b\operatorname{sh} x}} = \frac{1}{\sqrt[4]{a^2+b^2}}F(\alpha,\,r).$

BY (298.00)

17. $\int \sqrt{a + b\operatorname{sh} x}\,dx = \sqrt[4]{a^2+b^2}[F(\alpha,\,r) - 2E(\alpha,\,r)] + \frac{2b\operatorname{ch} x\sqrt{a+b\operatorname{sh} x}}{\sqrt{a^2+b^2} + a + b\operatorname{sh} x}.$

BY (298.02)

BY (298.03)

19. $$\int \frac{\operatorname{ch}^2 x\, dx}{[\sqrt{a^2+b^2}+a+b\operatorname{sh} x]^2 \sqrt{a+b\operatorname{sh} x}} = \frac{1}{b^2\sqrt[4]{a^2+b^2}} E(\alpha,\, r).$$

BY (298.01)

20. $$\int \frac{\sqrt{a+b\operatorname{sh} x}\, dx}{[\sqrt{a^2+b^2}-a-b\operatorname{sh} x]^2} = -\frac{1}{\sqrt[4]{a^2+b^2}(\sqrt{a^2+b^2}-a)} E(\alpha,\, r) + \\ + \frac{b}{\sqrt{a^2+b^2}-a} \cdot \frac{\operatorname{ch} x\sqrt{a+b\operatorname{sh} x}}{a^2+b^2-(a+b\operatorname{sh} x)^2}.$$

BY (298.04)

In 2.464 21.-2.464 31., we set $\alpha = \arcsin\left(\operatorname{th}\frac{x}{2}\right)$, $r = \sqrt{\frac{a-b}{a+b}}\ [0 < b < a,\ \ x > 0]$:

21. $$\int \frac{dx}{\sqrt{a+b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\alpha,\, r).$$

BY (297.25)

22. $$\int \sqrt{a+b\operatorname{ch} x}\, dx = 2\sqrt{a+b}[F(\alpha,\, r) - E(\alpha,\, r)] + 2\operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch} x}.$$

BY (297.29)

23. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a+b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\alpha,\, r) - \frac{2\sqrt{a+b}}{b} E(\alpha,\, r) + \frac{2}{b}\operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch} x}.$$

BY (297.33)

24. $$\int \frac{\operatorname{th}^2\frac{x}{2}}{\sqrt{a+b\ \operatorname{ch} x}}\, dx = \frac{2\sqrt{a+b}}{a-b}[F(\alpha,\, r) - E(\alpha, r)].$$

BY (297.28)

25. $$\int \frac{\operatorname{th}^4\frac{x}{2}}{\sqrt{a=b\ \operatorname{ch} x}}\, dx = \frac{2\sqrt{a+b}}{3(a-b)^2}[(3a+b)F(\alpha,\, r) - 4aE(\alpha,\, r)] + \frac{2}{3(a-b)}\ \frac{\operatorname{sh}\frac{x}{2}\sqrt{a+b\ \operatorname{ch} x}}{\operatorname{ch}^3\frac{x}{2}}.$$

26. $$\int \frac{\operatorname{ch} x - 1}{\sqrt{a + b\operatorname{ch} x}}\,dx = \frac{2}{b}\left[\operatorname{th}\frac{x}{2}\sqrt{a + b\operatorname{ch} x} - \sqrt{a+b}E(\alpha,\, r)\right].$$

BY (297.31)

27. $$\int \frac{(\operatorname{ch} x - 1)^2}{\sqrt{a + b\operatorname{ch} x}}\,dx = \frac{4\sqrt{a+b}}{3b^2}[(a+3b)E(\alpha,\, r) - bF(\alpha, r)] + \\ + \frac{4}{3b^2}\left[b\operatorname{ch}^2\frac{x}{2} - (a+3b)\right]\operatorname{th}\frac{x}{2}\sqrt{a + b\operatorname{ch} x}.$$

BY (297.31)

28. $$\int \frac{\sqrt{a + b\operatorname{ch} x}}{\operatorname{ch} x + 1}\,dx = \sqrt{a+b}E(\alpha,\, r).$$

BY (297.26)

29. $$\int \frac{dx}{(\operatorname{ch} x + 1)\sqrt{a + b\operatorname{ch} x}} = \frac{\sqrt{a+b}}{a-b}E(\alpha,\, r) - \frac{2b}{(a-b)\sqrt{a+b}}F(\alpha,\, r).$$

BY (297.30)

30. $$\int \frac{dx}{(\operatorname{ch} x + 1)^2\sqrt{a + b\operatorname{ch} x}} = \frac{1}{3(a-b)^2\sqrt{a+b}}[b(5b-a)F(\alpha,\, r) + \\ +(a-3b)(a+b)E(\alpha, r)] + \frac{1}{6(a-b)}\cdot\frac{\operatorname{sh}\frac{x}{2}}{\operatorname{ch}^3\frac{x}{2}}\sqrt{a + b\operatorname{ch} x}.$$

BY (297.30)

31. $$\int \frac{(1 + \operatorname{ch} x)\,dx}{[1 + p^2 + (1-p^2)\operatorname{ch} x]\sqrt{a + b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}}\Pi(\alpha,\, p^2,\, r).$$

BY (297.27)

In 2.464 32.-2.464 40., we set $\alpha = \arcsin\sqrt{\frac{a-b\operatorname{ch} x}{a-b}}, r = \sqrt{\frac{a-b}{a+b}}$ $\left[0 < b < a,\ \ 0 < x < \operatorname{Arch}\frac{a}{b}\right]$:

32. $$\int \frac{dx}{\sqrt{a - b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}}F(\alpha,\, r).$$

33. $$\int \sqrt{a - b \operatorname{ch} x}\, dx = 2\sqrt{a+b}\ [F(\alpha,\, r) - E(\alpha,\, r)].$$

BY (297.54)

34. $$\int \frac{\operatorname{ch} x\, dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\alpha,\, r) - \frac{2}{\sqrt{a+b}} F(\alpha,\, r).$$

BY (297.56)

35. $$\int \frac{\operatorname{ch}^2 x\, dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha,\, r) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha,\, r) + \frac{2}{3b} \operatorname{sh} x \sqrt{a - b \operatorname{ch} x}.$$

BY (297.56)

36. $$\int \frac{(1 + \operatorname{ch} x)\, dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\alpha,\, r).$$

BY (297.51)

37. $$\int \frac{dx}{\operatorname{ch} x \sqrt{a - b \operatorname{ch} x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\alpha,\, \frac{a-b}{a},\, r\right).$$

BY (297.57)

38. $$\int \frac{dx}{(1 + \operatorname{ch} x)\sqrt{a - b \operatorname{ch} x}} = \frac{1}{\sqrt{a+b}} E(\alpha,\, r) - \frac{1}{a+b} \operatorname{th} \frac{x}{2} \sqrt{a - b \operatorname{ch} x}.$$

BY (297.58)

39. $$\int \frac{dx}{(1 + \operatorname{ch} x)^2 \sqrt{a - b \operatorname{ch} x}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+3b)E(\alpha,\, r) - bF(\alpha,\, r)] - \\ - \frac{1}{3(a+b)^2} \frac{\operatorname{th} \frac{x}{2} \sqrt{a - b \operatorname{ch} x}}{\operatorname{ch} x + 1} [2a + 4b + (a+3b) \operatorname{ch} x].$$

BY (297.58)

40. $$\int \frac{dx}{(a - b - ap^2 + bp^2 \operatorname{ch} x)\sqrt{a - b \operatorname{ch} x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\alpha,\, p^2,\, r).$$

In 2.464 41.-2.464 47., we set $\alpha = \arcsin\sqrt{\frac{b(\mathrm{ch}\, x-1)}{b\,\mathrm{ch}\, x-a}}$, $r = \sqrt{\frac{a+b}{2b}}$ $[0 < a < b,\;\; x > 0]$:

41. $$\int \frac{dx}{\sqrt{b\,\mathrm{ch}\; - a}} = \sqrt{\frac{2}{b}} F(\alpha,\, r).$$

BY (297.00)

42. $$\int \sqrt{b\,\mathrm{ch}\, x - a}\, dx = (b-a)\sqrt{\frac{2}{b}} F(\alpha,\, r) - 2\sqrt{2b} E(\alpha,\, r) + \frac{2b\,\mathrm{sh}\, x}{\sqrt{b\,\mathrm{ch}\, x - a}}.$$

BY (297.05)

43. $$\int \frac{dx}{\sqrt{(b\,\mathrm{ch}\, x - a)^3}} = \frac{1}{b^2-a^2} \cdot \sqrt{\frac{2}{b}} [2bE(\alpha,\, r) - (b-a)F(\alpha,\, r)].$$

BY (297.06)

44. $$\int \frac{dx}{\sqrt{(b\,\mathrm{ch}\, x - a)^5}} = \frac{1}{3(b^2-a^2)^2}\sqrt{\frac{2}{b}} [(b-3a)(b-a)F(\alpha,\, r) + 8abE(\alpha,\, r)] + \\ + \frac{2b}{3(b^2-a^2)} \cdot \frac{\mathrm{sh}\, x}{\sqrt{(b\,\mathrm{ch}\, x - a)^3}}.$$

BY (297.06)

45. $$\int \frac{\mathrm{ch}\, x\, dx}{\sqrt{b\,\mathrm{ch}\, x - a}} = \sqrt{\frac{2}{b}} [F(\alpha,\, r) - 2E(\alpha,\, r)] + \frac{2\,\mathrm{sh}\, x}{\sqrt{b\,\mathrm{ch}\, x - a}}.$$

BY (297.03)

46. $$\int \frac{(\mathrm{ch}\, x + 1)\, dx}{\sqrt{(b\,\mathrm{ch}\, x - a)^3}} = \frac{2}{b-a}\sqrt{\frac{2}{b}} E(\alpha, r).$$

BY (297.01)

47. $$\int \frac{\sqrt{b\,\mathrm{ch}\, x - a}\, dx}{p^2 b - a + b(1-p^2)\mathrm{ch}\, x} = \sqrt{\frac{2}{b}} \Pi(\alpha,\, p^2,\, r).$$

BY (297.02)

In 2.464 48.-2.464 55., we set $\alpha = \arcsin\sqrt{\frac{b \operatorname{ch} x - a}{b(\operatorname{ch} x - 1)}}$, $r = \sqrt{\frac{2b}{a+b}}$ $\left[0 < b < a, \quad x > \operatorname{Arch}\frac{a}{b}\right]$:

48. $$\int \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{\sqrt{a+b}} F(\alpha, r).$$

BY (297.75)

49. $$\int \sqrt{b \operatorname{ch} x - a}\, dx = -2\sqrt{a+b}\, E(\alpha, r) + 2 \operatorname{cth}\frac{x}{2}\sqrt{b \operatorname{ch} x - a}.$$

BY (297.79)

50. $$\int \frac{\operatorname{cth}^2 \frac{x}{2}\, dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\alpha, r).$$

BY (297.76)

51. $$\int \frac{\sqrt{b \operatorname{ch} x - a}}{\operatorname{ch} x - 1}\, dx = \sqrt{a+b}[F(\alpha, r) - E(\alpha, r)].$$

BY (297.77)

52. $$\int \frac{dx}{(\operatorname{ch} x - 1)\sqrt{b \operatorname{ch} x - a}} = \frac{\sqrt{a+b}}{a-b} E(\alpha, r) - \frac{1}{\sqrt{a+b}} F(\alpha, r).$$

BY (297.78)

53. $$\int \frac{dx}{(\operatorname{ch} x - 1)^2\sqrt{b \operatorname{ch} x - a}} = \frac{1}{3(a-b)^2\sqrt{a+b}}[(a-2b)(a-b)F(\alpha, r) + $$
$$+(3a-b)(a+b)E(\alpha, r)] + \frac{a+b}{6b(a-b)} \cdot \frac{\operatorname{ch}\frac{x}{2}}{\operatorname{sh}^3\frac{x}{2}}\sqrt{b \operatorname{ch} x - a}.$$

BY (297.78)

54. $$\int \frac{dx}{(\operatorname{ch} x + 1)\sqrt{b \operatorname{ch} x - a}} = \frac{1}{\sqrt{a+b}}[F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b \operatorname{ch} x - a}}{(a+b)\operatorname{sh} x}.$$

BY (297.80)

BY (297.80)

55. $$\int \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{b \operatorname{ch} x - a}} = \frac{1}{3\sqrt{(a+b)^3}}[(a+b)F(\alpha,\, r) - (a+3b)E(\alpha,\, r)] + \frac{\sqrt{b \operatorname{ch} x - a}}{3(a+b)\operatorname{sh} x}\left(2\frac{a+3b}{a+b} - \operatorname{th}^2\frac{x}{2}\right).$$

BY (297.80)

In2.464 56.-2.464 60., we set $\alpha = \arccos\frac{\sqrt[4]{b^2-a^2}}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}},\ r = \frac{1}{\sqrt{2}}$ $\left[0 < a < b,\ \ -\operatorname{Arsh}\frac{a}{\sqrt{b^2-a^2}} < x\right]$:

56. $$\int \frac{dx}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}} = \sqrt[4]{\frac{4}{b^2-a^2}}F(\alpha,\, r).$$

BY (299.00)

57. $$\int \sqrt{a \operatorname{sh} x + b \operatorname{ch} x}\, dx = \sqrt[4]{4(b^2-a^2)}[F(\alpha,\, r) - 2E(\alpha,\, r)] + \frac{2(a \operatorname{ch} x + b \operatorname{sh} x)}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}.$$

BY (299.02)

58. $$\int \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = \sqrt[4]{\frac{4}{(b^2-a^2)^3}}[2E(\alpha,\, r) - F(\alpha,\, r)].$$

BY (299.03)

59. $$\int \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^5}} = \frac{1}{3}\sqrt[4]{\frac{4}{(b^2-a^2)^5}}F(\alpha,\, r) + \frac{2}{3(b^2-a^2)} \cdot \frac{a \operatorname{ch} x + b \operatorname{sh} x}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}}.$$

BY (299.03)

60. $$\int \frac{(\sqrt{b^2-a^2} + a \operatorname{sh} x + b \operatorname{ch} x)\, dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = 2\sqrt[4]{\frac{4}{b^2-a^2}}E(\alpha,\, r).$$

BY (299.01)

2.47 Combinations of hyperbolic functions and powers

2.471

1. $$\int x^r \operatorname{sh}^p x \operatorname{ch}^q x\,dx = \frac{1}{(p+q)^2}\Big[(p+q)x^r \operatorname{sh}^{p=1} x \operatorname{ch}^{q-1} x -$$
$$- rx^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r+1)\int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x\,dx +$$
$$+rp\int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\,dx + (q-1)(p+q)\int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x\,dx\Big];$$
$$= \frac{1}{(p+q)^2}\Big[(p+q)x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x -$$
$$- rx^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r-1)\int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x\,dx -$$
$$-rq\int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\,dx - (p-1)(p+q)\int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x\,dx\Big].$$

GU ((353))(1)

2. $$\int x^n \operatorname{sh}^{2m} x\,dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \operatorname{ch}(2m-2k)x\,dx.$$

3. $$\int x^n \operatorname{sh}^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int x^n \operatorname{sh}(2m-2k+1)x\,dx.$$

4. $$\int x^n \operatorname{ch}^{2m} x\,dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \operatorname{ch}(2m-2k)x\,dx.$$

5. $$\int x^n \operatorname{ch}^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \int x^n \operatorname{ch}(2m-2k+1)x\,dx.$$

2.472

1. $$\int x^n \operatorname{sh} x\,dx = x^n \operatorname{ch} x - n\int x^{n-1} \operatorname{ch} x\,dx =$$
$$= x^n \operatorname{ch} x - nx^{n-1} \operatorname{sh} x + n(n-1)\int x^{n-2} \operatorname{sh} x\,dx.$$

2.473

Notation: $z_1 = a + bx$

1. $$\int z_1 \operatorname{sh} kx \, dx = \frac{1}{k} z_1 \operatorname{ch} kx - \frac{b}{k^2} \operatorname{sh} kx.$$

2. $$\int z_1 \operatorname{ch} kx \, dx = \frac{1}{k} z_1 \operatorname{sh} kx - \frac{b}{k^2} \operatorname{ch} kx.$$

3. $$\int z_1^2 \operatorname{sh} kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{ch} kx - \frac{2bz_1}{k^2} \operatorname{sh} kx.$$

4. $$\int z_1^2 \operatorname{ch} kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{sh} kx - \frac{2bz_1}{k^2} \operatorname{ch} kx.$$

5. $$\int z_1^3 \operatorname{sh} kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{ch} kx - \frac{3b}{k^2} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{sh} kx.$$

6. $$\int z_1^3 \operatorname{ch} kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{sh} kx - \frac{3b}{k^2} \left(z_1^3 + \frac{2b^2}{k^2} \right) \operatorname{ch} kx.$$

7. $$\int z_1^4 \operatorname{sh} kx \, dx = \frac{1}{k} \left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \operatorname{ch} kx - \frac{4bz_1}{k^2} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{sh} kx.$$

8. $\int z_1^4 \operatorname{ch} kx\, dx = \frac{1}{k}\left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4}\right) \operatorname{sh} kx - \frac{4bz_1}{k^2}\left(z_1^2 + \frac{6b^2}{k^2}\right) \operatorname{ch} kx.$

9. $\int z_1^5 \operatorname{sh} kx\, dx = \frac{z_1}{k}\left(z_1^4 + \frac{20b^2}{k^2} z_1^2 + 120\frac{b^4}{k^4}\right) \operatorname{ch} kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2} z_1^2 + 24\frac{b^4}{k^4}\right) \operatorname{sh} kx.$

10. $\int z_1^5 \operatorname{ch} kx\, dx = \frac{z_1}{k}\left(z_1^4 + 20\frac{b^2}{k^2} z_1^2 + 120\frac{b^4}{k^4}\right) \operatorname{sh} kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2} z_1^2 + 24\frac{b^4}{k^4}\right) \operatorname{ch} kx.$

11. $$\int z_1^6 \operatorname{sh} kx\, dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2} z_1^4 + 360\frac{b^4}{k^4} z_1^2 + 720\frac{b^6}{k^6}\right) \operatorname{ch} kx - \\ - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2} z_1^2 + 120\frac{b^4}{k^4}\right) \operatorname{sh} kx.$$

12. $$\int z_1^6 \operatorname{ch} kx\, dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2} z_1^4 + 360\frac{b^4}{k^4} z_1^2 + 720\frac{b^6}{k^6}\right) \operatorname{sh} kx - \\ - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2} z_1 + 120\frac{b^4}{k^4}\right) \operatorname{ch} kx.$$

2.474

1. $\int x^n \operatorname{sh}^2 x\, dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{[n/2]}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x\right\}.$

GU ((353))(2b)

2. $\int x^n \operatorname{ch}^2 x\, dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{[n/2]}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x\right\}.$

GU ((353))(3e)

3. $\int x \operatorname{sh}^2 x\, dx = \frac{1}{4} x \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x - \frac{x^2}{4}.$

4. $\int x^2 \operatorname{sh}^2 x\, dx = \frac{1}{4}\left(x^2 + \frac{1}{2}\right) \operatorname{sh} 2x - \frac{x}{4} \operatorname{ch} 2x - \frac{x^3}{6}.$

5. $$\int x \operatorname{ch}^2 x\, dx = \frac{x}{4}\operatorname{sh} 2x - \frac{1}{8}\operatorname{ch} 2x + \frac{x^2}{4}.$$

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6. $$\int x^2 \operatorname{ch}^2 x\, dx = \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\operatorname{sh} 2x - \frac{x}{4}\operatorname{ch} 2x + \frac{x^3}{6}.$$

MZ 261

7. $$\int x^n \operatorname{sh}^3 x\, dx = \frac{n!}{4}\sum_{k=0}^{[n/2]}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\operatorname{ch} 3x}{3^{2k+1}} - 3\operatorname{ch} x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\operatorname{sh} 3x}{3^{2k+2}} - 3\ \operatorname{sh} x\right)\right\}.$$

GU ((353))(2f)

8. $$\int x^n \operatorname{ch}^3 x\, dx = \frac{n!}{4}\sum_{k=0}^{[n/2]}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\operatorname{sh} 3x}{3^{2k+1}} + 3\operatorname{sh} x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\operatorname{ch} 3x}{3^{2k+2}} + 3\ \operatorname{ch} x\right)\right\}.$$

GU ((353))(3f)

9. $$\int x \operatorname{sh}^3 x\, dx = \frac{3}{4}\operatorname{sh} x - \frac{1}{36}\operatorname{sh} 3x - \frac{3}{4}x\operatorname{ch} x - \frac{x}{12}\operatorname{ch} 3x.$$

10. $$\int x^2 \operatorname{sh}^3 x\, dx = -\left(\frac{3x^2}{4} + \frac{3}{2}\right)\operatorname{ch} x + \left(\frac{x^2}{12} + \frac{1}{54}\right)\operatorname{ch} 3x + \frac{3x}{2}\operatorname{sh} x - \frac{x}{18}\operatorname{sh} 3x.$$

MZ 257

11. $$\int x \operatorname{ch}^3 x\, dx = -\frac{3}{4}\operatorname{ch} x - \frac{1}{36}\operatorname{ch} 3x + \frac{3}{4}x\operatorname{sh} x + \frac{x}{12}\operatorname{sh} 3x.$$

12. $$\int x^2 \operatorname{ch}^3 x\, dx = \left(\frac{3}{4}x^2 + \frac{3}{2}\right)\operatorname{sh} x + \left(\frac{x^2}{12} + \frac{1}{54}\right)\operatorname{sh} 3x - \frac{3}{2}x\operatorname{ch} x - \frac{x}{18}\operatorname{ch} 3x.$$

MZ 262

2.475

GU ((353))(6a)

2. $$\int \frac{\operatorname{ch}^q x}{x^p}\,dx = -\frac{(p-2)\operatorname{ch}^q x + qx\operatorname{ch}^{q-1} x \operatorname{sh} x}{(p-1)(p-2)x^{p-1}} - \\ -\frac{q(q-1)}{(p-1)(p-2)}\int \frac{\operatorname{ch}^{q-2} x}{x^{p-2}}\,dx + \frac{q^2}{(p-1)(p-2)}\int \frac{\operatorname{ch}^q x}{x^{p-2}}\,dx \qquad [p>2].$$

GU ((353))(7a)

3. $$\int \frac{\operatorname{sh} x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\operatorname{ch} x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\operatorname{sh} x\right\} + \frac{1}{(2n-1)!}\operatorname{chi}(x).$$

GU ((353))(6b)

4. $$\int \frac{\operatorname{sh} x}{x^{2n+1}}\,dx = -\frac{1}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\operatorname{ch} x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\operatorname{sh} x\right\} + \frac{1}{(2n)!}\operatorname{shi}(x).$$

GU ((353))(6b)

5. $$\int \frac{\operatorname{ch} x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\operatorname{ch} x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\operatorname{ch} x\right\} + \frac{1}{(2n-1)!}\operatorname{shi}(x).$$

GU ((353))(7b)

6. $$\int \frac{\operatorname{ch} x}{x^{2n+1}}\,dx = -\frac{1}{(2n)!x}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\operatorname{sh} x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\operatorname{ch} x\right\} + \frac{1}{(2n)!}\operatorname{chi}(x).$$

GU ((353))(7b)

7. $$\int \frac{\operatorname{sh}^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\operatorname{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}}\binom{2m}{m}\ln x.$$

GU ((353))(6c)

8. $$\int \frac{\operatorname{sh}^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\operatorname{shi}(2m-2k+1)x.$$

9. $$\int \frac{\operatorname{ch}^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \operatorname{chi}(2m-2k)x + \frac{1}{2^{2m}} \binom{2m}{m} \ln\, x.$$

GU ((353))(7c)

10. $$\int \frac{\operatorname{ch}^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \operatorname{chi}(2m-2k+1)x.$$

GU ((353))(7c)

11. $$\int \frac{\operatorname{sh}^{2m} x}{x^2}\,dx = \frac{(-1)^{m-1}}{2^{2m}x} \binom{2m}{m} + \\ + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\operatorname{ch}(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x \right\}.$$

12. $$\int \frac{\operatorname{sh}^{2m+1} x}{x^2}\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+1} \binom{2m+1}{k} \times \\ \times \left\{ \frac{\operatorname{sh}(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{chi}(2m-2k+1)x \right\}.$$

13. $$\int \frac{\operatorname{ch}^{2m} x}{x^2}\,dx = -\frac{1}{2^{2m}x} \binom{2m}{m} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\operatorname{ch}(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x \right\}.$$

14. $$\int \frac{\operatorname{ch}^{2m+1} x}{x^2}\,dx = -\frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \left\{ \frac{\operatorname{ch}(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{shi}(2m-2k+1)x \right\}.$$

2.476

1. $$\int \frac{\operatorname{sh} kx}{a+bx}\,dx = \frac{1}{b}\left[\operatorname{ch}\frac{ka}{b}\operatorname{shi}(u) - \operatorname{sh}\frac{ka}{b}\operatorname{chi}(u)\right]; \\ = \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\operatorname{Ei}(u) - \exp\left(\frac{ka}{b}\right)\operatorname{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right].$$

2. $$\int \frac{\operatorname{ch} kx}{a+bx}\,dx = \frac{1}{b}\left[\operatorname{ch}\frac{ka}{b}\operatorname{chi}(u) - \operatorname{sh}\frac{ka}{b}\operatorname{shi}(u)\right]; \\ = \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\operatorname{Ei}(u) + \exp\left(\frac{ka}{b}\right)\operatorname{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right].$$

3. $$\int \frac{\operatorname{sh} kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\operatorname{sh} kx}{a+bx} + \frac{k}{b}\int \frac{\operatorname{ch} kx}{a+bx}\,dx$$ see **2.476** 2.).

2.476

4. $$\int \frac{\operatorname{ch} kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\operatorname{ch} kx}{a+bx} + \frac{k}{b}\int \frac{\operatorname{sh} kx}{a+bx}\,dx$$ see **2.476** 1.).

2.476

5. $$\int \frac{\operatorname{sh} kx}{(a+bx)^3}\,dx = -\frac{\operatorname{sh} kx}{2b(a+bx)^2} - \frac{k \operatorname{ch} kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int \frac{\operatorname{sh} kx}{a+bx}\,dx$$ see **2.476** 1.).

2.476

6. $$\int \frac{\operatorname{ch} kx}{(a+bx)^3}\,dx = -\frac{\operatorname{ch} kx}{2b(a+bx)^2} - \frac{k \operatorname{sh} kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int \frac{\operatorname{ch} kx}{a+bx}\,dx$$ see **2.476** 2.).

2.476

7. $$\int \frac{\operatorname{sh} kx}{(a+bx)^4}\,dx = -\frac{\operatorname{sh} kx}{3b(a+bx)^3} - \frac{k \operatorname{ch} kx}{6b^2(a+bx)^2} - \frac{k^2 \operatorname{sh} kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int \frac{\operatorname{ch} kx}{a+bx}\,dx$$ see **2.476** 2.).

2.476

8. $$\int \frac{\operatorname{ch} kx}{(a+bx)^4}\,dx = -\frac{\operatorname{ch} kx}{3b(a+bx)^3} - \frac{k \operatorname{sh} kx}{6b^2(a+bx)^2} - \frac{k^2 \operatorname{ch} kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int \frac{\operatorname{sh} kx}{a+bx}\,dx$$ see **2.476** 1.).

2.476

9. $$\int \frac{\operatorname{sh} kx}{(a+bx)^5}\,dx = -\frac{\operatorname{sh} kx}{4b(a+bx)^4} - \frac{k \operatorname{ch} kx}{12b^2(a+bx)^3} - \frac{k^2 \operatorname{sh} kx}{24b^3(a+bx)^2} - $$
$$- \frac{k^3 \operatorname{ch} kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int \frac{\operatorname{sh} kx}{a+bx}\,dx$$ see **2.476** 1.).

10. $$\int \frac{\operatorname{ch} kx}{(a+bx)^5}\,dx = -\frac{\operatorname{ch} kx}{4b(a+bx)^4} - \frac{k \operatorname{sh} kx}{12b^2(a+bx)^3} - \frac{k^2 \operatorname{ch} kx}{24b^3(a+bx)^2} - \frac{k^3 \operatorname{sh} kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int \frac{\operatorname{ch} kx}{a+bx}\,dx \qquad \text{see } \mathbf{2.476}\ 2.).$$

2.476

11. $$\int \frac{\operatorname{sh} kx}{(a+bx)^6}\,dx = -\frac{\operatorname{sh} kx}{5b(a+bx)^5} - \frac{k \operatorname{ch} kx}{20b^2(a+bx)^4} - \frac{k^2 \operatorname{sh} kx}{60b^3(a+bx)^3} - \frac{k^3 \operatorname{ch} kx}{120b^4(a+bx)^2} - \frac{k^4 \operatorname{sh} kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int \frac{\operatorname{ch} kx}{a+bx}\,dx \qquad \text{see } \mathbf{2.476}\ 2.).$$

2.476

12. $$\int \frac{\operatorname{ch} kx}{(a+bx)^6}\,dx = -\frac{\operatorname{ch} kx}{5b(a+bx)^5} - \frac{k \operatorname{sh} kx}{20b^2(a+bx)^4} - \frac{k^2 \operatorname{ch} kx}{60b^3(a+bx)^3} - \frac{k^3 \operatorname{sh} kx}{120b^4(a+bx)^2} - \frac{k^4 \operatorname{ch} kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int \frac{\operatorname{sh} kx}{a+bx}\,dx \qquad \text{see } \mathbf{2.476}\ 1.).$$

2.476

1. $$\int \frac{x^p\,dx}{\operatorname{sh}^q x} = \frac{-px^{p-1}\operatorname{sh} x - (q-2)x^p \operatorname{ch} x}{(q-1)(q-2)\operatorname{sh}^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}}{\operatorname{sh}^{q-2} x}\,dx - \frac{q-2}{q-1}\int \frac{x^p\,dx}{\operatorname{sh}^{q-2} x} \qquad [q > 2].$$

GU ((353))(8a)

2. $$\int \frac{x^p\,dx}{\operatorname{ch}^q x} = \frac{px^{p-1}\operatorname{ch} x + (q-2)x^p \operatorname{sh} x}{(q-1)(q-2)\operatorname{ch}^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\operatorname{ch}^{q-2} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\operatorname{ch}^{q-2} x} \qquad [q > 2].$$

GU ((353))(10a)

GU((353))(8b)

4. $$\int \frac{x^n}{\operatorname{ch} x}\,dx = \sum_{k=0}^{\infty} \frac{E_{2k} x^{n+2k+1}}{(n+2k+1)(2k)!} \qquad \left[|x| < \frac{\pi}{2}, \quad n \geq 0\right].$$

GU ((353))(10b)

5. $$\int \frac{dx}{x^n \operatorname{sh} x} = -[1+(-1)^n]\frac{2^{n-1}-1}{n!}B_n \ln x + \sum_{\substack{k=0 \\ k\neq \frac{n}{2}}}^{\infty} \frac{2-2^{2k}}{(2k-n)(2k)!}B_{2k}x^{2k-n} \qquad [|x| < \pi, \quad n \geq 1].$$

GU ((353))(9b)

6. $$\int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0 \\ k\neq \frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!}x^{2k-n+1} + \frac{1}{2}[1-(-1)^{n-1}] + \frac{E_{n-1}}{(n-1)!}\ln x \qquad \left[|x| < \frac{\pi}{2}\right].$$

GU ((353))(11b)

7. $$\int \frac{x^n}{\operatorname{sh}^2 x}\,dx = -x^n \operatorname{cth} x + n\sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1} \qquad [n > 1, \quad |x| < \pi].$$

GU ((353))(8c)

8. $$\int \frac{x^n}{\operatorname{ch}^2 x}\,dx = x^n \operatorname{th} x - n\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1} \qquad \left[n > 1, \quad |x| < \frac{\pi}{2}\right].$$

GU ((353))(10c)

9. $$\int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1-(-1)^n]\frac{2^n n}{(n+1)!}B_{n+1} \ln x - \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k\neq \frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!}(2x)^{2^k} \qquad [|x| < \pi].$$

GU ((353))(9c)

10. $$\int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + [1-(-1)^n] - \frac{2n(2^{n+1}-1)n}{(n+1)!} B_{n+1} \ln x + + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k\neq \frac{n+1}{2}}} \frac{(2^{2k}-1)B_{2k}}{(2k-n-1)(2k)!} (2x)^{2^k} \qquad \left[|x| < \frac{\pi}{2}\right].$$

GU ((353))(11c)

11. $$\int \frac{x}{\operatorname{sh}^{2n} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \times \left\{ \frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k+1} x} + \frac{1}{(2n-2k)\operatorname{sh}^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\,dx}{\operatorname{sh}^2 x}$$

(see **2.477** 17.).

2.477
GU ((353))(8e)

12. $$\int \frac{x}{\operatorname{sh}^{2n-1} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \times \left\{ \frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k} x} + \frac{1}{(2n-2k-1)\operatorname{sh}^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\,dx}{\operatorname{sh} x}$$

(see **2.477** 15.).

2.477
GU ((353))(8e)

13. $$\int \frac{x}{\operatorname{ch}^{2n} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \times \left\{ \frac{x \operatorname{sh} x}{\operatorname{ch}^{2n-2k+1} x} + \frac{1}{(2n-2k)\operatorname{ch}^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\,dx}{\operatorname{ch}^2 x}$$

(see **2.477** 18.).

2.477
GU ((353))(10e)

14. $$\int \frac{x}{\operatorname{ch}^{2n-1} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \times \left\{ \frac{x \operatorname{sh} x}{\operatorname{ch}^{2n-2k} x} + \frac{1}{(2n-2k-1)\operatorname{ch}^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\,dx}{\operatorname{ch} x}$$

(see **2.477** 16.).

2.477
GU ((353))(10e)

15. $$\int \frac{x\,dx}{\operatorname{sh} x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1} \qquad |x| < \pi.$$

GU ((353))(8b)a

16. $$\int \frac{x\,dx}{\operatorname{ch} x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!} \qquad |x| < \frac{\pi}{2}.$$

GU ((353))(10b)a

17. $$\int \frac{x\,dx}{\operatorname{sh}^2 x} = -x \operatorname{cth} x + \ln\, \operatorname{sh} x.$$

MZ 257

18. $$\int \frac{x\,dx}{\operatorname{ch}^2 x} = x \operatorname{th} x - \ln \operatorname{ch} x.$$

MZ 262

19. $$\int \frac{x\,dx}{\operatorname{sh}^3 x} = -\frac{x \operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{1}{2 \operatorname{sh} x} - \frac{1}{2} \int \frac{x\,dx}{\operatorname{sh} x} \qquad (\text{see } \mathbf{2.477}\ 15.).$$

2.477
MZ 257

20. $$\int \frac{x\,dx}{\operatorname{ch}^3 x} = \frac{x \operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{1}{2 \operatorname{ch} x} + \frac{1}{2} \int \frac{x\,dx}{\operatorname{ch} x} \qquad (\text{see } \mathbf{2.477}\ 16.).$$

2.477
MZ 262

21. $$\int \frac{x\,dx}{\operatorname{sh}^4 x} = -\frac{x \operatorname{ch} x}{3 \operatorname{sh}^3 x} - \frac{1}{6 \operatorname{sh}^2 x} + \frac{2}{3} x \operatorname{cth} x - \frac{2}{3} \ln \operatorname{sh} x.$$

22. $$\int \frac{x\,dx}{\operatorname{ch}^4 x} = \frac{x \operatorname{sh} x}{3 \operatorname{ch}^3 x} + \frac{1}{6 \operatorname{ch}^2 x} + \frac{2}{3} x \operatorname{th} x - \frac{2}{3} \ln \operatorname{ch} x.$$

MZ 262

23. $$\int \frac{x\,dx}{\operatorname{sh}^5 x} = -\frac{x \operatorname{ch} x}{4 \operatorname{sh}^4 x} - \frac{1}{12 \operatorname{sh}^3 x} + \frac{3x \operatorname{ch} x}{8 \operatorname{sh}^2 x} + \frac{3}{8 \operatorname{sh} x} + \frac{3}{8} \int \frac{x\,dx}{\operatorname{sh} x}$$ (see **2.477** 15.).

2.477
MZ 258

24. $$\int \frac{x\,dx}{\operatorname{ch}^5 x} = \frac{x \ \operatorname{sh} x}{4 \operatorname{ch}^4 x} + \frac{1}{12 \operatorname{ch}^3 x} + \frac{3x \operatorname{sh} x}{8 \operatorname{ch}^2 x} + \frac{3}{8 \operatorname{ch} x} + \frac{3}{8} \int \frac{x\,dx}{\operatorname{ch} x}$$ (see **2.477** 16.).

2.477
MZ 262

2.478

1. $$\int \frac{x^n \operatorname{ch} x\,dx}{(a + b \operatorname{sh} x)^m} = -\frac{x^n}{(m-1)b(a + b \operatorname{sh} x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1}\,dx}{(a + b \operatorname{sh} x)^{m-1}}$$ $[m \neq 1]$.

MZ 263

2. $$\int \frac{x^n \operatorname{sh} x\,dx}{(a + b \operatorname{ch} x)^m} = -\frac{x^n}{(m-1)b(a + b \operatorname{ch} x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1}\,dx}{(a + b \operatorname{ch} x)^{m-1}}$$ $[m \neq 1]$.

MZ 263

3. $$\int \frac{x\,dx}{1 + \operatorname{ch} x} = x \operatorname{th} \frac{x}{2} - 2 \ln \operatorname{ch} \frac{x}{2}.$$

4. $$\int \frac{x\,dx}{1 - \operatorname{ch} x} = x \operatorname{cth} \frac{x}{2} - 2 \ln \operatorname{sh} \frac{x}{2}.$$

5. $$\int \frac{x \operatorname{sh} x\,dx}{(1 + \operatorname{ch} x)^2} = -\frac{x}{1 + \operatorname{ch} x} + \operatorname{th} \frac{x}{2}.$$

6. $$\int \frac{x \operatorname{sh} x\, dx}{(1-\operatorname{ch} x)^2} = \frac{x}{1-\operatorname{ch} x} - \operatorname{cth}\frac{x}{2}.$$

MZ 262-264

7. $$\int \frac{x\, dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2\sin 2t}[L(u+t) - L(u-t) - 2L(t)]$$
$$[u = \operatorname{arctg}(\operatorname{th} x \operatorname{ctg} t), \quad t \neq \pm n\pi].$$

LO III 402

8. $$\int \frac{x \operatorname{ch} x\, dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2\sin t}\left[L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{\upsilon+t}{2}\right) + \right.$$
$$\left. + L\left(\frac{\upsilon-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right)\right]$$
$$\left[u = 2\operatorname{arctg}\left(\operatorname{th}\frac{x}{2}\cdot\operatorname{ctg}\frac{t}{2}\right), \ \ \upsilon = 2\operatorname{arctg}\left(\operatorname{cth}\frac{x}{2}\cdot\operatorname{ctg}\frac{t}{2}\right); \ \ t \neq \pm n\pi\right].$$

LO III 403

2.479

1. $$\int x^p \frac{\operatorname{sh}^{2m} x}{\operatorname{ch}^n x}\, dx = \sum_{k=0}^{m} (-1)^{m+k}\binom{m}{k}\int \frac{x^p\, dx}{\operatorname{ch}^{n-2k} x} \qquad (\text{see } \mathbf{2.477}\ 2.).$$

2.477

2. $$\int x^p \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch}^n x}\, dx = \sum_{k=0}^{m} (-1)^{m+k}\binom{m}{k}\int x^p \frac{\operatorname{sh} x}{\operatorname{ch}^{n-2k} x}\, dx \qquad [n>1] \qquad (\text{see } \mathbf{2.479}\ 3.)$$

2.479

3. $$\int x^p \frac{\operatorname{sh} x}{\operatorname{ch}^n x}\, dx = -\frac{x^p}{(n-1)\operatorname{ch}^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}\, dx}{\operatorname{ch}^{n-1} x} \qquad [n>1] \qquad (\text{see } \mathbf{2.477}\ 2.).$$

2.477

GU ((353))(12)

2.477

5. $$\int x^p \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh}^n x}\,dx = \sum_{k=0}^{m} \binom{m}{k} \int \frac{x^p \operatorname{ch} x}{\operatorname{sh}^{n-2k} x}\,dx \qquad (\text{see } \mathbf{2.479}\ 6.).$$

2.479

6. $$\int x^p \frac{\operatorname{ch} x}{\operatorname{sh}^n x}\,dx = -\frac{x^p}{(n-1)\operatorname{sh}^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}\,dx}{\operatorname{sh}^{n-1} x} \qquad [n > 1] \qquad (\text{see } \mathbf{2.477}\ 1.).$$

2.477

GU ((353))(13c)

7. $$\int x^p \operatorname{th} x\,dx = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k+p)(2k)!} x^{p+2k} \qquad \left[p > -1, \quad |x| < \frac{\pi}{2}\right].$$

GU ((353))(12d)

8. $$\int x^p \operatorname{cth} x\,dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \qquad [p \geq +1, \quad |x| < \pi].$$

GU ((353))(13d)

9. $$\int \frac{x \operatorname{ch} x}{\operatorname{sh}^2 x}\,dx = \ln \operatorname{th} \frac{x}{2} - \frac{x}{\operatorname{sh} x}.$$

10. $$\int \frac{x \operatorname{sh} x}{\operatorname{ch}^2 x}\,dx = -\frac{x}{\operatorname{ch} x} + \operatorname{arctg}(\operatorname{sh} x).$$

MZ 263

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.481

For $a^2 = b^2$:

3. $\int e^{ax}\,\mathrm{sh}(ax+c)\,dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}.$

4. $\int e^{-ax}\,\mathrm{sh}(ax+c)\,dx = \frac{1}{2}xe^{c} + \frac{1}{4a}e^{-(2ax+c)}.$

5. $\int e^{ax}\,\mathrm{ch}(ax+c)\,dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}.$

6. $\int e^{-ax}\,\mathrm{ch}(ax+c)\,dx = \frac{1}{2}xe^{c} - \frac{1}{4a}e^{-(2ax+c)}.$

MZ 275-277

156
2.482

1. $\int x^p e^{ax}\,\mathrm{sh}\,bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx - \int x^p e^{(a-b)x}\,dx\right\}$ $[a^2{\neq}b^2]$see **2.321**).

2.321

2. $\int x^p e^{ax}\,\mathrm{ch}\,bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx + \int x^p e^{(a-b)x}\,dx\right\}$ $[a^2{\neq}b^2] see 2.321$).

2.321

For $a^2 = b^2$:

3. $\int x^p e^{ax}\,\mathrm{sh}\,ax\,dx = \frac{1}{2}\int x^p e^{2ax}\,dx - \frac{x^{p+1}}{2(p+1)}$ (see **2.321**).

4. $$\int x^p e^{-ax}\operatorname{sh} ax\,dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2}\int x^p e^{-2ax}\,dx \qquad (\text{see } \mathbf{2.321}).$$

2.321

5. $$\int x^p e^{ax}\operatorname{ch} ax\,dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2}\int x^p e^{2ax}\,dx \qquad (\text{see } \mathbf{2.321})$$

2.321

MZ 276, 278

2.483

1. $$\int xe^{ax}\operatorname{sh} bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\operatorname{sh} bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\operatorname{ch} bx\right] \qquad [a^2 \neq b^2].$$

2. $$\int xe^{ax}\operatorname{ch} bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\operatorname{ch} bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\operatorname{sh} bx\right] \qquad [a^2 \neq b^2].$$

3. $$\int x^2 e^{ax}\operatorname{sh} bx\,dx = \frac{e^{ax}}{a^2-b^2}\left\{\left[ax^2 - \frac{2(a^2+b^2)}{a^2-b^2}x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2}\right]\operatorname{sh} bx - \right.$$
$$\left. - \left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2}\right]\operatorname{ch} x\right\} \qquad [a^2 \neq b^2].$$

4. $$\int x^2 e^{ax}\operatorname{ch} bx\,dx = \frac{e^{ax}}{a^2-b^2}\left\{\left[ax^2 - \frac{2(a^2+b^2)}{a^2-b^2}x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2}\right]\operatorname{ch} bx - \right.$$
$$\left. - \left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2}\right]\operatorname{sh} x\right\} \qquad [a^2 \neq b^2].$$

For $a^2 = b^2$:

5: $$\int xe^{ax}\operatorname{sh} ax\,dx = \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right) - \frac{x^2}{4}.$$

6. $\int xe^{-ax} \operatorname{sh} ax\, dx = \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right) + \frac{x^2}{4}.$

MZ 276, 278

157

7. $\int xe^{ax} \operatorname{ch} ax\, dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right).$

8. $\int xe^{-ax} \operatorname{ch} ax\, dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right).$

9. $\int x^2 e^{ax} \operatorname{sh} ax\, dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) - \frac{x^3}{6}.$

10. $\int x^2 e^{-ax} \operatorname{sh} ax\, dx = \frac{e^{-2ax}}{4a}\left(x^2 + \frac{x}{a} + \frac{1}{2a^2}\right) + \frac{x^3}{6}.$

11. $\int x^2 e^{ax} \operatorname{ch} ax\, dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right).$

2.484

1. $\int e^{ax} \operatorname{sh} bx \frac{dx}{x} = \frac{1}{2}\{\operatorname{Ei}[(a+b)x] - \operatorname{Ei}[(a-b)x]\} \qquad [a^2 \neq b^2].$

2. $\int e^{ax} \operatorname{ch} bx \frac{dx}{x} = \frac{1}{2}\{\operatorname{Ei}[(a+b)x] + \operatorname{Ei}[(a-b)x]\} \qquad [a^2 \neq b^2].$

3. $\int e^{ax} \operatorname{sh} bx \frac{dx}{x^2} = -\frac{e^{ax} \operatorname{sh} bx}{2x} + \frac{1}{2}\{(a+b)\operatorname{Ei}[(a+b)x] - (a-b)\operatorname{Ei}[(a-b)x]\} \qquad [a^2 \neq b^2].$

4. $\int e^{ax} \operatorname{ch} bx \frac{dx}{x^2} = -\frac{e^{ax} \operatorname{ch} bx}{2x} + \frac{1}{2}\{(a+b)\operatorname{Ei}[(a+b)x] + (a-b)\operatorname{Ei}[(a-b)x]\} \qquad [a^2 \neq b^2].$

For $a^2 = b^2$:

5. $$\int e^{ax} \operatorname{sh} ax \frac{dx}{x} = \frac{1}{2}[\operatorname{Ei}(2ax) - \ln x].$$

6. $$\int e^{-ax} \operatorname{sh} ax \frac{dx}{x} = \frac{1}{2}[\ln x - \operatorname{Ei}(-2ax)].$$

7. $$\int e^{ax} \operatorname{ch} ax \frac{dx}{x} = \frac{1}{2}[\ln x + \operatorname{Ei}(2ax)].$$

8. $$\int e^{ax} \operatorname{sh} ax \frac{dx}{x^2} = -\frac{1}{2x}(e^{2ax} - 1) + a \operatorname{Ei}(2ax).$$

9. $$\int e^{-ax} \operatorname{sh} ax \frac{dx}{x^2} = -\frac{1}{2x}(1 - e^{-2ax}) + a \operatorname{Ei}(-2ax).$$

10. $$\int e^{ax} \operatorname{ch} ax \frac{dx}{x^2} = -\frac{1}{2x}(e^{2ax} + 1) + a \operatorname{Ei}(2ax).$$

MZ 276, 278

2.5-2.6 Trigonometric Functions

2.50 Introduction

2.501

Integrals of the form $\int R(\sin x, \cos x)\, dx$ can always be reduced to integrals of rational functions by means of the substitution $t = \operatorname{tg} \frac{x}{2}$.

2.502

If $R(\sin x,\, \cos x)$ satisfies the relation

$$R(\sin x,\, \cos x) = -R(-\sin x,\, \cos x),$$

it is convenient to make the substitution $t = \cos x$.

2.503

If this function satisfies the relation

$$R(\sin x,\, \cos x) = -R(\sin x,\, -\cos x),$$

it is convenient to make the substitution $t = \sin x$.

2.504

If this function satisfies the relation

$$R(\sin x,\, \cos x) = R(-\sin x,\, -\cos\; x),$$

it is convenient to make the substitution $t = \operatorname{tg} x$.

2.51-2.52 Powers of trigonometric functions

2.510

$$\begin{aligned}
\int \sin^p x\; \cos^q x\, dx &= -\frac{\sin^{p-1} x\; \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x\; \cos^{q+2} x\, dx; \\
&= -\frac{\sin^{p-1} x\; \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x\; \cos^{q} x\, dx; \\
&= \frac{\sin^{p+1} x\; \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2} x\; \cos^{q} x\, dx; \\
&= \frac{\sin^{p+1} x\; \cos^{q-1} x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2} x\; \cos^{q-2} x\, dx; \\
&= \frac{\sin^{p+1} x\; \cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^{p} x\; \cos^{q-2} x\, dx; \\
&= -\frac{\sin^{p+1} x\; \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \sin^{p} x\; \cos^{q+2} x\, dx; \\
&= \frac{\sin^{p-1} x\; \cos^{q-1} x}{p+q}\left\{\sin^2 x - \frac{q-1}{p+q-2}\right\} + \\
&\quad + \frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2} x\; \cos^{q-2} x\, dx.
\end{aligned}$$

FI II 89, TI 214

2.511

$$1.\quad \int \sin^p x\,\cos^{2n} x\,dx = \frac{\sin^{p+1} x}{2n+p}\left\{\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\}+ \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \sin^p x\,dx.$$

This formula is applicable for arbitrary real p except for the following negative even integers:

$-2, -4, \dots, -2n$. If p is a natural number and $n=0$, we have:

$$2.\quad \int \sin^{2l} x\,dx = -\frac{\cos x}{2l}\left\{\sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)}\sin^{2l-2k-1} x\right\}+ \\ + \frac{(2l-1)!!}{2^l l!}x \qquad (\text{see also } \mathbf{2.513}\ 1.).$$

2.513
TI (232)

$$3.\quad \int \sin^{2l+1} x\,dx = -\frac{\cos x}{2l+1}\left\{\sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)}\sin^{2l-2k-2} x\right\} \\ (\text{see also } \mathbf{2.513}\ 2.).$$

2.513
TI (233)

$$4.\quad \int \sin^p x\cos^{2n+1} x\,dx = \frac{\sin^{p+1} x}{2n+p+1}\left\{\cos^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}.$$

This formula is applicable for arbitrary real p except for the negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.512

$$1.\quad \int \cos^p x\,\sin^{2n} x\,dx = \\ = -\frac{\cos^{p+1} x}{2n+p}\left\{\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\}+ \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \cos^p x\,dx.$$

p

$-2, -4, \ldots, -2n$. If p is a natural number and $n = 0$, we have

$$2. \quad \int \cos^{2l} x\, dx = \frac{\sin x}{2l} \left\{ \cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\ldots(2l-2k+1)}{2^k (l-1)(l-2)\ldots(l-k)} \cos^{2l-2k-1} x \right\} + \\ + \frac{(2l-1)!!}{2^l l!} x$$

TI (230)

$$3. \quad \int \cos^{2l+1} x\, dx = \frac{\sin x}{2l+1} \left\{ \cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\ldots(l-k)}{(2l-1)(2l-3)\ldots(2l-2k-1)} \cos^{2l-2k-2} x \right\}$$

2.513
TI (231)

$$4. \quad \int \cos^p x\, \sin^{2n+1} x\, dx = \\ = -\frac{\cos^{p+1} x}{2n+p+1} \left\{ \sin^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: $-1, -3, \ldots, -(2n+1)$.

2.513

$$1. \quad \int \sin^{2n} x\, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

2.511
TI (226)

$$2. \quad \int \sin^{2n+1} x\, dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$

2.511 TI (227)

3. $$\int \cos^{2n} x\,dx = \frac{1}{2^{2n}}\binom{2n}{n}x + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\frac{\sin(2n-2k)x}{2n-2k}$$

2.512 TI (224)

4. $$\int \cos^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}\frac{\sin(2n-2k+1)x}{2n-2k+1}$$ (see also **2.512** 3.).

2.512 TI (225)

5. $$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x = -\frac{1}{2}\sin x\,\cos x + \frac{1}{2}x.$$

6. $$\int \sin^3 x\,dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x = \frac{1}{3}\cos^3 x - \cos x.$$

7. $$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} = $$
$$= -\frac{3}{8}\sin x\,\cos x - \frac{1}{4}\sin^3 x\,\cos x + \frac{3}{8}x.$$

8. $$\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos\,3x - \frac{1}{80}\cos\,5x =$$
$$= -\frac{1}{5}\sin^4 x\,\cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x.$$

9. $$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x =$$
$$= -\frac{1}{6}\sin^5 x\,\cos x - \frac{5}{24}\sin^3 x\,\cos x - \frac{5}{16}\sin x\,\cos x + \frac{5}{16}x.$$

10. $$\int \sin^7 x\,dx = -\frac{35}{64}\cos x + \frac{7}{64}\cos 3x - \frac{7}{320}\cos 5x + \frac{1}{448}\cos 7x =$$
$$= -\frac{1}{7}\sin^6 x\,\cos x - \frac{6}{35}\sin^4 x\,\cos x + \frac{8}{35}\cos^3 x - \frac{24}{35}\cos x.$$

11. $$\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x\cos x + \frac{1}{2}x.$$

12. $$\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x.$$

13. $$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = \frac{3}{8}x + \frac{3}{8}\sin x\,\cos x + \frac{1}{4}\sin x\,\cos^3 x.$$

14. $$\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x = \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x\,\sin x.$$

15. $$\int \cos^6 x\,dx = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x =$$
$$= \frac{5}{16}x + \frac{5}{16}\sin x\,\cos x + \frac{5}{24}\sin x\,\cos^3 x + \frac{1}{6}\sin x\,\cos^5\,x.$$

16. $$\int \cos^7 x\,dx = \frac{35}{64}\sin x + \frac{7}{64}\sin 3x + \frac{7}{320}\sin 5x + \frac{1}{448}\sin 7x =$$
$$= \frac{24}{35}\sin x - \frac{8}{35}\sin^3 x + \frac{6}{35}\sin x\,\cos^4 x + \frac{1}{7}\sin x\,\cos^6 x.$$

17. $$\int \sin x\,\cos^2 x\,dx = -\frac{1}{4}\left\{\frac{1}{3}\cos 3x + \cos x\right\} = -\frac{\cos^3 x}{3}.$$

18. $$\int \sin x\,\cos^3 x\,dx = -\frac{\cos^4 x}{4}.$$

19. $$\int \sin x\,\cos^4 x\,dx = -\frac{\cos^5 x}{5}.$$

20. $$\int \sin^2 x \cos x \, dx = -\frac{1}{4}\left\{\frac{1}{3}\sin 3x - \sin x\right\} = \frac{\sin^3 x}{3}.$$

21. $$\int \sin^2 x \cos^2 x \, dx = -\frac{1}{8}\left\{\frac{1}{4}\sin 4x - x\right\}.$$

22. $$\int \sin^2 x \cos^3 x \, dx = -\frac{1}{16}\left\{\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right\} = $$
$$= \frac{\sin^3 x}{5}\left(\cos^2 x + \frac{2}{3}\right) = \frac{\sin^3 x}{5}\left(\frac{5}{3} - \sin^2 x\right).$$

23. $$\int \sin^2 x \cos^4 x \, dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x.$$

24. $$\int \sin^3 x \cos x \, dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{\sin^4 x}{4}.$$

25. $$\int \sin^3 x \cos^2 x \, dx = \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right) = $$
$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x.$$

26. $$\int \sin^3 x \cos^3 x \, dx = \frac{1}{32}\left(\frac{1}{6}\cos 6x - \frac{3}{2}\cos 2x\right).$$

27. $$\int \sin^3 x \cos^4 x \, dx = \frac{1}{7}\cos^3 x\left(-\frac{2}{5} - \frac{3}{5}\sin^2 x + \sin^4 x\right).$$

28. $$\int \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5}.$$

29. $\int \sin^4 x \cos^2 x \, dx = \frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x.$

30. $\int \sin^4 x \cos^3 x \, dx = \frac{1}{7}\sin^3 x \left(\frac{2}{5} + \frac{3}{5}\cos^2 x - \cos^4 x\right).$

31. $\int \sin^4 x \cos^4 x \, dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x.$

2.514

$$\int \frac{\sin^p x}{\cos^{2n} x}\, dx = \\ = \frac{\sin^{p+1} x}{2n-1}\left\{\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\sec^{2n-2k-1} x\right\} + \\ + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \sin^p x\, dx.$$

This formula is applicable for arbitrary real p. For $\int \sin^p x \, dx$, where p is a natural number, see 2.511 2., 3. and 2.513 1., 2. If $n = 0$ and p is a negative integer, we have for this integral:

2.515

1. $$\int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1}\left\{\operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k (l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)}\operatorname{cosec}^{2l-2k-1} x\right\}.$$

TI (242)

2. $$\int \frac{dx}{\sin^{2l+1} x} = -\frac{\cos x}{2l}\left\{\operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{28^k (l-1)(l-2)\dots(l-k)}\operatorname{cosec}^{2l-2k} x\right\} + \\ + \frac{(2l-1)!!}{2^l l!}\ln \operatorname{tg}\frac{x}{2}.$$

TI (243)

2.516

1. $$\int \frac{\sin^p x\, dx}{\cos^{2n+1} x} = \\ = \frac{\sin^{p+1}}{2n}\left\{\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)}\sec^{2n-2k} x\right\} + \\ + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\sin^p x}{\cos x}\, dx.$$

p $n = 0$ p

2. $$\int \frac{\sin^{2l+1} x\, dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k} x}{2k} - \ln \cos x.$$

3. $$\int \frac{\sin^{2l} x\, dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k-1} x}{2k-1} + \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

2.517

1. $$\int \frac{dx}{\sin^{2m+1} x\, \cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+2)\sin^{2m-2k+2} x} + \ln \operatorname{tg} x.$$

2. $$\int \frac{dx}{\sin^{2m} x\, \cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+1)\sin^{2m-2k+1} x} + \ln \operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

2.518

1. $$\int \frac{\sin^p x}{\cos^2 x}\, dx = \frac{\sin^{p-1} x}{\cos x} - (p-1)\int \sin^{p-2} x\, dx.$$

2. $$\int \frac{\cos^p x\, dx}{\sin^{2n} x} = -\frac{\cos^{p+1} x}{2n-1}\left\{\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x\right\} + \frac{(2n-p-2)(2n-p-4)\dots(2-p)(-p)}{(2n-1)!!}\int \cos^p x\, dx.$$

This formula is applicable for arbitrary real p. For $\int \cos^p x\, dx$ where p is a natural number, see 2.512 2., 3. and 2.513 3., 4. If $n = 0$ and p is a negative integer, we have for this integral:

2.519

1. $$\int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1}\left\{\sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k (l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)} \sec^{2l-2k-1} x\right\}.$$

2. $$\int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l} \left\{ \sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sec^{2l-2k} x \right\} + \\ + \frac{(2l-1)!!}{2^l l!} \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

TI (241)

2.521

1. $$\int \frac{\cos^p x\, dx}{\sin^{2n+1} x} = -\frac{\cos^{p+1} x}{2n} \left\{ \operatorname{cosec}^{2n} x + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \operatorname{cosec}^{2n-2k} x \right\} + \\ + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n \cdot n!} \int \frac{\cos^p x}{\sin x}\, dx.$$

This formula is applicable for arbitrary real p. For $n = 0$ and p a natural number, we have

2. $$\int \frac{\cos^{2l+1} x\, dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k} x}{2k} + \ln \sin x.$$

3. $$\int \frac{\cos^{2l} x\, dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k-1} x}{2k-1} + \ln \operatorname{tg} \frac{x}{2}.$$

2.522

1. $$\int \frac{dx}{\sin x\, \cos^{2m+1} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+2)\cos^{2m-2k+2} x} + \ln \operatorname{tg} x.$$

2. $$\int \frac{dx}{\sin x\, \cos^{2m} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+1)\cos^{2m-2k+1} x} + \ln \operatorname{tg} \frac{x}{2}.$$

GW ((331))(15)

2.523

$$\int \frac{\cos^m x}{\sin^2 x}\, dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1)\int \cos^{m-2} x\, dx.$$

2.524

1. $$\int \frac{\sin^{2n+1} x}{\cos^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} (-1)^{k+1}\binom{n}{k}\frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}}\binom{n}{\frac{m-1}{2}}\ln\cos x.$$

GU ((331))(11d)

2. $$\int \frac{\cos^{2n+1} x}{\sin^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{k} {}^{n}\,(-1)\binom{n}{k}\frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}}\binom{n}{\frac{m-1}{2}}\ln\sin x.$$

[In formulas 2.524 1. and 2.524 2., $s = 1$ for m odd and $m < 2n+1$; in other cases, $s = 0$.]

2.525

1. $$\int \frac{dx}{\sin^{2m} x\,\cos^{2n} x} = \sum_{k=0}^{m+n-1}\binom{m+n-1}{k}\frac{\operatorname{tg}^{2k-2m+1} x}{2k-2m+1}.$$

TI (267)

2. $$\int \frac{dx}{\sin^{2m+1} x\,\cos^{2n+1} x} = \sum_{k=0}^{m+n}\binom{m+n}{k}\frac{\operatorname{tg}^{2k-2m} x}{2k-2m} + \binom{m+n}{m}\ln\operatorname{tg} x.$$

TI (268), GU ((331))(15f)

2.526

1. $$\int \frac{dx}{\sin x} = \ln\operatorname{tg}\frac{x}{2}.$$

2. $$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

3. $$\int \frac{dx}{\sin^3 x} = -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln\operatorname{tg}\frac{x}{2}.$$

4. $$\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\,\sin^3 x} - \frac{2}{3}\operatorname{ctg} x = -\frac{1}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$$

5. $$\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\,\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8}\ln \operatorname{tg}\frac{x}{2}.$$

6. $$\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\,\sin^5 x} - \frac{4}{15}\operatorname{ctg}^3 x - \frac{4}{5}\operatorname{ctg} x;$$
$$= -\frac{1}{5}\operatorname{ctg}^5 x - \frac{2}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$$

7. $$\int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6\,\sin^2 x}\left(\frac{1}{\sin^4 x} + \frac{5}{4\,\sin^2 x} + \frac{15}{8}\right) + \frac{5}{16}\ln \operatorname{tg}\frac{x}{2}.$$

8. $$\int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7}\operatorname{ctg}^7 x + \frac{3}{5}\operatorname{ctg}^5 x + \operatorname{ctg}^3 x + \operatorname{ctg} x\right).$$

9. $$\int \frac{dx}{\cos x} = \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \ln \operatorname{ctg}\left(\frac{\pi}{4} - \frac{x}{2}\right) = \ln\sqrt{\frac{1+\sin x}{1-\sin x}}.$$

10. $$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

11. $$\int \frac{dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

12. $$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3\,\cos^3 x} + \frac{2}{3}\operatorname{tg} x = \frac{1}{3}\operatorname{tg}^3 x + \operatorname{tg} x.$$

13. $$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\,\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{8}\ln \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

14. $$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5\,\cos^5 x} + \frac{4}{15}\operatorname{tg}^3 x + \frac{4}{5}\operatorname{tg} x = \frac{1}{5}\operatorname{tg}^5 x + \frac{2}{3}\operatorname{tg}^3 x + \operatorname{tg} x.$$

15. $\int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\ \cos^6 x} + \frac{5\ \sin x}{24\ \cos^4 x} + \frac{5\ \sin x}{16\ \cos^2 x} + \frac{5}{16} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$

16. $\int \frac{dx}{\cos^8 x} = \frac{1}{7} \operatorname{tg}^7 x + \frac{3}{5} \operatorname{tg}^5 x + \operatorname{tg}^3 x + \operatorname{tg} x.$

17. $\int \frac{\sin x}{\cos x}\, dx = -\ln \cos x.$

18. $\int \frac{\sin^2 x}{\cos x}\, dx = -\sin x + \ln \operatorname{tg} \left(\frac{\pi}{4} = \frac{x}{2} \right).$

19. $\int \frac{\sin^3 x}{\cos x}\, dx = -\frac{\sin^2 x}{2} - \ln \cos x = \frac{1}{2} \cos^2 x - \ln \cos x.$

20. $\int \frac{\sin^4 x}{\cos x}\, dx = -\frac{1}{3} \sin^3 x - \sin x + \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$

21. $\int \frac{\sin^2 x\, dx}{\cos^2 x} = \frac{1}{\cos x}.$

22. $\int \frac{\sin^2 x\, dx}{\cos^2 x} = \operatorname{tg} x - x.$

23. $\int \frac{\sin^3 x\, dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}.$

24. $\int \frac{\sin^4 x\, dx}{\cos^2 x} = \operatorname{tg} x + \frac{1}{2} \sin x\ \cos x - \frac{3}{2} x.$

25. $$\int \frac{\sin x\,dx}{\cos^3 x} = \frac{1}{2\ \cos^2 x} = \frac{1}{2}\mathrm{tg}^2 x.$$

26. $$\int \frac{\sin^2 x\,dx}{\cos^3 x} = \frac{\sin x}{2\ \cos^2 x} - \frac{1}{2}\ln \mathrm{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

27. $$\int \frac{\sin^3 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \ln\cos x.$$

28. $$\int \frac{\sin^4 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2}\ln \mathrm{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

29. $$\int \frac{\sin x\,dx}{\cos^4 x} = \frac{1}{3\ \cos^3 x}.$$

30. $$\int \frac{\sin^2 x\,dx}{\cos^4 x} = \frac{1}{3}\ \mathrm{tg}^3 x.$$

31. $$\int \frac{\sin^3 x\,dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3\ \cos^3 x}.$$

32. $$\int \frac{\sin^4 x\,dx}{\cos^4 x} = \frac{1}{3}\ \mathrm{tg}^3 x - \mathrm{tg}\, x + x.$$

33. $$\int \frac{\cos x\,dx}{\sin x} = \ln\sin x.$$

34. $$\int \frac{\cos^2 x\,dx}{\sin x} = \cos x + \ln\ \mathrm{tg}\frac{x}{2}.$$

35. $\int \frac{\cos^3 x\, dx}{\sin x} = \frac{\cos^2 x}{2} + \ln \sin x.$

36. $\int \frac{\cos^4 x\, dx}{\sin x} = \frac{1}{3}\cos^3 x + \cos x + \ln\ \operatorname{tg}\left(\frac{x}{2}\right).$

37. $\int \frac{\cos x}{\sin^2 x}\, dx = -\frac{1}{\sin x}.$

38. $\int \frac{\cos^2 x}{\sin^2 x}\, dx = -\operatorname{ctg} x - x.$

39. $\int \frac{\cos^3 x}{\sin^2 x}\, dx = -\sin x - \frac{1}{\sin x}.$

40. $\int \frac{\cos^4 x}{\sin^2 x}\, dx = -\operatorname{ctg} x - \frac{1}{2}\sin x\, \cos x - \frac{3}{2}x.$

41. $\int \frac{\cos x}{\sin^3 x}\, dx = -\frac{1}{2\, \sin^2 x}.$

42. $\int \frac{\cos^2 x}{\sin^3 x}\, dx = -\frac{\cos x}{2\, \sin^2 x} - \frac{1}{2}\ln \operatorname{tg} \frac{x}{2}.$

43. $\int \frac{\cos^3 x}{\sin^3 x}\, dx = -\frac{1}{2\, \sin^2 x} - \ln \sin x.$

44. $\int \frac{\cos^4 x}{\sin^3 x}\, dx = -\frac{1}{2}\frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2}\ln \operatorname{tg}\frac{x}{2}.$

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62. $$\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

63. $$\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^2 x} + \frac{5}{2} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

64. $$\int \frac{dx}{\sin^4 x \cos^4 x} = -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x.$$

2.527

1. $$\int \operatorname{tg}^p x \, dx = \frac{\operatorname{tg}^{p-1} x}{p-1} - \int \operatorname{tg}^{p-2} x \, dx \qquad [p \neq 1].$$

2. $$\int \operatorname{tg}^{2n+1} x \, dx = \sum_{k=1}^{n} (-1)^{n+k} \binom{n}{k} \frac{1}{2k \cos^{2k} x} - (-1)^n \ln \cos x =$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k-1} \operatorname{tg}^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln \cos x.$$

3. $\int \operatorname{tg}^{2n} x\, dx = \sum_{k=1}^{n} (-1)^{k-1} \frac{\operatorname{tg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x.$

4. $\int \operatorname{ctg}^{p} x\, dx = -\frac{\operatorname{ctg}^{p-1} x}{p-1} - \int \operatorname{ctg}^{p-2} x\, dx \qquad [p \neq 1].$

5. $$\int \operatorname{ctg}^{2n+1} x\, dx = \sum_{k=1}^{n} (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k \sin^{2k} x} + (-1)^n \ln \sin x =$$
$$= \sum_{k=1}^{n} (-1)^{k} \frac{\operatorname{ctg}^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln \sin x.$$

6. $\int \operatorname{ctg}^{2n} x\, dx = \sum_{k=1}^{n} (-1)^{k} \frac{\operatorname{ctg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x.$

GU ((331))(14)

For special formulas for $p = 1,\ 2,\ 3,\ 4$, see 2.526 17., 2.526 33., 2.526 22., 2.526 38., 2.526 27., 2.526 43., 2.526 32., 2.526 48.

2.53-2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.531

1. $\int \sin(ax+b)\, dx = -\frac{1}{a} \cos(ax+b).$

2. $\int \cos(ax+b)\, dx = -\frac{1}{a} \sin(ax+b).$

2.532

1. $\int \sin(ax+b) \sin(cx+d)\, dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)} \qquad [a^2 \neq c^2].$

2.[8] $\int \sin(ax+b) \cos(cx+d)\, dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)} \qquad [a^2 \neq c^2].$

3. $$\int \cos(ax+b)\cos(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)} \qquad [a^2 \neq c^2].$$

For $c = a$:

4. $$\int \sin(ax+b)\sin(ax+d)\,dx = \frac{x}{2}\cos(b-d) - \frac{\sin(2ax+b+d)}{4a}.$$

5. $$\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d)}{4a}.$$

6. $$\int \cos(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\cos(b-d) + \frac{\sin(2ax+b+d)}{4a}.$$

GU ((332))(3)

2.533

1.[8] $$\int \sin ax \cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \qquad [a^2 \neq b^2].$$

2.[8] $$\int \sin ax \sin bx \sin cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right\}.$$

PE (376)

3. $$\int \sin ax \cos bx \cos cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right\}.$$

PE (378)

4. $$\int \cos ax \sin bx \sin cx\,dx = \frac{1}{4}\left\{\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right\}.$$

5. $$\int \cos ax \cos bx \cos cx \, dx = \frac{1}{4}\left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \right.$$
$$\left. + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right\}.$$

PE (377)

2.534

1. $$\int \frac{\cos px + i \sin px}{\sin nx}\, dx = -2\int \frac{z^{p+n-1}}{1-z^{2n}}\, dz$$
2. $$\int \frac{\cos px + i \sin px}{\cos nx}\, dx = -2i\int \frac{z^{p+n-1}}{1-z^{2n}}\, dz$$

$[z = \cos x + i \sin x]$.

PE (373)
PE (374)

2.535

1. $$\int \sin^p x \sin ax \, dx = \frac{1}{p+a}\left\{ -\sin^p x \cos ax + p\int \sin^{p-1} x \cos(a-1)x \, dx \right\}.$$

GU ((332))(5a)

2. $$\int \sin^p x \sin(2n+1)\, x \, dx =$$
$$= (2n+1)\left\{ \int \sin^{p+1} x \, dx + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2]\ldots \ldots[(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \times \right.$$
$$\left. \times \int \sin^{2k+p+1} x \, dx \right\};$$
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^{k-1}\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k+1)x + \right. \right.$$
$$\left. + (-1)^k \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k)\, x \right] +$$
$$\left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma(p-2n+1)} \int \sin^{p-2n+1} x \, dx \right\}.$$

1. $$\int \sin^p x \cos ax\, dx = \frac{1}{p+1}\left\{\sin^p x \sin ax - p\int \sin^{p-1} x \sin{(a-1)}\, x\, dx\right\}.$$

GU ((332))(6a)

2. $$\int \sin^p x \cos{(2n+1)}\, x\, dx =$$

$$= \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times$$
$$\times \sin^{2k+p+1} x;$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^k\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)}\sin^{p-2k} x \sin{(2n-2k+1)}x + \right.\right.$$
$$\left. + \frac{(-1)^k\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma(p-2k)}\sin^{p-2k-1} x \cos{(2n-2k)}\, x\right] +$$
$$\left. + \frac{(-1)^n\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma(p-2n+1)}\int \sin^{p-2n} x \cos x\, dx\right\};$$

$[p$ is not equal to $-3, -5, \dots, -(2n+1)]$.

GU ((332))(6c)
TI (301)

3. $$\int \sin^p x\ \cos 2nx\, dx = \int \sin^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2\cdot(4n^2-2^2)\dots[4n^2-(2k-2)^2]}{(2k)!}\int \sin^{2k+p} x\, dx;$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^k\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)}\sin^{p-2k} x \sin(2n-2k)x + \right.\right.$$
$$\left. + \frac{(-1)^k\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma(p-2k)}\sin^{p-2k-1} x \cos{(2n-2k-1)}x\right] +$$
$$\left. + \frac{(-1)^n\Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\Gamma(p-2n+1)}\int \sin^{p-2n} x\, dx\right\}.$$

GU ((332))(6c)
TI (300)

2.537

1. $$\int \cos^p x \sin ax\,dx = \frac{1}{p+a}\left\{-\cos^p x \cos ax + p\int \cos^{p-1} x \sin(a-1)x\,dx\right\}.$$

GU ((332))(7a)

2. $$\int \cos^p x \sin(2n+1)\,x\,dx =$$
$$= (-1)^{n+1}\left\{\frac{\cos^{p+1} x}{p+1} + \right.$$
$$\left. + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x\right\};$$
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{-\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{2k+1}\Gamma(p-2k+1)} \cos^{p-k} x \cos(2n-k+1)x + \right.$$
$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma(p-n+1)}\int \cos^{p-n} x \sin(n+1)x\,dx\right\};$$
$$[p \text{ is not equal to } -3, -5, \ldots, -(2n+1)].$$

GU ((332))(7b)a
TI (295)

3. $$\int \cos^p x\, \sin 2nx\,dx = (-1)^n\left\{\frac{\cos^{p+2} x}{p+2} + \right.$$
$$\left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2)\ldots[4n^2-(2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x\right\};$$
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{-\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)} \cos^{p-k} x \cos(2n-k)x + \right.$$
$$\left. + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\Gamma(p-n+1)}\int \cos^{p-n} x \sin nx\,dx\right\};$$
$$[p \text{ is not equal to } -2, -4, \ldots, -2n].$$

GU ((332))(7b)a
TI (297)

2.538

1. $$\int \cos^p x \cos ax\,dx = \frac{1}{p+a}\left\{\cos^p x \sin ax + p\int \cos^{p-1} x \cos(a-1)x\,dx\right\}.$$

2. $$\int \cos^p x\, \cos(2n+1)x\, dx =$$

$$= (-1)^n (2n+1) \left\{ \int \cos^{p+1} x\, dx + \right.$$

$$+ \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \times$$

$$\left. \times \int \cos^{2k+p+1} x\, dx \right\};$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x + \right.$$

$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x\, dx \right\}.$$

GU ((332))(8b)a
TI (293)

3. $$\int \cos^p x\, \cos 2nx\, dx =$$

$$= (-1)^n \left\{ \int \cos^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2[4n^2 - 2^2] \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \cos^{2k+p} x\, dx \right\};$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x + \right.$$

$$\left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos nx\, dx \right\}.$$

GU ((332))(8b)a
TI (294)

2.539

1. $$\int \frac{\sin(2n+1)x}{\sin x}\, dx = 2 \sum_{k=1}^{n} \frac{\sin 2kx}{2k} + x.$$

2. $$\int \frac{\sin 2nx}{\sin x}\, dx = 2 \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}.$$

3. $\int \frac{\cos(2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos 2kx}{2k} + \ln \sin x.$

4. $\int \frac{\cos 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos(2k-1)x}{2k-1} + \ln \operatorname{tg} \frac{x}{2}.$

GI ((332))(6e)

5. $\int \frac{\sin(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1} \ln \cos x.$

6. $\int \frac{\sin 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1}.$

GU ((332))(7d)

7. $\int \frac{\cos(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^{n} x.$

8. $\int \frac{\cos 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^{n} \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$

GU ((332))(8d)

2.541

1. $\int \sin(n+1)x \sin^{n-1} x\,dx = \frac{1}{n} \sin^n x \sin nx.$

BI ((71))(1)a

2. $\int \sin(n+1)x \cos^{n-1} x\,dx = -\frac{1}{n} \cos^n x \cos nx.$

BI ((71))(2)a

3. $\int \cos(n+1)x \sin^{n-1} x\,dx = \frac{1}{n} \sin^n x \cos nx.$

4. $\int \cos{(n+1)x} \cos^{n-1} x \, dx = \frac{1}{n} \cos^n x \sin nx.$

BI ((71))(4)a

5. $\int \sin\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x \, dx = \frac{1}{n} \sin^n x \cos n\left(\frac{\pi}{2} - x\right).$

BI ((71))(5)a

6. $\int \cos\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x \, dx = -\frac{1}{n} \sin^n x \sin n\left(\frac{\pi}{2} - x\right).$

BI ((71))(6)a

2.542

1. $\int \frac{\sin 2x}{\sin^n x} \, dx = -\frac{2}{(n-2)\sin^{n-2} x}.$

For $n = 2$:

2. $\int \frac{\sin 2x}{\sin^2 x} \, dx = 2 \ln \sin x.$

2.543

1. $\int \frac{\sin 2x \, dx}{\cos^n x} = \frac{2}{(n-2)\cos^{n-2} x}.$

For $n = 2$:

2. $\int \frac{\sin 2x}{\cos^2 x} \, dx = -2 \ln \cos x.$

2.544

1. $\int \frac{\cos 2x \, dx}{\sin x} = 2 \cos x + \ln \operatorname{tg} \frac{x}{2}.$

2. $$\int \frac{\cos 2x \, dx}{\sin^2 x} = -\operatorname{ctg} x - 2x.$$

3. $$\int \frac{\cos 2x \, dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} - \frac{3}{2} \ln \operatorname{tg} \frac{x}{2}.$$

4. $$\int \frac{\cos 2x \, dx}{\cos x} = 2\sin x - \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

5. $$\int \frac{\cos 2x \, dx}{\cos^2 x} = 2x - \operatorname{tg} x.$$

6. $$\int \frac{\cos 2x \, dx}{\cos^3 x} = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

7. $$\int \frac{\sin 3x \, dx}{\sin x} = x + \sin 2x.$$

8. $$\int \frac{\sin 3x}{\sin^2 x} \, dx = 3 \ln \operatorname{tg} \frac{x}{2} + 4\cos x.$$

9. $$\int \frac{\sin 3x}{\sin^3 x} \, dx = -3 \operatorname{ctg} x - 4x.$$

2.545

1. $$\int \frac{\sin 3x}{\cos^n x} \, dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}.$$

For $n = 1$ and $n = 3$:

2. $$\int \frac{\sin 3x}{\cos x} \, dx = 2\sin^2 x + \ln \cos x.$$

3. $\displaystyle\int \frac{\sin 3x}{\cos^3 x}\,dx = -\frac{1}{2\cos^2 x} - 4\ln\ \cos x.$

2.546

1. $\displaystyle\int \frac{\cos 3x}{\sin^n x}\,dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}.$

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For $n = 1$ and $n = 3$:

2. $\displaystyle\int \frac{\cos 3x}{\sin x}\,dx = -2\sin^2 x + \ln\sin x.$

3. $\displaystyle\int \frac{\cos 3x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - 4\ \ln\sin x.$

2.547

1. $\displaystyle\int \frac{\sin nx}{\cos^p x}\,dx = 2\int \frac{\sin (n-1)x\,dx}{\cos^{p-1} x} - \int \frac{\sin (n-2)\,x\,dx}{\cos^p x}.$

2. $\displaystyle\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x.$

3. $\displaystyle\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right).$

4. $\displaystyle\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\operatorname{tg} x.$

2.548

1. $$\int \frac{\sin^m x\,dx}{\sin (2n+1)x} = \frac{1}{2n+1}\sum_{k=0}^{2n} (-1)^{n+k}\cos^m\left[\frac{2k+1}{2(2n+1)}\pi\right]\ln\frac{\sin\left[\dfrac{(k-n)\pi}{2(2n+1)} + \dfrac{x}{2}\right]}{\sin\left[\dfrac{k+n+1}{(2n+1)}\pi - \dfrac{x}{2}\right]}$$

$[m -\ \text{a natural number}\ \leq 2n].$

2. $$\int \frac{\sin^{2m} x\,dx}{\sin 2nx} = \frac{(-1)^n}{2n}\left\{\ln\cos x + \sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left(\cos^2 x - \sin^2\frac{k\pi}{2n}\right)\right\}$$

[$m -$ a natural number $\leq n$].

TI (379)

3. $$\int \frac{\sin^{2m+1} x}{\sin 2nx}\,dx = \frac{(-1)^n}{2n}\left\{\ln\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right) + \right.$$
$$\left. + \sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left[\operatorname{tg}\left(\frac{n+k}{4n}\pi-\frac{x}{2}\right)\operatorname{tg}\left(\frac{n-k}{4n}\pi-\frac{x}{2}\right)\right]\right\}$$

[$m -$ a natural number $< n$].

TI (380)

4. $$\int \frac{\sin^{2m} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1}\left\{\ln\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right) + \sum_{k=1}^{n}(-1)^k \times \right.$$
$$\left. \times\cos^{2m}\frac{k\pi}{2n+1}\ln\left[\operatorname{tg}\left(\frac{2n+2k+1}{4(2n+1)}\pi-\frac{x}{2}\right)\operatorname{tg}\left(\frac{2n-2k+1}{2(2n+1)}\pi-\frac{x}{2}\right)\right]\right\}$$

[$m -$ a natural number $\leq n$].

TI (381)

5. $$\int \frac{\sin^{2m+1} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1}\left\{\ln\cos x + \sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left(\cos^2 x - \sin^2\frac{k\pi}{2n+1}\right)\right\}$$

[$m -$ a natural number $\leq n$].

TI (382)a

6. $$\int \frac{\sin^{m} x\,dx}{\cos 2nx} = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^{n+k}\cos^m\left[\frac{2k+1}{4n}\pi\right]\ln\frac{\sin\left[\dfrac{2k-2n+1}{8n}\pi+\dfrac{x}{2}\right]}{\sin\left[\dfrac{2k+2n+1}{8n}\pi-\dfrac{x}{2}\right]}$$

[$m -$ a natural number $< 2n$].

TI (377)

7. $$\int \frac{\cos^{2m+1} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\left\{\ln\sin x + \sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left(\sin^2 x - \sin^2\frac{k\pi}{2n+1}\right)\right\}$$

[$m -$ a natural number $\leq n$].

8. $$\int \frac{\cos^{2m} x\, dx}{\sin (2n+1)x} = \frac{1}{2n+1}\left\{\ln \operatorname{tg}\frac{x}{2} + \right.$$
$$\left. + \sum_{k=1}^{n} (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4n+2}\right) \operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n+2}\right)\right]\right\}$$
$[m-$ a natural number $\leq n]$.

TI (375)

9. $$\int \frac{\cos^{2m+1} x}{\sin 2nx}\, dx = \frac{1}{2n}\left\{\ln \operatorname{tg}\frac{x}{2} + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4}\right) \operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n}\right)\right]\right\}$$
$[m-$ a natural number $< n]$.

TI (374)

10. $$\int \frac{\cos^{2m} x}{\sin 2nx}\, dx = \frac{1}{2n}\left\{\ln \sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n}\right)\right\}$$
$[m-$ a natural number $\leq n]$.

TI (373)

11. $$\int \frac{\cos^{m} x}{\cos nx}\, dx = \frac{1}{n}\sum_{k=0}^{n-1} (-1)^k \cos^{m} \frac{2k+1}{2n}\pi \ln \frac{\sin\left[\frac{2k+1}{4n}\pi + \frac{x}{2}\right]}{\sin\left[\frac{2k+1}{4n}\pi - \frac{x}{2}\right]}$$
$[m-$ a natural number $\leq n]$.

TI (372)

2.549

1. $$\int \sin x^2\, dx = \sqrt{\frac{\pi}{2}}\, S(x).$$

2. $$\int \cos x^2\, dx = \sqrt{\frac{\pi}{2}}\, C(x).$$

3. $$\int \sin (ax^2+2bx+c)\, dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{ac-b^2}{a} S\left(\frac{ax+b}{\sqrt{a}}\right) + \sin\frac{ac-b^2}{a} C\left(\frac{ax+b}{\sqrt{a}}\right)\right\}.$$

PE (444)

6. $\int \cos \ln x \, dx = \frac{x}{2}(\sin \ln x + \cos \ln x).$

PE (445)

2.55-2.56 Rational functions of the sine and cosine

2.551

1. $$\int \frac{A + B\sin x}{(a + b\sin x)^n}\, dx = \frac{1}{(n-1)(a^2 - b^2)}\left[\frac{(Ab - aB)\cos x}{(a + b\sin x)^{n-1}} + \right.$$
$$\left. + \int \frac{(Aa - Bb)(n-1) + (aB - bA)(n-2)\sin x}{(a + b\sin x)^{n-1}}\, dx\right].$$

TI (358)a

For $n = 1$:

2. $$\int \frac{A + B\sin x}{a + b\sin x}\, dx = \frac{B}{b}x + \frac{Ab - aB}{b}\int \frac{dx}{a + b\sin x} \qquad (\text{see } \mathbf{2.551}\ 3.).$$

2.551
TI (342)

3. $$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{a \operatorname{tg}\frac{x}{2} + b}{\sqrt{a^2 - b^2}} \qquad [a^2 > b^2];$$
$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \operatorname{tg}\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \operatorname{tg}\frac{x}{2} + b + \sqrt{b^2 - a^2}} \qquad [a^2 < b^2].$$

2.552

1. $$\int \frac{A + B\cos x}{(a + b\sin x)^n}\, dx = -\frac{B}{(n-1)b(a + b\sin x)^{n-1}} + A\int \frac{dx}{(a + b\sin x)^n} \qquad (\text{see } \mathbf{2.552}\ 3.).$$

2.552

TI (361)

For $n = 1$:

$$2. \quad \int \frac{A + B\cos x}{a + b\sin x}\,dx = \frac{B}{b}\ln(a + b\sin x) + A\int \frac{dx}{a + b\sin x} \qquad (\text{see } \mathbf{2.551}\ 3.).$$

2.551
TI (344)

$$3. \quad \int \frac{dx}{(a + b\sin x)^n} = \frac{1}{(n-1)(a^2 - b^2)}\left\{\frac{b\cos x}{(a + b\sin x)^{n-1}} + \right.$$
$$\left. + \int \frac{(n-1)a - (n-2)b\sin x}{(a + b\sin x)^{n-1}}\,dx\right\} \qquad (\text{see } \mathbf{2.551}\ 1.).$$

2.551
TI (359)

2.553

$$1. \quad \int \frac{A + B\sin x}{(a + b\cos x)^n}\,dx = \frac{B}{(n-1)b(a + b\cos x)^{n-1}} + A\int \frac{dx}{(a + b\cos x)^n} \qquad (\text{see } \mathbf{2.554}\ 3.).$$

2.554
TI (355)

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For $n = 1$:

$$2. \quad \int \frac{A + B\sin x}{a + b\cos x}\,dx = -\frac{B}{b}\ln(a + b\cos x) + A\int \frac{dx}{a + b\cos x} \qquad (\text{see2.5533.}).$$

2.553
TI (343)

$$3. \quad \int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\operatorname{arctg}\frac{\sqrt{a^2 - b^2}\operatorname{tg}\frac{x}{2}}{a + b} \qquad [a^2 > b^2];$$
$$= \frac{1}{\sqrt{b^2 - a^2}}\ln\frac{\sqrt{b^2 - a^2}\operatorname{tg}\frac{x}{2} + a + b}{\sqrt{b^2 - a^2}\operatorname{tg}\frac{x}{2} - a - b} \qquad [a^2 < b^2].$$

2.554

$$1. \quad \int \frac{A + B\cos x}{(a + b\cos x)^n}\,dx = \frac{1}{(n-1)(a^2-b^2)}\left[\frac{(aB - Ab)\sin x}{(a+b\cos x)^{n-1}} + \right.$$
$$\left. + \int \frac{(Aa - bB)(n-1) + (n-2)(aB - bA)\cos x}{(a+b\cos x)^{n-1}}\,dx\right].$$

TI (353)

For $n = 1$:

$$2. \quad \int \frac{A + B\cos x}{a + b\cos x}\,dx = \frac{B}{b}x + \frac{Ab - aB}{b}\int \frac{dx}{a + b\cos x} \qquad (\text{see} 2.5533.).$$

2.553
TI (341)

$$3. \quad \int \frac{dx}{(a + b\cos x)^n} = -\frac{1}{(n-1)(a^2-b^2)}\left\{\frac{b\sin x}{(a+b\cos x)^{n-1}} - \right.$$
$$\left. - \int \frac{(n-1)a - (n-2)b\cos x}{(a+b\cos x)^{n-1}}\,dx\right\} \qquad (\text{see} 2.5541.).$$

2.554
TI (354)

In integrating the functions in formulas 2.551 3. and 2.553 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points $x = \arcsin\left(-\frac{a}{b}\right)$ in formula 2.551 3. or over the points $x = \arccos\left(-\frac{a}{b}\right)$ in formula 2.553 3.

2.555

Formulas 2.551 3. and 2.553 3 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

$$1. \quad \int \frac{A + B\sin x}{(1 \pm \sin x)^n}\,dx = -\frac{1}{2^{n-1}}\left\{2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\operatorname{tg}^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1} \pm \right.$$
$$\left. \pm (A \mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\operatorname{tg}^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1}\right\}.$$

2. $$\int \frac{A + B\cos x}{(1 \pm \cos x)^n}\, dx = \frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \pm \right.$$
$$\left. \pm\,(A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \right\}.$$

TI (356)

For $n = 1$:

3. $$\int \frac{A + B\sin x}{1 \pm \sin x}\, dx = \pm Bx + (A \mp B)\operatorname{tg}\left(\frac{\pi}{4} \mp \frac{x}{2}\right).$$

TI (250)

4. $$\int \frac{A + B\cos x}{1 \pm \cos x}\, dx = \pm Bx \pm (A \mp B)\operatorname{tg}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right].$$

TI (248)

2.556

1. $$\int \frac{(1-a^2)\, dx}{1 - 2a\cos x + a^2} = 2\operatorname{arctg}\left(\frac{1+a}{1-a}\operatorname{tg}\frac{x}{2}\right) \qquad [0 < a < 1, \quad |x| < \pi].$$

FI II 93

2. $$\int \frac{(1 - a\cos x)\, dx}{1 - 2a\cos x + a^2} = \frac{x}{2} + \operatorname{arctg}\left(\frac{1+a}{1-a}\operatorname{tg}\frac{x}{2}\right) \qquad [0 < a < 1, \quad |x| < \pi].$$

FI II 93

2.557

1. $$\int \frac{dx}{(a\cos x + b\sin x)^n} = \frac{1}{\sqrt{(a^2+b^2)^n}} \int \frac{dx}{\sin^n\left(x + \operatorname{arctg}\frac{a}{b}\right)} \qquad (\text{see} 2.515).$$

2.515
MZ 173a

2.[6] $$\int \frac{\sin x\, dx}{a\sin x + b\cos x} = \frac{ax - b\ln\sin\left(x + \operatorname{arctg}\frac{b}{a}\right)}{a^2 + b^2}.$$

3. $$\int \frac{\cos x\,dx}{a\cos x + b\sin x} = \frac{ax + b\ln\sin\left(x + \operatorname{arctg}\dfrac{a}{b}\right)}{a^2+b^2}.$$

MZ 174a

4. $$\int \frac{dx}{a\cos x + b\sin x} = \frac{\ln\operatorname{tg}\left[\dfrac{1}{2}\left(x + \operatorname{arctg}\dfrac{a}{b}\right)\right]}{\sqrt{a^2+b^2}}.$$

5. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = -\frac{\operatorname{ctg}\left(x + \operatorname{arctg}\dfrac{a}{b}\right)}{a^2+b^2} = +\frac{1}{a^2+b^2}\cdot\frac{a\sin x - b\cos x}{a\cos x + b\sin x}.$$

MZ 174a

1. $$\int \frac{A + B\,\cos x + C\,\sin x}{(a + b\,\cos x + c\,\sin x)^n}\,dx =$$
$$= \frac{(Bc - Cb) + (Ac - Ca)\,\cos x - (Ab - Ba)\,\sin x}{(n-1)(a^2 - b^2 - c^2)(a + b\,\cos x + c\,\sin x)^{n-1}} + \frac{1}{(n-1)(a^2-b^2-c^2)}\times$$
$$\times\int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba)\,\cos x - (Ac - Ca)\,\sin x]}{(a + b\,\cos x + c\,\sin x)^{n-1}}\,dx$$
$$[n \neq 1, \quad a^2 \neq b^2 + c^2];$$
$$= \frac{Cb - Bc + Ca\,\cos x - Ba\,\sin x}{(n-1)a(a + b\,\cos x + c\,\sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right)(-c\,\cos x + b\,\sin x)\times$$
$$\times\frac{(n-1)!}{(2n-1)!!}\sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!a^k}\cdot\frac{1}{(a + b\,\cos x + c\,\sin x)^{n-k}}\,[n \neq 1, \quad a^2 = b^2 + c^2].$$

For $n = 1$:

2. $$\int \frac{A + B\,\cos x + C\,\sin x}{a + b\,\cos x + c\,\sin x}\,dx = \frac{Bc - Cb}{b^2 + c^2}\,\ln(a + b\,\cos x + c\,\sin x) + \frac{Bb + Cc}{b^2 + c^2}x +$$
$$+\left(A - \frac{Bb + Cc}{B^2 + c^2}a\right)\int \frac{dx}{a + b\,\cos x + c\,\sin x} \qquad (\text{see } \mathbf{2.558}\ 4.).$$

2.558
GU ((331))(18)

3. $$\int \frac{dx}{(a+b\,\cos x+c\,\sin x)^n}=\int \frac{d(x-\alpha)}{[a+r\,\cos(x-\alpha)]^n},$$

where $b=r\,\cos\alpha$, $c=r\,\sin\alpha$ (see 2.554 3.).

4. $$\int \frac{dx}{a+b\,\cos x+c\,\sin x}=$$
$$=\frac{2}{\sqrt{a^2-b^2-c^2}}\,\operatorname{arctg}\frac{(a-b)\,\operatorname{tg}\dfrac{x}{2}+c}{\sqrt{a^2-b^2-c^2}}\qquad [a^2>b^2+c^2];$$
$$=\frac{1}{\sqrt{b^2+c^2-a^2}}\,\ln\frac{(a-b)\,\operatorname{tg}\dfrac{x}{2}+c-\sqrt{b^2+c^2-a^2}}{(a-b)\,\operatorname{tg}\dfrac{x}{2}+c+\sqrt{b^2+c^2-a^2}}\qquad [a^2<b^2+c^2];$$
$$=\frac{1}{c}\,\ln\left(a+c\cdot\operatorname{tg}\frac{x}{2}\right)\qquad [a=b];$$
$$=\frac{-2}{c+(a-b)\,\operatorname{tg}\dfrac{x}{2}}\qquad [a^2=b^2+c^2].$$

TI (253)a
TI (253)a
TI (253), FI II 94

2.559

1. $$\int \frac{dx}{[a(1+\cos x)+c\,\sin x]^2}=\frac{1}{c^3}\left[\frac{c(a\,\sin x-c\,\cos x)}{a(1+\cos x)+c\,\sin x}-a\,\ln\left(a+c\,\operatorname{tg}\frac{x}{2}\right)\right].$$

2. $$\int \frac{A+B\,\cos x+C\,\sin x}{(a_1+b_1\,\cos x+c_1\,\sin x)(a_2+b_2\,\cos x+c_2\,\sin x)}\,dx=$$
$$=A_0\,\ln\frac{a_1+b_1\,\cos x+c_1\,\sin x}{a_2+b_2\,\cos x+c_2\,\sin x}+A_1\int\frac{dx}{a_1+b_1\,\cos x+c_1\,\sin x}+A_2\int\frac{dx}{a_2+b_2\,\cos x+c_2\,\sin x},$$

where

$$A_0=\frac{\begin{vmatrix} A & B & C\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2-\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2},\qquad A_1=\frac{\begin{vmatrix} \begin{vmatrix} B & C\\ b_1 & c_1\end{vmatrix} & \begin{vmatrix} A & C\\ a_1 & c_1\end{vmatrix} & \begin{vmatrix} B & A\\ b_1 & a_1\end{vmatrix}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2-\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2};$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \right] \qquad \text{(see } \mathbf{2.558}\ 4.)\text{.}$$

2.558

GU ((331))(19)

3. $$\int \frac{A\,\cos^2\,x + 2B\,\sin x\,\cos x + C\,\sin^2\,x}{a\,\cos^2\,x + 2b\,\sin x\,\cos x + c\,\sin^2\,x}\,dx =$$
$$= \frac{1}{4b^2 + (a-c)^2}\{[4Bb + (A-C)(a-c)]x + [(A-C)b - B(a-c)] \times$$
$$\times \ln(a\,\cos^2\,x + 2b\,\sin x\,\cos x + c\,\sin^2\,x) +$$
$$+ [2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)]f(x)\},$$

where

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}}\,\ln\frac{c\,\operatorname{tg} x + b - \sqrt{b^2 - ac}}{c\,\operatorname{tg} x + b + \sqrt{b^2 - ac}} \qquad [b^2 > ac];$$
$$= \frac{1}{\sqrt{ac - b^2}}\,\operatorname{arctg}\frac{c\,\operatorname{tg} x + b}{\sqrt{ac - b^2}} \qquad [b^2 < ac];$$
$$= -\frac{1}{c\,\operatorname{tg} x + b} \qquad [b^2 = ac].$$

GU ((331))(24)

184

2.561

1. $$\int \frac{(A + B\,\sin x)\,dx}{\sin x(a + b\,\sin x)} = \frac{A}{a}\,\ln\operatorname{tg}\frac{x}{2} + \frac{Ba - Ab}{a}\int\frac{dx}{a + b\,\sin x} \qquad \text{(see 2.5513.).}$$

2.551

TI (348)

2. $$\int \frac{(A + B\,\sin x)\,dx}{\sin x(a + b\,\cos x)} = \frac{A}{a^2 - b^2}\left\{a\,\ln\operatorname{tg}\frac{x}{2} + b\,\ln\frac{a + b\,\cos x}{\sin x}\right\} +$$
$$+ B\int\frac{dx}{a + b\,\cos x} \qquad \text{(see 2.5533.).}$$

For $a^2 = b^2 (= 1)$:

$$3. \quad \int \frac{(A + B \sin x)\, dx}{\sin x (a + b \cos x)} = \frac{A}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} + \frac{1}{1 + \cos x} \right\} + B \operatorname{tg} \frac{x}{2}.$$

$$4. \quad \int \frac{(A + B \sin x)\, dx}{\sin x (1 - \cos x)} = \frac{A}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} - \frac{1}{1 - \cos x} \right\} - B \operatorname{ctg} \frac{x}{2}.$$

$$5. \quad \int \frac{(A + B \sin x)\, dx}{\cos x (a + b \sin x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - (Ab - aB) \ln \frac{a + b \sin x}{\cos x} \right\}.$$

TI (346)

For $a^2 = b^2 (= 1)$:

$$6. \quad \int \frac{(A + B \sin x)\, dx}{\cos x (1 \pm \sin x)} = \frac{A \pm B}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}.$$

$$7. \quad \int \frac{(A + B \sin x)\, dx}{\cos x (a + b \cos x)} = \frac{A}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{B}{a} \ln \frac{a + b \cos x}{\cos x} - \\ - \frac{Ab}{a} \int \frac{dx}{a + b \cos x} \qquad \text{(see } \mathbf{2.553}\ 3.).$$

2.553
TI (351)a

$$8. \quad \int \frac{(A + B \cos x)\, dx}{\sin x (a + b \sin x)} = \frac{A}{a} \ln \operatorname{tg} \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} - \frac{Ab}{a} \int \frac{dx}{a + b \sin x} \qquad \text{(see } 2.5513.).$$

2.551
TI (352)

$$9. \quad \int \frac{(A + B \cos x)\, dx}{\sin x (a + b \cos x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \operatorname{tg} \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right\}.$$

For $a^2 = b^2 (= 1)$:

10. $$\int \frac{(A + B\,\cos x)\,dx}{\sin x(1 \pm \cos x)} = \pm\frac{A \mp B}{2(1 \pm \cos x)} + \frac{A \pm B}{2}\,\ln\operatorname{tg}\frac{x}{2}.$$

11. $$\int \frac{(A + B\,\cos x)\,dx}{\cos x(a + b\,\sin x)} = \frac{A}{a^2 - b^2}\left\{a\,\ln\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right) - b\,\ln\frac{a + b\,\sin x}{\cos x}\right\} + B\int \frac{dx}{a + b\,\sin x}$$
(see **2.551** 3.).

2.551
TI (350)

For $a^2 = b^2 (= 1)$:

12. $$\int \frac{(A + B\,\sin x)\,dx}{\cos x(1 \pm \sin x)} = \frac{A \pm B}{2}\,\ln\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right) \mp \frac{A \mp B}{2(1 \pm \sin x)}.$$

13. $$\int \frac{(A + B\,\cos x)\,dx}{\cos x(a + b\,\cos x)} = \frac{A}{a}\,\ln\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right) + \frac{Ba - Ab}{a}\int \frac{dx}{a + b\,\cos x}$$
(see **2.553** 3.).

2.553
TI (347)

2.562

1. $$\int \frac{dx}{a + b\,\sin^2\,x} = \frac{\operatorname{sign} a}{\sqrt{a(a + b)}}\,\operatorname{arctg}\left(\sqrt{\frac{a + b}{a}}\,\operatorname{tg} x\right) \qquad \left[\frac{b}{a} > -1\right];$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a + b)}}\,\operatorname{Arth}\left(\sqrt{-\frac{a + b}{a}}\,\operatorname{tg} x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2\,x < -\frac{a}{b}\right];$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a + b)}}\,\operatorname{Arcth}\left(\sqrt{-\frac{a + b}{a}}\,\operatorname{tg} x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2\,x > -\frac{a}{b}\right].$$

2. $$\int \frac{dx}{a + b \cos^2 x} = \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg}\left(\sqrt{\frac{a+b}{a}} \operatorname{ctg} x\right) \qquad \left[\frac{b}{a} > -1\right];$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth}\left(\sqrt{-\frac{a+b}{a}} \operatorname{ctg} x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b}\right];$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth}\left(\sqrt{-\frac{a+b}{a}} \operatorname{ctg} x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b}\right].$$

MZ 162

3. $$\int \frac{dx}{1 + \sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x).$$

4. $$\int \frac{dx}{1 - \sin^2 x} = \operatorname{tg} x.$$

5. $$\int \frac{dx}{1 + \cos^2 x} = -\frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{ctg} x).$$

6. $$\int \frac{dx}{1 - \cos^2 x} = -\operatorname{ctg} x.$$

2.563

1. $$\int \frac{dx}{(a + b \sin^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b \sin^2 x} + \frac{b \sin x \cos x}{a + b \sin^2 x}\right] \qquad \text{(see 2.5621.)}.$$

2.562
MZ 155

186

2. $$\int \frac{dx}{(a + b \cos^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b \cos^2 x} - \frac{b \sin x \cos x}{a + b \cos^2 x}\right] \qquad \text{(see 2.5622.)}.$$

2.562
MZ 163

3. $$\int \frac{dx}{(a + b \sin^2 x)^3} = \frac{1}{8pa^3} \Bigg[\left(3 + \frac{2}{p^2} + \frac{3}{p^4}\right) \operatorname{arctg}(p \operatorname{tg} x) +$$
$$+ \left(3 + \frac{2}{p^2} - \frac{3}{p^4}\right) \frac{p \operatorname{tg} x}{1 + p^2 \operatorname{tg}^2 x} +$$
$$+ \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{tg}^2 x\right) \frac{2p \operatorname{tg} x}{\left(1 + p^2 \operatorname{tg}^2 x\right)^2} \Bigg] \qquad \left[p^2 = 1 + \frac{b}{a} > 0\right];$$
$$= \frac{1}{8qa^3} \Bigg[\left(3 - \frac{2}{q^2} + \frac{3}{q^4}\right) \operatorname{Arth}(q \operatorname{tg} x) +$$

4. $$\int \frac{dx}{(a+b\cos^2 x)^3} = -\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{ctg}x) + \right.$$
$$\left. +\left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{ctg}x}{1+p^2\operatorname{ctg}^2 x} + \left(1-\frac{2}{p^2}-\frac{1}{p^2}\operatorname{ctg}^2 x\right)\frac{2p\operatorname{ctg}x}{\left(1+p^2\operatorname{ctg}^2 x\right)^2}\right]$$
$$\left[p^2 = 1+\frac{b}{a} > 0\right];$$
$$= -\frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q\operatorname{ctg}x) + \right.$$
$$\left. +\left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\operatorname{ctg}x}{1-q^2\operatorname{ctg}^2 x} + \left(1+\frac{2}{q^2}+\frac{1}{q^2}\operatorname{ctg}^2 x\right)\frac{2p\operatorname{ctg}x}{\left(1-q^2\operatorname{ctg}^2 x\right)^2}\right]$$
$$\left[q^2 = -1-\frac{b}{a} < 0, \quad \cos^2 x < -\frac{a}{b}, \quad \text{for } cos^2 x > -\frac{a}{b}, \quad \text{one should replace} \right.$$
$$\left. \operatorname{Arth}(q\operatorname{ctg}x) \quad \text{with} \quad \operatorname{Arcth}(q\operatorname{ctg}x)\right]$$

MZ 163a

2.564

1. $$\int \frac{\operatorname{tg}x\,dx}{1+m^2\operatorname{tg}^2 x} = \frac{\ln(\cos^2 x + m^2\sin^2 x)}{2(m^2-1)}.$$

LA 210 (10)

2. $$\int \frac{\operatorname{tg}\alpha - \operatorname{tg}x}{\operatorname{tg}\alpha + \operatorname{tg}x}\,dx = \sin 2\alpha \ln\sin(x+\alpha) - x\cos 2\alpha.$$

LA 210 (11)a

3. $$\int \frac{\operatorname{tg}x\,dx}{a+b\operatorname{tg}x} = \frac{1}{a^2+b^2}\{bx - a\ln(a\cos x + b\sin x)\}.$$

PE (335)

4. $$\int \frac{dx}{a+b\operatorname{tg}^2 x} = \frac{1}{a-b}\left[x - \sqrt{\frac{b}{a}}\operatorname{arctg}\left(\sqrt{\frac{b}{a}}\operatorname{tg}x\right)\right].$$

PE (334)

2.57 Forms containing $\sqrt{a \pm b\sin x}$ or $\sqrt{a \pm b\cos x}$ and forms reducible to such expressions

Notations: $\alpha = \arcsin\sqrt{\frac{1-\sin x}{2}}$, $\beta = \arcsin\sqrt{\frac{b(1-\sin x)}{a+b}}$,

$\gamma = \arcsin\sqrt{\frac{b(1-\cos x)}{a+b}}$, $\delta = \arcsin\sqrt{\frac{(a+b)(1-\cos x)}{2(a-b\,\cos x)}}$, $r = \sqrt{\frac{2b}{a+b}}$.

2.571

$$1.\quad \int \frac{dx}{\sqrt{a+b\,\sin x}} = \frac{-2}{\sqrt{a+b}} F(\alpha,\, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right];$$
$$= -\sqrt{\frac{2}{b}} F\left(\beta,\, \frac{1}{r}\right) \qquad \left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.00, 288.50)

$$2.\quad \int \frac{\sin x\, dx}{\sqrt{a+b\,\sin x}} = \frac{2a}{b\sqrt{a+b}} F(\alpha,\, r) - \frac{2\sqrt{a+b}}{b} E(\alpha,\, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right];$$
$$= \sqrt{\frac{2}{b}} \left\{F\left(\beta,\, \frac{1}{r}\right) - 2E\left(\beta,\, \frac{1}{r}\right)\right\} \qquad \left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.54)
BY (288.03)

$$3.\quad \int \frac{\sin^2\, x\, dx}{\sqrt{a+b\,\sin x}} = \frac{4a\sqrt{a+b}}{3b^2} E(\alpha,\, r) - \frac{2(2a^2+b^2)}{3b^2\sqrt{a+b}} F(\alpha,\, r) -$$
$$-\frac{2}{3b}\, \cos x\sqrt{a+b\,\sin x} \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right];$$
$$= \sqrt{\frac{2}{b}} \left\{\frac{4a}{3b} E\left(\beta,\, \frac{1}{r}\right) - \frac{2a+b}{3b} F\left(\beta,\, \frac{1}{r}\right)\right\} - \frac{2}{3b}\, \cos x\, \sqrt{a+b\,\sin x}$$
$$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.03, 288.54)

$$4.\quad \int \frac{dx}{\sqrt{a+b\,\cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2},\, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi];$$
$$= \sqrt{\frac{2}{b}} F\left(\gamma,\, \frac{1}{r}\right)$$
$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right].$$

BY (290.00)

BY (289.00)

5. $$\int \frac{dx}{\sqrt{a - b\cos x}} = \frac{2}{\sqrt{a+b}} F(\delta, r) \qquad [a > b > 0, \quad 0 \le x \le \pi].$$

BY (291.00)

6. $$\int \frac{\cos x\, dx}{\sqrt{a + b\cos x}} = \frac{2}{b\sqrt{a+b}} \left\{ (a+b) E\left(\frac{x}{2}, r\right) - aF\left(\frac{x}{2}, r\right) \right\} \qquad [a > b > 0, \quad 0 \le x \le \pi];$$
$$= \sqrt{\frac{2}{b}} \left\{ 2E\left(\gamma, \frac{1}{r}\right) - F\left(\gamma, \frac{1}{r}\right) \right\} \qquad \left[b > |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right) \right].$$

BY (290.04)
BY (289.03)

7.[6] $$\int \frac{\cos x\, dx}{\sqrt{a - b\cos x}} = \frac{2}{b\sqrt{a+b}} \{ (b-a)\Pi(\delta, r^2, r) + aF(\delta, r) \} \qquad [a > b > 0, \quad 0 \le x \le \pi].$$

BY (291.03)

8. $$\int \frac{\cos^2 x\, dx}{\sqrt{a + b\cos x}} = \frac{2}{3b^2\sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, r\right) - 2a(a+b) E\left(\frac{x}{2}, r\right) \right\} + \frac{2}{3b} \sin x \sqrt{a + b\cos x}$$
$$[a > b > 0, \quad 0 \le x \le \pi];$$
$$= \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\gamma, \frac{1}{r}\right) - 4aE\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2}{3b} \sin x \sqrt{a + b\cos x}$$
$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right) \right].$$

BY (290.04)
BY (289.03)

9. $$\int \frac{\cos^2 x\, dx}{\sqrt{a - b\cos x}} = \frac{2}{3b^2\sqrt{a+b}} \{ (2a^2 + b^2) F(\delta, r) - 2a(a+b) E(\delta, r) \} +$$
$$+ \frac{2}{3b} \sin x \frac{a + b\cos x}{\sqrt{a - b\cos x}} [a > b > 0, \quad 0 \le x < \pi].$$

BY (291.04)a

2.572

3. $$\int \frac{\operatorname{tg}^2 x\, dx}{\sqrt{a + b\sin x}} = \frac{1}{\sqrt{a+b}} F(\alpha, r) + \frac{a}{(a-b)\sqrt{a+b}} E(\alpha, r) -$$
$$- \frac{b - a\sin x}{(a^2 - b^2)\cos x} \sqrt{a + b\sin x} \qquad \left[0 < b < a, -\frac{\pi}{2} < x < \frac{\pi}{2} \right];$$
$$= \sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a+b)} F\left(\beta, \frac{1}{r}\right) + \frac{ab}{a^2 - b^2} E\left(\beta, \frac{1}{r}\right) \right\} -$$
$b - a\sin x$ …

2.573

1. $$\int \frac{1-\sin x}{1+\sin x} \cdot \frac{dx}{\sqrt{a+b\,\sin x}} = \frac{2}{a-b}\left\{\sqrt{a+b}E(\alpha,\,r) - \right.$$
$$\left. -\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\sqrt{a+b\,\sin x}\right\} \qquad \left[0<b<a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right].$$

BY (288.07)

2. $$\int \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b\,\cos x}} = \frac{2}{a-b}\ \operatorname{tg}\frac{x}{2}\sqrt{a+b\,\cos x} -$$
$$-\frac{2\sqrt{a+b}}{a-b}E\left(\frac{x}{2},\,r\right) \qquad [a>b>0, \quad 0 \le x < \pi].$$

BY (289.07)

2.574

1. $$\int \frac{dx}{(2-p^2+p^2\,\sin x)\sqrt{a+b\,\sin x}} = -\frac{1}{a+b}\Pi(\alpha,\,p^2,\,r) \qquad \left[0<b<a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right].$$

BY (288.02)

2. $$\int \frac{dx}{(a+b-p^2b+p^2b\,\sin x)\sqrt{a+b\,\sin x}} = -\frac{1}{a+b}\sqrt{\frac{2}{b}}\Pi\left(\beta,\,p^2,\,\frac{1}{r}\right)$$
$$\left[0<|a|<b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.52)

3. $$\int \frac{dx}{(2-p^2+p^2\,\cos x)\sqrt{a+b\,\cos x}} = \frac{1}{\sqrt{a+b}}\Pi\left(\frac{x}{2},\,p^2,\,r\right) \qquad [a>b>0, \quad 0 \le x < \pi].$$

BY (289.02)

4. $$\int \frac{dx}{(a+b-p^2b+p^2b\,\cos x)\sqrt{a+b\,\cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}}\Pi\left(\gamma,\,p^2,\,\frac{1}{r}\right)$$
$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right].$$

BY (290.02)

1. $$\int \frac{dx}{\sqrt{(a+b\,\sin x)^3}} = \frac{2b\,\cos x}{(a^2-b^2)\sqrt{a+b\,\sin x}} - \frac{2}{(a-b)\sqrt{a+b}}E(\alpha,\,r)$$
$$\left[0<b<a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right];$$
$$= \sqrt{\frac{2}{b}}\left\{\frac{2b}{b^2-a^2}E\left(\beta,\,\frac{1}{r}\right) - \frac{1}{a+b}F\left(\beta,\,\frac{1}{r}\right)\right\} + \frac{2b}{b^2-a^2}\cdot\frac{\cos x}{\sqrt{a+b\,\sin x}}$$
$$\left[0<|a|<b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.56)
BY (288.05)

2. $$\int \frac{dx}{\sqrt{(a+b\,\sin x)^5}} = \frac{2}{3(a^2-b^2)^2\sqrt{a+b}}\{(a^2-b^2)F(\alpha,\,r) - 4a(a+b)E(\alpha,\,r)\} +$$
$$+ \frac{2b(5a^2-b^2+4ab\,\sin x)}{3(a^2-b^2)^2\sqrt{(a+b\,\sin x)^3}}\cos x$$
$$\left[0<b<a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right];$$
$$= -\frac{1}{3(a^2-b^2)^2}\sqrt{\frac{2}{b}}\left\{(3a-b)(a-b)F\left(\beta,\,\frac{1}{r}\right) + \right.$$
$$\left. + 8abE\left(\beta,\,\frac{1}{r}\right)\right\} + \frac{2b[a^2-b^2+4a(a+b\,\sin x)]}{3(a^2-b^2)^2\sqrt{(a+b\,\sin x)^3}}\cos x$$
$$\left[0<|a|<b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

BY (288.56)
BY (288.05)

3. $$\int \frac{dx}{\sqrt{(a+b\,\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}E\left(\frac{x}{2},\,r\right) - \frac{2b}{a^2-b^2}\cdot\frac{\sin x}{\sqrt{a+b\,\cos x}}$$
$$[a>b>0, \quad 0 \le x \le \pi];$$
$$= \frac{1}{a^2-b^2}\sqrt{\frac{2}{b}}\left\{(a-b)F\left(\gamma,\,\frac{1}{r}\right) + 2bE\left(\gamma,\,\frac{1}{r}\right)\right\} + \frac{2b}{b^2-a^2}\cdot\frac{\sin x}{\sqrt{a+b\,\cos x}}$$
$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right].$$

BY (290.06)
BY (289.05)

4. $$\int \frac{dx}{\sqrt{(a-b\,\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}E(\delta,\,r) \qquad [a>b>0, \quad 0 \le x \le \pi].$$

5. $$\int \frac{dx}{\sqrt{(a+b\cos x)^5}} = \frac{2\sqrt{a+b}}{3(a^2-b^2)^2}\left\{4aE\left(\frac{x}{2},\, r\right) - (a-b)\,F\left(\frac{x}{2},\, r\right)\right\} - \\ -\frac{2b}{3(a^2-b^2)^2}\cdot\frac{5a^2-b^2+4ab\cos x}{\sqrt{(a+b\cos x)^3}}\sin x \qquad [a > b > 0, \quad 0 \le x \le \pi];$$

$$= \frac{1}{3(a^2-b^2)^2}\sqrt{\frac{2}{b}}\left\{(a-b)(3a-b)F\left(\gamma,\, \frac{1}{r}\right) + \right. \\ \left. + 8abE\left(\gamma,\, \frac{1}{r}\right)\right\} + \frac{2b(5a^2-b^2+4ab\cos x)\sin x}{3(a^b-b^2)^2\sqrt{(a+b\cos x)^3}} \\ \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right].$$

BY (290.06)
BY (289.05)

2.576

1. $$\int \sqrt{a+b\cos x}\,dx = 2\sqrt{a+b}E\left(\frac{x}{2},\, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi];$$

$$= \sqrt{\frac{2}{b}}\left\{(a-b)F\left(\gamma,\, \frac{1}{r}\right) + 2bE\left(\gamma,\, \frac{1}{r}\right)\right\} \\ \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right].$$

BY (290.03)
BY (289.01)

2. $$\int \sqrt{a-b\cos x}\,dx = 2\sqrt{a+b}E(\delta,\, r) - \frac{2b\sin x}{\sqrt{a-b\cos x}} \qquad [a > b > 0, \quad 0 \le x \le \pi].$$

BY (291.05)

2.577

1.[3] $$\int \frac{\sqrt{a-b\cos x}}{1+p\cos x}\,dx = \frac{2(a-b)}{(1+p)\sqrt{a+b}}\Pi\left(\delta,\, \frac{2ap}{(a+b)(1+p)},\, r\right) \\ [a > b > 0, \quad 0 \le x \le \pi, \quad p \ne -1].$$

BY (291.02)

2.[3] $$\int \sqrt{\frac{a-b\cos x}{1+p\cos x}}\,dx = \frac{2(a-b)}{\sqrt{(1+p)(a+b)}}\Pi\left(\delta,\, -r^2,\, \sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right) \\ [a > b > 0, \quad 0 \le x \le \pi, \quad p \ne -1].$$

2.578

$$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a+b\ \operatorname{tg}^2\ x}} = \frac{1}{\sqrt{b-a}}\ \arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}}\ \cos x\right) \qquad [b>a, \quad b>0].$$

PE (333)

2.58-2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

1. $$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi}} = 2\int \frac{d\psi}{\sqrt{a-p+2p\cos^2\psi}} \quad \left[\varphi=2\psi+\alpha,\ \ \operatorname{tg}\alpha=\frac{c}{b},\ \ p=\sqrt{b^2+c^2}\right]$$

2. $$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi+d\cos^2\varphi+e\sin\varphi\cos\varphi+f\sin^2\varphi}} = 2\int \frac{dx}{\sqrt{A+Bx+Cx^2-Dx^3+Ex^4}}$$
$$\left[\operatorname{tg}\frac{\varphi}{2}=x,\ \ A=a+b+d,\ \ B=2c+2e,\ \ C=2a-2d+4f,\ \ D=2c-2e,\ \ E=a-b+d\right]$$

Forms containing $\sqrt{1-k^2\sin^2 x}$; Notations: $\Delta=\sqrt{1-k^2}$, $\sin^2 x$,

$k'=\sqrt{1-k^2}$

192
2.581

1. $$\int \sin^m\ x\ \cos^n\ x\Delta^r\,dx =$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m-3}\ x\ \cos^{n+1}\ x\Delta^{r+2}+[(m+n-2)+(m+r-1)k^2]\times\right.$$
$$\left.\times\int\ \sin^{m-2}\ x\ \cos^n\ x\Delta^r\,dx-(m-3)\int\ \sin^{m-4}\ x\ \cos^n\ x\Delta^r\,dx\right\} =$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m+1}\ x\ \cos^{n-3}\ x\Delta^{r+2}+[(n+r-1)k^2-(m+n-2)k'^2]\times\right.$$
$$\left.\times\int\ \sin^m\ x\ \cos^{n-2}\ x\Delta^r\,dx+(n-3)k'^2\int\ \sin^m\ x\ \cos^{n-4}\ x\Delta^r\,dx\right\}[m+n+r\neq 0]$$

For $r=-3$ and $r=-5$:

2. $$\int \frac{\sin^m\ x\ \cos^n\ x}{\Delta^3}\,dx = \frac{\sin^{m-1}\ x\ \cos^{n-1}\ x}{k^2\Delta} -$$
$$-\frac{m-1}{k^2}\int \frac{\sin^{m-2}\ x\ \cos^n\ x}{\Delta}\,dx+\frac{n-1}{k^2}\int \frac{\sin^m\ x\ \cos^{n-2}\ x}{\Delta}\,dx.$$

3. $$\int \frac{\sin^m x \cos^n x}{\Delta^5}\,dx = \frac{\sin^{m-1} x \cos^{n-1} x}{3k^2\Delta^3} - \\ -\frac{m-1}{3k^2}\int \frac{\sin^{m-2} x \cos^n x}{\Delta^3}\,dx + \frac{n-1}{3k^2}\int \frac{\sin^m x \cos^{n-2} x}{\Delta^3}\,dx.$$

For $m = 1$ or $n = 1$:

4. $$\int \sin x \cos^n x\Delta^r\,dx = -\frac{\cos^{n-1} x\Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1)k'^2}{(n+r+1)k^2}\int \cos^{n-2} x \sin x\Delta^r\,dx.$$

5. $$\int \sin^m x \cos x\Delta^r\,dx = -\frac{\sin^{m-1} x\Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2}\int \sin^{m-2} x \cos x\Delta^r\,dx.$$

For $m = 3$ or $n = 3$:

6. $$\int \sin^3 x \cos^n x\Delta^r\,dx = \frac{(n+r+1)k^2 \cos^2 x - [(r+2)k^2+n+1]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x\Delta^{r+2} - \\ -\frac{[(r+2)k^2+n+1](n-1)k'^2}{(n+r+1)(n+r+3)k^4}\int \cos^{n-2} x \sin x\Delta^r\,dx.$$

7. $$\int \sin^m x \cos^3 x\Delta^r\,dx = \frac{(m+r+1)k^2 \sin^2 x - [(r+2)k^2-(m+1)k'^2]}{(m+r+1)(m+r+3)k^4}\times \\ \times\sin^{m-1} x\Delta^{r+2} + \frac{[(r+2)k^2-(m-1)k'^2](m-1)}{(m+r+1)(m+r+3)k^4}\int \sin^{m-2} x \cos x\Delta^r\,dx.$$

2.582

1. $$\int \Delta^n\,dx = \frac{n-1}{n}(2-k^2)\int \Delta^{n-2}\,dx - \frac{n-2}{n}(1-k^2)\int \Delta^{n-4}\,dx + \\ + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2}.$$

LA (316)(1)a

2. $$\int \frac{dx}{\Delta^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1)k'^2\Delta^{n-1}} + \frac{n-2}{n-1}\frac{2-k^2}{k'^2}\int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1}\frac{1}{k'^2}\int \frac{dx}{\Delta^{n-3}}$$

3. $$\int \frac{\sin^n x}{\Delta}\,dx = \frac{\sin^{n-3} x}{(n-1)k^2}\cos x\cdot\Delta + \frac{n-2}{n-1}\frac{1+k^2}{k^2}\int\frac{\sin^{n-2} x}{\Delta}\,dx - \frac{n-3}{(n-1)k^2}\int\frac{\sin^{n-4} x}{\Delta}\,dx.$$

LA 316(1)a

4. $$\int \frac{\cos^n x}{\Delta}\,dx = \frac{\cos^{n-3} x}{(n-1)k^2}\sin x\cdot\Delta + \frac{n-2}{n-1}\frac{2k^2-1}{k^2}\int\frac{\cos^{n-2} x}{\Delta}\,dx + \frac{n-3}{n-1}\frac{k'^2}{k^2}\int\frac{\cos^{n-4} x}{\Delta}\,dx.$$

LA 316(2)a

5. $$\int \frac{\operatorname{tg}^n x}{\Delta}\,dx = \frac{\operatorname{tg}^{n-3} x}{(n-1)k'^2}\frac{\Delta}{\cos^2 x} - \frac{(n-2)(2-k^2)}{(n-1)k'^2}\int\frac{\operatorname{tg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{(n-1)k'^2}\int\frac{\operatorname{tg}^{n-4} x}{\Delta}\,dx.$$

LA 317(3)

6. $$\int \frac{\operatorname{ctg}^n x}{\Delta}\,dx = -\frac{\operatorname{ctg}^{n-1} x}{n-1}\frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1}(2-k^2)\int\frac{\operatorname{ctg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{n-1}k'^2\int\frac{\operatorname{ctg}^{n-4} x}{\Delta}\,dx.$$

LA 317(6)

2.583

1. $$\int \Delta\,dx = E(x,\,k).$$

2. $$\int \Delta\sin x\,dx = -\frac{\Delta\cos x}{2} - \frac{k'^2}{2k}\ln(k\cos x + \Delta).$$

3. $$\int \Delta\cos x\,dx = \frac{\Delta\sin x}{2} + \frac{1}{2k}\arcsin(k\sin x).$$

6. $$\int \Delta \cos^2 x\, dx = \frac{\Delta}{3} \sin x \cos x - \frac{k'^2}{3k^2} F(x,\ k) + \frac{k^2+1}{3k^2} E(x,\ k).$$

7. $$\int \Delta \sin^3 x\, dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k \cos x + \Delta).$$

8. $$\int \Delta \sin^2 x \cos x\, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x).$$

9. $$\int \Delta \sin x \cos^2 dx = -\frac{2k^2 \cos^2 x + k'^2}{8k^2} \Delta \cos x + \frac{k'^4}{8k^3} \ln(k \cos x + \Delta).$$

10. $$\int \Delta \cos^3 x\, dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x).$$

11. $$\int \Delta \sin^4 x\, dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x - \\ - \frac{2(2k^4 - k^2 - 1)}{15k^4} F(x,\ k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x,\ k).$$

12. $$\int \Delta \sin^3 x \cos x\, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$$

13. $$\int \Delta \sin^2 x \cos^2 x\, dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x - \\ - \frac{k'^2(1 + k'^2)}{15k^4} F(x,\ k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x,\ k).$$

14. $$\int \Delta \sin x \cos^3 x\, dx = -\frac{3k^4 \sin^4 x - k^2 (5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta.$$

15. $$\int \Delta \cos^4 x\,dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2}\Delta \sin x \cos x + \\ + \frac{2k'^2(k'^2 - 2k^2)}{15k^4}F(x,\ k) + \frac{3k^4 + 7k^2 - 2}{15k^4}E(x,\ k).$$

16. $$\int \Delta \sin^5 x\,dx = \frac{-8k^4 \sin^4 x - 2k^2(5k^2 - 1)\sin^2 x - 15k^4 + 4k^2 + 3}{48k^4}\Delta \cos x + \\ + \frac{5k^6 - 3k^4 - k^2 - 1}{16k^5}\ \ln(k \cos x + \Delta).$$

17. $$\int \Delta \sin^4 x \cos x\,dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4}\Delta \sin x + \frac{1}{16k^5}\ \arcsin(k \sin x).$$

18. $$\int \Delta \sin^3 x \cos^2 x\,dx = \frac{8k^4 \sin^4 x - 2k^2\,(k^2 + 1)\sin^2 x - 3k^4 + 2k^2 - 3}{48k^4}\Delta \cos x + \\ + \frac{k'^4(k^2 + 1)}{16k^5}\ \ln(k \cos x + \Delta).$$

19. $$\int \Delta \sin^2 x \cos^3 x\,dx = \frac{-8k^4 \sin^4 x + 2k^2\,(6k^2 + 1)\sin^2 x - 6k^2 + 3}{48k^4}\Delta \sin x + \\ + \frac{2k^2 - 1}{16k^5}\ \arcsin(k \sin x).$$

20. $$\int \Delta \sin x \cos^4 x\,dx = \frac{-8k^4 \sin^4 x + 2k^2\,(7k^2 + 1)\sin^2 x - 3k^4 - 8k^2 + 3}{48k^4}\Delta \cos x - \\ - \frac{k'^6}{16k^5}\ \ln(k \cos x + \Delta).$$

21. $$\int \Delta \cos^5 x\,dx = \frac{8k^4 \sin^4 x - 2k^2\,(12k^2 + 1)\sin^2 x + 24k^4 + 12k^2 - 3}{48k^4}\Delta \sin x + \\ + \frac{8k^4 - 4k^2 + 1}{16k^5}\ \arcsin(k \sin x).$$

22. $$\int \Delta^3\,dx = \frac{2}{3}(1 + k'^2)E(x,\ k) - \frac{k'^2}{3}F(x,\ F) + \frac{k^2}{3}\Delta \sin x \cos x.$$

23. $$\int \Delta^3 \sin x\,dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8}\Delta \cos x - \frac{3k'^4}{8k} \ln(k \cos x + \Delta).$$

24. $$\int \Delta^3 \cos x\,dx = \frac{-2k^2 \sin^2 x + 5}{8}\Delta \sin x + \frac{3}{8k} \arcsin(k \sin x).$$

25. $$\int \Delta^3 \sin^2 x\,dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15}\Delta \sin x \cos x + \frac{k'^2(3-4k^2)}{15k^2} F(x, k) - \\ - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k).$$

26. $$\int \Delta^3 \sin x \cos dx = -\frac{\Delta^5}{5k^2}.$$

27. $$\int \Delta^3 \cos^2 x\,dx = \frac{-3k^2 \sin^2 x + k^2 + 5}{15}\Delta \sin x \cos x - \frac{k'^2(k^2+3)}{15k^2} F(x, k) - \\ - \frac{2k^4 - 7k^2 - 3}{15k^2} E(x, k).$$

28. $$\int \Delta^3 \sin^3 x\,dx = \frac{8k^4 \sin^4 x + 2k^2(5k^2-7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2}\Delta \cos x - \\ - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln(k \cos x + \Delta).$$

29. $$\int \Delta^3 \sin^2 x \cos x\,dx = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2}\Delta \sin x + \\ + \frac{1}{16k^3} \arcsin(k \sin x).$$

30. $$\int \Delta^3 \sin x \cos^2 x\,dx = \frac{-8k^4 \sin^4 x + 2k^2(k^2+7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \times \\ \times \Delta \cos x + \frac{k'^6}{16k^3} \ln(k \cos x + \Delta).$$

31. $$\int \Delta^3 \cos^3 x\,dx = \frac{8k^4 \sin^4 x - 2k^2(6k^2+7)\sin^2 x + 30k^2 + 3}{48k^2}\Delta \sin x + + \frac{6k^2-1}{16k^3} \arcsin(k \sin x).$$

32. $$\int \frac{\Delta\,dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln k(k \cos x + \Delta).$$

33. $$\int \frac{\Delta\,dx}{\cos x} = \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} + k \arcsin(k \sin x).$$

34. $$\int \frac{\Delta\,dx}{\sin^2 x} = k'^2 F(x,\, k) - E(x,\, k) - \Delta \operatorname{ctg} x.$$

35. $$\int \frac{\Delta\,dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}.$$

36. $$\int \frac{\Delta\,dx}{\cos^2 x} = F(x,\, k) - E(x,\, k) + \Delta \operatorname{tg} x.$$

37. $$\int \frac{\sin x}{\cos x}\Delta\,dx = \int \Delta \operatorname{tg} x\,dx = -\Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}.$$

38. $$\int \frac{\cos x}{\sin x}\Delta\,dx = \int \Delta \operatorname{ctg} x\,dx = \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$$

39. $$\int \frac{\Delta\,dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

40. $$\int \frac{\Delta\,dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1+k^2}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

41. $$\int \frac{\Delta\,dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

42. $$\int \frac{\Delta\, dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

43. $$\int \frac{\Delta \sin x\, dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k \ln(k \cos x + \Delta).$$

44. $$\int \frac{\Delta \cos x\, dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k \arcsin(k \sin x).$$

45. $$\int \frac{\Delta \sin^2 x\, dx}{\cos x} = -\frac{\Delta \sin x}{2} + \frac{2k^2 - 1}{2k} \arcsin(k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

46. $$\int \frac{\Delta \cos^2 x\, dx}{\sin x} = \frac{\Delta \cos x}{2} + \frac{k^2 + 1}{2k} \ln(k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

47. $$\int \frac{\Delta\, dx}{\sin^4 x} = \frac{1}{3}\{-\Delta \operatorname{ctg}^3 x + (k^2 - 3)\Delta \operatorname{ctg} x + 2k'^2 F(x, k) + (k^2 - 2)E(x, k)\}.$$

48. $$\int \frac{\Delta\, dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$$

49. $$\int \frac{\Delta\, dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{k'^2} \operatorname{tg} x - \operatorname{ctg} x\right) \Delta + 2F(x, k) - \frac{1 + k'^2}{k'^2} E(x, k).$$

50. $$\int \frac{\Delta\, dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

51. $$\int \frac{\Delta\, dx}{\cos^4 x} = \frac{1}{3k'^2}\{[k'^2 \operatorname{tg}^2 x - (2k^2 - 3) \operatorname{tg} x]\Delta + 2k'^2 F(x, k) + (k^2 - 2)E(x, k)\}.$$

52. $$\int \frac{\sin dx}{\cos^3 x} \Delta\, dx = \frac{\Delta}{2\,\cos^2\, x} + \frac{k^2}{4k'}\, \ln \frac{\Delta + k'}{\Delta - k'}.$$

53. $$\int \frac{\cos x}{\sin^3\, x} \Delta\, dx = -\frac{\Delta}{2\,\sin^2\, x} + \frac{k^2}{4}\, \ln \frac{1 + \Delta}{1 - \Delta}.$$

54. $$\int \frac{\sin^2\, x}{\cos^2\, x} \Delta\, dx = \int\, \operatorname{tg}^2\, x \Delta\, dx = \Delta\, \operatorname{tg} x + F(x,\, k) - 2E(x,\, k).$$

55. $$\int \frac{\cos^2\, x}{\sin^2\, x} \Delta\, dx = \int\, \operatorname{ctg}^2\, x \Delta\, dx = -\Delta\, \operatorname{ctg} x + k'^2 F(x,\, k) - 2E(x,\, k).$$

56. $$\int \frac{\sin^3\, x}{\cos x} \Delta\, dx = -\frac{k^2\,\sin^2\, x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2}\, \ln \frac{\Delta + k'}{\Delta - k'}.$$

57. $$\int \frac{\cos^3\, x}{\sin x} \Delta\, dx = -\frac{k^2\,\sin^2\, x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2}\, \ln \frac{1 - \Delta}{1 + \Delta}.$$

58. $$\int \frac{\Delta\, dx}{\sin^5\, x} = \frac{(k^2 - 3)\,\sin^2\, x + 2}{8\,\sin^4\, x}\, \cos x\, \Delta + \frac{k'^2(k^2 + 3)}{16}\, \ln \frac{\Delta + \cos x}{\Delta - \cos\, x}.$$

59. $$\int \frac{\Delta\, dx}{\sin^4\, x\, \cos x} = -\frac{(3 - k^2)\,\sin^2\, x + 1}{3\,\sin^3\, x} \Delta - \frac{k'}{2}\, \ln \frac{\Delta - k'\,\sin x}{\Delta + k'\,\sin x}.$$

60. $$\int \frac{\Delta\, dx}{\sin^3\, x\, \cos^2\, x} = \frac{3\,\sin^2\, x - 1}{2\,\sin^2\, x\, \cos x} \Delta + \frac{k^2 - 3}{4}\, \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$$

61. $$\int \frac{\Delta\, dx}{\sin^2\, x\, \cos^3\, x} = \frac{3\,\sin^2\, x - 2}{2\,\sin x\, \cos^2\, x} \Delta - \frac{2k^2 - 3}{4k'}\, \ln \frac{\Delta + k'\,\sin x}{\Delta - k'\,\sin x}.$$

64. $$\int \frac{\sin x}{\cos^4 x} \Delta\, dx = \frac{-(2k^2+1)k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta.$$

65. $$\int \frac{\cos x}{\sin^4 x} \Delta\, dx = -\frac{\Delta^3}{3 \sin^3 x}.$$

66. $$\int \frac{\sin^2}{\cos^3 x} \Delta\, dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2-1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin(k \sin x).$$

67. $$\int \frac{\cos^2 x}{\sin^3 x} \Delta\, dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2+1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln(k \cos x + \Delta).$$

68. $$\int \frac{\sin^3 x}{\cos^2 x} \Delta\, dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2-1}{2k} \ln(k \cos x + \Delta).$$

69. $$\int \frac{\cos^3 x}{\sin^2 x} \Delta\, dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2+1}{2k} \arcsin(k \sin x).$$

70. $$\int \frac{\sin^4 x}{\cos x} \Delta\, dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \sin x \Delta + \\ + \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin(k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

71. $$\int \frac{\cos^4 x}{\sin x} \Delta\, dx = \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \cos x \Delta + \\ + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln(k \cos x + \Delta).$$

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8. $$\int \frac{\sin^2 x \cos x \, dx}{\Delta} = -\frac{\sin x \Delta}{2k^2} + \frac{\arcsin(k \sin x)}{2k^3}.$$

9. $$\int \frac{\sin x \cos^2 x \, dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{k'^2}{2k^3} \ln(k \cos x + \Delta).$$

10. $$\int \frac{\cos^3 x \, dx}{\Delta} = \frac{\sin x \Delta}{2k^2} + \frac{2k^2 - 1}{2k^3} \arcsin(k \sin x).$$

11. $$\int \frac{\sin^4 x \, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{2 + k^2}{3k^4} F(x, k) - \frac{2(1 + k^2)}{3k^4} E(x, k).$$

12. $$\int \frac{\sin^3 x \cos x \, dx}{\Delta} = -\frac{1}{3k^4}(2 + k^2 \sin^2 x)\Delta.$$

13. $$\int \frac{\sin^2 x \cos^2 x \, dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2 - k^2}{3k^4} E(x, k) + \frac{2k^2 - 2}{3k^4} F(x, k).$$

14. $$\int \frac{\sin x \cos^3 x \, dx}{\Delta} = -\frac{1}{3k^4}(k^2 \cos^2 x - 2k'^2)\Delta.$$

15. $$\int \frac{\cos^4 x \, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2 - 2}{3k^4} E(x, k) + \frac{3k^4 - 5k^2 + 2}{3k^4} F(x, k).$$

16. $$\int \frac{\sin^5\ x\,dx}{\Delta} = \frac{2k^2\ \sin^2\ x + 3k^2 + 3}{8k^4}\ \cos x\Delta - \frac{3 + 2k^2 + 3k^4}{8k^5}\ \ln(k\ \cos x + \Delta).$$

17. $$\int \frac{\sin^4\ x\ \cos x\,dx}{\Delta} = -\frac{2k^2\ \sin^2\ x + 3}{8k^4}\ \sin x\Delta + \frac{3}{8k^5}\ \arcsin(k\ \sin x).$$

18. $$\int \frac{\sin^3\ x\ \cos x\,dx}{\Delta} = \frac{2k^2\ \cos^2\ x - k^2 - 3}{8k^4}\ \cos x\Delta - \frac{k^4 + 2k^2 - 3}{8k^5}\ \ln(k\ \cos x + \Delta).$$

19. $$\int \frac{\sin^2\ x\ \cos^3\ x\,dx}{\Delta} = -\frac{2k^2\ \cos^2\ x + 2k^2 - 3}{8k^4}\ \sin x\Delta + \frac{4k^2 - 3}{8k^5}\ \arcsin(k\ \sin x).$$

20. $$\int \frac{\sin x\ \cos^4\ x\,dx}{\Delta} = \frac{3 - 5k^2 + 2k^2\ \sin^2\ x}{8k^4}\ \cos x\Delta - \frac{3k^4 - 6k^2 + 3}{8k^5}\ \ln(k\ \cos x + \Delta).$$

21. $$\int \frac{\cos^5\ x\,dx}{\Delta} = \frac{2k^2\ \cos^2\ x + 6k^2 - 3}{8k^4}\ \sin x\Delta + \frac{8k^4 - 8k^2 + 3}{8k^5}\ \arcsin(k\ \sin x).$$

22. $$\int \frac{\sin^6\ x\,dx}{\Delta} = \frac{3k^2\ \sin^2\ x + 4k^2 + 4}{15k^4}\ \sin x\ \cos x\Delta + \\ + \frac{4k^4 + 3k^2 + 8}{15k^6} F(x,\ k) - \frac{8k^4 + 7k^2 + 8}{15k^6} E(x,\ k).$$

23. $$\int \frac{\sin^5\ x\ \cos x\,dx}{\Delta} = -\frac{3k^4\ \sin^4\ x + 4k^2\ \sin^2\ x + 8}{15k^6}\Delta.$$

24. $$\int \frac{\sin^4\ x\ \cos x\,dx}{\Delta} = \frac{3k^2\ \cos^2\ x - 2k^2 - 4}{15k^4}\ \sin x\ \cos x\Delta + \\ + \frac{k^4 + 7k^2 - 8}{15k^6} F(x,\ k) - \frac{2k^4 + 3k^2 - 8}{15k^6} E(x,\ k).$$

25. $$\int \frac{\sin^3\ x\ \cos^3\ x\,dx}{\Delta} = \frac{3k^4\ \sin^4\ x - (5k^4 - 4k^2)\ \sin^2\ x - 10k^2 + 8}{15k^6}\Delta.$$

26. $$\int \frac{\sin^2 x \cos^4 x\,dx}{\Delta} = -\frac{3k^2\cos^2 x + 3k^2 - 4}{15k^4}\sin x\cos x\Delta + \\ + \frac{9k^4 - 17k^2 + 8}{15k^6}F(x,\,k) - \frac{3k^4 - 13k^2 + 8}{15k^6}E(x,\,k).$$

27. $$\int \frac{\sin x \cos^5 x\,dx}{\Delta} = \frac{-3k^4\cos^4 x + 4k^2k'^2\cos^2 x - 8k^4 + 16k^2 - 8}{15k^6}\Delta.$$

28. $$\int \frac{\cos^6 x\,dx}{\Delta} = \frac{3k^2\cos^2 x + 8k^2 - 4}{15k^4}\sin x\cos x\Delta + \\ + \frac{15k^6 - 34k^4 + 27k^2 - 8}{15k^6}F(x,\,k) + \frac{23k^4 - 23k^2 + 8}{15k^6}E(x,\,k).$$

29. $$\int \frac{\sin^7 x\,dx}{\Delta} = \frac{8k^4\sin^4 x + 10k^2(k^2+1)\sin^2 x + 15k^4 + 14k^2 + 15}{48k^6}\cos x\,\Delta - \\ - \frac{(5k^4 - 2k^2 + 5)(k^2+1)}{16k^7}\ln(k\cos x + \Delta).$$

30. $$\int \frac{\sin^6 x\cos x\,dx}{\Delta} = -\frac{8k^4\sin^4 x + 10k^2\sin^2 x + 15}{48k^6}\sin x\Delta + \frac{5}{16k^7}\arcsin(k\sin x).$$

31. $$\int \frac{\sin^5 x\cos^2 x\,dx}{\Delta} = \frac{-8k^4\sin^4 x + 2k^2(k^2-5)\sin^2 x + 3k^4 + 4k^2 - 15}{48k^6}\cos x\Delta - \\ - \frac{k^6 + k^4 + 3k^2 - 5}{16k^7}\ln(k\cos x + \Delta).$$

32. $$\int \frac{\sin^4 x\cos^3 x\,dx}{\Delta} = \frac{8k^4\sin^4 x - 2k^2(6k^2-5)\sin^2 x - 18k^2 + 15}{48k^6}\sin x\Delta + \\ + \frac{6k^2 - 5}{16k^7}\arcsin(k\sin x).$$

33. $$\int \frac{\sin^3 x\cos^4 x\,dx}{\Delta} = \frac{8k^4\sin^4 x - 2k^2(6k^2-5)\sin^2 x + 3k^4 - 22k^2 + 15}{48k^6}\cos x\Delta - \\ - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7}\ln(k\cos x + \Delta).$$

34. $$\int \frac{\sin^2 x \cos^5 x\,dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2(12k^2-5)\sin^2 x - 24k^4 + 36k^2 - 15}{48k^6} \sin x\Delta + \\ + \frac{8k^4 - 12k^2 + 5}{16k^7} \arcsin(k \sin x).$$

35. $$\int \frac{\sin x \cos^6 x\,dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2(13k^2-5)\sin^2 x - 33k^4 + 40k^2 - 15}{48k^6} \cos x\Delta + \\ + \frac{5k'^6}{16k^7} \ln(k \cos x + \Delta).$$

36. $$\int \frac{\cos^7 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(18k^2-5)\sin^2 x + 72k^4 - 54k^2 + 15}{48k^6} \sin x\Delta + \\ + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7} \arcsin(k \sin x).$$

37. $$\int \frac{dx}{\Delta^3} = \frac{1}{k'^2} E(x,\,k) - \frac{k^2}{k'^2} \frac{\sin x \cos x}{\Delta}.$$

38. $$\int \frac{\sin x\,dx}{\Delta^3} = -\frac{\cos x}{k'^2 \Delta}.$$

39. $$\int \frac{\cos x\,dx}{\Delta^3} = \frac{\sin x}{\Delta}.$$

40. $$\int \frac{\sin x\,dx}{\Delta^3} = \frac{1}{k'^2 k^2} E(x,\,k) - \frac{1}{k^2} F(x,\,k) - \frac{1}{k'^2} \frac{\mathrm{si}\,x \cos x}{\Delta}.$$

41. $$\int \frac{\sin x \cos x\,dx}{\Delta^3} = \frac{1}{k^2 \Delta}.$$

42. $$\int \frac{\cos^2 x\,dx}{\Delta^3} = \frac{1}{k^2} F(x,\,k) - \frac{1}{k^2} E(x,\,k) + \frac{\sin x \cos x}{\Delta}.$$

53. $$\int \frac{\sin^4 x \cos x\, dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k \sin x).$$

54. $$\int \frac{\sin^3 x \cos^2 x\, dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta).$$

55. $$\int \frac{\sin^2 x \cos^3 x\, dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k \sin x).$$

56. $$\int \frac{\sin x \cos^4 x\, dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3k'^2}{2k^5} \ln(k \cos x + \Delta).$$

57. $$\int \frac{\cos^5 x\, dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4 \Delta} \sin x + \frac{4k^2 - 3}{2k^5} \arcsin(k \sin x).$$

58. $$\int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2 (k'^2 + 1) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2} F(x,\ k) + \\ + \frac{2(k'^2 + 1)}{3k'^4} E(x,\ k).$$

59. $$\int \frac{\sin x\, dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x.$$

60. $$\int \frac{\cos x\, dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x.$$

61. $$\int \frac{\sin^2 x\, dx}{\Delta^5} = \frac{k^2+1}{3k'^4 k^2} E(x,\, k) - \frac{1}{3k'^2 k^2} F(x,\, k) + + \frac{k^2(k^2+1)\sin^2 x - 2}{3k'^4\Delta^3} \sin x \cos x.$$

62. $$\int \frac{\sin x \cos x\, dx}{\Delta^5} = \frac{1}{3k^2\Delta^3}.$$

63. $$\int \frac{\cos^2 x\, dx}{\Delta^5} = \frac{1}{3k^2} F(x,\, k) + \frac{2k^2-1}{3k^2 k'^2} E(x,\, k) + \frac{k^2(2k^2-1)\sin^2 x - 3k^2 + 2}{2k'^2\Delta} \sin x \cos x.$$

64. $$\int \frac{\sin^3 x}{\Delta^5}\, dx = \frac{(3k^2-1)\sin^2 x - 2}{3k'^4\Delta^3} \cos x.$$

65. $$\int \frac{\sin^2 x \cos x}{\Delta^5}\, dx = \frac{\sin^3 x}{3\Delta^3}.$$

66. $$\int \frac{\sin x \cos^2 x}{\Delta^5}\, dx = -\frac{\cos^3 x}{3k'^2\Delta^3}.$$

67. $$\int \frac{\cos^3 x\, dx}{\Delta^5} = \frac{-(2k^2+1)\sin^2 x + 3}{3\Delta^3} \sin x.$$

68. $$\int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

69. $$\int \frac{dx}{\Delta \cos x} = -\frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

70. $$\int \frac{dx}{\Delta \sin^2 x} = \int \frac{1+\operatorname{ctg}^2 x}{\Delta} dx = F(x,k) - E(x,\, k) - \Delta \operatorname{ctg} x.$$

71. $$\int \frac{dx}{\Delta \sin x \cos x} = \int (\operatorname{tg} x + \operatorname{ctg} x)\frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

72. $$\int \frac{dx}{\Delta \cos^2 x} = \int (1+\operatorname{tg}^2 x)\frac{dx}{\Delta} = F(x,\, k) - \frac{1}{k'^2}E(x,\, k) + \frac{1}{k'^2}\Delta \operatorname{tg} x.$$

73. $$\int \frac{\sin x}{\cos x}\frac{dx}{\Delta} = \int \operatorname{tg} x \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

74. $$\int \frac{\cos x}{\sin x}\frac{dx}{\Delta} = \int \operatorname{ctg} x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$$

75. $$\int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1+k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

76. $$\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

77. $$\int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$$

78. $$\int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} + \frac{2k^2-1}{4k'^3} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

79. $$\int \frac{\sin x}{\cos^2 x}\frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x}.$$

80. $$\int \frac{\cos x}{\sin^2 x}\frac{dx}{\Delta} = -\frac{\Delta}{\sin x}.$$

81. $$\int \frac{\sin^2 x}{\cos x}\frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{1}{k} \arcsin(k \sin x).$$

82. $$\int \frac{\cos^2 x}{\sin x}\frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

83. $$\int \frac{dx}{\Delta \sin^4 x} = \frac{1}{3}\{-\Delta \operatorname{ctg}^3 x - \Delta(2k^2+3) \operatorname{ctg} x + (k^2+2)F(x,\, k) - 2(k^2+1)E(x,\, k)\}.$$

84. $$\int \frac{dx}{\Delta \sin^3 x \cos x} = \int (\operatorname{tg} x + 2 \operatorname{ctg} x + \operatorname{ctg}^3 x)\frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} - \frac{k^2+2}{4} \ln \frac{1+\Delta}{1-\Delta}.$$

85. $$\int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \int (\operatorname{tg}^2 x + 2 + \operatorname{ctg}^2 x)\frac{dx}{\Delta} = \left(\frac{\operatorname{tg} x}{k'^2} - \operatorname{ctg} x\right)\Delta + \frac{k^2-2}{k'^2}E(x,\, k) + 2F(x, k).$$

86. $$\int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\operatorname{ctg} x + 2 \operatorname{tg} x + \operatorname{tg}^3 x)\frac{dx}{\Delta} = -\frac{\Delta}{2k'^2 \cos^2 x} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta} + \frac{2-3k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}.$$

87. $$\int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3k'^2}\left\{\Delta \operatorname{tg}^3 x - \frac{5k^2-3}{k'^2}\Delta \operatorname{tg} x - (3k^2-2)F(x,\, k) + \frac{2(2k^2-1)}{k'^2}E(x,\, k)\right\}.$$

88. $$\int \frac{\sin x}{\cos^3 x}\frac{dx}{\Delta} = \int \operatorname{tg} x(1+\operatorname{tg}^2 x)\frac{dx}{\Delta} = \frac{\Delta}{2k'^2 \cos^2 x} - \frac{k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}.$$

89. $$\int \frac{\cos x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta}{2\,\sin^2\,x} - \frac{k^2}{4}\,\ln\frac{1+\Delta}{1-\Delta}.$$

90. $$\int \frac{\sin^2\,x}{\cos^2\,x}\frac{dx}{\Delta} = \int\frac{\mathrm{tg}^2\,x}{\Delta}\,dx = \frac{\Delta}{k'^2}\,\mathrm{tg}\,x - \frac{1}{k'^2}\,E(x,\,k).$$

91. $$\int \frac{\cos^2\,x}{\sin^2\,x}\frac{dx}{\Delta} = \int\frac{\mathrm{ctg}^2\,x}{\Delta}\,dx = -\Delta\,\mathrm{ctg}\,x - E(x,\,k).$$

92. $$\int \frac{\sin^3\,x}{\cos x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}.$$

93. $$\int \frac{\cos^3\,x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2}\,\ln\frac{1+\Delta}{1-\Delta}.$$

94. $$\int \frac{dx}{\Delta\,\sin^5\,x} = -\frac{[3(1+k^2)\,\sin^2\,x+2]}{8\,\sin^2\,x}\Delta\,\cos x + \frac{3k^4+2k^2+3}{16}\,\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

95. $$\int \frac{dx}{\Delta\,\sin^4\,x\,\cos x} = -\frac{(3+2k^2)\,\sin^2\,x+1}{3\,\sin^3\,x}\Delta - \frac{1}{2k'}\,\ln\frac{\Delta-k'\,\sin x}{\Delta+k'\,\sin x}.$$

96. $$\int \frac{dx}{\Delta\,\sin^3\,x\,\cos^2\,x} = \frac{(3-k^2)\,\sin^2\,x-k'^2}{2k'^2\,\sin^2\,x\,\cos x}\Delta + \frac{k^2+3}{4}\,\ln\frac{\Delta-\cos\,x}{\Delta+\cos x}.$$

97. $$\int \frac{dx}{\Delta\,\sin^2\,x\,\cos^3\,x} = \frac{(3-2k^2)\,\sin^2\,x-2k'^2}{2k'^2\,\sin x\,\cos^2\,x}\Delta - \frac{4k^2-3}{4k'^3}\,\ln\frac{\Delta+k'\,\sin x}{\Delta-k'\,\sin x}.$$

98. $$\int \frac{dx}{\Delta\,\sin x\,\cos^4\,x} = \frac{(5k^2-3)\,\sin^2\,x-6k^2+4}{3k'^4\,\cos^3\,x}\Delta - \frac{1}{2}\,\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

99. $$\int \frac{dx}{\Delta\,\cos^5\,x} = \frac{3(2k^2-1)\,\sin^2\,x-8k^2+5}{8k'^4\,\cos^4\,x}\Delta\,\sin x + \frac{8k^4-8k^2+3}{16k'^5}\,\ln\frac{\Delta+k'\,\sin x}{\Delta-k'\,\sin x}.$$

100. $$\int \frac{\sin x}{\cos^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \cos^2 x - k'^2}{2k'^4 \cos^3 x} \Delta.$$

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101. $$\int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta.$$

102. $$\int \frac{\sin^2 x}{\cos^3 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} - \frac{1}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

103. $$\int \frac{\cos^3 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

104. $$\int \frac{\sin^3 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

105. $$\int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k} \arcsin(k \sin x).$$

106. $$\int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{2k^2 + 1}{2k^3} \arcsin(k \sin x).$$

107. $$\int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2 - 1}{2k^3} \ln(k \cos x + \Delta).$$

2.585

1. $$\int \frac{(a + \sin x)^{p+3}\, dx}{\Delta} =$$
$$= \frac{1}{(p+2)k^2} \Bigg[(a + \sin x)^p \cos x \Delta +$$
$$+2(2p+3)ak^2 \int \frac{(a + \sin x)^{p+2}\, dx}{\Delta} + (p+1)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p+1}\, dx}{\Delta} -$$
$$- a(2p+1)(1 + k^2 - 2a^2k^2) \int \frac{(a + b \sin x)^p\, dx}{\Delta} -$$
$$-p(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^{p-1}\, dx}{\Delta} \Bigg] \qquad \left[p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k}\right].$$

$p = n$

2. $$\int \frac{a+\sin x}{\Delta} dx = aF(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

3. $$\int \frac{(a+\sin x)^2}{\Delta} dx = \frac{1+k^2a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

4.[6] $$\int \frac{dx}{(a+\sin x)\Delta} = \frac{1}{a} \Pi\left(x, \frac{1}{a^2}, k\right) - \int \frac{\sin x\, dx}{(a^2 - \sin^2 x)\Delta},$$

where

5. $$\int \frac{\sin x\, dx}{(a^2 - \sin^2 x)\Delta} = \frac{-1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2} \cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2} \cos x}.$$

2.586

1. $$\int \frac{dx}{(a+\sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a+\sin x)^{n-1}} - \right.$$
$$- (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a+\sin x)^{n-1}\Delta} -$$
$$- (n-2)(6a^2k^2 - k^2 - 1) \int \frac{dx}{(a+\sin x)^{n-2}\Delta} -$$
$$\left. -(10-4n)ak^2 \int \frac{dx}{(a+\sin x)^{n-3}\Delta} - (n-3)k^2 \int \frac{dx}{(a+\sin x)^{n-4}\Delta} \right]$$
$$\left[n \neq 1, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right].$$

This integral can be reduced to the integrals:

2. $$\int \frac{dx}{(a+\sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{a+\sin x} - \right.$$
$$-a(1+k^2-2a^2k^2) \int \frac{dx}{(a+\sin x)\Delta} - 2ak^2 \int \frac{(a+\sin x)\, dx}{\Delta} +$$
$$\left. + k^2 \int \frac{(a+\sin x)^2\, dx}{\Delta} \right] \qquad \text{(see } \mathbf{2.585}\ 2., 3., 4.).$$

2.585

3. $$\int \frac{dx}{(a+\sin x)^3 \Delta} = \frac{1}{2(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a+\sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a+\sin x)^2 \Delta} - \right.$$
$$\left. - (6a^2k^2 - k^2 - 1) \int \frac{dx}{(a+\sin x)\Delta} + 2ak^2 F(x,\, k) \right]$$
(see **2.585** 4. and **2.586** 2.).

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2.585

For $a = \pm 1$, we have:

4. $$\int \frac{dx}{(1 \pm \sin x)^n \Delta} = \frac{1}{(2n-1)k'^2} \left[\mp \frac{\cos x \Delta}{(1 \pm \sin x)^n} + \right.$$
$$+ (n-1)(1-5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} + 2(2n-3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} -$$
$$\left. - (n-2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right].$$
GU ((241))(6a)

This integral can be reduced to the integrals

5. $$\int \frac{dx}{(1 \pm \sin x)\Delta} = \frac{\mp \cos x \Delta}{k'^2 (1 \pm \sin x)} + F(x,\, k) - \frac{1}{k'^2} E(x,\, k).$$
GU ((241))(6c)

6. $$\int \frac{dx}{(1 \pm \sin x)^2 \Delta} = \frac{1}{3k'^4} \left[\mp \frac{k'^2 \cos x \Delta}{(1 \pm \sin x)^2} \mp \frac{(1-5k^2) \cos x \Delta}{1 \pm \sin x} + \right.$$
$$\left. + (1-3k^2)k'^2 F(x,\, k) - (1-5k^2) E(x,\, k) \right].$$
GU ((241))(6b)

For $a = \pm \frac{1}{k}$, we have

GU ((241))(7a)

This integral can be reduced to the integrals

8. $$\int \frac{dx}{(1 \pm k\ \sin x)\Delta} = \pm \frac{k\ \cos x\Delta}{k'^2(1 \pm k\ \sin x)} + \frac{1}{k'^2} E(x,\ k).$$

GU ((241))(7b)

9. $$\int \frac{dx}{(1 \pm k\ \sin x)^2\Delta} = \frac{1}{3k'^4}\left[\pm \frac{kk'^2\ \cos x\Delta}{(1 \pm k\ \sin x)^2} \pm \frac{k(5-k^2)\ \cos x\Delta}{1 \pm k\ \sin x} - \right.$$
$$\left. - 2k'^2 F(x,\ k) + (5-k^2)E(x,\ k)\right].$$

GU((241))(7c)

2.587

1. $$\int \frac{(b+\cos x)^{p+3}\ dx}{\Delta} = \frac{1}{(p+2)k^2}\left[(b{+}\cos x)^p\ \sin x\Delta{+}2(2p{+}3)bk^2\int \frac{(b+\cos x)^{p+2}\ dx}{\Delta} - \right.$$
$$- (p+1)(k'^2 - k^2 + 6b^2k^2)\int \frac{(b+\cos x)^{p+1}\ dx}{\Delta} +$$
$$+ (2p+1)b(k'^2 - k^2 + b^2k^2)\int \frac{(b+\cos x)^{p}\ dx}{\Delta} +$$
$$\left. + p(1-b^2)(k'^2 + k^2b^2)\int \frac{(b+\cos x)^{p-1}\ dx}{\Delta}\right]$$
$$\left[p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k}\right].$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{b+\cos x}{\Delta}\,dx = bF(x,\ k) + \frac{1}{k}\ \arcsin(k\ \sin\ x).$$

3. $$\int \frac{(b+\cos x)^2}{\Delta}\,dx = \frac{b^2k^2 - k'^2}{k^2} F(x,\ k) + \frac{1}{k^2} E(x,\ k) + \frac{2b}{k}\ \arcsin(k\ \sin x).$$

4. $$\int \frac{dx}{(b+\cos x)\Delta} = \frac{b}{b^2-1}\Pi\left(x,\ \frac{1}{b^2-1},\ k\right) + \int \frac{\cos x\,dx}{(1-b^2-\sin^2\ x)\Delta},$$

where

5. $$\int \frac{\cos x\,dx}{(1-b^2-\sin^2\ x)\Delta} = \frac{1}{2\sqrt{(1-b^2)(k'^2+k^2b^2)}}\ \ln \frac{\sqrt{1-b^2}\Delta + k\sqrt{k'^2+k^2b^2}\ \sin x}{\sqrt{1-b^2}\Delta - k\sqrt{k'^2+k^2b^2}\ \sin x}.$$

2.588

1. $$\int \frac{dx}{(b+\cos x)^n\Delta} = \frac{1}{(n-1)(1-b^2)(k'^2+b^2k^2)}\left[\frac{-k'^2\ \sin x\Delta}{(b+\cos x)^{-1}} - \right.$$
$$- (2n-3)(1-2k^2+2b^2k^2)b\int \frac{dx}{(b+\cos x)^{n-1}\Delta} -$$
$$- (n-2)(2k^2-1-6b^2k^2)\int \frac{dx}{(b+\cos x)^{n-2}\Delta} -$$
$$\left. -(4n-10)bk^2\int \frac{dx}{(b+\cos x)^{n-3}\Delta} + (n-3)k^2\int \frac{dx}{(b+\cos x)^{n-4}\Delta}\right]$$
$$\left[n\neq 1,\quad b\neq \pm 1,\quad b\neq \pm \frac{ik'}{k}\right].$$

This integral can be reduced to the following integrals:

2. $$\int \frac{dx}{(b+\cos x)^2\Delta} =$$
$$= \frac{1}{(1-b^2)(k'^2+b^2k^2)}\left[\frac{-k'^2\ \sin x\Delta}{b+\cos x} - (1-2k^2+2b^2k^2)b\int \frac{dx}{(b+\cos x)\Delta} + \right.$$
$$\left. +2bk^2\int \frac{b+\cos x}{\Delta}dx - k^2\int \frac{(b+\cos x)^2}{\Delta}\ dx\right] \qquad \text{(see } \mathbf{2.587}\ 2.,\ 3.,\ 4.).$$

2.587

3. $$\int \frac{dx}{(b+\cos x)^3\Delta} = \frac{1}{2(1-b^2)(k'^2+b^2k^2)}\left[\frac{-k'^2\ \sin x\Delta}{(b+\cos x)^2} - \right.$$
$$- 3b(1-2k^2+2k^2b^2)\int \frac{dx}{(b+\cos x)^2\Delta} -$$
$$\left. - (2k^2-1-6b^2k^2)\int \frac{dx}{(b+\cos x)\Delta} - 2bk^2F(x,\ k)\right]$$
$$\text{(see } \mathbf{2.588}\ 2. \text{ and } \mathbf{2.587}\ 4.).$$

2.589

1. $$\int \frac{(c+\operatorname{tg} x)^{p+3}\,dx}{\Delta} =$$
$$= \frac{1}{(p+2)k'^2}\left[\frac{(c+\operatorname{tg} x)^p\Delta}{\cos^2 x} + 2(2n+3)ck'^2\int\frac{(c+\operatorname{tg} x)^{p+2}\,dx}{\Delta} - \right.$$
$$- (p+1)(1+k'^2+6c^2k'^2)\int\frac{(c+\operatorname{tg} x)^{p+1}\,dx}{\Delta} +$$
$$+ (2p+1)c(1+k'^2+2c^2k'^2)\int\frac{(c+\operatorname{tg} x)^{p}\,dx}{\Delta} -$$
$$\left. - p(1+c^2)(1+k'^2c^2)\int\frac{(c+\operatorname{tg} x)^{p-1}\,dx}{\Delta}\right] \qquad [p\neq -2].$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{c+\operatorname{tg} x}{\Delta}\,dx = cF(x,\,k) + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}.$$

3. $$\int \frac{(c+\operatorname{tg} x)^2}{\Delta}\,dx = \frac{1}{k'^2}\operatorname{tg} x\Delta + c^2F(x,\,k) - \frac{1}{k'^2}E(x,\,k) + \frac{c}{k'}\ln\frac{\Delta+k'}{\Delta-k'}.$$

4. $$\int \frac{dx}{(c+\operatorname{tg} x)\Delta} = \frac{c}{1+c^2}F(x,\,k) + \frac{1}{c(1+c^2)}\Pi\left(x,\,-\frac{1+c^2}{c^2},\,k\right) -$$
$$- \int\frac{\sin x\,\cos x\,dx}{[c^2-(1+c^2)\sin^2 x]\Delta},$$

where

5. $$\int \frac{\sin x\,\cos x\,dx}{[c^2-(1+c^2)\sin^2 x]\Delta} = \frac{1}{2\sqrt{(1+c^2)(1+c^2k'^2)}}\ln\frac{\sqrt{1+c^2k'^2}+\sqrt{1+c^2}\Delta}{\sqrt{1+c^2k'^2}-\sqrt{1+c^2}\Delta}.$$

2.591

1. $$\int \frac{dx}{(c+\operatorname{tg} x)^n\Delta} = \frac{1}{(n-1)(1+c^2)(1+k'^2c^2)}\left[-\frac{\Delta}{(c+\operatorname{tg} x)^{n-1}\cos^2 x} + \right.$$
$$+ (2n-3)c(1+k'^2+2c^2k'^2)\int\frac{dx}{(c+\operatorname{tg} x)^{n-1}\Delta} -$$
$$- (n-2)(1+k'^2+6c^2k'^2)\int\frac{dx}{(c+\operatorname{tg} x)^{n-2}\Delta} +$$
$$\left. + (4n-10)ck'^2\int\frac{dx}{(c+\operatorname{tg} x)^{n-3}\Delta} - (n-3)k'^2\int\frac{dx}{(c+\operatorname{tg} x)^{n-4}\Delta}\right].$$

This integral can be reduced to the integrals:

$$2.\quad \int \frac{dx}{(c+\operatorname{tg} x)^2\Delta} = \frac{1}{(1+c^2)(1+k'^2c^2)}\left[\frac{-\Delta}{(c+\operatorname{tg} x)\cos^2 x} + \right.$$
$$+ c(1+k'^2+2c^2k'^2)\int \frac{dx}{(c+\operatorname{tg} x)\Delta} -$$
$$\left. -2ck'^2\int \frac{c+\operatorname{tg} x}{\Delta}dx + k'^2\int \frac{(c+\operatorname{tg} x)^2}{\Delta}\,dx\right] \text{(see } \mathbf{2.589}\ 2., 3., 4.).$$

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$$3.\quad \int \frac{dx}{(c+\operatorname{tg} x)^3\Delta} = \frac{1}{2(1+c^2)(1+k'^2c^2)}\left[\frac{-\Delta}{(c+\operatorname{tg} x)^2\cos^2 x} + \right.$$
$$+ 3c(1+k'^2+2c^2k'^2)\int \frac{dx}{(c+\operatorname{tg} x)^2\Delta} -$$
$$\left. -(1+k'^2+6c^2k'^2)\int \frac{dx}{(c+\operatorname{tg} x)\Delta} + 2ck'^2F(x,k)\right]$$
$$\text{(see } \mathbf{2.591}\ 2. \text{ and } \mathbf{2.589}\ 4.).$$

2.589

2.591

2.592

$$1.\quad P_n = \int \frac{(a+\sin^2 x)^n}{\Delta}\,dx.$$

The recursion formula

$$P_{n+2} = \frac{1}{(2n+3)k^2}\left\{(a+\sin^2 x)^n \sin x \cos x\Delta + (2n+2)(1+k^2+3ak^2)P_{n+1} - \right.$$
$$\left. -(2n+1)[1+2a(1+k^2)+3a^2k^2]P_n + 2na(1+a)(1+k^2a)P_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals

$$2.\quad P_1 \text{ see } \mathbf{2.584}\ 1. \text{ and } \mathbf{2.584}\ 4.$$

2.584

3. P_0 see **2.584** 1.

2.584

4. $P_{-1} = \int \frac{dx}{(a + \sin^2 x)\Delta} = \frac{1}{a}\Pi\left(x, \frac{1}{a}, k\right).$

For $a = 0$

5. $\int \frac{dx}{\sin^2 x\Delta}$ see **2.584** 70.

2.584
ZH (124)a

6. $T_n = \int \frac{dx}{(h + g \sin^2 x)^n \Delta}$

can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2}\left\{\frac{-g^2 \sin x \cos x\Delta}{(h + g \sin^2 x)^{n-1}} + 2(n-2)[g(1+k^2) + 3hk^2]T_{n-2} - \right.$$
$$\left. - (2n-3)[g^2 + 2hg(1+k^2) + 3h^2k^2]T_{n-1} + 2(n-1)h(g+h)(g+hk^2)T_n\right\}.$$

2.593

1. $Q_n = \int \frac{(b + \cos^2 x)^n}{\Delta}\, dx.$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \{(b + \cos^2 x)^n \sin x \cos x\Delta - (2n+2)(1 - 2k^2 - 3bk^2)Q_{n+1} + \\ + (2n+1)[k'^2 + 2b(k'^2 - k^2) - 3b^2k^2]Q_n - 2nb(1-b)(k'^2 - k^2b)Q_{n-1}\Big\}$$

reduces this integral (for n an integer) to the integrals:

2. Q_1 see **2.584** 1. and **2.584** 6.

2.584

2.584

3. Q_0 see **2.584** 1.

2.584

4. $Q_{-1} = \int \frac{dx}{(b + \cos^2 x)\Delta} = \frac{1}{b+1}\Pi\left(x, -\frac{1}{b+1}, k\right).$

For $b = 0$

5. $\int \frac{dx}{\cos^2 x\Delta}$ see **2.584** 72.

2.584
ZH (123)

2.594

1. $R_n = \int \frac{(c + \operatorname{tg}^2 x)^n \, dx}{\Delta}.$

$$R_{n+2} = \frac{1}{(2n+3)k'^2}\left\{\frac{(c+\operatorname{tg}^2 x)^n \operatorname{tg} x\Delta}{\cos^2 x} - (2n+2)(1+k'^2-3ck'^2)R_{n+1} + \right.$$
$$\left. + (2n-1)[1-2c(1+k'^2)+3c^2k'^2]R_n + 2nc(1-c)(1-k'^2c)R_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals:

2. R_1 see **2.584** 1. and **2.584** 90.

2.584

2.584

3. R_0 see **2.584** 1.

2.584

4. $R_{-1} = \int \frac{dx}{(c+\operatorname{tg}^2 x)\Delta} = \frac{1}{c-1}F(x,\, k) + \frac{1}{c(1-c)}\Pi\left(x,\, \frac{1-c}{c}, k\right).$

For $c = 0$ see 2.582 5.

2.595

Integrals of the type $\int R(\sin x, \cos x, \sqrt{1-p^2 \sin^2 x})\, dx$ for $p^2 > 1$.

Notation: $\alpha = \arcsin(p \sin x)$.

Basic formulas

BY (283.00)

2. $$\int \sqrt{1-p^2\ \sin^2\ x}\,dx = pE\left(\alpha,\ \frac{1}{p}\right) - \frac{p^2-1}{p}F\left(\alpha,\ \frac{1}{p}\right) \qquad [p^2>1].$$

BY (283.03)

3. $$\int \frac{dx}{(1-r^2\ \sin^2\ x)\sqrt{1-p^2\ \sin^2\ x}} = \frac{1}{p}\Pi\left(\alpha,\ \frac{r^2}{p^2},\ \frac{1}{p}\right) \qquad [p^2>1].$$

BY (283.02)

To evaluate integrals of the form $\int R(\sin x, \cos x, \sqrt{1-p^2\ \sin^2\ x})\,dx$ for $p^2>1$, we may use formulas 2.583 and 2.584, making the following modifications in them.

We replace (1) k with p, (2) k'^2 with $1-p^2$, (3) $F(x,\ k)$ with $\frac{1}{p}F\left(\alpha,\ \frac{1}{p}\right)$, and (4) $E(x,\ k)$ with $pE\left(\alpha,\ \frac{1}{p}\right) - \frac{p^2-1}{p}F\left(\alpha,\ \frac{1}{p}\right)$.

For example (see 2.584 15.):

2.596

1. $$\frac{\cos^4\ x\,dx}{\sqrt{1-p^2\ \sin^2\ x}} = \frac{\sin x\ \cos x\sqrt{1-p^2\ \sin^2\ x}}{3p^2} + \frac{4p^2-2}{3p^4}\left[pE\left(\alpha,\ \frac{1}{p}\right) - \right.$$
$$\left. - \frac{p^2-1}{p}F\left(\alpha,\ \frac{1}{p}\right)\right] + \frac{2-5p^2+3p^4}{3p^4}\cdot\frac{1}{p}F\left(\alpha,\ \frac{1}{p}\right) =$$
$$= \frac{\sin x\ \cos x\sqrt{1-p^2\ \sin^2\ x}}{3p^2} - \frac{p^2-1}{3p^3}F\left(\alpha,\ \frac{1}{p}\right) + \frac{4p^2-2}{3p^3}E\left(\alpha,\ \frac{1}{p}\right) \qquad [p^2>1];$$

(see 2.583 36.):

2. $$\int \frac{\sqrt{1-p^2\ \sin^2\ x}}{\cos^2\ x}\,dx = \operatorname{tg} x\sqrt{1-p^2\ \sin^2\ x} + \frac{1}{p}F\left(\alpha,\ \frac{1}{p}\right) -$$
$$- \left[pE\left(\alpha,\ \frac{1}{p}\right) - \frac{p^2-1}{p}F\left(\alpha,\ \frac{1}{p}\right)\right] =$$
$$= p\left[F\left(\alpha,\ \frac{1}{p}\right) - E\left(\alpha,\ \frac{1}{p}\right)\right] + \operatorname{tg} x\sqrt{1-p^2\ \sin^2\ x} \qquad [p^2>1];$$

3. $$\int \frac{dx}{\sqrt{(1-p^2\sin^2 x)^3}} = \frac{-1}{p^2-1}\left[pE\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p}F\left(\alpha, \frac{1}{p}\right)\right] -$$
$$-\frac{p^2}{1-p^2}\cdot\frac{\sin x\cos x}{\sqrt{1-p^2\sin^2 x}} = \frac{p^2}{p^2-1}\cdot\frac{\sin x\cos x}{\sqrt{1-p^2\sin^2 x}} + \frac{1}{p}F\left(\alpha, \frac{1}{p}\right) - \frac{p}{p^2-1}E\left(\alpha, \frac{1}{p}\right) \qquad [p^2>1].$$

2.597

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1+p^2\sin^2 x})\,dx$.

Notation: $\alpha = \arcsin\left(\frac{\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}}\right)$.

Basic formulas

1. $$\int \frac{dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{\sqrt{1+p^2}}F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right).$$

BY (282.00)

2. $$\int \sqrt{1+p^2\sin^2 x}\,dx = \sqrt{1+p^2}E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}.$$

BY (282.03)

3. $$\frac{\sqrt{1+p^2\sin^2 x}\,dx}{1+(p^2-r^2p^2-r^2)\sin^2 x} = \frac{1}{\sqrt{1+p^2}}\Pi\left(\alpha, r^2, \frac{p}{\sqrt{1+p^2}}\right).$$

BY (282.02)

4. $$\int \frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}} = -\frac{1}{p}\arcsin\left(\frac{p\cos x}{\sqrt{1+p^2}}\right).$$

5. $$\int \frac{\cos x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{p}\ln(p\sin x + \sqrt{1+p^2\sin^2 x}).$$

6. $$\int \frac{dx}{\sin x\sqrt{1+p^2\sin^2 x}} = \frac{1}{2}\ln\frac{\sqrt{1+p^2\sin^2 x}-\cos x}{\sqrt{1+p^2\sin^2 x}+\cos x}.$$

$$7.\quad \int \frac{dx}{\cos x\sqrt{1+p^2\ \sin^2\ x}} = \frac{1}{2\sqrt{1+p^2}}\ \ln\frac{\sqrt{1+p^2\ \sin^2\ x}+\sqrt{1+p^2}\ \sin x}{\sqrt{1+p^2\ \sin^2\ x}-\sqrt{1+p^2}\ \sin x}.$$

$$8.\quad \int \frac{\operatorname{tg} x\, dx}{\sqrt{1+p^2\ \sin^2\ x}} = \frac{1}{2\sqrt{1+p^2}}\ \ln\frac{\sqrt{1+p^2\ \sin^2\ x}+\sqrt{1+p^2}}{\sqrt{1+p^2\ \sin^2\ x}-\sqrt{1+p^2}}.$$

$$9.\quad \int \frac{\operatorname{ctg} x\, dx}{\sqrt{1+p^2\ \sin^2\ x}} = \frac{1}{2}\ \ln\frac{1-\sqrt{1+p^2\ \sin^2\ x}}{1+\sqrt{1+p^2\ \sin^2\ x}}.$$

2.598

To calculate integrals of the form $\int\ R(\sin x, \cos x, \sqrt{1+p^2\ \sin^2\ x})\, dx$, we may use formulas 2.583 and 2.584, making the following modifications in them:

We replace 1) k^2 with $-p^2$; 2) k'^2 with $1+p^2$;

$$3)\quad F(x,\ k)\quad \text{with}\quad \frac{1}{\sqrt{1+p^2}}F\left(\alpha,\ \frac{p}{\sqrt{1+p^2}}\right);$$

$$4)\quad E(x,\ k)\quad \text{with}\quad \sqrt{1+p^2}E\left(\alpha,\ \frac{p}{\sqrt{1+p^2}}\right)-p^2\frac{\sin\ x\ \cos x}{\sqrt{1+p^2\ \sin^2\ x}};$$

$$5)\quad \frac{1}{k}\ \ln(k\ \cos x+\Delta)\quad \text{with}\quad \frac{1}{p}\ \arcsin\frac{p\ \cos x}{\sqrt{1+p^2}};$$

$$6)\quad \frac{1}{k}\ \arcsin(k\ \sin x)\quad \text{with}\quad \frac{1}{p}\ln(p\ \sin x+\sqrt{1+p^2\ \sin^2\ x}).$$

For example (see 2.584 90.):

(see 2.584 37.):

2. $$\int \frac{dx}{\sqrt{(1+p^2\,\sin^2\,x)^3}} = \frac{1}{\sqrt{1+p^2}} E\left(\alpha,\, \frac{p}{\sqrt{1+p^2}}\right)$$

2.599

Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2\,\sin^2\,x - 1})\,dx \qquad [a^2 > 1]$

Notation: $\alpha = \arcsin\left(\frac{\alpha\,\cos x}{sqrta^2-1}\right)$.

Basic formulas:

1. $$\int \frac{dx}{\sqrt{a^2\,\sin^2\,x-1}} = -\frac{1}{a} F\left(\alpha,\, \frac{\sqrt{a^2-1}}{a}\right) \qquad [a^2 > 1].$$

BY (285.00)a

2. $$\int \sqrt{a^2\,\sin^2\,x-1}\,dx = \frac{1}{a} F\left(\alpha,\, \frac{\sqrt{a^2-1}}{a}\right) - aE\left(\alpha,\, \frac{\sqrt{a^2-1}}{a}\right) \qquad [a^2 > 1].$$

BY (285.06)a

3. $$\int \frac{dx}{(1-r^2\,\sin^2\,x)\sqrt{a^2\,\sin^2\,x-1}} = \frac{1}{a(r^2-1)} \Pi\left(\alpha,\, \frac{r^2(a^2-1)}{a^2(r^2-1)},\, \frac{\sqrt{a^2-1}}{a}\right)$$
$$[a^2 > 1, \quad r^2 > 1].$$

BY (285.02)a

4. $$\int \frac{\sin x\,dx}{\sqrt{a^2\,\sin^2\,x-1}} = -\frac{\alpha}{a} \qquad [a^2 > 1].$$

5. $$\int \frac{\cos x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln(a \sin x + \sqrt{a^2 \sin^2 x - 1}) \qquad [a^2 > 1].$$

6. $$\int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}} \qquad [a^2 > 1].$$

7. $$\int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 2}} \ln \frac{\sqrt{a^2 - 1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} \sin x - \sqrt{a^2 \sin^2 x - 1}} \qquad [a^2 > 1].$$

8. $$\int \frac{\operatorname{tg} x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}} \qquad [a^2 > 1].$$

9. $$\int \frac{\operatorname{ctg} x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin\left(\frac{1}{a \sin x}\right) \qquad [a^2 > 1].$$

2.611

To calculate integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1})\, dx$ (for $a^2 > 1$), we may use formulas 2.583 and 2.584. In doing so, we should follow the procedure outlined below:

1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$\begin{aligned}
F(x, k) &\quad \text{should be replaced with} \quad \int \frac{dx}{\Delta}, \\
E(x, k) &\quad \text{should be replaced with} \quad \int \Delta dx, \\
-\frac{1}{k} \ln(k \cos x + \Delta) &\quad \text{should be replaced with} \quad \int \frac{\sin x\, dx}{\Delta}, \\
\frac{1}{k} \arcsin(k \sin x) &\quad \text{should be replaced with} \quad \int \frac{\cos x\, dx}{\Delta}, \\
\frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} &\quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \sin x}, \\
\frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} &\quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \cos x}, \\
\frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} &\quad \text{should be replaced with} \quad \int \frac{\operatorname{tg} x}{\Delta}\, dx, \\
\frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} &\quad \text{should be replaced with} \quad \int \frac{\operatorname{ctg} x}{\Delta}\, dx.
\end{aligned}$$

1. We rewrite equation 2.584 4. in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2 \sin^2 x - 1}}\, dx = \frac{1}{a^2}\int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{1}{a^2}\int i\sqrt{a^2 \sin^2 x - 1}\, dx,$$

from which we get

$$\int \frac{\sin^2 x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a^2}\left\{\int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} + \int \sqrt{a^2 \sin^2 x - 1}\, dx\right\} = -\frac{1}{a}E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) \qquad [a^2 > 1].$$

2. We rewrite equation 2.584 58. as follows:

$$\int \frac{dx}{i^5\sqrt{(a^2 \sin^2 x - 1)^5}} = -\frac{2a^4(a^2-2)\sin^2 x - (3a^2-5)a^2}{3(1-a^2)^2 i^3\sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x - \\ - \frac{1}{3(1-a^2)}\int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{2a^2-4}{3(1-a^2)^2}\int i\sqrt{a^2 \sin^2 x - 1}\, dx,$$

from which we obtain

$$\int \frac{dx}{\sqrt{(a^2 \sin^2 x - 1)^5}} = \frac{2a^4(a^2-2)\sin^2 x - (3a^2-5)a^2}{3(1-a^2)^2\sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x + \frac{1}{3(1-a^2)^2 a} \times \\ \times \left\{(a^2-3)F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) - 2a^2\left(a^2-2\right) E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)\right\} \qquad [a^2 > 1].$$

3. We rewrite equation 2.584 71. in the form

$$\int \frac{dx}{\sin x \cos x i\sqrt{a^2 \sin^2 x - 1}} = \int \frac{\operatorname{ctg} x\, dx}{i\sqrt{a^2 \sin^2 x - 1}} + \int \frac{\operatorname{tg} x\, dx}{i\sqrt{a^2 \sin^2 x - 1}},$$

from which we obtain

2.612

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1-k^2\ \cos^2\ x})\, dx$

To find integrals of the form $\int R(\sin x, \cos x, \sqrt{1-k^2\ \cos^2\ x})\, dx$ we make the substitution $x = \frac{\pi}{2} - y$, which yields

$$\int R\left(\sin x, \cos x, \sqrt{1-k^2\ \cos^2\ x}\right) dx = -\int R\left(\cos y, \sin y, \sqrt{1-k^2\ \sin^2\ y}\right) dy.$$

The integrals $\int R(\cos y, \sin y, \sqrt{1-k^2\ \sin^2\ y})\, dy$ are found from formulas 2.583 and 2.584. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions $F(x,\ k)$ and $E(x,\ k)$ with the corresponding integrals, we obtain an expression of the form

$$-g(\cos y, \sin y) - A\int \frac{dy}{\sqrt{1-k^2\ \sin^2\ y}} - B\int \sqrt{1-k^2\ \sin^2\ y}\, dy.$$

Returning now to the original variable x, we obtain

$$\int R\left(\sin x, \cos x, \sqrt{1-k^2\ \cos^2\ x}\right) dx = -g(\sin x, \cos x) - A\int \frac{dx}{\sqrt{1-k^2\ \cos^2\ x}} - B\int \sqrt{1-k^2\ \cos^2\ x}\, dx.$$

The integrals appearing in this expression are found from the formulas

1. $$\int \frac{dx}{\sqrt{1-k^2\ \cos^2\ x}} = F\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\ \cos^2\ x}}\right), k\right).$$

2. $$\int \sqrt{1-k^2\ \cos^2\ x}\, dx = E\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\ \cos^2\ x}}\right), k\right) - \frac{k^2\ \sin x\ \cos x}{\sqrt{1-k^2\ \cos^2\ x}}.$$

2.613

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2\ \cos^2\ x})\, dx \qquad [p > 1]$

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1 - p^2\ \cos^2\ x})\, dx$, where $[p > 1]$, we proceed as in 2.612. Here, we use the formulas

1. $$\int \frac{dx}{\sqrt{1 - p^2\ \cos^2\ x}} = -\frac{1}{p} F\left(\arcsin(p\ \cos x), \frac{1}{p}\right) \qquad [p > 1].$$

2. $$\int \sqrt{1 - p^2\ \cos^2\ x}\, dx = \frac{p^2 - 1}{p} F\left(\arcsin(p\ \cos x),\ \frac{1}{p}\right) - pE\left(\arcsin(p\ \cos x),\ \frac{1}{p}\right).$$

2.614

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 + p^2\ \cos^2\ dx})\, dx$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1 + p^2\ \cos^2\ x})\, dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{1 + p^2\ \cos^2\ x})\, dx = -\int R(\cos y, \sin y, \sqrt{1 + p^2\ \sin^2\ y})\, dy.$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{1 + p^2\ \sin^2\ y})\, dy$, we need to use first what was said in 2.598 and 2.612 and then, after returning to the variable x, the formulas

1. $$\int \frac{dx}{\sqrt{1 + p^2\ \cos^2\ x}} = \frac{1}{\sqrt{1 + p^2}} F\left(x,\ \frac{p}{\sqrt{1 + p^2}}\right).$$

2. $$\int \sqrt{1 + p^2\ \cos^2\ x}\, dx = \sqrt{1 + p^2} E\left(x,\ \frac{p}{\sqrt{1 + p^2}}\right).$$

2.615

Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2\ \cos^2\ x - 1})\, dx \qquad [a > 1]$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2\ \cos^2\ x - 1})\, dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{a^2\ \cos^2\ x - 1})\, dx = -\int R(\cos y, \sin y, \sqrt{a^2\ \sin^2\ y - 1})\, dy.$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{a^2\ \sin^2\ y - 1})\, dy$, we use what was said in 2.611 and then,

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after returning to the variable x, we use the formulas

1. $$\int \frac{dx}{\sqrt{a^2\ \cos^2\ x - 1}} = \frac{1}{a} F\left(\arcsin\left(\frac{a\ \sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a > 1].$$

2. $$\int \sqrt{a^2\ \cos^2\ x - 1}\, dx = aE\left(\arcsin\left(\frac{a\ \sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) - \frac{1}{a} F\left(\arcsin\left(\frac{a\ \sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a > 1].$$

2.616

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2\ \sin^2\ x}, \sqrt{1 - q^2\ \sin^2\ x})\, dx$.

Notation: $\alpha = \arcsin\left(\frac{\sqrt{1-p^2}\ \sin x}{\sqrt{1-p^2\ \sin^2\ x}}\right)$.

1. $$\int \frac{dx}{\sqrt{(1 - p^2\ \sin^2\ x)(1 - q^2\ \sin^2\ x)}} = \frac{1}{\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \qquad \left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right].$$

BY (284.00)

BY (284.07)

3. $$\int \frac{\operatorname{tg}^4 x\,dx}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}} =$$
$$= \frac{1}{3(1-q^2)^2(1-p^2)^{\frac{3}{2}}} \times \left[2(2-p^2-q^2)E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \left(1-q^2\right)F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right] +$$
$$+ \frac{2p^2+q^2-3+\sin^2 x(4-3p^2-2q^2+p^2q^2)}{3(1-p^2)(1-q^2)^2}\frac{\sin x}{\cos^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}}$$
$$\left[0<p^2<q^2<1, \quad 0<x\le\frac{\pi}{2}\right].$$

BY (284.07)

4. $$\int \frac{\sin^2 x\,dx}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)^3}} =$$
$$= \frac{\sqrt{1-p^2}}{(1-q^2)(q^2-p^2)}E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1}{(q^2-p^2)\sqrt{1-p^2}}F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) -$$
$$- \frac{\sin x\cos x}{(1-q^2)\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}} \qquad \left[0<p^2<q^2<1, \quad 0<x\le\frac{\pi}{2}\right].$$

BY (284.06)

5. $$\int \frac{\cos^2 x\,dx}{\sqrt{(1-p^2\sin^2 x)^3(1-q^2\sin^2 x)}} =$$
$$= \frac{\sqrt{1-p^2}}{q^2-p^2}E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1-q^2}{(q^2-p^2)\sqrt{1-p^2}}F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0<p^2<q^2<1, \quad 0<x\le\frac{\pi}{2}\right].$$

BY (284.05)

6. $$\int \frac{\cos^4 x\,dx}{\sqrt{(1-p^2\sin^2 x)^5(1-q^2\sin^2 x)}} =$$
$$= \frac{(1-p^2)^{\frac{3}{2}}}{3(q^2-p^2)^2}\left[\frac{(2+p^2-3q^2)(1-q^2)}{(1-p^2)^2}F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) +\right.$$
$$\left. +2\frac{2q^2-p^2-1}{1-p^2}E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right] + \frac{(1-p^2)\sin x\cos x\sqrt{1-q^2\sin^2 x}}{3(q^2-p^2)\sqrt{(1-p^2\sin^2 x)^3}}$$
$$\left[0<p^2<q^2<1, \quad 0<x\le\frac{\pi}{2}\right].$$

7. $$\int \frac{dx}{1-p^2\,\sin^2\,x}\sqrt{\frac{1-q^2\,\sin^2\,x}{1-p^2\,\sin^2}\,x} = \frac{1}{\sqrt{1-p^2}}E\left(\alpha,\,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$

BY (284.01)

8. $$\int \sqrt{\frac{1-p^2\,\sin^2\,x}{(1-q^2\,\sin^2\,x)^3}}\,dx = \frac{\sqrt{1-p^2}}{1-q^2}E\left(\alpha,\,\sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2}\frac{\sin x\,\cos x}{\sqrt{(1-p^2\,\sin^2\,x)(1-q^2\,\sin^2\,x)}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right].$$

BY (284.04)

9. $$\int \frac{dx}{1+(p^2r^2-p^2-r^2)\,\sin^2\,x}\sqrt{\frac{1-p^2\,\sin^2\,x}{1-q^2\,\sin^2 x}} = \frac{1}{\sqrt{1-p^2}}\Pi\left(\alpha,\,r^2,\,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right].$$

BY (284.02)

2.617

Notation: $\alpha = \arcsin\sqrt{\frac{\sqrt{b^2+c^2}-b\,\sin x-c\,\cos x}{2\sqrt{b^2+c^2}}}, \quad r = \sqrt{\frac{2\sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}$

1. $$\int \frac{dx}{\sqrt{a+b\,\sin x+c\,\cos x}} = -\frac{2}{\sqrt{a+\sqrt{b^2+c^2}}}F(\alpha,\,r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right];$$
$$= -\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}}F(\alpha,\,r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right].$$

BY (293.00)
BY (294.00)

2. $$\int \frac{\sin x\,dx}{\sqrt{a+b\,\sin x+c\,\cos x}} = -\frac{\sqrt{2b}}{\sqrt[4]{(b^2+c^2)^3}}\{2E(\alpha,\,r)-F(\alpha,\,r)\} +$$
$$+\frac{2c}{b^2+c^2}\sqrt{a+b\,\sin x+c\,\cos x}$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right].$$

3. $$\int \frac{(b\cos x - c\sin x)\,dx}{\sqrt{a + b\sin x + c\cos x}} = 2\sqrt{a + b\sin x + c\cos x}.$$

4. $$\int \frac{\sqrt{b^2+c^2} + b\sin x + c\cos x}{\sqrt{a + b\sin x + c\cos x}}\,dx =$$
$$= -2\sqrt{a + \sqrt{b^2+c^2}}\,E(\alpha,\, r) + \frac{2(a - \sqrt{b^2+c^2})}{\sqrt{a + \sqrt{b^2+c^2}}}F(\alpha,\, r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right];$$
$$= -2\sqrt{2}\,\sqrt[4]{b^2+c^2}\,E(\alpha,\, r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right].$$

BY (293.01)
BY (294.04)

5. $$\int \sqrt{a + b\sin x + c\cos x}\,dx = -2\sqrt{a + \sqrt{b^2+c^2}}\,E(\alpha,\, r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right];$$
$$= -2\sqrt{2}\,\sqrt[4]{b^2+c^2}\,E(\alpha,\, r) + \frac{\sqrt{2}(\sqrt{b^2+c^2} - a)}{\sqrt[4]{b^2+c^2}}F(\alpha,\, r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(\frac{-a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right].$$

BY (293.03)
BY (294.01)

2.618

Integrals of the form $\int R(\sin ax,\, \cos ax,\, \sqrt{\cos 2ax})\,dx = \frac{1}{a}\int R(\sin t, \cos t, \sqrt{12\sin^2 t})\,dt(t = ax)$. .

$$\text{Notation: } \alpha = \arcsin(\sqrt{2}\,\sin ax)$$

The integrals $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax})\,dx$ are special cases of the integrals 2.595. for $(p = 2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}}F\left(\alpha,\, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

2. $$\int \frac{\cos^2 ax}{\sqrt{\cos 2ax}}\,dx = \frac{1}{a\sqrt{2}}E\left(\alpha,\, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

3. $$\int \frac{dx}{\cos^2 ax\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a}E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{tg} x}{a}\sqrt{\cos 2ax} \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

4. $$\int \frac{dx}{\cos^4 ax\sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a}E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a}F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{(6\cos^2 ax + 1)\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{4}\right].$$

5. $$\int \frac{\operatorname{tg}^2 ax\, dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a}E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a}\operatorname{tg} ax\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{2}\right].$$

6. $$\int \frac{\operatorname{tg}^4 ax\, dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax} \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

7. $$\int \frac{dx}{(1 - 2r^2\sin^2 ax)\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}}\Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

8. $$\int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a}E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}} \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

9. $$\int \frac{\sin^2 ax\, dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}}E\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

10. $$\int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \qquad \left[0 < ax \le \frac{\pi}{4}\right].$$

11. $$\int \sqrt{\cos 2ax}\, dx = \frac{\sqrt{2}}{a}E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

12. $$\int \frac{\sqrt{\cos 2ax}}{\cos^2 ax}\, dx = \frac{\sqrt{2}}{a}\left\{F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\} + \frac{1}{a}\operatorname{tg} ax\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{4}\right].$$

2.619

Integrals of the form

$$\int R(\sin ax,\ \cos ax,\ \sqrt{-\cos 2ax})\,dx = \frac{1}{a}\int R(\sin\ x,\ \cos x,\ \sqrt{2\ \sin^2\ x-1})\,dx.$$

$$\text{Notation: } \alpha = \arcsin(\sqrt{2}\ \cos ax).$$

The integrals $\int R(\sin x, \cos x, \sqrt{2\ \sin^2\ x-1})\,dx$ are special cases of the integrals 2.599 and 2.611 for $(a=2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} F\left(\alpha,\ \frac{1}{\sqrt{2}}\right).$$

2. $$\int \frac{\cos^2\ ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[E\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - F\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right].$$

3. $$\int \frac{\cos^4\ ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}}\left[3F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - \frac{5}{2}E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right] - \frac{1}{12a}\ \sin 2ax\sqrt{-\cos 2ax}.$$

4. $$\int \frac{dx}{\sin^2\ ax\sqrt{-\cos 2ax}} = \frac{1}{a}\ \operatorname{ctg} ax\sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a}E\left(\alpha,\ \frac{1}{\sqrt{2}}\right).$$

5. $$\int \frac{dx}{\sin^4\ ax\sqrt{-\cos 2ax}} = \frac{2}{3a\sqrt{2}}\left[F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 6E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right] + \\ + \frac{1}{3a}\frac{\cos ax}{\sin^3\ ax}(6\ \sin^2\ ax+1)\sqrt{-\cos 2ax}.$$

6. $$\int \frac{\operatorname{ctg}^2\ ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{a}\ \operatorname{ctg} ax\sqrt{-\cos 2ax}.$$

7. $$\int \frac{dx}{(1-2r^2\ \cos^2\ ax)\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\Pi\left(\alpha,\ r^2,\ \frac{1}{\sqrt{2}}\right).$$

8. $$\int \frac{dx}{\sqrt{-\cos^3\ 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}.$$

9. $$\int \frac{\cos^2\ ax\,dx}{\sqrt{-\cos^3\ 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}}E\left(\alpha,\ \frac{1}{\sqrt{2}}\right).$$

10. $$\int \frac{dx}{\sqrt{-\cos^5\ 2ax}} = -\frac{1}{3a\sqrt{2}}F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3\ 2ax}}.$$

11. $$\int \sqrt{-\cos 2ax}\,dx = \frac{1}{a\sqrt{2}}\left[F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right].$$

Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax})\,dx$.

$$\text{Notation: } \alpha = \arcsin\sqrt{\frac{2\ \sin ax}{1+\sin ax+\cos ax}}.$$

2.621

1. $$\int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}F\left(\alpha,\ \frac{1}{\sqrt{2}}\right).$$

BY (287.50)

2. $$\int \frac{\sin ax\,dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{\frac{1+i}{2}\Pi\left(\alpha,\ \frac{1+i}{2},\ \frac{1}{\sqrt{2}}\right) + \right.$$
$$\left. + \frac{1-i}{2}\Pi\left(\alpha,\ \frac{1-i}{2},\ \frac{1}{\sqrt{2}}\right) + F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right\}.$$

BY (287.57)

BY (287.54)

4. $$\int \frac{\sin ax\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{\sqrt{\operatorname{tg} ax} - E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right\} \qquad \left[ax{\neq}\frac{\pi}{2}\right].$$

BY (287.55)

5. $$\int \frac{(1+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}E\left(\alpha,\ \frac{1}{\sqrt{2}}\right).$$

BY (287.51)

6. $$\int \frac{(1+\cos ax)\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - E\left(\alpha,\ \frac{1}{\sqrt{2}}\right) + \sqrt{\operatorname{tg} ax}\right\} \qquad \left[ax{\neq}\frac{\pi}{2}\right].$$

BY (287.56)

7. $$\int \frac{(1-\sin ax+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin\ 2ax}} = \frac{\sqrt{2}}{a}\left\{2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - F\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right\}.$$

BY (287.53)

8. $$\int \frac{(1+\sin ax+\cos ax)\,dx}{[1+\cos ax+(1-2r^2)\ \sin\ ax]\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\Pi\left(\alpha,\ r^2,\ \frac{1}{\sqrt{2}}\right).$$

BY (287.52)

2.63-2.65 Products of trigonometric functions and powers

2.631

1. $$\int x^r\ \sin^p\ x\ \cos^q\ x\,dx = \frac{1}{(p+q)^2}\Big[(p+q)x^r\ \sin^{p+1}\ x\ \cos^{q-1}\ x +$$
$$+rx^{r-1}\ \sin^p\ x\ \cos^q\ x - r(r-1)\int x^{r-2}\ \sin^p\ x\ \cos^q\ x\,dx -$$
$$-rp\int x^{r-1}\ \sin^{p-1} x\ \cos^{q-1}\ x\,dx + (q-1)(p+q)\int x^r\ \sin^p\ x\ \cos^{q-2}\ x\,dx\Big];$$
$$= \frac{1}{(p+q)^2}\Big[-(p+q)x^r\ \sin^{p-1}\ x\ \cos^{q+1}\ x +$$
$$+rx^{r-1}\ \sin^p\ x\ \cos^q\ x - r(r-1)\int x^{r-2}\ \sin^p\ x\ \cos^q\ x\,dx +$$
$$+rq\int x^{r-1}\ \sin^{p-1} x\ \cos^{q-1} x\,dx + (p-1)(p+q)\int x^r\ \sin^{p-2}\ x\ \cos^q\ x\,dx\Big].$$

2. $$\int x^m \sin^n x\,dx = \frac{x^{m-1}\sin^{n-1}x}{n^2}\{m\sin x - nx\cos x\} + \frac{n-1}{n}\int x^m \sin^{n-2}x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin^n x\,dx.$$

3. $$\int x^m \cos^n x\,dx = \frac{x^{m-1}\cos^{n-1}x}{n^2}\{m\cos x + nx\sin x\} + \frac{n-1}{n}\int x^m \cos^{n-2}x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos^n x\,dx.$$

4. $$\int x^n \sin^{2m} x\,dx = \binom{2m}{m}\frac{x^{n+1}}{2^{2m}(n+1)} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\int x^n \cos(2m-2k)x\,dx$$ (see **2.633** 2.).

2.633
TI 333

5. $$\int x^n \sin^{2m+1} x\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\int x^n \sin(2m-2k+1)x\,dx$$ (see **2.633** 1.).

2.633
TI 333

6. $$\int x^n \cos^{2m} x\,dx = \binom{2m}{m}\frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\int x^n \cos(2m-2k)x\,dx$$ (see 2.6332.).

2.633
TI 333

7. $$\int x^n \cos^{2m+1} x\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\int x^n \cos(2m-2k+1)x\,dx.$$ (see 2.6332.).

2.633
TI 333

2.632

1. $$\int x^{\mu-1}\sin\beta x\,dx = \frac{i}{2}(i\beta)^{-\mu}\gamma(\mu,\,i\beta x) - \frac{i}{2}(-i\beta)^{-\mu}\gamma(\mu,\,-i\beta x). \qquad [\operatorname{Re}\mu > -1, \quad x > 0].$$

ET I 317(2)

2. $$\int x^{\mu-1}\sin ax\,dx = -\frac{1}{2a^{\mu}}\left\{\exp\left[\frac{\pi i}{2}(\mu-1)\right]\Gamma(\mu,\,-iax) + \right.$$
$$\left. + \exp\left[\frac{\pi i}{2}(1-\mu)\right]\Gamma(\mu,\,iax)\right\} \qquad [\operatorname{Re}\mu < 1, \quad a > 0, \quad x > 0].$$

ET I 317(3)

3. $$\int x^{\mu-1}\cos\beta x\,dx = \frac{1}{2}\{(i\beta)^{-\mu}\gamma(\mu,\,i\beta x) + (-i\beta)^{-\mu}\gamma(\mu,\,-i\beta x)\}$$
$$[\operatorname{Re}\mu > 0, \quad x > 0].$$

ET I 319(22)

4. $$\int x^{\mu-1}\cos ax\,dx = -\frac{1}{2a^{\mu}}\left\{\exp\left(i\mu\frac{\pi}{2}\right)\Gamma(\mu,\,-iax) + \exp\left(-i\mu\frac{\pi}{2}\right)\Gamma(\mu,\,iax)\right\}.$$

ET I 319(23)

2.633

1. $$\int x^{n}\sin ax\,dx = -\sum_{k=0}^{n} k!\binom{n}{k}\frac{x^{n-k}}{a^{k+1}}\cos\left(ax + \frac{1}{2}k\pi\right).$$

TI (487)

2.[8] $$\int x^{n}\cos ax\,dx = \sum_{k=0}^{n} k!\binom{n}{k}\frac{x^{n-k}}{a^{k+1}}\sin\left(ax + \frac{1}{2}k\pi\right).$$

TI (486)

3. $$\int x^{2n}\sin x\,dx = (2n)!\left\{\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k}}{(2n-2k)!}\cos x + \sum_{k=0}^{n-1}(-1)^{k}\frac{x^{2n-2k-1}}{(2n-2k-1)!}\sin x\right\}.$$

4. $$\int x^{2n+1}\sin x\,dx = (2n+1)!\left\{\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + \sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right\}.$$

5. $$\int x^{2n} \cos x\,dx = (2n)!\left\{\sum_{k=0}^{n}(-1)^k\frac{x^{2n-2k}}{(2n-2k)!}\sin x+\sum_{k=0}^{n-1}(-1)^k\frac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right\}.$$

6. $$\int x^{2n+1} \cos x\,dx = (2n{+}1)!\left\{\sum_{k=0}^{n}(-1)^k\frac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x+\sum_{k=0}^{n}\frac{x^{2n-2k}}{(2n-2k)!}\cos x\right\}.$$

2.634

1. $$\int P_n(x)\sin mx\,dx = -\frac{\cos mx}{m}\sum_{k=0}^{[n/2]}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}}+\frac{\sin mx}{m}\sum_{k=1}^{[(n+1)/2]}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

2. $$\int P_n(x)\cos mx\,dx = \frac{\sin mx}{m}\sum_{k=0}^{[n/2]}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}}+\frac{\cos mx}{m}\sum_{k=1}^{[(n+1)/2]}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

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In formulas 2.634, $P_n(x)$ is an nth-degree polynomial and $P_n^{(k)}(x)$ is its kth derivative with respect to x.

Notation: $z_1 = A + Bx$.

2.635

1. $$\int z_1 \sin kx\,dx = -\frac{1}{k}z_1\cos kx+\frac{b}{k^2}\sin kx.$$

2. $$\int z_1 \cos kx\,dx = \frac{1}{k}z_1\sin kx+\frac{b}{k^2}\cos kx.$$

3. $$\int z_1^2 \sin kx\,dx = \frac{1}{k}\left(\frac{2b^2}{k^2}-z_1^2\right)\cos kx+\frac{2bz_1}{k^2}\sin kx.$$

4. $$\int z_1^2 \cos kx\,dx = \frac{1}{k}\left(z_1^2-\frac{2b^2}{k^2}\right)\sin kx+\frac{2bz_1}{k^2}\cos kx.$$

5. $$\int z_1^3 \sin kx\,dx = \frac{z_1}{k}\left(\frac{6b^2}{k^2} - z_1^2\right)\cos kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\sin kx.$$

6. $$\int z_1^3 \cos kx\,dx = \frac{z_1}{k}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\cos kx.$$

7. $$\int z_1^4 \sin kx\,dx = -\frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx.$$

8. $$\int z_1^4 \cos kx\,dx = \frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\cos kx.$$

9. $$\int z_1^5 \sin kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx - \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx.$$

10. $$\int z_1^5 \cos kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx.$$

11. $$\int z_1^6 \sin kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx - $$
$$- \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\cos kx.$$

12. $$\int z_1^6 \cos kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx + $$
$$+ \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\sin kx.$$

2.636

1. $$\int x^n \sin^2 x\,dx = \frac{x^{n+1}}{2(n+1)} + $$
$$+\frac{n!}{4}\left\{\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}.$$

2. $$\int x^n \cos^2 x\,dx = \frac{x^{n+1}}{2(n+1)} - \\ -\frac{n!}{4}\left\{\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}.$$

GU ((333))(3e)

3. $$\int x \sin^2 x\,dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x.$$

4. $$\int x^2 \sin^2 x\,dx = \frac{x^3}{6} - \frac{x}{4}\cos 2x - \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x.$$

MZ 241

5. $$\int x \cos^2 x\,dx = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x.$$

6. $$\int x^2 \cos^2 x\,dx = \frac{x^3}{6} + \frac{x}{4}\cos 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x.$$

MZ 245

2.637

1. $$\int x^n \sin^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\cos 3x}{3^{2k+1}} - 3\cos x\right) - \\ - \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\sin 3x}{2^{2k+2}} - 3\sin x\right)\right\}.$$

GU((333))(2f)

2. $$\int x^n \cos^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\sin 3x}{3^{2k+1}} + 3\sin x\right) + \\ + \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\cos 3x}{3^{2k+2}} + 3\cos x\right)\right\}.$$

GU((333))(3f)

3. $$\int x \sin^3 x\,dx = \frac{3}{4}\sin x - \frac{1}{36}\sin 3x - \frac{3}{4}x\cos x + \frac{x}{12}\cos 3x$$

4. $$\int x^2 \sin^3 x\, dx = -\left(\frac{3}{4}x^2 + \frac{3}{2}\right) \cos x + \left(\frac{x^2}{12} + \frac{1}{54}\right) \cos 3x + \frac{3}{2}x \sin x - \frac{x}{18} \sin 3x.$$

MZ 241

229

5. $$\int x \cos^3 x\, dx = \frac{3}{4} \cos x + \frac{1}{36} \cos 3x + \frac{3}{4}x \sin x + \frac{x}{12} \sin 3x.$$

6. $$\int x^2 \cos^3 x\, dx = \left(\frac{3}{4}x^2 - \frac{3}{2}\right) \sin x + \left(\frac{x^2}{12} - \frac{1}{54}\right) \sin 3x + \frac{3}{2}x \cos x + \frac{x}{18} \cos 3x.$$

MZ 245, 246

2.638

1. $$\int \frac{\sin^q x}{x^p}\, dx = -\frac{\sin^{q-1} x[(p-2) \sin x + qx \cos x]}{(p-1)(p-2)x^{p-1}} - \\ - \frac{q^2}{(p-1)(p-2)} \int \frac{\sin^q x\, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sin^{q-2} x\, dx}{x^{p-2}} \\ [p \neq 1, \quad p \neq 2].$$

TI (496)

2. $$\int \frac{\cos^q x}{x^p}\, dx = -\frac{\cos^{q-1} x[(p-2) \cos x - qx \sin x]}{(p-1)(p-2)x^{p-1}} - \\ - \frac{q^2}{(p-1)(p-2)} \int \frac{\cos^q x\, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cos^{q-2} x\, dx}{x^{p-2}} \\ [p \neq 1, \quad p \neq 2].$$

TI (495)

3.[6] $$\int \frac{\sin x\, dx}{x^p} = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{\cos x\, dx}{x^{p-1}}; \\ = -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\sin x\, dx}{x^{p-2}} \qquad (p > 2).$$

TI (492)

4.[6] $$\int \frac{\cos x\, dx}{x^p} = -\frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1} \int \frac{\sin x\, dx}{x^{p-1}}; \\ = -\frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\cos x\, dx}{x^{p-2}} \qquad (p > 2).$$

2.639

1. $$\int \frac{\sin x\, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \cos x + \right.$$
$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{x^{2k}} \sin x\right\} + \frac{(-1)^{n+1}}{(2n-1)!} \operatorname{ci}(x).$$

GU ((333))(6b)a

2. $$\int \frac{\sin x}{x^{2n+1}} dx = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{x^{2k}} \cos x + \right.$$
$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}} \sin x\right\} + \frac{(-1)^n}{(2n)!} \operatorname{si}(x).$$

GU ((333))(6b)a

3. $$\int \frac{\cos x\, dx}{x^{2n}}\, dx = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{x^{2k}} \cos x - \right.$$
$$\left. - \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \sin x\right\} + \frac{(-1)^n}{(2n-1)!} \operatorname{si}(x).$$

GU ((333))(7b)

4. $$\int \frac{\cos x\, dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}} \cos x - \right.$$
$$\left. - \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{x^{2k}} \sin x\right\} + \frac{(-1)^n}{(2n)!} \operatorname{ci}(x).$$

GU ((333))(7b)

2.641

1. $$\int \frac{\sin kx}{a+bx}\, dx = \frac{1}{b}\left[\cos\frac{ka}{b} \operatorname{si}(u) - \sin\frac{ka}{b} \operatorname{ci}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right].$$

2. $$\int \frac{\cos kx}{a+bx}\, dx = \frac{1}{b}\left[\cos\frac{ka}{b} \operatorname{ci}(u) + \sin\frac{ka}{b} \operatorname{si}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right].$$

3. $$\int \frac{\sin kx}{(a+bx)^2}\, dx = -\frac{1}{b}\frac{\sin kx}{a+bx} + \frac{k}{b}\int \frac{\cos kx}{a+bx}\, dx \qquad (\text{see } 2.6412.).$$

2.641

4. $$\int \frac{\cos kx}{(a+bx)^2}\,dx = -\frac{1}{b}\frac{\cos kx}{a+bx} - \frac{k}{b}\int \frac{\sin kx}{a+bx}\,dx \qquad \text{(see 2.6411.)}.$$

2.641

5. $$\int \frac{\sin kx}{(a+bx)^3}\,dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k\ \cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2}\int \frac{\sin kx}{a+bx}\,dx \qquad \text{(see 2.6411.)}.$$

2.641

6. $$\int \frac{\cos kx}{(a+bx)^3}\,dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k\ \sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2}\int \frac{\cos kx}{a+bx}\,dx \qquad \text{(see 2.6412.)}.$$

2.641

7. $$\int \frac{\sin kx}{(a+bx)^4}\,dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k\ \cos kx}{6b^2(a+bx)^2} + \\ + \frac{k^2\ \sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3}\int \frac{\cos kx}{a+bx}\,dx \qquad \text{(see 2.6412.)}.$$

2.641

8. $$\int \frac{\cos kx}{(a+bx)^4}\,dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k\ \sin kx}{6b^2(a+bx)^2} + \\ + \frac{k^2\ \cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int \frac{\sin kx}{a+bx}\,dx \qquad \text{(see 2.6411.)}.$$

2.641

9. $$\int \frac{\sin kx}{(a+bx)^5}\,dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k\ \cos kx}{12b^2(a+bx)^3} + \\ + \frac{k^2\ \sin kx}{24b^3(a+bx)^2} + \frac{k^3\ \cos kx}{24b^4(a+bx)}\ \frac{k^4}{24b^4}\int \frac{\sin kx}{a+bx}\,dx \qquad \text{(see 2.6411.)}.$$

10. $$\int \frac{\cos kx}{(a+bx)^5}\,dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k\,\sin kx}{12b^2(a+bx)^3} + \frac{k^2\,\cos kx}{24b^3(a+bx)^2} - \frac{k^3\,\sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int \frac{\cos kx}{a+bx}\,dx \qquad \text{(see 2.6412.)}.$$

2.641

11. $$\int \frac{\sin kx}{(a+bx)^6}\,dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k\,\cos kx}{20b^2(a+bx)^4} + \frac{k^2\,\sin kx}{60b^3(a+bx)^3} + \frac{k^3\,\cos kx}{120b^4(a+bx)^2} - \frac{k^4\,\sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int \frac{\cos kx}{a+bx}\,dx \qquad \text{(see 2.6412.)}.$$

2.641

12. $$\int \frac{\cos kx}{(a+bx)^6}\,dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k\,\sin kx}{20b^2(a+bx)^4} + \frac{k^2\,\cos kx}{60b^3(a+bx)^3} - \frac{k^3\,\sin kx}{120b^4(a+bx)^2} - \frac{k^4\,\cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5}\int \frac{\sin kx}{a+bx}\,dx \qquad \text{(see 2.6411.)}.$$

2.641

2.642

1. $$\int \frac{\sin^{2m} x}{x}\,dx = \binom{2m}{m}\frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\operatorname{ci}[(2m-2k)x].$$

2. $$\int \frac{\sin^{2m+1} x}{x}\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\operatorname{si}[(2m-2k+1)x].$$

3. $$\int \frac{\cos^{2m} x}{x}\,dx = \binom{2m}{m}\frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\operatorname{ci}[(2m-2k)x].$$

4. $$\int \frac{\cos^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\operatorname{ci}[(2m-2k+1)x].$$

5. $$\int \frac{\sin^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\operatorname{si}[(2m-2k)x]\right\}.$$

6. $$\int \frac{\sin^{2m+1} x}{x^2}\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+1}\binom{2m+1}{k}\times \left\{\frac{\sin(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{ci}[(2m-2k+1)x]\right\}.$$

7. $$\int \frac{\cos^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} - \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\operatorname{si}[(2m-2k)x]\right\}.$$

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8. $$\int \frac{\cos^{2m+1} x}{x^2} = -\frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\left\{\frac{\cos(2m-2k+1)x}{x} + (2m-2k+1)\operatorname{si}[(2m-2k+1)x]\right\}.$$

2.643

1. $$\int \frac{x^p\,dx}{\sin^q x} = -\frac{x^{p-1}[p\sin x + (q-2)x\cos x]}{(q-1)(q-2)\sin^{q-1} x} + \frac{q-2}{q-1}\int\frac{x^p\,dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}\,dx}{\sin^{q-2} x}.$$

2. $$\int \frac{x^p\,dx}{\cos^q x} = -\frac{x^{p-1}[p\cos x - (q-2)x\sin x]}{(q-1)(q-2)\cos^{q-1} x} + \frac{q-2}{q-1}\int\frac{x^p\,dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}\,dx}{\cos^{q-2} x}.$$

3.[4] $$\int \frac{x^n}{\sin x}\,dx = \frac{x^n}{n} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2(2^{2k-1}-1)}{(n+2k)(2k)!}B_{2k}x^{n+2k} \qquad [|x| < \pi, \quad n > 0].$$

4. $$\int \frac{dx}{x^n \sin x} = -\frac{1}{nx^n} - [1+(-1)^n](-1)^{\frac{n}{2}} \frac{2^{n-1}-1}{n!} B_n \ln x - - \sum_{\substack{k=1 \\ k \neq \frac{n}{2}}}^{\infty} (-1)^k \frac{2(2^{2n}-1)}{(2k-n)\cdot(2k)!} B_{2k} x^{2k-n} \qquad [n>1, \quad |x|>\pi].$$

GU ((333))(9b)

5.[8] $$\int \frac{x^n\, dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{n+2k+1}}{(n+2k+1)(2k)!} \qquad \left[|x| < \frac{\pi}{2}, \quad n>0\right].$$

GU ((333))(10b)

6. $$\int \frac{dx}{x^n \cos x} = \frac{1}{2}[1-(-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{|E_{2k}| x^{2k-n+1}}{(2k-n+1)\cdot(2k)!} \qquad \left[|x| < \frac{\pi}{2}\right].$$

GU ((333))(11b)

7. $$\int \frac{x^n\, dx}{\sin^2 x} = -x^n \operatorname{ctg} x + \frac{n}{n-1} x^{n-1} + + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{n+2k-1}}{(n+2k-1)(2k)!} B_{2k} \qquad [|x| < \pi, \quad n>1].$$

GU ((333))(8c)

8. $$\int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} - -[1-(-1)^n](-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \frac{n}{2^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \qquad [|x| < \pi].$$

GU ((333))(9c)

9. $$\int \frac{x^n\, dx}{\cos^2 x} = x^n \operatorname{tg} x + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}(2^{2k}-1) x^{n+2k-1}}{(n+2k-1)\cdot(2k)!} B_{2k} \qquad \left[n>1, \quad |x| < \frac{\pi}{2}\right].$$

GU ((333))(10c)

10. $$\int \frac{dx}{x^n \cos^2 x} = \frac{\operatorname{tg} x}{x^n} - [1-(-1)^n](-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} (2^{n+1}-1) B_{n+1} \ln x - - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2^{2k}-1)(2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \qquad \left[|x| < \frac{\pi}{2}\right].$$

2.644

1. $$\int \frac{x\,dx}{\sin^{2n}\ x} = -\sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{\sin x + (2n-2k)x\ \cos x}{(2n-2k+1)(2n-2k)\ \sin^{2n-2k+1}\ x} + \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!}(\ln\sin x - x\ \operatorname{ctg} x).$$

2. $$\int \frac{x\,dx}{\sin^{2n+1}\ x} = -\sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{\sin x + (2n-2k-1)x\ \cos x}{(2n-2k)(2n-2k-1)\ \sin^{2n-2k}\ x} + \\ + \frac{(2n-1)!!}{2^n n!}\int \frac{x\,dx}{\sin x} \qquad \text{(see 2.6445.)}.$$

3. $$\int \frac{x\,dx}{\cos^{2n}\ x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{(2n-2k)x\ \sin x - \cos x}{(2n-2k+1)(2n-2k)\ \cos^{2n-2k+1}\ x} + \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!}(x\ \operatorname{tg} x + \ln\cos x).$$

4. $$\int \frac{x\,dx}{\cos^{2n+1}\ x} = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{(2n-2k+1)x\ \sin x - \cos x}{(2n-2k)(2n-2k-1)\ \cos^{2n-2k}\ x} + \\ + \frac{(2n-1)!!}{2^n n!}\int \frac{x\,dx}{\cos x} \qquad \text{(see 2.6446.)}.$$

5. $$\int \frac{x\,dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(2k+1)!} B_{2k} x^{2k+1}.$$

6. $$\int \frac{x\,dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}.$$

7. $$\int \frac{x\,dx}{\sin^2\ x} = -x\ \operatorname{ctg} x + \ln\sin x.$$

8. $\int \frac{x\,dx}{\cos^2 x} = x \operatorname{tg} x + \ln \cos x.$

9. $\int \frac{x\,dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x}{\sin x}\,dx$ (see 2.6445.).

10. $\int \frac{x\,dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x\,dx}{\cos x}$ (see 2.6446.).

11. $\int \frac{x\,dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \operatorname{ctg} x + \frac{2}{3} \ln(\sin x).$

12. $\int \frac{x\,dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \operatorname{tg} x - \frac{2}{3} \ln(\cos x).$

13. $\int \frac{x\,dx}{\sin^5 x} = -\frac{x \cos x}{4 \sin^4 x} - \frac{1}{12 \sin^3 x} - \frac{3x \cos x}{8 \sin^2 x} - \frac{3}{8 \sin x} + \frac{3}{8} \int \frac{x\,dx}{\sin x}$ (see 2.6445.).

14. $\int \frac{x\,dx}{\cos^5 x} = \frac{x \sin x}{4 \cos^4 x} - \frac{1}{12 \cos^3 x} + \frac{3x \sin x}{8 \cos^2 x} - \frac{3}{8 \cos x} + \frac{3}{8} \int \frac{x\,dx}{\cos x}$ (see 2.6446.).

2.645

1. $\int x^p \frac{\sin^{2m} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\cos^{n-2k} x}$ (see 2.6432.).

2.643

2. $$\int x^p \frac{\sin^{2m+1} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x}\,dx \qquad \text{(see 2.6453.)}.$$

2.645

3. $$\int x^p \frac{\sin x\,dx}{\cos^n x} = \frac{x^p}{(n-1)\cos^{n-1} x} - \frac{p}{n-1}\int \frac{x^{p-1}}{\cos^{n-1} x}\,dx$$
$$[n>1] \qquad \text{(see } \mathbf{2.643}\ 2.).$$

2.643
GU ((333))(12)

4. $$\int x^p \frac{\cos^{2m} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\sin^{n-2k} x} \qquad \text{(see 2.6431.)}.$$

2.643

5. $$\int x^p \frac{\cos^{2m+1} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x}\,dx \qquad \text{(see 2.6456.)}.$$

2.645

6. $$\int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1)\sin^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}\,dx}{\sin^{n-1} x}$$
$$[n>1] \qquad \text{(see } \mathbf{2.643}\ 1.).$$

2.643
GU ((333))(13)

7. $$\int \frac{x \cos x}{\sin^2 x}\,dx = -\frac{x}{\sin x} + \ln \operatorname{tg} \frac{x}{2}.$$

8. $\displaystyle\int \frac{x \sin x}{\cos^2 x}\, dx = \frac{x}{\cos x} - \ln \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right).$

2.646

1. $$\int x^p \operatorname{tg} x\, dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k}(2^{2k-1} - 1)}{(p + 2k)\cdot(2k)!} B_{2k} x^{p+2k}$$
$$\left[p \geq -1, \quad |x| < \frac{\pi}{2}\right].$$

GU ((333))(12d)

235

2. $$\int x^p \operatorname{ctg} x\, dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p + 2k)(2k)!} x^{p+2k} \qquad [p \geq 1, \quad |x| < \pi].$$

GU ((333))(13d)

3. $$\int x^p \operatorname{tg}^2 x\, dx = x \operatorname{tg} x + \ln \cos x - \frac{x^2}{2}.$$

4. $$\int x \operatorname{ctg}^2 x\, dx = -x \operatorname{ctg} x + \ln \sin x - \frac{x^2}{2}.$$

2.647

1. $$\int \frac{x^n \cos x\, dx}{(a + b \sin x)^m} = -\frac{x^n}{(m-1)b(a + b \sin x)^{m-1}} + \\ + \frac{n}{(m-1)b} \int \frac{x^{n-1}\, dx}{(a + b \sin x)^{m-1}} \qquad [m \neq 1].$$

MZ 247

2. $$\int \frac{x^n \sin x\, dx}{(a + b \cos x)^m} = \frac{x^n}{(m-1)b(a + b \cos x)^{m-1}} - \\ - \frac{n}{(m-1)b} \int \frac{x^{n-1}\, dx}{(a + b \cos x)^{m-1}} \qquad [m \neq 1].$$

MZ 247

3. $$\int \frac{x\, dx}{1 + \sin x} = -x \operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right) + 2 \ln \cos\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

4. $$\int \frac{x\,dx}{1-\sin x} = x\ \operatorname{ctg}\left(\frac{\pi}{4}-\frac{x}{2}\right) + 2\ln\sin\ \left(\frac{\pi}{4}-\frac{x}{2}\right).$$

PE (330)

5. $$\int \frac{x\,dx}{1+\cos x} = x\ \operatorname{tg}\frac{x}{2} + 2\ln\cos\frac{x}{2}.$$

PE (331)

6. $$\int \frac{x\,dx}{1-\cos x} = -x\ \operatorname{ctg}\frac{x}{2} + 2\ln\cos\frac{x}{2}.$$

PE (332)

7. $$\int \frac{x\ \cos x}{(1+\sin x)^2}\,dx = -\frac{x}{1+\sin x} + \operatorname{tg}\left(\frac{x}{2}-\frac{\pi}{4}\right).$$

8. $$\int \frac{x\ \cos x}{(1-\sin x)^2}\,dx = \frac{x}{1-\sin x} + \operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right).$$

9. $$\int \frac{x\ \sin x}{(1+\cos x)^2}\,dx = \frac{x}{1+\cos x} - \operatorname{tg}\frac{x}{2}.$$

10. $$\int \frac{x\ \sin x}{(1-\cos x)^2}\,dx = -\frac{x}{1-\cos x} - \operatorname{ctg}\frac{x}{2}.$$

MZ 247a

2.648

1. $$\int \frac{x+\sin x}{1+\cos x}\,dx = x\ \operatorname{tg}\frac{x}{2}.$$

2. $$\int \frac{x-\sin x}{1-\cos x}\,dx = -x\ \operatorname{ctg}\frac{x}{2}.$$

$$\int \frac{x^2 dx}{[(ax-b)\,\sin x+(a+bx)\,\cos x]^2}=\frac{x\,\sin x+\cos x}{b[(ax-b)\,\sin x+(a+bx)\,\cos x]}.$$

GU ((333))(17)

2.651

$$\int \frac{dx}{[a+(ax+b)\,\operatorname{tg} x]^2}=\frac{\operatorname{tg} x}{a[a+(ax+b)\,\operatorname{tg} x]}.$$

GU ((333))(18)

2.652

$$\int \frac{x\,dx}{\cos(x+t)\,\cos(x-t)}=\operatorname{cosec} 2t\left\{x\,\ln\frac{\cos(x-t)}{\cos(x+t)}-L(x+t)+L(x-t)\right\}$$
$$\left[t\neq n\pi;\quad |x|<\left|\frac{\pi}{2}-|t_0|\right|\right],$$

where t_0 is the value of the argument t, which is reduced by multiples of the argument π to lie in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

2.653

1. $\displaystyle\int \frac{\sin x}{\sqrt{x}}\,dx=\sqrt{2\pi}S(\sqrt{x})\qquad$ (cf.8.2511.).

8.251
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2. $\displaystyle\int \frac{\cos x}{\sqrt{x}}\,dx=\sqrt{2\pi}C(\sqrt{x})\qquad$ (cf.8.2512.).

8.251

2.654

Notation: $\Delta=\sqrt{1-k^2\,\sin^2\,x},\quad k'=\sqrt{1-k^2}$:

1. $\displaystyle\int \frac{x\,\sin x\,\cos x}{\Delta}\,dx=-\frac{x\Delta}{k^2}+\frac{1}{k^2}E(x,\,k).$

2. $\displaystyle\int \frac{x\,\sin^3 x\,\cos x}{\Delta}\,dx = -\frac{k'^2}{9k^4}F(x,k)+\frac{2k^2+5}{9k^4}E(x,k)-\frac{1}{9k^4}[3(3-\Delta^2)x+k^2\sin x\cos x]\Delta.$

3. $\displaystyle\int \frac{x\,\sin x\,\cos^3 x}{\Delta}\,dx = -\frac{k'^2}{9k^4}F(x,k)+\frac{7k^2-5}{9k^4}E(x,k)-\frac{1}{9k^4}[3(\Delta^2-3k'^2)x-k^2\sin x\cos x]\Delta.$

4. $\displaystyle\int \frac{x\,\sin x\,dx}{\Delta^3}\,dx = -\frac{x\,\cos x}{k'^2\Delta}+\frac{1}{kk'^2}\arcsin(k\sin x).$

5. $\displaystyle\int \frac{x\,\cos x\,dx}{\Delta^3} = \frac{x\,\sin x}{\Delta}+\frac{1}{k}\ln(k\cos x+\Delta).$

6. $\displaystyle\int \frac{x\,\sin x\,\cos x\,dx}{\Delta^3} = \frac{x}{k^2\Delta}-\frac{1}{k^2}F(x,k).$

7. $\displaystyle\int \frac{x\,\sin^3 x\,\cos x\,dx}{\Delta^3} = x\frac{2-k^2\sin^2 x}{k^4\Delta}-\frac{1}{k^4}[E(x,k)+F(x,k)].$

8. $\displaystyle\int \frac{x\,\sin x\,\cos^3 x\,dx}{\Delta^3} = x\frac{k^2\sin^2 x+k^2-2}{k^4\Delta}+\frac{k'^2}{k^4}F(x,k)+\frac{1}{k^4}E(x,k).$

Integrals containing $\sin x^2$ and $\cos x^2$

In integrals containing $\sin x^2$ and $\cos x^2$, it is expedient to make the substitution $x^2 = u$.

2.655

1. $\displaystyle\int x^p\sin x^2\,dx = -\frac{x^{p-1}}{2}\cos x^2+\frac{p-1}{2}\int x^{p-2}\cos x^2\,dx.$

2. $\displaystyle\int x^p\cos x^2\,dx = \frac{x^{p-1}}{2}\sin x^2-\frac{p-1}{2}\int x^{p-2}\sin x^2\,dx.$

3. $$\int x^n \sin x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^k\left[\frac{x^{n-4k+3}\cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1}\sin x^2}{2^{2k}(n-4k+1)!!}\right] + \right.$$
$$\left. + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\sin x^2\,dx\right\} \qquad \left[r = \left[\frac{n}{4}\right]\right].$$

GU ((336))(4a)

4. $$\int x^n \cos x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^{k-1}\left[\frac{x^{n-4k+3}\sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1}\cos x^2}{2^{2k}(n-4k+1)!!}\right] + \right.$$
$$\left. + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\cos x^2\,dx\right\} \qquad \left[r = \left[\frac{n}{4}\right]\right].$$

GU ((336))(5a)

5. $$\int x \sin x^2\,dx = -\frac{\cos x^2}{2}.$$

6. $$\int x \cos x^2\,dx = -\frac{\sin x^2}{2}.$$

7. $$\int x^2 \sin x^2\,dx = -\frac{x}{2}\cos x^2 + \frac{1}{2}\sqrt{\frac{\pi}{2}}C(x).$$

8. $$\int x^2 \cos x^2\,dx = \frac{x}{2}\sin x^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}S(x).$$

9. $$\int x^3 \sin x^2\,dx = -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2.$$

10. $$\int x^3 \cos x^2\,dx = \frac{x^2}{2}\sin x^2 + \frac{1}{2}\cos x^2.$$

2.66 Combinations of trigonometric functions and exponentials

2.661

$$\int e^{ax}\sin^p x\cos^q x\,dx =$$
$$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^p x\cos^{q-1}x[a\cos x + (p+q)\sin x] - \right.$$

$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] + \right.$$
$$\left. + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x \, dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\};$$
$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - p \cos^2 x] + \right.$$
$$\left. + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x \, dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\};$$
$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + \right.$$
$$+ q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x \, dx -$$
$$\left. - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x \, dx \right\};$$
$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + m \right.$$
$$+ p(p-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x \, dx +$$
$$\left. + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^{q-2} x \, dx \right\}.$$

GU ((334))(1a)
TI (526)
TI (525)
TI (524)

For $p = m$ and $q = n$ even integers, the integral $\int e^{ax} \sin^m x \cos^n x \, dx$ can be reduced by means of these formulas to the integral $\int e^{ax} \, dx$. However, when only m or only n is even, they can be reduced to integrals of the form $\int e^{ax} \cos^n x \, dx$ or $\int e^{ax} \sin^m x \, dx$ respectively.

2.662

1. $$\int e^{ax} \sin^n bx \, dx = \frac{1}{a^2 + n^2 b^2} \left[(a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + \right.$$
$$\left. + n(n-1) b^2 \int e^{ax} \sin^{n-2} bx \, dx \right].$$

2. $$\int e^{ax} \cos^n bx \, dx = \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + \right.$$
$$\left. + n(n-1) b^2 \int e^{ax} \cos^{n-2} bx \, dx \right].$$

3. $\int e^{ax} \sin^{2m} bx\, dx =$

$$= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \sin^{2m-2k-1} bx}{(2m-2k)![a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+(2m-2k)^2b^2]} \times$$

$$\times [a \sin bx - (2m-2k) b \cos bx] + \frac{(2m)! b^{2m} e^{ax}}{[a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+4b^2]a} =$$

$$= \binom{2m}{m} \frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^{m} (-1)^k \binom{2m}{m-k} \frac{1}{a^2+4b^2k^2} (a \cos 2bkx + 2bk \sin 2bkx).$$

4. $\int e^{ax} \sin^{2m+1} bx\, dx =$

$$= \sum_{k=0}^{m} \frac{(2m+1)! b^{2k} e^{ax} \sin^{2m-2k} bx [a \sin bx - (2m-2k+1) b \cos bx]}{(2m-2k+1)![a^2+(2m+1)^2b^2][a^2+(2m-1)^2b^2]\dots[a^2+(2m-2k+1)^2b^2]} =$$

$$= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^{m} \frac{(-1)^k}{a^2+(2k+1)^2b^2} \binom{2m+1}{m-k} [a \sin(2k+1)bx - (2k+1) b \cos(2k+1)bx].$$

5.[8] $\int e^{ax} \cos^{2m} bx\, dx =$

$$= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \cos^{2m-2k-1} bx [a \cos bx + (2m-2k) b \sin bx]}{(2m-2k)![a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+(2m-2k)^2b^2]} +$$

$$+ \frac{(2m)! b^{2m} e^{ax}}{[a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+4b^2]a}$$

$$= \binom{2m}{m} \frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^{m} \binom{2m}{m-k} \frac{1}{a^2+4b^2k^2} [a \cos 2kbx + 2kb \sin 2kbx].$$

6. $\int e^{ax} \cos^{2m+1} bx\, dx =$

$$= \sum_{k=0}^{m} \frac{(2m+1)! b^{2k} e^{ax} \cos^{2m-2k} bx}{(2m-2k+1)![a^2+(2m-1)^2b^2]\dots[a^2+(2m-2k+1)^2b^2]} =$$

$$= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{m-k} \frac{1}{a^2+(2k+1)^2b^2} [a \cos(2k+1)bx + (2k+1) b \sin(2k+1)\, bx].$$

2.663

1. $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}.$

2.664

1. $$\int e^{ax} \sin bx \cos cx \, dx = \frac{e^{ax}}{2} \left[\frac{a \sin(b+c)x - (b+c) \cos(b+c)x}{a^2 + (b+c)^2} + \right.$$
$$\left. + \frac{a \sin(b-c)x - (b-c) \cos(b-c)x}{a^2 + (b-c)^2} \right].$$

GU ((334))(6b)

2. $$\int e^{ax} \sin^2 bx \cos cx \, dx = \frac{e^{ax}}{4} \left[2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos(2b+c)x + (2b+c) \sin(2b+c)x}{a^2 + (2b+c)^2} - \right.$$
$$\left. - \frac{a \cos(2b-c)x + (2b-c) \sin(2b-c)x}{a^2 + (2b-c)^2} \right].$$

GU ((334))(6c)

3. $$\int e^{ax} \sin bx \cos^2 cx \, dx = \frac{e^{ax}}{4} \left[2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin(b+2c)x - (b+2c) \cos(b+2c)x}{a^2 + (b+2c)^2} + \right.$$
$$\left. + \frac{a \sin(b-2c)x - (b-2c) \cos(b-2c)x}{a^2 + (b-2c)^2} \right].$$

GU ((334))(6d)

2.665

1. $$\int \frac{e^{ax} \, dx}{\sin^p bx} = - \frac{e^{ax}[a \sin bx + (p-2)b \cos bx]}{(p-1)(p-2)b^2 \sin^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\sin^{p-2} bx}.$$

TI (530)a

2. $$\int \frac{e^{ax} \, dx}{\cos^p bx} = - \frac{e^{ax}[a \cos bx - (p-2)b \sin bx]}{(p-1)(p-2)b^2 \cos^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\cos^{p-2} bx}.$$

TI (529)a

By successive applications of formulas 2.665 for p a natural number, we obtain integrals of the form

$\int \frac{e^{ax}\,dx}{\sin bx}$, $\int \frac{e^{ax}\,dx}{\sin^2 bx}$, $\int \frac{e^{ax}\,dx}{\cos bx}$, $\int \frac{e^{ax}\,dx}{\cos^2 bx}$, which are not expressible in terms of a finite combination of elementary functions.

1. $$\int e^{ax}\ \operatorname{tg}^p\ x\,dx = \frac{e^{ax}}{p-1}\ \operatorname{tg}^{p-1}\ x - \frac{a}{p-1}\int e^{ax}\ \operatorname{tg}^{p-1}\ x\,dx - \int e^{ax}\ \operatorname{tg}^{p-2}\ x\,dx.$$

TI (527)

2. $$\int e^{ax}\ \operatorname{ctg}^p\ x\,dx = -\frac{e^{ax}\ \operatorname{ctg}^{p-1} x}{p-1} + \frac{a}{p-1}\int e^{ax}\ \operatorname{ctg}^{p-1}\ x\,dx - \int e^{ax}\ \operatorname{ctg}^{p-2}\ x\,dx.$$

TI (528)

3. $$\int e^{ax}\ \operatorname{tg} x\,dx = \frac{e^{ax}\ \operatorname{tg} x}{a} - \frac{1}{a}\int \frac{e^{ax}\,dx}{\cos^2\ x} \qquad \text{(see remark following 2.665).}$$

2.665

4. $$\int e^{ax}\ \operatorname{tg}^2\ x\,dx = \frac{e^{ax}}{a}(a\ \operatorname{tg} x - 1) - a\int e^{ax}\ \operatorname{tg} x\,dx \qquad \text{(see 2.6663.).}$$

2.666

TI 355

5. $$\int e^{ax}\ \operatorname{ctg} x\,dx = \frac{e^{ax}\ \operatorname{ctg} x}{a} + \frac{1}{a}\int \frac{e^{ax}\,dx}{\sin^2\ x} \qquad \text{(see remark following 2.665).}$$

2.665

6. $$\int e^{ax}\ \operatorname{ctg}^2\ x\,dx = -\frac{e^{ax}}{a}(a\ \operatorname{ctg} x + 1) + a\int e^{ax}\ \operatorname{ctg} x\,dx \qquad \text{(see 2.6665.).}$$

2.666

Integrals of type $\int R(x, e^{ax}, \sin bx, \cos cx)\,dx$

Notation: $\sin t = -\frac{b}{\sqrt{a^2+b^2}}$; $\quad cost = \frac{a}{\sqrt{a^2+b^2}}$.

$$\sin t = -\tfrac{b}{\sqrt{a^2+b^2}}; \quad cost = \tfrac{a}{\sqrt{a^2+b^2}}.$$

2.667

1. $$\int x^p e^{ax} \sin bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a \sin bx - b \cos bx)\,dx;$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}} \sin(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax} \sin(bx+t)\,dx.$$

2. $$\int x^p e^{ax} \cos bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a \cos bx + b \sin bx)\,dx;$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}} \cos(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax} \cos(bx+t)\,dx.$$

3. $$\int x^n e^{ax} \sin bx\,dx = e^{ax}\sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}} \sin(bx+kt).$$

4. $$\int x^n e^{ax} \cos bx\,dx = e^{ax}\sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}} \cos(bx+kt).$$

5. $$\int x e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right) \sin bx - \left(bx - \frac{2ab}{a^2+b^2}\right) \cos bx\right].$$

6. $$\int x e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right) \cos bx + \left(bx - \frac{2ab}{a^2+b^2}\right) \sin bx\right].$$

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7. $$\int x^2 e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left\{\left[ax^2 - \frac{2(a^2-b^2)}{a^2+b^2}x + \frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right] \sin bx - \right.$$
$$\left. - \left[bx^2 - \frac{4ab}{a^2+b^2}x + \frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right] \cos bx\right\}.$$

8. $$\int x^2 e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left\{\left[ax^2 - \frac{2(a^2-b^2)}{a^2+b^2}x + \frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right] \cos bx + \right.$$
$$\left. + \left[bx^2 - \frac{4ab}{a^2+b^2}x + \frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right] \sin bx\right\}.$$

2.67 Combinations of trigonometric and hyperbolic functions

2.671

1. $$\int \operatorname{sh}(ax+b)\,\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\,\operatorname{ch}(ax+b)\,\sin(cx+d) - \\ - \frac{c}{a^2+c^2}\operatorname{sh}(ax+b)\,\cos(cx+d).$$

2. $$\int \operatorname{sh}(ax+b)\,\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\,\operatorname{ch}(ax+b)\,\cos(cx+d) + \\ + \frac{c}{a^2+c^2}\operatorname{sh}(ax+b)\,\sin(cx+d).$$

3. $$\int \operatorname{ch}(ax+b)\,\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\,\operatorname{sh}(ax+b)\,\sin(cx+d) - \\ - \frac{c}{a^2+c^2}\,\operatorname{ch}(ax+b)\,\cos(cx+d).$$

4. $$\int \operatorname{ch}(ax+b)\,\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\,\operatorname{sh}(ax+b)\,\cos(cx+d) + \\ + \frac{c}{a^2+c^2}\,\operatorname{ch}(ax+b)\,\sin(cx+d).$$

GU ((354))(1)

2.672

1. $$\int \operatorname{sh} x\,\sin x\,dx = \frac{1}{2}(\operatorname{ch} x\,\sin x - \operatorname{sh} x\,\cos x).$$

2. $$\int \operatorname{sh} x\,\cos x\,dx = \frac{1}{2}(\operatorname{ch} x\,\cos x + \operatorname{sh} x\,\sin x).$$

3. $$\int \operatorname{ch} x\,\sin x\,dx = \frac{1}{2}(\operatorname{sh} x\,\sin x - \operatorname{ch} x\,\cos x).$$

4. $$\int \operatorname{ch} x\,\cos x\,dx = \frac{1}{2}(\operatorname{sh} x\,\cos x + \operatorname{ch} x\,\sin x).$$

2.673

1. $$\int \operatorname{sh}^{2m}(ax+b)\,\sin^{2n}(cx+d)\,dx =$$
$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{(-1)^{m+n}}{2^{2m+2n-1}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k)c}\binom{2n}{k}\,\sin[(2n-2k)(cx+d)] +$$
$$+ \frac{(-1)^n}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j)a\,\operatorname{sh}[(2m-2j)(ax+b)]\,\cos[(2n-2k)(cx+d)] +$$
$$+(2-2k)c\,\operatorname{ch}[(2m-2j)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(3a)

2. $$\int \operatorname{sh}^{2m}(ax+b)\,\sin^{2n-1}(cx+d)\,dx =$$
$$= \frac{(-1)^{m+n}}{2^{2m+2n-2}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k-1)c}\binom{2n-1}{k}\,\cos[(2n-2k-1)(cx+d)] +$$
$$+ \frac{(-1)^{n-1}}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}\times$$
$$\times\{(2m-2j)a\,\operatorname{sh}[(2m-2j)(ax+b)]\,\sin[(2n-2k-1)(cx+d)] -$$
$$-(2n-2k-1)c\,\operatorname{ch}[(2m-2j)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

GU ((354))(3b)

3. $$\int \operatorname{sh}^{2m-1}(ax+b)\,\sin^{2n}(cx+d)\,dx =$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\,\operatorname{ch}[(2m-2j-1)(ax+d)] +$$
$$+ \frac{(-1)^n}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j-1)a\,\operatorname{ch}[(2m-2j-1)(ax+b)]\,\cos[(2n-2k)(cx+d)] +$$
$$+(2n-2k)c\,\operatorname{sh}[(2m-2j-1)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(3c)

4. $$\int \operatorname{sh}^{2m-1}(ax+b)\,\sin^{2n-1}(cx+d)\,dx =$$
$$= \frac{(-1)^{n-1}}{2^{2m-2n-4}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}\times$$
$$\times\{(2m-2j-1)a\,\operatorname{ch}[(2m-2j-1)(ax+b)]\,\sin[(2n-2k-1)(cx+d)] -$$
$$-(2n-2k-1)c\,\operatorname{sh}[(2m-2j-1)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

5. $$\int \operatorname{sh}^{2m}(ax+b)\cos^{2n}(cx+d)\,dx =$$

$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{\binom{2n}{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m}{j}}{(2m-2j)a}\operatorname{sh}[(2m-2j)(ax+b)] +$$
$$+ \frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{\binom{2n}{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)] +$$
$$+ \frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j)a\operatorname{sh}[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)] +$$
$$+ (2-2k)c\operatorname{ch}[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(4a)

6. $$\int \operatorname{sh}^{2m}(ax+b)\cos^{2n-1}(cx+d)\,dx =$$

$$= \frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\sin[(2n-2k-2)(cx+d)] +$$
$$+ \frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}\times$$
$$\times\{(2m-2j)a\operatorname{sh}[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)] +$$
$$+ (2n-2k-1)c\operatorname{ch}[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}.$$

GU ((354))(4a)

7. $$\int \operatorname{sh}^{2m-1}(ax+b)\cos^{2n}(cx+d)\,dx =$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\operatorname{ch}[(2m-2j-1)(ax+d)] +$$
$$+ \frac{1}{2^{2m-2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j-1)a\operatorname{ch}[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)] +$$
$$+ (2n-2k)c\operatorname{sh}[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(4b)

8. $$\int \operatorname{sh}^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx =$$

$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \times$$
$$\times \{(2m-2j-1)a \ \text{ch}[(2m-2j-1)(ax+b)] \ \cos[(2n-2k-1)(cx+d)] +$$
$$+ (2n-2k-1)c \ \text{sh}[(2m-2j-1)(ax+b)] \ \sin[(2n-2k-1)(cx+d)]\}.$$

GU ((354))(4b)

9. $$\int \text{ch}^{2m}(ax+b) \ \sin^{2n}(cx+d) \, dx =$$
$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \ \sin[(2n-2k)(cx+d)] +$$
$$+ \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \ \text{sh}[(2m-2j)(ax+b)] +$$
$$+ \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times$$
$$\times \{(2m-2j)a \ \text{sh}[(2m-2j)(ax+b)] \ \cos[(2n-2k)(cx+d)] +$$
$$+ (2n-2k)c \ \text{ch}[(2m-2j)(ax+b)] \ \sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(5a)

10. $$\int \text{ch}^{2m-1}(ax+b) \ \sin^{2n}(cx+d) \, dx =$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \ \text{sh}[(2m-2j-1)(ax+b)] +$$
$$+ \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \times$$
$$\times \{(2m-2j-1)a \ \text{sh}[(2m-2j-1)(ax+b)] \ \cos[(2n-2k)(cx+d)] +$$
$$+ (2n-2k)c \ \text{ch}[(2m-2j-1)(ax+b)] \ \sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(5a)

11. $$\int \text{ch}^{2m}(ax+b) \ \sin^{2n-1}(cx+d) \, dx =$$
$$= \frac{(-1)^{n-1} \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1} \binom{2n-1}{k}}{(2n-2k-1)c} \ \cos[(2n-2k-1)(cx+d)] +$$
$$+ \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \times$$
$$\times \{(2m-2j)a \ \text{sh}[(2m-2j)(ax+b)] \ \sin[(2n-2k-1)(cx+d)] -$$
$$- (2n-2k-1)c \ \text{ch}[(2m-2j)(ax+b)] \ \cos[(2n-2k-1)(cx+d)]\}.$$

12. $\displaystyle\int \operatorname{ch}^{2m-1}(ax+b)\,\sin^{2n-1}(cx+d)\,dx =$

$$= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2} \times$$
$$\times \{(2m-2j-1)a\ \operatorname{sh}[(2m-2j-1)(ax+b)]\ \sin[(2n-2k-1)(cx+d)] -$$
$$-(2n-2k-1)c\ \operatorname{ch}[(2m-2j-1)(ax+b)]\ \cos[(2n-2k-1)(cx+d)]\}.$$

GU ((354))(5b)

13. $\displaystyle\int \operatorname{ch}^{2m}(ax+b)\,\cos^{2n}(cx+d)\,dx =$

$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2}{k}}{(2n-2k)c}\ \sin[(2n-2k)(cx+d)] +$$
$$+ \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a}\ \operatorname{sh}[(2m-2j)(ax+b)] +$$
$$+ \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2} \times$$
$$\times \{(2m-2j)a\ \operatorname{sh}[(2m-2j)(ax+b)]\ \cos[(2n-2k)(cx+d)] +$$
$$+(2n-2k)c\ \operatorname{ch}[(2m-2j)(ax+b)]\ \sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(6)

14. $\displaystyle\int \operatorname{ch}^{2m-1}(ax+b)\,\cos^{2n}(cx+d)\,dx =$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a}\ \operatorname{sh}[(2m-2j-1)(ax+b)] +$$
$$+ \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2} \times$$
$$\times \{(2m-2j-1)a\ \operatorname{sh}[(2m-2j-1)(ax+b)]\ \cos[(2n-2k)(cx+d)] +$$
$$+(2n-2k)c\ \operatorname{ch}[(2m-2j-1)(ax+b)]\ \sin[(2n-2k)(cx+d)]\}.$$

GU ((354))(6)

15. $\displaystyle\int \operatorname{ch}^{2m}(ax+b)\,\cos^{2n-1}(cx+d)\,dx =$

$$= \frac{\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c}\ \sin[(2n-2k-1)(cx+d)] +$$
$$+ \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2} \times$$
$$\times \{(2m-2j)a\ \operatorname{sh}[(2m-2j)(ax+b)]\ \cos[(2n-2k-1)(cx+d)] +$$
$$+(2n-2k-1)c\ \operatorname{ch}[(2m-2j)(ax+b)]\ \sin[(2n-2k-1)(cx+d)]\}.$$

16. $$\int \operatorname{ch}^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx =$$
$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2} \times$$
$$\times \{(2m-2j-1)a \operatorname{sh}[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]+$$
$$+(2n-2k-1)c \operatorname{ch}[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}.$$

GU ((354))(6)

2.674

1. $$\int e^{ax}\operatorname{sh} bx \sin cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\sin cx - c\cos cx] - \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\sin cx - c\cos cx].$$

2. $$\int e^{ax}\operatorname{sh} bx \cos cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\cos cx + c\sin cx] - \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\cos cx + c\sin cx].$$

3. $$\int e^{ax}\operatorname{ch} bx \sin cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\sin cx - c\cos cx] + \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\sin cx - c\cos cx].$$

4. $$\int e^{ax}\operatorname{ch} bx \cos cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\cos cx + c\sin cx] + \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\cos cx + c\sin cx].$$

MZ 379

2.7 Logarithms and Inverse-Hyperbolic Functions

2.71 The logarithm

2.711

$$\int \ln^m x\,dx = x\ln^m x - m\int \ln^{m-1} x\,dx =$$
$$= \frac{x}{m+1}\sum_{k=0}^{m}(-1)^k(m+1)m(m+1)\ldots(m-k+1)\ln^{m-k} x \qquad (m>0).$$

2.72-2.73 Combinations of logarithms and algebraic functions

2.721

1. $$\int x^n \ln^m x\,dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x\,dx \qquad (\text{see } 2.722).$$

2.722

For $n = -1$

2. $$\int \frac{\ln^m x\,dx}{x} = \frac{\ln^{m+1} x}{m+1}.$$

For $n = -1$ and $m = -1$

3. $$\int \frac{dx}{x \ln x} = \ln(\ln x).$$

2.722

$$\int x^n \ln^m x\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}(-1)^k (m+1)m(m-1)\ldots(m-k+1)\frac{\ln^{m-k} x}{(n+1)^{k+1}}.$$

TI (604)

2.723

1. $$\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right].$$

TI 375

2. $$\int x^n \ln^2 x\,dx = x^{n+1}\left[\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right].$$

TI 375

2.724

1. $$\int \frac{x^n\,dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\ln x)^{m-1}}.$$

For $m = 1$

2. $$\int \frac{x^n\,dx}{\ln x} = \mathrm{li}\,(x^{n+1}).$$

2.725

1. $$\int (a+bx)^m \,\ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1}\,\ln x - \int \frac{(a+bx)^{m+1}\,dx}{x}\right].$$

TI 374

2. $$\int (a+bx)^m \,\ln x\,dx = \frac{1}{(m+1)b}[(a+bx)^{m+1} - a^{m+1}]\,\ln x - \sum_{k=0}^{m} \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}.$$

For $m = -1$ see 2.727 2.

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2.726

1. $$\int (a+bx)\,\ln x\,dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b}\right]\ln x - \left(ax + \frac{1}{4}bx^2\right).$$

2. $$\int (a+bx)^2\,\ln x\,dx = \frac{1}{3b}[(a+bx)^3 - a^3]\,\ln x - \left(a^2x + \frac{abx^2}{2} + \frac{b^2x^3}{9}\right).$$

3. $$\int (a+bx)^3\,\ln x\,dx = \frac{1}{4b}[(a+bx)^4 - a^4]\,\ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4\right).$$

2.727

1. $$\int \frac{\mathrm{l}\,x\,dx}{(a+bx)^m} = \frac{1}{b(m-1)}\left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right].$$

TI 376

For $m = 1$

2. $$\int \frac{\ln x\,dx}{a+bx} = \frac{1}{b}\ \ln x\ \ln(a+bx) - \frac{1}{b}\int \frac{\ln(a+bx)\,dx}{x} \qquad \text{(see 2.7282.)}.$$

2.728

3. $$\int \frac{\ln x\,dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab}\ \ln \frac{x}{a+bx}.$$

4. $$\int \frac{\ln x\,dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b}\ \ln \frac{x}{a+bx}.$$

5. $$\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + \sqrt{a}\ \ln \frac{\sqrt{a+bx}+\sqrt{a}}{\sqrt{a+bx}-\sqrt{a}}\right\} \qquad [a > 0];$$
$$= \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a}\ \operatorname{arctg}\sqrt{\frac{a+bx}{-a}}\right\} \qquad [a < 0].$$

2.728

1. $$\int x^m\ \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1}\ \ln(a+bx) - b\int \frac{x^{m+1}\,dx}{a+bx}\right].$$

2.[6] $$\int \frac{\ln(a+bx)}{x} = \ln a + \ln x + \frac{bx}{a}\Phi\left(-\frac{bx}{a},\ 2,\ 1\right) \qquad [a > 0].$$

2.729

1. $$\int x^m\ \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}}\right]\ \ln(a+bx) +$$
$$+ \frac{1}{m+1}\sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^{k-1}}.$$

2. $$\int x \ln(a+bx)\,dx = \frac{1}{2}\left[x^2 - \frac{a^2}{b^2}\right] \ln(a+bx) - \frac{1}{2}\left[\frac{x^2}{2} - \frac{ax}{b}\right].$$

3. $$\int x^2 \ln(a+bx)\,dx = \frac{1}{3}\left[x^3 + \frac{a^3}{b^3}\right] \ln(a+bx) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right].$$

4. $$\int x^3 \ln(a+bx)\,dx = \frac{1}{4}\left[x^4 - \frac{a^4}{b^4}\right] \ln(a+bx) - \frac{1}{4}\left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3}\right].$$

2.731

$$\frac{1}{2n+1}\left\{x^{2n+1} \ln(x^2+a^2) + (-1)^n 2a^{2n+1} \operatorname{arctg} \frac{x}{a} - \right.$$
$$\left. - 2\sum_{k=0}^{n} \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1}\right\}.$$

2.732

$$\frac{1}{2n+1}\left\{(x^{2n+2} + (-1)^n a^{2n+2}) \ln(x^2+a^2) + \right.$$
$$\left. + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} a^{2n-2k+2} x^{2k}\right\}.$$

2.733

1. $$\int \ln(x^2+a^2)\,dx = x \ln(x^2+a^2) - 2x + 2a \operatorname{arctg} \frac{x}{a}.$$

DW

2. $$\int x \ln(x^2+a^2)\,dx = \frac{1}{2}[(x^2+a^2) \ln(x^2+a^2) - x^2].$$

DW

3. $$\int x^2 \ln(x^2+a^2)\,dx = \frac{1}{3}\left[x^3 \ln(x^2+a^2) - \frac{2}{3}x^3 + 2a^2x - 2a^3 \operatorname{arctg} \frac{x}{a}\right].$$

4. $\int x^3 \ln(x^2+a^2)\,dx = \frac{1}{4}\left[(x^4-a^4)\ln(x^2+a^2) - \frac{x^4}{2} + a^2x^2\right]$.

DW

5. $\int x^4 \ln(x^2+a^2)\,dx = \frac{1}{5}\left[x^5 \ln(x^2+a^2) - \frac{2}{5}x^5 + \frac{2}{3}a^2x^3 - 2a^4x + 2a^5 \operatorname{arctg}\frac{x}{a}\right]$.

DW

2.734

$$\int x^{2n} \ln|x^2-a^2|\,dx = \frac{1}{2n+1}\left\{x^{2n+1} \ln|x^2-a^2| + a^{2n+1} \ln\left|\frac{x+a}{x-a}\right| - 2\sum_{k=0}^{n}\frac{1}{2k+1}a^{2n-2k}x^{2k+1}\right\} .$$

2.735

$$\int x^{2n+1} \ln|x^2-a^2|\,dx = \frac{1}{2n+2}\left\{(x^{2n+2}-a^{2n+2})\ln|x^2-a^2| - \sum_{k=1}^{n+1}\frac{1}{k}a^{2n-2k+2}x^{2k}\right\} .$$

2.736

1. $\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2| - 2x + a\ln\left|\frac{x+a}{x-a}\right|$.

DW

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2. $\int x\ln|x^2-a^2|\,dx = \frac{1}{2}\{(x^2-a^2)\ln|x^2-a^2| - x^2\}$.

DW

3. $\int x^2 \ln|x^2-a^2|\,dx = \frac{1}{3}\left\{x^3 \ln|x^2-a^2| - \frac{2}{3}x^3 - 2a^2x + a^3 \ln\left|\frac{x+a}{x-a}\right|\right\}$.

DW

4. $\int x^3 \ln|x^2-a^2|\,dx = \frac{1}{4}\left\{(x^4-a^4)\ln|x^2-a^2| - \frac{x^4}{2} - a^2x^2\right\}$.

5. $$\int x^4 \ln|x^2-a^2|\,dx = \frac{1}{5}\left\{x^5 \ln|x^2-a^2| - \frac{2}{5}x^5 - \frac{2}{3}a^2x^3 - 2a^4x + a^5 \ln\left|\frac{x+a}{x-a}\right|\right\}.$$

DW

2.74 Inverse hyperbolic functions

2.741

1. $$\int \operatorname{Arsh}\frac{x}{a}\,dx = x\operatorname{Arsh}\frac{x}{a} - \sqrt{x^2+a^2}.$$

DW

2. $$\int \operatorname{Arch}\frac{x}{a}\,dx = x\operatorname{Arch}\frac{x}{a} - \sqrt{x^2-a^2}. \qquad \left[\operatorname{Arch}\frac{x}{a} > 0\right];$$
$$= x\operatorname{Arch}\frac{x}{a} + \sqrt{x^2-a^2}. \qquad \left[\operatorname{Arch}\frac{x}{a} < 0\right].$$

DW
DW

3. $$\int \operatorname{Arth}\frac{x}{a}\,dx = x\operatorname{Arth}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2).$$

DW

4. $$\int \operatorname{Arcth}\frac{x}{a}\,dx = x\operatorname{Arcth}\frac{x}{a} + \frac{a}{2}\ln(x^2-a^2).$$

DW

2.742

1. $$\int x\operatorname{Arsh}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\operatorname{Arsh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}.$$

DW

2. $$\int x\operatorname{Arch}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{Arch}\frac{x}{a} - \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{Arch}\frac{x}{a} > 0\right];$$
$$= \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{Arch}\frac{x}{a} + \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{Arch}\frac{x}{a} < 0\right].$$

DW

2.8 Inverse Trigonometric Functions

2.81 Arcsines and arccosines

2.811

$$\int \left(\arcsin\frac{x}{a}\right)^n dx = x\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arcsin\frac{x}{a}\right)^{n-2k} + \\ + \sqrt{a^2-x^2}\sum_{k=1}^{[(n+1)/2]} (-1)^{k-1}\binom{n}{2k-1} \cdot (2k-1)! \left(\arcsin\frac{x}{a}\right)^{n-2k+1}.$$

252

2.812

$$\int \left(\arccos\frac{x}{a}\right)^n dx = x\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arccos\frac{x}{a}\right)^{n-2k} + \\ + \sqrt{a^2-x^2}\sum_{k=1}^{[(n+1)/2]} (-1)^{k}\binom{n}{2k-1} \cdot (2k-1)! \left(\arccos\frac{x}{a}\right)^{n-2k+1}.$$

2.813

1.* $$\int \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[n \arcsin\frac{x}{|a|} + \sqrt{a^2-x^2}\right].$$

2.* $$\int \left(\arcsin\frac{x}{a}\right)^2 dx = x\left(\arcsin\frac{x}{|a|}\right)^2 + 2\sqrt{a^2-x^2}\arcsin\frac{x}{|a|} - 2x.$$

3.* $$\int \left(\arcsin\frac{x}{a}\right)^3 dx = \operatorname{sign}(a)\left[x\left(\arcsin\frac{x}{|a|}\right)^3 + 3\sqrt{a^2-x^2}\left(\arcsin\frac{x}{|a|}\right)^2 - \right. \\ \left. - 6x\arcsin\frac{x}{|a|} - 6\sqrt{a^2-x^2}\right].$$

2.814

1. $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}.$$

2. $\int \left(\arccos\frac{x}{a}\right)^2 dx = x\left(\arccos\frac{x}{a}\right)^2 - 2\sqrt{a^2-x^2}\ \arccos\frac{x}{a} - 2x.$

3. $\int \left(\arccos\frac{x}{a}\right)^3 dx = x\left(\arccos\frac{x}{a}\right)^3 - 3\sqrt{a^2-x^2}\left(\arccos\frac{x}{a}\right)^2 - 6x\ \arccos\frac{x}{a} + 6\sqrt{a^2-x^2}.$

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.821

1. $$\int \operatorname{arccosec}\frac{x}{a}\,dx = \int \arcsin\frac{a}{x}\,dx = x\ \arcsin\frac{x}{2} + a\ \ln(x+\sqrt{x^2-a^2}) \qquad \left[0 < \arcsin\frac{a}{x} < \frac{\pi}{2}\right];$$
$$= x\ \arcsin\frac{a}{x} - a\ \ln(x+\sqrt{x^2-a^2}) \qquad \left[-\frac{\pi}{2} < \arcsin\frac{a}{x} < 0\right].$$

DW

2. $$\int \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = x\ \arccos\frac{a}{x} - a\ \ln(x+\sqrt{x^2-a^2}) \qquad \left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right];$$
$$= x\ \arccos\frac{a}{x} - a\ \ln(x+\sqrt{x^2-a^2}) \qquad \left[-\frac{\pi}{2} < \arccos\frac{a}{x} < 0\right].$$

DW

2.822

1. $\int \operatorname{arctg}\frac{x}{q}\,dx = x\ \operatorname{arctg}\frac{x}{a} - \frac{a}{2}\ \ln(a^2+x^2).$

DW

2. $\int \operatorname{arcctg}\frac{x}{a}\,dx = x\ \operatorname{arcctg}\frac{x}{a} - \frac{a}{2}\ \ln(a^2+x^2).$

DW

3.* $\int x\ \operatorname{arctg}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arctg}\frac{x}{a} - \frac{ax}{2}$

4.* $\int x\ \operatorname{arcctg}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2}(x^2+a^2)\ \operatorname{arctg}\frac{x}{a}$

5.* $\int x^2 \operatorname{arctg} \frac{x}{a}\, dx = \frac{1}{3}x^3 \operatorname{arctg} \frac{x}{a} + \frac{1}{6}a^3 \ln(x^2+a^2) - \frac{ax^2}{6}$

6.* $\int x^2 \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{3}x^3 \operatorname{arctg} \frac{x}{a} - \frac{1}{6}a^3 \ln(x^2+a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$

2.83 Combinations of arcsine or arccosine and algebraic functions

2.831

$$\int x^n \arcsin \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{a^2-x^2}} \text{(see } \mathbf{2.263}\ 1.,\ \mathbf{2.264},\ \mathbf{2.27}).$$

2.27

2.264

2.263

2.832

$$\int x^n \arccos \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \arccos \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{a^2-x^2}} \text{(see } \mathbf{2.263}\ 1.,\ \mathbf{2.264},\ \mathbf{2.27}).$$

2.27

2.264

2.263

1. For $n=-1$, these integrals (that is, $\int \frac{\arcsin x}{x}\, dx$ and $\int \frac{\arccos x}{x}\, dx$) cannot be expressed as a finite combination of elementary functions.

2. $\int \frac{\arccos x}{x}\, dx = -\frac{\pi}{2} \ln \frac{1}{x} - \int \frac{\arcsin x}{x}\, dx.$

1. $$\int x \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{|a|}+\frac{x}{4}\sqrt{a^2-x^2}\right].$$

2. $$\int x \arccos\frac{x}{a}\,dx = \frac{\pi x^2}{4}-\operatorname{sign}(a)\left[\frac{1}{4}(2x^2-a^2)\arcsin\frac{x}{|a|}+\frac{x}{4}\sqrt{a^2-x^2}\right].$$

3. $$\int x^2 \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|}+\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}\right].$$

4. $$\int x^2 \arccos\frac{x}{a}\,dx = \frac{\pi x^3}{6}-\operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|}+\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}\right].$$

5. $$\int x^3 \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^4}{4}-\frac{3a^4}{32}\right)\arcsin\frac{x}{|a|}+\frac{1}{32}x(2x^2+3a^2)\sqrt{a^2-x^2}\right].$$

6. $$\int x^3 \arccos\frac{x}{a}\,dx = \frac{\pi x^4}{8}-\operatorname{sign}(a)\left[\frac{(8x^4-3a^4)}{32}\arcsin\frac{x}{|a|}+\frac{1}{32}x(2x^2+3a^2)\sqrt{a^2-x^2}\right].$$

2.834

1. $$\int \frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a}-\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$$

2. $$\int \frac{1}{x^2}\arccos\frac{x}{a}\,dx = -\frac{1}{x}\arccos\frac{x}{a}-\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$$

2.835

$$\int \frac{\arcsin x}{(a+bx)^2}\,dx = -\frac{\arcsin x}{b(a+bx)}-\frac{2}{b\sqrt{a^2-b^2}}\operatorname{arctg}\sqrt{\frac{(a-b)(1-x)}{(a+b)(1+x)}} \qquad [a^2>b^2];$$

$$= -\frac{\arcsin x}{b(a+bx)}-\frac{1}{b\sqrt{b^2-a^2}}\ln\frac{\sqrt{(a+b)(1+x)}+\sqrt{(b-a)(1-x)}}{\sqrt{(a+b)(1+x)}-\sqrt{(b-a)(1-x)}} \qquad [a^2<b^2].$$

$$\int \frac{x \arcsin x}{(1+cx^2)^2}\,dx = -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{2c\sqrt{c+1}} \operatorname{arctg} \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \qquad [c > -1];$$
$$= -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \qquad [c < -1].$$

2.837

1. $$\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx = x - \sqrt{1-x^2} \arcsin x.$$

2. $$\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^2}{4} - \frac{x}{2}\sqrt{1-x^2} \arcsin x + \frac{1}{4} (\arcsin x)^2.$$

3. $$\int \frac{x^3 \arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3}(x^2+2)\sqrt{1-x^2} \arcsin x.$$

2.838

1. $$\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}}\,dx = \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln(1-x^2).$$

2. $$\int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}}\,dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1-x}{1+x}.$$

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.841

1. $$\int x \operatorname{arcsec} \frac{x}{a}\,dx = \int \arccos \frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arccos \frac{a}{x} - a\sqrt{x^2-a^2}\right\} \qquad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2}\right];$$
$$= \frac{1}{2}\left\{x^2 \arccos \frac{a}{x} + a\sqrt{x^2-a^2}\right\} \qquad \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi\right].$$

DW

2. $$\int x^2 \operatorname{arcsec} \frac{x}{a}\,dx = \int \arccos \frac{a}{x}\,dx = \frac{1}{3}\left\{x^3 \arccos \frac{a}{x} - \frac{a}{2}x\sqrt{x^2-a^2} - \frac{a^3}{2} \ln(x + \sqrt{x^2-a^2})\right\}$$
$$\left[0 < \arccos \frac{a}{x} < \frac{\pi}{2}\right];$$
$$= \frac{1}{3}\left\{x^3 \arccos \frac{a}{x} + \frac{a}{2}x\sqrt{x^2-a^2} + \frac{a^3}{2} \ln(x + \sqrt{x^2-z^2})\right\}$$
$$\left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi\right].$$

3. $\int x \operatorname{arccosec} \frac{x}{a}\, dx = \int \arcsin \frac{a}{x}\, dx = \frac{1}{2}\left\{x^2 \arcsin \frac{a}{x} + a\sqrt{x^2 - a^2}\right\} \qquad \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2}\right];$

$$= \frac{1}{2}\left\{x^2 \arcsin \frac{a}{x} - a\sqrt{x^2 - a^2}\right\} \qquad \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0\right].$$

DW

2.85 Combinations of the arctangent and arccotangent with algebraic functions

2.851

$$\int x^n \operatorname{arctg} \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \operatorname{arctg} \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{a^2 + x^2}.$$

2.852

1. $\int x^n \operatorname{arcctg} \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \operatorname{arcctg} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{a^2 + x^2}.$

For $n = -1$

$\int \frac{\operatorname{arctg} x}{x}\, dx$cannot be expressed as a finite combination of elementary functions.

2. $\int \frac{\operatorname{arcctg} x}{x}\, dx = \frac{\pi}{2} \ln x - \int \frac{\operatorname{arctg} x}{x}\, dx.$

2.853

1. $\int x \operatorname{arctg} \frac{x}{a}\, dx = \frac{1}{2}(x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{ax}{2}.$

2. $\int x \operatorname{arcctg} \frac{x}{a}\, dx = \frac{1}{2}(x^2 + a^2) \operatorname{arcctg} \frac{x}{a} + \frac{ax}{2}.$

3.* $\int x^2 \operatorname{arctg} \frac{x}{a}\, dx = \frac{x^3}{3} \operatorname{arctg} \frac{x}{a} + \frac{a^3}{6} \ln(x^2 + a^2) - \frac{ax^2}{6}.$

2.854

$$\int \frac{1}{x^2} \operatorname{arctg} \frac{x}{a}\, dx = -\frac{1}{x} \operatorname{arctg} \frac{x}{a} - \frac{1}{2a} \ln \frac{a^2+x^2}{x^2}.$$

2.855

$$\int \frac{\operatorname{arctg} x}{(\alpha+\beta x)^2}\, dx = \frac{1}{\alpha^2+\beta^2} \left\{ \ln \frac{\alpha+\beta x}{\sqrt{1+x^2}} - \frac{\beta-\alpha x}{\alpha+\beta x} \operatorname{arctg} x \right\}.$$

2.856

1. $$\int \frac{x \operatorname{arctg} x}{1+x^2}\, dx = \frac{1}{2} \operatorname{arctg} x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)\, dx}{1+x^2}.$$

TI (689)

2. $$\int \frac{x^2 \operatorname{arctg} x}{1+x^2}\, dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\operatorname{arctg} x)^2.$$

TI (405)

3. $$\int \frac{x^3 \operatorname{arctg} x}{1+x^2}\, dx = -\frac{1}{2} x + \frac{1}{2} (1+x^2) \operatorname{arctg} x - \int \frac{x \operatorname{arctg} x}{1+x^2}\, dx. \qquad \text{(see 2.851.)}$$

2.851

4. $$\int \frac{x^4 \operatorname{arctg} x}{1+x^2}\, dx = -\frac{1}{6} x^2 + \frac{2}{3} \ln(1+x^2) + \left(\frac{x^3}{3} - x \right) \operatorname{arctg} x + \frac{1}{2} (\operatorname{arctg} x)^2.$$

2.857

$$\int \frac{\operatorname{arctg} x\, dx}{(1+x^2)^{n+1}} = \left[\sum_{k=1}^{n} \frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!} \frac{x}{(1+x^2)^{n-k+1}} + \frac{1}{2} \frac{(2n-1)!!}{(2)!!} \operatorname{arctg} x \right] \operatorname{arctg} x + \\ + \frac{1}{2} \sum_{k=1}^{n} \frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)} \frac{1}{(1+x^2)^{n-k+1}}.$$

$$\int \frac{x \operatorname{arctg} x}{\sqrt{1-x^2}}\, dx = -\sqrt{1-x^2}\ \operatorname{arctg} x + \sqrt{2}\ \operatorname{arctg} \frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x.$$

2.859

$$\int \frac{\operatorname{arctg} x}{\sqrt{(a+bx^2)^3}}\, dx = \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}}\ \operatorname{arctg} \sqrt{\frac{a+bx^2}{b-a}} \qquad [a<b];$$
$$= \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}}\ \ln \frac{\sqrt{a+bx^2}-\sqrt{a-b}}{\sqrt{a+bx^2}+\sqrt{a-b}} \qquad [a>b].$$

3.-4. Definite Integrals of Elementary Functions

3.0 Introduction

* We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

3.01 Theorems of a general nature

3.011

Suppose that $f(x)$ is integrable** A function $f(x)$ is said to be integrable over the interval (p, q), if the integral $\int_p^q f(x)\, dx$ exists. Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

over the largest of the intervals $(p,\ q),\ (p,\ r),\ (r,\ q)$. Then (depending on the relative positions of the points p, q, and r) it is also integrable over the other two intervals and we have

$$\int_p^q f(x)\, dx = \int_p^r f(x)\, dx + \int_r^q f(x)\, dx.$$

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3.012

The first mean-value theorem. Suppose (1) that $f(x)$ is continuous and that $g(x)$ is integrable over the interval $(p,\ q)$, (2) that $m \leq f(x) \leq M$

and (3) that $g(x)$ does not change sign anywhere in the interval $(p,\ q)$. Then, there exists at least one point $\xi(p \le \xi \le q)$ such that

$$\int_p^q f(x)g(x)\,dx = f(\xi)\int_p^q g(x)\,dx.$$

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3.013

The second mean-value theorem. If $f(x)$ is monotonic and non-negative throughout the interval $(p,\ q)$, where $p < q$, and if $g(x)$ is integrable over that interval, then there exists at least one point $\xi\ [p \le \xi \le q]$ such that

$$1. \quad \int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx.$$

Under the conditions of Theorem 3.013 1, if $f(x)$ is nondecreasing, then

$$2. \quad \int_p^q f(x)g(x)\,dx = f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q].$$

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If $f(x)$ is monotonic in the interval $(p,\ q)$, where $p < q$, and if $g(x)$ is integrable over that interval, then

$$3. \quad \int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx + f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q],$$

or

$$4. \quad \int_p^q f(x)g(x)\,dx = A\int_p^\xi g(x)\,dx + B\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q],$$

where A and B are any two numbers satisfying the conditions

$$\begin{aligned} &A \ge f(p+0) \quad \text{and} \quad B \le f(q-0) \qquad [\text{if } f \text{ decreases}],\\ &A \le f(p+0) \quad \text{and} \quad B \ge f(q-0) \qquad [\text{if } f \text{ increases}]. \end{aligned}$$

In particular,

$$5.\quad \int_p^q f(x)g(x)\,dx = f(p+0)\int_p^\xi g(x)\,dx + f(q-0)\int_\xi^q g(x)\,dx.$$

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3.02 Change of variable in a definite integral

3.020

$$\int_\alpha^\beta f(x)\,dx = \int_\varphi^\psi f[g(t)]g'(t)\,dt; \qquad x = g(t).$$

This formula is valid under the following conditions:

1. $f(x)$ is continuous on some interval $A \le x \le B$ containing the original limits of integration α and β.
2. The equalities $\alpha = g(\varphi)$ and $\beta = g(\psi)$ hold.
3. $g(t)$ and its derivative $g'(t)$ are continuous on the interval $\varphi \le t \le \psi$.
4. As t varies from φ to ψ, the function $g(t)$ always varies in the same direction from $g(\varphi) = \alpha$ to $g(\psi) = \beta$.* If this last condition is not satisfied, the interval $\varphi \le t \le \psi$ should be partitioned into subintervals throughout each of which the condition is satisfied: $\int_\alpha^\beta f(x)\,dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t)\,dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t)\,dt + \cdots + \int_{\varphi_{n-1}}^{\psi} f[g(t)]g'(t)\,dt$.

3.021

The integral $\int_\alpha^\beta f(x)\,dx$ can be transformed into another integral with given limits φ and ψ by means of the linear substitution

$$x = \frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}:$$

$$1.\quad \int_\alpha^\beta f(x)\,dx = \frac{\beta-\alpha}{\psi-\varphi}\int_\varphi^\psi f\left(\frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}\right)\,dt.$$

In particular, for $\varphi = 0$ and $\psi = 1$,

$$2. \quad \int_\alpha^\beta f(x)\,dx = (\beta-\alpha)\int_0^1 f((\beta-\alpha)t+\alpha)\,dt.$$

For $\varphi = 0$ and $\psi = \infty$,

$$3. \quad \int_\alpha^\beta f(x)\,dx = (\beta-\alpha)\int_0^\infty f\left(\frac{\alpha+\beta t}{1+t}\right)\frac{dt}{(1+t)^2}$$

3.022

The following formulas also hold:

$$1. \quad \int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(\alpha+\beta-x)\,dx.$$

$$2. \quad \int_0^\beta f(x)\,dx = \int_0^\beta f(\beta-x)\,dx.$$

$$3. \quad \int_{-\alpha}^\alpha f(x)\,dx = \int_{-\alpha}^\alpha f(-x)\,dx.$$

3.03 General formulas

3.031

1. Suppose that a function $f(x)$ is integrable over the interval $(-p,\ p)$ and satisfies the relation $f(-x) = f(x)$ on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^p f(x)\,dx = 2\int_0^p f(x)\,dx.$$

2. Suppose that $f(x)$ is a function that is integrable on the interval $(-p,\ p)$ and satisfies the relation $f(-x) = -f(x)$ on that interval. (A function satisfying the latter condition is called an *odd* function). Then,

$$\int_{-p}^{p} f(x)\, dx = 0.$$

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3.032

$$1. \quad \int_0^{\frac{\pi}{2}} f(\sin x)\, dx = \int_0^{\frac{\pi}{2}} f(\cos x)\, dx,$$

where $f(x)$ is a function that is integrable on the interval $(0, 1)$.

FI II 159

$$2. \quad \int_0^{2\pi} f(p\ \cos x + q\ \sin x)\, dx = 2\int_0^{\pi} f(\sqrt{p^2+q^2}\ \cos x)\, dx,$$

where $f(x)$ is integrable on the interval $(-\sqrt{p^2+q^2}, \sqrt{p^2+q^2})$.

FI II 160

$$3. \quad \int_0^{\frac{\pi}{2}} f(\sin 2x)\ \cos x\, dx = \int_0^{\frac{\pi}{2}} f(\cos^2 x)\ \cos x\, dx,$$

where $f(x)$ is integrable on the interval $(0, 1)$.

FI II 161

3.033

$$1. \quad \text{If} \quad f(x+\pi) = f(x) \quad \text{and} \quad f(-x) = f(x), \quad \text{then}$$

$$\int_0^{\infty} f(x) \frac{\sin x}{x}\, dx = \int_0^{\frac{\pi}{2}} f(x)\, dx.$$

2. If $f(x+\pi) = -f(x)$ and $f(-x) = f(x)$, then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\,\cos x\,dx.$$

LO V 279(4)

In formulas 3.033, it is assumed that the integrals in the left members of the formulas exist.

3.034

$$\int_0^\infty \frac{f(px) - f(qx)}{x}\,dx = [f(0) - f(+\infty)]\,\ln\frac{q}{p},$$

if $f(x)$ is continuous for $x \geq 0$ and if there exists a finite limit $f(+\infty) = \lim_{x\to+\infty} f(x)$.

FI II 633

3.035

1. $$\int_0^\pi \frac{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})}{1 + 2p\,\cos x + p^2}\,dx = \frac{2\pi}{1-p^2} f(\alpha + p) \qquad [|p| < 1].$$

LA 230(16)

2. $$\int_0^\pi \frac{1 - p\,\cos x}{1 - 2p\,\cos x + p^2}\{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})\}\,dx = \pi\{f(\alpha+p) + f(\alpha)\} \qquad [|p| < 1].$$

BE 169

3. $$\int_0^\pi \frac{f(\alpha + e^{-xi}) - f(\alpha + e^{xi})}{1 - 2p\,\cos\,x + p^2}\,\sin\,x\,dx = \frac{\pi}{pi}\{f(\alpha+p) - f(\alpha)\} \qquad [|p| < 1].$$

BE 169

In formulas 3.035, it is assumed that the function f is analytic in the closed unit circle with its center at the point α.

3.036

1. $$\int_0^\pi f\left(\frac{\sin^2\,x}{1 + 2p\,\cos\,x + p^2}\right)\,dx = \int_0^\pi f(\sin^2\,x)\,dx \qquad [p^2 \geq 1];$$
$$= \int_0^\pi f\left(\frac{\sin^2 x}{p^2}\right)\,dx \qquad [p^2 < 1].$$

2. $\int_0^{\pi} F^{(n)}(\cos\, x)\, \sin^{2n}\, x\, dx = (2n-1)!! \int_0^{\pi} F(\cos\, x)\, \cos\, nx\, dx.$

BE 174

3.037

If f is analytic in the circle of radius r and if

$$f[r(\cos\, x + i\, \sin\, x)] = f_1(r,\, x) + i f_2(r,\, x),$$

then

1. $\int_0^{\infty} \frac{f_1(r,\, x)}{p^2 + x^2}\, dx = \frac{\pi}{2p} f(re^{-p}).$

LA 230(19)

2. $\int_0^{\infty} f_2(r,\, x) \frac{x\, dx}{p^2 + x^2} = \frac{\pi}{2}[f(re^{-p}) - f(0)].$

LA 230(20)

3. $\int_0^{\infty} \frac{f_2(r,\, x)}{x}\, dx = \frac{\pi}{2}[f(r) - f(0)].$

LA 230(21)

4. $\int_0^{\infty} \frac{f_2(r,\, x)}{x(p^2 + x^2)}\, dx = \frac{\pi}{2p^2}[f(r) - f(re^{-p})].$

LA 230(22)

3.038

$$\int_{-\infty}^{\infty} \frac{x\, dx}{\sqrt{1+x^2}} F(qx + p\sqrt{1+x^2}) = \int_{-\infty}^{\infty} F(p\ \mathrm{ch}\, x + q\ \mathrm{sh}\, x)\ \mathrm{sh}\, x\, dx =$$
$$= 2q \int_0^{\infty} F'(\mathrm{sign}\, p \cdot \sqrt{p^2 - q^2}\ \ \mathrm{ch}\, x)\, \mathrm{sh}^2\, x\, dx$$

[F is a function with a continuous derivative in the interval $(-\infty,\ \infty)$; all these integrals converge.]

3.04 Improper integrals

3.041

Suppose that a function $f(x)$ is defined on an interval $(p, +\infty)$ and that it is integrable over an arbitrary finite subinterval of the form (p, P). Then, by definition

$$\int_p^{+\infty} f(x)\,dx = \lim_{P\to+\infty}\int_p^P f(x)\,dx,$$

if this limit exists. If it does exist, we say that the integral $\int_p^{+\infty} f(x)\,dx$ exists or that it converges. Otherwise, we say that the integral diverges.

3.042

Suppose that a function $f(x)$ is bounded and integrable in an arbitrary interval $(p,\ q-\eta)$ (for $0<\eta<q-p$) but is unbounded in every interval $(q-\eta,\ q)$ to the left of the point q. The point q is then called a *singular point*. Then, by definition,

$$\int_p^q f(x)\,dx = \lim_{\eta\to 0}\int_p^{q-\eta} f(x)\,dx,$$

if this limit exists. In this case, we say that the integral $\int_p^q f(x)\,dx$ *exists* or that it *converges*.

3.043

If not only the integral of $f(x)$ but also the integral of $|f(x)|$ exists, we say that the integral of $f(x)$ converges *absolutely*.

3.044

The integral $\int_p^{+\infty} f(x)\,dx$ converges absolutely if there exists a number $\alpha>1$ such that the limit

$$\lim_{x\to+\infty}\{x^\alpha|f(x)|\}$$

exists. On the other hand, if

$$\lim_{x\to+\infty}\{x|f(x)|\} = L > 0,$$

the integral $\int_p^{+\infty} |f(x)|\,dx$ diverges.

3.045

Suppose that the upper limit q of the integral $\int_p^q f(x)\,dx$ is a singular point. Then, this integral converges absolutely if there exists a number $\alpha < 1$ such that the limit

$$\lim_{x\to q}[(q-x)^\alpha |f(x)|]$$

exists. On the other hand, if

$$\lim_{x\to q}[(q-x)|f(x)|] = L > 0,$$

the integral $\int_p^q f(x)\,dx$ diverges.

3.046

Suppose that the functions $f(x)$ and $g(x)$ are defined on the interval $(p,\ +\infty)$, that $f(x)$ is integrable over every finite interval of the form $(p,\ P)$, that the integral

$$\int_p^P f(x)\,dx$$

is a bounded function of P, that $g(x)$ is monotonic, and that $g(x) \to 0$ as $x \to +\infty$. Then, the integral

$$\int_p^{+\infty} f(x)g(x)\,dx$$

converges.

FI II 577

3.05 The principal values of improper integrals

3.051

Suppose that a function $f(x)$ has a singular point r somewhere inside the interval $(p,\ q)$, that $f(x)$ is defined at r, and that $f(x)$ is

$f(x)$ r $(p,\, q)$ $f(x)$ r $f(x)$

integrable over every portion of this interval that does not contain

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the point r. Then, by definition

$$\int_p^q f(x)\,dx = \lim_{\substack{\eta\to 0\\ \eta'\to 0}} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta'}^q f(x)\,dx \right\},$$

Here, the limit must exist for *independent* modes of approach of η and η' to zero. If this limit does not exist but the limit

$$\lim_{\eta\to 0} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta}^q f(x)\,dx \right\},$$

does exist, we say that this latter limit is the Cauchy *principal value* of the improper integral $\int_p^q f(x)\,dx$ and we say that the integral $\int_p^q f(x)\,dx$ exists in the sense of principal values.

FI II 603

3.052

Suppose that the function $f(x)$ is continuous over the interval $(p,\, q)$ and vanishes at only one point r inside this interval. Suppose that the first derivative $f'(x)$ exists in a neighborhood of the point r. Suppose that $f'(r) \neq 0$ and that the second derivative $f''(r)$ exists at the point r itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$

FI II 605

diverges, but exists in the sense of principal values.

3.053

A divergent integral of a positive function cannot exist in the sense of principal values.

3.054

Suppose that the function $f(x)$ has no singular points in the interval $(-\infty,\, +\infty)$. Then, by definition

Here, the limit must exist for independent approach of P and Q to $\pm\infty$. If this limit does not exist but the limit

$$\lim_{P\to+\infty}\int_{-P}^{+P} f(x)\,dx,$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,dx.$$

FI II 607

264

3.055

The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense).

FI II 607

3.1-3.2 Power and Algebraic Functions

3.11 Rational functions

3.111

$$\int_{-\infty}^{\infty} \frac{p+qx}{r^2+2rx\,\cos\lambda+x^2}\,dx = \frac{\pi}{r\,\sin\lambda}(p-qr\,\cos\lambda) \qquad (\text{principal value}^*)$$

$$(\text{see also } 3.194\ 8. \text{ and } 3.252\ 1. \text{ and } 2.\ ).$$

3.252

3.194

* We give the values of proper and improper convergent integrals and also the principal values of divergent integrals (see 3.05) if the latter exist. Henceforth, we make no special indication of principal values.

BI ((22))(14)

3.112

Integrals of the form $\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)}$,

where

$$g_n(x) = b_0 x^{2n-2} + b_1 x^{2n-4} + \cdots + b_{n-1},$$
$$h_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

[All roots of $h_n(x)$ lie in the upper half-plane.]

1. $$\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0} \frac{M_n}{\Delta n},$$

JE

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix},$$

$$M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}.$$

2. $$\int_{-\infty}^{\infty} \frac{g_1(x)\,dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0 a_1}.$$

3.[8] $$\int_{-\infty}^{\infty} \frac{g_2(x)\,dx}{h_2(x)h_2(-x)} = \pi i \frac{-b_0 + \frac{a_0 b_1}{a_2}}{a_0 a_1}.$$

4. $$\int_{-\infty}^{\infty} \frac{g_3(x)\,dx}{h_3(x)h_3(-x)} = \pi i \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{a_0(a_0 a_3 - a_1 a_2)}.$$

JE

5. $$\int_{-\infty}^{\infty} \frac{g_4(x)\,dx}{h_4(x)h_4(-x)} = \pi i \frac{b_0(-a_1 a_4 + a_2 a_3) - a_0 a_3 b_1 + a_0 a_1 b_2 + \frac{a_0 b_3}{a_4}(a_0 a_3 - a_1 a_2)}{a_0(a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}.$$

JE

6. $$\int_{-\infty}^{\infty} \frac{g_5(x)\,dx}{h_5(x)h_5(-x)} = \pi i \frac{M_5}{a_0 \Delta_5},$$

where

$$\begin{aligned} M_5 &= b_0(-a_0 a_4 a_5 + a_1 a_4^2 + a_2^2 a_5 - a_2 a_3 a_4) + a_0 b_1(-a_2 a_5 + a_3 a_4) + \\ &\quad + a_0 b_2(a_0 a_5 - a_1 a_4) + a_0 b_3(-a_0 a_3 + a_1 a_2) + \frac{a_0 b_4}{a_5}(-a_0 a_1 a_5 + a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3), \\ \Delta_5 &= a_0^2 a_5^2 - 2 a_0 a_1 a_4 a_5 - a_0 a_2 a_3 a_5 + a_0 a_3^2 a_4 + a_1^2 a_4^2 + a_1 a_2^2 a_5 - a_1 a_2 a_3 a_4. \end{aligned}$$

JE

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.121

1. $$\int_0^1 \frac{1}{1 - 2x\cos\lambda + x^2} \frac{dx}{\sqrt{x}} = 2 \operatorname{cosec} \lambda \sum_{k=1}^{\infty} \frac{\sin k\lambda}{2k-1}.$$

BI ((10))(17)

2. $$\int_0^1 \frac{1}{q - px} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \qquad [0 < p < q].$$

3. $\int_0^1 \frac{dx}{1-2rx+r^2}\sqrt{\frac{1\mp x}{1\pm x}} = \pm\frac{\pi}{4r} \mp \frac{1}{r}\frac{1\mp r}{1\pm r}\ \mathrm{arctg}\ \frac{1+r}{1-r}.$

LI ((14))(5, 16)

3.13-3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

In 3.131-3.137 we set: $\alpha = \arcsin\ \sqrt{\frac{a-c}{a-u}}, \beta = \arcsin\ \sqrt{\frac{c-u}{b-u}},$

$$\gamma = \arcsin\sqrt{\frac{u-c}{b-c}}, \qquad \delta = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\{ = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \lambda = \arcsin\sqrt{\frac{a-u}{a-b}},$$

$$\mu = \arcsin\sqrt{\frac{u-a}{u-b}}, \qquad \nu = \arcsin\sqrt{\frac{a-c}{u-c}}, \qquad p = \sqrt{\frac{a-b}{a-c}}, \qquad q = \sqrt{\frac{b-c}{a-c}}.$$

3.131

1. $\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\alpha,\ p) \qquad [a > b > c \geq u].$

BY (231.00)

2. $\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\beta,\ p) \qquad [a > b > c > u].$

BY (232.00)

3. $\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\gamma,\ q) \qquad [a > b \geq u > c].$

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\delta,\, q) \qquad [a > b > u \geq c].$$

BY (234.00)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\{,\, p) \qquad [a \geq u > b > c].$$

>BY (235.00)>

6. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda,\, p) \qquad [a > u \geq b > c].$$

BY (236.00)

7. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\mu,\, q) \qquad [u > a > b > c].$$

BY (237.00)

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\nu,\, q) \qquad [u \geq a > b > c].$$

BY (238.00)

3.132

1. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} [cF(\beta,\, p) + (a-c)E(\beta,\, p)] - 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$$[a > b > c > u].$$

BY (232.19)

2. $$\int_c^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\gamma,\, q) - 2\sqrt{a-c}\,E(\gamma,\, q) \qquad [a > b \geq u > c].$$

BY (233.17)

3. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-a)\Pi(\delta,\, q^2,\, q) + aF(\delta,\, q)] \qquad [a > b > u \geq c].$$

4. $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}[(b-c)\Pi(\{,\ p^2,\ p)+cF(\{,\ p)] \qquad [a \geq u > b > c].$$

BY (235.16)

5. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}}F(\lambda,\ p)+2\sqrt{a-c}E(\lambda,\ p) \qquad [a > u \geq b > c].$$

BY (236.16)

6. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}}[a(a-b)\Pi(\mu,\ 1,\ q)+b^2F(\mu,\ q)] \qquad [u > a > b > c].$$

BY (237.16)

3.133

1.[8] $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}[F(\alpha,\ p)-E(\alpha,\ p)] \qquad [a > b > c \geq u].$$

BY (231.08)

2. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}[F(\beta,\ p)-E(\beta,\ p)]+\frac{2}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$$[a > b > c > u].$$

BY (232.13)

3. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}E(\gamma,\ q) -$$
$$-\frac{2}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a > b \geq u > c].$$

BY (233.09)

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}E(\delta,\ q) \qquad [a > b > u \geq c].$$

BY (234.05)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}[F(\{,\ p)-E(\{,\ p)]+\frac{2}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$$[a > u > b > c].$$

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\nu,\, q) + \frac{2}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}} \quad [u > a > b > c].$$

BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\alpha,\, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\alpha,\, p) - \frac{2}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c \geq u].$$

BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\beta,\, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\beta,\, p) \quad [a > b > c > u].$$

BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}} F(\gamma,\, q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\gamma,\, q) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{b-u}} \quad [a > b > u > c].$$

BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} F(\lambda,\, p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\lambda,\, p) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{u-b}} \quad [a > u > b > c].$$

BY (236.09)

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\mu,\, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\mu,\, q) \quad [u > a > b > c].$$

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\nu,\, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\nu,\, q) - \frac{2}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u \geq a > b > c].$$

BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}} E(\alpha,\, p) + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}} \qquad [a > b > c > u].$$

BY (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\delta,\, q) - E(\delta,\, q)] + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}} \qquad [a > b > u > c].$$

BY (234.04)

15. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\{,\, p) \qquad [a \geq u > b > c].$$

BY (235.01)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\lambda,\, p) - \frac{2}{(b-c)(a-c)}\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a > u \geq b > c].$$

BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\mu,\, q) - E(\mu,\, q)] + \frac{2}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u > a > b > c].$$

BY (237.13)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\nu,\, q) - E(\nu,\, q)] \qquad [u \geq a > b > c].$$

BY (238.03)

3.134

1. $$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} =$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}[(3a-b-2c)F(\alpha,\, p) - 2(2a-b-c)E(\alpha,\, p)] +$$
$$+ \frac{2}{3(a-c)(a-b)}\sqrt{\frac{(c-u)(b-u)}{(a-u)^3}} \qquad [a > b > c \geq u].$$

BY (231.08)

2. $$\int_{u}^{c} \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} =$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}[(3a-b-2c)F(\beta,\, p) - 2(2a-b-c)E(\beta,\, p)] +$$
$$+ \frac{2[4a^2-3ab-2ac+bc-u(3a-2b-c)]}{3(a-b)(a-c)^2}\sqrt{\frac{c-u}{(a-u)^3(b-u)}} \qquad [a > b > c > u].$$

BY (232.13)

3. $$\int_{c}^{u} \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} =$$
$$= \frac{2}{3(a-b)^3\sqrt{(a-c)^3}}[2(2a-b-c)E(\gamma,\, q) - (a-b)F(\gamma,\, q)] -$$
$$- \frac{2[5a^2-3ab-3ac+bc-2u(2a-b-c)]}{3(a-b)^2(a-c)^2}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \qquad [a > b \geq u > c].$$

BY (233.09)

4. $$\int_{u}^{b} \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}[2(2a-b-c)E(\delta,\, q) - (a-b)F(\delta,\, q)] -$$
$$- \frac{2}{3(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \qquad [a > b > u \geq c].$$

BY (234.05)

5. $$\int_{b}^{u} \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}} =$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}[(3a-b-2c)F(\{,\, p) - 2(2a-b-c)E(\{,\, p)] +$$
$$+ \frac{2[4a^2-2ab-3ac+bc-u(3a-b-2c)]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(a-u)^3(u-c)}} \qquad [a > u > b > c].$$

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} =$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\left[2(2a-b-c)E(\nu,\, q) - (a-b)F(\nu,\, q)\right] +$$
$$+ \frac{2[4a^2-2ab-3ac+bc+u(b+2c-3a)]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(u-a)^3(u-c)}} \qquad [u > a > b > c].$$

BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times \left[2(a-c)(a+c-2b)E(\alpha,\, p) + (b-c)(3b-a-2c)F(\alpha,\, p)\right] -$$
$$- \frac{2[3ab-ac+2bc-4b^2-u(2a-3b+c)]}{3(a-b)(b-c)^2}\sqrt{\frac{c-u}{(a-u)(b-u)^3}} \qquad [a > b > c \geq u].$$

BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times \left[(b-c)(3b-a-2c)F(\beta,\, p) + 2(a-c)(a-2b+c)E(\beta,\, p)\right] +$$
$$+ \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(a-u)(c-u)}{(b-u)^3}} \qquad [a > b > c > u].$$

BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times \left[(a-b)(2a-3b+c)F(\gamma,\, q) + 2(a-c)(2b-a-c)E(\gamma,\, q)\right] +$$
$$+ \frac{2[3ab+3bc-ac-5b^2-2u(a-2b+c)]}{3(a-b)^2(b-c)^2}\sqrt{\frac{(a-u)(u-c)}{(b-u)^3}} \qquad [a > b > u > c].$$

BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times \left[(b-c)(3b-2c-a)F(\lambda,\, p) + 2(a-c)(a+c-2b)E(\lambda,\, p)\right] +$$

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times [(a-b)(2a+c-3b)F(\mu,\, q) + 2(a-c)(2b-a-c)E(\mu,\, q)] +$$
$$+ \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(u-a)(u-c)}{(u-b)^3}} \qquad [u > a > b > c].$$

BY (237.12)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} =$$
$$= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times$$
$$\times [(a-b)(2a+c-3b)F(\nu,\, q) + 2(a-c)(2b-c-a)E(\nu,\, q)] -$$
$$- \frac{2[3bc+2ab-ac-4b^2+u(3b-a-2c)]}{3(a-b)^2(b-c)}\sqrt{\frac{u-a}{(u-b)^3(u-c)}} \qquad [u \geq a > b > c].$$

BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} =$$
$$= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}[2(a+b-2c)E(\alpha,\, p) - (b-c)F(\alpha,\, p)] +$$
$$+ \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(a-c)(b-c)^2}\sqrt{\frac{b-u}{(a-u)(c-u)^3}} \qquad [a > b > c > u].$$

BY (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} =$$
$$= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}[(2a+b-3c)F(\delta,\, q) - 2(a+b-2c)E(\delta,\, q)] +$$
$$+ \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(b-c)^2(a-c)}\sqrt{\frac{b-u}{(a-u)(u-c)^3}} \qquad [a > b > u > c].$$

BY (234.04)

BY (235.20)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} =$$
$$= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}[2(a+b-2c)E(\lambda,\, p)-(b-c)F(\lambda,\, p)]-$$
$$-\frac{2[ab-3ac-3bc+5c^2+2u(a+b-2c)]}{3(b-c)^2(a-c)^2}\sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \qquad [a>u\geq b>c].$$

BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} =$$
$$= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}[(2a+b-3c)F(\mu,\, q)-2(a+b-2c)E(\mu,\, q)]+$$
$$+\frac{2[4c^2-ab-2ac-bc+u(3a+2b-5c)]}{3(b-c)(a-c)^2}\sqrt{\frac{u-a}{(u-b)(u-c)^3}} \qquad [u>a>b>c].$$

BY (237.13)

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18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} =$$
$$= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}[(2a+b-3c)F(\nu,\, q)-2(a+b-2c)E(\nu,\, q)]+$$
$$+\frac{2}{3(a-c)(b-c)}\sqrt{\frac{(u-a)(u-b)}{(u-c)^3}} \qquad [u\geq a>b>c].$$

BY (238.03)

3.135

1.[6] $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\times$$
$$\times[(b-c)F(\alpha,\, p)-(2a-b-c)E(\alpha,\, p)]+$$
$$+\frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \qquad [a>b>c>u].$$

BY (231.13)}

2. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\times$$
$$\times[(b-c)F(\lambda,\, p)-2(2a-b-c)E(\lambda,\, p)]+$$
$$+\frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)}\sqrt{\frac{a-u}{(u-b)(u-c)}} \qquad [a>u>b>c].$$

3. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times$$
$$\times [(2a-b-c)E(\mu,\, q) - 2(a-b)F(\mu,\, q)] +$$
$$+ \frac{2}{(a-c)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u > a > b > c].$$

BY (236.14)

4. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times$$
$$\times [(2a-b-c)E(\nu,\, q) - 2(a-b)F(\nu,\, q)] -$$
$$- \frac{2}{(a-b)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u \geq a > b > c].$$

BY (238.13)

5. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times$$
$$\times [(2b-a-c)E(\alpha,\, p) - (b-c)F(\alpha,\, p)] +$$
$$+ \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(c-u)}} \qquad [a > b > c > u].$$

BY(231.12)

6. $$\int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}} = \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}} \times$$
$$\times [(a-b)F(\delta,\, q) + (2b-a-c)E(\delta,\, q)] +$$
$$+ \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)}} \qquad [a > b > u > c].$$

BY (234.03)

7. $$\int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times$$
$$\times [(b-c)F(\{,\, p) - (2b-a-c)E(\{,\, p)] +$$
$$+ \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(a-u)(u-c)}} \qquad [a > u > b > c].$$

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times$$
$$\times\left[(a+c-2b)E(\nu,\, q)-(a-b)F(\nu,\, q)\right]+$$
$$+\frac{2}{(a-b)(a-c)}\sqrt{\frac{u-b}{(u-a)(u-c)}} \qquad [u>a>b>c].$$

BY (238.14)

9.[8] $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(b-c)(a-b)^2\sqrt{a-c}} \times$$
$$\times[(a+b-2c)E(\alpha,\, p)-2(b-c)F(\alpha,\, p)]-$$
$$-\frac{2}{(a-b)(b-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}} \qquad [a>b>c\geq u].$$

BY (231.11)

10. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times$$
$$\times\left[(a+b-2c)E(\beta,\, p)-2(b-c)F(\beta,\, p)\right]+$$
$$+\frac{2}{(a-b)(a-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}} \qquad [a>b>c>u].$$

BY (232.15)

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11. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times$$
$$\times\left[(a-b)F(\gamma,\, q)-(a+b-2c)E(\gamma,\, q)\right]+$$
$$+\frac{2[a^2+b^2-ac-bc-u(a+b-2c)]}{(a-b)^2(b-c)(a-c)}\sqrt{\frac{u-c}{(a-u)(b-u)}}$$
$$[a>b>u>c].$$

BY (233.11)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times$$
$$\times\left[(a-b)F(\nu,\, q)-(a+b-2c)E(\nu,\, q)\right]+$$
$$+\frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}} \qquad [u>a>b>c].$$

BY (238.15)

3.136

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} =$$
$$= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times$$
[illegible]

2. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} = \\ = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times \\ \times [(a-b)(2a-b-c)F(\nu,\ q) - 2(a^2+b^2+c^2-ab-ac-bc)E(\nu,\ q)] + \\ + \frac{2[u(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}} \qquad [u > a > b > c].$$

BY (238.16}

3.137

1. $$\int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-r)\sqrt{a-c}}\left[\Pi\left(\alpha,\ \frac{a-r}{a-c},\ p\right) - F\left(\alpha,\ p\right)\right] \\ [a > b > c \geq u].$$

BY (231.15)

2. $$\int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \times \\ \times \Pi\left(\beta,\ \frac{r-b}{r-c},\ p\right) + \frac{2}{(r-b)\sqrt{a-c}} F(\beta,\ p) \\ [a > b > c > u, \quad r \neq 0].$$

BY (232.17)

3. $$\int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \Pi\left(\gamma,\ \frac{b-c}{r-c},\ q\right) \\ [a > b \geq u > c, \quad r \neq c].$$

BY (233.02)

4. $$\int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-a)(r-b)\sqrt{a-c}} \times \\ \times \left[(b-a)\Pi\left(\delta,\ q^2\frac{r-a}{r-b},\ q\right) + (r-b)\,F\left(\delta,\ q\right)\right] \\ [a > b > u \geq c, \quad r \neq b].$$

BY (234.18)

BY (235.17)

6.[8] $$\int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}}\Pi\left(\lambda,\ \frac{a-b}{a-r},\ p\right)$$
$$[a > u \geq b > c, \quad r \neq a].$$

BY (236.02)

7. $$\int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \times$$
$$\times\left[(b-a)\Pi\left(\mu,\ \frac{b-r}{a-b},\ q\right) + (a-p)\,F\,(\mu,\ q)\right]$$
$$[u > a > b > c, \quad r \neq a].$$

BY (237.17)

8. $$\int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}}\left[\Pi\left(\nu,\ \frac{r-c}{a-c},\ q\right) - F\,(\nu,\ q)\right]$$
$$[u \geq a > b > c].$$

BY (238.06)

3.138

1. $$\int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2F(\arcsin\sqrt{u},\ k) \qquad [0 < u < 1].$$

PE (532), JA

2. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} = 2F(\arccos\sqrt{u},\ k) \qquad [0 < u < 1].$$

PE(533)

3. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2F\left(\arcsin\frac{\sqrt{1-u}}{k},\ k\right) \qquad [0 < u < 1].$$

PE (534)

4. $$\int_0^u \frac{dx}{\sqrt{x(1+x)(1+k'^2x)}} = 2F(\operatorname{arctg}\sqrt{u},\ k) \qquad [0 < u < 1].$$

5. $$\int_0^u \frac{dx}{\sqrt{x[1+x^2+2(k'^2-k^2)x]}} = F(2\ \operatorname{arctg}\sqrt{u},\ k) \qquad [0<u<1].$$

JA

6. $$\int_u^1 \frac{dx}{\sqrt{x[k'^2(1+x^2)+2(1+k^2)x]}} = F\left(\frac{\pi}{2} - 2\ \operatorname{arctg}\sqrt{u},\ k\right) \qquad [0<u<1].$$

JA

7. $$\int_a^u \frac{dx}{\sqrt{(x-\alpha)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2\ \operatorname{arctg}\sqrt{\frac{u-\alpha}{p}},\ \sqrt{\frac{p+m-\alpha}{2p}}\right) \qquad [\alpha<u],$$

8. $$\int_u^a \frac{dx}{\sqrt{(\alpha-x)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2\operatorname{arcctg}\sqrt{\frac{\alpha-u}{p}},\ \sqrt{\frac{p-m+\alpha}{2p}}\right) \qquad [u<\alpha],$$

where $p = \sqrt{(m-\alpha)^2+n^2}$.

3.139

Notation:[8] $$\alpha = \arccos\frac{1-\sqrt{3}-u}{1+\sqrt{3}-u}, \quad \beta = \arccos\frac{\sqrt{3}-1+u}{\sqrt{3}+1-u},$$
$$\gamma = \arccos\frac{\sqrt{3}+1-u}{\sqrt{3}-1+u}, \quad \delta = \arccos\frac{u-1-\sqrt{3}}{u-1+\sqrt{3}}.$$

1.[8] $$\int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\alpha,\ \sin 75^\circ).$$

ZH 66 (285)

2. $$\int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\beta,\ \sin 75^\circ).$$

ZH 65 (284)

3. $$\int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\gamma,\ \sin 15^\circ).$$

4. $$\int_u^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\delta, \sin 15^\circ).$$

ZH 65 (282)

5. $$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}} \left\{\Gamma\left(\frac{1}{3}\right)\right\}^3.$$

MO 9

6. $$\int_0^1 \frac{x\,dx}{\sqrt{1-x^3}} = \frac{1}{\pi}\frac{\sqrt{3}}{\sqrt[3]{4}} \left\{\Gamma\left(\frac{2}{3}\right)\right\}^3.$$

MO 9

7. $$\int_u^1 \sqrt{1-x^3}\,dx = \frac{1}{5}\{\sqrt[4]{27}F(\beta, \sin 75^\circ) - 2u\sqrt{1-u^3}\}.$$

BY (244.01)

8. $$\int_u^1 \frac{x\,dx}{\sqrt{1-x^3}} = (3^{-\frac{1}{4}} - 3^{\frac{1}{4}})F(\beta, \sin 75^\circ) + 2\sqrt[4]{3}E(\beta, \sin 75^\circ) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}.$$

BY (244.05)

9. $$\int_u^1 \frac{x^m\,dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1}\int_u^1 \frac{x^{m-3}\,dx}{\sqrt{1-x^3}}.$$

BY (244.07)

10. $$\int_1^u \frac{x\,dx}{\sqrt{x^3-1}} = (3^{-\frac{1}{4}} + 3^{\frac{1}{4}})F(\gamma, \sin 15^\circ) - 2\sqrt[4]{3}E(\gamma, \sin 15^\circ) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}.$$

BY (240.05)

11. $$\int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}}[F(\alpha, \sin 75^\circ) - 2E(\alpha, \sin 75^\circ)] + \frac{2}{\sqrt{3}}\frac{\sqrt{1+u+u^2}}{(1+\sqrt{3}-u)\sqrt{1-u}}$$

$[u \neq 1]$.

12. $$\int_u^{\infty} \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}}[F(\delta, \sin 15^\circ) - 2E(\delta, \sin 15^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(u-1+\sqrt{3})\sqrt{u-1}}$$
$$[u \neq 1].$$

BY (242.03)

13. $$\int_{-\infty}^{u} \frac{(1-x)\,dx}{(1+\sqrt{3}-x)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}[F(\alpha, \sin 75^\circ) - E(\alpha, \sin 75^\circ)].$$

BY (246.07)

14. $$\int_u^{1} \frac{(1-x)\,dx}{(1+\sqrt{3}-x)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}[F(\beta, \sin 75^\circ) - E(\beta, \sin 75^\circ)].$$

BY (244.04)

15. $$\int_1^{u} \frac{(x-1)\,dx}{(1+\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ).$$

BY (240.08)

16. $$\int_u^{\infty} \frac{(x-1)\,dx}{(1+\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\delta, \sin 15^\circ).$$

BY (242.07)

17. $$\int_{-\infty}^{u} \frac{(1-x)\,dx}{(1-\sqrt{3}-x)^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} \left[\frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E(\alpha, \sin 75^\circ) \right].$$

BY (246.08)

18. $$\int_1^{u} \frac{(x-1)\,dx}{(1-\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}[F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ)].$$

BY (240.04)

19. $$\int_u^{\infty} \frac{(x-1)\,dx}{(1-\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}[F(\delta, \sin 15^\circ) - E(\delta, \sin 15^\circ)].$$

BY (242.05)

BY (242.05)

20. $$\int_{-\infty}^{u} \frac{(x^2+x+1)\,dx}{(1+\sqrt{3}-x)^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\alpha,\, \sin\, 75^\circ).$$

BY (246.01)

21. $$\int_{u}^{1} \frac{(x^2+x+1)\,dx}{(x-1+\sqrt{3})^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\beta,\, \sin\, 75^\circ).$$

BY (244.02)

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22. $$\int_{1}^{u} \frac{(x^2+x+1)\,dx}{(\sqrt{3}+x-1)^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\gamma,\, \sin\, 15^\circ).$$

BY (240.01)

23. $$\int_{u}^{\infty} \frac{(x^2+x+1)\,dx}{(x-1+\sqrt{3})^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\delta,\, \sin 15^\circ).$$

BY (242.01)

24. $$\int_{1}^{u} \frac{(x-1)\,dx}{(x^2+x+1)\sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}} E(\gamma,\, \sin 15^\circ) - \frac{2+\sqrt{3}}{\sqrt[4]{27}} F(\gamma,\, \sin 15^\circ) - \\ - \frac{2-\sqrt{3}}{\sqrt{3}} \frac{2(u-1)(\sqrt{3}+1-u)}{(\sqrt{3}-1+u)\sqrt{u^3-1}}$$

BY (240.09)

25. $$\int_{-\infty}^{u} \frac{(1+\sqrt{3}-x)^2\,dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\alpha,\, p^2,\, \sin\, 75^\circ).$$

BY (246.02)

26. $$\int_{u}^{1} \frac{(1+\sqrt{3}-x)^2\,dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\beta,\, p^2,\, \sin\, 75^\circ).$$

BY (244.03)

BY (240.02)

28. $$\int_u^\infty \frac{(1-\sqrt{3}-x)^2\,dx}{[(1-\sqrt{3}-x)^2-4\sqrt{3}p^2(x-1)]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\Pi(\delta,\,p^2,\,\sin\,15^\circ).$$

BY (242.02)

In 3.141 and 3.142 we set: $\alpha = \arcsin\sqrt{\frac{a-c}{a-u}}$, $\beta = \arcsin\sqrt{\frac{c-u}{b-u}}$, $\gamma = \arcsin\sqrt{\frac{u-c}{b-c}}$, $\delta = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}$, $\{ = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}$, $\lambda = \arcsin\sqrt{\frac{a-u}{a-b}}$, $\mu = \arcsin\sqrt{\frac{u-a}{u-b}}$, $\nu = \arcsin\sqrt{\frac{a-c}{u-c}}$, $p = \sqrt{\frac{a-b}{a-c}}$, $q = \sqrt{\frac{b-c}{a-c}}$.

3.141

1. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}}\,dx = 2\sqrt{a-c}[F(\beta,\,p)-E(\beta,\,p)]+2\sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a>b>c>u].$$

BY (232.06)

2. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}E(\gamma,\,q) \qquad [a>b\ge u>c].$$

BY (233.01)

3. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}E(\delta,\,q)-2\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a>b>u\ge c].$$

BY (234.06)

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4. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}[F(\{,\,p)-E(\{,\,p)]+2\sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a\ge u>b>c].$$

BY (235.07)

5. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}[F(\lambda,\,p)-E(\lambda,\,p)] \qquad [a>u\ge b>c]$$

BY (236.04)

BY (237.03)

7. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}}\,dx = \frac{2(b-c)}{\sqrt{a-c}}F(\beta,\,p) - 2\sqrt{a-c}\,E(\beta,\,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \qquad [a > b > c > u].$$

BY (232.07)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\gamma,\,q) - \frac{2(a-b)}{\sqrt{a-c}}F(\gamma,\,q) \qquad [a > b \geq u > c].$$

BY (233.04)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\delta,\,q) - \frac{2(a-b)}{\sqrt{a-c}}F(\delta,\,q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a > b > u \geq c].$$

BY (234.07)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\{,\,p) - \frac{2(b-c)}{\sqrt{a-c}}F(\{,\,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a \geq u > b > c].$$

BY (235.06)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\lambda,\,p) - \frac{2(b-c)}{\sqrt{a-c}}F(\lambda,\,p) \qquad [a > u \geq b > c].$$

BY (236.03)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}}\,dx = \frac{2(a-b)}{\sqrt{a-c}}F(\mu,\,q) - 2\sqrt{a-c}\,E(\mu,\,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}} \qquad [u > a > b > c].$$

BY (237.04)

13. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}}\,dx = -2\sqrt{a-c}\,E(\beta,\,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \qquad [a > b > c > u].$$

14. $\displaystyle\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}[F(\gamma,\,q)-E(\gamma,\,q)] \qquad [a > b \geq u > c].$

BY (233.03)

15. $\displaystyle\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}[F(\delta,\,q)-E(\delta,\,q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$
$[a > b > u \geq c].$

BY (234.08)

16. $\displaystyle\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}E(\{,\,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a \geq u > b > c].$

BY (235.07)

17. $\displaystyle\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}E(\lambda,\,p) \qquad [a > u \geq b > c].$

BY (236.01)

18. $\displaystyle\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}}\,dx = 2\sqrt{a-c}[F(\mu,\,q)-E(\mu,\,q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$
$[u > a > b > c].$

BY (237.05)

19. $\displaystyle\int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\beta,\,p)-(b-c)F(\beta,\,p)] +$
$\displaystyle + \frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}} \qquad [a > b > c > u].$

BY (232.11)

20. $\displaystyle\int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\gamma,\,q)-2(a-b)F(\gamma,\,q)] -$
$\displaystyle - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)} \qquad [a > b \geq u > c].$

21. $$\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}[2(b-a)F(\delta,\,q)+(2a-b-c)E(\delta,\,q)]+$$
$$+\frac{2}{3}(2c-b-u)\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a>b>u\geq c].$$

BY (234.11)

22. $$\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\{,\,p)-(b-c)F(\{,\,p)]+$$
$$+\frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a\geq u>b>c].$$

BY (235.10)

22. $$\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\{,\,p)-(b-c)F(\{,\,p)]+$$
$$+\frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a\geq u>b>c].$$

BY (236.07)

24. $$\int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}}\,dx = \frac{2}{3}\sqrt{a-c}[2(a-b)F(\mu,\,q)+(b+c-2a)E(\mu,\,q)]+$$
$$+\frac{2}{3}(u+2a-2b-c)\sqrt{\frac{(u-a)(u-b)}{u-c}} \qquad [u>a>b>c].$$

BY (237.08)

25. $$\int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2b-a-c)E(\beta,\,p)-(b-c)F(\beta,\,p)]+$$
$$+\frac{2}{3}(a+c-b-u)\sqrt{\frac{(a-u)(c-u)}{b-u}} \qquad [a>b>c>u].$$

BY (232.10)

26. $$\int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(2b-a-c)E(\gamma,\,q)+(a-b)F(\gamma,\,q)]-$$
$$-\frac{2}{3}\sqrt{(a-u)(b-u)(u-c)} \qquad [a>b\geq u>c].$$

BY (233.05)

27. $$\int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}[(a-b)F(\delta,\,q)+(2b-a-c)E(\delta,\,q)]+ \\ +\frac{2}{3}(2a+c-2b-u)\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a>b>u\geq c].$$

BY (234.10)

28. $$\int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}[(b-c)F(\{,\,p)+(a+c-2b)E(\{,p)]+ \\ +\frac{2}{3}(2b-a-2c+u)\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a\geq u>b>c].$$

BY (235.11)

29. $$\int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+c-2b)E(\lambda,\,p)+(b-c)F(\lambda,\,p)]- \\ -\frac{2}{3}\sqrt{(a-u)(u-b)(u-c)} \qquad [a>u\geq b>c].$$

BY (236.06)

30. $$\int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}}\,dx = \frac{2}{3}\frac{\sqrt{(a-c)^3}}{b-c}[(a+c-2b)E(\mu,\,q)-(a-b)F(\mu,\,q)]+ \\ +\frac{2}{3}\frac{a-c}{b-c}(u+b-a-c)\sqrt{\frac{(u-a)(u-c)}{u-b}} \qquad [u>a>b>c].$$

BY (237.06)

31. $$\int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}}\,dx = \frac{2}{3}\sqrt{a-c}[2(b-c)F(\beta,\,p)+(2c-a-b)E(\beta,\,p)]+ \\ +\frac{2}{3}(a+2b-2c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}} \qquad [a>b>c>u].$$

BY (232.09)

32. $$\int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\gamma,\,q)-(a-b)F(\gamma,\,q)]+ \\ +\frac{2}{3}\sqrt{(a-u)(b-u)(u-c)} \qquad [a>b\geq u>c].$$

33. $$\int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\delta,\,q)-(a-b)F(\delta,\,q)]+ \\ +\frac{2}{3}(2c-2a-b+u)\sqrt{\frac{(b-u)(u-c)}{a-u}} \qquad [a>b>u\geq c].$$

BY (234.09)

34. $$\int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\{,\,p)-2(b-c)F(\{,\,p)]+ \\ +\frac{2}{3}(u+c-a-b)\sqrt{\frac{(a-u)(u-b)}{u-c}} \qquad [a\geq u>b>c].$$

BY (235.09)

35. $$\int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\lambda,\,p)-2(b-c)F(\lambda,\,p)]- \\ -\frac{2}{3}\sqrt{(a-u)(u-b)(u-c)} \qquad [a>u\geq b>c].$$

BY (236.05)

36. $$\int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\mu,\,q)-(a-b)F(\mu,\,q)]+ \\ +\frac{2}{3}(u+2c-a-2b)\sqrt{\frac{(u-a)(u-c)}{u-b}} \qquad [u>a>b>c].$$

BY (237.07)

3.142

1. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}F(\alpha,\,p)-\frac{2\sqrt{a-c}}{b-c}E(\alpha,\,p)+\frac{2(a-c)}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a>b>c>u].$$

BY (231.05)

2. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}}\,dx = 2\frac{a-b}{(b-c)\sqrt{a-c}}F(\delta,\,q)-\frac{2\sqrt{a-c}}{b-c}E(\delta,\,q)+ \\ +2\frac{a-c}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}} \qquad [a>b>u>c].$$

3. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\{,\ p) - \frac{2}{\sqrt{a-c}} F(\{,\ p) \qquad [a \geq u > b > c].$$

BY (235.12)

4. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\lambda,\ p) - \frac{2}{\sqrt{a-c}} F(\lambda,\ p) - \frac{2}{b-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$$[a > u \geq b > c].$$

BY (236.12)

5. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\mu,\ q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\mu,\ q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$$[u > a > b > c].$$

BY (237.10)

6. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\nu,\ q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\nu,\ q) \qquad [u \geq a > b > c].$$

BY (238.09)

7. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\alpha,\ p) - 2\frac{a-b}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}} \qquad [a > b > c \geq u].$$

BY (231.03)

8. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\beta,\ p) \qquad [a > b > c > u].$$

BY (232.01)

9. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}}\, dx = \frac{2\sqrt{a-c}}{b-c} [F(\gamma,\ q) - E(\gamma,\ q)] + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$$[a > b > u > c].$$

BY (233.15)

10. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}}\, dx = \frac{2\sqrt{a-c}}{c-b} E(\lambda,\ p) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{u-b}} \qquad [a > u > b > c].$$

11. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}[F(\mu,\,q) - E(\mu,\,q)] \qquad [u > a > b > c].$$

BY (237.09)

12. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}[F(\nu,\,q) - E(\nu,\,q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$$[u \geq a > b > c].$$

BY (238.10)

13. $$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}} E(\alpha,\,p) \qquad [a > b > c \geq u].$$

BY (231.01)

14. $$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}} E(\beta,\,p) - \frac{2(a-b)}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}} \qquad [a > b > c > u].$$

BY (232.05)

15. $$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}[F(\gamma,\,q) - E(\gamma,\,q)] + \frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$$[a > b \geq u > c].$$

BY (233.13)

16. $$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}[F(\delta,\,q) - E(\delta,\,q)] \qquad [a > b > u \geq c].$$

BY (234.15)

17. $$\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}}\,dx = -\frac{2}{\sqrt{a-c}} E(\{,\,p) +$$
$$+\,2\sqrt{\frac{u-b}{(a-u)(u-c)}} \qquad [a > u > b > c].$$

BY (235.08)

BY (238.07)

19. $$\int_{-\infty}^{u} \sqrt{\frac{b-x}{(a-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}[F(\alpha,\, p) - E(\alpha,\, p)] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$$[a > b > c > u].$$

BY (231.04)

20. $$\int_{u}^{b} \sqrt{\frac{b-x}{(a-x)(x-c)^3}}\,dx = -\frac{2}{\sqrt{a-c}}E(\delta,\, q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}} \qquad [a > b > u > c].$$

BY (234.14)

21. $$\int_{b}^{u} \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}[F(\{,\, p) - E(\{,\, p)] \qquad [a \geq u > b > c].$$

BY (235.03)

22. $$\int_{u}^{a} \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}[F(\lambda,\, p) - E(\lambda,\, p)] + \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$$[a > u \geq b > c].$$

BY (236.14)

23. $$\int_{a}^{u} \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}E(\mu,\, q) - 2\frac{b-c}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u > a > b > c].$$

BY (237.11)

24. $$\int_{u}^{\infty} \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}E(\nu,\, q) \qquad [u \geq a > b > c].$$

BY (238.01)

25. $$\int_{-\infty}^{u} \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\alpha,\, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}F(\alpha,\, p) \qquad [a > b > c \geq u].$$

26. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\beta,\,p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}F(\beta,\,p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$$[a > b > c > u].$$

BY (232.03)

27. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\gamma,\,q) - \frac{2}{\sqrt{a-c}}F(\gamma,\,q) - \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$$[a > b \geq u > c].$$

BY (233.14)

28. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\delta,\,q) - \frac{2}{\sqrt{a-c}}F(\delta,\,q) \qquad [a > b > u \geq c].$$

BY (234.20)

29. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}}F(\{,\,p) - \frac{2\sqrt{a-c}}{a-b}E(\{,\,p) +$$
$$+2\frac{a-c}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}} \qquad [a > u > b > c].$$

BY (235.13)

30. $$\int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}}\,dx = \frac{2}{\sqrt{a-c}}F(\nu,\,q) - \frac{2\sqrt{a-c}}{a-b}E(\nu,\,q) + \frac{2(a-c)}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$$[u > a > b > c].$$

BY (238.08)

31. $$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}[F(\alpha,\,p) - E(\alpha,\,p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$$[a > b > c \geq u].$$

BY (231.06)

32. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}[F(\beta,\,p) - E(\beta,\,p)] \qquad [a > b > c > u].$$

BY (232.04)

$$33. \quad \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}}\,dx = -\frac{2\sqrt{a-c}}{a-b}E(\gamma,\, q) + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{b-u}} \quad [a > b > u > c].$$

BY (233.16)

$$34. \quad \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}[F(\lambda,\, p) - E(\lambda,\, p)] + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{u-b}} \quad [a > u > b > c].$$

BY (236.13)

$$35. \quad \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\mu,\, q) \qquad [u > a > b > c].$$

BY (237.01)

$$36. \quad \int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\nu,\, q) - 2\frac{b-c}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}} \qquad [u \geq a > b > c].$$

BY (238.11)

3.143

$$1.^6 \quad \int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}F\left(\operatorname{arctg}\frac{(1+\sqrt{2})(1-u)}{(1+u)},\, 2\sqrt[4]{2}(\sqrt{2}-1)\right)$$

ZH 66 (286)

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$$2. \quad \int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}F\left(\arccos\frac{u^2-1}{u^2+1},\, \frac{\sqrt{2}}{2}\right).$$

ZH 66 (287)

3.144

Notation: $\alpha = \arcsin\frac{1}{\sqrt{u^2-u+}}$.

BY (261.50)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)(u^2-u+1)}} - 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \qquad [u > 1].$$

BY (261.54)

3. $$\int_u^\infty \frac{(2x-1)^2\, dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = 4\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u\,(u-1)\,(u^2-u-1)}}\right]$$
$[u > 1]$.

BY (261.56)

4. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right] \qquad u \geq 1.$$

BY (261.52)

5. $$\int_u^\infty \frac{(2x-1)^2\, dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \qquad [u > 1].$$

BY (261.51)

6. $$\int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}}\, dx = \frac{4}{3}E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\alpha, \frac{\sqrt{3}}{2}\right) \qquad [u > 1].$$

BY (261.53)

7. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right] + \frac{1}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$[u > 1]$.

BY (261.57)

8. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}} \qquad [u > 1].$$

BY (261.58)

BY (261.58)

9. $$\int_u^\infty \frac{dx}{(2x-1)^2\sqrt{x(x-1)(x^2-x+1)}} = \frac{4}{3}E\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$$[u>1].$$

BY (261.55)

10. $$\int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5(x^2-x+1)}} = \frac{40}{3}E\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{4}{3}F\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)(9u^2-9u-1)}{3\sqrt{u^3(u-1)^3(u^2-u+1)}}$$
$$[u>1].$$

BY (261.54)

11. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^5}} = \frac{44}{27}F\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{56}{27}E\left(\alpha,\ \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}}$$
$$[u>1].$$

BY (261.52)

12. $$\int_u^\infty \frac{dx}{(2x-1)^4\sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27}E\left(\alpha,\ \frac{\sqrt{3}}{2}\right) - \frac{1}{27}F\left(\alpha,\ \frac{\sqrt{3}}{2}\right) -$$
$$- \frac{8(5u^2-5u+2)}{9(2u-1)^3}\sqrt{\frac{u(u-1)}{u^2-u+1}} \qquad [u>1].$$

BY (261.55)

3.145

1. $$\int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} =$$
$$= \frac{1}{\sqrt{pq}}F\left(2\ \text{arctg}\sqrt{\frac{q(u-\alpha)}{p(u-\beta)}},\ \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right) \qquad [\beta<\alpha<u].$$

2. $$\int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)[(x-m)^2+n^2]}} =$$
$$= \frac{1}{\sqrt{pq}}F\left(2\,\text{arcctg}\sqrt{\frac{q(\alpha-u)}{p(u-\beta)}},\ \frac{1}{2}\sqrt{\frac{-(p-q)^2+(\alpha-\beta)^2}{pq}}\right) \qquad [\beta<u<\alpha].$$

3. $$\int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \\ = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arctg} \sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right) \qquad [u < \beta < \alpha],$$

where $(m-\alpha)^2+n^2=p^2$, $(m-\beta)^2+n^2=q^2$ *). * Formulas 3.145 are not valid for $\alpha+\beta=2m$. In this case, we make the substitution $x-m=z$, which leads to one of the formulas 3.152.

4. Set

$$(m_1-m)^2+(n_1+n)^2=p^2, \qquad (m_1-m)^2+(n_1-n)^2=p_1^2,$$

$$\operatorname{ctg}\alpha = \sqrt{\frac{(p+p_1)^2-4n^2}{4n^2-(p-p_1)^2}};$$

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then

$$\int_{m-n \operatorname{tg}\alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2][(x-m_1)^2+n_1^2]}} = \frac{2}{p+p_1} F\left(\alpha + \operatorname{arctg}\frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right) \\ [m - n \operatorname{tg}\alpha < u < m + n \operatorname{ctg}\alpha].$$

3.146

1. $$\int_0^1 \frac{1}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4}\sqrt{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right).$$

BI ((13))(6)

2. $$\int_0^1 \frac{x^2}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}.$$

3. $$\int_0^1 \frac{x^4}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right).$$

BI ((13))(8)

In 3.147-3.151 we set: $\alpha = \arcsin\sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$
$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \{ = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$
$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

3.147

1. $$\int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\alpha, q) \qquad [a > b > c > d > u].$$

BY (251.00)

2. $$\int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\beta, r) \qquad [a > b > c \geq u > d].$$

BY (254.00)

3. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\gamma, r) \qquad [a > b > c > u \geq d].$$

BY (253.00)

4. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\delta, q) \qquad [a > b \geq u > c > d].$$

5. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\{,\, q) \qquad [a > b > u \geq c > d].$$

BY (255.00)

6. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\lambda,\, r) \qquad [a \geq u > b > c > d].$$

BY (256.00)

7. $$\int_u^\alpha \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\mu,\, r) \qquad [a > u \geq b > c > d].$$

BY (257.00)

8. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\nu,\, q) \qquad [u > a > b > c > d].$$

BY (258.00)

3.148

1.[8] $$\int_u^d \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-c)\,\Pi\left(\alpha,\, \frac{a-d}{a-c},\, q\right) + c\,F\,(\alpha,\, q)\right\}$$
$$[a > b > c > d > u].$$

BY (251.03)

2. $$\int_d^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-a)\Pi\left(\beta,\, \frac{d-c}{a-c},\, r\right) + aF\,(\beta,\, r)\right\}$$
$$[a > b > c \geq u > d].$$

BY (252.11)

3. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(c-b)\Pi\left(\gamma,\, \frac{c-d}{b-d},\, r\right) + bF\,(\gamma,\, r)\right\}$$
$$[a > b > c > u \geq d].$$

BY (253.11)

BY (254.10)

5. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-a)\Pi\left(\{,\ \frac{b-c}{a-c},\ q\right) + aF\left(\{,\ q\right)\right\}$$
$$[a > b > u \geq c > d].$$

BY (255.17)

6.[8] $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-c)\Pi\left(\lambda,\ \frac{a-b}{a-c},\ r\right) + cF\left(\lambda,\ r\right)\right\}$$
$$[a \geq u > b > c > d].$$

BY (256.11)

7. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-d)\Pi\left(\mu,\ \frac{b-a}{b-d},\ r\right) + dF\left(\mu,\ r\right)\right\}$$
$$[a > u \geq b > c > d].$$

BY (257.11)

8. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-b)\Pi\left(\nu,\ \frac{a-d}{b-d},\ q\right) + bF\left(\nu,\ q\right)\right\}$$
$$[u > a > b > c > d].$$

BY (258.11)

3.149

1. $$\int_u^d \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$
$$= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{(c-d)\Pi\left(\alpha,\ \frac{c(a-d)}{d(a-c)},\ q\right) + dF\left(\alpha,\ q\right)\right\} \qquad [a > b > c > d > u].$$

BY (251.04)

2. $$\int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$
$$= \frac{2}{ad\sqrt{(a-c)(b-d)}} \left\{(a-d)\Pi\left(\beta,\ \frac{a(d-c)}{d(a-c)},\ r\right) + dF\left(\beta,\ r\right)\right\} \qquad [a > b > c \geq u > d].$$

3. $$\int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$
$$= \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{(b-c)\Pi\left(\gamma,\ \frac{b(c-d)}{c(b-d)},\ r\right) + cF\left(\gamma,\ r\right)\right\} \qquad [a > b > c > u \geq d].$$

BY (253.12)

4. $$\int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$
$$= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{(d-c)\Pi\left(\delta,\ \frac{d(b-c)}{c(b-d)},\ q\right) + cF\left(\delta,\ q\right)\right\} \qquad [a > b \geq u > c > d].$$

BY (254.11)

5. $$\int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$
$$= \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{(a-b)\Pi\left(\{,\ \frac{a(b-c)}{b(a-c)},\ q\right) + bF\left(\{,\ q\right)\right\} \qquad [a > b > u \geq c > d].$$

BY (255.18)

6. $$\int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{(c-b)\Pi\left(\lambda,\ \frac{c(a-b)}{b(a-c)},\ r\right) + bF\left(\lambda,\ r\right)\right\} \qquad [a \geq u > b > c > d].$$

BY (256.12)

7. $$\int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{(d-a)\Pi\left(\mu,\ \frac{d(b-a)}{a(b-d)},\ r\right) + aF\left(\mu,\ r\right)\right\} \qquad [a > u \geq b > c > d].$$

BY (257.12)

8. $$\int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{ab\sqrt{(a-c)(b-d)}} \left\{(b-a)\Pi\left(\nu,\ \frac{b(a-d)}{a(b-d)},\ q\right) + aF\left(\nu,\ q\right)\right\} \qquad [u > a > b > c > d].$$

BY (258.12)

3.151

1. $$\int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$
$$= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(d-c)\Pi\left(\alpha,\, \frac{(a-d)(p-c)}{(a-c)(p-d)},\, q\right) + (p-d)\,F\,(\alpha,\, q)\right] \qquad [a > b > c > d > u, \quad p \neq d].$$

BY (251.39)

2. $$\int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$
$$= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(d-a)\Pi\left(\beta,\, \frac{(d-c)(p-a)}{(a-c)(p-d)},\, r\right) + (p-d)\,F\,(\beta,\, r)\right] \qquad [a > b > c \geq u > d, \quad p \neq d].$$

BY (252.39)

3. $$\int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$
$$= \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(c-b)\Pi\left(\gamma,\, \frac{(c-d)(p-b)}{(b-d)(p-c)},\, r\right) + (p-c)\,F\,(\gamma,\, r)\right] \qquad [a > b > c > u \geq d, \quad p \neq c].$$

BY (253.39)

4. $$\int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$
$$= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(c-d)\Pi\left(\delta,\, \frac{(b-c)(p-d)}{(b-d)(p-c)},\, q\right) + (p-c)\,F\,(\delta,\, q)\right] \qquad [a > b \geq u > c > d, \quad p \neq c].$$

BY (254.39)

5. $$\int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$
$$= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times$$

6. $$\int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(b-c)\Pi\left(\lambda,\ \frac{(a-b)(p-c)}{(a-c)(p-b)},\ r\right) + (p-b)\,F\,(\lambda,\ r)\right] \qquad [a \geq u > b > c > d, \quad p \neq b].$$

BY (256.39)

7. $$\int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(a-d)\Pi\left(\mu,\ \frac{(b-a)(p-d)}{(b-d)(p-a)},\ r\right) + (p-a)\,F\,(\mu,\ r)\right] \qquad [a > u \geq b > c > d, \quad p \neq a].$$

BY (257.39)

8. $$\int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$
$$= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times$$
$$\times \left[(a-b)\Pi\left(\nu,\ \frac{(a-d)(p-b)}{(b-d)(p-a)},\ q\right) + (p-a)\,F\,(\nu,\ q)\right] \qquad [u > a > b > c > d, \quad p \neq a].$$

BY (258.39)

In 3.152–3.163 we set: $\alpha = \operatorname{arctg}\frac{u}{b}$, $\beta = \operatorname{arcctg}\frac{u}{a}$,

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \qquad \delta = \arccos\frac{u}{b}, \qquad \varepsilon = \arccos\frac{b}{u}, \qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
$$\eta = \arcsin\frac{u}{b}, \qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \qquad \{ = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \qquad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \qquad \nu = \arcsin\frac{a}{u}, \qquad q = \frac{\sqrt{a^2-b^2}}{a},$$
$$r = \frac{b}{\sqrt{a^2+b^2}}, \qquad s = \frac{a}{\sqrt{a^2+b^2}}, \qquad t = \frac{b}{a}.$$

3.152

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}F(\alpha,\, q) \qquad [a > b > 0].$$

ZH 62(258), BY (221.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}F(\beta,\, q) \qquad [a > b > 0].$$

ZH 63 (259), BY (222.00)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\gamma,\, r) \qquad [b \geq u > 0].$$

ZH 63 (260)

4. $$\int_u^b \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\delta,\, r) \qquad [b > u \geq 0].$$

ZH 63 (261), BY (213.00)

5. $$\int_b^u \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\varepsilon,\, s) \qquad [u > b > 0].$$

ZH 63 (262), BY (211.00)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\xi,\, s) \qquad [u > b > 0].$$

ZH 63 (263), BY (212.00)

7. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}F(\eta,\, t) \qquad [a > b \geq u > 0].$$

8. $$\int_u^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\zeta, t) \qquad [a > b > u \geq 0].$$

ZH 63 (265), BY (220.00)

9. $$\int_b^u \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\{, q) \qquad [a \geq u > b > 0].$$

ZH 63 (266), BY (217.00)

10. $$\int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\lambda, q) \qquad [a > u \geq b > 0].$$

ZH 63 (257), BY (218.00)

11. $$\int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\mu, t) \qquad [u > a > b > 0].$$

ZH 63 (268), BY (216.00)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\nu, t) \qquad [u \geq a > b > 0].$$

ZH 64(269), BY (215.00)

3.153

1. $$\int_0^u \frac{x^2\, dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = u\sqrt{\frac{a^2 + u^2}{b^2 + u^2}} - aE(\alpha, q) \qquad [u > 0, \quad a > b].$$

BY (221.09)

2. $$\int_0^u \frac{x^2\, dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\gamma, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\gamma, r) - u\sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$
$$[b \geq u > 0].$$

BY (214.05)

3. $$\int_u^b \frac{x^2\, dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\delta, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\delta, r) \qquad [b > u \geq 0].$$

BY (213.06)

4. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{b^2}{\sqrt{a^2+b^2}}F(\varepsilon,\, s) - \sqrt{a^2+b^2}E(\varepsilon,\, s) + \frac{1}{u}\sqrt{(u^2+a^2)(u^2-b^2)}$$
$$[u > b > 0].$$

BY (211.09)

5. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = a\{F(\eta,\, t) - E(\eta,\, t)\} \qquad [a > b \geq u > 0].$$

BY (219.05)

6. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = a\{F(\zeta,\, t) - E(\zeta,\, t)\} + u\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u \geq 0].$$

BY (220.06)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = aE(\{,\, q) - \frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u > b > 0].$$

BY (217.05)

8. $$\int_u^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = aE(\lambda,\, q) \qquad [a > u \geq b > 0].$$

BY (218.06)

9.[6] $$\int_a^u \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = a\{F(\mu,\, t) - E(\mu,\, t)\} + u\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.06)

10. $$\int_0^1 \frac{x^2\,dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2}\left\{\sqrt{\frac{1+k^2}{2}} - E\left(\frac{\pi}{4},\, \sqrt{1-k^2}\right)\right\}.$$

BI ((14))(9)

3.154

1. $$\int_0^u \frac{x^4\,dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{a}{3}\{2(a^2+b^2)E(\alpha,\,q)-b^2F(\alpha,\,q)\}+\frac{u}{3}(u^2-2a^2-b^2)\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$$[a>b,\quad u>0].$$

BY (221.09)

2. $$\int_0^u \frac{x^4\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\{(2a^2-b^2)a^2F(\gamma,\,r)-2(a^4-b^4)E(\gamma,\,r)\}-$$
$$-\frac{u}{3}(2b^2-a^2+u^2)\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [a\ge u>0].$$

BY (214.05)

3. $$\int_u^b \frac{x^4\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\{(2a^2-b^2)a^2F(\delta,\,r)-2(a^4-b^4)E(\delta,\,r)\}+$$
$$+\frac{u}{3}\sqrt{(a^2+u^2)(b^2-u^2)} \qquad [b>u\ge 0].$$

BY (213.06)

4. $$\int_b^u \frac{x^4\,dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\{(2b^2-a^2)b^2F(\varepsilon,\,s)+2(a^4-b^4)E(\varepsilon,\,s)\}+$$
$$+\frac{2b^2-2a^2+u^2}{3u}\sqrt{(u^2+a^2)(u^2-b^2)} \qquad [u>b>0].$$

BY (211.09)

5. $$\int_0^u \frac{x^4\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3}\{(2a^2+b^2)F(\eta,\,t)-2(a^2+b^2)E(\eta,\,t)\}+\frac{u}{3}\sqrt{(a^2-u^2)(b^2-u^2)}$$
$$[a>b\ge u>0].$$

BY (219.05)

6. $$\int_u^b \frac{x^4\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3}\{(2a^2+b^2)F(\zeta,\,t)-2(a^2+b^2)E(\zeta,\,t)\}+$$
$$+\frac{u}{3}(u^2+a^2+2b^2)\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a>b>u\ge 0].$$

BY (220.06)

7. $$\int_b^u \frac{x^4\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{a}{3}\{2(a^2+b^2)E(\{,\,q) - b^2F(\{,\,q)\} - \\ -\frac{u^2+2a^2+2b^2}{3u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u > b > 0].$$

BY (217.05)

8. $$\int_u^a \frac{x^4\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{a}{3}\{2(a^2+b^2)E(\lambda,\,q) - b^2F(\lambda,\,q)\} + \frac{u}{3}\sqrt{(a^2-u^2)(u^2-b^2)} \\ [a > u \geq b > 0].$$

BY (218.06)

9. $$\int_a^u \frac{x^4\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{a}{3}\{(2a^2+b^2)F(\mu,\,t) - 2(a^2+b^2)E(\mu,\,t)\} + \\ +\frac{u}{3}(u^2+2a^2+b^2)\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.06)

3.155

1. $$\int_u^a \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}\{(a^2+b^2)E(\lambda,\,q) - 2b^2F(\lambda,\,q)\} - \frac{u}{3}\sqrt{(a^2-u^2)(u^2-b^2)} \\ [a > u \geq b > 0].$$

BY (218.11)

2. $$\int_a^u \sqrt{(x^2-a^2)(x^2-b^2)}\,dx = \frac{a}{3}\{(a^2+b^2)E(\mu,\,t) - (a^2-b^2)F(\mu,\,t)\} + \\ +\frac{u}{3}(u^2-a^2-2b^2)\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.10)

3. $$\int_0^u \sqrt{(x^2+a^2)(x^2+b^2)}\,dx = \frac{a}{3}\{2b^2F(\alpha,\,q) - (a^2+b^2)E(\alpha,\,q)\} + \\ +\frac{u}{3}(u^2+a^2+2b^2)\sqrt{\frac{a^2+u^2}{b^2+u^2}} \qquad [a > b, \quad u > 0].$$

4. $$\int_0^u \sqrt{(a^2+x^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\{a^2F(\gamma,\, r)-(a^2-b^2)E(\gamma,\, r)\}+ + \frac{u}{3}(u^2+2a^2-b^2)\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [a \geq u > 0].$$

BY (214.12)

5. $$\int_u^b \sqrt{(a^2+x^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\{a^2F(\delta,\, r)+2(b^2-a^2)E(\delta,\, r)\}+ + \frac{u}{3}\sqrt{(a^2+u^2)(b^2-u^2)} \qquad [b > u \geq 0].$$

BY (213.13)

6. $$\int_b^u \sqrt{(a^2+x^2)(x^2-b^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\{(b^2-a^2)E(\varepsilon,\, s)-b^2F(\varepsilon,\, s)\}+ + \frac{u^2+a^2-b^2}{3u}\sqrt{(a^2+u^2)(u^2-b^2)} \qquad [u > b > 0].$$

BY (211.08)

7. $$\int_0^u \sqrt{(a^2-x^2)(b^2-x^2)}\,dx = \frac{a}{3}\{(a^2+b^2)E(\eta,\, t)-(a^2-b^2)F(\eta,\, t)\}+ + \frac{u}{3}\sqrt{(a^2-u^2)(b^2-u^2)} \qquad [a > b \geq u > 0].$$

BY (219.11)

8. $$\int_u^b \sqrt{(a^2-x^2)(b^2-x^2)}\,dx = \frac{a}{3}\{(a^2+b^2)E(\zeta,\, t)-(a^2-b^2)F(\zeta,\, t)\}+ + \frac{u}{3}(u^2-2a^2-b^2)\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u \geq 0].$$

BY (220.05)

9. $$\int_b^u \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}\{(a^2+b^2)E(\{,\, q)-2b^2F(\{,\, q)\}+ + \frac{u^2-a^2-b^2}{3u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u > b > 0].$$

BY (217.09)

3.156

1.[6] $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{ub^2}\sqrt{\frac{b^2+u^2}{a^2+u^2}}-\frac{1}{ab^2}E(\beta,\, q) \qquad [a \geq b, \quad u > 0].$$

2. $$\int_u^b \frac{dx}{x^2\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\{a^2F(\delta, r)-(a^2+b^2)E(\delta, r)\}+ + \frac{1}{a^2b^2u}\sqrt{(a^2+u^2)(b^2-u^2)} \qquad [b > u > 0].$$

BY (213.09)

3. $$\int_b^u \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\{(a^2+b^2)E(\varepsilon, s)-b^2F(\varepsilon, s)\} \qquad [u > b > 0].$$

BY (211.11)

4.[6] $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\{(a^2+b^2)E(\xi, s)-b^2F(\xi, s)\}- - \frac{1}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}} \qquad [u \geq b > 0].$$

BY (212.06)

5. $$\int_u^b \frac{dx}{x^2\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ab^2}\{F(\zeta, t)-E(\zeta, t)\}+\frac{1}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u > 0].$$

BY (220.09)

6. $$\int_b^u \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\{, q) \qquad [a \geq u > b > 0].$$

BY (217.01)

7. $$\int_u^a \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\lambda, q)-\frac{1}{a^2b^2u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a > u \geq b > 0].$$

BY (218.12)

8. $$\int_a^u \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\{F(\mu, t)-E(\mu, t)\}+\frac{1}{a^2u}\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

9. $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\{F(\nu,\, t) - E(\nu,\, t)\} \qquad [u \geq a > b > 0].$$

BY (215.07)

3.157

1. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p+b^2)}\left\{\frac{b^2}{p}\Pi\left(\alpha,\, \frac{p+b^2}{p},\, q\right) + F\left(\alpha,\, q\right)\right\} \qquad [p \neq 0].$$

BY (221.13)

2. $$\int_u^\infty \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)}\left\{\Pi\left(\beta,\, \frac{a^2+p}{a^2},\, q\right) - F\left(\beta,\, q\right)\right\}.$$

BY (222.11)

3. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{p(p+a^2)\sqrt{a^2+b^2}}\left\{a^2\Pi\left(\gamma,\, \frac{b^2(p+a^2)}{p(a^2+b^2)},\, r\right) + pF\left(\gamma,\, r\right)\right\}$$
$$[b \geq u > 0, \quad p \neq 0].$$

BY (214.13)a

4. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{(p-b^2)\sqrt{a^2+b^2}}\Pi\left(\delta,\, \frac{b^2}{b^2-p},\, r\right)$$
$$[b > u \geq 0, \quad p \neq b^2].$$

BY (213.02)

5. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(x^2-b^2)}} =$$
$$= \frac{1}{p(p-b^2)\sqrt{a^2+b^2}}\left\{b^2\Pi\left(\varepsilon,\, \frac{p}{p-b^2},\, s\right) + \left(p-b^2\right)F\left(\varepsilon,\, s\right)\right\} \qquad [u > b > 0, \quad p \neq b^2].$$

BY (211.14)

6. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{(a^2+p)\sqrt{a^2+b^2}}\left\{\Pi\left(\xi,\, \frac{a^2+p}{a^2+b^2},\, s\right) - F\left(\xi,\, s\right)\right\}$$
$$[u \geq b > 0].$$

BY (212.12)

7. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ap}\Pi\left(\eta,\, \frac{b^2}{p},\, t\right) \qquad [a > b \geq u > 0; \quad p \neq b].$$

8. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times$$
$$\times\left\{(b^2-a^2)\Pi\left(\zeta,\ \frac{b^2(p-a^2)}{a^2(p-b^2)},\ t\right) + \left(p-b^2\right)F\left(\zeta,\ t\right)\right\}$$
$$[a > b > u \geq 0; \quad p \neq b^2].$$

BY (220.13)

9. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)}\left\{b^2\Pi\left(\boldsymbol{\{},\ \frac{p(a^2-b^2)}{a^2(p-b^2)},\ q\right) + \left(p-b^2\right)F\left(\boldsymbol{\{},\ q\right)\right\}$$
$$[a \geq u > b > 0; \quad p \neq b^2].$$

BY (217.12)

10. $$\int_u^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)}\Pi\left(\lambda,\ \frac{a^2-b^2}{a^2-p},\ q\right) \qquad [a > u \geq b > 0;\ \ p \neq a^2].$$

BY (218.02)

11. $$\int_a^u \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} =$$
$$= \frac{1}{a(p-a^2)(p-b^2)}\left\{(a^2-b^2)\Pi\left(\mu,\ \frac{p-b^2}{p-a^2},\ t\right) + \left(p-a^2\right)F\left(\mu,\ t\right)\right\}$$
$$[u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2].$$

BY (216.12)

12. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap}\left\{\Pi\left(\nu,\ \frac{p}{a^2},\ t\right) - F\left(\nu,\ t\right)\right\} \qquad [u \geq a > b > 0; \quad p \neq 0].$$

BY (215.12)

3.158

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\{a^2E(\alpha,\ q) - b^2F(\alpha,\ q)\} \qquad [a > b; \quad u > 0].$$

BY (221.05)

BY (222.05)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\{F(\alpha,\, q)-E(\alpha,\, q)\}+\frac{u}{a^2\sqrt{(u^2+a^2)(u^2+b^2)}}$$
$$[a > b; \quad u > 0].$$

BY (221.06)

4. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\{F(\beta,\, q)-E(\beta,\, q)\} \qquad [a > b, \quad u \geq 0].$$

BY (222.03)

301

5. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}E(\gamma,\, r) \qquad [b \geq u > 0].$$

BY (214.01)a

6. $$\int_u^b \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}E(\delta,\, r)-\frac{u}{a^2(a^2+b^2)}\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [b > u \geq 0].$$

BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\{F(\varepsilon,\, s)-E(\varepsilon,\, s)\}+\frac{1}{(a^2+b^2)u}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$
$$[u > b > 0].$$

BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\{F(\xi,\, s)-E(\xi,\, s)\} \qquad [u \geq b > 0].$$

BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)^3}} = \frac{1}{b^2\sqrt{a^2+b^2}}\{F(\gamma,\, r)-E(\gamma,\, r)\}+\frac{u}{b^2\sqrt{(a^2+u^2)(b^2-u^2)}}$$
$$[b > u > 0].$$

10. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^3}} = \frac{u}{b^2\sqrt{(a^2+u^2)(u^2-b^2)}} - \frac{1}{b^2\sqrt{a^2+b^2}} E(\xi,\, s) \qquad [u \geq b > 0].$$

BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a^2(a^2-b^2)} \left\{ aE(\eta,\, t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}} \right\} \qquad [a > b \geq u > 0].$$

BY (219.07)

12. $$\int_u^b \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a(a^2-b^2)} E(\zeta,\, t) \qquad [a > b > u \geq 0].$$

BY (220.10)

13. $$\int_b^u \frac{dx}{\sqrt{(a^2-x^2)^3(x^2-b^2)}} = \frac{1}{a(a^2-b^2)} \left\{ F(\{,\, q) - E(\{,\, q) + \frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}} \right\} \qquad [a > u > b > 0].$$

BY (217.10)

14. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)}} = \frac{1}{a(b^2-a^2)} \left\{ E(\nu,\, t) - \frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} \right\} \qquad [u > a > b > 0].$$

BY (215.04)

15. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)^3}} = \frac{1}{ab^2} F(\eta,\, t) - \frac{1}{b^2(a^2-b^2)} \left\{ aE(\eta,\, t) - u\sqrt{\frac{a^2-u^2}{b^2-u^2}} \right\}$$
$$[a > b > u > 0].$$

BY (219.06)

16. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{ab^2(a^2-b^2)} \left\{ b^2F(\lambda,\, q) - a^2E(\lambda,\, q) + au\sqrt{\frac{a^2-u^2}{u^2-b^2}} \right\}$$
$$[a > u > b > 0].$$

BY (218.04)

17. $$\int_a^u \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{b^2(a^2-b^2)} E(\mu,\, t) - \frac{1}{ab^2} F(\mu,\, t) \qquad [u > a > b > 0].$$

18. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{1}{b^2(a^2-b^2)}\left\{aE(\nu,\, t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\} - \frac{1}{ab^2}F(\nu,\, t)$$
$$[u \geq a > b > 0].$$

BY (215.06)

3.159

1. $$\int_0^u \frac{x^2\, dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2}\{F(\alpha,\, q) - E(\alpha,\, q)\} \qquad [a > b, \quad u > 0].$$

BY (221.12)

2. $$\int_u^\infty \frac{x^2\, dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2}\{F(\beta,\, q) - E(\beta,\, q)\} + \frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$
$$[a > b, \quad u \geq 0].$$

BY (222.10)

3. $$\int_0^u \frac{x^2\, dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\{a^2E(\alpha,\, q) - b^2F(\alpha,\, q)\} - \frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$
$$[a > b, \quad u > 0].$$

BY (221.11)

4. $$\int_u^\infty \frac{x^2\, dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\{a^2E(\beta,\, q) - b^2F(\beta,\, q)\} \qquad [a > b, \quad u \geq 0].$$

BY (222.07)

5. $$\int_0^u \frac{x^2\, dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\{F(\gamma,\, r) - E(\gamma,\, r)\} \qquad [b \geq u > 0].$$

BY (214.04)

6. $$\int_u^b \frac{x^2\, dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\{F(\delta,\, r) - E(\delta,\, r)\} + \frac{u}{a^2+b^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$$[b > u \geq 0].$$

BY (213.07)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}E(\varepsilon,\, s) - \frac{a^2}{u(a^2+b^2)}\sqrt{\frac{u^2-b^2}{u^2+a^2}} \qquad [u > b > 0].$$

BY (211.13)

8. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}E(\xi, s) \qquad [u \geq b > 0].$$

BY (212.01)

303

9. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)^3}} = \frac{u}{\sqrt{(a^2+u^2)(b^2-u^2)}} - \frac{1}{\sqrt{a^2+b^2}}E(\gamma,\, r) \qquad [b > u > 0].$$

BY (214.07)

10. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(a^2+x^2)(x^2-b^2)^3}} = \frac{1}{\sqrt{a^2+b^2}}\{F(\xi,\, s) - E(\xi,\, s)\} + \frac{u}{\sqrt{(a^2+u^2)(u^2-b^2)}}$$
$$[u > b > 0].$$

BY (212.10)

11. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a^2-b^2}\left\{aE(\eta,\, t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} - \frac{1}{a}F(\eta,\, t) \qquad [a > b \geq u > 0].$$

BY (219.04)

12. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{a}{a^2-b^2}E(\zeta,\, t) - \frac{1}{a}F(\zeta,\, t) \qquad [a > b > u \geq 0].$$

BY (220.08)

13. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(x^2-b^2)}} = \frac{1}{a(a^2-b^2)}\left\{b^2F(\{,\, q) - a^2E(\{,\, q) + \frac{a^3}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}}\right\}$$
$$[a > u > b > 0].$$

BY (217.06)

14. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)}} = \frac{a}{a^2-b^2}\left\{\frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} - E(\nu,\,t)\right\} + \frac{1}{a}F(\nu,\,t)$$
$[u > a > b > 0]$.

BY (215.09)

15. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)^3}} = \frac{1}{a^2-b^2}\left\{u\sqrt{\frac{a^2-u^2}{b^2-u^2}} - aE(\eta,\,t)\right\} \qquad [a > b > u > 0].$$

BY (219.12)

16. $$\int_u^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left\{aF(\lambda,\,q) - aE(\lambda,\,q) + u\sqrt{\frac{a^2-u^2}{u^2-b^2}}\right\}$$
$[a > u > b > 0]$.

BY (218.07)

17. $$\int_a^u \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{a^2-b^2}E(\mu,\,t) \qquad [u > a > b > 0].$$

BY (216.01)

18. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left\{aE(\nu,\,t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\} \qquad [u \ge a > b > 0].$$

BY (215.11)

3.161

1. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{3a^3b^4}\{2(a^2+b^2)E(\beta,\,q) - b^2F(\beta,\,q)\} + \frac{a^2b^2-u^2(2a^2+b^2)}{3a^2b^4u^3}$$
$[a > b, \quad u > 0]$.

BY (222.04)

2. $$\int_u^b \frac{dx}{x^4\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\{a^2(2a^2-b^2)F(\delta,\,r) - 2(a^4-b^4)E(\delta,\,r)\} + \frac{a^2b^2+2u^2(a^2-b^2)}{3a^4b^4u^3}\sqrt{(b^2-u^2)(a^2+u^2)} \qquad [b > u > 0].$$

3. $$\int_b^u \frac{dx}{x^4\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{2b^2-a^2}{3a^4b^2\sqrt{a^2+b^2}}F(\varepsilon,\ s)+\frac{2}{3}\frac{(a^2-b^2)\sqrt{a^2+b^2}}{a^4b^4}E(\varepsilon,\ s)+ \\ +\frac{1}{3a^2b^2u^3}\sqrt{(u^2+a^2)(u^2-b^2)} \qquad [u>b>0].$$

BY (211.11)

4. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\{2(a^4-b^4)E(\xi,\ s)+b^2(2b^2-a^2)F(\xi,\ s)\}- \\ -\frac{a^2b^2+u^2(2a^2-b^2)}{3a^2b^4u^3}\sqrt{\frac{u^2-b^2}{u^2+a^2}} \qquad [u \geq b>0].$$

BY (212.06)

5. $$\int_u^b \frac{dx}{x^4\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{3a^3b^4}\left\{(2a^2+b^2)F(\zeta,\ t)-2(a^2+b^2)E(\zeta,\ t)+ \right. \\ \left. +\frac{[(2a^2+b^2)u^2+a^2b^2]a}{u^3}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} \qquad [a>b>u>0].$$

BY (220.09)

6. $$\int_b^u \frac{dx}{x^4\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\{2(a^2+b^2)E(\boldsymbol{\{},\ q)-b^2F(\boldsymbol{\{},\ q)\}+ \\ +\frac{1}{3a^2b^2u^3}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u>b>0].$$

BY (217.14)

7. $$\int_u^a \frac{dx}{x^4\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{2(a^2+b^2)E(\lambda,\ q)-b^2F(\lambda,\ q)- \right. \\ \left. -\frac{2(a^2+b^2)u^2+a^2b^2}{au^3}\sqrt{(a^2-u^2)(u^2-b^2)}\right\} \qquad [a>u \geq b>0].$$

BY (218.12)

8. $$\int_a^u \frac{dx}{x^4\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{(2a^2+b^2)F(\mu,\ t)-2(a^2+b^2)E(\mu,\ t)+ \right. \\ \left. +\frac{[(a^2+2b^2)u^2+a^2b^2]b^2}{au^3}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\} \qquad [u>a>b>0].$$

9. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{(2a^2+b^2)F(\nu,\, t) - 2(a^2+b^2)E(\nu,\, t) + \right.$$
$$\left. + \frac{ab^2}{u^3}\sqrt{(u^2-a^2)(u^2-b^2)}\right\} \qquad [u \geq a > b > 0].$$

BY (215.07)

3.162

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^5(x^2+b^2)}} = \frac{1}{3a^3(a^2-b^2)^2}\{(3a^2-b^2)F(\alpha,\, q) - 2(2a^2-b^2)E(\alpha,\, q)\} +$$
$$+ \frac{u[a^2(4a^2-3b^2)+u^2(3a^2-2b^2)]}{3a^4(a^2-b^2)\sqrt{(u^2+a^2)^3(u^2+b^2)}} \qquad [a > b, \quad u > 0].$$

BY (221.06)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)^5(x^2+b^2)}} = \frac{1}{3a^3(a^2-b^2)^2}\{(3a^2-b^2)F(\beta,\, q) - 2(2a^2-b^2)E(\beta,\, q)\} +$$
$$+ \frac{u}{3a^2(a^2-b^2)}\sqrt{\frac{u^2+b^2}{(a^2+u^2)^3}} \qquad [a > b, \quad u \geq 0].$$

BY (222.03)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^5}} = \frac{3b^2-a^2}{3ab^2(a^2-b^2)^2}F(\alpha,\, q) + \frac{a(2a^2-4b^2)}{3b^4(a^2-b^2)^2}E(\alpha,\, q) +$$
$$+ \frac{u}{3b^2(a^2-b^2)}\sqrt{\frac{u^2+a^2}{(u^2+b^2)^3}} \qquad [a > b, \quad u > 0].$$

BY (221.05)

4. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^5}} = \frac{1}{3ab^4(a^2-b^2)^2}\{2a^2(a^2-2b^2)E(\beta,\, q) + b^2(3b^2-a^2)F(\beta,\, q)\} -$$
$$- \frac{u[b^2(3a^2-4b^2)+u^2(2a^2-3b^2)]}{3b^4(a^2-b^2)\sqrt{(u^2+a^2)(u^2+b^2)^3}} \qquad [a > b, \quad u \geq 0].$$

BY (222.05)

5. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)^5(b^2-x^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\{2(b^2+2a^2)E(\gamma,\, r) - a^2F(\gamma,\, r)\} +$$
$$+ \frac{u}{3a^2(a^2+b^2)}\sqrt{\frac{b^2-u^2}{(a^2+u^2)^3}} \qquad [b \geq u > 0].$$

6. $$\int_u^b \frac{dx}{\sqrt{(a^2+x^2)^5(b^2-x^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\{(4a^2+2b^2)E(\delta,\, r)-a^2F(\delta,\, r)\}- -\frac{u[a^2(5a^2+3b^2)+u^2(4a^2+2b^2)]}{3a^4(a^2+b^2)^2}\sqrt{\frac{b^2-u^2}{(a^2+u^2)^3}} \quad [b>u>0].$$

BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{(a^2+x^3)^5(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{(3a^2+2b^2)F(\varepsilon,\, s)-(4a^2+2b^2)E(\varepsilon,\, s)\right\} + + \frac{(3a^2+b^2)u^2+2(2a^2+b^2)a^2}{3a^2(a^2+b^2)^2u}\sqrt{\frac{u^2-b^2}{(u^2+a^2)^3}} \qquad [u>b>0].$$

BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^5(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{(3a^2+2b^2)F(\xi,\, s)-(4a^2+2b^2)E(\xi,\, s)\right\} + + \frac{u}{3a^2(a^2+b^2)}\sqrt{\frac{u^2-b^2}{(a^2+u^2)^3}} \qquad [u>b>0].cr$$

BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)^5}} = \frac{1}{3b^4\sqrt{(a^2+b^2)^3}}\left\{(2a^2+3b^2)F(\gamma,\, r)-(2a^2+4b^2)E(\gamma,\, r)\right\} + + \frac{u[(3a^3+4b^2)b^2-(2a^2+3b^2)u^2]}{3b^4(a^2+b^2)\sqrt{(a^2+u^2)(b^2-u^2)^3}} \qquad [b>u>0].$$

BY (214.10)

10. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^5}} = \frac{1}{3b^4\sqrt{(a^2+b^2)^3}}\left\{(2a^2+4b^2)E(\xi,\, s)-b^2F(\xi,\, s)\right\} + + \frac{u[(3a^2+4b^2)b^2-(2a^2+3b^2)u^2]}{3b^4(a^2+b^2)\sqrt{(a^2+u^2)(u^2-b^2)^3}} \qquad [u>b>0].$$

BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)^5}} = \frac{2a^2-3b^2}{3ab^4(a^2-b^2)}F(\eta,\, t)+\frac{2a(2b^2-a^2)}{3b^4(a^2-b^2)^2}E(\eta,\, t)+ + \frac{u[(3a^2-5b^2)b^2-2(a^2-2b^2)u^2]}{3b^4(a^2-b^2)^2(b^2-u^2)}\sqrt{\frac{a^2-u^2}{b^2-u^2}} \quad [a>b>a>0].$$

12. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)^5}} = \frac{3b^2-a^2}{3ab^2(a^2-b^2)^2} F(\lambda,\, q) + \frac{2a(a^2-2b^2)}{3b^4(a^2-b^2)^2} E(\lambda,\, q) + \\ + \frac{u[2(2b^2-a^2)u^2+(3a^2-5b^2)b^2]}{3b^4(a^2-b^2)^2(u^2-b^2)} \sqrt{\frac{a^2-u^2}{u^2-b^2}} \\ [a > u > b > 0].$$

BY (218.04)

13. $$\int_a^u \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^5}} = \frac{2a^2-3b^2}{3ab^4(a^2-b^2)} F(\mu,\, t) + \frac{2a(2b^2-a^2)}{3b^4(a^2-b^2)^2} E(\mu,\, t) + \\ + \frac{u}{3b^2(a^2-b^2)(u^2-b^2)} \sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.11)

14. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^5}} = \frac{(4b^2-2a^2)a}{3b^4(a^2-b^2)^2} E(\nu,\, t) + \frac{2a^2-3b^2}{3ab^4(a^2-b^2)} F(\nu,\, t) - \\ - \frac{(3b^2-a^2)u^2-(4b^2-2a^2)b^2}{3b^2u(a^2-b^2)^2(u^2-b^2)} \sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u \geq a > b > 0].$$

BY (215.06)

15. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)^5(b^2-x^2)}} = \frac{1}{3a^3(a^2-b^2)^2} \left\{(4a^2-2b^2)E(\eta,\, t) - (a^2-b^2)F(\eta,\, t) - \right. \\ \left. - \frac{u[(5a^2-3b^2)a^2-(4a^2-2b^2)u^2]}{a(a^2-u^2)} \sqrt{\frac{b^2-u^2}{a^2-u^2}} \right\} \qquad [a > b \geq u > 0].$$

BY (219.07)

16. $$\int_u^b \frac{dx}{\sqrt{(a^2-x^2)^5(b^2-x^2)}} = \frac{2(2a^2-b^2)}{3a^3(a^2-b^2)^2} E(\zeta,\, r) - \frac{1}{3a^3(a^2-b^2)} F(\zeta,\, t) + \\ + \frac{u}{3a^2(a^2-b^2)(a^2-u^2)} \sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u \geq 0].$$

BY (220.10)

17. $$\int_b^u \frac{dx}{\sqrt{(a^2-x^2)^5(x^2-b^2)}} = \frac{1}{3a^3(a^2-b^2)^2} \left\{(3a^2-b^2)F(\{,\, q) - (4a^2-2b^2)E(\{,\, q)\right\} + \\ + \frac{2(2a^2-b^2)a^2+(b^2-3a^2)u^2}{3a^2u(a^2-b^2)^2(a^2-u^2)} \sqrt{\frac{u^2-b^2}{a^2-u^2}}, \qquad [a > u > b > 0].$$

18. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)^5(x^2-b^2)}} = \frac{1}{3a^3(a^2-b^2)^2}\left\{(4a^2-2b^2)E(\nu,\,t)-(a^2-b^2)F(\nu,\,t)\right\} + \\ + \frac{(4a^2-2b^2)a^2+(b^2-3a^2)u^2}{3a^2u(a^2-b^2)^2(u^2-a^2)}\sqrt{\frac{u^2-b^2}{u^2-a^2}} \qquad [u>a>b>0].$$

BY (215.04)

3.163

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)^2}\left\{(a^2+b^2)E(\alpha,\,q)-2b^2F(\alpha,\,q)\right\} - \\ - \frac{u}{a^2(a^2-b^2)\sqrt{(a^2+u^2)(b^2+u^2)}} \qquad [a>b, \quad u>0].$$

BY (221.07)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)^2}\left\{(a^2+b^2)E(\beta,\,q)-2b^2F(\beta,\,q)\right\} - \\ - \frac{u}{b^2(a^2-b^2)\sqrt{(a^2+u^2)(b^2+u^2)}} \qquad [a>b, \quad u\ge 0].$$

BY (222.12)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^3(b^3-x^2)^3}} = \frac{1}{a^2b^2\sqrt{(a^2+b^2)^3}}\left\{a^2F(\gamma,\,r)-(a^2-b^2)E(\gamma,\,r)\right\} + \\ + \frac{u}{b^2(a^2+b^2)\sqrt{(a^2+u^2)(b^2-u^2)}} \qquad [b>u>0].$$

BY (214.15)

4. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)^3}} = \frac{b^2-a^2}{a^2b^2\sqrt{(a^2+b^2)^3}}E(\xi,\,s) - \frac{1}{a^2\sqrt{(a^2+b^2)^3}}F(\xi,\,s) + \\ + \frac{u}{b^2(a^2+b^2)\sqrt{(u^2+a^2)(u^2-b^2)}} \qquad [u>b>0].$$

BY (212.05)

5. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)^3}} = \frac{1}{ab^2(a^2-b^2)}F(\eta,\,t) - \frac{a^2+b^2}{ab^2(a^2-b^2)^2}E(\eta,\,t) + \\ + \frac{[a^4+b^4-(a^2+b^2)u^2]u}{a^2b^2(a^2-b^2)^2\sqrt{(a^2-u^2)(b^2-u^2)}} \qquad [a>b>u>0].$$

BY (279.08)

BY (215.10)

3.164

Notations:

$\alpha = \arccos \frac{u^2-\rho\overline{\rho}}{u^2+\rho\overline{\rho}}, r = \frac{1}{2}\sqrt{-\frac{(\rho-\overline{\rho})^2}{\rho\overline{\rho}}}$.

1. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+\rho^2)(x^2+\overline{\rho}^2)}} = \frac{1}{\sqrt{\rho\overline{\rho}}} F(\alpha,\, r).$$

BY (225.00)

2. $$\int_u^\infty \frac{x^2\, dx}{(x^2-\rho\overline{\rho})^2\sqrt{(x^2+\rho^2)(x^2+\overline{\rho}^2)}} = \frac{2u\sqrt{(u^2+\rho^2)(u^2+\overline{\rho}^2)}}{(\rho+\overline{\rho})^2(u^4-\rho^2\,\overline{\rho}^2)} - \frac{1}{(\rho+\overline{\rho})^2\sqrt{\rho\overline{\rho}}} E(\alpha,\, r).$$

BY (225.03)

3. $$\int_u^\infty \frac{x^2\, dx}{(x^2+\rho\overline{\rho})^2\sqrt{(x^2+\rho^2)(x^2+\overline{\rho}^2)}} = -\frac{1}{(\rho-\overline{\rho})^2\sqrt{\rho\overline{\rho}}}[F(\alpha,\, r) - E(\alpha,\, r)].$$

BY (225.07)

4. $$\int_u^\infty \frac{x^2\, dx}{\sqrt{(x^2+\rho^2)^3(x^2+\overline{\rho}^2)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{(\rho^2-\overline{\rho}^2)^2} E(\alpha,\, r) + \frac{1}{(\rho-\overline{\rho})^2\sqrt{\rho\overline{\rho}}} F(\alpha,\, r) - \\ - \frac{2u(u^2-\rho\overline{\rho})}{(\rho+\overline{\rho})^2(u^2+\rho\overline{\rho})\sqrt{(u^2+\rho^2)(u^2+\overline{\rho}^2)}}.$$

BY (225.05)

5. $$\int_u^\infty \frac{(x^2-\rho\overline{\rho})^2\, dx}{\sqrt{(x^2+\rho^2)^3(x^2+\overline{\rho}^2)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{(\rho-\overline{\rho})^2}[F(\alpha,\, r) - E(\alpha,\, r)] + \\ + \frac{2u(u^2-\rho\overline{\rho})}{(u^2+\rho\overline{\rho})\sqrt{(u^2+\rho^2)(u^2+\overline{\rho}^2)}}.$$

BY (225.06)

6. $$\int_u^\infty \frac{\sqrt{(x^2+\rho^2)(x^2+\overline{\rho}^2)}}{(x^2+\rho\overline{\rho})^2}\, dx = \frac{1}{\sqrt{\rho\overline{\rho}}} E(\alpha,\, r).$$

7. $$\int_u^\infty \frac{(x^2-\varrho\overline{\varrho})^2\,dx}{(x^2+\varrho\overline{\varrho})^2\sqrt{(x^2+\varrho^2)(x^2+\overline{\varrho}^2)}} = -\frac{4\sqrt{\varrho\overline{\varrho}}}{(\varrho-\overline{\varrho})^2}E(\alpha,\,r)+\frac{(\varrho+\overline{\varrho})^2}{(\varrho-\overline{\varrho})^2\sqrt{\varrho\overline{\varrho}}}F(\alpha,\,r).$$

BY (225.08)

8. $$\int_u^\infty \frac{(x^2+\varrho\overline{\varrho})^2\,dx}{[(x^2+\varrho\overline{\varrho})^2-4p^2\varrho\overline{\varrho}x^2]\sqrt{(x^2+\varrho^2)(x^2+\overline{\varrho}^2)}} = \frac{1}{\sqrt{\varrho\overline{\varrho}}}\Pi(\alpha,\,p^2,\,r).$$

BY (225.02)

3.165

Notations:

$\alpha = \arccos\frac{u^2-a^2}{u^2+a^2}, r = \frac{\sqrt{a^2-b^2}}{a\sqrt{2}}$.

1. $$\int_u^a \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{\sqrt{2}}{a\sqrt{2}+\sqrt{a^2+b^2}} \times$$
$$\times F\left[\operatorname{arctg}\left(\frac{a\sqrt{2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}}\,\frac{a-u}{a+u}\right),\,\frac{2\sqrt{a\sqrt{2\,(a^2-b^2)}}}{a\sqrt{2}+\sqrt{a^2-b^2}}\right]$$
$$[a>b,\quad a>u\geq 0].$$

BY (264.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}F(\alpha,\,r) \qquad [a^2>b^2>-\infty,\quad a^2>0,\quad u\geq 0].$$

BY (263.00, 266.00)

3. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a^3}[F(\alpha,\,r)-2E(\alpha,\,r)]+\frac{\sqrt{u^4+2b^2u^2+a^4}}{a^2u(u^2+a^2)}$$
$$[a>b>0,\quad u>0].$$

BY (263.06)

4. $$\int_u^\infty \frac{x^2\,dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{4a(a^2-b^2)}[F(\alpha,\,r)-E(\alpha,\,r)]$$
$$[a^2>b^2>-\infty,\quad a^2>0,\quad u\geq 0].$$

5. $$\int_u^\infty \frac{x^2\,dx}{(x^2-a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{u\sqrt{u^4+2b^2u^2+a^4}}{2(a^2+b^2)(u^4-a^4)} - \frac{1}{4a(a^2+b^2)}E(\alpha,\,r)$$
$$[a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0].$$

BY (263.05, 266.02)

6. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^4+2b^2x^2+a^4)^3}} = \frac{a}{2(a^4-b^4)}E(\alpha,\,r) - \frac{1}{4a(a^2-b^2)}F(\alpha,\,r) - $$
$$- \frac{u(u^2-a^2)}{2(a^2+b^2)(u^2+a^2)\sqrt{u^4+2b^2u^2+a^4}}$$
$$[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

BY (263.08, 266.03)

7. $$\int_u^\infty \frac{(x^2-a^2)^2\,dx}{\sqrt{(x^4+2b^2x^2+a^4)^3}} = \frac{a}{a^2-b^2}[F(\alpha,\,r) - E(\alpha,\,r)] + \frac{u^2-a^2}{u^2+a^2}\frac{u}{\sqrt{u^4+2b^2u^2+a^4}}$$
$$[|b^2| < a^2, \quad u \geq 0].$$

BY (266.08)

8. $$\int_u^\infty \frac{(x^2+a^2)^2\,dx}{\sqrt{(x^2+2b^2x^2+a^4)^3}} = \frac{a}{a^2+b^2}E(\alpha,\,r) - \frac{a^2-b^2}{a^2+b^2}\cdot\frac{u^2-a^2}{u^2+a^2}\cdot\frac{u}{\sqrt{u^4+2b^2u^2+a^4}}$$
$$[|b^2| < a^2, \quad u \geq 0].$$

BY (266.06)a

9. $$\int_u^\infty \frac{(x^2-a^2)^2\,dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{a}{a^2-b^2}E(\alpha,\,r) - \frac{a^2+b^2}{2a(a^2-b^2)}F(\alpha,\,r)$$
$$[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$$

BY (263.04, 266.07)

10. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{(x^2+a^2)^2}\,dx = \frac{1}{2a}E(\alpha,\,r) \qquad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

BY (263.01, 266.01)

11. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{(x^2-a^2)^2}\,dx = \frac{1}{2a}[F(\alpha,\,r) - E(\alpha,\,r)] + \frac{u}{u^4-a^4}\sqrt{u^4+2b^2u^2+a^4}$$
$$[a > b > 0, \quad u > a].$$

12. $$\int_u^\infty \frac{(x^2+a^2)^2\,dx}{[(x^2+a^2)^2-4a^2p^2x^2]\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}\Pi(\alpha,\,p^2,\,r) \qquad [a>b>0, \quad u\ge 0].$$

BY (263.02)

3.166

Notations:

$\alpha = \arccos \frac{u^2-1}{u^2+1}, \beta = \operatorname{arctg}\left\{\frac{(1+\sqrt{2})\{1-u}{1+u}\right\},$

$$\gamma = \arccos u, \quad \delta = \arccos\frac{1}{u}, \quad \varepsilon = \arccos\frac{1-u^2}{1+u^2},$$
$$r = \frac{\sqrt{2}}{2}, \quad q = 2\sqrt{3\sqrt{2}-4} = 2\sqrt[4]{2}(\sqrt{2}-1) \approx 0,985171.$$

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1. $$\int_u^\infty \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}F(\alpha,\,r) \qquad [u\ge 0].$$

ZH (287), BY (263.50)

2. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4+1}} = \frac{1}{2}[F(\alpha,\,r)-2E(\alpha,\,r)] + \frac{\sqrt{u^4+1}}{u(u^2+1)} \qquad [u>0].$$

BY (263.57)

3. $$\int_u^\infty \frac{x^2\,dx}{(x^4+1)\sqrt{x^4+1}} = \frac{1}{2}E(\alpha,\,r) - \frac{1}{4}F(\alpha,\,r) - \frac{u(u^2-1)}{2(u^2+1)\sqrt{u^4+1}} \qquad [u\ge 0].$$

BY (263.59)

4. $$\int_u^\infty \frac{x^2\,dx}{(x^2+1)^2\sqrt{x^4+1}} = \frac{1}{4}[F(\alpha,\,r)-E(\alpha,\,r)] \qquad [u\ge 0].$$

BY (263.53)

5. $$\int_u^\infty \frac{x^2\,dx}{(x^2-1)^2\sqrt{x^4+1}} = \frac{u\sqrt{u^4+1}}{2(u^4-1)} - \frac{1}{4}E(\alpha,\,r) \qquad [u>1].$$

BY (263.55)

BY (263.58)

7. $$\int_u^\infty \frac{(x^2-1)^2\,dx}{(x^2+1)^2\sqrt{x^4+1}} = E(\alpha,\, r) - \frac{1}{2}F(\alpha,\, r) \qquad [u \geq 0].$$

BY (263.54)

8. $$\int_u^\infty \frac{\sqrt{x^4+1}\,dx}{(x^2+1)^2} = \frac{1}{2}E(\alpha,\, r) \qquad [u \geq 0].$$

BY (263.51)

9. $$\int_u^\infty \frac{(x^2+1)^2\,dx}{[(x^2+1)^2-4p^2x^2]\sqrt{x^4+1}} = \frac{1}{2}\Pi(\alpha,\, p^2,\, r) \qquad [u \geq 0].$$

BY (263.52)

10. $$\int_0^u \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}F(\varepsilon,\, r).$$

ZH 66(288)

11. $$\int_u^1 \frac{dx}{\sqrt{x^4+1}} = (2-\sqrt{2})F(\beta,\, q) \qquad [0 \leq u < 1].$$

BY (264.50)

12. $$\int_u^1 \frac{(x^2+x\sqrt{2}+1)\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}} = (2+\sqrt{2})E(\beta,\, q) \qquad [0 \leq u < 1].$$

BY (264.51)

13. $$\int_u^1 \frac{(1-x)^2\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}} = \frac{1}{\sqrt{2}}[F(\beta,\, q) - E(\beta,\, q)] \qquad [0 \leq u < 1].$$

BY (264.55)

14. $$\int_u^1 \frac{(1+x)^2\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}} = \frac{3\sqrt{2}+4}{2}E(\beta,\, q) - \frac{3\sqrt{2}-4}{2}F(\beta,\, q) \qquad [0 \leq u < 1].$$

15. $$\int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}} F(\gamma,\, r) \qquad [u < 1].$$

ZH 66 (290), BY (259.75)

16. $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}} \left\{\Gamma\left(\frac{1}{4}\right)\right\}^2 .$$

17. $$\int_1^u \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}} F(\delta,\, r) \qquad [u > 1].$$

ZH 66 (289), BY (260.75)

18.[8] $$\int_u^1 \frac{x^2\, dx}{\sqrt{1-x^4}} = \sqrt{2}E(\gamma,\, r) - \frac{1}{\sqrt{2}} F(\gamma,\, r) \qquad [u < 1];$$
$$= \frac{1}{\sqrt{2\pi}} \left\{\Gamma\left(\frac{3}{4}\right)\right\}^2 \qquad [u = 0].$$

BY (259.76)

19. $$\int_1^u \frac{x^2\, dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}} F(\delta,\, r) - \sqrt{2}E(\delta,\, r) + \frac{1}{u}\sqrt{u^4-1} \qquad [u > 1].$$

BY (260.77)

20. $$\int_u^1 \frac{x^4\, dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}} F(\gamma,\, r) + \frac{u}{3}\sqrt{1-u^4} \qquad [u < 1].$$

BY (259.76)

21.[3] $$\int_1^u \frac{x^4\, dx}{\sqrt{x^4-1}} = \frac{1}{3\sqrt{2}} F(\delta,\, r) + \frac{1}{3}u\sqrt{u^4-1} \qquad [u > 1].$$

BY (260.77)

22. $$\int_0^u \frac{dx}{\sqrt{x(1+x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1+(1-\sqrt{3})u}{1+(1+\sqrt{3})u},\, \frac{\sqrt{2+\sqrt{3}}}{2}\right) \qquad [u > 0].$$

23. $$\int_0^u \frac{dx}{\sqrt{x(1-x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos\frac{1-(1+\sqrt{3})u}{1+(\sqrt{3}-1)u}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) \qquad [1 \geq u > 0].$$

BY (259.50)

In 3.167 and 3.168 we set: $\alpha = \arcsin\sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \{ = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

3.167

1. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) - F(\alpha, q)\right\}$$
$$[a > b > c > d > u].$$

BY (251.05)

2. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - F(\beta, r)\right\}$$
$$[a > b > c \geq u > d].$$

BY (252.14)

3. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(c-b)\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (b-d)F(\gamma, r)\right\}$$
$$[a > b > c > u \geq d].$$

BY (253.14)

4. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\Pi\left(\delta, \frac{b-c}{b-d}, q\right)$$
$$[a > b \geq u > c > d].$$

5. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-a)\Pi\left(\{,\ \frac{b-c}{a-c},\ q\right) + (a-d)F(\{,\ q)\right\}$$
$$[a > b > u \geq c > d].$$

BY (255.20)

6. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-c)\Pi\left(\lambda,\ \frac{a-b}{a-c},\ r\right) + (c-d)F(\lambda,\ r)\right\}$$
$$[a \geq u > b > c > d].$$

BY (256.13)

7. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\Pi\left(\mu,\ \frac{b-a}{b-d},\ r\right) \qquad [a > u \geq b > c > d].$$

BY (257.02)

8. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(a-b)\Pi\left(\nu,\ \frac{a-d}{b-d},\ q\right) + (b-d)F(\nu,\ q)\right\}$$
$$[u > a > b > c > d].$$

BY (258.14)

9. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\Pi\left(\alpha,\ \frac{a-d}{a-c},\ q\right)$$
$$[a > b > c > d > u].$$

BY (251.02)

10. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\Pi\left(\beta,\ \frac{d-c}{a-c},\ r\right) - (a-c)F(\beta,\ r)\right]$$
$$[a > b > c \geq u > d].$$

BY (252.13)

11. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\gamma,\ \frac{c-d}{b-d},\ r\right) - F(\gamma,\ r)\right]$$
$$[a > b > c > u \geq d].$$

12. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\, dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\delta,\, \frac{b-c}{b-d},\, q\right) - F(\delta,\, q)\right]$$
$$[a > b \geq u > c > d].$$

BY (254.12)

13. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\, dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a)\Pi\left(\{,\, \frac{b-c}{a-c},\, q\right) + (a-c)F(\{,\, q)\right]$$
$$[a > b > u \geq c > d].$$

BY (259.19)

14. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\, dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\lambda,\, \frac{a-b}{a-c},\, r\right) \qquad [a \geq u > b > c > d].$$

BY (256.02)

15. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\, dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d)\Pi\left(\mu,\, \frac{b-a}{b-d},\, r\right) + (d-c)F(\mu,\, r)\right]$$
$$[a > u \geq b > c > d].$$

BY (257.13)

16. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}}\, dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b)\Pi\left(\nu,\, \frac{a-d}{b-d},\, q\right) + (b-c)F(\nu,\, q)\right]$$
$$[u > a > b > c > d].$$

BY (258.13)

17. $$\int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}}\, dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d)\Pi\left(\alpha,\, \frac{a-d}{a-c},\, q\right) + (b-c)F(\alpha,\, q)\right]$$
$$[a > b > c > d > u].$$

BY (251.07)

18. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\, dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d)\Pi\left(\beta,\, \frac{d-c}{a-c},\, r\right) - (a-b)F(\beta,\, r)\right]$$
$$[a > b > c \geq u > d].$$

BY (252.15)

19. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\Pi\left(\gamma,\ \frac{c-d}{b-d},\ r\right) \qquad [a > b > c > u \geq d].$$

BY (253.02)

20. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\Pi\left(\delta,\ \frac{b-c}{b-d},\ q\right) + (b-d)F(\delta,\ q)\right]$$
$$[a > b \geq u > c > d].$$

BY (254.14)

21. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\{,\ \frac{b-c}{a-c},\ q\right) - F(\{,\ q)\right]$$
$$[a > b > u \geq c > d].$$

BY (255.21)

22. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\lambda,\ \frac{a-b}{a-c},\ r\right) - F(\lambda,\ r)\right]$$
$$[a \geq u > b > c > d].$$

BY (256.15)

23.[8] $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\Pi\left(\mu,\ \frac{b-a}{b-d},\ r\right) - (b-d)F(\mu,\ r)\right]$$
$$[a > u \geq b > c > d].$$

BY (257.15)

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24. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\Pi\left(\nu,\ \frac{a-d}{b-d},\ q\right) \qquad [u > a > b > c > d].$$

BY (258.02)

25. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\Pi\left(\alpha,\ \frac{a-d}{a-c},\ q\right) + (a-c)F(\alpha,\ q)\right]$$
$$[a > b > c > d > u].$$

26. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\Pi\left(\beta,\, \frac{d-c}{a-c},\, r\right) \qquad [a > b > c \geq u > d].$$

BY (252.02)

27. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-c)\Pi\left(\gamma,\, \frac{c-d}{b-d},\, r\right) + (a-b)F(\gamma,\, r)\right]$$
$$[a > b > c > u \geq d].$$

BY (253.15)

28. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\Pi\left(\delta,\, \frac{b-c}{b-d},\, q\right) + (a-d)F(\delta,\, q)\right]$$
$$[a > b \geq u > c > d].$$

BY (254.13)

29. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\Pi\left(\{,\, \frac{b-c}{a-c},\, q\right) \qquad [a > b > u \geq c > d].$$

BY (255.02)

30. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-b)\Pi\left(\lambda,\, \frac{a-b}{a-c},\, r\right) + (a-c)F(\lambda,\, r)\right]$$
$$[a \geq u > b > c > d].$$

BY (256.14)

31. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\mu,\, \frac{b-a}{b-d},\, r\right) - F(\mu,\, r)\right]$$
$$[a > u \geq b > c > d].$$

BY (257.14)

32. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\nu,\, \frac{a-d}{b-d},\, q\right) - F(\nu,\, q)\right]$$
$$[u > a > b > c > d].$$

BY (258.15)

3.168

1. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{d-a}\left[\sqrt{\frac{a-c}{b-d}}E(\gamma,\,r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}\right] \quad [a > b > c > u > d].$$

BY (253.06)

2. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}[F(\delta,\,q) - E(\delta,\,q)] \qquad [a > b \geq u > c > d].$$

BY (254.04)

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3. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}[F(\{,\,q) - E(\{,\,q)] + \\ + \frac{2}{b-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \qquad [a > b > u \geq c > d].$$

BY (255.09)

4. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\left[\sqrt{\frac{a-c}{b-d}}E(\lambda,\,r) - \frac{c-d}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}\right] \quad [a \geq u > b > c > d].$$

BY (256.06)

5. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}E(\mu,\,r) \qquad [a > u \geq b > c > d].$$

BY (257.01)

6. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}[F(\nu,\,q) - E(\nu,\,q)] + \\ + \frac{2}{a-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} \qquad [u > a > b > c > d].$$

BY (258.10)

BY (253.03)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times\,[(a-c)(b-d)E(\delta,\,q) - (a-b)(c-d)F(\delta,\,q)]$$
$$[a > b \geq u > c > d].$$

BY (254.15)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times\,[(a-c)(b-d)E(\{,\,q) - (a-b)(c-d)F(\{,\,q)] -$$
$$-\,\frac{2}{c-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \qquad [a > b > u \geq c > d].$$

BY (255.06)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times$$
$$\times\,[(a-c)(b-d)E(\lambda,\,r) - (a-d)(b-c)F(\lambda,\,r)] -$$
$$-\,\frac{2}{a-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \qquad [a \geq u > b > c > d].$$

BY (256.03)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}E(\mu,\,r) -$$
$$-\,\frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}}F(\mu,\,r)$$
$$[a > u \geq b > c > d].$$

BY (257.09)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}}\,dx = \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} +$$
$$+\,\frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}}F(\nu,\,q) +$$
$$+\,2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}E(\nu,\,q) \qquad [u > a > b > c > d].$$

13. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}[F(\gamma,\,r)-E(\gamma,\,r)] + \\ + \frac{2}{c-d}\sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \qquad [a>b>c>u>d].$$

BY (253.04)

14. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}E(\delta,\,q) \qquad [a>b\geq u>c>d].$$

BY (254.01)

15. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}E(\{,\,q)-\frac{2(a-d)}{(b-d)(c-d)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\ [a>b>u\geq c>d].$$

BY (255.08)

16. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}[F(\lambda,\,r)-E(\lambda,\,r)] + \\ + \frac{2}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \qquad [a\geq u>b>c>d].$$

BY (256.05)

17. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}[F(\mu,\,r)-E(\mu,\,r)] \qquad [a>u\geq b>c>d].$$

BY (257.06)

18. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{-2}{c-d}\sqrt{\frac{a-c}{b-d}}E(\nu,\,q)+\frac{2}{c-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} \\ [u>a>b>c>d].$$

BY (258.05)

19. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}[F(\alpha,\,q)-E(\alpha,\,q)] \qquad [a>b>c>d>u].$$

20. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{-2}{b-c}\sqrt{\frac{b-d}{a-c}}E(\beta,\, r)+\frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$$[a > b > c \geq u > d].$$

BY (252.06)

21. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}[F(\{,\, q) - E(\{,\, q)] +$$
$$+\frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \qquad [a > b > u > c > d].$$

BY (255.05)

22. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}E(\lambda,\, r) \qquad [a \geq u > b > c > d].$$

BY (256.01)

23. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}E(\mu,\, r)-\frac{2(c-d)}{(a-c)(b-c)}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$$[a > u \geq b > c > d].$$

BY (257.06)

24. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}[F(\nu,\, q) - E(\nu,\, q)] +$$
$$+\frac{2}{a-c}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \qquad [u > a > b > c > d].$$

BY (258.06)

25. $$\int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}E(\alpha,\, q) \qquad [a > b > c > d > u].$$

BY (251.01)

26. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}[F(\beta,\, r) - E(\beta,\, r)] +$$
$$+\frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \qquad [a > b > c > u > d].$$

27. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{d-c}\sqrt{\frac{b-d}{a-c}}E(\{,\, q) + \frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$$[a > b > u > c > d].$$

BY (255.03)

28. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}[F(\lambda,\, r) - E(\lambda,\, r)] \qquad [a \geq u > b > c > d].$$

BY (256.08)

29. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}[F(\mu,\, r) - E(\mu,\, r)] +$$
$$+ \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \qquad [a > u \geq b > c > d].$$

BY (257.03)

30. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}E(\nu,\, q) - \frac{2(b-c)}{(a-c)(c-d)}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$$[u > a > b > c > d].$$

BY (258.03)

31. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}E(\alpha,\, q) -$$
$$- \frac{a-b}{b-c}\frac{2}{\sqrt{(a-c)(b-d)}}F(\alpha,\, q) \qquad [a > b > c > d > u].$$

BY (251.08)

32. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}}\,dx =$$
$$= \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}F(\beta,\, r) - 2\frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}E(\beta,\, r) +$$
$$+ 2\frac{a-c}{(b-c)(c-d)}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \qquad [a > b > c > u > d].$$

33. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}}\,dx =$$
$$= \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}} F(\{,\, q) - 2\sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\{,\, q) +$$
$$+ \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \qquad [a > b > u > c > d].$$

BY (255.04)

34. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\lambda,\, r) -$$
$$- \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\lambda,\, r) \qquad [a \geq u > b > c > d].$$

BY (256.09)

35. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\mu,\, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu,\, r) -$$
$$- \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \qquad [a > u \geq b > c > d].$$

BY (257.04)

36. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\nu,\, q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu,\, q) -$$
$$- \frac{2}{c-d} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \qquad [u > a > b > c > d].$$

BY (258.04)

37. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\alpha,\, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\alpha,\, q) -$$
$$- \frac{2}{a-b} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \qquad [a > b > c > d > u].$$

BY (251.11)

BY (252.07)

39. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\gamma,\, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma,\, r)$$
$$[a > b > c > u \geq d].$$

BY (253.07)

40. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}}\,dx =$$
$$= \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\delta,\, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\delta,\, q) +$$
$$+ \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \qquad [a > b > u > c > d].$$

BY (254.05)

41. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}}\,dx =$$
$$= \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\mu,\, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\mu,\, r) +$$
$$+ \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \qquad [a > u > b > c > d].$$

BY (257.07)

42. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\nu,\, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu,\, q)$$
$$[u > a > b > c > d].$$

BY (258.07)

43. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}}\,dx =$$
$$= \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}} E(\alpha,\, q) - \frac{2(b-c)}{(a-b)(b-d)} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$

44. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}[F(\beta,\,r)-E(\beta,\,r)] + \\ + \frac{2}{b-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \qquad [a > b > c \geq u > d].$$

BY (252.10)

45. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}[F(\gamma,\,r)-E(\gamma,\,r)] \qquad [a > b > c > u \geq d].$$

BY (254.08)

46. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{b-a}\sqrt{\frac{a-c}{b-d}}E(\delta,\,q)+\frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \\ [a > b \geq u > c > d].$$

BY (254.08)

47. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}[F(\mu,\,r)-E(\mu,\,r)] + \\ + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \qquad [a > u \geq b > c > d].$$

BY (257.10)

48. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}E(\nu,\,q) \qquad [u > a > b > c > d].$$

BY (258.01)

49. $$\int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}[F(\alpha,\,q)-E(\alpha,\,q)] + \\ + \frac{2}{b-d}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \qquad [a > b > c > d > u].$$

BY (251.12)

50. $$\int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}E(\beta,\,r) - \frac{2(a-b)}{(b-c)(b-d)}\sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}}$$
$$[a > b > c \geq u > d].$$

BY (252.09)

51. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}E(\gamma,\,r) \qquad [a > b > c > u \geq d].$$

BY (253.01)

52. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}[F(\delta,\,q) - E(\delta,\,q)] +$$
$$+ \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \qquad [a > b > u > c > d].$$

BY (254.06)

53. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{c-b}\sqrt{\frac{a-c}{b-d}}E(\mu,\,r) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$$[a > u > b > c > d].$$

BY (257.08)

54. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}[F(\nu,\,q) - E(\nu,\,q)] \qquad [u > a > b > c > d].$$

BY (258.08)

55. $$\int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{b-a}\sqrt{\frac{b-d}{a-c}}E(\alpha,\,q) + \frac{2}{a-b}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$$[a > b > c > d > u].$$

BY (251.09)

56. $$\int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}[F(\beta,\,q) - E(\beta,\,q)] \qquad [a > b > c \geq u > d].$$

57. $$\int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}[F(\gamma,\, r) - E(\gamma,\, r)] + \\ + \frac{2}{a-c}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \qquad [a > b > c > u \geq d].$$

BY (253.05)

58. $$\int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \\ = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}E(\delta,\, q) - \frac{2(a-d)}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \qquad [a > b \geq u > c > d].$$

BY (254.03)

59. $$\int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}E(\boldsymbol{\{},\, q) \qquad [a > b > u \geq c > d].$$

BY (255.01)

60. $$\int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}[F(\lambda,\, r) - E(\lambda,\, r)] + \\ + \frac{2}{a-b}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \qquad [a > u > b > c > d].$$

BY (256.10)

61. $$\int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}}\,dx = \\ = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}F(\alpha,\, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}E(\alpha,\, q) + \\ + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \qquad [a > b > c > d > u].$$

BY (251.15)

62. $$\int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}E(\beta,\, r) - \\ - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}F(\beta,\, r) \\ [a > b > c \geq u > d].$$

63. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\gamma,\,r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma,\,r) -$$
$$- \frac{2}{a-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \qquad [a > b > c > u \geq d].$$

BY (253.10)

64. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\delta,\,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\delta,\,q) -$$
$$- \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \qquad [a > b \geq u > c > d].$$

BY (254.09)

65. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx =$$
$$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\{,\,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\{,\,q)$$
$$[a > b > u \geq c > d].$$

BY (255.10)

66. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}}\,dx =$$
$$= \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\lambda,\,r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\lambda,\,r) +$$
$$+ \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \qquad [a > u > b > c > d].$$

BY (256.07)

67. $$\int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}[F(\alpha,\,q) - E(\alpha,\,q)] +$$
$$+ \frac{2}{a-d}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \qquad [a > b > c > d > u].$$

$$68.\quad \int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}E(\beta,\,r) \qquad [a>b>c\geq u>d].$$

BY (252.01)

$$69.\quad \int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}E(\gamma,\,r) - \frac{2(a-b)}{(a-c)(a-d)}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$$[a>b>c>u\geq d].$$

BY (253.08)

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$$70.\quad \int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}[F(\delta,\,q)-E(\delta,\,q)] +$$
$$+\frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \qquad [a>b\geq u>c>d].$$

BY (254.07)

$$71.\quad \int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}[F(\{,\,q)-E(\{,\,q)]$$
$$[a>b>u\geq c>d].$$

BY (255.07)

$$72.\quad \int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{-2}{a-d}\sqrt{\frac{b-d}{a-c}}E(\lambda,\,r) + \frac{2}{a-d}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$$[a\geq u>b>c>d].$$

BY (256.04)

In 3.169- 3.172, we set: $\alpha = \operatorname{arctg}\frac{u}{b}$, $\beta = \operatorname{arctg}\frac{a}{u}$,

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \qquad \delta = \arccos\frac{u}{b}, \qquad \varepsilon = \arccos\frac{b}{u}, \qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
$$\eta = \arcsin\frac{u}{b}, \qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \qquad \{ = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \qquad u = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \qquad \nu = \arcsin\frac{a}{u}, \qquad q = \frac{\sqrt{a^2-b^2}}{a},$$
$$r = \frac{b}{\sqrt{a^2+b^2}}, \qquad s = \frac{a}{\sqrt{a^2+b^2}} \qquad t = \frac{b}{a}.$$

1. $\int_0^u \sqrt{\frac{x^2+a^2}{x^2+b^2}}\,dx = a\left\{F(\alpha, q) - E(\alpha, q)\right\} + u\sqrt{\frac{a^2+u^2}{b^2+u^2}} \qquad [a > b, \quad u > 0].$

2.[6] $\int_0^u \sqrt{\frac{x^2+b^2}{x^2+a^2}}\,dx = \frac{b^2}{a}F(\alpha, q) - aE(\alpha, q) + u\sqrt{\frac{a^2+u^2}{b^2+u^2}} \qquad [a > b, \quad u > 0].$

BY (221.04)

3. $\int_0^u \sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\gamma, r) - u\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [b \ge u > 0].$

BY (214.11)

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4. $\int_u^b \sqrt{\frac{a^2+x^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\delta, r) \qquad [b > u \ge 0].$

BY (213.01), ZH 64 (273)

5. $\int_b^u \sqrt{\frac{a^2+x^2}{x^2-b^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\varepsilon, s) - E(\varepsilon, s)\right\} + \frac{1}{u}\sqrt{(u^2+a^2)(u^2-b^2)} \qquad [u > b > 0].$

BY (211.03)

6. $\int_0^u \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\gamma, r) - E(\gamma, r)\right\} + u\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [b \ge u > 0].$

BY (214.03)

7. $\int_u^b \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\delta, r) - E(\delta, r)\right\} \qquad [b > u \ge 0].$

BY (213.03)

8. $\int_b^u \sqrt{\frac{x^2-b^2}{a^2+x^2}}\,dx = \frac{1}{u}\sqrt{(a^2+u^2)(u^2-b^2)} - \sqrt{a^2+b^2}\,E(\varepsilon, s) \qquad [u > b > 0].$

BY (211.04)

9. $\int_0^u \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = aE(\eta, t) - \frac{a^2-b^2}{a}F(\eta, t) \qquad [a > b \ge u > 0].$

10. $$\int_u^b \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = aE(\zeta,\,t) - \frac{a^2-b^2}{a}F(\zeta,\,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u \geq 0].$$

BY (220.04)

11. $$\int_b^u \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = aE(\{,\,q) - \frac{b^2}{a}F(\{,\,q) - \frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u > b > 0].$$

BY (217.04)

12. $$\int_u^a \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = aE(\lambda,\,q) - \frac{b^2}{a}F(\lambda,\,q) \qquad [a > u \geq b > 0].$$

BY (218.03)

13. $$\int_a^u \sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{a^2-b^2}{a}F(\mu,\,t) - aE(\mu,\,t) + \mu\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.03)

14. $$\int_0^u \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = aE(\eta,\,t) \qquad [a > b \geq u > 0].$$

ZH 64 (276), BY (219.01)

15. $$\int_u^b \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\left\{E(\zeta,\,t) - \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} \qquad [a > b > u \geq 0].$$

BY (220.03)

16. $$\int_b^u \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\{,\,q) - E(\{,\,q)\right\} + \frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)} \qquad [a \geq u > b > 0].$$

BY (217.03)

17. $$\int_u^a \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\lambda,\,q) - E(\lambda,\,q)\right\} \qquad [a > u \geq b > 0].$$

18. $\displaystyle\int_a^u \sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = u\sqrt{\frac{u^2-a^2}{u^2-b^2}} - aE(\mu,\,t) \qquad [u > a > b > 0].$

BY (216.04)

327

3.171

1. $\displaystyle\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}E(\varepsilon,\,s) \qquad [u > b > 0].$

BY (211.01), ZH 64 (274)

2. $\displaystyle\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}E(\xi,\,s) - \frac{a^2}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}} \qquad [u \geq b > 0].$

BY (212.09)

3. $\displaystyle\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}} = \frac{a^2-b^2}{ab^2}F(\zeta,\,t) - \frac{a}{b^2}E(\zeta,\,t) + \frac{a^2}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}} \qquad [a > b > u > 0].$

BY (220.12)

4. $\displaystyle\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}E(\{,\,q) - \frac{1}{a}F(\{,\,q) \qquad [a \geq u > b > 0].$

BY (217.11)

5. $\displaystyle\int_u^a \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}E(\lambda,\,q) - \frac{1}{a}f(\lambda,\,q) - \frac{\sqrt{(a^2-u^2)(u^2-b^2)}}{b^2u} \qquad [a > u \geq b > 0].$

BY (218.10)

6. $\displaystyle\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}E(\mu,\,t) - \frac{a^2-b^2}{ab^2}F(\mu,\,t) - \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$

BY (216.08)

7. $\displaystyle\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+a^2}{x^2+b^2}} = \frac{1}{a}F(\beta,\,q) - \frac{a}{b^2}E(\beta,\,q) + \frac{a^2}{b^2u}\sqrt{\frac{b^2+u^2}{a^2+u^2}} \qquad [a > b, \quad u > 0].$

8. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+b^2}{x^2+a^2}} = \frac{1}{a}\left\{F(\beta,\, q) - E(\beta,\, q)\right\} + \frac{1}{u}\sqrt{\frac{b^2+u^2}{a^2+u^2}} \qquad [a > b, \quad u > 0].$$

BY (222.09)

9. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2+x^2}} = \frac{\sqrt{(b^2-u^2)(a^2+u^2)}}{a^2 u} - \frac{\sqrt{a^2+b^2}}{a^2}E(\delta,\, r) \qquad [b > u > 0].$$

BY (213.10)

10. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\left\{F(\varepsilon,\, s) - E(\varepsilon,\, s)\right\} \qquad [a > b > 0].$$

BY (211.07)

11. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\left\{F(\xi,\, s) - E(\xi,\, s)\right\} + \frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2+u^2}} \qquad [u \geq b > 0].$$

BY (212.11)

12. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{b^2-x^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\left\{F(\delta,\, r) - E(\delta,\, r)\right\} + \frac{\sqrt{(b^2-u^2)(a^2+u^2)}}{b^2 u} \qquad [b > u > 0].$$

BY (213.05)

13. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}E(\nu,\, t) - \frac{a^2-b^2}{ab^2}F(\nu,\, t) \qquad [u \geq a > b > 0].$$

BY (215.08)

14. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2-x^2}} = \frac{1}{u}\sqrt{\frac{b^2-u^2}{a^2-u^2}} - \frac{1}{a}E(\zeta,\, t) \qquad [a > b > u > 0].$$

BY (220.11)

15. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2-x^2}} = \frac{1}{a}\left\{F(\{,\, q) - E(\{,\, q)\right\} \qquad [a \geq u > b > 0].$$

BY (217.08)

BY (217.08)

16. $$\int_u^a \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{u^2-x^2}} = \frac{1}{a}\left\{F(\lambda,\, q) - E(\lambda,\, q)\right\} + \frac{\sqrt{(a^2-u^2)(u^2-b^2)}}{a^2u} \qquad [a > u \geq b > 0].$$

BY (218.08)

17. $$\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\mu,\, t) - \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u > a > b > 0].$$

BY (216.07)

18. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\nu,\, t) \qquad [u \geq a > b > 0].$$

BY (215.01), ZH 65 (281)

3.172

1. $$\int_0^u \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\, dx = \frac{1}{a}E(\alpha,\, q) - \frac{a^2-b^2}{a^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}} \qquad [a > b, \quad u > 0].$$

BY (221.10)

2. $$\int_u^\infty \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\, dx = \frac{1}{a}E(\beta,\, q) \qquad [a > b, \quad u \geq 0].$$

ZH 64 (271)

3. $$\int_0^u \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\, dx = \frac{a}{b^2}E(\alpha,\, q) \qquad [a > b, \quad u > 0].$$

ZH 64 (270)

4. $$\int_u^\infty \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\, dx = \frac{a}{b^2}E(\beta,\, q) - \frac{a^2-b^2}{b^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}} \qquad [a > b, \quad u \geq 0].$$

BY (222.06)

5. $$\int_0^u \sqrt{\frac{b^2-x^2}{(a^2+x^2)^3}}\, dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\gamma,\, r) - \frac{1}{\sqrt{a^2+b^2}}F(\gamma,\, r) \qquad [b \geq u > 0].$$

BY (214.08)

$$6. \quad \int_u^b \sqrt{\frac{b^2-x^2}{(a^2+x^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\delta,\,r) - \frac{1}{\sqrt{a^2+b^2}}F(\delta,\,r) - \frac{u}{a^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}} \qquad [b>u\geq 0].$$

BY (213.04)

$$7. \quad \int_b^u \sqrt{\frac{x^2-b^2}{(a^2+x^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\varepsilon,\,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}F(\varepsilon,\,s) - \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2+a^2}} \qquad [u>b>0].$$

BY (211.06)

$$8. \quad \int_u^\infty \sqrt{\frac{x^2-b^2}{(a^2+x^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\xi,\,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}F(\xi,\,s) \qquad [u\geq b>0].$$

BY (212.08)

$$9. \quad \int_0^u \sqrt{\frac{x^2+a^2}{(b^2-x^2)^3}}\,dx = \frac{a^2}{b^2\sqrt{a^2+b^2}}F(\gamma,\,r) - \frac{\sqrt{a^2+b^2}}{b^2}E(\gamma,\,r) + \frac{(a^2+b^2)u}{b^2\sqrt{(a^2+u^2)(b^2-u^2)}} \qquad [b>u>0].$$

BY (214.09)

$$10. \quad \int_u^\infty \sqrt{\frac{x^2+a^2}{(x^2-b^2)^3}}\,dx = \frac{1}{\sqrt{a^2+b^2}}F(\xi,\,s) - \frac{\sqrt{a^2+b^2}}{b^2}E(\xi,\,s) + \frac{(a^2+b^2)u}{b^2\sqrt{(a^2+u^2)(u^2-b^2)}} \qquad [u>b>0].$$

BY (212.07)

$$11. \quad \int_0^u \sqrt{\frac{b^2-x^2}{(a^2-x^2)^3}}\,dx = \frac{1}{a}\left\{F(\eta,\,t) - E(\eta,\,t) + \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} \qquad [a>b\geq u>0].$$

BY (219.09)

$$12. \quad \int_u^b \sqrt{\frac{b^2-x^2}{(a^2-x^2)^3}}\,dx = \frac{1}{a}\left\{F(\zeta,\,t) - E(\zeta,\,t)\right\} \qquad [a>b>u\geq 0].$$

13. $\displaystyle\int_b^u \sqrt{\frac{x^2-b^2}{(a^2-x^2)^3}}\,dx = \frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}} - \frac{1}{a}E(\{,\, q) \qquad [a > u > b > 0].$

BY (217.07)

14. $\displaystyle\int_u^\infty \sqrt{\frac{x^2-b^2}{(x^2-a^2)^3}}\,dx = \frac{1}{a}[F(\nu,\, t) - E(\nu,\, t)] + \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} \qquad [u > a > b > 0].$

BY (215.05)

15. $\displaystyle\int_0^u \sqrt{\frac{a^2-x^2}{(b^2-x^2)^3}}\,dx = \frac{a}{b^2}[F(\eta,\, t) - E(\eta,\, t)] + \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{b^2-u^2}} \qquad [a > b > u > 0].$

BY (219.10)

16. $\displaystyle\int_u^a \sqrt{\frac{a^2-x^2}{(x^2-b^2)^3}}\,dx = \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{u^2-b^2}} - \frac{a}{b^2}E(\lambda,\, q) \qquad [a > u > b > 0].$

BY (218.05)

17. $\displaystyle\int_a^u \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}}\,dx = \frac{a}{b^2}[F(\mu,\, t) - E(\mu,\, t)] \qquad [u > a > b > 0].$

BY (216.05)

18. $\displaystyle\int_u^\infty \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}}\,dx = \frac{a}{b^2}[F(\nu,\, t) - E(\nu,\, t)] + \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u \geq a > b > 0].$

BY (215.03)

3.173

1. $\displaystyle\int_u^1 \frac{dx}{x^2}\sqrt{\frac{x^2+1}{1-x^2}} = \sqrt{2}\left[F\left(\arccos u,\, \frac{\sqrt{2}}{2}\right) - E\left(\arccos u,\, \frac{\sqrt{2}}{2}\right)\right] + \frac{\sqrt{1-u^4}}{u} \qquad [u < 1].$

BY (259.77)

330

2. $\displaystyle\int_1^u \frac{dx}{x^2}\sqrt{\frac{x^2+1}{x^2-1}} = \sqrt{2}E\left(\arccos \frac{1}{u},\, \frac{\sqrt{2}}{2}\right) \qquad [u > 1].$

In 3.174 and 3.175, we take: $\alpha = \arccos \frac{1+(1-\sqrt{3})u}{1+(1+\sqrt{3})u}$,

$$\beta = \arccos \frac{1-(1+\sqrt{3})u}{1+(\sqrt{3}-1)u}, \qquad p = \frac{\sqrt{2+\sqrt{3}}}{2}, \qquad q = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

3.174

$$1. \quad \int_0^u \frac{dx}{[1+(1+\sqrt{3})x]^2} \sqrt{\frac{1-x+x^2}{x(1+x)}} = \frac{1}{\sqrt[4]{3}} E(\alpha,\, p) \qquad [u > 0].$$

BY (260.51)

$$2. \quad \int_0^u \frac{dx}{[1+(\sqrt{3}-1)x]^2} \sqrt{\frac{1+x+x^2}{x(1-x)}} = \frac{1}{\sqrt[4]{3}} E(\beta,\, q) \qquad [1 \geq u > 0].$$

BY (259.51)

$$3. \quad \int_0^u \frac{dx}{1-x+x^2} \sqrt{\frac{x(1+x)}{1-x+x^2}} = \frac{1}{\sqrt[4]{27}} E(\alpha,\, p) + -\frac{2-\sqrt{3}}{\sqrt[4]{27}} F(\alpha,\, p) - \frac{2(2+\sqrt{3})}{\sqrt{3}} \frac{1+(1-\sqrt{3})u}{1+(1+\sqrt{3})u} \times \\ \times \sqrt{\frac{u(1+u)}{1-u+u^2}} \qquad [u > 0].$$

BY (260.54)

$$4. \quad \int_0^u \frac{dx}{1+x+x^2} \sqrt{\frac{x(1-x)}{1+x+x^2}} = \frac{4}{\sqrt[4]{27}} E(\beta,\, q) - \frac{2+\sqrt{3}}{\sqrt[4]{27}} F(\beta,\, q) - \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{1-(1+\sqrt{3})u}{1+(\sqrt{3}-1)u} \times \\ \times \sqrt{\frac{u(1-u)}{1+u+u^2}} \qquad [1 \geq u > 0].$$

BY (259.55)

3.175

$$1. \quad \int_0^u \frac{dx}{1+x} \sqrt{\frac{x}{1+x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha,\, p) - 2E(\alpha,\, p)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1-u+u^2)}}{\sqrt{1+u}[1+(1+\sqrt{3})u]} \qquad [u > 0].$$

BY (260.55)

$$2. \quad \int_0^u \frac{dx}{1-x} \sqrt{\frac{x}{1-x^3}} = \frac{1}{\sqrt[4]{27}} [F(\beta,\, q) - 2E(\beta,\, q)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1+u+u^2)}}{\sqrt{1-u}[1+(\sqrt{3}-1)u]} \qquad [0 < u < 1].$$

3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

3.181

$$1. \quad \int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b}\left\{2\left[\boldsymbol{E}\left(\frac{1}{\sqrt{2}}\right) + E\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right] - \right.$$
$$\left. - \left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]\right\} \qquad [a \geq u > b].$$

BY (271.05)

$$2. \quad \int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} = \sqrt{\frac{a-b}{2}} F\left[\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right) - \right.$$
$$\left. - 2E\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right)\right] +$$
$$+ \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}} \qquad [u > a > b].$$

BY (272.05)

3.182

$$1. \quad \int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}}\left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]$$
$$[a \geq u > b].$$

BY (271.01)

$$2. \quad \int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right) \qquad [u > a > b].$$

BY (272.00)

In 3.183- 3.186 we set: $\alpha = \arccos\frac{1}{\sqrt[4]{u^2+1}}$,

$$\beta = \arccos\sqrt[4]{1-u^2}, \qquad \gamma = \arccos\frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

3.183

1. $$\int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2}\left[F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\ \frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}} \qquad [u > 0].$$

BY (273.55)

2. $$\int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2}\left[2E\left(\beta,\ \frac{1}{\sqrt{2}}\right) - F\left(\beta,\ \frac{1}{\sqrt{2}}\right)\right] \qquad [0 < u \le 1].$$

BY (271.55)

3. $$\int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\gamma,\ \frac{1}{\sqrt{2}}\right) - 2E\left(\gamma,\ \frac{1}{\sqrt{2}}\right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}} \qquad [u > 1].$$

BY (272.55)

3.184

1. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5}\left[2E\left(\beta,\ \frac{1}{\sqrt{2}}\right) - F\left(\beta,\ \frac{1}{\sqrt{2}}\right)\right] - \frac{2u}{5}\sqrt[4]{(1-u^2)^3} \qquad [0 < u \le 1].$$

BY (271.59)

2. $$\int_1^u \frac{dx}{x^2\sqrt[4]{x^2-1}} = E\left(\gamma,\ \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\gamma,\ \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}} \cdot \frac{\sqrt{u^2-1}}{u} \qquad [u > 1].$$

BY (272.54)

3.185

1. $$\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2}F\left(\alpha,\ \frac{1}{\sqrt{2}}\right) \qquad [u > 0].$$

BY (273.50)

2. $$\int_0^u \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2}F\left(\beta,\ \frac{1}{\sqrt{2}}\right) \qquad [0 < u \le 1].$$

BY (271.51)

BY (272.50)

4. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\beta,\, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} u \sqrt[4]{1-u^2} \qquad [0 < u \le 1].$$

BY (271.54)

5. $$\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} E\left(\alpha,\, \frac{1}{\sqrt{2}}\right) - \sqrt{2} F\left(\alpha,\, \frac{1}{\sqrt{2}}\right) \qquad [u > 0].$$

BY (273.54)

6. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2}\left[F\left(\alpha,\, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha,\, \frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}} \qquad [u > 0].$$

BY (273.56)

7. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(x^2+1)^7}} = \frac{1}{3\sqrt{2}} F\left(\alpha,\, \frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{(u^2+1)^3}} \qquad [u > 0].$$

BY (273.53)

3.186

1. $$\int_0^u \frac{1+\sqrt{x^2+1}}{(x^2+1)\sqrt[4]{x^2+1}}\,dx = 2\sqrt{2} E\left(\alpha,\, \frac{1}{\sqrt{2}}\right) \qquad [u > 0].$$

BY (273.51)

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2. $$\int_0^u \frac{dx}{\left(1+\sqrt{1-x^2}\right)\sqrt[4]{1-x^2}} = \sqrt{2}\left[F\left(\beta,\, \frac{1}{\sqrt{2}}\right) - E\left(\beta,\, \frac{1}{\sqrt{2}}\right)\right] + \frac{u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}} \qquad [0 < u \le 1].$$

BY (271.58)

3. $$\int_1^u \frac{dx}{\left(x^2+2\sqrt{x^2-1}\right)\sqrt[4]{x^2-1}} = \frac{1}{2}\left[F\left(\gamma,\, \frac{1}{\sqrt{2}}\right) - E\left(\gamma,\, \frac{1}{\sqrt{2}}\right)\right] \qquad [u > 1].$$

BY (272.53)}

BY (271.57)}

5. $$\int_1^u \frac{x^2\,dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{(x^2-1)^3}} = E\left(\gamma,\,\frac{1}{\sqrt{2}}\right) \qquad [u>1].$$

BY (272.51)}

3.19- 3.23 Combinations of powers of x and powers of binomials of the form (α + βx)

3.191

1. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\,dx = u^{\mu+\nu-1}\mathrm{B}(\mu,\,\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET II 185(7)

2. $$\int_u^\infty x^{-\nu}(x-u)^{\mu-1}\,dx = u^{\mu-\nu}\mathrm{B}(\nu-\mu,\,\mu) \qquad [\operatorname{Re}\nu>\operatorname{Re}\mu>0].$$

ET II 201(6)

3. $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\,dx = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\,\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

FI II 774(1)

3.192

1. $$\int_0^1 \frac{x^p\,dx}{(1-x)^p} = p\pi\operatorname{cosec}p\pi \qquad [p^2<1].$$

BI ((3))(4)

2. $$\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\pi\operatorname{cosec}p\pi \qquad [-1<p<0].$$

BI ((3))(5)

3. $$\int_0^1 \frac{(1-x)^p}{x^{p+1}}\,dx = -\pi\operatorname{cosec}p\pi \qquad [-1<p<0].$$

BI ((4))(6)

BI ((4))(6)

4. $$\int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \pi \sec p\pi \qquad \left[-\frac{1}{2} < p < \frac{1}{2}\right].$$

BI ((23))(7)

3.193

$$\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n!n^{\nu+n}}{\nu(\nu+1)(\nu+2)\dots(\nu+n)} \qquad [\operatorname{Re}\nu > 0].$$

EH I 2

3.194

1. $$\int_0^u \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu}\, {}_2F_1(\nu,\,\mu;\,1+\mu;\,-\beta u) \qquad [|\arg(1+\beta u)| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 310(20)

2.[6] $$\int_u^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu(\nu-\mu)}\, {}_2F_1\left(\nu,\,\nu-\mu; \quad \nu-\mu+1; \quad -\frac{1}{\beta u}\right) \qquad [\operatorname{Re}\nu > \operatorname{Re}\mu].$$

ET I 310(21)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \beta^{-\mu}\mathrm{B}(\mu,\,\nu-\mu) \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0].$$

FI II 775a, ET I 310(19)

4.[7] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu} \binom{\mu-1}{n} \operatorname{cosec}(\mu\pi) \qquad [|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < n+1].$$

ET I 308(6)

5. $$\int_0^u \frac{x^{\mu-1}\,dx}{1+\beta x} = \frac{u^\mu}{\mu}\, {}_2F_1(1,\,\mu;\,1+\mu;\,-u\beta) \qquad [|\arg(1+u\beta)| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 308(5)

BI ((16))(4)

7. $$\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+\frac{1}{2}}} = 2^{m+1} m! \frac{(2n-2m-3)!!}{(2n-1)!!} \frac{a^{m-n+\frac{1}{2}}}{b^{m+1}} \qquad \left[m < n - \frac{1}{2}, \quad a > 0, \quad b > 0\right].$$

BI ((21))(2)

8. $$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^m} = 2^{-n} \sum_{k=0}^{\infty} \binom{m-n-1}{k} \frac{(-2)^{-k}}{n+k}.$$

BI ((3))(1)

3.195

$$\int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}}\,dx = \frac{1-a^{-p}}{p(a-1)} \qquad [a > 0].$$

LI ((19))(6)

3.196

1. $$\int_0^u (x+\beta)^\nu (u-x)^{\mu-1}\,dx = \frac{\beta^\nu u^\mu}{\mu}\, {}_2F_1\left(1, -\nu; 1+\mu; -\frac{u}{\beta}\right) \qquad \left[\left|\arg \frac{u}{\beta}\right| < \pi\right].$$

ET II 185(8)

2. $$\int_u^\infty (x+\beta)^{-\nu} (x-u)^{\mu-1}\,dx = (u+\beta)^{\mu-\nu} \mathrm{B}(\nu-\mu,\, \mu) \qquad \left[\left|\arg \frac{u}{\beta}\right| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0\right].$$

ET II 201(7)

3. $$\int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1}\,dx = (b-a)^{\mu+\nu-1} \mathrm{B}(\mu,\, \nu) \qquad [b > a, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

EH I 10(13)

4. $$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{b-a}\right)^\nu \qquad [a < b, \quad b > 0, \quad 0 < \nu < 1].$$

LI ((23))(5)

5. $$\int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{a-b}\right)^\nu \qquad [a > b > 0, \quad 0 < \nu < 1].$$

1. $$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho}\,dx = \beta^{-\mu}\gamma^{\nu-\varrho}\mathrm{B}(\nu,\,\mu-\nu+\varrho)\,{}_2F_1\left(\mu,\,\nu;\,\mu+\varrho;\,1-\frac{\gamma}{\beta}\right)$$
$$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > \operatorname{Re}(\nu-\varrho)].$$

ET II 233(9)

2. $$\int_u^\infty x^{-\lambda}(x+\beta)^{\nu}(x-u)^{\mu-1}\,dx = u^{\mu+\nu-\lambda}\mathrm{B}(\lambda-\mu-\nu,\,\mu)\,{}_2F_1\left(-\nu,\,\lambda-\mu-\nu;\,\lambda-\nu;\,-\frac{\beta}{u}\right)$$
$$\left[\left|\arg\frac{u}{\beta}\right| < \pi \quad \text{or} \quad \left|\frac{\beta}{u}\right| < 1, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}(\lambda-\nu)\right].$$

ET II 201(8)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}\,dx = \mathrm{B}(\lambda,\,\mu)\,{}_2F_1(\nu,\,\lambda;\,\lambda+\mu;\,\beta)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0, \quad |\beta| < 1].$$

WH

4. $$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu}\,dx = (1+a)^{-\mu}\mathrm{B}(\mu,\,\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad a > -1].$$

BI((5))4, EH I 10(11)

5. $$\int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+\alpha x)^{\mu}\,dx = \mathrm{B}(\lambda,\,-\mu-\nu-\lambda)\,{}_2F_1(-\mu,\,\lambda;\,-\mu-\nu;\,1-\alpha)$$
$$[|\arg\alpha| < \pi, \quad -\operatorname{Re}(\mu+\nu) > \operatorname{Re}\lambda > 0].$$

EH I 60(12), ET I 310(23)

6. $$\int_1^\infty x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda}\,dx = \alpha^{-\lambda}\mathrm{B}(\mu,\,\nu-\mu)\,{}_2F_1(\nu,\,\mu;\,\lambda;\,\alpha^{-1})$$
$$[1+\operatorname{Re}\nu > \operatorname{Re}\lambda > \operatorname{Re}\mu, \quad |\arg(\alpha-1)| < \pi].$$

EH I 115(6)

7. $$\int_0^\infty x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu}\,dx = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2\mu}\frac{\Gamma\left(\mu-\frac{1}{2}\right)}{\Gamma(\mu)} \qquad [\operatorname{Re}\mu > 0].$$

8. $$\int_0^u x^{\nu-1}(x+\alpha)^\lambda(u-x)^{\mu-1}\,dx = \alpha^\lambda u^{\mu+\nu-1}\mathrm{B}(\mu,\,\nu)\,{}_2F_1\left(-\lambda,\,\nu;\,\mu+\nu;\,-\frac{u}{\alpha}\right)$$
$$\left[\left|\arg\left(\frac{u}{\alpha}\right)\right| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0\right].$$

ET II 186(9)

9. $$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu}\,dx = \mathrm{B}(\mu-\lambda,\,\lambda)\,{}_2F_1(\nu,\,\mu-\lambda;\,\mu;\,1-\beta) \qquad [\operatorname{Re}\mu > \operatorname{Re}\lambda > 0].$$

EH I 205

10. $$\int_0^1 \frac{x^{q-1}\,dx}{(1-x)^q(1+px)} = \frac{\pi}{(1+p)^q}\operatorname{cosec} q\pi \qquad [0 < q < 1, \quad p > -1].$$

BI ((5))(1)

11. $$\int_0^1 \frac{x^{p-\frac{1}{2}}\,dx}{(1-x)^p(1+qx)^p} = \frac{2\Gamma\left(p+\frac{1}{2}\right)\Gamma(1-p)}{\sqrt{\pi}}\cos^{2p}(\operatorname{arctg}\sqrt{q})\frac{\sin[(2p-1)\,\operatorname{arctg}(\sqrt{q})]}{(2p-1)\,\sin[\operatorname{arctg}(\sqrt{q})]}$$
$$\left[-\frac{1}{2} < p < 1, \quad q > 0\right].$$

BI ((11))(1)

12. $$\int_0^1 \frac{x^{p-\frac{1}{2}}\,dx}{(1-x)^p(1-qx)^p} = \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(1-p)}{\sqrt{\pi}}\frac{(1-\sqrt{q})^{1-2p}-(1+\sqrt{q})^{1-2p}}{(2p-1)\sqrt{q}}$$
$$\left[-\frac{1}{2} < p < 1, \quad 0 < q < 1\right].$$

BI ((11))(2)

3.198

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}[ax+b(1-x)+c]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}\mathrm{B}(\mu,\,\nu)$$
$$[a \geq 0, \quad b \geq 0, \quad c > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

FI II 787

3.199

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\,dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu}\mathrm{B}(\mu,\,\nu)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad c < a < b].$$

EH I 10(14)

3.211

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\varrho}(1-vx)^{-\sigma}\,dx = \mathrm{B}(\mu,\,\lambda)F_1(\lambda,\,\varrho,\,\sigma,\,\lambda+\mu;\,u,\,v)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

EH I 231(5)

3.212

$$\int_0^\infty [(1+ax)^{-p}+(1+bx)^{-p}]x^{q-1}\,dx = 2(ab)^{-\frac{q}{2}}\mathrm{B}(q,\,p-q)\cos\left\{q\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$
$$[p > q > 0].$$

BI ((19))(9)

3.213

$$\int_0^\infty [(1+ax)^{-p}-(1+bx)^{-p}]x^{q-1}\,dx = -2i(ab)^{-\frac{q}{2}}\mathrm{B}(q,\,p-q)\sin\left\{q\,\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$
$$[p > q > 0].$$

BI ((19))(10)

3.214

$$\int_0^1 [(1+x)^{\mu-1}(1-x)^{\nu-1}+(1+x)^{\nu-1}(1-x)^{\mu-1}]\,dx = 2^{\mu+\nu-1}\mathrm{B}(\mu,\,\nu)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

LI((1))(15), EH I 10(10)

3.215

$$\int_0^1 \left\{a^\mu x^{\mu-1}(1-ax)^{\nu-1}+(1-a)^\nu x^{\nu-1}[1-(1-a)x]^{\mu-1}\right\}\,dx = \mathrm{B}(\mu,\,\nu)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad |a| < 1].$$

BI ((1))(16)

3.216

1. $$\int_0^1 \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\,\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

FI II 775

2. $$\int_1^\infty \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\,\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

FI II 775

3.217

$$\int_0^\infty \left\{\frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1}x^p}\right\} dx = \pi\ \operatorname{ctg} p\pi \qquad [0<p<1, \quad b>0].$$

BI((18))(13)

3.218

$$\int_0^\infty \frac{x^{2p-1}-(a+x)^{2p-1}}{(a+x)^p x^p}\,dx = \pi\ \operatorname{ctg} p\pi \qquad [p<1] \qquad (\text{cf. } 3.217).$$

3.217
BI ((18))(7)

3.219

$$\int_0^\infty \left\{\frac{x^\nu}{(x+1)^{\nu+1}} - \frac{x^\mu}{(x+1)^{\mu+1}}\right\} dx = \psi(\mu+1)-\psi(\nu+1) \qquad [\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1].$$

BI ((19))(13)

3.221

1. $$\int_a^\infty \frac{(x-a)^{p-1}}{x-b}\,dx = \pi(a-b)^{p-1}\operatorname{cosec} p\pi \qquad [a>b, \quad 0<p<1].$$

LI ((24))(8)

2. $$\int_{-\infty}^a \frac{(a-x)^{p-1}}{x-b}\,dx = -\pi(b-a)^{p-1}\operatorname{cosec} p\pi \qquad [a<b, \quad 0<p<1].$$

3.222

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x} = \beta(\mu) \qquad [\operatorname{Re}\mu > 0].$$

WH

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = \begin{cases} \pi\operatorname{cosec}(\mu\pi)a^{\mu-1} & \text{for } a>0, \\ -\pi\ \operatorname{ctg}(\mu\pi)(-a)^{\mu-1} & \text{for } a<0, \end{cases}$$
$$[0 < \operatorname{Re}\mu < 1].$$

FI II 718, FI II 737
BI((18))(2), ET II 249(28)

3.223

1. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta}(\beta^{\mu-1}-\gamma^{\mu-1})\operatorname{cosec}(\mu\pi)$$
$$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad 0 < \operatorname{Re}\mu < 2].$$

ET I 309(7)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta}[\beta^{\mu-1}\operatorname{cosec}(\mu\pi)+\alpha^{\mu-1}\ \operatorname{ctg}(\mu\pi)]$$
$$[|\arg\beta| < \pi, \quad \alpha > 0, \quad 0 < \operatorname{Re}\mu < 2].$$

ET I 309(8)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)} = \pi\operatorname{ctg}(\mu\pi)\frac{a^{\mu-1}-b^{\mu-1}}{b-a} \qquad [a > b > 0, \quad 0 < \operatorname{Re}\mu < 2].$$

ET I 309(9)

3.224

$$\int_0^\infty \frac{(x+\beta)x^{\mu-1}\,dx}{(x+\gamma)(x+\delta)} = \pi\operatorname{cosec}(\mu\pi)\left\{\frac{\gamma-\beta}{\gamma-\delta}\gamma^{\mu-1}+\frac{\delta-\beta}{\delta-\gamma}\delta^{\mu-1}\right\}$$
$$[|\arg\gamma| < \pi, \quad |\arg\delta| < \pi, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 309(10)

3.225

1. $$\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx = (1-p)\pi\operatorname{cosec}p\pi \qquad [-1 < p < 1].$$

2. $$\int_1^\infty \frac{(x-1)^{1-p}}{x^3}\,dx = \frac{1}{2}p(1-p)\pi \operatorname{cosec} p\pi \qquad [0<p<1].$$

BI ((23))(1)

3. $$\int_0^\infty \frac{x^p\,dx}{(1+x)^3} = \frac{\pi}{2}p(1-p)\operatorname{cosec} p\pi \qquad [-1<p<2].$$

BI ((16))(5)

3.226

1. $$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = 2\frac{(2n)!!}{(2n+1)!!}$$

BI ((8))(1)

2. $$\int_0^1 \frac{x^{n-\frac{1}{2}}\,dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!}\pi.$$

BI ((8))(2)

3.227

1. $$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x}\,dx = \beta^{1-\mu}\gamma^{\nu-1}\mathrm{B}(\nu,\,\mu-\nu)\,{}_2F_1\left(\mu-1,\,\nu;\,\mu;\,1-\frac{\gamma}{\beta}\right)$$
$$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu].$$

ET II 217(9)

2. $$\int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = \pi\gamma^{-\varrho}(\beta-\gamma)^{-\sigma}\operatorname{cosec}(\varrho\pi)I_{1-\gamma/\beta}(\sigma,\,\varrho)$$
$$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad -\operatorname{Re}\sigma<\operatorname{Re}\varrho<1].$$

ET II 217(10)

3.228

1. $$\int_a^b \frac{(x-a)^\nu(b-x)^{-\nu}}{x-c}\,dx = \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{a-c}{b-c}\right)^\nu\right] \qquad \text{for} \quad c<a;$$
$$= \pi\operatorname{cosec}(\nu\pi)\left[1-\cos(\nu\pi)\left(\frac{c-a}{b-c}\right)^\nu\right] \qquad \text{for} \quad a<c<b;$$
$$= \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{c-a}{c-b}\right)^\nu\right] \qquad \text{for} \quad c>b \qquad [|\operatorname{Re}\nu|<1].$$

2. $$\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c}\,dx = \frac{\pi \operatorname{cosec}(\nu\pi)}{b-c}\left|\frac{a-c}{b-c}\right|^{\nu-1} \qquad \text{for} \quad c<a \quad \text{or} \quad c>b;$$
$$= -\frac{\pi(c-a)^{\nu-1}}{(b-c)^{\nu}}\operatorname{ctg}(\nu\pi) \qquad \text{for} \quad a<c<b \qquad [0<\operatorname{Re}\nu<1].$$

ET II 250(32)

339

3. $$\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x-c}\,dx = \frac{(b-a)^{\mu+\nu-1}}{b-c}\mathrm{B}(\mu,\nu)\,{}_2F_1\left(1,\,\mu;\,\mu+\nu;\,\frac{b-a}{b-c}\right)$$
$$\text{for} \quad c<a \quad \text{or} \quad c>b;$$
$$= \pi(c-a)^{\nu-1}(b-c)^{\mu-1}\operatorname{ctg}\mu\pi - (b-a)^{\mu+\nu-2}\mathrm{B}(\mu-1,\,\nu)\times$$
$$\times\,{}_2F_1\left(2-\mu-\nu,\,1;\,2-\mu;\,\frac{b-c}{b-a}\right) \qquad \text{for} \quad a<c<b$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad \mu+\nu\neq 1, \quad \mu\neq 1,\,2,\ldots].$$

ET II 250(33)

4. $$\int_0^1 \frac{(1-x)^{\nu-1}x^{-\nu}}{a-bx}\,dx = \frac{\pi(a-b)^{\nu-1}}{a^{\nu}}\operatorname{cosec}(\nu\pi) \qquad [0<\operatorname{Re}\nu<1, \quad 0<b<a].$$

BI ((5))(8)

5. $$\int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c}\,dx = a^{1-\mu}(-c)^{\nu-1}\mathrm{B}(\mu-\nu,\,\nu)\,{}_2F_1\left(\mu-1,\,\nu;\,\mu;\,1+\frac{c}{a}\right) \qquad \text{for} \quad c<0;$$
$$= \pi c^{\nu-1}(a+c)^{1-\mu}\operatorname{ctg}[(\mu-\nu)\pi] - \frac{a^{1-\mu-\nu}}{a+c}\mathrm{B}(\mu-\nu-1,\,\nu)\times$$
$$\times\,{}_2F_1\left(2-\mu,\,1;\,2-\mu+\nu;\,\frac{a}{a+c}\right) \qquad \text{for} \quad c>0$$
$$[a>0, \quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu].$$

ET II 251(34)

3.229

$$\int_0^1 \frac{x^{\mu-1}\,dx}{(1-x)^{\mu}(1+ax)(1+bx)} = \frac{\pi\operatorname{cosec}\mu\pi}{a-b}\left[\frac{a}{(1+a)^{\mu}} - \frac{b}{(1+b)^{\mu}}\right] \qquad [0<\operatorname{Re}\mu<1].$$

BI ((5))(7)

3.231

1. $$\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\operatorname{ctg}p\pi \qquad [p^2<1].$$

2. $\int_0^1 \frac{x^{p-1} - x^{-p}}{1+x}\,dx = \pi \operatorname{cosec} p\pi \qquad [p^2 < 1].$

BI ((4))(1)

3. $\int_0^1 \frac{x^p - x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi \operatorname{ctg} p\pi \qquad [p^2 < 1].$

BI ((4))(3)

4. $\int_0^1 \frac{x^p - x^{-p}}{1+x}\,dx = \frac{1}{p} - \pi \operatorname{cosec} p\pi \qquad [p^2 < 1].$

BI ((4))(2)

5. $\int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x}\,dx = \psi(\nu) - \psi(\mu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$

FI II 815, BI((4))(5)

6. $\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x}\,dx = \pi(\operatorname{ctg} p\pi - \operatorname{ctg} q\pi) \qquad [p > 0, \quad q > 0].$

FI II 718

3.232

$$\int_0^\infty \frac{(c+ax)^{-\mu} - (c+bx)^{-\mu}}{x}\,dx = c^{-\mu}\ \ln\frac{b}{a} \qquad [\operatorname{Re}\mu > -1; \quad a > 0; \quad b > 0; \quad c > 0].$$

BI ((18))(14)

340

3.233

$$\int_0^\infty \left\{\frac{1}{1+x} - (1+x)^{-\nu}\right\}\frac{dx}{x} = \psi(\nu) + \boldsymbol{C} \qquad [\operatorname{Re}\nu > 0].$$

EH I 17, WH

3.234

1. $\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x}\right)dx = \pi a^{-q} \operatorname{ctg} q\pi \qquad [0 < q < 1, \quad a > 0].$

2. $$\int_0^1 \left(\frac{x^{q-1}}{1+ax} + \frac{x^{-q}}{a+x}\right) dx = \pi a^{-q} \operatorname{cosec} q\pi \qquad [0 < q < 1, \quad a > 0].$$

BI ((5))(10)

3.235

$$\int_0^\infty \frac{(1+x)^\mu - 1}{(1+x)^\nu} \frac{dx}{x} = \psi(\nu) - \psi(\nu - \mu) \qquad [\operatorname{Re}\nu > \operatorname{Re}\mu > 0].$$

BI ((18))(5)

3.236

$$\int_0^1 \frac{\mu \frac{\mu}{2}\, dx}{[(1-x)(1-a^2x)]^{\frac{\mu+1}{2}}} = \frac{(1-a)^{-\mu} - (1+a)^{-\mu}}{2a\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right)$$
$$[-2 < \mu < 1, \quad |a| < 1].$$

BI ((12))(32)

3.237

$$\sum_{n=0}^{\infty} (-1)^{n+1} \int_n^{n+1} \frac{dx}{x+u} = \ln \frac{u\left[\Gamma\left(\frac{u}{2}\right)\right]^2}{2\left[\Gamma\left(\frac{u+1}{2}\right)\right]^2} \qquad [|\arg u| < \pi].$$

ET II 216(1)

3.238

1. $$\int_{-\infty}^\infty \frac{|x|^{\nu-1}}{x-u}\, dx = -\pi \operatorname{ctg} \frac{\nu\pi}{2} |u|^{\nu-1} \operatorname{sign} u \qquad [0 < \operatorname{Re}\nu < 1] \qquad [u \text{ real}, \quad u \neq 0].$$

ET II 249(29)

2. $$\int_{-\infty}^\infty \frac{|x|^{\nu-1}}{x-u} \operatorname{sign} x\, dx = \pi \operatorname{tg} \frac{\nu\pi}{2} |u|^{\nu-1} \qquad [0 < \operatorname{Re}\nu < 1] \qquad [u \text{ real}, \quad u \neq 0].$$

ET II 249(30)

3. $$\int_a^b \frac{(b-x)^{\mu-1}(x-a)^{\nu-1}}{|x-u|^{\mu+\nu}}\, dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^\mu |b-u|^\nu} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad 0 < u < a < b \quad \text{or} \quad 0 < a < b < u].$$

3.241

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x^p} = \frac{1}{p}\beta\left(\frac{\mu}{p}\right) \qquad [\operatorname{Re}\mu > 0, \quad p > 0].$$

WH, BI ((2))(13)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{1+x^\nu} = \frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu} = \frac{1}{\nu}\mathrm{B}\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right) \qquad [\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$$

ET I 309(15)A, BI ((17))(10)

341

3. $$\int_0^\infty \frac{x^{p-1}\,dx}{1-x^q} = \frac{\pi}{q}\operatorname{ctg}\frac{p\pi}{q} \qquad [p < q].$$

BI ((17))(11)

4. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\frac{\mu}{\nu}}\frac{\Gamma\left(\frac{\mu}{\nu}\right)\Gamma\left(1+n-\frac{\mu}{\nu}\right)}{\Gamma(1+n)} \qquad \left[0 < \frac{\mu}{\nu} < n+1, \quad p \neq 0, \quad q \neq 0\right].$$

BI ((17))(22)a

5. $$\int_0^\infty \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2}\operatorname{cosec}\frac{(p-q)\pi}{q} \qquad [p < 2q].$$

BI ((17))(18)

3.242

$$\int_{-\infty}^\infty \frac{x^{2m}\,dx}{x^{4n}+2x^{2n}\cos t+1} = \frac{\pi}{n}\sin\left[\frac{(2n-2m-1)}{2n}t\right]\operatorname{cosec} t \operatorname{cosec}\frac{(2m+1)\pi}{2n}$$
$$[m < n, \quad t^2 < \pi^2].$$

FI II 642

3.243

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+x^{2\nu})(1+x^{3\nu})} = -\frac{\pi}{8\nu}\frac{\operatorname{cosec}\left(\frac{\mu\pi}{3\nu}\right)}{1-4\cos^2\left(\frac{\mu\pi}{3\nu}\right)} \qquad [0 < \operatorname{Re}\mu < 5\operatorname{Re}\nu].$$

1. $$\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\,dx = \frac{\pi}{q}\ \operatorname{cosec}\ \frac{p\pi}{q} \qquad [q>p>0].$$

BI ((2))(14)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^q}\,dx = \frac{\pi}{q}\ \operatorname{ctg}\ \frac{p\pi}{q} \qquad [q>p>0].$$

BI ((2))(16)

3. $$\int_0^1 \frac{x^{\nu-1}-x^{\mu-1}}{1-x^\nu}\,dx = \frac{1}{\nu}\left[\boldsymbol{C}+\psi\left(\frac{\mu}{\nu}\right)\right] \qquad [\operatorname{Re}\mu>\operatorname{Re}\nu>0].$$

BI ((2))(17)

4. $$\int_{-\infty}^{\infty} \frac{x^{2m}-x^{2n}}{1-x^{2l}}\,dx = \frac{\pi}{l}\left[\operatorname{ctg}\left(\frac{2m+1}{2l}\pi\right)-\operatorname{ctg}\left(\frac{2n+1}{2l}\pi\right)\right] \qquad [m<l, \quad n<l].$$

FI II 640

3.245

$$\int_0^\infty \left[x^{\nu-\mu}-x^\nu(1+x)^{-\mu}\right]dx = \frac{\nu}{\nu-\mu+1}\,\mathrm{B}(\nu,\,\mu-\nu) \qquad [\operatorname{Re}\mu>\operatorname{Re}\nu>0].$$

BI ((16))(13)

3.246

$$\int_0^\infty \frac{1-x^q}{1-x^r}x^{p-1}\,dx = \frac{\pi}{r}\ \sin\frac{q\pi}{r}\ \operatorname{cosec}\ \frac{p\pi}{r}\operatorname{cosec}\frac{(p+q)\pi}{r} \qquad [p+q<r, \quad p>0].$$

ET I 331(33), BI ((17))(12)

Integrals of the form $\int f(x^p \pm x^{-p},\ x^q \pm x^{-q}, \dots)\frac{dx}{x}$ can be transformed by the substitution $x=e^t$ or $x=e^{-t}$. For example, instead of $\int_0^1 (x^{1+p}+x^{1-p})^{-1}\,dx$, we should seek to evaluate $\int_0^\infty \operatorname{sech} px\,dx$ and, instead of $\int_0^1 \frac{x^{n-m-1}+x^{n+m-1}}{1+2x^n\ \cos a+x^{2n}}\,dx$, we should seek to evaluate $\int_0^\infty \operatorname{ch} mx(\operatorname{ch} nx-\cos a)^{-1}\,dx$ (see 3.514 2.).

3.247

1. $$\int_0^1 \frac{x^{\alpha-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = (n-1)!\sum_{k=0}^{\infty}\frac{\xi^k}{(\alpha+kb),\ (\alpha+kb+1)\dots(\alpha+kb+k-1)}$$
$$[b>0, \quad |\xi|<1].$$

2. $$\int_0^\infty \frac{(1-x^p)x^{\nu-1}}{1-x^{np}}\,dx = \frac{\pi}{np}\,\sin\left(\frac{\pi}{n}\right)\,\mathrm{cosec}\,\frac{(p+\nu)\pi}{np}\,\mathrm{cosec}\,\frac{\pi\nu}{np} \qquad [0 < \operatorname{Re}\nu < (n-1)p].$$

ET I 311(33)

3.248

1.⁰ $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu}\mathrm{B}\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right) \qquad [\operatorname{Re}\nu > \operatorname{Re}2\mu > 0].$$

BI ((21))(9)

2. $$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}.$$

BI ((8))(14)

3. $$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}.$$

BI ((8))(13)

4.* $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}.$$

5.* $$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[1+\frac{4x^2}{3(1+x^2)^2}+\left(1+\frac{4x^2}{3(1+x^2)^2}\right)^{1/2}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}.$$

3.249

1.⁰ $$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2\cdot(2n-2)!!}\frac{\pi}{a^{2n-1}}.$$

FI II 743

2. $$\int_0^a (a^2-x^2)^{n-\frac{1}{2}}\,dx = a^{2n}\frac{(2n-1)!!}{2(2n)!!}\pi.$$

FI II 156

3. $$\int_{-1}^{1} \frac{(1-x^2)^n\,dx}{(a-x)^{n+1}} = 2^{n+1} Q_n(a).$$

EH II 181(31)

4. $$\int_0^1 \frac{x^\mu\,dx}{1+x^2} = \frac{1}{2}\beta\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1].$$

BI ((2))(7)

5. $$\int_0^1 (1-x^2)^{\mu-1}\,dx = 2^{2\mu-2}\mathrm{B}(\mu,\,\mu) = \frac{1}{2}\mathrm{B}\left(\frac{1}{2},\,\mu\right) \qquad [\operatorname{Re}\mu > 0].$$

FI II 784

343

6. $$\int_0^1 (1-\sqrt{x})^{p-1}\,dx = \frac{2}{p(p+1)} \qquad [p > 0].$$

BI ((7))(7)

7. $$\int_0^1 (1-x^\mu)^{-\frac{1}{\nu}}\,dx = \frac{1}{\mu}\mathrm{B}\left(\frac{1}{\mu},\,1-\frac{1}{\nu}\right) \qquad [\operatorname{Re}\mu > 0, \quad |\nu| > 1].$$

8.* $$\int_{-\infty}^{\infty} \left(1+\frac{x^2}{n-1}\right)^{-n/2} dx = \sqrt{\pi(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n}{2}\right) \qquad [\text{integer}\, n > 1].$$

3.251

1. $$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1}\,dx = \frac{1}{\lambda}\mathrm{B}\left(\frac{\mu}{\lambda},\,\nu\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \lambda > 0].$$

FI II 787

2. $$\int_0^\infty x^{\mu-1}(1+x^2)^{\nu-1}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\mu}{2},\,1-\nu-\frac{\mu}{2}\right) \qquad \left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\left(\nu+\frac{1}{2}\mu\right) < 1\right].$$

3. $$\int_1^\infty x^{\mu-1}(x^p-1)^{\nu-1}\,dx = \frac{1}{p}\mathrm{B}\left(1-\nu-\frac{\mu}{p},\ \nu\right) \qquad [p>0, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu<p-p\operatorname{Re}\nu].$$

ET I 311(32)

4. $$\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+c)^n} = \frac{(2m-1)!!(2n-2m-3)!!\pi}{2\cdot(2n-2)!!a^m c^{n-m-1}\sqrt{ac}} \qquad [a>0, \quad c>0, \quad n>m+1].$$

GU ((141))(8a)

5. $$\int_0^\infty \frac{x^{2m+1}\,dx}{(ax^2+c)^n} = \frac{m!(n-m-2)!}{2(n-1)!a^{m+1}c^{n-m-1}} \qquad [ac>0, \quad n>m+1\geq 1].$$

GU ((141))(8b)

6. $$\int_0^\infty \frac{x^{\mu+1}}{(1+x^2)^2}\,dx = \frac{\mu\pi}{4\sin\frac{\mu\pi}{2}} \qquad [-2<\operatorname{Re}\mu<2].$$

WH

7. $$\int_0^1 \frac{x^\mu\,dx}{(1+x^2)^2} = -\frac{1}{4}+\frac{\mu-1}{4}\beta\left(\frac{\mu-1}{2}\right) \qquad [\operatorname{Re}\mu>1].$$

LI ((3))(11)

8. $$\int_0^1 x^{q+p-1}(1-x^q)^{-\frac{p}{q}}\,dx = \frac{p\pi}{q^2}\ \operatorname{cosec}\ \frac{p\pi}{q} \qquad [q>p].$$

BI ((9))(22)

9. $$\int_0^1 x^{\frac{q}{p}-1}(1-x^q)^{-\frac{1}{p}}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{\pi}{p} \qquad [p>1, \quad q>0].$$

BI ((9))(23)a

10. $$\int_0^1 x^{p-1}(1-x^q)^{-\frac{p}{q}}\,dx = \frac{\pi}{q}\ \operatorname{cosec}\ \frac{p\pi}{q} \qquad [q>p>0].$$

BI ((9))(20)

11. $$\int_0^\infty x^{\mu-1}(1+\beta x^p)^{-\nu}\,dx = \frac{1}{p}\beta^{-\frac{\mu}{p}}\mathrm{B}\left(\frac{\mu}{p},\ \nu-\frac{\mu}{p}\right)$$
$$[|\arg\beta|<\pi, \quad p>0, \quad 0<\operatorname{Re}\mu<p\operatorname{Re}\nu].$$

1. $$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{1}{\sqrt{ac-b^2}}\,\operatorname{arcctg}\,\frac{b}{\sqrt{ac-b^2}}\right] \qquad [a>0, \quad ac>b^2].$$

GW ((131))(4)

2. $$\int_{-\infty}^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}} \qquad [a>0, \quad ac>b^2].$$

GW ((131))(5)

3. $$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left\{\frac{1}{\sqrt{c}(\sqrt{ac}+b)}\right\} \qquad [a\ge 0, \quad c>0, \quad b>-\sqrt{ac}].$$

GW ((213))(4)

4. $$\int_0^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2(ac-b^2)} - \frac{b}{2(ac-b^2)^{\frac{3}{2}}}\,\operatorname{arcctg}\,\frac{b}{\sqrt{ac-b^2}}\right\}$$
$$\text{for} \quad ac>b^2;$$
$$= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2(ac-b^2)} + \frac{b}{4(b^2-ac)^{\frac{3}{2}}}\,\ln\frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}}\right\}$$
$$\text{for} \quad b^2>ac>0;$$
$$= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \qquad \text{for} \quad ac=b^2 \qquad [a>0, \quad b>0, \quad n\ge 2].$$

GW ((141))(5)

5. $$\int_{-\infty}^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = -\frac{(2n-3)!!\pi b a^{n-2}}{(2n-2)!!(ac-b^2)^{\frac{(2n-1)}{2}}} \qquad [ac>b^2, \quad a>0, \quad n\ge 2].$$

GW ((141))(6)

6. $$\int_{-\infty}^\infty \frac{x^m\,dx}{(ax^2+2bx+c)^n} = \frac{(-1)^m\pi a^{n-m-1}b^m}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}} \times$$
$$\times \sum_{k=0}^{[m/2]}\binom{m}{2k}(2k-1)!!(2n-2k-3)!!\left(\frac{ac-b^2}{b^2}\right)^k$$
$$[ac>b^2, \quad 0\le m\le 2n-2].$$

7. $$\int_0^\infty \frac{x^n\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{c}(\sqrt{ac}+b)^{n+1}} \qquad [a \ge 0, \quad c > 0, \quad b > -\sqrt{ac}].$$

GW ((213))(5a)

8. $$\int_0^\infty \frac{x^{n+1}\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{a}(\sqrt{ac}+b)^{n+1}} \qquad [a > 0, \quad c \ge 0, \quad b > -\sqrt{ac}].$$

GW ((213))(5b)

9. $$\int_0^\infty \frac{x^{n+\frac{1}{2}}\,dx}{(ax^2+2bx+c)^{n+1}} = \frac{(2n-1)!!\pi}{2^{2n+\frac{1}{2}}(b+\sqrt{ac})^{n+\frac{1}{2}}n!\sqrt{a}} \qquad [a > 0, \quad c > 0, \quad b+\sqrt{ac} > 0].$$

LI ((21))(19)

10.[6] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+2x\cos t+x^2)^\nu} = 2^{\nu-\frac{1}{2}}(\sin t)^{\frac{1}{2}-\nu}t\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{B}(\mu,\, 2\nu-\mu)P^{\frac{1}{2}-\nu}_{\mu-\nu-\frac{1}{2}}(\cos t)$$
$$[0 < t < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}2\nu].$$

ET I 310(22)

11. $$\int_0^\infty (1+2\beta x+x^2)^{\mu-\frac{1}{2}}x^{-\nu-1}\,dx = 2^{-\mu}(\beta^2-1)^{\frac{\mu}{2}}\Gamma(1-\mu)\mathrm{B}(\nu-2\mu+1,\,-\nu)P^\mu_{\nu-\mu}(\beta)$$
$$[\operatorname{Re}\nu < 0, \quad \operatorname{Re}(2\mu-\nu) < 1, \quad |\arg(\beta\pm 1)| < \pi].$$
$$= -\pi\operatorname{cosec}\nu\pi C_\nu^{\frac{1}{2}-\mu}(\beta)$$
$$\left[-2 < \operatorname{Re}\left(\frac{1}{2}-\mu\right) < \operatorname{Re}\nu < 0, \quad |\arg(\beta\pm 1)| < \pi\right].$$

EH I 178(24)
EH I 160(33)

12. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x^2+2ax\cos t+a^2} = -\pi a^{\mu-2}\operatorname{cosec}t\operatorname{cosec}(\mu\pi)\sin[(\mu-1)t]$$
$$[a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re}\mu < 2].$$

FI II 738, BI((20))(3)

13. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(x^2+2ax\cos t+a^2)^2} = \frac{\pi a^{\mu-4}}{2}\operatorname{cosec}\mu\pi\operatorname{cosec}^3 t\,\times$$
$$\times\{(\mu-1)\sin t\cos[(\mu-2)t]-\sin[(\mu-1)t]\}$$
$$[a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re}\mu < 4].$$

LI((20))(8)A, ET I 309(13)

14. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+2x\,\cos t+x^2}} = \pi\operatorname{cosec}(\mu\pi)\,\mathrm{P}_{\mu-1}\left(\cos t\right) \qquad [-\pi < t < \pi, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 310(17)

3.253

$$\int_{-1}^1 \frac{(1+x)^{2\mu-1}(1-x)^{2\nu-1}}{(1+x^2)^{\mu+\nu}}\,dx = 2^{\mu+\nu-2}\,\mathrm{B}\left(\mu,\,\nu\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

FI II 787

3.254

1. $$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}(x^2+\beta^2)^\nu\,dx =$$
$$= \beta^{2\nu}u^{\lambda+\mu-1}\mathrm{B}(\lambda,\,\mu)\,{}_3F_2\left(-\nu,\,\frac{\lambda}{2},\,\frac{\lambda+1}{2};\,\frac{\lambda+\mu}{2},\,\frac{\lambda+\mu+1}{2};\,\frac{-u^2}{\beta^2}\right)$$
$$\left[\operatorname{Re}\left(\frac{u}{\beta}\right) > 0, \quad \lambda > 0, \quad \operatorname{Re}\mu > 0\right].$$

ET II 186(10)

2.[6] $$\int_u^\infty (x^{-\lambda}(x-u)^{\mu-1}(x^2+\beta^2))^\nu\,dx =$$
$$= u^{\mu-\lambda+2\nu}\frac{\Gamma(\mu)\Gamma(\lambda-\mu-2\nu)}{\Gamma(\lambda-2\nu)} \times$$
$$\times\ {}_3F_2\left(-\nu,\,\frac{\lambda-\mu}{2}-\nu,\,\frac{1+\lambda-\mu}{2}-\nu;\,\frac{\lambda}{2}-\nu,\,\frac{1+\lambda}{2}-\nu;\,-\frac{\beta^2}{u^2}\right)$$
$$\left[|u| > |\beta| \quad \text{or} \quad \operatorname{Re}\left(\frac{\beta}{u}\right) > 0, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}\left(\lambda-2\nu\right)\right].$$

ET II 202(9)

3.255

$$\int_0^1 \frac{x^{\mu+\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}}{(c+2bx-ax^2)^{\mu+1}}\,dx =$$
$$= \frac{\sqrt{\pi}}{\left\{a+(\sqrt{c+2b-a}+\sqrt{c})^2\right\}^{\mu+\frac{1}{2}}\sqrt{c+2b-a}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)}$$
$$\left[a+(\sqrt{c+2b-a}+\sqrt{c})^2 > 0, \quad c+2b-a > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right].$$

1. $$\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\cos\left(\frac{q-p}{4}\pi\right)\sec\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\,\frac{q}{2}\right) \qquad [p>0,\ q>0,\ p+q<2].$$

BI ((8))(25)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\sin\left(\frac{q-p}{4}\pi\right)\operatorname{cosec}\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\,\frac{q}{2}\right) \qquad [p>0,\ q>0,\ p+q<2].$$

BI ((8))(26)

3.257

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\,dx = \frac{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}{2ac^{p+\frac{1}{2}}\Gamma(p+1)}.$$

BI ((20))(4)

3.258

1. $$\int_b^\infty (x-\sqrt{x^2-a^2})^n\,dx = \frac{a^2}{2(n-1)}(b-\sqrt{b^2-a^2})^{n-1}-\frac{1}{2(n+1)}(b-\sqrt{b^2-a^2})^{n+1}$$
$$[0<a\le b, \quad n\ge 2].$$

GW ((215))(5)

2. $$\int_b^\infty (\sqrt{x^2+1}-x)^n\,dx = \frac{(\sqrt{b^2+1}-b)^{n-1}}{2(n-1)}+\frac{(\sqrt{b^2+1}-b)^{n+1}}{2(n+1)} \qquad [n\ge 2].$$

GW ((214))(7)

347

3. $$\int_0^\infty (\sqrt{x^2+a^2}-x)^n\,dx = \frac{na^{n+1}}{n^2-1} \qquad [n\ge 2].$$

GW ((214))(6a)

4. $$\int_0^\infty \frac{dx}{(x+\sqrt{x^2+a^2})^n} = \frac{n}{a^{n-1}(n^2-1)} \qquad [n\ge 2].$$

GW ((214))(5a)

5. $$\int_0^\infty x^m(\sqrt{x^2+a^2}-x)^n\,dx = \frac{n\cdot m!a^{m+n+1}}{(n-m-1)(n-m+1)\dots(m+n+1)} \qquad [a>0, \quad 0\le m\le n-2].$$

6. $$\int_0^\infty \frac{x^m\,dx}{(x+\sqrt{x^2+a^2})^n} = \frac{n\cdot m!}{(n-m-1)(n-m+1)\dots(m+n+1)a^{n-m-1}}$$
$$[a>0, \quad 0\le m\le n-2].$$

GW ((214))(5)

7. $$\int_a^\infty (x-a)^m(x-\sqrt{x^2-a^2})^n\,dx = \frac{n\cdot(n-m-2)!(2m+1)!a^{m+n+1}}{2^m(n+m+1)!} \qquad [a>0, \quad n\ge m+2].$$

GW ((215))(6)

3.259

1.[6] $$\int_0^1 x^{p-1}(1-x)^{n-1}(1+bx^m)^l\,dx = (n-1)!\sum_{k=0}^\infty \binom{l}{k}\frac{b^k\Gamma(p+km)}{\Gamma(p+n+km)}$$
$$[|b|<1 \text{ unless } l=0,\,1,\,2,\dots;\, p,\, n,\, p+ml>0].$$

BI ((1))(14)

2. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}(x^m+\beta^m)^\lambda\,dx = \beta^{m\lambda}u^{\mu+\nu+1}\mathrm{B}(\mu,\,\nu)\times$$
$$\times\ {}_{m+1}F_m\left(-\lambda,\,\frac{\nu}{m},\,\frac{\nu+1}{m},\dots,\frac{\nu+m-1}{m};\,\frac{\mu+\nu}{m},\,\frac{\mu+\nu+1}{m},\dots,\frac{\mu+\nu+m-1}{m};\,\frac{-u^m}{\beta^m}\right)$$
$$\left[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad \left|\arg\left(\frac{u}{\beta}\right)\right|<\frac{\pi}{m}\right].$$

ET II 186(11)

3. $$\int_0^\infty x^{\lambda-1}(1+\alpha x^p)^{-\mu}(1+\beta x^p)^{-\nu}\,dx = \frac{1}{p}\alpha^{-\frac{\lambda}{p}}\mathrm{B}\left(\frac{\lambda}{p},\,\mu+\nu-\frac{\lambda}{p}\right)\,{}_2F_1\left(\nu,\,\frac{\lambda}{p};\,\mu+\nu;\,1-\frac{\beta}{a}\right)$$
$$[|\arg\alpha|<\pi, \quad |\arg\beta|<\pi, \quad p>0, \quad 0<\operatorname{Re}\lambda<2\operatorname{Re}(\mu+\nu)].$$

ET I 312(35)

3.261

1. $$\int_0^1 \frac{(1-x\cos t)x^{\mu-1}\,dx}{1-2x\cos t+x^2} = \sum_{h=0}^\infty \frac{\cos kt}{\mu+k} \qquad [\operatorname{Re}\mu>0, \quad t\ne 2n\pi].$$

BI ((6))(9)

2. $$\int_0^1 \frac{(x^\nu+x^{-\nu})\,dx}{1+2x\cos t+x^2} = \frac{\pi\sin\nu t}{\sin t\sin\nu\pi} \qquad [\nu^2<1, \quad t\ne(2n+1)\pi].$$

3. $$\int_0^1 \frac{(x^{1+p}+x^{1-p})\,dx}{(1+2x\,\cos t+x^2)^2}=\frac{\pi(p\,\sin t\,\cos pt-\cos t\,\sin pt)}{2\sin^3\,t\,\sin p\pi}\qquad [p^2<1,\quad t\neq(2n+1)\pi].$$

BI ((6))(18)

4. $$\int_0^1 \frac{x^{\mu-1}}{1+2ax\,\cos t+a^2x^2}\cdot\frac{dx}{(1-x)^{\mu}}=\frac{\pi\,\mathrm{cosec}\,t\,\mathrm{cosec}\,\mu\pi}{(1+2a\,\cos t+a^2)^{\frac{\mu}{2}}}\,\sin\left(t-\mu\,\mathrm{arctg}\,\frac{a\,\sin t}{1+a\,\cos t}\right)$$
$$[a>0,\quad 0<\mathrm{Re}\,\mu<1].$$

BI ((6))(21)

3.262

$$\int_0^\infty \frac{x^{-p}\,dx}{1+x^3}=\frac{\pi}{3}\,\mathrm{cosec}\,\frac{(1-p)\pi}{3}\qquad [-2<p<1].$$

LI ((18))(3)

3.263

$$\int_0^\infty \frac{x^{\nu}\,dx}{(x+\gamma)(x^2+\beta^2)}=\frac{\pi}{2(\beta^2+\gamma^2)}\left[\gamma\beta^{\nu-1}\,\sec\frac{\nu\pi}{2}+\beta^{\nu}\,\mathrm{cosec}\,\frac{\nu\pi}{2}-2\gamma^{\nu}\,\mathrm{cosec}(\nu\pi)\right]$$
$$[\mathrm{Re}\,\beta>0,\quad |\arg\gamma|<\pi,\quad -1<\mathrm{Re}\,\nu<2,\quad \nu\neq 0].$$

ET II 216(7)

3.264

1. $$\int_0^\infty \frac{x^{p-1}\,dx}{(a^2+x^2)(b^2-x^2)}=\frac{\pi}{2}\,\frac{a^{p-2}+b^{p-2}\,\cos\frac{p\pi}{2}}{a^2+b^2}\,\mathrm{cosec}\,\frac{p\pi}{2}\qquad [0<p<4,\quad a>0,\quad b>0].$$

BI ((19))(14)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x^2)(\gamma+x^2)}=\frac{\pi}{2}\,\frac{\gamma^{\frac{\mu}{2}-1}-\beta^{\frac{\mu}{2}-1}}{\beta-\gamma}\,\mathrm{cosec}\,\frac{\mu\pi}{2}$$
$$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\mathrm{Re}\,\mu<4].$$

ET I 309(4)

3.265

$$\int_0^1 \frac{1-x^{\mu-1}}{1-x}\,dx=\psi(\mu)+\boldsymbol{C}\qquad [\mathrm{Re}\,\mu>0];$$
$$=\psi(1-\mu)+\boldsymbol{C}-\pi\,\mathrm{ctg}(\mu\pi)\qquad [\mathrm{Re}\,\mu>0].$$

3.266

$$\int_0^\infty \frac{(x^\nu - a^\nu)\,dx}{(x-a)(\beta+x)} = \frac{\pi}{a+\beta}\left\{\beta^\nu \operatorname{cosec}(\nu\pi) - a^\nu\ \operatorname{ctg}(\nu\pi) - \frac{a^\nu}{\pi}\ \ln\frac{\beta}{a}\right\}$$
$$[|\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < 1, \quad \nu \neq 0].$$

ET II 216(8)

3.267

1. $$\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n+1)}.$$

BI ((9))(6)

349

2. $$\int_0^1 \frac{x^{3n-1}\,dx}{\sqrt[3]{1-x^3}} = \frac{(n-1)!\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(n+\frac{2}{3}\right)}.$$

BI ((9))(7)

3.268

1. $$\int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p}\right)\,dx = \ln p.$$

BI ((5))(14)

2. $$\int_0^1 \frac{1-x^\mu}{1-x}x^{\nu-1}\,dx = \psi(\mu+\nu) - \psi(\nu) \qquad [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0].$$

BI ((2))(3)

3. $$\int_0^1 \left[\frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}}\right]\,dx = n\boldsymbol{C} + \sum_{k=1}^{n}\psi\left(\mu+\frac{n-k}{n}\right) \qquad [\operatorname{Re}\mu > 0].$$

BI ((13))(10)

3.269

BI ((4))(12)

2. $$\int_0^1 \frac{x^p - x^{-p}}{1+x^2} x\,dx = \frac{1}{p} - \frac{\pi}{2} \operatorname{cosec} \frac{p\pi}{2} \qquad [p^2 < 1].$$

BI ((4))(8)

3. $$\int_0^1 \frac{x^\mu - x^\nu}{1-x^2}\,dx = \frac{1}{2}\psi\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\psi\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1].$$

BI ((2))(9)

3.271

1. $$\int_0^\infty \frac{x^p - x^q}{x-1} \frac{dx}{x+a} = \frac{\pi}{1+a}\left(\frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi}\right) \qquad [p^2 < 1, \quad q^2 < 1, \quad a > 0].$$

BI ((19))(2)

2. $$\int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^p - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{\frac{a^{2p}-1}{\sin(2p\pi)} - \frac{1}{\pi} a^p \ln a\right\} \qquad \left[p^2 < \frac{1}{4}\right].$$

BI ((19))(3)

3. $$\int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^{-p} - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{2(a^p - 1)\operatorname{ctg} p\pi - \frac{1}{\pi}(a^p+1)\ln a\right\} \qquad [p^2 < 1].$$

BI ((18))(9)

4. $$\int_0^\infty \frac{x^p - a^p}{x-a} \frac{1 - x^{-p}}{1-x} x^q\,dx = \frac{\pi}{a-1}\left\{\frac{a^{p+q}-1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]}\right\}\frac{\sin p\pi}{\sin q\pi}$$
$$[(p+q)^2 < 1, \quad (p-q)^2 < 1].$$

BI ((19))(4)

5. $$\int_0^\infty \left(\frac{x^p - x^{-p}}{1-x}\right)^2 dx = 2(1 - 2p\pi \operatorname{ctg} 2p\pi) \qquad \left[0 < p^2 < \frac{1}{4}\right].$$

BI ((16))(3)

3.272

1. $$\int_0^1 \frac{x^{n-1}+x^{n-\frac{1}{2}}-2x^{2n-1}}{1-x}\,dx = 2\ln 2.$$

BI ((8))(8)

2. $$\int_0^1 \frac{x^{n-1}+x^{n-\frac{2}{3}}+x^{n-\frac{1}{3}}-3x^{3n-1}}{1-x}\,dx = 3\ln 3.$$

BI ((8))(9)

3.273

1. $$\int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1}x^{n+1}\sin nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\sin kt}{\Gamma(p+k)} \qquad [p>0].$$

BI ((6))(13)

2. $$\int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1}x^{n+1}\cos nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \\ = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\cos kt}{\Gamma(p+k)} \qquad [p>0].$$

BI ((6))(14)

3. $$\int_0^1 x\frac{\sin t - x^n\sin[(n+1)t] + x^{n+1}\sin nt}{1-2x\cos t + x^2}\,dx = \sum_{k=1}^{n}\frac{\sin kt}{k+1}.$$

BI ((6))(12)

4. $$\int_0^1 \frac{1 - x\cos t - x^{n+1}\cos[(n+1)t] + x^{n+2}\cos nt}{1-2x\cos t + x^2}\,dx = \sum_{k=0}^{n}\frac{\cos kt}{k+1}.$$

BI ((6))(11)

3.274

1. $$\int_0^\infty \frac{x^{\mu-1}(1-x)}{1-x^n}\,dx = \frac{\pi}{n}\sin\frac{\pi}{n}\operatorname{cosec}\frac{\mu\pi}{n}\operatorname{cosec}\frac{(\mu+1)\pi}{n} \qquad [0<\operatorname{Re}\mu<n-1].$$

2. $\int_0^1 \frac{1-x^n}{(1+x)^{n+1}} \frac{dx}{1-x} = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}.$

BI ((5))(3)

3. $\int_0^\infty \frac{x^q - 1}{x^p - x^{-p}} \frac{dx}{x} = \frac{\pi}{2p} \operatorname{tg} \frac{q\pi}{2p} \qquad [p > q].$

BI ((18))(6)

3.275

1. $\int_0^1 \left\{ \frac{x^{n-1}}{1 - x^{\frac{1}{p}}} - \frac{px^{np-1}}{1-x} \right\} dx = p \ln p \qquad [p > 0].$

BI ((13))(9)

2. $\int_0^1 \left\{ \frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x} \right\} dx = \boldsymbol{C} + \frac{1}{n} \sum_{k=1}^{n} \psi \left(m + \frac{n-k}{n} \right).$

BI ((5))(13)

3. $\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q} \right) dx = \ln q \qquad [q > 0].$

BI ((5))(12)

4. $\int_0^\infty \left\{ \frac{1}{1+x^{2^n}} - \frac{1}{1+x^{2^m}} \right\} \frac{dx}{x} = 0.$

BI ((18))(17)

3.276

1. $\int_0^\infty \frac{\left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx}{x^2} = \frac{\sqrt{\pi}}{2bc^{p+\frac{1}{2}}} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p+1)} \qquad \left[p > -\frac{1}{2} \right].$

BI ((20))(19)

2. $\int_0^\infty \left(a + \frac{b}{x^2} \right) \left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx = \frac{\sqrt{\pi}}{c^{p+\frac{1}{2}}} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p+1)} \qquad \left[p > -\frac{1}{2} \right].$

3.277

1. $$\int_0^\infty \frac{x^{\mu-1}[\sqrt{1+x^2}+\beta]^\nu}{\sqrt{1+x^2}}\,dx = 2^{\frac{\mu}{2}-1}(\beta^2-1)^{\frac{\nu}{2}+\frac{\mu}{4}}\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)P_{\frac{\mu}{2}-1}^{\frac{\nu+\mu}{2}}(\beta)$$
$$[\operatorname{Re}\beta > -1, \quad 0 < \operatorname{Re}\mu < 1-\operatorname{Re}\nu].$$

ET I 310(25)

2. $$\int_0^\infty \frac{x^{\mu-1}[\sqrt{\beta^2+x^2}+x]^\nu}{\sqrt{\beta^2+x^2}}\,dx = \frac{\beta^{\mu+\nu-1}}{2^\mu}\mathrm{B}\left(\mu,\,\frac{1-\mu-\nu}{2}\right) \qquad [\operatorname{Re}\beta > 0, \quad 0 < \operatorname{Re}\mu < 1-\operatorname{Re}\nu].$$

ET I 311(28)

3. $$\int_0^\infty \frac{x^{\mu-1}\left[\cos t \pm i\,\sin t\sqrt{1+x^2}\right]^\nu}{\sqrt{1+x^2}}\,dx = 2^{\frac{\mu-1}{2}}\,\sin^{\frac{1-\mu}{2}}t\,\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}\times$$
$$\times\left[\pi^{-\frac{1}{2}}\mathrm{Q}_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu-1}{2}}(\cos t) \mp \frac{i}{2}\pi\frac{1}{2}\mathrm{P}_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu-1}{2}}(\cos t)\right]$$
$$[\operatorname{Re}\mu > 0].$$

ET I 311 (27)

4. $$\int_0^\infty \frac{x^{\mu-1}[\sqrt{(\beta^2-1)(x^2+1)}+\beta]^\nu}{\sqrt{x^2+1}}\,dx$$
$$= \frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}}e^{-\frac{1}{2}i\pi(\mu-1)}\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}(\beta^2-1)\frac{1-\mu}{4}Q\frac{\mu-1}{2}_{-\nu-\frac{\mu+1}{2}}(\beta)$$
$$[\operatorname{Re}\beta > 1, \quad \operatorname{Re}\nu < 0, \quad \operatorname{Re}\mu < 1-\operatorname{Re}\nu].$$

ET I 311(26)

5. $$\int_u^\infty \frac{(x-u)^{\mu-1}(\sqrt{x+1}-\sqrt{x-1})^{2\nu}}{\sqrt{x^2-1}}\,dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}}e^{(\mu-\frac{1}{2})\pi i}(u^2-1)^{\frac{2\mu-1}{4}}Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u)$$
$$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1+\operatorname{Re}\nu].$$

ET II 202(10)

6. $$\int_1^\infty \frac{x^{\mu-1}\left[(x-\sqrt{x^2-1})^\nu+(x-\sqrt{x^2-1})^{-\nu}\right]}{\sqrt{x^2-1}}\,dx = 2^{-\mu}\mathrm{B}\left(\frac{1-\mu+\nu}{2},\,\frac{1-\mu-\nu}{2}\right)$$
$$[\operatorname{Re}\mu < 1+\operatorname{Re}\nu].$$

ET I 311(29)

7. $$\int_0^u \frac{(u-x)^{\mu-1}\left[(\sqrt{x+2}+\sqrt{x})^{2\nu}+(\sqrt{x+2}-\sqrt{x})^{2\nu}\right]}{\sqrt{x(x+2)}}\,dx = 2\frac{2\mu+1}{2}\sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u+1)$$
$$[|\arg u| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET II 186(12)

3.278

$$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^q \frac{dx}{1-x^2} = 0.$$

3.3- 3.4 Exponential Functions

3.31 Exponential functions

3.310

$$\int_0^\infty e^{-px}\,dx = \frac{1}{p} \qquad [\operatorname{Re} p > 0].$$

3.311

1. $$\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}.$$

LO III 284a

2. $$\int_0^\infty \frac{e^{-\mu x}}{1+e^{-x}}\,dx = \beta(\mu) \qquad [\operatorname{Re}\mu > 0].$$

EH I 20(3), ET I 144(7)

3. $$\int_{-\infty}^\infty \frac{e^{-px}}{1+e^{-qx}}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q} \qquad [q > p > 0 \quad \text{or} \quad 0 > p > q] \qquad (\text{cf. } 3.241\ 2.).$$

3.241

BI ((28))(7)

4. $$\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \sum_{k=0}^\infty \frac{a^k}{q+kp} \qquad [0 < a < 1].$$

BI ((27))(7)

5. $$\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \psi(\nu) + \boldsymbol{C} + \pi\ \operatorname{ctg}(\pi\nu) \qquad [\operatorname{Re}\nu < 1] \qquad (\text{cf. } 3.265).$$

3.265
EH I 16(16)

6. $$\int_0^\infty \frac{e^{-x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) + \boldsymbol{C} \qquad [\operatorname{Re}\nu > 0].$$

WH, EH I 16(14)

7. $$\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) - \psi(\mu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad (\text{cf. } 3.231\ 5.).$$

3.231
BI ((27))(8)

8. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = \pi b^{\mu-1}\ \operatorname{ctg}(\mu\pi) \qquad [b > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 120(14)a

9. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b+e^{-x}} = \pi b^{\mu-1}\operatorname{cosec}(\mu\pi) \qquad [|\arg b| < \pi, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 120(15)a

10. $$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\ \operatorname{ctg}\frac{p\pi}{p+q} \qquad [p > 0, \quad q > 0].$$

GW ((311))(16c)

11. $$\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\,dx = \frac{1}{r-s}\left[\psi\left(\frac{r-q}{r-s}\right)-\psi\left(\frac{r-p}{r-s}\right)\right] \qquad [r>s,\ r>p,\ r>q].$$

GW ((311))(16)

12. $$\int_0^\infty \frac{a^x-b^x}{c^x-d^x}\,dx = \frac{1}{\ln\frac{c}{d}}\left\{\psi\left(\frac{\ln\frac{c}{b}}{\ln\frac{c}{d}}\right)-\psi\left(\frac{\ln\frac{c}{a}}{\ln\frac{c}{d}}\right)\right\} \qquad [c>a>0, \quad b>0, \quad d>0].$$

GW ((311))(16a)

3.312

1. $$\int_0^\infty \left(1-e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x}\,dx = \beta\,\mathrm{B}\left(\beta\mu,\, \nu\right) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0].$$

LI((25))(13), EH I 11(24)

2. $$\int_0^\infty (1-e^{-x})^{-1}(1-e^{-\alpha x})(1-e^{-\beta x})e^{-px}\,dx = \psi(p+\alpha)+\psi(p+\beta)-\psi(p+\alpha+\beta)-\psi(p)$$
$$[\operatorname{Re}p>0, \quad \operatorname{Re}p>-\operatorname{Re}\alpha, \quad \operatorname{Re}p>-\operatorname{Re}\beta, \quad \operatorname{Re}p>-\operatorname{Re}(\alpha+\beta)].$$

ET I 145(15)

3. $$\int_0^\infty (1-e^{-x})^{\nu-1}(1-\beta e^{-x})^{-\varrho}e^{-\mu x}\,dx = \mathrm{B}\left(\mu,\, \nu\right){}_2F_1(\varrho,\, u;\, \mu+\nu;\, \beta)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad |\arg\left(1-\beta\right)|<\pi].$$

EH I 116(15)

3.313

1.[7] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{1-e^{-x}} = \pi\cot\pi\mu \qquad [0<\operatorname{Re}\mu<1].$$

2.[7] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{(1+e^{-x})^\nu} = B(\mu,\, \nu-\mu) \qquad [0<\operatorname{Re}\mu<\operatorname{Re}\nu].$$

3.314

ET I 120(21)

3.315

1. $$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\, dx}{(e^{\beta}+e^{-x})^{\nu}(e^{\gamma}+e^{-x})^{\varrho}} = \exp[\gamma(\mu-\varrho)-\beta\nu]\mathrm{B}(\mu,\, \nu+\varrho-\mu)\, {}_2F_1(\nu,\, \mu;\, \nu+\varrho;\, 1-e^{\nu-\beta})$$
$$[|\operatorname{Im}\beta| < \pi, \quad |\operatorname{Im}\gamma| < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}(\nu+\varrho)].$$

ET I 121(22)

2. $$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\, dx}{(\beta+e^{-x})(\gamma+e^{-x})} = \frac{\pi(\beta^{\mu-1}-\gamma^{\mu-1})}{\gamma-\beta} \operatorname{cosec}(\mu\pi)$$
$$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad \beta\neq\gamma, \quad 0 < \operatorname{Re}\mu < 2].$$

ET I 120(18)

3.316

$$\int_{-\infty}^{\infty} \frac{(1+e^{-x})^{\nu}-1}{(1+e^{-x})^{\mu}}\, dx = \psi(\mu)-\psi(\mu-\nu) \qquad [\operatorname{Re}\mu > \operatorname{Re}\nu > 0] \qquad (\text{cf. } 3.235).$$

3.235

BI ((28))(8)

3.317

1. $$\int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^{\mu}}\right\} dx = \boldsymbol{C}+\psi(\mu) \qquad [\operatorname{Re}\mu > 0] \qquad (\text{cf. } 3.233).$$

3.233

BI ((28))(10)

2. $$\int_{-\infty}^{\infty} \left\{\frac{1}{(1+e^{-x})^{\nu}} - \frac{1}{(1+e^{-x})^{\mu}}\right\} dx = \psi(\mu)-\psi(\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad (\text{cf. } 3.219).$$

3.219

BI ((28))(11)

3.318

3.318

1. $$\int_0^\infty \frac{[\beta+\sqrt{1-e^{-x}}]^{-\nu}+[\beta-\sqrt{1-e^{-x}}]^{-\nu}}{\sqrt{1-e^{-x}}} e^{-\mu x}\,dx = = \frac{2^{\mu+1}e^{(\mu-\nu)\pi i}(\beta^2-1)^{(\mu-\nu)/2}\Gamma(\mu)Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)} \qquad [\operatorname{Re}\mu>0].$$

ET I 145(18)

2.[7] $$\int_u^\infty \frac{1}{\sqrt{1-e^{-2x}}}\left\{e^{-u}\sqrt{1-e^{-2x}}-e^{-x}\sqrt{1-e^{-2u}}\right\}^\nu e^{-\mu x}\,dx = = \frac{2^{-\frac{1}{2}(\mu+\nu)}\sqrt{\pi}e^{-\frac{u}{2}(\mu+\nu)}\Gamma(\mu)\Gamma(\nu+1)P_{-\frac{1}{2}(\mu-\nu)}^{-\frac{1}{2}(\mu+\nu)}(\sqrt{1-e^{-2u}})}{\Gamma[(\mu+\nu+1)/2]}$$
$$[u>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1].$$

ET I 145(19)

3.32- 3.34 Exponentials of more complicated arguments

3.321

1.[7] $$\Phi(u) = \frac{\sqrt{\pi}}{2}\operatorname{erf}(u) = \int_0^u e^{-x^2}\,dx = \sum_{k=0}^\infty \frac{(-1)^k u^{2k+1}}{k!(2k+1)};$$
$$= e^{-u^2}\sum_{k=0}^\infty \frac{2^k u^{2k+1}}{(2k+1)!!}. \qquad [cf.\ \mathbf{8}.25]$$

8.25
AD 6.700

2. $$\int_0^u e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q}\Phi(qu) \qquad [q>0].$$

3. $$\int_0^\infty e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q} \qquad [q>0].$$

FI II 624

3.322

1.[7] $$\int_u^\infty \exp\left(-\frac{x^2}{4\beta}-\gamma x\right)\,dx = \sqrt{\pi\beta}e^{\beta\gamma^2}\left[1-\Phi\left(\gamma\sqrt{\beta}+\frac{u}{2\sqrt{\beta}}\right)\right] \qquad [\operatorname{Re}\beta>0, \quad u\ge 0].$$

2. $$\int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right)\, dx = \sqrt{\pi\beta}\exp(\beta\gamma^2)\left[1 - \Phi(\gamma\sqrt{\beta})\right] \qquad [\operatorname{Re}\beta > 0].$$

NT 27(1)a

3.323

1.[7] $$\int_1^\infty \exp(-qx - x^2)\, dx = \frac{\pi^{1/2}}{2} e^{q^2/4}\left[1 - \Phi\left(1 + \frac{1}{2}q\right)\right].$$

BI ((29))(4)

2.[7] $$\int_{-\infty}^\infty \exp(-p^2x^2 \pm qx)\, dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{|p|}.$$

BI ((28))(1)

3.[6] $$\int_0^\infty \exp(-\beta^2x^4 - 2\gamma^2x^2)\, dx = 2^{-\frac{3}{2}}\frac{\gamma}{\beta} e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}\right].$$

ET I 147(34)a

3.324

1. $$\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right)\, dx = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}) \qquad [\operatorname{Re}\beta \ge 0, \quad \operatorname{Re}\gamma > 0].$$

ET I 146(25)

2. $$\int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right]\, dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right).$$

3.325

$$\int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp(-2\sqrt{ab}) \qquad [a > 0, \quad b > 0].$$

FI II 644

3.326[8]

BI ((26))(4)

Exponentials of exponentials

3.327

$$\int_0^\infty \exp(-ae^{nx})\,dx = -\frac{1}{n}\operatorname{Ei}(-a) \qquad [n \geq 1, \quad \operatorname{Re} a \geq 0, \quad a \neq 0].$$

LI ((26))(5)

3.328

$$\int_{-\infty}^\infty \exp(-e^x)e^{\mu x}\,dx = \Gamma(\mu) \qquad [\operatorname{Re}\mu > 0].$$

NH 145(14)

3.329

$$\int_0^\infty \left[\frac{a\exp(-ce^{\alpha x})}{1-e^{-\alpha x}} - \frac{b\exp(-ce^{bx})}{1-e^{-bx}}\right] dx = e^{-c}\,\ln\frac{b}{a} \qquad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((27))(12)

3.331

1. $$\int_0^\infty \exp(-\beta e^{-x} - \mu x)\,dx = \beta^{-\mu}\gamma(\mu,\,\beta) \qquad [\operatorname{Re}\mu > 0].$$

ET I 147(36)

2. $$\int_0^\infty \exp(-\beta e^{x} - \mu x)\,dx = \beta^{\mu}\Gamma(-\mu,\,\beta) \qquad [\operatorname{Re}\beta > 0].$$

ET I 147(37)

3. $$\int_0^\infty (1-e^{-x})^{\nu-1}\,\exp\left(\beta e^{-x} - \mu x\right)dx = \mathrm{B}(\mu,\,\nu)\beta^{-\frac{\mu-\nu}{2}}e^{\frac{\beta}{2}}M_{\frac{\nu-\mu}{2},\,\frac{\nu+\mu-1}{2}}(\beta) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET I 147(38)

4. $$\int_0^\infty (1-e^{-x})^{\nu-1}\,\exp\left(-\beta e^{x} - \mu x\right)dx = \Gamma(\nu)\beta^{\frac{\mu-1}{2}}e^{-\frac{\beta}{2}}W_{\frac{1-\mu-2\nu}{2},\,\frac{-\mu}{2}}(\beta) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0].$$

3.332

$$\int_0^\infty (1-e^{-x})^{\nu-1}(1-\lambda e^{-x})^{-\varrho}\exp(\beta e^{-x}-\mu x)\,dx = \mathrm{B}(\mu,\,\nu)\Phi_1(\mu,\,\varrho,\,\nu,\,\lambda,\,\beta)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad |\arg(1-\lambda)|<\pi].$$

ET I 147(40)

3.333

1. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)-1} = \Gamma(\mu)\zeta(\mu) \qquad [\operatorname{Re}\mu>1].$$

ET I 121(24)

2.[3] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)+1} = (1-2^{1-\mu})\Gamma(\mu)\zeta(\mu) \qquad [\operatorname{Re}\mu>0, \quad \mu\neq 1];$$
$$= \ln 2 \qquad [\mu=1].$$

ET I 121(25)

3.334

$$\int_0^\infty (e^x-1)^{\nu-1}\exp\left[-\frac{\beta}{e^x-1}-\mu x\right]\,dx = \Gamma(\mu-\nu+1)e^{\frac{\beta}{2}}\beta^{\frac{\nu-1}{2}}W_{\frac{\nu-2\mu-1}{2},\,\frac{\nu}{2}}(\beta)$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\,\mu>\operatorname{Re}\nu-1].$$

ET I 137(41)

Exponentials of hyperbolic functions

3.335

$$\int_0^\infty \left(e^{\nu x}+e^{-\nu x}\,\cos\nu\pi\right)\,\exp\left(-\beta\,\operatorname{sh}\,x\right)dx = -\pi[\mathbf{E}_\nu(\beta)+N_\nu(\beta)] \qquad [\operatorname{Re}\beta>0].$$

EH II 35(34)

3.336

1. $$\int_0^\infty \exp(-\nu x-\beta\,\operatorname{sh}\,x)\,dx = \pi\operatorname{cosec}\nu\pi[\mathbf{J}_\nu(\beta)-J_\nu(\beta)]$$
$$\left[|\arg\beta|<\frac{\pi}{2} \quad \text{and} \quad |\arg\beta|=\frac{\pi}{2} \quad \text{for} \quad \operatorname{Re}\nu>0; \quad \nu-\text{not an integer}\right].$$

2. $\int_0^\infty \exp(nx-\beta \text{ sh } x)\,dx = \frac{1}{2}[S_n(\beta)-\pi\mathbf{E}_n(\beta)-\pi N_n(\beta)] \qquad [\operatorname{Re}\beta > 0; \quad n = 0,\,1,\,2,\dots].$

WA 342(6)

3. $\int_0^\infty \exp(-nx-\beta \text{ sh } x)\,dx = \frac{1}{2}(-1)^{n+1}[S_n(\beta)+\pi\mathbf{E}_n(\beta)+\pi N_n(\beta)]$
$[\operatorname{Re}\beta > 0; \quad n = 0,\,1,\,2,\dots].$

EH II 84(47)

3.337

1. $\int_{-\infty}^\infty \exp(-\alpha x-\beta\operatorname{ch} x)\,dx = 2K_\alpha(\beta) \qquad \left[|\arg\beta| < \frac{\pi}{2}\right].$

WA 201(7)

2. $\int_{-\infty}^\infty \exp(-\nu x+i\beta\operatorname{ch} x)\,dx = i\pi e^{\frac{i\nu\pi}{2}}H_\nu^{(1)}(\beta) \qquad [0 < \arg z < \pi].$

EH II 21(27)

3. $\int_{-\infty}^\infty \exp(-\nu x-i\beta\operatorname{ch} x)\,dx = -i\pi e^{-\frac{i\nu\pi}{2}}H_\nu^{(2)}(\beta) \qquad [-\pi < \arg z < 0].$

EH II 21(30)

Exponentials of trigonometric functions and logarithms

3.338

1. $\int_0^\pi \{\exp\ i[(\nu-1)x-\beta\sin x]-\exp\ i[(\nu+1)x-\beta\sin x]\}\ \ dx = 2\pi[\mathbf{J}'_\nu(\beta)+i\mathbf{E}'_\nu(\beta)]$
$[\operatorname{Re}\beta > 0].$

EH II 36

2. $\int_0^\pi \exp[\pm i(\nu x-\beta\sin x)]\,dx = \pi[\mathbf{J}_\nu(\beta)\pm i\mathbf{E}_\nu(\beta)] \qquad [\operatorname{Re}\beta > 0].$

EH II 35(32)

3. $\int_0^\infty \exp[-\gamma(x-\beta\sin x)]\,dx = \frac{1}{\gamma}+2\sum_{k=1}^\infty \frac{\gamma_k^{J(k\beta)}}{\gamma^2+k^2} \qquad [\operatorname{Re}\gamma > 0].$

4.[6] $$\int_{-\pi}^{\pi} \frac{\exp\left[\frac{a+b\sin x+c\cos x}{1+p\sin x+q\cos x}\right]}{1+p\sin x+q\cos x}\,dx = \frac{2\pi}{\sqrt{1-p^2-q^2}} e^{-\alpha} I_0(\beta),$$

$$\text{with } \alpha = \frac{bp+cq-a}{1-p^2-q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2-b^2-c^2}{1-p^2-q^2}}; \qquad [p^2+q^2<1].$$

3.339[6]

$$\int_0^{\pi} \exp(z\cos x)\,dx = \pi I_0(z).$$

BI ((277))(2)a

3.341

$$\int_0^{\frac{\pi}{2}} \exp(-p\operatorname{tg} x)\,dx = \operatorname{ci}(p)\sin p - \operatorname{si}(p)\cos(p) \qquad [p>0].$$

BI ((271))(2)a

3.342

$$\int_0^1 \exp(-px\ln x)\,dx = \int_0^1 x^{-px}\,dx = \sum_{k=1}^{\infty} \frac{p^k-1}{k^k}.$$

BI ((29))(1)

3.35 Combinations of exponentials and rational functions

3.351

1. $$\int_0^u x^n e^{-\mu x}\,dx = \frac{n!}{\mu^{n+1}} - e^{-u\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} \qquad [u>0].$$

ET I 134(5)

2. $$\int_u^{\infty} x^n e^{-\mu x}\,dx = e^{-u\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} \qquad [u>0, \quad \operatorname{Re}\mu>0].$$

ET I 33(4)

3. $$\int_0^{\infty} x^n e^{-\mu x}\,dx = n!\mu^{-n-1} \qquad [\operatorname{Re}\mu>0].$$

4. $$\int_u^\infty \frac{e^{-px}\,dx}{x^{n+1}} = (-1)^{n+1}\frac{p^n\,\mathrm{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n}\sum_{k=0}^{n-1}\frac{(-1)^k p^k u^k}{n(n-1)\dots(n-k)} \qquad [p>0].$$

NT 21(3)

5. $$\int_1^\infty \frac{e^{-\mu x}\,dx}{x} = -\,\mathrm{Ei}\,(-\mu) \qquad [\mathrm{Re}\,\mu>0].$$

BI ((104))(10)

6. $$\int_{-\infty}^u \frac{e^x}{x}dx = \mathrm{li}(e^u) = \mathrm{Ei}\,(u) \qquad [u<0].$$

7.* $$\int_0^u xe^{-\mu x}\,dx = \frac{1}{\mu^2} - \frac{1}{\mu^2}e^{-\mu u}(1+\mu u) \qquad [u>0]$$

8.* $$\int_0^u x^2e^{-\mu x}\,dx = \frac{2}{\mu^3} - \frac{1}{\mu^3}e^{-\mu u}(2+2\mu u-\mu^2u^2) \qquad [u>0]$$

9.* $$\int_0^u x^3e^{-\mu x}\,dx = \frac{6}{\mu^4} - \frac{1}{\mu^6}e^{-\mu u}(6+6\mu u+3\mu^2u^2+\mu^3u^3) \qquad [u>0]$$

3.352

1. $$\int_0^u \frac{e^{-\mu x}\,dx}{x+\beta} = e^{\mu\beta}[\mathrm{Ei}\,(-\mu u-\mu\beta) - \mathrm{Ei}\,(-\mu\beta)] \qquad [|\arg\beta|<\pi].$$

ET II 217(12)

2. $$\int_u^\infty \frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu}\ \mathrm{Ei}\,(-\mu u - \mu\beta) \qquad [u\ge 0, \quad |\arg(u+\beta)|<\pi, \quad \mathrm{Re}\,\mu>0].$$

ET I 134(6), JA

3. $$\int_u^v \frac{e^{-\mu x}\,dx}{x+\alpha} = e^{\alpha\mu}\left\{\mathrm{Ei}\,[-(\alpha+v)\mu] - \mathrm{Ei}\,[-(\alpha+u)\mu]\right\} \qquad [-\alpha<n, \quad \text{or} \quad -\alpha>v, \quad \mathrm{Re}\,\mu>0].$$

ET I 134 (7)

4. $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu}\ \mathrm{Ei}\,(-\mu\beta) \qquad [|\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0].$

ET II 217(11)

5.[7] $\displaystyle\int_u^\infty \frac{e^{-px}\,dx}{a-x} = e^{-pa}\ \mathrm{Ei}\,(pa-pu) \qquad [p > 0, \quad a < u]$

ET II 251(37)

6.[8] $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{a-x} = e^{-\mu a}\ \mathrm{Ei}\,(a\mu) \qquad [a < 0, \quad \mathrm{Re}\,\mu > 0].$

BI ((91))(4)

7. $\displaystyle\int_{-\infty}^\infty \frac{e^{ipx}\,dx}{x-a} = i\pi e^{iap} \qquad [p > 0].$

ET II 251(38)

3.353

1. $\displaystyle\int_u^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = e^{-u\mu}\sum_{k=1}^{n-1}\frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\ \mathrm{Ei}\,[-(u+\beta)\mu]$

$[n \geq 2, \quad |\arg(u+\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0].$

ET I 134(10)

2.[7] $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(k-1)!(-\mu)^{n-k-1}\beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\ \mathrm{Ei}\,(-\beta\mu)$

$[n \geq 2, \quad |\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0].$

ET I 134(9), BI ((92))(2)

3. $\displaystyle\int_0^\infty \frac{e^{-px}\,dx}{(a+x)^2} = pe^{\alpha p}\ \mathrm{Ei}\,(-ap) + \frac{1}{a} \qquad [p > 0, \quad a > 0].$

LI ((281))(28), LI ((281))(29)

BI ((80))(6)

5.[7] $$\int_0^\infty \frac{x^n e^{-\mu x}}{x+\beta}\,dx = (-1)^{n-1}\beta^n e^{\beta\mu}\ \mathrm{Ei}\,(-\beta\mu) + \sum_{k=1}^{n} (k-1)!(-\beta)^{n-k}\mu^{-k}$$
$$[|\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0;\ n \geq 0].$$

BI ((91))(3)A, LET I 135(11)

3.354

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2+x^2} = \frac{1}{\beta}[\mathrm{ci}\,(\beta\mu)\ \sin\beta\mu - \mathrm{si}\,(\beta\mu)\ \cos\beta\mu] \qquad [\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > 0].$$

BI ((91))(7)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2+x^2} = -\,\mathrm{ci}\,(\beta\mu)\ \cos\beta\mu - \mathrm{si}\,(\beta\mu)\ \sin\beta\mu \qquad [\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > 0].$$

BI ((91))(8)

3.[7] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2-x^2} = \frac{1}{2\beta}[e^{-\beta\mu}\ \mathrm{Ei}\,(\beta\mu) - e^{\beta\mu}\,\mathrm{Ei}\,(-\beta\mu)] \qquad [|\arg(\pm\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0]$$

BI ((91))(14)

4. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2-x^2} = \frac{1}{2}[e^{-\beta\mu}\ \mathrm{Ei}\,(\beta\mu) + e^{\beta\mu}\ \mathrm{Ei}\,(-\beta\mu)] \qquad [|\arg(\pm\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0;$$
for $\beta > 0$ one should replace $\mathrm{Ei}\,(\beta\mu)$ in this formula with $\overline{\mathrm{Ei}}\,(\beta\mu)$].

BI ((91))(15)

5.[8] $$\int_{-\infty}^\infty \frac{e^{-ipx}\,dx}{a^2+x^2} = \frac{\pi}{a}e^{-|ap|} \qquad [a \neq 0], \qquad p \quad \text{real}.$$

ET I 118(1)a

3.355

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{(\beta^2+x^2)^2} = \frac{1}{2\beta^3}\{\mathrm{ci}\,(\beta\mu)\ \sin\beta\mu - \mathrm{si}\,(\beta\mu)\ \cos\beta\mu -$$
$$- \beta\mu[\mathrm{ci}\,(\beta\mu)\ \cos\beta\mu + \mathrm{si}\,(\beta\mu)\ \sin\beta\mu]\}.$$

LI ((92))(6)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{(\beta^2+x^2)^2} = \frac{1}{2\beta^2}\left\{-\beta\mu[\operatorname{ci}(\beta\mu)\,\sin\beta\mu - \operatorname{si}(\beta\mu)\,\cos\beta\mu]\right\} \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>0].$$

BI ((92))(7)

360

3.[7] $$\int_0^\infty \frac{e^{-px}\,dx}{(a^2-x^2)^2} = \frac{1}{4a^3}[(ap-1)e^{ap}\operatorname{Ei}(-ap)+(1+ap)e^{-ap}\operatorname{Ei}(ap)] \qquad [\operatorname{Im}(a^2)\neq 0, \quad p>0].$$

BI ((92))(8)

4.[7] $$\int_0^\infty \frac{xe^{-px}\,dx}{(a^2-x^2)^2} = \frac{1}{4a^2}\left\{-2+ap[e^{-ap}\operatorname{Ei}(ap)-e^{ap}\operatorname{Ei}(-ap)]\right\} \qquad [\operatorname{Im}(a^2)\neq 0, \quad p>0].$$

LI ((92))(9)

3.356

1. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2+x^2}\,dx = (-1)^{n-1}a^{2n}[\operatorname{ci}(ap)\,\cos ap + \operatorname{si}(ap)\,\sin ap] + \\ + \frac{1}{p^{2n}}\sum_{k=1}^{n}(2n-2k+1)!(-a^2p^2)^{k-1} \qquad [p>0].$$

BI ((91))(12)

2. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2+x^2}\,dx = (-1)^n a^{2n-1}[\operatorname{ci}(ap)\,\sin ap - \operatorname{si}(ap)\,\cos ap] + \\ + \frac{1}{p^{2n-1}}\sum_{k=1}^{n}(2n-2k)!(-a^2p^2)^{k-1} \qquad [p>0].$$

BI ((91))(11)

3. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2-x^2}\,dx = \frac{1}{2}a^{2n}[e^{ap}\operatorname{Ei}(-ap)+e^{-ap}\operatorname{Ei}(ap)] - \frac{1}{p^{2n}}\sum_{k=1}^{n}(2n-2k+1)!(a^2p^2)^{k-1} \\ [p>0].$$

BI ((91))(17)

4. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2-x^2}\,dx = \frac{1}{2}a^{2n-1}[e^{-ap}\operatorname{Ei}(ap)-e^{ap}\operatorname{Ei}(-ap)] - \frac{1}{p^{2n-1}}\sum_{k=1}^{n}(2n-2k)!(a^2p^2)^{k-1} \\ [p>0].$$

3.357

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a^2}\{\operatorname{ci}(a\mu)(\sin a\mu+\cos a\mu) + \operatorname{si}(a\mu)(\sin a\mu-\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

BI ((92))(18)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a}\{\operatorname{ci}(a\mu)(\sin a\mu-\cos a\mu) - \operatorname{si}(a\mu)(\sin a\mu+\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

BI ((92))(19)

3. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2}\{-\operatorname{ci}(a\mu)(\sin a\mu+\cos a\mu) - \operatorname{si}(a\mu)(\sin a\mu-\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

BI ((92))(20)

361

4. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a^2}\{\operatorname{ci}(a\mu)(\sin a\mu-\cos a\mu) - \operatorname{si}(a\mu)(\sin a\mu+\cos a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

BI ((92))(21)

5. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a}\{-\operatorname{ci}(a\mu)(\sin a\mu+\cos a\mu) - \operatorname{si}(a\mu)(\sin a\mu-\cos a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

BI ((92))(22)

6. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2}\{\operatorname{ci}(a\mu)(\cos a\mu-\sin a\mu) + \operatorname{si}(a\mu)(\cos a\mu+\sin a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\} \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

1. $$\int_0^\infty \frac{e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4a^3}\left\{e^{-ap}\ \mathrm{Ei}\,(ap) - e^{ap}\ \mathrm{Ei}\,(-ap) + 2\ \mathrm{ci}\,(ap)\ \sin ap - 2\ \mathrm{si}\,(ap)\ \cos ap\right\}$$
$$[p > 0, \quad a > 0].$$

BI ((91))(18)

2. $$\int_0^\infty \frac{xe^{-px}dx}{a^4 - x^4} = \frac{1}{4a^2}\left\{e^{ap}\ \mathrm{Ei}\,(-ap) + e^{-ap}\ \mathrm{Ei}\,(ap) - 2\ \mathrm{ci}\,(ap)\ \cos ap - 2\ \mathrm{si}\,(ap)\ \sin ap\right\}$$
$$[p > 0, \quad a > 0].$$

BI ((91))(19)

3. $$\int_0^\infty \frac{x^2 e^{-px}dx}{a^4 - x^4} = \frac{1}{4a}\left\{e^{-ap}\ \mathrm{Ei}\,(ap) - e^{ap}\ \mathrm{Ei}\,(-ap) - 2\ \mathrm{ci}\,(ap)\ \sin ap + 2\ \mathrm{si}\,(ap)\ \cos ap\right\}$$
$$[p > 0, \quad a > 0].$$

BI ((91))(20)

4. $$\int_0^\infty \frac{x^3 e^{-px}dx}{a^4 - x^4} = \frac{1}{4}\left\{e^{ap}\ \mathrm{Ei}\,(-ap) + e^{-ap}\ \mathrm{Ei}\,(ap) + 2\ \mathrm{ci}\,(ap)\ \cos ap + 2\ \mathrm{si}\,(ap)\ \sin ap\right\}$$
$$[p > 0, \quad a > 0].$$

BI ((91))(21)

5. $$\int_0^\infty \frac{x^{4n} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4}a^{4n-3}[e^{-ap}\ \mathrm{Ei}\,(ap) - e^{ap}\ \mathrm{Ei}\,(-ap) + 2\ \mathrm{ci}\,(ap)\ \sin ap - 2\ \mathrm{si}\,(ap)\ \cos ap] -$$
$$- \frac{1}{p^{4n-3}}\sum_{k=1}^{n}(4n-4k)!(a^4p^4)^{k-1} \qquad [p > 0, \quad a > 0].$$

BI ((91))(22)

6. $$\int_0^\infty \frac{x^{4n+1} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4}a^{4n-2}[e^{ap}\ \mathrm{Ei}\,(-ap) + e^{-ap}\ \mathrm{Ei}\,(ap) - 2\ \mathrm{ci}\,(ap)\ \cos ap - 2\ \mathrm{si}\,(ap)\ \sin ap] -$$
$$- \frac{1}{p^{4n-2}}\sum_{k=1}^{n}(4n-4k+1)!(a^4p^4)^{k-1} \qquad [p > 0, \quad a > 0].$$

BI ((91))(23)

7. $$\int_0^\infty \frac{x^{4n+2} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4}a^{4n-1}[e^{-ap}\ \mathrm{Ei}\,(ap) - e^{ap}\ \mathrm{Ei}\,(-ap) - 2\ \mathrm{ci}\,(ap)\ \sin ap + 2\ \mathrm{si}\,(ap)\ \cos ap] -$$
$$- \frac{1}{p^{4n-1}}\sum_{k=1}^{n}(4n-4k+2)!(a^4p^4)^{k-1} \qquad [p > 0, \quad a > 0].$$

BI ((91))(24)

8. $$\int_0^\infty \frac{x^{4n+3}e^{-px}}{a^4-x^4}\,dx = \frac{1}{4}a^{4n}[e^{ap}\ \mathrm{Ei}\,(-ap)+e^{-ap}\ \mathrm{Ei}\,(ap)+2\ \mathrm{ci}\,(ap)\ \cos ap + 2\ \mathrm{si}\,(ap)\ \sin ap] - - \frac{1}{p^{4n}}\sum_{k=1}^{n}(4n-4k+3)!(a^4p^4)^{k-1} \qquad [p>0, \quad a>0].$$

BI ((91))(25)

3.359

$$\int_{-\infty}^\infty \frac{(i-x)^n}{(i+x)^n}\ \frac{e^{-ipx}}{i+x^2}\,dx = (-1)^{n-1}2\pi pe^{-p}L_{n-1}(2p) \qquad \text{for} \quad p>0;$$
$$= 0 \qquad \text{for} \quad p<0.$$

ET I 118(2)

3.36- 3.37 Combinations of exponentials and algebraic functions

3.361[8]

1. $$\int_0^u \frac{e^{-qx}}{\sqrt{x}}dx = \sqrt{\frac{\pi}{q}}\Phi(\sqrt{qu}).$$

2. $$\int_0^\infty \frac{e^{-qx}}{\sqrt{x}}dx = \sqrt{\frac{\pi}{q}} \qquad [q>0].$$

BI((98))(10)

3. $$\int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}}dx = e^q\sqrt{\frac{\pi}{q}} \qquad [q>0].$$

BI ((104))(16)

3.362

1. $$\int_1^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x-1}} = \sqrt{\frac{\pi}{\mu}}e^{-\mu} \qquad [\mathrm{Re}\,\mu>0].$$

BI ((104))(11)a

2. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{\mu}}e^{\beta\mu}[1-\Phi(\sqrt{\beta\mu})] \qquad [\mathrm{Re}\,\mu>0, \quad |\arg\beta|<\pi].$$

3.363

1. $$\int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x}\,dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u}[1-\Phi(\sqrt{u\mu})] \qquad [u>0, \quad \operatorname{Re}\mu>0].$$

ET I 136(23)

2. $$\int_u^\infty \frac{e^{-\mu x}\,dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}[1-\Phi(\sqrt{u\mu})] \qquad [u>0, \quad \operatorname{Re}\mu \geq 0].$$

ET I 136(26)

3.364

1. $$\int_0^2 \frac{e^{-px}\,dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p) \qquad [p>0].$$

GW ((312))(7a)

2. $$\int_{-1}^1 \frac{e^{2x}\,dx}{\sqrt{1-x^2}} = \pi I_0(2).$$

BI ((277))(2)a

3. $$\int_0^\infty \frac{e^{-px}\,dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}} K_0\left(\frac{ap}{2}\right) \qquad [a>0, \quad p>0].$$

GW ((312))(8a)

3.365

1. $$\int_0^u \frac{xe^{-\mu x}\,dx}{\sqrt{u^2-x^2}} = \frac{\pi u}{2}[\mathbf{L}_1(\mu u) - I_1(\mu u)] + u \qquad [u>0, \quad \operatorname{Re}\mu>0].$$

ET I 136(28)

2. $$\int_u^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2-u^2}} = uK_1(u\mu) \qquad [u>0, \quad \operatorname{Re}\mu>0].$$

ET I 136(29)

3.366

1. $$\int_0^{2u} \frac{(u-x)e^{-\mu x}\,dx}{\sqrt{2ux-x^2}} = \pi u e^{-u\mu} I_1(u\mu) \qquad [\operatorname{Re}\mu > 0].$$

ET I 136(31)

2. $$\int_0^{\infty} \frac{(x+\beta)e^{-\mu x}\,dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu) \qquad [\operatorname{Re}\mu > 0, \quad |\arg\beta| < \pi].$$

ET I 136(30)

3. $$\int_0^{\infty} \frac{xe^{-\mu x}\,dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2}[\mathbf{H}_1(\beta\mu) - N_1(\beta\mu)] - \beta \qquad \left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET I 136(27)

3.367

$$\int_0^{\infty} \frac{e^{-\mu x}\,dx}{(1+\cos t+x)\sqrt{x^2+2x}} = \frac{\exp\left(2\mu\cos^2\frac{t}{2}\right)}{\sin t}\left(t - \sin t\int_0^u K_0(v)e^{-v\cos t}\,dv\right) \qquad [\operatorname{Re}\mu > 0].$$

ET I 136(33)

3.368

$$\int_0^{\infty} \frac{e^{-\mu x}\,dx}{x+\sqrt{x^2+\beta^2}} = \frac{\pi}{2\beta\mu}[\mathbf{H}_1(\beta\mu) - N_1(\beta\mu)] - \frac{1}{\beta^2\mu^2} \qquad \left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET I 136(32)

3.369

$$\int_0^{\infty} \frac{e^{-\mu x}\,dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu}e^{\alpha\mu}(1-\Phi(\sqrt{a\mu})) \qquad [|\arg a| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 135(20)

3.371

$$\begin{aligned}\int_0^{\infty} x^{n-\frac{1}{2}} e^{-\mu x}\,dx &= \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2} m u^{-n-\frac{1}{2}} \\ &= \sqrt{\pi}2^{-n}\mu^{-n-1/2}(2n-1)!! \qquad [n \geq 0] \qquad [\operatorname{Re}\mu > 0].\end{aligned}$$

$$\int_0^\infty x^{n-\frac{1}{2}}(2+x)^{n-\frac{1}{2}}e^{-px}\,dx = \frac{(2n-1)!!}{p^n}e^p K_n(p) \qquad [p>0, \quad n=0,\,1,\,2,\dots].$$

GW ((312))(8)

3.373

$$\int_0^\infty \left[\left(x+\sqrt{x^2+\beta^2}\right)^n + (x-\sqrt{x^2+\beta^2})^n\right] e^{-\mu x}\,dx = 2\beta^{n+1}O_n(\beta\mu) \qquad [\operatorname{Re}\mu>0].$$

WA 05(1)

3.374

1. $$\int_0^\infty \frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \frac{1}{2}[S_n(\mu) - \pi\mathbf{E}_n(\mu) - \pi N_n(\mu)] \qquad [\operatorname{Re}\mu>0].$$

ET I 37(35)

2. $$\int_0^\infty \frac{(x-\sqrt{1+x^2})^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = -\frac{1}{2}[S_n(\mu) + \pi\mathbf{E}_n(\mu) + \pi N_n(\mu)] \qquad [\operatorname{Re}\mu>0].$$

ET I 137(36)

3.38- 3.39 Combinations of exponentials and arbitrary powers

3.381

1. $$\int_0^u x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\gamma(\nu,\,\mu u) \qquad [\operatorname{Re}\nu>0]$$

EH I 266(22), EH II 133(1)

2. $$\int_0^u x^{p-1}e^{-x}\,dx = \sum_{k=0}^\infty (-1)^k \frac{u^{p+k}}{k!(p+k)}; = e^{-u}\sum_{k=0}^\infty \frac{u^{p+k}}{p(p+1)\dots(p+k)}.$$

AD 6.705

3.[8] $$\int_u^\infty x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\Gamma(\nu,\,\mu u) \qquad [u>0, \quad \operatorname{Re}\mu>0].$$

4. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\mu^\nu}\Gamma(\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

FI II 779

5. $$\int_0^\infty x^{\nu-1}e^{-(p+iq)x}\,dx = \Gamma(\nu)(p^2+q^2)^{-\frac{\nu}{2}}\exp\left(-i\nu\operatorname{arctg}\frac{q}{p}\right)$$
$$[p>0, \quad \operatorname{Re}\nu > 0 \quad \text{or} \quad p=0, \quad 0<\operatorname{Re}\nu<1].$$

EH I 12(32)

6. $$\int_u^\infty \frac{e^{-x}}{x^\nu}\,dx = u^{-\frac{\nu}{2}}e^{-\frac{u}{2}}W_{-\frac{\nu}{2},\,\frac{(1-\nu)}{2}}(u) \qquad [u>0].$$

WH

3.382

1.[6] $$\int_0^u (u-x)^\nu e^{-\mu x}\,dx = (-\mu)^{-\nu-1}e^{-u\mu}\gamma(\nu+1,\,-u\mu) \qquad [\operatorname{Re}\nu > -1, \quad u>0].$$

ET I 137(6)

2. $$\int_u^\infty (x-u)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1}e^{-u\mu}\Gamma(\nu+1) \qquad [u>0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0].$$

ET I 137(5), ET II 202(11)

3. $$\int_0^\infty (1+x)^{-\nu}e^{-\mu x}\,dx = \mu^{\frac{\nu}{2}-1}e^{\frac{\mu}{2}}W_{-\frac{\nu}{2},\,\frac{(1-\nu)}{2}}(\mu) \qquad [\operatorname{Re}\mu > 0].$$

WH

4. $$\int_0^\infty (x+\beta)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1}e^{\beta\mu}\Gamma(\nu+1,\,\beta\mu) \qquad [|\arg\beta|<\pi, \quad \operatorname{Re}\mu > 0].$$

ET I 137(4), ET II 233(10)

5. $$\int_0^u (a+x)^{\mu-1}e^{-x}\,dx = e^a[\gamma(\mu,\,a+u)-\gamma(\mu,\,a)] \qquad [\operatorname{Re}\mu > 0].$$

EH II 139

6. $$\int_{-\infty}^{\infty} (\beta + ix)^{-\nu} e^{-ipx}\,dx = 0 \qquad \text{for} \quad p > 0;$$
$$= \frac{2\pi(-p)^{\nu-1}e^{\beta p}}{\Gamma(\nu)} \qquad \text{for} \quad p < 0 \qquad [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0].$$

ET I 118(4)

7. $$\int_{-\infty}^{\infty} (\beta - ix)^{-\nu} e^{-ipx}\,dx = \frac{2\pi p^{\nu-1}e^{-\beta p}}{\Gamma(\nu)} \qquad \text{for} \quad p > 0;$$
$$= 0 \qquad \text{for} \quad p < 0 \qquad [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0].$$

ET I 118(3)

3.383

1. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x}\,dx = \mathrm{B}(\mu,\,\nu)u^{u+\nu-1}\,{}_1F_1(\nu;\,\mu+\nu;\,\beta u) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 187(14)

2. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}e^{\beta x}\,dx = \sqrt{\pi}\left(\frac{u}{\beta}\right)^{u-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right)\Gamma(\mu)I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right) \qquad [\operatorname{Re}\mu > 0].$$

ET II 187(13)

3. $$\int_u^{\infty} x^{\mu-1}(x-u)^{\mu-1}e^{-\beta x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right)K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta u > 0].$$

ET II 202(12)

4. $$\int_u^{\infty} x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x}\,dx = \beta^{-\frac{\mu+\nu}{2}}u^{\frac{\mu+\nu-2}{2}}\Gamma(\mu)\,\exp\left(-\frac{\beta u}{2}\right)W_{\frac{\nu-\mu}{2},\,\frac{1-\mu-\nu}{2}}(\beta\mu)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta u > 0].$$

ET II 202(13)

5.[7] $$\int_0^{\infty} e^{-px}x^{q-1}(1+ax)^{-\nu}\,dx = a^{-q}\Gamma(q)\Psi(q,\,q+1-\nu,\,p/a)$$
$$= p^{-q}\Gamma(q)\left[\sum_{k=0}^{n}(q)_k(\nu)_k(a/p)^k/k! + O(a/p)^{N+1}\right]$$

$\operatorname{Re} q > 0, \operatorname{Re} p > 0, \operatorname{Re} a > 0, \nu$ complex, $N = 0,\ 1, \ldots$. For $\nu = 0$ the integral equals $p^{-q}\Gamma(q)$.

6. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu+\frac{1}{2}} e^{-\mu x}\,dx = 2^{\nu-\frac{1}{2}}\Gamma(\nu)\mu^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{1-2\nu}(\sqrt{2\beta\mu})$$
$$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu \geq 0, \quad \mu \neq 0].$$

ET I 39(20), EH II 119(2)a

7. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu-\frac{1}{2}} e^{-\mu x}\,dx = 2^{\nu}\Gamma(\nu)\beta^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{-2\nu}(\sqrt{2\beta\mu})$$
$$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu \geq 0].$$

ET I 139(21), EH II 119(1)a

8. $$\int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1} e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}}\Gamma(\nu) K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right)$$
$$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 233(11), EH II 19(16)A, EH II 82(22)a

9. $$\int_u^\infty \frac{(x-u)^\nu e^{-\mu x}}{x}\,dx = u^\nu\Gamma(\nu+1)\Gamma(-\nu,\ u\mu) \qquad [u > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0].$$

ET I 138(8)

10. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{x+\beta}\,dx = \beta^{\nu-1}e^{\beta\mu}\Gamma(\nu)\Gamma(1-\nu,\ \beta\mu) \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

EH II 137(3)

3.384

1. $$\int_{-1}^1 (1-x)^{\nu-1}(1+x)^{\mu-1}e^{-ipx}\,dx = 2^{\mu+\nu-1}\mathrm{B}(\mu,\ \nu)e^{ip}\,{}_1F_1(\mu;\ \nu+\mu;\ -2ip)$$
$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0].$$

ET I 119(13)

2. $$\int_u^v (x-u)^{2\mu-1}(v-x)^{2\nu-1}e^{-px}\,dx =$$
$$= \mathrm{B}(2\mu,\ 2\nu)(v-u)^{\mu+\nu-1}p^{-\mu-\nu}\ \exp\left(-p\frac{u+v}{2}\right) M_{\mu-\nu,\ \mu+\nu-\frac{1}{2}}(vp-up)$$
$$[v > u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

3. $$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\varrho-1}e^{-\mu x}\,dx = \\ = \frac{(u+\beta)^{\nu+\varrho-1}}{\mu^{\nu+\varrho}}\exp\left[\frac{(\beta-u)\mu}{2}\right]\Gamma(2\varrho)W_{\nu-\varrho,\,\nu+\varrho-\frac{1}{2}}(u\mu+\beta\mu)$$
$$[u>0, \quad |\arg(\beta+u)|<\pi, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\varrho>0].$$

ET I 139(22)

4. $$\int_u^\infty (x+\beta)^{\nu}(x-u)^{-\nu}e^{-\mu x}\,dx = \frac{1}{\mu}\nu\pi\operatorname{cosec}(\nu\pi)e^{-\frac{(\beta+u)\mu}{2}}k_{2\nu}\left[\frac{(\beta+u)\mu}{2}\right]$$
$$[\nu\neq 0, \quad u>0, \quad |\arg(u+\beta)|<\pi, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu<1].$$

ET I 139(17)

5. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu+\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{\mu}}2^{\nu-\frac{1}{2}}\Gamma(\nu)D_{1-2\nu}(2\sqrt{u\mu})$$
$$[u>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET I 139(18)

6. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu-\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{u}}2^{\nu-\frac{1}{2}}\Gamma(\nu)D_{-2\nu}(2\sqrt{u\mu})$$
$$[u>0, \quad \operatorname{Re}\mu\geq 0, \quad \operatorname{Re}\nu>0].$$

ET I 139(19)

7.[6] $$\int_{-\infty}^\infty (\beta-ix)^{-\mu}(\gamma-ix)^{-\nu}e^{-ipx}\,dx = \frac{2\pi e^{-\beta p}p^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\,{}_1F_1(\nu;\,\mu+\nu;\,(\beta-\gamma)p) \qquad \text{for} \quad p>0;$$
$$= 0 \qquad \text{for} \quad p<0$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad \operatorname{Re}(\mu+\nu)>1].$$

ET I 119(10)

8.[6] $$\int_{-\infty}^\infty (\beta+ix)^{-\mu}(\gamma+ix)^{-\nu}e^{-ipx}\,dx = 0 \qquad \text{for} \quad p>0;$$
$$= \frac{2\pi e^{\gamma p}(-p)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\,{}_1F_1[\mu;\,\mu+\nu;\,(\beta-\gamma)p] \qquad \text{for} \quad p<0$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad \operatorname{Re}(\mu+\nu)>1].$$

ET I 19(11)

ET I 19(12)

3.385

$$\int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\varrho}e^{-\mu x}\,dx = \mathrm{B}(\nu,\,\lambda)\Phi_1(\nu,\,\varrho,\,\lambda+\nu,\,\beta,\,-\mu)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > 0, \quad |\arg(1-\beta)| < \pi].$$

ET I 39(24)

3.386

1. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0}\prod_{k=1}^{n}(\beta_k+ix)^{\nu_k}e^{-ipx}\,dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p}\beta_0^{\nu_0}\prod_{k=1}^{n}(\beta_0+\beta_k)^{\nu_k}$$
$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n}\operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign}x, \quad p > 0\right].$$

ET I 118(8)

2. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0}\prod_{k=1}^{n}(\beta_k+ix)^{\nu_k}e^{-ipx}\,dx}{\beta_0 + ix} = 0$$
$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n}\operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign}x, \quad p > 0\right].$$

ET I 119(9)

3.387

1.[6] $$\int_{-1}^{1}(1-x^2)^{\nu-1}e^{-\mu x}\,dx = \sqrt{\pi}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)I_{\nu-\frac{1}{2}}(\mu) \qquad \left[\operatorname{Re}\nu > 0, \quad |\arg\mu| < \frac{\pi}{2}\right].$$

WA 172(2)a

2.[6] $$\int_{-1}^{1}(1-x^2)^{\nu-1}e^{i\mu x}\,dx = \sqrt{\pi}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)J_{\nu-\frac{1}{2}}(\mu) \qquad [\operatorname{Re}\nu > 0].$$

WA 25(3), WA 48(4)a

3. $$\int_{1}^{\infty}(x^2-1)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)K_{\nu-\frac{1}{2}}(\mu) \qquad \left[|\arg\mu| < \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right].$$

4. $$\int_1^\infty (x^2-1)^{\nu-1} e^{i\mu x}\,dx = i\frac{\sqrt{\pi}}{2}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)H^{(1)}_{\frac{1}{2}-\nu}(\mu) \qquad [\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 0];$$
$$= -i\frac{\sqrt{\pi}}{2}\left(-\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)H^{(2)}_{\frac{1}{2}-\nu}(-\mu) \qquad [\operatorname{Im}\mu < 0, \quad \operatorname{Re}\nu > 0].$$

EH II 83(29)a
EH II 83(28)a

5. $$\int_0^u (u^2-x^2)^{\nu-1} e^{\mu x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)[I_{\nu-\frac{1}{2}}(u\mu)+\mathbf{L}_{\nu-\frac{1}{2}}(u\mu)]. \qquad [u > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 188(20)a

6. $$\int_u^\infty (x^2-u^2)^{\nu-1} e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)K_{\nu-\frac{1}{2}}(u\mu) \qquad [u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 203(17)a

7. $$\int_0^\infty (x^2+u^2)^{\nu-1} e^{-\mu x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)[\mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - N_{\nu-\frac{1}{2}}(u,\mu)]$$
$$[|\arg u| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 138(10)

3.388

1. $$\int_0^{2u} (2ux-x^2)^{\nu-1} e^{-\mu x}\,dx = \sqrt{\pi}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} e^{-u\mu}\Gamma(\nu)I_{\nu-\frac{1}{2}}(u\mu) \qquad [u > 0, \quad \operatorname{Re}\nu > 0].$$

ET I 138(14)

2. $$\int_0^\infty (2\beta x+x^2)^{\nu-1} e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\beta\mu}\Gamma(\nu)K_{\nu-\frac{1}{2}}(\beta\mu)$$
$$[|\arg\beta| < \pi; \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0].$$

ET I 138(13)

3. $$\int_0^\infty (x^2+ix)^{\nu-1} e^{-\mu x}\,dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\Gamma(\nu)H^{(2)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

4. $$\int_0^\infty (x^2-ix)^{\nu-1} e^{-\mu x}\,dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\Gamma(\nu) H^{(1)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET I 138(16)

3.389

1. $$\int_0^u x^{2\nu-1}(u^2-x^2)^{\varrho-1} e^{\mu x}\,dx = \frac{1}{2}\mathrm{B}(\nu,\,\varrho)u^{2\nu+2\varrho-2}\,{}_1F_2\left(\nu;\,\frac{1}{2},\,\nu+\varrho;\,\frac{\mu^2u^2}{4}\right) + \\ + \frac{\mu}{2}\mathrm{B}\left(\nu+\frac{1}{2},\,\varrho\right)u^{2\nu+2\varrho-1}\,{}_1F_2\left(\nu+\frac{1}{2};\,\frac{3}{2},\,\nu+\varrho+\frac{1}{2};\,\frac{\mu^2u^2}{4}\right) \\ [\operatorname{Re}\varrho > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 188(21)

2.[7] $$\int_0^\infty x^{2\nu-1}(u^2+x^2)^{\varrho-1} e^{-\mu x}\,dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi}\Gamma(1-\varrho)} G^{31}_{13}\left(\frac{\mu^2u^2}{4}\,\middle|\,\begin{matrix}1-\nu\\ 1-\varrho-\nu,\,0,\,\frac{1}{2}\end{matrix}\right) \\ \left[|\arg u| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0\right].$$

ET II 234(15)a

3.[7] $$\int_0^u x(u^2-x^2)^{\nu-1} e^{\mu x}\,dx = \frac{u^{2\nu}}{2\nu} + \frac{\sqrt{\pi}}{2}\left(\frac{\mu}{2}\right)^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}}\Gamma(\nu)[I_{\nu+\frac{1}{2}}(\mu u) + \mathbf{L}_{\nu+\frac{1}{2}}(\mu u)] \qquad [\operatorname{Re}\nu > 0].$$

ET II 188(19)a

4. $$\int_u^\infty x(x^2-u^2)^{\nu-1} e^{-\mu x}\,dx = 2^{\nu-\frac{1}{2}}(\sqrt{\pi})^{-1}\mu^{\frac{1}{2}-\nu}u^{\nu+\frac{1}{2}}\Gamma(\nu)K_{\nu+\frac{1}{2}}(u\mu) \qquad [\operatorname{Re}(u\mu) > 0].$$

ET II 203(16)a

5. $$\int_{-\infty}^\infty \frac{(ix)^{-\nu}e^{-ipx}\,dx}{\beta^2+x^2} = \pi\beta^{-\nu-1}e^{-|p|\beta} \qquad \left[|\nu| < 1, \quad \operatorname{Re}\beta > 0, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x\right].$$

ET I 118(5)

6. $$\int_0^\infty \frac{x^\nu e^{-\mu x}}{\beta^2+x^2}\,dx = \frac{1}{2}\Gamma(\nu)\beta^{\nu-1}\left[\exp\left(i\mu\beta + i\frac{(\nu-1)\pi}{2}\right)\times \right. \\ \left. \times\,\Gamma(1-\nu,\,i\beta\mu) + \exp\left(-i\beta\mu - i\frac{(\nu-1)\pi}{2}\right)\Gamma\left(1-\nu,\,-i\beta\mu\right)\right] \\ [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1].$$

7. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1+x^2} = \pi \operatorname{cosec}(\nu\pi)V_\nu(2\mu,\,0) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET I 138(9)

8. $$\int_{-\infty}^\infty \frac{(\beta+ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{-p\gamma} \qquad [\operatorname{Re}\nu > -1, \quad p > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

ET I 118(6)

9.[6] $$\int_{-\infty}^\infty \frac{(\beta-ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{\gamma p} \qquad [p < 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > -1].$$

ET I 118(7)

3.391

$$\int_0^\infty \left[\left(\sqrt{x+2\beta}+\sqrt{x}\right)^{2\nu} - \left(\sqrt{x+2\beta}-\sqrt{x}\right)^{2\nu}\right] e^{-\mu x}\,dx = 2^{\nu+1}\frac{\nu}{\mu}\beta^\nu e^{\beta\mu}K_\nu(\beta\mu)$$
$$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 140(30)

3.392

1. $$\int_0^\infty (x+\sqrt{1+x^2})^\nu e^{-\mu x}\,dx = \frac{1}{\mu}S_{1,\,\nu}(\mu) + \frac{\nu}{\mu}S_{0,\,\nu}(\mu)[\operatorname{Re}\mu > 0].$$

ET I 140(25)

2. $$\int_0^\infty (\sqrt{1+x^2}-x)^\nu e^{-\mu x}\,dx = \frac{1}{\mu}S_{1,\,\nu}(\mu) - \frac{\nu}{\mu}S_{0,\,\nu}(\mu) \qquad [\operatorname{Re}\mu > 0].$$

ET I 140(26)

3. $$\int_0^\infty \frac{(x+\sqrt{1+x^2})^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \pi \operatorname{cosec}\nu\pi[\mathbf{J}_{-\nu}(\mu) - J_{-\nu}(\mu)] \qquad [\operatorname{Re}\mu > 0].$$

ET I 140(27), EH II 35(33)

4. $$\int_0^\infty \frac{(\sqrt{1+x^2}-x)^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = S_{0,\,\nu}(\mu) - \nu S_{-1,\,\nu}(\mu) \qquad [\operatorname{Re}\mu > 0].$$

3.393

$$\int_0^\infty \frac{(x+\sqrt{x^2+4\beta^2})^{2\nu}}{\sqrt{x^3+4\beta^2 x}} e^{-\mu x}\,dx = \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2}\beta^{2\nu}}[J_{\nu+1/4}(\beta\mu)N_{\nu-1/4}(\beta\mu)-J_{\nu-1/4}(\beta\mu)N_{\nu+1/4}(\beta\mu)]$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>0].$$

ET I 140(33)

3.394

$$\int_0^\infty \frac{(1+\sqrt{1+x^2})^{\nu+1/2}}{x^{\nu+1}\sqrt{1+x^2}} e^{-\mu x}\,dx = \sqrt{2}\Gamma(-\nu)D_\nu(\sqrt{2i\mu})D_\nu(\sqrt{-2i\mu}) \qquad [\operatorname{Re}\mu\geq 0,\quad \operatorname{Re}\nu<0].$$

ET I 140(32)

3.395

1. $$\int_1^\infty \frac{(\sqrt{x^2-1}+x)^\nu+(\sqrt{x^2-1}+x)^{-\nu}}{\sqrt{x^2-1}} e^{-\mu x}\,dx = 2K_\nu(\mu) \qquad [\operatorname{Re}\mu>0].$$

ET I 140(29)

2. $$\int_1^\infty \frac{(x+\sqrt{x^2-1})^{2\nu}+(x-\sqrt{x^2-1})^{2\nu}}{\sqrt{x(x^2-1)}} e^{-\mu x}\,dx = \sqrt{\frac{2\mu}{\pi}} K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right)$$
$$[\operatorname{Re}\mu>0].$$

ET I 140(34)

3. $$\int_0^\infty \frac{(x+\sqrt{x^2+1})^\nu+\cos\nu\pi(x+\sqrt{x^2+1})^{-\nu}}{\sqrt{x^2+1}} e^{-\mu x}\,dx = -\pi[\mathbf{E}_\nu(\mu)+N_\nu(\mu)] \qquad [\operatorname{Re}\mu>0].$$

EH II 35(34)

3.41- 3.44 Combinations of rational functions of powers and exponentials

3.411

1. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}-1} = \frac{1}{\mu^\nu}\Gamma(\nu)\zeta(\nu) \qquad [\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>1].$$

FI II 792a

2. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n} \qquad [n=1,\,2,\ldots].$$

3. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}+1} = \frac{1}{\mu^\nu}(1-2^{1-\nu})\Gamma(\nu)\zeta(\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

FI II 792a, WH

4. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (1-2^{1-2n})\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n} \qquad [n=1,\,2,\dots].$$

BI((83))(2), EH I 39(25)

5. $$\int_0^{\ln 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12}.$$

BI ((104))(5)

6.[8] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-\beta e^{-x}}\,dx = \Gamma(\nu)\sum_{n=0}^\infty (\mu+n)^{-\nu}\beta^n = \Gamma(\nu)\Phi(\beta,\,\nu,\,\mu)$$
$[\operatorname{Re}\mu>0 \quad \text{and either} \quad |\beta|\le 1, \quad \beta\ne 1, \quad \operatorname{Re}\nu>0; \quad \text{or} \quad \beta=1, \quad \operatorname{Re}\nu>1].$

EH I 27(3)

7. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-e^{-\beta x}}\,dx = \frac{1}{\beta^\nu}\Gamma(\nu)\zeta\left(\nu,\,\frac{\mu}{\beta}\right) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>1].$$

ET I 144(10)

8.[7] $$\int_0^\infty \frac{x^{n-1}e^{-px}}{1+e^x}\,dx = (n-1)!\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(p+k)^n} = (n-1)!\Phi(-1,n,p+1)$$
$[p>-1; \quad n=1,\,2,\dots].$

BI ((83))(9)

9. $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \frac{\pi^2}{6}-1 \qquad (\text{cf. } 4.231\ 3.).$$

4.231
BI ((82))(1)

4.251
BI ((82))(2)

11. $$\int_0^\infty \frac{xe^{-3x}}{e^{-x}+1}\,dx = \frac{\pi^2}{12} - \frac{3}{4} \qquad \text{(cf. 4.251 5.).}$$

4.251
BI ((82))(3)

12. $$\int_0^\infty \frac{xe^{-2nx}}{1+e^x}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2} \qquad \text{(cf. 4.251 6.).}$$

4.251
BI ((82))(5)

13. $$\int_0^\infty \frac{xe^{-(2n-1)x}}{1+e^x}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} \qquad \text{(cf. 4.251 5.).}$$

4.251
BI ((82))(4)

14.[7] $$\int_0^\infty \frac{x^2e^{-nx}}{1-e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{1}{k^3} = 2\left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3}\right) \qquad [n = 1,\, 2, \dots]. \qquad \text{(cf. 4.261 12.).}$$

4.261
BI ((82))(9)

15.[7] $$\int_0^\infty \frac{x^2e^{-nx}}{1+e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1}\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n-1} \frac{(-1)^k}{k^3}\right) \qquad [n = 1,\, 2, \dots].$$

(cf. 4.261 11.).

4.261
LI ((82))(10)

16. $$\int_{-\infty}^{\infty} \frac{x^2 e^{-\mu x}}{1+e^{-x}}\,dx = \pi^3 \csc^3\ \mu\pi(2-\sin^2\ \mu\pi) \qquad [0 < \operatorname{Re}\mu < 1].$$

ET I 120(17)a

17. $$\int_0^{\infty} \frac{x^3 e^{-nx}}{1-e^{-x}}\,dx = \frac{\pi^4}{15} - 6\sum_{k=1}^{n-1}\frac{1}{k^4} \qquad \text{(cf. 4.262 5.).}$$

4.262
BI ((82))(12)

18.[7] $$\int_0^{\infty} \frac{x^3 e^{-nx}}{1+e^{-x}}\,dx = 6\sum_{k=n}^{\infty}\frac{(-1)^{n+k}}{k^4} = (-1)^{n+1}\left(\frac{7}{120}\pi^4 - 6\sum_{k=1}^{n-1}\frac{(-1)^k}{k^4}\right) \qquad \text{(cf. 4.262 4.).}$$

4.262
LI ((82))(13)

19.* $$\int_0^{\infty} e^{-px}(e^{-x}-1)^n \frac{dx}{x} = -\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(p+n-k)$$

LI ((89))(10)

20.* $$\int_0^{\infty} e^{-px}(e^{-x}-1)^n \frac{dx}{x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(p+n-k)\ln(p+n-k).$$

LI ((89))(15)

21. $$\int_0^{\infty} x^{n-1}\frac{1-e^{-mx}}{1-e^{x}}\,dx = (n-1)!\sum_{k=1}^{m}\frac{1}{k^n} \qquad \text{(cf. 4.272 11.).}$$

4.272
LI ((83))(8)

22.[7] $\int_0^\infty \frac{x^{p-1}}{e^{rx}-q}\,dx = \frac{1}{qr^p}\Gamma(p)\sum_{k=1}^{\infty}\frac{q^k}{k^p} = \Gamma(p)r^{-p}\Phi(q,p,1) \qquad [p>0, \quad r>0, \quad -1<q<1].$

BI ((83))(5)

23. $\int_{-\infty}^\infty \frac{xe^{\mu x}\,dx}{\beta+e^x} = \pi\beta^{\mu-1}\operatorname{cosec}(\mu\pi)[\ln\beta - \pi\ \operatorname{ctg}(\mu\pi)] \qquad [|\arg\beta|<\pi, \quad 0<\operatorname{Re}\mu<1].$

BI ((101))(5), ET I 120(16)a

24. $\int_{-\infty}^\infty \frac{xe^{\mu x}}{e^{\nu x}-1}\,dx = \left(\frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu}\right)^2 \qquad [\operatorname{Re}\nu>\operatorname{Re}\mu>0]. \qquad (\text{cf. } 4.254\ 2.).$

4.254
LI ((101))(3)

25. $\int_0^\infty x\frac{1+e^{-x}}{e^x-1}\,dx = \frac{\pi^2}{3}-1 \qquad (\text{cf. } 4.231\ 4.).$

4.231
BI ((82))(6)

26. $\int_0^\infty x\frac{1-e^{-x}}{1+e^{-3x}}e^{-x}\,dx = \frac{2\pi^2}{27}.$

LI ((82))(7)a

27. $\int_0^\infty \frac{1-e^{-\mu x}}{1+e^x}\,\frac{dx}{x} = \ln\left[\frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\sqrt{\pi}\right] \qquad [\operatorname{Re}\mu>-1].$

BI ((93))(4)

28. $\int_0^\infty \frac{e^{-\nu x}-e^{-\mu x}}{e^{-x}+1}\,\frac{dx}{x} = \ln\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$

29. $$\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 + e^{rx}} \frac{dx}{x} = \ln\left[\operatorname{tg}\frac{p\pi}{2r} \operatorname{ctg}\frac{q\pi}{2r}\right] \qquad [|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0].$$

(cf. 4.267 18.).

4.267
BI ((103))(3)

30. $$\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^{rx}} \frac{dx}{x} = \ln\left[\sin\frac{p\pi}{r} \operatorname{cosec}\frac{q\pi}{r}\right] \qquad [|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0]$$

(cf. 4.267 19.).

4.267
BI ((103))(4)

31. $$\int_0^{\infty} \frac{e^{-qx} + e^{(q-p)x}}{1 - e^{-px}} x\, dx = \left(\frac{\pi}{p} \operatorname{cosec}\frac{q\pi}{p}\right)^2 \qquad [0 < q < p].$$

BI ((82))(8)

32. $$\int_0^{\infty} \frac{e^{-px} - e^{(p-q)x}}{e^{-qx} + 1} \frac{dx}{x} = \ln\operatorname{ctg}\frac{p\pi}{2q} \qquad [0 < p < q].$$

BI ((93))(7)

3.412

$$\int_0^{\infty} \left\{\frac{a + be^{-px}}{ce^{px} + g + he^{-px}} - \frac{a + be^{-qx}}{ce^{qx} + g + he^{-qx}}\right\} \frac{dx}{x} = \frac{a + b}{c + g + h} \ln\frac{p}{q} \qquad [p > 0, \quad q > 0].$$

BI ((96))(7)

3.413

1. $$\int_0^{\infty} \frac{(1 - 3^{-\beta x})(1 - e^{-\gamma x})e^{\mu x}}{1 - e^{-x}} \frac{dx}{x} = \ln\frac{\Gamma(\mu)\Gamma(\beta + \gamma + \mu)}{\Gamma(\mu + \beta)\Gamma(\mu + \gamma)}$$

$$[\operatorname{Re}\mu + 0, \quad \operatorname{Re}\mu > -\operatorname{Re}\beta, \quad \operatorname{Re}\mu > -\operatorname{Re}\gamma, \quad \operatorname{Re}\mu > -\operatorname{Re}(\beta + \gamma)]$$

(cf. 4.267 25.)

4.267
BI ((93))(13)

BI ((95))(6)

3. $$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-x}} \frac{1+e^{-(2n+1)x}}{x} dx = $$
$$= \ln\left\{\frac{q(q+2)(q+4)\dots(q+2n)(p+1)(p+3)\dots(p+2n-1)}{p(p+2)(p+4)\dots(p+2n)(q+1)(q+3)\dots(q+2n-1)}\right\}$$
$$[\operatorname{Re} p > -2n, \quad \operatorname{Re} q >= 2n] \qquad \text{(cf. 4.267 14.)}.$$

4.267
BI ((93))(11)

3.414

$$\int_0^\infty \frac{(1-e^{-\beta x})(1-e^{-\gamma x})(1-e^{-\delta x})e^{-\mu x}}{1-e^{-x}} \frac{dx}{x} =$$
$$= \ln \frac{\Gamma(\mu)\Gamma(\mu+\beta+\gamma)\Gamma(\mu+\beta+\delta)\Gamma(\mu+\gamma+\delta)}{\Gamma(\mu+\beta)\Gamma(\mu+\gamma)\Gamma(\mu+\delta)\Gamma(\mu+\beta+\gamma+\delta)}$$
$$[2\operatorname{Re}\mu > |\operatorname{Re}\beta| + |\operatorname{Re}\gamma| + |\operatorname{Re}\delta|] \qquad \text{(cf. 4.267 31.)}.$$

4.267
BI ((93))(14)
ET I 145(17)

3.415

1. $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}-1)} = \frac{1}{2}\left[\ln\left(\frac{\beta\mu}{2\pi}\right) - \frac{\pi}{\beta\mu} - \psi\left(\frac{\beta\mu}{2\pi}\right)\right] \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0].$$

BI ((97))(20), EH I 18(27)

2.[7] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2(e^{2\pi x}-1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta}\psi'(\beta);$$
$$= \frac{1}{4\beta^4}\sum_{k=0}^{\infty}\frac{|B_{2k+2}|}{\beta^{2k}} \qquad [\operatorname{Re}\beta > 0].$$

BI((97))(22), EH I 22(12)

3.[8] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}-1)} = \frac{1}{2}\left[\Psi\left(\frac{\beta\mu}{2\pi}+\frac{1}{2}\right) - \ln\left(\frac{\beta\mu}{2\pi}\right)\right] \qquad [\operatorname{Re}\beta > 0, \operatorname{Re}\mu > 0]$$

4.[8] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2(e^{2\pi x}+1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta}\Psi'\left(\beta+\frac{1}{2}\right) \qquad [\operatorname{Re}\beta > 0, \operatorname{Re}\mu > 0]$$

1. $\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i} \frac{dx}{e^{2\pi x}-1} = \frac{1}{2} \frac{2n-1}{2n+1} \qquad [n = 1, 2, \ldots].$

BI ((88))(4)

2. $\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i} \frac{dx}{e^{\pi x}+1} = \frac{1}{2n+1} \qquad [n = 1, 2, \ldots].$

BI ((87))(1)

3.[7] $\int_0^\infty \frac{(1+ix)^{2n-1}-(1-ix)^{2n-1}}{i} \frac{dx}{e^{\pi x}+1} = \frac{1}{2n}[1-2^{2n}\mathrm{B}_{2n}] \qquad [n = 1, 2, \ldots].$

BI ((87))(2)

3.417

1. $\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x+b^2e^{-x}} = \frac{\pi}{2ab} \ln\frac{b}{a} \qquad [ab > 0] \qquad \text{(cf. 4.231 8.).}$

4.231
BI ((101))(1)

2. $\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x-b^2e^{-x}} = \frac{\pi^2}{4ab} \qquad \text{(cf. 4.231 10.).}$

4.231
LI ((101))(2)

3.418

1.[6] $\int_0^\infty \frac{x\,dx}{e^x+e^{-x}-1} = 1.171\,953\,6193\ldots = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right]$

LI ((88))(1)

2.[6] $\int_0^\infty \frac{xe^{-x}\,dx}{e^x+e^{-x}-1} = 0.311\,821\,1319\ldots = \frac{1}{6}\left[\psi'\left(\frac{1}{3}\right) - \frac{5}{6}\pi^2\right]$

3. $\int_0^{\ln 2} \frac{x\,dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2.$

BI ((104))(7)

3.419

1. $\int_{-\infty}^{\infty} \frac{x\,dx}{(\beta + e^x)(1 + e^{-x})} = \frac{(\ln \beta)^2}{2(\beta - 1)} \qquad [|\arg \beta| < \pi] \qquad \text{(cf. 4.232 2.)}.$

4.232
BI ((101))(16)

2. $\int_{-\infty}^{\infty} \frac{x\,dx}{(\beta + e^x)(1 - e^{-x})} = \frac{\pi^2 + (\ln \beta)^2}{2(\beta + 1)} \qquad [|\arg \beta| < \pi] \qquad \text{(cf. 4.232 3.)}.$

4.232
BI ((101))(17)

3. $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2] \ln \beta}{3(\beta + 1)} \qquad [|\arg \beta| < \pi] \qquad \text{(cf. 4.261 4.)}.$

4.261
BI ((102))(6)

4. $\int_{-\infty}^{\infty} \frac{x^3\,dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{4(\beta + 1)} \qquad [|\arg \beta| < \pi] \qquad \text{(cf. 4.262 3.)}.$

4.262
BI ((102))(9)

5. $\int_{-\infty}^{\infty} \frac{x^4\,dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{15(\beta + 1)} [7\pi^2 + 3(\ln \beta)^2] \ln \beta \qquad \text{(cf. 4.263 1.)}.$

4.263

BI ((102))(10)

6. $$\int_{-\infty}^{\infty} \frac{x^5\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{[\pi^2+(\ln\beta)^2]^2}{6(\beta+1)}[3\pi^2+(\ln\beta)^2]^2 \qquad \text{(cf. 4.264 3).}$$

4.264
BI ((102))(11)

7. $$\int_{-\infty}^{\infty} \frac{(x-\ln\beta)x\,dx}{(\beta-e^x)(1-e^{-x})} = \frac{-[4\pi^2+(\ln\beta)^2]\,\ln\beta}{6(\beta-1)} \qquad [|\arg\beta|<\pi] \qquad \text{(cf. 4.257 4.).}$$

4.257
BI ((102))(7)

3.421

1. $$\int_0^{\infty} (e^{-\nu x}-1)^n(e^{-\rho x}-1)^m e^{-\mu x}\frac{dx}{x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{l=0}^{m}(-1)^l\binom{m}{l}\times$$
$$\times\{(m-l)\rho+(n-k)\nu+\mu\}\ \ln[(m-l)\rho+(n-k)\nu+\mu]$$
$$[\operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\rho>0].$$

BI ((89))(17)

2. $$\int_0^{\infty} (1-e^{-\nu x})^n(1-e^{-\rho x})e^{-x}\frac{dx}{x^3} = \frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\rho+k\nu+1)^2\times$$
$$\times\ln(\rho+k\nu+1)+\frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(k\nu+1)^2\ \ln(k\nu+1)$$
$$[n\geq 2, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\rho>0].$$

BI ((89))(31)

3. $$\int_{-\infty}^{\infty} \frac{xe^{-\mu x}\,dx}{(\beta+e^{-x})(\gamma+e^{-x})} = \frac{\pi(\beta^{\mu-1}\ \ln\beta-\gamma^{\mu-1}\ \ln\gamma)}{(\beta-\gamma)\ \sin\mu\pi}+\frac{\pi^2(\beta^{\mu-1}-\gamma^{\mu-1})\ \cos\mu\pi}{(\gamma-\beta)\ \sin^2\ \mu\pi}$$
$$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad \beta\neq\gamma. \quad 0<\operatorname{Re}\mu<2].$$

ET I 120(19)

4.267
BI ((89))(11)

5. $$\int_0^\infty (1-e^{-px})(1-e^{-qx})(1-e^{-rx})e^{-x}\frac{dx}{x} =$$
$$= (p+q+1)\ \ln(p+q+1) +$$
$$+ (p+r+1)\ \ln(p+r+1) + (q+r+1)\ \ln(q+r+1) -$$
$$- (p+1)\ \ln(p+1) - (q+1)\ \ln(q+1) - (r+1)\ \ln(r+1) -$$
$$- (p+q+r)\ \ln(p+q+r) \qquad [p>0, \quad q>0, \quad r>0] \qquad \text{(cf. 4.268 3.).}$$

4.268
BI ((89))(14)

3.422

$$\int_{-\infty}^\infty \frac{x(x-a)e^{\mu x}\,dx}{(\beta-e^x)(1-e^{-x})} = \frac{-\pi^2}{e^a-1}\ \operatorname{cosec}^2\ \mu\pi[(e^{\alpha\mu}+1)\ \ln\mu - 2\pi\ \operatorname{ctg}\mu\pi(e^{\alpha\mu}-1)]$$
$$[a>0, \quad |\arg\beta<\pi, \quad |\operatorname{Re}\mu|<1] \qquad \text{(cf. 4.257 5.).}$$

4.257
BI ((102))(8)a

3.423

1. $$\int_0^\infty \frac{x^{\nu-1}}{(e^x-1)^2}\,dx = \Gamma(\nu)[\zeta(\nu-1)-\zeta(\nu)] \qquad [\operatorname{Re}\nu>2].$$

ET I 313(10)

2.[6] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{(e^x-1)^2}\,dx = \Gamma(\nu)[\zeta(\nu-1,\ \mu+2)-(\mu+1)\zeta(\nu,\ \mu+2)] \qquad [\operatorname{Re}\mu>-2, \quad \operatorname{Re}\nu>2].$$

ET I 313(11)

3.[7] $$\int_0^\infty \frac{x^q e^{-px}\,dx}{(1-ae^{-px})^2} = \Gamma(p+1)p^{-q-1}\Phi(a,q,1) \qquad [-1\le a<1, q>-1, p>0]$$

BI ((85))(13)

4.[7] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{(1-\beta e^{-x})^2}\,dx = \Gamma(\nu)[\Phi(\beta,\,\nu-1,\,\mu)-(\mu-1)\Phi(\beta,\,\nu,\,\mu)]$$
$$[\operatorname{Re}\nu>0,\quad \operatorname{Re}\mu>0,\quad |\arg(1-\beta)|<\pi]\qquad (\text{cf. } 9.550)$$

9.550
ET I 313(12)

5. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{(\beta+e^x)^2} = \frac{1}{\beta}\,\ln\beta \qquad [|\arg\beta|<\pi]\qquad (\text{cf. } 4.231\ 5.).$$

4.231
BI ((101))(10)

3.424

1.[7] $$\int_0^\infty \frac{(1+a)e^x-a}{(1-e^x)^2}e^{-ax}x^n\,dx = n!\zeta(n,\,a)\qquad [a>-1, n=1,2,\dots]$$

BI ((85))(15)

2. $$\int_0^\infty \frac{(1+a)e^x+a}{(1+e^x)^2}e^{-ax}x^n\,dx = n!\Phi(-1,n,a+1)\qquad [a>-1, n=1,2,\dots]$$

BI ((85))(14)

3. $$\int_{-\infty}^\infty \frac{a^2e^x+b^2e^{-x}}{(a^2e^x-b^2e^{-x})^2}x^2\,dx = \frac{\pi^2}{2ab}\qquad [ab>0].$$

BI ((102))(3)a

4. $$\int_{-\infty}^\infty \frac{a^2e^x-b^2e^{-x}}{(a^2e^x+b^2e^{-x})^2}x^2\,dx = \frac{\pi}{ab}\,\ln\frac{b}{a}\qquad [ab>0].$$

BI ((102))(1)

5. $$\int_0^\infty \frac{e^x-e^{-x}+2}{(e^x-1)^2}x^2\,dx = \frac{2}{3}\pi^2-2.$$

1.[7] $$\int_{-\infty}^{\infty} \frac{xe^x\,dx}{(a^2+b^2e^{2x})^n} = \frac{\sqrt{\pi}\Gamma\left(n-\frac{1}{2}\right)}{4a^{2n-1}b\Gamma(n)}\left[2\ln\frac{a}{2b} - \boldsymbol{C} - \psi\left(n-\frac{1}{2}\right)\right] \qquad [ab>0, \quad n>0]$$
(cf. 4.231 7.).

4.231
BI((101))(13), LI((101))(13)

2.[7] $$\int_{-\infty}^{\infty} \frac{(a^2e^x - e^{-x})x^2\,dx}{(a^2e^x+e^{-x})^{p+1}} = -\frac{1}{a^{p+1}}B\left(\frac{p}{2}, \frac{p}{2}\right)\ln a \qquad [a>0, \quad p>0].$$

BI ((102))(5)

3.426

1. $$\int_{-\infty}^{\infty} \frac{(e^x - ae^{-x})x^2\,dx}{(a+e^x)^2(1+e^{-x})^2} = \frac{(\ln a)^2}{a-1}.$$

BI ((102))(12)

2. $$\int_{-\infty}^{\infty} \frac{(e^x - ae^{-x})x^2\,dx}{(a+e^x)^2(1-e^{-x})^2} = \frac{\pi^2+(\ln a)^2}{a+1}.$$

BI ((102))(13)

3.427

1. $$\int_0^{\infty} \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x}-1}\right)dx = \psi(\mu) \qquad [\operatorname{Re}\mu > 0] \qquad \text{(cf. 4.281 4.).}$$

4.281
WH

2.[7] $$\int_0^{\infty} \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right)e^{-x}\,dx = \boldsymbol{C} \qquad \text{(cf. 4.281 1.).}$$

4.281

BI ((94))(1)

3. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{1+e^{-x}}\right) \frac{e^{-2x}}{x}\, dx = \frac{1}{2}\ \ln \frac{\pi}{4}.$$

BI ((94))(5)

4. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right) \frac{e^{-\mu x}}{x}\, dx = \ln\Gamma(\mu) - \left(\mu - \frac{1}{2}\right)\ \ln\mu + \mu - \frac{1}{2}\ \ln(2\pi) \qquad [\operatorname{Re}\ \mu > 0].$$

WH

5. $$\int_0^\infty \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1}\right) \frac{dx}{x} = -\frac{1}{2}\ \ln\pi.$$

BI ((94))(6)

6. $$\int_0^\infty \left(\frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu\right) \frac{e^{-x}}{x}\, dx = -\ln\Gamma(\mu) - \ln\sin(\pi\mu) + \ln\pi \qquad [\operatorname{Re}\mu < 1].$$

EH I 21(6)

7. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1 - e^{-x}} - \frac{e^{-\mu x}}{x}\right) dx = \ln\mu - \psi(\nu) \qquad \text{(cf. 4.281 5.).}$$

4.281
BI ((94))(3)

8. $$\int_0^\infty \left(\frac{n}{x} - \frac{e^{-\mu x}}{1 - e^{-x/n}}\right) e^{-x}\, dx = n\psi(n\mu + n) - n\ \ln n \qquad [\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots].$$

BI ((94))(4)

9. $$\int_0^\infty \left(\mu - \frac{1 - e^{-\mu x}}{1 - e^{-x}}\right) \frac{e^{-x}}{x}\, dx = \ln\Gamma(\mu + 1) \qquad [\operatorname{Re}\mu > -1].$$

WH

10. $$\int_0^\infty \left(\nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1}\right) \frac{dx}{x} = \ln\frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1)} \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > 0]$$

(cf. 4.267 33.).

11. $$\int_0^\infty [(1-e^x)^{-1} + x^{-1} - 1]e^{-xz}\,dx = \psi(z) - \ln z \qquad [\operatorname{Re} z > 0].$$

EH I 18(24)

3.428

1. $$\int_0^\infty \left(\nu e^{-\mu x} - \frac{1}{\mu}e^{-x} - \frac{1}{\mu}\,\frac{e^{-1} - e^{-\mu\nu x}}{1-e^{-x}}\right)\frac{dx}{x} = \frac{1}{\mu}\ln\Gamma(\mu\nu) - \nu\ln\mu \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((94))(18)

2. $$\int_0^\infty \left(\frac{n-1}{2} + \frac{n-1}{1-e^{-x}} + \frac{e^{(1-\mu)x}}{1-e^{x/n}} + \frac{e^{-n\mu x}}{1-e^{-x}}\right)e^{-x}\frac{dx}{x} = \frac{n-1}{2}\ln 2\pi - \left(n\mu + \frac{1}{2}\right)\ln n$$
$$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots].$$

BI ((94))(14)

3. $$\int_0^\infty \left(n\mu - \frac{n-1}{2} - \frac{n}{1-e^{-x}} - \frac{e^{(1-\mu)x}}{1-e^{x/n}}\right)\frac{e^{-x}}{x}\,dx = \sum_{k=0}^{n-1}\ln\Gamma\left(\mu - \frac{k}{n} + 1\right)$$
$$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots].$$

BI ((94))(13)

4. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1-e^x} - \frac{e^{-\mu\nu x}}{1-e^{\mu x}} - \frac{e^x}{1-e^x} + \frac{e^{\mu x}}{1-e^{\mu x}}\right)\frac{dx}{x} = \nu\ln\mu \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

LI ((94))(15)

5. $$\int_0^\infty \left[\frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1-e^{-\mu x}} + \left(a\mu - \frac{\mu+1}{2}\right)e^{-\mu x} + (1-a\mu)\,e^{-x}\right]\frac{dx}{x} =$$
$$= \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - a\mu\right)\ln\mu \qquad [\operatorname{Re}\mu > 0].$$

BI ((94))(16)

6. $$\int_0^\infty \left[\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu\nu x}}{1-e^{-\mu x}} - \frac{(\mu-1)e^{-\mu x}}{1-e^{-\mu x}} - \frac{\mu-1}{2}e^{-\mu}x\right]\frac{dx}{x} = \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - \mu\nu\right)\ln\mu$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]. \qquad \text{(cf. 4.267 37.).}$$

4.267
BI ((94))(17)

7. $$\int_0^\infty \left[1 - e^{-x} - \frac{(1-e^{-\nu x})(1-e^{-\mu x})}{1-e^{-x}}\right] \frac{dx}{x} = \ln \mathrm{B}(\mu,\, \nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
(cf. 4.267 35.).

4.267
BI ((94))(12)

3.429

$$\int_0^\infty [e^{-x} - (1+x)^{-\mu}] \frac{dx}{x} = \psi(\mu) \qquad [\operatorname{Re}\mu > 0].$$

NH 184(7)

3.431

1. $$\int_0^\infty \left(e^{-\mu x} - 1 + \mu x - \frac{1}{2}\mu^2 x^2\right) x^{\nu-1}\, dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu}\Gamma(\nu+3)$$
$[\operatorname{Re}\mu > 0, \quad -2 > \operatorname{Re}\nu > -3]$.

LI ((90))(5)

2. $$\int_0^\infty \left[x^{-1} - \frac{1}{2}x^{-2}(x+2)(1-e^{-x})\right] e^{-px}\, dx = -1 + \left(p + \frac{1}{2}\right) \ln\left(1 + \frac{1}{p}\right) \qquad [\operatorname{Re} p > 0].$$

ET I 144(6)

3.432

1. $$\int_0^\infty x^{\nu-1} e^{-mx} (e^{-x} - 1)^n\, dx = \Gamma(\nu) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(n+m-k)^\nu} \qquad [n = 0,\, 1, \ldots, \operatorname{Re}\nu > 0].$$

LI ((90))(10)

2. $$\int_0^\infty [x^{\nu-1} e^{-x} - e^{-\mu x}(1-e^{-x})^{\nu-1}]\, dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

LI ((81))(14)

3.433

$$\int_0^\infty x^{p-1}\left[e^{-x}+\sum_{k=1}^{n}(-1)^k\frac{x^{k-1}}{(k-1)!}\right]dx=\Gamma(p)\qquad[-n<p<-n+1,\quad n=0,\,1,\ldots].$$

FI II 805

3.434

1. $$\int_0^\infty \frac{e^{-\nu x}-e^{-\mu x}}{x^{\rho+1}}\,dx=\frac{\mu^\rho-\nu^\rho}{\rho}\Gamma(1-\rho)\qquad[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\rho<1].$$

BI ((90))(6)

2. $$\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{x}\,dx=\ln\frac{\nu}{\mu}\qquad[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0].$$

FI II 634

3.435

1. $$\int_0^\infty \left\{(x+1)e^{-x}-e^{-\frac{x}{2}}\right\}\frac{dx}{x}=1-\ln 2.$$

LI ((89))(19)

2.[7] $$\int_0^\infty \frac{1-e^{-\mu x}}{x(x+\beta)}\,dx=\frac{1}{\beta}[\ln(\beta\mu \boldsymbol{C})-e^{\beta\mu}\operatorname{Ei}(-\beta\mu)]\qquad[|\arg\beta|<\pi,\quad \operatorname{Re}\mu>0].$$

ET II 217 (18)

3. $$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}=\boldsymbol{C}.$$

FI II 7 95, 802

4. $$\int_0^\infty \left(e^{-\mu x}-\frac{1}{1+ax}\right)\frac{dx}{x}=\ln\frac{a}{\mu}-\boldsymbol{C}\qquad[a>0,\quad \operatorname{Re}\mu>0].$$

BI ((92))(10)

3.436

$$\int_0^\infty \left\{ \frac{e^{-npx} - e^{-nqx}}{n} - \frac{e^{-mpx} - e^{-mqx}}{m} \right\} \frac{dx}{x^2} = (q - p) \ln \frac{m}{n} \qquad [p > 0, \quad q > 0].$$

BI ((89))(28)

3.437

$$\int_0^\infty \left\{ pe^{-x} - \frac{1 - e^{-px}}{x} \right\} \frac{dx}{x} = p \ln p - p \qquad [p > 0].$$

BI ((89))(24)

3.438

1. $$\int_0^\infty \left\{ \left(\frac{1}{2} + \frac{1}{x} \right) e^{-x} - \frac{1}{x} e^{-\frac{x}{2}} \right\} \frac{dx}{x} = \frac{\ln 2 - 1}{2}.$$

BI ((89))(19)

2.[7] $$\int_0^\infty \left\{ \frac{p^3}{6} e^{-x} - \frac{p^2}{2x} + \frac{p}{x^2} - \frac{1 - e^{-px}}{x^3} \right\} \frac{dx}{x} = \frac{p^3}{6} \ln p - \frac{11}{36} p^3 \qquad [p > 0].$$

BI ((89))(33)

3. $$\int_0^\infty \left(e^{-x} - e^{-2x} - \frac{1}{x} e^{-2x} \right) \frac{dx}{x} = 1 - \ln 2.$$

BI ((89))(25)

4. $$\int_0^\infty \left\{ \left(p - \frac{1}{2} \right) e^{-x} + \frac{x + 2}{2x} \left(e^{-px} - e^{-\frac{x}{2}} \right) \right\} \frac{dx}{x} = \left(p - \frac{1}{2} \right) (\ln p - 1) \qquad [p > 0].$$

BI ((89))(22)

3.439

$$\int_0^\infty \left\{ (p - q) e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right\} \frac{dx}{x} =$$
$$= p \ln p - q \ln q - (p - q) \left(1 + \ln \frac{r}{m} \right) \qquad [p > 0, \quad q > 0, \quad r > 0].$$

LI((89))(26), LI((89))(27)

3.441

4.268
BI ((89))(18)

3.442

1. $$\int_0^\infty \left\{1 - \frac{x+2}{2x}(1-e^{-x})\right\} e^{-qx}\,\frac{dx}{x} = -1 + \left(q + \frac{1}{2}\right)\,\ln\frac{q+1}{q} \qquad [q>0].$$

BI ((89))(23)

2. $$\int_0^\infty \left(\frac{e^{-x}-1}{x} + \frac{1}{1+x}\right)\,\frac{dx}{x} = \boldsymbol{C} - 1.$$

BI ((92))(16)

3. $$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2}\right)\,\frac{dx}{x} = -\boldsymbol{C} + \ln\frac{a}{p} \qquad [p>0].$$

BI ((92))(11)

380
3.443

1. $$\int_0^\infty \left\{\frac{e^{-x}p^2}{2} - \frac{p}{x} + \frac{1-e^{-px}}{x^2}\right\}\,\frac{dx}{x} = \frac{p^2}{2}\,\ln p - \frac{3}{4}p^2 \qquad [p>0].$$

BI ((89))(32)

2. $$\int_0^\infty \frac{(1-e^{-px})^n e^{-qx}}{x^3}\,dx = \frac{1}{2}\sum_{Kk=2}^{n}(-1)^{k-1}\binom{n}{k}(q+kp)^2\,\ln(q+kp)$$
$$[n>2, \quad q>0, \quad pn+q>0] \qquad \text{(cf. 4.268 4.)}.$$

4.268
BI ((89))(30)

3. $$\int_0^\infty (1-e^{-px})^2 e^{-qx}\,\frac{dx}{x^2} = (2p+q)\,\ln(2p+q) - 2(p+q)\,\ln(p+q) + q\,\ln q$$
$$[q>0, \quad 2p>-q] \qquad \text{(cf. 4.268 2.)}.$$

3.45 Combinations of powers and algebraic functions of exponentials

3.451

1. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{3}\left(\frac{4}{3}-\ln 2\right).$$

BI ((99))(1)

2. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-2x}}\,dx = \frac{\pi}{4}\left(\frac{1}{2}+\ln 2\right) \qquad \text{(cf. 4.241 9.).}$$

4.241
BI ((99))(2)

3.452

1. $$\int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = 2\pi\ \ln 2.$$

FI II 643a,BI((99))(4)

2. $$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = 4\pi\left\{(\ln 2)^2+\frac{\pi^2}{12}\right\}.$$

BI ((99))(5)

3. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = \frac{\pi}{2}[2\ \ln 2-1].$$

BI ((99))(6)

4. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1-\ln 2.$$

BI ((99))(8)

5. $$\int_0^\infty \frac{xe^{-2x}\,dx}{\sqrt{e^x-1}} = \frac{3}{4}\pi\left(\ln 2-\frac{7}{12}\right).$$

BI ((99))(7)

3.453

1. $$\int_0^\infty \frac{xe^x}{a^2e^x-(a^2-b^2)}\,\frac{dx}{\sqrt{e^x-1}} = \frac{2\pi}{ab}\,\ln\left(1+\frac{b}{a}\right) \qquad [ab>0] \qquad \text{(cf. 4.298 17.).}$$

4.298
BI ((99))(16)

2. $$\int_0^\infty \frac{xe^x\,dx}{[a^2e^x-(a^2+b^2)]\sqrt{e^x-1}} = \frac{2\pi}{ab}\,\operatorname{arctg}\,\frac{b}{a} \qquad [ab>0] \qquad \text{(cf. 4.298 18.).}$$

4.298
BI ((99))(17)

3.454

1. $$\int_0^\infty \frac{xe^{-2nx}\,dx}{\sqrt{e^{2x}+1}} = \frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2}\left\{\ln 2+\sum_{k=1}^{2n}\frac{(-1)^k}{k}\right\}.$$

LI ((99))(10)

2. $$\int_0^\infty \frac{xe^{-(2n-1)x}\,dx}{\sqrt{e^{2x}-1}} = -\frac{(2n-2)!!}{(2n-1)!!}\left\{\ln 2+\sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right\}.$$

LI ((99))(9)

3.455

1. $$\int_0^\infty \frac{x^2e^x\,dx}{\sqrt{(e^x-1)^3}} = 8\pi\,\ln 2.$$

BI ((99))(11)

2. $$\int_0^\infty \frac{x^3e^x\,dx}{\sqrt{(e^x-1)^3}} = 24\pi\left[(\ln 2)^2+\frac{\pi^2}{12}\right].$$

BI ((99))(12)

3.456

1. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{e^{3x}-1}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3 + \frac{\pi}{3\sqrt{3}}\right].$$

BI ((99))(13)

2. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{(e^{3x}-1)^2}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3 - \frac{\pi}{3\sqrt{3}}\right] \qquad \text{(cf. 4.244 3.).}$$

4.244
BI ((99))(14)

3.457

1. $$\int_0^\infty x e^{-x}(1-e^{-2x})^{n-1/2}\,dx = \frac{(2n-1)!!}{4\cdot(2n)!!}\pi[\boldsymbol{C}+\psi(n+1)+2\ln 2] \qquad \text{(cf. 4.241 5.).}$$

4.241
BI ((99))(3)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{(a+e^x)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}}[\ln(4a) - 3\boldsymbol{C} - 2\psi(2n) - \psi(n)].$$

BI ((101))(12)

3. $$\int_{-\infty}^\infty \frac{x\,dx}{(a^2e^x+e^{-x})^\mu} = \frac{-1}{2a^\mu}\mathrm{B}\left(\frac{\mu}{2}, \frac{\mu}{2}\right)\ln a \qquad [a>0, \quad \operatorname{Re}\mu > 0].$$

BI ((101))(14)

3.458

1.[7] $$\int_0^{\ln 2} xe^x(e^x-1)^{p-1}\,dx = \frac{1}{p}\left[\ln 2 + \sum_{k=0}^\infty \frac{(-1)^{k-1}}{p+k+1}\right] \qquad [p>-1]$$

BI ((104))(4)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{(a+e^x)^{\nu+1}} = \frac{1}{\nu a^\nu}[\ln a - \boldsymbol{C} - \psi(\nu)] \qquad [a>0];$$
$$= \frac{1}{\nu a^\nu}\left[\ln a - \sum_{k=1}^{\nu-1}\frac{1}{k}\right] \qquad [\nu = 1, 2, \ldots].$$

3.46- 3.48 Combinations of exponentials of more complicated arguments and powers

3.461

$$1.\quad \int_u^\infty \frac{e^{-p^2x^2}}{x^{2n}}\,dx = \frac{(-1)^n 2^{n-1} p^{2n-1}\sqrt{\pi}}{(2n-1)!!}[1-\Phi(pu)] + \\ + \frac{e^{-p^2u^2}}{2u^{2n-1}}\sum_{k=0}^{n-1}\frac{(-1)^k 2^{k+1}(pu)^{2k}}{(2n-1)(2n-3)\dots(2n-2k-1)} \qquad [p>0].$$

NT 21(4)

$$2.\quad \int_0^\infty x^{2n}e^{-px^2}\,dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \qquad [p>0, \quad n=0,\,1,\dots].$$

FI II 743

$$3.\quad \int_0^\infty x^{2n+1}e^{-px^2}\,dx = \frac{n!}{2p^{n+1}} \qquad [p>0].$$

BI ((81))(7)

$$4.\quad \int_{-\infty}^\infty (x+ai)^{2n}e^{-x^2}\,dx = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\sum_{k=0}^{n}(-1)^k\frac{(2a)^{2k}n!}{(2k)!(n-k)!}.$$

BI ((100))(12)

$$5.^3\quad \int_u^\infty e^{-\mu x^2}\frac{dx}{x^2} = \frac{1}{u}e^{-\mu u^2} - \sqrt{\mu\pi}[1-\Phi(\sqrt{\mu}u)] \qquad \left[|\arg\mu| < \frac{\pi}{2}, \quad u>0\right].$$

ET I 135(19)a

3.462

$$1.\quad \int_0^\infty x^{\nu-1}e^{-\beta x^2-\gamma x}\,dx = (2\beta)^{-\nu/2}\Gamma(\nu)\,\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0].$$

EH II 119(3)A, ET I 313(13)

$$2.\quad \int_{-\infty}^\infty x^n e^{-px^2+2qx}\,dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\,\frac{d^{n-1}}{dq^{n-1}}\left(qe^{q^2/p}\right) \qquad [p>0]; \\ = n!e^{q^2/p}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n\sum_{k=0}^{E(n/2)}\frac{1}{(n-2k)!(k)!}\left(\frac{p}{4q^2}\right)^k \qquad [p>0].$$

3. $$\int_{-\infty}^{\infty} (ix)^{\nu} e^{-\beta^2 x^2 - iqx}\, dx = 2^{-\frac{\nu}{2}} \sqrt{\pi} \beta^{-\nu-1} \exp\left(-\frac{q^2}{8\beta^2}\right) D_{\nu}\left(\frac{q}{\beta\sqrt{2}}\right)$$
$$\left[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x\right].$$

ET I 121(23)

4. $$\int_{-\infty}^{\infty} x^n \exp[-(x-\beta)^2]\, dx = (2i)^{-n} \sqrt{\pi} H_n(i\beta).$$

EH II 195(31)

5. $$\int_0^{\infty} x e^{-\mu x^2 - 2\nu x}\, dx = \frac{1}{2\mu} - \frac{\nu}{2\mu} \sqrt{\frac{\pi}{\mu}} e^{\frac{\nu^2}{\mu}} \left[1 - \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \qquad \left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET I 146(31)a

6. $$\int_{-\infty}^{\infty} x e^{-px^2 + 2qx}\, dx = \frac{q}{p} \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{p}\right) \qquad [\operatorname{Re} p > 0].$$

BI ((100))(7)

7. $$\int_0^{\infty} x^2 e^{-\mu x^2 - 2\nu x}\, dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}} \frac{2\nu^2 + \mu}{4} e^{\frac{\nu^2}{\mu}} \left[1 - \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \qquad \left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET I 146(32)

8. $$\int_{-\infty}^{\infty} x^2 e^{-\mu x^2 + 2\nu x}\, dx = \frac{1}{2\mu} \sqrt{\frac{\pi}{\mu}} \left(1 + 2\frac{\nu^2}{\mu}\right) e^{\frac{\nu^2}{\mu}} \qquad [|\arg\nu| < \pi, \quad \operatorname{Re}\mu > 0].$$

BI ((100))(8)a

3.463

$$\int_0^{\infty} \left(e^{-x^2} - e^{-x}\right) \frac{dx}{x} = \frac{1}{2} \boldsymbol{C}.$$

BI ((89))(5)

3.464

$$\int_0^\infty (e^{-\mu x^2} - e^{-\nu x^2})\frac{dx}{x^2} = \sqrt{\pi}(\sqrt{\nu} - \sqrt{\mu}) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

FI II 645

3.465

$$\int_0^\infty (1 + 2\beta x^2)e^{-\mu x^2}\,dx = \frac{\mu + \beta}{2}\sqrt{\frac{\pi}{\mu^3}} \qquad [\operatorname{Re}\mu > 0].$$

ET I 136(24)a

3.466

1. $$\int_0^\infty \frac{e^{-\mu^2 x^2}}{x^2 + \beta^2}\,dx = [1 - \Phi(\beta\mu)]\frac{\pi}{2\beta}e^{\beta^2\mu^2} \qquad \left[\operatorname{Re}\beta > 0, \quad |\arg\mu| < \frac{\pi}{4}\right].$$

NT 19(13)

2. $$\int_0^\infty \frac{x^2 e^{-\mu^2 x^2}}{x^2 + \beta^2}\,dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi\beta}{2}e^{\mu^2\beta^2}[1 - \Phi(\beta\mu)] \qquad \left[\operatorname{Re}\beta > 0, \quad |\arg\mu| < \frac{\pi}{4}\right].$$

ET II 217(16)

3. $$\int_0^1 \frac{e^{x^2} - 1}{x^2}\,dx = \sum_{k=1}^\infty \frac{1}{k!(2k-1)}.$$

FI II 683

3.467

$$\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}.$$

BI ((92))(12)

3.468

1. $$\int_{u\sqrt{2}}^\infty \frac{e^{-x^2}}{\sqrt{x^2 - u^2}}\,\frac{dx}{x} = \frac{\pi}{4u}[1 - \Phi(u)]^2 \qquad [u > 0].$$

NT 33(17)

NT 19(11)

3.469

1. $$\int_0^\infty e^{-\mu x^4 - 2\nu x^2}\, dx = \frac{1}{4}\sqrt{\frac{2\nu}{\mu}}\, \exp\left(\frac{\nu^2}{2\mu}\right) K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right) \qquad [\operatorname{Re}\mu \geq 0].$$

ET I 146(23)

2. $$\int_0^\infty (e^{-x^4} - e^{-x})\frac{dx}{x} = \frac{3}{4}\boldsymbol{C}.$$

BI ((89))(7)

3. $$\int_0^\infty (e^{-x^4} - e^{-x^2})\frac{dx}{x} = \frac{1}{4}\boldsymbol{C}.$$

BI ((89))(6)

384

3.471

1. $$\int_0^u \exp\left(-\frac{\beta}{x}\right)\frac{dx}{x^2} = \frac{1}{\beta}\, \exp\left(-\frac{\beta}{u}\right).$$

ET II 188(22)

2. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1} e^{-\frac{\beta}{x}}\, dx = \beta^{\frac{\nu-1}{2}} u^{\frac{2\mu+\nu-1}{2}}\, \exp\left(-\frac{\beta}{2u}\right)\Gamma(\mu) W_{\frac{1-2\mu-\nu}{2}, \frac{\nu}{2}}\left(\frac{\beta}{u}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0, \quad u > 0].$$

ET II 187(18)

3. $$\int_0^u x^{-\mu-1}(u-x)^{\mu-1} e^{-\frac{\beta}{x}}\, dx = \beta^{-\mu} u^{\mu-1}\Gamma(\mu)\, \exp\left(-\frac{\beta}{u}\right) \qquad [\operatorname{Re}\mu > 0, \quad u > 0].$$

ET II 187(16)

4. $$\int_0^u x^{-2\mu}(u-x)^{\mu-1} e^{-\frac{\beta}{x}}\, dx = \frac{1}{\sqrt{\pi u}}\beta^{\frac{1}{2}-\mu} e^{-\frac{\beta}{2u}}\Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$$
$$[u > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0].$$

5. $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \mathrm{B}(1-\mu-\nu,\,\mu)u^{\mu+\nu-1}\,{}_1F_1\left(1-\mu-\nu;\,1-\nu;\,\frac{\beta}{u}\right)$$
$$[0<\operatorname{Re}\mu<\operatorname{Re}(1-\nu),\quad u>0].$$

ET II 203(15)

6. $$\int_u^\infty x^{-2\mu}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \sqrt{\frac{\pi}{u}}\beta^{\frac{1}{2}-\mu}\Gamma(\mu)\,\exp\left(\frac{\beta}{2u}\right)I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)\qquad[\operatorname{Re}\mu>0,\quad u>0].$$

ET II 202(14)

7. $$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\Gamma(1-\mu-\nu)e^{\frac{\beta}{2\gamma}}W_{\frac{\nu-1}{2}+\mu,\,-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right)$$
$$[|\arg\gamma|<\pi,\quad \operatorname{Re}(1-\mu)>\operatorname{Re}\nu>0].$$

ET II 234(13)a

8. $$\int_0^u x^{-2\mu}(u^2-x^2)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}}u^{\mu-\frac{3}{2}}\Gamma(\mu)K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right)$$
$$[\operatorname{Re}\beta>0,\quad u>0,\quad \operatorname{Re}\mu>0].$$

ET II 188(23)a

9. $$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu(2\sqrt{\beta\gamma})\qquad[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0].$$

ET II 82(23)A, LET I 146(29)

10. $$\int_0^\infty x^{\nu-1}\,\exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right]dx = 2\beta^\nu e^{\frac{i\nu\pi}{2}}K_{-\nu}(\beta\mu)$$
$$[\operatorname{Im}\mu>0,\quad \operatorname{Im}(\beta^2\mu)<0;\quad \text{note }K_{-\nu}\equiv K_\nu].$$

EH II 82(24)

11. $$\int_0^\infty x^{\nu-1}\,\exp\left[\frac{i\mu}{2}\left(x+\frac{\beta^2}{x}\right)\right]dx = i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}}H^{(1)}_{-\nu}(\beta\mu)\qquad[\operatorname{Im}\mu>0,\quad \operatorname{Im}(\beta^2\mu)>0].$$

EH II 21(33)

12. $$\int_0^\infty x^{\nu-1}\,\exp\left(-x-\frac{\mu^2}{4x}\right)dx = 2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu)\qquad\left[|\arg\mu|<\frac{\pi}{2},\ \operatorname{Re}\mu^2>0;\ \ \text{note }K_{-\nu}\equiv K_\nu\right].$$

13. $$\int_0^\infty \frac{x^{\nu-1}e^{-\frac{\beta}{x}}}{x+\gamma}\,dx = \gamma^{\nu-1}e^{\frac{\beta}{\gamma}}\Gamma(1-\nu)\Gamma\left(\nu,\,\frac{\beta}{\gamma}\right) \qquad [|\arg\gamma|<\pi, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<1].$$

ET II 218(19)

14. $$\int_0^1 \frac{\exp\left(1-\frac{1}{x}\right)-x^\nu}{x(1-x)}\,dx = \psi(\nu) \qquad [\operatorname{Re}\nu>0].$$

BI ((80))(7)

3.472

1. $$\int_0^\infty \left(\exp\left(-\frac{a}{x^2}\right)-1\right)e^{-\mu x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}[\exp(-2\sqrt{a\mu})-1] \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}a>0].$$

ET I 146(30)

2. $$\int_0^\infty x^2\,\exp\left(-\frac{a}{x^2}-\mu x^2\right)\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu^3}}(1+2\sqrt{a\mu})\,\exp(-2\sqrt{a\mu}) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}a>0].$$

ET I 146(26)

3. $$\int_0^\infty \exp\left(-\frac{a}{x^2}-\mu x^2\right)\frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\exp(-2\sqrt{a\mu}) \qquad [\operatorname{Re}\mu>0, \quad a>0].$$

ET I 146(28)a

4. $$\int_0^\infty \exp\left[-\frac{1}{2a}\left(x^2+\frac{1}{x^2}\right)\right]\frac{dx}{x^4} = \sqrt{\frac{a\pi}{2}}(1+a)e^{-1/a} \qquad [a>0].$$

BI ((98))(14)

3.473

$$\int_0^\infty \exp(-x^n)x^{(m+1/2)n-1}\,dx = \frac{(2m-1)!!}{2^m n}\sqrt{\pi}.$$

BI ((98))(6)

3.474

1. $$\int_0^1 \left\{\frac{n\,\exp(1-x^{-n})}{1-x^n}-\frac{x^{np}}{1-x}\right\}\frac{dx}{x} = \frac{1}{n}\sum_{k=1}^{n}\psi\left(p+\frac{k-1}{n}\right) \qquad [p>0].$$

2. $\int_0^1 \left\{ \frac{n\ \exp(1-x^{-n})}{1-x^n} - \frac{\exp\left(1-\frac{1}{x}\right)}{1-x} \right\} \frac{dx}{x} = -\ln n.$

BI ((80))(9)

3.475

1.[7] $\int_0^\infty \left\{ \exp(-x^2) - \frac{1}{1+x^{2^n}} \right\} \frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}. \qquad [n \in]$

BI ((92))(14)

2. $\int_0^\infty \left\{ \exp(-x^{2^n}) - \frac{1}{1+x^2} \right\} \frac{dx}{x} = -2^{-n}\boldsymbol{C}.$

BI ((92))(13)

3. $\int_0^\infty \{\exp(-x^{2^n}) - e^{-x}\} \frac{dx}{x} = (1-2^{-n})\boldsymbol{C}.$

BI ((89))(8)

3.476

1. $\int_0^\infty [\exp(-\nu x^p) - \exp(-\mu x^p)] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$

BI ((89))(3)

2. $\int_0^\infty [\exp(-x^p) - \exp(-x^q)] \frac{dx}{x} = \frac{p-q}{pq}\boldsymbol{C} \qquad [p > 0, \quad q > 0].$

BI ((89))(9)

3.477

1.[7] $\int_{-\infty}^\infty \frac{\exp(-a|x|)}{x-u}\,dx = \frac{\operatorname{sign}\ u}{\pi}[\exp(a|u|)\ \operatorname{Ei}(-a|u|) - \exp(-a|u|)\ \operatorname{Ei}(a|u|)] \qquad [a > 0].$

ET II 251(35)

ET II 251(36)

3.478

1. $$\int_0^\infty x^{\nu-1}\,\exp(-\mu x^p)\,dx = \frac{1}{p}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad p > 0].$$

BI((81))(8)A, ET I 313(15, 16)

2. $$\int_0^\infty x^{\nu-1}[1-\exp(-\mu x^p)]\,dx = -\frac{1}{|p|}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right)$$
$$[\operatorname{Re}\mu > 0 \quad \text{and} \quad -p < \operatorname{Re}\nu < 0 \qquad \text{for} \quad p > 0, \quad 0 < \operatorname{Re}\nu < -p \qquad \text{for} \quad p < 0].$$

ET I 313(18, 19)

3. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\,\exp(\beta x^n)\,dx = \mathrm{B}(\mu,\,\nu)u^{\mu+\nu-1}{}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \dots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad n = 2, 3, \dots].$$

ET II 187(15)

4. $$\int_0^\infty x^{\nu-1}\,\exp(-\beta x^p - \gamma x^{-p})\,dx = \frac{2}{p}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}(2\sqrt{\beta\gamma}) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

ET I 313(17)

3.479

1. $$\int_0^\infty \frac{x^{\nu-1}\,\exp(-\beta\sqrt{1+x})}{\sqrt{1+x}}\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu}\Gamma(\nu)K_{\frac{1}{2}-\nu}(\beta) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0].$$

ET I 313(14)

2.[8] $$\int_0^\infty \frac{x^{\nu-1}\,\exp(i\mu\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = i\frac{\sqrt{\pi}}{2}\left(\frac{\mu}{2}\right)^{\frac{1-\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)H^{(1)}_{\frac{1-\nu}{2}}(\mu) \qquad [\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 2].$$

EH II 83(30)

3.481

1. $$\int_{-\infty}^{\infty} xe^x \, \exp(-\mu e^x)\, dx = -\frac{1}{\mu}(\boldsymbol{C} + \ln\mu) \qquad [\operatorname{Re}\mu > 0].$$

BI ((100))(13)

2. $$\int_{-\infty}^{\infty} xe^x \, \exp(-\mu e^{2x})\, dx = -\frac{1}{4}[\boldsymbol{C} + \ln(4\mu)]\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$$

BI ((100))(14)

3.482

1.[3] $$\int_0^{\infty} \exp(nx - \beta \operatorname{sh} x)\, dx = \frac{1}{2}[S_n(\beta) - \pi\, \mathbf{E}_n(\beta) - \pi N_n(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 168(11)

2. $$\int_0^{\infty} \exp(-nx - \beta \operatorname{sh} x)\, dx = (-1)^{n+1}\frac{1}{2}[S_n(\beta) + \pi\mathbf{E}_n(\beta) + \pi N_n(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 168(12)

3. $$\int_0^{\infty} \exp(-\nu x - \beta \operatorname{sh} x)\, dx = \frac{\pi}{\sin\nu\pi}[\mathbf{J}_\nu(\beta) - J_\nu(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 168(13)

3.483[3]

$$\int_{-\infty}^{\infty} \frac{\exp(\nu \operatorname{Arsh} x - iax)}{\sqrt{1+x^2}}\, dx = \begin{cases} 2\exp\left(-\dfrac{i\nu\pi}{2}\right) K_\nu(a) & \text{for} \quad a > 0, \\ 2\exp\left(\dfrac{i\nu\pi}{2}\right) K_\nu(-a) & \text{for} \quad a < 0 \end{cases} \qquad [|\operatorname{Re}\nu| < 1].$$

ET I 122(32)

3.484

$$\int_0^{\infty} \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px}\right] \frac{dx}{x} = (e^a - 1)\, \ln\frac{q}{p} \qquad [p > 0, \quad q > 0].$$

BI ((89))(34)

3.485

$$\int_0^{\frac{\pi}{2}} \exp(-\operatorname{tg}^2 x)\, dx = \frac{\pi e}{2}[1 - \Phi(1)].$$

3.486[6]

$$\int_0^1 x^{-x}\, dx = \int_0^1 e^{-x \ln x}\, dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627\ldots .$$

FI II 483

3.5 Hyperbolic Functions

3.51 Hyperbolic functions

3.511

1. $$\int_0^{\infty} \frac{dx}{\operatorname{ch} ax} = \frac{\pi}{2a} \qquad [a > 0].$$

2. $$\int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} bx}\, dx = \frac{\pi}{2b} \operatorname{tg} \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((27))(10)a

3. $$\int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} - \frac{1}{b}\beta\left(\frac{a+b}{2b}\right) \qquad [b > |a|].$$

GW ((351))(3b)

4. $$\int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{ch} bx}\, dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((4))(14)a

5. $$\int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{ch} bx}{\operatorname{sh} cx}\, dx = \frac{\pi}{2c} \frac{\sin \frac{a\pi}{c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \qquad [c > |a| + |b|].$$

6. $$\int_0^\infty \frac{\operatorname{ch} ax \operatorname{ch} bx}{\operatorname{ch} cx}\,dx = \frac{\pi}{c}\,\frac{\cos\frac{a\pi}{2c}\,\cos\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}} \qquad [c > |a| + |b|].$$

BI ((27))(5)a

7. $$\int_0^\infty \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} cx}\,dx = \frac{\pi}{c}\,\frac{\sin\frac{a\pi}{2c}\,\sin\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}} \qquad [c > |a| + |b|].$$

BI ((27))(6)a

8. $$\int_0^\infty \frac{dx}{\operatorname{ch} x^2} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}$$

BI ((98))(25)

9. $$\int_{-\infty}^\infty \frac{\operatorname{sh}^2 ax}{\operatorname{sh}^2 x}\,dx = 1 - a\pi \operatorname{ctg} a\pi \qquad [a^2 < 1].$$

BI ((16))(3)a

10. $$\int_0^\infty \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch}^2 bx}\,dx = \frac{a\pi}{2b^2}\sec\frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((27))(16)a

3.512

1. $$\int_0^\infty \frac{\operatorname{ch} 2\beta x}{\operatorname{ch}^{2\nu} ax}\,dx = \frac{4^{\nu-1}}{a}\mathrm{B}\left(\nu+\frac{\beta}{a},\ \nu-\frac{\beta}{a}\right) \qquad [\operatorname{Re}(\nu\pm\beta) > 0, \quad a > 0, \quad \beta > 0].$$

LI((27))(17)A, EH I 11(26)

2. $$\int_0^\infty \frac{\operatorname{sh}^\mu x}{\operatorname{ch}^\nu x}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\mu+1}{2},\ \frac{\nu-\mu}{2}\right) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) < 0].$$

EH I 11(23)

3.513

1. $$\int_0^\infty \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \qquad [ab\neq 0].$$

2. $$\int_0^\infty \frac{dx}{a+b\,\operatorname{ch}x} = \frac{2}{\sqrt{b^2-a^2}}\operatorname{arctg}\frac{\sqrt{b^2-a^2}}{a+b} \qquad [b^2>a^2];$$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \qquad [b^2<a^2].$$

GW ((351))(7)

3. $$\int_0^\infty \frac{dx}{a\,\operatorname{sh}x+b\,\operatorname{ch}x} = \frac{2}{\sqrt{b^2-a^2}}\operatorname{arctg}\frac{\sqrt{b^2-a^2}}{a+b} \qquad [b^2>a^2];$$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \qquad [a^2>b^2].$$

GW ((351))(9)

4. $$\int_0^\infty \frac{dx}{a+b\,\operatorname{ch}x+c\,\operatorname{sh}x} = \frac{2}{\sqrt{b^2-a^2-c^2}}\left[\operatorname{arctg}\frac{\sqrt{b^2-a^2-c^2}}{a+b+c}+\epsilon\pi\right]$$
$$[b^2>a^2+c^2; \quad \epsilon=0 \quad \text{for} \quad (b-a)(a+b+c)>0,$$
$$|\epsilon|=1 \quad \text{for} \quad (b-a)(a+b+c)<0,$$
$$\text{also } \epsilon=1 \quad \text{for} \quad a<b+c \quad \text{and} \quad \epsilon=-1 \quad \text{for} \quad a>b+c];$$
$$= \frac{1}{\sqrt{a^2-b^2+c^2}}\ln\frac{a+b+c+\sqrt{a^2-b^2+c^2}}{a+b+c-\sqrt{a^2-b^2+c^2}} [b^2<a^2+c^2, \quad a^2\neq b^2];$$
$$= \frac{1}{c}\ln\frac{a+c}{a} \qquad [a=b\neq 0, \quad c\neq 0];$$
$$= \frac{2(a-b)}{c(a-b-c)} \qquad [b^2=a^2+c^2, \quad c(a-b-c)<0].$$

GW ((351))(6)

3.514

1. $$\int_0^\infty \frac{dx}{\operatorname{ch}ax+\cos t} = \frac{t}{a}\operatorname{cosec}t \qquad [0<t<\pi, \quad a>0].$$

BI ((27))(22)a

2. $$\int_0^\infty \frac{\operatorname{ch}ax-\cos t_1}{\operatorname{ch}bx-\cos t_2}\,dx = \frac{\pi}{b}\,\frac{\sin\dfrac{a(\pi-t_2)}{b}}{\sin t_2\,\sin\frac{a}{b}\pi} - \frac{\pi-t_2}{b\,\sin t_2}\cos t_1 \qquad [0<|a|<b, \quad 0<t_2<\pi].$$

BI ((6))(20)a

3. $$\int_0^\infty \frac{\operatorname{ch}ax\,dx}{(\operatorname{ch}x+\cos t)^2} = \frac{\pi(-\cos t\,\sin at+a\,\sin t\,\cos at)}{\sin^3 t\,\sin a\pi} \qquad [0<a^2<1, \quad 0<t<\pi].$$

BI ((6))(18)a

4. $$\int_0^\infty \frac{\operatorname{sh} ax \, \operatorname{sh} bx}{(\operatorname{ch} ax + \cos t)^2} \, dx = \frac{b\pi}{a^2} \operatorname{cosec} t \operatorname{cosec} \frac{b\pi}{a} \sin \frac{bt}{a} \qquad [0 < |b| < a, \quad 0 < t < \pi].$$

BI ((27))(27)a

3.515

$$\int_{-\infty}^\infty \left(1 - \frac{\sqrt{2} \operatorname{ch} x}{\sqrt{\operatorname{ch} 2x}}\right) dx = -\ln 2.$$

BI ((21))(12)a

3.516

1. $$\int_0^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^\mu} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^\mu} = Q_{\mu-1}(z) \qquad [\operatorname{Re} \mu > -1].$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of z in the z-plane cut from -1 to $+1$ provided $\mu < 0$. If $\mu > 0$, this formula ceases to be valid for points at which the denominator vanishes.

CO, WH

2. $$\int_0^\infty \frac{dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{n+1}} = Q_n(\beta).$$

EH II 181(32)

3. $$\int_0^\infty \frac{\operatorname{ch} \gamma x \, dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{\nu+1}} = \frac{e^{-i\gamma\pi} \Gamma(\nu - \gamma + 1) Q_\nu^\gamma(\beta)}{\Gamma(\nu + 1)}$$
$$[\operatorname{Re}(\nu \pm \gamma) > -1, \quad \nu \neq -1, \ -2, \ -3, \ldots].$$

EH I 157(12)

4. $$\int_0^\infty \frac{\operatorname{sh}^{2\mu} x \, dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi} \Gamma(\nu - 2\mu + 1) \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\beta)$$
$$[\operatorname{Re}(\nu - 2\mu + 1) > 0, \quad \operatorname{Re}(\nu + 1) > 0].$$

EH I 155(2)

3.517

1. $$\int_0^\infty \frac{\operatorname{ch}\left(\gamma+\frac{1}{2}\right)x\,dx}{(\beta+\operatorname{ch}x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}(\beta^2-1)^{-\frac{\nu}{2}}\frac{\Gamma(\nu+\gamma+1)\Gamma(\nu-\gamma)P_\gamma^{-\nu}(\beta)}{\Gamma\left(\nu+\frac{1}{2}\right)}$$
$$[\operatorname{Re}(\nu-\gamma)>0, \quad \operatorname{Re}(\nu+\gamma+1)>0].$$

EH I 156(11)

2. $$\int_0^a \frac{\operatorname{ch}\left(\gamma+\frac{1}{2}\right)x\,dx}{(\operatorname{ch}a-\operatorname{ch}x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\operatorname{sh}^\nu a}P_\gamma^\nu(\operatorname{ch}a) \qquad \left[\operatorname{Re}\nu<\frac{1}{2}, \quad a>0\right].$$

EH I 156(8)

3.518

1. $$\int_0^\infty \frac{\operatorname{sh}^{2\mu}x\,dx}{(\operatorname{ch}a+\operatorname{sh}a\operatorname{ch}x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi}\operatorname{sh}^\mu a}\frac{\Gamma(\nu-2\mu+1)\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\nu+1)}Q_{\nu-\mu}^\mu(\operatorname{ch}a)$$
$$[\operatorname{Re}(\nu+1)>0, \quad \operatorname{Re}(\nu-2\mu+1)>0, \quad a>0].$$

EH I 155(3)a

2.* $$\int_0^\infty \frac{\operatorname{sh}^{2\mu+1}x\,dx}{(\beta+\operatorname{ch}x)^{\nu+1}} = 2^\mu(\beta^2-1)^{\frac{\mu-\nu}{2}}\Gamma(\nu-2\mu)\Gamma(\mu+1)P_\mu^{\mu-\nu}(\beta)$$
$$[\operatorname{Re}(\nu-\mu)>\operatorname{Re}\mu>-1, \quad \beta \text{ does not lie on the ray } (-\infty,+1) \text{ of the real axis}].$$

EH I 155(1)

3. $$\int_0^\infty \frac{\operatorname{sh}^{2\mu-1}x\operatorname{ch}x\,dx}{(1+a\operatorname{sh}^2x)^\nu} = \frac{1}{2}a^{-\mu}\mathrm{B}(\mu,\nu-\mu) \qquad [\operatorname{Re}\nu>\operatorname{Re}\mu>0, \quad a>0].$$

EH I 11(22)

4.[7] $$\int_0^\infty \frac{\operatorname{sh}^{\mu-1}x(\operatorname{ch}x+1)^{\nu-1}\,dx}{(\beta+\operatorname{ch}x)^\varrho} = 2^{\mu+\nu-\varrho-2}\mathrm{B}\left(\frac{1}{2}\mu,\ \varrho+2-\mu-\nu\right)\times$$
$$\times\ {}_2F_1\left(\varrho,\ \varrho+2-\mu-\nu;\varrho+2-\frac{1}{2}\mu-2;\ \frac{1}{2}-\frac{1}{2}\beta\right)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}(\varrho-\mu-\nu)>-2, \quad |\arg(1+\beta)|<\pi].$$

$$5.^6\quad \int_0^\infty \frac{\operatorname{sh}^{\mu-1}\ x(\operatorname{ch} x-1)^{\nu-1}\,dx}{(\beta+\operatorname{ch} x)^\varrho} = 2^{-(2-\mu-\nu+\varrho)}\ {}_2F_1\left(\varrho,\ 2-\mu-\nu+\varrho;\ 1+\varrho-\frac{\mu}{2};\ \frac{1-\beta}{2}\right) \times \mathrm{B}\left(2-\mu-\nu+\varrho,\ -1+\nu+\frac{\mu}{2}\right)$$
$$[\beta \notin (-\infty,\ -1),\quad \operatorname{Re}(2+\varrho) > \operatorname{Re}(\mu+\nu),\quad \operatorname{Re}(2\nu+\mu) > 2].$$

EH I 115(10)

$$6.^7\quad \int_0^\infty \frac{\operatorname{sh}^{\mu-1}\ x\ \operatorname{ch}^{\nu-1}\ x}{(\operatorname{ch}^2\ x-\beta)^\varrho}\,dx = \frac{1}{2}\,{}_2F_1\left(\varrho,\ 1+\varrho-\frac{\mu+\nu}{2};\ 1+\varrho-\frac{\nu}{2};\ \beta\right)\mathrm{B}\left(\frac{\mu}{2},\ 1+\varrho-\frac{\mu+\nu}{2}\right)$$
$$[\beta \notin (1,\ \infty),\quad \operatorname{Re}\mu > 0,\quad 2\ \operatorname{Re}(1+\varrho) > \operatorname{Re}(\mu+\nu)].$$

EH I 115(9)

3.519

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{sh}[(r-p)\ \operatorname{tg} x]}{\operatorname{sh}(r\ \operatorname{tg} x)}\,dx = \pi\sum_{k=1}^\infty \frac{1}{k\pi+r}\ \sin\frac{pk\pi}{r} \qquad [p^2 < r^2].$$

BI ((274))(13)

3.52- 3.53 Combinations of hyperbolic functions and algebraic functions

3.521

$$1.\quad \int_0^\infty \frac{x\,dx}{\operatorname{sh} ax} = \frac{\pi^2}{4a^2} \qquad [a>0].$$

GW ((352))(2b)

$$2.\quad \int_0^\infty \frac{x\,dx}{\operatorname{ch} x} = 2\boldsymbol{G} = \pi\ \ln 2 - 4L\left(\frac{\pi}{4}\right) = 1.831931188\ldots$$

LI III 225(103a), BI((84))(1)a

$$3.\quad \int_1^\infty \frac{dx}{x\ \operatorname{sh} ax} = -2\sum_{k=0}^\infty \operatorname{Ei}[-(2k+1)a] \qquad [a>0].$$

LI ((104))(14)

$$4.\quad \int_1^\infty \frac{dx}{x\ \operatorname{ch} ax} = 2\sum_{k=0}^\infty (-1)^{k+1}\ \operatorname{Ei}[-(2k+1)a] \qquad [a>0].$$

LI ((104))(13)

3.522

1. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\ \mathrm{sh}\,ax} = \frac{\pi}{2ab} + \pi\sum_{k=1}^\infty \frac{(-1)^k}{ab+k\pi} \qquad [a>0, \quad b>0].$$

2. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\ \mathrm{sh}\,\pi x} = \frac{1}{2b} - \beta(b+1) \qquad [b>0].$$

BI((97))(16), GW((352))(8)

3. $$\int_0^\infty \frac{dx}{(b^2+x^2)\ \mathrm{ch}\,ax} = \frac{2\pi}{b}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2ab+(2k-1)\pi} \qquad [a>0, \quad b>0].$$

BI ((97))(5)

4. $$\int_0^\infty \frac{dx}{(b^2+x^2)\ \mathrm{ch}\,\pi x} = \frac{1}{b}\beta\left(b+\frac{1}{2}\right) \qquad [b>0].$$

BI ((97))(4)

5. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\ \mathrm{sh}\,\pi x} = \ln 2 - \frac{1}{2}.$$

BI ((97))(7)

6. $$\int_0^\infty \frac{dx}{(1+x^2)\ \mathrm{ch}\,\pi x} = 2 - \frac{\pi}{2}.$$

BI ((97))(1)

7. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\ \mathrm{sh}\,\frac{\pi x}{2}} = \frac{\pi}{2} - 1.$$

BI ((97))(8)

8. $$\int_0^\infty \frac{dx}{(1+x^2)\ \mathrm{ch}\,\frac{\pi x}{2}} = \ln 2.$$

9. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\,\operatorname{sh}\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}[\pi + 2\,\ln(\sqrt{2}+1)] - 2.$$

BI ((97))(9)

10. $$\int_0^\infty \frac{dx}{(1+x^2)\,\operatorname{ch}\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}[\pi - 2\,\ln(\sqrt{2}+1)].$$

BI ((97))(3)

3.523

1. $$\int_0^\infty \frac{x^{\beta-1}}{\operatorname{sh} ax}\,dx = \frac{2^\beta - 1}{2^{\beta-1}a^\beta}\Gamma(\beta)\zeta(\beta) \qquad [\operatorname{Re}\beta > 1, \quad a > 0].$$

WH

2. $$\int_0^\infty \frac{x^{2n-1}}{\operatorname{sh} ax}\,dx = \frac{2^{2n}-1}{2n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}| \qquad [a > 0, \quad n = 1, 2, \ldots].$$

WH, GW((352))(2a)

3. $$\int_0^\infty \frac{x^{\beta-1}}{\operatorname{ch} ax}\,dx = \frac{2}{(2a)^\beta}\Gamma(\beta)\Phi\left(-1,\,\beta,\,\frac{1}{2}\right) = \\ = \frac{2}{(2a)^\beta}\Gamma(\beta)\sum_{k=0}^{\infty}(-1)^k\left(\frac{2}{2k+1}\right)^\beta \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

EH I 35, ET I 322(1)

4. $$\int_0^\infty \frac{x^{2n}}{\operatorname{ch} ax}\,dx = \left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}| \qquad [a > 0].$$

BI((84))(12)A, GW((352))(1a)

5. $$\int_0^\infty \frac{x^2\,dx}{\operatorname{ch} x} = \frac{\pi^3}{8} \qquad (\text{cf. } 4.261\ 6.).$$

4.261
BI ((84))(3)

6. $$\int_0^\infty \frac{x^3\,dx}{\operatorname{sh} x} = \frac{\pi^4}{8} \qquad (\text{cf. } 4.262\ 1. \text{ and } 2.).$$

7. $$\int_0^\infty \frac{x^4\,dx}{\operatorname{ch} x} = \frac{5}{32}\pi^5.$$

BI ((84))(7)

8. $$\int_0^\infty \frac{x^5}{\operatorname{sh} x}\,dx = \frac{\pi^6}{4}.$$

BI ((84))(8)

9. $$\int_0^\infty \frac{x^6}{\operatorname{ch} x}\,dx = \frac{61}{128}\pi^7.$$

BI ((84))(9)

10. $$\int_0^\infty \frac{x^7}{\operatorname{sh} x}\,dx = \frac{17}{16}\pi^8.$$

BI ((84))(10)

11. $$\int_0^\infty \frac{\sqrt{x}\,dx}{\operatorname{ch} x} = \sqrt{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{(2k+1)^3}}.$$

BI ((98))(7)a

12. $$\int_0^\infty \frac{dx}{\sqrt{x}\,\operatorname{ch} x} = 2\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((98))(25)a

3.524

1. $$\int_0^\infty x^{\mu-1}\frac{\operatorname{sh}\beta x}{\operatorname{sh}\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\zeta\left[\mu,\ \frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right] - \zeta\left[\mu,\ \frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]\right\}$$
$$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > -1].$$

2. $$\int_0^\infty x^{2^m} \frac{\operatorname{sh} ax}{\operatorname{sh} bx}\, dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \operatorname{tg} \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((112))(20)a

3. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{sh} bx} \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}} \right\} \qquad [b > |a|, \quad p < 1].$$

BI ((131))(2)a

4. $$\int_0^\infty x^{2^m+1} \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \sec \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((112))(18)a

5. $$\int_0^\infty x^{\mu-1} \frac{\operatorname{ch} \beta x}{\operatorname{sh} \gamma x}\, dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta\left[\mu, \frac{1}{2}\left(1 - \frac{\beta}{\gamma}\right)\right] + \zeta\left[\mu, \frac{1}{2}\left(1 + \frac{\beta}{\gamma}\right)\right] \right\}$$
$$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > 1].$$

ET I 323(12)

6. $$\int_0^\infty x^{2m} \frac{\operatorname{ch} ax}{\operatorname{ch} bx}\, dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \sec \frac{a\pi}{2b} \qquad [b > |a|].$$

BI((112))(17)

7. $$\int_0^\infty \frac{\operatorname{ch} ax}{\operatorname{ch} bx} \cdot \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}} \right\}$$
$$[b > |a|, \quad p < 1].$$

BI((131))(1)a

8. $$\int_0^\infty x^{2m+1} \frac{\operatorname{ch} ax}{\operatorname{sh} bx}\, dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \operatorname{tg} \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((112))(19)a

9.[8] $$\int_0^\infty x^2 \frac{\operatorname{sh} ax}{\operatorname{sh} bx}\, dx = \frac{\pi^3}{4b^3} \sin \frac{a\pi}{2b} \sec^3 \frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((84))(18)

BI ((84))(18)

10. $$\int_0^\infty x^4 \frac{\operatorname{sh} ax}{\operatorname{sh} bx}\, dx = 8\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^5 \cdot \sin\frac{a\pi}{2b} \cdot \left(2 + \sin^2\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(17)a

11. $$\int_0^\infty x^6 \frac{\operatorname{sh} ax}{\operatorname{sh} bx}\, dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^7 \sin\frac{a\pi}{2b}\left(45 - 30\cos^2\frac{a\pi}{2b} + 2\cos^4\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(21)a

12. $$\int_0^\infty x \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \frac{\pi^2}{4b^2}\sin\frac{a\pi}{2b}\sec^2\frac{a\pi}{2b} \qquad [b > |a|].$$

BI ((84))(15)a

13. $$\int_0^\infty x^3 \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4 \sin\frac{a\pi}{2b} \cdot \left(6 - \cos^2\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(14)a

14. $$\int_0^\infty x^5 \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^6 \sin\frac{a\pi}{2b}\left(120 - 60\cos^2\frac{a\pi}{2b} + \cos^4\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(18)a

15. $$\int_0^\infty x^7 \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\, dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8 \sin\frac{a\pi}{2b}\left(5040 - 4200\cos^2\frac{a\pi}{2b} + 546\cos^4\frac{a\pi}{2b} - \cos^6\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(22)a

16. $$\int_0^\infty x \frac{\operatorname{ch} ax}{\operatorname{sh} bx}\, dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^2 \qquad [b > |a|].$$

BI ((84))(16)a

17. $$\int_0^\infty x^3 \frac{\operatorname{ch} ax}{\operatorname{sh} bx}\, dx = 2\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4 \left(1 + 2\sin^2\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

18. $$\int_0^\infty x^5 \frac{\operatorname{ch} ax}{\operatorname{sh} bx}\,dx = 8\left(\frac{\pi}{2b}\,\sec\,\frac{a\pi}{2b}\right)^6\left(15 - 15\,\cos^2\,\frac{a\pi}{2b} + 2\,\cos^4\,\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(19)a

19. $$\int_0^\infty x^7 \frac{\operatorname{ch} ax}{\operatorname{sh} bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8\left(315 - 420\cos^2\frac{a\pi}{2b} + 126\cos^4\frac{a\pi}{2b} - 4\cos^6\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI((82))(23)a

20. $$\int_0^\infty x^2 \frac{\operatorname{ch} ax}{\operatorname{ch} bx}\,dx = \frac{\pi^3}{8b^3}\left(2\,\sec^3\,\frac{a\pi}{2b} - \sec\,\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((84))(17)a

21. $$\int_0^\infty x^4 \frac{\operatorname{ch} ax}{\operatorname{ch} bx}\,dx = \left(\frac{\pi}{2b}\,\sec\,\frac{a\pi}{2b}\right)^5\left(24 - 20\,\cos^2\,\frac{a\pi}{2b} + \cos^4\,\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(16)a

22. $$\int_0^\infty x^6 \frac{\operatorname{ch} ax}{\operatorname{ch} bx}\,dx = \left(\frac{\pi}{2b}\,\sec\,\frac{a\pi}{2b}\right)^7\left(720 - 840\,\cos^2\,\frac{a\pi}{2b} + 182\,\cos^4\,\frac{a\pi}{2b} - \cos^6\,\frac{a\pi}{2b}\right) \qquad [b > |a|].$$

BI ((82))(20)a

23. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{ch} bx}\cdot\frac{dx}{x} = \ln\operatorname{tg}\left(\frac{a\pi}{4b} + \frac{\pi}{4}\right) \qquad [b > |a|].$$

BI ((95))(3)a

3.525

1. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{sh} \pi x}\cdot\frac{dx}{1+x^2} = -\frac{a}{2}\,\cos a + \frac{1}{2}\,\sin a\,\ln[2(1+\cos a)] \qquad [\pi \geq |a|].$$

BI ((97))(10)a

2. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{sh} \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\,\sin a + \frac{1}{2}\,\cos a\,\ln\frac{1-\sin a}{1+\sin a} \qquad [\pi \geq 2|a|].$$

BI ((97))(11)a

BI ((97))(12)a

4. $$\int_0^\infty \frac{\operatorname{ch} ax}{\operatorname{sh} \frac{\pi}{2}x} \cdot \frac{x\,dx}{1+x^2} = \frac{\pi}{2}\cos a - 1 + \frac{1}{2}\sin a \,\ln\frac{1+\sin a}{1-\sin a} \qquad \left[\frac{\pi}{2} > |a|\right].$$

BI ((97))(13)a

5. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{ch} \pi x} \cdot \frac{x\,dx}{1+x^2} = -2\sin\frac{a}{2} + \frac{\pi}{2}\sin a - \cos a \,\ln\operatorname{tg}\frac{a+\pi}{4} \qquad [\pi > |a|].$$

GW ((352))(12)

6. $$\int_0^\infty \frac{\operatorname{ch} ax}{\operatorname{ch} \pi x} \cdot \frac{dx}{1+x^2} = 2\cos\frac{a}{2} - \frac{\pi}{2}\cos a - \sin a \,\ln\operatorname{tg}\frac{a+\pi}{4} \qquad [\pi > |a|].$$

GW ((352))(11)

7. $$\int_0^\infty \frac{\operatorname{sh} ax}{\operatorname{sh} bx} \cdot \frac{dx}{c^2+x^2} = \frac{\pi}{c}\sum_{k=1}^\infty \frac{\sin\frac{k(b-a)}{b}\pi}{bc+k\pi} \qquad [b \geq |a|].$$

BI ((97))(18)

8. $$\int_0^\infty \frac{\operatorname{ch} ax}{\operatorname{sh} bx} \cdot \frac{x\,dx}{c^2+x^2} = \frac{\pi}{2bc} + \pi\sum_{k=1}^\infty \frac{\cos\frac{k(b-a)}{b}\pi}{bc+k\pi} \qquad [b > |a|].$$

BI ((97))(19)

3.526

1. $$\int_0^\infty \frac{\operatorname{sh} ax \,\operatorname{ch} bx}{\operatorname{ch} cx} \cdot \frac{dx}{x} = \frac{1}{2}\ln\left\{\operatorname{tg}\frac{(a+b+c)\pi}{4c}\,\operatorname{ctg}\frac{(b+c-a)\pi}{4c}\right\} \qquad [c > |a|+|b|].$$

BI ((93))(10)a

2. $$\int_0^\infty \frac{\operatorname{sh}^2 ax}{\operatorname{sh} bx} \cdot \frac{dx}{x} = \frac{1}{2}\ln\sec\frac{a}{b}\pi \qquad [b > |2a|].$$

BI ((95))(5)a

3. $$\int_0^\infty \frac{x^{\mu-1}}{\operatorname{sh}\beta x\,\operatorname{ch}\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\Phi\left[-1,\,\mu,\,\frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right] + \Phi\left[-1,\,\mu,\,\frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right]\right\}$$
$$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > 0].$$

3.527

1. $$\int_0^\infty \frac{x^{\mu-1}}{\operatorname{sh}^2\, ax}\, dx = \frac{4}{(2a)^\mu}\Gamma(\mu)\zeta(\mu-1) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\mu > 2].$$

BI ((86))(7)a

2. $$\int_0^\infty \frac{x^{2m}}{\operatorname{sh}^2\, ax}\, dx = \frac{\pi^{2m}}{a^{2m+1}}|B_{2m}| \qquad [a > 0, \quad m = 1,\, 2, \dots].$$

BI((86))(5)a

3.[6] $$\int_0^\infty \frac{x^{\mu-1}}{\operatorname{ch}^2\, ax}\, dx = \frac{4}{(2a)^\mu}(1-2^{2-\mu})\Gamma(\mu)\zeta(\mu-1)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\mu > 0, \quad \mu \neq 2; \quad \text{integral equals}\,(1/a^2)\,\ln 2 \text{if} \mu = 2].$$

BI ((86))(6)a

396

4. $$\int_0^\infty \frac{x\, dx}{\operatorname{ch}^2\, ax} = \frac{\ln 2}{a^2} \qquad [a \neq 0].$$

LO III 396

5. $$\int_0^\infty \frac{x^{2m}}{\operatorname{ch}^2\, ax}\, dx = \frac{(2^{2m}-2)\pi^{2m}}{(2a)^{2m}a}|B_{2m}| \qquad [a > 0, \quad m = 1,\, 2, \dots].$$

BI((86))(2)a

6. $$\int_0^\infty x^{\mu-1}\frac{\operatorname{sh} ax}{\operatorname{ch}^2\, ax}\, dx = \frac{2\Gamma(\mu)}{a^\mu}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}} \qquad [\operatorname{Re}\mu > 1, \quad a > 0].$$

BI ((86))(15)a

7. $$\int_0^\infty \frac{x\, \operatorname{sh} ax}{\operatorname{ch}^2\, ax}\, dx = \frac{\pi}{2a^2} \qquad [a > 0].$$

BI ((86))(8)a

8. $$\int_0^\infty x^{2m+1}\frac{\operatorname{sh} ax}{\operatorname{ch}^2\, ax}\, dx = \frac{2m+1}{a}\left(\frac{\pi}{2a}\right)^{2m+1}|E_{2m}| \qquad [a > 0, \quad m = 0,\, 1, \dots].$$

9. $$\int_0^\infty x^{2m+1} \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax}\, dx = \frac{2^{2m+1}-1}{a^2(2a)^{2m}}(2m+1)!\zeta(2m+1) \qquad [a \neq 0, \quad m = 1, 2, \dots].$$

BI ((86))(13)a

10. $$\int_0^\infty x^{2m} \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax}\, dx = \frac{2^{2m}1}{a}\left(\frac{\pi}{a}\right)^{2m} |B_{2m}| \qquad [a > 0, \quad m = 1, 2, \dots].$$

BI ((86))(14)a

11. $$\int_0^\infty \frac{x \operatorname{sh} ax}{\operatorname{ch}^{2\mu+1} ax}\, dx = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\left(\mu + \frac{1}{2}\right)} \qquad [\mu > 0, \quad a > 0].$$

LI ((86))(9)

12. $$\int_{-\infty}^\infty \frac{x^2\, dx}{\operatorname{sh}^2 x} = \frac{\pi^2}{3}.$$

BI ((102))(2)a

13. $$\int_0^\infty x^2 \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax}\, dx = \frac{\pi^2}{2a^3} \qquad [a > 0].$$

BI ((86))(11)a

14. $$\int_0^\infty x^2 \frac{\operatorname{sh} ax}{\operatorname{ch}^2 ax}\, dx = \frac{\ln 2}{2a^3} \qquad [a \neq 0].$$

BI ((86))(10)a

15.* $$\int_0^\infty \frac{\theta \frac{x}{2}\, dx}{\operatorname{ch} x} = \ln 2.$$

BI ((93))(17)a

3.528

1. $$\int_0^\infty \frac{(1+xi)^{2n-1} - (1-xi)^{2n-1}}{i \operatorname{sh} \frac{\pi x}{2}}\, dx = 2.$$

2. $\int_0^\infty \frac{(1+xi)^{2n} - (1-xi)^{2n}}{i \text{ sh } \frac{\pi x}{2}} \, dx = (-1)^{n+1} 2|E_{2n}| + 2 \qquad [n = 0, 1, \dots].$

BI ((87))(7)

3.529

1. $\int_0^\infty \left(\frac{1}{\text{sh } x} - \frac{1}{x}\right) \frac{dx}{x} = -\ln 2.$

BI ((94))(10)a

2. $\int_0^\infty \frac{\text{ch } ax - 1}{\text{sh } bx} \cdot \frac{dx}{x} = -\ln \cos \frac{a\pi}{2b} \qquad [b > |a|].$

GW ((352))(66)

3. $\int_0^\infty \left(\frac{a}{\text{sh } ax} - \frac{b}{\text{sh } bx}\right) \frac{dx}{x} = (b-a) \ln 2.$

BI ((94))(11)a

3.531

1.[7] $\int_0^\infty \frac{x \, dx}{2 \text{ ch } x - 1} = \frac{4}{\sqrt{3}} \left[\frac{\pi}{3} \ln 2 - L(\pi/3)\right] \qquad 1.1719536193\dots$

LI ((88))(1)

2.* $\int_0^\infty \frac{x \, dx}{\text{ch } 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t}.$

LO III 402

3. $\int_0^\infty \frac{x^2 \, dx}{\text{ch } x + \cos t} = \frac{t}{3} \cdot \frac{\pi^2 - t^2}{\sin t} \qquad [0 < t < \pi].$

BI ((88))(3)a

4. $\int_0^\infty \frac{x^4 \, dx}{\text{ch } x + \cos t} = \frac{t}{15} \frac{(\pi^2 - t^2)(7\pi^2 - 3t^2)}{\sin t} \qquad [0 < t < \pi].$

5.[3] $$\int_0^\infty \frac{x^{2m}\,dx}{\operatorname{ch} x - \cos 2a\pi} = 2(2m)!\ \operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \qquad \left[0 < a < 1, \quad a \neq \frac{1}{2}\right]$$
$$= 2(2^{2m-1} - 1)\pi^{2m} |B_{2m}| \qquad \left[a = \frac{1}{2}\right].$$

BI ((88))(5)a

6.[3] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\operatorname{ch} x - \cos t} = \frac{i\Gamma(\mu)}{\sin t}[e^{-it}\Phi(e^{-it}, \mu, 1) - e^{it}\Phi(e^{it}, \mu, 1)]$$
$$[\operatorname{Re}\mu > 0, \quad 0 < t < 2\pi, \quad t \neq \pi].$$
$$= (2 - 2^{3-\mu})\Gamma(\mu)\zeta(\mu - 1) \qquad [\mu \neq 2, \quad t = \pi];$$
$$= 2\ \ln 2 \qquad [\mu = 2, \quad t = \pi].$$

ET I 323(5)

7. $$\int_0^\infty \frac{x^\mu\,dx}{\operatorname{ch} x + \cos t} = \frac{2\Gamma(\mu+1)}{\sin t} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin kt}{k^{\mu+1}} \qquad [\mu > -1, \quad 0 < t < \pi].$$

BII ((96))(14)a

8. $$\int_0^u \frac{x\,dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2}\ \operatorname{cosec} 2t[L(\theta+t) - L(\theta-t) - 2L(t)] \qquad [\theta = \operatorname{arctg}(\theta u\ \operatorname{ctg} t), \quad t \neq n\pi].$$

LO III 402

3.532

1. $$\int_0^\infty \frac{x^n\,dx}{a\ \operatorname{ch} x + b\ \operatorname{sh} x} = \frac{(2n)!}{a+b} \sum_{k=0}^\infty \frac{1}{(2k+1)^{n+1}} \left(\frac{b-a}{b+a}\right)^k \qquad [a > 0, \quad b > 0, \quad n > -1].$$

GW ((352))(5)

2. $$\int_0^u \frac{x\ \operatorname{ch} x\,dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2}\ \operatorname{cosec} t \left\{ L\left(\frac{\theta+t}{2}\right) - L\left(\frac{\theta-t}{2}\right) + L\left(\pi - \frac{\psi+t}{2}\right) \right.$$
$$\left. + L\left(\frac{\psi-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right\}$$
$$\left[\operatorname{tg}\frac{\theta}{2} = \theta\frac{u}{2}\ \operatorname{ctg}\frac{t}{2}, \quad \operatorname{tg}\frac{\psi}{2} = \operatorname{cth}\ \frac{u}{2}\ \operatorname{ctg}\frac{t}{2}; \quad t \neq n\pi\right].$$

LO III 288a

3.533

1. $$\int_0^\infty \frac{x \operatorname{ch} x\, dx}{\operatorname{ch} 2x - \cos 2t} = \operatorname{cosec} t \left[\frac{\pi}{2} \ln 2 - L\left(\frac{t}{2}\right) - L\left(\frac{(\pi - t)}{2}\right)\right] \qquad [t \neq m\pi].$$

LO III 403

2.[6] $$\int_0^\infty x \frac{\operatorname{sh} ax\, dx}{(\operatorname{ch} ax - \cos t)^2} = \frac{\pi - t}{a^2} \operatorname{cosec} t \qquad [a > 0, \quad 0 < t < \pi] \qquad (\text{cf. } 3.5141.).$$

3.514
BI ((88))(11)a

3. $$\int_0^\infty x^3 \frac{\operatorname{sh} x\, dx}{(\operatorname{ch} x + \cos t)^2} = \frac{t(\pi^2 - t^2)}{\sin t} \qquad [0 < t < \pi] \qquad (\text{cf. } 3.531\ 3.).$$

3.531
BI ((88))(13)

4.* $$\int_0^\infty x^{2^m+1} \frac{\operatorname{sh} x\, dx}{(\operatorname{ch} x - \cos 2a\pi)^2} = 2(2m+1)! \operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \qquad [0 < a < 1, \quad a \neq \frac{1}{2}].$$
$$= 2(2m+1)(2^{2m-1} - 1)\pi^{2m}|B_{2m}|, \qquad [a = \frac{1}{2}].$$

BI ((88))(14)

3.534

1. $$\int_0^1 \sqrt{1 - x^2} \operatorname{ch} ax\, dx = \frac{\pi}{2a} I_1(a).$$

WA 94(9)

2. $$\int_0^1 \frac{\operatorname{ch} ax}{\sqrt{1 - x^2}}\, dx = \frac{\pi}{2} I_0(a).$$

WA 94(9)

3.535

$$\int_0^1 \frac{x}{\sqrt{\operatorname{ch} 2a - \operatorname{ch} 2ax}} \cdot \frac{dx}{\operatorname{sh} ax} = \frac{\pi}{2\sqrt{2}a^2} \cdot \frac{\arcsin(\operatorname{th} a)}{\operatorname{sh} a} \qquad [a > 0].$$

3.536

1. $$\int_0^\infty \frac{x^2}{\operatorname{ch} x^2}\,dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

BI ((98))(7)

2. $$\int_0^\infty \frac{x^2\,\theta x^2\,dx}{\operatorname{ch} x^2} = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((98))(8)

3. $$\int_0^\infty \operatorname{sh}(\nu\ \operatorname{Arsh} x)\frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx = \frac{\sin\frac{\mu\pi}{2}\ \sin\frac{\nu\pi}{2}}{2^\mu\pi}\Gamma(\mu)\Gamma\left(\frac{1-\mu-\nu}{2}\right)\times\Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$$[-1<\operatorname{Re}\mu<1-|\operatorname{Re}\nu|].$$

ET I 324(14)

4. $$\int_0^\infty \operatorname{ch}(\nu\ \operatorname{Arch} x)\frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx = \frac{\cos\dfrac{\mu\pi}{2}\ \cos\dfrac{\nu\pi}{2}}{2^\mu\pi}\Gamma(\mu)\Gamma\left(\frac{1-\mu-\nu}{2}\right)\times\Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$$[0<\operatorname{Re}\mu<1-|\operatorname{Re}\nu|].$$

ET I 324(15)

3.54 Combinations of hyperbolic functions and exponentials

3.541

1. $$\int_0^\infty e^{-\mu x}\ \operatorname{sh}^\nu\ \beta x\,dx = \frac{1}{2^{\nu+1}\beta}\,\mathrm{B}\left(\frac{\mu}{2\beta}-\frac{\nu}{2},\ \nu+1\right) \qquad [\operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-1,\ \operatorname{Re}\mu>\operatorname{Re}\beta\nu].$$

EH I 11(25), ET I 163(5)

2. $$\int_0^\infty e^{-\mu x}\frac{\operatorname{sh}\beta x}{\operatorname{sh} bx}\,dx = \frac{1}{2b}\left[\psi\left(\frac{1}{2}+\frac{\mu+\beta}{2b}\right)-\psi\left(\frac{1}{2}+\frac{\mu-\beta}{2b}\right)\right] \qquad [\operatorname{Re}(\mu+b\pm\beta)>0].$$

EH I 16(14)a

3. $$\int_{-\infty}^\infty e^{-\mu x}\frac{\operatorname{sh}\mu x}{\operatorname{sh}\beta x}\,dx = \frac{\pi}{2\beta}\ \operatorname{tg}\frac{\mu\pi}{\beta} \qquad [\operatorname{Re}\beta>2|\operatorname{Re}\mu|].$$

BI ((18))(6)

4. $\int_0^\infty e^{-x} \frac{\operatorname{sh} ax}{\operatorname{sh} x}\, dx = \frac{1}{a} - \frac{\pi}{2} \operatorname{ctg} \frac{a\pi}{2} \qquad [0 < a < 2].$

BI ((4))(3)

5. $\int_0^\infty \frac{e^{-px}\, dx}{(\operatorname{ch} px)^{2q+1}} = \frac{2^{2q-2}}{p} \mathrm{B}\,(q,\, q) - \frac{1}{2qp} \qquad [p > 0, \quad q > 0].$

LI ((27))(19)

6. $\int_0^\infty e^{-\mu x} \frac{dx}{\operatorname{ch} x} = \beta \left(\frac{\mu + 1}{2} \right) \qquad [\operatorname{Re} \mu > -1].$

ET I 163(7)

7. $\int_0^\infty e^{-\mu x}\, \theta x\, dx = \beta \left(\frac{\mu}{2} \right) - \frac{1}{\mu} \qquad [\operatorname{Re} \mu > 0].$

ET I 163(9)

8. $\int_0^\infty \frac{e^{-\mu x}}{\operatorname{ch}^2\, x}\, dx = \mu\beta \left(\frac{\mu}{2} \right) - 1 \qquad [\operatorname{Re} \mu > 0].$

ET I 163(8)

9. $\int_0^\infty e^{-\mu x} \frac{\operatorname{sh} \mu x}{\operatorname{ch}^2\, \mu x}\, dx = \frac{1}{\mu}(1 - \ln 2) \qquad [\operatorname{Re} \mu > 0].$

LI ((27))(15)

10. $\int_0^\infty e^{-qx} \frac{\operatorname{sh} px}{\operatorname{sh} qx}\, dx = \frac{1}{p} - \frac{\pi}{2q} \operatorname{ctg} \frac{p\pi}{2q} \qquad [0 < p < 2q].$

BI ((27))(9)a

3.542

1. $\int_0^\infty e^{-\mu x} (\operatorname{ch} \beta x - 1)^\nu\, dx = \frac{1}{2^\nu \beta} \mathrm{B} \left(\frac{\mu}{\beta} - \nu,\, 2\nu + 1 \right)$

$$\left[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > \operatorname{Re} \beta\nu \right].$$

2. $$\int_0^\infty e^{-\mu x}(\operatorname{ch} x - \operatorname{ch} u)^{\nu-1}\,dx = -i\sqrt{\frac{2}{\pi}}e^{i\pi\nu}\Gamma(\nu)\ \operatorname{sh}^{\nu-\frac{1}{2}}\ uQ^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\operatorname{ch} u)$$
$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > \operatorname{Re}\nu - 1].$$

EH I 155(4), ET I 164(23)

3.543

1. $$\int_{-\infty}^\infty \frac{e^{-ibx}\,dx}{\operatorname{sh} x + \operatorname{sh} t} = -\frac{i\pi e^{itb}}{\operatorname{sh}\pi b\ \operatorname{ch} t}(\operatorname{ch}\pi b - e^{-2itb}) \qquad [t > 0].$$

ET I 121(30)

2. $$\int_0^\infty \frac{e^{-\mu x}}{\operatorname{ch} x - \cos t}\,dx = 2\ \operatorname{cosec} t \sum_{k=1}^\infty \frac{\sin kt}{\mu + k} \qquad [\operatorname{Re}\mu > -1, \quad t \neq 2n\pi].$$

BI ((6))(10)a

3. $$\int_0^\infty \frac{1 - e^{-x}\ \cos t}{\operatorname{ch} x - \cos t}e^{-(\mu-1)x}\,dx = 2\sum_{k=0}^\infty \frac{\cos kt}{\mu + k} \qquad [\operatorname{Re}\mu > 0, \quad t \neq 2n\pi].$$

BI ((6))(9)a

4. $$\int_0^\infty \frac{e^{px} - \cos t}{(\operatorname{ch} px + \cos t)^2}\,dx = \frac{1}{p}\left(t\ \operatorname{cosec} t + \frac{1}{1 + \cos t}\right) \qquad [p > 0].$$

BI ((27))(26)a

3.544

$$\int_u^\infty \frac{\exp\left[-\left(n + \frac{1}{2}\right)x\right]}{\sqrt{2(\operatorname{ch} x - \operatorname{ch} u)}}\,dx = Q_n(\operatorname{ch} u), \qquad [u > 0].$$

EH II 181(33)

3.545

1. $$\int_0^\infty \frac{\operatorname{sh} ax}{e^{px} + 1}\,dx = \frac{\pi}{2p}\ \operatorname{cosec}\frac{a\pi}{p} - \frac{1}{2a} \qquad [p > a, \quad p > 0].$$

BI ((27))(3)

BI ((27))(9)

3.546

1. $$\int_0^\infty e^{-\beta x^2} \operatorname{sh} ax\, dx = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} \exp \frac{a^2}{4\beta} \Phi\left(\frac{a}{2\sqrt{\beta}}\right) \qquad [\operatorname{Re} \beta > 0].$$

ET I166(38)a

2. $$\int_0^\infty e^{-\beta x^2} \operatorname{ch} ax\, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \frac{a^2}{4\beta} \qquad [\operatorname{Re} \beta > 0].$$

FI II 720a

3. $$\int_0^\infty e^{-\beta x^2} \operatorname{sh}^2 ax\, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} - 1\right) \qquad [\operatorname{Re} \beta > 0].$$

ET I 166(40)

4. $$\int_0^\infty e^{-\beta x^2} \operatorname{ch}^2 ax\, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} + 1\right) \qquad [\operatorname{Re} \beta > 0].$$

ET I 166(41)

3.547

1. $$\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{sh} \gamma x\, dx = \frac{\pi}{2} \operatorname{ctg} \frac{\gamma\pi}{2} [J_\gamma(\beta) - \mathbf{J}_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + N_\gamma(\beta)] = \gamma S_{-1,\gamma}(\beta) \qquad [\operatorname{Re} \beta > 0].$$

WA 341(5), ET I 168(14)a

2. $$\int_0^\infty \exp(-\beta \operatorname{ch} x) \operatorname{sh} \gamma x \operatorname{sh} x\, dx = \frac{\gamma}{\beta} K_\gamma(\beta).$$

3. $$\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{ch} \gamma x\, dx = \frac{\pi}{2} \operatorname{tg} \frac{\pi\gamma}{2} [\mathbf{J}_\gamma(\beta) - J_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + N_\gamma(\beta)] = S_{0,\gamma}(\beta) \qquad [\operatorname{Re} \beta > 0, \quad \gamma \text{ not an integer}].$$

ET I 168(16)A, WA 341(4), EH II 84(50)

ET I 168(16)A, WA 201(5)

5. $\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{sh} \gamma x \operatorname{ch} x \, dx = \frac{\gamma}{\beta} S_{0,\gamma}(\beta) \qquad [\operatorname{Re} \beta > 0].$

ET I 168(7), EH II 85(51)

6. $\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{sh}[(2n+1)x] \operatorname{ch} x \, dx = O_{2n+1}(\beta) \qquad [\operatorname{Re} \beta > 0].$

ET I 167(5)

7. $\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{ch} \gamma x \operatorname{ch} x \, dx = \frac{1}{\beta} S_{1,\gamma}(\beta) \qquad [\operatorname{Re} \beta > 0].$

8. $\int_0^\infty \exp(-\beta \operatorname{sh} x) \operatorname{ch} 2nx \operatorname{ch} x \, dx = O_{2n}(\beta) \qquad [\operatorname{Re} \beta > 0].$

ET I 168(6)

9. $\int_0^\infty \exp(-\beta \operatorname{ch} x) \operatorname{sh}^{2\nu} x \, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) K_\nu(\beta) \qquad \left[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$

EH II 82(20)

10. $$\int_0^\infty \exp[-2(\beta \operatorname{cth} x + \mu x)] \operatorname{sh}^{2\nu} x \, dx = \frac{1}{4} \beta^{\frac{\nu-1}{2}} \Gamma(\mu - \nu) \times$$
$$\times [W_{-\mu+\frac{1}{2},\nu}(4\beta) - (\mu - \nu) W_{-\mu-\frac{1}{2},\nu}(4\beta)]$$
$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu].$$

ET I 165(31)

11. $\int_0^\infty \exp\left(-\frac{\beta^2}{2} \operatorname{sh} x\right) \operatorname{sh}^{\nu-1} x \operatorname{ch}^\nu x \, dx = -\pi D_\nu\left(\beta e^{\frac{i\pi}{4}}\right) D_\nu\left(\beta e^{-\frac{i\pi}{4}}\right) \quad \left[\operatorname{Re} \nu > 0, \ |\arg \beta| \le \frac{\pi}{4}\right].$

EH II 120(10)

12. $\int_0^\infty \frac{\exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}} \, dx = \frac{1}{2} \sqrt{\pi^3 \beta} [J_{\nu+\frac{1}{4}}(\beta) J_{\nu-\frac{1}{4}}(\beta) + N_{\nu+\frac{1}{4}}(\beta) N_{\nu-\frac{1}{4}}(\beta)] \qquad [\operatorname{Re} \beta > 0].$

13. $$\int_0^\infty \frac{\exp(-2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}}\, dx = \frac{1}{2}\sqrt{\pi^3\beta}[J_{\nu+\frac{1}{4}}(\beta)N_{\nu-\frac{1}{4}}(\beta) - J_{\nu-\frac{1}{4}}(\beta)N_{\nu+\frac{1}{4}}(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 169(21)

14. $$\int_0^\infty \frac{\exp(-2\beta \operatorname{sh} x)\operatorname{sh} 2\nu x}{\sqrt{\operatorname{sh} x}}\, dx = \frac{1}{4i}\sqrt{\frac{\pi^3\beta}{2}}[e^{\nu\pi i} H^{(1)}_{\frac{1}{2}+\nu}(\beta) H^{(2)}_{\frac{1}{2}-\nu}(\beta) - e^{-\nu\pi i} H^{(1)}_{\frac{1}{2}-\nu}(\beta) H^{(2)}_{\frac{1}{2}+\nu}(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 170(24)

15. $$\int_0^\infty \frac{\exp(-2\beta \operatorname{sh} x)\operatorname{ch} 2\nu x}{\sqrt{\operatorname{sh} x}}\, dx = \frac{1}{4}\sqrt{\frac{\pi^3\beta}{2}}[e^{\nu\pi i} H^{(1)}_{\frac{1}{2}+\nu}(\beta) H^{(2)}_{\frac{1}{2}-\nu}(\beta) + e^{-\nu\pi i} H^{(1)}_{\frac{1}{2}-\nu}(\beta) H^{(2)}_{\frac{1}{2}+\nu}(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 170(25)

16. $$\int_0^\infty \frac{\exp(-2\beta \operatorname{ch} x)\operatorname{ch} 2\nu x}{\sqrt{\operatorname{ch} x}}\, dx = \sqrt{\frac{\beta}{\pi}} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \qquad [\operatorname{Re}\beta > 0].$$

ET I 170(26)

17.[8] $$\int_0^\infty \frac{\exp[-2\beta(\operatorname{ch} x - 1)]\operatorname{ch} 2\nu x}{\sqrt{\operatorname{ch} x}}\, dx = \sqrt{\frac{\beta}{\pi}} \cdot e^{2\beta} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \qquad [\operatorname{Re}\beta > 0].$$

ET I 170(27)

18. $$\int_0^\infty \frac{\cos\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(-2\nu x - 2\beta \operatorname{sh} x) + \sin\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}}\, dx = \frac{1}{2}\sqrt{\pi^3\beta}[J_{\frac{1}{4}+\nu}(\beta)J_{\frac{1}{4}-\nu}(\beta) + N_{\frac{1}{4}+\nu}(\beta)N_{\frac{1}{4}-\nu}(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

ET I 169(22)

19. $$\int_0^\infty \frac{\sin\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(-2\nu x - 2\beta \operatorname{sh} x) - \cos\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}}\, dx = \frac{1}{2}\sqrt{\pi^3\beta}[J_{\frac{1}{4}+\nu}(\beta)N_{\frac{1}{4}-\nu}(\beta) - J_{\frac{1}{4}-\nu}(\beta)N_{\frac{1}{4}+\nu}(\beta)] \qquad [\operatorname{Re}\beta > 0].$$

20. $$\int_0^\infty \frac{\exp[-\beta(\operatorname{ch} x - 1)]\ \operatorname{ch} \nu x\ \operatorname{sh} x}{\sqrt{\operatorname{ch} x(\operatorname{ch} x - 1)}}\, dx = e^\beta K_\nu(\beta) \qquad [\operatorname{Re} \beta > 0].$$

ET I 169(19)

3.548

1. $$\int_0^\infty e^{-\mu x^4}\ \operatorname{sh} ax^2\, dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\ \exp\left(\frac{a^2}{8\mu}\right) I_{\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \qquad [\operatorname{Re} \mu > 0, \quad a \geq 0].$$

ET I 166(42)

2. $$\int_0^\infty e^{-\mu x^4}\ \operatorname{ch} ax^2\, dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\ \exp\left(\frac{a^2}{8\mu}\right) I_{-\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \qquad [\operatorname{Re} \mu > 0, \quad a > 0],$$

ET I 166(43)

3.549

1. $$\int_0^\infty e^{-\beta x}\ \operatorname{sh}[(2n{+}1)\ \operatorname{Arsh} x]\, dx = O_{2n+1}(\beta) \qquad [\operatorname{Re} \beta > 0] \qquad (\text{cf. } 3.547\ 6.).$$

3.547
ET I 167(5)

2. $$\int_0^\infty e^{-\beta x}\ \operatorname{ch}(2n\ \operatorname{Arsh} x)\, dx = O_{2n}(\beta) \qquad [\operatorname{Re} \beta > 0] \qquad (\text{cf. } 3.547\ 8.).$$

3.547
ET I 168(6)

3. $$\int_0^\infty e^{-\beta x}\ \operatorname{sh}(\nu\ \operatorname{Arsh} x)\, dx = \frac{\nu}{\beta} S_{0,\nu}(\beta) \qquad [\operatorname{Re} \beta > 0] \qquad (\text{cf. } 3.5475.).$$

3.547
ET I 168(7)

4. $$\int_0^\infty e^{-\beta x}\ \operatorname{ch}(\nu\ \operatorname{Arsh} x)\, dx = \frac{1}{\beta} S_{1,\nu}(\beta) \qquad [\operatorname{Re} \beta > 0] \qquad (\text{cf. } 3.547\ 7.).$$

A number of other integrals containing hyperbolic functions and exponentials, depending on Arsh x or $\operatorname{Arch} x$ can be found by first making the substitution $x = \operatorname{sh} t$ or $x = \operatorname{ch} t$.

3.55- 3.56 Combinations of hyperbolic functions, exponentials, and powers

3.551

1. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \operatorname{sh} \gamma x \, dx = \frac{1}{2}\Gamma(\mu)[(\beta-\gamma)^{-\mu} - (\beta+\gamma)^{-\mu}] \qquad [\operatorname{Re}\beta > -1, \quad \operatorname{Re}\beta > |\operatorname{Re}\gamma|].$$

ET I 164(18)

2. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \operatorname{ch} \gamma x \, dx = \frac{1}{2}\Gamma(\mu)[(\beta-\gamma)^{-\mu} + (\beta+\gamma)^{-\mu}] \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > |\operatorname{Re}\gamma|].$$

ET I 164(19)

3. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \operatorname{cth} x \, dx = \Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu, \frac{\beta}{2}\right) - \beta^{-\mu}\right] \qquad [\operatorname{Re}\mu > 1, \quad \operatorname{Re}\beta > 0].$$

ET I 164(21)

4. $$\int_0^\infty x^n e^{-(p+mq)x} \operatorname{sh}^m qx \, dx = 2^{-m} n! \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(p+2kq)^{n+1}} \qquad [p > 0, \quad q > 0, \quad m < p+qm].$$

LI ((81))(4)

5. $$\int_0^1 \frac{e^{-\beta x}}{x} \operatorname{sh} \gamma x \, dx = \frac{1}{2}\left[\ln\frac{\beta+\gamma}{\beta-\gamma} \operatorname{Ei}(\gamma-\beta) - \operatorname{Ei}(-\gamma-\beta)\right] \qquad [\beta > \gamma].$$

BI ((80))(4)

6. $$\int_0^\infty \frac{e^{-\beta x}}{x} \operatorname{sh} \gamma x \, dx = \frac{1}{2} \ln\frac{\beta+\gamma}{\beta-\gamma} \qquad [\operatorname{Re}\beta > |\operatorname{Re}\gamma|].$$

ET I 163(12)

7. $$\int_1^\infty \frac{e^{-\beta x}}{x} \operatorname{ch} \gamma x \, dx = \frac{1}{2}[-\operatorname{Ei}(\gamma-\beta) - \operatorname{Ei}(-\gamma-\beta)] \qquad [\operatorname{Re}\beta > |\operatorname{Re}\gamma|].$$

ET I 164(15)

8.[6] $$\int_0^\infty xe^{-x} \operatorname{cth} x\,dx = \frac{\pi^2}{4} - 1.$$

BI ((82))(6)

9. $$\int_0^\infty e^{-\beta x}\,\theta x \frac{dx}{x} = \ln\frac{\beta}{4} + 2\ln\frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4}+\frac{1}{2}\right)} \qquad [\operatorname{Re}\beta > 0].$$

ET I 164(16)

10.[6] $$\int_0^\infty xe^{-x} \operatorname{cth}(x/2)\,dx = \frac{\pi^2}{3} - 1.$$

3.552

1. $$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\operatorname{sh} x}\,dx = 2^{1-\mu}\Gamma(\mu)\zeta\left[\mu, \frac{1}{2}(\beta+1)\right] \qquad [\operatorname{Re}\mu > 1, \quad \operatorname{Re}\beta > -1].$$

ET I 164(20)

2. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\operatorname{sh} ax}\,dx = \frac{1}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m} \qquad [a > 0, \quad m = 1, 2, \ldots].$$

EH I 38(24)a

3. $$\int_0^\infty \frac{x^{\mu-1}e^{-x}}{\operatorname{ch} x}\,dx = 2^{1-\mu}(1-2^{1-\mu})\Gamma(\mu)\zeta(\mu) \qquad [\operatorname{Re}\mu > 0, \quad \mu \neq 1] \qquad (= \ln 2 \text{ if } \mu = 1).$$

EH I 32(5)

4. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\operatorname{ch} ax}\,dx = \frac{1-2^{1-2m}}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m} \qquad [a > 0, \quad m = 1, 2, \ldots].$$

EH I 39(25)a

4.261
BI((84))(4)

6. $$\int_0^\infty \frac{x^3 e^{-2nx}}{\operatorname{sh} x}\,dx = \frac{\pi^4}{8} - 12\sum_{k=1}^{n} \frac{1}{(2k+1)^4} \qquad [n = 0, 1, \dots]. \qquad \text{(cf. 4.262 6.).}$$

4.262
BI ((84))(6)

3.553

1. $$\int_0^\infty \frac{\operatorname{sh}^2\, ax}{\operatorname{sh} x}\,\frac{e^{-x}\,dx}{x} = \frac{1}{2}\,\ln(a\pi\,\operatorname{cosec} a\pi) \qquad [a < 1].$$

BI ((95))(7)

2. $$\int_0^\infty \frac{\operatorname{sh}^2\,\frac{\pi}{2}}{\operatorname{ch} x}\cdot\frac{e^{-x}\,dx}{x} = \frac{1}{2}\,\ln\frac{4}{\pi} \qquad \text{(cf. 4.267 2.).}$$

4.267
BI ((95))(4)

3.554

1. $$\int_0^\infty e^{-\beta x}(1 - \operatorname{sech}\, s)\frac{dx}{x} = 2\,\ln\frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)} - \ln\frac{\beta}{4} \qquad [\operatorname{Re}\beta > 0].$$

ET I 164(17)

2. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x} - \operatorname{cosech}\, x\right)\,dx = \psi\left(\frac{\beta+1}{2}\right) - \ln\frac{\beta}{2} \qquad [\operatorname{Re}\beta > 0].$$

ET I 163(10)

3. $$\int_0^\infty \left[\frac{\operatorname{sh}\left(\frac{1}{2}-\beta\right)x}{\operatorname{sh}\frac{x}{2}} - (1-2\beta)e^{-x}\right]\frac{dx}{x} = 2\,\ln\Gamma(\beta) - \ln\pi + \ln(\sin\pi\beta) \qquad [0 < \operatorname{Re}\beta < 1].$$

4. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x}-\operatorname{cth}\,x\right)dx=\psi\left(\frac{\beta}{2}\right)-\ln\frac{\beta}{2}+\frac{1}{\beta}\qquad [\operatorname{Re}\beta>0].$$

ET I 163(11)

5. $$\int_0^\infty\left\{-\frac{\operatorname{sh} qx}{\operatorname{sh}\frac{x}{2}}+2qe^{-x}\right\}\frac{dx}{x}=2\,\ln\Gamma\left(q+\frac{1}{2}\right)+\ln\cos\pi q-\ln\pi\qquad\left[q^2<\frac{1}{2}\right].$$

WH

6. $$\int_0^\infty x^{\mu-1}e^{-\beta x}(\operatorname{cth}\,x-1)\,dx=2^{1-\mu}\Gamma(\mu)\zeta\left(\mu,\,\frac{\beta}{2}+1\right)\qquad[\operatorname{Re}\beta>0;\quad\operatorname{Re}\mu>1].$$

ET I 164(22)

3.555

1. $$\int_0^\infty\frac{\operatorname{sh}^2\,ax}{1-e^{px}}\cdot\frac{dx}{x}=\frac{1}{4}\,\ln\left(\frac{p}{2a\pi}\,\sin\frac{2a\pi}{p}\right)\qquad[0<2|a|<p].\qquad(\text{cf. } 3.545\ 2.).$$

3.545
BI ((93))(15)

2. $$\int_0^\infty\frac{\operatorname{sh}^2\,ax}{e^x+1}\cdot\frac{dx}{x}=-\frac{1}{4}\,\ln(a\pi\,\operatorname{ctg}a\pi)\qquad\left[a<\frac{1}{2}\right]\qquad(\text{cf. } 3.545\ 1.).$$

3.545
BI ((93))(9)

3.556

1. $$\int_{-\infty}^\infty x\frac{1-e^{px}}{\operatorname{sh}x}\,dx=-\frac{\pi^2}{2}\,\operatorname{tg}^2\frac{p\pi}{2}\qquad[p<1]\qquad(\text{cf. } 4.255\ 3.).$$

4.255
BI ((101))(4)

2. $$\int_0^\infty\frac{1-e^{-px}}{\operatorname{sh}x}\cdot\frac{1-e^{-(p+1)x}}{x}\,dx=2p\,\ln 2\qquad[p>-1].$$

3.557

1. $$\int_0^\infty \frac{e^{-px}-e^{-qx}}{\operatorname{ch} x - \cos\frac{m}{n}\pi}\cdot\frac{dx}{x} = 2\operatorname{cosec}\frac{m}{n}\pi\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q+k}{2n}\right)\Gamma\left(\frac{p+k}{2n}\right)}{\Gamma\left(\frac{n+p+k}{2n}\right)\Gamma\left(\frac{q+k}{2n}\right)}$$
$$[m+n \quad \text{odd}];$$
$$= 2\operatorname{cosec}\frac{m}{n}\pi\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q-k}{n}\right)\Gamma\left(\frac{p+k}{n}\right)}{\Gamma\left(\frac{n+p-k}{n}\right)\Gamma\left(\frac{q+k}{n}\right)}$$
$$[m+n \quad \text{even}]; \qquad [p>-1, \quad q>-1].$$

BI ((96))(1)

2. $$\int_0^\infty \frac{(1-e^{-x})^2}{\operatorname{ch} x + \cos\frac{m}{n}\pi}\cdot\frac{dx}{x} =$$
$$= 2\operatorname{cosec}\frac{m}{n}\pi\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n+k+1}{2n}\right)\right]^2\Gamma\left(\frac{k+2}{2n}\right)\Gamma\left(\frac{k}{2n}\right)}{\left[\Gamma\left(\frac{k+1}{2n}\right)\right]^2\Gamma\left(\frac{n+k}{2n}\right)\Gamma\left(\frac{n+k+2}{2n}\right)}$$
$$[m+n \quad \text{odd}];$$
$$= 2\operatorname{cosec}\frac{m}{n}\pi\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n-k+1}{n}\right)\right]^2\Gamma\left(\frac{k+2}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\left[\Gamma\left(\frac{k+1}{n}\right)\right]^2\Gamma\left(\frac{n-k}{n}\right)\Gamma\left(\frac{n-k+2}{n}\right)}$$
$$[m+n \quad \text{even}].$$

BI ((96))(2)

3. $$\int_0^\infty \left[e^{-x}\operatorname{tg}\frac{m}{2n}\pi - \frac{e^{-px}\sin\frac{m}{n}\pi}{\operatorname{ch} x + \cos\frac{m}{n}\pi}\right]\cdot\frac{dx}{x} =$$
$$= \operatorname{tg}\left(\frac{m}{2n}\pi\right)\ln(2n) + 2\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n+k}{2n}\right)}{\Gamma\left(\frac{p+k}{2n}\right)} \qquad [m+n \quad \text{odd}];$$
$$= \operatorname{tg}\left(\frac{m}{2n}\pi\right)\ln n + 2\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n-k}{n}\right)}{\Gamma\left(\frac{p+k}{n}\right)} \qquad [m+n \quad \text{even}].$$

BI ((96))(3)

4. $$\int_0^\infty \frac{1+e^{-x}}{\operatorname{ch} x + \cos a} \cdot \frac{dx}{x^{1-p}} = 2 \sec \frac{a}{2} \Gamma(p) \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos\left(k-\frac{1}{2}\right) a}{k^p} \qquad [p > 0].$$

LI ((96))(5)

5. $$\int_0^\infty \frac{x^q e^{-\frac{x}{2}} \operatorname{ch} \frac{x}{2}}{\operatorname{ch} x + \cos \lambda}\, dx = \frac{\Gamma(q+1)}{\cos \frac{\lambda}{2}} \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos\left(k-\frac{1}{2}\right) \lambda}{k^{q+1}} \qquad [q > -1].$$

LI ((96))(5)a

6. $$\int_0^\infty x \frac{e^{-x} - \cos a}{\operatorname{ch} x - \cos a}\, dx = |a|\pi - \frac{a^2}{2} - \frac{\pi^2}{3}.$$

BI ((88))(8)

7. $$\int_0^\infty x^{2m+1} \frac{e^{-x} - \cos a\pi}{\operatorname{ch} x - \cos a\pi}\, dx = 2 \cdot (2m+1)! \sum_{k=1}^\infty \frac{\cos ka\pi}{k^{2m+2}}.$$

BI ((88))(6)

3.558

1. $$\int_0^\infty x \frac{1-e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}}\, dx = \frac{2n\pi^2}{3} - 4 \sum_{k=1}^{n-1} \frac{n-k}{k^2}.$$

BI ((85))(3)

2. $$\int_0^\infty x \frac{1-(-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}}\, dx = \frac{n\pi^2}{3} + 4 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^2}.$$

LI ((85))(1)

3. $$\int_0^\infty x^2 \frac{1-e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}}\, dx = 8n\zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}.$$

4. $\displaystyle\int_0^\infty x^2 e^x \frac{1-e^{-2nx}}{\operatorname{sh}^2 x}\,dx = 8n\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} - 8\sum_{k=1}^{n-1}\frac{n-k}{(2k-1)^3}.$

LI ((85))(6)

5. $\displaystyle\int_0^\infty x^2 \frac{1+(-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}}\,dx = 6n\zeta(3) - 8\sum_{k=1}^{n-1}\frac{n-k}{k^3}.$

LI ((85))(4)

6. $\displaystyle\int_0^\infty x^3 \frac{1-e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}}\,dx = \frac{4}{15}n\pi^4 - 24\sum_{k=1}^{n-1}\frac{n-k}{k^4}.$

BI ((85))(9)

7. $\displaystyle\int_0^\infty x^3 \frac{1+(-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}}\,dx = \frac{7}{30}n\pi^4 + 24\sum_{k=1}^{n-1}(-1)^k\frac{n-k}{k^4}.$

BI ((85))(8)

3.559

$$\int_0^\infty e^{-x}\left[a-\frac{1}{2}+\frac{(1-e^{-x})(1-ax)-xe^{-x}}{4\operatorname{sh}^2\frac{x}{2}}e^{(2-a)x}\right]\frac{dx}{x} = a-\frac{1}{2}+\ln\Gamma(a)-\frac{1}{2}\ln(2\pi) \qquad [a>0].$$

BI ((96))(6)

3.561

$$\int_0^\infty \frac{e^{-2x}\,\theta\frac{x}{2}}{x\operatorname{ch} x}\,dx = 2\ln\frac{\pi}{2\sqrt{2}}.$$

BI ((93))(18)

3.562

1. $\displaystyle\int_0^\infty x^{2\mu-1}e^{-\beta x^2}\operatorname{sh}\gamma x\,dx = \frac{1}{2}\Gamma(2\mu)(2\beta)^{-\mu}\exp\left(\frac{\gamma^2}{8\beta}\right)\left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right)-D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right]$
$\displaystyle\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\beta > 0\right].$

ET I 166(44)

ET I 166(45)

3. $$\int_0^\infty xe^{-\beta x^2} \operatorname{sh} \gamma x \, dx = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}} \exp \frac{\gamma^2}{4\beta} \qquad [\operatorname{Re} \beta > 0].$$

BI((81))(12)A,ET I 165(34)

4. $$\int_0^\infty xe^{-\beta x^2} \operatorname{ch} \gamma x \, dx = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}} \exp \frac{\gamma^2}{4\beta} \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2\beta} \qquad [\operatorname{Re} \beta > 0].$$

ET I 166(35)

5. $$\int_0^\infty x^2 e^{-\beta x^2} \operatorname{sh} \gamma x \, dx = \frac{\sqrt{\pi}(2\beta + \gamma^2)}{8\beta^2\sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{\gamma}{4\beta^2} \qquad [\operatorname{Re} \beta > 0].$$

ET I 166(36)

6. $$\int_0^\infty x^2 e^{-\beta x^2} \operatorname{ch} \gamma x \, dx = \frac{\sqrt{\pi}(2\beta + \gamma^2)}{8\beta^2\sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \qquad [\operatorname{Re} \beta > 0].$$

ET I 166(37)

3.6- 4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

1. $$\int_0^{2\pi} (1 - \cos x)^n \sin nx \, dx = 0.$$

BI ((68))(10)

2. $$\int_0^{2\pi} (1 - \cos x)^n \cos nx \, dx = (-1)^n \frac{\pi}{2^{n-1}}.$$

3. $\int_0^{\pi} (\cos t + i \sin t \cos x)^n \, dx = \int_0^{\pi} (\cos t + i \sin t \cos x)^{-n-1} \, dx = \pi P_n(\cos t).$

EH I 158(23)a

3.612

1.[6] $\int_0^{\pi} \frac{\sin nx \cos mx}{\sin x} \, dx = 0 \qquad \text{for} \quad n \le m;$

$= \pi \qquad \text{for} \quad n > m \qquad \text{if} \quad m + n \text{ is odd and positive}$

$= 0 \qquad \text{for} \quad n > m, \qquad \text{if} \quad m + n \text{ is even.}$

LI ((64))(3)

2. $\int_0^{\pi} \frac{\sin nx}{\sin x} \, dx = 0 \qquad \text{for} \quad n \text{ even};$

$= \pi \qquad \text{for} \quad n \text{ odd}.$

BI ((64))(1, 2)

3. $\int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} \, dx = \frac{\pi}{2}.$

FI II 145

4. $\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x} \, dx = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1} \right).$

GW ((332))(21b)

5. $\int_0^{\pi} \frac{\sin 2nx}{\cos x} \, dx = 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\cos x} \, dx = (-1)^{n-1} 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} \right).$

GW ((332))(22a)

6. $\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x} \, dx = 2 \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} \, dx = (-1)^n \pi.$

GW ((332))(22b)

7.[7] $\int_0^{\frac{\pi}{2}} \frac{\sin 2nx \cos^{2m+1} x}{\sin x} \, dx = \frac{\pi}{2} \qquad [n > m \ge 0]$

3.613

1.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1+a\,\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad [a^2<1, \quad n\geq 0].$$

BI ((64))(12)

2.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1-2a\,\cos x+a^2} = \frac{\pi a^n}{1-a^2} \qquad [a^2<1, \quad n\geq 0].$$
$$= \frac{\pi}{(a^2-1)a^n} \qquad [a^2>1, \quad n\geq 0].$$

BI ((65))(3)

3. $$\int_0^\pi \frac{\sin nx\,\sin x\,dx}{1-2a\,\cos x+a^2} = \frac{\pi}{2}a^{n-1} \qquad [a^2<1, \quad n\geq 1];$$
$$= \frac{\pi}{2a^{n+1}} \qquad [a^2>1, \quad n\geq 1].$$

BI((65))(4), GW((332))(34a)

4.* $$\int_0^\pi \frac{\cos nx\,\cos x\,dx}{1-2a\,\cos x+a^2} = \frac{\pi}{2}\cdot\frac{1+a^2}{1-a^2}a^{n-1} \qquad [a^2<1, \quad n\geq 1];$$
$$= \frac{\pi}{2a^{n+1}}\cdot\frac{a^2+1}{a^2-1} \qquad [a^2>1, \quad n\geq 1];$$
$$= \frac{\pi a}{1-a^2} \qquad [n=0, \quad a^2<1].$$
$$= \frac{\pi}{a(a^2-1)} \qquad [n=0, \quad a^2>1].$$

BI((65))(5), GW((332))(34b)

5. $$\int_0^\pi \frac{\cos(2n-1)x\,dx}{1-2a\,\cos 2x+a^2} = \int_0^\pi \frac{\cos 2nx\,\cos x\,dx}{1-2a\,\cos 2x+a^2} = 0 \qquad [a^2\neq 1].$$

BI ((65))(9, 10)

6. $$\int_0^\pi \frac{\cos(2n-1)x\,\cos 2x\,dx}{1-2a\,\cos 2x+a^2} = 0 \qquad [a^2\neq 1].$$

BI ((65))(12)

7. $$\int_0^\pi \frac{\sin 2nx\,\sin x\,dx}{1-2a\,\cos 2x+a^2} = \int_0^\pi \frac{\sin(2n-1)x\,\sin 2x\,dx}{1-2a\,\cos 2x+a^2} = 0 \qquad [a^2\neq 1].$$

8. $$\int_0^\pi \frac{\sin(2n-1)x \sin x \, dx}{1-2a \cos 2x + a^2} = \frac{\pi}{2} \cdot \frac{a^{n-1}}{1+a} \qquad [a^2 < 1];$$
$$= \frac{\pi}{2} \cdot \frac{1}{(1+a)a^n} \qquad [a^2 > 1].$$

BI ((65))(8)

9. $$\int_0^\pi \frac{\cos(2n-1)x \cos x \, dx}{1-2a \cos 2x + a^2} = \frac{\pi}{2} \cdot \frac{a^{n-1}}{1-a} \qquad [a^2 < 1];$$
$$= \frac{\pi}{2} \cdot \frac{1}{(a-1)a^n} \qquad [a^2 > 1].$$

BI ((65))(11)

10. $$\int_0^\pi \frac{\sin nx - a \sin(n-1)x}{1-2a \cos x + a^2} \sin mx \, dx = 0 \qquad \text{for} \quad m < n;$$
$$= \frac{\pi}{2} a^{m-n} \qquad \text{for} \quad m \geq n; \qquad [a^2 < 1].$$

LI ((65))(13)

410

11.[6] $$\int_0^\pi \frac{\cos nx - a \cos(n-1)x}{1-2a \cos x + a^2} \cos mx \, dx = \frac{\pi}{2}(a^{|m|-n} - 1) \qquad [a^2 < 1].$$

BI ((65))(14)

12. $$\int_0^\pi \frac{\sin nx - a \sin[(n+1)x]}{1-2a \cos x + a^2} \, dx = 0 \qquad [a^2 < 1].$$

BI ((68))(13)

13. $$\int_0^\pi \frac{\cos nx - a \cos[(n+1)x]}{1-2a \cos x + a^2} \, dx = \pi a^n \qquad [a^2 < 1].$$

BI ((68))(14)

3.614[7]

$$\int_0^\pi \frac{\sin x}{a^2 - 2ab \cos x + b^2} \cdot \frac{\sin px \cdot dx}{1 - 2a^p \cos px + a^{2p}} =$$
$$= \frac{\pi b^{p-1}}{2a^{p+1}(1-b^p)} \qquad [0 < b \leq a \leq 1, \quad p = 1, 2, 3, \ldots];$$
$$= \frac{\pi a^{p-1}}{2b(b^p - a^{2p})} \qquad [0 < a \leq 1, \quad a^2 < b, a^2 \quad p = 1, 2, 3, \ldots].$$

1. $$\int_0^{\pi/2} \frac{\cos 2nx\,dx}{1 - a^2\,\sin^2\,x} = \frac{(-1)^n \pi}{2\sqrt{1-a^2}} \left(\frac{1 - \sqrt{1-a^2}}{a} \right)^{2n} \qquad [a^2 < 1].$$

BI ((47))(27)

2. $$\int_0^{\pi} \frac{\cos x\,\sin 2nx\,dx}{1 + (a + b\,\sin x)^2} = -\frac{\pi}{b}\,\sin\left\{ 2n\,\operatorname{arctg}\,\sqrt{\frac{s}{2}} \right\}\,\operatorname{tg}^{2n}\,\left(\frac{1}{2}\,\arccos\sqrt{\frac{s}{2a^2}} \right).$$

3. $$\int_0^{\pi} \frac{\cos x\,\cos(2n+1)x\,dx}{1 + (a + b\,\sin x)^2} = \frac{\pi}{b}\,\cos\left\{ (2n+1)\,\operatorname{arctg}\,\sqrt{\frac{s}{2}} \right\}\,\operatorname{tg}^{2n+1}\,\left(\frac{1}{2}\,\arccos\,\sqrt{\frac{s}{2a^2}} \right),$$

$$\text{where } s = -(1 + b^2 - a^2) + \sqrt{(1 + b^2 - a^2)^2 + 4a^2}.$$

BI ((65))(21, 22)

3.616

1. $$\int_0^{\pi} (1 - 2a\,\cos x + a^2)^n\,dx = \pi \sum_{k=0}^{n} \binom{n}{k}^2 a^{2k}.$$

BI ((63))(1)

2.* $$\int_0^{\pi} \frac{dx}{(1 - 2a\,\cos x + a^2)^n} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{(1 - 2a\,\cos x + a^2)^n} =$$

$$= \frac{\pi}{(1-a^2)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!} \left(\frac{a^2}{1-a^2} \right)^k \qquad [a^2 < 1]$$

$$= \frac{\pi}{(a^2-1)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!}\,\frac{1}{(a^2-1)^k} \qquad [a^2 > 1].$$

BI ((331))(63)

3. $$\int_0^{\pi} (1 - 2a\,\cos x + a^2)^n\,\cos nx\,dx = (-1)^n \pi a^n.$$

BI ((63))(2)

4. $$\int_0^{\pi} (1 - 2a\,\cos x + a^2)^n\,\cos mx\,dx = \frac{1}{2}\int_0^{2\pi} (1 - 2a\,\cos x + a^2)^n\,\cos mx\,dx =$$

$$= 0 [n < m];$$

$$= \pi(-a)^m (1+a^2)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k} \binom{n-k}{m+k} \left(\frac{a}{1+a^2} \right)^{2k}$$

$$[n \geq m]$$

5. $$\int_0^{2\pi} \frac{\sin nx\,dx}{(1-2a\cos 2x+a^2)^m} = 0.$$

GW ((332))(32a)

6. $$\int_0^{\pi} \frac{\sin x\,dx}{(1-2a\cos 2x+a^2)^m} = \frac{1}{2(m-1)a}\left[\frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}}\right] \qquad [a\neq 0, \quad \pm 1],$$

GW ((332))(32c)

7. $$\int_0^{\pi} \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m} =$$
$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m} =$$
$$= \frac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(\frac{1-a^2}{a^2}\right)^k \qquad [a^2<1];$$
$$= \frac{\pi}{a^n(a^2-1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}(a^2-1)^k \qquad [a^2>1].$$

GW ((332))(31)

8. $$\int_0^{\frac{\pi}{2}} \frac{\cos 2nx\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^{n+1}} = \binom{2n}{n}\frac{(b^2-a^2)^n}{(2ab)^{2n+1}}\pi \qquad [a>0, \quad b>0].$$

GW ((332))(30b)

3.617*

$$\int_0^{\pi} \frac{dx}{(1-2a\cos x+a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n\left(\frac{2\sqrt{|a|}}{|1+a|}\right), \qquad |a|\neq 1$$

with

$$F_n(k) = \int_0^{\pi/2} \frac{dx}{(1-k^2\sin^2 x)^{n+1/2}},$$

where the $F_n(k)$ satisfy the recurrence relation

$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1}\frac{dF_n(k)}{dk}, \qquad n = 0,\ 1,\ 2, \ldots$$

and

$$F_0(k) = K(k) \equiv \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}$$

is the complete elliptic integral of the first kind.

Introducing the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{1/2}\, dx,$$

the derivatives

$$\frac{dK}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \qquad \frac{dE}{dk} = \frac{E(k) - K(k)}{k}$$

combined with the recurrence relation lead to

$$\begin{aligned} F_1(k) &= F_0(k) + k\frac{dF_0(k)}{dk} \\ &= K(k) + \frac{E(k)}{1-k^2} - K(k) = \frac{E(k)}{1-k^2}, \\ F_2(k) &= \frac{E(k)}{1-k^2} + \frac{k}{3}\,\frac{d}{dk}\left[\frac{E(k)}{1-k^2}\right] \\ &= \frac{1}{3(1-k^2)}\left[\left(\frac{4-2k^2}{1-k^2}\right) E\,(k) - K\,(k)\right]. \end{aligned}$$

3.62 Powers of trigonometric functions

3.621

$$1. \quad \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{\mu-1} x\, dx = 2^{\mu-2}\, \mathrm{B}\left(\frac{\mu}{2}, \frac{\mu}{2}\right).$$

FI II 789

2. $$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}} x\,dx = \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

3. $$\int_0^{\frac{\pi}{2}} \sin^{2m} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2m} x\,dx = \frac{(2m-1)!!}{(2m)!!}\,\frac{\pi}{2}.$$

FI II 151

4. $$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2m+1} x\,dx = \frac{(2m)!!}{(2m+1)!!}.$$

FI II 151

413

5. $$\int_0^{\frac{\pi}{2}} \sin^{\mu-1} x\,\cos^{\nu-1} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2},\,\frac{\nu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

LO V 113(50), LO V 122, FI II 788

3.622

1. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x\,dx = \frac{\pi}{2}\,\sec\,\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((42))(1)

2. $$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu} x\,dx = \frac{1}{2}\beta\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1].$$

BI ((34))(1)

3. $$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n} x\,dx = (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}.$$

BI ((34))(2)

4. $$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n+1} x\,dx = (-1)^{n+1}\frac{\ln 2}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}.$$

3.623

1. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\mu-1}\ x\ \cos^{2\nu-2}\ x\,dx = \int_0^{\frac{\pi}{2}} \operatorname{ctg}^{\mu-1}\ x\ \sin^{2\nu-2}\ x\,dx = \\ = \frac{1}{2}\,\mathrm{B}\,\left(\frac{\mu}{2},\ \nu - \frac{\mu}{2}\right) \qquad [0 < \operatorname{Re}\mu < 2\ \operatorname{Re}\nu].$$

BI((42))(6), BI((45))(22)

2.[6] $$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu}\ x\ \sin^2\ x\,dx = \frac{1+\mu}{4}\beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \qquad [\operatorname{Re}\mu > -1].$$

BI ((34))(4)

3.[6] $$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu}\ x\ \cos^2\ x\,dx = \frac{1-\mu}{4}\beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \qquad [\operatorname{Re}\mu > -1].$$

BI ((34))(5)

3.624

1. $$\int_0^{\frac{\pi}{4}} \frac{\sin^p\ x}{\cos^{p+2}\ x}\,dx = \frac{1}{p+1} \qquad [p > -1].$$

GW ((331))(34b)

2.[3] $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-\frac{1}{2}}\ x}{\cos^{2\mu-1}\ x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-\frac{1}{2}}\ x}{\sin^{2\mu-1}\ x}\,dx = \frac{1}{2}\left\{\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}\right)\Gamma\left(1-\mu\right)}{\Gamma\left(\frac{5}{4}-\frac{\mu}{2}\right)}\right\} \qquad \left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right].$$

LI ((55))(12)

3. $$\int_0^{\frac{\pi}{4}} \frac{\cos^{n-\frac{1}{2}}\ 2x}{\cos^{2n+1}\ x}\,dx = \frac{(2n-1)!!}{2\cdot(2n)!!}\pi.$$

BI ((38))(3)

4. $$\int_0^{\frac{\pi}{4}} \frac{\cos^{\mu}\ 2x}{\cos^{2(\mu+1)}\ x} + dx = 2^{2\mu}\,\mathrm{B}\left(\mu+1,\ \mu+1\right) \qquad [\operatorname{Re}\mu > -1].$$

5. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2\mu-2} x}{\cos^{\mu} 2x}\, dx = 2^{1-2\mu}\, \mathrm{B}\,(2\mu-1,\ 1-\mu) = \frac{\Gamma\left(\mu-\frac{1}{2}\right)\Gamma(1-\mu)}{2\sqrt{\pi}} \qquad \left[\frac{1}{2} < \operatorname{Re}\mu < 1\right].$$

BI ((35))(4)

6.[6] $$\int_0^{\frac{\pi}{2}} \left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{a\pi}{2} - \frac{1}{2}\sin \pi a[2a\beta(a) - 1], \qquad [a > 0].$$

3.625

1. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x \cos^p 2x}{\cos^{2p+2n+1} x}\, dx = \frac{(n-1)!}{2} \cdot \frac{\Gamma(p+1)}{\Gamma(p+n+1)} = \\ = \frac{(n-1)!}{2(p+n)(p+n-1)\dots(p+1)} = \frac{1}{2}\,\mathrm{B}\,(n,\ p+1) \qquad [p > -1],$$

(cf. 3.251 1.).

3.251

BI ((35))(2)

2. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x \cos^p 2x}{\cos^{2p+2n+2} x}\, dx = \frac{1}{2}\,\mathrm{B}\left(n + \frac{1}{2},\ p+1\right) \qquad [p > -1] \qquad \text{(cf. 3.251 1.)}.$$

3.251

BI ((35))(3)

3. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x}\, dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!}.$$

BI ((38))(6)

4. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x} + dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \cdot \frac{\pi}{2}.$$

BI ((38))(7)

3.626

1. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x}{\cos^{2n+2} x}\sqrt{\cos 2x}\, dx = \frac{(2n-2)!!}{(2n+1)!!} \qquad \text{(cf. 3.251 1.)}.$$

3.251
BI ((38))(4)

$$2. \quad \int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos 2x}\, dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \qquad \text{(cf. 3.251 1.).}$$

3.251
BI ((38))(5)

415
3.627

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^\mu x}{\cos^\mu x}\, dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^\mu x}{\sin^\mu x}\, dx = \frac{\Gamma(\mu)\Gamma\left(\frac{1}{2}-\mu\right)}{2^\mu\sqrt{\pi}} \sin \frac{\mu\pi}{2} \qquad \left[-1 < \operatorname{Re}\mu < \frac{1}{2}\right].$$

BI ((55))(12)a

3.628

$$\int_0^{\frac{\pi}{2}} \sec^{2p+1} x \frac{d \sin^{2p} x}{dx}\, dx = \frac{1}{\sqrt{\pi}} \Gamma(p+1)\Gamma\left(\frac{1}{2}-p\right) \qquad \left[\frac{1}{2} > p > 0\right].$$

WA 691

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.631

$$1. \quad \int_0^\pi \sin^{\nu-1} x \sin ax\, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1}\nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \qquad [\operatorname{Re}\nu > 0].$$

LO V 121(67a), WA 337a

$$2.^7 \quad \int_0^{\pi/2} \sin^{\nu-2} x \sin \nu x\, dx = \frac{1}{1-\nu} \cos \frac{\nu\pi}{2} \qquad [\operatorname{Re}\nu > 1].$$

GW((332))(16d), FI I 152

$$3.^6 \quad \int_0^\pi \sin^\nu x \sin \nu x\, dx = 2^{-\nu}\pi \sin \frac{\nu\pi}{2} \qquad [\operatorname{Re}\nu > -1].$$

4. $\int_0^{\pi} \sin^n x \sin 2mx\, dx = 0.$

GW ((332))(11a)

5. $\int_0^{\pi} \sin^{2n} x \sin(2m+1)x\, dx = \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x\, dx =$

$$= \frac{(-1)^m 2^{n+1} n!(2n-1)!!}{(2n-2m-1)!!(2m+2n+1)!!} \qquad [m \le n]*;$$

$$= \frac{(-1)^n 2^{n+1} n!(2m-2n-1)!!(2n-1)!!}{(2m+2n+1)!!} \qquad [m \ge n].*$$

* For $m = n$ we should set $(2n-2m-1)!! = 1$.

GW ((332))(11b)

6. $\int_0^{\pi} \sin^{2n+1} x \sin(2m+1)x\, dx = 2\int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x\, dx =$

$$= \frac{(-1)^m \pi}{2^{2n+1}} \binom{2n+1}{n-m} \qquad [n \ge m];$$

$$= 0 \qquad [n < m].$$

BI((40))(12), GW((332))(11c)

416

7. $\int_0^{\pi} \sin^n x \cos(2m+1)x\, dx = 0.$

GW ((332))(12a)

8. $\int_0^{\pi} \sin^{\nu-1} x \cos ax\, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu \operatorname{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \qquad [\operatorname{Re} \nu > 0].$

LO V 121(68)A, WA 337a

9. $\int_0^{\pi/2} \cos^{\nu-1} x \cos ax\, dx = \frac{\pi}{2^{\nu} \nu \operatorname{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \qquad [\operatorname{Re} \nu > 0].$

10. $$\int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x \, dx = \frac{1}{\nu-1} \sin \frac{\nu\pi}{2} \qquad [\operatorname{Re} \nu > 1].$$

GW((332))(16b), FI II 15 2

11. $$\int_0^{\pi} \sin^{\nu} x \cos \nu x \, dx = \frac{\pi}{2^\nu} \cos \frac{\nu\pi}{2} \qquad [\operatorname{Re} \nu > -1].$$

LO V 121(70)a

12. $$\int_0^{\pi} \sin^{2n} x \cos 2mx \, dx = 2\int_0^{\pi/2} \sin^{2n} x \cos 2mx \, dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi \qquad [n \geq m];$$
$$= 0 \qquad [n < m].$$

BI((40))(16), GW((332))(12b)

13.[7] $$\int_0^{\pi} \sin^{2n+1} x \cos 2mx \, dx =$$
$$= 2\int_0^{\pi/2} \sin^{2n+1} x \cos 2mx \, dx = \frac{(-1)^m 2^{n+1} n!(2n+1)!!}{(2n-2m+1)!!(2m+2n+1)!!} \qquad [n \geq m-1];$$
$$= \frac{(-1)^{n+1} 2^{n+1} n!(2m-2n-3)!!(2n+1)!!}{(2m+2n+1)!!} \qquad [n < m-1].$$

GW ((332))(12c)

14. $$\int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x \, dx = \frac{1}{\nu-1} \qquad [\operatorname{Re} \nu > 1].$$

GW((332))(16c), FI II 152

15. $$\int_0^{\pi} \cos^m x \sin nx \, dx = [1-(-1)^{m+n}] \int_0^{\pi/2} \cos^m x \sin nx \, dx =$$
$$= [1-(-1)^{m+n}] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s\frac{m!(n-m-2)!!}{(m+n)!!} \right\}$$
$$\left[r = \begin{cases} m & [m \leq n], \\ n & [m \geq n], \end{cases} \quad s = \begin{cases} 2 & [n-m = 4l+2 > 0], \\ 1 & [n-m = 2l+1 > 0], \\ 0 & [n-m = 4l \quad \text{or} \quad n-m < 0] \end{cases} \right].$$

GW ((332))(13a)

16. $$\int_0^{\pi/2} \cos^n x \sin nx \, dx = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}.$$

17. $\int_0^{\pi} \cos^n x \cos mx\, dx =$

$$= [1+(-1)^{m+n}] \int_0^{\frac{\pi}{2}} \cos^n x \cos mx\, dx =$$

$$= [1+(-1)^{m+n}] \begin{cases} s \dfrac{n!}{(m-n)(m-n+2)\dots(m+n)} & [n < m]; \\ \dfrac{\pi}{2^{n+1}} \dbinom{n}{k} & [m \le n \quad \text{and} \quad n-m=2k]\ ; \\ \dfrac{n!}{(2k+1)!!\,(2m+2k+1)!!} & [m < n \quad \text{and} \quad n-m=2k+1]; \end{cases}$$

$$\text{where } s = \begin{cases} 0 & [m-n=2k], \\ 1 & [m-n=4k+1], \\ -1 & [m-n=4k-1]. \end{cases}$$

GW ((332))(15a)

417

18.[6] $\int_0^{\pi} \cos^m x \cos ax\, dx = \dfrac{(-1)^m \sin a\pi}{2^m(m+a)}\, {}_2F_1\left(-m, -\dfrac{a+m}{2}; 1-\dfrac{a+m}{2}; -1\right)$

$[a \neq 0, \pm 1, \pm 2, \dots]$.

WA 313

19. $\int_0^{\frac{\pi}{2}} \cos^{\nu-2} x \cos \nu x\, dx = 0 \qquad [\operatorname{Re} \nu > 1]$.

GW((332))(16a), FI II 152

20.* $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \dfrac{\pi}{2^{n+1}} \qquad [\operatorname{Re} n > -1]$.

LO V 122(78), FI II 153

3.632

1. $\int_0^{\pi} \sin^{p-1} x \cos\left[a\left(\dfrac{\pi}{2} - x\right)\right] dx = 2^{p-1} \dfrac{\Gamma\left(\dfrac{p-a}{2}\right)\Gamma\left(\dfrac{p+a}{2}\right)}{\Gamma(p-a)\Gamma(p+a)} \Gamma(p) \qquad [p^2 < a^2]$.

BI ((62))(11)

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin\left[a\left(x + \dfrac{\pi}{2}\right)\right] dx = \dfrac{\pi \sin \dfrac{a\pi}{2}}{2^{\nu-1} \nu\, \mathrm{B}\left(\dfrac{\nu+a+1}{2}, \dfrac{\nu-a+1}{2}\right)} \qquad [\operatorname{Re} \nu > 0]$.

3.* $\int_0^{\frac{\pi}{2}} \cos^p x \sin[(p+2n)x]\,dx = (-1)^{n-1}\sum_{k=0}^{n-1}\frac{(-1)^k 2^k}{p+k+1}\binom{n-1}{k} \qquad [n>0].$

LI ((41))(12)

4. $\int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)]\,dx = [1-(-1)^{n+m}] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)]\,dx =$

$$= \frac{[1-(-1)^{n+m}]\pi \cos ma}{2^{n-1} n\,\mathrm{B}\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}\right)} \qquad [n \ge m].$$

LO V 123(80), LO V 139(94a)

5. $\int_0^{\frac{\pi}{2}} \cos^{p+q-2} x \cos[(p-q)x]\,dx = \frac{\pi}{2^{p+q-1}(p+q-1)\,\mathrm{B}(p, q)} \qquad [p+q>1].$

WH

3.633

1. $\int_0^{\frac{\pi}{2}} \cos^{p-1} x \sin ax \sin x\,dx = \frac{a\pi}{2^{p+1}p(p+1)\,\mathrm{B}\left(\frac{p+a}{2}+1, \frac{p-a}{2}+1\right)}.$

LO V 150(110)

2. $\int_0^{\frac{\pi}{2}} \cos^n x \sin nx \sin 2mx\,dx = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \cos 2mx\,dx = \frac{\pi}{2^{n+2}}\binom{n}{m}.$

BI ((42))(19, 20)

3. $\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos[(n+1)x] \cos 2mx\,dx = \frac{\pi}{2^{n+1}}\binom{n-1}{m-1} \qquad [n > m-1].$

BI ((42))(21)

4. $\int_0^{\frac{\pi}{2}} \cos^{p+q} x \cos px \cos qx\,dx = \frac{\pi}{2^{p+q+2}}\left[1+\frac{1}{(p+q+1)\,\mathrm{B}(p+1, q+1)}\right] \qquad [p+q>-1].$

5.[6] $$\int_0^{\frac{\pi}{2}} \cos^{p+q} x \sin px \sin qx \, dx = \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} = \frac{\pi}{2^{p+q+2}} \left[\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} - 1 \right]$$ $$[p+q > -1].$$

BI ((42))(16)

3.634

1. $$\int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x \, dx = \sin \frac{\mu\pi}{2} \mathrm{B}(\mu, \nu) \qquad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

BI((42))(23), FI II 814a

2. $$\int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu+\nu)x \, dx = \cos \frac{\mu\pi}{2} \mathrm{B}(\mu, \nu) \qquad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

BI((42))(24), FI II 814a

3. $$\int_0^{\frac{\pi}{2}} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x \, dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n!\Gamma(p)} \qquad [p > -n].$$

BI ((42))(15)

3.635

1. $$\int_0^{\frac{\pi}{4}} \cos^{\mu-1} 2x \operatorname{tg} x \, dx = \frac{1}{4} \left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu}{2}\right) \right] \qquad [\operatorname{Re} \mu > 0].$$

BI ((34))(7)

2.[7] $$\int_0^{\frac{\pi}{2}} \cos^{p+2n} x \sin px \operatorname{tg} x \, dx = \frac{\pi}{2^{p+2n+1}\Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!} =$$ $$= \frac{p\pi}{2^{p+2n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1)\Gamma(p+n+1)} \qquad [p > -2n].$$

BI ((42))(22)

3. $$\int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin[(n+1)x] \operatorname{ctg} x \, dx = \frac{\pi}{2}.$$

BI ((45))(18)

3.636

1. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x \sin 2x\,dx = \frac{\mu\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \qquad [0 < \operatorname{Re}\mu < 2].$$

BI ((45))(20)a

2. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x \cos 2x\,dx = \mp\frac{\mu\pi}{2} \sec \frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((45))(21)

3. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{2\mu} x}{\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{2\mu} x}{\sin x}\,dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(-\mu)}{2\sqrt{\mu}} \qquad \left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right], \qquad \text{(cf. 3.251 1.).}$$

3.251
BI ((45))(13, 14)

3.637

1. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \sin^{q-2} x \sin qx\,dx = -\cos\frac{(p+q)\pi}{2}\,\mathrm{B}\,(p+q-1,\ 1-p) \qquad [p+q > 1 > p].$$

GW ((332))(15d)

2. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \sin^{q-2} x \cos qx\,dx = \sin\frac{(p+q)\pi}{2}\,\mathrm{B}\,(p+q-1,\ 1-p) \qquad [p+q > 1 > p].$$

GW ((332))(15b)

3. $$\int_0^{\frac{\pi}{2}} \operatorname{ctg}^p x \cos^{q-2} x \sin qx\,dx = \cos\frac{p\pi}{2}\mathrm{B}(p+q-1,\ 1-p) \qquad [p+q > 1 > p].$$

GW ((332))(15c)

4. $$\int_0^{\frac{\pi}{2}} \operatorname{ctg}^p x \cos^{q-2} x \cos qx\,dx = \sin\frac{p\pi}{2}\,\mathrm{B}\,(p+q-1,\ 1-p) \qquad [p+q > 1 > p].$$

1. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{2\mu} x\, dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2} \sec \mu\pi \qquad \left[|\operatorname{Re}\mu| < \frac{1}{2}\right], \qquad \text{(cf. 3.192 2.).}$$

3.192
BI ((38))(8)

2. $$\int_0^{\frac{\pi}{4}} \frac{\sin^{\mu-\frac{1}{2}} 2x\, dx}{\cos^{\mu} 2x \cos x} = \frac{2}{2\mu-1} \cdot \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(1-\mu)}{\sqrt{\pi}} \sin\left(\frac{2\mu-1}{4}\pi\right) \qquad \left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right].$$

BI ((38))(17)

3. $$\int_0^{\frac{\pi}{2}} \frac{\cos^{p-1} x \sin px}{\sin x}\, dx = \frac{\pi}{2} \qquad [p > 0].$$

GW((332))(17), BI((45))(5)

3.64- 3.65 Powers and rational functions of trigonometric functions

3.641

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{p-1} x \cos^{-p} x}{a \cos x + b \sin x}\, dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{-p} x \cos^{p-1} x}{a \sin x + b \cos x}\, dx = \frac{\pi \operatorname{cosec} p\pi}{a^{1-p} b^{p}} \qquad [ab > 0, \quad 0 < p < 1].$$

GW ((331))(62)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{1-p} x \cos^{p} x}{(\sin x + \cos x)^3}\, dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{p} x \cos^{1-p} x}{(\sin x + \cos x)^3}\, dx = \frac{(1-p)p}{2}\pi \operatorname{cosec} p\pi \qquad [-1 < p < 2].$$

BI((48))(5)

3.642

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x\, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} = \frac{1}{2a^{2\mu} b^{2\nu}} \mathrm{B}(\mu, \nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((48))(28)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{n-1} x \cos^{n-1} x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n}} = \frac{\mathrm{B}\left(\frac{n}{2}, \frac{n}{2}\right)}{2(ab)^{n}} \qquad [ab > 0].$$

GW ((331))(59a)

3. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\ x\,dx}{(a^2\ \cos^2 x + b^2\ \sin^2\ x)^{n+1}} = \\ = \frac{1}{2}\int_0^{\pi} \frac{\sin^{2n}\ x\,dx}{(a^2\ \cos^2\ x + b^2\ \sin^2\ x)^{n+1}} = \\ = \int_0^{\frac{\pi}{2}} \frac{\cos^{2n}\ x\,dx}{(a^2\ \sin^2\ x + b^2\ \cos^2\ x)^{n+1}} = \frac{1}{2}\int_0^{\pi} \frac{\cos^{2n}\ x\,dx}{(a^2\ \sin^2\ x + b^2\ \cos^2\ x)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!ab^{2n+1}} \\ [ab > 0].$$

GW ((331))(58)

4. $$\int_0^{\frac{\pi}{2}} \frac{\cos^{p+2n}\ x\ \cos px\,dx}{(a^2\ \cos^2\ x + b^2\ \sin^2\ x)^{n+1}} = \pi\sum_{k=0}^{n} \binom{2n-k}{n}\binom{p+k-1}{k}\frac{b^{p-1}}{(2a)^{2n-k+1}(a+b)^{p+k}} \\ [a > 0, \quad b > 0, \quad p > -2n-1].$$

GW ((332))(30)

3.643

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos^p\ x\ \cos px\,dx}{1-2a\ \cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \qquad [a^2 < 1, \quad p > -1].$$

GW ((332))(33c)

421

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\ x\ \cos^{\mu}\ x\ \cos\beta x}{(1-2a\ \cos 2x + a^2)^m}\,dx = \frac{(-1)^n\pi(1-a)^{2n-2m+1}}{2^{2m-\beta-1}(1+a)^{2m+\beta+1}}\sum_{k=0}^{m-1}\sum_{l=0}^{m-k-1}\binom{\beta}{k}\binom{2n}{l}\times \\ \times\binom{2m-k-l-2}{m-1(-2)^l}(a-1)^k \\ [a^2 < 1, \quad \beta = 2m-2n-\mu-2, \quad \mu > -1].$$

GW ((332))(33)

3.644

1. $$\int_0^{\pi} \frac{\sin^m\ x}{p+q\ \cos x}\,dx = 2^{m-2}\frac{p}{q^2}\sum_{\nu=1}^{k}\left(\frac{p^2-q^2}{-4q^2}\right)^{\nu-1}\ \mathrm{B}\left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2}\right) + \left(\frac{p^2-q^2}{-q^2}\right)^k A; \\ A = \frac{\pi p}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right) \qquad [m = 2k+2]; \\ A = \frac{1}{q}\ \ln\frac{p+q}{p-q} \qquad [m = 2k+1] \qquad [k \geq 1, \quad q \neq 0, \quad p^2-q^2 \geq 0].$$

2. $$\int_0^{\pi} \frac{\sin^m x}{1+\cos x}\,dx = 2^{m-1}\,\mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \geq 2].$$

3. $$\int_0^{\pi} \frac{\sin^m x}{1-\cos x}\,dx = 2^{m-1}\,\mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \geq 2].$$

4. $$\int_0^{\pi} \frac{\sin^2 x}{p+q\,\cos x}\,dx = \frac{p\pi}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right).$$

5. $$\int_0^{\pi} \frac{\sin^3 x}{p+q\,\cos x}\,dx = 2\frac{p}{q^2}+\frac{1}{q}\left(1-\frac{p^2}{q^2}\right)\,\ln\frac{p+q}{p-q}.$$

3.645

$$\int_0^{\pi} \frac{\cos^n x\,dx}{(a+b\,\cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n}(-1)^k\frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!}\left(\frac{a+b}{a-b}\right)^k \qquad [a^2 > b^2].$$

LI ((64))(16)

3.646

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos^n x\,\sin nx\,\sin 2x}{1-2a\,\cos 2x+a^2}\,dx = \frac{\pi}{4a}\left[\left(\frac{1+a}{2}\right)^n - \frac{1}{2^n}\right] \qquad [a^2 < 1].$$

BI ((50))(6)

2. $$\int_0^{\frac{\pi}{2}} \frac{1-a\,\cos 2nx}{1-2a\,\cos 2nx+a^2}\,\cos^m x\,\cos mx\,dx = \frac{\pi}{2^{m+2}}\sum_{k=1}^{\infty}\binom{m}{kn}a^k+\frac{\pi}{2^{m+1}} \qquad [a^2 < 1].$$

LI ((50))(7)

3.647

$$\int_0^{\frac{\pi}{2}} \frac{\cos^p x\,\cos px\,dx}{a^2\,\sin^2 x+b^2\,\cos^2 x} = \frac{\pi}{2b}\cdot\frac{a^{p-1}}{(a+b)^p} \qquad [p > -1, \quad a > 0, \quad b > 0].$$

BI ((47))(20)

3.648

1. $$\int_0^{\frac{\pi}{4}} \frac{\mathrm{tg}^l\,x\,dx}{1+\cos\frac{m}{n}\pi\,\sin 2x} = \frac{1}{2n}\,\mathrm{cosec}\,\frac{m}{n}\pi\sum_{k=0}^{n-1}(-1)^{k-1}\,\sin\frac{km}{n}\pi \times$$
$$\times\left[\psi\left(\frac{n+l+k}{2n}\right)-\psi\left(\frac{l+k}{2n}\right)\right] \qquad [m+n \quad \text{is odd}];$$
$$\frac{1}{\ldots}\,\mathrm{cosec}\,\frac{m}{\ldots}\sum^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km}{\ldots}\times$$

2. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x\, dx}{1+\cos t \sin 2x} = \pi \operatorname{cosec} t \sin \mu t \operatorname{cosec}(\mu\pi) \qquad [|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

BI ((47))(4)

3.649

1. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \sin 2x\, dx}{1 \mp 2a \cos 2x + a^2} = \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 - \left(\frac{1-a}{1+a}\right)^{\mu}\right] \qquad [a^2 < 1];$$
$$= \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 + \left(\frac{a-1}{a+1}\right)^{\mu}\right] \qquad [a^2 > 1] \qquad [-2 < \operatorname{Re}\mu < 1].$$

BI ((50))(3)

2. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x(1 \mp a \cos 2x)}{1 \mp 2a \cos 2x + a^2}\, dx = \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 + \left(\frac{1-a}{1+a}\right)^{\mu}\right] \qquad [a^2 < 1];$$
$$= \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 - \left(\frac{a-1}{a+1}\right)^{\mu}\right] \qquad [a^2 > 1] \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((50))(4)

3.651

1. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu} x\, dx}{1+\sin x \cos x} = \frac{1}{3}\left[\psi\left(\frac{\mu+2}{3}\right) - \psi\left(\frac{\mu+1}{3}\right)\right] \qquad [\operatorname{Re}\mu > -1].$$

BI ((36))(3)

423

2. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu} x\, dx}{1-\sin x \cos x} = \frac{1}{3}\left[\beta\left(\frac{\mu+2}{3}\right) + \beta\left(\frac{\mu+1}{3}\right)\right] \qquad [\operatorname{Re}\mu > -1].$$

BI ((36))(4)a

3.652

1. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x\, dx}{(\sin x + \cos x) \sin x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x\, dx}{(\sin x + \cos x) \cos x} = \pi \operatorname{cosec} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((49))(1)

2. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x\, dx}{(\sin x - \cos x) \sin x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x\, dx}{(\cos x - \sin x) \cos x} = -\pi \operatorname{ctg} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$$

3. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu+\frac{1}{2}} x\,dx}{(\sin x+\cos x)\,\cos x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu-\frac{1}{2}} x\,dx}{(\sin x+\cos x)\,\cos x} = \pi\,\sec\,\mu\pi \qquad \left[|\operatorname{Re}\mu| < \frac{1}{2}\right].$$

BI ((61))(1, 2)

3.653

1. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{1-2\mu} x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{1-2\mu} x\,dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2a^{2\mu}b^{2-2\mu}\sin\mu\pi} \qquad [0 < \operatorname{Re}\mu < 1].$$

GW ((331))(59b)

2.[7] $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x\,dx}{1-a\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x\,dx}{1-a\cos^2 x} = \frac{\pi\,\sec\frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}} \qquad [|\operatorname{Re}\mu| < 1, \quad a < 1].$$

BI ((49))(6)

3. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x\,dx}{1-\cos^2 t\,\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} t\,\sec\frac{\mu\pi}{2}\,\cos\left[\left(\frac{\pi}{2}-t\right)\mu\right] \qquad [|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

BI((49))(7), BI((47))(21)

4. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x\,\sin 2x}{1-\cos^2 t\,\sin^2 2x}\,dx = \pi\,\operatorname{cosec} 2t\,\operatorname{cosec}\frac{\mu\pi}{2}\,\sin\left[\left(\frac{\pi}{2}-t\right)\mu\right] \qquad [|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

BI ((47))(22)a

5. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x\,\sin^2 x\,dx}{1-\cos^2 t\,\sin^2 2x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x\,\cos^2 x\,dx}{1-\cos^2 t\,\sin^2 2x} =$$
$$= \frac{\pi}{2}\operatorname{cosec} 2t\,\sec\frac{\mu\pi}{2}\,\cos\left[\frac{\mu\pi}{2}-(\mu+1)t\right] \qquad [|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

BI((47))(23)A, BI((49))(10)

6. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x\,\cos^2 x\,dx}{1-\cos^2 t\,\sin^2 2x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x\,\sin^2 x\,dx}{1-\cos^2 t\,\sin^2 2x} =$$
$$= \frac{\pi}{2}\operatorname{cosec} 2t\,\sec\frac{\mu\pi}{2}\,\cos\left[\frac{\mu\pi}{2}-(\mu-1)t\right] \qquad [|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

1. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu+1} x \cos^2 x \, dx}{(1+\cos t \sin 2x)^2} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu+1} x \sin^2 x \, dx}{(1+\cos t \sin 2x)^2} = \frac{\pi(\mu \sin t \cos \mu t - \cos t \sin \mu t)}{2 \sin \mu\pi \sin^3 t}$$
$$[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2].$$

BI((48))(3), BI((49))(22)

2. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \, dx}{(\sin x + \cos x)^2} = \frac{\mu\pi}{\sin \mu\pi} \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((56))(9)a

3. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm(\mu-1)} x \, dx}{\cos^2 x - \sin^2 x} = \pm\frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \qquad [0 < \operatorname{Re}\mu < 2].$$

BI ((45))(27, 29)

3.655

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{2\mu-1} x \, dx}{1 - 2a(\cos t_1 \sin^2 x + \cos t_2 \cos^2 x) + a^2} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{2\mu-1} x \, dx}{1 - 2a(\cos t_1 \cos^2 x + \cos t_2 \sin^2 x) + a^2} =$$
$$= \frac{\pi \operatorname{cosec} \mu\pi}{(1 - 2a \cos t_2 + a^2)^{\mu}(1 - 2a \cos t_1 + a^2)1 - \mu}$$
$$[0 < \operatorname{Re}\mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2].$$

BI ((50))(18)

3.656

1. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu} x \, dx}{1 - \sin^2 x \cos^2 x} = \frac{1}{12}\left\{-\psi\left(\frac{\mu+1}{6}\right) - \psi\left(\frac{\mu+2}{6}\right) + \right.$$
$$\left. + \psi\left(\frac{\mu+4}{6}\right) + \psi\left(\frac{\mu+5}{6}\right) + 2\psi\left(\frac{\mu+2}{3}\right) - 2\psi\left(\frac{\mu+1}{3}\right)\right\}$$
$$[\operatorname{Re}\mu > -1], \qquad (\text{cf. } 3.651 \text{ 1. and 2.}).$$

3.651

LI ((36))(10)

2. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu-1} x \cos^2 x \, dx}{1 - \sin^2 x \cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu-1} x \sin^2 x \, dx}{1 - \sin^2 x \cos^2 x} = \frac{\pi}{4\sqrt{3}} \operatorname{cosec} \frac{\mu\pi}{6} \operatorname{cosec}\left(\frac{2+\mu}{6}\pi\right)$$
$$[0 < \operatorname{Re}\mu < 4].$$

LI ((47))(26)

3.66 Forms containing powers of linear functions of trigonometric functions

3.661

1. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n+1}\,dx = 0.$$

BI ((68))(9)

2. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot 2\pi(a^2+b^2)^n.$$

BI ((68))(8)

3. $$\int_0^{\pi} (a+b\cos x)^n\,dx = \frac{1}{2}\int_0^{2\pi} (a+b\cos x)^n\,dx = \pi(a^2-b^2)^{\frac{n}{2}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) = \\ = \frac{\pi}{2^n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k}(a^2-b^2)^k \qquad [a^2>b^2].$$

GW ((332))(37a)

4. $$\int_0^{\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{(a^2-b^2)^{\frac{n+1}{2}}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) = \\ = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n} \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!}\cdot\left(\frac{a+b}{a-b}\right)^k \\ [a>|b|].$$

GW((332))(38), LI((64))(14)

3.662

1. $$\int_0^{\frac{\pi}{2}} (\sec x - 1)^\mu \sin x\,dx = \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu \cos x\,dx = \mu\pi\operatorname{cosec}\mu\pi \qquad [|\operatorname{Re}\mu|<1].$$

BI ((55))(13)

2. $$\int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu \sin 2x\,dx = (1-\mu)\mu\pi\operatorname{cosec}\mu\pi \qquad [-1<\operatorname{Re}\mu<2].$$

3. $$\int_0^{\frac{\pi}{2}} (\sec\, x - 1)^\mu\, \operatorname{tg} x\, dx = \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu\, \operatorname{ctg} x\, dx = -\pi\, \operatorname{cosec} \mu\pi \qquad [-1 < \operatorname{Re} \mu < 0],$$

(cf. 3.192 2.).

3.192
BI ((46))(4,6)

4. $$\int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2}\, \operatorname{cosec} \mu\pi \qquad [-1 < \operatorname{Re} \mu < 0].$$

BI ((38))(22)a

5. $$\int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^\mu \frac{dx}{\cos^2\, x} = \mu\pi\, \operatorname{cosec} \mu\pi \qquad [|\operatorname{Re} \mu| < 1].$$

BI ((38))(11)a

3.663

1. $$\int_0^u (\cos x - \cos u)^{\nu - \frac{1}{2}}\, \cos ax\, dx = \sqrt{\frac{\pi}{2}}\, \sin^\nu\, u \Gamma\left(\nu + \frac{1}{2}\right)\, \mathrm{P}^{-\nu}_{a-\frac{1}{2}}\,(\cos u)$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}; \quad a > 0, \quad 0 < u < \pi\right].$$

EH I 159(27), ET I 22(28)

426

2. $$\int_0^u (\cos x - \cos u)^{\nu - 1}\, \cos[(\nu + \beta)x]\, dx = \frac{\sqrt{\pi}\Gamma(\beta + 1)\Gamma(\nu)\Gamma(2\nu)\, \sin^{2\nu - 1}\, u}{2^\nu \Gamma(\beta + 2\nu)\Gamma\left(\nu + \frac{1}{2}\right)} C^\nu_\beta(\cos u)$$

$$[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > -1, \quad 0 < u < \pi].$$

EH I 178(23)

3.664

1. $$\int_0^\pi (z + \sqrt{z^2 - 1}\, \cos x)^q\, dx = \pi P_q(z)$$

$$\left[\operatorname{Re} z > 0, \quad \arg(z + \sqrt{z^2 - 1}\, \cos x) = \arg z \quad \text{for} \quad x = \frac{\pi}{2}\right].$$

SM 482

WH

3. $$\int_0^{\pi} (z + \sqrt{z^2-1}\,\cos x)^q \cos nx\,dx = \frac{\pi}{(q+1)(q+2)\dots(q+n)} P_q^n(z).$$

$$\left[\operatorname{Re} z > 0, \quad \arg(z + \sqrt{z^2-1}\,\cos x) = \arg z \quad \text{for} \quad x = \frac{\pi}{2}, \right.$$
$$\left. z \text{ lies outside the interval } (-1,1) \text{ of the real axis} \right].$$

WH, SM 483(15)

4. $$\int_0^{\pi} (z + \sqrt{z^2-1}\,\cos x)^{\mu} \sin^{2\nu-1} x\,dx =$$
$$= \frac{2^{2\nu-1}\Gamma(\mu+1)[\Gamma(\nu)]^2}{\Gamma(2\nu+\mu)} C_{\mu}^{\nu}(z) =$$
$$= \frac{\sqrt{\pi}\Gamma(\nu)\Gamma(2\nu)\Gamma(\mu+1)}{\Gamma(2\nu+\mu)\Gamma\left(\nu+\frac{1}{2}\right)} C_{\mu}^{\nu}(z) = 2^{\nu}\sqrt{\frac{\pi}{2}}(z^2-1)^{\frac{1}{4}-\frac{\nu}{2}}\Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(z)$$
$$[\operatorname{Re} \nu > 0].$$

EH I 155(6)A, EH I 178(22)

5. $$\int_0^{2\pi} [\beta + \sqrt{\beta^2-1}\,\cos(a-x)]^{\nu} (\gamma + \sqrt{\gamma^2-1}\,\cos x)^{\nu-1}\,dx =$$
$$= 2\pi P_{\nu}[\beta\gamma - \sqrt{\beta^2-1}\sqrt{\gamma^2-1}\,\cos a] \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

EH I 157(18)

3.665

1. $$\int_0^{\pi} \frac{\sin^{\mu-1} x\,dx}{(a + b\cos x)^{\mu}} = \frac{2^{\mu-1}}{\sqrt{(a^2-b^2)^{\mu}}} \mathrm{B}\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad 0 < b < a].$$

FI II 790a

2. $$\int_0^{\pi} \frac{\sin^{2\mu-1} x\,dx}{(1 + 2a\cos x + a^2)^{\nu}} = \mathrm{B}\left(\mu, \frac{1}{2}\right) F\left(\nu, \nu-\mu+\frac{1}{2}; \mu+\frac{1}{2}; a^2\right) \qquad [\operatorname{Re}\mu > 0, \quad |a| < 1].$$

EH I 81(9)

3.666

1. $$\int_0^{\pi} (\beta + \cos x)^{\mu-\nu-\frac{1}{2}} \sin^{2\nu} x\,dx = \frac{2^{\nu+\frac{1}{2}} e^{-i\mu\pi} (\beta^2-1)^{\frac{\mu}{2}} \Gamma\left(\nu+\frac{1}{2}\right) Q_{\nu-\frac{1}{2}}^{\mu}(\beta)}{\Gamma\left(\nu+\mu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\left(\nu+\mu+\frac{1}{2}\right) > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

2.[6] $\int_0^{\pi} (\operatorname{ch}\beta + \operatorname{sh}\beta\, \cos x)^{\mu+\nu} \sin^{-2\nu} x\, dx = \frac{\sqrt{\pi}}{2^{\nu}} \operatorname{sh}^{\nu}(\beta) \Gamma\left(\frac{1}{2} - \nu\right) P_{\mu}^{\nu}(\operatorname{ch}\beta) \qquad \left[\operatorname{Re}\nu < \frac{1}{2}\right].$

EH I 156(7)

3. $\int_0^{\pi} (\cos t + i \sin t\, \cos x)^{\mu} \sin^{2\nu-1} x\, dx = 2^{\nu-\frac{1}{2}} \sqrt{\pi} \sin^{\frac{1}{2}-\nu} t \Gamma(\nu)\, \mathrm{P}_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t)$
$[\operatorname{Re}\nu > 0, \quad t^2 < \pi^2].$

EH I 158(23)

4. $\int_0^{2\pi} [\cos t + i \sin t\, \cos(a-x)]^{\nu} \cos mx\, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \cos ma\, \mathrm{P}_{\nu}^{m}(\cos t) \qquad \left[0 < t < \frac{\pi}{2}\right].$

EH I 159(25)

5. $\int_0^{2\pi} [\cos t + i + \sin t\, \cos(a-x)]^{\nu} \sin mx\, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \sin ma\, \mathrm{P}_{\nu}^{m}(\cos t) \qquad \left[0 < t < \frac{\pi}{2}\right].$

EH I 159(26)

3.667

1. $\int_0^{\frac{\pi}{4}} \frac{\sin^{\mu-1} 2x\, dx}{(\cos x + \sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\mu + \frac{1}{2}\right)} \qquad [\operatorname{Re}\mu > 0].$

BI ((37))(1)

2. $\int_0^{\frac{\pi}{4}} \frac{\sin^{\mu} x\, dx}{(\cos x - \sin x)^{\mu+1} \cos x} = -\pi \operatorname{cosec} \mu\pi \qquad [-1 < \operatorname{Re}\mu < 0], \qquad$ (cf. 3.192 2.).

3.192
BI ((37))(16)

3. $\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^{\mu}}{\sin^{\mu} x \sin 2x} dx = -\frac{\pi}{2} \operatorname{cosec} \mu\pi \qquad [-1 < \operatorname{Re}\mu < 0].$

4. $$\int_0^{\frac{\pi}{4}} \frac{\sin^\mu x\,dx}{(\cos x - \sin x)^\mu \sin 2x} = \frac{\pi}{2} \operatorname{cosec} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$$

LI ((37))(20)a

5. $$\int_0^{\frac{\pi}{4}} \frac{\sin^\mu x\,dx}{(\cos x - \sin x)^\mu \cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((37))(17)

6. $$\int_0^{\frac{\pi}{4}} \frac{\sin^\mu x\,dx}{(\cos x - \sin x)^{\mu-1} \cos^3 x} = \frac{1-\mu}{2}\mu\pi \operatorname{cosec} \mu\pi \qquad [|\operatorname{Re}\mu| < 1].$$

BI((35))(24), BI((37))(18)

7. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(\sin x + \cos x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu, \nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((48))(8)

3.668

1. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^{\cos 2t} dx = \frac{\pi}{2 \sin(\pi \cos^2 t)}.$$

FI II 788

2. $$\int_u^v \frac{(\cos u - \cos x)^{\mu-1}}{(\cos x - \cos v)^\mu} \cdot \frac{\sin x\,dx}{1 - 2a \cos x + a^2} = \frac{(1 - 2a \cos u + a^2)^{\mu-1}}{(1 - 2a \cos v + a^2)^\mu} \cdot \frac{\pi}{\sin \mu\pi}$$
$$[0 < \operatorname{Re}\mu < 1, \quad a^2 < 1].$$

BI ((73))(2)

3.669

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{p-1} x \cos^{q-p-1} x\,dx}{(a \cos x + b \sin x)^q} = \int_0^{\frac{\pi}{2}} \frac{\sin^{q-p-1} x \cos^{p-1} x}{(a \sin x + b \cos x)^q}\,dx = \frac{\mathrm{B}(p, q-p)}{a^{q-p} b^p}$$
$$[q > p > 0, \quad ab > 0].$$

BI ((331))(9)

3.67 Square roots of expressions containing trigonometric functions

3.671

1. $$\int_0^{\frac{\pi}{2}} \sin^\alpha x \cos^\beta x \sqrt{1-k^2 \sin^2 x}\, dx = \frac{1}{2} \mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$
$$[\alpha > -1, \quad \beta > -1, \quad |k| < 1].$$

GW ((331))(93)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^\alpha x \cos^\beta x}{\sqrt{1-k^2 \sin^2 x}}\, dx = \frac{1}{2} \mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$
$$[\alpha > -1, \quad \beta > -1, \quad |k| < 1].$$

GW ((331))(92)

3. $$\int_0^{\pi} \frac{\sin^{2n} x\, dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{\pi}{2^n} \sum_{j=0}^{\infty} \frac{(2j-1)!!(2n+2j-1)!!}{2^{2j} j!(n+j)!} k^{2j} \qquad [k^2 < 1];$$
$$= \frac{(2n-1)!!\pi}{2^n \sqrt{1-k^2}} \sum_{j=0}^{\infty} \frac{[(2j-1)!!]^2}{2^{2j} j!(n+j)!} \left(\frac{k^2}{k^2-1}\right)^j \qquad \left[k^2 < \frac{1}{2}\right].$$

LI ((67))(2)

429

3.672

1. $$\int_0^{\frac{\pi}{4}} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\cos x(\cos x - \sin x)}} = 2 \cdot \frac{(2n)!!}{(2n+1)!!}.$$

BI ((39))(5)

2. $$\int_0^{\frac{\pi}{4}} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\sin x(\cos x - \sin x)}} = \frac{(2n-1)!!}{(2n)!!}\pi.$$

BI ((39))(6)

3.673

$$\int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2}\boldsymbol{K}\left(\sin \frac{\pi - 2u}{4}\right).$$

BI ((74))(11)

3.674

1. $$\int_0^\pi \frac{dx}{\sqrt{1 \pm 2p\,\cos x + p^2}} = 2\boldsymbol{K}(p) \qquad [p^2 < 1].$$

BI ((67))(5)

2. $$\int_0^\pi \frac{\sin x\,dx}{\sqrt{1 - 2p\,\cos x + p^2}} = 2 \qquad [p^2 \leq 1];$$
$$= \frac{2}{p} \qquad [p^2 \geq 1].$$

BI ((67))(6)

3. $$\int_0^\pi \frac{\cos x\,dx}{\sqrt{1 - 2p\,\cos x + p^2}} = \frac{2}{p}[\boldsymbol{K}(p) - \boldsymbol{E}(p)] \qquad [p^2 < 1].$$

BI ((67))(7)

3.675

1. $$\int_u^\pi \frac{\sin\left(n + \frac{1}{2}\right) x\,dx}{\sqrt{2(\cos u - \cos x)}} = \frac{\pi}{2} P_n(\cos u).$$

WH

2. $$\int_0^u \frac{\cos\left(n + \frac{1}{2}\right) x\,dx}{\sqrt{2(\cos x - \cos u)}} = \frac{\pi}{2} P_n(\cos u).$$

FI II 684, WH

3.676

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin x\,dx}{\sqrt{1 + p^2\,\sin^2\,x}} = \frac{1}{p}\,\operatorname{arctg}\,p.$$

BI ((60))(5)

2. $$\int_0^{\frac{\pi}{2}} \operatorname{tg}^2\,x\sqrt{1 - p^2\,\sin^2\,x}\,dx = \infty.$$

3. $$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{p^2\,\cos^2\,x + q^2\,\sin^2\,x}} = \frac{1}{p}\boldsymbol{K}\left(\frac{\sqrt{p^2-q^2}}{p}\right) \qquad [0 < q < p].$$

FI II 165

3.677

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin^2\,x\,dx}{\sqrt{1+\sin^2\,x}} = \sqrt{2}\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right).$$

BI ((60))(2)

2. $$\int_0^{\frac{\pi}{2}} \frac{\cos^2\,x\,dx}{\sqrt{1+\sin^2\,x}} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((60))(3)

3.678

1. $$\int_0^{\frac{\pi}{4}} (\sec^{\frac{1}{2}}\,2x - 1)\frac{dx}{\operatorname{tg} x} = \ln 2.$$

BI ((38))(23)

2. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^2\,x\,dx}{\sqrt{1-k^2\,\sin^2\,2x}} = \sqrt{1-k^2} - \boldsymbol{E}(k) + \frac{1}{2}\boldsymbol{K}(k).$$

BI ((39))(2)

3. $$\int_0^{u} \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}}\,dx = \frac{\pi}{2}(1-\cos u) \qquad \left[u^2 < \frac{\pi^2}{4}\right].$$

LI ((74))(6)

4. $$\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1}\,x}\sqrt{\operatorname{cosec} x}\,dx = \frac{(2n-1)!!}{(2n)!!}\pi.$$

BI ((38))(24)

5. $$\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1}\,x}\,\operatorname{tg}^m\,x\sqrt{\operatorname{cosec} x}\,dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!}\pi.$$

3.679

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1-\cos^2 \beta \cos^2 x} \cdot \frac{dx}{\sqrt{1-k^2 \sin^2 x}} =$$
$$= \frac{1}{\sin\beta \cos\beta\sqrt{1-k'^2 \sin^2 \beta}} \left\{\frac{\pi}{2} - \boldsymbol{K}E(\beta, k') - \boldsymbol{E}F(\beta, k') + \boldsymbol{K}F(\beta, k')\right\} .^*$$

*

$k' = \sqrt{1-k^2}$

MO 138

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-(1-k'^2 \sin^2 \beta) \sin^2 x} \cdot \frac{dx}{\sqrt{1-k^2 \sin^2 x}} =$$
$$= \frac{1}{k'^2 \sin\beta \cos\beta\sqrt{1-k'^2 \sin^2 \beta}} \left\{\frac{\pi}{2} - \boldsymbol{K}E(\beta, k') - \boldsymbol{E}F(\beta, k') + \boldsymbol{K}F(\beta, k')\right\} .^*$$

MO 138

431

3. $$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-k^2 \sin^2 \beta \sin^2 x} \cdot \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{\boldsymbol{K}E(\beta, k) - \boldsymbol{E}F(\beta, k)}{k^2 \sin\beta \cos\beta\sqrt{1-k^2 \sin^2 \beta}}.$$

MO 138

3.68 Various forms of powers of trigonometric functions

3.681

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x \, dx}{(1-k^2 \sin^2 x)^{\varrho}} = \frac{1}{2} \mathrm{B}\,(\mu, \nu) F(\varrho, \mu; \mu+\nu; k^2) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

EH I 115(7)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x \, dx}{(1-k^2 \sin^2 x)^{\mu+\nu}} = \frac{\mathrm{B}(\mu, \nu)}{2(1-k^2)^{\mu}} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

EH I 10(20)

BI ((54))(10)

4.[8] $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu+1} x\, dx}{\cos^{\mu} x(1-k^2 \sin^2 x)^{\frac{\mu+1}{2}}} = \frac{(1+k)^{-\mu}-(1+k)^{-\mu}}{2k\mu\sqrt{\pi}}\Gamma\left(1+\frac{\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)$$
$$[-2<\operatorname{Re}\mu<1].$$

BI ((61))(5)

3.682

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x \cos^{\nu} x}{(a-b\cos^2 x)^{\varrho}}\, dx = \frac{1}{2a^{\varrho}}\,\mathrm{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2}, \varrho; \frac{\mu+\nu}{2}+1; \frac{b}{a}\right)$$
$$[\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1, \quad a>|b|\ge 0].$$

GW ((331))(64)

3.683

1. $$\int_0^{\frac{\pi}{4}} (\sin^n 2x-1)\operatorname{tg}\left(\frac{\pi}{4}+x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^n 2x-1)\operatorname{ctg} x\, dx = -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} =$$
$$= -\frac{1}{2}[\boldsymbol{C}+\psi(n+1)] \qquad [n\ge 0].$$

BI((34))(8), BI((35))(11)

2. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x-1)\operatorname{cosec}^{\mu} 2x\operatorname{tg}\left(\frac{\pi}{4}+x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x-1)\sec^{\mu} 2x\operatorname{ctg} x\, dx =$$
$$= \frac{1}{2}[\boldsymbol{C}+\psi(1-\mu)]; \qquad [\operatorname{Re}\mu<1].$$

BI ((35))(20)

3. $$\int_0^{\frac{\pi}{4}} (\sin^{2\mu} 2x-1)\operatorname{cosec}^{\mu} 2x\operatorname{tg}\left(\frac{\pi}{4}+x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^{2\mu} 2x-1)\sec^{\mu} 2x\operatorname{ctg} x\, dx =$$
$$= -\frac{1}{2\mu}+\frac{\pi}{2}\operatorname{ctg}\mu\pi.$$

BI ((35))(21)

4. $$\int_0^{\frac{\pi}{4}} (1-\sec^{\mu} 2x)\operatorname{ctg} x\, dx = \int_0^{\frac{\pi}{4}} (1-\operatorname{cosec}^{\mu} 2x)\operatorname{tg}\left(\frac{\pi}{4}+x\right) dx = \frac{1}{2}[\boldsymbol{C}+\psi(1-\mu)]$$
$$[\operatorname{Re}\mu<1].$$

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg}^\mu x - 1)\,dx}{(\cos x - \sin x)\,\sin x} = \int_0^{\frac{\pi}{2}} \frac{(\operatorname{tg}^\mu x - 1)\,dx}{(\sin x - \cos x)\,\cos x} = -\boldsymbol{C} - \psi(1-\mu) \qquad [\operatorname{Re}\mu < 1].$$

BI ((37))(9)

3.685

1. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x - \sin^{\nu-1} 2x)\,\operatorname{tg}\left(\frac{\pi}{4} + x\right)\,dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x - \cos^{\nu-1} 2x)\,\operatorname{ctg} x\,dx = \\ = \frac{1}{2}[\psi(\nu) - \psi(\mu)] \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI((34))(9), BI((35))(12)

2. $$\int_0^{\frac{\pi}{2}} (\sin^{\mu-1} x - \sin^{\nu-1} x)\frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} (\cos^{\mu-1} x - \cos^{\nu-1} x)\frac{dx}{\sin x} = \frac{1}{2}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\mu}{2}\right)\right] \\ [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((46))(2)

3. $$\int_0^{\frac{\pi}{2}} (\sin^{\mu} x - \operatorname{cosec}^{\mu} x)\frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} (\cos^{\mu} x - \sec^{\mu} x)\frac{dx}{\sin x} = -\frac{\pi}{2}\,\operatorname{tg}\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((46))(1, 3)

4. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x - \operatorname{cosec}^{\mu} 2x)\,\operatorname{ctg}\left(\frac{\pi}{4} + x\right)\,dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x - \sec^{\mu} 2x)\,\operatorname{tg} x\,dx = \\ = \frac{1}{2\mu} - \frac{\pi}{2}\,\operatorname{cosec}\mu\pi \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((35))(19, 22)

5. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x - \operatorname{cosec}^{\mu} 2x)\,\operatorname{tg}\left(\frac{\pi}{4} + x\right)\,dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x - \sec^{\mu} 2x)\,\operatorname{ctg} x\,dx = \\ = -\frac{1}{2\mu} + \frac{\pi}{2}\,\operatorname{ctg}\mu\pi \qquad [|\operatorname{Re}\mu| < 1].$$

BI ((35))(14)

6. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x + \operatorname{cosec}^{\mu} 2x)\,\operatorname{ctg}\left(\frac{\pi}{4} + x\right)\,dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x + \sec^{\mu} 2x)\,\operatorname{tg} x\,dx = \frac{\pi}{4}\,\operatorname{cosec}\mu\pi \\ [0 < \operatorname{Re}\mu < 1].$$

7. $$\int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x - \operatorname{cosec}^{\mu} 2x) \operatorname{tg}\left(\frac{\pi}{4}+x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x - \sec^{\mu} 2x) \operatorname{ctg} x\, dx = \frac{\pi}{2} \operatorname{ctg} \mu\pi$$
$$[0 < \operatorname{Re}\mu < 1].$$

BI((35))(7), LI((34))(10)

3.686

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg} x\, dx}{\cos^{\mu} x + \sec^{\mu} x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg} x\, dx}{\sin^{\mu} x + \operatorname{cosec}^{\mu} x} = \frac{\pi}{4\mu}.$$

BI((47))(28), BI((49))(14)

3.687

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\cos\left(\frac{\nu-\mu}{4}\pi\right)}{2 \cos\left(\frac{\nu+\mu}{4}\pi\right)} \mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 2].$$

BI ((46))(7)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\sin\left(\frac{\nu-\mu}{4}\pi\right)}{2 \sin\left(\frac{\nu+\mu}{4}\pi\right)} \mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 4].$$

BI((46))(8)

3. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x + \sin^{\nu} x}{\sin^{\mu+\nu} x + 1} \operatorname{ctg} x\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x + \cos^{\nu} x}{\cos^{\mu+\nu} x + 1} \operatorname{tg} x\, dx = \frac{\pi}{\mu+\nu} \sec\left(\frac{\mu-\nu}{\mu+\nu} \cdot \frac{\pi}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((49))(15)A, BI ((47))(29)

4. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x - \sin^{\nu} x}{\sin^{\mu+\nu} x - 1} \operatorname{ctg} x\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x - \cos^{\nu} x}{\cos^{\mu+\nu} x - 1} \operatorname{tg} x\, dx = \frac{\pi}{\mu+\nu} \operatorname{tg}\left(\frac{\mu-\nu}{\mu+\nu} \cdot \frac{\pi}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI((149))(16)A, BI((47))(30)

BI ((49))(12)

6. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^\mu x - \sec^\mu x}{\cos^\nu x - \sec^\nu x} \operatorname{tg} x \, dx = \frac{\pi}{2\nu} \operatorname{tg}\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right) \qquad [|\operatorname{Re}\nu| > |\operatorname{Re}\mu|].$

BI ((49))(13)

3.688

1. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\nu x - \operatorname{tg}^\mu x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$

BI ((37))(10)

2. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x - \operatorname{tg}^{1-\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \pi \operatorname{ctg} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$

BI ((37))(11)

3. $\displaystyle\int_0^{\frac{\pi}{4}} (\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x) \, dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1].$

BI ((35))(9)

4. $\displaystyle\int_0^{\frac{\pi}{4}} (\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x) \operatorname{tg} x \, dx = \frac{1}{\mu} - \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \qquad [0 < \operatorname{Re}\mu < 2].$

BI ((35))(15)

5. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1} x - \operatorname{ctg}^{\mu-1} x}{\cos 2x} \, dx = \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 2].$

BI ((35))(10)

6. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x}{\cos 2x} \operatorname{tg} x \, dx = -\frac{1}{\mu} + \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \qquad [-2 < \operatorname{Re}\mu < 0].$

BI ((35))(23)

7. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x}{1 + \cos t \sin 2x} \, dx = \pi \operatorname{cosec} t \operatorname{cosec} \mu\pi \sin \mu t \qquad [t \neq n\pi, \quad |\operatorname{Re}\mu| < 1].$

8. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1}\, x + \operatorname{ctg}^{\mu}\, x}{(\sin x + \cos x)\, \cos x}\, dx = \pi\, \operatorname{cosec} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((37))(3)

9. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu}\, x - \operatorname{ctg}^{\mu}\, x}{(\sin x + \cos x)\, \cos x}\, dx = -\pi\, \operatorname{cosec} \mu\pi + \frac{1}{\mu} \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((37))(4)

10. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\nu}\, x - \operatorname{ctg}^{\mu}\, x}{(\cos x - \sin x)\, \cos x}\, dx = \psi(1-\mu) - \psi(1+\nu) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}\nu > -1].$$

BI ((37))(5)

11. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1}\, x - \operatorname{ctg}^{\mu}\, x}{(\cos x - \sin x)\, \cos x}\, dx = \pi\, \operatorname{ctg} \mu\pi \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((37))(7)

12. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu}\, x - \operatorname{ctg}^{\mu}\, x}{(\cos x - \sin x)\, \cos x}\, dx = \pi\, \operatorname{ctg} \mu\pi - \frac{1}{\mu} \qquad [0 < \operatorname{Re}\mu < 1].$$

BI ((37))(8)

13. $$\int_0^{\frac{\pi}{4}} \frac{1}{\operatorname{tg}^{\mu}\, x + \operatorname{ctg}^{\mu}\, x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \qquad [\operatorname{Re}\mu \neq 0].$$

BI ((37))(12)

14. $$\int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^{\mu}\, x + \operatorname{ctg}^{\mu}\, x)^{\nu}} \cdot \frac{dx}{\operatorname{tg} x} = \int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^{\mu}\, x + \operatorname{ctg}^{\mu}\, x)^{\nu}} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1}\mu}\, \frac{\Gamma(\nu)}{\Gamma\left(\nu + \frac{1}{2}\right)} \qquad [\nu > 0].$$

BI((49))(25), BI((49))(26)

15. $$\int_0^{\frac{\pi}{4}} (\operatorname{tg}^{\mu}\, x - \operatorname{ctg}^{\mu}\, x)(\operatorname{tg}^{\nu}\, x - \operatorname{ctg}^{\nu}\, x)\, dx = \frac{2\pi\, \sin \frac{\mu\pi}{2}\, \sin \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \qquad [|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1].$$

16. $$\int_0^{\frac{\pi}{4}} (\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x)(\operatorname{tg}^\nu x + \operatorname{ctg}^\nu x)\, dx = \frac{2\pi \cos\frac{\mu\pi}{2} \cos\frac{\nu\pi}{2}}{\cos\mu\pi + \cos\nu\pi} \qquad [|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1].$$

BI ((35))(16)

435

17. $$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x)(\operatorname{tg}^\nu x + \operatorname{ctg}^\nu x)}{\cos 2x}\, dx = -\pi \frac{\sin\mu\pi}{\cos\mu\pi + \cos\nu\pi} \qquad [|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1].$$

BI ((35))(25)

18. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\nu x - \operatorname{ctg}^\nu x}{\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \operatorname{tg}\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1].$$

BI ((37))(14)

19. $$\int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\nu x + \operatorname{ctg}^\nu x}{\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \sec\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1].$$

BI ((37))(13)

20. $$\int_0^{\frac{\pi}{2}} \frac{(1 + \operatorname{tg} x)^\nu - 1}{(1 + \operatorname{tg} x)^{\mu+\nu}} \frac{dx}{\sin x \cos x} = \psi(\mu+\nu) - \psi(\mu) \qquad [\mu > 0, \quad \nu > 0].$$

BI ((49))(29)

3.689

1. $$\int_0^{\frac{\pi}{2}} \frac{(\sin^\mu x + \operatorname{cosec}^\mu x) \operatorname{ctg} x\, dx}{\sin^\nu x - 2\cos t + \operatorname{cosec}^\nu x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec}\frac{\mu\pi}{\nu} \sin\frac{\mu t}{\nu} \qquad [\mu < \nu].$$

LI ((50))(14)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^\mu x - 2\cos t_1 + \operatorname{cosec}^\mu x}{\sin^\nu x + 2\cos t_2 + \operatorname{cosec}^\nu x} \cdot \operatorname{ctg} x\, dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec}\frac{\mu\pi}{\nu} \sin\frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1$$
$$[\nu > \mu > 0 \quad \text{or} \quad \nu < \mu < 0 \quad \text{or} \quad \mu > 0, \quad \nu < 0 \quad \text{and} \quad \mu + \nu < 0$$
$$\text{or} \quad \mu < 0, \quad \nu > 0 \quad \text{and} \quad \mu + \nu > 0].$$

BI ((50))(15)

3.69- 3.71 Trigonometric functions of more complicated arguments

3.69- 3.71 Trigonometric functions of more complicated arguments

3.691

1. $$\int_0^\infty \sin(ax^2)\,dx = \int_0^\infty \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \qquad [a>0].$$

FI II 743a, ET I 64(7)a

2. $$\int_0^1 \sin(ax^2)\,dx = \sqrt{\frac{\pi}{2a}}S(\sqrt{a}) \qquad [a>0].$$

3. $$\int_0^1 \cos(ax^2)\,dx = \sqrt{\frac{\pi}{2a}}C(\sqrt{a}) \qquad [a>0].$$

ET I 8(5)a

4. $$\int_0^\infty \sin(ax^2)\,\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right) + \sin\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right\} \qquad [a>0, \quad b>0].$$

ET I 82(1)a

5. $$\int_0^\infty \sin(ax^2)\,\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(\frac{b^2}{a}+\frac{\pi}{4}\right)$$
$$[a>0, \quad b>0].$$

ET I 82(18), BI((70))(13) GW((334))(5a)

436

6. $$\int_0^\infty \cos ax^2\,\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\sin\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right) - \cos\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right\} \qquad [a>0, \quad b>0].$$

ET I 83(3)a

7. $$\int_0^\infty \cos ax^2\,\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right\} \qquad [a>0, \quad b>0].$$

GW((334))(5a), BI((70))(14), ET I 24(7)

8. $$\int_0^\infty (\cos ax + \sin ax)\,\sin(b^2x^2)\,dx =$$
$$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1+2C\left(\frac{a}{2b}\right)\right)\cos\left(\frac{a^2}{4b^2}\right) - \left(1-2S\left(\frac{a}{2b}\right)\right)\sin\left(\frac{a^2}{4b^2}\right)\right\} \qquad [a>0, \quad b>0].$$

ET I 85(22)

9. $$\int_0^\infty (\cos ax + \sin ax)\, \cos(b^2x^2)\, dx =$$
$$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1 + 2C\left(\frac{a}{2b}\right)\right)\,\sin\left(\frac{a^2}{4b^2}\right) + \left(1 - 2S\left(\frac{a}{2b}\right)\right)\,\cos\left(\frac{a^2}{4b^2}\right)\right\} \qquad [a > 0, \quad b > 0].$$

ET I 25(21)

10. $$\int_0^\infty \sin(a^2x^2)\,\sin 2bx\,\sin 2cx\,dx = \frac{\sqrt{\pi}}{2a}\,\sin\frac{2bc}{a^2}\,\cos\left(\frac{b^2+c^2}{a^2} - \frac{\pi}{4}\right) \qquad [a > 0,\ b > 0,\ c > 0].$$

ET I 84(15)

11. $$\int_0^\infty \sin(a^2x^2)\,\cos 2bx\,\cos 2cx\,dx = \frac{\sqrt{\pi}}{2a}\,\cos\frac{2bc}{a^2}\,\cos\left(\frac{b^2+c^2}{a^2} + \frac{\pi}{4}\right) \qquad [a > 0,\ b > 0,\ c > 0].$$

ET I 84(21)

12. $$\int_0^\infty \cos(a^2x^2)\,\sin 2bx\,\sin 2cx\,dx = \frac{\sqrt{\pi}}{2a}\,\sin\frac{2bc}{a^2}\,\sin\left(\frac{b^2+c^2}{a^2} - \frac{\pi}{4}\right) \qquad [a > 0,\ b > 0,\ c > 0].$$

ET I 25(19)

13. $$\int_0^\infty \sin(ax^2)\,\cos(bx^2)\,dx = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{a-b}}\right) \qquad [a > b > 0];$$
$$= \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{b+a}} - \frac{1}{\sqrt{b-a}}\right) \qquad [b > a > 0].$$

BI ((177))(21)

14. $$\int_0^\infty (\sin^2\, ax^2 - \sin^2\, bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}}\right) \qquad [a > 0, \quad b > 0].$$

BI ((178))(1)

15. $$\int_0^\infty (\cos^2\, ax^2 - \sin^2\, bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{a}}\right) \qquad [a > 0, \quad b > 0].$$

BI ((178))(3)

BI ((178))(5)

437

17. $$\int_0^\infty (\sin^4 ax^2 - \sin^4 bx^2)\, x = \frac{1}{64}(8-\sqrt{2}) \left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) \qquad [a > 0, \quad b > 0].$$

BI ((178))(2)

18. $$\int_0^\infty (\cos^4 ax^2 - \sin^4 bx^2)\, dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{b}} \right) + \frac{1}{32} \left(\sqrt{\frac{\pi}{2a}} - \sqrt{\frac{\pi}{2b}} \right) \qquad [a > 0, \quad b > 0].$$

BI ((178))(4)

19. $$\int_0^\infty (\cos^4 ax^2 - \cos^4 bx^2)\, dx = \frac{1}{64}(8+\sqrt{2}) \left(\sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) \qquad [a > 0, \quad b > 0].$$

BI ((178))(6)

20. $$\int_0^\infty \sin^{2n} ax^2\, dx = \int_0^\infty \cos^{2n} ax^2\, dx = \infty.$$

BI ((177))(5, 6)

21. $$\int_0^\infty \sin^{2n+1}(ax^2)\, dx = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \qquad [a > 0].$$

BI ((70))(9)

22. $$\int_0^\infty \cos^{2n+1}(ax^2)\, dx = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \qquad [a > 0].$$

BI((177))(7)A, BI((70))(10)

3.692

1. $$\int_0^\infty [\sin(a - x^2) + \cos(a - x^2)]\, dx = \sqrt{\frac{\pi}{a}} \sin a.$$

GW((333))(30c), BI((178))(7)a

GW((333))(30c), BI((178))(7)a

2. $$\int_0^\infty \cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cos ax \, dx = \sqrt{\frac{\pi}{2}} \cos\left(\frac{a^2}{2} - \frac{\pi}{8}\right) \qquad [a > 0].$$

ET I 24(8)

3. $$\int_0^\infty \sin[a(1-x^2)] \cos bx \, dx = -\frac{1}{2}\sqrt{\frac{\pi}{a}} \cos\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \qquad [a > 0].$$

ET I 23(2)

4. $$\int_0^\infty \cos[a(1-x^2)] \cos bx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \sin\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \qquad [a > 0].$$

ET I 24(10)

5. $$\int_0^\infty \sin\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \int_0^\infty \cos\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \qquad [a > 0].$$

BI ((70))(19, 20)

3.693

1. $$\int_0^\infty \sin(ax^2+2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{\cos\frac{b^2}{a}\left(\frac{1}{2} - S_2\left(\frac{b^2}{a}\right)\right) - \sin\frac{b^2}{a}\left(\frac{1}{2} - C_2\left(\frac{b^2}{a}\right)\right)\right\} \qquad [a > 0]$$

[see Use of Tables]

BI ((70))(3)

2. $$\int_0^\infty \cos(ax^2+2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{\cos\frac{b^2}{a}\left(\frac{1}{2} - C_2\left(\frac{b^2}{a}\right)\right) + \sin\frac{b^2}{a}\left(\frac{1}{2} - S_2\left(\frac{b^2}{a}\right)\right)\right\} \qquad [a > 0]$$

[see Use of Tables]

BI ((70))(4)

3.694

1. $$\int_0^\infty \sin(ax^2+2bx+c) \, dx = \sqrt{\frac{\pi}{2a}} \cos\frac{b^2}{a} \left\{\left(\frac{1}{2} - C_2\left(\frac{b^2}{a}\right)\right) \sin c + \left(\frac{1}{2} - S_2\left(\frac{b^2}{a}\right)\right) \cos c\right\} + \\ + \sqrt{\frac{\pi}{2a}} \sin\frac{b^2}{a} \left\{\left(\frac{1}{2} - S_2\left(\frac{b^2}{a}\right)\right) \sin c - \left(\frac{1}{2} - C_2\left(\frac{b^2}{a}\right)\right) \cos c\right\}$$

$[a > 0]$ [see Use of Tables]

2. $$\int_0^\infty \cos(ax^2+2bx+c)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c - \left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c\right\} + \\ + \sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c + \left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$[a > 0]$ [see Use of Tables]

GW ((334))(4b)

3.695

1. $$\int_0^\infty \sin(a^3x^3)\sin(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) + J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) - \frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a > 0, \quad b > 0].$

ET I 83(5)

2. $$\int_0^\infty \cos(a^3x^3)\cos(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) + J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) + \frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a > 0, \quad b > 0].$

ET I 24(11)

3.696

1. $$\int_0^\infty \sin(ax^4)\sin(bx^2)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET I 83(2)

2. $$\int_0^\infty \sin(ax^4)\cos(bx^2)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET I 84(19)

3. $$\int_0^\infty \cos(ax^4)\sin(bx^2)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET I 83(4), ET I 25(24)

4. $$\int_0^\infty \cos(ax^4)\cos(bx^2)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET I 25(25)

3.697

$$\int_0^\infty \sin\left(\frac{a^2}{x}\right)\sin(bx)\,dx = \frac{a\pi}{2\sqrt{b}}J_1(2a\sqrt{b}) \qquad [a>0, \quad b>0].$$

ET I 83(6)

3.698

1. $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\sin(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}[\sin 2ab - \cos 2ab + e^{-2ab}] \qquad [a>0, \quad b>0].$$

ET I 83(9)

2.[8] $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\cos(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}[\sin 2ab + \cos 2ab - e^{-2ab}]$$

ET I 24(13)

3. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\sin(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}[\sin 2ab + \cos 2ab + e^{-2ab}] \qquad [a>0, \quad b>0].$$

ET I 84(12)

4. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\cos(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}[\cos 2ab - \sin 2ab + e^{-2ab}] \qquad [a>0, \quad b>0].$$

ET I 24(14)

3.699

1. $$\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}(\cos 2ab + \sin 2ab) \qquad [a>0, \quad b>0].$$

BI ((70))(27)

2. $$\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}(\cos 2ab - \sin 2ab) \qquad [a>0, \quad b>0].$$

BI ((70))(28)

BI((179))(11, 12)A, ET I 83(6)

4. $$\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}e^{-2ab} \qquad [a>0, \quad b>0].$$

GW ((334))(9b)a

5. $$\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}e^{-2ab} \qquad [a>0, \quad b>0].$$

GW ((334))(9b)a

3.711

$$\int_0^u \sin(a\sqrt{u^2-x^2})\cos bx\,dx = \frac{\pi a u}{2\sqrt{a^2+b^2}}J_1(u\sqrt{a^2+b^2}) \qquad [a>0, \quad b>0, \quad u>0].$$

ET I 27(37)

3.712

1. $$\int_0^\infty \sin(ax^p)\,dx = \frac{\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}}{pa^{\frac{1}{p}}} \qquad [a>0, \quad p>1].$$

EH I 13(40)

2. $$\int_0^\infty \cos(ax^p)\,dx = \frac{\Gamma\left(\frac{1}{p}\right)\cos\frac{\pi}{2p}}{pa^{\frac{1}{p}}} \qquad [a>0, \quad p>1].$$

EH I 13(39)

3.713

1. $$\int_0^\infty \sin(ax^p+bx^q)\,dx = \frac{1}{p}\sum_{k=0}^\infty \frac{(-b)^k}{k!}a^{-\frac{kq+1}{p}}\Gamma\left(\frac{kq+1}{p}\right)\sin\left[\frac{k(q-p)+1}{2p}\pi\right]$$
$$[a>0, \quad b>0, \quad p>0, \quad q>0].$$

BI ((70))(7)

2. $$\int_0^\infty \cos(ax^p+bx^q)\,dx = \frac{1}{p}\sum_{k=0}^\infty \frac{(-b)^k}{k!}a^{-(kq+1)/p}\Gamma\left(\frac{kq+1}{p}\right)\cos\left[\frac{k(q-p)+1}{2p}\pi\right]$$
$$[a>0, \quad b>0, \quad p>0, \quad q>0].$$

BI ((70))(8)

3.714

1. $$\int_0^\infty \cos(z \operatorname{sh} x)\, dx = K_0(z) \qquad [\operatorname{Re} z > 0].$$

WA 202(14)

2. $$\int_0^\infty \sin(z \operatorname{ch} x)\, dx = \frac{\pi}{2} J_0(z) \qquad [\operatorname{Re} z > 0].$$

MO 36

3. $$\int_0^\infty \cos(z \operatorname{ch} x)\, dx = -\frac{\pi}{2} N_0(z) \qquad [\operatorname{Re} z > 0].$$

MO 37

4. $$\int_0^\infty \cos(z \operatorname{sh} x) \operatorname{ch} \mu x\, dx = \cos\frac{\mu\pi}{2} K_\mu(z) \qquad [\operatorname{Re} z > 0, \quad |\operatorname{Re}\mu| < 1].$$

WA 202(13)

5. $$\int_0^\pi \cos(z \operatorname{ch} x) \sin^{2\mu} x\, dx = \sqrt{\pi}\left(\frac{2}{z}\right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) I_\mu(z) \qquad \left[\operatorname{Re} z > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right].$$

WH

3.715

1. $$\int_0^\pi \sin(z \sin x) \sin ax\, dx = \sin a\pi s_{0,\,a}(z) = \sin a\pi \sum_{k=1}^\infty \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2)\dots[(2k-1)^2 - a^2]}$$
$$[a > 0].$$

WA 338(13)

2. $$\int_0^\pi \sin(z \sin x) \sin nx\, dx = \frac{1}{2}\int_{-\pi}^\pi \sin(z \sin x) \sin nx\, dx =$$
$$= [1 - (-1)^n]\int_0^{\frac{\pi}{2}} \sin(z \sin x) \sin nx\, dx = [1 - (-1)^n]\frac{\pi}{2} J_n(z)$$
$$[n = 0,\ \pm 1,\ \pm 2, \dots].$$

3. $$\int_0^{\frac{\pi}{2}} \sin(z \sin x) \sin 2x \, dx = \frac{2}{z^2}(\sin z - z \cos z).$$

LI ((43))(14)

4. $$\int_0^{\pi} \sin(z \sin x) \cos ax \, dx = (1 + \cos a\pi) s_{0,\,a}(z) =$$
$$= (1 + \cos a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \ldots [(2k-1)^2 - a^2]} \qquad [a > 0].$$

WA 338(14)

5. $$\int_0^{\pi} \sin(z \sin x) \cos[(2n+1)x] \, dx = 0.$$

GW ((334))(53b)

6. $$\int_0^{\pi} \cos(z \sin x) \sin ax \, dx = -a(1 - \cos a\pi) s_{-1,\,a}(z) =$$
$$= -a(1 - \cos a\pi) \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \ldots [(2k)^2 - a^2]} \right\}$$
$$[a > 0].$$

WA 338(12)

7. $$\int_0^{\pi} \cos(z \sin x) \sin 2nx \, dx = 0.$$

GW ((334))(54a)

8. $$\int_0^{\pi} \cos(z \sin x) \cos ax \, dx = -a \sin a\pi s_{-1,\,a}(z) =$$
$$= -a \sin a\pi \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \ldots [(2k)^2 - a^2]} \right\}$$
$$[a > 0].$$

WA 338(11)

9. $$\int_0^{\pi} \cos(z \sin x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(z \sin x) \cos nx \, dx =$$
$$= [1 + (-1)^n] \int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos nx \, dx = [1 + (-1)^n] \frac{\pi}{2} J_n(z).$$

10. $$\int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos^{2n} x \, dx = \frac{\pi}{2} \frac{(2n-1)!!}{z^n} J_n(z) \qquad \left[\operatorname{Re} n > -\frac{1}{2}\right].$$

FI II 486, WA 35a

11. $$\int_0^{\frac{\pi}{2}} \sin(z \cos x) \sin 2x \, dx = \frac{2}{z^2}(\sin z - z \cos z).$$

LI ((43))(15)

12. $$\int_0^{\frac{\pi}{2}} \sin(z \cos x) \cos ax \, dx = \cos \frac{a\pi}{2} s_{0,\,a}(z) = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{J}_\nu(z) - \mathbf{J}_-\nu(z)] =$$
$$= -\frac{\pi}{4} \sec \frac{a\pi}{4} [\mathbf{E}_\nu(z) + \mathbf{E}_{-\nu}(z)] =$$
$$= \cos \frac{a\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \ldots [(2k-1)^2 - a^2]} \qquad [a > 0].$$

WA 339

13. $$\int_0^{\pi} \sin(z \cos x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(z \cos x) \cos nx \, dx = \pi \sin \frac{n\pi}{2} J_n(z).$$

GW ((334))(55b)

14. $$\int_0^{\frac{\pi}{2}} \sin(z \cos x) \cos[(2n+1)x] \, dx = (-1)^n \frac{\pi}{2} J_{2n+1}(z).$$

WA 30(8)

15. $$\int_0^{\frac{\pi}{2}} \sin(a \cos x) \, tg \, x \, dx = \operatorname{si}(a) + \frac{\pi}{2} \qquad [a > 0].$$

BI ((43))(17)

16. $$\int_0^{\frac{\pi}{2}} \sin(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_\nu(z) \qquad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

WA 358(1)

17. $$\int_0^{\frac{\pi}{2}} \cos(z \cos x) \cos ax \, dx = -a \sin \frac{a\pi}{2} s_{-1,\,a}(z) =$$
$$= \frac{\pi}{4} \sec \frac{a\pi}{2} [\mathbf{J}_\nu(z) + \mathbf{J}_{-\nu}(z)] = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{E}_\nu(z) - \mathbf{E}_{-\nu}(z)] =$$
$$= -a \sin \frac{a\pi}{2} \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \ldots [(2k)^2 - a^2]} \right\}$$
$$[a > 0].$$

18. $\int_0^{\pi} \cos(z \cos x) \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(z \cos x) \cos nx \, dx = \pi \cos \frac{n\pi}{2} J_n(z).$

GW ((334))(56b)

19. $\int_0^{\frac{\pi}{2}} \cos(z \cos x) \cos 2nx \, dx = (-1)^n \cdot \frac{\pi}{2} J_{2n}(z).$

WA 30(9)

20. $\int_0^{\frac{\pi}{2}} \cos(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(z) \qquad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$

WA 35, WH

21. $\int_0^{\pi} \cos(z \cos x) \sin^{2\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{z}\right)^{\mu} \Gamma\left(\mu + \frac{1}{2}\right) J_{\mu}(z) \qquad \left[\operatorname{Re} \mu > -\frac{1}{2}\right].$

WH

3.716

1. $\int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \, dx = \frac{1}{2}[e^{-a} \overline{\operatorname{Ei}}(a) - e^{a} \operatorname{Ei}(-a)] \qquad [a > 0] \qquad (\text{cf. } 3.723\ 1.).$

3.723

BI ((43))(1)

2. $\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \, dx = \frac{\pi}{2} e^{-a} \qquad [a \geq 0].$

BI ((43))(2)

3. $\int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \qquad [a \geq 0].$

BI ((43))(7)

4. $\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \sin^2 x \, dx = \frac{1-a}{4} \pi e^{-a} \qquad [a \geq 0].$

5. $$\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \cos^2 x \, dx = \frac{1+a}{4}\pi e^{-a} \qquad [a \geq 0].$$

BI ((43))(9)

6. $$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \operatorname{tg} x \, dx = \frac{\pi}{2}e^{-a} \qquad [a > 0].$$

BI ((43))(5)

7. $$\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \operatorname{tg} x \, dx = -\frac{1}{2}[e^{-a}\, \overline{\mathrm{Ei}}(a) + e^{a}\, \mathrm{Ei}(-a)] \qquad [a > 0] \qquad \text{(cf. 3.723 5.)}.$$

3.723
BI ((43))(6)

8. $$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \sin^2 x \operatorname{tg} x \, dx = \frac{2-a}{4}\pi e^{-a} \qquad [a > 0].$$

BI ((43))(11)

9. $$\int_0^{\frac{\pi}{2}} \sin^2(a \operatorname{tg} x) \, dx = \frac{\pi}{4}(1 - e^{-2a}) \qquad [a \geq 0] \qquad \text{(cf. 3.742 1.)}.$$

3.742
BI ((43))(3)

10. $$\int_0^{\frac{\pi}{2}} \cos^2(a \operatorname{tg} x) \, dx = \frac{\pi}{4}(1 + e^{-2a}) \qquad [a \geq 0] \qquad \text{(cf. 3.742 3.)}.$$

3.742
BI ((43))(4)

11. $$\int_0^{\frac{\pi}{2}} \sin^2(a \operatorname{tg} x) \operatorname{ctg}^2 x \, dx = \frac{\pi}{4}(e^{-2a} + 2a - 1) \qquad [a \geq 0].$$

12. $$\int_0^{\frac{\pi}{2}} [1 - \sec^2 x \cos(\operatorname{tg} x)] \frac{dx}{\operatorname{tg} x} = \boldsymbol{C}.$$

BI ((51))(14)

13. $$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{ctg} x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \qquad [a \geq 0] \qquad (\text{cf. } 3.716\ 3.),$$

3.716

and in general, formulas 3.716 remain valid if we replace $\operatorname{tg} x$ in the argument of the sine or cosine with $\operatorname{ctg} x$ if we also replace $\sin x$ with $\cos x$, $\cos x$ with $\sin x$, hence $\operatorname{tg} x$ with $\operatorname{ctg} x$, $\operatorname{ctg} x$ with $\operatorname{tg} x$, $\sec x$ with $\operatorname{cosec} x$, and $\operatorname{cosec} x$ with $\sec x$ in the factors. Analogously,

3.717

$$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{cosec} x) \sin(a \operatorname{ctg} x) \frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} \sin(a \sec x) \sin(a \operatorname{tg} x) \frac{dx}{\sin x} = \frac{\pi}{2} \sin a \qquad [a \geq 0].$$

BI ((52))(11, 12)

3.718

1. $$\int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2} p - a \operatorname{tg} x\right) \operatorname{tg}^{p-1} x \, dx = \int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2} p - a \operatorname{tg} x\right) \operatorname{tg}^p x \, dx = \frac{\pi}{2} e^{-a}$$
$$[p^2 < 1, \quad p \neq 0, \quad a \geq 0].$$

BI ((44))(5, 6)

2. $$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x - \nu x) \sin^{\nu-2} x \, dx = 0 \qquad [\operatorname{Re} \nu > 0, \quad a > 0].$$

NH 157(15)

3. $$\int_0^{\frac{\pi}{2}} \sin(n \operatorname{tg} x + \nu x) \frac{\cos^{\nu-1} x}{\sin x} \, dx = \frac{\pi}{2} \qquad [\operatorname{Re} \nu > 0].$$

BI ((51))(15)

LO V 153(112), NT 157(14)

5. $$\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x + \nu x) \cos^{\nu} x \, dx = 2^{-\nu-1}\pi e^{-a} \qquad [\operatorname{Re}\nu > -1, \quad a \geq 0].$$

BI ((44))(4)

6. $$\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x - \gamma x) \cos^{\nu} x \, dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2}, -\frac{\nu+1}{2}}(2a)}{\Gamma\left(1 + \frac{\gamma+\nu}{2}\right)}$$
$$\left[a > 0, \quad \operatorname{Re}\nu > -1, \quad \frac{\nu+\gamma}{2} \neq -1, -2, \ldots\right].$$

EH I 274(13)a

7. $$\int_0^{\frac{\pi}{2}} \frac{\sin nx - \sin(nx - a \operatorname{tg} x)}{\sin x} \cos^{n-1} x \, dx = \begin{cases} \pi/2 & [n = 0, \quad a > 0], \\ \pi(1 - e^{-a}) & [n = 1, \quad a \geq 0]. \end{cases}$$

LO V 153(114)

3.719

1.[6] $$\int_0^{\pi} \sin(\nu x - z \sin x) \, dx = \pi \mathbf{E}_{\nu}(z).$$

WA 336(2)

2. $$\int_0^{\pi} \cos(nx - z \sin x) \, dx = \pi J_n(z).$$

WH

3. $$\int_0^{\pi} \cos(\nu x - z \sin x) \, dx = \pi \mathbf{J}_{\nu}(z).$$

WA 336(1)

3.72- 3.74 Combinations of trigonometric and rational functions

3.721

1. $$\int_0^{\infty} \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2} \operatorname{sign} a.$$

2. $$\int_1^\infty \frac{\sin(ax)}{x}\,dx = -\operatorname{si}(a).$$

BI 203(1)

3. $$\int_1^\infty \frac{\cos(ax)}{x}\,dx - \operatorname{ci}(a).$$

BI 203(5)

3.722

1. $$\int_0^\infty \frac{\sin(ax)}{x+\beta}\,dx = \operatorname{ci}(a\beta)\,\sin(a\beta) - \cos(a\beta)\,\operatorname{si}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0].$$

BI((16))(1), FI II 646a

445

2.[7] $$\int_{-\infty}^\infty \frac{\sin(ax)}{x+\beta}\,dx = \pi e^{ia\beta} \qquad [a > 0, \quad \operatorname{Im}\beta > 0].$$

3. $$\int_0^\infty \frac{\cos(ax)}{x+\beta}\,dx = -\sin(a\beta)\,\operatorname{si}(a\beta) - \cos(a\beta)\,\operatorname{ci}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0].$$

ET I 8(7), BI((160))(2)

4.* $$\int_{-\infty}^\infty \frac{\cos(ax)}{x+\beta}\,dx = -i\pi e^{iab} \qquad [a > 0, \quad \operatorname{Im}\beta > 0].$$

5.* $$\int_0^\infty \frac{\sin(ax)}{\beta - x}\,dx = \sin(\beta a)\,\operatorname{ci}(\beta a) - \cos(\beta a)[\operatorname{si}(\beta a) + \pi]$$
$$[a > 0, \quad \beta \text{ not real and positive}].$$

FI II 646, BI((161))(1)

6.[8] $$\int_{-\infty}^\infty \frac{\sin(ax)}{\beta - x}\,dx = -\pi e^{ia\beta} \qquad [a > 0, \quad \operatorname{Im}\beta > 0].$$

7.* $\int_0^\infty \frac{\cos(ax)}{\beta - x}\,dx = -\cos(a\beta)\ \mathrm{ci}(a\beta) + \sin(a\beta)[\mathrm{si}(a\beta) + \pi]$

$[a > 0, \quad \beta \text{ not real and positive}].$

ET I 8(8), BI((161))(2)a

8.7 $\int_{-\infty}^\infty \frac{\cos(ax)}{\beta - x}\,dx = -\pi e^{ia\beta} \qquad [a > 0, \quad \operatorname{Im}\beta > 0].$

3.723

1.7 $\int_0^\infty \frac{\sin(ax)}{\beta^2 + x^2}\,dx = \frac{1}{2\beta}[e^{-a\beta}\ \mathrm{Ei}(a\beta) - e^{a\beta}\ \mathrm{Ei}(-a\beta)] \qquad [a > 0, \quad \beta > 0].$

ET I 65(14), BI((160))(3)

2. $\int_0^\infty \frac{\cos(ax)}{\beta^2 + x^2}\,dx = \frac{\pi}{2\beta}e^{-a\beta} \qquad [a \geq 0, \quad \operatorname{Re}\beta > 0].$

FI II 741, 750, ET I 8(11), WH

3. $\int_0^\infty \frac{x\ \sin(ax)}{\beta^2 + x^2}\,dx = \frac{\pi}{2}e^{-a\beta} \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$

FI II 741, 750, ET I 65(15), WH

4. $\int_{-\infty}^\infty \frac{x\ \sin(ax)}{\beta^2 + x^2}\,dx = \pi e^{-a\beta} \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$

BI ((202))(10)

5.7 $\int_0^\infty \frac{x\ \cos(ax)}{\beta^2 + x^2}\,dx = -\frac{1}{2}[e^{-a\beta}\ \mathrm{Ei}(a\beta) + e^{a\beta}\ \mathrm{Ei}(-a\beta)] \qquad [a > 0, \quad \beta > 0].$

BI ((160))(6)

6. $\int_{-\infty}^\infty \frac{\sin[a(b - x)]}{c^2 + x^2}\,dx = \frac{\pi}{c}e^{-ac}\ \sin(ab) \qquad [a > 0, \quad b > 0, \quad c > 0].$

LI ((202))(9)

7. $\int_{-\infty}^\infty \frac{\cos[a(b - x)]}{c^2 + x^2}\,dx = \frac{\pi}{c}e^{-ac}\ \cos(ab) \qquad [a > 0, \quad b > 0, \quad c > 0].$

8. $$\int_0^\infty \frac{\sin(ax)}{\beta^2 - x^2}\,dx = \frac{1}{\beta}\left[\sin(a\beta)\ \mathrm{ci}(a\beta) - \cos(a\beta)\left(\mathrm{si}(a\beta) + \frac{\pi}{2}\right)\right] \qquad [|\arg\beta| < \pi, \quad a > 0].$$

BI ((161))(3)

9. $$\int_0^\infty \frac{\cos(ax)}{b^2 - x^2}\,dx = \frac{\pi}{2b}\ \sin(ab) \qquad [a > 0, \quad b > 0].$$

BI((161))(5), ET I 9(15)

10. $$\int_0^\infty \frac{x\ \sin(ax)}{b^2 - x^2}\,dx = -\frac{\pi}{2}\ \cos(ab) \qquad [a > 0].$$

FI II 647, ET II 252(45)

11. $$\int_0^\infty \frac{x\ \cos(ax)}{\beta^2 + x^2}\,dx = \cos(a\beta)\ \mathrm{ci}(a\beta) + \sin(a\beta)\left[\mathrm{si}(a\beta) + \frac{\pi}{2}\right] \qquad [|\arg\beta| < \pi, \quad a > 0].$$

BI ((161))(6)

12. $$\int_{-\infty}^\infty \frac{\sin(ax)}{x(x-b)}\,dx = \pi\frac{\cos(ab) - 1}{b} \qquad [a > 0, \quad b > 0].$$

ET II 252(44)

3.724

1. $$\int_{-\infty}^\infty \frac{b + cx}{p + 2qx + x^2}\ \sin(ax)\,dx = \left(\frac{cq - b}{\sqrt{p - q^2}}\ \sin(aq) + c\ \cos(aq)\right)\pi e^{-a\sqrt{p-q^2}}$$
$$[a > 0, \quad p > q^2].$$

BI ((202))(12)

2. $$\int_{-\infty}^\infty \frac{b + cx}{p + 2qx + x^2}\ \cos(ax)\,dx = \left(\frac{b - cq}{\sqrt{p - q^2}}\ \cos(aq) + c\ \sin(aq)\right)\pi e^{-a\sqrt{p-q^2}}$$
$$[a > 0, \quad p > q^2].$$

BI ((202))(13)

BI ((202))(14)

3.725

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(\beta^2+x^2)} = \frac{\pi}{2\beta^2}(1-e^{-a\beta}) \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

BI ((172))(1)

2. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(b^2-x^2)} = \frac{\pi}{2b^2}(1-\cos(ab)) \qquad [a > 0].$$

BI ((172))(4)

3. $$\int_0^\infty \frac{\sin(ax)\,\cos(bx)}{x(x^2+\beta^2)}\,dx = \frac{\pi}{2\beta^2}e^{-\beta b}\,\operatorname{sh}(a\beta) \qquad [0 < a < b]:$$
$$= -\frac{\pi}{2\beta^2}e^{-a\beta}\,\operatorname{ch}(b\beta) + \frac{\pi}{2\beta^2} \qquad [a > b > 0].$$

ET I 19(4)

447

3.726

1.[7] $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{b^3 \pm b^2x + bx^2 \pm x^3} =$$
$$= \pm\frac{1}{4b}\left[e^{-ab}\,\operatorname{Ei}(ab) - e^{ab}\,\operatorname{Ei}(-ab) - 2\,\operatorname{ci}(ab)\,\sin(ab) + 2\,\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right] +$$
$$+\frac{\pi e^{-ab} - \pi\,\cos(ab)}{4b} \qquad [a > 0, \quad b > 0]$$

ET I 65(21)A, BI((176))(10, 13)

2.[7] $$\int_0^\infty \frac{x^2\,\sin(ax)\,dx}{b^3 \pm b^2x + bx^2 \pm x^3} =$$
$$= \frac{1}{4}\left[e^{ab}\,\operatorname{Ei}(-ab) - e^{-ab}\operatorname{Ei}(ab) + 2\,\operatorname{ci}(ab)\,\sin(ab) - 2\,\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right] \pm$$
$$\pm\pi(e^{-ab} + \cos(ab)) \qquad [a > 0, \quad b > 0]$$

ET I 66(22), BI((176))(11, 14)

3.727

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{b^4+x^4} = \frac{\pi\sqrt{2}}{4b^3}\,\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}} + \sin\frac{ab}{\sqrt{2}}\right) \qquad [a > 0, \quad b > 0].$$

2.[8] $$\int_0^\infty \frac{\sin(ax)\,dx}{b^4 - x^4} = \frac{1}{4b^3}\left[2\sin(ab)\,\mathrm{ci}(ab) - 2\cos(ab)\left(\mathrm{si}(ab) + \frac{\pi}{2}\right) + \right.$$
$$\left. + e^{-ab}\,\mathrm{Ei}(ab) - e^{ab}\,\mathrm{Ei}(-ab)\right] \qquad [a > 0, \quad b > 0],$$

BI ((161))(12)

3. $$\int_0^\infty \frac{\cos(ax)\,dx}{b^4 - x^4} = \frac{\pi}{4b^3}[e^{-ab} + \sin(ab)] \qquad [a > 0, \quad b > 0] \qquad \text{(cf. 3.723 2. and 3.723 9.).}$$

3.723
BI ((161))(16)

4. $$\int_0^\infty \frac{x\sin(ax)\,dx}{b^4 + x^4} = \frac{\pi}{2b^2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\sin\frac{ab}{\sqrt{2}} \qquad [a > 0, \quad b > 0].$$

BI ((160))(23)a

5. $$\int_0^\infty \frac{x\sin(ax)}{b^4 - x^4}\,dx = \frac{\pi}{4b^2}[e^{-ab} - \cos(ab)] \qquad [a > 0, \quad b > 0],$$
(cf. 3.723 3. and 3.723 10.).

3.723
BI ((161))(13)

6.[7] $$\int_0^\infty \frac{x\cos(ax)\,dx}{b^4 - x^4} = \frac{1}{4b^2}\left[2\cos(ab)\,\mathrm{ci}(ab) + 2\sin(ab)\left(\mathrm{si}(ab) + \frac{\pi}{2}\right) - \right.$$
$$\left. - e^{-ab}\,\mathrm{Ei}(ab) - e^{ab}\,\mathrm{Ei}(-ab)\right] \qquad [a > 0, \quad b > 0],$$
(cf. 3.723 5. and 3.723 11.).

3.723
BI ((161))(17)

7. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{b^4 + x^4} = \frac{\pi\sqrt{2}}{4b}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}} - \sin\frac{ab}{\sqrt{2}}\right) \qquad [a > 0, \quad b > 0].$$

8.[7] $$\int_0^\infty \frac{x^2 \sin(ax)\,dx}{b^4 - x^4} = \frac{1}{4b}[2 \sin(ab) \operatorname{ci}(ab) - - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2}\right) - e^{-ab} \operatorname{Ei}(ab) + e^{ab} \operatorname{Ei}(-ab)\Big] \qquad [a > 0, \quad b > 0],$$

3.723
BI ((161))(14)

9. $$\int_0^\infty \frac{x^2 \cos(ax)\,dx}{b^4 - x^4} = \frac{\pi}{4b}(\sin(ab) - e^{-ab}) \qquad [a > 0, \quad b > 0],$$

BI ((161))(18)

10. $$\int_0^\infty \frac{x^3 \sin(ax)}{b^4 + x^4}\,dx = \frac{\pi}{2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos\frac{ab}{\sqrt{2}} \qquad [a > 0, \quad b > 0].$$

BI ((160))(24)

11. $$\int_0^\infty \frac{x^3 \sin(ax)}{b^4 - x^4}\,dx = \frac{-\pi}{4}[e^{-ab} - \cos(ab)] \qquad [a > 0, \quad b > 0],$$

BI ((161))(15)

12.[7] $$\int_0^\infty \frac{x^3 \cos(ax)\,dx}{b^4 - x^4} = \frac{1}{4}\Big[2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2}\right) + + e^{-ab} \operatorname{Ei}(ab) + e^{ab} \operatorname{Ei}(-ab)] \qquad [a > 0, \quad b > 0],$$

3.723
BI((161))(19)

3.728

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(\beta e^{-a\gamma} - \gamma e^{-a\beta})}{2\beta\gamma(\beta^2 - \gamma^2)} \qquad [a > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

BI ((175))(1)

2. $$\int_0^\infty \frac{x \sin(ax)\,dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(e^{-a\beta} - e^{-a\gamma})}{2(\gamma^2 - \beta^2)} \qquad [a > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

3. $$\int_0^\infty \frac{x^2 \cos(ax)\,dx}{(\beta^2+x^2)(\gamma^2+x^2)} = \frac{\pi(\beta e^{-a\beta}-\gamma e^{-a\gamma})}{2(\beta^2-\gamma^2)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0].$$

BI ((175))(2)

4. $$\int_0^\infty \frac{x^3 \sin(ax)\,dx}{(\beta^2+x^2)(\gamma^2+x^2)} = \frac{\pi(\beta^2 e^{-a\beta}-\gamma^2 e^{-a\gamma})}{2(\beta^2-\gamma^2)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0].$$

BI ((174))(2)

5. $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(b\sin(ac)-c\sin(ab))}{2bc(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((175))(3)

6. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(\cos(ab)-\cos(ac))}{2(b^2-c^2)} \qquad [a>0].$$

BI ((174))(3)

7. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(c\sin(ac)-b\sin(ab))}{2(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((175))(4)

8. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(b^2\cos(ab)-c^2\cos(ac))}{2(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((174))(4)

3.729

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2+x^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab} \qquad [a>0, \quad b>0].$$

BI ((170))(7)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(b^2+x^2)^2} = \frac{\pi}{4b}ae^{-ab} \qquad [a>0, \quad b>0].$$

3. $$\int_0^\infty \cos(px)\,\frac{1-x^2}{(1+x^2)^2}\,dx = \frac{\pi p}{2} e^{-p}.$$

BI ((43))(10)a

4. $$\int_0^\infty \frac{x^3\,\sin(ax)\,dx}{(b^2+x^2)^2} = \frac{\pi}{4}(2-ab)e^{-ab} \qquad [a>0, \quad b>0].$$

BI ((170))(4)

3.731

Notations: $2A^2 = \sqrt{b^4+c^2}+b^2, 2B^2 = \sqrt{b^4+c^2}-b^2$,

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(x^2+b^2)^2+c^2} = \frac{\pi}{2c}\,\frac{e^{-aA}(B\,\cos(aB)+A\,\sin(aB))}{\sqrt{b^4+c^2}} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((176))(3)

2. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(x^2+b^2)^2+c^2} = \frac{\pi}{2c} e^{-aA}\,\sin(aB) \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((176))(1)

3. $$\int_0^\infty \frac{(x^2+b^2)\,\cos(ax)\,dx}{(x^2+b^2)^2+c^2} = \frac{\pi}{2}\,\frac{e^{-aA}(A\,\cos(aB)-B\,\sin(aB))}{\sqrt{b^4+c^2}} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((176))(4)

4. $$\int_0^\infty \frac{x(x^2+b^2)\,\sin(ax)\,dx}{(x^2+b^2)^2+c^2} = \frac{\pi}{2} e^{-aA}\,\cos(aB) \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((176))(2)

3.732

1. $$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right] \sin(ax)\,dx = \frac{\pi}{\beta} e^{-a\beta}\,\sin(a\gamma)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{is not real}].$$

ET I 65(16)

ET I 8(13)

3. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2} - \frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right] \sin(ax)\,dx = \pi e^{-a\beta}\ \cos(a\gamma)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{is not real}].$$

LI ((175))(17)

4. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2} + \frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right] \cos(ax)\,dx = \pi e^{-a\beta}\ \sin(a\gamma) \qquad [a>0, \quad |\operatorname{Im} a|<\operatorname{Re}\beta].$$

LI ((176))(21)

3.733

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{x^4+2b^2x^2\ \cos 2t+b^4} = \frac{\pi}{2b^3}\ \exp(-ab\ \cos t)\frac{\sin(t+ab\ \sin t)}{\sin 2t}$$
$$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right].$$

BI ((176))(7)

2. $$\int_0^\infty \frac{x\ \sin(ax)\,dx}{x^4+2b^2x^2\ \cos 2t+b^4} = \frac{\pi}{2b^2}\ \exp(-ab\ \cos t)\frac{\sin(ab\ \sin t)}{\sin 2t} \qquad \left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right].$$

BI((176))(5), ET I 66(23)

3. $$\int_0^\infty \frac{x^2\ \cos(ax)\,dx}{x^4+2b^2x^2\ \cos 2t+b^4} = \frac{\pi}{2b}\ \exp(-ab\ \cos t)\frac{\sin(t-ab\ \sin t)}{\sin 2t}$$
$$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right].$$

BI ((176))(8)

4. $$\int_0^\infty \frac{x^3\ \sin(ax)\,dx}{x^4+2b^2x^2\ \cos 2t+b^4} = \frac{\pi}{2}\ \exp(-ab\ \cos t)\frac{\sin(2t-ab\ \sin t)}{\sin 2t}$$
$$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right].$$

BI ((176))(6)

5. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(x^4+2b^2x^2\ \cos 2t+b^4)} = \frac{\pi}{2b^4}\left[1-\exp(-ab\ \cos t)\frac{\sin(2t+ab\ \sin t)}{\sin 2t}\right]$$
$$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right].$$

3.734

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(b^4+x^4)} = \frac{\pi}{2b^4}\left[1-\exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}\right] \qquad [a>0, \quad b>0].$$

BI ((172))(7)

2. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(b^4-x^4)} = \frac{\pi}{4b^4}[2-e^{-ab}-\cos(ab)] \qquad [a>0, \quad b>0].$$

BI ((172))(10)

3.735

$$\int_0^\infty \frac{\sin(ax)\,dx}{x(b^2+x^2)^2} = \frac{\pi}{2b^4}\left[1-\frac{1}{2}e^{-ab}(2+ab)\right] \qquad [a>0, \quad b>0].$$

WH, BI ((172))(22)

3.736

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2+x^2)(b^4-x^4)} = \frac{\pi}{8b^5}[\sin(ab)+(2+ab)e^{-ab}] \qquad [a>0, \quad b>0]$$

(cf. 3.723 3. and 3.729 1.).

3.729

3.723

BI ((176))(5)

2. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(b^2+x^2)(b^4-x^4)} = \frac{\pi}{8b^4}[(1+ab)e^{-ab}-\cos(ab)] \qquad [a>0, \quad b>0]$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723

BI ((174))(5)

3.729

3.723
BI ((175))(6)

4. $$\int_0^\infty \frac{x^3 \sin(ax)\,dx}{(b^2+x^2)(b^4-x^4)} = \frac{\pi}{8b^2}[(1-ab)e^{-ab} - \cos(ab)] \qquad [a>0, \quad b>0]$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723
BI ((174))(6)}

5. $$\int_0^\infty \frac{x^4 \cos(ax)\,dx}{(b^2+x^2)(b^4-x^4)} = \frac{\pi}{8b}[\sin(ab) + (ab-2)e^{-ab}] \qquad [a>0, \quad b>0]$$

(cf. 3.723 2. and 9. and 3.729 1.).

3.729

3.723
BI ((175))(7)

6. $$\int_0^\infty \frac{x^5 \sin(ax)\,dx}{(b^2+x^2)(b^4-x^4)} = \frac{\pi}{8}[(ab-3)e^{-ab} - \cos(ab)] \qquad [a>0, \quad b>0],$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723
BI ((174))(7)

3.737

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2+x^2)^n} = \frac{\pi e^{-ab}}{(2b)^{2n-1}(n-1)!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2ab)^k}{k!(n-k-1)!};$$

$$= \frac{(-1)^{n-1}\pi}{b^{2n-1}(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{e^{-ab\sqrt{p}}}{\sqrt{p}}\right)\right]_{p=1};$$

2. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(x^2+\beta^2)^{n+1}} = \frac{\pi a e^{-a\beta}}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!} \qquad [a>0, \quad \operatorname{Re}\beta > 0].$$

GW ((333))(66c)

3. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(\beta^2+x^2)^{n+1}} = \frac{\pi}{2\beta^{2n+2}}\left[1 - \frac{e^{-a\beta}}{2^n n!} F_n(a\beta)\right]$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad F_0(z)=1, \quad F_1(z)=z+2, \dots, F_n(z)=(z+2n)F_{n-1}(z) - zF'_{n-1}(z)\right].$$

GW ((333))(66e)

4. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(b^2+x^2)^3} = \frac{\pi a}{16b^3}(1+ab)e^{-ab} \qquad [a>0, \quad b>0].$$

BI((170))(5), ET I 67(35)a

5. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(b^2+x^2)^4} = \frac{\pi a}{96b^5}(3+3ab+a^2b^2)e^{-ab} \qquad [a>0, \quad b>0].$$

BI((170))(6), ET I 67(35)a

3.738

1. $$\int_0^\infty \frac{x^{m-1}\,\sin(ax)}{x^{2n}+\beta^{2n}}\,dx = -\frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n} \exp\left[-a\beta\,\sin\frac{(2k-1)\pi}{2n}\right] \times$$
$$\times \cos\left\{\frac{(2k-1)m\pi}{2n} + a\beta\,\cos\frac{(2k-1)\pi}{2n}\right\} \qquad [m \text{ is even}];$$
$$\left[a>0, \quad |\arg\beta| < \frac{\pi}{2n}, \quad 0<m\le 2n\right].$$

ET I 67(38)

2. $$\int_0^\infty \frac{x^{m-1}\,\cos(ax)}{x^{2n}+\beta^{2n}}\,dx = \frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n} \exp\left[-a\beta\,\sin\frac{(2k-1)\pi}{2n}\right] \times$$
$$\times \sin\left\{\frac{(2k-1)m\pi}{2n} + a\beta\,\cos\frac{(2k-1)\pi}{2n}\right\} \qquad [m \text{ is odd}];$$
$$\left[a>0, \quad |\arg\beta| < \frac{\pi}{2n}, \quad 0<m<2n+1\right].$$

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} =$$
$$= \frac{\pi(-1)^n}{(2n)!2^{2n+1}}\left[2\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}e^{2(k-n)a} + (-1)^n\binom{2n}{n}\right] \qquad [a>0, \quad n\ge 0].$$

LI ((174))(8)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} = \frac{(-1)^n}{(2n+1)!}\,\frac{\pi}{2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}e^{(2k-2n-1)a}$$
$$[a\ge 0, \quad n\ge 0]$$
$$= \frac{\pi 2^{-2n-1}}{(2n+1)(n!)^2} \qquad [a=0, \quad n\ge 0].$$

BI((175))(8)

3. $$\int_0^\infty \frac{x\,\sin(ax)\,dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} =$$
$$= \frac{\pi(-1)^n}{(2n+1)!2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}(2n-2k+1)e^{(2k-2n-1)a} \qquad [a>0, \quad n\ge 0].$$

LI ((174))(9)

453

4. $$\int_0^\infty \frac{\cos ax\,dx}{(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} = \frac{\pi 2^{1-2n}}{(2n)!}\sum_{k=1}^{n}(-1)^k k\binom{2n}{n-k}e^{-2ak} \qquad [n\ge 1, \quad a\ge 0].$$

3.741

1. $$\int_0^\infty \frac{\sin(ax)\,\sin(bx)}{x}\,dx = \frac{1}{4}\ln\left(\frac{a+b}{a-b}\right)^2 \qquad [a>0, \quad b>0, \quad a\neq b].$$

FI II 647

2. $$\int_0^\infty \frac{\sin(ax)\,\cos(bx)}{x}\,dx = \frac{\pi}{2} \qquad [a>b\ge 0];$$
$$= \frac{\pi}{4} \qquad [a=b>0];$$
$$= 0 \qquad [b>a\ge 0].$$

FI II 645

3. $$\int_0^\infty \frac{\sin(ax)\,\sin(bx)}{x^2}\,dx = \frac{a\pi}{2} \qquad [0<a\le b];$$
$$= \frac{b\pi}{2} \qquad [0<b\le a].$$

1. $$\int_0^\infty \frac{\sin(ax)\,\sin(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}(e^{-|a-b|\beta} - e^{-(a+b)\beta}) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\beta>0].$$

BI((162))(1)A, GW((333))(71a)

2. $$\int_0^\infty \frac{\sin(ax)\,\cos(bx)}{\beta^2+x^2}\,dx = \frac{1}{4\beta}e^{-a\beta}\{e^{b\beta}\ \mathrm{Ei}[\beta(a-b)] + e^{-b\beta}\ \mathrm{Ei}[\beta(a+b)]\} - \\ -\frac{1}{4\beta}e^{a\beta}\{e^{b\beta}\ \mathrm{Ei}[-\beta(a+b)] + e^{-b\beta}\ \mathrm{Ei}[\beta(b-a)]\}.$$

BI ((162))(3)

3. $$\int_0^\infty \frac{\cos(ax)\,\cos(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}[e^{-|a-b|\beta} + e^{-(a+b)\beta}] \qquad [a>0, \quad b>0, \quad \operatorname{Re}\beta>0].$$

BI((163))(1)A, GW((333))(71c)

4. $$\int_0^\infty \frac{x\,\cos(ax)\,\cos(bx)}{\beta^2+x^2}\,dx = -\frac{1}{4}e^{a\beta}\{e^{b\beta}\ \mathrm{Ei}[-\beta(a+b)] + e^{-b\beta}\ \mathrm{Ei}[\beta(b-a)]\} - \\ -\frac{1}{4}e^{-a\beta}\{e^{b\beta}\ \mathrm{Ei}[\beta(a-b)] + e^{-b\beta}\ \mathrm{Ei}[\beta(a+b)]\} \qquad [a \neq b]; \\ = \infty \qquad [a=b].$$

BI ((163))(2)

5. $$\int_0^\infty \frac{x\,\sin(ax)\,\cos(bx)}{x^2+\beta^2}\,dx = \frac{\pi}{2}e^{-a\beta}\ \mathrm{ch}(b\beta) \qquad [0<b<a]; \\ = \frac{\pi}{4}e^{-2a\beta} \qquad [0<b=a]; \\ = -\frac{\pi}{2}e^{-b\beta}\ \mathrm{sh}(a\beta) \qquad [0<a<b].$$

BI ((162))(4)

6. $$\int_0^\infty \frac{\sin(ax)\,\sin(bx)}{p^2-x^2}\,dx = -\frac{\pi}{2p}\ \cos(ap)\ \sin(bp) \qquad [a>b>0]; \\ = -\frac{\pi}{4p}\ \sin(2ap) \qquad [a=b>0]; \\ = -\frac{\pi}{2p}\ \sin(ap)\ \cos(bp) \qquad [b>a>0].$$

BI ((166))(1)

BI ((166))(1)

7. $$\int_0^\infty \frac{\sin(ax)\,\cos(bx)}{p^2-x^2}\,x\,dx = -\frac{\pi}{2}\,\cos(ap)\,\cos(bp) \qquad [a>b>0];$$
$$= -\frac{\pi}{4}\,\cos(2ap) \qquad [a=b>0];$$
$$= \frac{\pi}{2}\,\sin(ap)\,\sin(bp) \qquad [b>a>0].$$

BI ((166))(2)

8. $$\int_0^\infty \frac{\cos(ax)\,\cos(bx)}{p^2-x^2}\,dx = \frac{\pi}{2p}\,\sin(ap)\,\cos(bp) \qquad [a>b>0];$$
$$= \frac{\pi}{4p}\,\sin(2ap) \qquad [a=b>0];$$
$$= \frac{\pi}{2p}\,\cos(ap)\,\sin(bp) \qquad [b>a>0].$$

BI ((166))(3)

3.743

1. $$\int_0^\infty \frac{\sin(ax)}{\sin(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\mathrm{sh}(a\beta)}{\mathrm{sh}(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0].$$

ET I 80(21)

2. $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = -\frac{\pi}{2}\cdot\frac{\mathrm{sh}(a\beta)}{\mathrm{ch}(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0].$$

ET I 81(30)

3. $$\int_0^\infty \frac{\cos(ax)}{\sin(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = \frac{\pi}{2}\cdot\frac{\mathrm{ch}(a\beta)}{\mathrm{sh}(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0].$$

ET I 23(37)

4. $$\int_0^\infty \frac{\cos(ax)}{\cos(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\mathrm{ch}(a\beta)}{\mathrm{ch}(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0].$$

ET I 23(36)

5.[6] P.V. $$\int_0^\infty \frac{\sin(ax)}{\sin x}\cdot\frac{dx}{b^2-x^2} = 0 \qquad \text{if } 0\le a\le 1$$
$$= \frac{\pi}{b}\,\sin(a-1)b \qquad \text{if } 1\le a\le 2 \qquad [b \text{ real}, \quad b/\pi\notin \mathrm{Z}].$$

$$\int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(x^2+\beta^2)} = \frac{\pi}{2\beta^2} \cdot \frac{\operatorname{sh}(a\beta)}{\operatorname{ch}(b\beta)} \qquad [0 < a < b, \quad \operatorname{Re}\beta > 0].$$

ET I 82(32)

3.745

$$\int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(c^2-x^2)} = 0 \qquad [0 < a < b, \quad c > 0].$$

ET I 82(31)

3.746

1. $$\int_0^\infty \frac{dx}{x^{n+1}} \prod_{k=0}^{n} \sin(a_k x) = \frac{\pi}{2} \prod_{k=1}^{n} a_k \qquad \left[a_0 > \sum_{k=1}^{n} a_k, \quad a_k > 0\right].$$

FI II 646

2. $$\int_0^\infty \frac{\sin(ax)}{x^{n+1}}\, dx \prod_{k=1}^{n} \sin(a_k x) \prod_{j=1}^{m} \cos(b_j x) = \frac{\pi}{2} \prod_{k=1}^{n} a_k \qquad \left[a > \sum_{k=1}^{n} |a_k| + \sum_{j=1}^{m} |b_j|\right].$$

WH

3.747

1.[7] $$\int_0^{\frac{\pi}{2}} \frac{x^m}{\sin x}\, dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{4^{2k-1}(m+2k)} \zeta(2k)\right] = 2\pi\mathbf{G} - \frac{7}{2}\zeta(3). \qquad [m = 2].$$

LI ((206))(2)

2. $$\int_0^{\frac{\pi}{2}} \frac{x\, dx}{\sin x} = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\, dx}{\cos x} = 2\boldsymbol{G}.$$

BI((204))(18)

3. $$\int_0^\infty \frac{x\, dx}{(x^2+b^2)\, \sin(ax)} = \frac{\pi}{2\, \operatorname{sh}(ab)} \qquad [b > 0].$$

GW ((333))(79c)

BI ((218))(4)

5. $$\int_0^{\frac{\pi}{2}} x \operatorname{tg} x \, dx = \infty.$$

BI ((205))(2)

6. $$\int_0^{\frac{\pi}{4}} x \operatorname{tg} x \, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G} = 0.1857845358\ldots$$

BI ((204))(1)

7. $$\int_0^{\frac{\pi}{2}} x \operatorname{ctg} x \, dx = \frac{\pi}{2} \ln 2.$$

FI II 623

8. $$\int_0^{\frac{\pi}{4}} x \operatorname{ctg} x \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G} = 0.730\,181\,0584\ldots$$

BI ((204))(2)

9. $$\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \operatorname{tg} x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \operatorname{tg} x \, dx = \frac{\pi}{2} \ln 2.$$

GW((333))(33b), BI((218))(12)

10. $$\int_0^{\infty} \operatorname{tg} ax \frac{dx}{x} = \frac{\pi}{2} \qquad [a > 0].$$

LO V 279(5)

11. $$\int_0^{\frac{\pi}{2}} \frac{x \operatorname{ctg} x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2.$$

BI ((206))(12)

3.748

1. $$\int_0^{\frac{\pi}{4}} x^m \operatorname{tg} x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \sum_{k=1}^{\infty} \frac{(4^k - 1)\zeta(2k)}{4^{2k-1}(m + 2k)}.$$

2. $$\int_0^{\frac{\pi}{2}} x^p \operatorname{ctg} x\,dx = \left(\frac{\pi}{2}\right)^p \left\{\frac{1}{p} - 2\sum_{k=1}^{\infty} \frac{1}{4^k(p+2k)}\zeta(2k)\right\}.$$

LI ((205))(7)

3. $$\int_0^{\frac{\pi}{4}} x^m \operatorname{ctg} x\,dx = \frac{1}{2}\left(\frac{\pi}{4}\right)^m \qquad \left[\frac{2}{m} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(m+2k)}\right].$$

LI ((204))(6)

3.749

1. $$\int_0^{\infty} \frac{x \operatorname{tg}(ax)\,dx}{x^2+b^2} = \frac{\pi}{e^{2ab}+1} \qquad [a>0, \quad b>0].$$

GW ((333))(79a)

2. $$\int_0^{\infty} \frac{x \operatorname{ctg}(ax)\,dx}{x^2+b^2} = \frac{\pi}{e^{2ab}-1} \qquad [a>0, \quad b>0].$$

GW ((333))(79b)

3. $$\int_0^{\infty} \frac{x \operatorname{tg}(ax)\,dx}{b^2-x^2} = \int_0^{\infty} \frac{x \operatorname{ctg}(ax)\,dx}{b^2-x^2} = \int_0^{\infty} \frac{x \operatorname{cosec}(ax)\,dx}{b^2-x^2} = \infty.$$

BI ((161))(7, 8, 9)

3.75 Combinations of trigonometric and algebraic functions

3.751

1. $$\int_0^{\infty} \frac{\sin(ax)\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}}[\cos(a\beta) - \sin(a\beta) + 2C(\sqrt{a\beta})\,\sin(a\beta) - 2S(\sqrt{a\beta})\,\cos(a\beta)]$$
$$[a>0, \quad |\arg\beta|<\pi].$$

ET I 65(12)a

2.* $$\int_0^{\infty} \frac{\cos(ax)\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}}[\cos(a\beta) + \sin(a\beta) - 2C(\sqrt{a\beta})\,\cos(a\beta) - 2S(\sqrt{a\beta})\,\sin(a\beta)]$$
$$[a>0, \quad |\arg\beta|<\pi].$$

3. $$\int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}[\sin(au)+\cos(au)] \qquad [a>0, \quad u>0].$$

ET I 65(13)

4. $$\int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}[\cos(au)-\sin(au)] \qquad [a>0, \quad u>0].$$

ET I 8(10)

3.752

1.[8] $$\int_0^1 \sin(ax)\sqrt{1-x^2}\,dx = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a}\,\mathbf{H}_i,\,(a) \qquad [a>0].$$

BI ((149))(6)

2. $$\int_0^1 \cos(ax)\sqrt{1-x^2}\,dx = \frac{\pi}{2a}J_1(a).$$

KU 65(6)a

3.753

1.[8] $$\int_0^1 \frac{\sin(ax)\,dx}{\sqrt{1-x^2}} = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} \qquad = \frac{\pi}{2}\,\mathbf{H}_0,\,(a)[a>0].$$

BI ((149))(9)

2. $$\int_0^1 \frac{\cos(ax)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}J_0(a).$$

WA 30(7)a

3. $$\int_1^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2-1}} = \frac{\pi}{2}J_0(a). \qquad [a>0].$$

WA 200(14)

4. $$\int_1^\infty \frac{\cos(ax)}{\sqrt{x^2-1}}\,dx = -\frac{\pi}{2}N_0(a).$$

5. $$\int_0^1 \frac{x\,\sin(ax)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}J_1(a) \qquad [a>0].$$

WA 30(6)

3.754

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{\beta^2+x^2}} = \frac{\pi}{2}[I_0(a\beta)-\mathbf{L}_0(a\beta)] \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

ET I 66(26)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

WA 191(1), GW((333))(78a)

3. $$\int_0^\infty \frac{x\,\sin(ax)}{\sqrt{(\beta^2+x^2)^3}}\,dx = aK_0(a\beta) \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

ET I 66(27)

3.755

1. $$\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}-\beta}\,\sin(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta} \qquad [a>0].$$

ET I 66(31)

2. $$\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}+\beta}\,\cos(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta} \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

ET I 10(25)

3.756

1. $$\int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}}\prod_{k=2}^{n}\sin(a_kx)\,dx = 0 \qquad \left[a_k>0, \quad a>\sum_{k=2}^{n}a_k\right]$$

ET I 80(22)

2. $$\int_0^\infty x^{\frac{n}{2}-1}\cos(ax)\prod_{k=1}^{n}\cos(a_kx)\,dx = 0 \qquad \left[a_k>0, \quad a>\sum_{k=1}^{n}a_k\right].$$

3.757

1. $$\int_0^\infty \frac{\sin(ax)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}.$$

BI ((177))(1)

2. $$\int_0^\infty \frac{\cos(ax)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}.$$

BI ((177))(2)

3.76- 3.77 Combinations of trigonometric functions and powers

3.761

1. $$\int_0^1 x^{\mu-1}\,\sin(ax)\,dx = \frac{-i}{2\mu}\left[{}_1F_1(\mu;\,\mu+1;\,ia) - {}_1F_1(\mu;\,\mu+1;\,-ia)\right]$$
$$[a>0,\quad \operatorname{Re}\mu>-1,\quad \mu\neq 0].$$

ET I 68(2)a

2. $$\int_u^\infty x^{\mu-1}\,\sin x\,dx = \frac{i}{2}\left[e^{-\frac{\pi}{2}i\mu}\Gamma(\mu,\,iu) - e^{\frac{\pi}{2}i\mu}\Gamma(\mu,\,-iu)\right] \qquad [\operatorname{Re}\mu>-1].$$

EH II 149(2)

3. $$\int_1^\infty \frac{\sin(ax)}{x^{2n}}\,dx = \frac{a^{2n-1}}{(2n-1)!}\left[\sum_{k=1}^{2n-1}\frac{(2n-k-1)!}{a^{2n-k}}\,\sin\left(a+(k-1)\frac{\pi}{2}\right) + (-1)^n\,\operatorname{ci}(a)\right]$$
$$[a>0].$$

LI ((203))(15)

4. $$\int_0^\infty x^{\mu-1}\,\sin(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\,\sin\frac{\mu\pi}{2} = \frac{\pi\,\sec\frac{\mu\pi}{2}}{2a^\mu\Gamma(1-\mu)} \qquad [a>0;\quad 0<|\operatorname{Re}\mu|<1].$$

FI II 809a, BI((150))(1)

5. $$\int_0^\pi x^m\,\sin(nx)\,dx = \frac{(-1)^{n+1}}{n^{m+1}}\sum_{k=0}^{[m/2]}(-1)^k\frac{m!}{(m-2k)!}(n\pi)^{m-2k} -$$
$$-(-1)^{[m/2]}\frac{m![m-2E[m/2]-1]}{n^{m+1}}$$

6.[7] $$\int_0^1 x^{\mu-1} \cos(ax)\,dx = \frac{1}{2\mu}\left[{}_1F_1(\mu;\, \mu+1;\, ia) + {}_1F_1(\mu,\, \mu+1;\, -ia)\right] \qquad [a > 0, \quad \operatorname{Re}\mu > 0].$$

ET I 11(2)

7. $$\int_u^\infty x^{\mu-1} \cos x\,dx = \frac{1}{2}\left[e^{-\frac{\pi}{2}i\mu}\Gamma(\mu,\, iu) + e^{\frac{\pi}{2}i\mu}\Gamma(\mu,\, -iu)\right] \qquad [\operatorname{Re}\mu < 1].$$

EH II 149(1)

8. $$\int_1^\infty \frac{\cos(ax)}{x^{2n+1}}\,dx = \frac{a^{2n}}{(2n)!}\left[\sum_{k=1}^{2n} \frac{(2n-k)!}{a^{2n-k+1}} \cos\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^{n+1} \operatorname{ci}(a)\right] \qquad [a > 0].$$

LI ((203))(16)

9. $$\int_0^\infty x^{\mu-1} \cos(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu} \cos\frac{\mu\pi}{a} = \frac{\pi \operatorname{cosec}\frac{\mu\pi}{2}}{2a^\mu\Gamma(1-\mu)} \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

FI II 809a, BI((150))(2)

10. $$\int_0^\pi x^m \cos(nx)\,dx = \frac{(-1)^n}{n^{m+1}} \sum_{k=0}^{[(m-1)/2]} (-1)^k \frac{m!}{(m-2k-1)!}(n\pi)^{m-2k-1} + \\ + (-1)^{[(m+1)/2]} \frac{2[(m+1)/2] - m}{n^{m+1}} \cdot m!$$

GW ((333))(7)

11. $$\int_0^{\frac{\pi}{2}} x^m \cos x\,dx = \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k)!}\left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{[m/2]}(2[m/2] - m)m!.$$

GW ((333))(9c)

12. $$\int_0^{2n\pi} x^m \cos kx\,dx = -\sum_{j=0}^{m-1} \frac{j!}{k^{j+1}} \binom{m}{j} (2n\pi)^{m-j} \cos\frac{j+1}{2}\pi.$$

BI ((226))(2)

3.762

1. $$\int_0^\infty x^{\mu-1} \sin(ax) \sin(bx)\, dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu)[|b-a|^{-\mu} - (b+a)^{-\mu}]$$
$$[a > 0, \quad b > 0, \quad a \neq b, \quad -2 < \operatorname{Re}\mu < 1]$$
$$(\text{for } \mu = 0, \text{ see } 3.741\ 1., \quad \text{for } \mu = -1, \text{see } 3.741\ 3.).$$

3.741
BI((149))(7), ET I 321(40)

2. $$\int_0^\infty x^{\mu-1} \sin(ax) \cos(bx)\, dx = \frac{1}{2} \sin \frac{\mu\pi}{2} \Gamma(\mu)[(a+b)^{-\mu} + |a-b|^{-\mu} \operatorname{sign}(a-b)]$$
$$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1] \qquad (\text{for } \mu = 0 \text{ see } 3.741\ 2.).$$

3.741
BI((159))(8)A, ET I 321(41)

3. $$\int_0^\infty x^{\mu-1} \cos(ax) \cos(bx)\, dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu)[(a+b)^{-\mu} + |a-b|^{-\mu}]$$
$$[a > 0, \quad b > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 20(17)

3.763

1. $$\int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^\nu}\, dx =$$
$$= \frac{1}{4} \cos \frac{\nu\pi}{2} \Gamma(1-\nu)[(c+a-b)^{\nu-1} - (c+a+b)^{\nu-1} -$$
$$- |c-a+b|^{\nu-1} \operatorname{sign}(a-b-c) + |c-a-b|^{\nu-1} \operatorname{sign}(a+b-c)]$$
$$[c > 0, \quad 0 < \operatorname{Re}\nu < 4, \quad \nu \neq 1, 2, 3, \quad a \geq b > 0].$$

GW((333))(26a)A, ET I 79(13)

2. $$\int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x}\, dx = 0 \qquad [c < a-b \quad \text{and} \quad c > a+b];$$
$$= \frac{\pi}{8} \qquad [c = a-b \quad \text{and} \quad c = a+b]; \qquad [a \geq b > 0, \quad c > 0].$$
$$= \frac{\pi}{4} [a-b < c < a+b].$$

FI II 645

BI((157))(8)A, ET I 79(11)

4. $$\int_0^\infty \frac{\sin(ax)\,\sin(bx)\,\sin(cx)}{x^3}\,dx = \frac{\pi bc}{2} \qquad [0 < c < a-b \quad \text{and} \quad c > a+b];$$
$$= \frac{\pi bc}{2} - \frac{\pi(a-b-c)^2}{8} \qquad [a-b < c < a+b];$$
$$[a \geq b > 0, \quad c > 0].$$

BI((157))(20), ET I 79(12)

3.764

1. $$\int_0^\infty x^p\,\sin(ax+b)\,dx = \frac{1}{a^{p+1}}\Gamma(1+p)\,\cos\left(b+\frac{p\pi}{2}\right) \qquad [a > 0, \quad -1 < p < 0].$$

GW ((333))(30a)

2. $$\int_0^\infty x^p\,\cos(ax+b)\,dx = -\frac{1}{a^{p+1}}\Gamma(1+p)\,\sin\left(b+\frac{\pi p}{2}\right) \qquad [a > 0, \quad -1 < p < 0].$$

GW ((333))(30b)

3.765

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x^\nu(x+\beta)} = \frac{i}{2\beta^\nu}\Gamma(1-\nu)[e^{-ia\beta}\Gamma(\nu,\,-ia\beta) - e^{ia\beta}\Gamma(\nu,\,ia\beta)]$$
$$[a > 0, \quad -1 < \operatorname{Re}\nu < 2, \quad |\arg\beta| < \pi].$$

ET I 219(34)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{x^\nu(x+\beta)} = \frac{\Gamma(1-\nu)}{2\beta^\nu}[e^{ia\beta}\Gamma(\nu,\,ia\beta) + e^{-ia\beta}\Gamma(\nu,\,-ia\beta)]$$
$$[a > 0, \quad |\operatorname{Re}\nu| < 1, \quad |\arg\beta| < \pi].$$

ET II 221(52)

3.766

1. $$\int_0^\infty \frac{x^{\mu-1}\,\sin(ax)}{1+x^2}\,dx = \frac{\pi}{2}\,\sec\frac{\mu\pi}{2}\,\operatorname{sh} a +$$
$$+\frac{1}{2}\,\sin\frac{\mu\pi}{2}\Gamma(\mu)\{\exp[-a+i\pi(1-\mu)]\gamma(1-\mu,\,-a) - e^a\gamma(1-\mu,\,a)\}$$
$$[a > 0, \quad -1 < \operatorname{Re}\mu < 3].$$

2. $$\int_0^\infty \frac{x^{\mu-1}\cos(ax)}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}\frac{\mu\pi}{2}\operatorname{ch} a + \\ + \frac{1}{2}\cos\frac{\mu\pi}{2}\Gamma(\mu)\{\exp[-a+i\pi(1-\mu)]\gamma(1-\mu,-a) - e^a\gamma(1-\mu,a)\} \\ [a>0, \quad 0<\operatorname{Re}\mu<3].$$

ET I 319(24)

3.* $$\int_0^\infty \frac{x^{2\mu+1}\sin(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2\mu}\sec(\mu\pi)\operatorname{sh}(ab) + \\ + \frac{\sin(\mu\pi)}{2a^{2\mu}}\Gamma(2\mu)[{}_1F_1(1;1-2\mu;ab) + {}_1F_1(1;1-2\mu;-ab)] \\ \left[a>0, \quad -\frac{3}{2}<\operatorname{Re}\mu<\frac{1}{2}\right].$$

ET II 220(39)

461

4.* $$\int_0^\infty \frac{x^{2\mu+1}\cos(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2(\mu+\frac{1}{2})}\operatorname{cosec}[(\mu+\frac{1}{2})\pi]\operatorname{ch}(ab) + \\ + \frac{\cos[(\mu+\frac{1}{2})\pi]}{2a^{2(\mu+\frac{1}{2})}}\Gamma[2(\mu+\frac{1}{2})][{}_1F_1(1;1-2(\mu+\frac{1}{2});ab) + \\ + {}_1F_1(1;1-2(\mu+\frac{1}{2});-ab)]. \\ \left[a>0, \quad -1<\operatorname{Re}\mu<\frac{1}{2}\right].$$

ET II 221(56)

3.767

1. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = -\frac{\pi}{2}\gamma^{\beta-2}e^{-a\gamma} \qquad [a>0, \quad \operatorname{Re}\gamma>0, \quad 0<\operatorname{Re}\beta<2].$$

BI ((160))(20)

2. $$\int_0^\infty \frac{x^{\beta}\cos\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = \frac{\pi}{2}\gamma^{\beta-1}e^{-a\gamma} \qquad [a>0, \quad \operatorname{Re}\gamma>0, \quad |\operatorname{Re}\beta|<1].$$

BI ((160))(21)

3. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{x^2-b^2}\,dx = \frac{\pi}{2}b^{\beta-2}\cos\left(ab-\frac{\pi\beta}{2}\right) \qquad [a>0, \quad b>0, \quad 0<\operatorname{Re}\beta<2].$$

4. $$\int_0^\infty \frac{x^\beta \cos\left(ax - \frac{\beta\pi}{2}\right)}{x^2 - b^2}\, dx = -\frac{\pi}{2} b^{\beta-1} \sin\left(ab - \frac{\pi\beta}{2}\right) \qquad [a > 0, \quad b > 0, \quad |\beta| < 1].$$

GW ((333))(82)

3.768

1. $$\int_u^\infty (x-u)^{\mu-1} \sin(ax)\, dx = \frac{\Gamma(\mu)}{a^\mu} \sin\left(au + \frac{\mu\pi}{2}\right) \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET II 203(19)

2. $$\int_u^\infty (x-u)^{\mu-1} \cos(ax)\, dx = \frac{\Gamma(\mu)}{a^\mu} \cos\left(au + \frac{\mu\pi}{2}\right) \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET II 204(24)

3.* $$\int_0^1 (1-x)^\nu \sin(ax)\, dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}} C_\nu(a) = a^{-\nu-1/2} s_{\nu+1/2,\,1/2}(a)$$

with $C_\nu(a)$ the Young's function

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)} [{}_1F_1(1;\, \nu+1;\, ia) + {}_1F_1(1;\, \nu+1;\, -ia)]$$

$$= \sum_{n=0}^\infty (-1)^n a^{\nu+2n} / \Gamma(\nu+2n+1) \qquad [a > 0, \quad \operatorname{Re}\nu > -1].$$

ET I 11(3)a

4.[3] $$\int_0^1 (1-x)^\nu \cos(ax)\, dx = \frac{i}{2} a^{-\nu-1} \left\{ \exp\left[\frac{i}{2}(\nu\pi - 2a)\right] \gamma\,(\nu+1,\, -ia) - \right.$$

$$\left. - \exp\left[-\frac{i}{2}(\nu\pi - 2a)\right] \gamma\,(\nu+1,\, ia) \right\} =$$

$$= \Gamma(\nu+1) \sum_{n=0}^\infty (-a^2)^n / \Gamma(\nu+2+2n) \qquad [a > 0, \quad \operatorname{Re}\nu > -1].$$

ET I 11(3)a

5. $$\int_0^u x^{\nu-1} (u-x)^{\mu-1} \sin(ax)\, dx = \frac{u^{\mu+\nu-1}}{2i} \mathrm{B}\,(\mu,\, \nu) [{}_1F_1(\nu;\, \mu+\nu;\, iau) - {}_1F_1(\nu;\, \mu+\nu;\, -iau)]$$

$$[a > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1, \quad \nu \neq 0].$$

ET II 189(26)

ET II 189(32)

7. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\sin(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-1/2}\sin\frac{au}{2}\Gamma(\mu)J_{\mu-1/2}\left(\frac{au}{2}\right) \qquad [\operatorname{Re}\mu>0].$$

ET II 189(25)

8. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\sin(ax)\,dx =$$
$$= \frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-1/2}\Gamma(\mu)\left[\cos\frac{au}{2}J_{1/2-\mu}\left(\frac{au}{2}\right)-\sin\frac{au}{2}N_{1/2-\mu}\left(\frac{au}{2}\right)\right]$$
$$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right].$$

ET II 203(20)

9. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\cos(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\cos\frac{au}{2}\Gamma(\mu)J_{\mu-\frac{1}{2}}\left(\frac{au}{2}\right) \qquad [\operatorname{Re}\mu>0].$$

ET II 189(31)

10. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\Gamma(\mu)\left[\sin\frac{au}{2}J_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right)-\cos\frac{au}{2}N_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right)\right]$$
$$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right].$$

ET II 204(25)

11.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\sin(ax)\,dx = -\frac{i}{2}\,\mathrm{B}(\mu,\nu)[{}_1F_1(\nu;\nu+\mu;ia)-{}_1F_1(\nu;\nu+\mu;-ia)]$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1, \quad \nu\neq 0]. \quad ET I 68(5)a,$$

ET I 317(5)

12.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}\,\mathrm{B}(\mu,\nu)[{}_1F_1(\nu;\nu+\mu;ia)+{}_1F_1(\nu;\nu+\mu;-ia)]$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET I 11(5)

13. $$\int_0^1 x^\mu (1-x)^\mu \sin(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a)\sin a \qquad [a>0, \quad \operatorname{Re}\mu > -1].$$

ET I 68(4)

14. $$\int_0^1 x^\mu (1-x)^\mu \cos(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a)\cos a \qquad [a>0, \quad \operatorname{Re}\mu > -1].$$

ET I 11(4)

3.769

1. $$\int_0^\infty [(\beta+ix)^{-\nu}-(\beta-ix)^{-\nu}]\sin(ax)\,dx = -\frac{\pi i a^{\nu-1}e^{-a\beta}}{\Gamma(\nu)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0].$$

ET I 70(15)

2. $$\int_0^\infty [(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}]\cos(ax)\,dx = \frac{\pi a^{\nu-1}e^{-a\beta}}{\Gamma(\nu)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0].$$

ET I 13(19)

3. $$\int_0^\infty x[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}]\sin(ax)\,dx = -\frac{\pi a^{\nu-2}(\nu-1-a\beta)}{\Gamma(\nu)}e^{-a\beta}$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0].$$

ET I 70(16)

4. $$\int_0^\infty x^{2n}[(\beta-ix)^{-\nu}-(\beta+ix)^{-\nu}]\sin(ax)\,dx = \frac{(-1)^n i}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}L_{2n}^{\nu-2n-1}(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu].$$

ET I 70(17)

5. $$\int_0^\infty x^{2n}[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}]\cos(ax)\,dx = \frac{(-1)^n}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}L_{2n}^{\nu-2n-1}(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu].$$

ET I 13(20)

ET I 70(18)

7. $$\int_0^\infty x^{2n+1}[(\beta+ix)^{-\nu}-(\beta-ix)^{-\nu}]\,\cos(ax)\,dx = \\ = \frac{(-1)^{n+1}}{\Gamma(\nu)}(2n+1)!\pi a^{\nu-2n-2}e^{-a\beta}L_{2n+1}^{\nu-2n-2}(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu-1].$$

ET I 13(21)

1. $$\int_0^\infty (\beta^2+x^2)^{\nu-\frac{1}{2}}\,\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)[I_{-\nu}(a\beta)-\mathbf{L}_\nu(a\beta)]$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}, \quad \nu\neq-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\dots\right].$$

EH II 38a, ET I 68(6)

2. $$\int_0^\infty (\beta^2+x^2)^{\nu-\frac{1}{2}}\,\cos(ax)\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu\cos(\pi\nu)\Gamma\left(\nu+\frac{1}{2}\right)K_{-\nu}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}\right].$$

WA 191(1)A, GW((333))(78)a

3. $$\int_0^u x^{2\nu-1}(u^2-x^2)^{\mu-1}\,\sin(ax)\,dx = \\ = \frac{a}{2}u^{2\mu+2\nu-1}\,\mathrm{B}\left(\mu,\,\nu+\frac{1}{2}\right)\,{}_1F_2\left(\nu+\frac{1}{2};\,\frac{3}{2},\,\mu+\nu+\frac{1}{2};\,-\frac{a^2u^2}{4}\right)$$
$$\left[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 189(29)

4. $$\int_0^u x^{2\nu-1}(u^2-x^2)^{\mu-1}\,\cos(ax)\,dx = \frac{1}{2}u^{2\mu+2\nu-2}\,\mathrm{B}\,(\mu,\,\nu)\,{}_1F_2\left(\nu;\,\frac{1}{2},\,\mu+\nu;\,-\frac{a^2u^2}{4}\right)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET II 190(35)

ET I 69(11)

6. $$\int_0^u (u^2-x^2)^{\nu-\frac{1}{2}} \sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_{\nu}(au) \qquad \left[a>0,\ u>0,\ \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET I 69(7), WA 358(1)a

7. $$\int_u^\infty (x^2-u^2)^{\nu-\frac{1}{2}} \sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_{-\nu}(au) \qquad \left[a>0,\ u>0,\ |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

EH II 81(12)A, ET I 69(8), WA 187(3)a

8. $$\int_0^u (u^2-x^2)^{\nu-\frac{1}{2}} \cos(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_{\nu}(au) \qquad \left[a>0,\ u>0,\ \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET I 11(8)

9. $$\int_u^\infty (x^2-u^2)^{\nu-\frac{1}{2}} \cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)N_{-\nu}(au)$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

WA 187(4)A, EH II 82(13)A, ET I 11(9)

10. $$\int_0^u x(u^2-x^2)^{\nu-\frac{1}{2}} \sin(ax)\,dx = \frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_{\nu+1}(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET I 69(9)

11. $$\int_u^\infty x(x^2-u^2)^{\nu-\frac{1}{2}} \sin(ax)\,dx = \frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)N_{-\nu-1}(au)$$
$$\left[a>0, \quad u>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<0\right].$$

ET I 69(10)

12.[7] $$\int_0^u x(u^2-x^2)^{\nu-\frac{1}{2}} \cos(ax)\,dx = -\frac{u^{\nu+1}}{a^{\nu}}s_{\nu-1,\nu+1}(au) =$$
$$= \frac{1}{2}\left(\nu+\frac{1}{2}\right)^{-1}u^{2\nu+1} - \frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_{\nu+1}(au)$$

13. $$\int_u^\infty x(x^2-u^2)^{\nu-1/2}\,\cos(ax)\,dx = \frac{\sqrt{\pi}u}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) J_{-\nu-1}(au)$$
$$\left[a>0, \quad u>0, \quad 0<\operatorname{Re}\nu<\frac{1}{2}\right].$$

ET I 12(11)

3.772

1. $$\int_0^\infty (x^2+2\beta x)^{\nu-1/2}\,\sin(ax)\,dx =$$
$$= \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(a\beta)\,\cos(a\beta)+N_{-\nu}(a\beta)\,\sin(a\beta)\right]$$
$$\left[a>0, \quad |\arg\beta|<\pi, \quad \frac{1}{2}>\operatorname{Re}\nu>-\frac{3}{2}\right].$$

ET I 69(12)

2. $$\int_0^\infty (x^2+2\beta x)^{\nu-1/2}\,\cos(ax)\,dx =$$
$$= -\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\left[N_{-\nu}(a\beta)\,\cos(a\beta)-J_{-\nu}(a\beta)\,\sin(a\beta)\right]$$
$$\left[a>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET I 12(13)

3. $$\int_0^{2u} (2ux-x^2)^{\nu-1/2}\,\sin(ax)\,dx = \sqrt{\pi}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\,\sin(au)J_\nu(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET I 69(13)a

4. $$\int_{2u}^\infty (x^2-2ux)^{\nu-1/2}\,\sin(ax)\,dx =$$
$$= \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(au)\,\cos(au)-N_{-\nu}(au)\,\sin(au)\right]$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

5. $$\int_0^{2u} (2ux - x^2)^{\nu-1/2} \cos(ax)\, dx = \sqrt{\pi} \left(\frac{2u}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(au) \cos(au)$$
$$\left[a > 0, \quad u > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET I 12(4)

6. $$\int_{2u}^{\infty} (x^2 - 2ux)^{\nu-1/2} \cos(ax)\, dx =$$
$$= -\frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[J_{-\nu}(au) \sin(au) + N_{-\nu}(au) \cos(au)\right]$$
$$\left[a > 0, \quad u > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET I 12(12)

3.773

1.[6] $$\int_0^{\infty} \frac{x^{2\nu}}{(x^2+\beta^2)^{\mu+1}}\, sin\,(ax)\, dx =$$
$$= \frac{1}{2}\beta^{2\nu-2\mu} a\, \mathrm{B}\,(1+\nu,\, \mu-\nu)\, {}_1F_2\left(\nu+1;\, \nu+1-\mu;\, \frac{3}{2};\, \frac{\beta^2 a^2}{4}\right) +$$
$$+ \frac{\sqrt{\pi} a^{2u-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu-\mu)}{\Gamma\left(\mu-\nu+\frac{3}{2}\right)}\, {}_1F_2\left(\mu+1;\, \mu-\nu+\frac{3}{2},\, \mu-\nu+1;\, \frac{\beta^2 a^2}{4}\right) =$$
$$= \frac{\sqrt{\pi}}{2\Gamma(\mu+1)} \beta^{2\nu-2\mu-1} G_{13}^{21}\left(\frac{a^2\beta^2}{4}\middle|\begin{matrix} -\nu+\frac{1}{2} \\ \mu-\nu+\frac{1}{2},\, \frac{1}{2},\, 0\end{matrix}\right)$$
$$[a > 0, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \operatorname{Re}\mu + 1].$$

ET I 71(28)A, ET II 234(17)

2. $$\int_0^{\infty} \frac{x^{2m+1} \sin(ax)}{(z+x^2)^{n+1}}\, dx = \frac{(-1)^{n+m}}{n!} \cdot \frac{\pi}{2} \frac{d^n}{dz^n} (z^m e^{-a\sqrt{z}}) \qquad [a > 0, \quad 0 \le m \le n, \quad |\arg z| < \pi].$$

ET I 68(39)

467

3. $$\int_0^{\infty} \frac{x^{2m+1} \sin(ax)\, dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^{m+1}\sqrt{\pi}}{2^n \beta^n \Gamma\left(n+\frac{1}{2}\right)} \frac{d^{2m+1}}{da^{2m+1}} [a^n K_n(a\beta)]$$
$$[a > 0, \quad \operatorname{Re}\beta > 0, \quad -1 \le m \le n].$$

ET I 67(37)

4. $$\int_0^{\infty} \frac{x^{2\nu} \cos(ax)\, dx}{(x^2+\beta^2)^{\mu+1}} =$$
$$= \frac{1}{2}\beta^{2\nu-2\mu-1}\, \mathrm{B}\left(\nu+\frac{1}{2},\, \mu-\nu+\frac{1}{2}\right)\, {}_1F_2\left(\nu+\frac{1}{2};\, \nu-\mu+\frac{1}{2},\, \frac{1}{2};\, \frac{\beta^2 a^2}{4}\right) +$$
$$+ \frac{\sqrt{\pi} a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma\left(\nu-\mu-\frac{1}{2}\right)}{\Gamma(\mu-\nu+1)}\, {}_1F_2\left(\mu+1;\, \mu-\nu+1,\, \mu-\nu+\frac{3}{2};\, \frac{\beta^2 a^2}{4}\right) =$$
$$= \frac{\sqrt{\pi}}{\ldots} \beta^{2\nu-2\mu-1} G^{21}\left(\frac{a^2\beta^2}{\ldots}\middle|\begin{matrix} -\nu+\frac{1}{2} \\ \ldots \end{matrix}\right)$$

5. $$\int_0^\infty \frac{x^{2m}\ \cos(ax)\,dx}{(z+x^2)^{n+1}} = (-1)^{m+n}\frac{\pi}{2\cdot n!}\cdot\frac{d^n}{dz^n}(z^{m-\frac{1}{2}}e^{-a\sqrt{z}})$$
$$[a>0, \quad n+1>m\geq 0, \quad |\arg z|<\pi].$$

ET I 10(28)

6.[7] $$\int_0^\infty \frac{x^{2m}\ \cos(ax)\,dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^m\sqrt{\pi}}{2^n\beta^n\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{d^{2m}}{da^{2m}}\{a^nK_n(a\beta)\}$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\leq m\leq n]\ .$$

ET I 14(28)

3.774

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2+b^2}(x+\sqrt{x^2+b^2})^\nu} = \frac{\pi}{b^\nu\ \sin(\nu\pi)}\left[\sin\frac{\nu\pi}{2}I_\nu(ab)+\frac{i}{2}\mathbf{J}_\nu(iab)-\frac{i}{2}\mathbf{J}_\nu(-iab)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1].$$

ET I 70(19)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{x^2+b^2}(x+\sqrt{x^2+b^2})^\nu} = \frac{\pi}{b^\nu\ \sin(\nu\pi)}\left[\frac{1}{2}\mathbf{J}_\nu(iab)+\frac{1}{2}\mathbf{J}_\nu(-iab)-\cos\frac{\nu\pi}{2}I_\nu(ab)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1].$$

ET I 12(15)

3. $$\int_0^\infty \frac{(x+\sqrt{x^2+\beta^2})^\nu}{\sqrt{x(x^2+\beta^2)}}\ \sin(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{3}{2}\right].$$

ET I 71(23)

468

4. $$\int_0^\infty \frac{(\sqrt{x^2+\beta^2}-x)^\nu}{\sqrt{x(x^2+\beta^2)}}\ \cos(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\ \nu>-\frac{3}{2}\right].$$

ET I 12(17)

5. $$\int_0^\infty \frac{(\beta+\sqrt{x^2+\beta^2})^\nu}{x^{\nu+\frac{1}{2}}\sqrt{x^2+\beta^2}}\ \sin(ax)\,dx = \frac{1}{\beta}\sqrt{\frac{2}{a}}\Gamma\left(\frac{3}{4}-\frac{\nu}{2}\right)W_{\frac{\nu}{2},\,\frac{1}{4}}(a\beta)M_{-\frac{\nu}{2},\,\frac{1}{4}}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{3}{2}\right].$$

6. $$\int_0^\infty \frac{(\beta+\sqrt{x^2+\beta^2})^\nu}{x^{\nu+\frac{1}{2}}\sqrt{\beta^2+x^2}}\cos(ax)\,dx = \frac{1}{\beta\sqrt{2a}}\Gamma\left(\frac{1}{4}-\frac{\nu}{2}\right)W_{\frac{\nu}{2},-\frac{1}{4}}(a\beta)M_{-\frac{\nu}{2},-\frac{1}{4}}(a\beta)$$
$$\left[a>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\nu<\frac{1}{2}\right].$$

ET I 12(18)

3.775

1. $$\int_0^\infty \frac{(\sqrt{x^2+\beta^2}+x)^\nu-(\sqrt{x^2+\beta^2}-x)^\nu}{\sqrt{x^2+\beta^2}}\sin(ax)\,dx = 2\beta^\nu\sin\frac{\nu\pi}{2}K_\nu(a\beta)$$
$$[a>0,\quad \operatorname{Re}\beta>0,\quad |\operatorname{Re}\nu|<1].$$

ET I 70(20)

2. $$\int_0^\infty \frac{(\sqrt{x^2+\beta^2}+x)^\nu+(\sqrt{x^2+\beta^2}-x)^\nu}{\sqrt{x^2+\beta^2}}\cos(ax)\,dx = 2\beta^\nu\cos\frac{\nu\pi}{2}K_\nu(a\beta)$$
$$[a>0,\quad \operatorname{Re}\beta>0,\quad |\operatorname{Re}\nu|<1].$$

ET I 13(22)

3. $$\int_u^\infty \frac{(x+\sqrt{x^2-u^2})^\nu+(x-\sqrt{x^2-u^2})^\nu}{\sqrt{x^2-u^2}}\sin(ax)\,dx = \pi u^\nu\left[J_\nu(au)\cos\frac{\nu\pi}{2}-N_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$$[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<1].$$

ET I 70(22)

4. $$\int_u^\infty \frac{(x+\sqrt{x^2-u^2})^\nu+(x-\sqrt{x^2-u^2})^\nu}{\sqrt{x^2-u^2}}\cos(ax)\,dx = -\pi u^\nu\left[N_\nu(au)\cos\frac{\nu\pi}{2}+J_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$$[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<1].$$

ET I 13(25)

5. $$\int_0^u \frac{(x+i\sqrt{u^2-x^2})^\nu+(x-i\sqrt{u^2-x^2})^\nu}{\sqrt{u^2-x^2}}\sin(ax)\,dx = \frac{\pi}{2}u^\nu\operatorname{cosec}\frac{\nu\pi}{2}[\mathbf{J}_\nu(au)-\mathbf{J}_{-\nu}(au)]$$
$$[a>0,\quad u>0].$$

ET I 70(21)

6. $$\int_0^u \frac{(x+i\sqrt{u^2-x^2})^\nu+(x-i\sqrt{u^2-x^2})^\nu}{\sqrt{u^2-x^2}}\cos(ax)\,dx = \frac{\pi}{2}u^\nu\sec\frac{\nu\pi}{2}[\mathbf{J}_\nu(au)+\mathbf{J}_{-\nu}(au)]$$
$$[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<1].$$

7.[6] $$\int_u^\infty \frac{(x+\sqrt{x^2-u^2})^\nu + (x-\sqrt{x^2-u^2})^\nu}{\sqrt{x(x^2-u^2)}} \sin(ax)\,dx =$$
$$= -\sqrt{\left(\frac{\pi}{2}\right)^3 au^\nu} \left[J_{1/4+\nu/2}\left(\frac{au}{2}\right) N_{1/4-\nu/2}\left(\frac{au}{2}\right) + J_{1/4-\nu/2}\left(\frac{au}{2}\right) N_{1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu| < \frac{3}{2}\right].$$

ET I 71(25)

8.[6] $$\int_u^\infty \frac{(x+\sqrt{x^2-u^2})^\nu + (x-\sqrt{x^2-u^2})^\nu}{\sqrt{x(x^2-u^2)}} \cos(ax)\,dx =$$
$$= -\sqrt{\left(\frac{\pi}{2}\right)^3 au^\nu} \left[J_{-1/4+\nu/2}\left(\frac{au}{2}\right) N_{-1/4-\nu/2}\left(\frac{au}{2}\right) + J_{-1/4-\nu/2}\left(\frac{au}{2}\right) N_{-1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\ \nu| < \frac{3}{2}\right].$$

ET I 13(26)

9. $$\int_0^\infty \frac{(x+\beta+\sqrt{x^2+2\beta x})^\nu + (x+\beta-\sqrt{x^2+2\beta x})^\nu}{\sqrt{x^2+2\beta x}} \sin(ax)\,dx =$$
$$= \pi\beta^\nu \left[N_\nu(\beta a)\, \sin\left(\beta a - \frac{\nu\pi}{2}\right) + J_\nu(\beta a)\, \cos\left(\beta a - \frac{\nu\pi}{2}\right)\right]$$
$$[a>0, \quad |\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < 1].$$

ET I 71(26)

10. $$\int_0^\infty \frac{(x+\beta+\sqrt{x^2+2\beta x})^\nu + (x+\beta-\sqrt{x^2+2\beta x})^\nu}{\sqrt{x^2+2\beta x}} \cos(ax)\,dx =$$
$$= \pi\beta^\nu \left[J_\nu(\beta a)\, \sin\left(\beta a - \frac{\nu\pi}{2}\right) - N_\nu(\beta a)\, \cos\left(\beta a - \frac{\nu\pi}{2}\right)\right]$$
$$[a>0, \quad |\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < 1].$$

ET I 13(23)

11. $$\int_0^{2u} \frac{(\sqrt{2u+x}+i\sqrt{2u-x})^{4\nu} + (\sqrt{2u+x}-i\sqrt{2u-x})^{4\nu}}{\sqrt{4u^2x-x^3}} \cos(ax)\,dx =$$
$$= (4u)^{2\nu}\pi^{3/2}\sqrt{\frac{a}{2}} J_{\nu-1/4}(au) J_{-\nu-1/4}(au) \qquad [a>0, \quad u>0].$$

ET I 14(27)

3.776

1. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}} \sin(ax)\,dx = \frac{a}{b^p} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((170))(1)

2. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}} \cos(ax)\,dx = \frac{p}{b^{p+1}} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((170))(2)

3.78- 3.81 Rational functions of x and of trigonometric functions

3.781

1. $$\int_0^\infty \left(\frac{\sin x}{x} - \frac{1}{1+x}\right)\frac{dx}{x} = 1 - \boldsymbol{C} \qquad \text{(cf. 3.784 4. and 3.781 2.).}$$

BI ((173))(7)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x}\right)\frac{dx}{x} = -\boldsymbol{C}.$$

BI ((173))(8)

3.782

1. $$\int_0^u \frac{1-\cos x}{x}\,dx - \int_u^\infty \frac{\cos x}{x}\,dx = \boldsymbol{C} + \ln u \qquad [u>0].$$

GW ((333))(31)

2. $$\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{a\pi}{2} \qquad [a \ge 0].$$

BI ((158))(1)

ET II 253(48)

3.783

1. $$\int_0^\infty \left[\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)}\right]\frac{dx}{x} = \frac{1}{2}\boldsymbol{C} - \frac{3}{4}.$$

BI ((173))(19)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\boldsymbol{C}.$$

EH I 17, BI((273))(21)

3.784

1. $$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{b}{a} \qquad [a > 0, \quad b > 0].$$

FI II 635, GW((333))(20)

2. $$\int_0^\infty \frac{a\,\sin bx - b\,\sin ax}{x^2}\,dx = ab\,\ln\frac{a}{b} \qquad [a > 0, \quad b > 0].$$

FI II 647

3. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{(b-a)\pi}{2} \qquad [a \ge 0, \quad b \ge 0].$$

BI((158))(12), FI II 645

4. $$\int_0^\infty \frac{\sin x - x\,\cos x}{x^2}\,dx = 1.$$

BI ((158))(3)

5. $$\int_0^\infty \frac{\cos ax - \cos bx}{x(x+\beta)}\,dx = \frac{1}{\beta}\Big[\operatorname{ci}(a\beta)\,\cos a\beta + \operatorname{si}(a\beta)\,\sin a\beta - {}$$
$$- \operatorname{ci}(b\beta)\,\cos b\beta - \operatorname{si}(b\beta)\,\sin b\beta + \ln\frac{b}{a}\Big]$$
$$[a > 0, \quad b > 0, \quad |\arg\beta| < \pi].$$

6. $$\int_0^\infty \frac{\cos ax + x\ \sin ax}{1+x^2}\,dx = \pi e^{-a} \qquad [a>0].$$

GW ((333))(73)

7. $$\int_0^\infty \frac{\sin ax - ax\ \cos ax}{x^3}\,dx = \frac{\pi}{4}a^2\ \operatorname{sign} a.$$

LI ((158))(5)

8. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2(x^2+\beta^2)}\,dx = \frac{\pi[(b-a)\beta + e^{-b\beta} - e^{-a\beta}]}{2\beta^3} \qquad [a>0, \quad b>0, \quad |\arg\beta|<\pi].$$

BI((173))(20)A, ET II 222(59)

3.785

$$\int_0^\infty \frac{1}{x}\sum_{k=1}^n a_k\ \cos b_k\, x\, dx = -\sum_{k=1}^n a_k\ \ln b_k \qquad \left[b_k>0, \quad \sum_{k=1}^n a_k = 0\right].$$

FI II 649

3.786

1. $$\int_0^\infty \frac{(1-\cos ax)\ \sin bx}{x^2}\,dx = \frac{b}{2}\ \ln\frac{b^2-a^2}{b^2} + \frac{a}{2}\ \ln\frac{a+b}{a-b} \qquad [a>0, \quad b>0].$$

ET I 81(29)

2. $$\int_0^\infty \frac{(1-\cos ax)\ \cos bx}{x}\,dx = \ln\frac{\sqrt{|a^2-b^2|}}{b} \qquad [0>0, \quad b>0, \quad a\neq b].$$

FI II 647

3. $$\int_0^\infty \frac{(1-\cos ax)\ \cos bx}{x^2}\,dx = \frac{\pi}{2}(a-b) \qquad [a<b\le a];$$
$$= 0 \qquad [0<a\le b].$$

ET I 20(16)

3.787

1. $$\int_0^\infty \frac{(\cos a - \cos nax)\ \sin mx}{x}\,dx = \frac{\pi}{2}(\cos a - 1) \qquad [m>na>0];$$
$$= \frac{\pi}{2}\ \cos a \qquad [na>m].$$

2. $$\int_0^\infty \frac{\sin^2 ax - \sin^2 bx}{x}\, dx = \frac{1}{2}\, \ln \frac{a}{b} \qquad [a > 0, \quad b > 0].$$

GW ((333))(20b)

3. $$\int_0^\infty \frac{x^3 - \sin^3 x}{x^5}\, dx = \frac{13}{32}\pi.$$

BI ((158))(6)

4. $$\int_0^\infty \frac{(3 - 4\, \sin^2 ax)\, \sin^2 ax}{x}\, dx = \frac{1}{2}\, \ln 2 \qquad [a \text{ real}, \quad a \neq 0].$$

BI ((155))(6)

3.788

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{x} - \operatorname{ctg} x\right)\, dx = \ln \frac{\pi}{2}.$$

GW ((333))(61)a

3.789

$$\int_0^{\frac{\pi}{2}} \frac{4x^2\, \cos x + (\pi - x)x}{\sin x}\, dx = \pi^2\, \ln 2.$$

LI ((206))(10)

3.791

1. $$\int_0^{\frac{\pi}{2}} \frac{x\, dx}{1 + \sin x} = \ln 2.$$

GW ((333))(55a)

2. $$\int_0^\pi \frac{x\, \cos x}{1 + \sin x}\, dx = \pi\, \ln 2 - 4\boldsymbol{G}.$$

GW ((333))(55c)

3. $$\int_0^{\frac{\pi}{2}} \frac{x\, \cos x}{1 + \sin x}\, dx = \pi\, \ln 2 - 2\boldsymbol{G}.$$

4. $$\int_0^{\pi} \frac{\left(\frac{\pi}{2} - x\right)\ \cos x}{1 - \sin x}\, dx = 2\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\ \cos x}{1 - \sin x}\, dx = \pi\ \ln 2 + 4\boldsymbol{G} = 5.8414484669\ldots$$

BI((207))(3), GW((333))(56c)

5. $$\int_0^{\frac{\pi}{2}} \frac{x^2\, dx}{1 - \cos x} = -\frac{\pi^2}{4} + \pi\ \ln 2 + 4\boldsymbol{G} = 3.3740473667\ldots$$

BI ((207))(3)

6. $$\int_0^{\pi} \frac{x^2\, dx}{1 - \cos x} = 4\pi\ \ln 2.$$

BI ((219))(1)

7. $$\int_0^{\frac{\pi}{2}} \frac{x^{p+1}\, dx}{1 - \cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^{p} (p+1) \left\{\frac{2}{p} - \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k)\right\} \qquad [p > 0].$$

LI ((207))(4)

8. $$\int_0^{\frac{\pi}{2}} \frac{x\, dx}{1 + \cos x} = \frac{\pi}{2} - \ln 2.$$

GW ((333))(55a)

9. $$\int_0^{\frac{\pi}{2}} \frac{x\ \sin x\, dx}{1 - \cos x} = \frac{\pi}{2}\ \ln 2 + 2\boldsymbol{G}.$$

GW ((333))(56a)

10. $$\int_0^{\pi} \frac{x\ \sin x\, dx}{1 - \cos x} = 2\pi\ \ln 2.$$

GW ((333))(56b)

11. $$\int_0^{\pi} \frac{x - \sin x}{1 - \cos x}\, dx = \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \frac{x - \sin x}{1 - \cos x}\, dx = 2.$$

GW ((333))(57a)

GW ((333))(55b)

3.792

1. $$\int_{-\pi}^{\pi} \frac{dx}{1-2a\,\cos x+a^2} = \frac{2\pi}{1-a^2} \qquad [a^2<1].$$

FI II 485

473

2. $$\int_0^{\frac{\pi}{2}} \frac{x\,\cos x\,dx}{1+2a\,\sin x+a^2} = \frac{\pi}{2a}\,\ln(1+a) - \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k}}{(2k+1)^2} \qquad [a^2<1].$$

LI ((241))(2)

3. $$\int_0^{\pi} \frac{x\,\sin x\,dx}{1-2a\,\cos x+a^2} = \frac{\pi}{a}\,\ln(1+a) \qquad [a^2<1, \quad a\neq 0];$$
$$= \frac{\pi}{a}\,\ln\left(1+\frac{1}{a}\right) \qquad [a^2<1];$$

BI ((221))(2)

4. $$\int_0^{2\pi} \frac{x\,\sin x\,dx}{1-2a\,\cos x+a^2} = \frac{2\pi}{a}\,\ln(1-a) \qquad [a^2<1, \quad a\neq 0];$$
$$= \frac{2\pi}{a}\,\ln\left(1-\frac{1}{a}\right) \qquad [a^2>1].$$

BI ((223))(4)

5. $$\int_0^{2\pi} \frac{x\,\sin nx\,dx}{1-2a\,\cos x+a^2} = \frac{2\pi}{1-a^2}\left[\left(a^{-n}-a^n\right)\,\ln(1-a) + \sum_{k=1}^{n-1}\frac{a^{-k}-a^k}{n-k}\right] \qquad [a^2<1, \quad a\neq 0].$$

BI ((223))(5)

6. $$\int_0^{\infty} \frac{\sin x}{1-2a\,\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{4a}\left[\left|\frac{1+a}{1-a}\right|-1\right] \qquad [a \text{ real}, \quad a\neq 0, \quad a\neq 1].$$

GW ((333))(62b)

7. $$\int_0^{\infty} \frac{\sin bx}{1-2a\,\cos x+a^3}\cdot\frac{dx}{x} = \frac{\pi}{2}\,\frac{1+a-2a^{[b]+1}}{(1-a)^2(1-a)} \qquad [b\neq 0,\ 1,\ 2,\dots];$$
$$= \frac{\pi}{2}\,\frac{1+a-a^b-a^{b+1}}{(1-a^2)(1-a)} \qquad [b=0,\ 1,\ 2,\dots]; \qquad [0<a<1].$$

8. $$\int_0^\infty \frac{\sin x \cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{2(1-a)}a^{[b]} \qquad [b\neq 0,\ 1,\ 2,\dots];$$
$$= \frac{\pi}{2(1-a)}a^b + \frac{\pi}{4}a^{b-1} \qquad [b=1,\ 2,\ 3,\dots].;$$
$$[0<a<1,\quad b>0;\quad \text{for } b=0, \text{ see } \mathbf{3}.7926].$$

3.792
ET I 19(5)

9. $$\int_0^\infty \frac{(1-a\cos x)\sin bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{2}\cdot\frac{1-a^{[b]+1}}{1-a} \qquad [b\neq 1, 2, 3, \dots];$$
$$= \frac{\pi}{2}\cdot\frac{1-a^b}{1-a} + \frac{\pi a^b}{4} \qquad [b=1,\ 2,\ 3,\dots]. \qquad [0<a<1,\ b>0].$$

ET I 82(33)

10.[3] $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\,\frac{dx}{\beta^2+x^2} = \frac{\pi}{2\beta(1-a^2)}\,\frac{1+ae^{-b\beta}}{1-ae^{-b\beta}} \qquad [a^2<1,\quad b\geq 0].$$

BI ((192))(1)

11. $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\,\frac{dx}{\beta^2-x^2} = \frac{a\pi}{\beta(1-a^2)}\,\frac{\sin b\beta}{1-2a\cos b\beta+a^2} \qquad [a^2<1,\quad b>0].$$

BI ((193))(1)

12. $$\int_0^\infty \frac{\sin bcx}{1-2a\cos bx+a^2}\,\frac{x\,dx}{\beta^2+x^2} = \frac{\pi}{2}\,\frac{e^{-\beta bc}-a^c}{(1-ae^{-b\beta})(1-ae^{b\beta})} \qquad [a^2<1,\quad b>0,\quad c>0].$$

BI ((192))(8)

13. $$\int_0^\infty \frac{\sin bx}{1-2a\cos bx+a^2}\,\frac{x\,dx}{\beta^2+x^2} = \frac{\pi}{2}\,\frac{1}{e^{b\beta}-a} \qquad [a^2<1,\quad b>0];$$
$$= \frac{\pi}{2a}\,\frac{1}{ae^{b\beta}-1} \qquad [a^2>1,\quad b>0].$$

BI ((192))(2)

BI ((193))(5)

15. $$\int_0^\infty \frac{\cos bcx}{1 - 2a\,\cos bx + a^2}\,\frac{dx}{\beta^2 - x^2} = \frac{\pi}{2\beta(1-a^2)}\frac{(1-a^2)\,\sin\beta bc + 2a^{c+1}\,\sin\beta b}{1 - 2a\,\cos\beta b + a^2}$$
$$[a^2 < 1, \quad b > 0, \quad c > 0].$$

BI ((193))(9)

16. $$\int_0^\infty \frac{1 - a\,\cos bx}{1 - 2a\,\cos bx + a^2}\,\frac{dx}{1 + x^2} = \frac{\pi}{2}\,\frac{e^b}{e^b - a} \qquad [a^2 < 1, \quad b > 0].$$

FI II 719

17. $$\int_0^\infty \frac{\cos bx}{1 - 2a\,\cos x + a^2}\cdot\frac{dx}{x^2 + \beta^2} = \frac{\pi(e^{\beta-\beta b} + ae^{\beta b})}{2\beta(1-a^2)(e^\beta - a)} \qquad [0 \le b < 1, \quad |a| < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 21(21)

18. $$\int_0^\infty \frac{\sin bx\,\sin x}{1 - 2a\,\cos x + a^2}\cdot\frac{dx}{x^2 + \beta^2} = \frac{\pi}{2\beta}\,\frac{\operatorname{sh} b\beta}{e^\beta - a} \qquad [0 \le b < 1];$$
$$= \frac{\pi}{4\beta(ae^\beta - 1)}[a^m e^{\beta(m+1-b)} - e^{(1-b)\beta}] -$$
$$- \frac{\pi}{4\beta(ae^{-\beta} - 1)}[a^m e^{-(m+1-b)\beta} - e^{-(1-b)\beta}]$$
$$[m \le b \le m+1] \qquad [0 < a < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 81(27)

19. $$\int_0^\infty \frac{(\cos x - a)\,\cos bx}{1 - 2a\,\cos x + a^2}\cdot\frac{dx}{x^2 + \beta^2} = \frac{\pi\,\operatorname{ch}\beta b}{2\beta(e^\beta - a)} \qquad [0 \le b < 1, \quad |a| < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 21(23)

20. $$\int_0^\infty \frac{\sin x}{(1 - 2a\,\cos 2x + a^2)^{n+1}}\,\frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{(1 - 2a\,\cos 2x + a^2)^{n+1}}\,\frac{dx}{x} =$$
$$= \int_0^\infty \frac{\operatorname{tg} x}{(1 - 2a\,\cos 4x + a^2)^{n+1}}\,\frac{dx}{x} = \frac{\pi}{2(1-a^2)^{2n+1}}\sum_{k=0}^{n}\binom{n}{k}^2 a^{2^k}.$$

BI ((187))(14)

3.793

BI ((223))(9)

2. $$\int_0^{2\pi} \frac{\cos nx - a\,\cos[(n+1)x]}{1-2a\,\cos x + a^2} x\,dx = 2\pi a^n \qquad [a^2<1].$$

BI ((223))(13)

3.794

1.[3] $$\int_0^{\pi} \frac{x\,dx}{1+a^2+2a\,\cos x} = \frac{\pi^2}{2(1-a^2)} + \frac{4}{(1-a^2)}\sum_{k=0}^{\infty}\frac{a^{2k+1}}{(2k+1)^2} \qquad [a^2<1].$$

2. $$\int_0^{2\pi} \frac{x\,\sin nx}{1\pm a\,\cos x}\,dx = \frac{2\pi}{\sqrt{1-a^2}}\left[(\mp1)^n\frac{(1+\sqrt{1-a^2})^n-(1-\sqrt{1-a^2})^n}{a^n}\times\right.$$
$$\left.\times\ \ln\frac{2\sqrt{1\pm a}}{\sqrt{1+a}+\sqrt{1-a}} + \sum_{k=0}^{n-1}\frac{(\mp1)^k}{n-k}\ \frac{(1+\sqrt{1-a^2})^k-(1-\sqrt{1-a^2})^k}{a^k}\right]$$
$$[a^2<1].$$

BI ((223))(2)

3.[3] $$\int_0^{2\pi} \frac{x\,\cos nx}{1\pm a\,\cos x}\,dx = \frac{2\pi^2}{\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{\mp a}\right)^n \qquad [a^2<1].$$

BI ((223))(3)

4. $$\int_0^{\pi} \frac{x\,\sin x\,dx}{a+b\,\cos x} = \frac{\pi}{b}\ \ln\frac{a+\sqrt{a^2-b^2}}{2(a-b)} \qquad [a>|b|>0].$$

GW ((333))(53a)

5. $$\int_0^{2\pi} \frac{x\,\sin x\,dx}{a+b\,\cos x} = \frac{2\pi}{b}\ \ln\frac{a+\sqrt{a^2-b^2}}{2(a+b)} \qquad [a>|b|>0].$$

GW ((333))(53b)

6. $$\int_0^{\infty} \frac{\sin x}{a\pm b\,\cos 2x}\cdot\frac{dx}{x} = \frac{\pi}{2\sqrt{a^2-b^2}} \qquad [a^2>b^2];$$
$$= 0 \qquad [a^2<b^2].$$

$$\int_{-\infty}^{\infty} \frac{(b^2+c^2+x^2)x\,\sin ax - (b^2-c^2-x^2)c\,\operatorname{sh} ac}{[x^2+(b-c)^2][x^2+(b+c)^2](\cos ax + \operatorname{ch} ac)}\,dx = \pi \qquad [c>b>0];$$

$$= \frac{2\pi}{e^{ab}+1} \qquad [b>c>0] \qquad [a>0].$$

BI ((202))(18)

3.796

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x\,dx = \mp \frac{\pi}{4}\ \ln 2 - \boldsymbol{G}.$$

BI ((207))(8, 9)

2. $$\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x\,dx = \frac{\pi}{4}\ \ln 2 - \frac{1}{2}\boldsymbol{G}.$$

BI ((204))(23)

3.797

1. $$\int_0^{\pi/4} \left(\frac{\pi}{4} - x\ \operatorname{tg} x\right)\ \operatorname{tg} x\,dx = \frac{1}{2}\ \ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8}\ \ln 2.$$

BI ((204))(8)

2. $$\int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x\right)\ \operatorname{tg} x\,dx}{\cos 2x} = -\frac{\pi}{8}\ \ln 2 + \frac{1}{2}\boldsymbol{G}.$$

BI ((204))(19)

3. $$\int_0^{\pi/4} \frac{\frac{\pi}{4} - x\ \operatorname{tg} x}{\cos 2x}\,dx = \frac{\pi}{8}\ \ln 2 + \frac{1}{2}\boldsymbol{G}.$$

BI ((204))(20)

3.798

1. $$\int_0^{\infty} \frac{\operatorname{tg} x}{a + b\ \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2-b^2}} \qquad [a^2 > b^2];$$

$$= 0 \qquad [a^2 < b^2] \qquad [a>0].$$

2. $$\int_0^\infty \frac{\operatorname{tg} x}{a + b\cos 4x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \qquad [a^2 > b^2];$$
$$= 0 \qquad [a^2 < b^2] \qquad [a > 0].$$

BI ((181))(3)

3.799

1. $$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{(\sin x + a\cos x)^2} = \frac{a}{1+a^2}\,\frac{\pi}{2} - \frac{\ln a}{1+a^2} \qquad [a > 0].$$

BI ((208))(5)

2. $$\int_0^{\frac{\pi}{4}} \frac{x\,dx}{(\cos x + a\sin x)^2} = \frac{1}{1+a^2}\,\ln\frac{1+a}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1-a}{(1+a)(1+a^2)} \qquad [a > 0].$$

BI ((204))(24)

3. $$\int_0^{\pi} \frac{a\cos x + b}{(a + b\cos x)^2} x^2\,dx = \frac{2\pi}{b}\,\ln\frac{2(a-b)}{a + \sqrt{a^2 - b^2}} \qquad [a > |b| > 0].$$

GW ((333))(58a)

3.811

1. $$\int_0^{\pi} \frac{\sin x}{1 - \cos t_1\,\cos x} \cdot \frac{x\,dx}{1 - \cos t_2\,\cos x} = \pi\,\operatorname{cosec}\frac{t_1 + t_2}{2}\,\operatorname{cosec}\frac{t_1 - t_2}{2}\,\ln\frac{1 + \operatorname{tg}\frac{t_1}{2}}{1 + \operatorname{tg}\frac{t_2}{2} m}$$
$$\text{(cf. 3.794 4.).}$$

3.794
BI ((222))(5)

2. $$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{(\cos x \pm \sin x)\,\sin x} = \frac{\pi}{4}\,\ln 2 + \boldsymbol{G}.$$

BI ((208))(16, 17)

3. $$\int_0^{\frac{\pi}{4}} \frac{x\,dx}{(\cos x + \sin x)\,\sin x} = -\frac{\pi}{8}\,\ln 2 + \boldsymbol{G}.$$

BI ((204))(29)}

BI ((204))(28)

5. $$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} \frac{x\,dx}{\cos^2 x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

BI ((204))(30)

3.812

1. $$\int_0^{\pi} \frac{x \sin x\,dx}{a + b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{b}{a}} \qquad [a > 0, \quad b > 0];$$
$$= \frac{\pi}{2\sqrt{-ab}} \ln \frac{\sqrt{a} + \sqrt{-b}}{\sqrt{a} - \sqrt{-b}} \qquad [a > -b > 0].$$

GW ((333))(60a)

2. $$\int_0^{\frac{\pi}{2}} \frac{x \sin 2x\,dx}{1 + a \cos^2 x} = \frac{\pi}{a} \ln \frac{1 + \sqrt{1+a}}{2} \qquad [a > -1, \quad a \neq 0].$$

BI ((207))(10)

3. $$\int_0^{\frac{\pi}{2}} \frac{x \sin 2x\,dx}{1 + a \sin^2 x} = \frac{\pi}{a} \ln \frac{2(1 + a - \sqrt{1+a})}{2} \qquad [a > -1, \quad a \neq 0].$$

BI ((207))(2)

4.[7] $$\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}} \qquad [a^2 > 1];$$
$$= 0 \qquad [0 < a^2 < 1]$$

BI ((219))(10)

5.[7] $$\int_0^{\pi} \frac{x \sin x\,dx}{a^2 - \cos^2 x} = \frac{\pi}{2a} \ln \left|\frac{1+a}{1-a}\right| \qquad [0 < a^2 < 1]$$

BI ((219))(13)

6.[6] $$\int_0^{\pi} \frac{x \sin 2x\,dx}{a^2 - \cos^2 x} = \pi \ln\{4(1-a^2)\} \qquad [0 \le a^2 < 1];$$
$$= 2\pi \ln[2(1 - a^2 + a\sqrt{a^2-1})] \qquad [a^2 > 1].$$

7. $$\int_0^{\frac{\pi}{2}} \frac{x\,\sin x\,dx}{\cos^2 t - \sin^2 x} = -2\,\operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}.$$

BI ((207))(1)

8. $$\int_0^{\pi} \frac{x\,\sin x\,dx}{1 - \cos^2 t\,\sin^2 x} = \pi(\pi - 2t)\,\operatorname{cosec} 2t.$$

BI ((219))(12)

9. $$\int_0^{\pi} \frac{x\,\cos x\,dx}{\cos^2 t - \cos^2 x} = 4\,\operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}.$$

BI ((219))(17)

10. $$\int_0^{\pi} \frac{x\,\sin x\,dx}{\operatorname{tg}^2 t + \cos^2 x} = \frac{\pi}{2}(\pi - 2t)\,\operatorname{ctg} t.$$

BI ((219))(14)

11. $$\int_0^{\infty} \frac{x(a\,\cos x + b)\,\sin x\,dx}{\operatorname{ctg}^2 t + \cos^2 x} = 2a\pi\,\ln\cos\frac{t}{2} + \pi bt\,\operatorname{tg} t.$$

BI ((219))(18)

3.813

1. $$\int_0^{\pi} \frac{x\,dx}{a^2\,\cos^2 x + b^2\,\sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{x\,dx}{a^2\,\cos^2 x + b^2\,\sin^2 x} = \frac{\pi^2}{2ab} \qquad [a > 0, \quad b > 0].$$

GW ((333))(36)

2. $$\int_0^{\infty} \frac{1}{\beta^2\,\sin^2 ax + \gamma^2\,\cos^2 ax} \cdot \frac{dx}{x^2 + \delta^2} = \frac{\pi\,\operatorname{sh}(2a\delta)}{4\delta(\beta^2\,\operatorname{sh}^2(a\delta) - \gamma^2\,\operatorname{ch}^2(a\delta))}\left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{2}{\operatorname{sh}(2a\delta)}\right]$$
$$\left[\left|\arg\frac{\beta}{\gamma}\right| < \pi, \quad \operatorname{Re}\delta > 0, \quad a > 0\right].$$

GW((333))(81), ET II 222(63)

3. $$\int_0^{\infty} \frac{\sin x\,dx}{x(a^2\,\sin^2 x + b^2\,\cos^2 x)} = \frac{\pi}{2ab} \qquad [ab > 0].$$

4. $$\int_0^\infty \frac{\sin^2 x\,dx}{x(a^2\cos^2 x + b^2\sin^2 x)} = \frac{\pi}{2b(a+b)} \qquad [a>0, \quad b>0].$$

BI ((181))(11)

5. $$\int_0^{\frac{\pi}{2}} \frac{x\sin 2x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{a^2-b^2}\ln\frac{a+b}{2b} \qquad [a>0, \quad b>0, \quad a\neq b].$$

GW ((333))(52a)

6. $$\int_0^{\pi} \frac{x\sin 2x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{2\pi}{a^2-b^2}\ln\frac{a+b}{2a} \qquad [a>0, \quad b>0, \quad a\neq b].$$

GW ((333))(52b)

7. $$\int_0^\infty \frac{\sin 2x}{a^2\cos^2 x + b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{a(a+b)} \qquad [a>0, \quad b>0].$$

BI ((182))(3)

8. $$\int_0^\infty \frac{\sin 2ax}{\beta^2\sin^2 ax + \gamma^2\cos^2 ax}\cdot\frac{x\,dx}{x^2+\delta^2} = \frac{\pi}{2(\beta^2\operatorname{sh}^2(a\delta) - \gamma^2\operatorname{ch}^2(a\delta))}\left[\frac{\beta-\gamma}{\beta+\gamma} - e^{-2a\delta}\right]$$
$$\left[a>0, \quad \left|\arg\frac{\beta}{\gamma}\right| < \pi, \quad \operatorname{Re}\delta > 0\right].$$

ET II 222(64), GW((333))(80)

9. $$\int_0^\infty \frac{(1-\cos x)\sin x}{a^2\cos^2 x + b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{2b(a+b)} \qquad [a>0, \quad b>0].$$

BI ((182))(7)a

10. $$\int_0^\infty \frac{\sin x\cos^2 x}{a^2\cos^2 x + b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{2a(a+b)} \qquad [a>0, \quad b>0].$$

BI ((182))(4)

11. $$\int_0^\infty \frac{\sin^3 x}{a^2\cos^2 x + b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{2b}\cdot\frac{2}{a+b} \qquad [a>0, \quad b>0].$$

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{(1 - x\ \operatorname{ctg} x)\, dx}{\sin^2 x} = \frac{\pi}{4}.$

BI ((206))(9)

2. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x\ \operatorname{tg} x\, dx}{(\sin x + \cos x)\ \cos x} = -\frac{\pi}{8}\ \ln 2 + \frac{\pi}{4} - \frac{1}{2}\ \ln 2.$

BI ((204))(30)

3. $\displaystyle\int_0^{\infty} \frac{\operatorname{tg} x}{a^2\ \cos^2 x + b^2\ \sin^2 x}\ \frac{dx}{x} = \frac{\pi}{2ab} \qquad [a > 0, \quad b > 0].$

BI ((181))(9)

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x\ \operatorname{ctg} x\, dx}{a^2\ \cos^2 x + b^2\ \sin^2 x} = \frac{\pi}{2a^2}\ \ln \frac{a+b}{b} \qquad [a > 0, \quad b > 0].$

LI ((208))(20)

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5. $$\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\ \operatorname{tg} x\, dx}{a^2\ \cos^2 x + b^2\ \sin^2 x} = \frac{1}{2}\int_0^{\pi} \frac{\left(\frac{\pi}{2} - x\right)\ \operatorname{tg} x\, dx}{a^2\ \cos^2 x + b^2\ \sin^2 x} = \frac{\pi}{2b^2}\ \ln \frac{a+b}{a} \qquad [a > 0, \quad b > 0].$$

GW ((333))(59)

6. $\displaystyle\int_0^{\infty} \frac{\sin^2 x\ \operatorname{tg} x}{a^2\ \cos^2 x + b^2\ \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)} \qquad [a > 0, \quad b > 0].$

BI ((182))(6)

7. $\displaystyle\int_0^{\infty} \frac{\operatorname{tg} x}{a^2\ \cos^2 2x + b^2\ \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab} \qquad [a > 0, \quad b > 0].$

BI ((181))(10)a

8. $\displaystyle\int_0^{\infty} \frac{\sin^2 2x\ \operatorname{tg} x}{a^2\ \cos^2 2x + b^2\ \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b} \qquad [a > 0, \quad b > 0].$

9. $$\int_0^\infty \frac{\cos^2 2x \ \operatorname{tg} x}{a^2 \ \cos^2 2x + b^2 \ \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b} \qquad [a > 0, \quad b > 0].$$

BI ((182))(5)a

10. $$\int_0^\infty \frac{\sin^2 x \ \cos x}{a^2 \ \cos^2 2x + b^2 \ \sin^2 2x} \cdot \frac{dx}{x \ \cos 4x} = -\frac{\pi}{8b} \frac{a}{a^2 + b^2} \qquad [a > 0, \quad b > 0].$$

BI ((186))(12)a

11. $$\int_0^\infty \frac{\sin x}{a^2 \ \cos^2 x + b^2 \ \sin^2 x} \cdot \frac{dx}{x \ \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2 - a^2}{b^2 + a^2} \qquad [a > 0, \quad b > 0].$$

BI ((186))(4)a

12. $$\int_0^\infty \frac{\sin x \ \cos x}{a^2 \ \cos^2 x + b^2 \ \sin^2 x} \cdot \frac{dx}{x \ \cos 2x} = \frac{\pi}{2a} \cdot \frac{b}{a^2 + b^2} \qquad [a > 0, \quad b > 0].$$

BI ((186))(7)a

13. $$\int_0^\infty \frac{\sin x \ \cos^2 x}{a^2 \ \cos^2 x + b^2 \ \sin^2 x} \cdot \frac{dx}{x \ \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2}{a^2 + b^2} \qquad [a > 0, \quad b > 0].$$

BI ((186))(8)a

14. $$\int_0^\infty \frac{\sin^3 x}{a^2 \ \cos^2 x + b^2 \ \sin^2 x} \cdot \frac{dx}{x \ \cos 2x} = -\frac{\pi}{2b} \cdot \frac{a}{a^2 + b^2} \qquad [a > 0, \quad b > 0].$$

BI ((186))(10)

15. $$\int_0^\infty \frac{1 - \cos x}{a^2 \ \cos^2 x + b^2 \ \sin^2 x} \cdot \frac{dx}{x \ \sin x} = \frac{\pi}{2ab} \qquad [a > 0, \quad b > 0].$$

BI ((186))(3)a

3.815

1. $$\int_0^{\frac{\pi}{2}} \frac{x \ \sin 2x \, dx}{(1 + a \ \sin^2 x)(1 + b \ \sin^2 x)} = \frac{\pi}{a - b} \ \ln \left\{ \frac{1 + \sqrt{1+b}}{1 + \sqrt{1+a}} \cdot \frac{\sqrt{1+a}}{\sqrt{1+b}} \right\} \qquad [a > 0, \quad b > 0],$$

(cf. 3.812 3.).

3.812
BI ((208))(22)

2. $$\int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1 + a \sin^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a + ab + b} \ln \frac{(1 + \sqrt{1+n})\sqrt{1+a}}{1 + \sqrt{1+a}} \qquad [a > 0, \quad b > 0],$$
(cf. 3.812 2. and 3.).

3.812
BI ((208))(24)

3. $$\int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1 + a \cos^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a - b} \ln \frac{1 + \sqrt{1+a}}{1 + \sqrt{1+b}} \qquad [a > 0, \quad b > 0], \qquad \text{(cf. 3.812 2.)}.$$

3.812
BI ((208))(23)

4. $$\int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1 - \sin^2 t_1 \cos^2 x)(1 - \sin^2 t_2 \cos^2 x)} = \frac{2\pi}{\cos^2 t_1 - \cos^2 t_2} \ln \frac{\cos \frac{t_1}{2}}{\cos \frac{t_2}{2}}$$
$$[-\pi < t_1 < \pi, \quad -\pi < t_2 < \pi].$$

BI ((208))(21)

3.816

1. $$\int_0^{\pi} \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} \, dx = \pi^2 \frac{\sqrt{a^2 - 1} - a}{a(a^2 - 1)} \qquad [a > 1].$$

LI ((220))(9)

2.[7] $$\int_0^{\pi} \frac{(a^2 - 1 - \sin^2 x) \cos x}{(a^2 - \cos^2 x)^2} x^2 \, dx = \frac{\pi}{a} \ln \left| \frac{1 - a}{1 + a} \right| \qquad [a^2 > 1] \qquad \text{(cf. 3.812 5.)}.$$

3.812
BI ((220))(12)

LI ((220))(10)

4. $$\int_0^\pi \frac{a\ \cos 2x + \sin^2 x}{(a - \sin^2 x)^2} x^2\, dx = \pi\ \ln(4a) \qquad [a > 1], \qquad \text{(cf. 3.812 6.).}$$

3.812
LI ((220))(11)

3.817

1. $$\int_0^\infty \frac{\sin x}{(a^2\ \cos^2 x + b^2\ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \qquad [ab > 0].$$

BI ((181))(12)

2. $$\int_0^\infty \frac{\sin x\ \cos x}{(a^2\ \cos^2 x + b^2\ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \qquad [ab > 0].$$

BI ((182))(8)

3. $$\int_0^\infty \frac{\sin^3 x}{(a^2\ \cos^2 x + b^2\ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \qquad [ab > 0].$$

BI ((181))(15)

4. $$\int_0^\infty \frac{\sin x\ \cos^2 x}{(a^2\ \cos^2 x + b^2\ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \qquad [ab > 0].$$

BI ((182))(9)

5. $$\int_0^\infty \frac{\operatorname{tg} x}{(a^2\ \cos^2 x + b^2\ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \qquad [ab > 0].$$

BI ((181))(13)

6. $$\int_0^\infty \frac{\operatorname{tg} x}{(a^2\ \cos^2 2x + b^2\ \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \frac{a^2 + b^2}{a^3 b^3} \qquad [ab > 0].$$

BI ((181))(14)

7. $$\int_0^\infty \frac{\sin^2 x \ \operatorname{tg} x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \qquad [ab > 0].$$

BI ((182))(11)

8. $$\int_0^\infty \frac{\operatorname{tg} x \ \cos^2 2x}{(a^2 \ \cos^2 2x + b^2 \ \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \qquad [ab > 0].$$

BI ((182))(10)

3.818

1. $$\int_0^\infty \frac{\sin x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5} \qquad [ab > 0].$$

BI ((181))(16)

2. $$\int_0^\infty \frac{\sin x \ \cos x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \qquad [ab > 0].$$

BI ((182))(13)

3. $$\int_0^\infty \frac{\sin x \ \cos^2 x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \qquad [ab > 0].$$

BI ((182))(14)

4. $$\int_0^\infty \frac{\sin^3 x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3 b^5} \qquad [ab > 0].$$

LI ((181))(19)

5. $$\int_0^\infty \frac{\sin^3 \ x \ \cos x}{(a^2 \ \cos^2 2x + b^2 \ \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{64} \cdot \frac{3a^2 + b^2}{a^3 b^5} \qquad [ab > 0].$$

BI ((182))(17)

6. $$\int_0^\infty \frac{\operatorname{tg} x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \ \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5} \qquad [ab > 0].$$

7. $$\int_0^\infty \frac{\sin^2 x \ \operatorname{tg} x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3 b^5} \qquad [ab > 0].$$

BI ((182))(16)

8. $$\int_0^\infty \frac{\operatorname{tg} x}{(a^2 \ \cos^2 2x + b^2 \ \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5 b^5} \qquad [ab > 0].$$

BI ((181))(18)

9. $$\int_0^\infty \frac{\operatorname{tg} x \ \cos^2 2x}{(a^2 \ \cos^2 2x + b^2 \ \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \qquad [ab > 0].$$

BI ((182))(15)

3.819

1. $$\int_0^\infty \frac{\sin x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^6 + 3a^4b^2 + 3a^2b^4 + 5b^6}{a^7 b^7} \qquad [ab > 0].$$

BI ((181))(20)

2. $$\int_0^\infty \frac{\sin x \ \cos x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7 b^5} \qquad [ab > 0].$$

BI ((182))(18)

3. $$\int_0^\infty \frac{\sin x \ \cos^2 x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7 b^5} \qquad [ab > 0].$$

BI ((182))(19)

4. $$\int_0^\infty \frac{\sin^3 x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + a^2b^2 + b^4}{a^5 b^7} \qquad [ab > 0].$$

BI ((181))(23)

5. $$\int_0^\infty \frac{\sin^3 x \ \cos x}{(a^2 \ \cos^2 x + b^2 \ \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5 b^5} \qquad [ab > 0].$$

6. $$\int_0^\infty \frac{\sin x \, \cos^3 x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3} \qquad [ab > 0].$$

BI ((182))(23)

7. $$\int_0^\infty \frac{\sin^3 x \, \cos^2 x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5 b^5} \qquad [ab > 0].$$

BI ((182))(27)

8. $$\int_0^\infty \frac{\sin x \, \cos^4 x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3} \qquad [ab > 0].$$

BI ((182))(24)

9. $$\int_0^\infty \frac{\sin^5 x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3 b^7} \qquad [ab > 0].$$

BI ((181))(24)

10. $$\int_0^\infty \frac{\sin^3 x \, \cos x}{(a^2 \, \cos^2 2x + b^2 \, \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{128} \cdot \frac{5a^4 + 2a^2 b^2 + b^4}{a^5 b^7} \qquad [ab > 0].$$

BI ((182))(22)

11. $$\int_0^\infty \frac{\sin^5 x \, \cos^3 x}{(a^2 \, \cos^2 2x + b^2 \, \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{512} \cdot \frac{5a^2 + b^2}{a^3 b^7} \qquad [ab > 0].$$

BI ((182))(30)

12. $$\int_0^\infty \frac{\sin^2 x \, \operatorname{tg} x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 \, x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + 2a^2 b^2 + b^4}{a^5 b^7} \qquad [ab > 0].$$

BI ((182))(21)

13. $$\int_0^\infty \frac{\sin^4 \, x \, \operatorname{tg} x}{(a^2 \, \cos^2 x + b^2 \, \sin^2 \, x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3 b^7} \qquad [ab > 0].$$

BI ((182))(29)

14. $$\int_0^\infty \frac{\cos^2 2x \, \operatorname{tg} x}{(a^2 \, \cos^2 2x + b^2 \, \sin^2 \, 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5} \qquad [ab > 0].$$

15. $$\int_0^\infty \frac{\sin^3 4x \ \operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{8} \cdot \frac{a^2+b^2}{a^5 b^5} \qquad [ab > 0].$$

BI ((182))(28)

16. $$\int_0^\infty \frac{\cos^4 2x \ \operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2+5b^2}{a^7 b^3} \qquad [ab > 0].$$

BI ((182))(25)

3.82- 3.83 Powers of trigonometric functions combined with other powers

3.821

1. $$\int_0^\pi x \sin^p x \, dx = \frac{\pi^2}{2^{p+1}} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2} \qquad [p > -1].$$

BI((218))(7), LO V 121(71)

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2. $$\int_0^{r\pi} x \sin^n x \, dx = \frac{\pi^2}{2} \cdot \frac{(2m-1)!!}{(2m)!!} r^2 \qquad [n = 2m];$$
$$= (-1)^{r+1} \pi \frac{(2m)!!}{(2m+1)!!} r \qquad [n = 2m+1], \qquad [r \text{ is a natural number}].$$

GW ((333))(8c)

3. $$\int_0^{\frac{\pi}{2}} x \cos^n x \, dx = -\sum_{k=0}^{m-1} \frac{(n-2k+1)(n-2k+3)\dots(n-1)}{(n-2k)(n-2k+2)\dots n} \frac{1}{n-2k} + \\ + \begin{cases} \frac{\pi}{2} \cdot \frac{(2m-2)!!}{(2m-1)!!} & [n = 2m-1]; \\ \frac{\pi^2}{8} \cdot \frac{(2m-1)!!}{(2m)!!} & [n = 2m]. \end{cases}$$

GW ((333))(9b)

4. $$\int_0^\pi x \cos^{2m} x \, dx = \frac{\pi^2}{2} \frac{(2m-1)!!}{(2m)!!}.$$

5. $\int_{r\pi}^{s\pi} x \cos^{2m} x\, dx = \frac{\pi^2}{2}(s^2 - r^2)\, \frac{(2m-1)!!}{(2m)!!}.$

BI ((226))(3)

6. $\int_0^\infty \frac{\sin^p x}{x}\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2}\, \mathrm{B}\left(\frac{p}{2}, \frac{p}{2}\right);$

[p is a fraction with odd numerator and denominator].

LO V 278, FI II 808

7. $\int_0^\infty \frac{\sin^{2n+1} x}{x}\, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}.$

BI ((151))(4)

8. $\int_0^\infty \frac{\sin^{2n} x}{x}\, dx = \infty.$

BI ((151))(3)

9. $\int_0^\infty \frac{\sin^2 ax}{x^2}\, dx = \frac{a\pi}{2} \qquad [a > 0].$

LO V 307, 312, FI II 632

10. $\int_0^\infty \frac{\sin^{2m} ax}{x^2}\, dx = \frac{(2m-3)!!}{(2m-2)!!} \cdot \frac{a\pi}{2} \qquad [a > 0].$

GW ((333))(14b)

11. $\int_0^\infty \frac{\sin^{2m+1} ax}{x^3}\, dx = \frac{(2m-3)!!}{(2m)!!}(2m+1)\frac{a^2\pi}{4} \qquad [a > 0].$

GW ((333))(14d)

12. $\int_0^\infty \frac{\sin^p x}{x^m}\, dx = \frac{p}{m-1}\int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}} \cos x\, dx \qquad [p > m-1 > 0];$

$= \frac{p(p-1)}{(m-1)(m-2)}\int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}}\, dx - \frac{p^2}{(m-1)(m-2)}\int_0^\infty \frac{\sin^p x}{x^{m-2}}\, dx$

$[p > m-1 > 1].$

13. $\int_0^\infty \frac{\sin^{2n} px}{\sqrt{x}}\,dx = \infty.$

BI ((177))(5)

14. $\int_0^\infty \sin^{2n+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}(-1)^k \binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$

BI ((177))(7)

3.822

1. $\int_0^{\frac{\pi}{2}} x^p \cos^m x\,dx = -\frac{p(p-1)}{m^2}\int_0^{\frac{\pi}{2}} x^{p-2} \cos^m x\,dx + \frac{m-1}{m}\int_0^{\frac{\pi}{2}} x^p \cos^{m-2} x\,dx$
$[m > 1, \quad p > 1].$

GW ((333))(9a)

2. $\int_0^\infty x^{-\frac{1}{2}} \cos^{2n+1}(px)\,dx = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}.$

BI ((177))(8)

3.823

$\int_0^\infty x^{\mu-1} \sin^2 ax\,dx = -\frac{\Gamma(\mu)\cos\frac{\mu\pi}{2}}{2^{\mu+1}a^\mu} \qquad [a > 0, \quad -2 < \operatorname{Re}\mu < 0].$

ET I 319(15), GW((333))(19c)a

3.824

1. $\int_0^\infty \frac{\sin^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}(1 - e^{-2a\beta}) \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$

BI ((160))(10)

2. $\int_0^\infty \frac{\cos^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}(1 + e^{-2a\beta}) \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$

BI ((160))(11)

3.[7] $\int_0^\infty \sin^{2m} x \frac{dx}{a^2+x^2} = \frac{(-1)^m}{2^{2m+1}}\cdot\frac{\pi}{a}\left\{2^{2m}\operatorname{sh}^{2m} a - 2\sum_{k=0}^{m}(-1)^k\binom{2m}{k}\operatorname{sh}[2(m-k)a]\right\} \qquad [a > 0].$

$$4.^{7}\quad \int_0^\infty \sin^{2m+1} x \frac{dx}{a^2+x^2} = \frac{(-1)^{m-1}}{2^{2m+2}a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}\left[(2k-2m-1)a\right] + \right.$$
$$\left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}\left[(2m+1-2k)a\right] \right\}$$
$$[a > 0].$$

BI ((160))(14)

$$5.^{7}\quad \int_0^\infty \sin^{2m+1} x \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{2m+1} e^{-(2m+1)a} \sum_{k=0}^{m} (-1)^{m+k} \binom{2m+1}{k} e^{2ka}}$$
$$[|\arg a| < \pi/2],\ m = 0,\ 1,\ 2, \ldots$$

BI ((160))(15)

$$6.^{7}\quad \int_0^\infty \cos^{2m} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \binom{2m}{m} + \frac{\pi}{2^{2m}a} \sum_{k=1}^{m} \binom{2m}{m+k} e^{-2ka} \qquad [a > 0].$$

BI ((160))(16)

$$7.\quad \int_0^\infty \cos^{2m+1} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \sum_{k=1}^{m} \binom{2m+1}{m+k+1} e^{-(2k+1)a} \qquad [a > 0].$$

BI ((160))(17)

$$8.\quad \int_0^\infty \cos^{2m+1} x \frac{x\,dx}{a^2+x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m-2k+1)a] -$$
$$- \frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a].$$

BI ((160))(18)

$$9.\quad \int_0^\infty \frac{\cos^2 ax}{b^2-x^2}\, dx = \frac{\pi}{4b} \sin 2ab \qquad [a > 0, \quad b > 0].$$

BI ((161))(10)

$$10.\quad \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2+x^2}\, dx = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \qquad [a > b];$$
$$= \frac{\pi}{16\beta} [1 - e^{-4a\beta}] \qquad [a = b];$$

BI ((162))(6)

11. $$\int_0^\infty \frac{x \sin 2ax \cos^2 bx}{\beta^2+x^2}\,dx = \frac{\pi}{8}[2e^{-2a\beta}+e^{-2(a+b)\beta}+e^{2(b-a)\beta}] \qquad [a>0];$$
$$= \frac{\pi}{8}[e^{-4a\beta}+2e^{-2a\beta}] \qquad [a=b];$$
$$= \frac{\pi}{8}[2e^{-2a\beta}+e^{-2(a+b)\beta}-e^{2(a-b)\beta}] \qquad [a<b].$$

LI ((162))(5)

3.825

1. $$\int_0^\infty \frac{\sin^2 ax\,dx}{(b^2+x^2)(c^2+x^2)} = \frac{\pi(b-c+ce^{-2ab}-be^{-2ac})}{4bc(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((174))(15)

2. $$\int_0^\infty \frac{\cos^2 ax\,dx}{(b^2+x^2)(c^2+x^2)} = \frac{\pi(b-c+be^{-2ac}-ce^{-2ab})}{4bc(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0].$$

BI ((175))(14)

3.[3] $$\int_0^\infty \frac{\sin^2 ax\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(c\sin 2ab-b\sin 2ac)}{4bc(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0, \quad b\neq c].$$

LI ((174))(16)

4.[3] $$\int_0^\infty \frac{\cos^2 ax\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi(b\sin 2ac-c\sin 2ab)}{4bc(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0, \quad b\neq c].$$

LI ((175))(15)

3.826

1. $$\int_0^\infty \frac{\sin^2 ax\,dx}{x^2(b^2+x^2)} = \frac{\pi}{4b^2}\left[2a-\frac{1}{b}(1-e^{-2ab})\right] \qquad [a>0, \quad b>0].$$

BI ((172))(13)

BII ((172))(14)

3.827

1.* $\int_0^\infty \frac{\sin^3 ax}{x^\nu}\,dx = \frac{3-3^{\nu-1}}{4}a^{\nu-1}\cos\frac{\nu\pi}{2}\Gamma(1-\nu) \qquad [a>0, \quad 0<\operatorname{Re}\nu<4].$

GW ((333))(19f)

2. $\int_0^\infty \frac{\sin^3 ax}{x}\,dx = \frac{\pi}{4}\ \operatorname{sign} a.$

LO V 277

3. $\int_0^\infty \frac{\sin^3 ax}{x^2}\,dx = \frac{3}{4}a\ \ln 3.$

BI ((156))(2)

4. $\int_0^\infty \frac{\sin^3 ax}{x^3}\,dx = \frac{3}{8}a^2\pi\ \operatorname{sign} a.$

BI((156))(7)A,LO V 312

5. $\int_0^\infty \frac{\sin^4 ax}{x^2}\,dx = \frac{a\pi}{4} \qquad [a>0].$

BI ((156))(3)

6. $\int_0^\infty \frac{\sin^4 ax}{x^3}\,dx = a^2\ \ln 2.$

BI ((156))(8)

7. $\int_0^\infty \frac{\sin^4 ax}{x^4}\,dx = \frac{a^3\pi}{3} \qquad [a>0].$

BI((156))(11), LO V 312

8. $\int_0^\infty \frac{\sin^5 ax}{x^2}\,dx = \frac{5}{16}a(3\ \ln 3-\ln 5).$

9. $\int_0^\infty \frac{\sin^5 ax}{x^3}\, dx = \frac{5}{32} a^2 \pi \qquad [a > 0].$

BI ((156))(9)

10. $\int_0^\infty \frac{\sin^5 ax}{x^4}\, dx = \frac{5}{96} a^3 (25 \ln 5 - 27 \ln 3).$

BI ((156))(12)

11. $\int_0^\infty \frac{\sin^5 ax}{x^5}\, dx = \frac{115}{384} a^4 \pi \qquad [a > 0].$

BI((156))(13), LO V 312

12. $\int_0^\infty \frac{\sin^6 ax}{x^2}\, dx = \frac{3}{16} a \pi \qquad [a > 0].$

BI ((156))(5)

13. $\int_0^\infty \frac{\sin^6 ax}{x^3}\, dx = \frac{3}{16} a^2 (8 \ln 2 - 3 \ln 3).$

BI ((156))(10)

14. $\int_0^\infty \frac{\sin^6 ax}{x^5}\, dx = \frac{1}{16} a^4 (27 \ln 3 - 32 \ln 2).$

BI ((156))(14)

15. $\int_0^\infty \frac{\sin^6 ax}{x^6}\, dx = \frac{11}{40} a^5 \pi \qquad [a > 0].$

LO V 312

3.828

1. $\int_0^\infty \frac{\sin px \sin qx}{x}\, dx = \ln \sqrt{\frac{p+q}{|p-q|}} \qquad [p \neq q].$

2. $\int_0^\infty \sin qx \, \sin px \frac{dx}{x^2} = \frac{1}{2} p\pi \qquad [p \le q];$

$= \frac{1}{2} q\pi \qquad [p \ge q].$

BI ((157))(1)

3. $\int_0^\infty \frac{\sin^2 ax \, \sin bx}{x} \, dx = \frac{\pi}{4} \qquad [0 < b < 2a];$

$= \frac{\pi}{8} \qquad [b = 2a];$

$= 0 \qquad [b > 2a].$

BI ((151))(10)

4. $\int_0^\infty \frac{\sin^2 ax \, \cos bx}{x} \, dx = \frac{1}{4} \, \ln \frac{4a^2 - b^2}{b^2}$

BI ((151))(12)

5. $\int_0^\infty \frac{\sin^2 ax \, \cos 2bx}{x^2} \, dx = \frac{\pi}{2}(a - b) \qquad [b < a];$

$= 0 \qquad [b \ge a].$

FI III 648a, BI((157))(5)A}\Cr

6. $\int_0^\infty \frac{\sin 2ax \, \cos^2 bx}{x} \, dx = \frac{\pi}{2} \qquad [a > b];$

$= \frac{3}{8}\pi \qquad [a = b];$

$= \frac{\pi}{4} \qquad [a < b].$

BI ((151))(9)

7. $\int_0^\infty \frac{\sin^2 ax \, \sin bx \, \sin cx}{x^2} \, dx = \frac{\pi}{16}(|b-2a-c|-|2a-b-c|+2c) \qquad [a > 0, \quad b > 0, \quad c > 0].$

BI((157))(9)A, ET I 79(15)

8. $\int_0^\infty \frac{\sin^2 ax \, \sin bx \, \sin cx}{x} \, dx = \frac{1}{4} \, \ln \frac{b + c}{b - c} + \frac{1}{8} \, \ln \frac{(2a - b + c)(2a + b - c)}{(2a + b + c)(2a - b - c)}$

$[a > 0, \quad b > 0, \quad c > 0, \quad b \neq c].$

9. $$\int_0^\infty \frac{\sin^2 ax \, \sin^2 bx}{x^2}\, dx = \frac{\pi}{4}a \qquad [0 \le a \le b];$$
$$= \frac{\pi}{4}b \qquad [0 \le b \le a].$$

BI ((157))(3)

10. $$\int_0^\infty \frac{\sin^2 ax \, \sin^2 bx}{x^4}\, dx = \frac{1}{6}a^2\pi(3b-a) \qquad [0 \le a \le b];$$
$$= \frac{1}{6}b^2\pi(3a-b) \qquad [0 \le b \le a].$$

BI ((157))(27)

11. $$\int_0^\infty \frac{\sin^2 ax \, \cos^2 bx}{x^2}\, dx = \frac{2a-b}{4}\pi \qquad [a \ge b > 0];$$
$$= \frac{a\pi}{4} \qquad [0 < a \le b].$$

BI ((157))(6)

12. $$\int_0^\infty \frac{\sin^3 ax \, \sin 3bx}{x^4}\, dx = \frac{a^3\pi}{2} \qquad [b > a];$$
$$= \frac{\pi}{16}[8a^3 - 9(a-b)^3] \qquad [a \le 3b \le 3a];$$
$$= \frac{9b\pi}{8}(a^2 - b^2) \qquad [3b \le a].$$

LI ((157))(28)
BI ((157))(28)

13. $$\int_0^\infty \frac{\sin^3 ax \, \cos bx}{x}\, dx = 0 \qquad [b > 3a];$$
$$= -\frac{\pi}{16} \qquad [b = 3a];$$
$$= -\frac{\pi}{8} \qquad [3a > b > a];$$
$$= \frac{\pi}{16} \qquad [b = a];$$
$$= \frac{\pi}{4} \qquad [a > b] \qquad [a > 0, \quad b > 0].$$

BI ((151))(15)

14. $$\int_0^\infty \frac{\sin^3 ax \, \cos 3bx}{x^2}\, dx = \frac{3}{8}\left\{(a+b)\, \ln[3(a+b)] + (b-a)\, \ln[3(b-a)] - \right.$$
$$\left. - \frac{1}{3}(a+3b)\, \ln(a+3b) - \frac{1}{3}(3b-a)\, \ln(3b-a)\right\} \qquad [a > 0,\ b > 0].$$

15. $$\int_0^\infty \frac{\sin^3 ax \cos bx}{x^3}\,dx = \frac{\pi}{8}(3a^2 - b^2) \qquad [b < a];$$
$$= \frac{\pi b^2}{4} \qquad [a = b];$$
$$= \frac{\pi}{16}(3a - b)^2 \qquad [a < b < 3a];$$
$$= 0 \qquad [3a < b]; \qquad [a > 0, \quad b > 0].$$

BI((157))(19), ET I 19(10)

16. $$\int_0^\infty \frac{\sin^3 ax \sin bx}{x^4}\,dx = \frac{b\pi}{24}(9a^2 - b^2) \qquad [0 < b \le a];$$
$$= \frac{\pi}{48}[24a^3 - (3a - b)^3] \qquad [0 < a \le b \le 3a];$$
$$= \frac{\pi a^3}{2} \qquad [0 < 3a \le b].$$

ET I 79(16)

17. $$\int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x}\,dx = \frac{\pi}{8} \qquad [2b > 3a];$$
$$= \frac{5\pi}{32} \qquad [2b = 3a];$$
$$= \frac{3\pi}{16} \qquad [3a > 2b > a];$$
$$= \frac{3\pi}{32} \qquad [2b = a];$$
$$= 0 \qquad [a > 2b]; \qquad [a > 0, \quad b > 0].$$

BI ((151))(14)

18. $$\int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x}\,dx = \frac{1}{16} \ln \frac{(2a+b)^3(b-2a)^3(2a+3b)(3b-2a)}{9b^8}$$
$$[b > 2a > 0 \quad \text{or} \quad 2a > 3b > 0];$$
$$= \frac{1}{16} \ln \frac{(2a+b)^3(2a-b)^3(2a+3b)(3b-2a)}{9b^8} \qquad [3b > 2a > b].$$

BI ((151))(13)

19. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx}{x}\,dx =$$
$$= \frac{\pi}{16}[1 + \operatorname{sign}(c - a + b) + \operatorname{sign}(c + a - b) - 2\operatorname{sign}(c - a) - 2\operatorname{sign}(c - b)]$$
$$[a > 0, \quad b > 0, \quad c > 0].$$

20. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx\, dx}{x^2} = \frac{a-b-c}{16} \ln 4(a-b-c)^2 - \\ - \frac{a+b+c}{16} \ln 4(a+b+c)^2 + \frac{a+b-c}{16} \ln 4(a+b-c)^2 - \\ - \frac{a-b+c}{16} \ln 4(a-b+c)^2 + \frac{a+c}{8} \ln 4(a+c)^2 - \\ - \frac{a-c}{8} \ln 4(a-c)^2 + \frac{b+c}{8} \ln 4(b+c)^2 - \frac{b-c}{8} \ln 4(b-c)^2 - \\ - \frac{1}{2} c \ln 2c \qquad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((157))(10)

21. $$\int_0^\infty \frac{\sin^2 ax \, \sin^3 bx}{x^3}\, dx = \frac{3b^2\pi}{16} \qquad [2a > 3b]; \\ = \frac{a^2\pi}{12} \qquad [2a = 3b]; \\ = \frac{6b^2 - (3b-2a)^2}{32}\pi \qquad [3b > 2a > b]; \\ = \frac{a^2\pi}{4} \qquad [b > 2a]; \qquad [a > 0, \quad b > 0].$$

BI ((157))(18)

3.829

1. $$\int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}}\, dx = \frac{\pi}{2^n (n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n+1}$$

GW ((333))(63)

2. $$\int_0^\infty \left(1 - \cos^{2m-1} x\right) \frac{dx}{x^2} = \int_0^\infty \left(1 - \cos^{2m} x\right) \frac{dx}{x^2} = \frac{m\pi}{2^{2m}} \binom{2m}{m}.$$

BI ((158))(7, 8)

3.831

1. $$\int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x}\, dx = \frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} \qquad [ab > 0, \quad n = 1, 2, \ldots].$$

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2. $$\int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x}\, dx = \left[1 - \frac{(2n-1)!!}{(2n)!!}\right] \ln \frac{b}{a} \qquad [ab > 0, \quad n = 0, 1, \ldots].$$

3. $$\int_0^\infty \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x}\, dx = \ln \frac{b}{a} \qquad [ab > 0, \quad m = 0, 1, \ldots].$$

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4. $$\int_0^\infty \frac{\cos^m ax \, \cos max - \cos^m bx \, \cos mbx}{x}\, dx = \left(1 - \frac{1}{2^m}\right) \ln \frac{b}{a} \qquad [ab > 0, \quad m = 0, 1, \ldots].$$

LI ((155))(8)

3.832

1. $$\int_0^{\frac{\pi}{2}} x \cos^{p-1} x \sin ax \, dx = \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right)\Gamma\left(\frac{p-a+1}{2}\right)}$$
$$[p > 0, \quad -(p+1) < a < p+1].$$

BI ((205))(6)

2.[3] $$\int_0^\infty \sin^{2m+1} x \sin 2mx \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a}[(1-e^{-2a})^{2m} - 1] \operatorname{sh} a \qquad [a > 0, \quad m = 0, 1 \ldots]$$
$$[\text{for } m = -1 \text{ see 3.723.2}].$$

3.723
BI ((162))(17)

3. $$\int_0^\infty \sin^{2m-1} x \sin[(2m-1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m+1}\pi}{2^{2m} a}(1-e^{-2a})^{2m-1} \qquad [a > 0, \quad m = 1, 2, \ldots].$$

BI ((162))(11)

4. $$\int_0^\infty \sin^{2m-1} x \sin[(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1}\pi}{2^{2m} a} e^{-2a}(1 - e^{-2a})^{2m-1}$$
$$[a > 0, \quad m = 1, 2, \ldots].$$

BI ((162))(12)

5. $$\int_0^\infty \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \operatorname{sh}^{2m+1} a \qquad [a > 0].$$

6.[3] $\int_0^\infty \sin^{2m} x \, \sin[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a [(1-e^{-2a})^{2m} - (1+e^{-2a})]$

$[a \geq 0, \quad m = 0, 1, \dots].$

BI ((162))(13)

7. $\int_0^\infty \sin^{2m} x \, \sin(2mx) \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} [(1-e^{-2a})^{2m} - 1] \qquad [a > 0, \quad m = 0, 1, \dots].$

BI ((162))(14)

8. $\int_0^\infty \sin^{2m} x \, \sin[(2m+2)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} (1-e^{-2a})^{2m} \qquad [a > 0, \quad m = 0, 1, \dots].$

BI ((162))(15)

9. $\int_0^\infty \sin^{2m} x \, \sin 4mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \, \mathrm{sh}^{2m} a \qquad [a > 0, \quad m = 1, 2, \dots].$

BI ((162))(16)

10. $\int_0^\infty \sin^{2m} x \, \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2} \qquad [m = 1, 2, \dots].$

GW ((333))(15a)

11. $\int_0^\infty \sin^{2m} x \, \cos[(2m-1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m} a} [(1-e^{-2a})^{2m-1} - 1] \, \mathrm{sh}\, a$

$[a > 0, \quad m = 1, 2, \dots].$

BI ((162))(25)

12. $\int_0^\infty \sin^{2m} x \, \cos(2mx) \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} (1-e^{-2a})^{2m} \qquad [a > 0, \quad m = 0, 1, \dots].$

BI ((162))(26)

13. $\int_0^\infty \sin^{2m} x \, \cos[(2m+2)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} (1-e^{-2a})^{2m} \qquad [a > 0, \quad m = 0, 1, \dots].$

14. $$\int_0^\infty \sin^{2m} x \cos 4mx \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \operatorname{sh}^{2m} a \qquad [a > 0, \quad m = 0, 1, \dots].$$

BI ((162))(28)

15. $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2} \qquad [m = 0, 1, \dots].$$

GW ((333))(15)

16.[3] $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2} \qquad [m = 1, 2, \dots].$$

GW ((333))(15b)

17. $$\int_0^\infty \sin^{2m-1} x \cos[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} [(1-e^{-2a})^{2m-1} - 1]$$
$$[m = 1, 2, \dots, \quad a > 0].$$

BI ((162))(23)

18.[3] $$\int_0^\infty \sin^{2m+1} x \cos 2mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \left\{e^a [(1-e^{-2a})^{2m+1} - 1] - e^{-a}\right\}$$
$$[m = 0, 1, \dots, \quad a \geq 0].$$

BI ((162))(29)

19. $$\int_0^\infty \sin^{2m-1} x \cos[(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} (1-e^{-2a})^{2m-1}$$
$$[m = 1, 2, \dots, \quad a > 0].$$

BI ((162))(24)

20. $$\int_0^\infty \sin^{2m+1} x \cos[2(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2} e^{-2(2m+1)a} \operatorname{sh}^{2m+1} a$$
$$[m = 0, 1, \dots, \quad a > 0].$$

BI ((162))(30)

21. $$\int_0^\infty \cos^m x \sin mx \frac{x\,dx}{a^2+x^2} = \frac{1}{2^{m+1} a} \sum_{k=1}^m \binom{m}{k} [e^{-2ka} \operatorname{Ei}(2ka) - e^{2ka} \operatorname{Ei}(-2ka)] \qquad [a > 0].$$

22. $$\int_0^\infty \cos^n sx \, \sin nsx \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{n+1}}[(1+e^{-2as})^n - 1] \qquad [s>0, \quad \operatorname{Re} a > 0, \quad n \geq 0].$$

BI ((163))(9)

23. $$\int_0^\infty \cos^n sx \, \sin nsx \frac{x\,dx}{a^2-x^2} = \frac{\pi}{2}(2^{-n} - \cos^n as \, \cos nas) \qquad [n = 0,\ 1, \ldots].$$

BI ((166))(10)

24. $$\int_0^\infty \cos^{m-1} x \, \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} e^{-2a}(1+e^{-2a})^{m-1} \qquad [a>0, \quad m = 1,\ 2, \ldots].$$

BI ((163))(6)

25. $$\int_0^\infty \cos^m x \, \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{m+1}} e^{-a}(1+e^{-2a})^m \qquad [m = 0,\ 1, \ldots, \quad a>0].$$

BI ((163))(10)

26.[3] $$\int_0^\infty \cos^m x \, \sin[(m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} \operatorname{ch} a[(1+e^{-2a})^{m-1} - 1] \qquad [m = 0,\ 1, \ldots, \quad a \geq 0].$$

BI ((163))(7)

27. $$\int_0^\infty \cos^m x \, \sin(3mx) \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2} e^{-3a} \operatorname{ch}^m a \qquad [a>0, \quad m = 1,\ 2, \ldots].$$

BI ((163))(11)

28. $$\int_0^\infty \cos^n sx \, \cos nxs \frac{dx}{a^2+x^2} = \frac{\pi}{2^{n+1}a}(1+e^{-2as})^n \qquad [n = 0,\ 1, \ldots].$$

BI ((163))(16)

29. $$\int_0^\infty \cos^n sx \, \cos nsx \frac{dx}{a^2-x^2} = \frac{\pi}{2a} \cos^n as \, \sin nas \qquad [n = 0,\ 1, \ldots].$$

30. $$\int_0^\infty \cos^{m-1} x \, \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^m a} e^{-2a} (1+e^{-2a})^{m-1} \qquad [m = 1,\, 2, \dots, \quad a > 0].$$

BI ((163))(14)

31. $$\int_0^\infty \cos^m x \, \cos[(m-1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1} a} e^a [(1+e^{-2a})^m - (1-e^{-2a})] \qquad [m = 0,\, 1, \dots, \quad a > 0].$$

BI ((163))(15)

32. $$\int_0^\infty \cos^m x \, \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1} a} e^{-a} (1+e^{-2a})^m \qquad [m = 0,\, 1, \dots, \quad a > 0].$$

BI ((163))(17)

33. $$\begin{aligned} \int_0^\infty \sin^p x \, \cos x \frac{dx}{x^q} &= \frac{p}{q-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^\infty \frac{\sin^{p+1} x}{x^{q-1}} \, dx \qquad [p > q-1 > 0]; \\ &= \frac{p(p-1)}{(q-1)(q-2)} \int_0^\infty \sin^{p-2} x \, \cos x \frac{dx}{x^{q-2}} - \\ &\quad - \frac{(p+1)^2}{(q-1)(q-2)} \int_0^\infty \sin^p x \, \cos x \frac{dx}{x^{q-2}} \qquad [p > q-1 > 1]. \end{aligned}$$

GW ((333))(18)

34. $$\int_0^\infty \cos^{2m} x \, \cos 2nx \, \sin x \frac{dx}{x} = \int_0^\infty \cos^{2m-1} x \, \cos 2nx \, \sin x \frac{dx}{x} = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}.$$

BI ((152))(5, 6)

35. $$\int_0^\infty \cos^p ax \, \sin bx \, \cos x \frac{dx}{x} = \frac{\pi}{2} \qquad [b > ap, \quad p > -1].$$

BI ((153))(12)

36. $$\int_0^\infty \cos^p ax \, \sin pax \, \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} (2^p - 1) \qquad [p > -1].$$

BI ((153))(2)

BI ((157))(15)

3.833

1. $$\int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} = \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!!}\pi.$$
$$= \frac{1}{2}\mathrm{B}\left(m+\frac{1}{2},\, n+\frac{1}{2}\right).$$

GW ((333))(24)
BI ((151))(24, 25)

2. $$\int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}.$$

LI ((152))(4)

494

3.834

1. $$\int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left| \frac{1-a}{1+a} \right|^{2m-1} - \right.$$
$$\left. - \sum_{k=0}^{2m} (-1)^k \binom{m-\frac{1}{2}}{k} \left(\frac{4a}{(1+a)^2} \right)^k \right\} \qquad [|a| \neq 1].$$

GW ((333))(62a)

2. $$\int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1 - 2a \cos x + a^2)^p} \cdot \frac{dx}{x} =$$
$$= \frac{n!\pi}{2^{n+1}(2m+n+1)!(1+a)^{2p}} \sum_{k=0}^{n} \frac{(-1)^k (2m+2n-2k+1)!!(2m+2k-1)!!}{k!(n-k)!} \times$$
$$\times F\left(m+n-k+\frac{3}{2},\, p;\, 2m+n+2;\, \frac{4a}{(1+a)^2}\right) \qquad [a \neq \pm 1].$$

GW ((333))(62)

3.835

1. $$\int_0^\infty \frac{\cos^{2m} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{b^{2m-1}}{a(a+b)^{2m}} \qquad [ab > 0].$$

BI ((182))(31)a

LI ((182))(32)a

3.836

1. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin mx}{x}\,dx = \frac{\pi}{2} \qquad [m \geq n].$$

LI ((159))(12)

2.[3] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos mx\,dx = \frac{n\pi}{2^n} \sum_{0 \leq k < \frac{m+n}{2}} \frac{(-1)^k (n+m-2k)^{n-1}}{k!(n-k)!} \qquad [0 \leq m < n];$$
$$= 0 \qquad [m \geq n] \qquad [n \geq 2];$$
$$= \frac{\pi}{4} \qquad [m = n = 1].$$

GI((159))(14), ET I 20(11)

3. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin nx\,\cos x \frac{dx}{x} = \frac{\pi}{2} \qquad [n \geq 1].$$

BI ((159))(20)

4.* $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin(anx)}{x}\,dx = \frac{\pi}{2}\left[1 - \frac{1}{2^{n-1}n!} \sum_{0 \leq k < \frac{n}{2}(1-a)} (-1)^k \binom{n}{k} (n+an-2k)^n\right]$$
$$[\text{all real } a, \quad n \geq 1].$$

ET I 20(11)

5.[7] $$\frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx\,dx = \frac{\pi}{2^{n-2}(n-1)!} \sum_{k=0}^{[r]} (-1)^k \binom{n}{k} (n-b-2k)^{n-1},$$
where $0 \leq b < n, n \geq 1, r = (n-b)/2$, and $[r]$ is the largest integer contained in r.

LO V 340(14)

6. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos anx\,dx = 0 \qquad [a \leq -1 \text{ or } a \geq 1, n \geq 2; \quad \text{for } n = 1 \text{see } 3.741.2].$$

1. $$\int_0^{\frac{\pi}{2}} \frac{x^2\,dx}{\sin^2 x} = \pi\,\ln 2.$$

BI ((206))(9)

2. $$\int_0^{\frac{\pi}{4}} \frac{x^2\,dx}{\sin^2 x} = -\frac{\pi^2}{16} + \frac{\pi}{4}\,\ln 2 + \boldsymbol{G} = 0.8435118417\ldots$$

BI ((204))(10)

3. $$\int_0^{\frac{\pi}{4}} \frac{x^2\,dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4}\,\ln 2 - \boldsymbol{G}.$$

GW ((333))(35a)

4. $$\int_0^{\frac{\pi}{4}} \frac{x^{p+1}}{\sin^2 x}\,dx = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1)\left(\frac{\pi}{4}\right)^p \left\{\frac{1}{p} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)}\zeta(2k)\right\} \qquad [p>0].$$

LI ((204))(14)

5. $$\int_0^{\frac{\pi}{2}} \frac{x^2\,\cos x}{\sin^2 x}\,dx = -\frac{\pi^2}{4} + 4\boldsymbol{G} = 1.1964612764\ldots$$

BI ((206))(7)

6. $$\int_0^{\frac{\pi}{2}} \frac{x^3\,\cos x}{\sin^3 x}\,dx = -\frac{\pi^3}{16} + \frac{3}{2}\pi\,\ln 2.$$

BI ((206))(8)

7. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\,\sin^{2n} x\,\frac{dx}{x^m} = 0 \qquad \left[n > \frac{m-1}{2}, \quad m>0\right].$$

BI ((180))(16)

8. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\,\sin^{2n+1} x\,\frac{dx}{x^m} = 0 \qquad \left[n > \frac{m-2}{2}, \quad m>0\right].$$

BI ((180))(17)

BI ((149))(20)

$$10.^3 \int_0^{\pi} \frac{x \sin(2n+1)x}{\sin x}\,dx = \frac{1}{2}\pi^2 \qquad [n = 0, 1, 2, \dots].$$

$$11.^3 \int_0^{\pi} \frac{x \sin 2nx}{\sin x}\,dx = -4\sum_{k=1}^{n}(2k-1)^{-2} \qquad [n = 1, 2, 3, \dots].$$

3.838

$$1. \quad \int_0^{\frac{\pi}{2}} \frac{x \cos^{p-1} x}{\sin^{p+1} x}\,dx = \frac{\pi}{2p} \sec\frac{\pi p}{2} \qquad [p < 1].$$

BI ((206))(13)a

$$2. \quad \int_0^{\frac{\pi}{4}} \frac{x \sin^{p-1} x}{\cos^{p+1} x}\,dx = \frac{\pi}{4p} - \frac{1}{2p}\beta\left(\frac{p+1}{2}\right) \qquad [p > -1].$$

LI ((204))(15)

$$3. \quad \int_0^{\frac{\pi}{4}} \frac{x \sin^{2m-1} x}{\cos^{2m+1} x}\,dx = \frac{\pi}{8m}(1 - \cos m\pi) + \frac{1}{2m}\sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{2m-2k-1}.$$

BI ((204))(17)

$$4. \quad \int_0^{\frac{\pi}{4}} \frac{x \sin^{2m} x}{\cos^{2m+2} x}\,dx = \frac{1}{2(2m+1)}\left[\frac{\pi}{2} + (-1)^{m-1}\ln 2 + \sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{m-k}\right].$$

BI ((204))(16)

3.839

$$1. \quad \int_0^{\frac{\pi}{4}} x \operatorname{tg}^2 x\,dx = \frac{\pi}{4} - \frac{\pi^3}{32} - \frac{1}{2}\ln 2.$$

BI ((204))(3)

BI ((204))(7)

3. $$\int_0^{\frac{\pi}{4}} \frac{x^2 \ \operatorname{tg} x}{\cos^2 x} \, dx = \frac{1}{2} \ \ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16} \qquad \text{(cf. 3.839 1.).}$$

3.839
BI ((204))(13)

4. $$\int_0^{\frac{\pi}{4}} \frac{x^2 \ \operatorname{tg}^2 x}{\cos^2 x} \, dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + \boldsymbol{G} \right) \qquad \text{(cf. 3.839 2.).}$$

3.839
BI ((204))(12)

5. $$\int_0^{\frac{\pi}{2}} x \ \cos^p x \ \operatorname{tg} x \, dx = \frac{\pi}{2^{p+1} p} \cdot \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2} \qquad [p > -1].$$

BI ((205))(3)

6. $$\int_0^{\frac{\pi}{2}} x \ \sin^p x \ \operatorname{ctg} x \, dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p} \operatorname{B}\left(\frac{p+1}{2}, \frac{p+1}{2}\right) \qquad [p > -1].$$

BI ((206))(11)

7. $$\int_0^{\infty} \sin^{2n} x \ \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}.$$

GW ((333))(16)

8. $$\int_0^{\infty} \cos^s rx \ \operatorname{tg} qx \frac{dx}{x} = \frac{\pi}{2} \qquad [s > -1].$$

BI ((151))(26)

9. $$\int_0^{\infty} \frac{\cos[(2n-1)x]}{\cos x} \cdot \left(\frac{\sin x}{x}\right)^{2n} dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1} \pi |B_{2n}|.$$

10. $$\int_0^\infty \operatorname{tg}^r px \frac{dx}{q^2+x^2} = \frac{\pi}{2q} \sec\frac{r\pi}{2}\, \theta^r pq \qquad [r^2<1].$$

BI ((160))(19)

3.84 Integrals containing the expressions $\sqrt{1-k^2\, sin^2\, x}$, $\sqrt{1-k^2\, cos^2\, x}$, and similar expressions

Notation: $k' = 1 - k^2$

3.841

1. $$\int_0^\infty \sin x \sqrt{1-k^2\, \sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k).$$

BI ((154))(8)

2. $$\int_0^\infty \sin x \sqrt{1-k^2\, \cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k).$$

BI ((154))(20)

3. $$\int_0^\infty \operatorname{tg} x \sqrt{1-k^2\, \sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k).$$

BI ((154))(9)

4. $$\int_0^\infty \operatorname{tg} x \sqrt{1-k^2\, \cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k).$$

BI ((154))(21)

3.842

1. $$\int_0^\infty \frac{\sin x}{\sqrt{1+\sin^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \\ = \int_0^\infty \frac{\sin x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right).$$

BI ((183))(4, 5, 9, 10)

BI ((226))(4)

3. $$\int_0^\infty \frac{\sin x}{\sqrt{1-k^2\ \sin^2 x}} \frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1-k^2\ \sin^2 x}} \frac{dx}{x} = \\ = \int_0^\infty \frac{\sin x}{\sqrt{1-k^2\ \cos^2 x}} \frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1-k^2\ \cos^2 x}} = \frac{dx}{x} = \boldsymbol{K}(k).$$

BI ((183))(12, 13, 21, 22)

4. $$\int_0^{\frac{\pi}{2}} \frac{x\ \sin x\ \cos x}{\sqrt{1-k^2\ \sin^2 x}}\, dx = \frac{1}{2k^2}[-\pi k' + 2\boldsymbol{E}(k)].$$

BI ((211))(1)

5. $$\int_0^{\frac{\pi}{2}} \frac{x\ \sin x\ \cos x}{\sqrt{1-k^2\ \cos^2 x}}\, dx = \frac{1}{2k^2}[\pi - 2\boldsymbol{E}(k)].$$

BI ((214))(1)

6. $$\int_0^\alpha \frac{x\ \sin x\, dx}{\cos^2 x\sqrt{\sin^2\ \alpha - \sin^2\ x}} = \frac{\pi\ \sin^2\ \frac{\alpha}{2}}{\cos^2\ \alpha}.$$

LO III 284

7. $$\int_0^\beta \frac{x\ \sin x\, dx}{(1-\sin^2\ \alpha\ \sin^2\ x)\sqrt{\sin^2\ \beta - \sin^2\ x}} = \frac{\pi\ \ln \dfrac{\cos\alpha + \sqrt{1-\sin^2\ \alpha\ \sin^2\ \beta}}{2\ \cos\beta\ \cos^2\ \frac{\alpha}{2}}}{2\ \cos\alpha\sqrt{1-\sin^2\ \alpha\ \sin^2\ \beta}}.$$

LO III 284

3.843

1. $$\int_0^\infty\ \operatorname{tg} x\sqrt{1-k^2\ \sin^2 2x}\frac{dx}{x} = \boldsymbol{E}(k).$$

BI ((154))(10)

2. $$\int_0^\infty\ \operatorname{tg} x\sqrt{1-k^2\ \cos^2 2x}\frac{dx}{x} = \boldsymbol{E}(k).$$

3. $$\int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1+\sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1+\cos^2 2x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right).$$

BI ((183))(6, 11)

4. $$\int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \frac{dx}{x} = \boldsymbol{K}(k).$$

BI ((183))(14, 23)

3.844

1. $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

BI ((185))(20)

2. $$\int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

BI ((185))(21)

3. $$\int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2+k^2)\boldsymbol{K}(k) - 2(1+k^2)\boldsymbol{E}(k)].$$

BI ((185))(22)

4. $$\int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2+k^2)\boldsymbol{K}(k) - 2(1+k^2)\boldsymbol{E}(k)].$$

BI ((185))(23)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(1+k'^2)\boldsymbol{E}(k) - 2k'^2 \boldsymbol{K}(k)].$$

BI ((185))(24)

6. $$\int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(1+k'^2)\boldsymbol{E}(k) - 2k'^2 \boldsymbol{K}(k)].$$

BI ((185))(25)

BI ((184))(16)

8. $$\int_0^\infty \frac{\sin^4 x \ \operatorname{tg} x}{\sqrt{1-k^2 \ \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2+3k^2)k'^2 \boldsymbol{K}(k) - 2(k'^2-k^2)\boldsymbol{E}(k)].$$

BI ((184))(18)

3.845

1. $$\int_0^\infty \frac{\sin x \ \cos x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((185))(6)

2. $$\int_0^\infty \frac{\sin x \ \cos^2 x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((185))(7)

3. $$\int_0^\infty \frac{\sin^2 x \ \operatorname{tg} x}{\sqrt{1+\cos^2 \ x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BU ((184))(8)

3.846

1. $$\int_0^\infty \frac{\sin x \ \cos x}{\sqrt{1-k^2 \ \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{E}(k) - k'^2 \boldsymbol{K}(k)].$$

BI ((185))(9)

2. $$\int_0^\infty \frac{\sin x \ \cos^2 x}{\sqrt{1-k^2 \ \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{E}(k) - k'^2 \boldsymbol{K}(k)].$$

BI ((185))(10)

3. $$\int_0^\infty \frac{\sin x \ \cos^3 x}{\sqrt{1-k^2 \ \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2-3k^2)k'^2 \boldsymbol{K}(k) - 2(k'^2-k^2)\boldsymbol{E}(k)].$$

4. $$\int_0^\infty \frac{\sin x \, \cos^4 x}{\sqrt{1-k^2 \, \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2-3k^2)k'^2 \boldsymbol{K}(k) - 2(k'^2 - k^2)\boldsymbol{E}(k)].$$

BI ((185))(12)

5. $$\int_0^\infty \frac{\sin^3 x \, \cos x}{\sqrt{1-k^2 \, \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(1+k'^2)\boldsymbol{E}(k) - 2k'^2 \boldsymbol{K}(k)].$$

BI ((185))(13)

6. $$\int_0^\infty \frac{\sin^3 x \, \cos^2 x}{\sqrt{1-k^2 \, \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(1+k'^2)\boldsymbol{E}(k) - 2k'^2 \boldsymbol{K}(k)].$$

BI ((185))(14)

7. $$\int_0^\infty \frac{\sin^2 x \, \operatorname{tg} x}{\sqrt{1-k^2 \, \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

BI ((184))(9)

8. $$\int_0^\infty \frac{\sin^4 x \, \operatorname{tg} x}{\sqrt{1-k^2 \, \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2+k^2)\boldsymbol{K}(k) - 2(1+k^2)\boldsymbol{E}(k)].$$

BI ((184))(11)

3.847

$$\int_0^\infty \frac{\sin x \, \cos x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^\infty \frac{\sin x \, \cos^2 x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((185))(3, 4)

3.848

1. $$\int_0^\infty \frac{\sin^3 x \, \cos x}{\sqrt{1-k^2 \, \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

BI ((185))(15)

2. $$\int_0^\infty \frac{\cos^2 2x \, \operatorname{tg} x}{\sqrt{1-k^2 \, \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{E}(k) - k'^2 \boldsymbol{K}(k)].$$

3. $$\int_0^\infty \frac{\cos^4 2x \ \operatorname{tg} x}{\sqrt{1-k^2 \ \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2-3k^2)k'^2 \boldsymbol{K}(k) - 2(k'^2-k^2)\boldsymbol{E}(k)].$$

BI ((184))(13)

4. $$\int_0^\infty \frac{\sin^2 4x \ \operatorname{tg} x}{\sqrt{1-k^2 \ \sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4}[(1+k'^2)\boldsymbol{E}(k) - 2k'^2 \boldsymbol{K}(k)].$$

BI ((184))(17)

5. $$\int_0^\infty \frac{\sin^3 x \ \cos x}{\sqrt{1-k^2 \ \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}[\boldsymbol{E}(k) - k'^2 \boldsymbol{K}(k)].$$

BI ((185))(26)

500

6. $$\int_0^\infty \frac{\cos^2 2x \ \operatorname{tg} x}{\sqrt{1-k^2 \ \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

BI ((184))(19)

7. $$\int_0^\infty \frac{\cos^4 2x \ \operatorname{tg} x}{\sqrt{1-k^2 \ \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}[(2+k^2)\boldsymbol{K}(k) - 2(1+k^2)\boldsymbol{E}(k)].$$

BI ((184))(20)

3.849

1. $$\int_0^\infty \frac{\sin^3 x \ \cos x}{\sqrt{1+\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((185))(8)

2. $$\int_0^\infty \frac{\sin^3 x \ \cos x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8}\left[2\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right].$$

BI ((185))(5)

3. $$\int_0^\infty \frac{\cos^2 2x \ \operatorname{tg} x}{\sqrt{1+\sin^2 \ 2x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right].$$

powers

3.851

1. $$\int_0^\infty x\,\sin(ax^2)\,\sin(2bx)\,dx = \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a}+\sin\frac{b^2}{a}\right) \qquad [a \geq 0, \quad b>0].$$

BI ((150))(4)

2. $$\int_0^\infty x\,\sin(ax^2)\,\cos(2bx)\,dx = \frac{1}{2a} - \frac{b}{a}\sqrt{\frac{\pi}{2a}}\left[\sin\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right) - \cos\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right].$$

BI ((150))(5)a

3. $$\int_0^\infty x\,\cos(ax^2)\,\sin(2bx)\,dx = \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left(\sin\frac{b^2}{a}-\cos\frac{b^2}{a}\right) \qquad [a>0, \quad b>0], \qquad \text{(cf. 3.691 7.).}$$

3.691
BI ((150))(7)

4. $$\int_0^\infty x\,\cos(ax^2)\,\cos(2bx)\,dx = \frac{b}{a}\sqrt{\frac{\pi}{2a}}\left[\cos\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right) + \sin\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right].$$

BI ((150))(6)a

5. $$\int_0^\infty \sin(ax^2)\,\cos(bx)\frac{dx}{x^2} = \frac{b\pi}{2}\left\{S\left(\frac{b}{2\sqrt{a}}\right) - C\left(\frac{b}{2\sqrt{a}}\right) + \sqrt{a\pi}\,\sin\left(\frac{b^2}{4a}+\frac{\pi}{4}\right)\right\}$$
$$[a>0, \quad b>0], \qquad \text{(cf. 3.691 7.).}$$

3.691
ET I 23(3)a

3.852

1. $$\int_0^\infty \frac{\sin(ax^2)}{x^2}\,dx = \sqrt{\frac{a\pi}{2}} \qquad [a \geq 0].$$

BI ((177))(10)a

2. $$\int_0^\infty \sin(ax^2)\,\cos(bx^2)\frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}(\sqrt{a+b}+\sqrt{a-b}) \qquad [a>b>0];$$
$$= \frac{1}{2}\sqrt{\pi a} \qquad [b=a\ge 0];$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}(\sqrt{a+b}-\sqrt{b-a}) \qquad [b>a>0], \qquad \text{(cf. 3.852 1.).}$$

3.852
BI ((177))(23)

3. $$\int_0^\infty \frac{\sin^2(a^2x^2)}{x^4}\,dx = \frac{2\sqrt{\pi}}{3}a^3 \qquad [a\ge 0].$$

GW ((333))(19e)

4. $$\int_0^\infty \frac{\sin^3(a^2x^2)}{x^2}\,dx = \frac{3-\sqrt{3}}{8}\sqrt{\pi}a \qquad [a\ge 0].$$

GW ((333))(19g)

5. $$\int_0^\infty (\sin x^2 - x^2\,\cos x^2)\frac{dx}{x^4} = \frac{1}{3}\sqrt{\frac{\pi}{2}}.$$

BI ((178))(8)

6. $$\int_0^\infty \left\{\cos x^2 - \frac{1}{1+x^2}\right\}\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}.$$

BI ((173))(22)

3.853

1. $$\int_0^\infty \frac{\sin(ax^2)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\sqrt{2}\,\sin\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\,\cos\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right) - \sin\left(a\beta^2\right)\right]$$
$$[a>0, \quad \operatorname{Re}\beta>0].$$

ET II 219(33)a

2. $$\int_0^\infty \frac{\cos(ax^2)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\cos(a\beta^2) - \sqrt{2}\,\cos\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right)\sqrt{2}\,\sin\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right]$$
$$[a>0, \quad \operatorname{Re}\beta>0].$$

3. $$\int_0^\infty \frac{x^2 \sin(ax^2)}{\beta^2+x^2} dx = \frac{\beta\pi}{2}\left[\sin(a\beta^2) - \sqrt{2}\sin\left(a\beta^2 + \frac{\pi}{4}\right) C\left(\sqrt{a}\beta\right) + \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right) S(\sqrt{a}\beta)\right] - \\ -\frac{1}{2}\sqrt{\frac{\pi}{2a}} \qquad [a>0,\ \operatorname{Re}\beta>0].$$

ET II 219(32)a

4. $$\int_0^\infty \frac{x^2 \cos(ax^2)}{\beta^2+x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2}\left[\cos(a\beta^2) - \sqrt{2}\,\cos\left(a\beta^2+\frac{\pi}{4}\right) C\left(\sqrt{a}\beta\right) - \\ - \sqrt{2}\,\sin\left(a\beta^2+\frac{\pi}{4}\right) S\left(\sqrt{a}\beta\right)\right] \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

ET II 221(50)a

3.854

1. $$\int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b^3\sqrt{2}} \qquad [a>0, \quad b>0].$$

LI ((178))(11)A, BI ((168))(25)

502

2. $$\int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2\, dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b\sqrt{2}} \qquad [a>0, b>0].$$

LI ((178))(12)

3. $$\int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2\, dx}{(x^4+b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3}\left(a + \frac{1}{2b^2}\right) \qquad [a>0, \quad b>0].$$

LI ((178))(14)

4. $$\int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{x^4\, dx}{(x^4+b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b}\left(\frac{1}{2b^2} - a\right) \qquad [a>0, \quad b>0].$$

BI ((178))(15)

3.855

1. $$\int_0^\infty \frac{\sin(ax^2)}{\sqrt{\beta^2+x^4}}\, dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}} I_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0, \quad \operatorname{Re}\beta>0].$$

2. $$\int_0^\infty \frac{\cos(ax^2)}{\sqrt{\beta^2+x^4}}\,dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}} I_{-\frac{1}{4}}\left(\frac{a\beta}{2}\right) K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0, \quad \operatorname{Re}\beta > 0].$$

ET I 9(22)

3. $$\int_0^u \frac{\sin(a^2x^2)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\right]^2 \qquad [a>0].$$

ET I 66(29)

4. $$\int_u^\infty \frac{\sin\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}} J_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) N_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) \qquad [a>0].$$

ET I 66(30)

5. $$\int_0^u \frac{\cos(a^2x^2)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\right]^2.$$

ET I 9(23)

6. $$\int_u^\infty \frac{\cos\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}} J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) N_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right).$$

ET I 10(24)

3.856

1. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}}\,\sin(a^2x^2)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu} I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{3}{2}, \quad |\arg\beta| < \frac{\pi}{4}\right].$$

ET I 71(23)

2. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}}\,\cos(a^2x^2)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu} I_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{3}{2}, \quad |\arg\beta| < \frac{\pi}{4}\right].$$

ET I 12(16)

3. $$\int_0^\infty \frac{(\sqrt{\beta^4+x^4}-x^2)^\nu}{\sqrt{\beta^4+x^4}} \cos(a^2x^2)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu} I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\,\nu > -\frac{3}{2}, \quad |\arg\beta| < \frac{\pi}{4}\right].$$

ET I 12(17)

4. $$\int_0^\infty \frac{\sin(a^2x^2)\,dx}{\sqrt{\beta^4+x^4}\sqrt{x^2+\sqrt{\beta^4+x^4}}} = \frac{\operatorname{sh}\frac{a^2\beta^2}{2}}{\sqrt{2}\beta^2} K_0\left(\frac{a^2\beta^2}{2}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}\right].$$

ET I 66(32)

5. $$\int_0^\infty \frac{\cos(a^2x^2)\,dx}{\sqrt{\beta^4+x^4}\sqrt{(x^2+\sqrt{\beta^4+x^4})^3}} = \frac{\operatorname{sh}\frac{a^2\beta^2}{2}}{2\sqrt{2}\beta^4} K_1\left(\frac{a^2\beta^2}{2}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}\right].$$

ET I 10(27)

6. $$\int_0^\infty \frac{\sqrt{\sqrt{\beta^4+x^4}+x^2}}{\sqrt{\beta^4+x^4}} \sin(a^2x^2)\,dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2\beta^2}{2}} I_0\left(\frac{a^2\beta^2}{2}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}\right],$$

ET I 67(33)

3.857

1. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2-R_1}{R_2+R_1}} \sin(ax^2)\,dx = \frac{1}{2\sqrt{b}} K_0(ac)\sin ab$$
$$\left[R_1 = \sqrt{c^2+(b-x^2)^2}, \quad R_2 = \sqrt{c^2+(b+x^2)^2}, \quad a>0, \quad c>0\right].$$

ET I 67(34)

2. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2+R_1}{R_2-R_1}} \cos(ax^2)\,dx = \frac{1}{2\sqrt{b}} K_0(ac)\cos ab$$
$$\left[R_1 = \sqrt{c^2+(b-x^2)^2}, \quad R_2 = \sqrt{c^2+(b+x^2)^2}, \quad a>0, \quad c>0\right].$$

ET I 10(26)

3.858

1. $$\int_u^\infty \frac{(x^2+\sqrt{x^4-u^4})^\nu + (x^2-\sqrt{x^4-u^4})^\nu}{\sqrt{x^4-u^4}} \sin(a^2x^2)\,dx =$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}} u^{2\nu}\left[J_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) N_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) + J_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) N_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)\right]$$

2. $$\int_u^\infty \frac{(x^2+\sqrt{x^4-u^4})^\nu + (x^2-\sqrt{x^4-u^4})^\nu}{\sqrt{x^4-u^4}} \cos(a^2x^2)\,dx =$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}}u^{2\nu}\left[J_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)N_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) + J_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)N_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)\right]$$
$$\left[\operatorname{Re}\nu < \frac{3}{2}\right].$$

ET I 13(26)

3.859

$$\int_0^\infty \left[\cos(x^{2^n}) - \frac{1}{1+x^{2^{n+1}}}\right]\frac{dx}{x} = -\frac{1}{2^n}\boldsymbol{C}.$$

BI ((173))(24)

3.861

1. $$\int_0^\infty \sin^{2n+1}(ax^2)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac{1}{2}}}{2^{2n-m+\frac{1}{2}}(2m-1)!!}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{2n+1}{n+k}(2k-1)^{m-\frac{1}{2}}$$
[the sign $+$ is taken when $m \equiv 0 \pmod 4$ or $m \equiv 1 \pmod 4$, the sign $-$ is taken when $m \equiv 2 \pmod 4$ or $m \equiv 3 \pmod 4$].

BI ((177))(19)a

2. $$\int_0^\infty \sin^{2n}(ax^2)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac{1}{2}}}{2^{2n-2m+1}(2m-1)!!}\sum_{k=1}^{n}(-1)^{k}\binom{2n}{n+k}k^{m-\frac{1}{2}}$$
[the sign $+$ is taken when $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$, the sign $-$ is taken when $m \equiv 2 \pmod 4$ or $m \equiv 1 \pmod 4$].

BI ((177))(18)A, LI ((177))(18)

3.862

$$\int_0^\infty [\cos(ax^2\sqrt{n}) + \sin(ax^2\sqrt{n})]\left(\frac{\sin x^2}{x^2}\right)^n dx =$$
$$= \frac{\sqrt{\pi}}{(2n-1)!!\sqrt{2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-2k+a\sqrt{n})^{n-\frac{1}{2}} \qquad [a > \sqrt{n} > 0].$$

BI ((178))(9)

3.863

1. $$\int_0^\infty x^2\ \cos(ax^4)\ \sin(2bx^2)\,dx = -\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right)+\cos\left(\frac{b^2}{2a}-\frac{\pi}{8}\right)J\frac{3}{4}\left(\frac{b^2}{2a}\right)\right]$$
$$[a>0, \quad b>0].$$

ET I 25(22)

2. $$\int_0^\infty x^2\ \cos(ax^4)\ \cos(2bx^2)\,dx = -\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a}+\frac{\pi}{8}\right)J_{-\frac{3}{4}}\left(\frac{b^2}{2a}\right)+\cos\left(\frac{b^2}{2a}+\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right)\right]$$
$$[a>0, \quad b>0].$$

ET I 25(23)

3.864

1. $$\int_0^\infty \sin\frac{b}{x}\ \sin ax\frac{dx}{x} = \frac{\pi}{2}N_0(2\sqrt{ab})+K_0(2\sqrt{ab}) \qquad [a>0, \quad b>0].$$

WA 204(3)a

2. $$\int_0^\infty \cos\frac{b}{x}\ \cos ax\frac{dx}{x} = -\frac{\pi}{2}N_0(2\sqrt{ab})+K_0(2\sqrt{ab}) \qquad [a>0, \quad b>0].$$

WA 204(4)A, ET I 24 (12)

3.865

1. $$\int_0^u \frac{(u^2-x^2)^{\mu-1}}{x^{2\mu}}\ \sin\frac{a}{x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}}\Gamma(\mu)J_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$
$$[a>0, \quad u>0, \quad 0<\operatorname{Re}\ \mu<1].$$

ET II 189(30)

2. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}}\ \sin\frac{a}{x}\,dx = \sqrt{\frac{\pi}{u}}a^{\frac{1}{2}-\mu}\Gamma(\mu)\ \sin\frac{a}{2u}J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right) \qquad [a>0, \quad u>0, \quad \operatorname{Re}\mu>0].$$

ET II 203(21)

3. $$\int_0^u \frac{(u^2-x^2)^{\mu-1}}{x^{2\mu}}\ \cos\frac{a}{x}\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac{1}{2}}\Gamma(\mu)u^{\mu-\frac{3}{2}}N_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$
$$[a>0, \quad u>0, \quad 0<\operatorname{Re}\mu<1].$$

4. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \cos\frac{a}{x}\,dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu}\Gamma(\mu) \cos\frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right) \qquad [a>0, \quad u>0, \quad \operatorname{Re}\mu>0].$$

ET II 204(26)

3.866

1. $$\int_0^\infty x^{\mu-1} \sin\frac{b^2}{x} \sin(a^2x)\,dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \operatorname{cosec}\frac{\mu\pi}{2}[J_\mu(2ab)-J_{-\mu}(2ab)+I_{-\mu}(2ab)-I_\mu(2ab)]$$
$$[a>0, \quad b>0, \quad |\operatorname{Re}\mu|<1].$$

ET I 322(42)

2. $$\int_0^\infty x^{\mu-1} \sin\frac{b^2}{x} \cos(a^2x)\,dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \sec\frac{\mu\pi}{2}[J_\mu(2ab)+J_{-\mu}(2ab)+I_\mu(2ab)-I_{-\mu}(2ab)]$$
$$[a>0, \quad b>0, \quad |\operatorname{Re}\mu|<1].$$

ET I 322(43)

3. $$\int_0^\infty x^{\mu-1} \cos\frac{b^2}{x} \cos(a^2x)\,dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \operatorname{cosec}\frac{\mu\pi}{2}[J_{-\mu}(2ab)-J_\mu(2ab)+I_{-\mu}(2ab)-I_\mu(2ab)]$$
$$[a>0, \quad b>0, \quad |\operatorname{Re}\mu|<1].$$

ET I 322(44)

3.867

1. $$\int_0^1 \frac{\cos ax - \cos\frac{a}{x}}{1-x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos ax - \cos\frac{a}{x}}{1-x^2}\,dx = \frac{\pi}{2}\sin a \qquad [a>0].$$

GW ((334))(7a)

2. $$\int_0^1 \frac{\cos ax + \cos\frac{a}{x}}{1+x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos ax + \cos\frac{a}{x}}{1+x^2}\,dx = \frac{\pi}{2}e^{-a} \qquad [a>0].$$

GW ((334))(7b)

3.868

1. $$\int_0^\infty \sin\left(a^2x+\frac{b^2}{x}\right)\frac{dx}{x} = \pi J_0(2ab) \qquad [a>0, \quad b>0].$$

2. $$\int_0^\infty \cos\left(a^2x + \frac{b^2}{x}\right)\frac{dx}{x} = -\pi N_0(2ab). \qquad [a > 0, \quad b > 0].$$

GW ((334))(11a)

3. $$\int_0^\infty \sin\left(a^2x - \frac{b^2}{x}\right)\frac{dx}{x} = 0 \qquad [a > 0, \quad b > 0].$$

GW ((334))(11b)

4. $$\int_0^\infty \cos\left(a^2x - \frac{b^2}{x}\right)\frac{dx}{x} = 2K_0(2ab) \qquad [a > 0, \quad b > 0].$$

GW ((334))(11b)

3.869

1. $$\int_0^\infty \sin\left(ax - \frac{b}{x}\right)\frac{x\,dx}{\beta^2 + x^2} = \frac{\pi}{2}\exp\left(-\alpha\beta - \frac{b}{\beta}\right) \qquad [a > 0, \quad b > 0, \quad \operatorname{Re}\beta > 0].$$

ET II 220(42)

2. $$\int_0^\infty \cos\left(ax - \frac{b}{x}\right)\frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta}\exp\left(-a\beta - \frac{b}{\beta}\right) \qquad [a > 0, \quad b > 0, \quad \operatorname{Re}\beta > 0].$$

ET II 222(58)

3.871

1. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x + \frac{b^2}{x}\right)\right]dx = \pi b^\mu\left[J_\mu(2ab)\cos\frac{\mu\pi}{2} - N_\mu(2ab)\sin\frac{\mu\pi}{2}\right]$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}\mu < 1].$$

ET I 319(17)

2. $$\int_0^\infty x^{\mu-1}\cos\left[a\left(x + \frac{b^2}{x}\right)\right]dx = -\pi b^\mu\left[J_\mu(2ab)\sin\frac{\mu\pi}{2} + N_\mu(2ab)\cos\frac{\mu\pi}{2}\right]$$
$$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1].$$

ET I 321(35)

3. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x - \frac{b^2}{x}\right)\right]dx = 2b^\mu K_\mu(2ab)\sin\frac{\mu\pi}{2} \qquad [a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1].$$

4. $$\int_0^\infty x^{\mu-1} \cos\left[a\left(x - \frac{b^2}{x}\right)\right] dx = 2b^\mu K_\mu(2ab) \cos\frac{\mu\pi}{2} \qquad [a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1].$$

ET I 321(36)

3.872

1. $$\int_0^1 \sin\left[a\left(x+\frac{1}{x}\right)\right] \sin\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1-x^2} = \\ = \frac{1}{2}\int_0^\infty \sin\left[a\left(x+\frac{1}{x}\right)\right] \sin\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1-x^2} = -\frac{\pi}{4} \sin 2a \qquad [a \geq 0].$$

BI ((149))(15), GW ((334))(8a)

507

2. $$\int_0^1 \cos\left[a\left(x+\frac{1}{x}\right)\right] \cos\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1+x^2} = \\ = \frac{1}{2}\int_0^\infty \cos\left[a\left(x+\frac{1}{x}\right)\right] \cos\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1+x^2} = \frac{\pi}{4}e^{-2a} \qquad [a \geq 0].$$

GW ((334))(8b)

3.873

1. $$\int_0^\infty \sin\frac{a^2}{x^2} \cos b^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2a}}[\sin(2ab)+\cos(2ab)+e^{-2ab}] \qquad [a > 0, \quad b > 0].$$

ET I 24(15)

2. $$\int_0^\infty \cos\frac{a^2}{x^2} \cos b^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2a}}[\cos(2ab)-\sin(2ab)+e^{-2ab}] \qquad [a > 0, \quad b > 0].$$

ET I 24(16)

3.874

1. $$\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \sin\left(2ab + \frac{\pi}{4}\right) \qquad [a > 0, \quad b > 0].$$

BI ((179))(6)A, GW((334))(10a)

2. $$\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \cos\left(2ab + \frac{\pi}{4}\right) \qquad [a > 0, \quad b > 0].$$

3. $\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right)\frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab} \qquad [a \geq 0, \quad b > 0].$

GW ((335))(10b)

4. $\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab} \qquad [a \geq 0, \quad b > 0].$

GW ((334))(10b)

5. $\int_0^\infty \sin\left(ax - \frac{b}{x}\right)^2\frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \qquad [a > 0, \quad b > 0].$

BI ((179))(13)a

6. $\int_0^\infty \cos\left(ax - \frac{b}{x}\right)^2\frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \qquad [a > 0, \quad b > 0].$

BI ((179))(14)a

3.875

1. $\int_u^\infty \frac{x\,\sin(p\sqrt{x^2-u^2})}{x^2+a^2}\cos bx\,dx = \frac{\pi}{2}\,\exp(-p\sqrt{a^2+u^2})\,\operatorname{ch} ab \qquad [0 < b < p].$

ET I 27(39)

2. $\int_u^\infty \frac{x\,\sin(p\sqrt{x^2-u^2})}{a^2+x^2-u^2}\cos bx\,dx = \frac{\pi}{2}e^{-ap}\,\cos(b\sqrt{u^2-a^2}) \qquad [0 < b < p, \quad a > 0].$

ET I 27(38)

3.[6] $\int_0^\infty \frac{\sin(p\sqrt{a^2+x^2})}{(a^2+x^2)^{3/2}}\cos bx\,dx = \frac{\pi p}{2a}e^{-ab} \qquad [0 < p < b, \quad a > 0].$

ET I 26(29)

3.876

1. $\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\cos bx\,dx = \frac{\pi}{2}J_0(a\sqrt{p^2-b^2}) \qquad [0 < b < p];$

$= 0 \qquad [b > p > 0]; \qquad [a > 0].$

2. $\displaystyle\int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\cos bx\,dx = -\frac{\pi}{2}N_0(a\sqrt{p^2-b^2}) \qquad [0<b<p];$

$\displaystyle = K_0(a\sqrt{b^2-p^2}) \qquad [b>p>0]; \qquad [a>0].$

ET I 26(34)

3. $\displaystyle\int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{x^2+c^2}\cos bx\,dx = \frac{\pi}{2c}e^{-bc}\cos(p\sqrt{a^2-c^2}) \qquad [c>0, \quad b>p].$

ET I 26(33)

4. $\displaystyle\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{(x^2+c^2)\sqrt{x^2+a^2}}\cos bx\,dx = \frac{\pi}{2c}\frac{e^{-bc}\sin(p\sqrt{a^2-c^2})}{\sqrt{a^2-c^2}} \qquad [c\neq a];$

$\displaystyle = \frac{\pi}{2}e^{-ba}\frac{p}{a} \qquad [c=a]; \qquad [b>p, \quad c>0].$

ET I 26(31)a

5.[6] $\displaystyle\int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{x^2+a^2}\cos bx\,dx = \frac{\pi}{2a}e^{-ab} \qquad [b>p>0; \quad a>0].$

ET I 27(35)a

6.[6] $\displaystyle\int_0^\infty \frac{x\cos(p\sqrt{x^2+a^2})}{x^2+a^2}\sin bx\,dx = \frac{\pi}{2}e^{-ab} \qquad [a>0, \quad b>p>0].$

ET I 85(29)a

7. $\displaystyle\int_0^u \frac{\cos(p\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}}\cos bx\,dx = \frac{\pi}{2}J_0(u\sqrt{b^2+p^2}).$

ET I 28(42)

8. $\displaystyle\int_u^\infty \frac{\cos(p\sqrt{x^2-u^2})}{\sqrt{x^2-u^2}}\cos bx\,dx = K_0(u\sqrt{p^2-b^2}) \qquad [0<b<|p|].$

$\displaystyle = -\frac{\pi}{2}N_0(u\sqrt{b^2-p^2}) \qquad [b>|p|].$

ET I 28(43)

3.877

1. $\displaystyle\int_0^u \frac{\sin(p\sqrt{u^2-x^2})}{\sqrt[4]{(u^2-x^2)^3}}\cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}}J_{\frac{1}{4}}\left[\frac{u}{2}(\sqrt{b^2+p^2}-b)\right]J_{\frac{1}{4}}\left[\frac{u}{2}(\sqrt{b^2+p^2}+b)\right]$

$[b>0, \quad p>0].$

2. $$\int_u^\infty \frac{\sin(p\sqrt{x^2-u^2})}{\sqrt[4]{(x^2-u^2)^3}} \cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}} J\frac{1}{4}\left[\frac{u}{2}(b-\sqrt{b^2-p^2})\right] N\frac{1}{4}\left[\frac{u}{2}(b+\sqrt{b^2-p^2})\right]$$
$$[b > p > 0].$$

ET I 27(41)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^2-x^2}\right)}{\sqrt[4]{(u^2-x^2)^3}} \cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}} J_-\frac{1}{4}\left[\frac{u}{2}(\sqrt{p^2+b^2}-b)\right] J_{-\frac{1}{4}}\left[\frac{u}{2}(\sqrt{p^2+b^2}+b)\right]$$
$$[u > 0, \quad p > 0].$$

ET I 28(44)

509

4. $$\int_u^\infty \frac{\cos(p\sqrt{x^2-u^2})}{\sqrt[4]{(x^2-u^2)^3}} \cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}} J_-\frac{1}{4}\left[\frac{u}{2}(b-\sqrt{b^2-p^2})\right] N\frac{1}{4}\left[\frac{u}{2}(b+\sqrt{b^2-p^2})\right]$$
$$[b > p > 0].$$

ET I 28(45)

3.878

1. $$\int_0^\infty \frac{\sin(p\sqrt{x^4+a^4})}{\sqrt{x^4+a^4}} \cos bx^2\,dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b} J_{-\frac{1}{4}}\left[\frac{a^2}{2}(p-\sqrt{p^2-b^2})\right] J\frac{1}{4}\left[\frac{a^2}{2}(p+\sqrt{p^2-b^2})\right]$$
$$[p > b > 0].$$

ET I 26(32)

2. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^4+a^4}\right)}{\sqrt{x^4+a^4}} \cos bx^2\,dx =$$
$$= -\frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b} J_{-\frac{1}{4}}\left[\frac{a^2}{2}(p-\sqrt{p^2-b^2})\right] N\frac{1}{4}\left[\frac{a^2}{2}(p+\sqrt{p^2-b^2})\right] \qquad [a > 0, \quad p > b > 0].$$

ET I 27(36)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^4-x^4}\right)}{\sqrt{u^4-x^4}} \cos bx^2\,dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b} J_-\frac{1}{4}\left[\frac{u^2}{2}(\sqrt{p^2+b^2}-p)\right] J_{-\frac{1}{4}}\left[\frac{u^2}{2}(\sqrt{p^2+b^2}+p)\right]$$
$$[p > 0, \quad b > 0].$$

ET I 28(46)

3.879

$$\int_0^\infty \sin ax^p \frac{dx}{x} = \frac{\pi}{2p} \qquad [a > 0, \quad p > 0].$$

GW ((334))(6)

3.881

1. $$\int_0^{\frac{\pi}{2}} x \sin(a \operatorname{tg} x)\, dx = \frac{\pi}{4} e^{-a} [\boldsymbol{C} + \ln 2a - e^{2a} \operatorname{Ei}(-2a)] \qquad [a > 0].$$

BI ((205))(9)

2. $$\int_0^\infty \sin(a \operatorname{tg} x) \frac{dx}{x} = \frac{\pi}{2}(1 - e^{-a}) \qquad [a > 0].$$

BI ((151))(6)

3. $$\int_0^\infty \sin(a \operatorname{tg} x) \cos x \frac{dx}{x} = \frac{\pi}{2}(1 - e^{-a}) \qquad [a > 0].$$

BI ((151))(19)

4. $$\int_0^\infty \cos(a \operatorname{tg} x) \sin x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \qquad [a > 0].$$

BI ((151))(20)

5. $$\int_0^\infty \sin(a \operatorname{tg} x) \sin 2x \frac{dx}{x} = \frac{1 + a}{2} \pi e^{-a} \qquad [a > 0].$$

BI ((152))(11)

6. $$\int_0^\infty \cos(a \operatorname{tg} x) \sin^3 x \frac{dx}{x} = \frac{1 - a}{4} \pi e^{-a} \qquad [a > 0].$$

BI ((151))(23)

7. $$\int_0^\infty \sin(a \operatorname{tg} x) \operatorname{tg} \frac{x}{2} \cos^2 x \frac{dx}{x} = \frac{1 + a}{4} \pi e^{-a} \qquad [a > 0].$$

8. $\int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \frac{x\,dx}{\sin 2x} = -\frac{\pi}{4} \operatorname{Ei}(-a) \qquad [a > 0].$

BI ((206))(15)

9. $\int_0^{\frac{\pi}{2}} \sin(a \operatorname{ctg} x) \frac{x\,dx}{\sin^2 x} = \frac{1 - e^{-a}}{2a}\pi \qquad [a > 0].$

LI ((206))(14)

10. $\int_0^{\frac{\pi}{2}} x \cos(a \operatorname{tg} x) \operatorname{tg} x\,dx = -\frac{\pi}{4} e^{-a}[\boldsymbol{C} + \ln 2a + e^{2a} \operatorname{Ei}(-2a)] \qquad [a > 0].$

BI ((205))(10)

11. $\int_0^{\infty} \cos(a \operatorname{tg} x) \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \qquad [a > 0].$

BI ((151))(21)

12. $\int_0^{\infty} \cos(a \operatorname{tg} x) \sin^2 x \operatorname{tg} x \frac{dx}{x} = \frac{1 - a}{16}\pi e^{-a} \qquad [a > 0].$

BI ((152))(15)

13. $\int_0^{\infty} \sin(a \operatorname{tg} x) \operatorname{tg}^2 x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \qquad [a > 0].$

BI ((152))(9)

14. $\int_0^{\infty} \cos(a \operatorname{tg} 2x) \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \qquad [a > 0].$

BI ((151))(22)

15. $\int_0^{\infty} \sin(a \operatorname{tg} 2x) \cos^2 2x \operatorname{tg} x \frac{dx}{x} = \frac{1 + a}{4}\pi e^{-a} \qquad [a > 0].$

BI ((152))(13)

BI ((152))(10)

17. $$\int_0^\infty \sin(a\ \mathrm{tg}\, 2x)\ \mathrm{tg}\, x\ \mathrm{ctg}\, 2x \frac{dx}{x} = \frac{\pi}{2}(1 - e^{-a}) \qquad [a > 0].$$

BI ((180))(6)

3.882

1. $$\int_0^\infty \sin(a\ \mathrm{tg}^2\, x) \frac{x\, dx}{b^2 + x^2} = \frac{\pi}{2}[\exp(-a\,\theta b) - e^{-a}] \qquad [a > 0, \quad b > 0].$$

BI ((160))(22)

2. $$\int_0^\infty \cos(a\ \mathrm{tg}^2\, x)\ \cos x \frac{dx}{b^2 + x^2} = \frac{\pi}{2b}[\mathrm{ch}\, b\ \exp(-a\,\theta b) - e^{-a}\ \mathrm{sh}\, b] \qquad [a > 0, \quad b > 0].$$

BI ((163))(3)

3. $$\int_0^\infty \cos(a\ \mathrm{tg}^2\, x)\ \mathrm{cosec}\, 2x \frac{x\, dx}{b^2 + x^2} = \frac{\pi}{2\ \mathrm{sh}\, 2b}\ \exp(-a\,\theta b) \qquad [a > 0, \quad b > 0].$$

BI ((191))(10)

4. $$\int_0^\infty \cos(a\ \mathrm{tg}^2\, x)\ \mathrm{tg}\, x \frac{x\, dx}{b^2 + x^2} = \frac{\pi}{2\ \mathrm{ch}\, b}[e^{-a}\ \mathrm{ch}\, b - \exp(-a\,\theta b)\ \mathrm{sh}\, b] \qquad [a > 0, \quad b > 0].$$

BI ((163))(4)

5. $$\int_0^\infty \cos(a\ \mathrm{tg}^2\, x)\ \mathrm{ctg}\, x \frac{x\, dx}{b^2 + x^2} = \frac{\pi}{2}[\mathrm{cth}\ b\ \exp(-a\,\theta b) - e^{-a}] \qquad [a > 0, \quad b > 0].$$

BI ((163))(5)

6. $$\int_0^\infty \cos(a\ \mathrm{tg}^2\, x)\ \mathrm{ctg}\, 2x \frac{x\, dx}{b^2 + x^2} = \frac{\pi}{2}[\mathrm{cth}\ 2b\ \exp(-a\,\theta b) - e^{-a}] \qquad [a > 0, \quad b > 0].$$

BI ((191))(11)

3.883

BI ((404))(4)

2. $$\int_0^1 x^{\mu-1} \sin(\beta \ln x)\, dx = -\frac{\beta}{\beta^2+\mu^2} \qquad [\operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

ET I 319(19)

3. $$\int_0^1 x^{\mu-1} \cos(\beta \ln x)\, dx = \frac{\mu}{\beta^2+\mu^2} \qquad [\operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

ET I 321(38)

3.884

$$\int_{-\infty}^{\infty} \frac{\sin a\sqrt{|x|}}{x-b} \operatorname{sign} x\, dx = \cos a\sqrt{|b|} + exp\} \left(-a\sqrt{|b|}\right) \qquad [a > 0].$$

ET II 253(46)

3.89- 3.91 Trigonometric functions and exponentials

3.891

1. $$\int_0^{2\pi} e^{imx} \sin nx\, dx = 0 \qquad [m \neq n; \quad m = n = 0];$$
$$= \pi i \qquad [m = n \neq 0].$$

2. $$\int_0^{2\pi} e^{imx} \cos nx\, dx = 0 \qquad [m \neq n];$$
$$= \pi \qquad [m = n \neq 0];$$
$$= 2\pi \qquad [m = n = 0].$$

3.892

1. $$\int_0^{\pi} e^{i\beta x}\, sin^{\nu-1} x\, dx = \frac{\pi e^{i\beta\frac{\pi}{2}}}{2^{\nu-1}\nu \mathrm{B}\left(\frac{\nu+\beta+1}{2}, \frac{\nu-\beta+1}{2}\right)} \qquad [\operatorname{Re}\nu > -1].$$

NH 158, EH I 12(29)

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^{\nu-1} x\, dx = \frac{\pi}{2^{\nu-1}\nu \mathrm{B}\left(\frac{\nu+\beta+1}{2}, \frac{\nu-\beta+1}{2}\right)} \qquad [\operatorname{Re}\nu > -1].$$

$$3.^6 \quad \int_0^{\frac{\pi}{2}} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x \, dx = \frac{1}{2^{2\mu+2\nu+1}} \left\{ \exp\left[i\pi \left(\beta - \nu - \frac{1}{2} \right) \right] \mathrm{B}\left(\beta - \mu - \nu, \, 2\nu + 1 \right) \times \right.$$

$$\times F(-2\mu, \, \beta - \mu - \nu; \, 1 + \beta - \mu + \nu; \, -1) + \exp\left[i\pi \left(\mu + \frac{1}{2} \right) \right] \times$$

$$\left. \times \mathrm{B}(\beta - \mu - \nu, \, 2\mu + 1) F(-2\nu, \, \beta - \mu - \nu; \, 1 + \beta + \mu - \nu; \, -1) \right\}$$

$$\left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

EH I 80(6)

$$4. \quad \int_0^{\pi} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x \, dx = \frac{\pi \exp[i\pi(\beta - \nu)] F(-2\nu, \, \beta - \mu - \nu; \, 1 + \beta + \mu - \nu; \, -1)}{4^{\mu+\nu} (2\mu + 1) \mathrm{B}(1 - \beta + \mu + \nu, \, 1 + \beta + \mu - \nu)}.$$

EH I 80(8)

$$5. \quad \int_0^{\frac{\pi}{2}} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x \, dx = e^{i\mu\frac{\pi}{2}} \mathrm{B}(\mu, \, \nu) =$$

$$= \frac{1}{2^{\mu+\nu-1}} e^{i\mu\frac{\pi}{2}} \left\{ \frac{1}{\mu} F(1 - \nu, \, 1; \, \mu + 1; \, -1) + \frac{1}{\nu} F(1 - \mu, \, 1; \, \nu + 1; \, -1) \right\}$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

EH I 80(7)

3.893

$$1. \quad \int_0^{\infty} e^{-px} \sin(qx + \lambda) \, dx = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda) \qquad [p > 0].$$

BI ((261))(3)

$$2. \quad \int_0^{\infty} e^{-px} \cos(qx + \lambda) \, dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda) \qquad [p > 0].$$

BI ((261))(4)

$$3. \quad \int_0^{\infty} e^{-x \cos t} \cos(t - x \sin t) \, dx = 1.$$

BI ((261))(7)

GW ((335))(15)

5. $$\int_0^\infty \frac{e^{-2px} \ \sin[(2n+1)x]}{\sin x} \, dx = \frac{1}{2p} + \sum_{k=1}^{n} \frac{p}{p^2+k^2} \qquad [p > 0].$$

BI ((267))(15)

6. $$\int_0^\infty \frac{e^{-px} \ \sin 2nx}{\sin x} \, dx = 2p \sum_{k=0}^{n-1} \frac{1}{p^2+(2k+1)^2} \qquad [p > 0].$$

GW ((335))(15c)

7. $$\int_0^\infty e^{-px} \ \cos[(2n+1)x] \ \operatorname{tg} x \, dx = \frac{2n+1}{p^2+(2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k(2k+1)}{p^2+(2k+1)^2} \qquad [p > 0].$$

LI ((267))(16)

3.894

$$\int_{-\pi}^{\pi} [\beta + \sqrt{\beta^2-1} \ \cos x]^\nu e^{inx} \, dx = \frac{2\pi\Gamma(\nu+1) P_\nu^m(\beta)}{\Gamma(\nu+m+1)} \qquad [\operatorname{Re} \beta > 0].$$

ET I 157(15)

3.895

1. $$\int_0^\infty e^{-\beta x} \, sin^{2m} x \, dx = \frac{(2m)!}{\beta(\beta^2+2^2)(\beta^2+4^2)\dots[\beta^2+(2m)^2]}; \qquad [\operatorname{Re} \beta > 0].$$

FI II 615, WA 620a

2.* $$\int_0^\pi e^{-px} \ \sin^{2m} x \, dx = \frac{(2m)!(1-e^{-p\pi})}{p(p^2+2^2)(p^2+4^2)\dots[p^2+(2m)^2]}.$$

GW ((335))(4a)

3.* $$\int_0^{\frac{\pi}{2}} e^{-px} \ \sin^{2m} \ x \, dx = \frac{(2m)!}{p(p^2+2^2)(p^2+4^2)\dots[p^2+(2m)^2]} \times$$
$$\times \left\{ 1 - e^{-\frac{p\pi}{2}} \left[1 + \frac{p^2}{2!} + \frac{p^2(p^2+2^2)}{4!} + \dots + \frac{p^2(p^2+2^2)\dots[p^2+(2m-2)^2]}{(2m)!} \right] \right\}$$

BI ((270))(4)

4. $$\int_0^\infty e^{-\beta x}\ \sin^{2m+1} x\,dx = \frac{(2m+1)!}{(\beta^2+1^2)(\beta^2+3^2)\dots[\beta^2+(2m+1)^2]} \qquad [\operatorname{Re}\beta>0].$$

FI II 615, WA 620a

5.* $$\int_0^\pi e^{-px}\ \sin^{2m+1} x\,dx = \frac{(2m+1)!(1+e^{-p\pi})}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]}.$$

GW ((335))(4b)

6.* $$\int_0^{\frac{\pi}{2}} e^{-px}\ \sin^{2m+1}\ x\,dx = \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]}\times$$
$$\times\left\{1-pe\frac{p\pi}{2}\left[1+\frac{p^2+1^2}{3!}+\dots+\frac{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!}\right]\right\}$$

BI ((270))(5)

7. $$\int_0^\infty e^{-px}\ \cos^{2m} x\,dx = \frac{(2m)!}{p(p^2+2^2)\dots[p^2+(2m)^2]}\times$$
$$\times\left\{1+\frac{p^2}{2!}+\frac{p^2(p^2+2^2)}{4!}+\dots+\frac{p^2(p^2+2^2)\dots[p^2+(2m-2)^2]}{(2m)!}\right\}$$
$$[p>0].$$

BI ((262))(3)

8.* $$\int_0^{\frac{\pi}{2}} e^{-px}\ \cos^{2m}\ x\,dx = \frac{(2m)!}{p(p^2+2^2)\dots[p^2+(2m)^2]}\times$$
$$= \times\left\{-e^{-p\frac{\pi}{2}}+1+\frac{p^2}{2!}+\frac{p^2(p^2+2^2)}{4!}+\dots+\frac{p^2(p^2+2^2)\dots[p^2+(2m-2)^2]}{(2m)!}\right\}$$

BI ((270))(6)

9.[7] $$\int_0^\infty e^{-px}\ \cos^{2m+1} x\,dx = \frac{(2m+1)!p}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]}\times$$
$$= \times\left\{1+\frac{p^2+1^2}{3!}+\frac{(p^2+1^2)(p^2+3^2)}{5!}+\dots+\frac{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!}\right\}$$
$$[p>0].$$

BI ((262))(4)

10.[7] $$\int_0^{\frac{\pi}{2}} e^{-px} \cos^{2m+1} x\,dx = \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]} \times$$
$$= \times \left\{ e^{-p\frac{\pi}{2}} + p\left[1 + \frac{p^2+1^2}{3!} + \dots + \frac{(p^2+1)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!}\right]\right\}$$

BI ((270))(7)

11. $$\int_0^\infty e^{-\beta x} sin^{2n} x \sin ax\,dx = -\frac{1}{(-4)^{n+1}(2n+1)}\left\{\frac{1}{\binom{\frac{a}{2}+i\frac{\beta}{2}+n}{2n+1}} + \frac{1}{\binom{\frac{a}{2}-i\frac{\beta}{2}+n}{2n+1}}\right\}$$
$$[\operatorname{Re}\beta > 0, \quad a > 0].$$

ET I 80(19)

12.[7] $$\int_0^\infty e^{-\beta x} \sin^{2n-1} x \sin ax\,dx = \frac{-i}{(-4)^{n+1}n}\left\{\frac{1}{\binom{\frac{a}{2}-i\frac{\beta}{2}+n-\frac{1}{2}}{2n}} - \frac{1}{\binom{\frac{a}{2}+i\frac{\beta}{2}+n-\frac{1}{2}}{2n}}\right\}$$
$$[\operatorname{Re}\beta > 0, \quad a > 0 n > 0].$$

ET I 80(20)a

13.* $$\int_0^\infty e^{-\beta x} \sin^{2n} x \cos ax\,dx = \frac{(-1)^n i}{(2n+1)2^{2n+2}}\left\{\frac{1}{\binom{\frac{a}{2}+i\frac{\beta}{2}+n}{2n+1}} - \frac{1}{\binom{\frac{a}{2}-i\frac{\beta}{2}+n}{2n+1}}\right\}$$
$$[\operatorname{Re}\beta > 0, \quad a \geq 0].$$

ET I 20(12)a

14.* $$\int_0^\infty e^{-\beta x} \sin^{2n-1} x \cos ax\,dx = \frac{(-1)^n}{2^{2n+2}n}\left\{\frac{1}{\binom{\frac{a}{2}-i\frac{\beta}{2}+n-\frac{1}{2}}{2n}} + \frac{1}{\binom{\frac{a}{2}+i\frac{\beta}{2}+n-\frac{1}{2}}{2n}}\right\}$$
$$[\operatorname{Re}\beta > 0, \quad a \geq 0 n > 0].$$

1. $$\int_{-\infty}^{\infty} e^{-q^2x^2} \sin[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda.$$

BI ((269))(2)

2. $$\int_{-\infty}^{\infty} e^{-q^2x^2} \cos[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda.$$

BI ((269))(3)

515

3. $$\int_0^{\infty} e^{-ax^2} \sin bx\,dx = \frac{b}{2a} \exp\left(-\frac{b^2}{4a}\right) {}_1F_1\left(\frac{1}{2};\ \frac{3}{2};\ \frac{b^2}{4a}\right) = \\ = \frac{b}{2a}\, {}_1F_1\left(1;\ \frac{3}{2};\ -\frac{b^2}{4a}\right); \\ = \frac{b}{2a}\sum_{k=1}^{\infty} \frac{1}{(2k-1)!!}\left(-\frac{b^2}{2a}\right)^{k-1} \qquad [a>0].$$

FI II 720
ET I 73(18)

4. $$\int_0^{\infty} e^{-\beta x^2} \cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

BI ((263))(2)

3.897

1. $$\int_0^{\infty} e^{-\beta x^2 - \nu x} \sin bx\,dx = -\frac{i}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma - ib)^2}{4\beta}\left[1 - \Phi\left(\frac{\gamma - ib}{2\sqrt{\beta}}\right)\right] - \right. \\ \left. - \exp\frac{(\gamma + ib)^2}{4\beta}\left[1 - \Phi\left(\frac{\gamma + ib}{2\sqrt{\beta}}\right)\right]\right\} \qquad [\operatorname{Re}\beta > 0].$$

ET I 74(27)

2. $$\int_0^{\infty} e^{-\beta x^2 - \gamma x} \cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma - ib)^2}{4\beta}\left[1 - \Phi\left(\frac{\gamma - ib}{2\sqrt{\beta}}\right)\right] + \right. \\ \left. + \exp\frac{(\gamma + ib)^2}{4\beta}\left[1 - \Phi\left(\frac{\gamma + ib}{2\sqrt{\beta}}\right)\right]\right\} \qquad [\operatorname{Re}\beta > 0].$$

ET I 15(16)

3.898

1. $$\int_0^{\infty} e^{-\beta x^2} \sin ax \sin bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}}\right\} \qquad [\operatorname{Re}\beta > 0].$$

2. $$\int_0^\infty e^{-\beta x^2} \cos ax \cos bx \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}} \right\} \qquad [\operatorname{Re}\beta > 0].$$

BI ((263))(5)

3. $$\int_0^\infty e^{-px^2} \sin^2 ax \, dx = \frac{1}{4}\sqrt{\frac{\pi}{p}} \left(1 - e^{-\frac{a^2}{p}}\right) \qquad [p > 0].$$

BI ((263))(6)

3.899

1.[7] $$\int_0^\infty \frac{e^{-p^2x^2} \sin[(2n+1)x]}{\sin x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=1}^{n} e^{-\left(\frac{k}{p}\right)^2} \right] \qquad [p > 0].$$

BI ((267))(17)

2.* $$\int_0^\infty \frac{e^{-p^2x^2} \cos[(4n+1)x]}{\cos x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=1}^{2n} (-1)^{k-n} e^{-\left(\frac{k}{p}\right)^2} \right] \qquad [p > 0].$$

BI ((267))(18)

3. $$\int_0^\infty \frac{e^{-px^2} \, dx}{1 - 2a \cos x + a^2} = \frac{\sqrt{\frac{\pi}{p}}}{1 - a^2} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^k \exp\left(-\frac{k^2}{4p}\right) \right\} \qquad [a^2 < 1, \quad p > 0];$$
$$= \frac{\sqrt{\frac{\pi}{p}}}{a^2 - 1} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^{-k} \exp\left(-\frac{k^2}{4p}\right) \right\} \qquad [a^2 > 1, \quad p > 0].$$

LI ((266))(1)

3.911

1. $$\int_0^\infty \frac{\sin ax}{e^{\beta x} + 1} \, dx = \frac{1}{2a} - \frac{\pi}{2\beta \operatorname{sh} \frac{a\pi}{\beta}} \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$$

BI ((264))(1)

2. $$\int_0^\infty \frac{\sin ax}{e^{\beta x} - 1} \, dx = \frac{\pi}{2\beta} \operatorname{cth}\left(\frac{\pi a}{\beta}\right) - \frac{1}{2a} \qquad [a > 0, \quad \operatorname{Re}\beta > 0].$$

3.[6] $$\int_0^\infty \frac{\sin ax}{e^x - 1} e \frac{x}{2}\, dx = \frac{1}{2}\pi\, \theta(a\pi) \qquad [a > 0].$$

ET I 73(13)

4. $$\int_0^\infty \frac{\sin ax}{1 - e^{-x}} e^{-nx}\, dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a} - 1} - \sum_{k=1}^{n-1} \frac{a}{a^2 + k^2} \qquad [a > 0].$$

BI ((264))(8)

5. $$\int_0^\infty \frac{\sin ax}{e^{\beta x} - e^{\gamma x}}\, dx = \frac{1}{2i(\beta - \gamma)} \left[\psi\left(\frac{\beta + ia}{\beta - \gamma}\right) - \psi\left(\frac{\beta - ia}{\beta - \gamma}\right)\right] \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

GW ((335))(8)

6. $$\int_0^\infty \frac{\sin ax\, dx}{e^{\beta x}(e^{-x} - 1)} = \frac{i}{2}[\psi(\beta + ia) - \psi(\beta - ia)] \qquad [\operatorname{Re}\beta > -1].$$

ET 73(15)

3.912

1. $$\int_0^\infty e^{-\beta x}(1 - e^{-\gamma x})^{\nu - 1} \sin ax\, dx = -\frac{i}{2\gamma}\left[\mathrm{B}\left(\nu,\ \frac{\beta - ia}{\gamma}\right) - \mathrm{B}\left(\nu,\ \frac{\beta + ia}{\gamma}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > 0, \quad a > 0].$$

ET I 73(17)

2. $$\int_0^\infty e^{-\beta x}(1 - e^{-\gamma x})^{\nu - 1} \cos ax\, dx = \frac{1}{2\gamma}\left[\mathrm{B}\left(\nu,\ \frac{\beta - ia}{\gamma}\right) + \mathrm{B}\left(\nu,\ \frac{\beta + ia}{\gamma}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > 0, \quad a > 0].$$

ET I 15(10)

3.913

1. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^\nu x (\beta^2 e^{ix} + \nu^2 e^{-ix})^\mu\, dx = \frac{\pi\, {}_2F_1\left(-\mu,\ \frac{\beta}{2} - \frac{\nu}{2} - \frac{\mu}{2};\ 1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2};\ \frac{\beta^2}{\nu^2}\right)}{2^\nu(\nu + 1)\mathrm{B}\left(1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2},\ 1 - \frac{\beta}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)}$$
$$[\operatorname{Re}\nu > -1, \quad |\nu| > |\beta|].$$

EH I 81(11)a

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{iux} \cos^{\mu} x(a^2e^{ix}+b^2e^{-ix})^{\nu}\,dx =$$

$$= \frac{\pi b^{2\nu}\,{}_2F_1\left(-\nu,\,\frac{u+\mu+\nu}{2};\,1+\frac{\mu-\nu-u}{2};\,\frac{a^2}{b^2}\right)}{2^{\mu}(\mu+1)\mathrm{B}\left(1-\frac{u+\nu-\mu}{2},\,1+\frac{u+\mu+\nu}{2}\right)} \qquad \text{for} \quad a^2<b^2;$$

$$= \frac{\pi a^{2\nu}\,{}_2F_1\left(-\nu,\,\frac{u+\mu-\nu}{2};\,1+\frac{\mu-\nu+u}{2};\,\frac{b^2}{a^2}\right)}{2^{\mu}(\mu+1)\mathrm{B}\left(1+\frac{u+\mu-\nu}{2},\,1+\frac{\mu+\nu-u}{2}\right)} \qquad \text{for} \quad b^2<a^2 \qquad [\operatorname{Re}\mu>-1].$$

ET I 122(31)a

3.914

$$\int_0^{\infty} e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\,dx = \frac{\beta\gamma}{\sqrt{\beta^2+b^2}}K_1(\gamma\sqrt{\beta^2+b^2}) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0].$$

ET I 16(26)

3.915

1. $$\int_0^{\pi} e^{a\cos x} \sin x\,dx = \frac{2}{a}\operatorname{sh} a.$$

GW ((337))(15c)

2. $$\int_0^{\pi} e^{i\beta\cos x} \cos nx\,dx = i^n\pi J_n(\beta).$$

EH II 81(2)

3.[3] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\sin x} \cos^{2\nu} x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_{\nu}(\beta) \qquad \left[\operatorname{Re}\nu>-\frac{1}{2}\right].$$

EH II 81(6)

4. $$\int_0^{\pi} e^{\pm\beta\cos x} \sin^{2\nu} x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)I_{\nu}(\beta) \qquad \left[\operatorname{Re}\nu>-\frac{1}{2}\right].$$

GW ((337))(15b)

GW ((337))(15b)

5. $$\int_0^{\pi} e^{i\beta\cos x}\sin^{2\nu}x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 34(2), WA 60(6)

3.916

1. $$\int_0^{\frac{\pi}{2}} e^{-p^2\operatorname{tg}x}\frac{\sin\frac{x}{2}\sqrt{\cos x}}{\sin 2x}\,dx = \left[C(p)-\frac{1}{2}\right]^2+\left[S(p)-\frac{1}{2}\right]^2.m$$

NT 33(18)a

2. $$\int_0^{\frac{\pi}{2}} \frac{\exp(-p\operatorname{tg}x)\,dx}{\sin 2x + a\cos 2x + a} = -\frac{1}{2}e^{ap}\operatorname{Ei}(-ap) \qquad [p>0], \quad \text{(cf. 3552 4. and 6.)}$$

BI ((273))(11)

3. $$\int_0^{\frac{\pi}{2}} \frac{\exp(-p\operatorname{ctg}x)\,dx}{\sin 2x + a\cos 2x - a} = -\frac{1}{2}e^{-ap}\operatorname{Ei}(ap) \qquad [p>0], \qquad \text{(cf. 3.552 4. and 6.).}$$

3.552
BI ((273))(12)

518

4. $$\int_0^{\frac{\pi}{2}} \frac{\exp(-p\operatorname{tg}x)\sin 2x\,dx}{(1-a^2)-2a^2\cos 2x-(1+a^2)\cos^2 2x} = -\frac{1}{4}[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)] \qquad [p>0].$$

BI ((273))(13)

5. $$\int_0^{\frac{\pi}{2}} \frac{\exp(-p\operatorname{ctg}x)\sin 2x\,dx}{(1-a^2)+2a^2\cos 2x-(1+a^2)\cos^2 2x} = -\frac{1}{4}[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)] \qquad [p>0].$$

BI ((273))(14)

3.917

1. $$\int_0^{\frac{\pi}{2}} e^{-2\beta\operatorname{ctg}x}\cos^{\nu-1/2}x\sin^{-(\nu+1)}x\sin\left[\beta-\left(\nu-\frac{1}{2}\right)x\right]dx = \frac{\sqrt{\pi}}{2(2\beta)^{\nu}}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta)$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

2. $$\int_0^{\frac{\pi}{2}} e^{-2\beta \operatorname{ctg} x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \cos\left[\beta - \left(\nu - \frac{1}{2}\right) x\right] dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma\left(\nu + \frac{1}{2}\right) N_\nu(\beta)$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 186(8)

3.918

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos^\mu x}{\sin^{2\mu+2} x} e^{i\gamma(\beta-\mu x)-2\beta \operatorname{ctg} x}\, dx = \frac{i\gamma}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1) H^{(\varepsilon)}_{\mu+\frac{1}{2}}(\beta)$$
$$\left[\varepsilon = 1,\, 2; \quad \gamma = (-1)^{\varepsilon+1}; \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > -1\right].$$

GW ((337))(16)

2. $$\int_0^{\frac{\pi}{2}} \frac{\cos^\mu x \sin(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \operatorname{ctg} x}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1) J_{\mu+}\frac{1}{2}(\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > -1].$$

WH

3. $$\int_0^{\frac{\pi}{2}} \frac{\cos^\mu x \cos(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \operatorname{ctg} x}\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1) N_{\mu+}\frac{1}{2}(\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > -1].$$

GW ((337))(17b)

3.919

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp(2\pi \operatorname{ctg} x) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)}.$$

BI ((275))(6), LI ((275))(6)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp(\pi \operatorname{ctg} x) - 1} = (-1)^{n-1} \frac{n}{2n+1}.$$

BI ((275))(7), LI ((275))(7)

3.92 Trigonometric functions of more complicated arguments combined with exponentials

3.921

$$\int_0^\infty e^{-\gamma x} \cos ax^2 (\cos \gamma x - \sin \gamma x)\, dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right) \qquad [a > 0, \quad \operatorname{Re}\gamma \geq |\operatorname{Im}\gamma|].$$

ET I 26(28)

3.922

1. $$\int_0^\infty e^{-\beta x^2} \sin ax^2\, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\beta x^2} \sin ax^2\, dx = \sqrt{\frac{\pi}{8}}\sqrt{\frac{\sqrt{\beta^2+a^2}-\beta}{\beta^2+a^2}} = \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}} \sin\left(\frac{1}{2}\operatorname{arctg}\frac{a}{\beta}\right) \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

FI II 750, BI ((263))(8)

2. $$\int_0^\infty e^{-\beta x^2} \cos ax^2\, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\beta x^2} \cos ax^2\, dx = \sqrt{\frac{\pi}{8}}\sqrt{\frac{\sqrt{\beta^2+a^2}+\beta}{\beta^2+a^2}} = \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}} \cos\left(\frac{1}{2}\operatorname{arctg}\frac{a}{\beta}\right) \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

FI II 750, BI ((263))(9)

3. $$\int_0^\infty e^{-\beta x^2} \sin ax^2 \cos bx\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{\beta^2+a^2}} e^{-A\beta}(B \sin Aa - C \cos Aa) = \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2+a^2)}\right) \sin\left\{\frac{1}{2}\operatorname{arctg}\frac{a}{\beta} - \frac{ab^2}{4(\beta^2+a^2)}\right\}.$$

LI ((263))(10), GW ((337))(5)

4. $$\int_0^\infty e^{-\beta x^2} \cos ax^2 \cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^2+a^2}} e^{-A\beta}(B \cos Aa + C \sin Aa) = \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2+a^2)}\right) \cos\left\{\frac{1}{2}\operatorname{arctg}\frac{a}{\beta} - \frac{ab^2}{4(\beta^2+a^2)}\right\}.$$

LI ((263))(11), GW ((337))(5)

[In formulas 3.922 3 and 4. $a > 0, b > 0, \operatorname{Re}\beta > 0$,

$$A = \frac{b^2}{4(a^2+\beta^2)}, \qquad B = \sqrt{\frac{1}{2}(\sqrt{\beta^2+a^2}+\beta)}, \qquad C = \sqrt{\frac{1}{2}(\sqrt{\beta^2+a^2}-\beta)}.$$

If a is complex, $\operatorname{Re}\beta > |\operatorname{Im} a|$.]

3.923

1. $$\int_{-\infty}^{\infty} \exp[-(ax^2+2bx+c)] \sin(px^2+2qx+r)\,dx =$$
$$= \frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}} \exp\frac{a(b^2-ac)-(aq^2-2bpq+cp^2)}{a^2+p^2} \times$$
$$\times \sin\left\{\frac{1}{2}\operatorname{arctg}\frac{p}{a} - \frac{p(q^2-pr)-(b^2p-2abq+a^2r)}{a^2+p^2}\right\} \qquad [a>0].$$

GW ((337))(3), BI ((296))(6)

2. $$\int_{-\infty}^{\infty} \exp[-(ax^2+2bx+c)] \cos(px^2+2qx+r)\,dx =$$
$$= \frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}} \exp\frac{a(b^2-ac)-(aq^2-2bpq+cp^2)}{a^2+p^2} \times$$
$$\times \cos\left\{\frac{1}{2}\operatorname{arctg}\frac{p}{a} - \frac{p(q^2-pr)-(b^2p-2abq+a^2r)}{a^2+p^2}\right\} \qquad [a>0].$$

GW ((337))(3), BI ((269))(7)

3.924

1. $$\int_0^{\infty} e^{-\beta x^4} \sin bx^2\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I\frac{1}{4}\left(\frac{b^2}{8\beta}\right) \qquad [\operatorname{Re}\beta>0, \quad b>0].$$

ET 73(22)

2. $$\int_0^{\infty} e^{-\beta x^4} \cos bx^2\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_-\frac{1}{4}\left(\frac{b^2}{8\beta}\right) \qquad [\operatorname{Re}\beta>0, \quad b>0].$$

ET I 15(12)

3.925

1. $$\int_0^{\infty} e^{-\frac{p^2}{x^2}} \sin 2a^2x^2\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{p^2}{x^2}} \sin 2a^2x^2\,dx = \frac{\sqrt{\pi}}{4a} e^{-2ap}(\cos 2ap + \sin 2ap)$$
$$[a>0, \quad b>0].$$

BI ((268))(12)

BI ((268))(13)

3.926

1. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)} \sin ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}[v\cos(2v\sqrt{\gamma})+u\sin(2v\sqrt{\gamma})]$$ $$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0].$$

BI ((268))(14)

2. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}[u\cos(2v\sqrt{\gamma})-v\sin(2v\sqrt{\gamma})]$$ $$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0].$$

BI ((268))(15)

[In formulas 3.926 1., 3.926 2.

$$u=\sqrt{\frac{\sqrt{a^2+\beta^2}+\beta}{2}},\qquad v=\sqrt{\frac{\sqrt{a^2+\beta^2}-\beta}{2}}.\Bigg]$$

3.927

$$\int_0^\infty e^{-\frac{p}{x}}\sin^2\frac{a}{x}\,dx = a\operatorname{arctg}\frac{2a}{p}+\frac{p}{4}\ln\frac{p^2}{p^2+4a^2}\qquad [a>0,\quad p>0].$$

LI ((268))(4)

3.928

1. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\sin\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{\pi}}{2r}e^{-2rs\,\cos(A+B)}\,\sin\left\{A+2rs\,\sin(A+B)\right\}.$$

BI ((268))(22)

2. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\cos\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{\pi}}{2r}e^{-2rs\,\cos(A+B)}\,\cos\left\{A+2rs\,\sin(A+B)\right\}.$$

BI ((268))(23)

[In formulas 3.928 1., 3.928 2. $a^2+p^2>0$ and

$$r=\sqrt[4]{a^4+p^4}, \qquad s=\sqrt[4]{b^4+q^4}, \qquad A=\frac{1}{2}\operatorname{arctg}\frac{a^2}{p^2}, \qquad B=\frac{1}{2}\operatorname{arctg}\frac{b^2}{q^2}.\Big]$$

3.929

$$\int_0^\infty \left[e^{-x}\cos(p\sqrt{x})+pe^{-x^2}\sin px\right]dx=1.$$

LI ((268))(3)

3.93 Trigonometric and exponential functions of trigonometric functions

3.931

1. $$\int_0^{\frac{\pi}{2}} e^{-p\cos x}\sin(p\sin x)\,dx=\operatorname{Ei}(-p)-\operatorname{ci}(p).$$

NT 13(27)

2. $$\int_0^{\pi} e^{-p\cos x}\sin(p\sin x)\,dx=-\int_{-\pi}^{0} e^{-p\cos x}\sin(p\sin x)\,dx=-2\operatorname{si}(p).$$

GW ((337))(11b)

3. $$\int_0^{\frac{\pi}{2}} e^{-p\cos x}\cos(p\sin x)\,dx=-\operatorname{si}(p).$$

NT 13(26)

4. $$\int_0^{\frac{\pi}{2}} e^{-p\cos x}\cos(p\sin x)\,dx=\frac{1}{2}\int_0^{2\pi} e^{-p\cos x}\cos(p\sin x)\,dx=\pi.$$

GW ((337))(11a)

3.932

1. $$\int_0^{\pi} e^{p\cos x}\sin(p\sin x)\sin mx\,dx=\frac{1}{2}\int_0^{2\pi} e^{p\cos x}\sin(p\sin x)\sin mx\,dx=\frac{\pi}{2}\cdot\frac{p^m}{m!}.$$

2. $$\int_0^{\pi} e^{p\,\cos x}\,\cos(p\,\sin x)\,\cos mx\,dx = \frac{1}{2}\int_0^{2\pi} e^{p\,\cos x}\,\cos(p\,\sin x)\,\cos mx\,dx = \frac{\pi}{2}\cdot\frac{p^m}{m!}.$$

BI ((277))(8), GW ((337))(13b)

3.933

$$\int_0^{\pi} e^{p\,\cos x}\,\sin(p\,\sin x)\,\operatorname{cosec} x\,dx = \pi\,\operatorname{sh} p.$$

BI ((278))(1)

3.934

1. $$\int_0^{\pi} e^{p\,\cos x}\,\sin(p\,\sin x)\,\operatorname{tg}\frac{x}{2}\,dx = \pi(1-e^p).$$

BI ((271))(8)

2. $$\int_0^{\pi} e^{p\,\cos x}\,\sin(p\,\sin x)\,\operatorname{ctg}\frac{x}{2}\,dx = \pi(e^p-1).$$

BI ((272))(5)

3.935

$$\int_0^{\pi} e^{p\,\cos x}\,\cos(p\,\sin x)\frac{\sin 2nx}{\sin x}\,dx = \pi\sum_{k=0}^{n-1}\frac{p^{2k+1}}{(2k+1)!} \qquad [p>0].$$

LI ((278))(3)

3.936

1. $$\int_0^{2\pi} e^{p\,\cos x}\,\cos(p\,\sin x - mx)\,dx = 2\int_0^{\pi} e^{p\,\cos x}\,\cos(p\,\sin x - mx)\,dx = \frac{2\pi p^m}{m!}.$$

BI ((277))(9), GW ((337))(14a)

2. $$\int_0^{2\pi} e^{p\,\sin x}\,\sin(p\,\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\,\sin\frac{m\pi}{2} \qquad [p>0].$$

GW ((337))(14b)

GW ((337))(14b)

4. $$\int_0^{2\pi} e^{\cos x}\,\sin(mx - \sin x)\,dx = 0.$$

WH

5. $$\int_0^{\pi} e^{\beta\,\cos x}\,\cos(ax + \beta\,\sin x)\,dx = \beta^{-a}\,\sin(a\pi)\gamma(a,\,\beta).$$

EH II 137(2)

3.937

1. $$\int_0^{2\pi} \exp(p\,\cos x + q\,\sin x)\,\sin(a\,\cos x + b\,\sin x - mx)\,dx =$$
$$= i\pi[(b-p)^2 + (a+q)^2]^{-\frac{m}{2}}\left\{(A+iB)\frac{m}{2}I_m(\sqrt{C-iD}) - (A-iB)\frac{m}{2}I_m(\sqrt{C+iD})\right\}\,.$$

GW ((337))(9b)

2. $$\int_0^{2\pi} \exp(p\,\cos x + q\,\sin x)\,\cos(a\,\cos x + b\,\sin x - mx)\,dx =$$
$$= \pi[(b-p)^2 + (a+q)^2]^{-\frac{m}{2}}\left\{(A+iB)^{\frac{m}{2}}I_m(\sqrt{C-iD}) + (A-iB)^{\frac{m}{2}}I_m(\sqrt{C+iD})\right\}\,.$$

[In formulas 3.937 1. and 3.937 2. $(b-p)^2 + (a+q)^2 > 0, m = 0, 1,\,2, \ldots\,, A = p^2 - q^2 + a^2 - b^2,$

$B = 2(pq + ab), C = p^2 + q^2 - a^2 - b^2, D = 2(ap + bq)$.]

3. $$\int_0^{2\pi} \exp(p\,\cos x{+}q\,\sin x)\,\sin(q\,\cos x{-}p\,\sin x{+}mx)\,dx = \frac{2\pi}{m!}(p^2{+}q^2)\frac{m}{2}\,\sin\left(m\,\operatorname{arctg}\frac{q}{p}\right).$$

GW ((337))(12)

GW ((337))(9a)

4 $$\int_0^{2\pi} \exp(p\,\cos x{+}q\,\sin x)\,\cos(q\,\cos x{-}p\,\sin x{+}mx)\,dx = \frac{2\pi}{m!}(p^2{+}q^2)\frac{m}{2}\,\cos\left(m\,\operatorname{arctg}\frac{q}{p}\right).$$

1. $$\int_0^{\pi} e^{r(\cos px+\cos qx)} \sin(r \sin px) \sin(r \sin qx)\,dx = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{\Gamma(pk+1)\Gamma(qk+1)} r^{(p+q)k}.$$

BI ((277))(14)

2. $$\int_0^{\pi} e^{r(\cos px+\cos qx)} \cos(r \sin px) \cos(r \sin qx)\,dx = \frac{\pi}{2}\left(2+\sum_{k=1}^{\infty}\frac{r^{(p+q)k}}{\Gamma(pk+1)\Gamma(qk+1)}\right).$$

BI ((277))(15)

3.939

1. $$\int_0^{\pi} e^{q \cos x}\frac{\sin rx}{1-2p^r \cos rx+p^{2r}} \sin(q \sin x)\,dx = \frac{\pi}{2pr}\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)} \qquad [r>0, \quad 0<p<1].$$

BI ((278))(15)

2.[3] $$\int_0^{\pi} e^{q \cos x}\frac{1-p^r \cos rx}{1-2p^r \cos rx+p^{2r}} \cos(q \sin x)\,dx = \frac{\pi}{2}\left[2+\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)}\right] \qquad [r>0,\ 0<p<1].$$

BI ((278))(16)

3. $$\int_0^{\frac{\pi}{2}} \frac{e^{p \cos 2x} \cos(p \sin 2x)\,dx}{\cos^2 x+q^2 \sin^2 x} = \frac{\pi}{2q} \exp\left(p\frac{q-1}{q+1}\right).$$

BI ((273))(8)

3.94- 3.97 Combinations involving trigonometric functions, exponentials,and powers

3.941

1. $$\int_0^{\infty} e^{-px} \sin qx\frac{dx}{x} = \operatorname{arctg}\frac{q}{p} \qquad [p>0].$$

BI ((365))(1)

2. $$\int_0^{\infty} e^{-px} \cos qx\frac{dx}{x} = \infty.$$

1. $$\int_0^\infty e^{-px} \cos px \frac{x\,dx}{b^4+x^4} = \frac{\pi}{4b^2} \exp(-bp\sqrt{2}) \qquad [p>0, \quad b>0].$$

BI ((386))(6)a

2. $$\int_0^\infty e^{-px} \cos px \frac{x\,dx}{b^4-x^4} = \frac{\pi}{4b^2} e^{-bp} \sin bp \qquad [p>0, \quad b>0].$$

BI ((386))(7)a

3.943

$$\int_0^\infty e^{-\beta x}(1-\cos ax)\frac{dx}{x} = \frac{1}{2} \ln \frac{a^2+\beta^2}{\beta^2} \qquad [\operatorname{Re}\beta > 0].$$

BI ((367))(6)

3.944

1. $$\int_0^u x^{\mu-1} e^{-\beta x} \sin\delta x\,dx = \frac{i}{2}(\beta+i\delta)^{-\mu}\gamma[\mu,\,(\beta+i\delta)u] - \frac{i}{2}(\beta-i\delta)^{-\mu}\gamma[\mu,\,(\beta-i\delta)u]$$
$$[\operatorname{Re}\mu > -1].$$

ET I 318(8)

2. $$\int_u^\infty x^{\mu-1} e^{-\beta x} \sin\delta x\,dx = \frac{i}{2}(\beta+i\delta)^{-\mu}\Gamma[\mu,\,(\beta+i\delta)u] - \frac{i}{2}(\beta-i\delta)^{-\mu}\Gamma[\mu,\,(\beta-i\delta)u]$$
$$[\operatorname{Re}\beta > |\operatorname{Im}\delta|].$$

ET I 318(9)

3. $$\int_0^u x^{\mu-1} e^{-\beta x} \cos\delta x\,dx = \frac{1}{2}(\beta+i\delta)^{-\mu}\gamma[\mu,\,(\beta+i\delta)u] + \frac{1}{2}(\beta-i\delta)^{-\mu}\gamma[\mu,\,(\beta-i\delta)u]$$
$$[\operatorname{Re}\mu > 0].$$

ET I 320(28)

4. $$\int_u^\infty x^{\mu-1} e^{-\beta x} \cos\delta x\,dx = \frac{1}{2}(\beta+i\delta)^{-\mu}\Gamma[\mu,\,(\beta+i\delta)u] + \frac{1}{2}(\beta-i\delta)^{-\mu}\Gamma[\mu,\,(\beta-i\delta)u]$$
$$[\operatorname{Re}\beta > |\operatorname{Im}\delta|].$$

ET I 320(29)

5. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \sin\delta x\,dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\frac{\mu}{2}}} \sin\left(\mu \operatorname{arctg}\frac{\delta}{\beta}\right) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\beta > |\operatorname{Im}\delta|].$$

6. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \cos\delta x \, dx = \frac{\Gamma(\mu)}{(\delta^2+\beta^2)^{\frac{\mu}{2}}} \cos\left(\mu \operatorname{arctg} \frac{\delta}{\beta}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > |\operatorname{Im}\delta|].$$

FI II 812, BI ((361))(10)

7. $$\int_0^\infty x^{\mu-1} \exp(-ax \cos t) \sin(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \sin(\mu t)$$
$$\left[\operatorname{Re}\mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}\right].$$

EH I 13(36)

8. $$\int_0^\infty x^{\mu-1} \exp(-ax \cos t) \cos(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \cos(\mu t)$$
$$\left[\operatorname{Re}\mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}\right].$$

EH I 13(35)

9. $$\int_0^\infty x^{p-1} e^{-qx} \sin(qx \operatorname{tg} t) \, dx = \frac{1}{q^p}\Gamma(p) \cos^p t \sin pt \qquad \left[|t| < \frac{\pi}{2}, \quad q > 0\right].$$

LO V 288(16)

10. $$\int_0^\infty x^{p-1} e^{-qx} \cos(qx \operatorname{tg} t) \, dx = \frac{1}{q^p}\Gamma(p) \cos^p(t) \cos pt \qquad \left[|t| < \frac{\pi}{2}, \quad q > 0\right].$$

LO V 288(15)

11. $$\int_0^\infty x^n e^{-\beta x} \sin bx \, dx = n! \left(\frac{\beta}{\beta^2+b^2}\right)^{n+1} \sum_{0 \le 2k \le n} (-1)^k \binom{n+1}{2k+1} \left(\frac{b}{\beta}\right)^{2k+1} =$$
$$= (-1)^n \frac{\partial^n}{\partial\beta^n} \left(\frac{b}{b^2+\beta^2}\right) \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$$

GW ((336))(3), ET I 72(3)

12. $$\int_0^\infty x^n e^{-\beta x} \cos bx \, dx = n! \left(\frac{\beta}{\beta^2+b^2}\right)^{n+1} \sum_{0 \le 2k \le n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{\beta}\right)^{2k} =$$
$$= (-1)^n \frac{\partial^n}{\partial\beta^n} \left(\frac{\beta}{b^2+\beta^2}\right) \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$$

13. $$\int_0^\infty x^{n-1/2} e^{-\beta x} \sin bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \frac{\sqrt{\sqrt{\beta^2+b^2}-\beta}}{\sqrt{\beta^2+b^2}} \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$$

ET I 72(6)

14. $$\int_0^\infty x^{n-1/2} e^{-\beta x} \cos bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \frac{\sqrt{\sqrt{\beta^2+b^2}+\beta}}{\sqrt{\beta^2+b^2}} \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$$

ET I 15(6)

3.945

1. $$\int_0^\infty \left(e^{-\beta x} \sin ax - e^{-\gamma x} \sin bx\right) \frac{dx}{x^r} =$$
$$= \Gamma(1-r) \left\{ (b^2+\gamma^2) \frac{r-1}{2} \sin\left[(r-1) \operatorname{arctg} \frac{b}{\gamma}\right] - \left(a^2+\beta^2\right) \frac{r-1}{2} \sin\left[(r-1) \operatorname{arctg} \frac{a}{\beta}\right] \right\}$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad r < 2, \quad r \neq 1].$$

BI((371))(6)

2. $$\int_0^\infty \left(e^{-\beta x} \cos ax - e^{-\gamma x} \cos bx\right) \frac{dx}{x^r} =$$
$$= \Gamma(1-r) \left\{ (a^2+\beta^2) \frac{r-1}{2} \cos\left[(r-1) \operatorname{arctg} \frac{a}{\beta}\right] - \left(b^2+\gamma^2\right) \frac{r-1}{2} \cos\left[(r-1) \operatorname{arctg} \frac{b}{\gamma}\right] \right\}$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad r < 2, \quad r \neq 1].$$

BI ((371))(7)

3. $$\int_0^\infty \left(a e^{-\beta x} \sin bx - b e^{-\gamma x} \sin ax\right) \frac{dx}{x^2} = ab \left[\frac{1}{2} \ln \frac{a^2+\gamma^2}{b^2+\beta^2} + \frac{\gamma}{a} \operatorname{arcctg} \frac{\gamma}{a} - \frac{\beta}{b} \operatorname{arcctg} \frac{\beta}{b}\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

BI ((368))(22)

3.946

1. $$\int_0^\infty e^{-px} \sin^{2m+1} ax \frac{dx}{x} = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \operatorname{arctg} \frac{(2m-2k+1)a}{p}$$
$$[m = 0, 1, \ldots, \quad p > 0].$$

GW ((336))(9a)

GW ((336))(9b)

3.947

1. $$\int_0^\infty e^{-\beta x}\,\sin\gamma x\,\sin ax\frac{dx}{x}=\frac{1}{4}\,\ln\frac{\beta^2+(a+\gamma)^2}{\beta^2+(a-\gamma)^2}\qquad[\operatorname{Re}\beta>|\operatorname{Im}\gamma|,\quad a>0].$$

BI ((365))(5)

2. $$\int_0^\infty e^{-px}\,\sin ax\,\sin bx\frac{dx}{x^2}=\frac{a}{2}\,\operatorname{arctg}\frac{2pb}{p^2+a^2-b^2}+\frac{b}{2}\,\operatorname{arctg}\frac{2pa}{p^2+b^2-a^2}+\frac{p}{4}\,\ln\frac{p^2+(a-b)^2}{p^2+(a+b)^2}$$
$$[p>0].$$

BI ((368))(1), FI II 744

3. $$\int_0^\infty e^{-px}\,\sin ax\,\cos bx\frac{dx}{x}=\frac{1}{2}\,\operatorname{arctg}\frac{2pa}{p^2-a^2+b^2}+s\frac{\pi}{2}$$
$$[a\ge 0,\quad p>0,\quad s=0\quad\text{for}\quad p^2-a^2+b^2\ge 0\quad\text{and}\quad s=1\quad\text{for}\quad p^2-a^2+b^2<0].$$

GW ((336))(10b)

3.948

1. $$\int_0^\infty e^{-\beta x}\,(\sin ax-\sin bx)\frac{dx}{x}=\operatorname{arctg}\frac{(a-b)\beta}{ab+\beta^2}\qquad[\operatorname{Re}\beta>0],\qquad(\text{cf. } 3.951\ 2.).$$

3.951
BI ((367))(7)

2. $$\int_0^\infty e^{-\beta x}(\cos ax-\cos bx)\frac{dx}{x}=\frac{1}{2}\,\ln\frac{b^2+\beta^2}{a^2+\beta^2}\qquad[\operatorname{Re}\beta>0],\qquad(\text{cf. } 3.951\ 3.).$$

3.951
BI ((367))(8), FI II 748a

3. $$\int_0^\infty e^{-\beta x}(\cos ax-\cos bx)\frac{dx}{x^2}=\frac{\beta}{2}\,\ln\frac{a^2+\beta^2}{b^2+\beta^2}+b\,\operatorname{arctg}\frac{b}{\beta}-a\,\operatorname{arctg}\frac{a}{\beta}\qquad[\operatorname{Re}p>0].$$

4. $\int_0^\infty e^{-\beta x}(\sin^2 ax - \sin^2 bx)\frac{dx}{x^2} = a \operatorname{arctg}\frac{2a}{p} - b \operatorname{arctg}\frac{2b}{p} - \frac{p}{4}\ln\frac{p^2+4a^2}{p^2+4b^2} \qquad [p>0].$

BI ((368))(25)

5. $\int_0^\infty e^{-\beta x}(\cos^2 ax - \cos^2 bx)\frac{dx}{x^2} = -a \operatorname{arctg}\frac{2a}{p} + b \operatorname{arctg}\frac{2b}{p} + \frac{p}{4}\ln\frac{p^2+4a^2}{p^2+4b^2} \qquad [p>0].$

BI ((368))(26)

3.949

1. $\int_0^\infty e^{-px}\sin ax \sin bx \sin cx\frac{dx}{x} = -\frac{1}{4}\operatorname{arctg}\frac{a+b+c}{p} + \frac{1}{4}\operatorname{arctg}\frac{a+b-c}{p} + \frac{1}{4}\operatorname{arctg}\frac{a-b+c}{p} + \frac{1}{4}\operatorname{arctg}\frac{-a+b+c}{p} \qquad [p>0].$

BI ((365))(11)

2.[8] $\int_0^\infty e^{-\beta x}\sin^2 ax \sin bx\frac{dx}{x} = \frac{1}{2}\operatorname{arctg}\frac{b}{p} - \frac{1}{2}\left[\frac{1}{2}\operatorname{arctg}\frac{2pb}{p^2+4a^2-b^2} + s\frac{\pi}{2}\right] \left[s = \begin{cases} 1, & p^2+4a^2-b^2<0 \\ 0, & p^2+4a^2-b^2\ge 0\end{cases}\right]$

BI ((365))(8)

3. $\int_0^\infty e^{-\beta x}\sin^2 ax \cos bx\frac{dx}{x} = \frac{1}{8}\ln\frac{[p^2+(2a+b)^2][p^2+(2a-b)^2]}{(p^2+b^2)^2} \qquad [p>0].$

BI ((365))(9)

4. $\int_0^\infty e^{-\beta x}\sin ax \cos^2 bx\frac{dx}{x} = \frac{1}{2}\operatorname{arctg}\frac{a}{p}\frac{1}{2}\left[\operatorname{arctg}\frac{2pb}{p^2+4a^2-b^2} + s\frac{\pi}{2}\right] \left[s = \begin{cases} 1, & p^2+4a^2-b^2<0 \\ 0, & p^2+4a^2-b^2\ge 0\end{cases}\right]$

BI ((365))(10)

5. $\int_0^\infty e^{-px}\sin^2 ax \sin bx \sin cx\frac{dx}{x} = \frac{1}{8}\ln\frac{p^2+(b+c)^2}{p^2+(b-c)^2} + \frac{1}{16}\ln\frac{[p^2+(2a-b+c)^2][p^2+(2a+b-c)^2]}{[p^2+(2a+b+c)^2][p^2+(2a-b-c)^2]} \qquad [p>0].$

BI ((365))(15)

3.951

FI II 745

2. $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \operatorname{arctg} \frac{(\beta-\gamma)b}{b^2+\beta\gamma} \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma \geq 0].$$

BI ((367))(3)

3. $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \cos bx \, dx = \frac{1}{2} \ln \frac{b^2+\beta^2}{b^2+\gamma^2} \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma \geq 0].$$

BI ((367))(4)

4. $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \frac{b}{2} \ln \frac{b^2+\beta^2}{b^2+\gamma^2} + \beta \operatorname{arctg} \frac{b}{\beta} - \gamma \operatorname{arctg} \frac{b}{\gamma} \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

BI ((368))(21)a

5. $$\int_0^\infty \frac{x}{e^{\beta x}-1} \cos bx \, dx = \frac{1}{2b^2} - \frac{\pi^2}{2\beta^2} \operatorname{cosech}^2 \frac{b\pi}{\beta} \qquad [\operatorname{Re}\beta > 0].$$

ET I 15(18)

6. $$\int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} \right) \cos bx \, dx = \ln b - \frac{1}{2}[\psi(ib) + \psi(-ib)] \qquad [b > 0].$$

ET I 15(9)

7. $$\int_0^\infty \frac{1-\cos ax}{e^{2\pi x}-1} \cdot \frac{dx}{x} = \frac{a}{4} + \frac{1}{2} \ln \frac{1-e^{-a}}{a} \qquad [a > 0].$$

BI ((387))(10)

8. $$\int_0^\infty (e^{-\beta x} - e^{-\gamma x} \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2+\gamma^2}{\beta^2} \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

BI ((367))(10)

9. $$\int_0^\infty \frac{\cos px - e^{-px}}{b^4+x^4} \frac{dx}{x} = \frac{\pi}{2b^4} \exp\left(-\frac{1}{2}bp\sqrt{2}\right) \sin\left(\frac{1}{2}bp\sqrt{2}\right) \qquad [p > 0].$$

10. $$\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = \boldsymbol{C}.$$

NT 65(8)

11. $$\int_0^\infty \left(ae^{-px} - \frac{e^{-qx}}{x} \sin ax \right) \frac{dx}{x} = \frac{a}{2} \ln \frac{a^2+q^2}{p^2} + q \operatorname{arctg} \frac{a}{q} - a \qquad [p > 0, \quad q > 0].$$

BI ((368))(24)

12. $$\int_0^\infty \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{2} \operatorname{cth} b\pi - \frac{1}{2b} \right] \qquad [b > 0].$$

GW ((336))(15a)

13. $$\int_0^\infty \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{2} \operatorname{cth} b\pi - \frac{1}{2b} \right] \qquad [b > 0].$$

GW ((336))(15b)

14. $$\int_0^\infty \frac{x^{2m} \sin bx \, dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \theta \frac{b\pi}{2c} - \sum_{k=1}^{n} \frac{b}{b^2 + (2k-1)^2 c^2} \right] \qquad [b > 0].$$

GW ((336))(14a)

15. $$\int_0^\infty \frac{x^{2m+1} \cos bx \, dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \theta \frac{b\pi}{2c} - \sum_{k=1}^{n} \frac{b}{b^2 + (2k-1)^2 c^2} \right] \qquad [b > 0].$$

GW ((336))(14b)

16. $$\int_0^\infty \frac{x^{2m} \sin bx \, dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \operatorname{cth} \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right]$$
$$[b > 0, \quad c > 0].$$

GW ((336))(14c)

17. $$\int_0^\infty \frac{x^{2m+1} \cos bx \, dx}{e^{2ncx} - e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \operatorname{cth} \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right]$$
$$[b > 0, \quad c > 0].$$

18. $$\int_0^\infty \frac{\cos ax - \cos bx}{e^{(2m+1)px} - e^{(2m-1)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{\operatorname{ch} \frac{b\pi}{2p}}{\operatorname{ch} \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^{m} \ln \frac{b^2 + (2k-1)^2 p^2}{a^2 + (2k-1)^2 p^2} \qquad [p > 0].$$

GW ((336))(16a)

19. $$\int_0^\infty \frac{\cos ax - \cos bx}{e^{2mpx} - e^{(2m-2)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{a \operatorname{sh} \frac{b\pi}{2p}}{b \operatorname{sh} \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^{m-1} \ln \frac{b^2 + 4k^2 p^2}{a^2 + 4k^2 p^2} \qquad [p > 0].$$

GW ((336))(16b)

20. $$\int_0^\infty \frac{\sin x \, \sin bx}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{(b+1) \operatorname{sh}[(b-1)\pi]}{(b-1) \operatorname{sh}[(b+1)\pi]} \qquad [b^2 \neq 1].$$

LO V 305

529

21. $$\int_0^\infty \frac{\sin^2 ax}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{2a\pi}{\operatorname{sh} 2a\pi}.$$

LO V 306, BI ((387))(5)

3.952

1. $$\int_0^\infty x e^{-p^2 x^2} \sin ax \, dx = \frac{a\sqrt{\pi}}{4p^3} \exp\left(-\frac{a^2}{4p^2}\right).$$

BI ((362))(1)

2. $$\int_0^\infty x e^{-p^2 x^2} \cos ax \, dx = \frac{1}{2p^2} - \frac{a}{4p^3} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \qquad [a > 0].$$

BI ((362))(2)

3. $$\int_0^\infty x^2 e^{-p^2 x^2} \sin ax \, dx = \frac{a}{4p^4} + \frac{2p^2 - a^2}{8p^5} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \qquad [a > 0].$$

BI ((362))(4)

BI ((362))(5)

5. $$\int_0^\infty x^3 e^{-p^2x^2} \sin ax\,dx = \sqrt{\pi}\frac{6ap^2 - a^3}{16p^7} \exp\left(-\frac{a^2}{4p^2}\right).$$

BI ((362))(6)

6.[3] $$\int_0^\infty e^{-p^2x^2} \sin ax\frac{dx}{x} = \frac{a\sqrt{\pi}}{2p}\sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)}\left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2}\Phi\left(\frac{a}{2p}\right).$$

BI ((365))(21)

7. $$\int_0^\infty x^{\mu-1} e^{-\beta x^2} \sin \gamma x\,dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1-\frac{\mu}{2};\ \frac{3}{2};\ \frac{\gamma^2}{4\beta}\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > -1].$$

ET I 318(10)

8.* $$\int_0^\infty x^{\mu-1} e^{-\beta x^2} \cos ax\,dx = \frac{1}{2}\beta^{-\mu/2}\Gamma(\mu/2)e^{-a^2/4\beta}\, {}_1F_1(-\mu/2+1/2;\ 1/2;\ a^2/4\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad a > 0].$$

ET I 320(30)

9. $$\int_0^\infty x^{2n} e^{-\beta^2x^2} \cos ax\,dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+1}\beta^{2n+1}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right) =$$
$$= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} \exp\left(-\frac{a^2}{4\beta^2}\right) \mathrm{H}_{2n}\left(\frac{a}{2\beta}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}, \quad a > 0\right].$$

WH, ET I 15(13)

10. $$\int_0^\infty x^{2n+1} e^{-\beta^2x^2} \sin ax\,dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+\frac{3}{2}}\beta^{2n+2}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right) =$$
$$= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} \exp\left(-\frac{a^2}{4\beta^2}\right) \mathrm{H}_{2n+1}\left(\frac{a}{2\beta}\right) \qquad \left[|\arg\beta| < \frac{\pi}{4}, \quad a > 0\right].$$

WH, ET I 74(23)

ET I 318(11)

2. $$\int_0^\infty x^{\mu-1}e^{-\gamma x-\beta x^2}\ \cos ax\,dx =$$
$$= \frac{1}{2(2\beta)^{\frac{\mu}{2}}}\ \exp\left(\frac{\gamma^2-a^2}{8\beta}\right)\Gamma(\mu)\left\{\exp\left(-\frac{ia\gamma}{4\beta}\right)D_{-\mu}\left(\frac{\gamma-ia}{\sqrt{2\beta}}\right)+\{exp\}\ \frac{ia\gamma}{4\beta}D_{-\mu}\left(\frac{\gamma+ia}{\sqrt{2\beta}}\right)\right\}$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0,\quad a>0].$$

ET I 16(18)

3. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\ \sin ax\,dx = \frac{i\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\exp\left[-\frac{(\gamma-ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right]\ -\right.$$
$$\left.-\ (\gamma+ia)\exp\left[-\frac{(\gamma+ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\}\quad [\operatorname{Re}\beta>0, a>0].$$

ET I 74(28)

4. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\ \cos ax\,dx = -\frac{\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\ \exp\frac{(\gamma-ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right]\ +\right.$$
$$\left.+\ (\gamma+ia)\ \exp\frac{(\gamma+ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\}+\frac{1}{2\beta}$$
$$[\operatorname{Re}\beta>0,\quad a>0].$$

ET I 16(17)

3.954

1. $$\int_0^\infty e^{-\beta x^2}\ \sin ax\frac{x\,dx}{\gamma^2+x^2} = -\frac{\pi}{4}e^{\beta\gamma 2}\left[2\ \operatorname{sh}a\gamma+e^{-\gamma a}\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-e^{\gamma a}\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad a>0].$$

ET I 74(26)a

2. $$\int_0^\infty e^{-\beta x^2}\ \cos ax\frac{dx}{\gamma^2+x^2} = \frac{\pi}{4\gamma}e^{\beta\gamma 2}\left[2\ \operatorname{ch}a\gamma-e^{-\gamma a}\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-e^{\gamma a}\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad a>0].$$

ET I 15(15)

3.955

$$\int_0^\infty x^\nu e^{-\frac{x^2}{2}}\ \cos\left(\beta x-\nu\frac{\pi}{2}\right)\ dx = \sqrt{\frac{\pi}{2}}e^{-\frac{\beta^2}{4}}D_\nu(\beta)\qquad [\operatorname{Re}\nu>-1].$$

3.956

$$\int_0^\infty e^{-x^2}(2x\,\cos x - \sin x)\,\sin x\frac{dx}{x^2} = \sqrt{\pi}\frac{e-1}{2e}.$$

BI ((369))(19)

3.957

1. $$\int_0^\infty x^{\mu-1}\,\exp\left(\frac{-\beta^2}{4x}\right)\,\sin ax\,dx = \frac{i}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}} \times$$
$$\times\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\frac{\pi i}{4}}\sqrt{a}\right) - \{exp\}\,\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\frac{\pi i}{4}}\sqrt{a}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu < 1, \quad a > 0].$$

ET I 318(12)

2. $$\int_0^\infty x^{\mu-1}\,\exp\left(\frac{-\beta^2}{4x}\right)\,\cos ax\,dx =$$
$$= \frac{1}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}}\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\frac{\pi i}{4}}\sqrt{a}\right) + \{exp\}\,\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\frac{\pi i}{4}}\sqrt{a}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu < 1, \quad a > 0].$$

ET I 320(32)a

3.958

1. $$\int_{-\infty}^\infty x^n e^{-(ax^2+bx+c)}\,\sin(px+q)\,dx = -\left(\frac{-1}{2a}\right)^n\sqrt{\frac{\pi}{a}}\,\exp\left(\frac{b^2-p^2}{4a}-c\right)\sum_{k=0}^{[n/2]}\frac{n!}{(n-2k)!k!}a^k \times$$
$$\times\sum_{j=0}^{n-2k}\binom{n-2k}{j}b^{n-2k-j}p^j\,\sin\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right) \qquad [a>0].$$

GW ((37))(1b)

2. $$\int_{-\infty}^\infty x^n e^{-(ax^2+bx+c)}\,\cos(px+q)\,dx = \left(\frac{-1}{2a}\right)^n\sqrt{\frac{\pi}{a}}\,\exp\left(\frac{b^2-p^2}{4a}-c\right)\sum_{k=0}^{[n/2]}\frac{n!}{(n-2k)!k!}a^k \times$$
$$\times\sum_{j=0}^{n-2k}\binom{n-2k}{j}p^j\,\cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$
$$[a>0].$$

GW ((337))(1a)

3.959

$$\int_0^\infty xe^{-p^2x^2}\ \operatorname{tg} ax\,dx = \frac{a\sqrt{\pi}}{p^3}\sum_{k=1}^\infty (-1)^k k\ \exp\left(-\frac{a^2k^2}{p^2}\right) \qquad [p>0].$$

BI ((362))(15)

3.961

1. $$\int_0^\infty \exp(-\beta\sqrt{\gamma^2+x^2})\ \sin ax \frac{x\,dx}{\sqrt{\gamma^2+x^2}} = \frac{a\gamma}{\sqrt{a^2+\beta^2}} K_1(\gamma\sqrt{a^2+\beta^2})$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0].$$

ET I 75(36)

2. $$\int_0^\infty \exp[-\beta\sqrt{\gamma^2+x^2}]\ \cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} = K_0(\gamma\sqrt{a^2+\beta^2}) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0].$$

ET I 17(27)

3.962

1. $$\int_0^\infty \frac{\sqrt{\sqrt{\gamma^2+x^2}-\gamma}\ \exp(-\beta\sqrt{\gamma^2+x^2})}{\sqrt{\gamma^2+x^2}}\ \sin ax\,dx = \sqrt{\frac{\pi}{2}}\frac{a\ \exp(-\gamma\sqrt{a^2+\beta^2})}{\sqrt{\beta^2+a^2}\sqrt{\beta+\sqrt{a^2+\beta^2}}}$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0].$$

ET I 75(38)

2. $$\int_0^\infty \frac{x\ \exp(-\beta\sqrt{\gamma^2+x^2})}{\sqrt{\gamma^2+x^2}\sqrt{\sqrt{\gamma^2+x^2}-\gamma}}\ \cos ax\,dx = \sqrt{\frac{\pi}{2}}\frac{\sqrt{\beta+\sqrt{a^2+\beta^2}}}{\sqrt{a^2+\beta^2}}\ \exp[-\gamma\sqrt{a^2+\beta^2}]$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0].$$

ET I 17(29)

3.963

1. $$\int_0^\infty e^{-\operatorname{tg}^2 x}\frac{\sin x}{\cos^2 x}\frac{dx}{x} = \frac{\sqrt{\pi}}{2}.$$

BI ((391))(1)

3.339

BI ((396))(3)

3. $$\int_0^{\frac{\pi}{2}} x e^{-\operatorname{tg}^2 x} \sin 4x \frac{dx}{\cos^2 x} = -\frac{3}{2}\sqrt{\pi}.$$

BI ((396))(5)

4. $$\int_0^{\frac{\pi}{2}} x e^{-\operatorname{tg}^2 x} \sin^2 2x \frac{dx}{\cos^2 x} = 2\sqrt{\pi}.$$

BI ((396))(6)

3.964

1. $$\int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg} x} \frac{p \sin x - \cos x}{\cos^3 x}\, dx = -\sin p \operatorname{si}(p) - \operatorname{ci}(p) \cos p \qquad [p > 0].$$

LI ((396))(4)

2. $$\int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg}^2 x} \frac{p - \cos^2 x}{\cos^4 x \operatorname{ctg} x}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{p}} \qquad [p > 0].$$

BI ((396))(7)

3.[8] $$\int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg}^2 x} \frac{p - 2\cos^2 x}{\cos^6 x \operatorname{ctg} x}\, dx = \frac{1+2p}{8p}\sqrt{\frac{\pi}{p}} \qquad [p > 0].$$

BI ((396))(8)

3.965

1. $$\int_0^{\infty} x e^{-\beta x} \sin ax^2 \sin \beta x\, dx = \frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \qquad \left[|\arg \beta| < \frac{\pi}{4}, \quad a > 0\right].$$

ET I 84(17)

2. $$\int_0^{\infty} x e^{-\beta x} \cos ax^2 \cos \beta x\, dx = \frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \qquad [a > 0, \quad \operatorname{Re}\beta > |\operatorname{Im}\beta|].$$

1. $\int_0^\infty xe^{-px} \cos(2x^2 + px)\, dx = 0 \qquad [p > 0].$

BI ((361))(16)

2. $\int_0^\infty xe^{-px} \cos(2x^2 - px)\, dx = \frac{p\sqrt{\pi}}{8} \exp\left(-\frac{1}{4}p^2\right) \qquad [p > 0].$

BI ((361))(17)

3. $\int_0^\infty x^2 e^{-px}[\sin(2x^2 + px) + \cos(2x^2 + px)]\, dx = 0 \qquad [p > 0].$

BI ((361))(18)

4. $\int_0^\infty x^2 e^{-px}[\sin(2x^2 - px) - \cos(2x^2 - px)]\, dx = \frac{\sqrt{\pi}}{16}(2 - p^2) \exp\left(-\frac{1}{4}p^2\right).$

BI ((361))(19)

5.[3] $\int_0^\infty x^{\mu-1} e^{-x} \cos(x+ax^2)\, dx = \frac{e^{\frac{1}{4a}}\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \cos\frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right) \qquad [\operatorname{Re}\mu > 0, \quad a > 0].$

ET I 321(37)

6.[6] $\int_0^\infty x^{\mu-1} e^{-x} \sin(x+ax^2)\, dx = \frac{e^{\frac{1}{4a}}\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \sin\frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right) \qquad [\operatorname{Re}\mu > -1, \quad a > 0].$

ET I 319(18)

3.967

1. $\int_0^\infty e^{-\frac{\beta^2}{x^2}} \sin a^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \sin(\sqrt{2}a\beta) \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$

ET I 75(30)A, BI((369))(3)a

2. $\int_0^\infty e^{-\frac{\beta^2}{x^2}} \cos a^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \cos(\sqrt{2}a\beta) \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$

BI ((369))(4), ET I 16(20)

3. $$\int_0^\infty x^2 e^{-\beta x^2} \cos ax^2\,dx = \frac{\sqrt{\pi}}{4\sqrt[4]{(a^2+\beta^2)^3}} \cos\left(\frac{3}{2}\operatorname{arctg}\frac{a}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

ET I 14(3)a

3.968

1. $$\int_0^\infty e^{-\beta x^2} \sin ax^4\,dx = -\frac{\pi}{8}\sqrt{\frac{\beta}{a}}\left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\{cos\}\left(\frac{\beta^2}{8a}+\frac{\pi}{8}\right) + N_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\{sin\}\left(\frac{\beta^2}{8a}+\frac{\pi}{8}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad a > 0].$$

ET I 75(34)

2. $$\int_0^\infty e^{-\beta x^2} \cos ax^4\,dx = \frac{\pi}{8}\sqrt{\frac{\beta}{a}}\left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\sin\left(\frac{\beta^2}{8a}+\frac{\pi}{8}\right) - N_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\{cos\}\left(\frac{\beta^2}{8a}+\frac{\pi}{8}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad a > 0].$$

ET I 16(24)

3.969

1. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\cos(2pqx^3) + q\sin(2pqx^3)\right]dx = \frac{\sqrt{\pi}}{2}.$$

BI ((363))(7)

2. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\sin(2pqx^3) - q\cos(2pqx^3)\right]dx = 0.$$

BI ((363))(8)

3.971

1. $$\int_0^\infty \exp\left(-px^2-\frac{q}{x^2}\right)\sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2} =$$
$$= \frac{1}{2}\int_{-\infty}^\infty \exp\left(-px^2-\frac{q}{x^2}\right)\sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2} =$$
$$= \frac{\sqrt{\pi}}{2s}\exp[-2rs\cos(A+B)]\sin[A+2rs\sin(A+B)].$$

BI ((369))(16, 17)

3.972

1. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^4+x^4}\right]\,\sin ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}} = \\ = \sqrt{\frac{a\pi}{8}} I_{1/4}\left[\frac{\gamma^2}{2}(\sqrt{\beta^2+a^2}-\beta)\right] K_{1/4}\left[\frac{\gamma^2}{4}(\sqrt{\beta^2+a^2}+\beta)\right] \\ \left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right].$$

ET I 75(37)

2. $$\int_0^\infty \exp[-\beta\sqrt{\gamma^4+x^4}]\,\cos ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}} = \\ = \sqrt{\frac{a\pi}{8}} I_{-1/4}\left[\frac{\gamma^2}{2}(\sqrt{\beta^2+a^2}-\beta)\right] K_{1/4}\left[\frac{\gamma^2}{4}(\sqrt{\beta^2+a^2}+\beta)\right] \\ \left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right].$$

ET I 17(28)

3.973

1. $$\int_0^\infty \exp(p\cos ax)\,\sin(p\sin ax)\frac{dx}{x} = \frac{\pi}{2}(e^p-1) \qquad [p > 0, \quad a > 0].$$

WH, FI II 725

2. $$\int_0^\infty \exp(p\cos ax)\,\sin(p\sin ax+bx)\frac{x\,dx}{c^2+x^2} = \frac{\pi}{2}\,\exp(-cb+pe^{-ac}) \\ [a > 0, \quad b > 0, \quad c > 0, \quad p > 0].$$

BI ((372))(3)

3. $$\int_0^\infty \exp(p\cos ax)\,\cos(p\sin ax+bx)\frac{dx}{c^2+x^2} = \frac{\pi}{2c}\,\exp(-cb+pe^{-ac}) \\ [a > 0, \quad b > 0, \quad c > 0, \quad p > 0].$$

BI ((372))(4)

BI ((366))(2)

5. $$\int_0^\infty \exp(p\cos x)\sin(p\sin x)\cos nx\frac{dx}{x} = \frac{p^n}{n!}\cdot\frac{\pi}{4}+\frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{p^k}{k!} \qquad [p>0].$$

LI ((366))(3)

6. $$\int_0^\infty \exp(p\cos x)\cos(p\sin x)\sin nx\frac{dx}{x} = \frac{\pi}{2}\sum_{k=0}^{n-1}\frac{p^k}{k!}+\frac{p^n}{n!}\frac{\pi}{4} \qquad [p>0].$$

LI ((366))(4)

3.974

1. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi[e^p-\exp(pe^{-ab})]}{2b\operatorname{sh} ab} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((391))(4)

2. $$\int_0^\infty [1-\exp(p\cos ax)\cos(p\sin ax)]\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi[e^p-\exp(pe^{-ab})]}{2\operatorname{sh} ab} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((391))(5)

3. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax+ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi[e^p-\exp(pe^{-ab}-ab)]}{2b\operatorname{sh} ab} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((391))(6)

4. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax+ax)\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi[e^p-\exp(pe^{-ab}-ab)]}{2\operatorname{sh} ab} \qquad [a>0, \quad b>0, \quad p>0].$$

BI ((391))(7)

5. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\frac{x\,dx}{b^2-x^2} = \frac{\pi}{2}[1-\exp(p\cos ab)\cos(p\sin ab)] \qquad [p>0, \quad a>0].$$

6. $$\int_0^\infty \exp(p\ \cos ax)\ \cos(p\ \sin ax)\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\ \exp(p\ \cos ab)\ \sin(p\ \sin ab)$$
$$[a>0,\quad b>0,\quad p>0].$$

BI ((378))(2)

7. $$\int_0^\infty \exp(p\ \cos ax)\ \sin(p\ \sin ax)\ \operatorname{tg} ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\cdot\theta ab[\exp(pe^{-ab})-e^p]$$
$$[a>0,\quad b>0,\quad p>0].$$

BI ((372))(14)

8. $$\int_0^\infty \exp(p\ \cos ax)\ \sin(p\ \sin ax)\ \operatorname{ctg} ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\ \operatorname{cth}\ ab[e^p-\exp(pe^{-ab})]$$
$$[a>0,\quad b>0,\quad p>0].$$

BI ((372))(15)

9. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\operatorname{cosec} ab[e^p-\exp(p\cos ab)\cos(p\sin ab)]$$
$$[a>0,\quad b>0,\quad p>0].$$

BI ((391))(12)

10. $$\int_0^\infty [1-\exp(p\ \cos ax)\ \cos(p\ \sin ax)]\ \operatorname{cosec} ax\frac{x\,dx}{b^2-x^2} =$$
$$= -\frac{\pi}{2}\ \exp(p\ \cos ab)\ \sin(p\ \sin ab)\ \operatorname{cosec} ab \qquad [a>0,\quad b>0,\quad p>0].$$

BI ((391))(13)

3.975

1. $$\int_0^\infty \frac{\sin\left(\beta\ \operatorname{arctg}\dfrac{x}{\gamma}\right)}{(\gamma^2+x^2)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}-1} = \frac{1}{2}\zeta(\beta,\,\gamma)-\frac{1}{4\gamma^\beta}-\frac{\gamma^{1-\beta}}{2(\beta-1)} \qquad [\operatorname{Re}\beta>1,\quad \operatorname{Re}\gamma>0].$$

WH, ET I 26(7)

2. $$\int_0^\infty \frac{\sin(\beta\ \operatorname{arctg} x)}{(1+x^2)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}+1} = \frac{1}{2(\beta-1)}-\frac{\zeta(\beta)}{2^\beta} \qquad [\operatorname{Re}\beta>1].$$

$$\int_0^\infty (1+x^2)^{\beta-\frac{1}{2}} e^{-px^2} \cos[2px+(2\beta-1) \operatorname{arctg} x]\, dx = \frac{e^{-p}}{2p^\beta} \sin \pi\beta\Gamma(\beta) \qquad [\operatorname{Re}\beta > 0, \quad p > 0].$$

WH

3.98- 3.99 Combinations of trigonometric and hyperbolic functions

3.981

1. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh} \beta x}\, dx = \frac{\pi}{2\beta}\, \theta \frac{a\pi}{2\beta} \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

BI ((264))(16)

2. $$\int_0^\infty \frac{\sin ax}{\operatorname{ch} \beta x}\, dx = -\frac{\pi}{2\beta}\, \theta \frac{a\pi}{2\beta} - \frac{i}{2\beta}\left[\psi\left(\frac{\beta+ai}{4\beta}\right) - \psi\left(\frac{\beta-ai}{4\beta}\right)\right] \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

GW ((335))(12), ET I 88(1)

537

3. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch} \beta x}\, dx = \frac{\pi}{2\beta} \operatorname{sech} \frac{a\pi}{2\beta} \qquad [\operatorname{Re}\beta > 0, \quad \text{all real } a].$$

BI ((264))(14)

4. $$\int_0^\infty \sin ax \frac{\operatorname{sh} \beta x}{\operatorname{sh} \gamma x}\, dx = \frac{\pi}{2\gamma} \frac{\operatorname{sh} \frac{a\pi}{\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} + \frac{i}{2\gamma}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right) - \psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right]$$
$$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a > 0].$$

ET I 88(5)

5. $$\int_0^\infty \cos ax \frac{\operatorname{sh} \beta x}{\operatorname{sh} \gamma x}\, dx = \frac{\pi}{2\gamma} \frac{\sin \frac{\pi\beta}{\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \qquad [|\operatorname{Re}\beta| < \operatorname{Re}\gamma].$$

BI ((265))(7)

6. $$\int_0^\infty \sin ax \frac{\operatorname{sh} \beta x}{\operatorname{ch} \gamma x}\, dx = \frac{\pi}{\gamma} \frac{\sin \frac{\beta\pi}{2\gamma} \operatorname{sh} \frac{a\pi}{2\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \qquad [|\operatorname{Re}\beta| < \operatorname{Re}\gamma; \quad a > 0].$$

7. $$\int_0^\infty \cos ax \frac{\operatorname{sh}\beta x}{\operatorname{ch}\gamma x}\,dx = \frac{1}{4\gamma}\left\{\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta-ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ia}{4\gamma}\right)-\right.$$
$$\left.-\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)+\frac{2\pi\,\sin\dfrac{\pi\beta}{\gamma}}{\cos\dfrac{\pi\beta}{\gamma}+\operatorname{ch}\dfrac{\pi a}{\gamma}}\right\} \qquad [|\operatorname{Re}\beta|<\operatorname{Re}\gamma, \quad a>0].$$

ET I 31(13)

8. $$\int_0^\infty \sin ax \frac{\operatorname{ch}\beta x}{\operatorname{sh}\gamma x}\,dx = \frac{\pi}{2\gamma}\cdot\frac{\operatorname{sh}\dfrac{\pi a}{\gamma}}{\operatorname{ch}\dfrac{\pi a}{\gamma}+\cos\dfrac{\pi\beta}{\gamma}} \qquad [|\operatorname{Re}\beta|<\operatorname{Re}\gamma, \quad a>0].$$

BI ((265))(4)

9. $$\int_0^\infty \sin ax \frac{\operatorname{ch}\beta x}{\operatorname{ch}\gamma x}\,dx = \frac{i}{4\gamma}\left[\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ai}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)-\right.$$
$$\left.-\psi\left(\frac{3\gamma-\beta-ai}{4\gamma}\right)-\frac{2\pi i\,\operatorname{sh}\dfrac{\pi a}{\gamma}}{\operatorname{ch}\dfrac{a\pi}{\gamma}+\cos\dfrac{\beta\pi}{\gamma}}\right] \qquad [|\operatorname{Re}\beta|<\operatorname{Re}\gamma, \quad a>0].$$

ET I 88(6)

10. $$\int_0^\infty \cos ax \frac{\operatorname{ch}\beta x}{\operatorname{ch}\gamma x}\,dx = \frac{\pi}{\gamma}\frac{\cos\dfrac{\beta\pi}{2\gamma}\ \operatorname{ch}\dfrac{a\pi}{2\gamma}}{\operatorname{ch}\dfrac{a\pi}{\gamma}+\cos\dfrac{\beta\pi}{\gamma}} \qquad [|\operatorname{Re}\beta|<\operatorname{Re}\gamma, \quad \text{all real } a].$$

BI ((265))(6)

11. $$\int_0^{\frac{\pi}{2}} \cos^{2m} x\ \operatorname{ch}\beta x\,dx = \frac{(2m)!\ \operatorname{sh}\dfrac{\pi\beta}{2}}{\beta(\beta^2+2^2)\ldots[\beta^2+(2m)^2]} \qquad [\operatorname{Re}\beta>0].$$

WA 620a

WA 620a

3.982

1. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}^2 \beta x}\,dx = \frac{a\pi}{2\beta^2 \operatorname{sh}\frac{a\pi}{2\beta}} \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

BI ((264))(16)

2. $$\int_0^\infty \sin ax \frac{\operatorname{sh}\beta x}{\operatorname{ch}^2 \gamma x}\,dx = \frac{\pi\left(a \sin\frac{\beta\pi}{2\gamma} \operatorname{ch}\frac{a\pi}{2\gamma} - \beta \cos\frac{\beta\pi}{2\gamma} \operatorname{sh}\frac{a\pi}{2\gamma}\right)}{\gamma^2\left(\operatorname{ch}\frac{a\pi}{\gamma} - \cos\frac{\beta\pi}{\gamma}\right)} \qquad [|\operatorname{Re}\beta| < 2 \operatorname{Re}\gamma, \quad a > 0].$$

ET I 88(9)

3.[8] $$\int_0^\infty \frac{\sin^2 x \cos ax}{\sin h^2 x} dx = \frac{\pi}{4}\left\{\frac{a+2}{1-e^{-\pi(a+2)}} - \frac{2a}{1-e^{-\pi a}} + \frac{a-2}{1-e^{-\pi(a-2)}}\right\} = \mathrm{I}(a)$$
$$\left[\mathrm{I}(o) = \frac{1}{2}(\pi \coth \pi - 1),\ \mathrm{I}(\pm 2) = \frac{1}{4} + \frac{\pi}{2}(\coth 2\pi - \coth \pi)\right]$$

3.983

1.[6] $$\int_0^\infty \frac{\cos ax\,dx}{b \operatorname{ch}\beta x + c} = \frac{\pi \sin\left(\frac{a}{\beta} \operatorname{Arch}\frac{c}{b}\right)}{\beta\sqrt{c^2-b^2} \operatorname{sh}\frac{a\pi}{\beta}} \qquad [c > b > 0]$$
$$= \frac{\pi \operatorname{sh}\left(\frac{a}{\beta} \arccos\frac{c}{b}\right)}{\beta\sqrt{b^2-c^2} \operatorname{sh}\frac{a\pi}{\beta}} \qquad [b > |c| > 0]; \qquad [\operatorname{Re}\beta > 0, \quad a > 0].$$

GW ((335))(13a)

2. $$\int_0^\infty \frac{\cos ax\,dx}{\operatorname{ch}\beta x + \cos\gamma} = \frac{\pi}{\beta}\frac{\operatorname{sh}\frac{a\gamma}{\beta}}{\sin\gamma \operatorname{sh}\frac{a\pi}{\beta}} \qquad [\pi \operatorname{Re}\beta < \operatorname{Im}\overline{\beta}\gamma, \quad a > 0].$$

BI ((267))(3)

3.[3] $$\int_0^\infty \frac{\cos ax\,dx}{\operatorname{ch} x - \operatorname{ch} b} = -\pi \operatorname{cth} a\pi \frac{\sin ab}{\operatorname{sh} b} \qquad [a > 0, \quad b > 0].$$

4. $$\int_0^\infty \frac{\cos ax\,dx}{1+2\ \mathrm{ch}\left(\sqrt{\frac{2}{3}}\pi x\right)} = \frac{\sqrt{\frac{\pi}{2}}}{1+2\ \mathrm{ch}\left(\sqrt{\frac{2}{3}}\pi a\right)} \qquad [a>0].$$

ET I 30(9)

5. $$\int_0^\infty \frac{\sin ax\ \mathrm{sh}\,\beta x}{\mathrm{ch}\,\gamma x+\cos\delta}\,dx = \frac{\pi\left\{\sin\left[\frac{\beta}{\gamma}(\pi-\delta)\right]\ \mathrm{sh}\left[\frac{a}{\gamma}(\pi+\delta)\right]-\sin\left[\frac{\beta}{\gamma}(\pi+\delta)\right]\ \mathrm{sh}\left[\frac{a}{\gamma}(\pi-\delta)\right]\right\}}{\gamma\ \sin\delta\left(\mathrm{ch}\,\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}\right)}$$
$$[\pi\ \mathrm{Re}\,\gamma > |\,\mathrm{Re}\,\overline{\gamma}\delta|, \quad |\,\mathrm{Re}\,\beta| < \mathrm{Re}\,\gamma, \quad a>0].$$

BI ((267))(2)

6. $$\int_0^\infty \frac{\cos ax\ \mathrm{ch}\,\beta x}{\mathrm{ch}\,\gamma x+\cos b}\,dx = \frac{\pi\left\{\cos\left[\frac{\beta}{\gamma}(\pi-b)\right]\ \mathrm{ch}\left[\frac{a}{\gamma}(\pi+b)\right]-\{cos\}\left[\frac{\beta}{\gamma}(\pi+b)\right]\ \mathrm{ch}\left[\frac{a}{\gamma}(\pi-b)\right]\right\}}{\gamma\ \sin b\left(\mathrm{ch}\,\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}\right)}$$
$$[|\,\mathrm{Re}\,\beta| < \mathrm{Re}\,\gamma, \quad 0<b<\pi, \quad a<0].$$

BI ((267))(6)

7. $$\int_0^\infty \frac{\cos ax\,dx}{(\beta+\sqrt{\beta^2-1}\ \mathrm{ch}\,x)^{\nu+1}} = \Gamma(\nu+1-ai)e^{a\pi}\frac{Q_\nu^{ai}(\beta)}{\Gamma(\nu+1)}$$
$$[\mathrm{Re}\,\nu>-1, \quad |\arg(\beta+1)|<\pi, \quad a>0].$$

ET I 30(10)

3.984

1.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\sin ax\ \mathrm{sh}\,cx}{\mathrm{ch}\,x+\cos b}\,dx = \pi\frac{\mathrm{ch}\,ab}{\mathrm{sh}\,a\pi} \qquad [|b|\le\pi, \quad a \text{ real}].$$

BI ((267))(1)

2.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\cos ax\ \mathrm{ch}\,cx}{\mathrm{ch}\,x+\cos b}\,dx = -\pi\ \mathrm{ctg}\,b\frac{\mathrm{sh}\,ab}{\mathrm{sh}\,a\pi} \qquad [0<|b|<\pi, \quad a \text{ real}].$$

BI ((267))(5)

ET I 80(10)

4. $$\int_0^\infty \frac{\cos ax \ \mathrm{ch}\,\dfrac{\beta}{2}x}{\mathrm{ch}\,\beta x + \mathrm{ch}\,\gamma}\,dx = \frac{\pi\ \cos\dfrac{a\gamma}{\beta}}{2\beta\ \mathrm{ch}\,\dfrac{\gamma}{2}\ \mathrm{ch}\,\dfrac{a\pi}{\beta}} \qquad [\pi\ \mathrm{Re}\,\beta > |\,\mathrm{Im}(\overline{\beta}\gamma)|].$$

ET I 31(16)

5. $$\int_0^\infty \frac{\sin ax\ \mathrm{sh}\,\beta x}{\mathrm{ch}\,2\beta x + \cos 2ax}\,dx = \frac{a\pi}{4(a^2+\beta^2)} \qquad [a>0, \quad \mathrm{Re}\,\beta > 0].$$

BI ((267))(7)

6. $$\int_0^\infty \frac{\cos ax\ \mathrm{ch}\,\beta x}{\mathrm{ch}\,2\beta x + \cos 2ax}\,dx = \frac{\beta\pi}{4(a^2+\beta^2)} \qquad [\mathrm{Re}\,\beta > 0, \quad a>0].$$

BI ((267))(8)

7.[8] $$\int_0^\infty \frac{\mathrm{sh}^{2\mu-1}\,x\ \mathrm{ch}^{2\varrho-2\nu+1}\,x}{(\mathrm{ch}^2\,x - \beta\ \mathrm{sh}^2\,x)^\varrho}\,dx = \frac{1}{2}\mathrm{B}(\mu,\,\nu-\mu)\,{}_2F_1(\varrho,\,\mu;\,\nu;\,\beta) \qquad [\mathrm{Re}\,\nu > \mathrm{Re}\,\mu > 0].$$

EH I 115(12)

3.985

1. $$\int_0^\infty \frac{\cos ax\,dx}{\mathrm{ch}^\nu\ \beta x} = \frac{2^{\nu-2}}{\beta\Gamma(\nu)}\Gamma\left(\frac{\nu}{2}+\frac{ai}{2\beta}\right)\Gamma\left(\frac{\nu}{2}-\frac{ai}{2\beta}\right) \qquad [\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>0, \quad a>0].$$

ET I 30(5)

2. $$\int_0^\infty \frac{\cos ax\,dx}{\mathrm{ch}^{2n}\ \beta x} = \frac{4^{n-1}\pi a}{2(2n-1)!\beta^2\ \mathrm{sh}\,\dfrac{a\pi}{2\beta}}\prod_{k=1}^{n-1}\left(\frac{a^2}{4\beta^2}+k^2\right);$$
$$= \frac{\pi a(a^2+2^2\beta^2)(a^2+4^2\beta^2)\dots\left[a^2+(2n-2)^2\beta^2\right]}{2(2n-1)!\beta^{2n}\ \mathrm{sh}\,\dfrac{a\pi}{2\beta}} \qquad [n\ge 2, \quad a>0].$$

ET I 30(3)

ET I 30(4)

3.986

1. $$\int_0^\infty \frac{\sin\beta x\ \sin\gamma x}{\operatorname{ch}\delta x}\,dx = \frac{\pi}{\delta}\cdot\frac{\operatorname{sh}\dfrac{\beta\pi}{2\delta}\ \operatorname{sh}\dfrac{\gamma\pi}{2\delta}}{\operatorname{ch}\dfrac{\beta}{\delta}\pi + \operatorname{ch}\dfrac{\gamma}{\delta}\pi} \qquad [|\operatorname{Im}(\beta+\gamma)| < \operatorname{Re}\delta].$$

BI ((264))(19)

2. $$\int_0^\infty \frac{\sin\alpha x\ \cos\beta x}{\operatorname{sh}\gamma x}\,dx = \frac{\pi\ \operatorname{sh}\dfrac{\pi\alpha}{\gamma}}{2\gamma\left(\operatorname{ch}\dfrac{\alpha\pi}{\gamma} + \operatorname{ch}\dfrac{\beta\pi}{\gamma}\right)} \qquad [|\operatorname{Im}(\alpha+\beta)| < \operatorname{Re}\gamma].$$

LI ((264))(20)

3. $$\int_0^\infty \frac{\cos\beta x\ \cos\gamma x}{\operatorname{ch}\delta x}\,dx = \frac{\pi}{\delta}\cdot\frac{\operatorname{ch}\dfrac{\beta\pi}{2\delta}\ \operatorname{ch}\dfrac{\gamma\pi}{2\delta}}{\operatorname{ch}\dfrac{\beta\pi}{\delta} + \operatorname{ch}\dfrac{\gamma\pi}{\delta}} \qquad [|\operatorname{Im}(\beta+\gamma)| < \operatorname{Re}\delta].$$

BI ((264))(21)

4.[3] $$\int_0^\infty \frac{\sin^2\beta x}{\operatorname{sh}^2\pi x}\,dx = \frac{\beta}{\pi(e^{2\beta}-1)} + \frac{\beta-1}{2\pi} = \frac{\beta\ \operatorname{cth}\ \beta - 1}{2\pi} \qquad [|\operatorname{Im}\beta| < \pi].$$

EH I 44(3)

3.987

1. $$\int_0^\infty \sin ax(1-\theta\beta x)\,dx = \frac{1}{a} - \frac{\pi}{2\beta\ \operatorname{sh}\dfrac{\alpha\pi}{2\beta}} \qquad [\operatorname{Re}\beta > 0].$$

ET I 88(4)a

2. $$\int_0^\infty \sin ax(\operatorname{cth}\ \beta x - 1)\,dx = \frac{\pi}{2\beta}\ \operatorname{cth}\ \frac{a\pi}{2\beta} - \frac{1}{a} \qquad [\operatorname{Re}\beta > 0].$$

ET I 88(3)

3.988

ET I 37(66)

2. $$\int_0^{\frac{\pi}{2}} \frac{\cos ax\ \operatorname{ch}(2b\ \cos x)}{\sqrt{\cos x}}\,dx = \frac{\pi}{2}\sqrt{\pi b}\,I_{\frac{a}{2}-\frac{1}{4}}(b)\,I_{-\frac{a}{2}-\frac{1}{4}}(b) \qquad [a > 0].$$

ET I 37(67)

3. $$\int_0^{\infty} \frac{\cos ax\,dx}{\sqrt{\operatorname{ch} x \cos b}} = \frac{\pi P_{-\frac{1}{2}+ia}(\cos b)}{\sqrt{2}\ \operatorname{ch} a\pi} \qquad [a > 0, \quad b > 0].$$

ET I 30(7)

3.989

1. $$\int_0^{\infty} \frac{\sin \dfrac{a^2x^2}{\pi}\ \sin bx}{\operatorname{sh} ax}\,dx = \frac{\pi}{2a}\ \sin\frac{\pi b^2}{4a^2}\ \operatorname{cosech}\ \frac{\pi b}{2a} \qquad [a > 0, \quad b > 0].$$

ET I 93(44)

2. $$\int_0^{\infty} \frac{\cos \dfrac{a^2x^2}{\pi}\ \sin bx}{\operatorname{sh} ax}\,dx = \frac{\pi}{2a}\,\frac{\operatorname{ch}\dfrac{\pi b}{a} - \cos\dfrac{\pi b^2}{4a^2}}{\operatorname{sh}\dfrac{\pi b}{2a}} \qquad [a > 0, \quad b > 0].$$

ET I 93(45)

3. $$\int_0^{\infty} \frac{\sin \dfrac{x^2}{\pi}\ \cos ax}{\operatorname{ch} x}\,dx = \frac{\pi}{2}\,\frac{\cos\dfrac{a^2\pi}{4} - \dfrac{1}{\sqrt{2}}}{\operatorname{ch}\dfrac{a\pi}{2}}.$$

ET I 36(54)

4. $$\int_0^{\infty} \frac{\cos \dfrac{x^2}{\pi}\ \cos ax}{\operatorname{ch} x}\,dx = \frac{\pi}{2}\cdot\frac{\sin\dfrac{a^2\pi}{4} + \dfrac{1}{\sqrt{2}}}{\operatorname{ch}\dfrac{a\pi}{2}}.$$

ET I 36(55)

5. $$\int_0^{\infty} \frac{\sin(\pi a x^2)\ \cos bx}{\operatorname{ch} \pi x}\,dx = -\sum_{k=0}^{\infty} \exp\left[-\left(k+\frac{1}{2}\right)b\right]\ \sin\left[\left(k+\frac{1}{2}\right)^2 \pi a\right] +$$
$$+\frac{1}{\sqrt{a}}\sum^{\infty} \exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\ \sin\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right]$$

6. $$\int_0^\infty \frac{\cos(\pi a x^2)\,\cos bx}{\operatorname{ch}\pi x}\,dx = \sum_{k=0}^\infty (-1)^k \exp\left[-\left(k+\frac{1}{2}\right)b\right]\cos\left[\left(k+\frac{1}{2}\right)^2\pi a\right] + \\ + \frac{1}{\sqrt{a}}\sum_{k=0}^\infty \exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\cos\left[\frac{\pi}{4}-\frac{b^2}{4\pi a}+\frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right] \\ [a>0, \quad b>0].$$

ET I 36(57)

3.991

1. $$\int_0^\infty \sin\pi x^2\,\sin ax\,\operatorname{cth}\,\pi x\,dx = \frac{1}{2}\,\theta\frac{a}{2}\,\sin\left(\frac{\pi}{4}+\frac{a^2}{4\pi}\right).$$

ET I 93(42)

2. $$\int_0^\infty \cos\pi x^2\,\sin ax\,\operatorname{cth}\,\pi x\,dx = \frac{1}{2}\,\theta\frac{a}{2}\left[1\cos\left(\frac{\pi}{4}+\frac{a^2}{4\pi}\right)\right].$$

ET I 93(43)

3.992

1. $$\int_0^\infty \frac{\sin\pi x^2\,\cos ax}{1+2\,\operatorname{ch}\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = -\sqrt{3}+\frac{\cos\left(\frac{\pi}{12}-\frac{a^2}{4\pi}\right)}{4\,\operatorname{ch}\frac{a}{\sqrt{3}}-2}.$$

ET I 37(60)

2. $$\int_0^\infty \frac{\cos\pi x^2\,\cos ax}{1+2\,\operatorname{ch}\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = 1-\frac{\sin\left(\frac{\pi}{12}-\frac{a^2}{4\pi}\right)}{4\,\operatorname{ch}\frac{a}{\sqrt{3}}-2}.$$

ET I 37(61)

3.993

$$\int_0^\infty \frac{\sin x^2+\cos x^2}{\operatorname{ch}(\sqrt{\pi}x)}\,\cos ax\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\sin a^2+\cos a^2}{\operatorname{ch}(\sqrt{\pi}a)}.$$

1. $$\int_0^\infty \frac{\sin(2a\ \mathrm{ch}\, x)\ \cos bx}{\sqrt{\mathrm{ch}\, x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{\frac{1}{4}+\frac{ib}{2}}(a)N_{\frac{1}{4}-\frac{ib}{2}}(a)+J_{\frac{1}{4}-\frac{ib}{2}}(a)N_{\frac{1}{4}+\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 37(62)

2. $$\int_0^\infty \frac{\cos(2a\ \mathrm{ch}\, x)\ \cos bx}{\sqrt{\mathrm{ch}\, x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{-\frac{1}{4}+\frac{ib}{2}}(a)N_{-\frac{1}{4}-\frac{ib}{2}}(a)+J_{-\frac{1}{4}-\frac{ib}{2}}(a)N_{-\frac{1}{4}+\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 37(63)

3. $$\int_0^\infty \frac{\sin(2a\ \mathrm{sh}\, x)\ \sin bx}{\sqrt{\mathrm{sh}\, x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)K_{-\frac{1}{4}+\frac{ib}{2}}(a)-I_{\frac{1}{4}+\frac{ib}{2}}(a)K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 93(47)

4. $$\int_0^\infty \frac{\cos(2a\ \mathrm{sh}\, x)\ \sin bx}{\sqrt{\mathrm{sh}\, x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)K_{-\frac{1}{4}+\frac{ib}{2}}(a)-I_{-\frac{1}{4}+\frac{ib}{2}}(a)K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 93(48)

5. $$\int_0^\infty \frac{\sin(2a\ \mathrm{sh}\, x)\ \cos bx}{\sqrt{\mathrm{sh}\, x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)K_{\frac{1}{4}+\frac{ib}{2}}(a)+I_{\frac{1}{4}+\frac{ib}{2}}(a)K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 37(64)

6. $$\int_0^\infty \frac{\cos(2a\ \mathrm{sh}\, x)\ \cos bx}{\sqrt{\mathrm{sh}\, x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)K_{-\frac{1}{4}+\frac{ib}{2}}(a)+I_{-\frac{1}{4}+\frac{ib}{2}}(a)K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$$[a>0, \quad b>0].$$

ET I 37(65)

7. $$\int_0^\infty \sin(a\ \mathrm{ch}\, x)\ \sin(a\ \mathrm{sh}\, x)\frac{dx}{\mathrm{sh}\, x} = \frac{\pi}{2}\ \sin a \qquad [a>0].$$

BI ((264))(22)

3.995

1. $$\int_0^{\frac{\pi}{2}} \frac{\sin(2a\cos^2 x)\operatorname{ch}(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\sin\frac{2ac}{b+c} \qquad [b>0, \quad c>0].$$

BI ((273))(9)

2. $$\int_0^{\frac{\pi}{2}} \frac{\cos(2a\cos^2 x)\operatorname{ch}(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\cos\frac{2ac}{b+c} \qquad [b>0, \quad c>0].$$

BI ((273))(10)

3.996

1. $$\int_0^{\infty} \sin(a\operatorname{sh} x)\operatorname{sh}\beta x\,dx = \sin\frac{\beta\pi}{2}K_\beta(a) \qquad [|\operatorname{Re}\beta|<1, \quad a>0].$$

EH II 82(26)

2. $$\int_0^{\infty} \cos(a\operatorname{sh} x)\operatorname{ch}\beta x\,dx = \cos\frac{\beta\pi}{2}K_\beta(a) \qquad [|\beta|<1, \quad a>0].$$

WA 202(13)

3. $$\int_0^{\frac{\pi}{2}} \cos(a\sin x)\operatorname{ch}(\beta\cos x)\,dx = \frac{\pi}{2}J_0(\sqrt{a^2-\beta^2}).$$

MO 40

4. $$\int_0^{\infty} \sin\left(a\operatorname{ch} x - \frac{1}{2}\beta\pi\right)\operatorname{ch}\beta x\,dx = \frac{\pi}{2}J_\beta(a) \qquad [|\operatorname{Re}\beta|<1, \quad a>0].$$

WA 199(12)

5. $$\int_0^{\infty} \cos\left(a\operatorname{ch} x - \frac{1}{2}\beta\pi\right)\operatorname{ch}\beta x\,dx = -\frac{\pi}{2}N_\beta(a) \qquad [|\operatorname{Re}\beta|<1, \quad a>0].$$

WA 199(13)

3.997

EH II 38(53)

2. $$\int_0^{\pi} \sin^{\nu}\, x\, \operatorname{ch}(\beta\, \cos x)\, dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) I_{\frac{\nu}{2}}(\beta) \qquad [\operatorname{Re}\nu > -1].$$

WH

3. $$\int_0^{\frac{\pi}{2}} \frac{dx}{\operatorname{ch}(\operatorname{tg} x)\, \cos x\sqrt{\sin 2x}} = \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((276))(13)

4. $$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^q x}{\operatorname{ch}(\operatorname{tg} x) + \cos\lambda}\, \frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin\lambda}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin k\lambda}{k^q} \qquad [q > 0].$$

BI ((275))(20)

4.11- 4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.111

1. $$\int_0^{\infty} \frac{\sin ax}{\operatorname{sh}\beta x}\cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta}\cdot\frac{\partial^{2m}}{\partial a^{2m}}\left(\theta\frac{a\pi}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } 3.981\ 1.).$$

3.981
GW ((336))(17a)

2. $$\int_0^{\infty} \frac{\cos ax}{\operatorname{sh}\beta x}\cdot x^{2m+1}\, dx = (-1)^m \frac{\pi}{2\beta}\frac{\partial^{2m+1}}{\partial a^{2m+1}}\left(\theta\frac{a\pi}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } 3.981\ 1.).$$

3.981
GW ((336))(17b)

3. $$\int_0^{\infty} \frac{\sin ax}{\operatorname{ch}\beta x}\cdot x^{2m+1}\, dx = (-1)^{m+1} \frac{\pi}{2\beta}\cdot\frac{\partial^{2m+1}}{\partial a^{2m+1}}\left(\frac{1}{\operatorname{ch}\dfrac{a\pi}{2\beta}}\right) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } 3.981\ 3.).$$

4. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch} \beta x} \cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\frac{1}{\operatorname{ch} \frac{a\pi}{2\beta}} \right) \qquad [\operatorname{Re} \beta > 0] \qquad \text{(cf. 3.981 3.).}$$

3.981
GW ((336))(18a)

5. $$\int_0^\infty x \frac{\sin 2ax}{\operatorname{ch} \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\operatorname{sh} \frac{a\pi}{\beta}}{\operatorname{ch}^2 \frac{a\pi}{\beta}} \qquad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((364))(6)a

6. $$\int_0^\infty x \frac{\cos 2ax}{\operatorname{sh} \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\operatorname{ch}^2 \frac{a\pi}{\beta}} \qquad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((364))(1)a

7. $$\int_0^\infty \frac{\sin ax}{\operatorname{ch} \beta x} \frac{dx}{x} = 2 \operatorname{arctg} \left(\exp \frac{\pi a}{2\beta} \right) - \frac{\pi}{2} \qquad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((387))(1), ET I 89(13), LI ((298))(17)

4.112

1. $$\int_0^\infty (x^2 + \beta^2) \frac{\cos ax}{\operatorname{ch} \frac{\pi x}{2\beta}}\, dx = \frac{2\beta^3}{\operatorname{ch}^3 a\beta} \qquad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 32(19)

2. $$\int_0^\infty x(x^2 + 4\beta^2) \frac{\cos ax}{\operatorname{sh} \frac{\pi x}{2\beta}}\, dx = \frac{6\beta^4}{\operatorname{ch}^4 a\beta} \qquad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 32(20)

4.113

1. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh}\pi x}\cdot\frac{dx}{x^2+\beta^2} = -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta\,\sin\pi\beta} + \\ + \frac{1}{2\beta^2}\left[{}_2F_1(1,\,-\beta;\,1-\beta;\,-e^{-a}) + {}_2F_1(1,\,\beta;\,1+\beta\colon\,-e^{-a})\right] = \\ = \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta\,\sin\pi\beta} - \sum_{k=1}^{\infty}\frac{(-1)^k e^{-ak}}{k^2-\beta^2}$$
$$[\operatorname{Re}\beta > 0, \quad \beta\neq 0,\,1,\,2,\dots,\,a>0].$$

ET I 90(18)

2. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh}\pi x}\cdot\frac{dx}{x^2+m^2} = \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m}\sum_{k=1}^{m-1}\frac{(-1)^k e^{-ka}}{m-k} + \frac{(-1)^m e^{-ma}}{2m}\ \ln(1+e^{-a}) + \\ + \frac{1}{2m!}\frac{d^{m-1}}{dz^{m-1}}\left[\frac{(1+z)^{m-1}}{z}\ \ln(1+z)\right]_{z=e^{-a}} \qquad [a>0].$$

ET I 89(17)

3. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh}\pi x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin ax}{\operatorname{sh}\pi x}\frac{dx}{1+x^2} = -\frac{a}{2}\ \operatorname{ch}a + \operatorname{sh}a\ \ln\left(2\ \operatorname{ch}\frac{a}{2}\right).$$

GW ((336))(21b)

4. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh}\frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin ax}{\operatorname{sh}\frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\operatorname{sh}a - \operatorname{ch}a\ \operatorname{arctg}(\operatorname{sh}a).$$

GW ((336))(21a)

5. $$\int_0^\infty \frac{\sin ax}{\operatorname{sh}\frac{\pi}{4}x}\cdot\frac{dx}{1+x^2} = -\frac{\pi}{\sqrt{2}}e^{-a} + \frac{\operatorname{sh}a}{\sqrt{2}}\ \ln\frac{2\ \operatorname{ch}a+\sqrt{2}}{2\ \operatorname{ch}a-\sqrt{2}} + \sqrt{2}\ \operatorname{ch}a\ \operatorname{arctg}\frac{\sqrt{2}}{2\ \operatorname{sh}a} \qquad [a>0].$$

LI ((389))(1)

6. $$\int_0^\infty \frac{\sin ax}{\operatorname{ch}\frac{\pi}{4}x}\cdot\frac{x\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{\operatorname{sh}a}{\sqrt{2}}\ \ln\frac{2\ \operatorname{ch}a+\sqrt{2}}{2\ \operatorname{ch}a-\sqrt{2}} - \sqrt{2}\ \operatorname{ch}a\ \operatorname{arctg}\left(\frac{1}{\sqrt{2}\ \operatorname{sh}a}\right) \qquad [a>0].$$

BI ((388))(1)

BI ((389))(14), ET I 32(24)

8. $$\int_0^\infty \frac{\cos ax}{\operatorname{sh}\frac{\pi}{2}x}\cdot\frac{x\,dx}{1+x^2} = 2\operatorname{sh}a\operatorname{arctg}(e^{-a}) + \frac{\pi}{2}e^{-a} - 1 \qquad [a>0].$$

BI ((389))(11)

9. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}\pi x}\cdot\frac{x\,dx}{x^2+\beta^2} = \sum_{k=0}^{\infty}(-1)^k \frac{\left(k+\frac{1}{2}\right)^2 e^{-a\beta} - \beta e^{-(k+\frac{1}{2})a}}{\beta\left[\left(k+\frac{1}{2}\right)^2-\beta^2\right]} \qquad [\operatorname{Re}\beta>0, \quad a>0].$$

ET I 32(26)

10. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}\pi x}\cdot\frac{dx}{\left(m+\frac{1}{2}\right)^2+x^2} = \frac{(-1)^m e^{-(m+\frac{1}{2})a}}{2m+1}\left[a+\ln(1+e^{-a})\right] + \frac{e^{-\frac{a}{2}}}{2m+1}\sum_{k=0}^{m-1}\frac{(-1)^k e^{-ak}}{k-m} + \frac{e^{-\frac{a}{2}}}{(2m+1)(m+1)} \times {}_2F_1(1,\ m+1;\ m+2;\ -e^{-a}) \qquad [a>0].$$

ET I 32(25)

11. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}\pi x}\cdot\frac{dx}{1+x^2} = 2\operatorname{ch}\frac{a}{2} - \left[e^a\operatorname{arctg}\left(e^{-\frac{a}{2}}\right) + e^{-a}\operatorname{arctg}\left(e^{\frac{a}{2}}\right)\right] \qquad [a>0].$$

ET I 32(21)

12. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}\frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = ae^{-a} + \operatorname{ch}a\,\ln(1+e^{-2a}) \qquad [a>0].$$

BI ((388))(6)

13. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch}\frac{\pi}{4}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{2\operatorname{sh}a}{\sqrt{2}}\operatorname{arctg}\left(\frac{1}{\sqrt{2}\operatorname{sh}a}\right) - \frac{\operatorname{ch}a}{\sqrt{2}}\ln\frac{2\operatorname{ch}a+\sqrt{2}}{2\operatorname{ch}a-\sqrt{2}} \qquad [a>0].$$

BI ((388))(5)

4.114

1. $$\int_0^\infty \frac{\sin ax}{x}\frac{\operatorname{sh}\beta x}{\operatorname{sh}\gamma x}\,dx = \operatorname{arctg}\left(\operatorname{tg}\frac{\beta\pi}{2\gamma}\,\theta\frac{a\pi}{2\gamma}\right) \qquad [|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a > 0].$$

BI ((387))(6)a

2. $$\int_0^\infty \frac{\cos ax}{x}\frac{\operatorname{sh}\beta x}{\operatorname{ch}\gamma x}\,dx = \frac{1}{2}\ln\frac{\operatorname{ch}\dfrac{a\pi}{2\gamma} + \sin\dfrac{\beta\pi}{2\gamma}}{\operatorname{ch}\dfrac{a\pi}{2\gamma} - \sin\dfrac{\beta\pi}{2\gamma}} \qquad [|\operatorname{Re}\beta| < \operatorname{Re}\gamma].$$

ET I 33(34)

4.115

1. $$\int_0^\infty \frac{x\,\sin\,ax}{x^2+b^2}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{sh}\pi x}\,dx = \frac{\pi}{2}\frac{e^{-ab}\,\sin b\beta}{\sin b\pi} + \sum_{k=1}^{\infty}(-1)^k\frac{ke^{-ak}\,\sin k\beta}{k^2-b^2}$$
$$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0].$$

BI ((389))(23)

2. $$\int_0^\infty \frac{x\,\sin ax}{x^2+1}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{sh}\pi x}\,dx = \frac{1}{2}e^{-a}(a\,\sin\beta - \beta\,\cos\beta) - \frac{1}{2}\,\operatorname{sh}a\,\sin\beta\,\ln[1+2e^{-a}\,\cos\beta + e^{-2a}] +$$
$$+\operatorname{ch}a\,\cos\beta\,\operatorname{arctg}\frac{\sin\beta}{e^a+\cos\beta} \qquad [|\operatorname{Re}\beta| < \pi, \quad a > 0].$$

LI ((389))(10)

3. $$\int_0^\infty \frac{x\,\sin\,ax}{x^2+1}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{sh}\frac{\pi}{2}x}\,dx = \frac{\pi}{2}e^{-a}\,\sin\beta + \frac{1}{2}\,\cos\beta\,\operatorname{sh}a\,\ln\frac{\operatorname{ch}a+\sin\beta}{\operatorname{ch}a-\sin\beta} - \sin\beta\,\operatorname{ch}a\,\operatorname{arctg}\left(\frac{\cos\beta}{\operatorname{sh}a}\right)$$
$$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0\right].$$

BI ((389))(8)

4. $$\int_0^\infty \frac{\cos ax}{x^2+b^2}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{sh}\pi x}\,dx = \frac{\pi}{2b}\cdot\frac{e^{-ab}\,\sin b\beta}{\sin b\pi} + \sum_{k=1}^{\infty}(-1)^k\frac{e^{-ak}\,\sin k\beta}{k^2-b^2}$$
$$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0].$$

BI ((389))(22)

BI ((389))(20)a

6. $$\int_0^\infty \frac{\cos ax}{x^2+1}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{sh}\frac{\pi}{2}x}\,dx = \frac{\pi}{2}e^{-a}\sin\beta - \frac{1}{2}\operatorname{ch}a\cos\beta\ln\frac{\operatorname{ch}a+\sin\beta}{\operatorname{ch}a-\sin\beta} + \operatorname{sh}a\sin\beta\operatorname{arctg}\frac{\cos\beta}{\operatorname{sh}a}$$
$$\left[|\operatorname{Re}\beta| < \frac{\pi}{2},\quad a>0,\quad b>0\right].$$

BI ((389))(18)

7. $$\int_0^\infty \frac{\sin ax}{x^2+\frac{1}{4}}\cdot\frac{\operatorname{sh}\beta x}{\operatorname{ch}\pi x}\,dx = e^{-\frac{a}{2}}\left(a\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}\right) - \operatorname{sh}\frac{a}{2}\sin\frac{\beta}{2}\ln(1+2e^{-a}\cos\beta+e^{-2a}) +$$
$$+\operatorname{ch}\frac{a}{2}\cos\frac{\beta}{2}\operatorname{arctg}\frac{\sin\beta}{1+e^{-a}\cos\beta}\qquad [|\operatorname{Re}\beta|<\pi,\quad a>0].$$

ET I 91(26)

8. $$\int_0^\infty \frac{\sin ax}{x^2+\beta^2}\cdot\frac{\operatorname{ch}\gamma x}{\operatorname{sh}\pi x}\,dx = \frac{1}{2\beta^2} - \frac{\pi}{2\beta}\cdot\frac{e^{-a\beta}\cos\beta\gamma}{\sin\beta\pi} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-ak}\cos k\gamma}{k^2-\beta^2}$$
$$[0\le\operatorname{Re}\beta,\quad |\operatorname{Re}\gamma|<\pi,\quad a>0].$$

BI ((389))(21)

9. $$\int_0^\infty \frac{\sin ax}{x^2+1}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{sh}\pi x}\,dx = -\frac{1}{2}e^{-a}(a\cos\beta+\beta\sin\beta) + \frac{1}{2}\operatorname{sh}a\cos\beta\ln(1+2e^{-a}\cos\beta+e^{-2a}) +$$
$$+\operatorname{ch}a\sin\beta\operatorname{arctg}\frac{\sin\beta}{e^a+\cos\beta}\qquad [|\operatorname{Re}\beta|<\pi,\quad a>0].$$

ET I 91(25), LI ((389))(9)

10. $$\int_0^\infty \frac{\sin ax}{x^2+1}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{sh}\frac{\pi}{2}x}\,dx = -\frac{\pi}{2}e^{-a}\cos\beta + \frac{1}{2}\operatorname{sh}a\sin\beta\ln\frac{\operatorname{ch}a+\sin\beta}{\operatorname{ch}a-\sin\beta} + \operatorname{ch}a\cos\beta\operatorname{arctg}\frac{\cos\beta}{\operatorname{sh}a}$$
$$\left[|\operatorname{Re}\beta|<\frac{\pi}{2},\quad a>0\right],$$

BI ((389))(7)

11. $$\int_0^\infty \frac{x\cos ax}{x^2+b^2}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{sh}\pi x}\,dx = \frac{\pi}{2}\cdot\frac{e^{-ab}\cos b\beta}{\sin b\pi} + \sum_{k=1}^{\infty}(-1)^k\frac{ke^{-ak}\cos k\beta}{k^2-b^2}\qquad [|\operatorname{Re}\beta|<\pi,\quad a>0].$$

12. $$\int_0^\infty \frac{x\,\cos ax}{x^2+1}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{sh}\pi x}\,dx = \frac{1}{2}e^{-a}(a\,\cos\beta+\beta\,\sin\beta) - -\frac{1}{2}+\frac{1}{2}\,\operatorname{ch}a\,\cos\beta\,\ln[1+2e^{-a}\,\cos\beta+e^{-2a}] + +\operatorname{sh}a\,\sin\beta\,\operatorname{arctg}\frac{\sin\beta}{e^a+\cos\beta} \qquad [|\operatorname{Re}\beta|<\pi, \quad a>0].$$

BI ((389))(19)

13. $$\int_0^\infty \frac{x\,\cos ax}{x^2+1}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{sh}\frac{\pi}{2}x}\,dx = -1+\frac{\pi}{2}e^{-a}\,\cos\beta+\frac{1}{2}\,\operatorname{ch}a\,\sin\beta\,\ln\frac{\operatorname{ch}a+\sin\beta}{\operatorname{ch}a-\sin\beta} +\operatorname{sh}a\,\cos\beta\,\operatorname{arctg}\frac{\cos\beta}{\operatorname{sh}a} \qquad \left[|\operatorname{Re}\beta|<\frac{\pi}{2}, \quad a>0\right].$$

BI ((389))(17)

14. $$\int_0^\infty \frac{\cos ax}{x^2+1}\cdot\frac{\operatorname{ch}\beta x}{\operatorname{ch}\frac{\pi}{2}x}\,dx = ae^{-a}\,\cos\beta+\beta e^{-a}\,\sin\beta+\operatorname{sh}a\,\sin\beta\,\operatorname{arctg}\frac{e^{-2a}\,\sin 2\beta}{1+e^{-2a}\,\cos 2\beta} + +\frac{1}{2}\,\operatorname{ch}a\,\cos\beta\,\ln(1+2e^{-2a}\,\cos 2\beta+e^{-4a}) \qquad \left[|\operatorname{Re}\beta|<\frac{\pi}{2},\ a>0\right].$$

ET I 34(37)

4.116

1.[6] $\int_0^\infty x\,\cos 2ax\,\theta x\,dx$ the integral is divergent.

BI ((364))(2)

2. $$\int_0^\infty \cos ax\,\theta\beta x\frac{dx}{x} = \ln\operatorname{cth}\frac{a\pi}{4\beta} \qquad [\operatorname{Re}\beta>0, \quad a>0].$$

BI ((387))(8)

4.117

1. $$\int_0^\infty \frac{\sin ax}{1+x^2}\,\theta\frac{\pi x}{2}\,dx = a\,\operatorname{ch}a-\operatorname{sh}a\,\ln(2\,\operatorname{sh}a) \qquad [a>0].$$

BI ((388))(3)

2. $$\int_0^\infty \frac{\sin ax}{1+x^2}\,\theta\frac{\pi x}{4}\,dx = -\frac{\pi}{2}e^a+\operatorname{sh}a\,\ln\operatorname{cth}\frac{a}{2}+2\,\operatorname{ch}a\,\operatorname{arctg}(e^a).$$

3. $\int_0^\infty \frac{\sin ax}{1+x^2}\ \mathrm{cth}\ \pi x\, dx = \frac{a}{2}e^{-a} - \mathrm{sh}\, a\ \ln(1-e^{-a}) \qquad [a>0].$

BI ((389))(5)

4. $\int_0^\infty \frac{\sin ax}{1+x^2}\ \mathrm{cth}\ \frac{\pi}{2}x\, dx = \mathrm{sh}\, a\ \ln \mathrm{cth}\ \frac{a}{2} \qquad [a>0].$

BI ((389))(6)

5. $\int_0^\infty \frac{x\ \cos ax}{1+x^2}\ \theta\frac{\pi}{2}x\, dx = -ae^{-a} - \mathrm{ch}\, a\ \ln(1-e^{-2a}) \qquad [a>0].$

BI ((388))(7)

6. $\int_0^\infty \frac{x\ \cos ax}{1+x^2}\ \theta\frac{\pi}{4}x\, dx - \frac{\pi}{2}e^{a} + \mathrm{ch}\, a\ \ln \mathrm{cth}\ \frac{a}{2} + 2\ \mathrm{sh}\, a\ \mathrm{arctg}(e^{a}) \qquad [a>0].$

BI ((388))(8)

7. $\int_0^\infty \frac{x\ \cos ax}{1+x^2}\ \mathrm{cth}\ \pi x\, dx = -\frac{a}{2}e^{-a} - \frac{1}{2} - \mathrm{ch}\, a\ \ln(1-e^{-a}).$

BI ((389))(15)A, ET I 33(31)a

8. $\int_0^\infty \frac{x\ \cos ax}{1+x^2}\ \mathrm{cth}\ \frac{\pi}{2}x\, dx = -1 + \mathrm{ch}\, a\ \ln \mathrm{cth}\ \frac{a}{2} \qquad [a>0].$

BI ((389))(12)

9. $\int_0^\infty \frac{x\ \cos ax}{1+x^2}\ \mathrm{cth}\ \frac{\pi}{4}x\, dx = -2+\frac{\pi}{2}e^{-a}+\mathrm{ch}\, a\ \ln \mathrm{cth}\ \frac{a}{2}+2\ \mathrm{sh}\, a\ \mathrm{arctg}(e^{-a}) \qquad [a>0].$

BI ((389))(13)

4.118

$$\int_0^\infty \frac{x\ \sin ax}{\mathrm{ch}^2\ x}\, dx = -\frac{d}{da}\left(\frac{\pi a}{2\ \mathrm{sh}\ \frac{\pi a}{2}}\right) \qquad [a>0].$$

$$\int_0^\infty \frac{1-\cos px}{\operatorname{sh} qx} \cdot \frac{dx}{x} = \ln\left(\operatorname{ch}\frac{p\pi}{2q}\right).$$

BI ((387))(2)a

4.121

1. $$\int_0^\infty \frac{\sin ax - \sin bx}{\operatorname{ch} \beta x} \cdot \frac{dx}{x} = 2 \operatorname{arctg} \frac{\exp\dfrac{a\pi}{2\beta} - \exp\dfrac{b\pi}{2\beta}}{1+\exp\dfrac{(a+b)\pi}{2\beta}} \qquad [\operatorname{Re}\beta > 0].$$

GW ((336))(19b)

2. $$\int_0^\infty \frac{\cos ax - \cos bx}{\operatorname{sh} \beta x} \cdot \frac{dx}{x} = \ln \frac{\operatorname{ch}\dfrac{b\pi}{2\beta}}{\operatorname{ch}\dfrac{a\pi}{2\beta}} \qquad [\operatorname{Re}\beta > 0].$$

GW ((336))(19a)

4.122

1.[6] $$\int_0^\infty \frac{\cos \beta x \, \sin \gamma x}{\operatorname{ch} \delta x} \cdot \frac{dx}{x} = \operatorname{arctg} \frac{\operatorname{sh}\dfrac{\gamma\pi}{2\delta}}{\operatorname{ch}\dfrac{\beta\pi}{2\delta}} \qquad [\operatorname{Re}\delta > |\operatorname{Im}\beta| + |\operatorname{Im}\gamma|].$$

ET I 93(46)a

2. $$\int_0^\infty \sin^2 ax \frac{\operatorname{ch}\beta x}{\operatorname{sh} x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\operatorname{ch} 2a\pi + \cos\beta\pi}{1+\cos\beta\pi} \qquad [|\operatorname{Re}\beta| < 1].$$

BI ((387))(7)

4.123

1. $$\int_0^\infty \frac{\sin x}{\operatorname{ch} ax + \cos x} \cdot \frac{x\,dx}{x^2-\pi^2} = \operatorname{arctg}\frac{1}{a} - \frac{1}{a}.$$

BI ((390))(1)

2. $$\int_0^\infty \frac{\sin x}{\operatorname{ch} ax - \cos x} \cdot \frac{x\,dx}{x^2-\pi^2} = \frac{a}{1+a^2} - \operatorname{arctg}\frac{1}{a}.$$

3. $$\int_0^\infty \frac{\sin 2x}{\operatorname{ch} 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1 + 2a^2}{1 + a^2} - \operatorname{arctg} \frac{1}{a}.$$

BI ((390))(4)

4. $$\int_0^\infty \frac{\operatorname{ch} ax\, \sin x}{\operatorname{ch} 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{-1}{2a(1 + a^2)}.$$

LI ((390))(3)

5. $$\int_0^\infty \frac{\cos ax}{\operatorname{ch} \pi x + \cos \pi\beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma(\cos\gamma\pi + \cos\beta\pi)} + \\ + \frac{1}{\operatorname{sh}\beta\pi} \sum_{k=0}^\infty \left\{ \frac{\exp[-(2k+1-\beta)a]}{\gamma^2 - (2k+1-\beta)^2} - \frac{\exp[-(2k+1+\beta)a]}{\gamma^2 - (2k+1+\beta)^2} \right\}$$
$$[0 < \operatorname{Re}\beta < 1, \quad \operatorname{Re}\gamma > 0, \quad a > 0].$$

ET I 33(27)

6. $$\int_0^\infty \frac{\sin ax\, \operatorname{sh} bx}{\cos 2ax + \operatorname{ch} 2bx} x^{p-1}\,dx = \frac{\Gamma(p)}{(a^2 + b^2)^{\frac{p}{2}}} \sin\left(p \operatorname{arctg} \frac{a}{b}\right) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^p} \qquad [p > 0].$$

BI ((364))(8)

7. $$\int_0^\infty \sin ax^2 \frac{\sin \frac{\pi x}{2}\, \operatorname{sh} \frac{\pi x}{2}}{\cos \pi x + \operatorname{ch} \pi x} \cdot x\,dx = \frac{1}{4} \left[\frac{\partial \theta_1(z,\, q)}{\partial z} \right]_{z=0, q=e^{-2a}} \qquad [a > 0].$$

ET I 93(49)

4.124

1. $$\int_0^1 \frac{\cos px\, \operatorname{ch}(q\sqrt{1 - x^2})}{\sqrt{1 - x^2}}\,dx = \frac{\pi}{2} J_0(\sqrt{p^2 - q^2}).$$

MO (40)

2. $$\int_u^\infty \cos ax\, \operatorname{ch} \sqrt{\beta(u^2 - x^2)} \cdot \frac{dx}{\sqrt{u^2 - x^2}} = \frac{\pi}{2} J_0 \left(\frac{u}{\sqrt{a^2 - \beta^2}} \right).$$

ET I 34(38)

4.125

1. $$\int_0^\infty \operatorname{sh}(a\,\sin x)\,\cos(a\,\cos x)\,\sin x\,\sin 2nx\frac{dx}{x} = \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\,\frac{\pi}{8}\left[1+\frac{a^2}{2n(2n+1)}\right].$$

LI ((367))(14)

2. $$\int_0^\infty \operatorname{ch}(a\,\sin x)\,\cos(a\,\cos x)\,\sin x\,\cos(2n-1)x\frac{dx}{x} = \frac{(-1)^{n-1}a^{2(n-1)}}{[2(n-1)]!}\,\frac{\pi}{8}\left[1-\frac{a^2}{2n(2n-1)}\right].$$

LI ((367))(15)

3. $$\int_0^\infty \operatorname{sh}(a\,\sin x)\,\cos(a\,\cos x)\,\cos x\,\cos 2nx\frac{dx}{x} = \frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1}}{(2n+1)!}\,\frac{3\pi}{8} + \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\,\frac{\pi}{8}.$$

LI ((367))(21)

4.126

1. $$\int_0^\infty \sin(a\,\cos bx)\,\operatorname{sh}(a\,\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}[\cos(a\,\cos bc)\,\operatorname{ch}(a\,\sin bc)-1] \qquad [b>0].$$

BI ((381))(2)

2. $$\int_0^\infty \sin(a\,\cos bx)\,\operatorname{ch}(a\,\sin bx)\frac{dx}{c^2-x^2} = \frac{\pi}{2c}\,\cos(a\,\cos bc)\,\operatorname{sh}(a\,\sin bc) \qquad [b>0, \quad c>0].$$

BI ((381))(1)

3. $$\int_0^\infty \cos(a\,\cos bx)\,\operatorname{sh}(a\,\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}[a\,\cos bc-\sin(a\,\cos bc)\,\operatorname{ch}(a\,\sin bc)] \qquad [b>0].$$

BI ((381))(4)

4. $$\int_0^\infty \cos(a\,\cos bx)\,\operatorname{ch}(a\,\sin bx)\frac{dx}{c^2-x^2} = -\frac{\pi}{2c}\,\sin(a\,\cos bc)\,\operatorname{sh}(a\,\sin bc) \qquad [b>0].$$

BI ((381))(3)

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

$$1.\quad \int_0^\infty \sin ax \,\operatorname{sh}^\nu \gamma x e^{-\beta x}\,dx = -\frac{i\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu+ai}{2\gamma}+1\right)}\right\}$$
$$[\operatorname{Re}\nu > -2, \quad \operatorname{Re}\gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re}\beta].$$

ET I 91(30)a

$$2.\quad \int_0^\infty \cos ax \,\operatorname{sh}^\nu \gamma x e^{-\beta x}\,dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)}\right\}$$
$$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re}\beta].$$

ET I 34(40)a

$$3.\quad \int_0^\infty e^{-\beta x}\frac{\sin ax}{\operatorname{sh}\gamma x}\,dx = \sum_{k=1}^\infty \frac{2a}{a^2+[\beta+(2k-1)\gamma]^2};$$
$$= \frac{1}{2\gamma i}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right) - \psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right] \qquad [\operatorname{Re}\beta > |\operatorname{Re}\gamma|].$$

ET I 91(28)
BI ((264))(9)a

$$4.\quad \int_0^\infty e^{-x}\frac{\sin ax}{\operatorname{sh}x}\,dx = \frac{\pi}{2}\operatorname{cth}\frac{a\pi}{2} - \frac{1}{a}.$$

ET I 91(29)

4.132

$$1.\quad \int_0^\infty \frac{\sin ax \,\operatorname{sh}\beta x}{e^{\gamma x}-1}\,dx = -\frac{a}{2(a^2+\beta^2)} + \frac{\pi}{2\gamma}\cdot\frac{\operatorname{sh}\frac{2\pi a}{\gamma}}{\operatorname{ch}\frac{2\pi a}{\gamma} - \cos\frac{2\pi\beta}{\gamma}} +$$
$$+ \frac{i}{2\gamma}\left[\psi\left(\frac{\beta}{\gamma}+i\frac{a}{\gamma}+1\right) - \psi\left(\frac{\beta}{\gamma}-i\frac{a}{\gamma}+1\right)\right] \quad [\operatorname{Re}\gamma > |\operatorname{Re}\beta|,\ a>0].$$

2. $$\int_0^\infty \frac{\sin ax \ \mathrm{ch}\, \beta x}{e^{\gamma x} - 1}\, dx = -\frac{a}{2(a^2+\beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\mathrm{sh}\, \frac{2\pi a}{\gamma}}{\mathrm{ch}\, \frac{2\pi a}{\gamma} - \cos \frac{2\pi\beta}{\gamma}} \qquad [\mathrm{Re}\, \gamma > |\, \mathrm{Re}\, \beta|].$$

BI ((265))(5)a

3. $$\int_0^\infty \frac{\sin ax \ \mathrm{ch}\, \beta x}{e^{\gamma x} + 1}\, dx = \frac{a}{2(a^2+\beta^2)} - \frac{\pi}{\gamma} \cdot \frac{\mathrm{sh}\, \frac{a\pi}{\gamma} \ \cos \frac{\beta\pi}{\gamma}}{\mathrm{ch}\, \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \qquad [\mathrm{Re}\, \gamma > |\, \mathrm{Re}\, \beta|].$$

ET I 92(35)

4. $$\int_0^\infty \frac{\cos ax \ \mathrm{sh}\, \beta x}{e^{\gamma x} - 1}\, dx = \frac{\beta}{2(a^2+\beta^2)} - \frac{\pi}{2\gamma} \cdot \frac{\sin \frac{2\pi\beta}{\gamma}}{\mathrm{ch}\, \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \qquad [\mathrm{Re}\, \gamma > |\, \mathrm{Re}\, \beta|].$$

LI ((265))(8)

5. $$\int_0^\infty \frac{\cos ax \ \mathrm{sh}\, \beta x}{e^{\gamma x} + 1}\, dx = -\frac{\beta}{2(a^2+\beta^2)} + \frac{\pi}{\gamma} \cdot \frac{\sin \frac{\pi\beta}{\gamma} \ \mathrm{ch}\, \frac{\pi a}{\gamma}}{\mathrm{ch}\, \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \qquad [\mathrm{Re}\, \gamma > |\, \mathrm{Re}\, \beta|].$$

ET I 34(39)

4.133

1. $$\int_0^\infty \sin ax \ \mathrm{sh}\, \beta x \ \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma}\ \exp \gamma(\beta^2 - a^2)\ \sin(2a\beta\gamma) \qquad [\mathrm{Re}\, \gamma > 0].$$

ET I 92(37)

2. $$\int_0^\infty \cos ax \ \mathrm{ch}\, \beta x \ \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma}\ \exp \gamma(\beta^2 - a^2)\ \cos(2a\beta\gamma) \qquad [\mathrm{Re}\, \gamma > 0].$$

ET I 35(41)

4.134

1. $$\int_0^\infty e^{-\beta x^2} (\mathrm{ch}\, x - \cos x)\, dx = \sqrt{\frac{\pi}{\beta}}\ \mathrm{ch}\, \frac{1}{4\beta} \qquad [\mathrm{Re}\, \beta > 0].$$

ME 24

2. $$\int_0^\infty e^{-\beta x^2} (\mathrm{ch}\, x - \cos x)\, dx = \sqrt{\frac{\pi}{\beta}}\ \mathrm{sh}\, \frac{1}{4\beta} \qquad [\mathrm{Re}\, \beta > 0].$$

4.135

1. $$\int_0^\infty \sin ax^2 \operatorname{ch} 2\gamma x e^{-\beta x^2}\, dx = \frac{1}{2}\sqrt[4]{\frac{\pi^2}{a^2+\beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right) \sin\left(\frac{a\gamma^2}{a^2+\beta^2} + \frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

LI ((268))(7)

2. $$\int_0^\infty \cos ax^2 \operatorname{ch} 2\gamma x e^{-\beta x^2}\, dx = \frac{1}{2}\sqrt[4]{\frac{\pi^2}{a^2+\beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right) \cos\left(\frac{a\gamma^2}{a^2+\beta^2} + \frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

LI ((268))(8)

4.136

1. $$\int_0^\infty (\operatorname{sh} x^2 + \sin x^2) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{ch} \frac{1}{8\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

2. $$\int_0^\infty (\operatorname{sh} x^2 - \sin x^2) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{sh} \frac{1}{8\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

3. $$\int_0^\infty (\operatorname{ch} x^2 + \cos x^2) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{ch} \frac{1}{8\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

4. $$\int_0^\infty (\operatorname{ch} x^2 - \cos x^2) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{sh} \frac{1}{8\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

4.137

1. $$\int_0^\infty \sin 2x^2 \operatorname{sh} 2x^2 e^{-\beta x^4}\, dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} + \frac{\pi}{4}\right) \qquad [\operatorname{Re}\beta > 0].$$

2. $\int_0^\infty \sin 2x^2 \,\operatorname{ch} 2x^2 e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

554

3. $\int_0^\infty \cos 2x^2 \,\operatorname{sh} 2x^2 e^{-\beta x^4}\,dx = \frac{-\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

4. $\int_0^\infty \cos 2x^2 \,\operatorname{ch} 2x^2 e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} + \frac{\pi}{4}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

4.138

1. $\int_0^\infty (\sin 2x^2 \,\operatorname{ch} 2x^2 + \cos 2x^2 \,\operatorname{sh} 2x^2) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

2. $\int_0^\infty (\sin 2x^2 \,\operatorname{ch} 2x^2 - \cos 2x^2 \,\operatorname{sh} 2x^2) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

3. $\int_0^\infty (\cos 2x^2 \,\operatorname{ch} 2x^2 + \sin 2x^2 \,\operatorname{sh} 2x^2) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

4. $\int_0^\infty (\cos 2x^2 \,\operatorname{ch} 2x^2 - \sin 2x^2 \,\operatorname{sh} 2x^2) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

MI 32

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.141

1. $$\int_0^\infty x e^{-\beta x^2} \operatorname{ch} x \sin x \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta} + \sin\frac{1}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

MI 32

2. $$\int_0^\infty x e^{-\beta x^2} \operatorname{sh} x \cos x \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta} - \sin\frac{1}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

MI 32

3. $$\int_0^\infty x^2 e^{-\beta x^2} \operatorname{ch} x \cos x \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta} - \frac{1}{\beta}\sin\frac{1}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

MI 32

4. $$\int_0^\infty x^2 e^{-\beta x^2} \operatorname{sh} x \sin x \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\sin\frac{1}{2\beta} + \frac{1}{\beta}\cos\frac{1}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

MI 32

4.142

1. $$\int_0^\infty x e^{-\beta x^2} (\operatorname{sh} x + \sin x) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \operatorname{ch}\frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

2. $$\int_0^\infty x e^{-\beta x^2} (\operatorname{sh} x - \sin x) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \operatorname{sh}\frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0].$$

ME 24

3. $$\int_0^\infty x^2 e^{-\beta x^2} (\operatorname{ch} x + \cos x) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\left(\operatorname{ch}\frac{1}{4\beta} + \frac{1}{2\beta}\operatorname{sh}\frac{1}{4\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

ME 24

4. $$\int_0^\infty x^2 e^{-\beta x^2} (\operatorname{ch} x - \cos x) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\left(\operatorname{sh}\frac{1}{4\beta} + \frac{1}{2\beta}\operatorname{ch}\frac{1}{4\beta}\right) \qquad [\operatorname{Re}\beta > 0].$$

4.143

1. $$\int_0^\infty xe^{-\beta x^2}(\operatorname{ch} x\,\sin x+\operatorname{sh} x\,\cos x)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\cos\frac{1}{2\beta}\qquad[\operatorname{Re}\beta>0].$$

MI 32

2. $$\int_0^\infty xe^{-\beta x^2}(\operatorname{ch} x\,\sin x-\operatorname{sh} x\,\cos x)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\sin\frac{1}{2\beta}\qquad[\operatorname{Re}\beta>0].$$

MI 32

4.144

$$\int_0^\infty e^{-x^2}\operatorname{sh} x^2\,\cos ax\frac{dx}{x^2}=\sqrt{\frac{\pi}{2}}e^{-\frac{a^2}{8}}-\frac{\pi a}{4}\left[1-\Phi\left(\frac{a}{\sqrt{8}}\right)\right]\qquad[a>0].$$

ET I 35(44)

4.145

1. $$\int_0^\infty xe^{-\beta x^2}\operatorname{ch}(2ax\,\sin t)\,\sin(2ax\,\cos t)\,dx=\frac{a}{2}\sqrt{\frac{\pi}{\beta^3}}\exp\left(-\frac{a^2}{\beta}\cos 2t\right)\cos\left(t-\frac{a^2}{\beta}\sin 2t\right)\qquad[\operatorname{Re}\beta>0].$$

BI ((363))(5)

2. $$\int_0^\infty xe^{-\beta x^2}\operatorname{sh}(2ax\,\sin t)\,\cos(2ax\,\cos t)\,dx=\frac{a}{2}\sqrt{\frac{\pi}{\beta^3}}\exp\left(-\frac{a^2}{\beta}\cos 2t\right)\sin\left(t-\frac{a^2}{\beta}\sin 2t\right)\qquad[\operatorname{Re}\beta>0].$$

BI ((363))(6)

4.146*

1.[8] $$\int_0^\infty e^{-\beta x^2}\operatorname{sh} ax\,\sin bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{a^2-b^2}{4\beta}\right)\sin\frac{ab}{2\beta}\qquad[\operatorname{Re}\beta>0].$$

2.[8] $$\int_0^\infty e^{-\beta x^2}\operatorname{ch} ax\,\cos bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{a^2-b^2}{4\beta}\right)\cos\frac{ab}{2\beta}\qquad[\operatorname{Re}\beta>0].$$

3. $$\int_0^\infty xe^{-\beta x^2}\operatorname{ch} ax\,\sin ax\,dx=\frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta}+\sin\frac{a^2}{2\beta}\right)\qquad[\operatorname{Re}\beta>0].$$

4. $\int_0^\infty xe^{-\beta x^2} \operatorname{sh} ax \cos ax \, dx = \frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta} - \sin\frac{a^2}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

5.[8] $\int_0^\infty x^2 e^{-\beta x^2} \operatorname{ch} ax \sin ax \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\sin\frac{a^2}{2\beta} + \frac{a^2}{\beta}\cos\frac{a^2}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

6.[8] $\int_0^\infty x^2 e^{-\beta x^2} \operatorname{ch} ax \cos ax \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{a^2}{2\beta} - \frac{a^2}{\beta}\sin\frac{a^2}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0].$

4.2- 4.4 Logarithmic Functions

4.21 Logarithmic Functions

4.211

1. $\int_e^\infty \frac{dx}{\ln\frac{1}{x}} = -\infty$

BI((33))(9)

2. $\int_0^u \frac{dx}{\ln x} = \operatorname{li} u$

FI III 653, FI II 606

4.212

1.[7] $\int_0^1 \frac{dx}{a + \ln x} = e^{-a}\operatorname{Ei}(a)[a > 0].$ (cf. note after **4**.212.9)

BI((31))(4)

2. $\int_0^1 \frac{dx}{a - \ln x} = -e^{a}\operatorname{Ei}(-a)[a > 0].$

3.[7] $\int_0^1 \frac{dx}{(a\ln x)^2} = -\frac{1}{a} + e^{-a}\operatorname{Ei}(a) \qquad [a>0].$

BI((31))(14)

4. $\int_0^1 \frac{dx}{(a-\ln x)^2} = \frac{1}{a} + e^{a}\operatorname{Ei}(-a) \qquad [a>0].$

BI((31))(16)

5.[8] $\int_0^1 \frac{\ln x\,dx}{(a+\ln x)^2} = 1 + (1-a)e^{-a}\operatorname{Ei}(a) \qquad [a>0].$

BI((31))(15)

6. $\int_0^1 \frac{\ln x\,dx}{(a-\ln x)^2} = 1 + (1+a)e^{a}\operatorname{Ei}(-a) \qquad [a>0].$

BI((31))(17)

7. $\int_1^e \frac{\ln x\,dx}{(1+\ln x)^2} = \frac{e}{2} - 1.$

BI((33))(10)

8.[7] $$\int_0^1 \frac{dx}{(a+\ln x)^n} = \frac{1}{(n-1)!}e^{-a}\operatorname{Ei}(a) - \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(n-k-1)!a^{k-n} \qquad [a>0, n \text{ odd}].$$

BI((31))(22)

9. $$\int_0^1 \frac{dx}{(a-\ln x)^n} = \frac{(-1)^n}{(n-1)!}e^{a}\operatorname{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=1}^{n-1}(n-k-1)!(-a)^{k-n} \qquad [a>0, n \text{ odd}].$$

BI((31))(23)

In integrals of the form $\int \frac{(\ln x)^m}{[a^n+(\ln x)^n]^l}\,dx$ it is convenient to make the substitution $x = e^{-t}$.

Results 4.212.3, 4.212.5, and 4.212.8 [for $n > 1$] and 4.213.6., 4.213.8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z)\,dz}{(z-z_0)^n} = \frac{1}{(n-1)!}\left(\frac{d}{dz_0}\right)^{n-1} P\int_0^a \frac{f(z)}{z-z_0}\,dz$$

where $a > z_0 > 0,\ n = 1,\ 2,\ 3, \ldots$ and P indicates the Cauchy principal value.

4.213

1. $$\int_0^1 \frac{dx}{a^2+(\ln x)^2} = \frac{1}{a}[\operatorname{ci}(a)\sin a - \operatorname{si}(a)\cos a] \qquad [a>0].$$

BI ((31))(6)

2.[7] $$\int_0^1 \frac{dx}{a^2-(\ln x)^2} = \frac{1}{2a}[e^{-a}\operatorname{Ei}(a) - e^a \operatorname{Ei}(-a)] \qquad [a>0], \quad \text{(cf. 4.212 1. and 2.)}$$

4.212
BI ((31))(8)

3. $$\int_0^1 \frac{\ln x\,dx}{a^2+(\ln x)^2} = \operatorname{ci}(a)\cos a + \operatorname{si}(a)\sin a \qquad [a>0].$$

BI ((31))(7)

4.[7] $$\int_0^1 \frac{\ln x\,dx}{a^2-(\ln x)^2} = -\frac{1}{2}[e^{-a}\operatorname{Ei}(a) + e^a \operatorname{Ei}(-a)] \qquad [a>0], \quad \text{(cf. 4.212 1. and 2.).}$$

4.212
BI ((31))(9)

5. $$\int_0^1 \frac{dx}{[a^2+(\ln x)^2]^2} = \frac{1}{2a^3}[\operatorname{ci}(a)\sin a - \operatorname{si}(a)\cos a] - \frac{1}{2a^2}[\operatorname{ci}(a)\cos a + \operatorname{si}(a)\sin a] \qquad [a>0].$$

6.[7] $$\int_0^1 \frac{dx}{[a^2-(\ln x)^2]^2} = \frac{1}{4a^3}[(a-1)e^a \operatorname{Ei}(-a) + (1+a)e^{-a} \operatorname{Ei}(a)] \qquad [a>0].$$

BI ((31))(20)

7. $$\int_0^1 \frac{\ln x\, dx}{[a^2+(\ln x)^2]^2} = \frac{1}{2a}[\operatorname{ci}(a)\sin a - \operatorname{si}(a)\cos a] - \frac{1}{2a^2} \qquad [a>0].$$

BI ((31))(19)

8.[7] $$\int_0^1 \frac{\ln x\, dx}{[a^2-(\ln x)^2]^2} = \frac{1}{4a^2}\{2 + a[e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a)]\} \qquad [a>0].$$

LI ((31))(21)

4.214

1. $$\int_0^1 \frac{dx}{a^4-(\ln x)^4} = -\frac{1}{4a^3}[e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a) - 2\operatorname{ci}(a)\sin a + 2\operatorname{si}(a)\cos a] \qquad [a>0].$$

BI ((31))(10)

2. $$\int_0^1 \frac{\ln x\, dx}{a^4-(\ln x)^4} = -\frac{1}{4a^2}[e^a \operatorname{Ei}(-a) + e^{-a} \operatorname{Ei}(a) - 2\operatorname{ci}(a)\cos a - 2\operatorname{si}(a)\sin a] \qquad [a>0].$$

BI ((31))(11)

3. $$\int_0^1 \frac{(\ln x)^2\, dx}{a^4-(\ln x)^4} = -\frac{1}{4a}[e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a) + 2\operatorname{ci}(a)\sin a - 2\operatorname{si}(a)\cos a] \qquad [a>0].$$

BI ((31))(12)

4.[7] $$\int_0^1 \frac{(\ln x)^3\, dx}{a^4-(\ln x)^4} = -\frac{1}{4}[e^a \operatorname{Ei}(-a) + e^{-a} \operatorname{Ei}(a) + 2\operatorname{ci}(a)\cos a + 2\operatorname{si}(a)\sin a] \qquad [a>0].$$

BI ((31))(13)

4.215

1. $$\int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} dx = \Gamma(\mu) \qquad [\operatorname{Re}\mu > 0].$$

2. $$\int_0^1 \frac{dx}{\left(\ln \frac{1}{x}\right)^{\mu}} = \frac{\pi}{\Gamma(\mu)} \operatorname{cosec} \mu\pi \qquad [\operatorname{Re} \mu < 1].$$

BI ((31))(1)

3. $$\int_0^1 \sqrt{\ln \frac{1}{x}}\, dx = \frac{\sqrt{\pi}}{2}.$$

BI ((32))(1)

4. $$\int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi}.$$

BI ((32))(3)

4.216

$$\int_0^{\frac{1}{e}} \frac{dx}{\sqrt{(\ln x)^2 - 1}} = K_0(1).$$

GW ((32))(2)

4.22 Logarithms of more complicated arguments

4.221

1. $$\int_0^1 \ln x \ln(1-x)\, dx = 2 - \frac{\pi^2}{6}.$$

BI ((30))(7)

2. $$\int_0^1 \ln x \ln(1+x)\, dx = 2 - \frac{\pi^2}{12} - 2\ln 2.$$

BI ((30))(8)

3. $$\int_0^1 \ln \frac{1-ax}{1-a} \frac{dx}{\ln x} = -\sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \qquad [a < 1].$$

1. $\int_0^\infty \ln \frac{a^2+x^2}{b^2+x^2}\,dx = (a-b)\pi \qquad [a>0, \quad b>0].$

GW ((322))(20)

2. $\int_0^\infty \ln x \ln \frac{a^2+x^2}{b^2+x^2}\,dx = \pi(b-a) + \pi \ln \frac{a^a}{b^b} \qquad [a>0, \quad b>0].$

BI ((33))(1)

3. $\int_0^\infty \ln x \ln\left(1+\frac{b^2}{x^2}\right)\,dx = \pi b(\ln b - 1) \qquad [b>0].$

BI ((33))(2)

4. $\int_0^\infty \ln(1+a^2x^2) \ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab) - b\right] \qquad [a>0, \quad b>0].$

BI ((33))(3)

559

5. $\int_0^\infty \ln(a^2+x^2) \ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi[(a+b)\ln(a+b) - a\ln a - b] \qquad [a>0, \quad b>0].$

BI ((33))(4)

6. $\int_0^\infty \ln\left(1+\frac{a^2}{x^2}\right) \ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi[(a+b)\ln(a+b) - a\ln a - b\ln b] \qquad [a>0, \quad b>0].$

BI ((33))(5)

7. $\int_0^\infty \ln\left(a^2+\frac{1}{x^2}\right) \ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab) - b\ln b\right] \qquad [a>0, \quad b>0].$

BI ((33))(7)

4.223

1. $\int_0^\infty \ln(1+e^{-x})\,dx = \frac{\pi^2}{12}.$

2. $\int_0^{\infty} \ln(1 - e^{-x})\, dx = -\frac{\pi^2}{6}.$

BI ((256))(11)

3. $\int_0^{\infty} \ln(1 + 2e^{-x} \cos t + e^{-2x})\, dx = \frac{\pi^2}{6} - \frac{t^2}{2} \qquad [|t| < \pi].$

BI ((256))(18)

4.224

1. $\int_0^{u} \ln \sin x\, dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right).$

LO III 186(15)

2. $\int_0^{\frac{\pi}{4}} \ln \sin x\, dx = -\frac{\pi}{4} \ln 2 - \frac{1}{2} \boldsymbol{G}.$

BI ((285))(1)

3. $\int_0^{\frac{\pi}{2}} \ln \sin x\, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x\, dx = -\frac{\pi}{2} \ln 2.$

FI II 629,643

4. $\int_0^{u} \ln \cos x\, dx = -L(u).$

LO III 184(10)

5. $\int_0^{\frac{\pi}{4}} \ln \cos x\, dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} \boldsymbol{G}.$

BI ((286))(1)

6. $\int_0^{\frac{\pi}{2}} \ln \cos x\, dx = -\frac{\pi}{2} \ln 2.$

BI 306(1)

7. $\int_0^{\frac{\pi}{2}} (\ln \sin x)^2\, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12}\right].$

BI ((305))(19)

8. $$\int_0^{\frac{\pi}{2}} (\ln\cos x)^2\, dx = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right].$$

BI ((306))(14)

9. $$\int_0^{\pi} \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0].$$

GW ((322))(15)

10. $$\int_0^{\pi} \ln(1 \pm \sin x)\, dx = -\pi \ln 2 \pm 4\boldsymbol{G}.$$

GW ((322))(16a)

11.[7] $$\int_0^{\frac{\pi}{2}} \ln(1 + a\sin x)^2\, dx = \pi\ \ln(a/2) + 4\mathbf{G} + 4\sum_{k=1}^{\infty} \frac{bk}{k} \sum_{n=1}^{k} \frac{(-1)^{n+1}}{2n-1} \qquad b = \frac{(1-a)}{(1+a)} \qquad [a > 0]$$
$$= -\pi\ \ln 2 - 4\mathbf{G} \qquad [a = -1].$$

BI ((308))(5, 6, 7, 8)

12. $$\int_0^{\pi} \ln(1 + a\cos x)^2\, dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2}}{2} \qquad [a^2 \leq 1].$$

BI ((330))(1)

13. $$\int_0^{\frac{\pi}{2}} \ln(1 + 2a\sin x + a^2)\, dx = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k+1)\cdot(2k+1)!!} \left(\frac{2a}{1+a^2}\right)^{2k+1} \qquad [a^2 \leq 1].$$

BI ((308))(24)

14.[8] $$\int_0^{n\pi} \ln(b^2 - 2ab\cos x + a^2)\, dx = n\pi \ln a^2 \qquad [a^2 \geq b^2 > 0];$$
$$= n\pi \ln b^2 \qquad [b^2 > a^2 > 0].$$

FI II 142, 163, 688

4.225

1. $$\int_0^{\frac{\pi}{4}} \ln(\cos x - \sin x)\, dx = -\frac{\pi}{8} \ln 2 - \frac{1}{2} \boldsymbol{G}.$$

GW ((322))(9b)

2. $$\int_0^{\frac{\pi}{4}} \ln(\cos x + \sin x)\, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}.$$

GW ((322))(9a)

3. $$\int_0^{2\pi} \ln(1 + a \sin x + b \cos x)\, dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2 - b^2}}{2} \qquad [a^2 + b^2 < 1].$$

BI ((332))(2)

4. $$\int_0^{2\pi} \ln(1 + a^2 + b^2 + 2a \sin x + 2b \cos x)\, dx = 0 \qquad [a^2 + b^2 \leq 1];$$
$$= 2\pi \ln(a^2 + b^2) \qquad [a^2 + b^2 \geq 1].$$

BI ((322))(3)

4.226

1. $$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)^2 dx = -2\pi \ln 2 \qquad [a^2 \leq 1];$$
$$= 2\pi \ln \frac{a + \sqrt{a^2 - 1}}{2} = 2\pi(\operatorname{Arch}\ a - \ln 2) \qquad [a > 1].$$

FI II 644, 687

561

2. $$\int_0^{\frac{\pi}{2}} \ln(1 + a \sin^2 x)\, dx = \frac{1}{2} \int_0^{\pi} \ln(1 + a \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} \ln(1 + a \cos^2 x)\, dx =$$
$$= \frac{1}{2} \int_0^{\pi} \ln(1 + a \cos^2 x)\, dx = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \qquad [a \geq -1].$$

BI ((308))(15), GW((322))(12)

3. $$\int_0^{u} \ln(1 - \sin^2 \alpha \sin^2 x)\, dx = (\pi - 2\theta) \ln \operatorname{ctg} \frac{\alpha}{2} + 2u \ln\left(\frac{1}{2} \sin \alpha\right) - \frac{\pi}{2} \ln 2 +$$
$$+ L(\theta + u) - L(\theta - u) + L\left(\frac{\pi}{2} - 2u\right)$$
$$\left[\operatorname{ctg} \theta = \cos \alpha \operatorname{tg} u; \quad -\pi < \alpha < \pi, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}\right]$$

LO III 287

4. $$\int_0^{\frac{\pi}{2}} \ln[1-\cos^2 x(\sin^2\alpha-\sin^2\beta\sin^2 x)]\,dx = \pi\ln\left[\frac{1}{2}\left(\cos^2\frac{\alpha}{2}+\sqrt{\cos^4\frac{\alpha}{2}+\sin^2\ \frac{\beta}{2}\cos^2\frac{\beta}{2}}\right)\right]$$
$$[\alpha > \beta > 0].$$

LO III 283

5. $$\int_0^u \ln\left(1-\frac{\sin^2 x}{\sin^2 a}\right)\,dx = -u\ln\sin^2\alpha - L\left(\frac{\pi}{2}-\alpha+u\right)+L\left(\frac{\pi}{2}-\alpha-u\right)$$
$$\left[-\frac{\pi}{2}\le u\le\frac{\pi}{2},\quad |\sin u|\le|\sin\alpha|\right].$$

LO III 287

6. $$\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x+b^2\sin^2 x)\,dx = \frac{1}{2}\int_0^{\pi}\ln(a^2\ \cos^2 x+b^2\sin^2 x)\,dx = \pi\ln\frac{a+b}{2}$$
$$[a>0,\quad b>0].$$

GW ((322))(13)

7. $$\int_0^{\frac{\pi}{2}} \ln\frac{1+\sin t\cos^2 x}{1-\sin t\cos^2 x}\,dx = \pi\ln\frac{1+\sin\dfrac{t}{2}}{\cos\dfrac{t}{2}} = \pi\ln\operatorname{ctg}\frac{\pi-t}{4}\qquad\left[|t|<\frac{\pi}{2}\right].$$

LO III 283

4.227

1. $$\int_0^u \ln\operatorname{tg} x\,dx = L(u)+L\left(\frac{\pi}{2}-u\right)-L\left(\frac{\pi}{2}\right).$$

LO III 186(16)

2. $$\int_0^{\frac{\pi}{4}} \ln\operatorname{tg} x\,dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\operatorname{tg} x\,dx = -\boldsymbol{G}.$$

BI ((286))(11)

3. $$\int_0^{\frac{\pi}{2}} \ln(a\ \operatorname{tg} x)\,dx = \frac{\pi}{2}\ln a\qquad[a>0].$$

4.[7] $$\int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^n \, dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} = \frac{1}{2} \left(\frac{\pi}{2}\right)^{n+1} |E_n| \qquad [n \text{ even}].$$

BI ((286))(21)

5.[7] $$\int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2n} \, dx \left(\frac{\pi}{2}^{2n+1} |E_{2n}|\right).$$

BI ((307))(15)

6. $$\int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2n+1} \, dx = 0.$$

BI ((307))(14)

7. $$\int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^2 \, dx = \frac{\pi^3}{16}.$$

BI ((286))(16)

8. $$\int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^4 \, dx = \frac{5}{64}\pi^5.$$

BI ((286))(19)

9. $$\int_0^{\frac{\pi}{4}} \ln(1 + \operatorname{tg} x) \, dx = \frac{\pi}{8} \ln 2.$$

BI ((287))(1)

10. $$\int_0^{\frac{\pi}{2}} \ln(1 + \operatorname{tg} x) \, dx = \frac{\pi}{4} \ln 2 + \boldsymbol{G}.$$

BI ((308))(9)

11. $$\int_0^{\frac{\pi}{4}} \ln(1 - \operatorname{tg} x) \, dx = \frac{\pi}{8} \ln 2 - \boldsymbol{G}.$$

BI ((287))(2)

BI ((308))(10)

13. $$\int_0^{\frac{\pi}{4}} \ln(1+\operatorname{ctg} x)\,dx = \frac{\pi}{8}\ln 2 + \boldsymbol{G}.$$

BI ((287))(3)

14. $$\int_0^{\frac{\pi}{4}} \ln(\operatorname{ctg} x - 1)\,dx = \frac{\pi}{8}\ln 2.$$

BI ((287))(4)

15. $$\int_0^{\frac{\pi}{4}} \ln(\operatorname{tg} x + \operatorname{ctg} x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\operatorname{tg} x + \operatorname{ctg} x)\,dx = \frac{\pi}{2}\ln 2.$$

BI ((287))(5), BI ((308))(11)

16. $$\int_0^{\frac{\pi}{4}} \ln(\operatorname{ctg} x - \operatorname{tg} x)^2\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\operatorname{ctg} x - \operatorname{tg} x)^2\,dx = \frac{\pi}{2}\ln 2.$$

BI ((287))(6), BI ((308))(12)

17. $$\int_0^{\frac{\pi}{2}} \ln(a^2 + b^2\ \operatorname{tg}^2 x)\,dx = \frac{1}{2}\int_0^{\pi} \ln(a^2 + b^2\ \operatorname{tg}^2 x)\,dx = \pi\ln(a+b)$$

$$[a > 0, \quad b > 0].$$

GW ((322))(17)

563

4.228

1. $$\int_0^{\frac{\pi}{2}} \ln(\sin t \sin x + \sqrt{1-\cos^2 t \sin^2 x})\,dx = \frac{\pi}{2}\ln 2 - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi - t}{2}\right).$$

LO III 290

2. $$\int_0^{u} \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t})\,dx = -\left(\frac{\pi}{2} - t - \varphi\right)\ln\cos t + \frac{1}{2}L(u+\varphi) - \frac{1}{2}L(u-\varphi) - L(\varphi)$$

$$\left[\cos\varphi = \frac{\sin u}{\sin t}; \quad 0 \le u \le t \le \frac{\pi}{2}\right].$$

3. $\int_0^t \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t})\,dx = -\left(\frac{\pi}{2} - t\right)\ln\cos t.$

LO III 285

4. $\int_0^u \ln \frac{\sin u + \sin t \cos x\sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}}\,dx = \pi \ln\left[\operatorname{tg}\frac{t}{2}\sin u + \sqrt{\operatorname{tg}^2\frac{t}{2}\sin^2 u + 1}\right]$

$[t > 0, \quad u > 0].$

LO III 283

5. $\int_0^{\frac{\pi}{4}} \sqrt{\ln \operatorname{ctg} x}\,dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}.$

BI ((297))(9)

6. $\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\ln\operatorname{ctg} x}} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}.$

BI ((304))(24)

7. $\int_0^{\frac{\pi}{4}} \ln(\sqrt{\operatorname{tg} x} + \sqrt{\operatorname{ctg} x})\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sqrt{\operatorname{tg} x} + \sqrt{\operatorname{ctg} x})\,dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}.$

BI ((287))(7), BI ((308))(22)

8. $\int_0^{\frac{\pi}{4}} \ln(\sqrt{\operatorname{ctg} x} - \sqrt{\operatorname{tg} x})^2\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sqrt{\operatorname{ctg} x} - \sqrt{\operatorname{tg} x})^2\,dx = \frac{\pi}{4}\ln 2 - \boldsymbol{G}.$

BI ((287))(8), BI ((308))(23)

4.229

1. $\int_0^1 \ln\left(\ln\frac{1}{x}\right)\,dx = -\boldsymbol{C}.$

FI II 807

2. $\int_0^1 \frac{dx}{\ln\left(\ln\frac{1}{x}\right)} = 0.$

3. $\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{\sqrt{\ln\frac{1}{x}}} = -(\boldsymbol{C}+2\ln 2)\sqrt{\pi}.$

BI ((32))(4)

4. $\int_0^1 \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{u-1} dx = \psi(\mu)\,\Gamma(\mu) \qquad [\operatorname{Re}\mu > 0].$

BI ((30))(10)

If the integrand contains ln $(\ln\frac{1}{x})$, it is convenient to make the substitution $\ln\frac{1}{x} = u$, i.e., $x = e^{-u}$.

5.[7] $\int_0^1 \ln(a+\ln x)\,dx = \ln a - e^{-a}\operatorname{Ei}(a) \qquad [a > 0].$

BI ((30))(5)

6. $\int_0^1 \ln(a-\ln x)\,dx = \ln a - e^{a}\operatorname{Ei}(-a) \qquad [a > 0].$

BI ((30))(6)

7. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\ln\operatorname{tg} x\,dx = \frac{\pi}{2}\ln\left\{\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\sqrt{2\pi}\right\}.$

BI ((308))(28)

4.23 Combinations of logarithms and rational functions

4.231

1. $\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}.$

FI II 483a

2. $\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}.$

FI II 714

BI ((108))(7)

4. $$\int_0^1 \frac{1+x}{1-x} \ln x \, dx = 1 - \frac{\pi^2}{3}.$$

BI ((108))(9)

5.[7] $$\int_0^\infty \frac{\ln x \, dx}{(x+a)^2} = \frac{\ln a}{a} \qquad [a > 0].$$

BI ((139))(1)

6. $$\int_0^1 \frac{\ln x}{(1+x)^2} \, dx = -\ln 2.$$

BI ((111))(1)

7.[7] $$\int_0^\infty \ln x \frac{dx}{(a^2+b^2x^2)^n} = \frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{4(n-1)!a^{2n-1}b} \left[2\ln\frac{a}{2b} - \mathbf{C} - \psi\left(n - \frac{1}{2}\right)\right] \qquad [a > 0, \quad b > 0].$$

LI ((139))(3)

8. $$\int_0^\infty \frac{\ln x \, dx}{a^2+b^2x^2} = \frac{\pi}{2ab} \ln \frac{a}{b} \qquad [ab > 0].$$

BI ((135))(6)

9. $$\int_0^\infty \frac{\ln px}{q^2+x^2} \, dx = \frac{\pi}{2q} \ln pq \qquad [p > 0, \quad q > 0].$$

BI ((135))(4)

10. $$\int_0^\infty \frac{\ln x \, dx}{a^2-b^2x^2} = -\frac{\pi^2}{4ab} \qquad [ab > 0].$$

LI ((324))(7b)

11. $$\int_0^a \frac{\ln x \, dx}{x^2+a^2} = \frac{\pi \ln a}{4a} - \frac{\boldsymbol{G}}{a} \qquad [a > 0].$$

12. $\int_0^1 \frac{\ln x}{1+x^2}\,dx = -\int_1^\infty \frac{\ln x}{1+x^2}\,dx = -\boldsymbol{G}.$

FI II 482, 614

13. $\int_0^1 \frac{\ln x\,dx}{1-x^2} = -\frac{\pi^2}{8}.$

BI ((108))(11)

14. $\int_0^1 \frac{x\ln x}{1+x^2}\,dx = -\frac{\pi^2}{48}.$

GW ((324))(7b)

15. $\int_0^1 \frac{x\ln x}{1-x^2}\,dx = -\frac{\pi^2}{24}.$

16. $\int_0^1 \ln x \frac{1-x^{2n+2}}{(1-x^2)^2}\,dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^2}.$

BI ((111))(5)

17. $\int_0^1 \ln x \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^{n} (-1)^k \frac{n-k+1}{k^2}$

BI ((111))(2)

18. $\int_0^1 \ln x \frac{1-x^{n+1}}{(1-x)^2}\,dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^{n} \frac{n-k+1}{k^2}.$

BI ((111))(3)

4.232

1. $\int_u^v \frac{\ln x\,dx}{(x+u)(x+v)} = \frac{\ln uv}{2(v-u)} \ln \frac{(u+v)^2}{4uv}.$

2. $\displaystyle\int_0^\infty \frac{\ln x\,dx}{(x+\beta)(x+\gamma)} = \frac{(\ln\beta)^2 - (\ln\gamma)^2}{2(\beta-\gamma)} \qquad [|\arg\beta| < \pi, \quad |\arg\gamma| < \pi].$

ET II 218(24)

3. $\displaystyle\int_0^\infty \frac{\ln x}{x+a}\frac{dx}{x-1} = \frac{\pi^2 + (\ln a)^2}{2(a+1)} \qquad [a > 0].$

BI ((140))(10)

4.233

1.[3] $\displaystyle\int_0^1 \frac{\ln x\,dx}{1+x+x^2} = \frac{2}{9}\left[\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right] = -0.781\,302\,4129\ldots$

LI ((113))(1)

2.[3] $\displaystyle\int_0^1 \frac{\ln x\,dx}{1-x+x^2} = \frac{1}{3}\left[\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right] = -1.171\,953\,619\,34\ldots$

LI ((113))(2)

3.[7] $\displaystyle\int_0^1 \frac{x\ln x\,dx}{1+x+x^2} = -\frac{1}{9}\left[\frac{7\pi^2}{6} - \psi'\left(\frac{1}{3}\right)\right] = -0.157\,660\,149\,17\ldots$

LI ((113))(2)

4.[3] $\displaystyle\int_0^1 \frac{x\ln x\,dx}{1-x+x^2} = \frac{1}{6}\left[\frac{5\pi^2}{6} - \psi'\left(\frac{1}{3}\right)\right] = -0.311\,821\,131\,9\ldots$

LI ((113))(4)

5. $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2 + 2xa\cos t + a^2} = \frac{t\ln a}{a\sin t} \qquad [a > 0, \quad 0 < t < \pi].$

GW ((324))(13c)

566

4.234

1. $\displaystyle\int_1^\infty \frac{\ln x\,dx}{(1+x^2)^2} = \ln 2.$

2. $\int_0^1 \frac{x \ln x \, dx}{(1+x^2)^2} = -\frac{1}{4} \ln 2.$

BI ((111))(4)

3. $\int_0^\infty \frac{1+x^2}{(1-x^2)^2} \ln x \, dx = 0.$

BI ((142))(2)a

4. $\int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x \, dx = -\frac{\pi}{2}.$

BI ((142))(1)a

5. $\int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})}.$

BI ((112))(21)

6. $\int_0^\infty \frac{\ln x \, dx}{(a^2+b^2x^2)(1+x^2)} = \frac{b\pi}{2a(b^2-a^2)} \ln \frac{a}{b} \qquad [ab > 0].$

BI ((317))(16)a

7. $\int_0^\infty \frac{\ln x}{x^2+a^2} \cdot \frac{dx}{1+b^2x^2} = \frac{\pi}{2(1-a^2b^2)} \left(\frac{1}{a} \ln a + b \ln b \right) \qquad [a > 0, \quad b > 0].$

LI ((140))(12)

8. $\int_0^\infty \frac{x^2 \ln x \, dx}{(a^2+b^2x^2)(1+x^2)} = \frac{a\pi}{2b(b^2-a^2)} \ln \frac{b}{a} \qquad [ab > 0].$

LI ((140))(12), BI ((317))(15)a

4.235

1. $\int_0^\infty \ln x \frac{(1-x)x^{n-2}}{1-x^{2n}} \, dx = -\frac{\pi^2}{4n^2} \, \operatorname{tg}^2 \frac{\pi}{2n} \qquad [n > 1].$

BI ((135))(10)

2. $\int_0^\infty \ln x \frac{(1-x^2)x^{m-1}}{1-x^{2n}} \, dx = -\frac{\pi^2 \sin \frac{m+1}{n}\pi \sin \frac{\pi}{n}}{4n^2 \sin^2 \frac{m\pi}{2n} \, \sin^2 \left(\frac{m+2}{2n}\pi \right)}$

LI ((135))(12)

3. $$\int_0^\infty \ln x \frac{(1-x^2)x^{n-2}}{1-x^{2n}}\,dx = -\frac{\pi^2}{4n^2}\ \mathrm{tg}^2\,\frac{\pi}{n} \qquad [n>2].$$

BI ((135))(11)

4. $$\int_0^1 \ln x \frac{x^{m-1}+x^{n-m-1}}{1-x^n}\,dx = -\frac{\pi^2}{n^2\sin^2\left(\frac{m}{n}\pi\right)} \qquad [n>m].$$

BI ((108))(15)

4.236

1. $$\int_0^1 \left\{\frac{1+(p-1)\ln x}{1-x} + \frac{x\ln x}{(1-x)^2}\right\} x^{p-1}\,dx = -1+\psi'(p) \qquad [p>0].$$

BI ((111))(6)A, GW ((326))(13)

2. $$\int_0^1 \left[\frac{1}{1-x} + \frac{x\ln x}{(1-x)^2}\right]\,dx = \frac{\pi^2}{6} - 1.$$

GW ((326))(13a)

4.24 Combinations of logarithms and algebraic functions

4.241

1. $$\int_0^1 \frac{x^{2n}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right).$$

BI ((118))(5)a

2. $$\int_0^1 \frac{x^{2n+1}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n)!!}{(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right).$$

BI ((118))(5)a

3. $$\int_0^1 x^{2n}\sqrt{1-x^2}\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2\right).$$

4. $$\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n)!!}{(2n+3)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k} - \frac{1}{2n+3}\right).$$

BI ((117))(5), GW ((324))(53b)

5. $$\int_0^1 \ln x\cdot\sqrt{(1-x^2)^{2n-1}}\,dx = -\frac{(2n-1)!!}{4\cdot(2n)!!}\pi[\psi(n+1)+\boldsymbol{C}+\ln 4].$$

BI ((117))(3)

6. $$\int_0^{\sqrt{\frac{1}{2}}}\frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\boldsymbol{G}.$$

BI ((145))(1)

7. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\ln 2.$$

FI II 614, 643

8. $$\int_1^\infty \frac{\ln x\,dx}{x^2\sqrt{x^2-1}} = 1-\ln 2.$$

BI ((144))(17)

9. $$\int_0^1 \sqrt{1-x^2}\ln x\,dx = -\frac{\pi}{8}-\frac{\pi}{4}\ln 2.$$

BI ((117))(1), GW ((324))(53c)

10. $$\int_0^1 x\sqrt{1-x^2}\ln x\,dx = \frac{1}{3}\ln 2-\frac{4}{9}.$$

BI ((117))(2)

11. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

GW ((324))(54a)

4.242

1. $$\int_0^\infty \frac{\ln x\,dx}{\sqrt{(a^2+x^2)(x^2+b^2)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab \qquad [a>b>0].$$

BY (800.04)

2. $$\int_0^b \frac{\ln x\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\ln ab - \frac{\pi}{2}\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\right] \quad [a>0, \quad b>0].$$

BY (800.02)

568

3. $$\int_b^\infty \frac{\ln x\,dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right] \quad [a>0, \quad b>0].$$

BY (800.06)

4. $$\int_0^b \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab - \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right] \qquad [a>b>0].$$

BY (800.01)

5. $$\int_b^a \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab.$$

BY (800.03)

6. $$\int_a^\infty \frac{\ln x\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right] \qquad [a>b>0].$$

BY (800.05)

4.243

$$\int_0^1 \frac{x\ln x}{\sqrt{1-x^4}}\,dx = -\frac{\pi}{8}\ln 2.$$

4.244

1. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8}\left[\Gamma\left(\frac{1}{3}\right)\right]^3.$$

GW ((324))(54b)

2. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 + \frac{\pi}{3\sqrt{3}}\right).$$

BI ((118))(7)

3. $$\int_0^1 \frac{x\ln x\,dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}}\left(\frac{\pi}{3\sqrt{3}} - \ln 3\right).$$

BI ((118))(8)

4.245

1. $$\int_0^1 \frac{x^{4n+1}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{8}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right)$$

GW ((324))(56a)

2. $$\int_0^1 \frac{x^{4n+3}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n)!!}{4\cdot(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right).$$

GW ((324))(56c)

4.246

$$\int_0^1 (1-x^2)^{n-\frac{1}{2}}\ln x\,dx = -\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{4}\left[2\ln 2 + \sum_{k=1}^{n}\frac{1}{k}\right].$$

GW ((324))(55)

4.247

1.[6] $$\int_0^1 \frac{\ln x}{\sqrt[n]{1-x^{2n}}}\,dx = -\frac{\pi \mathrm{B}\left(\frac{1}{2n},\frac{1}{2n}\right)}{8n^2\sin\frac{\pi}{2n}} \qquad [n>1].$$

2.[6] $$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi \mathrm{B}\left(\frac{1}{2n}, \frac{1}{2n}\right)}{8\sin\frac{\pi}{2n}}.$$

GW ((324))(54)

4.25 Combinations of logarithms and powers

4.251

1. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{\beta+x}\,dx = \frac{\pi\beta^{\mu-1}}{\sin\mu\pi}(\ln\beta - \pi\operatorname{ctg}\mu\pi) \qquad [|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 1].$$

BI ((135))(1)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{a-x}\,dx = \pi a^{\mu-1}\left(\operatorname{ctg}\mu\pi\ln a - \frac{\pi}{\sin^2\mu\pi}\right) \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 314(5)

3.* $$\int_0^1 \frac{x^{\mu-1}\ln x}{x+1}\,dx = \beta'(\mu) \qquad [\operatorname{Re}\mu > 0].$$

GW ((324))(6), ET I 314(3)

4. $$\int_0^1 \frac{x^{\mu-1}\ln x}{1-x}\,dx = -\psi'(\mu) = -\zeta(2,\,\mu) \qquad [\operatorname{Re}\mu > 0].$$

BI ((108))(8)

5. $$\int_0^1 \ln x \frac{x^{2n}\,dx}{1+x} = -\frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k^2}.$$

BI ((108))(4)

6. $$\int_0^1 \ln x \frac{x^{2n-1}\,dx}{1+x} = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}.$$

BI ((108))(5)

4.252

BI ((140))(9)A, ET 314(6)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x\,dx}{(x+\beta)(x-1)} = \frac{\pi}{(\beta+1)\sin^2\mu\pi}[\pi - \beta^{\mu-1}(\sin\mu\pi\ln\beta - \pi\cos\mu\pi)]$$ $$[|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 2, \quad \mu\neq 1].$$

BI ((140))(11)

3. $$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^2}\,dx = -\frac{\pi^2}{4}\operatorname{cosec}^2\frac{p\pi}{2} \qquad [0 < p < 2] \qquad \text{(see also 4.254 2.)}.$$

4.254

4.[6] $$\int_0^\infty \frac{x^{\mu-1}\ln x}{(x+a)^2}\,dx = \frac{(1-\mu)a^{\mu-2}\pi}{\sin\mu\pi}\left(\ln a - \pi\operatorname{ctg}\mu\pi + \frac{1}{\mu-1}\right)$$ $$[|\arg a| < \pi \quad 0 < \operatorname{Re}\mu < 2 \quad (\mu\neq 1)].$$

GW ((324))(13b)

1. $$\int_0^1 x^{\mu-1}(1-x^r)^{\nu-1}\ln x\,dx = \frac{1}{r^2}\mathrm{B}\left(\frac{\mu}{r},\,\nu\right)\left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r}+\nu\right)\right]$$ $$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad r > 0].$$

GW ((324))(3b)A, BI ((107))(5)a

2. $$\int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}}\ln x\,dx = -\frac{\pi}{p}\operatorname{cosec}p\pi \qquad [0 < p < 1].$$

Bi ((319))(10)a

3. $$\int_u^\infty \frac{(x-u)^{\mu-1}\ln x\,dx}{x^\lambda} = u^{\mu-\lambda}B\,(\lambda-\mu,\,\mu)[\ln u + \psi(\lambda) - \psi(\lambda-\mu)]$$ $$[0 < \operatorname{Re}\mu < \operatorname{Re}\lambda].$$

ET II 203(18)

4.[8] $$\int_0^1 \ln x\left(\frac{x}{a^2+x^2}\right)^p\frac{dx}{x} = \frac{\ln a}{2a^p}\mathrm{B}\left(\frac{p}{2},\,\frac{p}{2}\right) \qquad [a > 0, \quad p > 0].$$

5. $$\int_1^\infty (x-1)^{p-1} \ln x \, dx = \frac{\pi}{p} \operatorname{cosec} \pi p \qquad [-1 < p < 0].$$

BI ((289))(12)a

6.[7] $$\int_0^\infty \ln x \frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu} (\ln a - \boldsymbol{C} - \psi(\mu))$$
$$[\operatorname{Re} \mu > 0, \quad a \neq 0, |\arg a| < \pi].$$

NT 68(7)

7.[7] $$\int_0^\infty \ln x \frac{dx}{(a+x)^{n+\frac{1}{2}}} = \frac{2}{(2n-1)a^{n-\frac{1}{2}}} \left(\ln a + 2 \ln 2 - 2 \sum_{k=1}^{n-1} \frac{1}{2k-1} \right)$$
$$[|\arg a| < \pi, n = 1, 2, \dots].$$

BI ((142))(5)

4.254

1. $$\int_0^1 \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{1}{q^2} \psi' \left(\frac{p}{q} \right) \qquad [p > 0, \quad q > 0].$$

GW ((324))(5)

2. $$\int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{\pi^2}{q^2 \sin^2 \dfrac{p\pi}{q}} \qquad [0 < p < q].$$

BI ((135))(8)

3. $$\int_0^\infty \frac{\ln x}{x^q - 1} \frac{dx}{x^p} = \frac{\pi^2}{q^2 \sin^2 \dfrac{p-1}{q}\pi} \qquad [p < 1, \quad p + q > 1].$$

BI ((140))(2)

4.[3] $$\int_0^1 \frac{x^{p-1} \ln x}{1+x^q} \, dx = \frac{1}{q^2} \beta' \left(\frac{p}{q} \right) \qquad [p > 0, \quad q > 0].$$

GW ((324))(7)

5. $$\int_0^\infty \frac{x^{p-1} \ln x}{1+x^q} \, dx = -\frac{\pi^2}{q^2} \frac{\cos \dfrac{p\pi}{q}}{\sin^2 \dfrac{p\pi}{q}} \qquad [0 < p < q].$$

BI ((135))(7)

6. $\int_0^1 \frac{x^{q-1}\ln x}{1-x^{2q}}\,dx = -\frac{\pi^2}{8q^2} \qquad [q>0].$

BI ((108))(12)

4.255

1. $\int_0^1 \ln x \frac{(1-x^2)x^{p-2}}{1+x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2 \frac{\sin\frac{\pi}{2p}}{\cos^2\frac{\pi}{2p}} \qquad [p>1].$

BI ((108))(13)

2. $\int_0^1 \ln x \frac{(1+x^2)x^{p-2}}{1-x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2\frac{\pi}{2p} \qquad [p>1].$

BI ((108))(14)

3. $\int_0^\infty \ln x \frac{1-x^p}{1-x^2}\,dx = \frac{\pi^2}{4}\ \mathrm{tg}^2\,\frac{p\pi}{2} \qquad [p<1].$

BI ((140))(3)

4.256

$$\int_0^1 \ln\frac{1}{x}\frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2}\mathrm{B}\left(\frac{\mu}{n},\frac{m}{n}\right)\left[\psi\left(\frac{\mu+m}{n}\right)-\psi\left(\frac{\mu}{n}\right)\right] \qquad [\mathrm{Re}\,\mu>0].$$

LI ((118))(12)

4.257

1. $\int_0^\infty \frac{x^\nu \ln\frac{x}{\beta}\,dx}{(x+\beta)(x+\gamma)} = \frac{\pi\left[\gamma^\nu\ln\frac{\gamma}{\beta}+\pi(\beta^\nu-\gamma^\nu)\,\mathrm{ctg}\,\nu\pi\right]}{\sin\nu\pi(\gamma-\beta)}$
$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad |\mathrm{Re}\,\nu|<1].$

ET II 219(30)

2. $\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)\frac{dx}{x} = 0 \qquad [q>0].$

3. $$\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)^r\frac{dx}{q^2+x^2} = 0 \qquad [q>0].$$

BI ((140))(4)a

4. $$\int_0^\infty \ln x \ln\frac{x}{a}\,\frac{dx}{(x-1)(x-a)} = \frac{[4\pi^2+(\ln a)^2]\ln a}{6(a-1)} \qquad [a>0], \qquad [a=1 \quad \text{see } 4.261\ 5.].$$

4.261
BI ((141))(5)

5. $$\int_0^\infty \ln x \ln\frac{x}{a}\,\frac{x^p\,dx}{(x-1)(x-a)} = \frac{\pi^2[(a^p+1)\ln a - 2\pi(a^p-1)\operatorname{ctg} p\pi]}{(a-1)\sin^2 p\pi} \qquad [p^2<1, \quad a>0].$$

BI ((141))(6)

4.26-4.27 Combinations involving powers of the logarithm and other powers

4.261

1. $$\int_0^1 (\ln x)^2\frac{dx}{1+2x\cos t+x^2} = \frac{t(\pi^2-t^2)}{6\sin t} \qquad [0<t<\pi].$$

BI ((113))(7)

2. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{10\pi^3}{81\sqrt{3}}.$$

GW ((324))(16c)

3. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{8\pi^3}{81\sqrt{3}}.$$

GW ((324))(16b)

4. $$\int_0^\infty (\ln x)^2\frac{dx}{(x-1)(x+a)} = \frac{[\pi^2+(\ln a)^2]\ln a}{3(1+a)} \qquad [a>0].$$

5. $$\int_0^\infty (\ln x)^2 \frac{dx}{(1-x)^2} = \frac{2}{3}\pi^2.$$

BI ((139))(4)

6. $$\int_0^1 (\ln x)^2 \frac{dx}{1+x^2} = \frac{\pi^3}{16}.$$

BI ((109))(3)

7. $$\int_0^1 (\ln x)^2 \frac{1+x^2}{1+x^4}\,dx = \frac{1}{2}\int_0^\infty (\ln x)^2 \frac{1+x^2}{1+x^4}\,dx = \frac{3\sqrt{2}}{64}\pi^3.$$

BI ((109))(5), BI ((135))(13)

8.[3] $$\int_0^1 (\ln x)^2 \frac{1-x}{1-x^6}\,dx = \frac{1}{36}\left(\frac{4\sqrt{3}\pi^3}{27} - \psi''\left(\frac{1}{3}\right)\right).$$

9. $$\int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right].$$

BI ((118))(13)

10. $$\int_0^\infty (\ln x)^2 \frac{x^{\mu-1}}{1+x}\,dx = \frac{\pi^3(2-\sin^2\mu\pi)}{\sin^3\mu\pi} \qquad [0 < \operatorname{Re}\mu < 1].$$

ET I 315(10)

11.[7] $$\int_0^1 (\ln x)^{2x^n} \frac{dx}{1+x} = (-1)^n\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\right) \qquad [n = 0, 1, \dots]$$

BI ((109))(1)

12.[7] $$\int_0^1 (\ln x)^2 \frac{x^n\,dx}{1-x} = 2\left(\zeta(3) - \sum_{k=1}^{n}\frac{1}{k^3}\right) \qquad [n = 0, 1, \dots]$$

BI ((109))(2)

13.[7] $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{1-x^2} = \frac{7}{4}\zeta(3) - 2\sum_{k=1}^{n}\frac{1}{(2k-1)^3} \qquad [n = 0,\ 1, \dots].$$

BI ((109))(4)

14. $$\int_0^\infty (\ln x)^2 \frac{x^{p-1}\,dx}{x^2+2x\cos t+1} = \frac{\pi\sin(1-p)t}{\sin t\sin p\pi}\{\pi^2-t^2+2\pi\operatorname{ctg} p\pi[\pi\operatorname{ctg} p\pi+t\operatorname{ctg}(1-p)t]\}$$
$$[0<t<\pi, \quad 0<p<2 \quad (p\neq 1)].$$

GW ((324))(17)

15. $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2\cdot(2n)!!}\pi\left\{\frac{\pi^2}{12}+\sum_{k=1}^{2n}\frac{(-1)^k}{k^2}+\left[\sum_{k=1}^{2n}\frac{(-1)^k}{k}+\ln 2\right]^2\right\}.$$

GW ((324))(60a)

16. $$\int_0^1 (\ln x)^2 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}\left\{-\frac{\pi^2}{12}-\sum_{k=1}^{2n+1}\frac{(-1)^k}{k^2}+\left[\sum_{k=1}^{2n+1}\frac{(-1)^k}{k}+\ln 2\right]^2\right\}.$$

GW ((324))(60b)

17.[7] $$\int_0^1 (\ln x)^2 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\nu)\left\{[\psi(\mu)-\psi(\nu+\mu)]^2+\psi^1(\mu)-\psi'(\mu+\nu)\right\}$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET I 315(11)

18. $$\int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2}\,dx = 2(n+1)\zeta(3)-2\sum_{k=1}^{n}\frac{n-k+1}{k^3}.$$

LI ((111))(8)

19. $$\int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = \frac{3}{2}(n+1)\zeta(3)-2\sum_{k=1}^{n}(-1)^{k-1}\frac{n-k+1}{k^3}.$$

LI ((111))(7)

20.[7] $$\int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{(1-x^2)^2}\,dx = \frac{7}{4}(n+1)\zeta(3)-2\sum_{k=1}^{n}\frac{n-k+1}{(2k-1)^3}.$$

21. $$\int_0^1 (\ln x)^2 x^{p-1} (1-x^r)^{q-1}\,dx = \frac{1}{r^3} \mathrm{B}\left(\frac{p}{r},\, q\right) \left\{ \psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r}+q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r}+q\right)\right]^2 \right\}$$
$$[p > 0, \quad q > 0, \quad r > 0].$$

GW ((324))(8a)

4.262

1. $$\int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4.$$

BI ((109))(9)

2. $$\int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15}.$$

BI ((109))(11)

3. $$\int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{[\pi^2 + (\ln a)^2]^2}{4(a+1)} \qquad [a > 0].$$

BI ((141))(2)

4. $$\int_0^1 (\ln x)^3 \frac{x^n\,dx}{1+x} = (-1)^{n+1} \left[\frac{7\pi^4}{120} - 6\sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4}\right] \qquad [n = 1, 2, \ldots].$$

BI ((109))(10)

5. $$\int_0^1 (\ln x)^3 \frac{x^n\,dx}{1-x} = -\frac{\pi^4}{15} + 6\sum_{k=0}^{n-1} \frac{1}{(k+1)^4} \qquad [n = 1, 2, \ldots].$$

BI ((109))(12)

6. $$\int_0^1 (\ln x)^3 \frac{x^{2n}\,dx}{1-x^2} = -\frac{\pi^4}{16} + 6\sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} \qquad [n = 1, 2, \ldots].$$

BI ((109))(14)

7. $$\int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2}\,dx = -\frac{(n+1)\pi^4}{15} + 6\sum_{k=1}^{n} \frac{n-k+1}{k^4}.$$

8. $$\int_0^1 (\ln x)^3 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\, dx = -\frac{7(n+1)\pi^4}{120} + 6\sum_{k=1}^{n} (-1)^{k-1} \frac{n-k+1}{k^4}.$$

BI ((111))(10)

9. $$\int_0^1 (\ln x)^3 \frac{1-x^{2n+2}}{(1-x^2)^2}\, dx = -\frac{(n+1)\pi^4}{16} + 6\sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^4}.$$

BI ((111))(12)

4.263

1.[8] $$\int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a[\pi^2 + (\ln a)^2][7\pi^2 + 3(\ln a)^2]}{15(1+a)} \qquad [a > 0].$$

BI ((141))(3)

2. $$\int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64}.$$

BI ((109))(17)

3. $$\int_0^1 (\ln x)^4 \frac{dx}{1+2x\cos t + x^2} = \frac{t(\pi^2 - t^2)(7\pi^2 - 3t^2)}{30 \sin t} \qquad [|t| < \pi].$$

BI ((113))(8)

4.264

1. $$\int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252}.$$

BI ((109))(20)

2. $$\int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63}.$$

BI ((109))(21)

3. $$\int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{[\pi^2 + (\ln a)^2]^2[3\pi^2 + (\ln a)^2]}{6(1+a)} \qquad [a > 0].$$

$$\int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256}.$$

BI ((109))(25)

4.266

1. $$\int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240}.$$

BI ((109))(28)

2. $$\int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15}.$$

BI ((109))(29)

4.267

1. $$\int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = \ln \frac{2}{\pi}.$$

BI ((127))(3)

2. $$\int_0^1 \frac{(1-x)^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{\pi}{4}.$$

BI ((128))(2)

3.[8] $$\int_0^1 \frac{(1-x)^2}{1+2x\cos\frac{mx}{n}+x^2} \cdot \frac{dx}{\ln x =}$$

$$= \frac{1}{\sin\frac{m\pi}{n}} \sum_{k=1}^{n-1} (-1)^k \sin\frac{km\pi}{n} \ln \frac{\left\{\Gamma\left(\frac{n+k+1}{2n}\right)\right\}^2 \Gamma\left(\frac{k+2}{2n}\right) \Gamma\left(\frac{k}{2n}\right)}{\left\{\Gamma\left(\frac{k+1}{2n}\right)\right\}^2 \Gamma\left(\frac{n+k}{2n}\right) \Gamma\left(\frac{n+k+2}{2n}\right)}$$

$[m+n \text{ is odd}];$

$$= \frac{1}{\sin\frac{m\pi}{n}} \sum_{k=1}^{\left[\frac{1}{2}(n-1)\right]} (-1)^k \sin\frac{km\pi}{n} \ln \frac{\left\{\Gamma\left(\frac{n-k+1}{n}\right)\right\}^2 \Gamma\left(\frac{k+2}{n}\right) \Gamma\left(\frac{k}{n}\right)}{\left\{\Gamma\left(\frac{k+1}{n}\right)\right\}^2 \Gamma\left(\frac{n-k}{n}\right) \Gamma\left(\frac{n-k+2}{n}\right)}$$

$[m+n \text{ is even}]; \quad [m < n].$

5. $$\int_0^1 \frac{1-x}{1+x} \cdot \frac{x^2}{1+x^2} \cdot \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi}.$$

6. $$\int_0^1 (4-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \frac{p}{k} \ln(1+k) \qquad [p \geq 1].$$

BI ((123))(2)

7. $$\int_0^1 \left(\frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} = \ln \Gamma(p+1).$$

GW ((326))(10)

8. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \, dx = \ln \frac{p}{q} \qquad [p > 0, \quad q > 0].$$

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9. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \cdot \frac{dx}{1+x} = \ln \frac{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q+1}{2}\right)} \qquad [p > 0, \quad q > 0].$$

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10. $$\int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x)\ln x} \, dx = \frac{1}{2} \int_0^{\infty} \frac{x^{p-1} - x^{-p}}{(1+x)\ln x} \, dx = \ln\left(\operatorname{tg} \frac{p\pi}{2} \right) \qquad [0 < p < 1].$$

FI II 816

11. $$\int_0^1 (x^p - x^q) x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q} \qquad [r > 0, \quad p > 0, \quad q > 0].$$

LI ((123))(5)

12. $$\int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \ln \frac{p}{q} + \sum_{k=1}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k} \qquad [p > 0, \quad q > 0, \quad a^2 < 1].$$

BI ((130))(15)

13. $$\int_0^1 (x^p - 1)(x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)} \qquad [p > -1, \quad q > -1, \quad p+q > -1]$$

14. $$\int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x}\, dx = \ln \frac{\Gamma\left(\frac{p}{2}+n+1\right)\Gamma\left(\frac{q+1}{2}+n\right)\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{p+1}{2}+n\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p}{2}\right)}$$
$$[p > 0, \quad q > 0].$$

BI ((127))(7)

15. $$\int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x}\, dx = \ln \frac{\Gamma(q+1)\Gamma(p+r+1)}{\Gamma(p+1)\Gamma(q+r+1)}$$
$$[p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1].$$

GW ((324))(23)

16. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r)\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right)\Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right)\Gamma\left(\frac{p}{2r}\right)} \qquad [p > 0, \quad q > 0, \quad r > 0].$$

GW ((324))(21)

17. $$\int_0^1 \frac{1 - x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1}\, dx}{\ln x} = \ln \operatorname{tg} \frac{q\pi}{4p} \qquad [0 < q < p] \qquad \text{(see also 3.524 27.)}.$$

BI ((128))(6)

18. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x^r)\ln x}\, dx = \ln\left(\operatorname{tg} \frac{p\pi}{2r} \operatorname{ctg} \frac{q\pi}{2r}\right) \qquad [0 < p < r,\ 0 < q < r].$$

GW ((324))(22), BI ((143))(2)

19. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1-x^r)\ln x}\, dx = \ln\left(\frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}}\right) \qquad [0 < p < r, \quad 0 < q < r].$$

BI ((143))(4)

20. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right)\Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right)\Gamma\left(\frac{p}{2n}\right)} \qquad [p > 0, \quad q > 0].$$

BI ((128))(11)

21. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{1+x^{2(2n+1)}}\frac{1+x^2}{\ln x}\,dx = \ln\frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right)\Gamma\left(\frac{q+2}{4(2n+1)}\right)\Gamma\left(\frac{p+4n+2}{4(2n+1)}\right)\Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right)\Gamma\left(\frac{p+2}{4(2n+1)}\right)\Gamma\left(\frac{q+4n+2}{4(2n+1)}\right)\Gamma\left(\frac{p}{4(2n+1)}\right)}$$
$$[p>0, \quad q>0].$$

BI ((128))(7)

22. $$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1+x^{2(2n+1)}}\cdot\frac{1+x^2}{\ln x}\,dx = \ln\left\{\operatorname{tg}\frac{p\pi}{4(2n+1)}\cdot\operatorname{tg}\frac{(p+2)\pi}{4(2n+1)}\cdot\operatorname{ctg}\frac{q\pi}{4(2n+1)}\cdot\operatorname{ctg}\frac{(q+2)\pi}{4(2n+1)}\right\}$$
$$[0<p<4n, \quad 0<q<4n].$$

BI ((143))(5)

23. $$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x^{2n}}\frac{1-x^2}{\ln x}\,dx = \ln\frac{\sin\frac{p\pi}{2n}\cdot\sin\frac{(q+2)\pi}{2n}}{\sin\frac{q\pi}{2n}\cdot\sin\frac{(p+2)\pi}{2n}} \qquad [0<p<2n, \quad 0<q<2n].$$

BI ((143))(6)

24. $$\int_0^1 (1-x^p)(1-x^q)\frac{x^{r-1}\,dx}{\ln x} = \ln\frac{(p+q+r)r}{(p+r)(q+r)} \qquad [p>0, \quad q>0, \quad r>0].$$

BI ((123))(8)

25. $$\int_0^1 (1-x^p)(1-x^q)\frac{x^{r-1}\,dx}{(1-x)\ln x} = \ln\frac{\Gamma(p+r)\Gamma(q+r)}{\Gamma(p+q+r)\Gamma(r)}$$
$$[r>0, \quad r+p>0, \quad r+q>0, \quad r+p+q>0].$$

FI II 815a

26. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r)\frac{dx}{\ln x} = \ln\frac{(p+q+1)(q+r+1)(r+p+1)}{(p+q+r+1)(p+1)(q+1)(r+1)}$$
$$[p>-1, \quad q>-1, \quad r>-1, \quad p+q>-1, \quad p+r>-1, \quad q+r>-1, \quad p+q+r>-1].$$

GW ((324))(19c)

FI II 815

28. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r)\frac{x^{s-1}\,dx}{\ln x} = \ln\frac{(p+q+s)(p+r+s)(q+r+s)s}{(p+s)(q+s)(r+s)(p+q+r+s)}$$
$$[p>0, \quad q>0, \quad r>0, \quad s>0].$$

BI ((123))(10)

29. $$\int_0^1 (1-x^p)(1-x^q)\frac{x^{s-1}\,dx}{(1-x^r)\ln x} = \ln\frac{\Gamma\left(\frac{p+s}{r}\right)\Gamma\left(\frac{q+s}{r}\right)}{\Gamma\left(\frac{s}{r}\right)\Gamma\left(\frac{p+q+s}{r}\right)}$$
$$[p>0, \quad q>0, \quad r>0, \quad s>0].$$

GW ((324))(23a)

30. $$\int_0^\infty (1-x^p)(1-x^q)\frac{x^{s-1}\,dx}{(1-x^{p+q+2s})\ln x} =$$
$$= 2\int_0^1 (1-x^p)(1-x^q)\frac{x^{s-1}\,dx}{(1-x^{p+q+2s})\ln x} = 2\ln\left\{\sin\frac{s\pi}{p+q+2s}\operatorname{cosec}\frac{(p+s)\pi}{p+q+2s}\right\}$$
$$[s>0, \quad s+p>0, \quad s+p+q>0].$$

GW ((324))(23b)a

31. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r)\frac{x^{s-1}\,dx}{(1-x)\ln x} = \ln\frac{\Gamma(p+s)\Gamma(q+s)\Gamma(r+s)\Gamma(p+q+r+s)}{\Gamma(p+q+s)\Gamma(p+r+s)\Gamma(q+r+s)\Gamma(s)}$$
$$[p>0, \quad q>0, \quad r>0, \quad s>0]^*. < ftn1 >$$

* These restrictions can be somewhat weakened by writing, for example, $s>0$, $p+s>0$, $q+s>0$,

$r+s>0$, $p+q+s>0$, $p+r+s>0$, $q+r+s>0$, $p+q+r+s>0$, in 4.267 31. and 32.

BI ((127))(11)

32. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r)\frac{x^{s-1}\,dx}{(1-x^t)\ln x} = \ln\frac{\Gamma\left(\frac{p+s}{t}\right)\Gamma\left(\frac{q+s}{t}\right)\Gamma\left(\frac{r+s}{t}\right)\Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right)\Gamma\left(\frac{q+r+s}{t}\right)\Gamma\left(\frac{p+r+s}{t}\right)\Gamma\left(\frac{s}{t}\right)}$$
$$[p>0, \quad q>0, \quad r>0, \quad s>0, \quad t>0]^*.$$

GW ((324))(23b)

BI ((127))(19)

34. $$\int_0^1 \left\{ \frac{x^\mu - x}{x-1} - x(\mu - 1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu) \qquad [\operatorname{Re} \mu > 0].$$

WH, BI ((127))(18)

35. $$\int_0^1 \left\{ 1 - x - \frac{(1-x^p)(1-x^q)}{1-x} \right\} \frac{dx}{x \ln x} = \ln\{\mathrm{B}\ (p,\ q)\} \qquad [p > 0, \quad q > 0].$$

BI ((130))(18)

36. $$\int_0^1 \left\{ \frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)} \right\} \frac{dx}{\ln x} = q \ln p \qquad [p > 0].$$

BI ((130))(20)

37. $$\int_0^1 \left\{ \frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1} \right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left(pq - \frac{1}{2} \right) \ln p$$
$$[p > 0, \quad q > 0].$$

BI ((130))(22)

38. $$\int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x)\ln x}\, dx = \ln \mathrm{B}\,(p,\ q) \qquad [p > 0, \quad q > 0].$$

GW ((324))(24)

39.[6] $$\int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^{n} \genfrac{}{}{0pt}{}{n}{n-k} (-1)^{n-k} \ln(pk+1) \qquad [n > 0, \quad pn > -1]$$

GW ((324))(19d), BI ((123))(12)a

40.[6] $$\int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln \Gamma[(n-k)p + 1] \qquad [n > 1, \quad pn > -1]$$

BI ((127))(12)

BI ((123))(12)

42.[6] $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x)\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln\Gamma[(n-k)p+q]$$
$$[n>1, \quad q>0, \quad pn>-q]$$

BI ((127))(13)

43.[6] $$\int_0^1 (x^p-1)^n (x^q-1)^m \frac{x^{r-1}\,dx}{\ln x} = \sum_{j=0}^{n} (-1)^j \genfrac{}{}{0pt}{}{n}{j} \sum_{k=0}^{m} (-1)^k \genfrac{}{}{0pt}{}{m}{k} [r+(m-k)q+(n-j)p]$$
$$[n \ge 0, \quad m \ge 0, \quad n+m>0, \quad r>0, \quad pn+qm+r>0].$$

BI ((123))(16)

4.268

1. $$\int_0^1 \frac{(x^p-x^q)(1-x^r)}{(\ln x)^2}\,dx = (p+1)\ln(p+1) - (q+1)\ln(q+1) - $$
$$-(p+r+1)\ln(p+r+1) + (q+r+1)\ln(q+r+1)$$
$$[p>-1, \quad q>-1, \quad p+r>-1, \quad q+r>-1].$$

GW ((324))(26)

2. $$\int_0^1 (x^p-x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1)\ln(2p+1)+(2q+1)\ln(2q+1)-2(p+q+1)\ln(p+q+1)$$
$$\left[p>-\frac{1}{2}, \quad q>-\frac{1}{2}\right]$$

GW ((324))(26a)

3. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{(\ln x)^2} =$$
$$= (p+q+1)\ln(p+q+1) + (q+r+1)\ln(q+r+1) + (p+r+1)\ln(p+r+1) -$$
$$-(p+1)\ln(p+1)-(q+1)\ln(q+1)-(r+1)\ln(r+1)-(p+q+r)\ln(p+q+r)$$
$$[p>-1, \quad q>-1, \quad r>-1, \quad p+q>-1, \quad p+r>-1, \quad q+r>-1, \quad p+q+r>0].$$

BI ((124))(4)

579

4. $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2}\sum_{k=0}^{n} (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q) \qquad \left[q>0, \quad p>-\frac{q}{n}\right].$$

5. $$\int_0^1 (1-x^p)^n(1-x^q)^m x^{r-1}\frac{dx}{(\ln x)^2} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\times$$
$$\times[(m-k)q+(n-j)p+r]\ln[(m-k)q+(n-j)p+r]$$
$$[r>0, \quad mq+r>0, \quad np+r>0, \quad mq+np+r>0].$$

BI ((124))(8)

6. $$\int_0^1 [(q-r)x^{p-1}+(r-p)x^{q-1}+(p-q)x^{r-1}]\frac{dx}{(\ln x)^2} =$$
$$= (q-r)p\ln p+(r-p)q\ln q+(p-q)r\ln r \qquad [p>0, \quad q>0, \quad r>0].$$

BI ((124))(9)

7. $$\int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)}+\frac{x^{q-1}}{(q-p)(q-r)(q-s)}+\frac{x^{r-1}}{(r-p)(r-q)(r-s)}+\right.$$
$$\left.+\frac{x^{s-1}}{(s-p)(s-q)(s-r)}\right]\frac{dx}{(\ln x)^2} = \frac{1}{2}\left[\frac{p^2\ln p}{(p-q)(p-r)(p-s)}+\frac{q^2\ln q}{(q-p)(q-r)(q-s)}+\right.$$
$$\left.+\frac{r^2\ln r}{(r-p)(r-q)(r-s)}+\frac{s^2\ln s}{(s-p)(s-q)(s-r)}\right]$$
$$[p>0, \quad q>0, \quad r>0, \quad s>0].$$

BI ((124))(16)

4.269

1. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\cdot\frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

BI ((115))(33)

2. $$\int_0^1 \frac{dx}{\sqrt{\ln\frac{1}{x}}\cdot(1+x)^2} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((133))(2)

3. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\cdot x^{p-1}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{p^3}} \qquad [p>0].$$

GW ((324))(1c)

BI ((133))(1)

5. $$\int_0^1 \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x\cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^{n} \frac{\sin kt}{\sqrt{k}} \qquad [|t| < \pi].$$

BI ((133))(5)

6. $$\int_0^1 \frac{\cos t - x - x^{n-1} \cos nt + x^n \cos[(n-1)t]}{1 - 2x\cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^{n-1} \frac{\cos kt}{\sqrt{k}} \qquad [|t| < \pi].$$

BI ((133))(6)

7. $$\int_u^v \frac{dx}{x \cdot \sqrt{\ln \frac{x}{u} \ \ln \frac{v}{x}}} = \pi \qquad [uv > 0].$$

BI ((145))(37)

4.271

1. $$\int_0^1 (\ln x)^{2n} \frac{dx}{1+x} = \frac{2^{2n} - 1}{2^{2n}} \cdot (2n)! \zeta(2n+1).$$

BI ((110))(1)

2. $$\int_0^1 (\ln x)^{2n-1} \frac{dx}{1+x} = \frac{1 - 2^{2n-1}}{2n} \pi^{2n} |B_{2n}| \qquad [n = 1, 2, \ldots].$$

BI ((110))(2)

3. $$\int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x} = -\frac{1}{n} 2^{2n-2} \pi^{2n} |B_{2n}| \qquad [n = 1, 2, \ldots].$$

BI (110)(5), GW((324))(9a)

4. $$\int_0^1 (\ln x)^{p-1} \frac{dx}{1-x} = e^{i(p-1)\pi} \Gamma(p) \zeta(p) \qquad [p > 1].$$

5. $$\int_0^1 (\ln x)^n \frac{dx}{1+x^2} = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}.$$

BI ((110))(11)

6. $$\int_0^1 (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{1}{2}\int_0^{\infty} (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}}|E_{2n}|.$$

GW ((324))(10)a

7. $$\int_0^{\infty} \frac{(\ln x)^{2n+1}}{1+bx+x^2}\, dx = 0 \qquad [|b| < 2].$$

BI ((135))(2)

8. $$\int_0^1 (\ln x)^{2n} \frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}} \cdot (2n)!\zeta(2n+1) \qquad [n = 1, 2, \ldots].$$

BI ((110))(12)

9. $$\int_0^{\infty} (\ln x)^{2n} \frac{dx}{1-x^2} = 0.$$

BI ((312))(7)a

10. $$\int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1}{2}\int_0^{\infty} (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n}\,\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots].$$

BI ((290))(17)A, BI((312))(6)a

11. $$\int_0^1 (\ln x)^{2n-1} \frac{x\, dx}{1-x^2} = -\frac{1}{4n}\,\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots].$$

BI ((290))(19)a

12. $$\int_0^1 (\ln x)^{2n} \frac{1+x^2}{(1-x^2)^2}\, dx = \frac{2^{2n}-1}{2}\,\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots].$$

BI ((296))(17)a

13. $$\int_0^1 (\ln x)^{2n+1} \frac{(\cos 2a\pi - x)\,dx}{1 - 2x\cos 2a\pi + x^2} = -(2n+1)! \sum_{k=1}^{\infty} \frac{\cos 2ak\pi}{k^{2n+2}} \qquad [a \text{ is not an integer}].$$

LI ((113))(10)

14.[6] $$\int_0^{\infty} (\ln x)^n \frac{x^{\nu-1}\,dx}{a^2 + 2ax\cos t + x^2} = -\pi \operatorname{cosec} t \frac{d^n}{d\nu^n}\left[a^{\nu-2}\frac{\sin(\nu-1)t}{\sin \nu\pi}\right]$$
$$[a > 0, \quad 0 < \operatorname{Re}\nu < 2, \quad 0 < |t| < \pi].$$

ET I 315(12)

15. $$\int_0^1 (\ln x)^n \frac{x^{p-1}}{1 - x^q}\,dx = -\frac{1}{q^{n+1}} \psi^{(n)}\left(\frac{p}{q}\right) \qquad [p > 0, \quad q > 0].$$

GW ((324))(9)

16.[3] $$\int_0^1 (\ln x)^n \frac{x^{p-1}}{1 + x^q}\,dx = \frac{1}{q^{n+1}} \beta^{(n)}\left(\frac{p}{q}\right) \qquad [p > 0, \quad q > 0].$$

GW ((324))(10)

4.272

1. $$\int_0^1 \frac{\left[\ln\left(\frac{1}{x}\right)\right]^{q-1} dx}{1 + 2x\cos t + x^2} = \operatorname{cosec} t\,\Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^q} \qquad [|t| < \pi,\ q < 1].$$

LI ((130))(1)

2. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{q-1} \frac{(1+x)\,dx}{1 + 2x\cos t + x^2} = \sec\frac{t}{2}\cdot\Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos\left[\left(k - \frac{1}{2}\right)t\right]}{k^q}$$
$$\left[|t| < \pi, \quad q < \frac{1}{2}\right].$$

LI ((130))(5)

3.* $$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\mu} \frac{x^{\nu-1}\,dx}{1 - 2ax\cos t + x^2a^2} = \frac{\Gamma(\mu+1)}{a\sin t} \sum_{k=1}^{\infty} \frac{a^k \sin kt}{(\nu+k-1)^{\mu+1}}$$
$$[a > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad -\pi < t < \pi].$$

4. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1} \frac{\cos\lambda - px}{1+p^2x^2-2px\cos\lambda}\, x^{q-1}\, dx = \Gamma(r)\sum_{k=1}^{\infty}\frac{p^{k-1}\cos k\lambda}{(q+k-1)^r} \qquad [r>0, \quad q>0].$$

BI ((113))(11)

5. $$\int_1^{\infty} (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p) \qquad [p>-1].$$

BI ((149))(1)

6. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} x^{\nu-1}\, dx = \frac{1}{\nu^{\mu}}\,\Gamma(\mu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

BI ((107))(3)

7. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-\frac{1}{2}} x^{\nu-1}\, dx = \frac{(2n-1)!!}{(2\nu)^n}\sqrt{\frac{\pi}{\nu}} \qquad [\operatorname{Re}\nu>0].$$

BI ((107))(2)

8. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1+x}\, dx = (n-1)!\sum_{k=0}^{\infty}\frac{(-1)^k}{(\nu+k)^n} \qquad [\operatorname{Re}\nu>0].$$

BI ((110))(4)

9. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1-x}\, dx = (n-1)!\zeta(n,\,\nu) \qquad [\operatorname{Re}\nu>0].$$

BI ((110))(7)

10. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} (x-1)^n \left(a+\frac{nx}{x-1}\right) x^{a-1}\, dx = \Gamma(\mu)\sum_{k=0}^{n}\frac{(-1)^k n(n-1)\dots(n-k+1)}{(a+n-k)^{\mu-1}k!}$$
$$[\operatorname{Re}\mu>0].$$

LI ((110))(10)

11. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{1-x^m}{1-x}\, dx = (n-1)!\sum_{k=1}^{m}\frac{1}{k^n}.$$

12. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} = \Gamma(\mu)\sum_{k=0}^{\infty}\frac{1}{(\nu+2k)^{\mu}} = \frac{1}{2^{\mu}}\Gamma(\mu)\zeta\left(\mu,\ \frac{\nu}{2}\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((110))(13)

13. $$\int_0^1 \frac{x^q - x^{-q}}{1-x^2}\left(\ln\frac{1}{x}\right)^{p} dx = \Gamma(p+1)\sum_{k=1}^{\infty}\left\{\frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}}\right\}$$
$$[p > -1, \quad q^2 < 1].$$

LI ((326))(12)a

14. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1} \frac{x^{p-1}\,dx}{(1+x^q)^s} = \Gamma(r)\sum_{k=0}^{\infty}\genfrac{}{}{0pt}{}{-s}{k}\frac{1}{(p+kq)^r}$$
$$[p > 0, \quad q > 0, \quad r > 0, \quad 0 < s < r+2].$$

GW ((324))(11)

15. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n} (1+x^q)^m x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{1}{(p+kq)^{n+1}} \qquad [p > 0, \quad q > 0].$$

BI ((107))(6)

16. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n} (1-x^q)^m x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(p+kq)^{n+1}} \qquad [p > 0, \quad q > 0].$$

BI ((107))(7)

17. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \frac{1}{aq^p}\Gamma(p)\sum_{k=1}^{\infty}\frac{a^k}{k^p} \qquad [p > 0, \quad q > 0, \quad a < 1].$$

LI ((110))(8)

18. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{2-\frac{1}{n}} (x^{p-1} - x^{q-1})\,dx = \frac{n}{n-1}\Gamma\left(\frac{1}{n}\right)\left(q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right) \qquad [q > p > 0].$$

BI ((133))(4)

LI ((110))(16)

4.273

$$\int_u^v \left(\ln\frac{x}{u}\right)^{p-1}\left(\ln\frac{v}{x}\right)^{q-1}\frac{dx}{x} = B(p,\,q)\left(\ln\frac{v}{u}\right)^{p+q-1} \qquad [p>0, \quad q>0, \quad uv>0].$$

BI ((145))(36)

4.274

$$\int_0^{\frac{1}{e}} \frac{\sqrt[q]{x}\,dx}{x\sqrt{-(1+\ln x)}} = \frac{\sqrt{q\pi}}{\sqrt[q]{e}} \qquad [q>0].$$

BI ((145))(4)

4.275

1. $$\int_0^1 \left[\left(\ln\frac{1}{x}\right)^{q-1} - x^{p-1}\,(1-x)^{q-1}\right] dx = \frac{\Gamma(q)}{\Gamma(p+q)}[\Gamma(p+q)-\Gamma(p)] \qquad [p>0, \quad q>0].$$

BI ((107))(8)

2. $$\int_0^1 \left[x - \left(\frac{1}{1-\ln x}\right)^q\right]\frac{dx}{x\ln x} = -\psi(q) \qquad [q>0].$$

BI ((126))(5)

4.28 Combinations of rational functions of ln x and powers

4.281

1. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1}{1-x}\right] dx = \boldsymbol{C}.$$

BI ((127))(15)

2. $$\int_1^\infty \frac{dx}{x^2(\ln p - \ln x)} = \frac{1}{p}\,\mathrm{li}(p).$$

LA 281(30)

LI ((144))(11,12)

4. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x}\right] dx = -\psi(\mu) \qquad [\operatorname{Re}\mu > 0].$$

WH

5. $$\int_0^1 \left[\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x}\right] dx = \ln p - \psi(q) \qquad [p > 0, \quad q > 0].$$

BI ((127))(17)

6. $$\int_0^1 \left[\frac{1}{1-x^2} + \frac{1}{2x\ln x}\right] \frac{dx}{\ln x} = \frac{\ln 2}{2}.$$

LI ((130))(19)

7. $$\int_0^1 \left[q - \frac{1}{2} + \frac{(1-x)(1+q\ln x) + x\ln x}{(1-x)^2}\, x^{q-1}\right] \frac{dx}{\ln x} = \frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2} \qquad [q > 0].$$

BI ((128))(15)

4.282

1. $$\int_0^1 \frac{\ln x}{4\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2}\boldsymbol{C}.$$

BI ((129))(1)

2. $$\int_0^1 \frac{1}{a^2 + (\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{2a}\beta\left(\frac{2a+\pi}{4\pi}\right) \qquad \left[a > -\frac{\pi}{2}\right].$$

BI ((129))(9)

3. $$\int_0^1 \frac{1}{\pi^2 + (\ln x)^2} \frac{dx}{1+x^2} = \frac{4-\pi}{4\pi}.$$

BI ((129))(6)

4. $$\int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{1}{2}\left(\frac{1}{2} - \ln 2\right).$$

5. $$\int_0^1 \frac{\ln x}{a^2+(\ln x)^2}\cdot\frac{x\,dx}{1-x^2} = \frac{1}{2}\left[\frac{\pi}{2a}+\ln\frac{\pi}{a}+\psi\left(\frac{a}{\pi}\right)\right] \qquad [a>0].$$

BI ((129))(14)

6. $$\int_0^1 \frac{\ln x}{\pi^2+(\ln x)^2}\cdot\frac{x\,dx}{1-x^2} = \frac{1}{2}\left(\frac{1}{2}-\boldsymbol{C}\right).$$

BI ((129))(13)

7. $$\int_0^1 \frac{1}{\pi^2+4(\ln x)^2}\cdot\frac{dx}{1+x^2} = \frac{\ln 2}{4\pi}.$$

BI ((129))(7)

8. $$\int_0^1 \frac{\ln x}{\pi^2+4(\ln x)^2}\cdot\frac{dx}{1-x^2} = \frac{2-\pi}{16}.$$

BI ((129))(11)

9.* $$\int_0^1 \frac{1}{\pi^2+16(\ln x)^2}\cdot\frac{dx}{1+x^2} = \frac{1}{8\pi\sqrt{2}} \quad [\pi+2\ln(\sqrt{2}-1)].$$

BI ((129))(8)

10. $$\int_0^1 \frac{\ln x}{\pi^2+16(\ln x)^2}\cdot\frac{dx}{1-x^2} = -\frac{\pi}{32\sqrt{2}}+\frac{1}{16}+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}-1).$$

BI ((129))(12)

11. $$\int_0^1 \frac{\ln x}{[a^2+(\ln x)^2]^2}\,\frac{dx}{1-x} = -\frac{\pi^2}{a^4}\sum_{k=1}^{\infty}|B_{2k}|\left(\frac{2\pi}{a}\right)^{2k-2}$$

BI ((129))(4)

12. $$\int_0^1 \frac{\ln x}{[a^2+(\ln x)^2]^2}\,\frac{x\,dx}{1-x^2} = -\frac{\pi^2}{4a^4}\sum_{k=1}^{\infty}|B_{2k}|\left(\frac{\pi}{a}\right)^{2k-2}$$

13. $$\int_0^1 \frac{x^p - x^{-p}}{x^2-1} \frac{dx}{q^2+(\ln x)^2} = \frac{2\pi}{q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kp\pi}{2q+k\pi} \qquad [p^2 < 1].$$

BI ((132))(13)a

4.283

1. $$\int_0^1 \left(\frac{x-1}{\ln x} - x\right) \frac{dx}{\ln x} = \ln 2 - 1.$$

BI ((132))(17)a

2. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} - \frac{1}{2}\right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} - 1.$$

BI ((127))(20)

3. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{x}{1-x} + \frac{x}{2}\right) \frac{dx}{x \ln x} = \frac{\ln 2\pi}{2}.$$

BI ((127))(23)

4. $$\int_0^1 \left[\frac{1}{(\ln x)^2} - \frac{x}{(1-x)^2}\right] dx = \boldsymbol{C} - \frac{1}{2}.$$

GW ((326))(8a)

5. $$\int_0^1 \left(\frac{1}{1-x^2} + \frac{1}{2\ln x} - \frac{1}{2}\right) \frac{dx}{\ln x} = \frac{\ln 2 - 1}{2}.$$

BI ((128))(14)

6. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{2} \cdot \frac{1+x}{1-x} - \ln x\right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2}.$$

BI ((127))(22)

7. $$\int_0^1 \left[\frac{1}{1-\ln x} - x\right] \frac{dx}{x \ln x} = -\boldsymbol{C}.$$

GW ((326))(11a)

8. $$\int_0^1 \left[\frac{x^q-1}{x(\ln x)^2} - \frac{q}{\ln x}\right] dx = q\ln q - q \qquad [q>0].$$

BI ((126))(2)

9. $$\int_0^1 \left[x + \frac{1}{a\ln x - 1}\right] \frac{dx}{x\ln x} = \ln\frac{a}{q} + \boldsymbol{C} \qquad [a>0, \quad q>0].$$

BI ((126))(8)

10. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)}\right] \frac{x^{p-1}}{\ln x}\, dx = -\ln\Gamma(p) + \left(p - \frac{1}{2}\right)\ln p - p + \frac{\ln 2\pi}{2} \qquad [p>0].$$

GW ((326))(9)

11. $$\int_0^1 \left[p - 1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x}\right) x^{p-1}\right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)\ln p + p - \frac{\ln 2\pi}{2} \qquad [p>0].$$

BI ((127))(25)

12. $$\int_0^1 \left[-\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2}\right] dx = -\psi(p) + p - \frac{3}{2} \qquad [p>0]$$

GW ((326))(8)

13. $$\int_0^1 \left[\left(p - \frac{1}{2}\right) x^3 + \frac{1}{2}\left(1 - \frac{1}{\ln x}\right)\left(x^{2p-1} - 1\right)\right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)(\ln p - 1) \qquad [p>0].$$

BI ((132))(23)a

14. $$\int_0^1 \left[\left(q - \frac{1}{2}\right) \frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r}\right] \frac{dx}{\ln x} = (p-r)\left[\frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2}\right] \qquad [q>0].$$

BI ((132))(13)

4.284

1. $$\int_0^1 \left[\frac{x^q-1}{x(\ln x)^3} - \frac{q}{x(\ln x)^2} - \frac{q^2}{2\ln x}\right] dx = \frac{q^2}{2}\ln q - \frac{3}{4}q^2 \qquad [q>0].$$

2. $$\int_0^1 \left[\frac{x^q-1}{x(\ln x)^4} - \frac{q}{x(\ln x)^3} - \frac{q^2}{2x(\ln x)^2} - \frac{q^3}{6\ln x}\right] dx = \frac{q^3}{6}\ln q - \frac{11}{36}q^3 \qquad [q>0].$$

BI ((126))(4)

4.285

$$\int_0^1 \frac{x^{p-1}\,dx}{(q+\ln x)^n} = \frac{p^{n-1}}{(n-1)!}\,e^{-pq}\,\mathrm{Ei}(pq) - \frac{1}{(n-1)!q^{n-1}}\sum_{k=1}^{n-1}(n-k-1)!(pq)^{k-1} \qquad [p>0, \quad q<0].$$

BI ((125))(21)

In integrals of the form $\int \frac{x^a(\ln x)^n\,dx}{[b\pm(\ln x)^m]^l}$, we should make the substitution $x=e^t$ or $x=e^{-t}$ and then seek the resulting integrals in 3.351-3.356.

4.29- 4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.291

1. $$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$$

FI II 483

2. $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}.$$

FI II 714

3. $$\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$$

BI ((145))(2)

4. $$\int_0^1 \ln\left(1-\frac{x}{2}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$$

5. $$\int_0^1 \frac{\ln \frac{1+x}{2}}{1-x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$$

BI ((115))(1)

6. $$\int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \frac{1}{2}(\ln 2)^2.$$

BI ((114))(14)a

7. $$\int_0^\infty \frac{\ln(1+ax)}{1+x}\,dx = \frac{\pi}{4}\ln(1+a^2) - \int_0^a \frac{\ln u\,du}{1+u^2} \qquad [a>0].$$

GI II (2209)

8. $$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2.$$

FI II 157

9. $$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2 + \boldsymbol{G}.$$

BI ((136))(1)

10. $$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2 - \boldsymbol{G}.$$

BI ((114))(17)

11. $$\int_1^\infty \frac{\ln(x-1)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2.$$

BI ((144))(4)

12. $$\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2.$$

BI ((144))(4)

BI ((141))(9)a

14. $$\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{1}{a(a-b)}\ln\frac{a+b}{b} + \frac{2\ln 2}{b^2-a^2} \qquad [a \neq b, \quad ab>0];$$
$$= \frac{1}{2a^2}(1-\ln 2) \qquad [a=b].$$

LI ((114))(5)a

15. $$\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{\ln\frac{a}{b}}{a(a-b)} \qquad [ab>0].$$

BI ((139))(5)

16. $$\int_0^1 \ln(a+x)\frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}}\ \operatorname{arcctg}\sqrt{a}\ln[(1+a)a] \qquad [a>0].$$

BI ((114))(20)

17. $$\int_0^\infty \ln(a+x)\frac{dx}{(b+x)^2} = \frac{a\ln a - b\ln b}{b(a-b)} \qquad [a>0, \quad b>0, \quad a \neq b].$$

LI ((139))(6)

18. $$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{1}{2}\operatorname{arctg} a\ln(1+a^2).$$

GI II (2195)

19. $$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\,dx = \frac{1}{2\sqrt{a}}\operatorname{arctg}\sqrt{a}\ln(1+a) \qquad [a>0].$$

BI ((114))(21)

20. $$\int_0^1 \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[\frac{1}{2}(a+b)\ln(a+b) - b\ln b - a\ln 2\right] \qquad [a>0, \quad b>0, \quad a \neq b].$$

BI ((114))(22)

BI ((139))(8)

22. $$\int_0^\infty \ln(a+x)\frac{x\,dx}{(b^2+x^2)^2} = \frac{1}{2(a^2+b^2)}\left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right) \qquad [a>0, \quad b>0].$$

BI ((139))(9)

23. $$\int_0^1 \ln(1+x)\frac{1+x^2}{(1+x)^4}\,dx = -\frac{1}{3}\ln 2 + \frac{23}{72}.$$

LI ((114))(12)

24. $$\int_0^1 \ln(1+x)\frac{1+x^2}{a^2+x^2}\cdot\frac{dx}{1+a^2x^2} = \frac{1}{2a(1+a^2)}\left[\frac{\pi}{2}\ln(1+a^2) - 2\operatorname{arctg} a\cdot\ln a\right] \qquad [a>0].$$

LI ((114))(11)

25. $$\int_0^1 \ln(1+x)\frac{1-x^2}{(ax+b)^2}\,\frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2}\left\{\frac{1}{a-b}\left[\frac{a+b}{ab}\ \ln(a+b) - \frac{1}{a}\ln b - \frac{1}{b}\ln a\right] + \right.$$
$$\left. + \ \frac{4\ln 2}{b^2-a^2}\right\} \qquad [a>0, \quad b>0, \quad a^2 \neq b^2].$$

LI ((114))(13)

26. $$\int_0^\infty \ln(1+x)\frac{1-x^2}{(ax+b)^2}\cdot\frac{dx}{(bx+a)^2} = \frac{1}{ab(a^2-b^2)}\ln\frac{b}{a} \qquad [a>0, \quad b>0].$$

LI ((139))(14)

27. $$\int_0^1 \ln(1+ax)\frac{1-x^2}{(1+x^2)^2}\,dx = \frac{1}{2}\frac{(1+a)^2}{1+a^2}\ln(1+a) - \frac{1}{2}\cdot\frac{a}{1+a^2}\ln 2 - \frac{\pi}{4}\cdot\frac{a^2}{1+a^2} \qquad [a>-1].$$

BI ((114))(23)}

28. $$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{1}{a^2+b^2}\left(a\ln\frac{b}{a} - \frac{b\pi}{2}\right) \qquad [a>0, \quad b>0].$$

BI ((139))

29. $$\int_0^\infty \ln(a-x)^2\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{b\pi}{2}\right) \qquad [a>0, \quad b>0].$$

30. $$\int_0^\infty \ln(a-x)^2 \frac{x\,dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2}\left(\ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right) \qquad [a>0, \quad b>0].$$

BI ((139))(10)

4.292

1. $$\int_0^1 \frac{\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln 2 \pm 2\boldsymbol{G}.$$

GW ((325))(20)

2. $$\int_0^1 \frac{x\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -1 \pm \frac{\pi}{2}.$$

GW ((325))(22c)

3. $$\int_{-a}^a \frac{\ln(1+bx)}{\sqrt{a^2-x^2}}\,dx = \pi\ln\frac{1+\sqrt{1-a^2b^2}}{2} \qquad \left[0 \le |b| \le \frac{1}{a}\right].$$

BI ((145))(16, 17)A, GW ((325))(21e)

4. $$\int_0^1 \frac{x\ln(1+ax)}{\sqrt{1-x^2}}\,dx = -1 + \frac{\pi}{2}\cdot\frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a}\arcsin a \qquad [|a|\le 1];$$
$$= -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a}\ln\left(a+\sqrt{a^2-1}\right) \qquad [a \ge 1].$$

GW ((325))(22)

5. $$\int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin a(\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2}(\arccos a)^2 \qquad [|a| \le 1].$$

BI ((120))(4), GW ((325))(21a)

4.293

1. $$\int_0^1 x^{\mu-1}\ln(1+x)\,dx = \frac{1}{\mu}[\ln 2 - \beta(\mu+1)] \qquad [\operatorname{Re}\mu > -1].$$

BI ((106))(4)a

ET I 315(17)

3. $$\int_0^\infty x^{\mu-1}\ln(1+x)\,dx = \frac{\pi}{\mu\sin\mu\pi} \qquad [-1 < \operatorname{Re}\mu < 0].$$

GW ((325))(3)a

4. $$\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}$$

GW ((325))(2b)

5. $$\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right].$$

GW ((325))(2c)

6. $$\int_0^1 x^{n-\frac{1}{2}}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1} + \frac{(-1)^n\cdot 4}{2n+1}\left[\pi - \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right].$$

GW ((325))(2f)

7. $$\int_0^\infty x^{\mu-1}\ln|1-x|\,dx = \frac{\pi}{\mu}\operatorname{ctg}(\mu\pi) \qquad [-1 < \operatorname{Re}\mu < 0].$$

BI ((134))(4), ET I 315(18)

8. $$\int_0^1 x^{\mu-1}\ln(1-x)\,dx = -\frac{1}{\mu}[\psi(\mu+1)-\psi(1)] \qquad [\operatorname{Re}\mu > -1].$$

ET I 316(19)

9.[7] $$\int_1^\infty x^{\mu-1}\ln(x-1)\,dx = \frac{1}{\mu}[\pi\operatorname{ctg}(\mu\pi)+\psi(\mu+1)+\boldsymbol{C}] \qquad [\operatorname{Re}\mu < 0].$$

ET I 316(20)

BI ((134))(3)

11. $$\int_0^\infty \frac{x^{\mu-1}\ \ln(1+x)}{1+x}\,dx = \frac{\pi}{\sin\mu\pi}[\boldsymbol{C}+\psi(1-\mu)] \qquad [-1<\operatorname{Re}\mu<1].$$

ET I 316(21)

12. $$\int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}}\,dx = \frac{-\ln 2}{2^\mu\mu} + \frac{2^\mu-1}{2^\mu\mu^2}.$$

BI ((114))(6)

13. $$\int_0^1 \frac{x^{\mu-1}\ \ln(1-x)}{(1-x)^{1-\nu}}\,dx = \mathrm{B}(\mu,\,\nu)[\psi(\nu)-\psi(\mu+\nu)] \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

ET I 316(122)

14. $$\int_0^\infty \frac{x^{\mu-1}\ \ln(\gamma+x)}{(\gamma+x)^\nu}\,dx = \gamma^{\mu-\nu}\mathrm{B}(\mu,\,\nu-\mu)[\psi(\nu)-\psi(\nu-\mu)+\ln\gamma] \qquad [0<\operatorname{Re}\mu<\operatorname{Re}\nu].$$

ET I 316(23)

4.294

1. $$\int_0^1 \ln(1+x)\frac{(p-1)x^{p-1}-px^{-p}}{x}\,dx = 2\ln 2 - \frac{\pi}{\sin p\pi} \qquad [0<p<1].$$

BI ((114))(2)

2. $$\int_0^1 \ln(1+x)\frac{1+x^{2n+1}}{1+x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1} - \sum_{j=1}^{2n+1}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}.$$

BI ((114))(7)

3. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1+x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1} - \sum_{j=1}^{2n}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}.$$

BI ((114))(8)

4. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1-x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1} + \sum_{i=1}^{2n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}.$$

5. $$\int_0^1 \ln(1+x)\frac{1-x^{2n+1}}{1-x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1} + \sum_{j=1}^{2n+1}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}.$$

BI ((114))(10)

6. $$\int_0^1 \ln(1-x)\frac{1-(-1)^n x^n}{1-x}\,dx = \sum_{j=1}^{n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{1}{k}.$$

BI ((114))(15)

7. $$\int_0^1 \ln(1-x)\frac{1-x^n}{1-x}\,dx = -\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}\frac{1}{k}.$$

BI ((114))(16)

8. $$\int_0^\infty \ln(1-x)^2 x^p\,dx = \frac{2\pi}{p+1}\operatorname{ctg} p\pi \qquad [-2<p<-1].$$

BI ((134))(13)a

9. $$\int_0^1 [\ln(1+x)]^n(1+x)^r\,dx = (-1)^{n-1}\frac{n!}{(r+1)^{n+1}} + 2^{r+1}\sum_{k=0}^{n}\frac{(-1)^k n!(\ln 2)^{n-k}}{(n-k)!(r+1)^{k+1}}.$$

LI ((106))(34)a

10. $$\int_0^1 [\ln(1-x)]^n(1-x)^r\,dx = (-1)^n\frac{n!}{(r+1)^{n+1}} \qquad [r>-1].$$

BI ((106))(35)a

11. $$\int_0^1 \left(\ln\frac{1}{1-x^2}\right)^n x^{2q-1}\,dx = \frac{n!}{2}\zeta(n+1,\,q+1) \qquad [-1<q<0].$$

BI ((311))(15)a

12. $$\int_0^1 (\ln x)^{2n}\ln(1-x^2)\frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)}|B_{2n+2}|.$$

BI ((309))(5)a

13.[6] $$\int_0^1 [\ln(1/x)]^m \ln(1-x^2)\,dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}} \qquad [m+1>0, \quad n+1>0].$$

4.295

1. $$\int_0^\infty \ln(\mu x^2+\beta)\,\frac{dx}{\gamma+x^2} = \frac{\pi}{\sqrt{\gamma}}\,\ln(\sqrt{\mu\gamma}+\sqrt{\beta}) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>0, \quad |\arg\gamma|<\pi].$$

ET II 218(27)

2. $$\int_0^1 \ln(1+x^2)\frac{dx}{x^2} = \frac{\pi}{2} - \ln 2.$$

GW ((325))(2g)

3. $$\int_0^\infty \ln(1+x^2)\frac{dx}{x^2} = \pi.$$

GW ((325))(4c)

4. $$\int_0^\infty \ln(1+x^2)\frac{dx}{(a+x)^2} = \frac{2a}{1+a^2}\left(\frac{\pi}{2a} + \ln a\right) \qquad [a>0].$$

BI ((319))(6)a

5. $$\int_0^1 \ln(1+x^2)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2 - \boldsymbol{G}.$$

BI ((114))(24)

6. $$\int_1^\infty \ln(1+x^2)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2 + \boldsymbol{G}.$$

BI ((114))(5)

7. $$\int_0^\infty \ln(a^2+b^2x^2)\frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\ln\frac{ag+bc}{g} \qquad [a>0, \quad b>0, \quad c>0, \quad g>0].$$

8. $$\int_0^\infty \ln(a^2+b^2x^2)\frac{dx}{c^2-g^2x^2} = -\frac{\pi}{cg}\operatorname{arctg}\frac{bc}{ag} \qquad [a>0, \quad b>0, \quad c>0, \quad g>0].$$

BI ((136))(15)a

9. $$\int_0^\infty \frac{\ln(1+p^2x^2)-\ln(1+q^2x^2)}{x^2}\,dx = \pi(p-q) \qquad [p>0, \quad q>0].$$

FI II 645

10. $$\int_0^1 \ln\frac{1+a^2x^2}{1+a^2}\,\frac{dx}{1-x^2} = -(\operatorname{arctg} a)^2.$$

BI ((115))(2)

591

11. $$\int_0^1 \ln(1-x^2)\frac{dx}{x} = -\frac{\pi^2}{12}.$$

12. $$\int_0^\infty \ln(1-x^2)^2\frac{dx}{x^2} = 0.$$

BI ((142))(9)a

13. $$\int_0^1 \ln(1-x^2)\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2 - \boldsymbol{G}.$$

GW ((325))(17)

14. $$\int_1^\infty \ln(x^2-1)\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2 + \boldsymbol{G}.$$

BI ((144))(6)

15. $$\int_0^\infty \ln(a^2-x^2)^2\,\frac{dx}{b^2+x^2} = \frac{\pi}{b}\ln(a^2+b^2) \qquad [b>0].$$

BI ((136))(16)

16. $$\int_0^\infty \ln(a^2-x^2)^2\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = -\frac{2b\pi}{a^2+b^2} \qquad [b>0].$$

17. $\int_0^1 \ln(1+x^2)\,\frac{dx}{x(1+x^2)} = \frac{1}{2}\left[\frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2\right].$

BI ((114))(25)

18. $\int_0^\infty \ln(1+x^2)\,\frac{dx}{x(1+x^2)} = \frac{\pi^2}{12}.$

BI ((141))(9)

19. $\int_0^1 \ln(\cos^2 t + x^2 \sin^2 t)\frac{dx}{1-x^2} = -t^2.$

BI ((114))(27)a

20. $\int_0^\infty \ln(a^2+b^2x^2)\frac{dx}{(c+gx)^2} = \frac{2\ln b}{cg} + \frac{b^2}{a^2g^2+b^2c^2}\left(\frac{a}{b}\pi + 2\frac{c}{g}\ln\frac{c}{g} + 2\frac{a^2g}{b^2c}\ln\frac{a}{b}\right)$
$[a>0, \quad b>0, \quad c>0, \quad g>0].$

21. $\int_0^1 \ln(a^2+b^2x^2)\frac{dx}{(c+gx)^2} =$
$= \frac{2}{c(c+g)}\ln a + \frac{b^2}{a^2g^2+b^2c^2}\left[\frac{2a}{b}\ \operatorname{arcctg}\frac{a}{b} + \frac{cb^2-ga^2}{b^2(c+g)}\ln\frac{a^2+b^2}{a^2} - 2\frac{c}{g}\ln\frac{c+g}{c}\right]$
$[a>0, \quad b>0, \quad c>0, \quad g>0].$

BI ((114))(28)a

22. $\int_0^\infty \frac{\ln(1+p^2x^2)}{r^2+q^2x^2}\,dx = \int_0^\infty \frac{\ln(p^2+x^2)}{q^2+r^2x^2}\,dx = \frac{\pi}{qr}\ln\frac{q+pr}{q} \qquad [qr>0, \quad p>0].$

FI II 745a, BI ((318))(1)A, BI ((318))(4)a

23. $\int_0^\infty \frac{\ln(1+a^2x^2)}{b^2+c^2x^2}\,\frac{dx}{d^2+g^2x^2} = \frac{\pi}{b^2g^2-c^2d^2}\left[\frac{g}{d}\ln\left(1+\frac{ad}{g}\right) - \frac{c}{b}\ln\left(1+\frac{ab}{c}\right)\right]$
$[a>0, \quad b>0, \quad c>0, \quad d>0, \quad g>0, \quad b^2g^2 \neq c^2d^2].$

BI ((141))(10)

24. $\int_0^\infty \frac{\ln(1+a^2x^2)}{b^2+c^2x^2}\,\frac{x^2\,dx}{d^2+g^2x^2} = \frac{\pi}{b^2g^2-c^2d^2}\left[\frac{b}{c}\ln\left(1+\frac{ab}{c}\right) - \frac{d}{g}\ln\left(1+\frac{ad}{g}\right)\right]$
$[a>0, \quad b>0, \quad c>0, \quad d>0, \quad g>0, \quad b^2g^2 \neq c^2d^2].$

25. $$\int_0^\infty \ln(a^2+b^2x^2)\,\frac{dx}{(c^2+g^2x^2)^2} = \frac{\pi}{2c^3g}\left(\ln\frac{ag+bc}{g} - \frac{bc}{ag+bc}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0].$$

GW ((325))(18a)

26. $$\int_0^\infty \ln(a^2+b^2x^2)\,\frac{x^2\,dx}{(c^2+g^2x^2)^2} = \frac{\pi}{2cg^3}\left(\ln\frac{ag+bc}{g} + \frac{bc}{ag+bc}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0].$$

GW ((325))(18b)

27. $$\int_0^1 \ln(1+ax^2)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\,\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\} \qquad [a>0].$$

BI ((117))(6)

28. $$\int_0^1 \ln(1+a-ax^2)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\,\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\} \qquad [a>0].$$

BI ((117))(7)

29. $$\int_0^1 \ln(1-a^2x^2)\,\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1-a^2}}{2} \qquad [a^2<1].$$

BI ((119))(1)

30.[6] $$\int_0^1 \ln(1-a^2x^2)\,\frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2$$

LI ((120))(11)

31. $$\int_0^1 \ln(1-x^2)\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln\frac{k'}{k}\boldsymbol{K}(k) - \frac{\pi}{2}\,\boldsymbol{K}(k').$$

BI ((120))(12)

32. $$\int_0^1 \ln(1\pm kx^2)\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\ln\frac{2\pm 2k}{\sqrt{k}}\boldsymbol{K}(k) - \frac{\pi}{8}\boldsymbol{K}(k').$$

33. $$\int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \ln k'\,\boldsymbol{K}(k).$$

BI ((119))(27)

34. $$\int_0^1 \ln(1-k^2x^2)\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = (2-k^2)\boldsymbol{K}(k) - (2-\ln k')\boldsymbol{E}(k).$$

BI ((119))(3)

35. $$\int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}}\ln(1-k^2x^2)\,dx = \frac{1}{k^2}(1+k'^2-k'^2\ \ln k')\boldsymbol{K}(k)-(2-\ln k')\boldsymbol{E}(k).$$

BI ((119))(7)

36. $$\int_{-1}^1 \ln(1-x^2)\,\frac{dx}{(a+bx)\sqrt{1-x^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \qquad [a>0, \quad b>0, \quad a\neq b].$$

BI ((145))(15)

37.[8] $$\int_0^1 \ln(1-x^2)(px^{p-1}-qx^{q-1})\,dx = \psi\left(\frac{q}{2}+1\right)-\psi\left(\frac{p}{2}+1\right) \qquad [p>-2, \quad q>-2].$$

BI ((106))(15)

38. $$\int_0^1 \ln(1+ax^2)\,\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1+a}}{2} \qquad [a\geq -1].$$

GW ((325))(21b)

39. $$\int_0^1 \ln(1+x^2)x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln 2 - \beta\left(\frac{\mu}{2}+1\right)\right] \qquad [\operatorname{Re}\mu > -2].$$

BI ((106))(12)

40. $$\int_0^\infty \ln(1+x^2)x^{\mu-1}\,dx = \frac{\pi}{\mu\sin\frac{\mu\pi}{2}} \qquad [-2<\operatorname{Re}\mu<0].$$

41. $$\int_0^\infty \ln(1+x^2)\frac{x^{\mu-1}\,dx}{1+x} = \frac{\pi}{\sin\mu\pi}\left\{\ln 2 - (1-\mu)\sin\frac{\mu\pi}{2}\beta\left(\frac{1-\mu}{2}\right) - (2-\mu)\cos\frac{\mu\pi}{2}\beta\left(\frac{2-\mu}{2}\right)\right\}$$
$$[-2 < \operatorname{Re}\mu < 1].$$

ET I 316(25)

4.296

1. $$\int_0^1 \ln(1+2x\cos t+x^2)\,\frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2}.$$

BI ((114))(34)

2. $$\int_{-\infty}^\infty \ln(a^2-2ax\cos t+x^2)\,\frac{dx}{1+x^2} = \pi\ln(1+2a\,|\sin t|+a^2).$$

BI ((145))(28)

3. $$\int_0^\infty \ln(1+2x\cos t+x^2)x^{\mu-1}\,dx = \frac{2\pi}{\mu}\,\frac{\cos\mu t}{\sin\mu\pi} \qquad [|t|<\pi, \quad -1<\operatorname{Re}\mu<0].$$

ET I 316(27)

4. $$\int_0^\infty \ln\left(\frac{x^2+2ax\cos t+a^2}{x^2-2ax\cos t+a^2}\right)\frac{x\,dx}{x^2+b^2} = \frac{1}{2}\pi^2-\pi t+\pi\operatorname{arctg}\frac{(a^2-b^2)\cos t}{(a^2+b^2)\sin t+2ab}$$
$$[a>0, \quad b>0, \quad 0<t<\pi]$$

4.297

1. $$\int_0^1 \ln\frac{ax+b}{bx+a}\,\frac{dx}{(1+x)^2} = \frac{1}{a-b}\left[(a+b)\ln\frac{a+b}{2} - a\ln a - b\,\ln b\right] \qquad [a>0, \quad b>0].$$

BI ((115))(16)

2. $$\int_0^\infty \ln\frac{ax+b}{bx+a}\,\frac{dx}{(1+x)^2} = 0 \qquad [ab>0].$$

BI ((139))(23)

3. $$\int_0^1 \ln\frac{1-x}{x}\,\frac{dx}{1+x^2} = \frac{\pi}{8}\ln 2.$$

4. $\int_0^1 \ln \frac{1+x}{1-x} \frac{dx}{1+x^2} = \boldsymbol{G}.$

BI ((115))(17)

5. $\int_0^\infty \ln \left(\frac{1+x}{1-x}\right)^2 \frac{dx}{x(1+x^2)} = \frac{\pi^2}{2}.$

BI ((141))(13)

6. $\int_u^v \ln \frac{v+x}{u+x} \frac{dx}{x} = \frac{1}{2}\left(\ln \frac{v}{u}\right)^2 \qquad [uv > 0].$

BI ((145))(33)

7. $\int_0^\infty \frac{b\ln(1+ax) - a\ \ln(1+bx)}{x^2}\, dx = ab \ln \frac{b}{a} \qquad [a > 0, \quad b > 0].$

FI II 647

8. $\int_0^1 \ln \frac{1+ax}{1-ax} \frac{dx}{x\sqrt{1-x^2}} = \pi\ \arcsin a \qquad [|a| \le 1]$

GW ((325))(21c), BI ((122))(2)

9. $\int_u^v \ln \left(\frac{1+ax}{1-ax}\right) \frac{dx}{\sqrt{(x^2-u^2)(v^2-x^2)}} = \frac{\pi}{v} F\left(\arcsin av,\ \frac{u}{v}\right) \qquad [|av| < 1].$

BI ((145))(35)

4.298

1. $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x}\, dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1).$

BI ((137))(1)

2. $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1+x}\, dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1).$

3. $$\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1-x}\, dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1).$$

BI ((137))(2)

4. $$\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1-x}\, dx = -\frac{\ln 2}{2n} - \frac{1}{4n^2} + \frac{1}{2n}\beta(2n+1).$$

BI ((137))(4)

5. $$\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x^2}\, dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1).$$

BI ((137))(10)

6. $$\int_0^1 \ln \frac{1+x^2}{x} x^{2n}\, dx = \frac{1}{2n+1}\left\{(-1)^n \frac{\pi}{2} + \ln 2 - \frac{1}{2n+1} + 2\sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1}\right\}.$$

BI ((294))(8)

7. $$\int_0^1 \ln \frac{1+x^2}{x} x^{2n-1}\, dx = \frac{1}{2n}\left\{(-1)^{n+1} \ln 2 + \ln 2 - \frac{1}{2n} + (-1)^{n+1}\sum_{k=1}^{n-1} \frac{(-1)^k}{k}\right\}.$$

BI ((294))(9)a

8. $$\int_0^1 \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2.$$

BI ((115))(7)

9. $$\int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \pi \ln 2.$$

BI ((137))(8)

10. $$\int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1-x^2} = 0.$$

BI ((137))(9)

11. $$\int_0^1 \ln \frac{1-x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2.$$

BI ((115))(9)

12. $$\int_1^\infty \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2.$$

BI ((144))(8)

13. $$\int_0^1 \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 - \boldsymbol{G}.$$

BI ((115))(18)

14. $$\int_1^\infty \ln \frac{1+x^2}{x-1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 + \boldsymbol{G}.$$

BI ((144))(9)

15. $$\int_0^1 \ln \frac{1+x^2}{1-x} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2.$$

BI ((115))(19)

16. $$\int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x\,dx}{1+x^2} = \frac{\pi^2}{12}.$$

BI ((138))(3)

17. $$\int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg} \ln \frac{ag+bc}{c} \qquad [a>0, \quad b>0, \quad c>0, \quad g>0].$$

BI ((138))(6, 7, 9, 10)a

18. $$\int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2-g^2x^2} = \frac{1}{cg} \operatorname{arctg} \frac{ag}{bc} \qquad [a>0,\ b>0,\ c>0,\ g>0].$$

BI ((138))(8, 11)a

19. $$\int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x^2\,dx}{(1+x^2)^2} = \frac{\pi}{4}(\ln 4 - 1).$$

20. $\int_0^1 \ln\left(\frac{1-x^2}{x^2}\right)^2 \sqrt{1-x^2}\,dx = \pi.$

FI II 643a

21. $\int_0^1 \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\,\frac{dx}{x} = \frac{1}{2}\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\,\frac{dx}{x} = -\frac{t^2}{2} \qquad [|t|<\pi].$

BI ((115))(23), BI ((134))(15)

22. $\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\,x^{p-1}\,dx = -\frac{2\pi(1-\cos pt)}{p\,\sin p\pi} \qquad [0<|p|<1, \quad |t|<\pi].$

BI ((134))(17)

23. $\int_0^1 \ln\frac{1+x^2\sin t}{1-x^2\sin t}\,\frac{dx}{\sqrt{1-x^2}} = \pi\ln\operatorname{ctg}\left(\frac{\pi-t}{4}\right) \qquad [|t|<\pi].$

GW ((325))(21d)

4.299

1. $\int_0^\infty \ln\frac{(x+1)(x+a^2)}{(x+a)^2}\,\frac{dx}{x} = (\ln a)^2 \qquad [a>0].$

BI ((134))(14)

2. $\int_0^1 \ln\frac{(1-ax)(1+ax^2)}{(1-ax^2)^2}\,\frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}}\operatorname{arctg}\sqrt{a}\ln(1+a) \qquad [a>0].$

BI ((115))(25)

3. $\int_0^1 \ln\frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2}\,\frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}}\operatorname{arctg}\sqrt{a}\ln(1+a) \qquad [a>0].$

BI ((115))(26)

4. $\int_0^1 \ln\frac{(x+1)(x+a^2)}{(x+a)^2}\,x^{\mu-1}\,dx = \frac{\pi(a^\mu-1)^2}{\mu\sin\mu\pi} \qquad [a>0, \quad \operatorname{Re}\mu>0].$

BI ((134))(16)

4.311

1.[3] $\int_0^\infty \ln(a^3 - x^3)\frac{dx}{x^3}$ integral divergent.

BI ((134))(7)

2. $\int_0^\infty \ln(1+x^3)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\ln 3.$

LI ((136))(8)

3. $\int_0^\infty \ln(1+x^3)\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9}.$

LI ((136))(6)

4. $\int_0^\infty \ln(1+x^3)\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 + \frac{\pi^2}{9}.$

LI ((136))(7)

5. $\int_0^\infty \ln(1+x^3)\frac{1-x}{1+x^3}\,dx = -\frac{2}{9}\,\pi^2.$

BI ((136))(9)

6.[8] $\int_0^\infty \ln\left|1-\frac{n^3}{a^3}\right|\frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$

4.312

1. $\int_0^\infty \ln\frac{1+x^3}{x^3}\,\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 + \frac{\pi^2}{9}.$

BI ((138))(12)

2. $\int_0^\infty \ln\frac{1+x^3}{x^3}\,\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9}.$

4.313

1. $$\int_0^\infty \ln x \, \ln(1+a^2x^2)\frac{dx}{x^2} = \pi a(1-\ln a) \qquad [a>0].$$

BI ((134))(18)

2. $$\int_0^\infty \ln(1+c^2x^2)\ln(a^2+b^2x^2)\frac{dx}{x^2} = 2\pi\left[\left(c+\frac{b}{a}\right)\ln(b+ac) - \frac{b}{a}\ln b - c\ln c\right]$$
$$[a>0, \quad b>0, \quad c>0].$$

BI ((134))(20, 21)a

3. $$\int_0^\infty \ln(1+c^2x^2)\ln\left(a^2+\frac{b^2}{x^2}\right)\frac{dx}{x^2} = 2\pi\left[\frac{a+bc}{b}\ln(a+bc) - \frac{a}{b}\ln a - c\right]$$
$$[a>0, \quad a+bc>0].$$

BI ((134))(22, 23)a

4. $$\int_0^\infty \ln x \ln\frac{1+a^2x^2}{1+b^2x^2}\,\frac{dx}{x^2} = \pi(a-b)+\pi\ln\frac{b^b}{a^a} \qquad [a>0, \quad b>0].$$

BI ((134))(24)

5. $$\int_0^\infty \ln x \ln\frac{a^2+2bx+x^2}{a^2-2bx+x^2}\,\frac{dx}{x} = 2\pi\ln a\,\arcsin\frac{b}{a} \qquad [a\ge|b|].$$

BI ((134))(25)

6. $$\int_0^\infty \ln(1+x)\frac{x\ln x - x - a}{(x+a)^2}\,\frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)} \qquad [a>0].$$

BI ((141))(7)

7. $$\int_0^\infty \ln(1-x)^2\,\frac{x\ln x - x - a}{(x+a)^2}\,\frac{dx}{x} = \frac{\pi^2+(\ln a)^2}{1+a} \qquad [a>0].$$

LI ((141))(8)

4.314

1. $$\int_0^1 \ln(1+ax)\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \sum_{k=1}^\infty \frac{a^k}{k}\ln\frac{p+k}{q+k}+\ln\frac{p}{q} \qquad [a>0, \quad p>0, \quad q>0].$$

2. $$\int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q}\right] \frac{dx}{x\ln(1+x)} = \ln\Gamma(q) \qquad [q>0].$$

BI ((143))(7)

3. $$\int_0^1 \frac{x\ln x + 1 - x}{x(\ln x)^2}\,\ln(1+x)dx = \ln\frac{4}{\pi}.$$

BI ((126))(12)

4. $$\int_0^1 \frac{\ln(1-x^2)dx}{x(q^2+(\ln x)^2)} = -\frac{\pi}{q}\ln\Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q}\ln 2q + \ln\frac{q}{\pi} - 1 \qquad [q>0].$$

LI ((327))(12)a

4.315

1. $$\int_0^1 \ln(1+x)(\ln x)^{n-1}\frac{dx}{x} = (-1)^{n-1}(n-1)!\left(1-\frac{1}{2^n}\right)\zeta(n+1).$$

BI ((116))(3)

2. $$\int_0^1 \ln(1+x)(\ln x)^{2n}\frac{dx}{x} = \frac{2^{2n+1}-1}{(2n+1)(2n+2)}\pi^{2n+2}|B_{2n+2}|.$$

BI ((116))(1)

3. $$\int_0^1 \ln(1-x)(\ln x)^{n-1}\frac{dx}{x} = (-1)^n(n-1)!\zeta(n+1).$$

BI ((116))(4)

4. $$\int_0^1 \ln(1-x)(\ln x)^{2n}\frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)}\pi^{2n+2}\ |B_{2n+2}|.$$

BI ((116))(2)

4.316

1. $$\int_0^1 \ln(1-ax^r)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -\frac{1}{r^{p+1}}\Gamma(p+1)\sum_{k=1}^\infty\frac{a^k}{k^{p+2}} \qquad [p>-1, \quad a<1, \quad r>0].$$

2. $$\int_0^1 \ln(1-2ax\cos t+a^2x^2)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -2\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k\cos kt}{k^{p+2}}.$$

LI ((116))(8)

4.317

1. $$\int_0^\infty \ln\frac{\sqrt{1+x^2}+a}{\sqrt{1+x^2}-a}\frac{dx}{\sqrt{1+x^2}} = \pi\ \arcsin a \qquad [|a|<1].$$

BI ((142))(11)

2. $$\int_0^1 \ln\frac{\sqrt{1-a^2x^2}-x\sqrt{1-a^2}}{1-x}\frac{dx}{x} = \frac{1}{2}(\arcsin a)^2.$$

BI ((115))(32)

3. $$\int_0^1 \ln\frac{1+\cos t\sqrt{1-x^2}}{1-\cos t\sqrt{1-x^2}}\frac{dx}{x^2+\operatorname{tg}^2 v} = \pi\operatorname{ctg} t\frac{\cos\frac{v-t}{2}}{\sin\frac{v+t}{2}}.$$

BI ((115))(30)

4. $$\int_0^1 \ln\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\frac{x\,dx}{1-x^2} = \frac{\pi^2}{2}.$$

BI ((115))(31)

5. $$\int_0^1 \ln\{\sqrt{1+kx}+\sqrt{1-kx}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{4}\ln(4k)\boldsymbol{K}(k)+\frac{\pi}{8}\boldsymbol{K}(k').$$

BI ((121))(8)

6. $$\int_0^1 \ln\{\sqrt{1+kx}-\sqrt{1-kx}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{4}\ \ln(4k)\boldsymbol{K}(k)+\frac{3}{8}\pi\boldsymbol{K}(k').$$

BI ((121))(9)

7. $$\int_0^1 \ln\{1+\sqrt{1-k^2x^2}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\ln k\boldsymbol{K}(k)+\frac{\pi}{4}\boldsymbol{K}(k').$$

8. $$\int_0^1 \ln\{1-\sqrt{1-k^2x^2}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\ln k\boldsymbol{K}(k) - \frac{3}{4}\pi\boldsymbol{K}(k').$$

BI ((121))(7)

9. $$\int_0^1 \ln\frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}}\,\frac{dx}{1-x} = \pi\,\arcsin p \qquad [p^2<1].$$

BI ((115))(29)

10. $$\int_0^1 \ln\frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \pi F(\arcsin q,\,k') \qquad [q^2<1].$$

BI ((122))(15)

11.* $$\int_{-\infty}^{\infty} \ln\left|\frac{1+2\sqrt{1+x^2}}{1-2\sqrt{1+x^2}}\right|\,\frac{dx}{\sqrt{1+x^2}} = \pi^2/3$$

4.318

1. $$\int_0^1 \frac{\ln(1-x^q)}{1+(\ln x)^2}\,\frac{dx}{x} = \pi\left[\ln\Gamma\left(\frac{q}{2\pi}+1\right) - \frac{\ln q}{2} + \frac{q}{2\pi}\left(\ln\frac{q}{2\pi}-1\right)\right] \qquad [q>0].$$

BI ((126))(11)

2. $$\int_0^\infty \ln(1+x^r)\left[\frac{(p-r)x^p-(q-r)x^q}{\ln x} + \frac{x^q-x^p}{(\ln x)^2}\right]\frac{dx}{x^{r+1}} = r\ln\left(\operatorname{tg}\frac{q\pi}{2r}\operatorname{ctg}\frac{p\pi}{2r}\right) \quad [p<r, q<r].$$

BI ((143))(9)

In integrals containing $\ln(a+bx^r)$, it is useful to make the substitution $x^r=t$ and then to seek the resulting integral in the tables.

For example,

$$\int_0^\infty x^{p-1}\ln(1+x^r)\,dx = \frac{1}{r}\int_0^\infty t^{\frac{p}{r}-1}\ln(1+t)\,dt = \frac{\pi}{p\sin\frac{p\pi}{r}} \qquad (\text{see } 4.293\ 3).$$

4.319

1. $$\int_0^\infty \ln(1-e^{-2a\pi x})\frac{dx}{1+x^2} = -\pi\left[\frac{1}{2}\ln 2a\pi + a(\ln a - 1) - \ln\Gamma(a+1)\right] \qquad [a>0].$$

BI ((354))(6)

2. $$\int_0^\infty \ln(1+e^{-2a\pi x})\frac{dx}{1+x^2} = \pi\left[\ln\Gamma(2a) - \ln\Gamma(a) + a(1-\ln a) - \left(2a-\frac{1}{2}\right)\ln 2\right] \qquad [a>0].$$

BI ((354))(7)

3. $$\int_0^\infty \ln\frac{a+be^{-px}}{a+be^{-qx}}\,\frac{dx}{x} = \ln\frac{a}{a+b}\ln\frac{p}{q} \qquad \left[\frac{b}{a} > -1, \quad pq > 0\right].$$

FI II 635, BI ((354))(1)

4.321

1. $$\int_{-\infty}^\infty x\ln\operatorname{ch}\, x\,dx = 0.$$

BI ((358))(2)a

2. $$\int_0^\infty \ln\operatorname{ch}\, x\,\frac{dx}{1-x^2} = 0.$$

BI ((138))(20)a

4.322

1.[7] $$\int_0^\pi x\ln\sin\, x\,dx = \frac{1}{2}\int_0^\pi x\ln\cos^2\, x\,dx = -\frac{\pi^2}{2}\ln 2.$$

BI ((432))(1, 2) FI II 643

2. $$\int_0^\infty \frac{\ln\sin^2\, ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\frac{1-e^{-2ab}}{2} \qquad [a>0, \quad b>0].$$

GW ((338))(28b)

GW ((338))(28a)

4. $\int_0^\infty \frac{\ln\sin^2 ax}{b^2-x^2}\,dx = -\frac{\pi^2}{2b} + a\pi \qquad [a>0, \quad b>0].$

BI ((418))(1)

5. $\int_0^\infty \frac{\ln\cos^2 ax}{b^2-x^2}\,dx = a\pi \qquad [a>0].$

BI ((418))(2)

6. $\int_0^\infty \frac{\ln\cos^2 x}{x^2}\,dx = -\pi.$

FI II 686

7.[7] $\int_0^{\frac{\pi}{4}} x^{\mu-1}\ln\sin x\,dx = -\frac{1}{2\mu}\left(\frac{\pi}{4}\right)^\mu \left[\ln 2 + \frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu+2k)}\right] \qquad [\operatorname{Re}\mu>0].$

LI ((425))(1)

8.[7] $\int_0^{\frac{\pi}{2}} x^{\mu-1}\ln\sin x\,dx = -\frac{1}{\mu}\left(\frac{\pi}{2}\right)^\mu \left[\frac{1}{\mu} - 2\sum_{k=1}^\infty \frac{\zeta(2k)}{4^{k}(\mu+2k)}\right] \qquad [\operatorname{Re}\mu>0].$

LI ((430))(1)

9. $\int_0^{\frac{\pi}{2}} \ln(1-\cos x)x^{\mu-1}\,dx = \frac{-1}{\mu}\left(\frac{\pi}{2}\right)^\mu \left[\frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu+2k)}\right] \qquad [\operatorname{Re}\mu>0].$

LI ((430))(2)

10. $\int_0^\infty \ln(1\pm 2p\cos\beta x + p^2)\frac{dx}{q^2+x^2} = \frac{\pi}{q}\ln(1\pm pe^{-\beta q}) \qquad [p^2<1];$

$= \frac{\pi}{q}\ln(p\pm e^{-\beta q}) \qquad [p^2>1]$

FI II 718a

4.323

1.[7] $$\int_0^{\pi} x \ln \operatorname{tg}^2 x \, dx = 0.$$

BI ((432))(3)

2. $$\int_0^{\infty} \frac{\ln \operatorname{tg}^2 ax}{b^2 + x^2}\, dx = \frac{\pi}{b} \ln \theta\, ab \qquad [a > 0, \quad b > 0].$$

GW ((338))(28c)

3. $$\int_0^{\infty} \ln \left(\frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x} \right)^2 \frac{dx}{x} = \frac{\pi^2}{2}.$$

GW ((338))(26)

4.324

1. $$\int_0^{\infty} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right)^2 \frac{dx}{x} = \pi^2.$$

GW ((338))(25)

2. $$\int_0^{\infty} \ln \frac{1 + 2a \cos px + a^2}{1 + 2a \cos qx + a^2} \, \frac{dx}{x} = \ln(1+a) \ln \frac{q^2}{p^2} \qquad [-1 < a \le 1];$$
$$= \ln \left(1 + \frac{1}{a}\right) \ln \frac{q^2}{p^2} \qquad [a < -1 \quad or \quad a \ge 1].$$

GW ((338))(27)

3. $$\int_0^{\infty} \ln(a^2 \sin^2 px + b^2 \cos^2 px) \frac{dx}{c^2 + x^2} = \frac{\pi}{c} [\ln(a \operatorname{sh} cp + b \operatorname{ch} cp) - cp]$$
$$[a > 0, \quad b > 0, \quad c > 0, \quad p > 0].$$

GW ((338))(29)

4.325

1.[3] $$\int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x} = -\boldsymbol{C} \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k} = -\boldsymbol{C} \ln 2 + 0.159\,868\,905 \ldots = -\frac{1}{2} (\ln 2)^2.$$

2. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{x+e^{i\lambda}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}e^{-ik\lambda}(\boldsymbol{C}+\ln k).$$

GW ((325))(26)

3. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x)^2} = \int_1^{\infty}\ln\ln\, x\frac{dx}{(1+x)^2} = \frac{1}{2}\left[\psi\left(\frac{1}{2}\right)+\ln 2\pi\right] = \frac{1}{2}\left(\ln\frac{\pi}{2}-\boldsymbol{C}\right).$$

BI ((147))(7)

4. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_1^{\infty}\ln\ln\, x\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

BI ((148))(1)

5. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \int_1^{\infty}\ln\ln\, x\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\frac{\sqrt[3]{2\pi}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

BI ((148))(2)

6. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \int_1^{\infty}\ln\ln\, x\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi-\ln\Gamma\left(\frac{1}{6}\right)\right].$$

BI ((148))(5)

7. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+2x\cos t+x^2} = \int_1^{\infty}\ln\ln\, x\frac{dx}{1+2x\cos t+x^2} = \frac{\pi}{2\sin t}\ln\frac{(2\pi)^{t/\pi}\,\Gamma\left(\frac{1}{2}+\frac{t}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{t}{2\pi}\right)}.$$

BI ((147))(9)

8. $$\int_0^1 \ln\ln\frac{1}{x}x^{\mu-1}\,dx = -\frac{1}{\mu}(\boldsymbol{C}+\ln\mu) \qquad [\operatorname{Re}\mu>0].$$

9. $$\int_1^\infty \ln\ln x \frac{x^{n-2}\,dx}{1+x^2+x^4+\cdots+x^{2n-2}} =$$

$$= \frac{\pi}{2n}\operatorname{tg}\frac{\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \qquad [n \text{ is even}].$$

$$= \frac{\pi}{2n}\operatorname{tg}\frac{\pi}{2n}\ln \pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \qquad [n \text{ is odd}].$$

BI ((148))(4)

10. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x^2)\sqrt{\ln\frac{1}{x}}} = \int_1^\infty \ln\ln x \frac{dx}{(1+x^2)\sqrt{\ln x}} =$$

$$= \sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}}[\ln(2k+1) + 2\ln 2 + \boldsymbol{C}].$$

BI ((147))(4)

11. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{x^{\mu-1}\,dx}{\sqrt{\ln\frac{1}{x}}} = -(\boldsymbol{C}+\ln 4\mu)\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$$

BI ((147))(3)

12. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\Gamma(\mu)[\psi(\mu)-\ln(\nu)]$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

BI ((147))(2)

4.326

1. $$\int_0^1 \ln(a-\ln x)x^{\mu-1}\,dx = \frac{1}{\mu}[\ln a - e^{a\mu}\operatorname{Ei}(-a\mu)] \qquad [\operatorname{Re}\mu > 0, \quad a > 0].$$

BI ((107))(23)

2. $$\int_0^{\frac{1}{e}} \ln\left(2\ln\frac{1}{x}-1\right)\frac{x^{2\mu-1}}{\ln x}\,dx = -\frac{1}{2}[\operatorname{Ei}(-\mu)]^2 \qquad [\operatorname{Re}\mu > 0].$$

BI ((145))(5)

4.327

1. $$\int_0^1 \ln[a^2 + (\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma\left(\dfrac{2a+3\pi}{4\pi}\right)}{\Gamma\left(\dfrac{2a+\pi}{4\pi}\right)} + \frac{\pi}{2} \ln \frac{\pi}{2} \qquad \left[a > -\frac{\pi}{2}\right].$$

BI ((147))(10)

2. $$\int_0^1 \ln[a^2 + 4(\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma\left(\dfrac{a+3\pi}{4\pi}\right)}{\Gamma\left(\dfrac{a+\pi}{4\pi}\right)} + \frac{\pi}{2} \ln \pi \qquad [a > -\pi].$$

BI ((147))(16)a

3. $$\int_0^\infty \ln[a^2 + (\ln x)^2] x^{\mu-1}\, dx = \frac{2}{\mu}[-\cos a\mu \ \mathrm{ci}(a\mu) - \sin a\mu \ \mathrm{si}(a\mu) + \ln a] \qquad [a > 0, \quad \operatorname{Re}\mu > 0].$$

GW ((325))(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains $\ln \ln \frac{1}{x}$, it is useful to make the substitution $\ln x = t$ and then seek the transformed integral in the tables.

4.33- 4.34 Combinations of logarithms and exponentials

4.331

1. $$\int_0^\infty e^{-\mu x} \ln x \, dx = -\frac{1}{\mu}(\boldsymbol{C} + \ln \mu) \qquad [\operatorname{Re}\mu > 0].$$

BI ((256))(2)

2. $$\int_1^\infty e^{-\mu x} \ln x \, dx = -\frac{1}{\mu} \operatorname{Ei}(-\mu) \qquad [\operatorname{Re}\mu > 0].$$

BI ((260))(5)

GW ((324))(81a)

4.332

1. $$\int_0^\infty \frac{\ln x\,dx}{e^x + e^{-x} - 1} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right] \qquad \text{(cf. 4.325 6.).}$$

4.325
BI ((257))(6)

2. $$\int_0^\infty \frac{\ln x\,dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}}\ln\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\sqrt{2\pi}\right] \qquad \text{(cf. 4.325 5.).}$$

4.325
BI ((257))(7)A, LI ((260))(3)

4.333

$$\int_0^\infty e^{-\mu x^2}\ln x\,dx = -\frac{1}{4}(\boldsymbol{C} + \ln 4\mu)\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$$

BI ((256))(8), FI II 807a

4.334

$$\int_0^\infty \frac{\ln x\,dx}{e^{x^2} + 1 + e^{-x^2}} = \frac{1}{2}\sqrt{\frac{\pi}{3}}\sum_{k=1}^\infty (-1)^k \frac{\boldsymbol{C} + \ln 4k}{\sqrt{k}}\sin\frac{k\pi}{3}.$$

BI ((357))(13)

603

4.335

1. $$\int_0^\infty e^{-\mu x}(\ln x)^2\,dx = \frac{1}{\mu}\left[\frac{\pi^2}{6} + (\boldsymbol{C} + \ln\mu)^2\right] \qquad [\operatorname{Re}\mu > 0].$$

ET I 149(13)

2. $$\int_0^\infty e^{-x^2}(\ln x)^2\,dx = \frac{\sqrt{\pi}}{8}\left[(\boldsymbol{C} + 2\ln 2)^2 + \frac{\pi^2}{2}\right].$$

3.[7] $$\int_0^\infty e^{-\mu x}(\ln x)^3\,dx = -\frac{1}{\mu}\left[(\boldsymbol{C}+\ln\mu)^3 + \frac{\pi^2}{2}(\boldsymbol{C}+\ln\mu) + 2\zeta(3)\right].$$

MI 26

4.336

1.[6] $$\int_0^\infty \frac{e^{-x}}{\ln x}\,dx = -0.154479567.$$

BI ((260))(9)

2. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\pi^2+(\ln x)^2} = \nu'(\mu) - e^\mu \qquad [\operatorname{Re}\mu > 0].$$

MI 26

4.337

1. $$\int_0^\infty e^{-\mu x}\ \ln(\beta+x)dx = \frac{1}{\mu}[\ln\beta - e^{\mu\beta}\operatorname{Ei}(-\beta\mu)] \qquad [|\arg\beta|<\pi, \quad \operatorname{Re}\mu>0].$$

BI ((256))(3)

2. $$\int_0^\infty e^{-\mu x}\ln(1+\beta x)dx = -\frac{1}{\mu}e^{\frac{\mu}{\beta}}\operatorname{Ei}\left(-\frac{\mu}{\beta}\right) \qquad [|\arg\beta|<\pi, \quad \operatorname{Re}\mu>0].$$

ET I 148(4)

3. $$\int_0^\infty e^{-\mu x}\ln|a-x|\,dx = \frac{1}{\mu}[\ln a - e^{-a\mu}\operatorname{Ei}(a\mu)] \qquad [a>0, \quad \operatorname{Re}\mu>0].$$

BI ((256))(4)

4.[7] $$\int_0^\infty e^{-\mu x}\ln\left|\frac{\beta}{\beta-x}\right|\,dx = \frac{1}{\mu}[e^{-\beta\mu}\operatorname{Ei}(\beta\mu)] \qquad [Re\,\mu>0].$$

MI 26

4.338

1. $$\int_0^\infty e^{-\mu x}\ln(\beta^2+x^2)\,dx = \frac{2}{\mu}[\ln\beta - \operatorname{ci}(\beta\mu)\cos(\beta\mu) - \operatorname{si}(\beta\mu)\sin(\beta\mu)]$$
$$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>0].$$

2. $$\int_0^\infty e^{-\mu x} \ln(x^2-\beta^2)^2\, dx = \frac{2}{\mu}[\ln \beta^2 - e^{\beta\mu} \operatorname{Ei}(-\beta\mu) - e^{\beta\mu} \operatorname{Ei}(\beta\mu)]$$
$$[\operatorname{Im} \beta > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((256))(5)

4.339

$$\int_0^\infty e^{-\mu x} \ln \left|\frac{x+1}{x-1}\right| \, dx = \frac{1}{\mu}[e^{-\mu}(\ln 2\mu + \gamma) - e^{\mu} \operatorname{Ei}(-2\mu)] \qquad [\operatorname{Re} \mu > 0].$$

MI 27

4.341

$$\int_0^\infty e^{-\mu x} \ln \frac{\sqrt{x+ai}+\sqrt{x-ai}}{\sqrt{2a}} \, dx = \frac{\pi}{4\mu}[\mathbf{H}_0(a\mu) - N_0(a\mu)] \qquad [a > 0, \quad \operatorname{Re} \mu > 0].$$

ET I 149(20)

4.342

1. $$\int_0^\infty e^{-2nx} \ln(\operatorname{sh} x)\, dx = \frac{1}{2n}\left[\frac{1}{n} + \ln 2 - 2\beta(2n+1)\right].$$

BI ((256))(17)

2. $$\int_0^\infty e^{-\mu x} \ln(\operatorname{ch} x)\, dx = \frac{1}{\mu}\left[\beta\left(\frac{\mu}{2}\right) - \frac{1}{\mu}\right] \qquad [\operatorname{Re} \mu > 0].$$

ET I 165(32)

3. $$\int_0^\infty e^{-\mu x} [\ln(shx) - \ln x]\, dx = \frac{1}{\mu}\left[\ln\frac{\mu}{2} - \frac{1}{2\mu} - \psi\left(\frac{\mu}{2}\right)\right] \qquad [\operatorname{Re} \mu > 0].$$

ET I 165(33)

4.343

$$\int_0^\pi e^{\mu \cos x} [\ln(2\mu \sin^2 x) + \boldsymbol{C}]\, dx = -\pi K_0(\mu).$$

WA 95(16)

4.35- 4.36 Combinations of logarithms, exponentials, and powers

4.351

1. $$\int_0^1 (1-x)e^{-x}\ln x\,dx = \frac{1-e}{e}.$$

BI ((352))(1)

2. $$\int_0^1 e^{\mu x}(\mu x^2+2x)\ln x\,dx = \frac{1}{\mu^2}[(1-\mu)e^{\mu}-1].$$

BI ((352))(2)

3. $$\int_1^\infty \frac{e^{-\mu x}\ln x}{1+x}\,dx = \frac{1}{2}e^{\mu}[\mathrm{Ei}(-\mu)]^2 \qquad [\operatorname{Re}\mu>0].$$

NT 32(10)

4.352

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\ln x\,dx = \frac{1}{\mu^\nu}\Gamma(\nu)[\psi(\nu)-\ln\mu] \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0].$$

BI ((353))(3), ET I 315(10)a

2. $$\int_0^\infty x^n e^{-\mu x}\ln x\,dx = \frac{n!}{\mu^{n+1}}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\boldsymbol{C}-\ln\mu\right] \qquad [\operatorname{Re}\mu>0].$$

ET I 148(7)

3. $$\int_0^\infty x^{n-\frac{1}{2}}e^{-\mu x}\ln x\,dx = \sqrt{\pi}\frac{(2n-1)!!}{2^n\mu^{n+\frac{1}{2}}}\left[2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)-\boldsymbol{C}-\ln 4\mu\right]$$
$$[\operatorname{Re}\mu>0].$$

ET I 148(10)

4. $$\int_0^\infty x^{\mu-1}e^{-x}\ln x\,dx = \Gamma'(\mu) \qquad [\operatorname{Re}\mu>0].$$

GW ((324))(83a)

4.353

1. $$\int_0^\infty (x-\nu)x^{\nu-1}e^{-x}\ln x\,dx = \Gamma(\nu) \qquad [\operatorname{Re}\nu > 0].$$

GW ((324))(84)

2. $$\int_0^\infty \left(\mu x - n - \frac{1}{2}\right) x^{n-\frac{1}{2}} e^{-\mu x}\ln x\,dx = \frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$$

BI ((357))(2)

3. $$\int_0^1 (\mu x+n+1)x^n e^{\mu x}\ln x\,dx = e^\mu \sum_{k=0}^{n} (-1)^{k-1}\frac{n!}{(n-k)!\mu^{k+1}} + (-1)^n \frac{n!}{\mu^{n+1}} \qquad [\mu \neq 0].$$

GW ((324))(82)

4.354

1.[6] $$\int_0^\infty \frac{x^{\nu-1}\ln x}{e^x+1}\,dx = \Gamma(\nu)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\nu}[\psi(\nu)-\ln k] \qquad [\operatorname{Re}\nu > 0].$$
$$= -\frac{1}{2}(\ln 2)^2 \quad \text{for } \nu = 1.$$

GW ((324))(86a)

2. $$\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^2}\,dx = \Gamma(\nu)\sum_{k=2}^{\infty}\frac{(-1)^k(k-1)}{k^\nu}[\psi(\nu)-\ln k] \qquad [\operatorname{Re}\nu > 0].$$

GW ((324))(86b)

3. $$\int_0^\infty \frac{(x-\nu)e^x - \nu}{(e^x+1)^2} x^{\nu-1}\ln x\,dx = \Gamma(\nu)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\nu} \qquad [\operatorname{Re}\nu > 0].$$

GW ((324))(87a)

4. $$\int_0^\infty \frac{(x-2n)e^x - 2n}{(e^x+1)^2} x^{2n-1}\ln x\,dx = \frac{2^{2n-1}-1}{2n}\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots].$$

GW ((324))(87b)

GW ((324))(86c)

4.355

1. $\int_0^\infty x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu}(2 - \ln 4\mu - \boldsymbol{C})\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$

BI ((357))(1)a

2. $\int_0^\infty x(\mu x^2 - \nu x - 1) e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu}{4\mu}\sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\nu^2}{\mu}\right)\left[1 + \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$
$[\operatorname{Re}\mu > 0].$

BI ((358))(1)

3. $\int_0^\infty (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n} \qquad [\operatorname{Re}\mu > 0].$

BI ((353))(4)

4. $\int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n}\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0].$

BI ((353))(5)

4.356

1. $\int_0^\infty \exp\left[-\mu\left(\frac{x}{a} + \frac{a}{x}\right)\right] \ln x \frac{dx}{x} = 2 \ln a \, K_0(2\mu) \qquad [a > 0, \quad \operatorname{Re}\mu > 0].$

GW ((324))(91)

2. $\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{1}{2}} \, dx =$
$= 2\left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k)!}{(n-k)!(2k)!!(2\sqrt{ab})^k} \qquad [a > 0, \quad b > 0].$

BI ((357))(4)

3. $\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 + (2n-1)x - 2b] \frac{dx}{x^{n+\frac{3}{2}}} =$
$= 2\left(\frac{a}{b}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-k-1)!(2k)!!(2\sqrt{ab})^k} \qquad [a > 0, \quad b > 0].$

For $n = \frac{1}{2}$:

$$4. \quad \int_0^\infty \exp\left(-ax - \frac{a}{x}\right) \ln x \frac{ax^2 - b}{x^2}\, dx = 2K_0(2\sqrt{ab}) \qquad [a > 0, \quad b > 0].$$

GW ((324))(92c)

For $n = 0$:

$$5. \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}}\, dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}. \qquad [a > 0, \quad b > 0].$$

BI ((357))(7), GW((324))(92a)

For $n = -1$:

$$6. \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}}\, dx = \frac{1 + 2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \qquad [a > 0, \quad b > 0].$$

LI ((357))(6), GW ((324))(92b)

$$7.^* \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left(a - \frac{b}{x^2}\right) dx = K_0(2\sqrt{ab}) \qquad [a > 0, \quad b > 0].$$

$$8.^* \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{3}{2}}\, dx =$$
$$= 4(b/a)^{(2n+1)/4} K_{n+\frac{1}{2}}(2\sqrt{ab}) =$$
$$= 2(b/a)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!(2k)!!(2\sqrt{ab})^k} \qquad [n = 0,\ 1, \ldots, a > 0, \quad b > 0].$$

$$9.^* \quad \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln[(ax^2 - b)\cos(\alpha \ln x) + \alpha x \sin(\alpha \ln x)] \frac{dx}{x^2} =$$
$$= 2\cos(\alpha \ln \sqrt{b/a}) K_{i\alpha}(2\sqrt{ab}) \qquad [a > 0, \quad b > 0, \quad -\infty < \alpha < \infty].$$

11.* $$q\int_0^\infty x^\alpha \ln x \left[a - \frac{\alpha}{x} - \frac{b}{x^2}\right] \exp\left(-ax - \frac{b}{x}\right) dx = 2\left(\frac{b}{a}\right)^{\alpha/2} K_\alpha(2\sqrt{ab})$$
$$[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty].$$

4.357

1. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{1 + ax^2 - x^4}{x^2}\, dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}} \qquad [a > 0].$$

BI ((357))(8)

2. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4 + ax^2 - 1}{x^4}\, dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}} \qquad [a > 0].$$

BI ((357))(9)

3. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4 + 3ax - 1}{x^6}\, dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[a]{e}} \qquad [a > 0].$$

BI ((357))(10)

4.358

1.[6] $$\int_1^\infty x^{\nu-1} e^{-\mu x} (\ln x)^m\, dx = \frac{\partial^m}{\partial \nu^m}\{\mu^{-\nu}\Gamma(\nu, \mu)\} \qquad [m = 0, 1, \ldots, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

MI 26

2. $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^2\, dx = \frac{\Gamma(\nu)}{\mu^\nu}\{[\psi(\nu) - \ln\mu]^2 + \zeta(2, \nu)\} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

MI 26

3.* $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^3\, dx = \frac{\Gamma(\nu)}{\mu^\nu}\{[\psi(\nu) - \ln\mu]^3 + 3\zeta(2,\nu)[\psi(\nu) - \ln\mu] - 2\zeta(3, \nu)\}$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

MI 26

4.359

1. $$\int_0^\infty e^{-\mu x}\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx=\frac{1}{\mu}[\lambda(\mu,\,p-1)-\lambda(\mu,\,q-1)]\qquad[\operatorname{Re}\mu>0,\quad p>0,\quad q>0].$$

MI 27

2. $$\int_0^1 e^{\mu x}\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx=\sum_{k=0}^\infty\frac{\mu^k}{k!}\ln\frac{p+k}{q+k}\qquad[\operatorname{Re}\mu>0,\quad p>0,\quad q>0].$$

BI ((352))(9)

608

4.361

1. $$\int_0^\infty\frac{(x+1)e^{-\mu x}}{\pi^2+(\ln x)^2}\,dx=\nu'(\mu)-\nu''(\mu)\qquad[\operatorname{Re}\mu>0].$$

MI 27

2. $$\int_0^\infty\frac{e^{-\mu x}\,dx}{x[\pi^2+(\ln x)^2]}=e^\mu-\nu(\mu)\qquad[\operatorname{Re}\mu>0].$$

MI 27

4.362

1. $$\int_0^1 xe^x\ln(1-x)\,dx=1-e.$$

BI ((352))(5)a

2. $$\int_1^\infty e^{-\mu x}\ln(2x-1)\frac{dx}{x}=\frac{1}{2}\left[\operatorname{Ei}\left(-\frac{\mu}{2}\right)\right]^2\qquad[\operatorname{Re}\mu>0].$$

ET I 148(8)

4.363

1. $$\int_0^\infty e^{-\mu x}\ln(a+x)\frac{\mu(x+a)\ln(x+a)-2}{x+a}\,dx=$$
$$=\frac{1}{4}\int_0^\infty e^{-\mu x}\ln(a-x)^2\frac{\mu(x-a)\ \ln(x-a)^2-4}{x-a}\,dx=(\ln a)^2\qquad[\operatorname{Re}\mu>0,\quad a>0].$$

2. $$\int_0^1 x(1-x)(2-x)e^{-(1-x)^2}\ln(1-x)\,dx = \frac{1-e}{4e}.$$

BI ((352))(4)

4.364

1. $$\int_0^\infty e^{-\mu x}\ln[(x+a)(x+b)]\frac{dx}{x+a+b} =$$
$$= e^{(a+b)\mu}\{\operatorname{Ei}(-a\mu)\operatorname{Ei}(-b\mu) - \ln(ab)\operatorname{Ei}[-(a+b)\mu]\} \qquad [a>0, \quad b>0, \quad \operatorname{Re}\mu>0].$$

BI ((354))(11)

2. $$\int_0^\infty e^{-\mu x}\ln(x+a+b)\left(\frac{1}{x+a}+\frac{1}{x+b}\right)dx =$$
$$= (1+\ln a\ln b)\ln(a+b) + e^{-(a+b)\mu}\{\operatorname{Ei}(-\alpha\mu)\operatorname{Ei}(-b\mu) +$$
$$+ (1-\ln(ab))\operatorname{Ei}[-(a+b)\mu]\} \qquad [a>0, \quad b>0, \quad \operatorname{Re}\mu>0].$$

BI ((354))(12)

4.365

$$\int_0^\infty \left[e^{-x} - \frac{x}{(1+x)^{p+1}\,\ln(1+x)}\right]\frac{dx}{x} = \ln p \qquad [p>0].$$

BI ((354))(15)

609

4.366

1. $$\int_0^\infty e^{-\mu x}\ln\left(1+\frac{x^2}{a^2}\right)\frac{dx}{x} = [\operatorname{ci}(a\mu)]^2 + [\operatorname{si}(a\mu)]^2 \qquad [\operatorname{Re}\mu>0].$$

NT 32(11)a

2. $$\int_0^\infty e^{-\mu x}\ln\left|1-\frac{x^2}{a^2}\right|\frac{dx}{x} = \operatorname{Ei}(a\mu)\operatorname{Ei}(-a\mu) \qquad [\operatorname{Re}\mu>0].$$

ME 18

3. $$\int_0^\infty xe^{-\mu x^2}\ln\left|\frac{1+x^2}{1-x^2}\right|dx = \frac{1}{\mu}[\operatorname{ch}\mu\ \operatorname{sh}i(\mu) - \operatorname{sh}\mu\ \operatorname{ch}i(\mu)] \qquad [\operatorname{Re}\mu>0]; \qquad (\text{cf. } 4.339).$$

4.367

$$\int_0^\infty xe^{-\mu x^2} \ln \frac{x+\sqrt{x^2+2\beta}}{\sqrt{2\beta}}\, dx = \frac{e^{\beta\mu}}{4\mu} K_0(\beta\mu) \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0].$$

ET I 149(19)

4.368

$$\int_0^{2u} e^{-\mu x^2} \ln \frac{x^2(4u^2-x^2)}{u^4} \frac{dx}{\sqrt{4u^2-x^2}} = \frac{\pi}{2} e^{-2u^2\mu} \left[\frac{\pi}{2} N_0(2iu^2\mu) - (\boldsymbol{C} - \ln 2) J_0(2iu^2\mu)\right] \qquad [\operatorname{Re}\mu > 0].$$

ET I 149(21)a

4.369

1. $$\int_0^\infty x^{\nu-1} e^{-\mu x} [\psi(\nu) - \ln x]\, dx = \frac{\Gamma(\nu)\ln\mu}{\mu^\nu} \qquad [\operatorname{Re}\nu > 0].$$

ET I 149(12)

2. $$\int_0^\infty x^n e^{-\mu x} \left\{\left[\ln x - \frac{1}{2}\psi(n+1)\right]^2 - \frac{1}{2}\psi'(n+1)\right\} dx = \frac{n!}{\mu^{n+1}} \left\{\left[\ln\mu - \frac{1}{2}\psi(n+1)\right]^2 + \frac{1}{2}\psi'(n+1)\right\} \qquad [\operatorname{Re}\mu > 0].$$

MI 26

4.37 Combinations of logarithms and hyperbolic functions

4.371

1. $$\int_0^\infty \frac{\ln x}{\operatorname{ch} x}\, dx = \pi \ln \left[\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right].$$

LI ((260))(1)a

2. $$\int_0^\infty \frac{\ln x\, dx}{\operatorname{ch} x + \cos t} = \frac{\pi}{\sin t} \ln \frac{(2\pi)^{\frac{t}{\pi}}\,\Gamma\left(\frac{\pi+t}{2\pi}\right)}{\Gamma\left(\frac{\pi-t}{2\pi}\right)} \qquad [t^2 < \pi^2].$$

3. $\int_0^\infty \frac{\ln x\,dx}{\operatorname{ch}^2 x} = \psi\left(\frac{1}{2}\right) + \ln\pi = \ln\pi - 2\ln 2 - \boldsymbol{C}.$

BI ((257))(4)a

4.372

1. $$\int_1^\infty \ln x \frac{\operatorname{sh} mx}{\operatorname{sh} nx}\,dx =$$
$$= \frac{\pi}{2n}\,\operatorname{tg}\frac{m\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \qquad [m+n \text{ is odd}];$$
$$= \frac{\pi}{2n}\,\operatorname{tg}\frac{m\pi}{2n}\ln\pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \qquad [m+n \text{ is even}].$$

BI ((148))(3)a

2. $$\int_1^\infty \ln x \frac{\operatorname{ch} mx}{\operatorname{ch} nx}\,dx =$$
$$= \frac{\pi}{2n}\frac{\ln 2\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{n}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \qquad [m+n \text{ is odd}];$$
$$= \frac{\pi}{2n}\frac{\ln\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \qquad [m+n \text{ is even}].$$

BI ((148))(6)a

4.373

1. $$\int_0^\infty \frac{\ln(a^2+x^2)}{\operatorname{ch} bx}\,dx = \frac{\pi}{b}\left[2\ln\frac{2\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln\frac{2b}{\pi}\right] \qquad \left[b>0, \quad a>-\frac{\pi}{2b}\right].$$

BI ((258))(11)a

2. $$\int_0^\infty \ln(1+x^2)\frac{dx}{\operatorname{ch}\frac{\pi x}{2}} = 2\ln\frac{4}{\pi}.$$

3. $$\int_0^\infty \ln(a^2+x^2)\frac{\operatorname{sh}\left(\frac{2}{3}\pi x\right)}{\operatorname{sh}\pi x}\,dx = 2\sin\frac{\pi}{3}\,\ln\frac{6\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)} \qquad [a > -1].$$

BI ((258))(12)

4. $$\int_0^\infty \ln(1+x^2)\frac{dx}{\operatorname{sh}^2 ax} = \frac{2}{a}\left[\ln\frac{a}{\pi}+\frac{\pi}{2a}-\psi\left(\frac{\pi+a}{\pi}\right)\right] \qquad [a > 0].$$

BI ((258))(5)

5. $$\int_0^\infty \ln(1+x^2)\frac{\operatorname{ch}\frac{\pi}{2}x}{\operatorname{sh}^2\frac{\pi}{2}x}\,dx = \frac{2\pi-4}{\pi}.$$

BI ((258))(3)

6. $$\int_0^\infty \ln(1+x^2)\frac{\operatorname{ch}\frac{\pi}{4}x}{\operatorname{sh}^2\frac{\pi}{4}x}\,dx = 4\sqrt{2}-\frac{16}{\pi}+\frac{8\sqrt{2}}{\pi}\ln(\sqrt{2}+1).$$

BI ((258))(2)

4.374

1. $$\int_0^\infty \ln(\cos^2 t+e^{-2x}\,\sin^2 t)\frac{dx}{\operatorname{sh} x} = -2t^2.$$

BI ((259))(10)a

2. $$\int_0^\infty \ln(a+be^{-2x})\frac{dx}{\operatorname{ch}^2 x} = \frac{2}{(b-a)}\left[\frac{a+b}{2}\ln(a+b)-a\ln a-b\ln 2\right] \qquad [a > 0, \quad a+b > 0].$$

LI ((259))(14)

4.375

1. $$\int_0^\infty \ln\operatorname{ch}\frac{x}{2}\frac{dx}{\operatorname{ch} x} = \boldsymbol{G}+\frac{\pi}{4}\ln 2.$$

2. $$\int_0^\infty \ln \operatorname{cth} x \frac{dx}{\operatorname{ch} x} = \frac{\pi}{2} \ln 2.$$

BI ((259))(16)

4.376

1. $$\int_0^\infty \frac{\ln x}{\sqrt{x}\ \operatorname{ch} x}\, dx = 2\sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}\{\ln(2k+1) + 2\ln 2 + \boldsymbol{C}\}.$$

BI ((147))(4)

2. $$\int_0^\infty \ln x \frac{(\mu+1)\ \operatorname{ch} x - x\ \operatorname{sh} x}{\operatorname{ch}^2 x} x^\mu\, dx = 2\Gamma(\mu+1)\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}} \qquad [\operatorname{Re}\mu > -1].$$

BI ((356))(10)

3. $$\int_0^\infty \ln x \frac{(n+1)\ \operatorname{ch} x - x\ \operatorname{sh} x}{\operatorname{ch}^2 x} x^n\, dx = \frac{(-1)^n}{2^n}\beta^{(n)}\left(\frac{1}{2}\right).$$

4. $$\int_0^\infty \ln 2x \frac{n\ \operatorname{sh} 2ax - ax}{\operatorname{sh}^2 ax} x^{2n-1}\, dx = -\frac{1}{n}\left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \qquad [n = 1,\ 2, \dots].$$

BI ((356))(9)a

5. $$\int_0^\infty \ln x \frac{ax\ \operatorname{ch} ax - (2n+1)\ \operatorname{sh} ax}{\operatorname{sh}^2 ax} x^{2n}\, dx = 2\frac{2^{2n+1}-1}{(2a)^{2n+1}}(2n)!\zeta(2n+1).$$

BI ((356))(14)

6. $$\int_0^\infty \ln x \frac{ax\ \operatorname{ch} ax - 2n\ \operatorname{sh} ax}{\operatorname{sh}^2 ax} x^{2n-1}\, dx = \frac{2^{2n-1}-1}{2n}|B_{2n}|\left(\frac{\pi}{a}\right)^{2n} \qquad [n = 1,\ 2, \dots, a > 0].$$

BI ((356))(15)

7. $$\int_0^\infty \ln \frac{(2n+1)\ \operatorname{ch} ax - ax\ \operatorname{sh} ax}{\operatorname{ch}^2 ax} x^{2n}\, dx = -\left(\frac{\pi}{2a}\right)^{2n+1} |E_{2n}| \qquad [a > 0].$$

BI ((356))(11)

BI ((356))(2)

9.[6] $$\int_0^\infty \ln x \frac{2ax\ \operatorname{ch} ax - (2n+1)\ \operatorname{sh} ax}{\operatorname{sh}^3\ ax} x^{2n}\,dx = \frac{1}{a}\left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \qquad [a > 0, \quad n = 1,\ 2, \dots].$$

BI ((356))(6)a

10. $$\int_0^\infty \ln x \frac{x\ \operatorname{sh} x - 6\ \operatorname{sh}^2\left(\frac{x}{2}\right) - 6\cos^2\ \frac{t}{2}}{(\operatorname{ch} x + \cos t)^2} x^2\,dx = \frac{(\pi - t^2)t}{3\sin t} \qquad [0 < t < \pi].$$

BI ((356))(16)a

11. $$\int_0^\infty \ln(1+x^2) \frac{\operatorname{ch}\pi x + \pi x\ \operatorname{sh}\pi x}{\operatorname{ch}^2\pi x}\frac{dx}{x^2} = 4 - \pi.$$

BI ((356))(12)

12. $$\int_0^\infty \ln(1+4x^2) \frac{\operatorname{ch}\pi x + \pi x\ \operatorname{sh}\pi x}{\operatorname{ch}^2\pi x}\frac{dx}{x^2} = 4\ln 2.$$

BI ((356))(13)

4.377

$$\int_0^\infty \ln 2x \frac{ax - n(1 - e^{-2ax})}{\operatorname{sh}^2 ax} x^{2n-1}\,dx = \frac{1}{2n}\left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \qquad [n = 1,\ 2, \dots].$$

LI ((356))(8)a

4.38- 4.41 Logarithms and trigonometric functions

4.381

1. $$\int_0^1 \ln x \sin ax\,dx = -\frac{1}{a}[\boldsymbol{C} + \ln a - \operatorname{ci}(a)] \qquad [a > 0].$$

GW ((338))(2a)

2. $$\int_0^1 \ln x \cos ax\,dx = -\frac{1}{a}\left[\operatorname{si}(a) + \frac{\pi}{2}\right] \qquad [a > 0].$$

3. $\displaystyle\int_0^{2\pi} \ln x \sin nx\,dx = -\frac{1}{n}[\boldsymbol{C} + \ln(2n\pi) - \operatorname{ci}(2n\pi)].$

GW ((338))(1a)

4. $\displaystyle\int_0^{2\pi} \ln x \cos nx\,dx = -\frac{1}{n}\left[\operatorname{si}(2n\pi) + \frac{\pi}{2}\right].$

GW ((338))(1b)

4.382

1. $\displaystyle\int_0^{\infty} \ln\left|\frac{x+a}{x-a}\right| \sin bx\,dx = \frac{\pi}{b}\sin ab \qquad [a < 0, \quad b > 0].$

ET I 77(11)

2. $\displaystyle\int_0^{\infty} \ln\left|\frac{x+a}{x-a}\right| \cos bx\,dx = \frac{2}{b}[\cos ab\ \operatorname{si}(ab) - \sin ab\ \operatorname{ci}(ab)] \qquad [a > 0, \quad b > 0].$

ET I 18(9)

613

3. $\displaystyle\int_0^{\infty} \ln\frac{a^2+x^2}{b^2+x^2} \cos cx\,dx = \frac{\pi}{c}(e^{-bc} - e^{-ac}) \qquad [a > 0, \quad b > 0, \quad c > 0].$

FI III 648a, BI ((337))(5)

4. $\displaystyle\int_0^{\infty} \ln\frac{x^2+x+a^2}{x^2-x+a^2} \sin bx\,dx = \frac{2\pi}{b}\ \exp\left(-b\sqrt{a^2-\frac{1}{4}}\right)\sin\frac{b}{2} \qquad [b > 0].$

ET I 77(12)

5. $\displaystyle\int_0^{\infty} \ln\frac{(x+\beta)^2+\gamma^2}{(x-\beta)^2+\gamma^2} \sin bx\,dx = \frac{2\pi}{b}e^{-\gamma b}\sin\beta b \qquad [\operatorname{Re}\gamma > 0, \quad |\operatorname{Im}\beta| \le \operatorname{Re}\gamma, \quad b > 0].$

ET I 77(13)

4.383

1. $\displaystyle\int_0^{\infty} \ln(1+e^{-\beta x}) \cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b\ \operatorname{sh}\left(\frac{\pi b}{\beta}\right)} \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$

2. $\int_0^\infty \ln(1-e^{-\beta x})\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b}\,\operatorname{cth}\left(\frac{\pi b}{\beta}\right) \qquad [\operatorname{Re}\beta > 0, \quad b > 0].$

ET I 18(14)

4.384

1. $\int_0^1 \ln(\sin\pi x)\sin 2n\pi x\,dx = 0.$

GW ((338))(3a)

2.[7] $\int_0^1 \ln(\sin\pi x)\sin(2n+1)\pi x\,dx = 2\int_0^{\frac{1}{2}} \ln(\sin\pi x)\sin(2n+1)\pi x\,dx =$

$$= \frac{2}{(2n+1)\pi}\left[\ln 2 - \frac{1}{2n+1} - 2\sum_{k=1}^{n}\frac{1}{2k-1}\right].$$

GW ((338))(3b)

3.[6] $\int_0^1 \ln(\sin\pi x)\cos 2n\pi x\,dx = \ = 2\int_0^{\frac{1}{2}} \ln(\sin\pi x)\cos 2n\pi x\,dx =$

$$= -\ln 2 \qquad [n=0];$$

$$= -\frac{1}{2n} \qquad [n>0].$$

GW ((338))(3c)

4. $\int_0^1 \ln(\sin\pi x)\cos(2n+1)\pi x\,dx = 0.$

GW ((338))(3d)

5. $\int_0^{\frac{\pi}{2}} ln\; sin\; x \sin x\,dx = \ln 2 - 1.$

BI ((305))(4)

6. $\int_0^{\frac{\pi}{2}} \ln\sin\ x\cos x\,dx = -1.$

BI ((305))(5)

7. $$\int_0^{\frac{\pi}{2}} ln\, sin\, x\, \cos 2nx\, dx = \begin{cases} -\dfrac{\pi}{4n}, & n > 0 \\ -\dfrac{\pi}{2}\ln 2, & n = 0. \end{cases}$$

LI ((305))(6)

8. $$\int_0^{\pi} \ln\sin\, x \cos[2m(x-n)]\, dx = -\frac{\pi\cos 2mn}{2m}.$$

LI ((330))(8)

9. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x\, \sin^2 x\, dx = \frac{\pi}{8}(1 - \ln 4).$$

BI ((305))(7)

10. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x\, \cos^2 x\, dx = -\frac{\pi}{8}(1 + \ln 4).$$

BI ((305))(8)

11. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x\, \sin x \cos^2 x\, dx = \frac{1}{9}(\ln 8 - 4).$$

BI ((305))(9)

12. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x\, \operatorname{tg} x\, dx = -\frac{\pi^2}{24}.$$

BI ((305))(11)

13. $$\int_0^{\frac{\pi}{2}} \ln\sin\, 2x\, \sin x\, dx = \int_0^{\frac{\pi}{2}} \ln\sin\, 2x\, \cos x\, dx = 2(\ln 2 - 1).$$

BI ((305))(16, 17)

14. $$\int_0^{\pi} \frac{\ln(1 + p\cos x)}{\cos x}\, dx = \pi\, \arcsin p \qquad [p^2 < 1].$$

15. $$\int_0^{\pi} \ln\sin\, x \frac{dx}{1-2a\cos x+a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^2}{2} \qquad [a^2<1];$$
$$= \frac{\pi}{a^2-1}\ln\frac{a^2-1}{2a^2} \qquad [a^2>1].$$

BI ((331))(8)

16. $$\int_0^{\pi} \ln\sin\, bx \frac{dx}{1-2a\cos x+a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^{2b}}{2} \qquad [a^2<1].$$

BI ((331))(10)

17. $$\int_0^{\pi} \ln\cos\, bx \frac{dx}{1-2a\cos x+a^2} = \frac{\pi}{1-a^2}\ln\frac{1+a^{2b}}{2} \qquad [a^2<1].$$

BI ((331))(11)

18. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x \frac{dx}{1-2a\cos 2x+a^2} = \frac{1}{2}\int_0^{\pi} \ln\sin\, x \frac{dx}{1-2a\cos 2x+a^2} =$$
$$= \frac{\pi}{2(1-a^2)}\ln\frac{1-a}{2} \qquad [a^2<1];$$
$$= \frac{\pi}{2(a^2-1)}\ln\frac{a-1}{2a} \qquad [a^2>1].$$

BI ((321))(1), BI ((331))(13)

19. $$\int_0^{\pi} \ln\sin\, bx \frac{dx}{1-2a\cos 2x+a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^{b}}{2} \qquad [a^2<1].$$

BI ((331))(18)

20. $$\int_0^{\pi} \ln\cos\, bx \frac{dx}{1-2a\cos 2x+a^2} = \frac{\pi}{1-a^2}\ln\frac{1+a^{b}}{2} \qquad [a^2<1].$$

BI ((331))(21)

21. $$\int_0^{\frac{\pi}{2}} \frac{\ln\cos\, x\, dx}{1-2p\cos 2x+p^2} = \frac{\pi}{2(1-p^2)}\ln\frac{1+p}{2} \qquad [p^2<1];$$
$$= \frac{\pi}{2(p^2-1)}\ln\frac{p+1}{2p} \qquad [p^2>1].$$

22. $$\int_0^{\pi} \ln \sin\, x \frac{\cos x\, dx}{1 - 2a\cos x + a^2} = \frac{\pi}{2a}\frac{1+a^2}{1-a^2}\ln(1-a^2) - \frac{a\pi \ln 2}{1-a^2} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2a}\frac{a^2+1}{a^2-1}\ln\frac{a^2-1}{a^2} - \frac{\pi\ln 2}{a(a^2-1)} \qquad [a^2 > 1].$$

LI ((331))(9)

23. $$\int_0^{\pi} \ln \sin\, bx \frac{\cos x\, dx}{1 - 2a\cos 2x + a^2} = \int_0^{\pi} \ln\cos\, bx \frac{\cos x\, dx}{1 - 2a\cos 2x + a^2} = 0 \qquad [0 < a < 1].$$

BI ((331))(19, 22)

24. $$\int_0^{\pi} \ln \sin\, x \frac{\cos^2 x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{4a}\frac{1+a}{1-a}\ln(1-a) - \frac{\pi\ln 2}{2(1-a)} \qquad [0 < a < 1];$$
$$= \frac{\pi}{4a}\frac{a+1}{a-1}\ln\frac{a-1}{a} - \frac{\pi\ln 2}{2a(a-1)} \qquad [a > 1].$$

BI ((331))(16)

25. $$\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{\cos 2x\, dx}{1 - 2a\cos 2x + a^2} = \frac{1}{2}\int_0^{\pi} \ln\sin\, x \frac{\cos 2x\, dx}{1 - 2a\cos 2x + a^2} =$$
$$= \frac{\pi}{2a(1-a^2)}\left\{\frac{1+a^2}{2}\ln(1-a) - a^2\ln 2\right\} \qquad [a^2 < 1];$$
$$= \frac{\pi}{2a(a^2-1)}\left\{\frac{1+a^2}{2}\ln\frac{a-1}{a} - \ln 2\right\} \qquad [a^2 > 1].$$
$$=$$

BI ((321))(2), BI ((331))(15), LI ((321))(2)

26. $$\int_0^{\frac{\pi}{2}} \ln \cos\, x \frac{\cos 2x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2a(1-a^2)}\left\{\frac{1+a^2}{2}\ln(1+a) - a^2\ln 2\right\} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2a(a^2-1)}\left\{\frac{1+a^2}{2}\ln\frac{1+a}{a} - \ln 2\right\} \qquad [a^2 > 1].$$

BI ((321))(9)

4.385

1. $$\int_0^{\pi} \ln\sin\, x \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \qquad [a > 0, \quad a > b].$$

BI ((331))(6)

2. $$\int_0^{\frac{\pi}{2}} \ln\sin\, x \frac{dx}{(a\sin x \pm b\cos x)^2} = \int_0^{\frac{\pi}{2}} \ln\cos\, x \frac{dx}{(a\cos x \pm b\sin x)^2} =$$
$$= \frac{1}{b(a^2+b^2)}\left(\mp a\ln\frac{a}{b} - \frac{b\pi}{2}\right) \qquad [a > 0, \quad b > 0].$$

3. $$\int_0^{\frac{\pi}{2}} \frac{\ln \sin\, x\, dx}{a^2 \sin^2\, x + b^2 \cos^2\, x} = \int_0^{\frac{\pi}{2}} \frac{\ln \cos x\, dx}{b^2 \sin^2 x + a^2 \cos^2\, x} = \frac{\pi}{2ab} \ln \frac{b}{a+b} \qquad [a > 0, \quad b > 0].$$

BI ((317))(4, 10)

4. $$\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{\sin 2x\, dx}{(a \sin^2 x + b \cos^2 x)^2} =$$
$$= \int_0^{\frac{\pi}{2}} \ln \cos\, x \frac{\sin 2x\, dx}{(b \sin^2 x + a \cos^2 x)^2} = \frac{1}{2b(b-a)} \ln \frac{a}{b} \qquad [a > 0, \quad b > 0].$$

BI ((319))(3, 7), LI ((319))(3)

5. $$\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2\, \cos^2 x)^2}\, dx =$$
$$= \int_0^{\frac{\pi}{2}} \ln \cos\, x \frac{a^2 \cos^2 x - b^2 \sin^2\, x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}\, dx = \frac{\pi}{2b(a+b)} \qquad [a > 0, \quad b > 0].$$

LI ((319))(2, 8)

4.386

1. $$\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{\sin x}{\sqrt{1 + \sin^2 x}}\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos x\, \ln \cos\, x}{\sqrt{1 + \cos^2 x}}\, dx = -\frac{\pi}{8}\, \ln 2.$$

BI ((322))(1, 6)

2. $$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x \ln \sin\, x}{\sqrt{1 + \sin^2 x}}\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x \ln \cos\, x}{\sqrt{1 + \cos^2 x}}\, dx = \frac{\ln 2 - 1}{4}.$$

BI ((322))(2, 7)

3. $$\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{2} \boldsymbol{K}(k) \ln k - \frac{\pi}{4} \boldsymbol{K}(k').$$

BI ((322))(3)

4. $$\int_0^{\frac{\pi}{2}} \frac{\ln \cos\, x\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \boldsymbol{K}(k) \ln \frac{k'}{k} - \frac{\pi}{4} \boldsymbol{K}(k').$$

1. $$\int_0^{\frac{\pi}{2}} \ln\sin x \sin^\mu x \cos^\nu x\,dx = \int_0^{\frac{\pi}{2}} \ln\cos x \cos^\mu x \sin^\nu x\,dx = $$
$$= \frac{1}{4} B\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right)\left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right)\right]$$
$$= \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1].$$

GW ((338))(6c)

2. $$\int_0^{\frac{\pi}{2}} \ln\sin x \sin^{\mu-1} x\,dx = \frac{\sqrt{\pi}\Gamma\left(\frac{\mu}{2}\right)}{4\Gamma\left(\frac{\mu+1}{2}\right)}\left[\psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right)\right] \qquad [\operatorname{Re}\mu > 0].$$

GW ((338))(6a)

3. $$\int_0^{\frac{\pi}{2}} \ln\sin x \cos^{\nu-1} x\,dx = \frac{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}{4\Gamma\left(\frac{\nu+1}{2}\right)}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right] \qquad [\operatorname{Re}\nu > 0].$$

GW ((338))(6b)

4. $$\int_0^{\frac{\pi}{2}} \ln\sin x \sin^{2n} x\,dx = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}\left\{\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k} - \ln 2\right\}.$$

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5. $$\int_0^{\frac{\pi}{2}} \ln\sin x \sin^{2n+1} x\,dx = \frac{(2n)!!}{(2n+1)!!}\left\{\sum_{k=1}^{2n+1}\frac{(-1)^k}{k} + \ln 2\right\}.$$

BI ((305))(13)

6. $$\int_0^{\frac{\pi}{2}} \ln\sin x \cos^{2n} x\,dx = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}\left[\sum_{k=1}^{n}\frac{1}{k} + \ln 4\right] =$$
$$= -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}[\boldsymbol{C} + \psi(n+1) + \ln 4].$$

BI ((305))(14)

7. $$\int_0^{\frac{\pi}{2}} \ln\sin x \cos^{2n+1} x\,dx = -\frac{(2n)!!}{(2n+1)!!}\sum_{k=0}^{n}\frac{1}{2k+1} =$$
$$= -\frac{(2n)!!}{2(2n+1)!!}\left[\psi\left(n+\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right)\right].$$

8. $$\int_0^{\frac{\pi}{2}} \ln\cos\, x\, \sin^{2n} x\, dx = -\frac{(2n-1)!!}{2^{n+1}\cdot n!}\frac{\pi}{2}\{\boldsymbol{C} + 2\ln 2 + \psi(n+1)\}.$$

BI ((306))(8)

9. $$\int_0^{\frac{\pi}{2}} \ln\cos\, x\, \cos^{2n} x\, dx = -\frac{(2n-1)!!}{2^n n!}\frac{\pi}{2}\left\{\ln 2 + \sum_{k=1}^{2n}\frac{(-1)^k}{k}\right\}.$$

BI ((306))(10)

10. $$\int_0^{\frac{\pi}{2}} \ln\cos\, x\, \cos^{2n-1} x\, dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!}\left[\ln 2 + \sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right].$$

BI ((306))(9)

4.388

1. $$\int_0^{\frac{\pi}{4}} \ln\sin\, x \frac{\sin^{2n} x}{\cos^{2n+2} x}\, dx = \frac{1}{2n+1}\left[\frac{1}{2}\ln 2 + (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right].$$

BI ((288))(1)

2. $$\int_0^{\frac{\pi}{4}} \ln\sin\, x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\, dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n\ln 2 + \sum_{k=1}^{n-1}\frac{(-1)^k}{n-k}\right].$$

LI ((288))(2)

3. $$\int_0^{\frac{\pi}{4}} \ln\cos\, x \frac{\sin^{2n} x}{\cos^{2n+2} x}\, dx = \frac{1}{2n+1}\left[-\frac{1}{2}\ln 2 + (-1)^{n+1}\frac{\pi}{4} + \sum_{k=0}^{n}\frac{(-1)^{k-1}}{2n-2k+1}\right].$$

BI ((288))(10)

618

4. $$\int_0^{\frac{\pi}{4}} \ln\cos\, x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\, dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n\ln 2 + \sum_{k=0}^{n-1}\frac{(-1)^k}{n-k}\right].$$

BI ((288))(11)

BI ((310))(4)

6. $\int_0^{\frac{\pi}{2}} \ln \sin\, x \frac{dx}{\mathrm{tg}^{p-1}\, x \sin 2x} = \frac{1}{4} \frac{\pi}{p-1}\, \sec \frac{p\pi}{2} \qquad [p^2 < 1].$

BI ((310))(3)

4.389

1. $\int_0^{\pi} \ln \sin\, x \sin^{2n} 2x \cos 2x\, dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4n+2}.$

BI ((330))(9)

2. $\int_0^{\frac{\pi}{4}} \ln \sin\, x\, \cos^n 2x \sin 2x\, dx = -\frac{1}{4(n+1)} \{\boldsymbol{C} + \psi(n+2) + \ln 2\}.$

BI ((285))(2)

3. $\int_0^{\frac{\pi}{4}} \ln \cos\, x\, \cos^{\mu-1} 2x\, \mathrm{tg}\, 2x\, dx = \frac{1}{4(1-\mu)} \beta(\mu) \qquad [\mathrm{Re}\, \mu > 0].$

BI ((286))(2)

4. $\int_0^{\frac{\pi}{2}} \ln \sin\, x\, \sin^{\mu-1} x \cos x\, dx = \int_0^{\frac{\pi}{2}} \ln \cos\, x \cos^{\mu-1} x \sin x\, dx = -\frac{1}{\mu^2} \qquad [\mathrm{Re}\, \mu > 0].$

BI ((306))(11)

5.[3] $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \cos\, x \cos^p x \cos px\, dx = \frac{\pi}{2^{p+1}} [\boldsymbol{C} + \psi(p+1) - 2 \ln 2] \qquad [p > -1].$

6. $\int_0^{\frac{\pi}{2}} \ln \cos\, x\, \cos^{p-1} x \sin px \sin x\, dx = \frac{\pi}{2^{p+2}} \left[\boldsymbol{C} + \psi(p) - \frac{1}{p} - 2 \ln 2\right] \qquad [p > 0].$

BI ((306))(12)

4.391

1. $\int_0^{\frac{\pi}{4}} (\ln \cos\, 2x)^n \cos^{p-1} 2x\, \mathrm{tg}\, x\, dx =$

$= \int_0^{\frac{\pi}{4}} (\ln \sin\, 2x)^n \sin^{p-1} 2x\, \mathrm{tg}\, \left(\frac{\pi}{4} - x\right)\, dx = \frac{1}{2} \beta^{(n)}(p) \qquad [p > 0].$

2. $\int_0^{\frac{\pi}{4}} (\ln \sin\, 2x)^n\, \sin^{p-1} 2x\, \operatorname{tg}\left(\frac{\pi}{4}+x\right)\, dx = \frac{(-1)^n n!}{2}\zeta(n+1,\, p).$

BI ((285))(17)

619

3. $\int_0^{\frac{\pi}{4}} (\ln \cos\, 2x)^{2n-1}\, \operatorname{tg} x\, dx = \frac{1-2^{2n-1}}{4n}\pi^{2n}|B_{2n}| \qquad [n=1,\, 2, \dots].$

BI ((286))(7)

4. $\int_0^{\frac{\pi}{4}} (\ln \cos\, 2x)^{2n}\, \operatorname{tg} x\, dx = \frac{2^{2n}-1}{2^{2n+1}}(2n)!\zeta(2n+1).$

BI ((286))(8)

4.392

1. $\int_0^{\frac{\pi}{4}} \ln(\sin x \cos x)\frac{\sin^{2n} x}{\cos^{2n+2}\, x}\, dx = \frac{1}{2n+1}\left[(-1)^{n+1}\frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2\sum_{k=0}^{n-1}\frac{(-1)^{k-1}}{2n-2k-1}\right].$

BI ((294))(8)

2. $\int_0^{\frac{\pi}{4}} \ln(\sin x \cos x)\frac{\sin^{2n-1} x}{\cos^{2n+1}\, x}\, dx = \frac{1}{2n}\left[(-1)^n\, \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right].$

BI ((294))(9)

4.393

1. $\int_0^{\frac{\pi}{2}} \ln \operatorname{tg}\, x \sin x\, dx = \ln 2.$

BI ((307))(3)

2. $\int_0^{\frac{\pi}{2}} \ln \operatorname{tg}\, x \cos x\, dx = -\ln 2.$

BI ((307))(4)

BI ((307))(5, \, 6)

4. $\int_0^{\frac{\pi}{4}} \frac{\ln \operatorname{tg}\, x}{\cos 2x}\, dx = -\frac{\pi^2}{8}.$

GW ((338))(10b)a

5. $\int_0^{\frac{\pi}{2}} \sin x \ln \operatorname{ctg}\, \frac{x}{2}\, dx = \ln 2.$

LO III 290

4.394

1. $$\int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg}\, x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2(1-a^2)} \ln \frac{1-a}{1+a} \qquad [a^2 < 1];$$
$$= \frac{\pi}{2(a^2-1)} \ln \frac{a-1}{a+1} \qquad [a^2 > 1].$$

BI ((321))(15)

2. $$\int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg}\, x \cos 2x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{4a} \frac{1+a^2}{1-a^2} \ln \frac{1-a}{1+a} \qquad [a^2 < 1];$$
$$= \frac{\pi}{4a} \frac{a^2+1}{a^2-1} \ln \frac{a-1}{a+1} \qquad [a^2 > 1].$$

BI ((321))(16)

3. $$\int_0^{\pi} \frac{\ln \operatorname{tg}\, bx\, dx}{1 - 2a\, \cos 2x + a^2} = \frac{\pi}{1-a^2} \ln \frac{1-a^b}{1+a^b} \qquad [0 < a < 1, \quad b > 0].$$

BI ((331))(24)

4. $$\int_0^{\pi} \frac{\ln \operatorname{tg}\, bx \cos x\, dx}{1 - 2a\cos 2x + a^2} = 0 \qquad [0 < a < 1].$$

BI ((331))(25)

5. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg}\, x \frac{\cos 2x\, dx}{1 - a\sin 2x} = -\frac{\arcsin a}{4a}(\pi + \arcsin a) \qquad [a^2 \leq 1].$$

6. $\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x\,dx}{1-a^2\sin^2 2x} = -\frac{\pi}{4a} \arcsin a \qquad [a^2<1].$

BI ((291))(9)

7. $\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x\,dx}{1+a^2\sin^2 x} = -\frac{\pi}{4a} \operatorname{Arsh} a = -\frac{\pi}{4a} \ln(a+\sqrt{1+a^2}) \qquad [a^2<1].$

BI ((291))(10)

8. $\int_0^{u} \frac{\sin x \ln \operatorname{ctg} \frac{x}{2}}{1-\cos^2\alpha\sin^2 x}\,dx = \operatorname{cosec} 2\alpha \left\{\frac{\pi}{2}\ln 2 + L(\varphi-\alpha) - L(\varphi+\alpha) - L\left(\frac{\pi}{2}-2\alpha\right)\right\}$
$[\operatorname{tg}\varphi = \operatorname{ctg}\alpha\cos u; \quad 0<u<\pi].$

LO III 290

9. $\int_0^{\frac{\pi}{4}} \frac{\ln \operatorname{tg} x \sin 2x\,dx}{1-\cos^2 t\sin^2 2x} = \operatorname{cosec} 2t \left[L\left(\frac{\pi}{2}-t\right) - \left(\frac{\pi}{2}-t\right)\ln 2\right].$

LO III 290a

4.395

1. $\int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg} x\,dx}{\sqrt{1-k^2\sin^2 x}} = -\ln k' \boldsymbol{K}(k).$

BI ((322))(11)

2. $\int_u^{\frac{\pi}{4}} \frac{\ln \operatorname{tg} x \sin 4x\,dx}{(\sin^2 u + \operatorname{tg}^2 v \sin^2 2x)\sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2}\frac{\cos^2 v}{\sin u \sin v} \ln \frac{\sin v + \sqrt{1-\cos^2 u\cos^2 v}}{\sin u(1+\sin v)}$
$\left[0<u<\frac{\pi}{2}, \quad 0<v<\frac{\pi}{2}\right].$

LO III 285a

4.396

1. $\int_0^{\frac{\pi}{2}} \ln(a \operatorname{tg} x) \sin^{\mu-1} 2x\,dx = 2^{\mu-2}\ln a \frac{\left\{\Gamma\left(\frac{a}{2}\right)\right\}^2}{\Gamma(a)} \qquad [a>0, \quad \operatorname{Re}\mu>0]$

LI ((307))(8)

BI ((307))(9)

3. $$\int_0^{\frac{\pi}{2}} \ln \operatorname{tg}\, x\, \cos^{q-1} x \operatorname{ctg} x \sin[(q+1)x]\, dx = -\frac{\pi}{2}[\boldsymbol{C}+\psi(q+1)] \qquad [q>-1].$$

BI ((307))(11)

621

4. $$\int_0^{\frac{\pi}{2}} \ln \operatorname{tg}\, x\, \cos^{q-1} x \cos[(q+1)x]\, dx = -\frac{\pi}{2q} \qquad [q>0].$$

BI ((307))(10)

5. $$\int_0^{\frac{\pi}{4}} (\ln \operatorname{tg}\, x)^n\, \operatorname{tg}^p x\, dx = \frac{1}{2^{n+1}} B^{(n)}\left(\frac{p+1}{2}\right) \qquad [p>-1].$$

LI ((286))(22)

6. $$\int_0^{\frac{\pi}{2}} (\ln \operatorname{tg}\, x)^{2n-1} \frac{dx}{\cos 2x} = \frac{1-2^{2n}}{2n} \pi^{2n} |B_{2n}| \qquad [n=1,\, 2, \dots].$$

BI ((312))(6)

7. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg}\, x\, \operatorname{tg}^{2n+1} x\, dx = \frac{(-1)^{n+1}}{4}\left[\frac{\pi^2}{12} + \sum_{k=1}^{n} \frac{(-1)^k}{k^2}\right].$$

GW ((338))(8a)

4.397

1. $$\int_0^{\frac{\pi}{2}} \ln(1+p\sin x) \frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2}(\arccos p^2) \qquad [p^2<1].$$

BI ((313))(1)

2. $$\int_0^{\frac{\pi}{2}} \ln(1+p\cos x) \frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2}(\arccos p)^2 \qquad [p^2<1].$$

BI ((313))(8)

3. $$\int_0^{\pi} \ln(1+p\cos x) \frac{dx}{\cos x} = \pi\, \arcsin p \qquad [p^2<1].$$

BI ((331))(1)

4. $$\int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1+\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\,dx = \frac{L\left(\frac{\pi}{2}-\alpha\right) - \alpha\ln\sin\,\alpha}{\sin\alpha\cos\alpha} \qquad \left[0<\alpha<\frac{\pi}{2}\right].$$

LO III 291

5. $$\int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1-\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\,dx = \frac{L\left(\frac{\pi}{2}-\alpha\right) + (\pi-\alpha)\ln\sin\,\alpha}{\sin\alpha\cos\alpha} \qquad \left[0<\alpha<\frac{\pi}{2}\right].$$

LO III 291

6. $$\int_0^{\pi} \ln(1-2a\cos x+a^2)\cos nx\,dx = \frac{1}{2}\int_0^{2\pi} \ln(1-2a\cos x+a^2)\cos nx\,dx =$$
$$= -\frac{\pi}{n}a^n \qquad [a^2<1];$$
$$= -\frac{\pi}{na^n} \qquad [a^2>1].$$

GW ((338))(13a)
BI ((330))(11), BI ((332))(5)

7. $$\int_0^{\pi} \ln(1-2a\cos x+a^2)\sin nx\sin x\,dx =$$
$$= \frac{1}{2}\int_0^{2\pi} \ln(1-2a\cos x+a^2)\sin nx\sin x\,dx = \frac{\pi}{2}\left(\frac{a^{n+1}}{n+1}-\frac{a^{n-1}}{n-1}\right)$$
$$[a^2>1].$$

BI ((330))(10), BI ((332))(4)

8. $$\int_0^{\pi} \ln(1-2a\cos x+a^2)\sin nx\sin x\,dx =$$
$$= \frac{1}{2}\int_0^{2\pi} \ln(1-2a\cos x+a^2)\,\cos nx\cos x\,dx = -\frac{\pi}{2}\left(\frac{a^{n+1}}{n+1}+\frac{a^{n-1}}{n-1}\right) \qquad [a^2<1].$$

BI ((330))(12), BI ((332))(6)

9. $$\int_0^{\pi} \ln(1-2a\cos 2x+a^2)\,\cos(2n-1)x\,dx = 0 \qquad [a^2<1].$$

BI ((330))(15)

BI ((330))(13)

11. $$\int_0^{\pi} \ln(1-2a\cos 2x+a^2)\, \sin(2n-1)\, x \sin x\, dx = \frac{\pi}{2}\left(\frac{a^n}{n} - \frac{a^{n-1}}{n-1}\right) \qquad [a^2 < 1].$$

BI ((330))(14)

12. $$\int_0^{\pi} \ln(1-2a\cos 2x+a^2)\, \cos 2nx \cos x\, dx = 0 \qquad [a^2 < 1].$$

BI ((330))(16)

13. $$\int_0^{\pi} \ln(1-2a\cos 2x+a^2)\, \cos(2n-1)\, x \cos x\, dx = -\frac{\pi}{2}\left(\frac{a^n}{n} + \frac{a^{n-1}}{n-1}\right) \qquad [a^2 < 1].$$

BI ((330))(17)

14. $$\int_0^{\frac{\pi}{2}} \ln(1+2a\cos 2x+a^2) \sin^2 x\, dx = -\frac{a\pi}{4} \qquad [a^2 < 1];$$
$$= \frac{\pi \ln a^2}{4} - \frac{\pi}{4a} \qquad [a^2 > 1].$$

BI ((309))(22), LI ((309))(22)

15. $$\int_0^{\frac{\pi}{2}} \ln(1+2a\cos 2x+a^2) \cos^2 x\, dx = \frac{a\pi}{4} \qquad [a^2 < 1];$$
$$= \frac{\pi \ln a^2}{4} + \frac{\pi}{4a} \qquad [a^2 > 1].$$

BI ((309))(23), LI ((309))(23)

16. $$\int_0^{\pi} \frac{\ln(1-2a\cos x+a^2)}{1-2b\cos x+b^2}\, dx = \frac{2\pi \ln(1-ab)}{1-b^2} \qquad [a^2 \leq 1, \quad b^2 < 1].$$

BI ((331))(26)

4.398

1. $$\int_0^{\pi} \ln \frac{1+2a\cos x+a^2}{1-2a\cos x+a^2} \sin(2n+1)x\, dx = (-1)^n \frac{2\pi a^{2n+1}}{2n+1} \qquad [a^2 < 1].$$

2. $$\int_0^{2\pi} \ln \frac{1 - 2a\cos x + a^2}{1 - 2a\cos nx + a^2} \cos mx\,dx = 2\pi\left(\frac{n}{m}a^{\frac{m}{n}} - \frac{a^m}{m}\right) \qquad [a^2 \le 1];$$
$$= 2\pi\left(\frac{n}{m}a^{-\frac{m}{n}} - \frac{a^{-m}}{m}\right) \qquad [a^2 \ge 1].$$

BI ((332))(9)

3. $$\int_0^{\pi} \ln \frac{1 + 2a\cos 2x + a^2}{1 + 2a\cos 2nx + a^2} \operatorname{ctg} x\,dx = 0.$$

BI ((331))(5), LI((331))(5)

623

4.399

1. $$\int_0^{\frac{\pi}{2}} \ln(1+a\sin^2 x)\sin^2 x\,dx = \frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad [a > -1].$$

BI ((309))(14)

2. $$\int_0^{\frac{\pi}{2}} \ln(1+a\sin^2 x)\cos^2 x\,dx = \frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad [a > -1].$$

BI ((309))(15)

3. $$\int_0^{\frac{\pi}{2}} \frac{\ln(1-\cos^2\beta\,\cos^2 x)}{1-\cos^2\alpha\cos^2 x}\,dx = -\frac{\pi}{\sin\alpha}\ln\frac{1+\sin\alpha}{\sin\alpha+\sin\beta} \qquad \left[0<\beta<\frac{\pi}{2},\quad 0<\alpha<\frac{\pi}{2}\right].$$

LO III 285

4.411

1. $$\int_0^{\pi} \ln\frac{1+\sin x}{1+\cos\lambda\sin x}\frac{dx}{\sin x} = \lambda^2 \qquad [\lambda^2 < \pi^2].$$

BI ((331))(2)

2. $$\int_0^{\frac{\pi}{2}} \ln\frac{p+q\sin ax}{p-q\sin ax}\frac{dx}{\sin ax} = \int_0^{\frac{\pi}{2}} \ln\frac{p+q\cos ax}{p-q\cos ax}\frac{dx}{\cos ax} = \int_0^{\frac{\pi}{2}} \ln\frac{p+q\operatorname{tg} ax}{p-q\operatorname{tg} ax}\frac{dx}{\operatorname{tg} ax} = \pi\arcsin\frac{q}{p}$$
$$[p > q > 0].$$

FI II 695a, BI ((315))(5, 13,17)a

LO III 284

4.412

1. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x\right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8}.$$

BI ((293))(1)

2. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x\right) \frac{dx}{\operatorname{tg} 2x} = \pm \frac{\pi^2}{16}.$$

BI ((293))(2)

3. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x\right) (\ln \operatorname{tg}\ x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2}-1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|.$$

BI ((294))(24)

4. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x\right) (\ln \operatorname{tg}\ x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1-2^{2n+1}}{2^{2n+2} n} (2n)! \zeta(2n+1).$$

BI ((294))(25)

5. $$\int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x\right) (\ln \sin\ 2x)^{n-1} \frac{dx}{\operatorname{tg} 2x} = \frac{(-1)^{n-1}}{2} (n-1)! \zeta(n+1).$$

LI ((294))(20)

624

4.413

1. $$\int_0^{\frac{\pi}{2}} \ln(p^2 + q^2\ \operatorname{tg}^2 x) \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{ab} \ln \frac{ap+bq}{a}$$
$$[a > 0, \quad b > 0, \quad p > 0, \quad q > 0].$$

BI ((318))(1--4)a

2. $$\int_0^{\frac{\pi}{2}} \ln(1 + q^2\ \operatorname{tg}^2 x) \frac{1}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} =$$
$$= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p^2 - r^2}{pr} \ln\left(1 + \frac{qr}{p}\right) + \frac{t^2 - s^2}{st} \ln\left(1 + \frac{qt}{s}\right) \right\}$$
$$[q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0].$$

3. $$\int_0^{\frac{\pi}{2}} \ln(1+q^2\ \operatorname{tg}^2 x)\frac{\sin^2 x}{p^2\sin^2 x + r^2\ \cos^2 x}\frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \\ = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{t}{s}\ln\left(1+\frac{qt}{s}\right) - \frac{r}{p}\ln\left(1+\frac{qr}{p}\right)\right\} \\ [q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0].$$

BI ((320))(20)

4. $$\int_0^{\frac{\pi}{2}} \ln(1+q^2\ \operatorname{tg}^2 x)\frac{\cos^2 x}{p^2\sin^2 x + r^2\ \cos^2 x}\frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \\ = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{p}{r}\ln\left(1+\frac{qr}{p}\right) - \frac{s}{t}\ln\left(1+\frac{qt}{s}\right)\right\} \\ [q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0].$$

BI ((320))(21)

5. $$\int_0^{\pi} \frac{\ln \operatorname{tg}\ rx\,dx}{1-2p\ \cos x + p^2} = \frac{\pi}{1-p^2}\ln\frac{1-p^{2r}}{1+p^{2r}} \qquad [p^2<1].$$

BI ((331))(12)

4.414

1. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\sin^2 x)\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \ln k'\,\boldsymbol{K}(k).$$

BI ((323))(1)

2. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\ \sin^2 x)\frac{\sin^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k^2}\{(k^2-2+\ln k')\boldsymbol{K}(k)+(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((323))(3)

3. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\ \sin^2 x)\frac{\cos^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k^2}[(1+k'^2-k'^2\ln k')\boldsymbol{K}(k)-(2-\ln k')\boldsymbol{E}(k)].$$

BI ((323))(6)

4. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\ \sin^2 x)\frac{dx}{\sqrt{(1-k^2\sin^2 x)^3}} = \frac{1}{k'^2}[(k^2-2)\boldsymbol{K}(k)+(2+\ln k')\boldsymbol{E}(k)].$$

5. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin^2 x \, dx}{\sqrt{(1-k^2\sin^2 x)^3}} = \frac{1}{k^2 k'^2}[(2+\ln k')\boldsymbol{E}(k)-(1+k'^2+k'^2 \ln k')\boldsymbol{K}(k)].$

BI ((323))(10)

6. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\cos^2 x \, dx}{\sqrt{(1-k^2\sin^2 x)^3}} = \frac{1}{k^2}[(1+k'^2+\ln k')\boldsymbol{K}(k)-(2+\ln k')\boldsymbol{E}(k)].$

BI ((323))(16)

7. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x)\sqrt{1-k^2\sin^2 x}\, dx = (1+k'^2)\boldsymbol{K}(k) - (2-\ln k')\boldsymbol{E}(k).$

BI ((324))(18)

8. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x)\sin^2 x\sqrt{1-k^2\sin^2 x}\, dx = \frac{1}{9k^2}\Big\{(-2+11k^2-6k^4+3k'^2\ln k')\boldsymbol{K}(k)+$
$\displaystyle + [2-10k^2-3(1-2k^2)\ln k']\boldsymbol{E}(k)\Big\}.$

BI ((324))(20)

9. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x)\cos^2 x\sqrt{1-k^2\sin^2 x}\, dx = \frac{1}{9k^2}\Big\{(2+7k^2-3k^4-3k'^2\ln k')\boldsymbol{K}(k)-$
$\displaystyle - [2+8k^2-3(1+k^2)\ln k']\boldsymbol{E}(k)\Big\}.$

BI ((324))(21), LI ((324))(21)

10. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x \, dx}{\sqrt{(1-k^2\sin^2 x)^{2n+1}}} = \frac{2}{(2n-1)^2 k^2}\{[1+(2n-1)\ln k']k'^{1-2n}-1\}.$

BI ((324))(17)

4.415

1. $\displaystyle\int_0^{\infty} \ln x \sin ax^2 \, dx = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left(\ln 4a + \boldsymbol{C} - \frac{\pi}{2}\right) \qquad [a>0].$

GW ((338))(19)

GW ((338))(19)

4.416

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos x \ln\left(1+\sqrt{\sin^2\beta - \cos^2\beta\ \operatorname{tg}^2\alpha \sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx =$$
$$= \operatorname{cosec} 2\alpha\{(2\alpha+2\gamma-\pi)\ln\cos\ \beta + 2L(\alpha) - 2L(\gamma) + L(\alpha+\gamma) - L(\alpha-\gamma)\}$$
$$\left[\cos\gamma = \frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right].$$

LO III 291

626

2. $$\int_0^{\frac{\pi}{2}} \frac{\cos x \ln\left(1-\sqrt{\sin^2\beta - \cos^2\beta\ \operatorname{tg}^2\alpha \sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx =$$
$$= \operatorname{cosec} 2\alpha\{(\pi+2\alpha-2\gamma)\ln\cos\ \beta + 2L(\alpha) + 2L(\gamma) - L(\alpha+\gamma) + L(\alpha-\gamma)\}$$
$$\left[\cos\gamma = \frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right].$$

LO III 291

3. $$\int_\beta^{\frac{\pi}{2}} \frac{\ln(\sin x + \sqrt{\sin^2 x - \sin^2\beta})}{1-\cos^2\alpha\cos^2 x}\,dx =$$
$$= -\operatorname{cosec}\alpha\left\{\operatorname{arctg}\left(\frac{\operatorname{tg}\beta}{\sin\alpha}\right)\ln\sin\ \beta + \frac{\pi}{2}\ln\frac{1+\sin\alpha}{\sin\alpha+\sqrt{1-\cos^2\alpha\cos^2\beta}}\right\}$$
$$\left[0<\alpha<\pi,\quad 0<\beta<\frac{\pi}{2}\right].$$

LO III 285

4.[7] $$\int_0^{\frac{\pi}{4}} \ln\operatorname{tg}\ x(\ln\cos\ 2x)^{n-1}\ \operatorname{tg} 2x\,dx = \frac{1}{2}(-1)^n(n-1)!(1-2^{-(n+1)})\zeta(n+1)$$

4.42- 4.43 Combinations of logarithms, trigonometric functions, and powers

4.421

1. $$\int_0^\infty \ln x \sin ax\frac{dx}{x} = -\frac{\pi}{2}(\boldsymbol{C}+\ln a) \qquad [a>0].$$

FI II 810a

ET I 76(5), NT 27(10)a

3. $$\int_0^\infty \ln ax \cos bx \frac{\beta' \, dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') + \frac{\pi}{4} [e^{b\beta'} \operatorname{Ei}(-b\beta') - e^{-b\beta'} \operatorname{Ei}(b\beta')]$$
$$[\beta' = \beta \operatorname{sign} \beta; \quad a > 0, \quad b > 0].$$

ET I 17(3), NT 27(11)a

4. $$\int_0^\infty \ln ax \sin bx \frac{x \, dx}{x^2 - c^2} = \frac{\pi}{2} \{-\operatorname{si}(bc) \sin bc + \cos bc [\ln ac - ci(bc)]\}$$
$$[a > 0, \quad b > 0, \quad c > 0].$$

BI ((422))(5)

5. $$\int_0^\infty \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c} \{\sin bc [\operatorname{ci}(bc) - \ln ac] - \cos bc \operatorname{si}(bc)\}$$
$$[a > 0, \quad b > 0, \quad c > 0].$$

BI ((422))(6)

4.422

1. $$\int_0^\infty \ln x \sin ax \, x^{\mu-1} \, dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a + \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2}\right] \qquad [a > 0, \quad |\operatorname{Re} \mu| < 1].$$

BI ((411))(5)

2. $$\int_0^\infty \ln x \cos ax \, x^{\mu-1} \, dx = \frac{\Gamma(\mu)}{a^\mu} \cos \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a - \frac{\pi}{2} \operatorname{tg} \frac{\mu\pi}{2}\right] \qquad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

BI ((411))(6)

4.423

1. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x} \, dx = \ln \frac{a}{b} \left(\boldsymbol{C} + \frac{1}{2} \ln ab\right) \qquad [a > 0, \quad b > 0].$$

GW ((338))(21a)

2. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2} \, dx = \frac{\pi}{2} [(a-b)(\boldsymbol{C}-1) + a \ln a - b \ln b] \qquad [a > 0, \quad b > 0].$$

3. $$\int_0^\infty \ln x \frac{\sin^2 ax}{x^2}\,dx = -\frac{a\pi}{2}(\boldsymbol{C} + \ln 2a - 1) \qquad [a > 0].$$

GW ((338))(20b)

4.424

1. $$\int_0^\infty (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2}\boldsymbol{C}^2 + \frac{\pi^3}{24} + \pi \boldsymbol{C} \ln a + \frac{\pi}{2}(\ln a)^2 \qquad [a > 0].$$

ET I 77(9), FI II 810a

2.[6] $$\int_0^\infty (\ln x)^2 \sin ax\, x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi'(\mu) + \psi^2(\mu) + \pi\psi(\mu)\operatorname{ctg}\frac{\mu\pi}{2} - 2\psi(\mu)\ln a - \right.$$
$$\left. - \pi \ln a \operatorname{ctg}\frac{\mu\pi}{2} + (\ln a)^2 - \frac{1}{4}\pi^2\right] \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

ET I 77(10)

4.425

1. $$\int_0^\infty \ln(1+x)\cos ax \frac{dx}{x} = \frac{1}{2}\{[\operatorname{si}(a)]^2 + [\operatorname{ci}(a)]^2\} \qquad [a > 0].$$

ET I 18(8)

2. $$\int_0^\infty \ln\left(\frac{b+x}{b-x}\right)^2 \cos ax \frac{dx}{x} = -2\pi\ \operatorname{si}(ab) \qquad [a \geq 0, \quad b > 0].$$

ET I 18(11)

3. $$\int_0^\infty \ln(1+b^2x^2)\sin ax \frac{dx}{x} = -\pi \operatorname{Ei}\left(-\frac{a}{b}\right) \qquad [a > 0, \quad b > 0].$$

GW ((338))(24), ET I 77(14)

4. $$\int_0^1 \ln(1-x^2)\cos(p\ln x)\frac{dx}{x} = \frac{1}{2p^2} + \frac{\pi}{2p}\ \operatorname{cth}\ \frac{p\pi}{2}.$$

LI ((309))(1)a

4.426

1. $$\int_0^\infty \ln\frac{b^2+x^2}{c^2+x^2}\sin ax\, x\, dx = \frac{\pi}{a^2}[(1+ac)e^{-ac}-(1+ab)e^{-ab}] \qquad [b \geq 0, \quad c \geq 0, \quad a > 0].$$

GW ((338))(23)

2. $$\int_0^\infty \ln\frac{b^2x^2+p^2}{c^2x^2+p^2}\sin ax\frac{dx}{x} = \pi\left[\operatorname{Ei}\left(-\frac{ap}{c}\right)-\operatorname{Ei}\left(-\frac{ap}{b}\right)\right] \qquad [b > 0, \quad c > 0, \quad p > 0, \quad a > 0].$$

ET I 77(15)

4.427

$$\int_0^\infty \ln(x+\sqrt{\beta^2+x^2})\frac{\sin ax}{\sqrt{\beta^2+x^2}}\, dx = \frac{\pi}{2}K_0(a\beta)+\frac{\pi}{2}\ln(\beta)[I_0(a\beta)-\mathbf{L}(a\beta)]$$
$$[\operatorname{Re}\beta > 0, \quad a > 0].$$

ET I 77(16)

4.428

1. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2}\, dx = \pi b\ln 2 - a\pi \qquad [a > 0, \quad b > 0].$$

ET I 22(29)

2. $$\int_0^\infty \ln(4\cos^2 ax)\frac{\cos bx}{x^2+c^2}\, dx = \frac{\pi}{c}\ \operatorname{ch}(bc)\ln(1+e^{-2ac}) \qquad \left[a < b < 2a < \frac{\pi}{c}\right].$$

ET I 22(30)

3. $$\int_0^\infty \ln\cos^2 ax\frac{\sin bx}{x(1+x^2)}\, dx = \pi\ln(1+e^{-2a})\ \operatorname{sh} b-\pi\ln 2(1-e^{-b}) \qquad [a > 0, \quad b > 0].$$

ET I 82(36)

4. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2(1+x^2)}\, dx = -\pi\ln(1+e^{-2a})\ \operatorname{ch} b+(b+e^{-b})\pi\ln 2-a\pi \qquad [a > 0, \quad b > 0].$$

ET I 22(31)

4.429

$$\int_0^1 \frac{(1+x)x}{\ln x} \sin(\ln x)\, dx = \frac{\pi}{4}.$$

BI ((326))(2)a

4.431

1. $$\int_0^\infty \ln(2\pm 2\cos x) \frac{\sin bx}{x^2+c^2}\, x\, dx = -\,\pi\ \operatorname{sh}(bc) \ln(1\pm e^{-c}) \qquad [b>0, \quad c>0].$$

ET I 22(32)

2. $$\int_0^\infty \ln(2\pm 2\cos x) \frac{\cos bx}{x^2+c^2}\, dx = \frac{\pi}{c}\ \operatorname{ch}(bc) \ln(1\pm e^{-c}) \qquad [b>0, \quad c>0].$$

ET I 22(32)

3. $$\int_0^\infty \ln(1+2a\cos x+a^2) \frac{\sin bx}{x}\, dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{(-a)^k}{k} [1+\operatorname{sign}(b-k)] \qquad [0<a<1, \quad b>0].$$

ET I 82(25)

4. $$\int_0^\infty \ln(1-2a\cos x+a^2) \frac{\cos bx}{x^2+c^2}\, dx = \frac{\pi}{c} \ln(1-ae^{-c})\ \operatorname{ch}(bc) + \frac{\pi}{c} \sum_{k=1}^{[b]} \frac{a^k}{k}\ \operatorname{sh}[c(b-k)]$$
$$[|a|<1, \quad b>0, \quad c>0].$$

ET I 22(33)

629

4.432

1. $$\int_0^\infty \ln(1-k^2\ \sin^2 x) \frac{\sin x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x) \frac{\sin x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \ln k' \boldsymbol{K}(k).$$

BI ((412, 414))(4)

2. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\sin^2 x) \frac{\sin x\cos x}{\sqrt{1-k^2\sin^2 x}} x\, dx =$$
$$= \frac{1}{k^2} \{\pi k'(1-\ln k') + (2-k^2)\boldsymbol{K}(k) - (4-\ln k')\boldsymbol{E}(k)\}.$$

3. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\ \cos^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2\cos^2 x}} x\, dx = \frac{1}{k^2}\{-\pi-(2-k^2)\boldsymbol{K}(k)+(4-\ln k')\boldsymbol{E}(k)\}.$$

BI ((426))(6)

4. $$\int_0^{\infty} \ln(1-k^2\ \sin^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln k')\boldsymbol{K}(k)-(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((412))(5)

5. $$\int_0^{\infty} \ln(1-k^2\ \cos^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln k')\boldsymbol{K}(k)+(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((414))(5)

6. $$\begin{aligned}\int_0^{\infty} \ln(1\pm k\sin^2 x) \frac{\sin x}{\sqrt{1-k^2\sin^2\ x}} \frac{dx}{x} &= \int_0^{\infty} \ln(1\pm k\cos^2 x) \frac{\sin x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1\pm k\sin^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1\pm k\cos^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1\pm k\sin^2 2x) \frac{\operatorname{tg} x}{\sqrt{1-k^2\sin^2 2x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1\pm k^2\cos^2 2x) \frac{\operatorname{tg} x}{\sqrt{1-k^2\cos^2 2x}} \frac{dx}{x} = \\ &= \frac{1}{2}\ln\frac{2(1\pm k)}{\sqrt{k}}\boldsymbol{K}(k) - \frac{\pi}{8}\boldsymbol{K}(k').\end{aligned}$$

BI ((413))(1--6), BI ((415))(1--6)

7. $$\int_0^{\infty} \ln(1-k^2\ \sin^2 x) \frac{\sin^3 x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln k')\boldsymbol{K}(k)+(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((412))(6)

8. $$\int_0^{\infty} \ln(1-k^2\ \cos^2 x) \frac{\sin^3 x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln k')\boldsymbol{K}(k)-(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((414))(6)a

BI ((412))(7)

10. $$\int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln k')\boldsymbol{K}(k)+(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((414))(7)

11. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\operatorname{tg} x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\operatorname{tg} x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \ln k'\boldsymbol{K}(k).$$

BI ((412, 414))(9)

12. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin^2 x\operatorname{tg} x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln k')\boldsymbol{K}(k)+(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((412))(8)

13. $$\int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin^2 x\operatorname{tg} x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln k')\boldsymbol{K}(k)-(2-\ln k')\boldsymbol{E}(k)\}.$$

BI ((414))(8)

14. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin^2 x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} =$$
$$= \frac{1}{k'^2}\{(k^2-2)\boldsymbol{K}(k)+(2+\ln k')\boldsymbol{E}(k)\}.$$

BI ((412, 414))(13)

15. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\sin^2 x)\frac{\sin x\cos x}{\sqrt{(1-k^2\sin^2 x)^3}}x\,dx = \frac{1}{k^2}\left\{(1+\ln k')\frac{\pi}{k'}-(2+\ln k')\boldsymbol{K}(k)\right\}.$$

BI ((426))(9)

16. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\cos^2 x)\frac{\sin x\cos x}{\sqrt{(1-k^2\cos^2 x)^3}}x\,dx = \frac{1}{k^2}\{-\pi+(2+\ln k')\boldsymbol{K}(k)\}.$$

BI ((426))(15)

17. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin x\cos x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin^3 x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2+\ln k')\boldsymbol{K}(k) - (2+\ln k')\boldsymbol{E}(k)\}.$$

BI ((412))(14), BI((414))(15)

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18. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin^3 x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\ cos^2 x)\frac{\sin x\cos x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} = \frac{1}{k^2k'^2}\{(2+\ln k')\boldsymbol{E}(k) - (2-k^2+k'^2\ln k')\boldsymbol{K}(k)\}.$$

BI ((412))(15), BI((414))(14)

19. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin x\cos^2 x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2\ x)\frac{\sin^2 x\ \operatorname{tg} x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2+\ln k')\boldsymbol{K}(k) - (2+\ln k')\boldsymbol{E}(k)\}.$$

BI ((412))(16), BI((414))(17)

20. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin^2 x\ \operatorname{tg} x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin x\ \cos^2 x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} = \frac{1}{k^2k'^2}\{(2+\ln k')\boldsymbol{E}(k) - (2-k^2+k'^2\ln k')\boldsymbol{K}(k)\}.$$

BI ((412))(17), BI((414))(16)

21. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\operatorname{tg} x}{\sqrt{(1-k^2\sin^2 x)^3}}\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\operatorname{tg} x}{\sqrt{(1-k^2\cos^2 x)^3}}\frac{dx}{x} = \frac{1}{k'^2}\{(k^2-2)\boldsymbol{K}(k) + (2+\ln k')\boldsymbol{E}(k)\}.$$

22. $$\int_0^\infty \ln(1-k^2\sin^2 x)\sqrt{1-k^2\sin^2 x}\sin x\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\sqrt{1-k^2\ \cos^2 x}\sin x\frac{dx}{x} = \\ = (2-k^2)\boldsymbol{K}(k) - (2-\ln k')\boldsymbol{E}(k)$$

BI ((412, 414))(1)

23. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\sin^2 x)\sqrt{1-k^2\sin^2 x}\sin x\cos x\cdot x\,dx = \\ = \frac{1}{27k^2}\{3\pi k'^3(1-3\ln k') + (22k'^2+6k^4-3k'^2\ln k')\boldsymbol{K}(k) - \\ -(2-k^2)(14-6\ln k')\boldsymbol{E}(k)\}$$

BI ((426))(1)

24. $$\int_0^{\frac{\pi}{2}} \ln(1-k^2\cos^2 x)\sqrt{1-k^2\cos^2 x}\sin x\cos x\cdot x\,dx = \\ = \frac{1}{27k^2}\{-3\pi - (22k'^2+6k^4-3k'^2\ln k')\boldsymbol{K}(k) + (2-k^2)(14-6\ln k')\boldsymbol{E}(k)\}.$$

BI ((426))(2)

25. $$\int_0^\infty \ln(1-k^2\sin^2 x)\sqrt{1-k^2\sin^2 x}\ \operatorname{tg} x\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\sqrt{1-k^2\ \cos^2 x}\ \operatorname{tg} x\frac{dx}{x} = \\ = (2-k^2)\boldsymbol{K}(k) - (2-\ln k')\boldsymbol{E}(k).$$

BI ((412, 414))(2)

26. $$\int_0^\infty ln(\sin^2 x + k'\cos^2 x)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \int_0^\infty \ln(\sin^2 x + k'\cos^2 x)\frac{\operatorname{tg} x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \\ = \int_0^\infty \ln(\sin^2 2x + k'\cos^2 2x)\frac{\operatorname{tg} x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x} = \\ = \frac{1}{2}\ln\left[\frac{2(\sqrt{k'})^3}{1+k'}\right]\boldsymbol{K}(k).$$

BI ((415))(19--21)

4.44 Combinations of logarithms, trigonometric functions, and exponentials

4.441

1.[7] $$\int_0^\infty e^{-qx}\sin px\ln x\,dx = \frac{1}{p^2+q^2}\left[q\operatorname{arctg}\frac{p}{q} - p\boldsymbol{C} - \frac{p}{2}\ \ln(p^2-q^2)\right] \qquad [q>0, \quad p>0].$$

2. $$\int_0^\infty e^{-qx}\cos px \ln x\,dx = -\frac{1}{p^2+q^2}\left[\frac{q}{2}\ln(p^2+q^2) + p\,\mathrm{arctg}\,\frac{p}{q} + q\boldsymbol{C}\right] \qquad [q>0].$$

BI ((467))(2)

4.442

$$\int_0^{\frac{\pi}{2}} \frac{e^{-p\ \mathrm{tg}\,x}\ln\cos\ x\,dx}{\sin x\cos x} = -\frac{1}{2}[\mathrm{ci}(p)]^2 + \frac{1}{2}[\mathrm{si}(p)]^2 \qquad [\mathrm{Re}\,p>0].$$

NT 32(11)

4.5 Inverse Trigonometric Functions

4.51 Inverse trigonometric functions

4.511

$$\int_0^\infty \mathrm{arcctg}\,px\ \mathrm{arcctg}\,qx\,dx = \frac{\pi}{2}\left\{\frac{1}{p}\ln\left(1+\frac{p}{q}\right) + \frac{1}{q}\ln\left(1+\frac{q}{p}\right)\right\} \qquad [p>0, \quad q>0].$$

BI ((77))(8)

4.512

$$\int_0^\pi \mathrm{arctg}(\cos x)\,dx = 0.$$

BI ((345))(1)

4.52 Combinations of arcsines, arccosines, and powers

4.521

1. $$\int_0^1 \frac{\arcsin x}{x}\,dx = \frac{\pi}{2}\ \ln 2.$$

FI II 614, 623

2. $$\int_0^1 \frac{\arccos x}{1\pm x}\,dx = \mp\frac{\pi}{2}\ln 2 + 2\boldsymbol{G}.$$

BI ((231))(7, 8)

3. $$\int_0^1 \arcsin x \frac{x}{1+qx^2}\,dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \qquad [q > -1].$$

BI ((231))(1)

4. $$\int_0^1 \arcsin x \frac{x}{1-p^2x^2}\,dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \qquad [p^2 < 1].$$

LI ((231))(3)

5. $$\int_0^1 \arccos x \frac{dx}{\sin^2\lambda - x^2} = 2\operatorname{cosec}\lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}.$$

BI ((231))(10)

6. $$\int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \qquad [q > -1].$$

BI ((235))(10)

7. $$\int_0^1 \arcsin x \frac{x}{(1+qx^2)^2}\,dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \qquad [q > -1].$$

BI ((234))(2)

8. $$\int_0^1 \arccos x \frac{x}{(1+qx^2)^2}\,dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \qquad [q > -1].$$

BI ((234))(4)

4.522

1. $$\int_0^1 x\sqrt{1-k^2x^2}\,\arccos x\,dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\boldsymbol{K}(k) - 2(1+k'^2)\boldsymbol{E}(k)\right].$$

BI ((236))(9)

2. $$\int_0^1 x\sqrt{1-k^2x^2}\,\arcsin x\,dx = \frac{1}{9k^2}\left[-\frac{3}{2}\pi k'^3 - k'^2\boldsymbol{K}(k) + 2(1+k'^2)\boldsymbol{E}(k)\right].$$

3. $$\int_0^1 x\sqrt{k'^2 + k^2x^2}\ \arcsin x\, dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\boldsymbol{K}(k) - 2(1 + k'^2)\boldsymbol{E}(k)\right].$$

BI((236))(5)

4. $$\int_0^1 \frac{x\ \arcsin x}{\sqrt{1 - k^2x^2}}\, dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right].$$

BI ((237))(1)

5. $$\int_0^1 \frac{x\ \arccos x}{\sqrt{1 - k^2x^2}}\, dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right].$$

BI ((240))(1)

6. $$\int_0^1 \frac{x\ \arcsin x}{\sqrt{k'^2 + k^2x^2}}\, dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right].$$

BI ((238))(1)

7. $$\int_0^1 \frac{x\ \arccos x}{\sqrt{k'^2 + k^2x^2}}\, dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right].$$

BI ((241))(1)

8. $$\int_0^1 \frac{x\ \arcsin x\, dx}{(x^2 - \cos^2\lambda)\sqrt{1 - x^2}} = \frac{2}{\sin\lambda}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\lambda]}{(2k+1)^2}.$$

BI ((243))(11)

9. $$\int_0^1 \frac{x\ \arcsin kx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}\, dx = -\frac{\pi}{2k}\ln k'.$$

BI ((239))(1)

10. $$\int_0^1 \frac{x\ \arccos kx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}\, dx = \frac{\pi}{2k}\ln(1 + k).$$

BI ((242))(1)

4.523

1. $$\int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1}\left[\frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!}\right].$$

BI ((229))(1)

2. $$\int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n}\left[1 - \frac{(2n-1)!!}{2^n n!}\right].$$

BI ((229))(2)

3. $$\int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!}.$$

BI ((229))(4)

4. $$\int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n}\frac{(2n-1)!!}{2^n n!}.$$

BI ((229))(5)

5. $$\int_{-1}^1 (1-x^2)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!}.$$

BI ((254))(2)

6. $$\int_{-1}^1 (1-x^2)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2}\frac{(2n-1)!!}{2^n n!}.$$

BI ((254))(3)

4.524

1. $$\int_0^1 (\arcsin x)^2 \frac{dx}{x^2\sqrt{1-x^2}} = \pi \ln 2.$$

BI ((243))(13)

2. $$\int_0^1 (\arccos x)^2 \frac{dx}{(\sqrt{1-x^2})^3} = \pi \ln 2.$$

4.53- 4.54 Combinations of arctangents, arccotangents, and powers

4.531

1. $$\int_0^1 \frac{\operatorname{arctg} x}{x}\,dx = \int_1^\infty \frac{\operatorname{arcctg} x}{x}\,dx = \boldsymbol{G}.$$

FI II 482, BI ((253))(8)

2. $$\int_0^\infty \frac{\operatorname{arcctg} x}{1 \pm x}\,dx = \pm\frac{\pi}{4}\ln 2 + \boldsymbol{G}.$$

BI ((248))(6, 7)

3. $$\int_0^1 \frac{\operatorname{arcctg} x}{x(1+x)}\,dx = -\frac{\pi}{8}\ln 2 + \boldsymbol{G}.$$

BI ((235))(11)

4. $$\int_0^\infty \frac{\operatorname{arctg} x}{1-x^2}\,dx = -\boldsymbol{G}.$$

BI ((248))(2)

5. $$\int_0^1 \operatorname{arctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)}\operatorname{arctg} q \qquad [p > -1].$$

BI ((243))(7)

6. $$\int_0^1 \operatorname{arcctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2}\operatorname{arctg} q + \frac{1}{1+p}\operatorname{arcctg} q \qquad [p > -1].$$

BI ((234))(10)

7. $$\int_0^1 \frac{\operatorname{arctg} x}{x(1+x^2)}\,dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}.$$

8. $\int_0^\infty \frac{x \operatorname{arctg} x}{1+x^4}\,dx = \frac{\pi^2}{16}.$

BI ((248))(3)

9. $\int_0^\infty \frac{x \operatorname{arctg} x}{1-x^4}\,dx = -\frac{\pi}{8}\ln 2.$

BI ((248))(4)

10. $\int_0^\infty \frac{x \operatorname{arctg} x}{1-x^4}\,dx = \frac{\pi}{8}\ln 2.$

BI ((248))(12)

11. $\int_0^\infty \frac{\operatorname{arcctg} x}{x\sqrt{1+x^2}}\,dx = \int_0^\infty \frac{\operatorname{arcctg} x}{\sqrt{1+x^2}}\,dx = 2\boldsymbol{G}.$

BI ((251))(3, 10)

12. $\int_0^1 \frac{\operatorname{arctg} x}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln(1+\sqrt{2}).$

FI II 694

13. $\int_0^1 \frac{x \operatorname{arctg} x\,dx}{\sqrt{(1+x^2)(1+k'^2x^2)}} = \frac{1}{k^2}\left[F\left(\frac{\pi}{4},\,k\right) - \frac{\pi}{2\sqrt{2\,(1+k'^2)}}\right].$

BI ((294))(14)

4.532

1. $\int_0^1 x^p \operatorname{arctg} x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} - \beta\left(\frac{p}{2}+1\right)\right] \qquad [p > -2].$

BI ((229))(7)

2. $\int_0^\infty x^p \operatorname{arctg} x\,dx = \frac{\pi}{2(p+1)} \operatorname{cosec}\frac{p\pi}{2} \qquad [-1 > p > -2].$

BI ((246))(1)

3. $\int_0^1 x^p \operatorname{arcctg} x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} + \beta\left(\frac{p}{2}+1\right)\right] \qquad [p > -1].$

4. $$\int_0^\infty x^p \operatorname{arcctg} x\,dx = -\frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2} \qquad [-1 < p < 0].$$

BI ((246))(2)

5. $$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^{2q} \operatorname{arctg} x \frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2}p} \frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)} \qquad [q > 0].$$

BI ((250))(10)

4.533

1. $$\int_0^\infty (1 - x \operatorname{arcctg} x)\,dx = \frac{\pi}{4}.$$

BI ((246))(3)

2. $$\int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x\right) \frac{dx}{1-x} = -\frac{\pi}{8} \ln 2 + \boldsymbol{G}.$$

BI ((232))(2)

3. $$\int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x\right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}.$$

BI ((235))(25)

4. $$\int_0^1 \left(x \operatorname{arctg} x - \frac{1}{x} \operatorname{arctg} x\right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2.$$

BI ((232))(1)

4.534

$$\int_0^\infty (\operatorname{arctg} x)^2 \frac{dx}{x^2\sqrt{1+x^2}} = \int_0^\infty (\operatorname{arcctg} x)^2 \frac{x\,dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4\boldsymbol{G}.$$

BI ((251))(9, 17)

4.535

1. $$\int_0^1 \frac{\operatorname{arctg} px}{1+p^2x}\,dx = \frac{1}{2p^2}\operatorname{arctg} p \ln(1+p^2).$$

BI ((231))(19)

2. $$\int_0^1 \frac{\operatorname{arcctg} px}{1+p^2x}\,dx = \frac{1}{p^2}\left\{\frac{\pi}{4} + \frac{1}{2}\operatorname{arcctg} p\right\}\ln(1+p^2) \qquad [p>0].$$

BI ((231))(24)

3. $$\int_0^\infty \frac{\operatorname{arctg} qx}{(p+x)^2}\,dx = -\frac{q}{1+p^2q^2}\left(\ln pq - \frac{\pi}{2}pq\right) \qquad [p>0, \quad q>0].$$

BI ((249))(1)

4. $$\int_0^\infty \frac{\operatorname{arcctg} qx}{(p+x)^2}\,dx = \frac{q}{1+p^2q^2}\left(\ln pq + \frac{\pi}{2pq}\right) \qquad [p>0, \quad q>0].$$

BI ((249))(8)

5. $$\int_0^\infty \frac{x\ \operatorname{arcctg} px}{q^2+x^2}\,dx = \frac{\pi}{2}\ln\frac{1+pq}{pq} \qquad [p>0, \quad q>0].$$

BI ((248))(9)

6. $$\int_0^\infty \frac{x\ \operatorname{arcctg} px\,dx}{x^2-q^2} = \frac{\pi}{4}\ln\frac{1+p^2q^2}{p^2q^2} \qquad [p>0, \quad q>0].$$

BI ((248))(10)

7. $$\int_0^\infty \frac{\operatorname{arctg} px}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(1+p) \qquad [p \geq 0].$$

FI II 745

8. $$\int_0^\infty \frac{\operatorname{arctg} px}{x(1-x^2)}\,dx = \frac{\pi}{4}\ln(1+p^2) \qquad [p \geq 0].$$

BI ((250))(6)

BI ((250))(3)

10. $\int_0^\infty \operatorname{arctg} qx \frac{dx}{x(1-p^2x^2)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \qquad [p \geq 0].$

BI ((250))(6)

11. $\int_0^\infty \frac{x \operatorname{arctg} qx}{(p^2+x^2)^2}\, dx = \frac{\pi q}{4p(1+pq)} \qquad [p > 0, \quad q \geq 0].$

BI ((252))(12)a

12. $\int_0^\infty \frac{x \operatorname{arcctg} qx}{(p^2+x^2)^2}\, dx = \frac{\pi}{4p^2(1+pq)} \qquad [p > 0, \quad q \geq 0].$

BI ((252))(20)a

13. $\int_0^1 \frac{\operatorname{arctg} qx}{x\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \ln\left(q + \sqrt{1+q^2}\right).$

BI ((244))(11)

14.* $\int_{-\infty}^\infty \frac{x \arctan(\alpha x)\, dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \frac{\pi}{\beta^2-\gamma^2} \log\left(\frac{1+|\alpha\beta|}{1+|\alpha\gamma|}\right) \operatorname{sign}(\alpha) & (\alpha, \beta, \gamma \text{real; } \beta \neq \gamma) \\ \frac{\pi\alpha}{2|\beta|\,(1+|\alpha\beta|)} & (\beta = \gamma) \end{cases}$

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15.* $\int_{-\infty}^\infty \frac{x \arctan(\alpha/x)\, dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \frac{\pi}{\beta^2-\gamma^2} \log\left(\frac{1+|\alpha/\gamma|}{1+|\alpha/\beta|}\right) \operatorname{sign}(\alpha) & (\alpha,\ \beta,\ \gamma \text{ real; } \beta \neq \gamma) \\ \frac{\pi\alpha}{2\beta^2(|\beta|+|\alpha|)} & (\beta = \gamma) \end{cases}$

4.536

1. $\int_0^\infty \operatorname{arctg} qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q\pi \ln \frac{1+\sqrt{1+q^2}}{\sqrt{1+q^2}} + \frac{\pi}{2} \ln\left(q + \sqrt{1+q^2}\right) - \frac{\pi}{2} - \operatorname{arctg} q.$

2. $\int_0^\infty \frac{\operatorname{arctg} px - \operatorname{arctg} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q} \qquad [p>0, \quad q>0].$

FI II 635

3. $\int_0^\infty \frac{\operatorname{arctg} px \operatorname{arctg} qx}{x^2}\,dx = \frac{\pi}{2}\,\ln\frac{(p+q)^{p+q}}{p^p q^q} \qquad [p>0, \quad q>0].$

FI II 745

4.537

1.[8] $\int_0^1 \operatorname{arctg}(\sqrt{1-x^2})\frac{dx}{1-x^2\cos^2\lambda} = \frac{\pi}{2\cos\lambda}\ln\left[\cos\left(\frac{\pi-4\lambda}{8}\right)\operatorname{cosec}\left(\frac{\pi+4\lambda}{8}\right)\right].$

BI ((245))(9)

2. $\int_0^1 \operatorname{arctg}\left(p\sqrt{1-x^2}\right)\frac{dx}{1-x^2} = \frac{1}{2}\pi\ln\left(p+\sqrt{1+p^2}\right) \qquad [p>0].$

BI ((245))(10)

3. $$\int_0^1 \operatorname{arctg}\left(\operatorname{tg}\lambda\sqrt{1-k^2x^2}\right)\sqrt{\frac{1-x^2}{1-k^2x^2}}\,dx = \frac{\pi}{2k^2}[E(\lambda,\,k) - k'^2F(\gamma,\,k)] - \\ -\frac{\pi}{2k^2}\operatorname{ctg}\gamma\left(1-\sqrt{1-k^2\,\sin^2\gamma}\right).$$

BI ((245))(12)

4. $\int_0^1 \operatorname{arctg}\left(\operatorname{tg}\lambda\sqrt{1-k^2x^2}\right)\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = \frac{\pi}{2}E(\lambda,\,k) - \frac{\pi}{2}\operatorname{ctg}\lambda\left(1-\sqrt{1-k^2\sin^2\lambda}\right).$

BI ((245))(11)

5. $\int_0^1 \frac{\operatorname{arctg}\left(\operatorname{tg}\lambda\sqrt{1-k^2x^2}\right)}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \frac{\pi}{2}F(\lambda,\,k).$

BI ((245))(13)

4.538

1. $$\int_0^\infty \operatorname{arctg} x^2\frac{dx}{1+x^2} = \int_0^\infty \operatorname{arctg} x^3\frac{dx}{1+x^2}; \\ = \int_0^\infty \operatorname{arcctg} x^2\frac{dx}{1+x^2} = \int_0^\infty \operatorname{arcctg} x^3\frac{dx}{1+x^2} = \frac{\pi^2}{8}.$$

2. $$\int_0^\infty \frac{1-x^2}{x^2} \operatorname{arctg} x^2 \, dx = \frac{\pi}{2}(\sqrt{2}-1).$$

BI ((244))(10)a

4.539

$$\int_0^\infty x^{s-1} \operatorname{arctg}(ae^{-x}) \, dx = 2^{-s-1}\Gamma(s)a\Phi\left(-a^2,\ s+1,\ \frac{1}{2}\right).$$

ET I 222(47)

4.541

$$\int_0^\infty \operatorname{arctg}\left(\frac{p\sin qx}{1+p\ \cos qx}\right)\frac{x\,dx}{1+x^2} = \frac{\pi}{2}\ln(1+pe^{-q}) \qquad [p > -e^q].$$

BI ((341))(14)a

4.55 Combinations of inverse trigonometric functions and exponentials

4.551

1. $$\int_0^1 (\arcsin x)e^{-bx} \, dx = \frac{\pi}{2b}[I_0(b) - \boldsymbol{L}_0(b)].$$

ET I 160(1)

2. $$\int_0^1 x(\arcsin x)e^{-bx} \, dx = \frac{\pi}{2b^2}[\boldsymbol{L}_0(b) - I_0(b) + b\boldsymbol{L}_1(b) - bI_1(b)] + \frac{1}{b}.$$

ET I 161(2)

3.* $$\int_0^\infty \left(\operatorname{arctg}\frac{x}{a}\right)e^{-bx} \, dx = \frac{1}{b}[\operatorname{ci}(ab)\sin(ab) - \operatorname{si}(ab)\cos(ab)] \qquad [\operatorname{Re} b > 0].$$

ET I 161(3)

4.* $$\int_0^\infty \left(\operatorname{arcctg}\frac{x}{a}\right)e^{-bx} \, dx = \frac{1}{b}\left[\frac{\pi}{2} - \operatorname{ci}(ab)\sin(ab) + \operatorname{si}(ab)\cos(ab)\right] \qquad [\operatorname{Re} b > 0].$$

ET I 161(4)

4.552

$$\int_0^\infty \frac{\operatorname{arctg} \frac{x}{q}}{e^{2\pi x} - 1}\, dx = \frac{1}{2}\left[\ln\Gamma(q) - \left(q - \frac{1}{2}\right)\ln q + q - \frac{1}{2}\ln 2\pi\right] \qquad [q > 0].$$

WH

4.553

$$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arcctg} x - e^{-px}\right)\frac{dx}{x} = \boldsymbol{C} + \ln p \qquad [p > 0].$$

NT 66(12)

4.56 A combination of the arctangent and a hyperbolic function

4.561

$$\int_{-\infty}^\infty \frac{\operatorname{arctg} e^{-x}}{\operatorname{ch}^{2q} px}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\Pi(x)}{\operatorname{ch}^{2q} px}\, dx = \frac{\sqrt{\pi^3}}{4p}\frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)} \qquad [q > 0].$$

LI ((282))(10)

4.57 Combinations of inverse and direct trigonometric functions

4.571

$$\int_0^{\frac{\pi}{2}} \arcsin(k\sin x)\frac{\sin x\, dx}{\sqrt{1 - k^2\sin^2 x}} = -\frac{\pi}{2k}\ln k'.$$

BI ((344))(2)

4.572

$$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arcctg} x - \cos px\right) dx = \boldsymbol{C} + \ln p \qquad [p > 0].$$

NT 66(12)

4.573

1. $$\int_0^\infty \operatorname{arcctg} qx \sin px\, dx = \frac{\pi}{2p}\left(1 - e^{-\frac{p}{q}}\right) \qquad [p > 0, \quad q > 0]$$

2. $\int_0^\infty \operatorname{arcctg} qx \cos px\, dx = \frac{1}{2p}\left[e^{-\frac{p}{q}} \operatorname{Ei}\left(\frac{p}{q}\right) - e^{\frac{p}{q}} \operatorname{Ei}\left(-\frac{p}{q}\right)\right] \qquad [p > 0, \quad q > 0]$

BI ((347))(2)a

3. $\int_0^\infty \operatorname{arcctg} rx \frac{\sin px\, dx}{1 \pm 2q \cos px + q^2} = \pm\frac{\pi}{2pq} \ln \frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}} \qquad [p^2 < 1, \quad r > 0, \quad p > 0];$

$= \pm\frac{\pi}{2pq} \ln \frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \qquad [q^2 > 1, \quad r > 0, \quad p > 0].$

BI ((347))(10)

4. $\int_0^\infty \operatorname{arcctg} px \frac{\operatorname{tg} x\, dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln\left(1 + \frac{r}{q}\theta\frac{1}{p}\right) \qquad [p > 0, \quad q > 0, \quad r > 0].$

BI ((347))(9)

4.574

1. $\int_0^\infty \operatorname{arctg}\left(\frac{2a}{x}\right) \sin(bx)\, dx = \frac{\pi}{b} e^{-ab} \operatorname{sh}(ab) \qquad [\operatorname{Re} a > 0, \quad b > 0].$

ET I 87(8)

2.[7] $\int_0^\infty \operatorname{arctg} \frac{a}{x} \cos(bx)\, dx = \frac{1}{2b}[e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab)] \qquad [a > 0, \quad b > 0].$

ET I 29(7)

3. $\int_0^\infty \operatorname{arctg}\left[\frac{2ax}{x^2 + c^2}\right] \sin(bx)\, dx = \frac{\pi}{b} e^{-b\sqrt{a^2+c^2}} \operatorname{sh}(ab) \qquad [b > 0].$

ET I 87(9)

4. $\int_0^\infty \operatorname{arctg}\left(\frac{2}{x^2}\right) \cos(bx)\, dx = \frac{\pi}{b} e^{-b} \sin b \qquad [b > 0].$

ET I 29(8)

4.575

1. $\int_0^\pi \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \sin nx\, dx = \frac{\pi}{2n} p^n \qquad [p^2 < 1].$

2. $$\int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p\, \cos x} \sin nx \cos x \, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} + \frac{p^{n-1}}{n-1} \right) \qquad [p^2 < 1].$$

BI ((345))(5)

3. $$\int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p\, \cos x} \cos nx \sin x \, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} - \frac{p^{n-1}}{n-1} \right) \qquad [p^2 < 1].$$

BI ((345))(6)

4.576

1. $$\int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p\, \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p} \qquad [p^2 < 1].$$

BI((346))(1)

2. $$\int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p\, \cos x} \frac{dx}{\operatorname{tg} x} = -\frac{\pi}{2} \ln(1 - p^2) \qquad [p^2 < 1].$$

BI((346))(3)

4.577

1. $$\int_0^{\frac{\pi}{2}} \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1 - k^2 \sin^2 x}) \frac{\sin^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \\ = \frac{\pi}{2k^2} [F(\lambda,\, k) - E(\lambda,\, k) + \operatorname{ctg} \lambda (1 - \sqrt{1 - k^2 \sin^2 \lambda})].$$

BI ((344))(4)

2. $$\int_0^{\frac{\pi}{2}} \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1 - k^2 \sin^2 x}) \frac{\cos^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \\ = \frac{\pi}{2k^2} [E(\lambda,\, k) - k'^2 F(\lambda,\, k) + \operatorname{ctg} \lambda (\sqrt{1 - k^2 \sin^2 \lambda} - 1)].$$

BI ((344))(5)

4.58 A combination involving an inverse and a direct trigonometric function and a power

4.581*

$$\int_0^\infty \operatorname{arctg} x \cos px \frac{dx}{x} = \int_0^\infty \operatorname{arctg} \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2} \operatorname{Ei}(-p) \qquad [\operatorname{Re}(p) > 0].$$

ET I 29(3), NT 25(13)

4.59 Combinations of inverse trigonometric functions and logarithms

4.591

$$1.\quad \int_0^1 \arcsin x \ln x \, dx = 2 - \ln 2 - \frac{1}{2}\pi.$$

BI ((339))(1)

$$2.\quad \int_0^1 \arccos x \ln x \, dx = \ln 2 - 2.$$

BI ((339))(2)

4.592

$$\int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^\infty \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1}.$$

BI ((339))(8)

4.593

$$1.\quad \int_0^1 \operatorname{arctg} x \ln x \, dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48}\pi^2.$$

BI ((339))(3)

$$2.\quad \int_0^1 \operatorname{arcctg} x \ln x \, dx = -\frac{1}{48}\pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

BI ((339))(4)

4.594

$$\int_0^1 \operatorname{arctg} x (\ln x)^{n-1} (\ln x + n) \, dx = \frac{n!}{(-2)^{n+1}} (2^{-n} - 1) \zeta(n+1).$$

4.6 Multiple Integrals

4.60 Change of variables in multiple integrals

4.601

$$1.\quad \iint\limits_{(\sigma)} f(x,y)\,dx\,dy = \iint\limits_{(\sigma')} f[\varphi(u,v),\ \psi(u,\ v)]\,|\Delta|\,du\,dv,$$

641

where $x = \varphi(u,\ v),\ y = \psi(u,\ v)$, and $\Delta = \frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v} - \frac{\partial\psi}{\partial u}\frac{\partial\varphi}{\partial v} \equiv \frac{\Delta(\varphi,\psi)}{\Delta(u,v)}$ is the Jacobian determinant of the functions φ and ψ.

$$2.\quad \iiint\limits_{(V)} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{(V')} f[\varphi(u,v,w),\ \psi(u,v,w),\ \mathrm{chi}(u,v,w)]\,|\Delta|\,du\,dv\,dw,$$

where $x = \varphi(u,\ v,\ w),\ y = \psi(u,\ v,\ w)$, and $z = \mathrm{chi}(u,\ v,\ w)$ and where

$$\Delta = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial w} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\,\mathrm{chi}}{\partial u} & \frac{\partial\,\mathrm{chi}}{\partial v} & \frac{\partial\,\mathrm{chi}}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi,\psi,\mathrm{chi})}{D(u,v,w)}$$

is the Jacobian determinant of the functions φ, ψ, and chi.

Here, we assume, both in (4.601 2.) and in (4.601 1.) that

(a) the functions $\varphi,\ \psi$, and chi and also their first partial derivatives are continuous in the region of integration;

(b) the Jacobian does not change sign in this region;

(c) there exists a one-to-one correspondence between the old variables $x,\ y,\ z$ and the new ones $u,\ v,\ w$ in the region of integration;

(d) when we change from the variables $x,\ y,\ z$ to the variables $u,\ v,\ w$, the region V (resp. σ) is mapped into the region V' (resp. σ').

4.602

Transformation to polar coordinates:

$$x = r\cos\varphi, \quad y = r\ \sin\varphi; \quad \frac{D(x,\ y)}{D(r,\ \varphi)} = r.$$

4.603

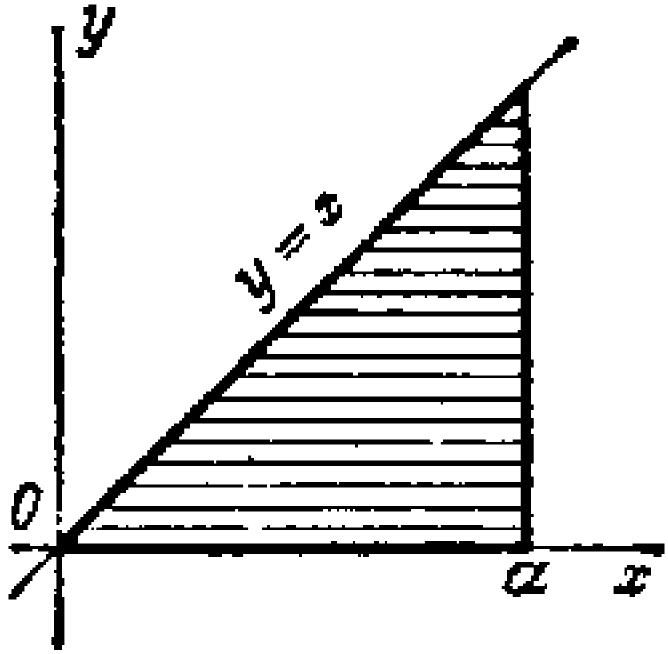
Transformation to spherical coordinates:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\ \sin\varphi, \quad z = r\cos\theta, \quad \frac{D(x,\ y,\ z)}{D(r,\ \theta,\ \varphi)} = r^2\sin\theta.$$

4.61 Change of the order of integration and change of variables

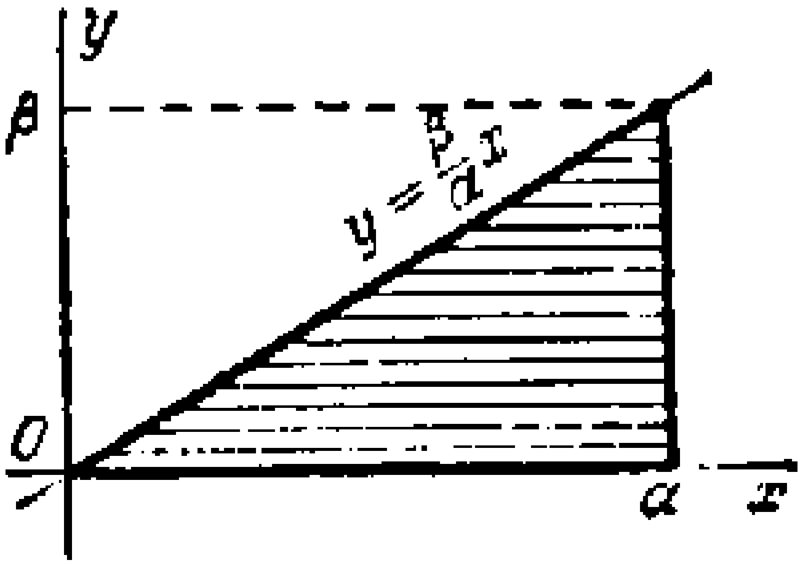
4.611

$$1. \quad \int_0^\alpha dx \int_0^x f(x,\ y)\,dy = \int_0^\alpha dy \int_y^\alpha f(x,\ y)\,dx.$$



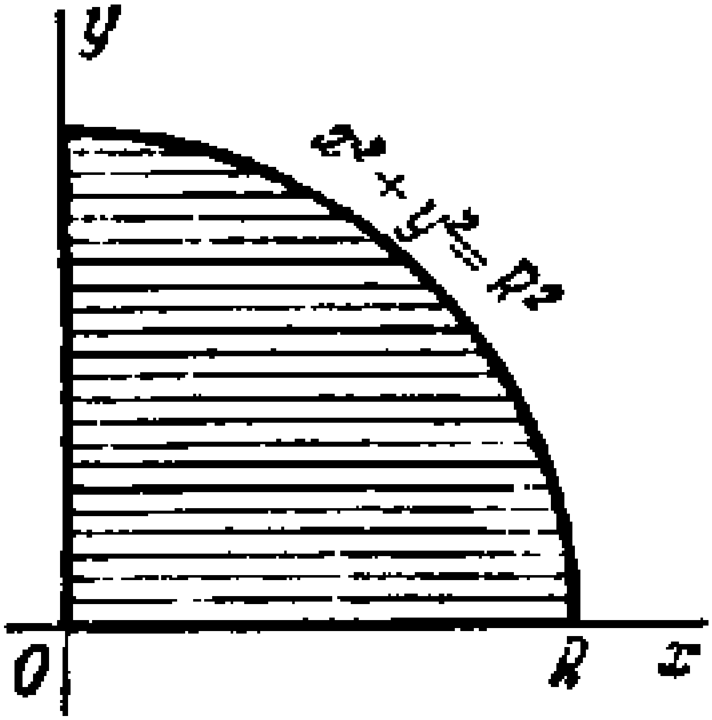
642

2. $$\int_0^{\alpha} dx \int_0^{\frac{\beta}{\alpha}x} f(x,\, y)\, dy = \int_0^{\beta} dy \int_{\frac{\alpha}{\beta}y}^{\alpha} f(x,\, y)\, dx.$$

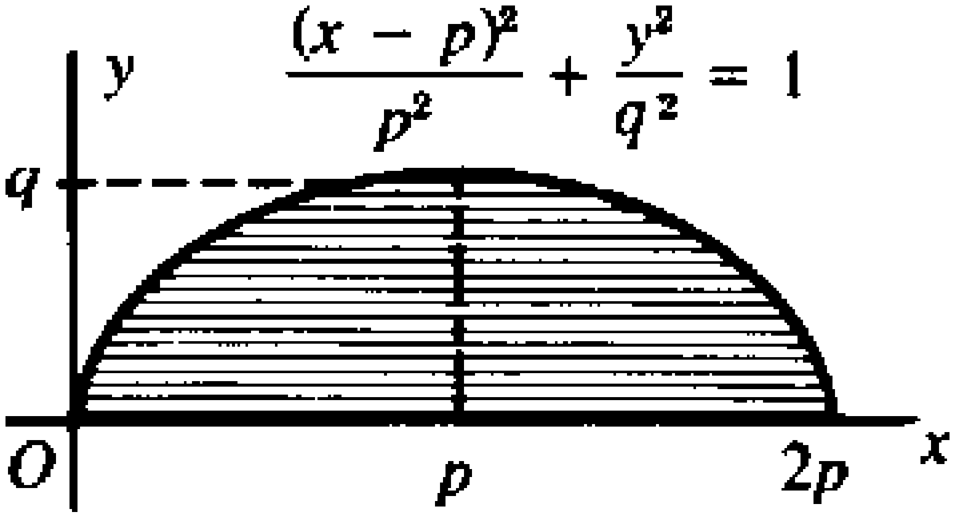


4.612

1. $$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,\, y)\, dy = \\ = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f(x,\, y)\, dx.$$

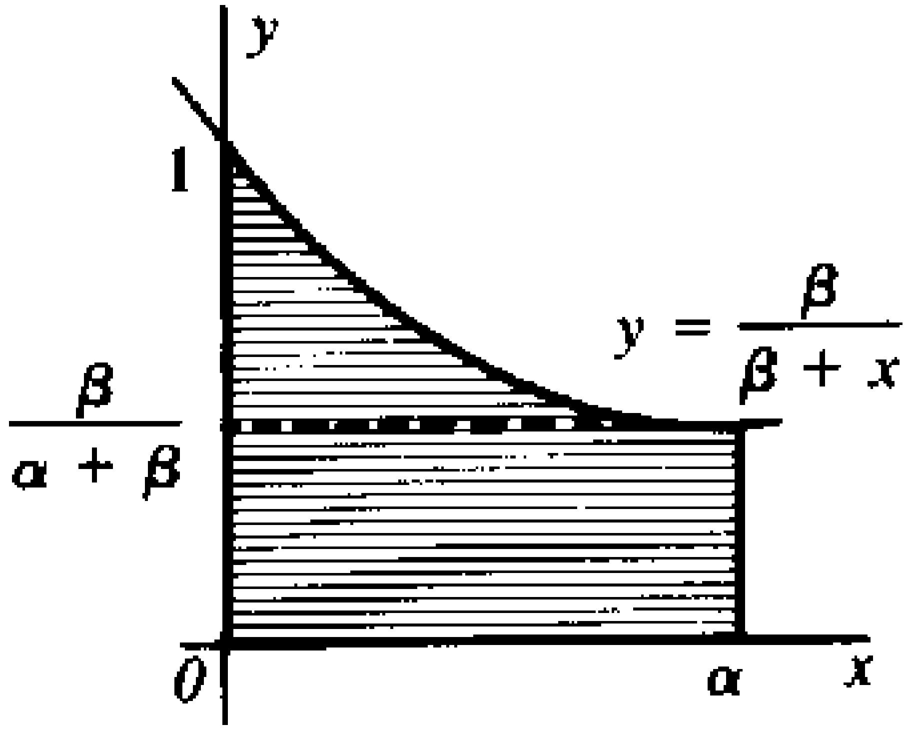


2. $$\int_0^{2p} dx \int_0^{q/p\sqrt{2px-x^2}} f(x,\, y)\, dy =$$
$$= \int_0^{q} dy \int_{p[1-\sqrt{1-(y/q)^2}]}^{p[1+\sqrt{1-(y/q)^2}]} f(x,\, y)\, dx.$$

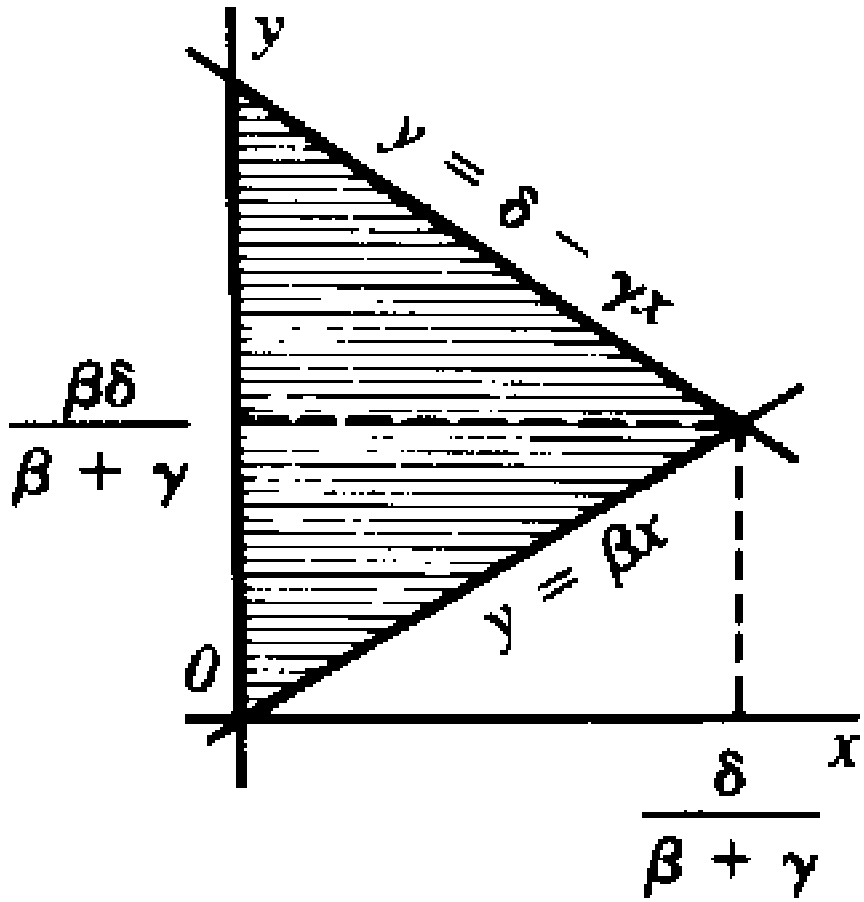


4.613

1. $$\int_0^\alpha dx \int_0^{\beta/(\beta+x)} f(x,\, y)\, dy =$$
$$= \int_0^{\beta/(\beta+\alpha)} dy \int_0^\alpha f(x,\, y)\, dx +$$
$$+ \int_{\beta/(\beta+\alpha)}^1 dy \int_0^{\beta(1-y)/y} f(x,\, y)\, dx.$$



2. $$\int_0^{\alpha} dx \int_{\beta x}^{\delta-\nu x} f(x,\, y)\, dy =$$
$$= \int_0^{\alpha\beta} dy \int_0^{y/\beta} f(x,\, y)\, dx +$$
$$+ \int_{\alpha\beta}^{\delta} dy \int_0^{(\delta-y)/\gamma} f(x,\, y)\, dx.$$
$$\left[\alpha = \frac{\delta}{\beta+\gamma}, \quad a > 0, \quad \beta > 0, \quad \gamma > 0\right].$$

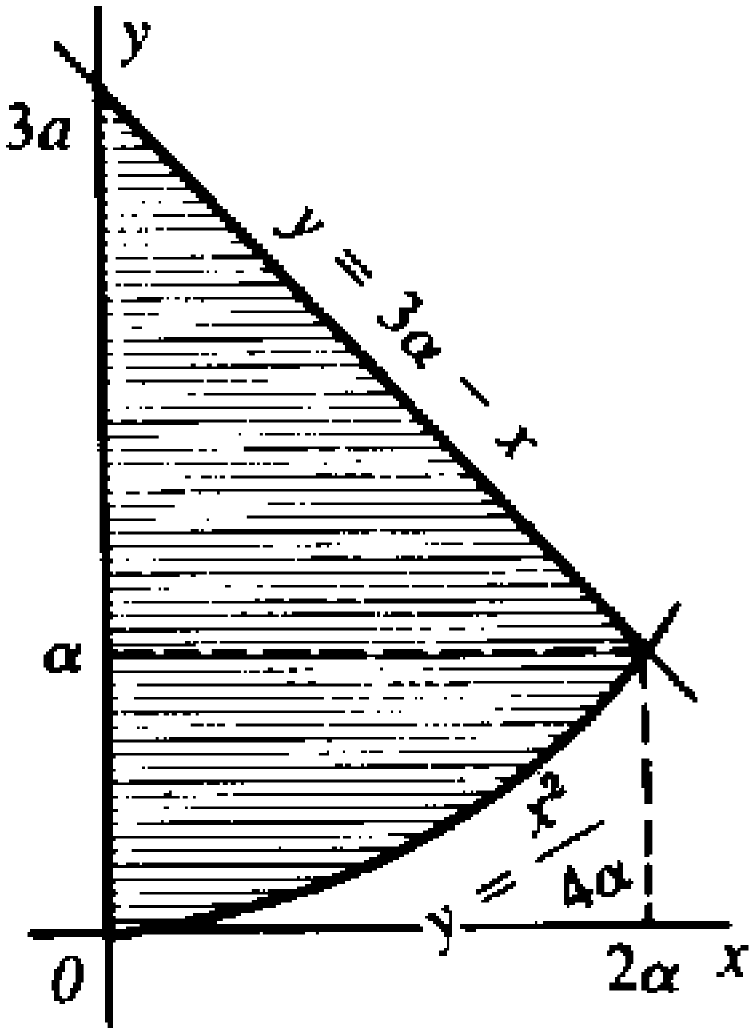


643

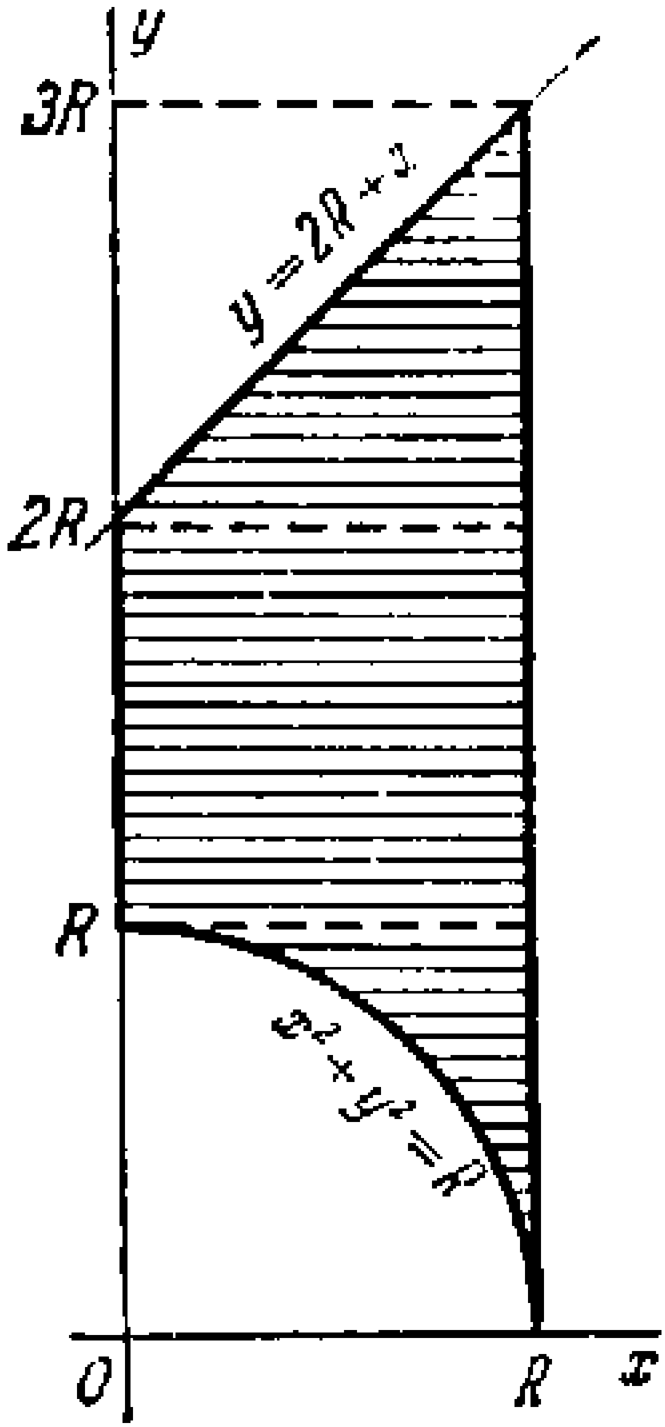
3. $\displaystyle\int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha - x} f(x,\, y)\, dy =$

$$= \int_0^{\alpha} dy \int_0^{2\sqrt{\alpha y}} f(x,\, y)\, dx +$$

$$+ \int_{\alpha}^{3\alpha} dy \int_0^{3\alpha - y} f(x,\, y)\, dx.$$



4. $$\int_0^R dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x,\,y)\,dy = \int_0^R dy \int_{\sqrt{R^2-y^2}}^{R} f(x,\,y)\,dx + \\ + \int_R^{2R} dy \int_0^R f(x,\,y)\,dx + \\ + \int_{2R}^{3R} dy \int_{y-2R}^{R} f(x,\,y)\,dx.$$

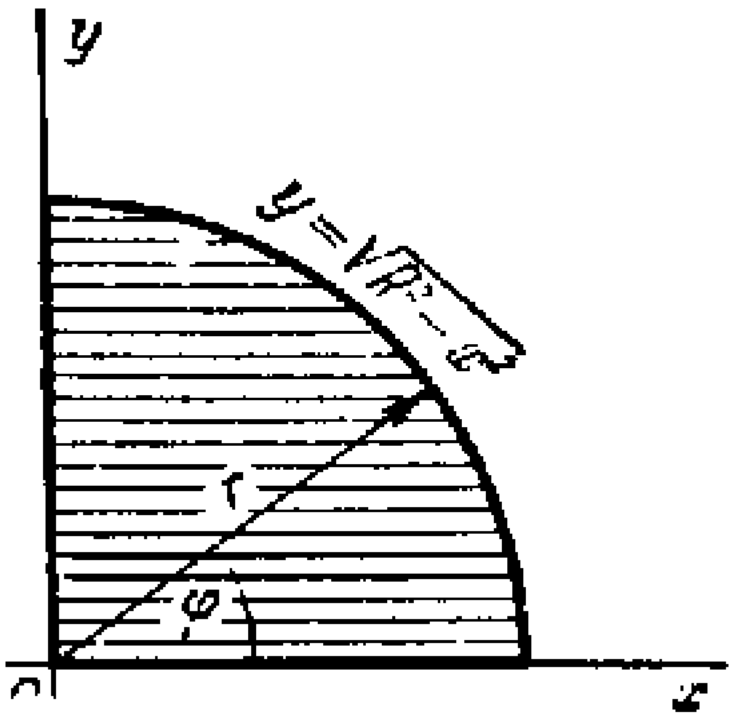


4.614

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2R\,\cos\varphi} f(r,\,\varphi)\,dr = \int_0^{2R} dr \int_0^{\arccos\frac{r}{2R}} f(r,\,\varphi)\,d\varphi.$$

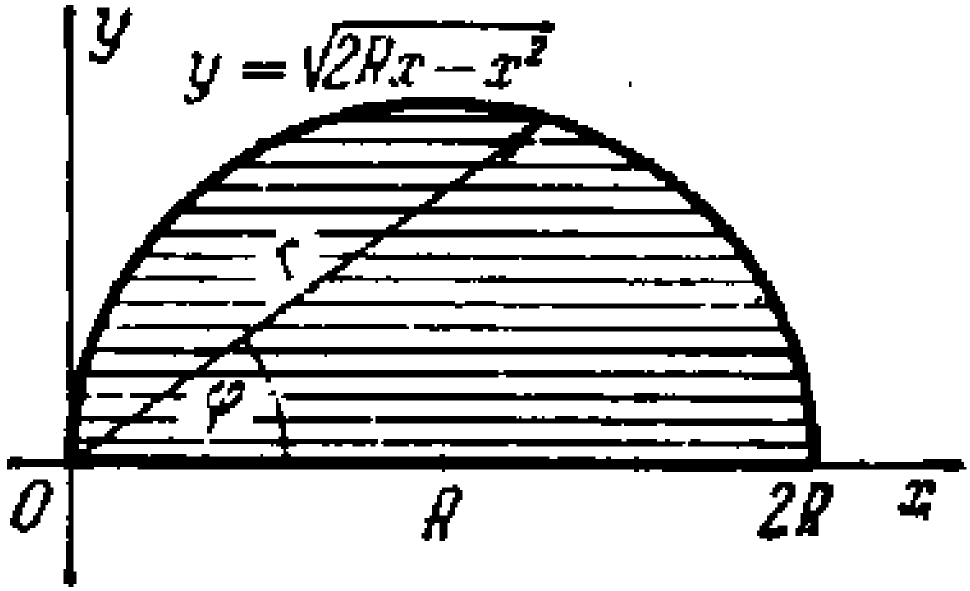
4.615

$$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,\ y)\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^R f(r\ \cos\varphi r, \sin\varphi)\,r\,dr.$$



4.616

$$\int_0^{2R} dx \int_0^{\sqrt{2R-x^2}} f(x,\ y)\,dy =$$
$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2R\cos\varphi} f(r\cos\varphi r, \sin\varphi)\,r\,dr.$$



4.617

$$\int_\alpha^\beta dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy = \int_0^\beta dx \int_0^{\varphi_2(x)} f(x,y)dy - \int_0^\beta dx \int_0^{\varphi_1(x)} f(x,y)dy - \int_0^\alpha dx \int_0^{\varphi_2(x)} f(x,y)dy + \\ + \int_0^\alpha dx \int_0^{\varphi_1(x)} f(x,\, y)\, dy [\varphi_1(x) \le \varphi_2(x) \text{ for } \alpha \le x \le \beta].$$

4.618

$$\int_0^\gamma dx \int_0^{\varphi(x)} f(x,\, y)\, dy = \int_0^\gamma dx \int_0^1 f[x,\, z\varphi(x)]\varphi(x) dz \qquad [y = z\varphi(x)]; \\ = \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z,\, y)\, dy \qquad [x = \gamma z].$$

4.619

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x,\, y)\, dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0) f[x,\, y_0 + (y_1 - y_0)t]\, dt \qquad [y = y_0 + (y_1 - y_0)t].$$

4.62 Double and triple integrals with constant limits

4.620

General formulas

1. $$\int_0^{\pi} d\omega \int_0^{\infty} f'(p \text{ ch } x + q \cos\omega \text{ sh } x) \text{ sh } x\, dx = -\frac{\pi \text{ sign } p}{\sqrt{p^2 - q^2}} f\left(\text{sign } p \sqrt{p^2 - q^2}\right)$$
$$\left[p^2 > q^2, \quad \lim_{x \to +\infty} f(x) = 0\right]$$

LO III 389

2. $$\int_0^{2\pi} d\omega \int_0^{\infty} f'[p \text{ ch } x + (q \cos\omega + r \sin\omega) \text{ sh } x] \text{ sh } x\, dx =$$
$$= -\frac{2\pi \text{ sign } p}{\sqrt{p^2 - q^2 - r^2}} f(\text{sign } p \sqrt{p^2 - q^2 - r^2}) \qquad \left[p^2 > q^2 + r^2, \quad \lim_{x \to +\infty} f(x) = 0\right].$$

LO III 390

3. $$\int_0^{\pi} \int_0^{\pi} \frac{dx\, dy}{\sin x \sin^2 y} f'\left[\frac{p - q\cos x}{\sin x \sin y} + r \text{ ctg } y\right] = -\frac{2\pi \text{ sign } p}{\sqrt{p^2 - q^2 - r^2}} f\left(\text{sign } p \sqrt{p^2 - q^2 - r^2}\right)$$
$$\left[p^2 > q^2 + r^2, \quad \lim_{x \to +\infty} f(x) = 0\right].$$

LO III 280

4. $$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f'(p \text{ ch } x \text{ ch } y + q \text{ sh } x \text{ ch } y + r \text{ sh } y) \text{ ch } y\, dy =$$
$$= -\frac{2\pi \text{ sign } p}{\sqrt{p^2 - q^2 - r^2}} f(\text{sign } p \sqrt{p^2 - q^2 - r^2}) \qquad \left[p^2 > q^2 + r^2, \quad \lim_{x \to +\infty} f(x) = 0\right].$$

LO III 390

5. $$\int_0^{\infty} dx \int_0^{\pi} f(p \text{ ch } x + q \cos\omega \text{ sh } x) \text{ sh}^2 x \sin\omega\, d\omega = 2\int_0^{\infty} f(\text{sign } p \sqrt{p^2 - q^2} \text{ ch } x) \text{ sh}^2 x\, dx$$
$$\left[\lim_{x \to +\infty} f(x) = 0\right].$$

LO III 391

6. $$\int_0^{\infty} dx \int_0^{2\pi} d\omega \int_0^{\pi} f[p \text{ ch } x + (q\cos\omega + r\sin\omega)\sin\theta \text{ sh } x] \text{ sh}^2 x \sin\theta\, d\theta =$$
$$= 4\int_0^{\infty} f(\text{sign } p \sqrt{p^2 - q^2 - r^2} \text{ ch } x) \text{ sh}^2 x\, dx \qquad \left[p^2 > q^2 + r^2, \quad \lim_{x \to +\infty} f(x) = 0\right].$$

7. $$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f\{p\ \operatorname{ch} x + [(q\cos\omega + r\sin\omega)\sin\theta + s\ \operatorname{ch}\theta]\ \operatorname{sh} x\}\operatorname{sh}^2 x \sin\theta\, d\theta =$$
$$= 4\pi \int_0^\infty f(\operatorname{sign} p\sqrt{p^2-q^2-r^2-s^2}\ \operatorname{ch} x)\operatorname{sh}^2 x\, dx \qquad \left[p^2 > q^2+r^2+s^2, \quad \lim_{x\to+\infty} f(x) = 0\right].$$

LO III 391

4.621

1. $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\, dx\, dy = \frac{\pi}{2\sqrt{1-k^2}}.$$

LO I 252(90)

2. $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\, dx\, dy = \boldsymbol{K}(k).$$

LO I 252(91)

3. $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin\alpha\sin y\, dx\, dy}{\sqrt{1-\sin^2\alpha\sin^2 x\sin^2 y}} = \frac{\pi\alpha}{2}.$$

LO I 253

4.622

1. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\, dy\, dz}{1-\cos x\cos y\cos z} = 4\pi\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right).$$

MO 137

2. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\, dy\, dz}{3-\cos y\cos z-\cos x\cos z-\cos x\cos y} = \sqrt{3}\pi\boldsymbol{K}^2\left(\sin\frac{\pi}{12}\right).$$

MO 137

3. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\, dy\, dz}{3-\cos x-\cos y-\cos z} = 4\pi[18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}]\boldsymbol{K}^2[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})].$$

MO 137

$$\int_0^\infty \int_0^\infty \varphi(a^2x^2 + b^2y^2)\,dx\,dy = \frac{\pi}{2ab}\int_0^\infty \varphi(x^2)\,x\,dx.$$

4.624

$$\int_0^\pi \int_0^{2\pi} f(\alpha\cos\theta + \beta\sin\theta\cos\psi + \gamma\sin\theta\sin\psi)\sin\theta\,d\theta\,d\psi =$$
$$= 2\pi\int_0^\pi f(R\cos p)\sin p\,dp = 2\pi\int_{-1}^1 f(Rt)\,dt \qquad \left[R = \sqrt{\alpha^2+\beta^2+\gamma^2}\right].$$

4.625*

$$P_l(a,\,b) = \int_0^a dx\int_0^b dy\,(x^2+y^2+1)^{-3/2}P_l\left(1/\sqrt{x^2+y^2+1}\right)$$

$$P_{2l}(a,b) = \frac{1}{l(2l+1)2^{2l}}\,\frac{ab}{\sqrt{a^2+b^2+1}}\sum_{k=0}^{l-1}\frac{(-1)^{l-k-1}2^{2k}\binom{2l+2k}{l+k}\binom{l+k}{l-k-1}}{\binom{2k}{k}(2k+1)}\times$$
$$\times(2l+2k+1)\sum_{j=0}^{k}\frac{\binom{2j}{j}}{2^{2j}}\,\frac{1}{(a^2+b^2+1)^j}\left(\frac{1}{(a^2+1)^{k-j+1}}+\frac{1}{(b^2+1)^{k-j+1}}\right).$$

$$p_{2l+1}(a,b) = \frac{1}{2^{2l+1}(2l+1)}\sum_{k=0}^{l}\frac{(-1)^{l+k}}{2^{2k}}\binom{l}{k}\binom{l+k+1}{k}\binom{2l+2k+1}{l+k}\times$$
$$\times\left\{\frac{1}{(b^2+1)^k}\,\frac{b}{\sqrt{b^2+1}}\operatorname{arctg}^{-1}\frac{a}{\sqrt{b^2+1}}+\frac{1}{(a^2+1)^k}\,\frac{a}{\sqrt{a^2+1}}\operatorname{arctg}^{-1}\frac{b}{\sqrt{a^2+1}}+\right.$$
$$\left.+ab\sum_{j=1}^{k}\frac{2^{2j-1}}{j\binom{2j}{j}}\cdot\frac{1}{(a^2+b^2+1)^j}\left(\frac{1}{(a^2+1)^{k-j+1}}+\frac{1}{(b^2+1)^{k-j+1}}\right)\right\}.$$

4.63-4.64 Multiple integrals

4.631

$$\int_p^x dt_{n-1}\int_p^{t_{n-1}} dt_{n-2}\ldots\int_p^{t_1} f(t)\,dt = \frac{1}{(n-1)!}\int_p^x (x-t)^{n-1}f(t)\,dt,$$

$$\int_p^x dt_{n-1} \int_p^{t_{n-1}} dt_{n-2} \dots \int_p^{t_1} f(t)\,dt = \frac{1}{(n-1)!} \int_p^x (x-t)^{n-1} f(t)\,dt,$$

where $f(t)$ is continuous on the interval $[p,\, q]$ and $p \le x \le q$.

4.632

1. $$\int\int\limits_{\substack{x_1 \ge 0,\, x_2 \ge 0, \dots,\, x_n \ge 0 \\ x_1 + x_2 + \dots + x_n \le h}} \int dx_1\, dx_2 \dots dx_n = \frac{h^n}{n!} \qquad \text{[the volume of an } n\text{-dimensional simplex]}$$

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FI II 692

2. $$\int\int\limits_{x_1^2 + x_2^2 + \dots + x_n^2 \le R^2} \cdots \int dx_1\, dx_2 \dots dx_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n \qquad \text{[the volume of an } n\text{-dimensional sphere]}$$

FI III 473

647

4.633

$$\int\int\limits_{x_1^2 + x_2^2 + \dots + x_n^2 \le 1} \cdots \int \frac{dx_1\, dx_2 \dots dx_n}{\sqrt{1 - x_1^2 - x_2^2 - \dots - x_n^2}} = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \qquad [n > 1]$$

[Half-area of the surface of an $(n+1)$-dimensional sphere $x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1$].

FI III {474}

4.634[8]

$$\int\int\limits_{\substack{x_1 \ge 0,\, x_2 \ge 0, \dots,\, x_n \ge 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \le 1}} \int x_1^{p_1 - 1} x_2^{p_2 - 1} \dots x_n^{p_n - 1}\, dx_1\, dx_2 \dots dx_n =$$

$$= \frac{q_1^{p_1} q_2^{p_2} \dots q_n^{p_n}}{\alpha_1 \alpha_2 \dots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} + 1\right)} \qquad [\alpha_i > 0, \quad p_i > 0, \quad q_i > 0, \quad i = 1, 2, \dots, n].$$

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4.635

1.[8] $$\int\int\limits_{\substack{x_1 \ge 0,\, x_2 \ge 0, \dots,\, x_n \ge 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \ge 1}} \cdots \int f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n}\right] \times$$

$$\times\, x_1^{p_1 - 1} x_2^{p_2 - 1} \dots x_n^{p_n - 1}\, dx_1\, dx_2 \dots dx_n =$$

$$= \frac{q_2^{p_1} q_2^{p_2} \dots q_n^{p_n}}{\alpha_1 \alpha_2 \dots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}\right)} \int_1^\infty f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} - 1}\, dx$$

2.[8] $$\underset{\substack{x_1\ge 0,\, x_2\ge 0,\dots,x_n\ge 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\dots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\le 1}}{\int\int\cdots\int} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\dots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]\times$$
$$\times\, x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}\,dx_1\,dx_2\dots dx_n =$$
$$=\frac{q_1^{p_1}q_2^{p_2}\dots q_n^{p_n}}{\alpha_1\alpha_2\dots\alpha_n}\,\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}\right)}\int_0^1 f(x)x^{\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}-1}\,dx$$

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under the assumptions that the one-dimensional integral on the right coverges absolutely and that the numbers $q_i,\ \alpha_i$, and p_i are positive.

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In particular,

3. $$\underset{\substack{x_1\ge 0,\, x_2\ge 0,\dots,x_n\ge 0\\ x_1+x_2+\dots+x_n\le 1}}{\int\int\cdots\int} x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}e^{-q(x_1+x_2+\dots x_n)}\,dx_1\,dx_2\dots dx_n =$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1+p_2+\dots+p_n)}\int_0^1 x^{p_1+p_2+\dots+p_n-1}e^{-qx}\,dx \qquad [n>0,\quad p_1>0,\quad p_2>0,\dots,p_n>0].$$

FI III 479

4.[8] $$\underset{\substack{x_1\ge 0,\, x_2\ge 0,\dots,x_n\ge 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\dots+x_n^{\alpha_n}\le 1}}{\int\int\cdots\int} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(1-x_1^{\alpha_1}-x_2^{\alpha_2}-\dots-x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n =$$
$$=\frac{1}{\alpha_1\alpha_2\dots\alpha_n}\,\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1-\mu+\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}\right)}$$
$$[p_1>0,\quad p_2>0,\dots,p_n>0,\quad \mu>1].$$

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4.636

1.[8] $$\underset{\substack{x_1\ge 0,\, x_2\ge 0,\dots,x_n\ge 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\dots+x_n^{\alpha_n}\ge 1}}{\int\int\cdots\int} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(x_1^{\alpha_1}+x_2^{\alpha_2}+\dots+x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n =$$
$$=\frac{1}{\alpha_1\alpha_2\dots\alpha_n\left(\mu-\frac{p_1}{\alpha_1}-\frac{p_2}{\alpha_2}-\dots-\frac{p_n}{\alpha_n}\right)}\,\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}\right)}$$

$$2.^8 \quad \underset{\substack{x_1 \geq 0,\, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}}{\int\int\cdots\int} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n})^{\mu}}\, dx_1\, dx_2 \dots dx_n =$$

$$= \frac{1}{\alpha_1 \alpha_2 \dots \alpha_n \left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} - \mu \right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}\right)}$$

$$\left[\mu < \frac{p_1}{\alpha_1} - \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}\right].$$

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$$3.^8 \quad \underset{\substack{x_1 \geq 0,\, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}}{\int\int\cdots\int} x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1} \sqrt{\frac{1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \dots - x_n^{\alpha_n}}{1 + x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n}}}\, dx_1\, dx_2 \dots dx_n =$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_1}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1 \alpha_2 \dots \alpha_n} \frac{1}{\Gamma(m)} \left\{ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \right\},$$

where $m = \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}$.

4.637

$$\underset{\substack{x_1 \leq 0, \quad x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}}{\int\int\cdots\int} f(x_1 + x_2 + \dots + x_n) \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1}\, dx_1\, dx_2 \dots dx_n}{(q_1 x_1 + q_2 x_2 + \dots + q_n x_n + r)^{p_1 + p_2 + \dots + p_n}} =$$

$$= \frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1 + p_2 + \dots + p_n)} \int_0^1 f(x) \frac{x^{p_1 + p_2 + \dots + p_n - 1}}{(q_1 x + r)^{p_1} (q_2 x + r)^{p_2} \dots (q_n x + r)^{p_n}} dx,$$

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where $f(x)$ is continuous on the interval $(0, 1)$

$$[q_1 \geq 0, \quad q_2 \geq 0, \dots, q_n \geq 0; \quad r > 0].$$

4.638

$$1. \quad \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1} e^{-(q_1 x_1 + q_2 x_2 + \dots + q_n x_n)}}{(r_0 + r_1 x_1 + r_2 x_2 + \dots + r_n x_n)^s}\, dx_1\, dx_2 \dots dx_n =$$

$$= \frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(s)} \int_0^\infty \frac{e^{-r_0 x} x^{s-1}\, dx}{(q_1 + r_1 x)^{p_1} (q_1 + r_2 x)^{p_2} \dots (q_n + r_n x)^{p_n}}$$

$p_i,\ q_i,\ r_i$ s $r_0 = 0$ $p_1 + p_2 + \cdots + p_n > s$

2. $$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \ldots x_n^{p_n-1}}{(r_0 + r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)^s} \, dx_1 \, dx_2 \ldots dx_n = \\ = \frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)\Gamma(s - p_1 - p_2 - \cdots - p_n)}{r_1^{p_1} r_2^{p_2} \ldots r_n^{p_n} r_0^{s-p_1-p_2-\cdots-p_n} \Gamma(s)} \qquad [p_i > 0, \quad r_i > 0, \quad s > 0].$$

3. $$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \ldots x_n^{p_n-1}}{[1 + (r_1 x_1)^{q_1} + (r_2 x_2)^{q_2} + \cdots + (r_n x_n)^{q_n}]^s} \, dx_1 \, dx_2 \ldots dx_n = \\ = \frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\ldots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1 q_2 \ldots q_n r_1^{p_1 q_1} r_2^{p_1 q_2} \ldots r_n^{p_n q_n}} \frac{\Gamma\left(s - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \cdots - \frac{p_n}{q_n}\right)}{\Gamma(s)} \\ [p_i > 0, \quad q_i > 0, \quad r_i > 0, \quad s > 0].$$

650

4.639

1. $$\underset{x_1^2+x_2^2+\cdots+x_n^2 \le 1}{\int\int\cdots\int} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m} \, dx_1 \, dx_2 \ldots dx_n = \\ = \frac{(2m-1)!!}{2^m} \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + m + 1\right)} (p_1^2 + p_2^2 + \cdots + p_n^2)^m.$$

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2. $$\underset{x_1^2+x_2^2+\cdots+x_n^2 \le 1}{\int\int\cdots\int} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m+1} \, dx_1 \, dx_2 \ldots dx_n = 0.$$

FI III 483

4.641

1. $$\underset{x_1^2+x_2^2+\cdots+x_n^2 \le 1}{\int\int\cdots\int} e^{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n} \, dx_1 \, dx_2 \ldots dx_n = \\ = \sqrt{\pi^n} \sum_{k=0}^\infty \frac{1}{k!\Gamma\left(\frac{n}{2} + k + 1\right)} \left(\frac{p_1^2 + p_2^2 + \cdots + p_n^2}{4}\right)^k.$$

FI III 483

2. $$\underset{x_1^2+x_2^2+\cdots+x_{2n}^2 \le 1}{\int\int\cdots\int} e^{p_1 x_1 + p_2 x_2 + \cdots + p_{2n} x_{2n}} \, dx_1 \, dx_2 \ldots dx_{2n} = \frac{(2\pi)^n I_n(\sqrt{p_1^2 + p_2^2 + \cdots + p_{2n}^2})}{(p_1^2 + p_2^2 + \cdots + p_{2n}^2)^{n/2}}.$$

4.642

$$\int\int\cdots\int_{x_1^2+x_2^2+\cdots+x_n^2\le R^2} f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})\,dx_1\,dx_2\ldots dx_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)}\int_0^R x^{n-1}f(x)\,dx,$$

where $f(x)$ is a function that is continuous on the interval $(0,R)$.

4.643

$$\begin{aligned}\int_0^1\int_0^1\ldots\int_0^1 f(x_1x_2\ldots x_n)(1-x_1)^{p_1-1}(1-x_2)^{p_2-1}\ldots(1-x_n)^{p_n-1}\times\\ x_2^{p_1}x_3^{p_1+p_2}\ldots x_n^{p_1+p_2+\cdots+p_{n-1}}\,dx_1\,dx_2\ldots dx_n=\\ =\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(x)(1-x)^{p_1+p_2+\cdots+p_n-1}\,dx\end{aligned}$$

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under the assumption that the integral on the right converges absolutely.

4.644

$$\begin{aligned}&\overbrace{\int\int\cdots\int}^{n-1}_{x_1^2+x_2^2+\cdots+x_n^2=1} f\left(p_1x_1+p_2x_2+\cdots+p_nx_n\right)\frac{dx_1\,dx_2\ldots dx_{n-1}}{|x_n|}=\\ &=2\int\int\cdots\int_{x_1^2+x_2^2+\cdots+x_{n-1}^2\le 1} f\left(p_1x_1+p_2x_2+\cdots+p_nx_n\right)\frac{dx_1\,dx_2\ldots dx_{n-1}}{\sqrt{1-x_1^2-x_2^2-\cdots-x_{n-1}^2}}=\\ &=\frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi f(\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\cos x)\sin^{n-2}x\,dx \qquad [n\ge 3],\end{aligned}$$

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where $f(x)$ is continuous on the interval $\left[-\sqrt{p_1^2+p_2^2+\cdots+p_n^2},\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\right]$.

4.645

Suppose that two functions $f(x_1,\,x_2,\ldots,x_n)$ and $g(x_1,\,x_2,\ldots,x_n)$ are continuous in a closed bounded region D and that the smallest and greatest values of the function g in D are m and M respectively. Let $\varphi(u)$ denote a function that is continuous for $m\le u\le M$

. We denote by $\psi(u)$ the integral

$$1. \quad \psi(u) = \underset{m \le g(x_1,\, x_2, \dots, x_n) \le u}{\int\int\cdots\int} f(x_1,\, x_2, \dots, x_n)\, dx_1\, dx_2 \dots dx_n,$$

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over that portion of the region D on which the inequality $m \le g(x_1,\, x_2, \dots, x_n) \le u$ is satisfied. Then

$$2. \quad \underset{m \le g(x_1,\, x_2, \dots, x_n) \le M}{\int\int\cdots\int} f(x_1,\, x_2, \dots, x_n)\varphi[g(x_1,\, x_2, \dots, x_n)]\, dx_1\, dx_2 \dots dx_n = \\ = (S)\int_m^M \varphi(u) d\psi(u) = (R)\int_m^M \varphi(u) \frac{d\psi(u)}{du}\, du,$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative $\frac{d\psi}{du}$ exists and is continuous, the Riemann integral on the right exists.

M may be $+\infty$ in formulas 4.645 2., in which case $\int_m^{+\infty}$ should be understood to mean $\lim_{M\to+\infty} \int_m^M$.

4.646[3]

$$\underset{\substack{x_1 \ge 0,\, x_2 \ge 0, \dots, x_n \ge 0 \\ x_1 + x_2 + \cdots + x_n \le 1}}{\int\int\cdots\int} \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1}}{(q_1x_1 + q_2x_2 + \cdots + q_nx_n)^r}\, dx_1\, dx_2 \dots dx_n = \\ = \frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1)\Gamma(r)} \int_0^\infty \frac{x^{r-1}\, dx}{(1+q_1x)^{p_1}(1+q_2x)^{p_2}\dots(1+q_nx)^{p_n}} \\ [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad q_2 > 0, \quad q_2 > 0, \dots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0].$$

FI III 493

4.647

$$\underset{0 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le 1}{\int\int\cdots\int} \exp\left\{\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}\right\} dx_1\, dx_2 \dots dx_n = \\ = \frac{2\sqrt{\pi^n}}{n(p_1^2 + p_2^2 + \cdots + p_n^2)^{\frac{n}{4} - \frac{1}{2}}} I_{\frac{n}{2}-1}(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}).$$

FI III 495

4.648

5. Indefinite Integrals of Special Functions

5.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

5.11 Complete elliptic integrals

5.111

1. $$\int \boldsymbol{K}(k) k^{2p+3}\, dk = \frac{1}{(2p+3)^2} \left\{ 4(p+1)^2 \int \boldsymbol{K}(k) k^{2p+1}\, dk + k^{2p+2} [\boldsymbol{E}(k) - (2p+3) \boldsymbol{K}(k) k'^2] \right\} .$$

BY (610.04)

2. $$\int \boldsymbol{E}(k) k^{2p+3}\, dk = = \frac{1}{4p^2+16p+15} \left\{ 4(p+1)^2 \int \boldsymbol{E}(k) k^{2p+1}\, dk - \right. \\ \left. - \boldsymbol{E}(k) k^{2p+2} [(2p+3)k'^2 - 2] - k^{2p+2} k'^2 \boldsymbol{K}(k) \right\}$$

BY (611.04)

5.112

1. $$\int \boldsymbol{K}(k)\, dk = \frac{\pi k}{2} \left[1 + \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(2j+1) 2^{4j} (j!)^4} \right] .$$

BY (610.00)

2.[6] $$\int \boldsymbol{E}(k)\, dk = \frac{\pi k}{2} \left[1 - \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(4j^2-1) 2^{4j} (j!)^4} \right] .$$

BY (611.00)

3. $$\int \boldsymbol{K}(k) k\, dk = \boldsymbol{E}(k) - k'^2 \boldsymbol{K}(k).$$

4. $\int \boldsymbol{E}(k)k\,dk = \frac{1}{3}[(1+k^2)\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)].$

BY (611.01)

5. $\int \boldsymbol{K}(k)k^3\,dk = \frac{1}{9}[(4+k^2)\boldsymbol{E}(k) - k'^2(4+3k^2)\boldsymbol{K}(k)].$

BY (610.02)

6. $\int \boldsymbol{E}(k)k^3\,dk = \frac{1}{45}[(4+k^2+9k^4)\boldsymbol{E}(k) - k'^2(4+3k^2)\boldsymbol{K}(k)].$

BY 611.02

7. $\int \boldsymbol{K}(k)k^5\,dk = \frac{1}{225}[(64+16k^2+9k^4)\boldsymbol{E}(k) - k'^2(64+48k^2+45k^4)\boldsymbol{K}(k)].$

BY (610.03)

8. $\int \boldsymbol{E}(k)k^5\,dk = \frac{1}{1575}[(64+16k^2+9k^4+225k^6)\boldsymbol{E}(k) - k'^2(64+48k^2+45k^4)\boldsymbol{K}(k)].$

BY (611.03)

9. $\int \frac{\boldsymbol{K}(k)}{k^2}\,dk = -\frac{\boldsymbol{E}(k)}{k}$

BY (612.05)

10. $\int \frac{\boldsymbol{E}(k)}{k^2}\,dk = \frac{1}{k}[k'^2\boldsymbol{K}(k) - 2\boldsymbol{E}(k)].$

BY (612.02)

11. $\int \frac{\boldsymbol{E}(k)}{k'^2}\,dk = k\boldsymbol{K}(k).$

BY (612.01)

BY (612.03)

13. $$\int \frac{k\boldsymbol{E}(k)}{k'^2}\,dk = \boldsymbol{K}(k) - \boldsymbol{E}(k).$$

BY (612.04)

5.113

1. $$\int [\boldsymbol{K}(k) - \boldsymbol{E}(k)]\frac{dk}{k} = -\boldsymbol{E}(k).$$

BY (612.06)

2. $$\int [\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)]\frac{dk}{k} = 2\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k).$$

BY (612.09)

3. $$\int [(1+k^2)\boldsymbol{K}(k) - \boldsymbol{E}(k)]\frac{dk}{k} = -k'^2\boldsymbol{K}(k).$$

BY (612.12)

4. $$\int [\boldsymbol{K}(k) - \boldsymbol{E}(k)]\frac{dk}{k^2} = \frac{1}{k}[\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)].$$

BY (612.07)

5. $$\int [\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)]\frac{dk}{k^2k'^2} = \frac{1}{k}[\boldsymbol{K}(k) - \boldsymbol{E}(k)].$$

6. $$\int [(1+k^2)\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)]\frac{dk}{kk'^4} = \frac{\boldsymbol{E}(k)}{k'^2}.$$

BY (612.13)

5.114

$$\int \frac{k\boldsymbol{K}(k)\,dk}{[\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)]^2} = \frac{1}{k'^2\boldsymbol{K}(k) - \boldsymbol{E}(k)}.$$

BY (612.11)

5.115

1. $$\int \Pi\left(\frac{\pi}{2},\, r^2,\, k\right) k\, dk = (k^2 - r^2)\Pi\left(\frac{\pi}{2},\, r^2,\, k\right) - \boldsymbol{K}(k) + \boldsymbol{E}(k).$$

BY (612.14)

2. $$\int \left[\boldsymbol{K}(k) - \Pi\left(\frac{\pi}{2},\, r^2,\, k\right)\right] k\, dk = k^2 \boldsymbol{K}(k) - (k^2 - r^2)\Pi\left(\frac{\pi}{2},\, r^2,\, k\right).$$

BY (612.15)

3. $$\int \left[\frac{\boldsymbol{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2},\, r^2,\, k\right)\right] k\, dk = (k^2 - r^2)\Pi\left(\frac{\pi}{2},\, r^2,\, k\right).$$

BY (612.16)

5.12 Elliptic integrals

5.121

$$\int_0^x \frac{F(x,\, k)\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{[F(x,\, k)]^2}{2} \qquad \left[0 < x \le \frac{\pi}{2}\right].$$

BY (630.01)

5.122

$$\int_0^x E(x,\, k)\sqrt{1 - k^2 \sin^2 x}\, dx = \frac{[\boldsymbol{E}(x,\, k)]^2}{2}.$$

BY (630.32)

5.123

1. $$\int_0^x F(x,\, k) \sin x\, dx = -\cos x F(x,\, k) + \frac{1}{k} \arcsin(k \sin x).$$

BY (630.11)

2. $$\int_0^x F(x,\, k) \cos x\, dx = \sin x F(x,\, k) + \frac{1}{k} \operatorname{Arch} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - \frac{1}{k} \operatorname{Arch}\left(\frac{1}{k'}\right).$$

5.124

1. $$\int_0^x E(x,\, k) \sin x \, dx = -\cos x E(x,\, k) + \frac{1}{2k}\left[k \sin x \sqrt{1 - k^2 \sin^2 x} + \arcsin(k \sin x)\right].$$

BY (630.12)

2. $$\int_0^x E(x,\, k) \cos x \, dx = \sin x E(x,\, k) + \frac{1}{2k}\left[k \cos x \sqrt{1 - k^2 \sin^2 x} - \right.$$
$$\left. - k'^2 \operatorname{Arch} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - k + k'^2 \operatorname{Arch}\left(\frac{1}{k'}\right)\right].$$

BY (630.22)

5.125

1. $$\int_0^x \Pi(x,\, \alpha^2,\, k) \sin x \, dx = -\cos x \Pi(x,\, \alpha^2,\, k) +$$
$$+ \frac{1}{\sqrt{k^2 - \alpha^2}} \operatorname{arctg}\left[\sqrt{\frac{k^2 - \alpha^2}{1 - k^2 \sin^2 x}} \sin x\right] \qquad [\alpha^2 < k^2];$$
$$= -\cos x \Pi(x,\, \alpha^2,\, k) +$$
$$+ \frac{1}{\sqrt{\alpha^2 - k^2}} \operatorname{Arth}\left[\sqrt{\frac{\alpha^2 - k^2}{1 - k^2 \sin^2 x}} \sin x\right] \qquad [\alpha^2 > k^2].$$

BY (630.13)

2. $$\int_0^x \Pi(x,\, \alpha^2,\, k) \cos x \, dx = \sin x \Pi(x,\, \alpha^2,\, k) - f + f_0,$$

where

$$f = \frac{1}{2\sqrt{(1 - \alpha^2)(\alpha^2 - k^2)}} \operatorname{arctg}\left[\frac{2(1 - \alpha^2)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(2k^2 - \alpha^2 - \alpha^2 k^2)}{2\alpha^2 \sqrt{(1 - \alpha^2)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}}\right]$$
$$\text{for } (1 - \alpha^2)(\alpha^2 - k^2) > 0;$$
$$= \frac{1}{2\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}} \ln\left[\frac{2(\alpha^2 - 1)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(\alpha^2 + \alpha^2 k^2 - 2k^2)}{1 - \alpha^2 \sin^2 x} + \right.$$
$$\left. + \frac{2\alpha^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}}{1 - \alpha^2 \sin^2 x}\right] \text{ for } (1 - \alpha^2)(\alpha^2 - k^2) < 0,$$

f_0 is the value of f at $x = 0$.

BY (630.23)

Integration with respect to the modulus

5.126

$$\int F(x,\, k) k\, dk = E(x,\, k) - k'^2 F(x,\, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x.$$

BY (613.01)

5.127

$$\int E(x,\, k) k\, dk = \frac{1}{3}\Big[(1+k^2)E(x,\, k) - k'^2 F(x,\, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x\Big].$$

BY (613.02)

5.128

$$\int \Pi(x,\, r^2,\, k) k\, dk = (k^2 - r^2)\Pi(x,\, r^2,\, k) - F(x,\, k) + E(x,\, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x.$$

BY (613.03)

5.13 Jacobian elliptic functions

5.131

1. $$\int \operatorname{sn}^m u\, du = \frac{1}{m+1}\left[\operatorname{sn}^{m+1} u \operatorname{cn} u \operatorname{dn} u + (m+2)(1+k^2)\int \operatorname{sn}^{m+2} u\, du - {} - (m+3)k^2 \int \operatorname{sn}^{m+4} u\, du\right].$$

SI 259, PE(567)

2. $$\int \operatorname{cn}^m u\, du = \frac{1}{(m+1)k'^2}\Big[-\operatorname{cn}^{m+1} u \operatorname{sn} u \operatorname{dn} u + {} + (m+2)(1-2k^2)\int \operatorname{cn}^{m+2} u\, du + (m+3)k^2 \int \operatorname{cn}^{m+4} u\, du\Big].$$

PE (568)

$$3. \quad \int \operatorname{dn}^m u\,du = \frac{1}{(m+1)k'^2}\Big[k^2 \operatorname{dn}^{m+1} u \operatorname{sn} u \operatorname{cn} u + + (m+2)(2-k^2)\int \operatorname{dn}^{m+2} u\,du - (m+3)\int \operatorname{dn}^{m+4} u\,du\Big] .$$

PE (569)

By using formulas 5.131, we can reduce the integrals $\int \operatorname{sn}^m udu, \int \operatorname{cn}^m udu, \int \operatorname{dn}^m udu$ to the integrals 5.132, 5.133 and 5.134.

5.132

$$1. \quad \int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}; = \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u}.$$

SI 266(4)
ZH 87(164)

$$2. \quad \int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u}.$$

SI 266(5)

$$3. \quad \int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \operatorname{arctg} \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u}; = \frac{1}{k'} \arccos \frac{\operatorname{cn} u}{\operatorname{dn} u}; = \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}; = \frac{1}{k'} \arcsin \frac{k' \operatorname{sn} u}{\operatorname{dn} u}.$$

JA
SI 266(6)
JA
ZH 88(166)

5.133

$$1. \quad \int \operatorname{sn} u\,du = \frac{1}{k} \ln(\operatorname{dn} u - k \operatorname{cn} u); = \frac{1}{k} \operatorname{Arch} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1-k^2}; = \frac{1}{k} \operatorname{Arsh}\left(k\frac{\operatorname{dn} u - \operatorname{cn} u}{1-k^2}\right);$$

1

JA
ZH 87(161)

$$2. \quad \int \operatorname{cn} u\, du = \frac{1}{k} \arccos(\operatorname{dn} u);$$
$$= \frac{i}{k} \ln(\operatorname{dn} u - ik \ \operatorname{sn} u);$$
$$= \frac{1}{k} \arcsin(k \ \operatorname{sn} u).$$

JA
SI 265(2)A, ZH 87(162)
ZH 87(162)

$$3. \quad \int \operatorname{dn} u\, du = \arcsin(\operatorname{sn} u);$$
$$= \operatorname{am} u = i \ln(\operatorname{cn} u - i \ \operatorname{sn} u).$$

SI 266(3), ZH 87(163)

5.134

$$1. \quad \int \operatorname{sn}^2 \ u\, du = \frac{1}{k^2}[u - E(\operatorname{am} u,\ k)].$$

PE (564)}

$$2. \quad \int \operatorname{cn}^2 \ u\, du = \frac{1}{k^2}[E(\operatorname{am} u,\ k) - k'^2 u].$$

PE (565)

$$3. \quad \int \operatorname{dn}^2 \ u\, du = E(\operatorname{am} u,\ k).$$

PE (566)

657

5.135

$$1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn} u}\, du = \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u};$$
$$= \frac{1}{2k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}.$$

2. $$\int \frac{\operatorname{sn} u}{\operatorname{dn} u}\, du = \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u};$$
$$= \frac{1}{kk'} \operatorname{arcctg} \frac{k \operatorname{cn} u}{k'}.$$

ZH 88(169)
SI 266(8)

3. $$\int \frac{\operatorname{cn} u}{\operatorname{sn} u}\, du = \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u};$$
$$= \frac{1}{2} \ln \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}.$$

ZH 88(168)
SI 266(10)

4. $$\int \frac{\operatorname{cn} u}{\operatorname{dn} u}\, du = -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u};$$
$$= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u}.$$

ZH 88(171)
SI 266(9)

5. $$\int \frac{\operatorname{dn} u}{\operatorname{cn} u}\, du = \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u};$$
$$= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}.$$

JA
ZH 88(172)

6. $$\int \frac{\operatorname{dn} u}{\operatorname{sn} u}\, du = \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}.$$

ZH 87(170)

5.136

1. $$\int \operatorname{sn} u \operatorname{cn} u \, du = -\frac{1}{k^2} \operatorname{dn} u.$$

2. $$\int \operatorname{sn} u \operatorname{dn} u \, du = -\operatorname{cn} u.$$

5.137

1. $\displaystyle\int \frac{\operatorname{sn} u}{\operatorname{cn}^2\, u}\, du = \frac{1}{k'^2}\,\frac{\operatorname{dn} u}{\operatorname{cn} u}.$

ZH 88(173)

2. $\displaystyle\int \frac{\operatorname{sn} u}{\operatorname{dn}^2\, u}\, du = -\frac{1}{k'^2}\,\frac{\operatorname{cn} u}{\operatorname{dn} u}.$

ZH 88(175)

3. $\displaystyle\int \frac{\operatorname{cn} u}{\operatorname{sn}^2\, u}\, du = -\frac{\operatorname{dn} u}{\operatorname{sn} u}.$

ZH 88(174)

4. $\displaystyle\int \frac{\operatorname{cn} u}{\operatorname{dn}^2\, u}\, du = \frac{\operatorname{sn} u}{\operatorname{dn} u}.$

ZH 88(177)

5. $\displaystyle\int \frac{\operatorname{dn} u}{\operatorname{sn}^2\, u}\, du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}.$

ZH 88(176)

6. $\displaystyle\int \frac{\operatorname{dn} u}{\operatorname{cn}^2\, u}\, du = \frac{\operatorname{sn} u}{\operatorname{cn} u}.$

ZH 88(178)

5.138

1. $\displaystyle\int \frac{\operatorname{cn} u}{\operatorname{sn} u\ \operatorname{dn} u}\, du = \ln\frac{\operatorname{sn} u}{\operatorname{dn} u}.$

ZH 88(183)

2. $\displaystyle\int \frac{\operatorname{sn} u}{\operatorname{cn} u\ \operatorname{dn} u}\, du = \frac{1}{k'^2}\ln\frac{\operatorname{dn} u}{\operatorname{cn} u}.$

3. $\displaystyle\int \frac{\operatorname{dn} u}{\operatorname{sn} u \ \operatorname{cn} u}\, du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u}.$

ZH 88(184)

5.139

1. $\displaystyle\int \frac{\operatorname{cn} u \ \operatorname{dn} u}{\operatorname{sn} u}\, du = \ln snu.$

ZH 88(179)

2. $\displaystyle\int \frac{\operatorname{sn} u \ \operatorname{dn} u}{\operatorname{cn} u}\, du = \ln \frac{1}{\operatorname{cn} u}.$

ZH 88(180)

3. $\displaystyle\int \frac{\operatorname{sn} u \ \operatorname{cn} u}{\operatorname{dn} u}\, du = -\frac{1}{k^2} \ln \operatorname{dn} \ u.$

ZH 88(181)

5.14 Weierstrass elliptic functions

5.141

1. $\displaystyle\int \wp(u)\, du = -\zeta(u).$

2. $\displaystyle\int \wp^2(u)\, du = \frac{1}{6}\, \wp'(u) + \frac{1}{12}\, g_2 u.$

ZH 120(192)

3. $\displaystyle\int \wp^3(u)\, du = \frac{1}{120}\, \wp'''(u) - \frac{3}{20}\, g_2 \zeta(u) + \frac{1}{10}\, g_3 u.$

ZH 120(193)

4. $\displaystyle\int \frac{du}{\mathcal{P}(u) - \mathcal{P}(v)} = \frac{1}{\mathcal{P}'(v)} \left[2u\zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right].$

5. $$\int \frac{\alpha\mathcal{P}(u)+\beta}{\gamma\mathcal{P}(u)+\delta}\,du = \frac{au}{\gamma} - \frac{\alpha\delta-\beta\gamma}{\gamma^2\mathcal{P}'(v)}\left[\ln\frac{\sigma(u+v)}{\sigma(u-v)} - 2u\zeta(v)\right], \text{where } \mathcal{P}'(v) = -\frac{\delta}{\gamma}.$$

ZH 120(195)

5.2 The Exponential-Integral Function

5.21 The exponential-integral function

5.211

$$\int_x^\infty \operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x)\,dx = \left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\operatorname{Ei}[-(\beta+\gamma)x] - \\ - x\operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta}\operatorname{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma}\operatorname{Ei}(-\beta x) \\ [\operatorname{Re}(\beta+\gamma)>0].$$

NT 53(2)

5.22 Combinations of the exponential-integral function and powers

5.221

1. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^{n+1}}\,dx = \left[\frac{1}{x^n} - \frac{(-1)^n}{b^n}\right]\frac{\operatorname{Ei}[-a(x+b)]}{n} + \\ + \frac{e^{-ab}}{n}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{b^{n-k}}\int_x^\infty \frac{e^{-ax}}{x^{k+1}}\,dx [a>0, \quad b>0].$$

NT 52(3)

2. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^2}\,dx = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{Ei}[-a(x+b)] - \frac{e^{-ab}\operatorname{Ei}(-ax)}{b} \qquad [a>0,\ b>0].$$

NT 52(4)

5.23 Combinations of the exponential-integral and the exponential

5.231

1. $$\int_0^x e^x\operatorname{Ei}(-x)\,dx = -\ln x - \boldsymbol{C} + e^x\operatorname{Ei}(-x).$$

2. $\int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x)\,dx = -\frac{1}{\beta}\left\{e^{-\beta x} \operatorname{Ei}(-\alpha x) + \ln\left(1+\frac{\beta}{\alpha}\right) - \operatorname{Ei}\left[-(\alpha+\beta)x\right]\right\}.$

ET II 308(12)

5.3 The Sine-Integral and the Cosine-Integral

5.31

1. $\int \cos\alpha x\ \operatorname{ci}(\beta x)\,dx = \frac{\sin\alpha x\ \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)}{2\alpha}.$

NT 49(1)

2. $\int \sin\alpha x\ \operatorname{ci}(\beta x)\,dx = -\frac{\cos\alpha x\ \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)+\operatorname{ci}(\alpha x-\beta x)}{2\alpha}.$

NT 49(2)

5.32

1. $\int \cos\alpha x\ \operatorname{si}(\beta x)\,dx = \frac{\sin\alpha x\ \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)-\operatorname{ci}(\alpha x-\beta x)}{2\alpha}.$

NT 49(3)

2. $\int \sin\alpha x\ \operatorname{si}(\beta x)\,dx = -\frac{\cos\alpha x\ \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x+\beta x)-\operatorname{si}(\alpha x-\beta x)}{2\alpha}.$

NT 49(4)

5.33

1. $$\int \operatorname{ci}(\alpha x)\operatorname{ci}(\beta x)\,dx = x\ \operatorname{ci}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{2\alpha}(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)) + \\ + \frac{1}{2\beta}(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x-\alpha x)) - \frac{1}{\alpha}\sin\alpha x\ \operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x\ \operatorname{ci}(\alpha x).$$

2. $\int \operatorname{si}(\alpha x) \operatorname{si}(\beta x)\, dx = x \operatorname{si}(\alpha x) \operatorname{si}(\beta x) - \frac{1}{2\beta}(\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)) -$

$- \frac{1}{2\alpha}(\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x + \alpha x)) + \frac{1}{\alpha} \cos \alpha x \operatorname{si}(\beta x) + \frac{1}{\beta} \cos \beta x \operatorname{si}(\alpha x).$

NT 54(6)

3. $\int \operatorname{si}(\alpha x) \operatorname{ci}(\beta x)\, dx = x \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{\alpha} \cos \alpha x \operatorname{ci}(\beta x) -$

$- \frac{1}{\beta} \sin \beta x \operatorname{si}(\alpha x) - \left(\frac{1}{2\alpha} + \frac{1}{2\beta}\right) \operatorname{ci}(\alpha x + \beta x) - \left(\frac{1}{2\alpha} - \frac{1}{2\beta}\right) \operatorname{ci}(\alpha x - \beta x).$

NT 54(10)

5.34

1. $\int_x^\infty \operatorname{si}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b}\right) \operatorname{si}[a(x+b)] - \frac{\cos ab \operatorname{si}(ax) + \sin ab \operatorname{ci}(ax)}{b} [a > 0, \quad b > 0].$

NT 52(6)

2. $\int_x^\infty \operatorname{ci}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b}\right) \operatorname{ci}[a(x+b)] + \frac{\sin ab \operatorname{si}(ax) - \cos ab \operatorname{ci}(ax)}{b} [a > 0, \quad b > 0].$

NT 52(5)

5.4 The Probability Integral and Fresnel Integrals

5.41

$$\int \Phi(\alpha x)\, dx = x\Phi(ax) + \frac{e^{-\alpha^2 x^2}}{\alpha\sqrt{\pi}}.$$

NT 12(20)a

5.42

$$\int S(\alpha x)\, dx = xS(\alpha x) + \frac{\cos \alpha^2 x^2}{\alpha\sqrt{2\pi}}.$$

5.43

$$\int C(\alpha x)\,dx = xC(\alpha x) - \frac{\sin\alpha^2 x^2}{\alpha\sqrt{2\pi}}.$$

NT 12(21)a

5.5 Bessel Functions

5.51

$$\int J_p(x)\,dx = 2\sum_{k=0}^{\infty} J_{p+2k+1}(x).$$

JA, MO 30

5.52

1. $$\int x^{p+1} Z_p(x)\,dx = x^{p+1} Z_{p+1}(x)^*$$

WA 146(1)

2. $$\int x^{-p+1} Z_p(x)\,dx = -x^{-p+1} Z_{p-1}(x)^*.$$

WA 146(2)

5.53

$$\int \left[(\alpha^2 - \beta^2)x - \frac{p^2 - q^2}{x}\right] Z_p(\alpha x)\mathsf{Z}_q(\beta x)\,dx = \beta x Z_p(\alpha x)\mathsf{Z}_{q-1}(\beta x) - \\ - \alpha x Z_{p-1}(\alpha x)\mathsf{Z}_q(\beta x) + (p-q)Z_p(\alpha x)\mathsf{Z}_q(\beta x)^*$$

JA, MO 30, WA 148(7)a

5.54

1.[6] $$\int xZ_p(\alpha x)\mathsf{Z}_p(\beta x)\,dx = \frac{\beta x Z_p(\alpha x)\mathsf{Z}_{p-1}(\beta x) + \alpha x Z_{p-1}(\alpha x)\mathsf{Z}_p(\beta x)^*}{\alpha^2 - \beta^2}.$$

WA 148(8)a

2. $$\int x[Z_p(\alpha x)]^2\,dx = \frac{x^2}{2}\{[Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x)Z_{p+1}(\alpha x)\}^*.$$

WA 149(11)

5.55

$$\int \frac{1}{x}\,Z_p(\alpha x)\mathsf{Z}_q(\alpha x)\,dx = \alpha x\,\frac{Z_{p-1}(\alpha x)\mathsf{Z}_q(\alpha x) - Z_p(\alpha x)\mathsf{Z}_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x)\mathsf{Z}_q(\alpha x)^*}{p+q}$$

* In formulas 5.52- 5.56, $Z_p(x)$ and Z are arbitrary Bessel functions.

WA 149(13)

5.56

1. $$\int Z_1(x)\,dx = -Z_0(x)^*.$$

JA

2. $$\int xZ_0(x)\,dx = xZ_1(x)^*.$$

JA

6.-7. Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

6.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

6.11 Forms containing F(x, k)

6.111

$$\int_0^{\frac{\pi}{2}} F(x,\ k) \operatorname{ctg} x \, dx = \frac{\pi}{4} \boldsymbol{K}(k') + \frac{1}{2} \ln k \boldsymbol{K}(k).$$

BI ((350))(1)

6.112

1. $$\int_0^{\frac{\pi}{2}} F(x,\ k) \frac{\sin x \cos x}{1 + k \sin^2 x} \, dx = \frac{1}{4k} \boldsymbol{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \boldsymbol{K}(k').$$

BI ((350))(6)

2. $$\int_0^{\frac{\pi}{2}} F(x,\ k) \frac{\sin x \cos x}{1 - k \sin^2 x} \, dx = \frac{1}{4k} \boldsymbol{K}(k) \ln \frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k} \boldsymbol{K}(k').$$

BI ((350))(7)

3. $$\int_0^{\frac{\pi}{2}} F(x,\ k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} \, dx = -\frac{1}{2k^2} \ln k' \boldsymbol{K}(k).$$

BI ((350))(2)A, BY(802.12)a

6.113

1. $$\int_0^{\frac{\pi}{2}} F(x,\ k') \frac{\sin x \cos x \, dx}{\cos^2 x + k \sin^2 x} = \frac{1}{4(1-k)} \ln \frac{2}{(1+k)\sqrt{k}} \boldsymbol{K}(k').$$

BI ((350))(5)

2. $$\int_0^{\frac{\pi}{2}} F(x,\ k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{k^2 \sin t \cos t} \left[\boldsymbol{K}(k) \operatorname{arctg}(k' \ \operatorname{tg} t) - \frac{\pi}{2} F(t,\ k) \right].$$

BI ((350))(12)

6.114

$$\int_u^v F(x,\, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2\cos u \sin v} \boldsymbol{K}(k) \boldsymbol{K}(\sqrt{1 - tg^2 u \operatorname{ctg}^2\ v})$$

$$[k^2 = 1 - \operatorname{ctg}^2\ u \cdot \operatorname{ctg}^2\ v].$$

BI ((351))(9)

663

6.115

$$\int_0^1 F(\arcsin x,\, k) \frac{x\, dx}{1 + kx^2} = \frac{1}{4k} \boldsymbol{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \boldsymbol{K}(k') \qquad (\text{cf. } \mathbf{6.112}\ 2.).$$

6.112

BI ((466))(1)

This and similar formulas can be obtained from formulas 6.111 6.113 by means of the substitution $x = \arcsin t$.

6.12 Forms containing E(x, k)

6.121

$$\int_0^{\frac{\pi}{2}} E(x,\, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x}\, dx = \frac{1}{2k^2} \{(1 + k'^2) \boldsymbol{K}(k) - (2 + \ln k') \boldsymbol{E}(k)\}.$$

BI ((350))(4)

6.122

$$\int_0^{\frac{\pi}{2}} E(x,\, k) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \{\boldsymbol{E}(k) \boldsymbol{K}(k) - \ln k'\}.$$

BI ((350))(10), BY (630.02)

6.123

$$\int_0^{\frac{\pi}{2}} E(x,\, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= -\frac{1}{k^2 \sin t \cos t} \left[\boldsymbol{E}(k) \operatorname{arctg}(k'\ \operatorname{tg} t) - \frac{\pi}{2} E(t,\, k) + \frac{\pi}{2} \operatorname{ctg} t (1 - \sqrt{1 - k^2 \sin^2 t}) \right] .$$

BI ((350))(13)

6.124

$$\int_u^n E(x,\,k)\frac{dx}{\sqrt{(\sin^2 x-\sin^2 u)(\sin^2 v-\sin^2 x)}}=\frac{1}{2\cos u\sin v}\boldsymbol{E}(k)\boldsymbol{K}\left(\sqrt{1-\frac{tg^2\,u}{tg^2\,v}}\right)+$$
$$+\frac{k^2\sin v}{2\cos u}\boldsymbol{K}\left(\sqrt{1-\frac{\sin^2 2u}{\sin^2 2v}}\right)$$
$$[k^2=1-\operatorname{ctg}^2\,u\operatorname{ctg}^2\,v].$$

BI ((351))(10)

6.13 Integration of elliptic integrals with respect to the modulus

6.131

$$\int_0^1 F(x,\,k)k\,dk=\frac{1-\cos x}{\sin x}=\operatorname{tg}\frac{x}{2}.$$

BY (616.03)

6.132

$$\int_0^1 E(x,\,k)k\,dk=\frac{\sin^2 x+1-\cos x}{3\sin x}.$$

BY (616.04)

6.133

$$\int_0^1 \Pi(x,\,r^2,\,k)k\,dk=\operatorname{tg}\frac{x}{2}-r\ln\sqrt{\frac{1+r\sin x}{1-r\sin x}}-r^2\prod(x,\,r^2,\,0).$$

BY (616.05)

6.14- 6.15 Complete elliptic integrals

6.141

1. $\displaystyle\int_0^1 \boldsymbol{K}(k)\,dk=2\boldsymbol{G}.$

FI II 755

2. $\displaystyle\int_0^1 \boldsymbol{K}(k')\,dk=\frac{\pi^2}{4}.$

BY (615.03)

6.142

$$\int_0^1 \left(\boldsymbol{K}(k) - \frac{\pi}{2}\right) \frac{dk}{k} = \pi \ln 2 - 2\boldsymbol{G}.$$

BY (615.05)

6.143

$$\int_0^1 \boldsymbol{K}(k) \frac{dk}{k'} = \boldsymbol{K}^2 \left(\frac{\sqrt{2}}{2}\right).$$

BY (615.08)

6.144

$$\int_0^1 \boldsymbol{K}(k) \frac{dk}{1+k} = \frac{\pi^2}{8}.$$

BY (615.09)

6.145

$$\int_0^1 \left(\boldsymbol{K}(k') - \ln \frac{4}{k}\right) \frac{dk}{k} = \frac{1}{12}[24(\ln 2)^2 - \pi^2].$$

BY (615.13)

6.146

$$n^2 \int_0^1 k^n \boldsymbol{K}(k)\, dk = (n-1)^2 \int_0^1 k^{n-2} \boldsymbol{K}(k)\, dk + 1.$$

BY (615.12)

6.147

$$n \int_0^1 k^n \boldsymbol{K}(k')\, dk = (n-1) \int_0^1 k^{n-2} \boldsymbol{E}(k)\, dk \qquad [n > 1] \qquad (\text{see } \mathbf{6.152}).$$

6.152

BY (615.11)

6.148

1. $$\int_0^1 \boldsymbol{E}(k)\,dk = \frac{1}{2} + \boldsymbol{G}.$$

BY (615.02)

2. $$\int_0^1 \boldsymbol{E}(k')\,dk = \frac{\pi^2}{8}.$$

BY (615.04)

6.149

1. $$\int_0^1 \left(\boldsymbol{E}(k) - \frac{\pi}{2}\right)\frac{dk}{k} = \pi\ln 2 - 2\boldsymbol{G} + 1 - \frac{\pi}{2}.$$

BY (615.06)

2. $$\int_0^1 (\boldsymbol{E}(k') - 1)\frac{dk}{k} = 2\ln 2 - 1.$$

BY (615.07)

6.151

$$\int_0^1 \boldsymbol{E}(k)\frac{dk}{k'} = \frac{1}{8}\left[4\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi^2}{\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right)}\right].$$

BY (615.10)

6.152

$$(n+2)\int_0^1 k^n\boldsymbol{E}(k')\,dk = (n+1)\int_0^1 k^n\boldsymbol{K}(k')\,dk \qquad [n>1] \qquad (\text{see } \mathbf{6.147}).$$

6.147
BY (615.14)

6.153[6]

LO I 252

6.154

$$\int_0^{\frac{\pi}{2}} \frac{\boldsymbol{E}(p\sin x)}{1-p^2\sin^2 x}\sin x\,dx = \frac{\pi}{2\sqrt{1-p^2}} \qquad [p^2 > 1].$$

FI II 489

6.16 The theta function

665

6.161

1. $$\int_0^\infty x^{s-1}\theta_2(0|ix^2)\,dx = 2^s(1-2^{-s})\pi^{-\frac{s}{2}}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \qquad [\operatorname{Re} s > 2].$$

ET I 339(20)

2. $$\int_0^\infty x^{s-1}[\theta_3(0|ix^2)-1]\,dx = \pi^{-\frac{s}{2}}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \qquad [\operatorname{Re} s > 2].$$

ET I 339(21)

3. $$\int_0^\infty x^{s-1}[1-\theta_4(0|ix^2)]\,dx = (1-2^{1-s})\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \qquad [\operatorname{Re} s > 2].$$

ET I 339(22)

4. $$\int_0^\infty x^{s-1}[\theta_4(0|ix^2)+\theta_2(0|ix^2)-\theta_3(0|ix^2)]\,dx = -(2^s-1)(2^{1-s}-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s).$$

ET I 339(24)

6.162

1. $$\int_0^\infty e^{-ax}\theta_4\left(\frac{b\pi}{2l}\middle|\frac{i\pi x}{l^2}\right)dx = \frac{l}{\sqrt{a}}\operatorname{ch}(b\sqrt{a})\operatorname{cosec}(l\sqrt{a}) \qquad [\operatorname{Re} a > 0, \quad |b| \le l].$$

ET I 224(1)a

2. $$\int_0^\infty e^{-ax}\theta_1\left(\frac{b\pi}{2l}\middle|\frac{i\pi x}{l^2}\right)dx = -\frac{l}{\sqrt{a}}\operatorname{sh}(b\sqrt{a})\operatorname{sech}(l\sqrt{a}) \qquad [\operatorname{Re} a > 0, \quad |b| \le l].$$

3. $$\int_0^\infty e^{-ax}\theta_2\left(\frac{(1+b)\pi}{2l}\middle|\frac{i\pi x}{l^2}\right)dx = -\frac{l}{\sqrt{a}}\,\mathrm{sh}(b\sqrt{a})\,\mathrm{sech}(l\sqrt{a}) \qquad [\mathrm{Re}\,a>0, \quad |b|\le l].$$

ET I 224(3)a

4. $$\int_0^\infty e^{-ax}\theta_3\left(\frac{(1+b)\pi}{2l}\middle|\frac{i\pi x}{l^2}\right)dx = \frac{l}{\sqrt{a}}\,\mathrm{ch}(b\sqrt{a})\,\mathrm{cosech}(l\sqrt{a}) \qquad [\mathrm{Re}\,a>0, \quad |b|\le l].$$

ET I 224(4)a

6.163*

1. $$\int_0^\infty e^{-(a-\mu)x}\theta_3(\pi\sqrt{\mu}x|\,i\pi x)\,dx = \frac{1}{2\sqrt{a}}[\coth(\sqrt{a}+\sqrt{\mu})+\coth(\sqrt{a}-\sqrt{\mu})] \qquad [\mathrm{Re}\,a>0].$$

ET I 224(7)a

2.* $$\int_0^\infty \theta_3(i\pi kx|i\pi x)e^{-(k^2+l^2)x}\,dx = \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}.$$

6.164

$$\int_0^\infty [\theta_4(0|ie^{2x}) + \theta_2(0|ie^{2x}) - \theta_3(0|ie^{2x})]e^{\frac{1}{2}x}\cos(ax)\,dx =$$
$$= \frac{1}{2}(2^{\frac{1}{2}+ia}-1)(1-2^{\frac{1}{2}-ia})\pi^{-\frac{1}{4}-\frac{1}{2}ia}\Gamma\left(\frac{1}{4}+\frac{1}{2}ia\right)\zeta\left(\frac{1}{2}+ia\right) \qquad [a>0].$$

ET I 61(11)

6.165

$$\int_0^\infty e^{\frac{1}{2}x}[\theta_3(0|ie^{2x})-1]\cos(ax)\,dx =$$
$$= 2(1+4a^2)^{-1}\left\{1+\left[\left(a^2+\frac{1}{4}\right)\pi^{-\frac{1}{2}ia-\frac{1}{4}}\Gamma\left(\frac{1}{2}ia+\frac{1}{4}\right)\zeta\left(ia+\frac{1}{2}\right)\right]\right\} \qquad [a>0].$$

ET I 61(12)

6.17 Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi [1-k^2\cos\phi]^{-(j+\frac{1}{2})}\,d\phi,$$
$$\alpha_m(j) = \frac{\pi}{(64)^m}\,\frac{j!}{(2j)!}\,\frac{(4m+2j)!}{(2m+j)!}\left(\frac{1}{m!}\right)^2, \qquad \lambda = (\pi/2)\sqrt{(2j+1)k^2/(1-k^2)},$$

then

$$\begin{aligned}\Omega_j(k) &= \sum_{m=0}^{\infty}\alpha_m(j)k^{4m}\\ &= \sqrt{\pi/(2j+1)k^2}(1-k^2)^{-j}\Bigg[\operatorname{erf}\lambda + (1/2)(2j+1)^{-1}\left(1+\frac{1}{2k^2}\right)\times\\ &\quad\times\left\{\operatorname{erf}\lambda - (2/\sqrt{\pi})(\lambda e^{-\lambda^2})\left(1+\frac{2}{3}\lambda^2\right)\right\} - (1/12)(2j+1)^{-2}\left(16+\frac{13}{k^2}+\frac{1}{k^4}\right)\times\\ &\quad\times\left\{\operatorname{erf}\lambda - (2/\sqrt{\pi})(\lambda e^{-\lambda^2})\left(1+\frac{2}{3}\lambda^2+\frac{4}{15}\lambda^4\right)\right\} + \ldots\Bigg],\end{aligned}$$

while for large λ

$$\lim_{j\to\infty}\Omega_j(k) = \sqrt{\pi/(2j+1)k^2}(1-k^2)^{-j}\times$$
$$\times\left[1+(1/2)(2j+1)^{-1}\left\{1+\frac{1}{2k^2}\right\} - (4/3)\,(2j+1)^{-2}\left\{1+\frac{13}{16k^2}+\frac{1}{16k^4}\right\}+\ldots\right].$$

2. Set

$$R_\mu(k,\,\alpha,\,\delta) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\delta-2\alpha-1}(\theta/2)\,d\theta}{[1-k^2\cos\theta]^{\mu+\frac{1}{2}}},$$
$$0<k<1, \quad \operatorname{Re}\delta > \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -1/2,$$
$$M_\nu(\mu,\,\alpha,\,\delta) = \left[(-1)^\nu 2^\nu \left(\mu+\frac{1}{2}\right)_\nu/\nu!\right]\Gamma(\alpha)\Gamma(\delta-\alpha+\nu)/\Gamma(\delta+\nu),$$
$$\text{with } (\lambda)_\nu = \Gamma(\lambda+\nu)/\Gamma(\lambda), \quad \text{and}$$
$$W_\nu(\mu,\,\alpha,\,\delta) = \left[2^\nu\left(\mu+\frac{1}{2}\right)_\nu \nu!\right]\Gamma(\alpha+\nu)\Gamma(\delta-\alpha)/\Gamma(\delta+\nu),$$

then:

for small k;

$$R_\mu(k,\,\alpha,\,\delta) = (1-k^2)^{-(\mu+\frac{1}{2})}\sum_{\nu=0}^{\infty}[k^2/(1-k^2)]^\nu M_\nu(\mu,\,\alpha,\,\delta)$$
$$= (1+k^2)^{-(\mu+\frac{1}{2})}\sum_{\nu=0}^{\infty}[k^2/(1+k^2)]^\nu W_\nu(\mu,\,\alpha,\,\delta),$$

for k^2 close to 1;

$$R_\mu(k,\,\alpha,\,\delta) = \left[\Gamma(\delta-\alpha)\Gamma\left(\mu+\alpha-\delta+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)\right](2k^2)^{\alpha-\delta}(1-k^2)^{\delta-\alpha-\mu-\frac{1}{2}}\times$$
$$\times\left\{\Gamma\left(\delta-\alpha-\mu-\frac{1}{2}\right)\Gamma(\alpha)\left[\Gamma\left(\delta-\mu-\frac{1}{2}\right)(2k^2)^{\mu+\frac{1}{2}}\right]\right\},$$
$$\left[\mathrm{Re}\left(\mu+\alpha-\delta+\frac{1}{2}\right)\qquad\text{not an integer}\right]$$
$$=\left[2^{\mu+\frac{1}{2}}k^{2\mu+1}\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(1-\alpha)\right]\times$$
$$\times\sum_{n=0}^{\infty}\left[\Gamma(\delta-\alpha+n)\Gamma(1-\alpha+n)\Gamma\left(\alpha-\delta+\mu-n+\frac{1}{2}\right)n!\right][2k^2/(1-k^2)]^{\alpha-\delta+\mu-n+\frac{1}{2}}$$
$$\left[\alpha-\delta+\mu+\frac{1}{2}=m,\quad\text{with } m \text{ a non-negative integer}\right]$$

6.2- 6.3 The Exponential-Integral Function and Functions Generated by It

6.21 The logarithm-integral

6.211

$$\int_0^1 \mathrm{li}\,(x)\,dx = -\ln 2.$$

BI ((79))(5)

6.212

1. $$\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)x\,dx = 0.$$

2. $\int_0^1 \operatorname{li}(x) x^{p-1}\,dx = -\frac{1}{p}\ln(p+1) \qquad [p > -1].$

BI ((255))(2)

3. $\int_0^1 \operatorname{li}(x) \frac{dx}{x^{q+1}} = \frac{1}{q}\ln(1-q) \qquad [q < 1].$

BI ((255))(3)

4. $\int_1^\infty \operatorname{li}(x) \frac{dx}{x^{q+1}} = -\frac{1}{q}\ln(q-1) \qquad [q > 1].$

BI ((255))(4)

6.213

1. $\int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x)\,dx = \frac{1}{1+a^2}\left(a \ln a - \frac{\pi}{2}\right) \qquad [a > 0].$

BI ((475))(1)

2. $\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x)\,dx = -\frac{1}{1+a^2}\left(\frac{\pi}{2} + a \ln a\right) \qquad [a > 0].$

BI ((475))(9)

3. $\int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x)\,dx = -\frac{1}{1+a^2}\left(\ln a + \frac{\pi}{2} a\right) \qquad [a > 0].$

BI ((475))(2)

4. $\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x)\,dx = \frac{1}{1+a^2}\left(\ln a - \frac{\pi}{2} a\right) \qquad [a > 0].$

BI ((475))(10)

5. $\int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x} = \frac{\ln(1+a^2)}{2a} \qquad [a > 0].$

6. $\int_0^1 \mathrm{li}(x)\cos(a\ln x)\frac{dx}{x} = -\frac{\operatorname{arctg} a}{a}.$

BI ((479))(2)

7. $\int_0^1 \mathrm{li}(x)\sin(a\ln x)\frac{dx}{x^2} = \frac{1}{1+a^2}\left(a\ln a + \frac{\pi}{2}\right) \qquad [a>0].$

BI ((479))(3)

8. $\int_1^\infty \mathrm{li}(x)\sin(a\ln x)\frac{dx}{x^2} = \frac{1}{1+a^2}\left(\frac{\pi}{2} - a\ln a\right) \qquad [a>0].$

BI ((479))(13)

9. $\int_0^1 \mathrm{li}(x)\cos(a\ln x)\frac{dx}{x^2} = \frac{1}{1+a^2}\left(\ln a - \frac{\pi}{2}a\right) \qquad [a>0].$

BI ((479))(4)

10. $\int_1^\infty \mathrm{li}(x)\ \cos(a\ln x)\frac{dx}{x^2} = -\frac{1}{1+a^2}\left(\ln a + \frac{\pi}{2}a\right) \qquad [a>0].$

BI ((479))(14)

11. $\int_0^1 \mathrm{li}(x)\sin(a\ln x)x^{p-1}\,dx = \frac{1}{a^2+p^2}\left\{\frac{a}{2}\ \ln[(1+p)^2+a^2] - p\operatorname{arctg}\frac{a}{1+p}\right\} \qquad [p>0].$

BI ((477))(1)

12. $\int_0^1 \mathrm{li}(x)\cos(a\ln x)x^{p-1}\,dx = -\frac{1}{a^2+p^2}\left\{a\operatorname{arctg}\frac{a}{1+p} + \frac{p}{2}\ln[(1+p)^2+a^2]\right\} \qquad [p>0].$

BI ((477))(2)

6.214

1. $\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{p-1}\,dx = -\pi\operatorname{ctg} p\pi\cdot\Gamma(p) \qquad [0<p<1].$

BI ((340))(1)

2. $$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right)(\ln x)^{p-1}\,dx = -\frac{\pi}{\sin p\pi}\Gamma(p) \qquad [0 < p < 1].$$

BI ((340))(9)

6.215

1. $$\int_0^1 \operatorname{li}(x)\frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}}\,dx = -2\sqrt{\frac{\pi}{p}}\ \operatorname{Arsh}\sqrt{p} = -2\sqrt{\frac{\pi}{p}}\ln(\sqrt{p}+\sqrt{p+1}) \qquad [p > 0].$$

BI ((444))(3)

2. $$\int_0^1 \operatorname{li}(x)\frac{dx}{x^{p+1}\sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}}\arcsin\sqrt{p} \qquad [1 > p > 0].$$

BI ((444))(4)

6.216

1. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1}\frac{ax}{x} = -\frac{1}{p}\Gamma(p) \qquad [0 < p \le 1].$$

BI ((444))(1)

2. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1}\frac{dx}{x^2} = -\frac{\pi\Gamma(p)}{\sin p\pi} \qquad [0 < p < 1].$$

BI ((444))(2)

6.22- 6.23 The exponential-integral function

6.221

$$\int_0^p \operatorname{Ei}(\alpha x)\,dx = p\operatorname{Ei}(\alpha p) + \frac{1-e^{\alpha p}}{\alpha}.$$

NT 11(7)

6.222

$$\int_0^\infty \operatorname{Ei}(-px)\operatorname{Ei}(-qx)\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q} \qquad [p > 0, \quad q > 0].$$

FI II 653, NT 53(3)

6.223

$$\int_0^\infty \mathrm{Ei}(-\beta x) x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu\beta^\mu} \qquad [\operatorname{Re}\beta \geq 0, \quad \operatorname{Re}\mu > 0].$$

NT 55(7), ET I 325(10)

6.224

1. $$\int_0^\infty \mathrm{Ei}(-\beta x) e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(1+\frac{\mu}{\beta}\right) \qquad [\operatorname{Re}(\beta+\mu) \geq 0, \quad \mu > 0];$$
$$= -1/\beta \qquad [\mu = 0].$$

FI II 652, NT 48(8)

2. $$\int_0^\infty \mathrm{Ei}(ax) e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(\frac{\mu}{a}-1\right) \qquad [a > 0, \quad \operatorname{Re}\mu > 0, \quad \mu > a].$$

ET I 178(23)A, BI ((283))(3)

6.225

1. $$\int_0^\infty \mathrm{Ei}(-x^2) e^{-\mu x^2}\,dx = -\sqrt{\frac{\pi}{\mu}}\operatorname{Arsh}\sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}}\ln(\sqrt{\mu}+\sqrt{1+\mu}) \qquad [\operatorname{Re}\mu > 0].$$

BI ((283))(5), ET I 178(25)a

2. $$\int_0^\infty \mathrm{Ei}(-x^2) e^{px^2}\,dx = -\sqrt{\frac{\pi}{p}}\arcsin\sqrt{p} \qquad [1 > p > 0].$$

NT 59(9)a

6.226

1. $$\int_0^\infty \mathrm{Ei}\left(-\frac{1}{4x}\right) e^{-\mu x}\,dx = -\frac{2}{\mu}K_0(\sqrt{\mu}) \qquad [\operatorname{Re}\mu > 0].$$

MI 34

2. $$\int_0^\infty \mathrm{Ei}\left(\frac{a^2}{4x}\right) e^{-\mu x}\,dx = -\frac{2}{\mu}K_0(a\sqrt{\mu}) \qquad [a > 0, \quad \operatorname{Re}\mu > 0].$$

3. $$\int_0^\infty \mathrm{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2}\,dx = \sqrt{\frac{\pi}{\mu}}\,\mathrm{Ei}(-\sqrt{\mu}) \qquad [\mathrm{Re}\,\mu > 0].$$

MI 34

4. $$\int_0^\infty \mathrm{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2+\frac{1}{4x^2}}\,dx = \sqrt{\frac{\pi}{\mu}}[\cos\sqrt{\mu}\ \mathrm{ci}\sqrt{\mu} - \sin\sqrt{\mu}\ \mathrm{si}\sqrt{\mu}] \qquad [\mathrm{Re}\,\mu > 0].$$

MI 34

6.227

1. $$\int_0^\infty \mathrm{Ei}(-x)e^{-\mu x}x\,dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2}\ln(1+\mu) \qquad [\mathrm{Re}\,\mu > 0].$$

MI 34

2. $$\int_0^\infty \left[\frac{e^{-ax}\,\mathrm{Ei}(ax)}{x-b} - \frac{e^{ax}\,\mathrm{Ei}(-ax)}{x+b}\right]dx = 0 \qquad [a>0, \quad b<0];$$
$$= \pi^2 e^{-ab} \qquad [a>0, \quad b>0].$$

ET II 253(1)a

6.228

1. $$\int_0^\infty \mathrm{Ei}(-x)e^{x}x^{\nu-1}\,dx = -\frac{\pi\Gamma(\nu)}{\sin\nu\pi} \qquad [0 < \mathrm{Re}\,\nu < 1].$$

ET II 308(13)

2. $$\int_0^\infty \mathrm{Ei}(-\beta x)e^{-\mu x}x^{\nu-1}\,dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu}\,{}_2F_1\left(1,\,\nu;\,\nu+1;\,\frac{\mu}{\beta+\mu}\right)$$
$$[|\arg\beta| < \pi, \quad \mathrm{Re}(\beta+\mu) > 0, \quad \mathrm{Re}\,\nu > 0].$$

ET II 308(14)

6.229

$$\int_0^\infty \mathrm{Ei}\left(-\frac{1}{4x^2}\right)\exp\left(-\mu x^2 + \frac{1}{4x^2}\right)\frac{dx}{x^2} = 2\sqrt{\pi}(\cos\sqrt{\mu}\ \mathrm{si}\sqrt{\mu} - \sin\sqrt{\mu}\ \mathrm{ci}\sqrt{\mu}) \qquad [\mathrm{Re}\,\mu > 0].$$

MI 34

6.231

$$\int_{-\ln a}^\infty [\mathrm{Ei}(-a) - \mathrm{Ei}(-e^{-x})]e^{-\mu x}\,dx = \frac{1}{\mu}\gamma(\mu,\,a) \qquad [a<1, \quad \mathrm{Re}\,\mu > 0].$$

6.232

1. $$\int_0^\infty \operatorname{Ei}(-ax)\sin bx\,dx = -\frac{\ln\left(1+\frac{b^2}{a^2}\right)}{2b} \qquad [a>0, \quad b>0].$$

BI ((473))(1)a

2. $$\int_0^\infty \operatorname{Ei}(-ax)\cos bx\,dx = -\frac{1}{b}\operatorname{arctg}\frac{b}{a} \qquad [a>0, \quad b>0].$$

BI ((473))(2)a

6.233

1. $$\int_0^\infty \operatorname{Ei}(-x)e^{-\mu x}\sin\beta x\,dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\beta}{2}\ln[(1+\mu)^2+\beta^2] - \mu\operatorname{arctg}\frac{\beta}{1+\mu}\right\}$$
$$[\operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

BI ((473))(7)a

2. $$\int_0^\infty \operatorname{Ei}(-x)e^{-\mu x}\cos\beta x\,dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\mu}{2}\ln[(1+\mu)^2+\beta^2] + \beta\operatorname{arctg}\frac{\beta}{1+\mu}\right\}$$
$$[\operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

BI ((473))(8)a

6.234

$$\int_0^\infty \operatorname{Ei}(-x)\ln x\,dx = \boldsymbol{C} + 1.$$

NT 56(10)

6.24- 6.26 The sine- and cosine-integral functions

6.241

1. $$\int_0^\infty \operatorname{si}(px)\ \operatorname{si}(qx)\,dx = \frac{\pi}{2p} \qquad [p \ge q].$$

BI II 653, NT 54(8)

FI II 653, NT 54(7)

3. $$\int_0^\infty \operatorname{si}(px)\ \operatorname{ci}(qx)\,dx = \frac{1}{4q}\ln\left(\frac{p+q}{p-q}\right)^2 + \frac{1}{4p}\ln\frac{(p^2-q^2)^2}{q^4} \qquad [p \neq q];$$
$$= \frac{1}{q}\ln 2 \qquad [p=q].$$

FI II 653, NT 54(10, 12)

6.242

$$\int_0^\infty \frac{\operatorname{ci}(ax)}{\beta+x}\,dx = -\frac{1}{2}\{[\operatorname{si}(a\beta)]^2 + [\operatorname{ci}(a\beta)]^2\} \qquad [a>0, \quad |\arg\beta| < \pi].$$

ET II 224(1)

6.243

1. $$\int_{-\infty}^\infty \frac{\operatorname{si}(a|x|)}{x-b}\operatorname{sign} x\,dx = \pi\ \operatorname{ci}(a|b|) \qquad [a>0, \quad b>0].$$

ET II 253(3)

2. $$\int_{-\infty}^\infty \frac{\operatorname{ci}(a|x|)}{x-b}\,dx = -\pi\ \operatorname{sign} b\cdot\operatorname{si}(a|b|) \qquad [a>0].$$

ET II 253(2)

6.244

1. $$\int_0^\infty [\operatorname{si}(px)]\frac{x\,dx}{q^2+x^2} = \frac{\pi}{2}\operatorname{Ei}(-pq) \qquad [p>0, \quad q>0].$$

BI ((255))(6)

2. $$\int_0^\infty [\operatorname{si}(px)]\frac{x\,dx}{q^2-x^2} = -\frac{\pi}{2}\ \operatorname{ci}(pq) \qquad [p>0, \quad q>0].$$

BI ((255))(6)

6.245

1. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2+x^2} = \frac{\pi}{2q}\operatorname{Ei}(-pq) \qquad [p>0, \quad q>0].$$

2. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \operatorname{si}(pq) \qquad [p > 0, \quad q > 0].$$

BI ((255))(8)

6.246

1. $$\int_0^\infty \operatorname{si}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \sin\frac{\mu\pi}{2} \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

NT 56(9), ET I 325(12)a

2. $$\int_0^\infty \operatorname{ci}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \cos\frac{\mu\pi}{2} \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1].$$

NT 56(8), ET I 325(13)a

6.247

1. $$\int_0^\infty \operatorname{si}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu} \operatorname{arctg}\frac{\mu}{\beta} \qquad [\operatorname{Re}\mu > 0].$$

NT 49(12), ET I 177(18)

2. $$\int_0^\infty \operatorname{ci}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu} \ln\sqrt{1 + \frac{\mu^2}{\beta^2}} \qquad [\operatorname{Re}\mu > 0].$$

NT 49(11), ET I 178(19)a

6.248

1.[8] $$\int_0^\infty \operatorname{si}(x)e^{-\mu x^2}x\,dx = \frac{\pi}{4\mu}\left[\Phi\left(\frac{1}{2\sqrt{\mu}}\right) - 1\right] \qquad [\operatorname{Re}\mu > 0].$$

MI 34

2. $$\int_0^\infty \operatorname{ci}(x)e^{-\mu x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}} \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \qquad [\operatorname{Re}\mu > 0].$$

MI 34

6.249

$$\int_0^\infty \left[\operatorname{si}(x^2) + \frac{\pi}{2}\right] e^{-\mu x}\,dx = \frac{\pi}{\mu}\left\{\left[S\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2 + \left[C\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2\right\} \qquad [\operatorname{Re}\mu > 0].$$

6.251

1. $$\int_0^\infty \operatorname{si}\left(\frac{1}{x}\right) e^{-\mu x}\, dx = \frac{2}{\mu} \operatorname{kei}(2\sqrt{\mu}) \qquad [\operatorname{Re} \mu > 0].$$

MI 34

2. $$\int_0^\infty \operatorname{ci}\left(\frac{1}{x}\right) e^{-\mu x}\, dx = -\frac{2}{\mu} \operatorname{ker}(2\sqrt{\mu}) \qquad [\operatorname{Re} \mu > 0].$$

MI 34

6.252

1. $$\int_0^\infty \sin px \operatorname{si}(qx)\, dx = |-\frac{\pi}{2p} \qquad [p^2 > q^2]$$
$$= -\frac{\pi}{4p} \qquad [p^2 = q^2];$$
$$= 0 \qquad [p^2 < q^2].$$

FI II 652, NT 50(8)

2.[6] $$\int_0^\infty \cos px \operatorname{si}(qx)\, dx = -\frac{1}{4p} \ln\left(\frac{p+q}{p-q}\right)^2 \qquad [p \neq 0, \quad p^2 \neq q^2];$$
$$= \frac{1}{q} \qquad [p = 0].$$

FI II 652, NT 50(10)

3. $$\int_0^\infty \sin px \operatorname{ci}(qx)\, dx = -\frac{1}{4p} \ln\left(\frac{p^2}{q^2} - 1\right)^2 \qquad [p \neq 0, \quad p^2 \neq q^2];$$
$$= 0 \qquad [p = 0].$$

FI II 652, NT 50(9)

4. $$\int_0^\infty \cos px \operatorname{ci}(qx)\, dx = -\frac{\pi}{2p} \qquad [p^2 > q^2];$$
$$= -\frac{\pi}{4p} \qquad [p^2 = q^2];$$
$$= 0 \qquad [p^2 < q^2].$$

FI II 654, NT 50(7)

6.253

$$\int_0^\infty \frac{\mathrm{si}(ax)\sin bx}{1-2r\cos x+r^2}\,dx = -\frac{\pi(r^m+r^{m+1})}{4b(1-r)(1-r^2)} \qquad [b=a-m];$$
$$= -\frac{\pi(2+2r-r^m-r^{m+1})}{4b(1-r)(1-r^2)} \qquad [b=a+m];$$
$$= -\frac{\pi r^{m+1}}{2b(1-r)(1-r^2)} \qquad [a-m-1<b<a-m];$$
$$= -\frac{\pi(1+r-r^{m+1})}{2b(1-r)(1-r^2)} \qquad [a+m<b<a+m+1].$$

ET I 97(10)

6.254

1. $$\int_0^\infty \left[\mathrm{si}(ax)+\frac{\pi}{2}\right]\sin bx\frac{dx}{x} = \frac{1}{2}\left[L_2\left(\frac{a}{b}\right)-L_2\left(-\frac{a}{b}\right)\right] \qquad [a>0, \quad b>0].$$

ET I 97(12)

2. $$\int_0^\infty \left[\mathrm{si}(ax)+\frac{\pi}{2}\right]\cos bx\cdot\frac{dx}{x} = \frac{\pi}{2}\ln\frac{a}{b} \qquad [a>0, \quad b>0].$$

ET I 41(11)

6.255

1. $$\int_{-\infty}^\infty [\cos ax\ \mathrm{ci}(a|x|)+\sin(a|x|)\ \mathrm{si}(a|x|)\frac{dx}{x-b} = -\pi[\mathrm{sign}\,b\cos ab\ \mathrm{si}(a|b|)-\sin ab\ \mathrm{ci}(a|b|)]$$
$$[a>0].$$

ET II 253(4)

2. $$\int_{-\infty}^\infty [\sin ax\ \mathrm{ci}(a|x|)-\mathrm{sign}\,x\cos ax\ \mathrm{si}(a|x|)]\frac{dx}{x-b} = -\pi[\sin(a|b|)\ \mathrm{si}(a|b|)+\cos ab\ \mathrm{ci}(a|b|)]$$
$$[a>0].$$

ET II 253(5)

6.256

$$\int_0^\infty [\mathrm{si}^2(x)+\mathrm{ci}^2(x)]\cos ax\,dx = \frac{\pi}{a}\ln(1+a) \qquad [a>0].$$

6.257

$$\int_0^\infty \operatorname{si}\left(\frac{a}{x}\right)\sin bx\,dx = -\frac{\pi}{2b}J_0(2\sqrt{ab}) \qquad [b>0].$$

ET I 96(9)

6.258

1. $$\int_0^\infty \left[\operatorname{si}(ax)+\frac{\pi}{2}\right]\sin bx\frac{dx}{x^2+c^2} =$$
$$= \frac{\pi}{4c}\{e^{-bc}[\operatorname{Ei}(bc)-\operatorname{Ei}(-ac)]+e^{bc}[\operatorname{Ei}(-ac)-\operatorname{Ei}(-bc)]\} \qquad [0<b\le a, \quad c>0];$$
$$= \frac{\pi}{4c}e^{-bc}[\operatorname{Ei}(ac)-\operatorname{Ei}(-ac)] \qquad [0<a\le b, \quad c>0].$$

BI ((460))(1)

2. $$\int_0^\infty \left[\operatorname{si}(ax)+\frac{\pi}{2}\right]\cos bx\frac{x\,dx}{x^2+c^2} = -\frac{\pi}{4}\{e^{-bc}[\operatorname{Ei}(bc)-\operatorname{Ei}(-ac)]+e^{bc}[\operatorname{Ei}(-bc)-\operatorname{Ei}(-ac)]\}$$
$$[0<b\le a, \quad c>0];$$
$$= \frac{\pi}{4}e^{-bc}[\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)] \qquad [0<a\le b, \quad c>0].$$

BI ((460))(2, 5)

6.259

1. $$\int_0^\infty \operatorname{si}(ax)\sin bx\frac{dx}{x^2+c^2} = \frac{\pi}{2c}\operatorname{Ei}(-ac)\operatorname{sh}(bc) \qquad [0<b\le a, \quad c>0];$$
$$= \frac{\pi}{4c}e^{-cb}[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc)-\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)]+$$
$$+\frac{\pi}{2c}\operatorname{Ei}(-bc)\operatorname{sh}(bc) \qquad [0<a\le b, \quad c>0].$$

ET I 96(8)

2. $$\int_0^\infty \operatorname{ci}(ax)\sin bx\frac{x\,dx}{x^2+c^2} = -\frac{\pi}{2}\operatorname{sh}(bc)\operatorname{Ei}(-ac) \qquad [0<b\le a, \quad c>0];$$
$$= -\frac{\pi}{2}\operatorname{sh}(bc)\operatorname{Ei}(-bc)+\frac{\pi}{4}e^{-bc}[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc)-$$
$$-\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)] \qquad [0<a\le b, \quad c>0].$$

BI ((460))(3)A, ET I 97(15)a

BI ((460))(4), ET I 41(15)

6.261

1. $$\int_0^\infty \operatorname{si}(bx)\cos ax\, e^{-px}\,dx = -\frac{1}{2(a^2+p^2)}\left[\frac{a}{2}\ln\frac{p^2+(a+b)^2}{p^2+(a-b)^2} + p \operatorname{arctg}\frac{2bp}{b^2-a^2-p^2}\right]$$
$$[a>0, \quad b>0, \quad p>0].$$

ET I 40(8)

2. $$\int_0^\infty \operatorname{si}(\beta x)\cos ax\, e^{-\mu x}\,dx = -\frac{\operatorname{arctg}\frac{\mu+ai}{\beta}}{2(\mu+ai)} - \frac{\operatorname{arctg}\frac{\mu-ai}{\beta}}{2(\mu-ai)} \qquad [a>0, \quad \operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

ET I 40(9)

6.262

1. $$\int_0^\infty \operatorname{ci}(bx)\sin ax\, e^{-\mu x}\,dx = \frac{1}{2(a^2+\mu^2)}\left\{\mu \operatorname{arctg}\frac{2a\mu}{\mu^2+b^2-a^2} - \frac{a}{2}\ln\frac{(\mu^2+b^2-a^2)^2+4a^2\mu^2}{b^4}\right\}$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0].$$

ET I 98(16)a

2. $$\int_0^\infty \operatorname{ci}(bx)\cos ax\, e^{-px}\,dx = \frac{-1}{2(a^2+p^2)}\left\{\frac{p}{2}\ln\frac{[(b^2+p^2-a^2)^2+4a^2p^2]}{b^4} + a \operatorname{arctg}\frac{2ap}{b^2+p^2-a^2}\right\}$$
$$[a>0, \quad b>0, \quad \operatorname{Re}p>0].$$

ET I 41(16)

3. $$\int_0^\infty \operatorname{ci}(\beta x)\cos ax\, e^{-\mu x}\,dx = \frac{-\ln\left[1+\frac{(\mu+ai)^2}{\beta^2}\right]}{4(\mu+ai)} - \frac{\ln\left[1+\frac{(\mu-ai)^2}{\beta^2}\right]}{4(\mu-ai)}$$
$$[a>0, \quad \operatorname{Re}\mu > |\operatorname{Im}\beta|].$$

ET I 41(17)

6.263

1. $$\int_0^\infty [\operatorname{ci}(x)\cos x + \operatorname{si}(x)\sin x]e^{-\mu x}\,dx = \frac{-\frac{\pi}{2} - \mu\ln\mu}{1+\mu^2} \qquad [\operatorname{Re}\mu>0].$$

ME 26a, ET I 178(21)a

2. $$\int_0^\infty [\operatorname{si}(x)\cos x - \operatorname{ci}(x)\sin x]e^{-\mu x}\,dx = \frac{-\frac{\pi}{2}\mu + \ln\mu}{1+\mu^2} \qquad [\operatorname{Re}\mu>0].$$

3. $$\int_0^\infty [\sin x - x\ \mathrm{ci}(x)] e^{-\mu x}\, dx = \frac{\ln(1+\mu^2)}{2\mu^2} \qquad [\operatorname{Re}\mu > 0].$$

ME 26

6.264

1. $$\int_0^\infty \mathrm{si}(x) \ln x\, dx = \boldsymbol{C} + 1.$$

NT 46(10)

2. $$\int_0^\infty \mathrm{ci}(x) \ln x\, dx = \frac{\pi}{2}.$$

NT 56(11)

6.27 The hyperbolic-sine- and cosine-integral functions

6.271

1. $$\int_0^\infty \mathrm{shi}(x) e^{-\mu x}\, dx = \frac{1}{2\mu} \ln \frac{\mu+1}{\mu-1} = \frac{1}{\mu} \operatorname{Arcth} \mu \qquad [\operatorname{Re}\mu > 1].$$

MI 34

2. $$\int_0^\infty \mathrm{chi}(x) e^{-\mu x}\, dx = -\frac{1}{2\mu} \ln(\mu^2 - 1) \qquad [\operatorname{Re}\mu > 1].$$

MI 34

6.272

$$\int_0^\infty \mathrm{chi}(x) e^{-px^2}\, dx = \frac{1}{4} \sqrt{\frac{\pi}{p}}\, \mathrm{Ei}\left(\frac{1}{4p}\right) \qquad [p > 0].$$

MI 35

6.273

1. $$\int_0^\infty [\operatorname{ch} x\, \mathrm{shi}(x) - \operatorname{sh} x\, \mathrm{chi}(x)] e^{-\mu x}\, dx = \frac{\ln \mu}{\mu^2 - 1} \qquad [\operatorname{Re}\mu > 0].$$

2. $\int_0^\infty [\operatorname{ch} x \operatorname{chi}(x) + \operatorname{sh} x \operatorname{shi}(x)] e^{-\mu x}\, dx = \frac{\mu \ln \mu}{1-\mu^2} \qquad [\operatorname{Re} \mu > 2].$

MI 35

6.274

$$\int_0^\infty [\operatorname{ch} x \operatorname{shi}(x) - \operatorname{sh} x \operatorname{chi}(x)] e^{-\mu x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}} e^{\frac{1}{4\mu}} \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \qquad [\operatorname{Re} \mu > 0].$$

MI 35

6.275

$$\int_0^\infty [x \operatorname{chi}(x) - \operatorname{sh} x] e^{-\mu x}\, dx = -\frac{\ln(\mu^2 - 1)}{2\mu^2} \qquad [\operatorname{Re} \mu > 1].$$

MI 35

6.276

$$\int_0^\infty [\operatorname{ch} x \operatorname{chi}(x) + \operatorname{sh} x \operatorname{shi}(x)] e^{-\mu x^2} x\, dx = \frac{1}{8}\sqrt{\frac{\pi}{\mu^3}} \exp\left(\frac{1}{4\mu}\right) \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \qquad [\operatorname{Re} \mu > 0].$$

MI 35

6.277

1. $\int_0^\infty [\operatorname{chi}(x) + \operatorname{ci}(x)] e^{-\mu x}\, dx = -\frac{\ln(\mu^4 - 1)}{2\mu} \qquad [\operatorname{Re} \mu > 1].$

MI 34

677

2. $\int_0^\infty [\operatorname{chi}(x) - \operatorname{ci}(x)] e^{-\mu x}\, dx = \frac{1}{2\mu} \ln \frac{\mu^2 + 1}{\mu^2 - 1} \qquad [\operatorname{Re} \mu > 1].$

MI 35

6.28- 6.31 The probability integral

6.281

1.[6] $\int_0^\infty [1 - \Phi(px)] x^{2q-1}\, dx = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2\sqrt{\pi} q p^{2q}} \qquad [\operatorname{Re} q > 0, \quad \operatorname{Re} p > 0].$

2.[6] $$\int_0^\infty \left[1 - \Phi\left(at^\alpha \pm \frac{b}{t^\alpha}\right)\right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a}\right)^{\frac{1-\alpha}{2\alpha}} \left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab)\right] e^{\pm 2ab}$$
$$[a > 0, \quad b > 0, \quad \alpha \neq 0].$$

6.282

1. $$\int_0^\infty \Phi(qt) e^{-pt}\, dt = \frac{1}{p}\left[1 - \Phi\left(\frac{p}{2q}\right)\right] \exp\left(\frac{p^2}{4q^2}\right) \qquad [\operatorname{Re} p > 0, \quad |\arg q| < \pi/4].$$

MO 175, EH II 148(11)

2. $$\int_0^\infty \left[\Phi\left(x + \frac{1}{2}\right) - \Phi\left(\frac{1}{2}\right)\right] e^{-\mu x + \frac{1}{4}}\, dx = \frac{1}{(\mu+1)(\mu+2)} \exp\frac{(\mu+1)^2}{4} \left[1 - \Phi\left(\frac{\mu+1}{2}\right)\right].$$

ME 27

6.283

1. $$\int_0^\infty e^{\beta x}[1 - \Phi(\sqrt{\alpha x})]\, dx = \frac{1}{\beta}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1\right] \qquad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta < \operatorname{Re} \alpha].$$

ET II 307(5)

2. $$\int_0^\infty \Phi(\sqrt{qt}) e^{-pt}\, dt = \frac{\sqrt{q}}{p} \frac{1}{\sqrt{p+q}} \qquad [\operatorname{Re} p > 0, \quad \operatorname{Re}(q+p) > 0].$$

EH II 148(12)

6.284

$$\int_0^\infty \left[1 - \Phi\left(\frac{q}{2\sqrt{x}}\right)\right] e^{-px}\, dx = \frac{1}{p} e^{-q\sqrt{p}} \qquad \left[\operatorname{Re} p > 0, \ |\arg q| < \frac{\pi}{4}\right].$$

EH II 148(13)
EF 147(235)

6.285

1. $$\int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2}\, dx = \frac{\operatorname{arctg} \mu}{\sqrt{\pi}\mu} \qquad [\operatorname{Re} \mu > 0].$$

MI 37

2. $$\int_0^\infty \Phi(iat) e^{-a^2t^2 - st}\, dt = \frac{-1}{2ai\sqrt{\pi}} \exp\left(\frac{s^2}{4a^2}\right) \operatorname{Ei}\left(-\frac{s^2}{4a^2}\right) \qquad \left[\operatorname{Re} s > 0, \quad |\arg a| < \frac{\pi}{4}\right].$$

6.286

1. $$\int_0^\infty [1-\Phi(\beta x)]e^{\mu^2 x^2} x^{\nu-1}\,dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\nu\beta^\nu}\,{}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; \frac{\mu^2}{\beta^2}\right)$$
$$[\operatorname{Re}\beta^2 > \operatorname{Re}\mu^2, \quad \operatorname{Re}\nu > 0].$$

ET II 306(2)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1}\,dx = 2^{\frac{\nu}{2}-1}\sec\frac{\nu\pi}{2}\Gamma\left(\frac{\nu}{2}\right) \qquad [0 < \operatorname{Re}\nu < 1].$$

ET I 325(9)

6.287

1. $$\int_0^\infty \Phi(\beta x)e^{-\mu x^2} x\,dx = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \qquad [\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0].$$

ME 27a, ET I 176(4)

2. $$\int_0^\infty [1-\Phi(\beta x)]e^{-\mu x^2} x\,dx = \frac{1}{2\mu}\left(1-\frac{\beta}{\sqrt{\mu+\beta^2}}\right) \qquad [\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0].$$

NT 49(14), ET I 177(9)

6.288

$$\int_0^\infty \Phi(iax)e^{-\mu x^2} x\,dx = \frac{ai}{2\mu\sqrt{\mu-a^2}} \qquad [a > 0, \quad \operatorname{Re}\mu > \operatorname{Re}a^2].$$

MI 37a

6.289

1. $$\int_0^\infty \Phi(\beta x)e^{(\beta^2-\mu^2)x^2} x\,dx = \frac{\beta}{2\mu(\mu^2-\beta^2)} \qquad \left[\operatorname{Re}\mu^2 > \operatorname{Re}\beta^2, \quad |\arg\mu| < \frac{\pi}{4}\right].$$

ET I 176(5)

2. $$\int_0^\infty [1-\Phi(\beta x)]e^{(\beta^2-\mu^2)x^2} x\,dx = \frac{1}{2\mu(\mu+\beta)} \qquad \left[\operatorname{Re}\mu^2 > \operatorname{Re}\beta^2, \quad \arg\mu < \frac{\pi}{4}\right]$$

3. $$\int_0^\infty \Phi(\sqrt{b-a}x)e^{-(a+\mu)x^2} x\,dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \qquad [\operatorname{Re}\mu > -a > 0, \quad b > a].$$

ME 27

6.291

$$\int_0^\infty \Phi(ix)e^{-(\mu x+x^2)} x\,dx = \frac{i}{\sqrt{\pi}}\left[\frac{1}{\mu} + \frac{\mu}{4}\operatorname{Ei}\left(-\frac{\mu^2}{4}\right)\right] \qquad [\operatorname{Re}\mu > 0].$$

MI 37

6.292

$$\int_0^\infty [1-\Phi(x)]e^{-\mu^2x^2} x^2\,dx = \frac{1}{2\sqrt{\pi}}\left\{\frac{\operatorname{arctg}\mu}{\mu^3} - \frac{1}{\mu^2(\mu^2+1)}\right\} \qquad \left[|\arg\mu| < \frac{\pi}{4}\right].$$

MI 37

6.293

$$\int_0^\infty \Phi(x)e^{-\mu x^2}\,\frac{dx}{x} = \frac{1}{2}\ln\frac{\sqrt{\mu+1}+1}{\sqrt{\mu+1}-1} = \operatorname{Arcth}\sqrt{\mu+1} \qquad [\operatorname{Re}\mu > 0].$$

MI 37a

6.294

1. $$\int_0^\infty \left[1-\Phi\left(\frac{\beta}{x}\right)\right] e^{-\mu^2x^2} x\,dx = \frac{1}{2\mu^2}\exp(-2\beta\mu) \qquad \left[|\arg\beta| < \frac{\pi}{4}, \quad |\arg\mu| < \frac{\pi}{4}\right].$$

ET I 177(11)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] e^{-\mu^2x^2}\,\frac{dx}{x} = -\operatorname{Ei}(-2\mu) \qquad \left[|\arg\mu| < \frac{\pi}{4}\right].$$

MI 37

6.295

1. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right]\exp\left(-\mu^2x^2+\frac{1}{x^2}\right)\,dx = \frac{1}{\sqrt{\pi}\mu}[\sin 2\mu\ \operatorname{ci}(2\mu) - \cos 2\mu\ \operatorname{si}(2\mu)]$$
$$\left[|\arg\mu| < \frac{\pi}{4}\right].$$

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2+\frac{1}{x^2}\right) x\,dx = \frac{\pi}{2\mu}[H_1(2\mu)-N_1(2\mu)]-\frac{1}{\mu^2} \qquad \left[|\arg\mu|<\frac{\pi}{4}\right].$$

MI 37

3. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2+\frac{1}{x^2}\right) \frac{dx}{x} = \frac{\pi}{2}[H_0(2\mu)-N_0(2\mu)] \qquad \left[|\arg\mu|<\frac{\pi}{4}\right].$$

MI 37

6.296

$$\int_0^\infty \left\{(x^2+a^2)\left[1-\Phi\left(\frac{a}{\sqrt{2}x}\right)\right]-\sqrt{\frac{2}{\pi}}ax\cdot e^{-\frac{a^2}{2x^2}}\right\} e^{-\mu^2x^2}\,x\,dx = \frac{1}{2\mu^4}e^{-a\mu\sqrt{2}}$$
$$\left[|\arg\mu|<\frac{\pi}{4},\quad a>0\right].$$

MI 38a

6.297

1. $$\int_0^\infty \left[1-\Phi\left(\gamma x+\frac{\beta}{x}\right)\right] e^{(\gamma^2-\mu)x^2}\,x\,dx = \frac{1}{2\sqrt{\mu}(\sqrt{\mu}+\gamma)}\exp[-2(\beta\gamma+\beta\sqrt{\mu})]$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>0].$$

ET I 177(12)a

2. $$\int_0^\infty \left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right] \exp[-(\mu^2-a^2)x^2+ab]x\,dx = \frac{e^{-b\mu}}{2\mu(\mu+a)}$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0].$$

MI 38

3. $$\int_0^\infty \left\{\left[1-\Phi\left(\frac{b-2ax^2}{2x}\right)\right]e^{-ab}+\left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right]e^{ab}\right\} e^{-\mu x^2}\,x\,dx = \frac{1}{\mu}\exp(-b\sqrt{a^2+\mu})$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0].$$

MI 38

6.298

$$\int_0^\infty \left\{2\operatorname{ch}ab - e^{-ab}\Phi\left(\frac{b-2ax^2}{2x}\right)-e^{ab}\Phi\left(\frac{b+2ax^2}{2x}\right)\right\} e^{-(\mu-a^2)x^2}\,x\,dx = \frac{1}{\mu-a^2}\exp(-b\sqrt{\mu})$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0].$$

MI 38

$$\int_0^\infty \operatorname{ch}(2\nu t)\exp[(a\ \operatorname{ch} t)^2][1-\Phi(a\ \operatorname{ch} t)]\,dt = \frac{1}{2\cos(\nu\pi)}\exp\left(\frac{1}{2}a^2\right)K_\nu(a^2)$$

$$\left[\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 308(10)

6.311

$$\int_0^\infty [1-\Phi(ax)]\sin bx\,dx = \frac{1}{b}\left(1-e^{-\frac{b^2}{4a^2}}\right) \qquad [a>0, \quad b>0].$$

ET I 96(4)

6.312

$$\int_0^\infty \Phi(ax)\sin bx^2\,dx = \frac{1}{4\sqrt{2\pi b}}\left(\ln\frac{b+a^2+a\sqrt{2b}}{b+a^2-a\sqrt{2b}}+2\operatorname{arctg}\frac{a\sqrt{2b}}{b-a^2}\right) \qquad [a>0, \quad b>0].$$

ET I 96(3)

6.313

1. $$\int_0^\infty \sin(\beta x)[1-\Phi(\sqrt{\alpha x})]\,dx = \frac{1}{\beta}-\left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}[(\alpha^2+\beta^2)^{\frac{1}{2}}-\alpha]^{-\frac{1}{2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

ET II 307(6)

2. $$\int_0^\infty \cos(\beta x)[1-\Phi(\sqrt{\alpha x})]\,dx = \left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}[(\alpha^2+\beta^2)^{\frac{1}{2}}+\alpha]^{-\frac{1}{2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

ET II 307(7)

6.314

1. $$\int_0^\infty \sin(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]\,dx = b^{-1}\exp[-(2ab)^{\frac{1}{2}}]\cos[(2ab)^{\frac{1}{2}}] \qquad [\operatorname{Re} a>0, \quad b>0].$$

2. $\int_0^\infty \cos(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]\,dx = -b^{-1}\exp[-(2ab)^{\frac{1}{2}}]\sin[(2ab)^{\frac{1}{2}}] \qquad [\operatorname{Re} a > 0, \quad b > 0].$

ET II 307(9)

6.315

1. $\int_0^\infty x^{\nu-1}\sin(\beta x)[1-\Phi(\alpha x)]\,dx = \frac{\Gamma\left(1+\frac{1}{2}\nu\right)\beta}{\sqrt{\pi}(\nu+1)\alpha^{\nu+1}}\, {}_2F_2\left(\frac{\nu+1}{2}, \frac{\nu}{2}+1; \frac{3}{2}, \frac{\nu+3}{2}; -\frac{\beta^2}{4\alpha^2}\right)$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1].$

ET II 307(3)

2. $\int_0^\infty x^{\nu-1}\cos(\beta x)[1-\Phi(\alpha x)]dx = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)}{\sqrt{\pi}\nu\alpha^\nu}\, {}_2F_2\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}, \frac{\nu}{2}+1; -\frac{\beta^2}{4\alpha^2}\right)$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > 0].$

ET II 307(4)

681

3. $\int_0^\infty [1-\Phi(ax)]\cos bx\cdot x\,dx = \frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right) - \frac{1}{b^2}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right] \qquad [a > 0, \quad b > 0].$

ET I 40(5)

4. $\int_0^\infty [\Phi(ax)-\Phi(bx)]\cos px\frac{dx}{x} = \frac{1}{2}\left[\operatorname{Ei}\left(-\frac{p^2}{4b^2}\right) - \operatorname{Ei}\left(\frac{p^2}{4a^2}\right)\right] \qquad [a > 0, \quad b > 0, \quad p > 0].$

ET I 40(6)

5. $\int_0^\infty x^{-\frac{1}{2}}\Phi(a\sqrt{x})\sin bx\,dx = \frac{1}{2\sqrt{2\pi b}}\left\{\ln\left[\frac{b+a\sqrt{2b}+a^2}{b-a\sqrt{2b}+a^2}\right] + 2\operatorname{arctg}\left[\frac{a\sqrt{2b}}{b-a^2}\right]\right\} \qquad [a > 0,\ b > 0].$

ET I 96(3)

6.316

$\int_0^\infty e^{\frac{1}{2}x^2}\left[1-\Phi\left(\frac{x}{\sqrt{2}}\right)\right]\sin bx\,dx = \sqrt{\frac{\pi}{2}}e^{\frac{b^2}{2}}\left[1-\Phi\left(\frac{b}{\sqrt{2}}\right)\right] \qquad [b > 0].$

$$\int_0^\infty e^{-a^2x^2}\,\Phi(iax)\sin bx\,dx = \frac{i}{a}\,\frac{\sqrt{\pi}}{2}e^{-\frac{b^2}{4a^2}} \qquad [b>0].$$

ET I 96(2)

6.318

$$\int_0^\infty [1-\Phi(x)]\ \mathrm{si}(2px)\,dx = \frac{2}{\pi p}(1-e^{-p^2}) - \frac{2}{\sqrt{\pi}}(1-\Phi(p)) \qquad [p>0].$$

NT 61(13)a

6.32 Fresnel integrals

6.321

1. $$\int_0^\infty \left[\frac{1}{2}-S(px)\right] x^{2q-1}\,dx = \frac{\sqrt{2}\Gamma\left(q+\frac{1}{2}\right)\sin\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}} \qquad \left[0<\operatorname{Re}q<\frac{3}{2},\quad p>0\right].$$

NT 56(14)a

2. $$\int_0^\infty \left[\frac{1}{2}-C(px)\right] x^{2q-1}\,dx = \frac{\sqrt{2}\Gamma\left(q+\frac{1}{2}\right)\cos\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}} \qquad \left[0<\operatorname{Re}q<\frac{3}{2},\quad p>0\right].$$

NT 56(13)a

6.322

1. $$\int_0^\infty S(t)e^{-pt}\,dt = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right]+\sin\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right]\right\}.$$

MO 173a

2. $$\int_0^\infty C(t)e^{-pt}\,dt = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right]-\sin\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right]\right\}.$$

MO 172a

6.323

1. $$\int_0^\infty S(\sqrt{t})e^{-pt}\,dx = \frac{(\sqrt{p^2+1}-p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}.$$

2. $$\int_0^\infty C(\sqrt{t})e^{-pt}\,dt = \frac{(\sqrt{p^2+1}+p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}.$$

EF 122(58)a

6.324

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \sin 2px\,dx = (1+\sin p^2 - \cos p^2)/4p \qquad [p>0].$$

NT 61(12)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \sin 2px\,dx = (1-\sin p^2 - \cos p^2)/4p \qquad [p>0].$$

NT 61(11)a

6.325

1. $$\int_0^\infty S(x)\sin b^2x^2\,dx = \frac{1}{b}\sqrt{\pi}2^{-\frac{5}{2}} \qquad [0<b^2<1];$$
$$= 0 \qquad [b^2>1].$$

ET I 98(21)a

2. $$\int_0^\infty C(x)\cos b^2x^2\,dx = \frac{\sqrt{\pi}}{b}2^{-\frac{5}{2}} \qquad [0<b^2<1];$$
$$= 0 \qquad [b^2>1].$$

ET I 42(22)

6.326

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \operatorname{si}(2px)\,dx = (\pi/8)^{1/2}(S(p)+C(p)-1)-(1+\sin p^2-\cos p^2)/4p$$
$$[p>0].$$

NT 61(15)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \operatorname{si}(2px)\,dx = (\pi/8)^{1/2}(S(p)-C(p))-(1-\sin p^2-\cos p^2)/4p \qquad [p>0].$$

NT 61(14)a

6.4 The Gamma Function and Functions Generated by It

6.41 The gamma function

6.411

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)\,dx = -i\pi 2^{1-\alpha-\beta}\Gamma(\alpha+\beta) \qquad [\operatorname{Re}(\alpha+\beta) < 1, \quad \operatorname{Im}\alpha, \quad \operatorname{Im}\beta > 0];$$

$$= i\pi 2^{1-\alpha-\beta}\Gamma(\alpha+\beta) \qquad [\operatorname{Re}(\alpha+\beta) < 1, \quad \operatorname{Im}\alpha, \quad \operatorname{Im}\beta < 0];$$

$$= 0 \qquad [\operatorname{Re}(\alpha+\beta) < 1, \quad \operatorname{Im}\alpha, \quad \operatorname{Im}\beta < 0].$$

ET II 297(1)
ET II 297(2)
ET II 297(3)

6.412

$$\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds = 2\pi i \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$
$$[\operatorname{Re}\alpha, \quad \operatorname{Re}\beta, \quad \operatorname{Re}\gamma, \quad \operatorname{Re}\delta > 0].$$

ET II 302(32)

6.413

1. $$\int_0^{\infty} |\Gamma(a+ix)\Gamma(b+ix)|^2\,dx = \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{2\Gamma\left(a+b+\frac{1}{2}\right)} \qquad [a > 0, \quad b > 0].$$

ET II 302(27)

2. $$\int_0^{\infty} \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2 dx = \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-a-\frac{1}{2}\right)}{2\Gamma(b)\Gamma\left(b-\frac{1}{2}\right)\Gamma(b-a)} \qquad \left[0 < a < b-\frac{1}{2}\right].$$

ET II 302(28)

6.414

ET II 297(4)

2. $$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad [\operatorname{Re}(\alpha+\beta) > 1].$$

ET II 297(5)

3. $$\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}\,dx = 0 \qquad [\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im}\gamma, \quad \operatorname{Im}\delta > 0].$$

ET II 299(18)

4. $$\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}\,dx = \frac{\pm 2\pi^2 i\Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)]\Gamma(\alpha-\gamma)\Gamma(\alpha-\delta)\Gamma(\beta-\gamma)\Gamma(\beta-\delta)}$$

[$\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1$, $\operatorname{Im}\gamma$, $\operatorname{Im}\delta < 0$. In the numerator, we take the plus sign of $\operatorname{Im}\gamma > \operatorname{Im}\delta$ and the minus sign if $\operatorname{Im}\gamma < \operatorname{Im}\delta$.]

5. $$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha-\beta-\gamma+x+1)\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)} = \frac{\pi \exp\left[\pm\frac{1}{2}\pi(\delta-\gamma)i\right]}{\Gamma(\beta+\gamma-1)\Gamma\left[\frac{1}{2}(\alpha+\beta)\right]\,\Gamma\left[\frac{1}{2}(\gamma-\delta+1)\right]}$$

ET II 300(19)

[$\operatorname{Re}(\beta+\gamma) > 1, \delta = \alpha-\beta-\gamma+1, \operatorname{Im}\delta \neq 0$. The sign is plus in the argument if the exponential for $\operatorname{Im}\delta > 0$ and minus for $\operatorname{Im}\delta < 0$].

6. $$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}$$
$$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3].$$

ET II 300(21)
ET II 300(20)

ET II 301(24)

2. $$\int_{-\infty}^{\infty} \frac{R(x)\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\int_0^1 R(t)\cos\left[\frac{1}{2}\pi(2t+\alpha-\beta)\right]\,dt}{\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$$[\alpha+\delta=\beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta)>2, \quad R(x+1)=-R(x)].$$

ET II 301(25)

6.42 Combinations of the gamma function, the exponential, and powers

6.421

1. $$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)\exp[2(\pi n+\theta)xi]\,dx = 2\pi i\Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta}\exp[(\beta-\alpha)i\theta] \times$$
$$\times[\eta_n(\beta)\exp(2n\pi\beta i)-\eta_n(-\alpha)\exp(-2n\pi\alpha i)]$$
$$\left[\operatorname{Re}(\alpha+\beta)<1; \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}; \quad n - \text{an integer}; \quad \eta_n(\zeta)=0,\right.$$
$$\left.\text{if} \quad \left(\frac{1}{2}-n\right)\operatorname{Im}\zeta>0, \quad \eta_n(\zeta)=\operatorname{sign}\left(\frac{1}{2}-n\right), \quad \text{if} \quad \left(\frac{1}{2}-n\right)\operatorname{Im}\zeta<0\right].$$

ET II 298(7)

2. $$\int_{-\infty}^{\infty} \frac{e^{\pi icx}\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+kx)\Gamma(\delta-kx)} = 0$$
$$[\operatorname{Re}(\alpha+\beta+\gamma+\delta)>2, \quad c, \quad k - \text{are real:} \quad |c|>|k|+1].$$

ET II 301(26)

3. $$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx = =$$
$$= 2\pi i\operatorname{sign}\left(n+\frac{1}{2}\right)\frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\exp[-(2\pi n+\pi-\theta)\alpha i+\theta i(\beta-1)]$$
$$\left[\operatorname{Re}(\beta-\alpha)>0, \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}, \quad n - \text{an integer}, \quad \left(n+\frac{1}{2}\right)\operatorname{Im}\alpha<0\right].$$

ET II 298(8)

4. $$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx = 0$$
$$\left[\operatorname{Re}(\beta-\alpha)>0, \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}, \quad n - \text{an integer}, \quad \left(n+\frac{1}{2}\right)\operatorname{Im}\alpha>0\right].$$

1. $$\int_{-i\infty}^{i\infty} \Gamma(s-k-\lambda)\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\frac{1}{2}\right) z^s\,ds =$$
$$= 2\pi i\Gamma\left(\frac{1}{2}-k-\mu\right)\Gamma\left(\frac{1}{2}-k+\mu\right) z^{\lambda} e^{\frac{z}{2}} W_{k,\,\mu}(z)$$
$$\left[\operatorname{Re}(k+\lambda)<0, \quad \operatorname{Re}\lambda > |\operatorname{Re}\mu| - \frac{1}{2}, \quad |\arg z| < \frac{3\pi}{2}\right].$$

ET II 302(29)

2. $$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+s)\Gamma(-s)\Gamma(1-c-s)x^s\,ds = 2\pi i\Gamma(\alpha)\Gamma(\alpha-c+1)\Psi(\alpha,\,c;\,x)$$
$$\left[-\operatorname{Re}\alpha<\gamma<\min(0,\,1-\operatorname{Re}c), \quad -\frac{3\pi}{2}<\arg x<\frac{3\pi}{2}\right].$$

EH I 256(5)

3. $$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s)\Gamma(\beta+s)t^s\,ds = 2\pi i\Gamma(\beta)(1+t)^{-\beta} \qquad [0>\gamma>\operatorname{Re}(1-\beta), \quad |\arg t|<\pi].$$

EH I 256, BU 75

4. $$\int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right)\Gamma(-t)(\sqrt{2})^{t-p-2} z^t\,dt = 2\pi i e^{\frac{1}{4}z^2}\Gamma(-p)D_p(z)$$
$$\left[|\arg z|<\frac{3}{4}\pi; \quad p - \text{not a positive integer}\right].$$

WH

5. $$\int_{-i\infty}^{i\infty} \Gamma(s)\Gamma\left(\frac{1}{2}\nu+\frac{1}{4}-s\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{4}-s\right)\left(\frac{z^2}{2}\right)^s ds = =$$
$$= 2\pi i\cdot 2^{\frac{1}{4}-\frac{1}{2}\nu} z^{-\frac{1}{2}} e^{\frac{3}{4}z^2}\Gamma\left(\frac{1}{2}\nu+\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{4}\right)D_\nu(z)$$
$$\left[|\arg z|<\frac{3}{4}\pi, \quad \nu\neq\frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2},\dots\right].$$

EH II 120

6.[3] $$\int_{c-i\infty}^{c+i\infty} \left(\frac{1}{2}x\right)^{-s}\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}s\right)\left[\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}s\right)\right]^{-1} ds = 4\pi i J_\nu(x) \quad [x>0,\ -\operatorname{Re}\nu<c<1].$$

EH II 21(34)

7. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\Gamma(-s)\left(-\frac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi} H_\nu^{(1)}(z)$$
$$\left[|\arg(-iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re}\nu < c\right].$$

EH II 83(34)

8. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\Gamma(-s)\left(\frac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi} H_\nu^{(2)}(z)$$
$$\left[|\arg(iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re}\nu < c\right].$$

EH II 83(35)

9. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)} ds = 2\pi i J_\nu(x) \qquad [x > 0, \quad \operatorname{Re}\nu > 0].$$

EH II 83(36)

10. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\Gamma(-2\nu-s)\Gamma\left(\nu+s+\frac{1}{2}\right)(-2iz)^s ds = -\pi^{\frac{5}{2}} e^{-i(z-\nu\pi)} \sec(\nu\pi)(2z)^{-\nu} H_\nu^{(1)}(z)$$
$$\left[|\arg(-iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3 \ldots\right].$$

EH II 83(37)

11. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\Gamma(-2\nu-s)\Gamma\left(\nu+s+\frac{1}{2}\right)(2iz)^s ds = \pi^{\frac{5}{2}} e^{i(z-\nu\pi)} \sec(\nu\pi)(2z)^{-\nu} H_\nu^{(2)}(z)$$
$$\left[|\arg(iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3 \ldots\right].$$

EH II 84(38)

12. $$\int_{-i\infty}^{i\infty} \Gamma(s)\Gamma\left(\frac{1}{2}-s-\nu\right)\Gamma\left(\frac{1}{2}-s+\nu\right)(2z)^s ds = 2^{\frac{3}{2}}\pi^{\frac{3}{2}} i z^{\frac{1}{2}} e^z \sec(\nu\pi) K_\nu(z)$$
$$\left[|\arg z| < \frac{3\pi}{2}, \quad 2\nu \neq \pm 1, \quad \pm 3, \ldots\right].$$

13. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s\Gamma(1+s)} x^{2s}\, ds = 4\pi \int_{2x}^{\infty} \frac{J_0(t)}{t}\, dt \qquad [x > 0].$$

MO 41

14. $$\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s\, ds = 2\pi i \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha,\, \beta;\, \gamma;\, z)$$

[For $\arg(-z) < \pi$, the path of integration must separate the poles of the integrand at the points $s = 0, 1, 2, 3, \ldots$ from the poles $s = -\alpha - n$ and $s = -\beta - n$ (for $n = 0, 1, 2, \ldots$)].

15. $$\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s\, ds = \frac{2\pi i \Gamma(\alpha)}{\Gamma(\gamma)}\, {}_1F_1(\alpha;\, \gamma;\, z)$$
$$\left[-\frac{\pi}{2} < \arg(-z) < \frac{\pi}{2}, \quad 0 > \delta > -\operatorname{Re}\alpha, \quad \gamma \neq 0,\, 1,\, 2, \ldots\right].$$

EH I 256(4)
EH I 62(15)

16. $$\int_{-i\infty}^{i\infty} \left[\frac{\Gamma\left(\frac{1}{2} - s\right)}{\Gamma(s)}\right]^2 z^s\, ds = 2\pi i z^{\frac{1}{2}} [2\pi^{-1} K_0(4z^{\frac{1}{4}}) - N_0(4z^{\frac{1}{4}})] \qquad [z > 0].$$

ET II 303(33)

17. $$\int_{-i\infty}^{i\infty} \frac{\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right) \Gamma\left(\lambda-\mu-s+\frac{1}{2}\right)}{\Gamma(\lambda-k-s+1)} z^s\, ds = 2\pi i z^{\lambda} e^{-\frac{z}{2}} W_{k,\,\mu}(z)$$
$$\left[\operatorname{Re}\lambda > |\operatorname{Re}\mu| - \frac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right].$$

ET II 302(30)

18. $$\int_{-i\infty}^{i\infty} \frac{\Gamma(k-\lambda+s)\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)}{\Gamma\left(\mu-\lambda+s+\frac{1}{2}\right)} z^s\, ds = 2\pi i \frac{\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma(2\mu+1)} z^{\lambda} e^{-\frac{z}{2}} M_{k,\,\mu}(z)$$
$$\left[\operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(\lambda+\mu) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right].$$

19. $$\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s\,ds = 2\pi i G_{pq}^{mn}\left(z\,\middle|\,\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$$
$$\left[p + q < 2(m+n); \quad |\arg z| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi; \right.$$
$$\left. \operatorname{Re} a_k < 1, \quad k = 1, \dots, n; \quad \operatorname{Re} b_j > 0, \quad j = 1, \dots, m\right].$$

ET II 303(34)

6.423

1. $$\int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(1+x)} = \nu(e^{-\alpha}).$$

MI 39, EH III 222(16)

2. $$\int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(x+\beta+1)} = e^{\beta\alpha}\nu(e^{-\alpha},\, \beta).$$

MI 39, EH III 222(16)

3. $$\int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+1)}\,dx = \mu(e^{-\alpha},\, m)\Gamma(m+1) \qquad [\operatorname{Re} m > -1].$$

MI 39, EH III 222(17)

4. $$\int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+n+1)}\,dx = e^{n\alpha}\mu(e^{-\alpha},\, m,\, n)\Gamma(m+1).$$

MI 39, EH III 222(17)

6.424

$$\int_{-\infty}^{\infty} \frac{R(x)\exp[(2\pi n + \theta)xi]\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{\left[2\cos\left(\frac{\theta}{2}\right)\right]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \exp\left[\frac{1}{2}\theta(\beta-\alpha)i\right] \int_0^1 R(t)\exp(2\pi n t i)\,dt$$
$$[\operatorname{Re}(\alpha+\beta) > 1, \quad -\pi < \theta < \pi, \quad n - \text{an integer}, \quad R(x+1) = R(x)].$$

ET II 299(16)

6.43 Combinations of the gamma function and trigonometric functions

6.431

1. $$\int_{-\infty}^{\infty} \frac{\sin rx\,dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}}{\Gamma(p+q-1)} \sin\frac{r(q-p)}{2} \quad [|r| < \pi];$$
$$= 0 \qquad [|r| > \pi]; \qquad [r - \text{real}; \quad \operatorname{Re}(p+q) > 1].$$

2. $$\int_{-\infty}^{\infty} \frac{\cos rx\, dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\cos\frac{r(q-p)}{2}}{\Gamma(p+q-1)} \qquad [|r| < \pi];$$
$$= 0 \qquad [|r| > \pi]; \qquad [r - \text{real}; \quad \operatorname{Re}(p+q) > 1].$$

MO 10a, ET II 299(13, 14)

6.432

$$\int_{-\infty}^{\infty} \frac{\sin(m\pi x)}{\sin(\pi x)} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = 0 \qquad [m-\text{ an even integer}];$$
$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad [m-\text{ an odd integer}]$$
$$[\operatorname{Re}(\alpha+\beta) > 1].$$

ET II 298(11, 12)

6.433

1. $$\int_{-\infty}^{\infty} \frac{\sin \pi x\, dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\sin\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$$[\alpha+\delta = \beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2].$$

ET II 300(22)

2. $$\int_{-\infty}^{\infty} \frac{\cos \pi x\, dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\cos\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$$[\alpha+\delta = \beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2].$$

ET II 301(23)

6.44 The logarithm of the gamma function Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.*

6.441

1. $$\int_{p}^{p+1} \ln \Gamma(x)\, dx = \frac{1}{2}\ln 2\pi + p\ln p - p.$$

2. $$\int_0^1 \ln\Gamma(x)\,dx = \int_0^1 \ln\Gamma(1-x)\,dx = \frac{1}{2}\ln 2\pi.$$

FI II 783

3. $$\int_0^1 \ln\Gamma(x+q)\,dx = \frac{1}{2}\ln 2\pi + q\ln q - q \qquad [q \geq 0].$$

NH 89(17), ET II 304(40)

4. $$\int_0^z \ln\Gamma(x+1)\,dx = \frac{z}{2}\ln 2\pi - \frac{z(z+1)}{2} + z\ln\Gamma(z+1) - \ln G(z+1),$$

where $$G(z+1) = (2\pi)^{\frac{z}{2}}\exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right)\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z+\frac{z^2}{2k}\right)\right\}$$

WH

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5. $$\int_0^n \ln\Gamma(\alpha+x)\,dx = \sum_{k=0}^{n-1}(a+k)\ln(a+k) - na + \frac{1}{2}n\ln(2\pi) - \frac{1}{2}n(n-1)$$
$$[a \geq 0; \quad n = 1, 2, \dots].$$

ET II 304(41)

6.442

$$\int_0^1 \exp(2\pi nxi)\ln\Gamma(a+x)\,dx = (2\pi ni)^{-1}[\ln a - \exp(-2\pi nai)\,\mathrm{Ei}(2\pi nai)]$$
$$[a > 0; \quad n = \pm 1, \pm 2, \dots].$$

ET II 304(38)

6.443

1. $$\int_0^1 \ln\Gamma(x)\sin 2\pi nx\,dx = \frac{1}{2\pi n}[\ln(2\pi n) + \boldsymbol{C}].$$

NH 203(5), ET II 304(42)

2. $$\int_0^1 \ln\Gamma(x)\sin(2n+1)\,\pi x\,dx = \frac{1}{(2n+1)\pi}\left[\ln\left(\frac{\pi}{2}\right) + 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) + \frac{1}{2n+1}\right].$$

ET II 305(43)

3. $$\int_0^1 \ln\Gamma(x)\cos 2\pi nx\,dx = \frac{1}{4n}.$$

NH 203(6), ET II 305(44)

4.[7] $$\int_0^1 \ln\Gamma(x)\cos(2n+1)\pi x\,dx = \frac{2}{\pi^2}\left[\frac{1}{(2n+1)^2}(\boldsymbol{C}+\ln 2\pi) + 2\sum_{k=2}^{\infty}\frac{\ln k}{4k^2-(2n+1)^2}\right].$$

5. $$\int_0^1 \sin(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}[\ln a + \cos(2\pi na)\ \mathrm{ci}(2\pi na) - \sin(2\pi na)\ \mathrm{si}(2\pi na)]$$
$$[a>0;\quad n=1,\,2,\ldots].$$

ET II 304(36)

6. $$\int_0^1 \cos(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}[\sin(2\pi na)\ \mathrm{ci}(2\pi na) + \cos(2\pi na)\ \mathrm{si}(2\pi na)]$$
$$[a>0;\quad n=1,\,2,\ldots].$$

ET II 304(37)

6.45 The incomplete gamma function

6.451

1. $$\int_0^\infty e^{-\alpha x}\gamma(\beta,\,x)\,dx = \frac{1}{\alpha}\Gamma(\beta)(1+\alpha)^{-\beta} \qquad [\beta>0].$$

MI 39

2. $$\int_0^\infty e^{-\alpha x}\Gamma(\beta,\,x)\,dx = \frac{1}{\alpha}\Gamma(\beta)\left[1-\frac{1}{(\alpha+1)^\beta}\right] \qquad [\beta>0].$$

MI 39

6.452

1. $$\int_0^\infty e^{-\mu x}\gamma\left(\nu,\,\frac{x^2}{8a^2}\right)\,dx = \frac{1}{\mu}2^{-\nu-1}\Gamma(2\nu)e^{(a\mu)^2}D_{-2\nu}(2a\mu)$$
$$\left[|\arg a|<\frac{\pi}{4},\quad \mathrm{Re}\,\nu>-\frac{1}{2},\quad \mathrm{Re}\,\mu>0\right].$$

2. $$\int_0^\infty e^{-\mu x}\gamma\left(\frac{1}{4},\,\frac{x^2}{8a^2}\right)\,dx = \frac{2^{\frac{3}{4}}\sqrt{a}}{\sqrt{\mu}}e^{(a\mu)^2}K_{\frac{1}{4}}(a^2\mu^2) \qquad \left[|\arg a| < \frac{\pi}{4},\quad \operatorname{Re}\mu > 0\right].$$

ET I 179(35)

6.453

$$\int_0^\infty e^{-\mu x}\Gamma\left(\nu,\,\frac{a}{x}\right)\,dx = 2a^{\frac{1}{2}\nu}\mu^{\frac{1}{2}\nu-1}K_\nu(2\sqrt{\mu a}) \qquad \left[|\arg a| < \frac{\pi}{2},\quad \operatorname{Re}\mu > 0\right].$$

ET I 179(32)

6.454

$$\int_0^\infty e^{-\beta x}\gamma(\nu,\,\alpha x^{\frac{1}{2}})\,dx = 2^{-\frac{1}{2}\nu}\alpha^\nu\beta^{-\frac{1}{2}\nu-1}\Gamma(\nu)\exp\left(\frac{\alpha^2}{8\beta}\right)D_{-\nu}\left(\frac{\alpha}{\sqrt{2\beta}}\right) \qquad [\operatorname{Re}\beta > 0,\quad \operatorname{Re}\nu > 0].$$

ET II 309(19), MI 39a

6.455

1. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\Gamma(\nu,\,\alpha x)\,dx = \frac{\alpha^\nu\Gamma(\mu+\nu)}{\mu(\alpha+\beta)^{\mu+\nu}}\,{}_2F_1\left(1,\,\mu+\nu;\,\mu+1;\,\frac{\beta}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta) > 0,\quad \operatorname{Re}\mu > 0,\quad \operatorname{Re}(\mu+\nu) > 0].$$

ET II 309(16)

2. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\gamma(\nu,\,\alpha x)\,dx = \frac{\alpha^\nu\Gamma(\mu+\nu)}{\nu(\alpha+\beta)^{\mu+\nu}}\,{}_2F_1\left(1,\,\mu+\nu;\,\nu+1;\,\frac{\alpha}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta) > 0,\quad \operatorname{Re}\beta > 0,\quad \operatorname{Re}(\mu+\nu) > 0].$$

(15)

6.456

1. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\gamma\left(\nu,\,\frac{1}{4x}\right)\,dx = \sqrt{\pi}\frac{\gamma(2\nu,\,\sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39a

2. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\Gamma\left(\nu,\,\frac{1}{4x}\right)\,dx = \frac{\sqrt{\pi}\Gamma(2\nu,\,\sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39a

6.457

1. $$\int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \gamma\left(\nu+1,\ \frac{1}{4x}\right)\, dx = \sqrt{\pi} \frac{\gamma(2\nu+1,\ \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39

2. $$\int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \Gamma\left(\nu+1,\ \frac{1}{4x}\right)\, dx = \sqrt{\pi} \frac{\Gamma(2\nu+1,\ \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39

6.458

$$\int_0^\infty x^{1-2\nu} \exp(\alpha x^2) \sin(bx) \Gamma(\nu,\ \alpha x^2)\, dx = \pi^{\frac{1}{2}} 2^{-\nu} \alpha^{\nu-1} \Gamma\left(\frac{3}{2}-\nu\right) \exp\left(\frac{b^2}{8\alpha}\right) D_{2\nu-2}\left[\frac{b}{(2\alpha)^{\frac{1}{2}}}\right]$$
$$\left[|\arg \alpha| < \frac{3\pi}{2}, \quad 0 < \operatorname{Re} \nu < 1\right].$$

ET II 309(18)

6.46-6.47 The function ψ(x)

6.461

$$\int_1^x \psi(x)\, dx = \ln \Gamma(x).$$

6.462

$$\int_0^1 \psi(\alpha + x)\, dx = \ln \alpha \qquad [\alpha > 0].$$

ET II 305(1)

6.463

$$\int_0^\infty x^{-\alpha} [\boldsymbol{C} + \psi(1+x)] = -\pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \qquad [1 < \operatorname{Re} \alpha < 2].$$

ET II 305(6)

6.464

ET II 305(2)

6.465

1.[7] $$\int_0^1 \psi(x) \sin \pi x \, dx = -\frac{2}{\pi} \left[\boldsymbol{C} + \ln 2\pi + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - 1} \right].$$

NH 204

2. $$\int_0^1 \psi(x) \sin(2\pi n x) \, dx = -\frac{1}{2}\pi \qquad [n = 1, 2, \dots].$$

ET II 305(3)

6.466

$$\int_0^\infty [\psi(\alpha + ix) - \psi(\alpha - ix)] \sin xy \, dx = i\pi e^{-\alpha y}(1 - e^{-y})^{-1} \qquad [\alpha > 0, \quad y > 0].$$

ET I 96(1)

6.467

1. $$\int_0^1 \sin(2\pi n x)\psi(\alpha + x) \, dx = \sin(2\pi n \alpha) \ \operatorname{ci}(2\pi n \alpha) + \cos(2\pi n \alpha) \ \operatorname{si}(2\pi n \alpha) \qquad [\alpha \geq 0; \quad n = 1, 2, \dots].$$

ET II 305(4)

2. $$\int_0^1 \cos(2\pi n x)\psi(\alpha + x) \, dx = \sin(2\pi n \alpha) \ \operatorname{si}(2\pi n \alpha) - \cos(2\pi n \alpha) \ \operatorname{ci}(2\pi n \alpha) \qquad [\alpha > 0; \quad n = 1, 2, \dots].$$

ET II 305(5)

6.468

$$\int_0^1 \psi(x) \sin^2 \pi x \, dx = -\frac{1}{2}[\boldsymbol{C} + \ln(2\pi)].$$

NH 204

6.469

1. $$\int_0^1 \psi(x) \sin \pi x \cos \pi x \, dx = -\frac{\pi}{4}.$$

2.[7] $$\int_0^1 \psi(x) \sin \pi x \sin(n\pi x)\, dx = \frac{n}{1-n^2} \qquad [n - \text{even}];$$
$$= \frac{1}{2} \ln \frac{n-1}{n+1} \qquad [n - \text{odd}].$$

NH 204(8)a

6.471

1. $$\int_0^\infty x^{-\alpha} [\ln x - \psi(1+x)]\, dx = \pi \operatorname{cosec}(\pi\alpha)\zeta(\alpha) \qquad [0 < \operatorname{Re}\alpha < 1].$$

ET II 306(7)

2. $$\int_0^\infty x^{-\alpha} [\ln(1+x) - \psi(1+x)]\, dx = \pi \operatorname{cosec}(\pi\alpha)[\zeta(\alpha) - (\alpha-1)^{-1}] \qquad [0 < \operatorname{Re}\alpha < 1].$$

ET II 306(8)

3. $$\int_0^\infty [\psi(x+1) - \ln x] \cos(2\pi x y)\, dx = \frac{1}{2}[\psi(y+1) - \ln y].$$

ET II 306(12)

6.472

1. $$\int_0^\infty x^{-\alpha} [(1+x)^{-1} - \psi'(1+x)]\, dx = -\pi\alpha \operatorname{cosec}(\pi\alpha)[\zeta(1+\alpha) - \alpha^{-1}] \qquad [|\operatorname{Re}\alpha| < 1].$$

ET II 306(9)

2. $$\int_0^\infty x^{-\alpha} [x^{-1} - \psi'(1+x)]\, dx = -\pi\alpha \operatorname{cosec}(\pi\alpha)\zeta(1+\alpha) \qquad [-2 < \operatorname{Re}\alpha < 0].$$

ET II 306(10)

6.473

$$\int_0^\infty x^{-\alpha} \psi^{(n)}(1+x)\, dx = (-1)^{n-1} \frac{\pi\Gamma(\alpha+n)}{\Gamma(\alpha)\sin\pi\alpha} \zeta(\alpha+n) \qquad [n = 1, 2, \dots; \quad 0 < \operatorname{Re}\alpha < 1].$$

ET II 306(11)

6.5- 6.7 Bessel Functions

6.51 Bessel functions

6.511

1. $$\int_0^\infty J_\nu(bx)\,dx = \frac{1}{b} \qquad [\operatorname{Re}\nu > -1, \quad b > 0].$$

ET II 22(3)

2. $$\int_0^\infty N_\nu(bx)\,dx = -\frac{1}{b}\,\operatorname{tg}\left(\frac{\nu\pi}{2}\right) \qquad [|\operatorname{Re}\nu| < 1, \quad b > 0].$$

WA 432(7), ET II 96(1)

3. $$\int_0^a J_\nu(x)\,dx = 2\sum_{k=0}^\infty J_{\nu+2k+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 333(1)

4. $$\int_0^a J_{\frac{1}{2}}(t)\,dt = 2S(\sqrt{a}).$$

WA 599(4)

5. $$\int_0^a J_{-\frac{1}{2}}(t)\,dt = 2C(\sqrt{a}).$$

WA 599(3)

6. $$\int_0^a J_0(x)\,dx = aJ_0(a) + \frac{\pi a}{2}[J_1(a)H_0(a) - J_0(a)H_1(a)] \qquad [a > 0].$$

ET II 7(2)

7. $$\int_0^a J_1(x)\,dx = 1 - J_0(a) \qquad [a > 0].$$

8. $$\int_a^\infty J_0(x)\,dx = 1 - aJ_0(a) + \frac{\pi a}{2}[J_0(a)H_1(a) - J_1(a)H_0(a)] \qquad [a > 0].$$

ET II 7(3)

693

9. $$\int_a^\infty J_1(x)\,dx = J_0(a) \qquad [a > 0].$$

ET II 18(2)

10. $$\int_a^b N_\nu(x)\,dx = 2\sum_{n=0}^\infty [N_{\nu+2n+1}(b) - N_{\nu+2n+1}(a)].$$

ET II 339(46)

11. $$\int_0^a I_\nu(x)\,dx = 2\sum_{n=0}^\infty (-1)^n I_{\nu+2n+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 364(1)

6.512

1. $$\int_0^\infty J_\mu(ax)J_\nu(bx)\,dx = b^\nu a^{-\nu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+1}{2}\right)}F\left(\frac{\mu+\nu+1}{2},\ \frac{\nu-\mu+1}{2};\ \nu+1;\ \frac{b^2}{a^2}\right)$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu) > -1, \quad b > a.$$
For $a < b$, the positions of μ and ν should be reversed].

ET II 48(6)

2.[7] $$\int_0^\infty J_{\nu+n}(\alpha t)J_{\nu-n-1}(\beta t)\,dt = \frac{\beta^{\nu-n-1}\Gamma(\nu)}{\alpha^{\nu-n}n!\Gamma(\nu-n)}F\left(\nu,\ -n;\ \nu-n;\ \frac{\beta^2}{\alpha^2}\right) \qquad [0 < \beta < \alpha];$$
$$= (-1)^n\frac{1}{2\alpha} \qquad [0 < \beta = \alpha];$$
$$= 0 \qquad [0 < \alpha < \beta] \qquad [\operatorname{Re}(\nu) > 0, n \geq 0].$$

MO 50

3. $$\left.\begin{aligned}\int_0^\infty J_\nu(ax)J_{\nu-1}(\beta x)\,dx &= \frac{\beta^{\nu-1}}{\alpha^\nu} && [\beta < \alpha];\\ &= \frac{1}{2\beta} && [\beta = \alpha];\\ &= 0 && [\beta > \alpha];\end{aligned}\right\} \qquad [\operatorname{Re}\nu > 0].$$

4. $$\int_0^\infty J_{\nu+2n+1}(ax) J_\nu(bx)\, dx = b^\nu a^{-\nu-1} P_n^{(\nu,\, 0)}\left(1 - \frac{2b^2}{a^2}\right) \qquad [\operatorname{Re}\nu > -1-n, \quad 0 < b < a];$$
$$= 0 \qquad [\operatorname{Re}\nu > -1-n, \quad 0 < a < b].$$

ET II 47(5)

5. $$\int_0^{\infty]} J_{\nu+n}(ax) N_{\nu-n}(ax)\, dx = (-1)^{n+1} \frac{1}{2a} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}; \quad a > 0; \quad n = 0, 1, 2, \dots\right].$$

ET II 347(57)

6. $$\int_0^\infty J_1(bx) N_0(ax)\, dx = -\frac{b^{-1}}{\pi} \ln\left(1 - \frac{b^2}{a^2}\right) \qquad [0 < b < a].$$

ET II 21(31)

7. $$\int_0^a J_\nu(x) J_{\nu+1}(x)\, dx = \sum_{n=0}^\infty [J_{\nu+n+1}(a)]^2 \qquad [\operatorname{Re}\nu > -1].$$

ET II 338(37)

6.513

1. $$\int_0^\infty [J_\mu(ax)]^2 J_\nu(bx)\, dx = a^{2\mu} b^{-2\mu-1} \frac{\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{[\Gamma(\mu+1)]^2 \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \times$$
$$\times \left[F\left(\frac{1-\nu+2\mu}{2}, \frac{1+\nu+2\mu}{2}; \mu+1; \frac{1 - \sqrt{1 - \frac{4a^2}{b^2}}}{2}\right)\right]^2$$
$$[\operatorname{Re}\nu + \operatorname{Re}2\mu > -1, \quad 0 < 2a < b].$$

ET II 52(33)

2. $$\int_0^\infty [J_\mu(ax)]^2 K_\nu(bx)\, dx = \frac{b^{-1}}{2} \Gamma\left(\frac{2\mu+\nu+1}{2}\right) \Gamma\left(\frac{2\mu-\nu+1}{2}\right) \left[P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1 + \frac{4a^2}{b^2}}\right)\right]^2$$
$$[2\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1, \quad \operatorname{Re}b > 2|\operatorname{Im}a|].$$

ET II 138(18)

ET II 65(20)

4. $$\int_0^\infty J_\mu(ax)J_{-\mu}(ax)K_\nu(bx)\,dx = \frac{\pi}{2b}\sec\left(\frac{\nu\pi}{2}\right)P^{\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)$$
$$[|\operatorname{Re}\nu| < 1, \quad \operatorname{Re} b > 2|\operatorname{Im} a|].$$

ET II 138(21)

5. $$\int_0^\infty [K_\mu(ax)]^2 J_\nu(bx)\,dx = \frac{e^{2\mu\pi i}\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b\Gamma\left(\frac{1+\nu-2\mu}{2}\right)}\left[Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) > -\frac{1}{2}\right].$$

ET II 66(28)

6. $$\int_0^z J_\mu(x)J_\nu(z-x)\,dx = 2\sum_{k=0}^\infty (-1)^k J_{\mu+\nu+2k+1}(z) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$$
[see also **6.683** 3.].

6.683
WA 414(2)

7. $$\int_0^z J_\mu(x)J_{-\mu}(z-x)\,dx = \sin z \qquad [-1 < \operatorname{Re}\mu < 1].$$

WA 415(4)

8. $$\int_0^z J_\mu(x)J_{1-\mu}(z-x)\,dx = J_0(z) - \cos(z) \qquad [-1 < \operatorname{Re}\mu < 2].$$

WA 415(4)

1. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right)J_\nu(bx)\,dx = b^{-1}J_{2\nu}(2\sqrt{ab}) \qquad \left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

2. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) N_\nu(bx)\,dx = b^{-1}\left[N_{2\nu}(2\sqrt{ab}) + \frac{2}{\pi}K_{2\nu}(\sqrt{2ab})\right]$$
$$\left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{3}{2}\right].$$

ET II 110(12)

3. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = b^{-1}e^{\frac{1}{2}i(\nu+1)\pi}K_{2\nu}[2e^{\frac{1}{4}i\pi}\sqrt{ab}]+b^{-1}e^{-\frac{1}{2}i(\nu+1)\pi}K_{2\nu}[2e^{-\frac{1}{4}\pi i}\sqrt{ab}]$$
$$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right].$$

ET II 141(31)

4. $$\int_0^\infty N_\nu\left(\frac{a}{x}\right) J_\nu(bx)\,dx = -\frac{2b^{-1}}{\pi}\left[K_{2\nu}(2\sqrt{ab}) - \frac{\pi}{2}N_{2\nu}(2\sqrt{ab})\right] \qquad \left[a > 0,\ b > 0,\ |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 62(37)a

5. $$\int_0^\infty N_\nu\left(\frac{a}{x}\right) N_\nu(bx)\,dx = -b^{-1}J_{2\nu}(2\sqrt{ab}) \qquad \left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 110(14)

6. $$\int_0^\infty N_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = -b^{-1}e^{\frac{1}{2}\nu\pi i}K_{2\nu}(2e^{\frac{1}{4}\pi i}\sqrt{ab})-b^{-1}e^{-\frac{1}{2}\nu\pi i}K_{2\nu}(2e^{-\frac{1}{4}\pi i}\sqrt{ab})$$
$$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right].$$

ET II 143(37)

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) N_\nu(bx)\,dx = -2b^{-1}\left[\sin\left(\frac{3\nu\pi}{2}\right)\operatorname{ker}_{2\nu}\left(2\sqrt{ab}\right) + \cos\left(\frac{3\nu\pi}{2}\right)\operatorname{kei}_{2\nu}\left(2\sqrt{ab}\right)\right]$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 113(28)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \pi b^{-1}K_{2\nu}(2\sqrt{ab}) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 146(54)

6.515

1. $$\int_0^\infty J_\mu\left(\frac{a}{x}\right) N_\mu\left(\frac{a}{x}\right) K_0(bx)\,dx = -2b^{-1} J_{2\mu}(2\sqrt{ab}) K_{2\mu}(2\sqrt{ab}) \qquad [a>0, \quad \operatorname{Re} b > 0].$$

ET II 143(42)

2. $$\int_0^\infty \left[K_\mu\left(\frac{a}{x}\right)\right]^2 K_0(bx)\,dx = 2\pi b^{-1} K_{2\mu}(2e^{\frac{1}{4}\pi i}\sqrt{ab}) K_{2\mu}(2e^{-\frac{1}{4}\pi i}\sqrt{ab}) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 147(59)

3. $$\int_0^\infty H_\mu^{(1)}\left(\frac{a^2}{x}\right) H_\mu^{(2)}\left(\frac{a^2}{x}\right) J_0(bx)\,dx = 16\pi^{-2} b^{-1} \cos\mu\pi K_{2\mu}(2e^{\pi i/4} a\sqrt{b}) K_{2\mu}(2e^{-\pi i/4} a\sqrt{b})$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad b>0, \quad |\operatorname{Re}\mu| < \frac{1}{4}\right].$$

ET II 17(36)

6.516

1. $$\int_0^\infty J_{2\nu}(a\sqrt{x}) J_\nu(bx)\,dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a>0, \quad b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 58(16)

2. $$\int_0^\infty J_{2\nu}(a\sqrt{x}) N_\nu(bx)\,dx = -b^{-1} \mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a>0, \quad b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 111(18)

3. $$\int_0^\infty J_{2\nu}(a\sqrt{x}) K_\nu(bx)\,dx = \frac{\pi}{2} b^{-1} \left[I_\nu\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right] \qquad \left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 144(45)

4. $$\int_0^\infty N_{2\nu}(a\sqrt{x}) J_\nu(bx)\,dx = 2\sec(\nu\pi) b^{-1} \left[\frac{1}{2}\cos(\nu\pi) N_\nu\left(\frac{a^2}{4b}\right) - N_{-\nu}\left(\frac{a^2}{4b}\right) + \mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right)\right]$$
$$\left[a>0, \quad b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

5. $$\int_0^\infty N_{2\nu}(a\sqrt{x})N_\nu(bx)\,dx =$$
$$= \frac{b^{-1}}{2}\left[\sec(\nu\pi)J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}\,(\nu\pi)\,H_{-\nu}\left(\frac{a^2}{4b}\right) - 2\operatorname{ctg}\,(2\nu\pi)\,H_\nu\left(\frac{a^2}{4b}\right)\right]$$
$$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 111(19)

6. $$\int_0^\infty N_{2\nu}(a\sqrt{x})K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[\operatorname{cosec}(2\nu\pi)L_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{ctg}\,(2\nu\pi)\,L_\nu\left(\frac{a^2}{4b}\right) - \right.$$
$$\left. - \operatorname{tg}(\nu\pi)I_\nu\left(\frac{a^2}{4b}\right) - \frac{\sec\,(\nu\pi)}{\pi}K_\nu\left(\frac{a^2}{4b}\right)\right]$$
$$\left[\operatorname{Re}b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 144(46)

7. $$\int_0^\infty K_{2\nu}(a\sqrt{x})J_\nu(bx)\,dx = \frac{1}{4}\pi b^{-1}\sec(\nu\pi)\left[H_{-\nu}\left(\frac{a^2}{4b}\right) - N_{-\nu}\left(\frac{a^2}{4b}\right)\right]$$
$$\left[\operatorname{Re}a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 70(22)

8. $$\int_0^\infty K_{2\nu}(a\sqrt{x})N_\nu(bx)\,dx =$$
$$= -\frac{1}{4}\pi b^{-1}\left[\sec(\nu\pi)J_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{cosec}\,(\nu\pi)\,H_{-\nu}\left(\frac{a^2}{4b}\right) + 2\operatorname{cosec}\,(2\nu\pi)\,H_\nu\left(\frac{a^2}{4b}\right)\right]$$
$$\left[\operatorname{Re}a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 114(34)

9. $$\int_0^\infty K_{2\nu}(a\sqrt{x})K_\nu(bx)\,dx = \frac{\pi b^{-1}}{4\cos(\nu\pi)}\left\{K_\nu\left(\frac{a^2}{4b}\right) + \frac{\pi}{2\sin\,(\nu\pi)}\left[L_{-\nu}\left(\frac{a^2}{4b}\right) - L_\nu\left(\frac{a^2}{4b}\right)\right]\right\}$$
$$\left[\operatorname{Re}b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 147(63)

10. $$\int_0^\infty I_{2\nu}(a\sqrt{x})K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[I_\nu\left(\frac{a^2}{4b}\right) + L_\nu\left(\frac{a^2}{4b}\right)\right] \qquad \left[\operatorname{Re}b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

6.517

$$\int_0^z J_0(\sqrt{z^2 - x^2})\, dx = \sin z.$$

MO 48

6.518

$$\int_0^\infty K_{2\nu}(2z\ \mathrm{sh}\, x)\, dx = \frac{\pi^2}{8\cos\nu\pi}(J_\nu^2(z) + N_\nu^2(z)) \qquad \left[\operatorname{Re} z > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right].$$

MO 45

6.519

1. $$\int_0^{\pi/2} J_{2\nu}(2z\cos x)\, dx = \frac{\pi}{2} J_\nu^2(z) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WH

2. $$\int_0^{\pi/2} J_{2\nu}(2z\sin x)\, dx = \frac{\pi}{2} J_\nu^2(z) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 42(1)a

6.52 Bessel functions combined with x and x^2

6.521

1. $$\int_0^1 x J_\nu(\alpha x) J_\nu(\beta x)\, dx = 0 \qquad [\alpha \neq \beta];$$
$$= \frac{1}{2}\{J_{\nu+1}(\alpha)\}^2 \qquad [\alpha = \beta] \qquad [J_\nu(\alpha) = J_\nu(\beta) = 0, \quad \nu > -1].$$

WH

2. $$\int_0^\infty x K_\nu(\alpha x) J_\nu(bx)\, dx = \frac{b^\nu}{a^\nu(b^2 + a^2)} \qquad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 63(2)

3. $$\int_0^\infty x K_\nu(ax) K_\nu(bx)\, dx = \frac{\pi(ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2\sin(\nu\pi)(a^2 - b^2)} \qquad [|\operatorname{Re}\nu| < 1, \quad \operatorname{Re}(a+b) > 0].$$

4. $$\int_0^a xJ_\nu(\lambda x)K_\nu(\mu x)\,dx = (\mu^2+\lambda^2)^{-1}\left[\left(\frac{\lambda}{\mu}\right)^\nu + \lambda a J_{\nu+1}(\lambda a)\,K_\nu(\mu a) - \mu a J_\nu(\lambda a)\,K_{\nu+1}(\mu a)\right]$$
$$[\operatorname{Re}\nu > -1].$$

ET II 367(26)

6.522

1. $$\int_0^\infty x[J_\mu(ax)]^2 K_\nu(bx)\,dx = \Gamma\left(\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\mu-\frac{1}{2}\nu+1\right)b^{-2}\times$$
$$\times(1+4a^2b^{-2})^{-\frac{1}{2}}P^{-\mu}_{\frac{1}{2}\nu}[(1+4a^2b^{-2})^{\frac{1}{2}}]P^{-\mu}_{\frac{1}{2}\nu-1}[(1+4a^2b^{-2})^{\frac{1}{2}}]$$
$$[\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re}\mu > |\operatorname{Re}\nu| - 2].$$

ET II 138(19)

2. $$\int_0^\infty x[K_\mu(ax)]^2 J_\nu(bx)\,dx = \frac{2e^{2\mu\pi i}\Gamma\left(1+\frac{1}{2}\nu+\mu\right)}{b(4a^2+b^2)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\nu-\mu\right)}\times$$
$$\times Q^{-\mu}_{\frac{1}{2}\nu}[(1+4a^2b^{-2})^{\frac{1}{2}}]Q^{-\mu}_{\frac{1}{2}\nu-1}[(1+4a^2b^{-2})^{\frac{1}{2}}]$$
$$\left[b>0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu\pm\mu\right) > -1\right].$$

ET II 66(27)a

3. $$\int_0^\infty xK_0(ax)J_\nu(bx)J_\nu(cx)\,dx = r_1^{-1}r_2^{-1}(r_2-r_1)^\nu(r_2+r_1)^{-\nu},$$
$$r_1 = [a^2+(b-c)^2]^{\frac{1}{2}}, \qquad r_2 = [a^2+(b+c)^2]^{\frac{1}{2}}$$
$$[c>0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re} a > |\operatorname{Im} b|].$$

ET II 63(6)

4. $$\int_0^\infty xI_0(ax)K_0(bx)J_0(cx)\,dx = (a^4+b^4+c^4-2a^2b^2+2a^2c^2+2b^2c^2)^{-\frac{1}{2}}$$
$$[\operatorname{Re} b > \operatorname{Re} a, \quad c>0].$$

ET II 16(27)

5. $$\int_0^\infty xJ_0(ax)K_0(bx)J_0(cx)\,dx = (a^4+b^4+c^4-2a^2c^2+2a^2b^2+2b^2c^2)^{-\frac{1}{2}}$$
$$[\operatorname{Re} b > |\operatorname{Im} a|, \quad c>0].$$

6. $$\int_0^\infty xJ_0(ax)N_0(ax)J_0(bx)\,dx = 0 \qquad [0 < b < 2a];$$
$$= -2\pi^{-1}b^{-1}[b^2-4a^2]^{-\frac{1}{2}} \qquad [0 < 2a < b < \infty].$$

ET II 15(21)

7. $$\int_0^\infty xJ_\mu(ax)J_{\mu+1}(ax)K_\nu(bx)\,dx = \Gamma\left(\mu+\frac{3+\nu}{2}\right)\Gamma\left(\mu+\frac{3-\nu}{2}\right)b^{-2}(1+4a^2b^{-2})^{-\frac{1}{2}} \times$$
$$\times P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2b^{-2})^{\frac{1}{2}}]P^{-\mu-1}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2b^{-2})^{\frac{1}{2}}]$$
$$[\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re}\mu > |\operatorname{Re}\nu| - 3].$$

ET II 138(20)

8. $$\int_0^\infty xK_{\mu-\frac{1}{2}}(ax)K_{\mu+\frac{1}{2}}(ax)J_\nu(bx)\,dx =$$
$$= -\frac{2e^{2\mu\pi i}\Gamma\left(\frac{1}{2}\nu+\mu+1\right)}{b\Gamma\left(\frac{1}{2}\nu-\mu\right)(b^2+4a^2)^{\frac{1}{2}}}Q^{-\mu+\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2b^{-2})^{\frac{1}{2}}]Q^{-\mu-\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2b^{-2})^{\frac{1}{2}}]$$
$$\left[b>0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1, \quad |\operatorname{Re}\mu| < 1 + \frac{1}{2}\operatorname{Re}\nu\right].$$

ET II 67(29)a

9. $$\int_0^\infty xI_{\frac{1}{2}\nu}(ax)K_{\frac{1}{2}}(ax)J_\nu(bx)\,dx = b^{-1}(b^2+4a^2)^{-\frac{1}{2}} \qquad [b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu > -1].$$

ET II 65(16)

10. $$\int_0^\infty xJ_{\frac{1}{2}\nu}(ax)N_{\frac{1}{2}\nu}(ax)J_\nu(bx)\,dx =$$
$$= 0 \qquad [a>0, \quad \operatorname{Re}\nu > -1; \quad 0<b<2a];$$
$$= -2\pi^{-1}b^{-1}(b^2-4a^2)^{-\frac{1}{2}} \qquad [a>0, \quad \operatorname{Re}\nu>-1, \quad 2a<b<\infty].$$

ET II 55(48)

11. $$\int_0^\infty xJ_{\frac{1}{2}(\nu+n)}(ax)J_{\frac{1}{2}(\nu-n)}(ax)J_\nu(bx)\,dx =$$
$$= 2\pi^{-1}b^{-1}(4a^2-b^2)^{-\frac{1}{2}}T_n\left(\frac{b}{2a}\right) \qquad [a>0, \quad \operatorname{Re}\nu>-1, \quad 0<b<2a];$$
$$= 0 \qquad [a>0, \quad \operatorname{Re}\nu>-1, \quad 2a<b<\infty].$$

12. $$\int_0^\infty xI_{\frac{1}{2}(\nu-\mu)}(ax)K_{\frac{1}{2}(\nu+\mu)}(ax)J_\nu(bx)\,dx = 2^{-\mu}a^{-\mu}b^{-1}(b^2+4a^2)^{-\frac{1}{2}}[b+(b^2+4a^2)^{\frac{1}{2}}]^\mu$$
$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu-\mu) > -2].$$

ET II 66(23)

13. $$\int_0^\infty xJ_\mu(xa\sin\varphi)K_{\nu-\mu}(ax\cos\varphi\cos\psi)J_\nu(xa\sin\psi)\,dx = \frac{(\sin\varphi)^\mu(\sin\psi)^\nu(\cos\varphi)^{\nu-\mu}(\cos\psi)^{\mu-\nu}}{a^2(1-\sin^2\varphi\sin^2\psi)}$$
$$\left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1\right].$$

ET II 64(10)

14.[6] $$\int_0^\infty 7xJ_\mu(xa\sin\varphi\cos\psi)J_{\nu-\mu}(ax)J_\nu(xa\cos\varphi\sin\psi)\,dx =$$
$$= -2\pi^{-1}a^{-2}\sin(\mu\pi)(\sin\varphi)^\mu(\sin\psi)^\nu(\cos\varphi)^{-\nu}(\cos\psi)^{-\mu}[\cos(\varphi+\psi)\cos(\varphi-\psi)]^{-1}$$
$$\left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{1}{2}\pi, \quad \operatorname{Re}\nu > -1\right].$$

ET II 54(39)

6.523

$$\int_0^\infty x[2\pi^{-1}K_0(ax) - N_0(ax)]K_0(bx)\,dx = 2\pi^{-1}[(a^2+b^2)^{-1} + (b^2-a^2)^{-1}]\ln\frac{b}{a}$$
$$[\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a+b) > 0].$$

ET II 145(50)

6.524

1. $$\int_0^\infty xJ_\nu^2(ax)J_\nu(bx)N_\nu(bx)\,dx = 0 \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{2}\right];$$
$$= -(2\pi ab)^{-1} \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 352(14)

2. $$\int_0^\infty x[J_0(ax)K_0(bx)]^2\,dx = \frac{\pi}{8ab} - \frac{1}{4ab}\arcsin\left(\frac{b^2-a^2}{b^2+a^2}\right) \qquad [a > 0, \quad b > 0].$$

ET II 373(9)

6.525

1. $$\int_0^\infty x^2J_1(ax)K_0(bx)J_0(cx)\,dx = 2a(a^2+b^2-c^2)[(a^2+b^2+c^2)^2 - 4a^2c^2]^{-\frac{3}{2}}$$
$$[c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0].$$

2. $$\int_0^\infty x^2 I_0(ax)K_1(bx)J_0(cx)\,dx = 2b(b^2+c^2-a^2)[(a^2+b^2+c^2)^2-4a^2b^2]^{-\frac{3}{2}}.$$

ET II 16(28)

6.526

1. $$\int_0^\infty xJ_{\frac{1}{2}\nu}(ax^2)J_\nu(bx)\,dx = (2a)^{-1}J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\nu>-1].$$

ET II 56(1)

2. $$\int_0^\infty xJ_{\frac{1}{2}\nu}(ax^2)N_\nu(bx)\,dx = (4a)^{-1}\left[N_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \operatorname{tg}\left(\frac{\nu\pi}{2}\right)J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \sec\left(\frac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1].$$

ET II 109(9)

3. $$\int_0^\infty xJ_{\frac{1}{2}\nu}(ax^2)K_\nu(bx)\,dx = \frac{\pi}{8a\cos\left(\frac{\nu\pi}{2}\right)}\left[\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - N_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$$[a>0, \quad \operatorname{Re}b>0, \quad \operatorname{Re}\nu>-1].$$

ET II 140(27)

4. $$\int_0^\infty xN_{\frac{1}{2}\nu}(ax^2)J_\nu(bx)\,dx = -(2a)^{-1}\mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad [a>0, \quad \operatorname{Re}b>0, \quad \operatorname{Re}\nu>-1].$$

ET II 61(35)

5. $$\int_0^\infty xN_{\frac{1}{2}\nu}(ax^2)K_\nu(bx)\,dx =$$
$$= \frac{\pi}{4a\sin(\nu\pi)}\left[\cos\left(\frac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right)J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$$[a>0, \quad \operatorname{Re}b>0, \quad |\operatorname{Re}\nu|<1].$$

ET II 141(28)

6. $$\int_0^\infty xK_{\frac{1}{2}\nu}(ax^2)J_\nu(bx)\,dx = \frac{\pi}{4a}\left[I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right] \qquad [\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-1].$$

ET II 68(9)

ET II 112(25)

8. $$\int_0^\infty xK_{\frac{1}{2}\nu}(ax^2)K_\nu(bx)dx = \frac{\pi}{8a}\left\{\sec\left(\frac{\nu\pi}{2}\right)K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi\operatorname{cosec}(\nu\pi)\left[\mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]\right\}$$
$$[\operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < 1].$$

ET II 146(52)

6.527

1. $$\int_0^\infty x^2 J_{2\nu}(2ax)J_{\nu-\frac{1}{2}}(x^2)\,dx = \frac{1}{2}aJ_{\nu+\frac{1}{2}}(a^2) \qquad \left[a > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 355(33)

2. $$\int_0^\infty x^2 J_{2\nu}(2ax)J_{\nu+\frac{1}{2}}(x^2)\,dx = \frac{1}{2}aJ_{\nu-\frac{1}{2}}(a^2) \qquad [a > 0, \quad \operatorname{Re}\nu > -2].$$

ET II 355(35)

3. $$\int_0^\infty x^2 J_{2\nu}(2ax)N_{\nu+\frac{1}{2}}(x^2)\,dx = -\frac{1}{2}aH_{\nu-\frac{1}{2}}(a^2) \qquad [a > 0, \quad \operatorname{Re}\nu > -2].$$

ET II 355(36)

6.528

$$\int_0^\infty xK_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right)I_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right)J_\nu(bx)\,dx = K_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right)I_{\frac{1}{4}\nu}\left(\frac{b^2}{4}\right) \qquad [b > 0, \quad \nu > -1].$$

MO 183a

6.529

1. $$\int_0^\infty xJ_\nu(2\sqrt{ax})K_\nu(2\sqrt{ax})J_\nu(bx)\,dx = \frac{1}{2}b^{-2}e^{-\frac{2a}{b}} \qquad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 70(23)

2. $$\int_0^a xJ_\lambda(2a)I_\lambda(2x)J_\mu(2\sqrt{a^2-x^2})I_\mu(2\sqrt{a^2-x^2})\,dx =$$
$$= \frac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)} \times$$
$$\times\ {}_1F_4\left(\frac{\lambda+\mu+1}{2};\ \lambda+1,\ \mu+1,\ \lambda+\mu+1,\ \frac{\lambda+\mu+3}{2};\ -a^4\right)$$
$$[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\mu > -1].$$

6.53- 6.54 Combinations of Bessel functions and rational functions

6.531

1. $$\int_0^\infty \frac{N_\nu(bx)}{x+a}\,dx = \frac{\pi}{\sin(\pi\nu)}[E_\nu(ab)+N_\nu(ab)] + 2\operatorname{ctg}(\pi\nu)[J_\nu(ab) - J_\nu(ab)]$$
$$\left[b>0, \quad |\arg a| < \pi, \quad |\operatorname{Re}\nu| < 1, \quad \nu \neq 0, \quad \pm\frac{1}{2}\right].$$

ET II 97(5)

2. $$\int_0^\infty \frac{N_\nu(bx)}{x-a}\,dx = \pi\{\operatorname{ctg}(\nu\pi)[N_\nu(ab)+E_\nu(ab)]+J_\nu(ab)+2[\operatorname{ctg}(\nu\pi)]^2[J_\nu(ab)-J_\nu(ab)]\}$$
$$[b>0, \quad a>0, \quad |\operatorname{Re}\nu >< 1].$$

ET II 98(9)

3. $$\int_0^\infty \frac{K_\nu(bx)}{x+a}\,dx = \frac{\pi^2}{2}[\operatorname{cosec}(\nu\pi)]^2[I_\nu(ab)+I_{-\nu}(ab)-e^{-\frac{1}{2}i\nu\pi}J_\nu(iab)-e^{\frac{1}{2}i\nu\pi}J_{-\nu}(iab)]$$
$$[\operatorname{Re} b>0, \quad |\arg a| < \pi, \quad |\operatorname{Re}\nu| < 1].$$

ET II 128(5)

6.532

1. $$\int_0^\infty \frac{J_\nu(x)}{x^2+a^2}\,dx = \frac{\pi[J_\nu(a) - J_\nu(a)]}{a\sin(\nu\pi)} \qquad [\operatorname{Re} a>0, \quad \operatorname{Re}\nu > -1].$$

ET II 340(2)

2. $$\int_0^\infty \frac{N_\nu(x)}{x^2+a^2}\,dx = \frac{1}{\cos\frac{\nu\pi}{2}}\left[-\frac{\pi}{2a}\operatorname{tg}\left(\frac{\nu\pi}{2}\right)I_\nu(ab) - \frac{1}{a}K_\nu(ab) + \right.$$
$$\left. + \frac{b\sin\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\,{}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2b^2}{4}\right)\right]$$
$$[b>0, \quad \operatorname{Re} a>0, \quad |\operatorname{Re}\nu| < 1].$$

ET II 99(13)

3. $$\int_0^\infty \frac{N_\nu(bx)}{x^2-a^2}\,dx = \frac{\pi}{2a}\left\{J_\nu(ab) + \operatorname{tg}\left(\frac{\nu\pi}{2}\right)\left\{\operatorname{tg}\left(\frac{\nu\pi}{2}\right)[J_\nu(ab) - J_\nu(ab)] - E_\nu(ab) - N_\nu(ab)\right\}\right\}$$
$$[b>0, \quad a>0, \quad |\operatorname{Re}\nu| < 1].$$

4. $$\int_0^\infty \frac{xJ_0(ax)}{x^2+k^2}\,dx = K_0(ak) \qquad [a>0, \quad \operatorname{Re} k > 0].$$

WA 466(5)

5. $$\int_0^\infty \frac{N_0(ax)}{x^2+k^2}\,dx = -\frac{K_0(ak)}{k} \qquad [a>0, \quad \operatorname{Re} k > 0].$$

WA 466(6)

6. $$\int_0^\infty \frac{J_0(ax)}{x^2+k^2}\,dx = \frac{\pi}{2k}[I_0(ak) - L_0(ak)] \qquad [a>0, \quad \operatorname{Re} k > 0].$$

WA 467(7)

6.533

1. $$\int_0^z J_p(x)J_q(z-x)\frac{dx}{x} = \frac{J_{p+q}(z)}{p} \qquad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > -1].$$

WA 415(3)

2. $$\int_0^z \frac{J_p(x)}{x}\,\frac{J_q(z-x)}{z-x}\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\frac{J_{p+q}(z)}{z} \qquad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > 0].$$

WA 415(5)

3. $$\int_0^\infty [J_0(ax)-1]J_1(bx)\frac{dx}{x} = \frac{-b}{4}\left[1+2\ln\frac{a}{b}\right] \qquad [0<b<a];$$
$$= -\frac{a^2}{4b} \qquad [0<a<b].$$

ET II 21(28)a

4. $$\int_0^\infty [1-J_0(ax)]J_0(bx)\frac{dx}{x} = 0 \qquad [0<a<b];$$
$$= \ln\frac{a}{b} \qquad [0<b<a].$$

ET II 14(16)

6.534

$$\int_0^\infty \frac{x^3J_0(x)}{x^4-a^4}\,dx = \frac{1}{2}K_0(a) - \frac{1}{4}\pi N_0(a) \qquad [a>0].$$

6.535

$$\int_0^\infty \frac{x}{x^2+a^2}\,[J_\nu(x)]^2\,dx = I_\nu(a)K_\nu(a) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 342(26)

703

6.536

$$\int_0^\infty \frac{x^3 J_0(bx)}{x^4+a^4}\,dx = \operatorname{ker}(ab) \qquad \left[b > 0, \quad |\arg a| < \frac{1}{4}\pi\right].$$

ET II 8(9), MO 46a

6.537

$$\int_0^\infty \frac{x^2 J_0(bx)}{x^4+a^4}\,dx = -\frac{1}{a^2}\operatorname{kei}(ab) \qquad \left[b > 0, \quad |\arg a| < \frac{\pi}{4}\right].$$

MO 46a

6.538

1. $$\int_0^\infty J_1(ax)J_1(bx)\frac{dx}{x^2} = \frac{a+b}{\pi}\left[E\left(\frac{2i\sqrt{ab}}{|b-a|}\right) - K\left(\frac{2i\sqrt{ab}}{|b-a|}\right)\right] \qquad [a > 0, \quad b > 0].$$

ET II 21(30)

2.[8] $$\int_0^\infty x^{-1} J_{\nu+2n+1}(x)J_{\nu+2m+1}(x)\,dx = 0 \qquad [m \neq n \text{ with } m,\, n \text{ integers}, \quad \nu > -1];$$
$$= (4n+2\nu+2)^{-1} \qquad [m = n, \quad \nu > -1].$$

EH II 64

6.539

1. $$\int_a^b \frac{dx}{x[J_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{N_\nu(b)}{J_\nu(b)} - \frac{N_\nu(a)}{J_\nu(a)}\right] \qquad [J_\nu(x) \neq 0 \quad \text{for } x \in [a,\, b]].$$

ET II 338(41)

2. $$\int_a^b \frac{dx}{x[N_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{J_\nu(a)}{N_\nu(a)} - \frac{J_\nu(b)}{N_\nu(b)}\right] \qquad [N_\nu(x) \neq 0 \quad \text{for } x \in [a,\, b]].$$

3. $$\int_a^b \frac{dx}{xJ_\nu(x)N_\nu(x)} = \frac{\pi}{2}\ln\left[\frac{J_\nu(a)N_\nu(b)}{J_\nu(b)N_\nu(a)}\right].$$

ET II 339(50)

6.541

1. $$\int_0^\infty xJ_\nu(ax)J_\nu(bx)\frac{dx}{x^2+c^2} = I_\nu(bc)K_\nu(ac) \qquad [0<b<a, \quad \operatorname{Re} c>0, \quad \operatorname{Re}\nu>-1];$$
$$= I_\nu(ac)K_\nu(bc) \qquad [0<a<b, \quad \operatorname{Re} c>0, \quad \operatorname{Re}\nu>-1].$$

ET II 49(10)

2.[8] $$\int_0^\infty x^{1-2n}J_\nu(ax)J_\nu(bx)\frac{dx}{x^2+c^2} =$$
$$= \left(-\frac{1}{c^2}\right)^n\left[I_\nu(bc)K_\nu(ac) - \frac{1}{2}\left(\frac{b}{a}\right)^\nu \frac{\pi}{\sin(\pi\nu)}\sum_{p=0}^{n-1}\frac{(a^2c^2/4)^p}{p!\Gamma(1-\nu+p)}\sum_{k=0}^{n-1-p}\frac{(b^2c^2/4)^k}{k!\Gamma(1+\nu+k)}\right],$$
$$[0<b<a]$$
$$= \left(-\frac{1}{c^2}\right)^n\left[I_\nu(bc)K_\nu(ac) - \frac{1}{2\nu}\left(\frac{b}{a}\right)^\nu \sum_{p=0}^{n-1}\frac{(a^2c^2/4)^p}{p!(1-\nu)_p}\sum_{k=0}^{n-1-p}\frac{(b^2c^2/4)^k}{k!(1+\nu)_k}\right],$$
$$[n=1,\,2,\dots, \quad \operatorname{Re}\nu>n-1, \quad \operatorname{Re}c>0, \quad 0<b<a].$$

3.[7] $$\int_0^\infty \frac{x^{\alpha-1}}{(x^2+z^2)^\rho}J_\mu(cx)J_\nu(cx)\,dx = \frac{1}{2}\left(\frac{c}{2}\right)^{2\rho-\alpha}\times$$
$$\times\Gamma\left[\begin{matrix}(\mu+\nu+\alpha)/2-\rho,\ 1+2\rho-\alpha\\ (\mu-\nu-\alpha)/2+\rho+1,\ (\mu+\nu-\alpha)/2+\rho+1,\ (\nu-\mu-\alpha)/2+\rho+1\end{matrix}\right]\times$$
$$\times\,{}_3F_4\left(\frac{1-\alpha}{2}+\rho,\ 1-\frac{\alpha}{2}+\rho,\ \rho;\ \rho+1-\frac{\mu+\nu+\alpha}{2},\ \rho+1+\frac{\mu-\nu-\alpha}{2},\right.$$
$$\left.\rho+1+\frac{\mu+\nu-\alpha}{2},\ \rho+1+\frac{\nu-\mu-\alpha}{2};\ c^2z^2\right)+\frac{z^{\alpha-2\rho}}{2}\left(\frac{cz}{2}\right)^{\mu+\nu}\times$$
$$\times\Gamma\left[\begin{matrix}\rho-(\alpha+\mu+\nu)/2,\ (\alpha+\mu+\nu)/2\\ \rho,\ \mu+1,\ \nu+1\end{matrix}\right]{}_3F_4\left(\frac{1+\mu+\nu}{2},\ 1+\frac{\mu+\nu}{2},\right.$$
$$\left.\frac{\alpha+\mu+\nu}{2};\ 1-\rho+\frac{\alpha+\mu+\nu}{2},\ \mu+1,\ \nu+1,\ \mu+\nu+1;\ c^2z^2\right)$$
$$\left[\Gamma\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right]=\frac{\Gamma(a_1)\dots\Gamma(a_p)}{\Gamma(b_1)\dots\Gamma(b_q)}\right]c, \quad \operatorname{Re} z, \quad \operatorname{Re}(\alpha+\mu+\nu)>0; \quad \operatorname{Re}(\alpha-2p)<1\right].$$

6.542

$$\int_0^\infty \frac{J_\nu(ax)N_\nu(bx)-J_\nu(bx)N_\nu(ax)}{x\left\{[J_\nu(bx)]^2+[N_\nu(bx)]^2\right\}}\,dx = -\frac{\pi}{2}\left(\frac{b}{a}\right)^\nu \qquad [0<b<a]$$

$$\int_0^\infty J_\mu(bx)\left\{\cos\left[\frac{1}{2}(\nu-\mu)\pi\right]J_\nu(ax)-\sin\left[\frac{1}{2}(\nu-\mu)\pi\right]N_\nu(ax)\right\}\frac{x\,dx}{x^2+r^2}=I_\mu(br)K_\nu(ar)$$
$$[\mathrm{Re}+>0,\quad a\geq b>0,\quad \mathrm{Re}\,\mu>|\,\mathrm{Re}\,\nu|-2].$$

6.544

1. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right)N_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=-\frac{1}{a}\left[\frac{2}{\pi}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)-N_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]\qquad\left[a>0,\ b>0,\ |\,\mathrm{Re}\,\nu|<\frac{1}{2}\right].$$

ET II 357(47)

2. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right)J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=\frac{1}{a}J_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\qquad\left[a>0,\quad b>0,\quad \mathrm{Re}\,\nu>-\frac{1}{2}\right].$$

ET II 57(10)

3. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right)K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=\frac{1}{a}e^{\frac{1}{2}i\nu\pi}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right)+\frac{1}{a}e^{-\frac{1}{2}i\nu\pi}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)$$
$$\left[\mathrm{Re}\,b>0,\quad a>0,\quad |\,\mathrm{Re}\,\nu|<\frac{1}{2}\right].$$

ET II 142(32)

4. $$\int_0^\infty N_\nu\left(\frac{a}{x}\right)J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=\frac{2}{a\pi}\left[K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)+\frac{\pi}{2}N_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]\qquad\left[a>0,\ b>0,\ |\,\mathrm{Re}\,\nu|<\frac{1}{2}\right].$$

ET II 62(38)

5. $$\int_0^\infty N_\nu\left(\frac{a}{x}\right)K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=\frac{4}{a}\left[e^{\frac{1}{2}i(\nu+1)\pi}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right)+e^{-\frac{1}{2}i(\nu+1)\pi}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)\right]$$
$$\left[\mathrm{Re}\,b>0,\quad a>0,\quad |\,\mathrm{Re}\,\nu|<\frac{1}{2}\right].$$

ET II 143(38)

6. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right)J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2}=\frac{i}{a}\left[e^{\frac{1}{2}\nu\pi i}K_{2\nu}\left(e^{\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right)-e^{-\frac{1}{2}\nu\pi i}K_{2\nu}\left(e^{-\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[\mathrm{Re}\,a>0,\quad b>0,\quad |\,\mathrm{Re}\,\nu|<\frac{5}{2}\right].$$

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) N_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{2}{a}\left[\sin\left(\frac{3}{2}\pi\nu\right)\operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right)\operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right].$$

ET II 113(29)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{\pi}{a}K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 146(55)

6.55 Combinations of Bessel functions and algebraic functions

6.551*

1. $$\int_0^1 x^{1/2} J_\nu(xy)\,dx = \sqrt{2}y^{-3/2}\frac{\Gamma\left(\frac{3}{4}+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\nu\right)} + $$
$$+ y^{-1/2}\left[\left(\nu - \frac{1}{2}\right) J_\nu(y) S_{-1/2,\,\nu-1}(y) - J_{\nu-1}(y) S_{1/2,\,\nu}(y)\right]$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 21(1)

2. $$\int_1^\infty x^{1/2} J_\nu(xy)\,dx = y^{-1/2}\left[J_{\nu-1}(y) S_{1/2,\,\nu}(y) + \left(\frac{1}{2} - \nu\right) J_\nu(y) S_{-1/2,\,\nu-1}(y)\right] \qquad [y > 0].$$

ET II 22(2)

6.552

1. $$\int_0^\infty J_\nu(xy)\frac{dx}{(x^2+a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \qquad [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 23(11)
WA 477(3)
MO 44

2. $$\int_0^\infty N_\nu(xy)\frac{dx}{(x^2+a^2)^{1/2}} = -\frac{1}{\pi}\sec\left(\frac{1}{2}\nu\pi\right) K_{\nu/2}\left(\frac{1}{2}ay\right)\left[K_{\nu/2}\left(\frac{1}{2}ay\right) + \pi\sin\left(\frac{1}{2}\nu\pi\right) I_{\nu/2}\left(\frac{1}{2}ay\right)\right]$$
$$[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < 1].$$

3. $\displaystyle\int_0^\infty K_\nu(xy)\frac{dx}{(x^2+a^2)^{1/2}} = \frac{\pi^2}{8}\sec\left(\frac{1}{2}\nu\pi\right)\left\{\left[J_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2 + \left[N_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2\right\}$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad |\operatorname{Re}\nu| < 1].$

ET II 128(6)

4. $\displaystyle\int_0^1 J_\nu(xy)\frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2}\left[J_{\nu/2}\left(\frac{1}{2}y\right)\right]^2 \qquad [y > 0, \quad \operatorname{Re}\nu > -1].$

ET II 24(22)a

5. $\displaystyle\int_0^1 N_0(xy)\frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2}J_0\left(\frac{1}{2}y\right)N_0\left(\frac{1}{2}y\right) \qquad [y > 0].$

ET II 102(26)a

6. $\displaystyle\int_1^\infty J_\nu(xy)\frac{dx}{(x^2-1)^{1/2}} = -\frac{\pi}{2}J_{\nu/2}\left(\frac{1}{2}y\right)N_{\nu/2}\left(\frac{1}{2}y\right) \qquad [y > 0].$

ET II 24(23)a

7. $\displaystyle\int_1^\infty N_\nu(xy)\frac{dx}{(x^2-1)^{1/2}} = \frac{\pi}{4}\left\{\left[J_{\nu/2}\left(\frac{1}{2}y\right)\right]^2 - \left[N_{\nu/2}\left(\frac{1}{2}y\right)\right]^2\right\} \qquad [y > 0].$

ET II 102(27)

6.553

$$\int_0^\infty x^{-1/2} I_\nu(x)K_\nu(x)K_\mu(2x)\,dx = \frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu-\frac{1}{2}\mu\right)}{4\Gamma\left(\frac{3}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}+\nu-\frac{1}{2}\mu\right)}$$

$$\left[|\operatorname{Re}\mu| < \frac{1}{2}, \quad 2\operatorname{Re}\nu > |\operatorname{Re}\mu| - \frac{1}{2}\right].$$

ET II 372(2)

6.554

1. $\displaystyle\int_0^\infty xJ_0(xy)\frac{dx}{(a^2+x^2)^{1/2}} = y^{-1}e^{-ay} \qquad [y > 0, \quad \operatorname{Re} a > 0].$

2. $$\int_0^1 xJ_0(xy)\frac{dx}{(1-x^2)^{1/2}} = y^{-1}\sin y \qquad [y>0].$$

ET II 7(5)a

3. $$\int_1^\infty xJ_0(xy)\frac{dx}{(x^2-1)^{1/2}} = y^{-1}\cos y \qquad [y>0].$$

ET II 7(6)a

4. $$\int_0^\infty xJ_0(xy)\frac{dx}{(x^2+a^2)^{3/2}} = a^{-1}e^{-ay} \qquad [y>0, \quad \operatorname{Re} a>0].$$

ET II 7(7)a

5. $$\int_0^\infty \frac{xJ_0(ax)}{\sqrt{x^4+4k^4}}\,dx = K_0(ak)J_0(ak) \qquad [a>0, \quad k>0].$$

WA 473(1)

6.555

$$\int_0^\infty x^{1/2}J_{2\nu-1}(ax^{1/2})N_\nu(xy)\,dx = -\frac{a}{2y^2}\mathbf{H}_{\nu-1}\left(\frac{a^2}{4y}\right) \qquad \left[a>0, \quad y>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 111(17)

6.556

$$\int_0^\infty J_\nu\left[a(x^2+1)^{1/2}\right]\frac{dx}{\sqrt{x^2+1}} = -\frac{\pi}{2}J_{\nu/2}\left(\frac{a}{2}\right)N_{\nu/2}\left(\frac{a}{2}\right) \qquad [\operatorname{Re}\nu>-1, \quad a>0].$$

MO 46

6.56- 6.58 Combinations of Bessel functions and powers

6.561

1. $$\int_0^1 x^\nu J_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)\left[J_\nu(a)\mathbf{H}_{\nu-1}(a)-\mathbf{H}_\nu(a)J_{\nu-1}(a)\right] \qquad \left[\operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 333(2)a

2. $$\int_0^1 x^\nu N_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)\left[N_\nu(a)\mathbf{H}_{\nu-1}(a)-\mathbf{H}_\nu(a)N_{\nu-1}(a)\right] \qquad \left[\operatorname{Re}\nu>-\frac{1}{2}\right].$$

3. $$\int_0^1 x^\nu I_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)$$
$$[I_\nu(a)\mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a)I_{\nu-1}(a)] \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 364(2)a

4. $$\int_0^1 x^\nu K_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)$$
$$[K_\nu(a)\mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a)K_{\nu-1}(a)] \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 367(21)a

5. $$\int_0^1 x^{\nu+1} J_\nu(ax)\,dx = a^{-1}J_{\nu+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 333(3)a

6. $$\int_0^1 x^{\nu+1} N_\nu(ax)\,dx = a^{-1}N_{\nu+1}(a) + 2^{\nu+1}a^{-\nu-2}\pi^{-1}\Gamma(\nu+1) \qquad [\operatorname{Re}\nu > -1].$$

ET II 339(44)a

7. $$\int_0^1 x^{\nu+1} I_\nu(ax)\,dx = a^{-1}I_{\nu+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 365(3)a

8. $$\int_0^1 x^{\nu+1} K_\nu(ax)\,dx = 2^\nu a^{-\nu-2}\Gamma(\nu+1) - a^{-1}K_{\nu+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 367(22)a

9. $$\int_0^1 x^{1-\nu} J_\nu(ax)\,dx = \frac{a^{\nu-2}}{2^{\nu-1}\Gamma(\nu)} - a^{-1}J_{\nu-1}(a).$$

ET II 333(4)a

ET II 339(45)a

11. $\int_0^1 x^{1-\nu} I_\nu(ax)\,dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1}\Gamma(\nu)}.$

ET II 365(4)a

12. $\int_0^1 x^{1-\nu} K_\nu(ax)\,dx = 2^{-\nu} a^{\nu-2}\Gamma(1-\nu) - a^{-1} K_{\nu-1}(a) \qquad [\operatorname{Re}\nu < 1].$

ET II 367(23)a

13.[7] $\int_0^1 x^\mu J_\nu(ax)\,dx = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{a^{\mu+1}\Gamma\left(\frac{\nu-\mu+1}{2}\right)} + a^{-\mu}\left\{(\mu+\nu-1)J_\nu(a)S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a)S_{\mu,\nu}(a)\right\}$

$[a > 0, \quad \operatorname{Re}(\mu+\nu) > -1].$

ET II 22(8)a

14. $\int_0^\infty x^\mu J_\nu(ax)\,dx = 2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)} \qquad \left[-\operatorname{Re}\nu - 1 < \operatorname{Re}\mu < \frac{1}{2}, \quad a > 0\right].$

EH II 49(19)

15. $\int_0^\infty x^\mu N_\nu(ax)\,dx = 2^\mu \operatorname{ctg}\left[\frac{1}{2}(\nu+1-\mu)\pi\right] a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$

$\left[|\operatorname{Re}\nu| - 1 < \mu < \frac{1}{2}, \quad a > 0\right].$

ET II 97(3)a

16. $\int_0^\infty x^\mu K_\nu(ax)\,dx = 2^{\mu-1} a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$

$[\operatorname{Re}(\mu+1\pm\nu) > 0, \quad \operatorname{Re} a > 0].$

17. $$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}}\,dx = \frac{\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{2^{\nu-q}a^{q-\nu+1}\Gamma\left(\nu-\frac{1}{2}q+\frac{1}{2}\right)} \qquad \left[-1<\operatorname{Re} q<\operatorname{Re}\nu-\frac{1}{2}\right].$$

WA 428(1), KU 144(5)

18. $$\int_0^\infty \frac{N_\nu(x)}{x^{\nu-\mu}}\,dx = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\nu\right)\sin\left(\frac{1}{2}\mu-\nu\right)\pi}{2^{\nu-\mu}\pi}$$
$$\left[|\operatorname{Re}\nu|<\operatorname{Re}(1+\mu-\nu)<\frac{3}{2}\right].$$

WA 430(5)

6.562

1. $$\int_0^\infty x^\mu N_\nu(bx)\frac{dx}{x+a} =$$
$$= (2a)^\mu\pi^{-1}\left\{\sin\left[\frac{1}{2}\pi(\mu-\nu)\right]\Gamma\left[\frac{1}{2}(\mu+\nu+1)\right]\Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]S_{-\mu,\,\nu}(ab) -\right.$$
$$\left. - 2\cos\left[\frac{1}{2}\pi(\mu-\nu)\right]\Gamma\left(1+\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(1+\frac{1}{2}\mu-\frac{1}{2}\nu\right)S_{-\mu-1,\,\nu}(ab)\right\}$$
$$\left[b>0, \quad |\arg a|<\pi, \quad \operatorname{Re}(\mu\pm\nu)>-1, \quad \operatorname{Re}\mu<\frac{3}{2}\right].$$

ET II 98(8)

2. $$\int_0^\infty \frac{x^\nu J_\nu(ax)}{x+k}\,dx = \frac{\pi k^\nu}{2\cos\nu\pi}[H_{-\nu}(ak)-N_{-\nu}(ak)] \qquad \left[-\frac{1}{2}<\operatorname{Re}\nu<\frac{3}{2}, \quad a>0, \quad |\arg k|<\pi\right].$$

WA 479(7)

3. $$\int_0^\infty x^\mu K_\nu(bx)\frac{dx}{x+a} =$$
$$= 2^{\mu-2}\Gamma\left[\frac{1}{2}(\mu+\nu)\right]\Gamma\left[\frac{1}{2}(\mu-\nu)\right]b^{-\mu}\,{}_1F_2\left(1;\,1-\frac{\mu+\nu}{2},\,1-\frac{\mu-\nu}{2};\,\frac{a^2b^2}{4}\right) -$$
$$- 2^{\mu-3}\Gamma\left[\frac{1}{2}(\mu-\nu-1)\right]\Gamma\left[\frac{1}{2}(\mu+\nu-1)\right]ab^{1-\mu}\,{}_1F_2\left(1;\,\frac{3-\mu-\nu}{2},\,\frac{3-\mu+\nu}{2};\,\frac{a^2b^2}{4}\right) -$$
$$- \pi a^\mu \operatorname{cosec}[\pi(\mu-\nu)]\{K_\nu(ab)+\pi\cos(\mu\pi)\operatorname{cosec}[\pi(\nu+\mu)]I_\nu(ab)\}$$
$$[\operatorname{Re} b>0, \quad |\arg a|<\pi, \quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-1].$$

ET II 127(4)

6.563

$$\int_0^\infty x^{\varrho-1} J_\nu(bx) \frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho+\nu-\mu)\pi]\Gamma(\mu+1)} \times \\ \times \left\{ \sum_{m=0}^\infty \frac{(-1)^m \left(\frac{1}{2}ab\right)^{\nu+2m} \Gamma(\varrho+\nu+2m)}{m!\Gamma(\nu+m+1)\Gamma(\varrho+\nu-\mu+2m)} - \right. \\ \left. - \sum_{m=0}^\infty \frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m} \Gamma(\mu+m+1)}{m!\Gamma\left[\frac{1}{2}(\mu+\nu-\varrho+m+3)\right]} \frac{\sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{\Gamma\left[\frac{1}{2}(\mu-\nu-\varrho+m+3)\right]} \right\}$$

$$\left[b>0, \quad |\arg a|<\pi, \quad \operatorname{Re}(\varrho+\nu)>0, \quad \operatorname{Re}(\varrho-\mu)<\frac{5}{2}\right].$$

ET II 23(10), WA 479

6.564

1. $$\int_0^\infty x^{\nu+1} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{2}{\pi b}} a^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(ab) \qquad \left[\operatorname{Re} a>0, \quad b>0, \quad -1<\operatorname{Re}\nu<\frac{1}{2}\right].$$

ET II 23(15)

2. $$\int_0^\infty x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{\pi}{2b}} a^{\frac{1}{2}-\nu} [I_{\nu-\frac{1}{2}}(ab) - L_{\nu-\frac{1}{2}}(ab)]$$
$$\left[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 23(16)

6.565

1. $$\int_0^\infty x^{-\nu}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right)$$
$$\left[\operatorname{Re} a>0,\ b>0,\ \operatorname{Re}\nu>-\frac{1}{2}\right].$$

WA 477(4), ET II 23(17)

2. $$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab}\Gamma\left(\nu+\frac{1}{2}\right)} \qquad \left[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

3. $$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{3}{2}} J_\nu(bx)\,dx = \frac{b^\nu\sqrt{\pi}}{2^{\nu+1}ae^{ab}\Gamma\left(\nu+\frac{3}{2}\right)} \qquad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 24(19)

4. $$\int_0^\infty \frac{J_\nu(bx)x^{\nu+1}}{(x^2+a^2)^{\mu+1}}\,dx = \frac{a^{\nu-\mu}b^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(ab) \qquad \left[-1 < \operatorname{Re}\nu < \operatorname{Re}\left(2\mu+\frac{3}{2}\right),\ a>0,\ b>0\right].$$

MO 43

5. $$\int_0^\infty x^{\nu+1}(x^2+a^2)^\mu N_\nu(bx)\,dx = 2^{\nu-1}\pi^{-1}a^{2\mu+2}(1+\mu)^{-1}\Gamma(\nu)b^{-\nu} \times$$
$$\times\ {}_1F_2\left(1;\ 1-\nu,\ 2+\mu;\ \frac{a^2b^2}{4}\right) - 2^\mu a^{\mu+\nu+1}[\sin(\nu\pi)]^{-1} \times$$
$$\times\ \Gamma(\mu+1)b^{-1-\mu}\left[I_{\mu+\nu+1}(ab) - 2\cos(\mu\pi)K_{\mu+\nu+1}(ab)\right]$$
$$[b>0, \quad \operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < -2\operatorname{Re}\mu].$$

ET II 100(19)

6. $$\int_0^\infty x^{1-\nu}(x^2+a^2)^\mu N_\nu(bx)\,dx = 2^\mu a^{\mu-\nu+1}b^{-1-\mu}\left\{\frac{\cos(\nu\pi)}{\pi}\Gamma(\mu+1)\ \times\right.$$
$$\left.\times\ \Gamma(\nu)I_{\nu-\mu-1}(ab) - 2\operatorname{cosec}(\nu\pi)[\Gamma(-\mu)]^{-1}K_{\nu-\mu-1}(ab)\right\}\ -$$
$$-\ \frac{a^{2\mu+2}\operatorname{ctg}(\nu\pi)b^\nu}{2^{\nu+1}(\mu+1)\Gamma(\nu+1)}\,{}_1F_2\left(1;\ \nu+1,\ \mu+2;\ \frac{a^2b^2}{4}\right)$$
$$\left[b>0, \quad \operatorname{Re} a>0, \quad \frac{1}{2}+2\operatorname{Re}\mu < \operatorname{Re}\nu < 1\right].$$

ET II 100(20)

7. $$\int_0^\infty x^{1+\nu}(x^2+a^2)^\mu K_\nu(bx)\,dx = 2^\nu\Gamma(\nu+1)a^{\nu+\mu+1}b^{-1-\mu}S_{\mu-\nu,\,\mu+\nu+1}(ab)$$
$$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\nu > -1].$$

ET II 128(8)

8. $$\int_0^\infty \frac{x^{\varrho-1}J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\,dx = \frac{a^\nu k^{\varrho+\nu-2\mu-2}\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu\right)\Gamma\left(\mu+1-\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{2^{\nu+1}\Gamma(\mu+1)\Gamma(\nu+1)} \times$$
$$\times\ {}_1F_2\left(\frac{\varrho+\nu}{2};\ \frac{\varrho+\nu}{2}-\mu,\ \nu+1;\ \frac{a^2k^2}{4}\right) +$$
$$+\ \frac{a^{2\mu+2-\varrho}\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\varrho-\mu-1\right)}{2^{2\mu+3-\varrho}\Gamma\left(\mu+2+\frac{1}{2}\nu-\frac{1}{2}\varrho\right)} \times$$

1. $$\int_0^\infty x^\mu N_\nu(bx)\frac{dx}{x^2+a^2} = 2^{\mu-2}\pi^{-1}b^{1-\mu}\times$$
$$\times\cos\left[\frac{\pi}{2}(\mu-\nu+1)\right]\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\right)\times$$
$$\times\ {}_1F_2\left(1;\ 2-\frac{\mu+1+\nu}{2},\ 2-\frac{\mu+1-\nu}{2};\ \frac{a^2b^2}{4}\right)-$$
$$-\frac{1}{2}\pi a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu+\nu+1)\right]\operatorname{ctg}\left[\frac{\pi}{2}(\mu-\nu+1)\right]I_\nu(ab)-$$
$$-a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu-\nu+1)\right]K_\nu(ab)$$
$$\left[b>0,\quad \operatorname{Re}a>0,\quad |\operatorname{Re}\nu|-1<\operatorname{Re}\mu<\frac{5}{2}\right].$$

ET II 100(17)

2. $$\int_0^\infty x^{\nu+1}J_\nu(ax)\frac{dx}{x^2+b^2} = b^\nu K_\nu(ab)\qquad\left[a>0,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\frac{3}{2}\right].$$

EH II 96(58)

3. $$\int_0^\infty x^\nu K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2 b^{\nu-1}}{4\cos\nu\pi}[H_{-\nu}(ab)-N_{-\nu}(ab)]\qquad\left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

WA 468(9)

4. $$\int_0^\infty x^{-\nu} K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2}{4b^{\nu+1}\cos\nu\pi}[H_\nu(ab)-N_\nu(ab)]\qquad\left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu<\frac{1}{2}\right].$$

WA 468(10)

5. $$\int_0^\infty x^{-\nu} J_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi}{2b^{\nu+1}}[I_\nu(ab)-L_\nu(ab)]\qquad\left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu>-\frac{5}{2}\right].$$

WA 468(11)

6.567

1. $$\int_0^1 x^{\nu+1}(1-x^2)^\mu J_\nu(bx)\,dx = 2^\mu\Gamma(\mu+1)b^{-(\mu+1)}J_{\nu+\mu+1}(b)$$
$$[b>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>-1].$$

ET II 26(33)a

2. $$\int_0^1 x^{\nu+1}(1-x^2)^\mu N_\nu(bx)\,dx =$$
$$= b^{-(\mu+1)}\left[2^\mu\Gamma(\mu+1)N_{\mu+\nu+1}(b)+2^{\nu+1}\pi^{-1}\Gamma(\nu+1)S_{\mu-\nu,\,\mu+\nu+1}(b)\right]$$
$$[b>0,\quad \operatorname{Re}\mu>-1,\quad \operatorname{Re}\nu>-1].$$

3. $$\int_0^1 x^{1-\nu}(1-x^2)^{\mu} J_\nu(bx)\,dx = \frac{2^{1-\nu} s_{\nu+\mu,\,\mu-\nu+1}(b)}{b^{\mu+1}\Gamma(\nu)} \qquad [b>0, \quad \operatorname{Re}\mu > -1].$$

ET II 25(31)a

4. $$\int_0^1 x^{1-\nu}(1-x^2)^{\mu} N_\nu(bx)\,dx = b^{-(\mu+1)}\left[2^{1-\nu}\pi^{-1}\cos(\nu\pi)\Gamma(1-\nu)\times\right.$$
$$\left.\times\, s_{\mu+\nu,\,\mu-\nu+1}(b) - 2^{\mu}\operatorname{cosec}(\nu\pi)\Gamma(\mu+1)J_{\mu-\nu+1}(b)\right]$$
$$[b>0, \quad \operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu < 1]. \quad cr$$

ET II 104(37)a

5. $$\int_0^1 x^{1-\nu}(1-x^2)^{\mu} K_\nu(bx)\,dx = 2^{-\nu-2}b^{\nu}(\mu+1)^{-1}\Gamma(-\nu)\,{}_1F_2\left(1;\nu+1,\,\mu+2;\,\frac{b^2}{4}\right) +$$
$$+\,\pi 2^{\mu-1}b^{-(\mu+1)}\operatorname{cosec}(\nu\pi)\Gamma(\mu+1)I_{\mu-\nu+1}(b)$$
$$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu < 1].$$

ET II 129(12)a

6. $$\int_0^1 x^{1-\nu} J_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\mathbf{H}_{\nu-\frac{1}{2}}(b) \qquad [b>0].$$

ET II 24(24)a

7. $$\int_0^1 x^{1+\nu} N_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\operatorname{cosec}(\nu\pi)[\cos(\nu\pi)J_{\nu+\frac{1}{2}}(b) - \mathbf{H}_{-\nu-\frac{1}{2}}(b)]$$
$$[b>0, \quad \operatorname{Re}\nu > -1].$$

ET II 102(28)a

8. $$\int_0^1 x^{1-\nu} N_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\left\{\operatorname{ctg}(\nu\pi)\left[\mathbf{H}_{\nu-\frac{1}{2}}(b) - N_{\nu-\frac{1}{2}}(b)\right] - J_{\nu-\frac{1}{2}}(b)\right\}$$
$$[b>0, \quad \operatorname{Re}\nu < 1].$$

ET II 102(30)a

9. $$\int_0^1 x^{\nu}(1-x^2)^{\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi}b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right)\left[J_\nu\left(\frac{b}{2}\right)\right]^2 \qquad \left[b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

10. $$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} N_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right) J_\nu\left(\frac{b}{2}\right) N_\nu\left(\frac{b}{2}\right) \qquad \left[b>0,\ \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 102(31)a

11. $$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right) I_\nu\left(\frac{b}{2}\right) K_\nu\left(\frac{b}{2}\right) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 129(10)a

12. $$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} I_\nu(bx)\,dx = 2^{-\nu-1}\sqrt{\pi} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right) \left[I_\nu\left(\frac{b}{2}\right)\right]^2$$

ET II 365(5)a

13. $$\int_0^1 x^{\nu+1} (1-x^2)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^{-\nu}\frac{b^{\nu-1}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}-\nu\right)\sin b \qquad \left[b>0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 25(27)a

14. $$\int_1^\infty x^\nu (x^2-1)^{\nu-\frac{1}{2}} N_\nu(bx)\,dx = 2^{\nu-2}\sqrt{\pi} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right) \left[J_\nu\left(\frac{b}{2}\right) J_{-\nu}\left(\frac{b}{2}\right) - N_\nu\left(\frac{b}{2}\right) N_{-\nu}\left(\frac{b}{2}\right)\right]$$
$$\left[|\operatorname{Re}\nu| < \frac{1}{2}, \quad b>0\right].$$

ET II 103(32)a

15. $$\int_1^\infty x^\nu (x^2-1)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx = \frac{2^{\nu-1}}{\sqrt{\pi}} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right) \left[K_\nu\left(\frac{b}{2}\right)\right]^2 \qquad \left[\operatorname{Re} b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 129(11)a

16. $$\int_1^\infty x^{-\nu} (x^2-1)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = -2^{-\nu-1}\sqrt{\pi} b^{\nu}\Gamma\left(\frac{1}{2}-\nu\right) J_\nu\left(\frac{b}{2}\right) N_\nu\left(\frac{b}{2}\right)$$
$$\left[b>0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 25(26)a

ET II 25(28)

6.568

1. $$\int_0^\infty x^\nu N_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\nu-1}J_\nu(ab) \qquad \left[a>0, \quad b>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{5}{2}\right].$$

ET II 101(22)

2. $$\int_0^\infty x^\mu N_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\mu-1}J_\nu(ab) + 2^\mu\pi^{-1}a^{\mu-1}\cos\left[\frac{\pi}{2}(\mu-\nu+1)\right] \times$$
$$\times\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)S_{-\mu,\,\nu}(ab)$$
$$\left[a>0, \quad b>0, \quad |\operatorname{Re}\nu|-1<\operatorname{Re}\mu<\frac{5}{2}\right].$$

ET II (101)(25)

6.569

$$\int_0^1 x^\lambda(1-x)^{\mu-1}J_\nu(ax)\,dx =$$
$$= \frac{\Gamma(\mu)\Gamma(1+\lambda+\nu)2^{-\nu}a^\nu}{\Gamma(\nu+1)\Gamma(1+\lambda+\mu+\nu)} \times$$
$$\times\ {}_2F_3\left(\frac{\lambda+1+\nu}{2}, \frac{\lambda+2+\nu}{2}; \nu+1, \frac{\lambda+1+\mu+\nu}{2}, \frac{\lambda+2+\mu+\nu}{2}; -\frac{a^2}{4}\right)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}(\lambda+\nu)>-1].$$

ET II 193(56)a

6.571

1. $$\int_0^\infty \left[(x^2+a^2)^{\frac{1}{2}} \pm x\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = a^\mu I_{\frac{1}{2}(\nu\mp\mu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\nu\pm\mu)}\left(\frac{ab}{2}\right)$$
$$\left[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu<\frac{3}{2}\right].$$

ET II 26(38)

2. $$\int_0^\infty [(x^2+a^2)^{\frac{1}{2}}-x]^\mu N_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} =$$
$$= a^\mu\left[\operatorname{ctg}(\nu\pi)I_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) - \operatorname{cosec}(\nu\pi)I_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)\right]$$
$$\left[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\mu>-\frac{3}{2}, \quad |\operatorname{Re}\nu|<1\right].$$

3. $$\int_0^\infty \left[(x^2+a^2)^{\frac{1}{2}}+x\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \\ = \frac{\pi^2}{4}a^\mu \operatorname{cosec}(\nu\pi)\left[J_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)N_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)-N_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)J_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)\right] \\ [\operatorname{Re} a>0, \quad \operatorname{Re} b>0].$$

ET II 130(15)

6.572

1. $$\int_0^\infty x^{-\mu}\left[(x^2+a^2)^{\frac{1}{2}}+a\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab\Gamma(\nu+1)}W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ab)M_{-\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ab) \\ [\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}(\nu-\mu)>-1].$$

ET II 26(40)

2. $$\int_0^\infty x^{-\mu}\left[(x^2+a^2)^{\frac{1}{2}}+a\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \\ = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab}W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(iab)W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(-iab) \\ [\operatorname{Re} a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\mu+|\operatorname{Re}\nu|<1].$$

ET II 130(18), BU 87(6a)

3. $$\int_0^\infty x^{-\mu}\left[(x^2+a^2)^{\frac{1}{2}}-a\right]^\mu N_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \\ = -\frac{1}{ab}W_{-\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ab)\left\{\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)}\operatorname{tg}\left(\frac{\nu-\mu}{2}\pi\right)M_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ab)\ + \right. \\ \left. +\ \sec\left(\frac{\nu-\mu}{2}\pi\right)W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ab)\right\} \qquad \left[\operatorname{Re} a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}+\frac{1}{2}\operatorname{Re}\mu\right].$$

ET II 105(42)

6.573

1. $$\int_0^\infty x^{\nu-M+1}J_\nu(bx)\prod_{i=1}^k J_{\mu_i}(a_i x)\,dx = 0, \qquad M=\sum_{i=1}^k \mu_i \\ \left[a_i>0, \quad \sum_{i=1}^k a_i<b<\infty, \quad -1<\operatorname{Re}\nu<\operatorname{Re} M+\frac{1}{2}k-\frac{1}{2}\right].$$

2. $$\int_0^\infty x^{\nu-M-1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x)\, dx = 2^{\nu-M-1} b^{-\nu} \Gamma(\nu) \prod_{i=1}^k \frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \qquad M = \sum_{i=1}^k \mu_i$$
$$\left[a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad 0 < \operatorname{Re}\nu < \operatorname{Re} M + \frac{1}{2}k + \frac{3}{2}\right].$$

WA 460(16)A, ET II 54(43)

6.574

1. $$\int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda}\, dt = \frac{\alpha^\nu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1} \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma(\nu+1)} \times$$
$$\times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{a^2}{\beta^2}\right)$$
$$[\operatorname{Re}(\nu+\mu-\lambda+1) > 0, \quad \operatorname{Re}\lambda > -1, \quad 0 < \alpha < \beta].$$

WA 439(2)A, MO 49

If we reverse the positions of ν and μ and at the same time reverse the positions of α and β, the function on the right hand side of this equation will change. Thus, the right hand side represents a function of $\frac{\alpha}{\beta}$ that is not analytic at $\frac{\alpha}{\beta} = 1$.

For $\alpha = \beta$, we have the following equation

2. $$\int_0^\infty J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda}\, dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)}$$
$$[\operatorname{Re}(\nu+\mu+1) > \operatorname{Re}\lambda > 0, \quad \alpha > 0].$$

MO 49, WA 441(2)a

3. $$\int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda}\, dt = \frac{\beta^\mu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \alpha^{\mu-\lambda+1} \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right) \Gamma(\mu+1) \times}$$
$$\times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{-\nu+\mu-\lambda+1}{2}; \mu+1; \frac{\beta^2}{\alpha^2}\right)$$
$$[\operatorname{Re}(\nu+\mu-\lambda+1) > 0, \quad \operatorname{Re}\lambda > -1, \quad 0 < \beta < \alpha].$$

MO 50, WA 440(3)a

716

If $\mu-\nu+\lambda+1$ (or $\nu-\mu+\lambda+1$) is a negative integer, the right hand side of equation 6.574 1. (or 6.574 3.) vanishes. The

$\mu - \nu + \lambda + 1 \qquad \nu - \mu + \lambda + 1$

cases in which the hypergeometric function F in 6.574 3. (or 6.574 1.) can be reduced to an elementary function are then especially important.

6.575

$$1. \quad \int_0^\infty J_{\nu+1}(\alpha t) J_\mu(\beta t) t^{\mu-\nu}\, dt = 0 \qquad [\alpha < \beta];$$

$$= \frac{(\alpha^2 - \beta^2)^{\nu-\mu}\beta^\mu}{2^{\nu-\mu}\alpha^{\nu+1}\Gamma(\nu - \mu + 1)} \qquad [\alpha \geq \beta] \qquad [\operatorname{Re}\mu > \operatorname{Re}(\nu + 1) > 0].$$

MO 51

$$2. \quad \int_0^\infty \frac{J_\nu(x) J_\mu(x)}{x^{\nu+\mu}}\, dx = \frac{\sqrt{\pi}\,\Gamma(\nu + \mu)}{2^{\nu+\mu}\Gamma\left(\nu + \mu + \frac{1}{2}\right)\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2}\right)} \qquad [\operatorname{Re}(\nu+\mu) > 0].$$

KU 147(17), WA 434(1)

6.576

$$1. \quad \int_0^\infty x^{\mu-\nu+1} J_\mu(x) K_\nu(x)\, dx = \frac{1}{2}\Gamma(\mu-\nu+1) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) > -1].$$

ET II 370(47)

$$2. \quad \int_0^\infty x^{-\lambda} J_\nu(ax) J_\nu(bx)\, dx = \frac{2^\nu b^\nu \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1}\Gamma(\nu+1)\Gamma\left(\frac{1+\lambda}{2}\right)} \times$$

$$\times F\left[\nu + \frac{1-\lambda}{2}, \ \nu + \frac{1}{2};\ 2\nu + 1;\ \frac{4ab}{(a+b)^2}\right]$$

$$[a > 0, \quad b > 0, \quad 2\operatorname{Re}\nu + 1 > \operatorname{Re}\lambda > -1].$$

ET II 47(4)

$$3. \quad \int_0^\infty x^{-\lambda} K_\mu(ax) J_\nu(bx)\, dx = \frac{b^\nu \Gamma\left(\frac{\nu - \lambda + \mu + 1}{2}\right)\Gamma\left(\frac{\nu - \lambda - \mu + 1}{2}\right)}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma(1+\nu)} \times$$

$$\times F\left(\frac{\nu - \lambda + \mu + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2};\ \nu + 1;\ -\frac{b^2}{a^2}\right)$$

$$[\operatorname{Re}(a \pm ib) > 0, \quad \operatorname{Re}(\nu - \lambda + 1) > |\operatorname{Re}\mu|].$$

EH II 52(31), ET II 63(4), WA 449(1)

ET II 145(49), EH II 93(36)

5. $$\int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx)\,dx = \\ = \frac{b^\nu \Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{2^{\lambda+1}\Gamma(\nu+1)a^{-\lambda+\nu+1}} \times \\ \times F\left(\frac{1}{2}-\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu,\ \frac{1}{2}-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu;\ \nu+1;\ \frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(\nu+1-\lambda\pm\mu)>0, \quad a>b]$.

EH II 93(35)

6. $$\int_0^\infty x^{-\lambda} N_\mu(ax) J_\nu(bx)\,dx = \frac{2}{\pi}\sin\frac{\pi(\nu-\mu-\lambda)}{2}\int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx)\,dx$$
$[a>b, \quad \operatorname{Re}\lambda>-1, \quad \operatorname{Re}(\nu-\lambda+1\pm\mu)>0]$; (see **6.576** 5.).

6.576

EH II 93(37)

7. $$\int_0^\infty x^{\mu+\nu+1} J_\mu(ax) K_\nu(bx)\,dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}$$ $[\operatorname{Re}\mu > |\operatorname{Re}\nu|-1, \quad \operatorname{Re} b > |\operatorname{Im} a|]$.

ET 137(16), EH II 93(36), B 449(2)

6.577

1.[8] $$\int_0^\infty x^{\nu-\mu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\nu-\mu+2n} I_\mu(ac) K_\nu(bc)$$
$[a>0, \quad b>a, \quad \operatorname{Re} c>0, \quad 2+\operatorname{Re}\mu-2n>\operatorname{Re}\nu>-1-n, \quad n\ge 0$ an integer$]$.

ET II 49(13)

2.[8] $$\int_0^\infty x^{\mu-\nu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\nu+2n} I_\nu(bc) K_\mu(ac)$$
$[b>0, \quad a>b, \quad \operatorname{Re}\nu-2n+2>\operatorname{Re}\mu>-n-1, \quad n\ge 0$ an integer$]$.

ET II 49(15)

6.578

1. $$\int_0^\infty x^{\varrho-1} J_\lambda(ax)\, J_\mu(bx) J_\nu(cx)\, dx = \frac{2^{\varrho-1} a^\lambda b^\mu c^{-\lambda-\mu-\varrho} \Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left(1-\frac{\lambda+\mu-\nu+\varrho}{2}\right)} \times$$
$$\times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1,\ \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right)$$
$$\left[\operatorname{Re}(\lambda+\mu+\nu+\varrho) > 0, \quad \operatorname{Re}\varrho < \frac{5}{2}, \quad a>0, \quad b>0, \quad c>0, \quad c>a+b\right].$$

ET II 351(9)

2. $$\int_0^\infty x^{\varrho-1} J_\lambda(ax)\, J_\mu(bx) K_\nu(cx)\, dx =$$
$$= \frac{2^{\varrho-2} a^\lambda b^\mu c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)} \Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right) \times$$
$$\times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1,\ \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right)$$
$$[\operatorname{Re}(\varrho+\lambda+\mu) > |\operatorname{Re}\nu|, \quad \operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

ET II 373(8)

3. $$\int_0^\infty x^{\lambda-\mu-\nu+1} J_\nu(ax) J_\mu(bx) J_\lambda(cx)\, dx = 0$$
$$\left[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{1}{2}, \quad c>b>0, \quad 0<a<c-b\right].$$

ET II 53(36)

4. $$\int_0^\infty x^{\lambda-\mu-\nu-1} J_\nu(ax) J_\mu(bx) J_\lambda(cx)\, dx = \frac{2^{\lambda-\mu-\nu-1} a^\nu b^\mu \Gamma(\lambda)}{c^\lambda \Gamma(\mu+1)\Gamma(\nu+1)}$$
$$\left[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{5}{2}, \quad c>b>0, \quad 0<a<c-b\right].$$

ET II 53(37)

5. $$\int_0^\infty x^{1+\mu} N_\mu(ax) J_\nu(bx) J_\nu(cx)\, dx = 0 \qquad [0<b<c, \quad 0<a<c-b].$$

ET II 352(13)

WA 452(2), ET II 64(12)

7. $\int_0^\infty x^{\mu+1} I_\nu(ax) K_\mu(bx) J_\nu(cx)\,dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{4})\pi i} (v^2+1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q^\mu + \frac{1}{2}_\nu - \frac{1}{2}(iv),$

$2acv = b^2 - a^2 + c^2 \qquad [\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1].$

ET II 66(22)

8. $\int_0^\infty x^{1-\mu} J_\mu(ax) J_\nu(bx) J_\nu(cx)\,dx = \frac{c^{\mu-1}(\operatorname{sh} u)^{\mu-\frac{1}{2}}}{\sqrt{\frac{1}{2}\pi^3} a^\mu b^{1-\mu}} e^{(\mu-\frac{1}{2})\pi i} \sin[(\mu-\nu)\pi] Q^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\operatorname{ch} u),$

$2bc \operatorname{ch} u = a^2 - b^2 - c^2 \qquad \left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad 0 < c < a - b, \quad b > 0\right];$

$= \frac{b^{\mu-1} c^{\mu-1}}{\sqrt{2\pi} a^\mu} (\sin v)^{\mu-\frac{1}{2}} P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\cos v), \qquad 2bc\cos v = b^2 + c^2 - a^2$

$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad |a-b| < c < a+b, \quad a > 0, \quad b > 0\right];$

$= 0$

$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad 0 < c < b - a \quad \text{or} \quad a + b < c < \infty, \quad a > 0, \quad b > 0\right].$

ET II 52(34)

9. $\int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu}\,dx = \frac{2^{\nu-1}\Delta^{2\nu-1}}{(abc)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)},$

where Δ is the area of a triangle whose sides are $a,\ b,$ and c. In the case in which the segments whose lengths are $a,\ b,$ and c cannot form a triangle, the value of the integral is zero $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$.

10. $\int_0^\infty x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx)\,dx =$

$= \frac{\sqrt{\pi} c^\nu \Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}{2^{\frac{3}{2}(ab)\nu+1}(u^2-1)^{\frac{1}{2}\nu+\frac{1}{4}}} P^{-\nu-\frac{1}{2}}_{\mu-\frac{1}{2}}(u), \qquad 2abu = a^2 + b^2 + c^2$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad \operatorname{Re}(\nu \pm \mu) > -1, \quad \operatorname{Re}\nu > -1].$

ET II 67(30)
MO 52, WA 451(3)

ET II 66(24)

12. $$\int_0^\infty x^{\nu+1}[J_\nu(ax)]^2 N_\nu(bx)\,dx = 0 \qquad \left[a>0, \quad 0<b<2a, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right];$$
$$= \frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}(b^2-4a^2)^{-\nu-\frac{1}{2}}$$
$$\left[a>0, \quad 2a<b<\infty, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET II 109(3)

13. $$\int_0^\infty x^{\nu+1}J_\nu(ax)N_\nu(ax)J_\nu(bx)\,dx = 0 \qquad \left[a>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}, \quad 0<b<2a\right];$$
$$= -\frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}(b^2-4a^2)^{-\nu-\frac{1}{2}}$$
$$\left[a>0, \quad 2a<b<\infty, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET II 55(49)

14. $$\int_0^\infty x^{\nu+1}J_\mu(xa\sin\psi)J_\nu(xa\sin\varphi)K_\mu(xa\cos\varphi\cos\psi)\,dx =$$
$$= \frac{2^\nu\Gamma(\mu+\nu+1)(\sin\varphi)^\nu\left(\cos\frac{\alpha}{2}\right)^{2\nu+1}}{a^{\nu+2}(\cos\psi)^{2\nu+2}}P_\nu^{-\mu}(\cos\alpha), \qquad \operatorname{tg}\frac{1}{2}\alpha = \operatorname{tg}\psi\cos\varphi$$
$$\left[a>0, \quad \frac{\pi}{2}>\varphi>0, \quad 0<\psi<\frac{\pi}{2}, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1\right].$$

ET II 64(11)

15. $$\int_0^\infty x^{\nu+1}J_\nu(ax)K_\nu(bx)J_\nu(cx)\,dx = \frac{2^{3\nu}(abc)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}[(a^2+b^2+c^2)^2-4a^2c^2]^{\nu+\frac{1}{2}}}$$
$$\left[\operatorname{Re}b>|\operatorname{Im}a|, \quad c>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 63(8)

16. $$\int_0^\infty x^{\nu+1}I_\nu(ax)K_\nu(bx)J_\nu(cx)\,dx = \frac{2^{3\nu}(abc)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}[(b^2-a^2+c^2)^2+4a^2c^2]^{\nu+\frac{1}{2}}}$$
$$\left[\operatorname{Re}b>\operatorname{Re}a, \quad c>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

6.579

1. $$\int_0^\infty x^{2\nu+1} J_\nu(ax) N_\nu(ax) J_\nu(bx) N_\nu(bx)\,dx = \frac{a^{2\nu}\Gamma(3\nu+1)}{2\pi b^{4\nu+2}\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(2\nu+\frac{3}{2}\right)} \times$$
$$\times F\left(\nu+\frac{1}{2},\, 3\nu+1;\, 2\nu+\frac{3}{2};\, \frac{a^2}{b^2}\right)$$
$$\left[0 < a < b, \quad -\frac{1}{3} < \operatorname{Re}\nu < \frac{1}{2}\right].$$

EH II 94(45), ET II 352(15)

2. $$\int_0^\infty x^{2\nu+1} J_\nu(ax) K_\nu(ax) J_\nu(bx) K_\nu(bx)\,dx = \frac{2^{\nu-3}a^{2\nu}\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{3\nu+1}{2}\right)}{\sqrt{\pi}\, b^{4\nu+2}\Gamma(\nu+1)} \times$$
$$\times F\left(\nu+\frac{1}{2},\, \frac{3\nu+1}{2};\, 2\nu+1;\, 1-\frac{a^4}{b^4}\right)$$
$$\left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{3}\right].$$

ET II 373(10)

3. $$\int_0^\infty x^{1-2\nu}[J_\nu(ax)]^4\,dx = \frac{\Gamma(\nu)\Gamma(2\nu)}{2\pi\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2\Gamma(3\nu)} \qquad [\operatorname{Re}\nu > 0].$$

ET II 342(25)

4. $$\int_0^\infty x^{1-2\nu}[J_\nu(ax)]^2[J_\nu(bx)]^2\,dx = \frac{a^{2\nu-1}\Gamma(\nu)}{2\pi b\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)} F\left(\nu,\, \frac{1}{2}-\nu;\, 2\nu+\frac{1}{2};\, \frac{a^2}{b^2}\right).$$

ET II 351(10)

6.581

1. $$\int_0^a x^{\lambda-1} J_\mu(x) J_\nu(a-x)\,dx = 2^\lambda \sum_{m=0}^\infty \frac{(-1)^m\Gamma(\lambda+\mu+m)\Gamma(\lambda+m)}{m!\Gamma(\lambda)\Gamma(\mu+m+1)} J_{\lambda+\mu+\nu+2m}(a)$$
$$[\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 354(25)

ET II 354(27)

3. $$\int_0^a x^\mu (a-x)^\nu J_\mu(x) J_\nu(a-x)\,dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 354(28), EH II 46(6)

4. $$\int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x)\,dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}\right].$$

ET II 354(29)

5. $$\int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x)\,dx = \frac{2^\mu\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu)}{\sqrt{\pi}\,\Gamma(\mu+\nu+1)} a^\mu J_\nu(a)$$
$$\left[\operatorname{Re}\nu > \operatorname{Re}\mu > -\frac{1}{2}\right].$$

ET II 355(30)

6.582

$$\int_0^\infty x^{\mu-1}|x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x)\,dx = \frac{1}{\sqrt{\pi}}(2b)^{-\mu}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma(\mu+\nu)\Gamma(\mu-\nu)K_\nu(b)$$
$$\left[b > 0, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|\right].$$

ET II 374(14)

6.583

$$\int_0^\infty x^{\mu-1}(x+b)^{-\mu} K_\mu(x+b) K_\nu(x)\,dx = \frac{\sqrt{\pi}\,\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{2^\mu b^\mu \Gamma\left(\mu+\frac{1}{2}\right)} K_\nu(b)$$
$$[|\arg b| < \pi, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|].$$

ET II 374(15)

6.584

1. $$\int_0^\infty \frac{x^{\varrho-1}[H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}(axe^{\pi i})]}{(x^2 - r^2)^{m+1}}\,dx = \frac{\pi i}{m!}\left(\frac{d}{dr}\right)^m [r^{\varrho-2} H_\nu^{(1)}(ar)]$$
$$\left[m = 0,\, 1,\, 2, \dots, \quad \operatorname{Im} r > 0, \quad a > 0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m + \frac{7}{2}\right].$$

WA 465

2. $$\int_0^\infty \left[\cos\frac{1}{2}(\varrho-\nu)\pi J_\nu(ax) + \sin\frac{1}{2}(\varrho-\nu)\pi N_\nu(ax)\right] \frac{x^{\varrho-1}}{(x^2-k^2)^{m+1}}\,dx =$$
$$= \frac{(-1)^{m+1}}{2^m \cdot m!}\left(\frac{d}{k\,dk}\right)^m [k^{\varrho-2} K_\nu(ak)]$$
$$\left[m = 0,\, 1,\, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m + \frac{7}{2}\right].$$

WA 466(2)

3. $$\int_0^\infty \{\cos\nu\pi J_\nu(ax) - \sin\nu\pi N_\nu(ax)\}\, \frac{x^{1-\nu}\,dx}{(x^2+k^2)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m \cdot m! k^{\nu+m}}$$
$$\left[m = 0,\, 1,\, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad -2m - \frac{3}{2} < \operatorname{Re}\nu < 1\right].$$

WA 466(3)

4. $$\int_0^\infty \left\{\cos\left[\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu\right)\pi\right] J_\nu(ax) + \sin\left[\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu\right)\pi\right] N_\nu(ax)\right\} \frac{x^{\varrho-1}}{(x^2+k^2)^{\mu+1}}\,dx =$$
$$= \frac{\pi k^{\varrho-2\mu-2}}{2\sin\nu\pi\cdot\Gamma(\mu+1)}\left[\frac{\left(\frac{1}{2}ak\right)^\nu \Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu\right)}{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu-\mu\right)}\, {}_1F_2\left(\frac{\varrho+\nu}{2};\ \frac{\varrho+\nu}{2}-\mu,\ \nu+1;\ \frac{a^2k^2}{4}\right) - \right.$$
$$\left. - \frac{\left(\frac{1}{2}ak\right)^{-\nu}\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu-\mu\right)}\, {}_1F_2\left(\frac{\varrho-\nu}{2};\ \frac{\varrho-\nu}{2}-\mu,\ 1-\nu;\ \frac{a^2k^2}{4}\right)\right]$$
$$\left[a > 0, \quad \operatorname{Re} k > 0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2\operatorname{Re}\mu + \frac{7}{2}\right].$$

WA 407(1)

5.[8] $$\int_0^\infty \left[\prod_{j,\,n} J_{\mu_j}(b_n x)\right] \left\{\cos\left[\frac{1}{2}\left(\varrho + \sum_j \mu_j - \nu\right)\pi\right] J_\nu(ax) + \right.$$
$$\left. + \sin\left[\frac{1}{2}\left(\varrho + \sum_j \mu_j - \nu\right)\pi\right] N_\nu(ax)\right\} \frac{x^{\varrho-1}}{x^2+k^2}\,dx =$$

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.591

1. $$\int_0^\infty x^{2\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} J_{1+2\nu}\left(\sqrt{2ab}\right) K_{1+2\nu}\left(\sqrt{2ab}\right)$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 142(35)

2. $$\int_0^\infty x^{2\nu+\frac{1}{2}} N_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} N_{2\nu+1}\left(\sqrt{2ab}\right) K_{2\nu+1}\left(\sqrt{2ab}\right)$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 143(41)

3. $$\int_0^\infty x^{2\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} K_{2\nu+1}\left(e^{\frac{1}{4}i\pi}\sqrt{2ab}\right) K_{2\nu+1}\left(e^{-\frac{1}{4}i\pi}\sqrt{2ab}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 146(56)

4. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} K_{2\nu-1}\left(\sqrt{2ab}\right) \times$$
$$\times \left[\sin(\nu\pi) J_{2\nu-1}\left(\sqrt{2ab}\right) + \cos(\nu\pi) N_{2\nu-1}\left(\sqrt{2ab}\right)\right]$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 142(34)

5. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} N_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = -\sqrt{\frac{\pi}{2}} b^{\nu-1} a^{\frac{1}{2}-\nu} \sec(\nu\pi) K_{2\nu-1}\left(\sqrt{2ab}\right) \times$$
$$\times \left[J_{2\nu-1}\left(\sqrt{2ab}\right) - J_{1-2\nu}\left(\sqrt{2ab}\right)\right] \qquad [a > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 143(40)

6. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\frac{1}{2}-\nu}\left(\frac{a}{x}\right) J_\nu(bx)\,dx =$$
$$= -\frac{1}{2} i \operatorname{cosec}(2\nu\pi) b^{\nu-1} a^{\frac{1}{2}-\nu} [e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v) - e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)],$$
$$u = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}; \qquad v = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i} \qquad \left[a > 0,\ b > 0,\ -\frac{1}{2} < \operatorname{Re} \nu < 3\right].$$

7. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) N_\nu(bx)\,dx = \sqrt{2\pi}b^{\nu-1}a^{\frac{1}{2}-\nu} N_{2\nu-1}\left(\sqrt{2ab}\right) K_{2\nu-1}\left(\sqrt{2ab}\right)$$
$$\left[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu>\frac{1}{6}\right].$$

ET II 113(30)

8. $$\int_0^\infty x^{\varrho-1} J_\mu(ax) J_\nu\left(\frac{b}{x}\right)\,dx = \frac{a^{\nu-\varrho}b^\nu\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\varrho+1\right)}\times$$
$$\times {}_0F_3\left(\nu+1,\ \frac{\nu-\mu-\varrho}{2}+1,\ \frac{\nu+\mu-\varrho}{2}+1;\ \frac{a^2b^2}{16}\right)+$$
$$+\frac{a^\mu b^{\mu+\varrho}\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1}\Gamma(\mu+1)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\varrho+1\right)}\times$$
$$\times {}_0F_3\left(\mu+1,\ \frac{\mu-\nu+\varrho}{2}+1,\ \frac{\nu+\mu+\varrho}{2}+1;\ \frac{a^2b^2}{16}\right)$$
$$\left[a>0, \quad b>0, \quad -\operatorname{Re}\left(\mu+\frac{3}{2}\right)<\operatorname{Re}\varrho<\operatorname{Re}\left(\nu+\frac{3}{2}\right)\right].$$

WA 480(1)

6.592

1. $$\int_0^\infty x^\lambda(1-x)^{\mu-1} N_\nu\left(a\sqrt{x}\right)\,dx = 2^{-\nu}a^\nu \operatorname{ctg}(\nu\pi)\frac{\Gamma(\mu)\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)}\times$$
$$\times\ {}_1F_2\left(\lambda+1+\frac{1}{2}\nu;\ 1+\nu,\ \lambda+1+\mu+\frac{1}{2}\nu;\ -\frac{a^2}{4}\right)-$$
$$-2^\nu a^{-\nu}\operatorname{cosec}(\nu\pi)\frac{\Gamma(\mu)\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}\times$$
$$\times\ {}_1F_2\left(\lambda-\frac{1}{2}\nu+1;\ 1-\nu,\ \lambda+1+\mu-\frac{1}{2}\nu;\ -\frac{a^2}{4}\right)$$
$$\left[\operatorname{Re}\lambda>-1+\frac{1}{2}|\operatorname{Re}\nu|, \quad \operatorname{Re}\mu>0\right].$$

ET II 197(76)a

2.* $$\int_0^1 x^\lambda(1-x)^{\mu-1} K_\nu\left(a\sqrt{x}\right)\,dx =$$
$$=2^{-\nu-1}a^{-\nu}\frac{\Gamma(\nu)\Gamma(\mu)\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}\,{}_1F_2\left(\lambda+1-\frac{1}{2}\nu;\ 1-\nu,\ \lambda+1+\mu-\frac{1}{2}\nu;\ \frac{a^2}{4}\right)+$$

3. $$\int_1^\infty x^\lambda (x-1)^{\mu-1} J_\nu\left(a\sqrt{x}\right)\,dx = 2^{2\lambda} a^{-2\lambda} G_{13}^{20}\left(\frac{a^2}{4}\middle|\begin{matrix}0\\ -\mu,\ \lambda+\frac{1}{2}\nu,\ \lambda-\frac{1}{2}\nu\end{matrix}\right)\Gamma(\mu)$$
$$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{4}-\operatorname{Re}\lambda\right].$$

ET II 205(36)a

4. $$\int_1^\infty x^\lambda (x-1)^{\mu-1} K_\nu\left(a\sqrt{x}\right)\,dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G_{13}^{30}\left(\frac{a^2}{4}\middle|\begin{matrix}0\\ -\mu,\ \frac{1}{2}\nu+\lambda,\ -\frac{1}{2}\nu+\lambda\end{matrix}\right)$$
$$[\operatorname{Re} a>0, \quad \operatorname{Re}\mu>0].$$

ET II 209(60)a

5. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} J_\nu\left(a\sqrt{x}\right)\,dx = \pi\left[J_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2 \qquad [\operatorname{Re}\nu>-1].$$

ET II 194(59)a

6. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} I_\nu\left(a\sqrt{x}\right)\,dx = \pi\left[I_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2 \qquad [\operatorname{Re}\nu>-1].$$

ET II 197(79)

7. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} K_\nu\left(a\sqrt{x}\right)\,dx = \frac{1}{2}\pi\sec\left(\frac{1}{2}\nu\pi\right)\left[I_{\frac{\nu}{2}}\left(\frac{a}{2}\right)+I_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)$$
$$[|\operatorname{Re}\nu|<1].$$

ET II 198(85)a

8. $$\int_1^\infty x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}} K_\nu\left(a\sqrt{x}\right)\,dx = \left[K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2 \qquad [\operatorname{Re} a>0].$$

ET II 208(56)a

9. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} N_\nu\left(a\sqrt{x}\right)\,dx = \pi\left\{\operatorname{ctg}(\nu\pi)\left[J_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2-\operatorname{cosec}(\nu\pi)\left[J_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$$[|\operatorname{Re}\nu|<1].$$

10. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} J_\nu\left(a\sqrt{x}\right)\,dx = \Gamma(\mu)2^\mu a^{-\mu} J_{\nu-\mu}(a) \qquad \left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right].$$

ET II 205(34)a

11. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} J_{-\nu}\left(a\sqrt{x}\right)\,dx = \Gamma(\mu)2^\mu a^{-\mu}[\cos(\nu\pi)J_{\nu-\mu}(a)-\sin(\nu\pi)N_{\nu-\mu}(a)]$$
$$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right].$$

ET II 205(35)a

12. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} K_\nu\left(a\sqrt{x}\right)\,dx = \Gamma(\mu)2^\mu a^{-\mu} K_{\nu-\mu}(a) \qquad [\operatorname{Re}a>0, \quad \operatorname{Re}\mu>0].$$

ET II 209(59)a

13. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} N_\nu\left(a\sqrt{x}\right)\,dx = 2^\mu a^{-\mu} N_{\nu-\mu}(a)\Gamma(\mu) \qquad \left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right].$$

ET II 206(40)a

14. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} H_\nu^{(1)}\left(a\sqrt{x}\right)\,dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(1)}(a)\Gamma(\mu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Im}a>0].$$

ET II 206(45)a

15. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} H_\nu^{(2)}\left(a\sqrt{x}\right)\,dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(2)}(a)\Gamma(\mu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Im}a<0].$$

ET II 207(48)a

16. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} J_\nu\left(a\sqrt{x}\right)\,dx = \frac{2^{2-\nu}a^{-\mu}}{\Gamma(\nu)} s_{\mu+\nu-1,\,\mu-\nu}(a) \qquad [\operatorname{Re}\mu>0].$$

ET II 194(64)a

17. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} N_\nu\left(a\sqrt{x}\right)\,dx = \frac{2^{2-\nu}a^{-\mu}\operatorname{ctg}(\nu\pi)}{\Gamma(\nu)} s_{\mu+\nu-1,\,\mu-\nu}(a)$$
$$- 2^\mu a^{-\mu}\operatorname{cosec}(\nu\pi) J_{\mu-\nu}(a)\Gamma(\mu)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu<1].$$

1. $$\int_0^\infty \sqrt{x} J_{2\nu-1}\left(a\sqrt{x}\right) J_\nu(bx)\, dx = \frac{1}{2} a b^{-2} J_{\nu-1}\left(\frac{a^2}{4b}\right) \qquad \left[b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 58(15)

2. $$\int_0^\infty \sqrt{x} J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \frac{\pi a}{4b^2}\left[I_{\nu-1}\left(\frac{a^2}{4b}\right) - L_{\nu-1}\left(\frac{a^2}{4b}\right)\right] \qquad \left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 144(44)

6.594

1. $$\int_0^\infty x^\nu I_{2\nu-1}\left(a\sqrt{x}\right) J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \sqrt{\pi} 2^{-\nu} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)$$
$$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 148(65)

2. $$\int_0^\infty x^\nu I_{2\nu-1}\left(a\sqrt{x}\right) N_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx =$$
$$= \sqrt{\pi} 2^{-\nu-1} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi)\left[H_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \right.$$
$$\left. + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi) N_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)\right] \qquad [\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 148(66)

3. $$\int_0^\infty x^\nu J_{2\nu-1}\left(a\sqrt{x}\right) K_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx =$$
$$= \pi^2 2^{-\nu-2} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi)\left[H_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - N_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right)\right] \qquad [\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 148(67)

6.595

1. $$\int_0^\infty x^{\nu+1} J_\nu(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i)\, dx = 0, \qquad z_i = \sqrt{x^2 + b_i^2}$$
$$\left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c; \quad \operatorname{Re}\left(\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2}\right) > \operatorname{Re}\nu > -1\right].$$

EH II 52(33), ET II 60(26)

EH II 52(34), ET II 60(27)

6.596

1. $$\int_0^\infty J_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}}\,dx = \frac{2^\mu\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}J_{\nu-\mu-1}(\alpha z)$$
$$\left[\alpha>0,\quad \operatorname{Re}\left(\frac{1}{2}\nu-\frac{1}{4}\right)>\operatorname{Re}\mu>-1\right].$$

WA 457(5)

2. $$\int_0^\infty \frac{J_\nu\left(\alpha\sqrt{t^2+1}\right)}{\sqrt{t^2+1}}\,dt = -\frac{\pi}{2}J_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right)N_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right)\qquad[\operatorname{Re}\nu>-1,\quad\alpha>0].$$

MO 46

3. $$\int_0^\infty K_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}}\,dx = \frac{2^\mu\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}K_{\nu-\mu-1}(\alpha z)\qquad[\alpha>0,\quad\operatorname{Re}\mu>-1].$$

WA 457(6)

4.[6] $$\int_0^\infty J_\nu(\beta x)\frac{J_{\mu-1}\left\{\alpha\sqrt{x^2+z^2}\right\}}{(x^2+z^2)^{\frac{1}{2}\mu+\frac{1}{2}}}x^{\nu+1}\,dx = \frac{\alpha^{\mu-1}z^\nu}{2^{\mu-1}\Gamma(\mu)}K_\nu(\beta x)$$
$$[\alpha<\beta,\quad\operatorname{Re}(\mu+2)>\operatorname{Re}\nu>-1].$$

WA 459(11)A, ET II 59(19)

5. $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left\{\alpha\sqrt{x^2+z^2}\right\}}{\sqrt{(x^2+z^2)^\mu}}x^{\nu-1}\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{\beta^\nu}\,\frac{J_{\mu(\alpha z)}}{z^\mu}$$
$$[\operatorname{Re}(\mu+2)>\operatorname{Re}\nu>0,\quad\beta>\alpha>0].$$

WA 459(12)

6.[6] $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{(x^2+z^2)^\mu}}x^{\nu+1}\,dx$$
$$=0\qquad[0<\alpha<\beta];$$
$$=\frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2-\beta^2}}{z}\right)^{\mu-\nu-1}J_{\mu-\nu-1}\left\{z\sqrt{\alpha^2-\beta^2}\right\}\qquad[\alpha>\beta>0];\qquad[\operatorname{Re}\mu>\operatorname{Re}\nu>-1].$$

7. $$\int_0^\infty J_\nu(\beta x)\frac{K_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{(x^2+z^2)^\mu}}x^{\nu+1}\,dx = \frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2+\beta^2}}{z}\right)^{\mu-\nu-1}K_{\mu-\nu-1}\left(z\sqrt{z\alpha^2+\beta^2}\right)$$
$$\left[\alpha>0,\quad \beta>0,\quad \operatorname{Re}\nu>-1,\quad |\arg z|<\frac{\pi}{2}\right].$$

KU 151(31), WA 416(2)

8.[6] $$\int_0^\infty J_\nu(\beta t)\frac{K_\mu\left(\alpha\sqrt{t^2-y^2}\right)}{\sqrt{(t^2-y^2)^\mu}}t^{\nu+1}\,dt = \frac{\pi}{2}\frac{\beta^\nu}{\alpha^\mu}\left\{\frac{\sqrt{\alpha^2+\beta^2}}{y}\right\}^{\mu-\nu-1}\exp\left[-\frac{\pi i}{2}\left(\mu-\nu-\frac{1}{2}\right)\right]\times$$
$$\times\left\{J_{\mu-\nu-1}\left[y\sqrt{\alpha^2+\beta^2}\right]-iN_{\mu-\nu-1}\left[y\sqrt{\alpha^2+\beta^2}\right]\right\}$$

[$\operatorname{Re}\mu<1$. Here, it is assumed that the integration contour does not contain the singularity $t=y$, which can be excluded by going *upwards* around it, and that the sign of $\sqrt{t^2-y^2}$ is chosen in such a way that the expression in question is positive for $t>y; \alpha>0, \beta>0, y>0$].

9. $$\int_0^\infty J_\nu(ux)K_\mu\left(v\sqrt{x^2-y^2}\right)\left(x^2-y^2\right)^{-\frac{\mu}{2}}x^{\nu+1}\,dx = \frac{\pi}{2}\exp\left[-i\pi\left(\mu-\nu-\frac{1}{2}\right)\right]\cdot\frac{u^\nu}{v^\mu}\cdot$$
$$\cdot\left[\frac{\sqrt{u^2+v^2}}{y}\right]^{\mu-\nu-1}H^{(2)}_{\mu-\nu-1}\left(y\sqrt{u^2+v^2}\right)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}\nu>-1,\ u>0,\quad v>0;\quad \arg\sqrt{x^2-y^2}=0\quad \text{for } x>y;\right.$$
$$\left.\text{if } x<y,\ \text{then}\ \arg(x^2-y^2)^\sigma=\pi\sigma,\quad \text{where } \sigma=\tfrac{1}{2}\ \text{or}\ \sigma=-\tfrac{\mu}{2}\right].$$

MO 43
WA 416(3)

10.[6] $$\int_0^\infty J_\nu(ux)H^{(2)}_\mu\left(v\sqrt{x^2+y^2}\right)(x^2+y^2)^{-\frac{\mu}{2}}x^{\nu+1}\,dx = \frac{u^\nu}{v^\mu}\left[\frac{\sqrt{v^2-u^2}}{y}\right]^{\mu-\nu-1}H^{(2)}_{\mu-\nu-1}\left(y\sqrt{v^2-u^2}\right)$$
$[u<v]$ $\left[\operatorname{Re}\mu<\operatorname{Re}\nu,\quad \operatorname{Re}\nu>-1,\quad u>0,\quad v>0,\quad y>0;\quad \arg\sqrt{v^2-u^2}=0\right.$
$\left.\text{for } v>u,\quad \arg(v^2-u^2)^\sigma=-\pi\sigma\quad \text{for } v<u,\quad \text{where } \sigma=\frac{1}{2}\quad\text{or}\quad \sigma=\frac{\mu-\nu-1}{2}\right].$

MO 43

11. $$\int_0^\infty J_\nu(\beta x)J_\mu\left(\alpha\sqrt{x^2+z^2}\right)J_\mu\left(\gamma\sqrt{x^2+z^2}\right)\frac{x^{\nu-1}}{(x^2+z^2)^\mu}\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{\beta^\nu}\frac{J_\mu(\alpha z)}{z^\mu}\frac{J_\mu(\gamma z)}{z^\mu}$$
$$\left[\alpha>0;\quad \beta>\alpha+\gamma;\quad \gamma>0,\quad \operatorname{Re}\left(2\mu+\frac{5}{2}\right)>\operatorname{Re}\nu>0\right].$$

12.[6] $$\int_0^\infty J_\nu(\beta t)t^{\nu-1}\prod_{k=1}^n J_\mu\left(\alpha_k\sqrt{t^2+x^2}\right)\sqrt{(t^2+x^2)^{-n\mu}}\,dt = 2^{\nu-1}\beta^{-\nu}\Gamma(\nu)\prod_{k=1}^n\left[x^{-\mu}J_\mu(\alpha_k x)\right]$$
$$\left[x>0,\quad \alpha_1>0,\quad \alpha_2>0,\dots,\alpha_n>0,\quad \beta>\prod_{k=1}^n\alpha_k;\quad \operatorname{Re}\left(n\mu+\frac{1}{2}n+\frac{1}{2}\right)>\operatorname{Re}\nu>0\right].$$

MO 43

13. $$\int_0^\infty \frac{J_\nu^2\left(\sqrt{a^2+x^2}\right)}{(a^2+x^2)^\nu}x^{2\nu-2}\,dx = \frac{\Gamma\left(\nu-\frac{1}{2}\right)}{2a^{\nu+1}\sqrt{\pi}}\mathbf{H}_\nu(2a) \qquad \left[\operatorname{Re}\nu>\frac{1}{2}\right].$$

WA 457(8)

6.597

$$\int_0^\infty t^{\nu+1}J_\mu\left[b(t^2+y^2)^{\frac{1}{2}}\right](t^2+y^2)^{-\frac{1}{2}\mu}(t^2+\beta^2)^{-1}J_\nu(at)\,dt =$$
$$= \beta^\nu J_\mu\left[b(y^2-\beta^2)^{\frac{1}{2}}\right](y^2-\beta^2)^{-\frac{1}{2}\mu}K_\nu(a\beta) \qquad [a\ge b,\quad \operatorname{Re}\beta>0,\quad -1<\operatorname{Re}\nu<2+\operatorname{Re}\mu].$$

EH II 95(56)

6.598

$$\int_0^1 x^{\frac{\mu}{2}}(1-x)^{\frac{\nu}{2}}J_\mu\left(a\sqrt{x}\right)J_\nu\left(b\sqrt{1-x}\right)\,dx = 2a^\mu b^\nu(a^2+b^2)^{-\frac{1}{2}(\nu+\mu+1)}J_{\nu+\mu+1}\left(\sqrt{a^2+b^2}\right)$$
$$[\operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>-1].$$

EH II 46a

6.61 Combinations of Bessel functions and exponentials

6.611

1. $$\int_0^\infty e^{-\alpha x}J_\nu(\beta x)\,dx = \frac{\beta^{-\nu}\left[\sqrt{\alpha^2+\beta^2}-\alpha\right]^\nu}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\nu>-1,\quad \operatorname{Re}(\alpha\pm i\beta)>0].$$

EH II 49(18), WA 422(8)

2. $$\int_0^\infty e^{-\alpha x}N_\nu(\beta x)\,dx = (\alpha^2+\beta^2)^{-\frac{1}{2}}\operatorname{cosec}(\nu\pi)\times$$
$$\times\left\{\beta^\nu\left[(\alpha^2+\beta^2)^{\frac{1}{2}}+\alpha\right]^{-\nu}\cos(\nu\pi)-\beta^{-\nu}\left[(\alpha^2+\beta^2)^{\frac{1}{2}}+\alpha\right]^\nu\right\}$$
$$[\operatorname{Re}\alpha>0,\quad \beta>0,\quad |\operatorname{Re}\nu|<1].$$

3. $$\int_0^\infty e^{-\alpha x} K_\nu(\beta x)\,dx = \frac{\pi}{\beta\sin(\nu\pi)}\,\frac{\sin(\nu\theta)}{\sin\theta} \qquad \left[\cos\theta = \frac{\alpha}{\beta};\quad \theta\to\frac{\pi}{2}\quad \text{for } \beta\to\infty\right];$$
$$= \frac{\pi\operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2-\beta^2}}\left[\beta^{-\nu}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^{\nu} - \beta^{\nu}\left(\sqrt{\alpha^2-\beta^2}+\alpha\right)^{-\nu}\right]$$
$$[|\operatorname{Re}\nu|<1,\quad \operatorname{Re}(\alpha+\beta)>0].$$

ET I 197(24), MO 180
ET II 131(22)

4. $$\int_0^\infty e^{-\alpha x} I_\nu(\beta x)\,dx = \frac{\beta^\nu}{\sqrt{\alpha^2-\beta^2}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^\nu} \qquad [\operatorname{Re}\nu > -1,\quad \operatorname{Re}\alpha > |\operatorname{Re}\beta|].$$

MO 180, ET I 195(1)

5. $$\int_0^\infty e^{-\alpha x} H_\nu^{(1,2)}(\beta x)\,dx = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\beta^\nu\sqrt{\alpha^2+\beta^2}}\left\{1 \pm \frac{i}{\sin(\nu\pi)}\left[\cos(\nu\pi) - \frac{\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)^{2\nu}}{b^{2\nu}}\right]\right\}$$

[$-1 < \operatorname{Re}\nu < 1$; a plus sign corresponds to the function $H_\nu^{(1)}$, a minus sign to the function $H_\nu^{(2)}$].

6. $$\int_0^\infty e^{-\alpha x} H_0^{(1)}(\beta x)\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\left\{1 - \frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

MO 180, ET I 188(53)
MO 180, ET I 188(54, 55)

7. $$\int_0^\infty e^{-\alpha x} H_0^{(2)}(\beta x)\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\left\{1 + \frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

MO 180, ET I 188(53)}

8. $$\int_0^\infty e^{-\alpha x} N_0(\beta x)\,dx = \frac{-2}{\pi\sqrt{\alpha^2+\beta^2}}\ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

MO 47, ET I 187(44)

9. $$\int_0^\infty e^{-\alpha x} K_0(\beta x)\,dx = \frac{\arccos\frac{\alpha}{\beta}}{\sqrt{\beta^2-\alpha^2}} \qquad [0<\alpha<\beta,\quad \operatorname{Re}(\alpha+\beta)>0];$$
$$= \frac{1}{\sqrt{\alpha^2-\beta^2}}\ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}-1}\right) \qquad [0\le\beta<\alpha,\quad \operatorname{Re}(\alpha+\beta)>0].$$

6.612

1. $$\int_0^\infty e^{-2\alpha x} J_0(x) N_0(x)\,dx = \frac{\boldsymbol{K}\left[\alpha(\alpha^2+1)^{-\frac{1}{2}}\right]}{\pi(\alpha^2+1)^{\frac{1}{2}}} \qquad [\operatorname{Re}\alpha > 0].$$

ET II 347(58)

2. $$\int_0^\infty e^{-2\alpha x} I_0(x) K_0(x)\,dx = \frac{1}{2}\boldsymbol{K}\left[(1-\alpha^2)^{\frac{1}{2}}\right] \qquad [0<\alpha<1];$$
$$= \frac{1}{2\alpha}\boldsymbol{K}\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right] \qquad [1<\alpha<\infty].$$

ET II 370(48)

3. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) J_\nu(\gamma x)\,dx = \frac{1}{\pi\sqrt{\gamma\beta}} Q_{\nu-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}\right)$$
$$\left[\operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 426(2), ET II 50(17)

4. $$\int_0^\infty e^{-\alpha x} [J_0(\beta x)]^2\,dx = \frac{2}{\pi\sqrt{\alpha^2+4\beta^2}}\boldsymbol{K}\left(\frac{2\beta}{\sqrt{\alpha^2+4\beta^2}}\right).$$

MO 178

5. $$\int_0^\infty e^{-2\alpha x} J_1^2(\beta x)\,dx = \frac{(2\alpha^2+\beta^2)\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) - 2(\alpha^2+\beta^2)\boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{\pi\beta^2\sqrt{\alpha^2+\beta^2}}.$$

WA 428(3)

6.613[7]

$$\int_0^\infty e^{-x^2} J_{\nu+\frac{1}{2}}\left(\frac{x^2}{2}\right)\,dx = \frac{\Gamma(\nu+1)}{\sqrt{\pi}} D_{-\nu-1}\left(ze^{\frac{\pi}{4}i}\right) D_{-\nu-1}\left(ze^{-\frac{\pi i}{4}}\right) \qquad [\operatorname{Re}\nu > -1].$$

MO 122

6.614

1. $$\int_0^\infty e^{-\alpha x} J_\nu\left(\beta\sqrt{x}\right)\,dx = \frac{\beta}{4}\sqrt{\frac{\pi}{\alpha^3}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\beta^2}{8\alpha}\right)\right].$$

2. $$\int_0^\infty e^{-\alpha x} N_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \left\{ \operatorname{ctg}(\nu\pi) \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{\frac{1}{2},\,\nu}\left(\frac{\beta}{\alpha}\right) - \operatorname{cosec}(\nu\pi)\, W_{\frac{1}{2},\,nu}\left(\frac{\beta}{\alpha}\right) \right\}$$
$$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1].$$

ET I 188(50)a

3. $$\int_0^\infty e^{-\alpha x} I_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2},\,\nu}\left(\frac{\beta}{\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1].$$

ET I 197(20)a

4. $$\int_0^\infty e^{-\alpha x} K_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}} \Gamma(\nu+1)\Gamma(1-\nu) W_{-\frac{1}{2},\,\nu}\left(\frac{\beta}{\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1].$$

ET I 199(37)a

5. $$\int_0^\infty e^{-\alpha x} K_1\left(\beta\sqrt{x}\right) dx = \frac{\beta}{8}\sqrt{\frac{\pi}{\alpha^3}} \exp\left(\frac{\beta^2}{8\alpha}\right) \left[K_1\left(\frac{\beta^2}{8\alpha}\right) - K_0\left(\frac{\beta^2}{8\alpha}\right)\right].$$

MO 181

6.615

$$\int_0^\infty e^{-\alpha x} J_\nu\left(2\beta\sqrt{x}\right) J_\nu\left(2\gamma\sqrt{x}\right) dx = \frac{1}{\alpha} I_\nu\left(\frac{2\beta\gamma}{\alpha}\right) \exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right) \qquad [\operatorname{Re}\nu > -1].$$

MO 178

6.616

1. $$\int_0^\infty e^{-\alpha x} J_0\left(\beta\sqrt{x^2+2\gamma x}\right) dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \exp\left[\gamma\left(\alpha - \sqrt{\alpha^2+\beta^2}\right)\right].$$

MO 179

2. $$\int_1^\infty e^{-\alpha x} J_0\left(\beta\sqrt{x^2-1}\right) dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \exp\left(-\sqrt{\alpha^2+\beta^2}\right).$$

MO 179

MO 49

4. $$\int_{-\infty}^{\infty} e^{-itx} H_0^{(2)}\left(r\sqrt{\alpha^2-t^2}\right)\,dt = 2i\frac{e^{-i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$
$$\left[-\pi < \arg\sqrt{\alpha^2-t^2} \le 0, \quad -\pi < \arg\alpha \le 0, \quad r \text{and } x \text{ are real}\right].$$

MO 49

5.[3] $$\int_{-1}^{1} e^{-ax} I_0\left(b\sqrt{1-x^2}\right)\,dx = 2(a^2+b^2)^{-1/2}\ \operatorname{sh}\sqrt{a^2+b^2} \qquad [a>0, \quad b>0].$$

6.617

1. $$\int_0^{\infty} K_{q-p}(2z\ \operatorname{sh}x)e^{(p+q)x}\,dx = \frac{\pi^2}{4\sin[(p-q)\pi]}\left[J_p(z)N_q(z) - J_q(z)N_p(z)\right]$$
$$[\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(p-q) < 1]\ .$$

MO 44

2. $$\int_0^{\infty} K_0(2z\ \operatorname{sh}x)e^{-2px}\,dx = -\frac{\pi}{4}\left\{J_p(z)\frac{\partial N_p(z)}{\partial p} - N_p(z)\frac{\partial J_p(z)}{\partial p}\right\} \qquad [\operatorname{Re} z > 0].$$

MO 44

6.618

1. $$\int_0^{\infty} e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\exp\left(-\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > -1].$$

WA 432(5), ET II 29(8)

2. $$\int_0^{\infty} e^{-\alpha x^2} N_\nu(\beta x)\,dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[\operatorname{tg}\frac{\nu\pi}{2} I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$$[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad |\operatorname{Re}\nu| < 1].$$

WA 432(6), ET II 106(3)

3. $$\int_0^{\infty} e^{-\alpha x^2} K_\nu(\beta x)\,dx = \frac{1}{4}\sec\left(\frac{\nu\pi}{2}\right)\frac{\sqrt{\pi}}{\sqrt{\alpha}}\exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1].$$

4. $$\int_0^\infty e^{-\alpha x^2} I_\nu(\beta x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\nu > -1, \quad \operatorname{Re}\alpha > 0].$$

EH II 92(27)

5. $$\int_0^\infty e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x)\,dx =$$
$$= 2^{-\nu-\mu-1}\alpha^{-\frac{\nu+\mu+1}{2}}\beta^{\nu+\mu}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)} \times$$
$$\times\, {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$$
$$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}\alpha > 0].$$

EH II 50(21)a

6.62- 6.63 Combinations of Bessel functions, exponentials, and powers

6.621

1. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\mu-1}\,dx =$$
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right);$$
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} \left(1+\frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2}+1; \nu+1; -\frac{\beta^2}{\alpha^2}\right);$$
$$= \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}}\Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$$[\operatorname{Re}(\nu+\mu) > 0, \quad \operatorname{Re}(\alpha+i\beta) > 0, \quad \operatorname{Re}(\alpha-i\beta) > 0];$$
$$= (\alpha^2+\beta^2)^{-\frac{1}{2}\mu}\Gamma(\nu+\mu) \mathrm{P}_{\mu-1}^{-\nu}\left[\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}\right] \qquad [\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}(\nu+\mu) > 0].$$

ET II 29(6)
WA 421(3)
WA 421(3)
WA 421(2)

2. $$\int_0^\infty e^{-\alpha x} N_\nu(\beta x) x^{\mu-1}\,dx =$$
$$= \operatorname{ctg}\nu\pi \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}}\Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right) -$$

3. $$\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)\,dx = \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)}F\left(\mu+\nu,\,\nu+\frac{1}{2};\,\mu+\frac{1}{2};\,\frac{\alpha-\beta}{\alpha+\beta}\right)$$
$$[\mathrm{Re}\ \mu > |\,\mathrm{Re}\ \nu|, \quad \mathrm{Re}\,(\alpha+\beta) > 0].$$

ET II 131(23)A, EH II 50(26)

4. $$\int_0^\infty x^{m+1}e^{-\alpha x}J_\nu(\beta x)\,dx = (-1)^{m+1}\beta^{-\nu}\frac{d^{m+1}}{d\alpha^{m+1}}\left[\frac{(\sqrt{\alpha^2+\beta^2}-\alpha)^\nu}{\sqrt{\alpha^2+\beta^2}}\right]$$
$$[\beta > 0, \quad \mathrm{Re}\ \nu > -m-2].$$

ET II 28(3)

6.622

1. $$\int_0^\infty (J_0(x) - e^{-\alpha x})\frac{dx}{x} = \ln 2\alpha \qquad [\alpha > 0].$$

NT 66(13)

2. $$\int_0^\infty \frac{e^{i(u+x)}}{u+x}J_0(x)\,dx = \frac{\pi}{2}iH_0^{(1)}(u).$$

MO 44

3. $$\int_0^\infty e^{-x\ \mathrm{ch}\,\alpha}I_p(x)\frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}Q_{p-\frac{1}{2}}(\mathrm{ch}\,\alpha).$$

WA 424(5)

6.623

1. $$\int_0^\infty e^{-\alpha x}J_\nu(\beta x)x^\nu\,dx = \frac{(2\beta)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}} \qquad \left[\mathrm{Re}\ \nu > -\frac{1}{2}, \quad \mathrm{Re}\ \alpha > |\,\mathrm{Im}\,\beta|\right].$$

WA 422(5)

2. $$\int_0^\infty e^{-\alpha x}J_\nu(\beta x)x^{\nu+1}\,dx = \frac{2\alpha(2\beta)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}} \qquad [\mathrm{Re}\ \nu > -1, \quad \mathrm{Re}\ \alpha > |\,\mathrm{Im}\,\beta|].$$

3. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) \frac{dx}{x} = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\nu\beta^\nu} \qquad [\operatorname{Re}\,\nu > 0; \quad \operatorname{Re}\,\alpha > |\operatorname{Im}\beta|] \qquad (\text{cf. } \mathbf{6.611}\ 1.).$$

6.611
WA 422(7)

6.624

1. $$\int_0^\infty x e^{-\alpha x} K_0(\beta x)\,dx = \frac{1}{\alpha^2-\beta^2}\left\{\frac{\alpha}{\sqrt{\alpha^2-\beta^2}} \ln\left[\frac{\alpha}{\beta}+\sqrt{\left(\frac{\alpha}{\beta}\right)^2-1}\right]-1\right\}.$$

MO 181

2. $$\int_0^\infty \sqrt{x} e^{-\alpha x} K_{\pm\frac{1}{2}}(\beta x)\,dx = \sqrt{\frac{\pi}{2\beta}}\frac{1}{\alpha+\beta}.$$

MO 181

3. $$\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}} K_\mu(t) t^\nu\,dt = \frac{\Gamma(\nu-\mu+1)}{(z^2-1)^{-\frac{1}{2}(\nu+1)}} e^{i\mu\pi} Q_\nu^\mu(z) \qquad [\operatorname{Re}\,(\nu\pm\mu) > -1].$$

EH II 57(7)

4. $$\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}} I_{-\mu}(t) t^\nu\,dt = \frac{\Gamma(-\nu-\mu)}{(z^2-1)^{\frac{1}{2}\nu}} P_\nu^\mu(z) \qquad [\operatorname{Re}(\nu+\mu) < 0].$$

EH II 57(8)

5. $$\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}} I_\mu(t) t^\nu\,dt = \frac{\Gamma(\nu+\mu+1)}{(z^2-1)^{-\frac{1}{2}(\nu+1)}} P_\nu^{-\mu}(z) \qquad [\operatorname{Re}(\nu+\mu) > -1].$$

EH II 57(9)

6. $$\int_0^\infty e^{-t\cos\theta} J_\mu(t\sin\theta) t^\nu\,dt = \Gamma(\nu+\mu+1) \mathrm{P}_\nu^{-\mu}(\cos\,\theta) \qquad \left[\operatorname{Re}\,(\nu+\mu) > -1, \quad 0 \le \theta < \frac{1}{2}\pi\right].$$

EH II 57(10)

7. $$\int_0^\infty \frac{J_\nu(bx)x^\nu}{e^{\pi x}-1}\,dx = \frac{(2b)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{n=1}^\infty \frac{1}{(n^2\pi^2+b^2)^{\nu+\frac{1}{2}}} \qquad [\operatorname{Re}\,\nu > 0, \quad |\operatorname{Im} b| < \pi].$$

1. $$\int_0^1 x^{\lambda-\nu-1}(1-x)^{\mu-1}e^{\pm i\alpha x}J_\nu(\alpha x)\,dx = \frac{2^{-\nu}\alpha^\nu\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)\Gamma(\nu+1)}\,{}_2F_2\left(\lambda,\,\nu+\frac{1}{2};\,\lambda+\mu,\,2\nu+1;\,\pm 2i\alpha\right)$$
$$[\mathrm{Re}\ \lambda > 0, \quad \mathrm{Re}\ \mu > 0].$$
ET II 194(58)a

2. $$\int_0^1 x^{\nu}(1-x)^{\mu-1}e^{\pm i\alpha x}J_\nu(\alpha x)\,dx = \frac{(2\alpha)^\nu\Gamma(\mu)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\mu+2\nu+1)}\,{}_1F_1\left(\nu+\frac{1}{2};\,\mu+2\nu+1;\,\pm 2i\alpha\right)$$
$$\left[\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\ \nu > -\frac{1}{2}\right].$$
ET II 194(57)a

3. $$\int_0^1 x^{\nu}(1-x)^{\mu-1}e^{\pm \alpha x}J_\nu(\alpha x)\,dx = \frac{(2\alpha)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\mu)}{\sqrt{\pi}\Gamma(\mu+2\nu+1)}\,{}_1F_1\left(\nu+\frac{1}{2};\,\mu+2\nu+1;\,\pm 2\alpha\right)$$
$$\left[\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\ \nu > -\frac{1}{2}\right].$$
BU 9(16a), ET II 197(77)a

4. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}e^{\pm\alpha x}I_\nu(\alpha x)\,dx = \frac{\left(\frac{1}{2}\alpha\right)^\nu\Gamma(\lambda+\nu)\Gamma(\mu)}{\Gamma(\nu+1)\Gamma(\lambda+\mu+\nu)}\times$$
$$\times\ {}_2F_2\left(\nu+\frac{1}{2},\,\lambda+\nu;\,2\nu+1,\,\mu+\lambda+\nu;\,\pm 2\alpha\right)$$
$$[\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\,(\lambda+\nu) > 0].$$
ET II 197(78)a

5. $$\int_0^1 x^{\mu-\{}(1-x)^{2\{-1}\,I_{\mu-\{}\left(\frac{1}{2}xz\right)e^{-\frac{1}{2}xz}\,dx = \frac{\Gamma(2\{)}{\sqrt{\pi}\Gamma(1+2\mu)}e^{\frac{x}{2}}z^{-\{-\frac{1}{2}}\,M_{\{,\,u}(z)$$
$$\left[\mathrm{Re}\ \left(\{-\frac{1}{2}-\mu\right) < 0, \quad \mathrm{Re}\ \{ > 0\right]$$
BU 129(14a)

ET II 207(50)a

7. $$\int_1^\infty x^{-\lambda}(x-1)^{\mu-1}e^{-\alpha x}K_\nu(\alpha x)\,dx = \Gamma(\mu)\sqrt{\pi}(2\alpha)^\lambda G_{23}^{30}\left(2\alpha\,\middle|\,\begin{matrix}0,\ \frac{1}{2}-\lambda\\ -\mu,\ \nu-\lambda,\ -\nu-\lambda\end{matrix}\right)$$
$$[\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\ \alpha > 0].$$

ET II 208(55)a

8. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}I_\nu(\alpha x)\,dx = \frac{(2\alpha)^{\nu-\mu}\Gamma\left(\frac{1}{2}-\mu+\nu\right)\Gamma(\mu)}{\sqrt{\pi}\Gamma(1-\mu+2\nu)} \times$$
$$\times\,{}_1F_1\left(\frac{1}{2}-\mu+\nu;\ 1-\mu+2\nu;\ -2\alpha\right)$$
$$\left[0 < \mathrm{Re}\ \mu < \frac{1}{2}+\mathrm{Re}\ \nu, \quad \mathrm{Re}\ \alpha > 0\right].$$

ET II 207(49)a

9. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}K_\nu(\alpha x)\,dx = \sqrt{\pi}\Gamma(\mu)(2\alpha)^{-\frac{1}{2}\mu-\frac{1}{2}}e^{-\alpha}W_{-\frac{1}{2}\mu,\,\nu-\frac{1}{2}\mu}(2\alpha)$$
$$[\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\ \alpha > 0].$$

ET II 208(53)a

10. $$\int_1^\infty x^{-\mu-\frac{1}{2}(x-1)\mu-1}e^{-\alpha x}K_\nu(\alpha x)\,dx = \sqrt{\pi}\Gamma(\mu)(2\alpha)^{-\frac{1}{2}}e^{-\alpha}W_{-\mu,\,\nu}(2\alpha) \qquad [\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\,\alpha > 0].$$

ET II 207(51)a

11.[3] $$\int_{-1}^1 (1-x^2)^{-1/2}xe^{-ax}I_1\left(b\sqrt{1-x^2}\right)\,dx = \frac{2}{b}\left\{\mathrm{sh}\,a - a(a^2+b^2)^{-1/2}\ \mathrm{sh}\,\sqrt{a^2+b^2}\right\}$$
$$[a > 0, \quad b > 0].$$

6.626

1. $$\int_0^\infty x^{\lambda-1}e^{-\alpha x}J_\mu(\beta x)J_\nu(\gamma x)\,dx = \frac{\beta^\mu\gamma^\nu}{\Gamma(\nu+1)}2^{-\nu-\mu}\alpha^{-\lambda-\mu-\nu}\sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\Gamma(\mu+m+1)} \times$$
$$\times\ F\left(-m,\ -\mu-m;\ \nu+1;\ \frac{\gamma^2}{\beta^2}\right)\left(-\frac{\beta^2}{4\alpha^2}\right)^m$$
$$[\mathrm{Re}\,(\lambda+\mu+\nu) > 0, \quad \mathrm{Re}\,(\alpha \pm i\beta \pm i\gamma) > 0].$$

2. $$\int_0^\infty e^{-2\alpha x} J_\nu(\beta x) J_\mu(\beta x) x^{\nu+\mu}\,dx = \frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\beta^{\nu+\mu}}{\sqrt{\pi^3}} \times$$
$$\times \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu+\mu}\ \varphi \cos(\nu-\mu)\varphi}{(\alpha^2+\beta^2\cos^2\ \varphi)^{\nu+\mu}\sqrt{\alpha^2+\beta^2\cos^2\ \varphi}}\,d\varphi$$
$$\left[\operatorname{Re}\ \alpha > |\operatorname{Im}\beta|, \quad \operatorname{Re}(\nu+\mu) > -\frac{1}{2}\right].$$

WA 427(1)

3. $$\int_0^\infty e^{-2\alpha x} J_0(\beta x) J_1(\beta x) x\,dx = \frac{\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) - \boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{2\pi\beta\sqrt{\alpha^2+\beta^2}}.6$$

WA 427(2)

4. $$\int_0^\infty e^{-2\alpha x} I_0(\beta x) I_1(\beta x) x\,dx = \frac{1}{2\pi\beta}\left\{\frac{\alpha}{\alpha^2-\beta^2}\boldsymbol{E}\left(\frac{\beta}{\alpha}\right) - \frac{1}{\alpha}\boldsymbol{K}\left(\frac{\beta}{\alpha}\right)\right\} \qquad [\operatorname{Re}\ \alpha > \operatorname{Re}\ \beta].$$

WA 428(5)

6.627

$$\int_0^\infty \frac{x^{-1/2}}{x+a} e^{-x} K_\nu(x)\,dx = \frac{\pi e^a K_\nu(a)}{\sqrt{a}\cos(\nu\pi)} \qquad \left[|\arg a| < \pi, \quad |\operatorname{Re}\ \nu| < \frac{1}{2}\right].$$

ET II 368(29)

6.628

1. $$\int_0^\infty e^{-x\cos\beta} J_{-\nu}(x\sin\beta) x^\mu\,dx = \Gamma(\mu-\nu+1) \mathrm{P}_\mu^\nu(\cos\beta) \qquad \left[0 < \beta < \frac{\pi}{2}, \quad \operatorname{Re}(\mu-\nu) > -1\right].$$

WA 424(3), WH

737

2. $$\int_0^\infty e^{-x\cos\beta} N_\nu(x\sin\beta) x^\mu\,dx = -\frac{\sin\mu\pi}{\sin(\mu+\nu)\pi}\frac{\Gamma(\mu-\nu+1)}{\pi} \times$$
$$\times \left[Q_\mu^\nu(\cos\beta+0\cdot i) e^{\frac{1}{2}\nu\pi i} + Q_\mu^\nu(\cos\beta-0\cdot i) e^{-\frac{1}{2}\nu\pi i}\right]$$
$$\left[\operatorname{Re}(\mu+\nu) > -1, \quad 0 < \beta < \frac{\pi}{2}\right].$$

WA 424(4)

3. $$\int_0^1 e^{\frac{xu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu} J_{\mu-\nu}\left(\frac{ixu}{2}\right)\,dx = 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{\mathrm{B}(2\nu,\ 2\mu-2\nu+1)}{\Gamma(\mu-\nu+1)} \frac{e^{\frac{u}{2}}}{u^{\nu+\frac{1}{2}}} M_{\nu,\,\mu}(u).$$

4. $$\int_0^\infty e^{-x\,\operatorname{ch}\alpha} I_\nu(x\,\operatorname{sh}\alpha) x^\mu\,dx = \Gamma(\nu+\mu+1) P_\mu^{-\nu}(\operatorname{ch}\alpha) \qquad [\operatorname{Re}\,\mu > -2].$$

WA 423(1)

5. $$\int_0^\infty e^{-x\,\operatorname{ch}\alpha} K_\nu(x\,\operatorname{sh}\alpha) x^\mu\,dx = \frac{\sin\mu\pi}{\sin(\nu+\mu)\pi}\Gamma(\mu-\nu+1) Q_\mu^\nu(\operatorname{ch}\alpha) \qquad [\operatorname{Re}(\mu+1) > |\operatorname{Re}\,\nu|].$$

WA 423(2)

6. $$\int_0^\infty e^{-x\,\operatorname{ch}\alpha} I_\nu(x) x^{\mu-1}\,dx = \frac{\frac{\cos\nu\pi}{\sin(\mu+\nu)\pi} Q^{\mu}-\frac{1}{2}{}_\nu-\frac{1}{2}(\operatorname{ch}\alpha)}{\sqrt{\frac{\pi}{2}}(\operatorname{sh}\alpha)^{\mu-\frac{1}{2}}} \qquad [\operatorname{Re}(\mu+\nu) > 0, \quad \operatorname{Re}(\operatorname{ch}\alpha) > 1].$$

WA 424(6)

7. $$\int_0^\infty e^{-x\,\operatorname{ch}\alpha} K_\nu(x) x^{\mu-1}\,dx = \sqrt{\frac{\pi}{2}}\Gamma(\mu-\nu)\Gamma(\mu+\nu)\frac{P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\operatorname{ch}\alpha)}{(\operatorname{sh}\alpha)^{\mu-\frac{1}{2}}}$$
$$[\operatorname{Re}\,\mu > |\operatorname{Re}\,\nu|, \quad \operatorname{Re}(\operatorname{ch}\alpha) > -1].$$

WA 424(7)

6.629

$$\int_0^\infty x^{-1/2} e^{-x\alpha\cos\varphi\cos\psi} J_\mu(\alpha x\sin\varphi) J_\nu(\alpha x\sin\psi)\,dx = \Gamma\left(\mu+\nu+\frac{1}{2}\right)\alpha^{-\frac{1}{2}} \mathrm{P}_{\nu-\frac{1}{2}}^{-\mu}(\cos\varphi)\mathrm{P}_{\mu-\frac{1}{2}}^{-\nu}(\cos\psi)$$
$$\left[\alpha > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re}(\mu+\nu) > -\frac{1}{2}\right].$$

ET II 50(19)

6.631

1. $$\int_0^\infty x^\mu e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\beta^\nu \Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{2^{\nu+1}\alpha^{\frac{1}{2}(\mu+\nu+1)}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{\nu+\mu+1}{2};\,\nu+1;\,-\frac{\beta^2}{4\alpha}\right);$$
$$= \frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{\beta\alpha^{\frac{1}{2}\mu}\Gamma(\nu+1)}\exp\left(-\frac{\beta^2}{8\alpha}\right) M_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$
$$[\operatorname{Re}\,\alpha > 0, \quad \operatorname{Re}(\mu+\nu) > -1].$$

EH II 50(22), ET II 30(14), BU 14(13b)
BU 8(15)

2. $$\int_0^\infty x^\mu e^{-\alpha x^2} N_\nu(\beta x)\,dx = -\alpha^{-\frac{1}{2}\mu}\beta^{-1}\sec\left(\frac{\nu-\mu}{2}\pi\right)\exp\left(-\frac{\beta^2}{8\alpha}\right)\times$$
$$\times\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)}\sin\left(\frac{\nu-\mu}{2}\pi\right)M_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)+W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)\right\}$$
$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-1,\quad \beta>0].$$

ET II 106(4)

3. $$\int_0^\infty x^\mu e^{-\alpha x^2} K_\nu(\beta x)\,dx = \frac{1}{2}\alpha^{-\frac{1}{2}\mu}\beta^{-1}\Gamma\left(\frac{1+\nu+\mu}{2}\right)\Gamma\left(\frac{1-\nu+\mu}{2}\right)\exp\left(\frac{\beta^2}{8\alpha}\right)W_{-\frac{1}{2}\mu,\,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$
$$[\operatorname{Re}\mu>|\operatorname{Re}\nu|-1].$$

ET II 132(25)

4. $$\int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right)\qquad[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-1].$$

WA 43(4), ET II 29(10)

5. $$\int_0^\infty x^{\nu-1} e^{-\alpha x^2} J_\nu(\beta x)\,dx = 2^{\nu-1}\beta^{-\nu}\gamma\left(\nu,\,\frac{\beta^2}{4\alpha}\right)\qquad[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>0].$$

ET II 30(11)

6. $$\int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2} J_\nu(\beta x)\,dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left[\pm i\left(\frac{\nu+1}{2}\pi-\frac{\beta^2}{4\alpha}\right)\right]$$
$$\left[\alpha>0,\quad -1<\operatorname{Re}\nu<\frac{1}{2},\quad \beta>0\right].$$

ET II 30(12)

7. $$\int_0^\infty x e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\sqrt{\pi}\beta}{8\alpha^{\frac{3}{2}}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right)-I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-2].$$

8. $$\int_0^1 x^{n+1} e^{-\alpha x^2} I_n(2\alpha x)\,dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=-n}^{n} I_r(2\alpha)\right] \qquad [n = 0,\, 1, \ldots].$$

ET II 365(8)a

9. $$\int_1^\infty x^{1-n} e^{-\alpha x^2} I_n(2\alpha x)\,dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=1-n}^{n-1} I_r(2\alpha)\right] \qquad [n = 1,\, 2, \ldots].$$

ET II 367(20)a

10. $$\int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{z})\,dx = \frac{n!}{2} e^{-z} z^{\frac{1}{2}\mu} L_n^\mu(z) \qquad [n = 0,\, 1, \ldots;\quad n+\mathrm{Re}\ \mu > -1].$$

BU 135(5)

6.632

$$\int_0^\infty x^{-\frac{1}{2}} \exp[-(x^2 + a^2 - 2ax\cos\varphi)^{\frac{1}{2}}][x^2 + a^2 - 2ax\cos\varphi]^{-\frac{1}{2}} K_\nu(x)\,dx =$$
$$= \pi a^{-\frac{1}{2}} \sec(\nu\pi) P_{\nu-\frac{1}{2}}(-\cos\varphi) K_\nu(a) \qquad \left[|\arg a| + |\mathrm{Re}\ \varphi| < \pi, \quad |\mathrm{Re}\ \nu| < \frac{1}{2}\right].$$

ET II 368(32)

739

6.633

1. $$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x)\,dx =$$
$$= \frac{\beta^\mu \gamma^\nu \alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1}\Gamma(\nu+1)} \sum_{m=0}^\infty \frac{\Gamma\left(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1\right)}{m!\Gamma(m+\mu+1)} \left(-\frac{\beta^2}{4\alpha}\right)^m \times$$
$$\times F\left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2}\right) \qquad [\mathrm{Re}\ \alpha > 0,\ \mathrm{Re}\,(\mu+\nu+\lambda) > -2,\ \beta > 0, \quad \gamma > 0].$$

EH II 49(20)A, ET II 51(24)a

2. $$\int_0^\infty e^{-\varrho^2 x^2} J_p(\alpha x) J_p(\beta x) x\,dx = \frac{1}{2\varrho^2}\exp\left(-\frac{\alpha^2+\beta^2}{4\varrho^2}\right) I_p\left(\frac{\alpha\beta}{2\varrho^2}\right)$$
$$\left[\mathrm{Re}\ p > -1, \quad |\arg\varrho| < \frac{\pi}{4}, \quad \alpha > 0, \quad \beta > 0\right].$$

KU 146(16)A, WA 433(1)

ET II 347(59)

4. $$\int_0^\infty xe^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x)\, dx = \frac{1}{2\alpha} \exp\left(\frac{\beta^2-\gamma^2}{4\alpha}\right) J_\nu\left(\frac{\beta\gamma}{2\alpha}\right) \qquad [\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \nu > -1].$$

ET II 63(1)

5. $$\int_0^\infty x^{\lambda-1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x)\, dx =$$
$$= 2^{-\nu-\mu-1} \alpha^{-\frac{1}{2}(\nu+\lambda+\mu)} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(\mu+1)\Gamma(\nu+1)} \times$$
$$\times\ {}_3F_3\left[\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2},\ \frac{\nu}{2} + \frac{\mu}{2} + 1,\ \frac{\nu+\mu+\lambda}{2};\ \mu+1,\ \nu+1,\ \mu+\nu+1;\ -\frac{\beta^2}{\alpha}\right]$$
$$[\mathrm{Re}\,(\nu+\lambda+\mu) > 0, \quad \mathrm{Re}\ \alpha > 0].$$

WA 434, EH II 50(21)

6.634

$$\int_0^\infty xe^{-\frac{x^2}{2a}} [I_\nu(x) + I_{-\nu}(x)] K_\nu(x)\, dx = ae^a K_\nu(a) \qquad [\mathrm{Re}\ a > 0, \quad -1 < \mathrm{Re}\ \nu < 1].$$

ET II 371(49)

6.635

1. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} J_\nu(\beta x)\, dx = 2J_\nu\left(\sqrt{2\alpha\beta}\right) K_\nu\left(\sqrt{2\alpha\beta}\right) \qquad [\mathrm{Re}\ \alpha > 0, \quad \beta > 0].$$

ET II 30(15)

2. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} N_\nu(\beta x)\, dx = 2N_\nu\left(\sqrt{2\alpha\beta}\right) K_\nu\left(\sqrt{2\alpha\beta}\right) \qquad [\mathrm{Re}\ \alpha > 0, \quad \beta > 0].$$

ET II 106(5)

3. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x} - \beta x} J_\nu(\gamma x)\, dx = 2J_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2} - \beta\right]^{\frac{1}{2}}\right\} K_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2} + \beta\right]^{\frac{1}{2}}\right\}$$
$$[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad \gamma > 0].$$

ET II 30(16)

6.636

$$\int_0^\infty x^{-\frac{1}{2}} e^{-\alpha\sqrt{x}} J_\nu(\beta x)\, dx = \frac{\sqrt{2}}{\sqrt{\pi\beta}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}}\alpha e^{\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}) D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}}\alpha e^{-\frac{1}{4}\pi i}\beta^{-\frac{1}{2}})$$

$$\left[\mathrm{Re}\ \alpha > 0, \quad \beta > 0, \quad \mathrm{Re}\ \nu > -\frac{1}{2}\right].$$

ET II 30(17)

6.637

1. $$\int_0^\infty (\beta^2+x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2+x^2)^{\frac{1}{2}}\right] J_\nu(\gamma x)\, dx = I_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[(\alpha^2+\gamma^2)^{\frac{1}{2}} - \alpha\right]\right\} \times$$
$$\times K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[(\alpha^2+\gamma^2)^{\frac{1}{2}} + \alpha\right]\right\}$$
$$[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad \gamma > 0, \quad \mathrm{Re}\ \nu > -1].$$

ET II 31(20)

2. $$\int_0^\infty (\beta^2+x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2+x^2)^{\frac{1}{2}}\right] N_\nu(\gamma x)\, dx =$$
$$= -\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[(\alpha^2+\gamma^2)^{\frac{1}{2}} + \alpha\right]\right\} \times$$
$$\times \left(\frac{1}{\pi} K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[(\alpha^2+\gamma^2)^{\frac{1}{2}} + \alpha\right]\right\} + \sin\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[(\alpha^2+\gamma^2)^{\frac{1}{2}} - \alpha\right]\right\}\right)$$
$$[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad \gamma > 0, \quad |\mathrm{Re}\ \nu| < 1].$$

ET II 106(6)

3. $$\int_0^\infty (x^2+\beta^2)^{-\frac{1}{2}} \exp\left[-\alpha(x^2+\beta^2)^{\frac{1}{2}}\right] K_\nu(\gamma x)\, dx =$$
$$= \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\alpha + (\alpha^2-\gamma^2)^{\frac{1}{2}}\right]\right\} K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\alpha - (\alpha^2-\gamma^2)^{\frac{1}{2}}\right]\right\}$$
$$[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad \mathrm{Re}(\gamma+\beta) > 0, \quad |\mathrm{Re}\ \nu| < 1].$$

ET II 132(26)

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and titles

6.641

$$\int_0^\infty \sqrt{x} e^{-\alpha x} J_{\pm\frac{1}{4}}(x^2)\, dx = \frac{\sqrt{\pi\alpha}}{4}\left[\mathbf{H}_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) - N_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right)\right].$$

6.642

1. $$\int_0^\infty x^{-1} e^{-\alpha x} N_\nu\left(\frac{2}{x}\right) dx = N_\nu\left(\sqrt{\alpha}\right) K_\nu\left(\sqrt{\alpha}\right).$$

MI 44

2. $$\int_0^\infty x^{-1} e^{-\alpha x} H_\nu^{(1,2)}\left(\frac{2}{x}\right) dx = H_\nu^{(1,2)}\left(\sqrt{\alpha}\right) K_\nu\left(\sqrt{\alpha}\right).$$

MI 44, EH II 91(26)

6.643

1. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} J_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\beta\Gamma(2\nu+1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \qquad \left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right)>0\right], \quad (\text{cf. } \mathbf{6.631}\ 1.).$$

6.631

BU 14(13a), MI 42a

2. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} I_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \quad \left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right)>0\right].$$

MI 45

3. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} K_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}{2\beta} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} W_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \quad \left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right)>0\right], \qquad (\text{cf. } \mathbf{6.631}\ 3.).$$

6.631

MI 47a

MO 178a

5. $$\int_0^\infty x^{-\frac{1}{2}} e^{-\alpha x} N_{2\nu}\left(\beta\sqrt{x}\right)\,dx = -\sqrt{\frac{\pi}{\alpha}}\frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)}\left[\sin(\nu\pi) I_\nu\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi}K_\nu\left(\frac{\beta^2}{8\alpha}\right)\right] \quad \left[|\operatorname{Re}\,\nu| < \frac{1}{2}\right].$$

MI 44

6. $$\int_0^\infty x^{\frac{1}{2}m} e^{-\alpha x} K_m\left(2\sqrt{x}\right)\,dx = \frac{\Gamma(m+1)}{2\alpha}\left(\frac{1}{\alpha}\right)^{\frac{1}{2}m-\frac{1}{2}} e^{\frac{1}{2\alpha}} W_{-\frac{1}{2}(m+1),\,-\frac{1}{2}m}\left(\frac{1}{\alpha}\right).$$

MI 48a

6.644

$$\int_0^\infty e^{-\beta x} J_{2\nu}\left(2a\sqrt{x}\right) J_\nu(bx)\,dx = \exp\left(-\frac{a^2\beta}{\beta^2+b^2}\right) J_\nu\left(\frac{a^2 b}{\beta^2+b^2}\right)\frac{1}{\sqrt{\beta^2+b^2}} \quad \left[\operatorname{Re}\,\beta > 0, \quad b > 0, \quad \operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

ET II 58(17)

6.645

1. $$\int_1^\infty (x^2-1)^{-\frac{1}{2}} e^{-\alpha x} J_\nu\left(\beta\sqrt{x^2-1}\right)\,dx = I_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right] K_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)\right].$$

MO 179a

2. $$\int_1^\infty (x^2-1)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu\left(\beta\sqrt{x^2-1}\right)\,dx = \sqrt{\frac{2}{\pi}}\beta^\nu\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} K_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2}\right).$$

MO 179a

3.[3] $$\int_{-1}^1 (1-x^2)^{-1/2} e^{-ax} I_1\left(b\sqrt{1-x^2}\right)\,dx = \frac{2}{b}\left\{\operatorname{ch}\sqrt{a^2+b^2} - \operatorname{ch} a\right\} \qquad [a > 0, \quad b > 0].$$

6.646

EF 89(52), MO 179

2. $$\int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} I_\nu\left(\beta\sqrt{x^2-1}\right)\, dx = \frac{\exp\left(-\sqrt{\alpha^2-\beta^2}\right)}{\sqrt{\alpha^2-\beta^2}} \left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^\nu$$
$$[\mathrm{Re}\ \nu > -1, \quad \alpha > \beta].$$

MO 180

3.[7] $$\int_b^\infty e^{-pt}\left(\frac{t-b}{t+b}\right)^{\nu/2} K_\nu\left[a(t^2-b^2)^{1/2}\right]\, dt = \frac{\Gamma(\nu+1)}{2sa^\nu}\left[x^\nu e^{-bs}\Gamma(-\nu,\, bx) - y^\nu e^{bs}\Gamma(-\nu,\, by)\right]$$
where $x = p-s, y = p+s, s = (p^2-a^2)^{1/2}$ $\quad [\mathrm{Re}\,(p+a) > 0, \quad |\mathrm{Re}\,(\nu)| < 1].$

ME 39a

6.647

1. $$\int_0^\infty x^{-\lambda-\frac{1}{2}}(\beta+x)^{\lambda-\frac{1}{2}} e^{-\alpha x} K_{2\mu}\left[\sqrt{x(\beta+x)}\right]\, dx = \frac{1}{\beta} e^{\frac{1}{2}\alpha\beta}\Gamma\left(\frac{1}{2}-\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda-\mu\right) W_{\lambda,\,\mu}(z_1)\times$$
$$\times W_{\lambda,\,\mu}(z_2),$$
$$z_1 = \frac{1}{2}\beta\left(\alpha+\sqrt{\alpha^2-1}\right),$$
$$z_2 = \frac{1}{2}\beta\left(\alpha-\sqrt{\alpha^2-1}\right),$$
$$\left[|\arg\beta| < \pi, \quad \mathrm{Re}\ \alpha > -1, \quad \mathrm{Re}\ \lambda + |\,\mathrm{Re}\ \mu| < \frac{1}{2}\right].$$

ET II 377(37)

2. $$\int_0^\infty (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x\ \mathrm{ch}\, t} K_\nu\left[\sqrt{x(\alpha+x)}\right]\, dx = \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right) e^{\frac{1}{2}\alpha\ \mathrm{ch}\, t} K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^t\right) K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^{-t}\right)$$
$$[-1 < \mathrm{Re}\ \nu < 1].$$

ET II 377(36)

3.[7] $$\int_0^\alpha x^{\lambda-\frac{1}{2}}(\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x\ \mathrm{sh}\, t} I_{2\mu}[\sqrt{x(\alpha-x)}]\, dx =$$
$$= e^{-(\alpha/2)\ \mathrm{sh}\, t} \frac{2\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha[\Gamma(2\mu+1)]^2} M_{\lambda,\,\mu}\left(\frac{1}{2}\alpha e^t\right) M_{-\lambda,\,\mu}\left(\frac{1}{2}\alpha e^{-t}\right)$$
$$\left[\mathrm{Re}\ \mu > |\,\mathrm{Re}\ \lambda| - \frac{1}{2}\right].$$

$$\int_{-\infty}^{\infty} e^{\varrho x} \left(\frac{\alpha+\beta e^x}{\alpha e^x+\beta}\right)^{\nu} K_{2\nu}\left[(\alpha^2+\beta^2+2\alpha\beta \operatorname{ch} x)^{\frac{1}{2}}\right]\, dx = 2K_{\nu+\varrho}(\alpha)K_{\nu-\varrho}(\beta)$$
$$[\operatorname{Re}\, \alpha > 0, \quad \operatorname{Re}\, \beta > 0].$$

ET II 379(45)

6.649

1. $$\int_0^{\infty} K_{\mu-\nu}(2z \operatorname{sh} x) e^{(\nu+\mu)x}\, dx = \frac{\pi^2}{4\sin[(\nu-\mu)\pi]}\left[J_\nu(z)N_\mu(z) - J_\mu(z)N_\nu(z)\right]$$
$$[\operatorname{Re}\, z > 0, \quad -1 < \operatorname{Re}(\nu-\mu) < 1].$$

MO 44

2. $$\int_0^{\infty} J_{\nu+\mu}(2x \operatorname{sh} t) e^{(\nu-\mu)t}\, dt = K_\nu(x) I_\mu(x) \qquad \left[\operatorname{Re}(\nu-\mu) < \frac{3}{2}, \quad \operatorname{Re}(\nu+\mu) > -1, \quad x > 0\right].$$

EH II 97(68)

3. $$\int_0^{\infty} N_{\nu-\mu}(2x \operatorname{sh} t) e^{-(\nu+\mu)t}\, dt = \frac{1}{\sin[\pi(\mu-\nu)]}\left\{I_\mu(x)K_\nu(x) - \cos[(\nu-\mu)\pi] I_\nu(x)K_\mu(x)\right\}$$
$$\left[|\operatorname{Re}(\nu-\mu)| < 1, \quad \operatorname{Re}(\nu+\mu) > -\frac{1}{2}, \quad x > 0\right].$$

EH II 97(73)

4. $$\int_0^{\infty} K_0(2z \operatorname{sh} x) e^{-2\nu x}\, dx = -\frac{\pi}{4}\left\{J_\nu(z)\frac{\partial N_\nu(z)}{\partial \nu} - N_\nu(z)\frac{\partial J_\nu(z)}{\partial \nu}\right\}.$$

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.651

1. $$\int_0^{\infty} x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} I_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\, dx = \frac{1}{\sqrt{2\pi}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{21}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\, \begin{matrix} 1-\mu,\ 1+\mu \\ h,\ \frac{1}{2},\ k \end{matrix}\right),$$
$$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu,$$
$$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu,$$
$$\left[|\arg \alpha| < \frac{\pi}{4}, \quad \beta > 0, \quad -\frac{3}{2} - \operatorname{Re}(2\mu+\nu) < \operatorname{Re}\, \lambda < 0\right].$$

ET II 68(8)

2. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\,dx = \sqrt{\frac{\pi}{2}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{12}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\,\begin{matrix}1-\mu,\ 1+\mu\\ h,\ \frac{1}{2},\ k\end{matrix}\right),$$
$$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu,$$
$$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu,$$
$$\left[|\arg\alpha| < \frac{\pi}{4}, \quad \mathrm{Re}\,(\lambda+\nu\pm 2\mu) > -\frac{3}{2}\right].$$

ET II 69(15)

3. $$\int_0^\infty x^{2\mu-\nu+1} e^{-\frac{1}{4}\alpha x^2} I_\mu\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x)\,dx =$$
$$= 2^{\mu-\nu+\frac{1}{2}} (\pi\alpha)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}+\mu\right) \frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)}\, {}_1F_1\left(\frac{1}{2}+\mu;\ \frac{1}{2}-\mu+\nu;\ -\frac{\beta^2}{2\alpha}\right)$$
$$\left[\mathrm{Re}\ \alpha > 0, \quad \beta > 0, \quad \mathrm{Re}\ \nu > 2\ \mathrm{Re}\ \mu + \frac{1}{2} > -\frac{1}{2}\right].$$

ET II 68(6)

4. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\,dx =$$
$$= \sqrt{\pi} 2^\mu \alpha^{-2\mu-2\nu-2} \beta^\nu \frac{\Gamma(1+2\mu+\nu)}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)}\, {}_1F_1\left(1+2\mu+\nu;\ \mu+\nu+\frac{3}{2};\ -\frac{\beta^2}{2\alpha^2}\right)$$
$$\left[|\arg\alpha| < \frac{1}{4}\pi, \quad \mathrm{Re}\ \nu > -1, \quad \mathrm{Re}\,(2\mu+\nu) > -1, \quad \beta > 0\right].$$

ET II 69(13)

5. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha x^2} I_\mu\left(\frac{1}{2}\alpha x^2\right) K_\nu(\beta x)\,dx =$$
$$= \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \beta^{-\mu-\frac{3}{2}} \alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} \Gamma(2\mu+\nu+1) \Gamma\left(\mu+\frac{1}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{k,\,m}\left(\frac{\beta^2}{4\alpha}\right),$$
$$2k = -3\mu-\nu-\frac{1}{2},$$
$$2m = \mu+\nu+\frac{1}{2}$$
$$\left[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \mu > -\frac{1}{2}, \quad \mathrm{Re}\,(2\mu+\nu) > -1\right].$$

ET II 146(53)

745

6. $$\int_0^\infty xe^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu}\left(\frac{1}{4}\beta x^2\right) J_\nu(\gamma x)\,dx = 2(\alpha^2+\beta^2)^{-\frac{1}{2}} \exp\left(-\frac{\alpha\gamma^2}{\alpha^2+\beta^2}\right) J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{\alpha^2+\beta^2}\right)$$
$$[\gamma > 0, \quad \operatorname{Re}\,\alpha > |\operatorname{Im}\beta|, \quad \operatorname{Re}\,\nu > -1].$$

ET II 56(2)

7. $$\int_0^\infty xe^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x)\,dx = \left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{2}} \beta^{-1} \exp\left(-\frac{\beta^2}{2\alpha}\right)$$
$$[\operatorname{Re}\,\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\,\nu > -1].$$

ET II 67(3)

8. $$\int_0^\infty x^{1-\nu} e^{-\frac{1}{4}\alpha^2 x^2} I_\nu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\,dx = \sqrt{\frac{2}{\pi}}\frac{\beta^{\nu-1}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu}\left(\frac{\beta}{\alpha}\right)$$
$$\left[|\arg\alpha| < \frac{1}{4}\pi, \quad \beta > 0, \quad \operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

ET II 67(1)

9. $$\int_0^\infty x^{-\nu-1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu+1}\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\,dx = \sqrt{\frac{2}{\pi}}\beta^\nu \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu-3}\left(\frac{\beta}{\alpha}\right)$$
$$\left[|\arg\alpha| < \frac{1}{4}\pi, \quad \operatorname{Re}\,\nu > -1, \quad \beta > 0\right].$$

ET II 67(2)

6.652

$$\int_0^\infty x^{2\nu} e^{-\left(\frac{x^2}{8}+\alpha x\right)} I_\nu\left(\frac{x^2}{8}\right)\,dx = \frac{\Gamma(4\nu+1)}{2^{4\nu}\Gamma(\nu+1)}\frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}} W_{-\frac{3}{2}\nu,\,\frac{1}{2}\nu}(\alpha^2) \qquad \left[\operatorname{Re}\left(\nu+\frac{1}{4}\right) > 0\right].$$

MI 45

6.653

1. $$\int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(a^2+b^2)\right] I_\nu\left(\frac{ab}{x}\right)\frac{dx}{x} = 2I_\nu(a)K_\nu(b) \qquad [0 < a < b];$$
$$= 2K_\nu(a)I_\nu(b) \qquad [0 < b < a] \qquad [\operatorname{Re}\,\nu > -1].$$

2. $$\int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(z^2+w^2)\right] K_\nu\left(\frac{zw}{x}\right)\frac{dx}{x} = 2K_\nu(z)K_\nu(w)$$
$$\left[|\arg z| < \pi, \quad |\arg w| < \pi, \quad |\arg(z+w)| < \frac{1}{4}\pi\right].$$

WA 483(1), EH II 53(36)

6.654

$$\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x}-\alpha x} K_\nu\left(\frac{\beta^2}{8x}\right) dx = \sqrt{4\pi}\alpha^{-\frac{1}{2}} K_{2\nu}\left(\beta\sqrt{\alpha}\right).$$

ME 39

6.655

$$\int_0^\infty x(\beta^2+x^2)^{-\frac{1}{2}} \exp\left(-\frac{\alpha^2\beta}{\beta^2+x^2}\right) J_\nu\left(\frac{\alpha^2 x}{\beta^2+x^2}\right) J_\nu(\gamma x)\, dx = \gamma^{-1} e^{-\beta\gamma} J_{2\nu}\left(2\alpha\sqrt{\gamma}\right)$$
$$\left[\operatorname{Re}\, \beta > 0, \quad \gamma > 0, \quad \operatorname{Re}\, \nu > -\frac{1}{2}\right].$$

ET II 58(14)

6.656

1. $$\int_0^\infty e^{-(\xi-z)\operatorname{ch} t} J_{2\nu}[2(z\xi)^{\frac{1}{2}} \operatorname{sh} t]\, dt = I_\nu(z)K_\nu(\xi) \qquad \left[\operatorname{Re}\, \nu > -\frac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0\right].$$

EH II 98(78)

2. $$\int_0^\infty e^{-(\xi+z)\operatorname{ch} t} K_{2\nu}[2(z\xi)^{\frac{1}{2}} \operatorname{sh} t]\, dt = \frac{1}{2}K_\nu(z)K_\nu(\xi)\sec(\nu\pi) \qquad \left[|\operatorname{Re}\nu| < \frac{1}{2},\ \operatorname{Re}(z^{\frac{1}{2}} + \xi^{\frac{1}{2}})^2 \geq 0\right].$$

EH II 98(79)

6.66 Combinations of Bessel, hyperbolic, and exponential functions

Bessel and hyperbolic functions

6.661

1. $$\int_0^\infty \operatorname{sh}(ax)K_\nu(bx)\, dx = \frac{\pi}{2}\frac{\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\sin\left[\nu \arcsin\left(\frac{a}{b}\right)\right]}{\sqrt{b^2-a^2}} \qquad [\operatorname{Re}\, b > |\operatorname{Re}\, a|, \quad |\operatorname{Re}\, \nu| < 2].$$

ET II 133(32)

ET II 133(32)

2. $$\int_0^\infty \operatorname{ch}(ax) K_\nu(bx)\,dx = \frac{\pi \cos\left[\nu \arcsin\left(\frac{a}{b}\right)\right]}{2\sqrt{b^2-a^2}\cos\left(\frac{\nu\pi}{2}\right)} \qquad [\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re}\nu| < 1].$$

ET II 134(33)

6.662

1. $$\int_0^\infty \operatorname{ch}(\beta x) K_0(\alpha x) J_0(\gamma x)\,dx = \frac{\boldsymbol{K}(k)}{\sqrt{u+v}},$$
$$u = \frac{1}{2}\left\{[(\alpha^2+\beta^2+\gamma^2)^2 - 4\alpha^2\beta^2]^{\frac{1}{2}} + \alpha^2 - \beta^2 - \gamma^2\right\},$$
$$v = \frac{1}{2}\left\{[(\alpha^2+\beta^2+\gamma^2)^2 - 4\alpha^2\beta^2]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2\right\},$$
$$k^2 = v(u+v)^{-1} \qquad [\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \gamma > 0].$$

ET II 15(23)

2. $$\int_0^\infty \operatorname{sh}(\beta x) K_1(\alpha x) J_0(\gamma x)\,dx = a^{-1}\left[u\boldsymbol{E}(k) - \boldsymbol{K}(k)E(u) + \frac{\boldsymbol{K}(k)\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right],$$
$$cn^2 u = 2\gamma^2\left\{\left[(\alpha^2+\beta^2+\gamma^2)^2 - 4\alpha^2\beta^2\right]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2\right\}^{-1},$$
$$k^2 = \frac{1}{2}\left\{1 - (\alpha^2-\beta^2-\gamma^2)[(\alpha^2+\beta^2+\gamma^2)^2 - 4\alpha^2\beta^2]^{-\frac{1}{2}}\right\} \qquad [\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \gamma > 0].$$

ET II 15(24)

6.663

1. $$\int_0^\infty K_{\nu\pm\mu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu\mp\nu)t]\,dt = \frac{1}{2}K_\mu(z)K_\nu(z) \qquad [\operatorname{Re} z > 0].$$

WA 484(1), EH II 54(39)

2. $$\int_0^\infty N_{\mu+\nu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu-\nu)t]\,dt = \frac{\pi}{4}[J_\mu(z)J_\nu(z) - N_\mu(z)N_\nu(z)] \qquad [z > 0].$$

EH II 96(64)

3. $$\int_0^\infty J_{\mu+\nu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu-\nu)t]\,dt = -\frac{\pi}{4}[J_\mu(z)N_\nu(z) + J_\nu(z)N_\mu(z)] \qquad [z > 0].$$

4. $$\int_0^\infty J_{\mu+\nu}(2z\,\operatorname{sh} t)\,\operatorname{ch}[(\mu-\nu)t]\,dt = \frac{1}{2}[I_\nu(z)K_\mu(z) + I_\mu(z)K_\nu(z)]$$
$$\left[\operatorname{Re}(\nu+\mu) > -1, \quad |\operatorname{Re}(\mu-\nu)| < \frac{3}{2}, \quad z > 0\right].$$

EH II 97(71)

5. $$\int_0^\infty J_{\mu+\nu}(2z\,\operatorname{sh} t)\,\operatorname{sh}[(\mu-\nu)t]\,dt = \frac{1}{2}[I_\nu(z)K_\mu(z) - I_\mu(z)K_\nu(z)]$$
$$\left[\operatorname{Re}(\nu+\mu) > -1, \quad |\operatorname{Re}(\mu-\nu)| < \frac{3}{2}, \quad z > 0\right].$$

EH II 97(72)

6.664

1. $$\int_0^\infty J_0(2z\,\operatorname{sh} t)\,\operatorname{sh}(2\nu t)\,dt = \frac{\sin(\nu\pi)}{\pi}[K_\nu(z)]^2 \qquad \left[|\operatorname{Re}\,\nu| < \frac{3}{4}, \quad z > 0\right].$$

EH II 97(69)

2. $$\int_0^\infty N_0(2z\,\operatorname{sh} t)\,\operatorname{ch}(2\nu t)\,dt = -\frac{\cos(\nu\pi)}{\pi}[K_\nu(z)]^2 \qquad \left[|\operatorname{Re}\,\nu| < \frac{3}{4}, \quad z > 0\right].$$

EH II 97(70)

3. $$\int_0^\infty N_0(2z\,\operatorname{sh} t)\,\operatorname{sh}(2\nu t)\,dt = \frac{1}{\pi}\left[I_\nu(z)\frac{\partial K_\nu(z)}{\partial\nu} - K_\nu(z)\frac{\partial I_\nu(z)}{\partial\nu}\right] - \frac{1}{\pi}\cos(\nu\pi)[K_\nu(z)]^2$$
$$\left[|\operatorname{Re}\,\nu| < \frac{3}{4}, \quad z > 0\right].$$

EH II 97(75)

4. $$\int_0^\infty K_0(2z\,\operatorname{sh} t)\,\operatorname{ch} 2\nu t\,dt = \frac{\pi^2}{8}\left\{J_\nu^2(z) + N_\nu^2(z)\right\} \qquad [\operatorname{Re}\,z > 0].$$

MO 44

5. $$\int_0^\infty K_{2\mu}(z\,\operatorname{sh} 2t)\,\operatorname{cth}^{2\nu} t\,dt = \frac{1}{4z}\Gamma\left(\frac{1}{2}+\mu-\nu\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)W_{\nu,\,\mu}(iz)W_{\nu,\,\mu}(-iz)$$
$$\left[|\arg z| \le \frac{\pi}{2}, \quad |\operatorname{Re}\,\mu| + \operatorname{Re}\,\nu < \frac{1}{2}\right].$$

MO 119

6. $\int_0^\infty \operatorname{ch}(2\mu x) K_{2\nu}(2a \operatorname{ch} x)\, dx = \frac{1}{2} K_{\mu+\nu}(a) K_{\mu-\nu}(a) \qquad [\operatorname{Re} a > 0].$

ET II 378(42)

$$\int_0^\infty \operatorname{sech} x \operatorname{ch}(2\lambda x) I_{2\mu}(a \operatorname{sech} x)\, dx = \frac{\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{2a[\Gamma(2\mu+1)]^2} M_{\lambda,\,\mu}(a) M_{-\lambda,\,\mu}(a)$$
$$\left[|\operatorname{Re} \lambda| - \operatorname{Re} \mu < \frac{1}{2}\right].$$

ET II 378(43)

Bessel, hyperbolic, and algebraic functions

6.666

$$\int_0^\infty x^{\nu+1} \operatorname{sh}(\alpha x) \operatorname{cosech} \pi x J_\nu(\beta x)\, dx = \frac{2}{\pi} \sum_{n=1}^\infty (-1)^{n-1} n^{\nu+1} \sin(n\alpha) K_\nu(n\beta)$$
$$[|\operatorname{Re} \alpha| < \pi, \quad \operatorname{Re} \nu > -1].$$

ET II 41(3), WA 469(12)

6.667

1.[3] $\int_0^a y^{-1} \operatorname{ch}(y \operatorname{sh} t) I_{2\nu}(x)\, dx = \frac{\pi}{2} I_\nu\left(\frac{1}{2} a e^t\right) I_\nu\left(\frac{1}{2} a e^{-t}\right),$
$$y = (a^2 - x^2)^{\frac{1}{2}} \qquad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 365(10)

2. $\int_0^a y^{-1} \operatorname{ch}(y \operatorname{sh} t) K_{2\nu}(x)\, dx = \frac{\pi^2}{4} \operatorname{cosec}(\nu\pi)[I_{-\nu}(ae^t) I_{-\nu}(ae^{-t}) - I_\nu(ae^t) I_\nu(ae^{-t})],$
$$y = (a^2 - x^2)^{\frac{1}{2}} \qquad \left[|\operatorname{Re} \nu| < \frac{1}{2}\right].$$

ET II 367(25)

6.668

1. $$\int_0^\infty e^{-\alpha x}\ \mathrm{sh}(\beta x) J_0(\gamma x)\, dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}},$$
$$r_1 = [\gamma^2 + (\beta - \alpha)^2]^{\frac{1}{2}}, \qquad r_2 = [\gamma^2 + (\beta + \alpha)^2]^{\frac{1}{2}} \qquad [\mathrm{Re}\ \alpha > |\,\mathrm{Re}\ \beta|, \quad \gamma > 0].$$

ET II 12(52)

2. $$\int_0^\infty e^{-\alpha x}\ \mathrm{ch}(\beta x) J_0(\gamma x)\, dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 + r_1)^{\frac{1}{2}} (r_2 - r_1)^{-\frac{1}{2}},$$
$$r_1 [\gamma^2 + (\beta - \alpha)^2]^{\frac{1}{2}}, \qquad r_2 = [\gamma^2 + (\beta + \alpha)^2]^{\frac{1}{2}} \qquad [\mathrm{Re}\ \alpha > |\,\mathrm{Re}\ \beta|, \quad \gamma > 0].$$

ET II 12(54)

6.669

1. $$\int_0^\infty \left[\mathrm{cth}\left(\frac{1}{2}x\right)\right]^{2\lambda} e^{-\beta\ \mathrm{ch}\, x} J_{2\mu}(\alpha\ \mathrm{sh}\, x)\, dx = \frac{\Gamma\left(\frac{1}{2} - \lambda + \mu\right)}{\alpha\Gamma(2\mu + 1)} M_{-\lambda,\,\mu}\left[(\alpha^2 + \beta^2)^{\frac{1}{2}} - \beta\right] \times$$
$$\times W_{\lambda,\,\mu}\left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \beta\right]$$
$$\left[\mathrm{Re}\ \beta > |\,\mathrm{Re}\ \alpha|, \quad \mathrm{Re}\,(\mu - \lambda) > -\frac{1}{2}\right].$$

BU 86(5b)A, ET II 363(34)

2. $$\int_0^\infty \left[\mathrm{cth}\left(\frac{1}{2}x\right)\right]^{2\lambda} e^{-\beta\ \mathrm{ch}\, x} N_{2\mu}(\alpha\ \mathrm{sh}\, x)\, dx =$$
$$= -\frac{\sec[(\mu + \lambda)\pi]}{\alpha} W_{\lambda,\,\mu}\left(\sqrt{\alpha^2 + \beta^2} + \beta\right) W_{-\lambda,\,\mu}\left(\sqrt{\alpha^2 + \beta^2} - \beta\right) -$$
$$- \frac{\mathrm{tg}[(\mu + \lambda)\pi]\Gamma\left(\frac{1}{2} - \lambda + \mu\right)}{\alpha\Gamma(2\mu + 1)} W_{\lambda,\,\mu}\left(\sqrt{\alpha^2 + \beta^2} + \beta\right) M_{-\lambda,\,\mu}\left(\sqrt{\alpha^2 + \beta^2} - \beta\right)$$
$$\left[\mathrm{Re}\ \beta > |\,\mathrm{Re}\ \alpha|, \quad \mathrm{Re}\ \lambda < \frac{1}{2} - |\,\mathrm{Re}\ \mu|\right].$$

ET II 363(35)

3. $$\int_0^\infty e^{-\frac{1}{2}(a_1 + a_2)t\ \mathrm{ch}\, x} \left[\mathrm{cth}\left(\frac{1}{2}x\right)\right]^{2\nu} K_{2\mu}\left(t\sqrt{a_1 a_2}\,\mathrm{sh}\, x\right)\, dx = \frac{\Gamma\left(\frac{1}{2} + \mu - \nu\right)\Gamma\left(\frac{1}{2} - \mu - \nu\right)}{2t\sqrt{a_1 a_2}} W_{\nu,\mu}(a_1 t) \times$$
$$\times W_{\nu,\,\mu}(a_2 t)$$
$$\left[\mathrm{Re}\,\nu < \mathrm{Re}\,\frac{1 \pm 2\mu}{2},\ \mathrm{Re}\left[t(\sqrt{a_1} + \sqrt{a_2})^2\right] > 0\right].$$

4. $$\int_0^\infty e^{-\frac{1}{2}(a_1+a_2)t\,\operatorname{ch} x}\left[\operatorname{cth}\left(\frac{x}{2}\right)\right]^{2\nu} I_{2\mu}\left(t\sqrt{a_1a_2}\,\operatorname{sh} x\right)\,dx = \frac{\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{t\sqrt{a_1a_2}\Gamma(1+2\mu)} W_{\nu,\,\mu}(a_1t)M_{\nu,\,\mu}(a_2t)$$
$$\left[\operatorname{Re}\left(\frac{1}{2}+\mu-\nu\right)>0, \quad \operatorname{Re}\,\mu>0, \quad a_1>a_2\right].$$

BU 86(5c)

5. $$\int_{-\infty}^\infty e^{2\nu s-\frac{x-y}{2}\,\theta s} I_{2\mu}\left(\frac{\sqrt{xy}}{\operatorname{ch} s}\right)\frac{ds}{\operatorname{ch} s} = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\sqrt{xy}[\Gamma(1+2\mu)]^2} M_{\nu,\,\mu}(x)M_{-\nu,\,\mu}(y)$$
$$\left[\operatorname{Re}\left(\pm\nu+\frac{1}{2}+\mu\right)>0\right].$$

BU 83(3a)a

6. $$\int_{-\infty}^\infty e^{2\nu s-\frac{x+y}{2}\,\theta s} J_{2\mu}\left(\frac{\sqrt{xy}}{\operatorname{ch} s}\right)\frac{ds}{\operatorname{ch} s} = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\sqrt{xy}[\Gamma(1+2\mu)]^2} M_{\nu,\,\mu}(x)M_{\nu,\,\mu}(y)$$
$$\left[\operatorname{Re}\left(\mp\nu+\frac{1}{2}+\mu\right)>0\right]$$

BU 84(3b)a

6.67- 6.68 Combinations of Bessel and trigonometric functions

6.671

1. $$\left.\begin{aligned}\int_0^\infty J_\nu(\alpha x)\sin\beta x\,dx &= \frac{\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} && [\beta<\alpha];\\ &=\infty \quad \text{or} \quad 0 && [\beta=\alpha];\\ &= \frac{\alpha^\nu\,\cos\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} && [\beta>\alpha].\end{aligned}\right\} \qquad [\operatorname{Re}\,\nu>-2].$$

WA 444(4)

2. $$\left.\begin{aligned}\int_0^\infty J_\nu(\alpha x)\cos\beta x\,dx &= \frac{\cos\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} && [\beta<\alpha];\\ &=\infty \quad \text{or} \quad 0 && [\beta=\alpha];\\ &= \frac{-\alpha^\nu\sin\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} && [\beta>\alpha].\end{aligned}\right\} \qquad [\operatorname{Re}\,\nu>-1].$$

3. $$\int_0^\infty N_\nu(ax)\sin(bx)\,dx = \operatorname{ctg}\left(\frac{\nu\pi}{2}\right)(a^2-b^2)^{-\frac{1}{2}}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a,\ |\operatorname{Re}\nu|<2];$$
$$= \frac{1}{2}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)(b^2-a^2)^{-\frac{1}{2}}\times$$
$$\times\left\{a^{-\nu}\cos(\nu\pi)\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{\nu} - a^{\nu}\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$$[0<a<b,\quad |\operatorname{Re}\nu|<2].$$

ET I 103(33)

4. $$\int_0^\infty N_\nu(ax)\cos(bx)\,dx = \frac{\operatorname{tg}\left(\frac{\nu\pi}{2}\right)}{(a^2-b^2)^{\frac{1}{2}}}\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a,\quad |\operatorname{Re}\nu|<1];$$
$$= -\sin\left(\frac{\nu\pi}{2}\right)(b^2-a^2)^{-\frac{1}{2}}\left\{a^{-\nu}\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{\nu} + \operatorname{ctg}(\nu\pi)+\right.$$
$$\left.+a^{\nu}\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{-\nu}\operatorname{cosec}(\nu\pi)\right\}$$
$$[0<a<b,\quad |\operatorname{Re}\nu|<1].$$

ET I 47(29)

5. $$\int_0^\infty K_\nu(ax)\sin(bx)\,dx = \frac{1}{4}\pi a^{-\nu}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)(a^2+b^2)^{-\frac{1}{2}}\left\{\left[(b^2+a^2)^{\frac{1}{2}}+b\right]^{\nu} - \left[(b^2+a^2)^{\frac{1}{2}}-b\right]^{\nu}\right\}$$
$$[\operatorname{Re} a>0,\quad b>0,\quad |\operatorname{Re}\nu|<2,\quad \nu\neq 0].$$

ET I 105(48)

6. $$\int_0^\infty K_\nu(ax)\cos(bx)dx = \frac{\pi}{4}(b^2+a^2)^{-\frac{1}{2}}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\left[b+(b^2+a^2)^{\frac{1}{2}}\right]^{\nu} + a^{\nu}\left[b+(b^2+a^2)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$$[\operatorname{Re} a>0,\ b>0,\ |\operatorname{Re}\nu|<1].$$

ET I 49(40)

7. $$\int_0^\infty J_0(ax)\sin(bx)\,dx = 0 \qquad [0<b<a];$$
$$= \frac{1}{\sqrt{b^2-a^2}} \qquad [0<a<b].$$

ET I 99(1)

8. $$\int_0^\infty J_0(ax)\cos(bx)\,dx = \frac{1}{\sqrt{a^2-b^2}} \qquad [0<b<a];$$
$$= \infty \qquad [a=b];$$
$$= 0 \qquad [0<a<b].$$

9. $$\int_0^\infty J_{2n+1}(ax)\sin(bx)\,dx = (-1)^n \frac{1}{\sqrt{a^2-b^2}} T_{2n+1}\left(\frac{b}{a}\right) \qquad [0 < b < a];$$
$$= 0 \qquad [0 < a < b].$$

ET I 99(2)

10. $$\int_0^\infty J_{2n}(ax)\cos(bx)\,dx = (-1)^n \frac{1}{\sqrt{a^2-b^2}} T_{2n}\left(\frac{b}{a}\right) \qquad [0 < b < a];$$
$$= 0 \qquad [0 < a < b].$$

ET I 43(2)

11. $$\int_0^\infty N_0(ax)\sin(bx)\,dx = \frac{2\arcsin\left(\frac{b}{a}\right)}{\pi\sqrt{a^2-b^2}} \qquad [0 < b < a];$$
$$= \frac{2}{\pi}\frac{1}{\sqrt{b^2-a^2}} \ln\left[\frac{b}{a} - \sqrt{\frac{b^2}{a^2}-1}\right] \qquad [0 < a < b].$$

ET I 103(31)

12. $$\int_0^\infty N_0(ax)\cos(bx)\,dx = 0 \qquad [0 < b < a];$$
$$= -\frac{1}{\sqrt{b^2-a^2}} \qquad [0 < a < b].$$

ET I 47(28)

13. $$\int_0^\infty K_0(\beta x)\sin\alpha x\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \ln\left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2}+1}\right) \qquad [\alpha > 0, \quad \beta > 0].$$

WA 425(11)A, MO 48

14. $$\int_0^\infty K_0(\beta x)\cos\alpha x\,dx = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}} \qquad [\alpha \text{ and } \beta \text{ are real; } \beta > 0].$$

WA 425(10)A, MO 48

6.672

1. $$\int_0^\infty J_\nu(ax)J_\nu(bx)\sin(cx)\,dx =$$
$$= 0 \qquad [\operatorname{Re}\nu > -1, \quad 0 < c < b-a, \quad 0 < a < b];$$
$$= \frac{1}{2\sqrt{ab}} P_{\nu-\frac{1}{2}}\left(\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu > -1, \quad b-a < c < b+a, \quad 0 < a < b];$$
$$= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}} Q_{\nu-\frac{1}{2}}\left(-\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu > -1, \quad b+a < c, \quad 0 < a < b]$$

2. $$\int_0^\infty J_\nu(x)J_{-\nu}(x)\cos(bx)\,dx = \frac{1}{2}P_{\nu-\frac{1}{2}}\left(\frac{1}{2}b^2-1\right) \qquad [0<b<2];$$
$$= 0 \qquad [2<b].$$

ET I 46(21)

3. $$\int_0^\infty K_\nu(ax)K_\nu(bx)\cos(cx)\,dx = \frac{\pi^2}{4\sqrt{ab}}\sec(\nu\pi)P_{\nu-\frac{1}{2}}[(a^2+b^2+c^2)(2ab)^{-1}]$$
$$\left[\mathrm{Re}\,(a+b)>0, \quad c>0, \quad |\,\mathrm{Re}\ \nu|<\frac{1}{2}\right].$$

ET I 50(51)

4. $$\int_0^\infty K_\nu(ax)I_\nu(bx)\cos(cx)\,dx = \frac{1}{2\sqrt{ab}}Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right)$$
$$\left[\mathrm{Re}\ a>|\,\mathrm{Re}\ b|, \quad c>0, \quad \mathrm{Re}\ \nu>-\frac{1}{2}\right].$$

ET I 49(47)

5. $$\int_0^\infty \sin(2ax)[J_\nu(x)]^2\,dx = \frac{1}{2}P_{\nu-\frac{1}{2}}(1-2a^2) \qquad [0<a<1, \quad \mathrm{Re}\ \nu>-1];$$
$$= \frac{1}{\pi}\cos(\nu\pi)Q_{\nu-\frac{1}{2}}(2a^2-1) \qquad [a>1, \quad \mathrm{Re}\ \nu>-1];$$

ET II 343(30)

6. $$\int_0^\infty \cos(2ax)[J_\nu(x)]^2\,dx = \frac{1}{\pi}Q_{\nu-\frac{1}{2}}(1-2a^2) \qquad \left[0<a<1, \quad \mathrm{Re}\ \nu>-\frac{1}{2}\right];$$
$$= -\frac{1}{\pi}\sin(\nu\pi)Q_{\nu-\frac{1}{2}}(2a^2-1) \qquad \left[a>1, \quad \mathrm{Re}\ \nu>-\frac{1}{2}\right].$$

ET II 344(32)

7. $$\int_0^\infty \sin(2ax)J_0(x)N_0(x)\,dx = 0 \qquad [0<a<1];$$
$$= -\frac{\boldsymbol{K}\left[(1-a^{-2})^{\frac{1}{2}}\right]}{\pi a} \qquad [a>1].$$

ET II 348(60)

8. $$\int_0^\infty K_0(ax)I_0(bx)\cos(cx)\,dx = \frac{1}{\sqrt{c^2+(a+b)^2}}\boldsymbol{K}\left\{\frac{2\sqrt{ab}}{\sqrt{c^2+(a+b)^2}}\right\} \qquad [\mathrm{Re}\ a>|\,\mathrm{Re}\ b|, \quad c>0].$$

9. $$\int_0^\infty \cos(2ax) J_0(x) N_0(x)\,dx = -\frac{1}{\pi}\boldsymbol{K}(a) \qquad [0 < a < 1];$$
$$= -\frac{1}{\pi a}\boldsymbol{K}\left(\frac{1}{a}\right) \qquad [a > 1].$$

ET II 348(61)

10. $$\int_0^\infty \cos(2ax)[N_0(x)]^2\,dx = \frac{1}{\pi}\boldsymbol{K}(\sqrt{1-a^2}) \qquad [0 < a < 1];$$
$$= \frac{2}{\pi a}\boldsymbol{K}\left(\sqrt{1-\frac{1}{a^2}}\right) \qquad [a > 1].$$

ET II 348(62)

6.673

1. $$\int_0^\infty \left[J_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) - N_\nu\,(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\sin(bx)\,dx =$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 2];$$
$$= \frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b + (b^2-a^2)^{\frac{1}{2}}\right]^\nu + \left[b - (b^2-a^2)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 2].$$

ET I 104(39)

2. $$\int_0^\infty \left[N_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) + J_\nu\,(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\cos(bx)\,dx =$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 1];$$
$$= -\frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b + (b^2-a^2)^{\frac{1}{2}}\right]^\nu + \left[b - (b^2-a^2)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 1].$$

ET I 48(32)

6.674

1. $$\int_0^a \sin(a-x) J_\nu(x)\,dx = aJ_{\nu+1}(a) - 2\nu\sum_{n=0}^\infty (-1)^n J_{\nu+2n+2}(a) \qquad [\operatorname{Re}\nu > -1].$$

ET II 334(12)

2. $$\int_0^a \cos(a-x) J_\nu(x)\,dx = aJ_\nu(a) - 2\nu\sum_{n=0}^\infty (-1)^n J_{\nu+2n+1}(a) \qquad [\operatorname{Re}\nu > -1].$$

3. $$\int_0^a \sin(a-x) J_{2n}(x)\,dx = aJ_{2n+1}(a) + (-1)^n 2n \left[\cos a - J_0(a) - 2\sum_{m=1}^{n} (-1)^m J_{2m}(a)\right]$$
$$[n = 0,\ 1,\ 2, \ldots].$$

ET II 334(10)

4. $$\int_0^a \cos(a-x) J_{2n}(x)\,dx = aJ_{2n}(a) - (-1)^n 2n \left[\sin a - 2\sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a)\right]$$
$$[n = 0,\ 1,\ 2, \ldots].$$

ET II 335(21)

5. $$\int_0^a \sin(a-x) J_{2n+1}(x)\,dx = aJ_{2n+2}(a) + (-1)^n (2n+1) \left[\sin a - 2\sum_{m=0}^{n} (-1)^m J_{2m+1}(a)\right]$$
$$[n = 0,\ 1,\ 2, \ldots].$$

ET II 334(11)

6. $$\int_0^a \cos(a-x) J_{2n+1}(x)\,dx = aJ_{2n+1}(a) + (-1)^n (2n+1) \left[\cos a - J_0(a) - 2\sum_{m=1}^{n} (-1)^m J_{2m}(a)\right]$$
$$[n = 0,\ 1,\ 2, \ldots].$$

ET II 336(22)

7. $$\int_0^z \sin(z-x) J_0(x)\,dx = zJ_1(z).$$

WA 415(2)

8. $$\int_0^z \cos(z-x) J_0(x)\,dx = zJ_0(z).$$

WA 415(1)

6.675

1. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) \sin(bx)\,dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -4].$$

2. $\int_0^\infty J_\nu\left(a\sqrt{x}\right)\cos(bx)\,dx =$

$$= -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}}\left[\sin\left(\frac{a^2}{8b}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right)+\cos\left(\frac{a^2}{8b}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$

$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-2].$

ET I 53(22)a

3. $\int_0^\infty J_0\left(a\sqrt{x}\right)\sin(bx)\,dx = \frac{1}{b}\cos\left(\frac{a^2}{4b}\right) \qquad [a>0, \quad b>0].$

ET I 110(22)

4. $\int_0^\infty J_0\left(a\sqrt{x}\right)\cos(bx)\,dx = \frac{1}{b}\sin\left(\frac{a^2}{4b}\right) \qquad [a>0, \quad b>0].$

ET I 53(21)

6.676

1. $\int_0^\infty J_\nu\left(a\sqrt{x}\right)J_\nu(b\sqrt{x})\sin(cx)\,dx = \frac{1}{c}J_\nu\left(\frac{ab}{2c}\right)\cos\left(\frac{a^2+b^2}{4c}-\frac{\nu\pi}{2}\right)$

$[a>0, \quad b>0, \quad c>0, \quad \operatorname{Re}\nu>-2].$

ET I 111(29)a

2. $\int_0^\infty J_\nu\left(a\sqrt{x}\right)J_\nu(b\sqrt{x})\cos(cx)\,dx = \frac{1}{c}J_\nu\left(\frac{ab}{2c}\right)\sin\left(\frac{a^2+b^2}{4c}-\frac{\nu\pi}{2}\right)$

$[a>0, \quad b>0, \quad c>0, \quad \operatorname{Re}\nu>-1].$

ET I 54(27)

3. $\int_0^\infty J_0\left(a\sqrt{x}\right)K_0\left(a\sqrt{x}\right)\sin(bx)\,dx = \frac{1}{2b}K_0\left(\frac{a^2}{2b}\right) \qquad [\operatorname{Re} a>0, \quad b>0].$

ET I 111(31)

4. $\int_0^\infty J_0(\sqrt{ax})K_0(\sqrt{ax})\cos(bx)\,dx = \frac{\pi}{4b}\left[I_0\left(\frac{a}{2b}\right)-L_0\left(\frac{a}{2b}\right)\right] \qquad [\operatorname{Re} a>0, \quad b>0].$

ET I 54(29)

5. $\int_0^\infty K_0(\sqrt{ax})N_0(\sqrt{ax})\cos(bx)\,dx = -\frac{1}{2b}K_0\left(\frac{a}{2b}\right) \qquad [\operatorname{Re}\sqrt{a}>0, \quad b>0].$

6. $\int_0^\infty K_0\left(\sqrt{ax}e^{\frac{1}{4}\pi i}\right) K_0\left(\sqrt{ax}e^{-\frac{1}{4}\pi i}\right) \cos(bx)\,dx = \frac{\pi^2}{8b}\left[H_0\left(\frac{a}{2b}\right) - N_0\left(\frac{a}{2b}\right)\right] \quad [\operatorname{Re} a > 0,\ b > 0].$

ET I 54(31)

6.677

1. $\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\sin(cx)\,dx = 0 \qquad [0 < c < b];$

$= \frac{\cos\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c].$

ET I 113(47)

2. $\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\exp\left(-a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b];$

$= \frac{-\sin\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c].$

ET I 57(48)a

3.[6] $\int_0^\infty J_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{\cos z\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-\beta^2}} \qquad [0 < \beta < \alpha, \quad z > 0];$

$= 0 \qquad [0 < \alpha < \beta, \quad z > 0].$

MO 47a

4. $\int_0^\infty N_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right) \qquad [0 < \beta < \alpha, \quad z > 0];$

$= -\frac{1}{\sqrt{\beta^2-\alpha^2}}\exp\left(-z\sqrt{\beta^2-\alpha^2}\right) \qquad [0 < \alpha < \beta, \quad z > 0].$

MO 47a

5. $\int_0^\infty K_0\left[\alpha\sqrt{x^2+\beta^2}\right]\cos(\gamma x)\,dx = \frac{\pi}{2\sqrt{\alpha^2+\gamma^2}}\exp\left(-\beta\sqrt{\alpha^2+\gamma^2}\right)$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0].$

ET I 56(43)

6. $\int_0^a J_0\left(b\sqrt{a^2-x^2}\right)\cos(cx)\,dx = \frac{\sin\left(a\sqrt{b^2+c^2}\right)}{\sqrt{b^2+c^2}} \qquad [b > 0].$

7. $$\int_0^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\operatorname{ch}\left(a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b, \quad a > 0];$$
$$= 0 \qquad [0 < b < c, \quad a > 0].$$

ET I 57(49)

8. $$\int_0^\infty H_0^{(1)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx = -i\frac{\exp\left(i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[\pi > \arg\sqrt{\beta^2-x^2} \geq 0, \quad \alpha > 0, \quad \gamma > 0\right].$$

ET I 59(59)

9. $$\int_0^\infty H_0^{(2)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx = \frac{i\,\exp\left(-i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[-\pi < \arg\sqrt{\beta^2-x^2} \leq 0, \quad \alpha > 0, \quad \gamma > 0\right].$$

ET I 58(58)

6.678

$$\int_0^\infty \left[K_0(2\sqrt{x}) + \frac{\pi}{2}N_0(2\sqrt{x})\right]\sin(bx)\,dx = \frac{\pi}{2b}\sin\left(\frac{1}{b}\right) \qquad [b > 0].$$

ET I 111(34)

6.679

1. $$\int_0^\infty J_{2\nu}\left[2b\,\operatorname{sh}\left(\frac{x}{2}\right)\right]\sin(bx)\,dx = -i[I_{\nu-ib}(a)K_{\nu+ib}(a) - I_{\nu+ib}(a)K_{\nu-ib}(a)]$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}\,\nu > -1].$$

ET I 115(59)

2. $$\int_0^\infty J_{2\nu}\left[2a\,\operatorname{sh}\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = I_{\nu-ib}(a)K_{\nu+ib}(a) + I_{\nu+ib}(a)K_{\nu-ib}(a)$$
$$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

3. $$\int_0^\infty J_{2\nu}\left[2a\ \mathrm{ch}\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = -\frac{\pi}{2}[J_{\nu+ib}(a)N_{\nu-ib}(a)+J_{\nu-ib}(a)N_{\nu+ib}(a)].$$

ET I 59(63)

4. $$\int_0^\infty J_0\left[2a\ \mathrm{sh}\left(\frac{x}{2}\right)\right]\sin(bx)\,dx = \frac{2}{\pi}\ \mathrm{sh}(\pi b)[K_{ib}(a)]^2 \qquad [a>0, \quad b>0].$$

ET I 115(58)

5. $$\int_0^\infty J_0\left[2a\ \mathrm{sh}\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = [I_{ib}(a)+I_{-ib}(a)]K_{ib}(a) \qquad [a>0, \quad b>0].$$

ET I 59(62)

6. $$\int_0^\infty N_0\left[2a\ \mathrm{sh}\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = -\frac{2}{\pi}\ \mathrm{ch}(\pi b)[K_{ib}(a)]^2 \qquad [a>0, \quad b>0].$$

ET I 59(65)

7. $$\int_0^\infty K_0\left[2a\ \mathrm{sh}\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = \frac{\pi^2}{4}\left\{[J_{ib}(a)]^2+[N_{ib}(a)]^2\right\} \qquad [\mathrm{Re}\ a>0, \quad b>0].$$

ET I 59(66)

6.681

1. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)J_{2\nu}(2a\cos x)\,dx = \frac{\pi}{2}J_{\nu+\mu}(a)J_{\nu-\mu}(a) \qquad \left[\mathrm{Re}\ \nu > -\frac{1}{2}\right].$$

ET II 361(23)

2. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)N_{2\nu}(2a\cos x)\,dx =$$
$$= \frac{\pi}{2}[\mathrm{ctg}(2\nu\pi)J_{\nu+\mu}(a)J_{\nu-\mu}(a) - \mathrm{cosec}(2\nu\pi)J_{\mu-\nu}(a)J_{-\mu-\nu}(a)] \qquad \left[|\,\mathrm{Re}\ \nu| < \frac{1}{2}\right].$$

ET II 361(24)

3. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)I_{2\nu}(2a\cos x)\,dx = \frac{\pi}{2}I_{\nu-\mu}(a)I_{\nu+\mu}(a) \qquad \left[\mathrm{Re}\ \nu > -\frac{1}{2}\right].$$

ET I 59(61)

4. $$\int_0^{\frac{\pi}{2}} \cos(\nu x)K_{\nu}(2a\cos x)\,dx = \frac{\pi}{2}I_0(a)K_{\nu}(a) \qquad [\mathrm{Re}\ \nu < 1].$$

5. $$\int_0^{\pi} J_0(2z\cos x)\cos 2nx\,dx = (-1)^n \pi J_n^2(z).$$

MO 45

6. $$\int_0^{\pi} J_0(2z\sin x)\cos 2nx\,dx = \pi J_n^2(z).$$

WA 43(3), MO 45

7. $$\int_0^{\frac{\pi}{2}} \cos(2n\pi) N_0(2a\sin x)\,dx = \frac{\pi}{2} J_n(a) N_n(a) \qquad [n = 0,\ 1,\ 2, \dots].$$

ET II 360(16)

8. $$\int_0^{\pi} \sin(2\mu x) J_{2\nu}(2a\sin x)\,dx = \pi\sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a) \qquad [\mathrm{Re}\ \nu > -1].$$

ET II 360(13)

9. $$\int_0^{\pi} \cos(2\mu x) J_{2\nu}(2a\sin x)\,dx = \pi\cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a) \qquad \left[\mathrm{Re}\ \nu > -\frac{1}{2}\right].$$

ET II 360(14)

10. $$\int_0^{\frac{\pi}{2}} J_{\nu+\mu}(2z\cos x)\cos[(\nu-\mu)x]\,dx = \frac{\pi}{2} J_\nu(z) J_\mu(z) \qquad [\mathrm{Re}\,(\nu+\mu) > -1].$$

MO 42

11. $$\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x] I_{\mu+\nu}(2a\cos x)\,dx = \frac{\pi}{2} I_\mu(a) I_\nu(a) \qquad [\mathrm{Re}\,(\mu+\nu) > -1].$$

WA 484(2), ET II 378(39)

12. $$\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x] K_{\mu+\nu}(2a\cos x)\,dx = \frac{\pi^2}{4}\,\mathrm{cosec}[(\mu+\nu)\pi][I_{-\mu}(a)I_{-\nu}(a) - I_\mu(a)I_\nu(a)]$$
$$[|\,\mathrm{Re}\,(\mu+\nu)| < 1].$$

13. $$\int_0^{\frac{\pi}{2}} K_{\nu-m}(2a\cos x)\cos[(m+\nu)x]\,dx = (-1)^m \frac{\pi^2}{4} I_m(a) K_\nu(a) \qquad [|\operatorname{Re}(\nu-m)| < 1].$$

WA 485(4)

6.682

1.[7] $$\int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}(x\sin t)\sin^{\nu+\frac{1}{2}} t\,dt = \sqrt{\frac{\pi}{2x}} J_\nu(x)$$

$[x > 0; \nu \geq 0$ integer or half-integer$]$

MO 42a

2. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin x)\sin^\nu x\cos^{2\nu} x\,dx = 2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right) z^{-\nu} J_\nu^2\left(\frac{z}{2}\right) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

MO 42a

6.683

1. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin x) I_\mu(z\cos x)\ \operatorname{tg}^{\nu+1} x\,dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} J_\mu(z) \qquad [\operatorname{Re}\nu > \operatorname{Re}\mu > -1].$$

WA 407(4)

2. $$\int_0^{\frac{\pi}{2}} J_\nu(z_1\sin x) J_\mu(z_2\cos x)\sin^{\nu+1} x\cos^{\mu+1} x\,dx = \frac{z_1^\nu z_2^\mu J_{\nu+\mu+1}\left(\sqrt{z_1^2+z_2^2}\right)}{\sqrt{(z_1^2+z_2^2)^{\nu+\mu+1}}}$$

$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1].$

WA 410(1)

3. $$\int_0^{\frac{\pi}{2}} J_\nu(z\cos^2 x) J_\mu(z\sin^2 x)\sin x\cos x\,dx = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k J_{\nu+\mu+2k+1}(z)$$

$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ (see also **6.513** 6.).

6.513
WA 414(1)

4. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}(\cos\theta)^{2\nu+1}\,d\theta = \frac{s_{\mu+\nu,\,\nu-\mu+1}(z)}{2^{\mu-1}z^{\nu+1}\Gamma(\mu)} \qquad [\operatorname{Re}\,\nu > -1].$$

WA 407(2)

5. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}\,d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\dfrac{2z}{\pi}}}.$$

WA 407(3)

6. $$\int_0^{\frac{\pi}{2}} J_\mu(a\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\varrho+1}\,d\theta = 2^\varrho\Gamma(\varrho+1)a^{-\varrho-1}J_{\varrho+\mu+1}(a)$$
$$[\operatorname{Re}\,\varrho > -1, \quad \operatorname{Re}\,\mu > -1].$$

WA 406(1), EH II 46(5)

7. $$\int_0^{\frac{\pi}{2}} J_\nu(2z\sin\theta)(\sin\theta)^{\nu}(\cos\theta)^{2\nu}\,d\theta = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m z^{\nu+2m}\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{m!\Gamma(\nu+m+1)\Gamma(2\nu+m+1)};$$
$$= \frac{1}{2}z^{-\nu}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)[J_\nu(z)]^2 \qquad \left[\operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

EH II 47(10)

8. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{-2\nu}\,d\theta = 2^{-\nu}\frac{z^{\nu-1}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}-\nu\right)\sin z \qquad \left[-1 < \operatorname{Re}\,\nu < \frac{1}{2}\right].$$

EH II 68(39)

9. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin^2\theta)J_\nu(z\cos^2\theta)(\sin\theta)^{2\nu+1}(\cos\theta)^{2\nu+1}\,d\theta = \frac{\Gamma\left(\frac{1}{2}+\nu\right)J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}}\Gamma(\nu+1)\sqrt{z}} \qquad \left[\operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

WA 409(1)

10. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin^2\theta)J_\nu(z\cos^2\theta)\sin^{2\mu+1}\theta\cos^{2\nu+1}\theta\,d\theta = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi}\Gamma(\mu+\nu+1)\sqrt{2z}}$$
$$\left[\operatorname{Re}\,\mu > -\frac{1}{2}, \quad \operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

1. $$\int_0^\pi (\sin x)^{2\nu} \frac{J_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\,dx = 2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{J_\nu(\beta)}{\beta^\nu} \qquad \left[\operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

ET II 362(27)

2. $$\int_0^\pi (\sin x)^{2\nu} \frac{N_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\,dx = 2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{N_\nu(\beta)}{\beta^\nu}$$
$$\left[|\alpha| < |\beta|, \quad \operatorname{Re}\,\nu > -\frac{1}{2}\right].$$

ET II 362(28)

6.685

$$\int_0^{\frac{\pi}{2}} \sec x\cos(2\lambda x)K_{2\mu}(a\sec x)\,dx = \frac{\pi}{2a}W_{\lambda,\,\mu}(a)W_{-\lambda,\,\mu}(a) \qquad [\operatorname{Re}\,a > 0].$$

ET II 378(41)

6.686

1. $$\int_0^\infty \sin(ax^2)J_\nu(bx)\,dx = -\frac{\sqrt{\pi}}{2\sqrt{a}}\sin\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right)J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \qquad [a>0,\ b>0,\ \operatorname{Re}\,\nu > -3].$$

ET II 34(13)

2. $$\int_0^\infty \cos(ax^2)J_\nu(bx)\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\cos\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right)J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\,\nu > -1].$$

ET II 38(38)

3. $$\int_0^\infty \sin(ax^2)N_\nu(bx)\,dx = -\frac{\sqrt{\pi}}{4\sqrt{a}}\sec\left(\frac{\nu\pi}{2}\right)\times$$
$$\times\left[\cos\left(\frac{b^2}{8a}-\frac{3\nu+1}{4}\pi\right)J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)-\sin\left(\frac{b^2}{8a}+\frac{\nu-1}{4}\pi\right)N_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)\right]$$
$$[a>0, \quad b>0, \quad -3 < \operatorname{Re}\,\nu < 3].$$

ET II 107(7)

ET II 107(8)

5. $$\int_0^\infty \sin(ax^2) J_1(bx)\, dx = \frac{1}{b} \sin \frac{b^2}{4a} \qquad [a > 0, \quad b > 0].$$

ET II 19(16)

6. $$\int_0^\infty \cos(ax^2) J_1(bx)\, dx = \frac{2}{b} \sin^2\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET II 20(20)

7. $$\int_0^\infty \sin^2(ax^2) J_1(bx)\, dx = \frac{1}{2b} \cos\left(\frac{b^2}{8a}\right) \qquad [a > 0, \quad b > 0].$$

ET II 19(17)

6.687

$$\int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}\left(xe^{i\frac{\pi}{4}}\right) K_{2\nu}\left(xe^{-i\frac{\pi}{4}}\right)\, dx = \frac{\Gamma\left(\frac{1}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)\sqrt{\pi}}{8\sqrt{a}} W_{\frac{1}{4},\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\frac{1}{4},\nu}\left(ae^{-i\frac{\pi}{2}}\right)$$
$$\left[a > 0, \quad |\operatorname{Re}\, \nu| < \frac{1}{4}\right].$$

ET II 372(1)

6.688

1. $$\int_0^{\frac{\pi}{2}} J_\nu(\mu z \sin t) \cos(\mu x \cos t)\, dt = \frac{\pi}{2} J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2+z^2}+x}{2}\right) J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2+z^2}-x}{2}\right)$$
$$[\operatorname{Re}\, \nu > -1, \quad \operatorname{Re}\, z > 0].$$

MO 46

2. $$\int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1} \cos(\beta \cos x) J_\nu(\alpha \sin x)\, dx = 2^{-\frac{1}{2}} \sqrt{\pi} \alpha^\nu (\alpha^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}} J_{\nu+\frac{1}{2}}\left[(\alpha^2+\beta^2)^{\frac{1}{2}}\right]$$
$$[\operatorname{Re}\, \nu > -1].$$

ET II 361(19)

EH II 47(8)

6.69- 6.74 Combinations of Bessel and trigonometric functions and powers

6.691

$$\int_0^\infty x\sin(bx)K_0(ax)\,dx = \frac{\pi b}{2}(a^2+b^2)^{-\frac{3}{2}} \qquad [\mathrm{Re}\ a>0, \quad b>0].$$

ET I 105(47)

6.692

1. $$\int_0^\infty xK_\nu(ax)I_\nu(bx)\sin(cx)\,dx = -\frac{1}{2}(ab)^{-\frac{3}{2}}c(u^2-1)^{-\frac{1}{2}}Q^1_{\nu-\frac{1}{2}}(u), \qquad u=(2ab)^{-1}(a^2+b^2+c^2)$$
$$\left[\mathrm{Re}\ a>|\,\mathrm{Re}\ b|, \quad c>0, \quad \mathrm{Re}\ \nu>-\frac{3}{2}\right].$$

ET I 106(54)

2. $$\int_0^\infty xK_\nu(ax)K_\nu(bx)\sin(cx)\,dx = \frac{\pi}{4}(ab)^{-\frac{3}{2}}c(u^2-1)^{-\frac{1}{2}}\Gamma\left(\frac{3}{2}+\nu\right)\Gamma\left(\frac{3}{2}-\nu\right)P^{-1}_{\nu-\frac{1}{2}}(u),$$
$$u=(2ab)^{-1}(a^2+b^2+c^2) \qquad \left[\mathrm{Re}\,(a+b)>0, \quad c>0, \quad |\,\mathrm{Re}\ \nu|<\frac{3}{2}\right].$$

ET I 107(61)

6.693

1. $$\left.\begin{aligned}\int_0^\infty J_\nu(\alpha x)\sin\beta x\frac{dx}{x} &= \frac{1}{\nu}\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right) && [\beta\le\alpha]\\ &= \frac{\alpha^\nu\sin\dfrac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} && [\beta\ge\alpha]\end{aligned}\right\} \qquad [\mathrm{Re}\ \nu>-1].$$

WA 443(2)

2. $$\left.\begin{aligned}\int_0^\infty J_\nu(\alpha x)\cos\beta x\frac{dx}{x} &= \frac{1}{\nu}\cos\left(\nu\arcsin\frac{\beta}{a}\right) && [\beta\le\alpha]\\ &= \frac{\alpha^\nu\cos\dfrac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} && [\beta\ge\alpha]\end{aligned}\right\} \qquad [\mathrm{Re}\ \nu>0].$$

WA 443(3)

3. $$\int_0^\infty N_\nu(ax)\sin(bx)\frac{dx}{x} = -\frac{1}{\nu}\ \operatorname{tg}\left(\frac{\nu\pi}{2}\right)\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0 < b < a, \quad |\operatorname{Re}\,\nu| < 1];$$
$$= \frac{1}{2\nu}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\cos(\nu\pi)\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{\nu} - a^{\nu}\left[b-(b^2-a^2)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$$[0 < a < b, \quad |\operatorname{Re}\,\nu| < 1].$$

ET I 103(35)

4. $$\int_0^\infty J_\nu(ax)\sin(bx)\frac{dx}{x^2} = \frac{\sqrt{a^2-b^2}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu^2-1} - \frac{b\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu(\nu^2-1)}$$
$$[0 < b < a, \quad \operatorname{Re}\,\nu > 0];$$
$$= \frac{-a^{\nu}\cos\left(\frac{\nu\pi}{2}\right)\left[b+\nu\sqrt{b^2-a^2}\right]}{\nu(\nu^2-1)\left[b+\sqrt{b^2-a^2}\right]^{\nu}} \qquad [0 < a < b, \quad \operatorname{Re}\,\nu > 0].$$

ET I 99(6)

5. $$\int_0^\infty J_\nu(ax)\cos(bx)\frac{dx}{x^2} = \frac{a\cos\left[(\nu-1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu-1)} + \frac{a\cos\left[(\nu+1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu+1)}$$
$$[0 < b < a, \quad \operatorname{Re}\,\nu > 1];$$
$$= \frac{a^{\nu}\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu-1)\left[b+\sqrt{b^2-a^2}\right]^{\nu-1}} - \frac{a^{\nu+2}\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu+1)\left[b+\sqrt{b^2-a^2}\right]^{\nu+1}}$$
$$[0 < a < b, \quad \operatorname{Re}\,\nu > 1].$$

ET I 44(6)

6. $$\int_0^\infty J_0(\alpha x)\sin x\frac{dx}{x} = \frac{\pi}{2} \qquad [0 < \alpha < 1];$$
$$= \operatorname{arccosec}\alpha \qquad [\alpha > 1].$$

WH

7. $$\int_0^\infty J_0(x)\sin\beta x\frac{dx}{x} = \frac{\pi}{2} \qquad [\beta > 1];$$
$$= \arcsin\beta \qquad [\beta^2 < 1];$$
$$= -\frac{\pi}{2} \qquad [\beta < -1].$$

NT 66(13)

9. $$\int_0^z J_\nu(x)\sin(z-x)\frac{dx}{x} = \frac{2}{\nu}\sum_{k=0}^{\infty}(-1)^k J_{\nu+2k+1}(z) \qquad [\mathrm{Re}\ \nu > 0].$$

WA 416(4)

10. $$\int_0^z J_\nu(x)\cos(z-x)\frac{dx}{x} = \frac{1}{\nu}J_\nu(z) + \frac{2}{\nu}\sum_{k=1}^{\infty}(-1)^k J_{\nu+2k}(z) \qquad [\mathrm{Re}\ \nu > 0].$$

WA 416(5)

6.694*

$$\int_0^\infty \left[\frac{J_1(ax)}{x}\right]^2 \sin(bx)\,dx =$$
$$= \frac{1}{2}b - \left(\frac{4a}{3\pi}\right)\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{b}{2a}\right) + \left(1-\frac{b^2}{4a^2}\right)\boldsymbol{K}\left(\frac{b}{2a}\right)\right] \qquad [0 \le b \le 2a].$$
$$= \frac{1}{2}b - \frac{2b}{3\pi}\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{2a}{b}\right) - \left(1-\left(\frac{4a^2}{b^2}\right)^{-1}\right)\boldsymbol{K}\left(\frac{2a}{b}\right)\right] \qquad [0 \le 2a \le b].$$

ET I 102(22)

6.695

1. $$\int_0^\infty \frac{\sin\alpha x}{\beta^2+x^2}J_0(ux)\,dx = \frac{\mathrm{sh}\,\alpha\beta}{\beta}K_0(\beta u) \qquad [\alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad u > \alpha].$$

MO 46

2. $$\int_0^\infty \frac{\cos\alpha x}{\beta^2+x^2}J_0(ux)\,dx = \frac{\pi}{2}\frac{e^{-\alpha\beta}}{\beta}I_0(\beta u) \qquad [\alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad -\alpha < u < \alpha].$$

MO 46

3. $$\int_0^\infty \frac{x}{x^2+\beta^2}\sin(\alpha x)J_0(\gamma x)\,dx = \frac{\pi}{2}e^{-\alpha\beta}I_0(\gamma\beta) \qquad [\alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad 0 < \gamma < \alpha].$$

ET II 10(36)

ET II 11(45)

6.696

$$\int_0^\infty [1-\cos(\alpha x)] J_0(\beta x) \frac{dx}{x} = \operatorname{Arch}\left(\frac{\alpha}{\beta}\right) \qquad [0<\beta<\alpha];$$
$$= 0 \qquad [0<\alpha<\beta].$$

ET II 11(43)

6.697

1. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_0(x)\,dx = 2\int_0^\alpha \frac{\cos\beta u}{\sqrt{1-u^2}}\,du \qquad [0\le\alpha\le 1];$$
$$= \pi J_0(\beta) \qquad [1\le\alpha<\infty].$$

WA 463(1), ET II 345(42)
WA 463(2)

2. $$\int_0^\infty \frac{\sin(x+t)}{x+t} J_0(t)\,dt = \frac{\pi}{2} J_0(x) \qquad [x>0].$$

WA 475(4)

3. $$\int_0^\infty \frac{\cos(x+t)}{x+t} J_0(t)\,dt = -\frac{\pi}{2} N_0(x) \qquad [x>0].$$

WA 475(5)

4. $$\int_{-\infty}^\infty \frac{|x|}{x+\beta} \sin[\alpha(x+\beta)] J_0(bx)\,dx = 0 \qquad [0\le\alpha<b].$$

WA 464(5), ET II 345(43)a

5. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} [J_{n+\frac{1}{2}}(x)]^2\,dx = \pi[J_{n+\frac{1}{2}}(\beta)]^2 \qquad [2\le\alpha<\infty, \quad n=0,\,1,\ldots].$$

ET II 346(45)

6. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x) J_{-n-\frac{1}{2}}(x)\,dx = \pi J_{n+\frac{1}{2}}(\beta) J_{-n-\frac{1}{2}}(\beta) \qquad [2\le\alpha<\infty, \quad n=0,\,1,\ldots].$$

ET II 346(46)

7. $$\int_{-\infty}^{\infty} \frac{J_\mu[a(z+x)]}{(z+x)^\mu} \frac{J_\nu[a(\zeta+x)]}{(\zeta+x)^\nu} \, dx = \frac{\Gamma(\mu+\nu)\sqrt{\pi}\sqrt{\frac{2}{a}}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu+\frac{1}{2})} \cdot \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}}$$
$$[\operatorname{Re}(\mu+\nu) > 0].$$

WA 463(3)

6.698

1. $$\int_0^\infty \sqrt{x} J_{\nu+\frac{1}{4}}(ax) J_{-\nu+\frac{1}{4}}(ax) \sin(bx) \, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a];$$
$$= 0 \qquad [0 < 2a < b].$$

ET I 102(26)

2. $$\int_0^\infty \sqrt{x} J_{\nu-\frac{1}{4}}(ax) J_{-\nu-\frac{1}{4}}(ax) \cos(bx) \, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a];$$
$$= 0 \qquad [0 < 2a < b].$$

ET I 46(24)

3. $$\int_0^\infty \sqrt{x} I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx) \, dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b+\sqrt{a^2+b^2})^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{5}{4}\right].$$

ET I 106(56)

4. $$\int_0^\infty \sqrt{x} I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx) \, dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b+\sqrt{a^2+b^2})^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{3}{4}\right].$$

ET I 50(49)

6.699

1. $$\int_0^\infty x^\lambda J_\nu(ax) \sin(bx) \, dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma\left(\frac{2+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda}{2}\right)} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right)$$
$$\left[0 < b < a, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2}\right];$$

2. $$\int_0^\infty x^\lambda J_\nu(ax)\cos(bx)\,dx =$$

$$= \frac{2^\lambda a^{-(1+\lambda)}\Gamma\left(\frac{1+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda+1}{2}\right)} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right)$$

$$\left[0 < b < a, \quad -\operatorname{Re}\nu < 1 + \operatorname{Re}\lambda < \frac{3}{2}\right];$$

$$= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)}\Gamma(1+\lambda+\nu)\cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right)$$

$$\left[0 < a < b, \quad -\operatorname{Re}\nu < 1 + \operatorname{Re}\lambda < \frac{3}{2}\right].$$

ET I 45(13)

3. $$\int_0^\infty x^\lambda K_\mu(ax)\sin(bx)dx = \frac{2^\lambda b\Gamma\left(\frac{2+\mu+\lambda}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$

$$[\operatorname{Re}(-\lambda \pm \mu) < 2, \quad \operatorname{Re} a > 0, \quad b > 0].$$

ET I 106(50)

4. $$\int_0^\infty x^\lambda K_\mu(ax)\cos(bx)\,dx = 2^{\lambda-1}a^{-\lambda-1}\Gamma\left(\frac{\mu+\lambda+1}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right)\times$$

$$\times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right)$$

$$[\operatorname{Re}(-\lambda \pm \mu) < 1, \quad \operatorname{Re} a > 0, \quad b > 0].$$

ET I 49(42)

5. $$\int_0^\infty x^\nu \sin(ax) J_\nu(bx)\,dx = \frac{\sqrt{\pi}2^\nu b^\nu (a^2-b^2)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0 < b < a, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right];$$

$$= 0 \qquad \left[0 < a < b, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 32(4)

6. $$\int_0^\infty x^\nu \cos(ax) J_\nu(bx)\,dx =$$

$$= -2^\nu \frac{\sin(\nu\pi)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\nu\right) b^\nu (a^2-b^2)^{-\nu-\frac{1}{2}} \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right];$$

$$= 2^\nu \frac{b^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\nu\right)(b^2-a^2)^{-\nu-\frac{1}{2}} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

7. $$\int_0^\infty x^{\nu+1}\sin(ax)J_\nu(bx)\,dx =$$
$$= -2^{1+\nu}a\frac{\sin(\nu\pi)}{\sqrt{\pi}}b^\nu\Gamma\left(\nu+\frac{3}{2}\right)(a^2-b^2)^{-\nu-\frac{3}{2}} \qquad \left[0<b<a, \quad -\frac{3}{2}<\operatorname{Re}\,\nu<-\frac{1}{2}\right];$$
$$= -\frac{2^{1+\nu}}{\sqrt{\pi}}ab^\nu\Gamma\left(\nu+\frac{3}{2}\right)(b^2-a^2)^{-\nu-\frac{3}{2}} \qquad \left[0<a<b, \quad -\frac{3}{2}<\operatorname{Re}\,\nu<-\frac{1}{2}\right].$$

ET II 32(3)

8. $$\int_0^\infty x^{\nu+1}\cos(ax)J_\nu(bx)\,dx = 2^{1+\nu}\sqrt{\pi}ab^\nu\frac{(a^2-b^2)^{-\nu-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}-\nu\right)} \qquad \left[0<b<a, \quad -1<\operatorname{Re}\,\nu<-\frac{1}{2}\right];$$
$$= 0 \qquad \left[0<a<b, \quad -1<\operatorname{Re}\,\nu<-\frac{1}{2}\right].$$

ET II 36(28)

9. $$\int_0^1 x^\nu\sin(ax)J_\nu(ax)\,dx = \frac{1}{2\nu+1}[\sin aJ_\nu(a)-\cos aJ_{\nu+1}(a)] \qquad [\operatorname{Re}\,\nu>-1].$$

ET II 334(9)a

10. $$\int_0^1 x^\nu\cos(ax)J_\nu(ax)\,dx = \frac{1}{2\nu+1}[\cos aJ_\nu(a)+\sin aJ_{\nu+1}(a)] \qquad \left[\operatorname{Re}\,\gamma>-\frac{1}{2}\right].$$

ET II 335(20)

11. $$\int_0^\infty x^{1+\nu}K_\nu(ax)\sin(bx)\,dx = \sqrt{\pi}(2a)^\nu\Gamma\left(\frac{3}{2}+\nu\right)b(b^2+a^2)^{-\frac{3}{2}-\nu}$$
$$\left[\operatorname{Re}\,a>0, \quad b>0, \quad \operatorname{Re}\,\nu>-\frac{3}{2}\right].$$

ET I 105(49)

12. $$\int_0^\infty x^\mu K_\mu(ax)\cos(bx)\,dx = \frac{1}{2}\sqrt{\pi}(2a)^\mu\Gamma\left(\mu+\frac{1}{2}\right)(b^2+a^2)^{-\mu-\frac{1}{2}}$$
$$\left[\operatorname{Re}\,a>0, \quad b>0, \quad \operatorname{Re}\,\mu>-\frac{1}{2}\right].$$

ET I 49(41)

ET I 104(36)

14. $$\int_0^\infty x^\nu N_\nu(ax)\cos(bx)\,dx = 0 \qquad \left[0 < b < a, \quad |\operatorname{Re}\,\nu| < \frac{1}{2}\right];$$
$$= -2^\nu\sqrt{\pi}a^\nu \frac{(b^2-a^2)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0 < a < b, \quad |\operatorname{Re}\,\nu| < \frac{1}{2}\right].$$

ET I 47(30)

6.711

1. $$\int_0^\infty x^{\nu-\mu} J_\mu(ax) J_\nu(bx)\sin(cx)\,dx = 0 \qquad [0 < c < b-a,\ -1 < \operatorname{Re}\nu < 1+\operatorname{Re}\mu].$$

ET I 103(28)

2. $$\int_0^\infty x^{\nu-\mu+1} J_\mu(ax) J_\nu(bx)\cos(cx)\,dx = 0 \qquad [0 < c < b-a,\ a > 0,\ b > 0,\ -1 < \operatorname{Re}\,\nu < \operatorname{Re}\,\mu].$$

ET I 47(25)

3. $$\int_0^\infty x^{\nu-\mu-2} J_\mu(ax) J_\nu(bx)\sin(cx)\,dx = 2^{\nu-\mu-1}a^\mu b^{-\nu}\frac{c\Gamma(\nu)}{\Gamma(\mu+1)}$$
$$[0 < a, \quad 0 < b, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\,\nu < \operatorname{Re}\,\mu+3].$$

ET I 103(29)

4. $$\int_0^\infty x^{\varrho-\mu-1} J_\mu(ax) J_\varrho(bx)\cos(cx)\,dx = 2^{\varrho-\mu-1}b^{-\varrho}a^\mu\frac{\Gamma(\varrho)}{\Gamma(\mu+1)}$$
$$[b > 0, \quad a > 0, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\,\varrho < \operatorname{Re}\,\mu+2].$$

ET I 47(26)

5. $$\int_0^\infty x^{1-2\nu}\sin(2ax) J_\nu(x) N_\nu(x)\,dx = -\frac{\Gamma\left(\frac{3}{2}-\nu\right)a}{2\Gamma\left(2\nu-\frac{1}{2}\right)\Gamma(2-\nu)} F\left(\frac{3}{2}-\nu,\ \frac{3}{2}-2\nu;\ 2-\nu;\ a^2\right)$$
$$\left[0 < \operatorname{Re}\,\nu < \frac{3}{2}, \quad 0 < a < 1\right].$$

1. $$\int_0^\infty x^\nu [J_\nu(ax)\cos(ax)+N_\nu(ax)\sin(ax)]\sin(bx)\,dx = \frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}(b^2+2ab)^{-\nu-\frac{1}{2}}$$
$$\left[b>0, \quad -1<\operatorname{Re}\,\nu<\frac{1}{2}\right].$$

ET I 104(40)

2. $$\int_0^\infty x^\nu [N_\nu(ax)\cos(ax)-J_\nu(ax)\sin(ax)]\cos(bx)\,dx = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}(b^2+2ab)^{-\nu-\frac{1}{2}}.$$

ET I 48(35)

3. $$\int_0^\infty x^\nu [J_\nu(ax)\cos(ax)-N_\nu(ax)\sin(ax)]\sin(bx)\,dx =$$
$$= 0 \qquad \left[0<b<2a, \quad -1<\operatorname{Re}\,\nu<\frac{1}{2}\right];$$
$$= \frac{2^\nu\sqrt{\pi}b^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}(b^2-2ab)^{-\nu-\frac{1}{2}} \qquad \left[2a<b, \quad -1<\operatorname{Re}\,\nu<\frac{1}{2}\right].$$

ET I 104(41)

4. $$\int_0^\infty x^\nu [J_\nu(ax)\sin(ax)+N_\nu(ax)\cos(ax)]\cos(bx)\,dx$$
$$= 0 \qquad \left[0<b<2a, \quad |\operatorname{Re}\,\nu|<\frac{1}{2}\right];$$
$$= -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}(b^2-2ab)^{-\nu-\frac{1}{2}} \qquad \left[0<2a<b, \quad |\operatorname{Re}\,\nu|<\frac{1}{2}\right].$$

ET I 48(33)

767
6.713

1. $$\int_0^\infty x^{1-2\nu}\sin(2ax)\left\{[J_\nu(x)]^2-[N_\nu(x)]^2\right\}\,dx =$$
$$= \frac{\sin(2\nu\pi)\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{3}{2}-2\nu\right)a}{\pi\Gamma(2-\nu)}F\left(\frac{3}{2}-\nu,\ \frac{3}{2}-2\nu;\ 2-\nu;\ a^2\right)$$
$$\left[0<\operatorname{Re}\nu<\frac{3}{4}, \quad 0<a<1\right].$$

ET II 348(64)

ET II 348(65)

3. $$\int_0^\infty x^{2-2\nu}\sin(2ax)\left[J_\nu(x)N_{\nu-1}(x)+N_\nu(x)J_{\nu-1}(x)\right]\,dx =$$
$$= -\frac{\Gamma\left(\frac{3}{2}-\nu\right)a}{\Gamma\left(2\nu-\frac{3}{2}\right)\Gamma(2-\nu)}F\left(\frac{3}{2}-\nu,\ \frac{5}{2}-2\nu;\ 2-\nu;\ a^2\right) \qquad \left[\frac{1}{2}<\operatorname{Re}\nu<\frac{5}{2},\quad 0<a<1\right].$$

ET II 349(66)

6.714

1. $$\int_0^\infty \sin(2ax)\left[x^\nu J_\nu(x)\right]^2\,dx =$$
$$= \frac{a^{-2\nu}\Gamma\left(\frac{1}{2}+\nu\right)}{2\sqrt{\pi}\Gamma(1-\nu)}F\left(\frac{1}{2}+\nu,\ \frac{1}{2};\ 1-\nu;\ a^2\right) \qquad \left[0<a<1,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right],$$
$$= \frac{a^{-4\nu-1}\Gamma\left(\frac{1}{2}+\nu\right)}{2\Gamma(1+\nu)\Gamma\left(\frac{1}{2}-2\nu\right)}F\left(\frac{1}{2}+\nu,\ \frac{1}{2}+2\nu;\ 1+\nu;\ \frac{1}{a^2}\right) \qquad \left[a>1,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET II 343(31)

2. $$\int_0^\infty \cos(2ax)\left[x^\nu J_\nu(x)\right]^2\,dx = \frac{a^{-2\nu}\Gamma(\nu)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}F\left(\nu+\frac{1}{2},\ \frac{1}{2};\ 1-\nu;\ a^2\right) +$$
$$+\frac{\Gamma(-\nu)\Gamma\left(\frac{1}{2}+2\nu\right)}{2\pi\Gamma\left(\frac{1}{2}-\nu\right)}F\left(\frac{1}{2}+\nu,\ \frac{1}{2}+2\nu;\ 1+\nu;\ a^2\right)$$
$$\left[0<a<1,\quad -\frac{1}{4}<\operatorname{Re}\nu<\frac{1}{2}\right];$$
$$= -\frac{\sin(\nu\pi)a^{-4\nu-1}\Gamma\left(\frac{1}{2}+2\nu\right)}{\Gamma(1+\nu)\Gamma\left(\frac{1}{2}-\nu\right)}F\left(\frac{1}{2}+\nu,\ \frac{1}{2}+2\nu;\ 1+\nu;\ \frac{1}{a^2}\right)$$
$$\left[a>1,\quad -\frac{1}{4}<\operatorname{Re}\nu<\frac{1}{2}\right].$$

ET II 344(33)

6.715

1. $$\int_0^\infty \frac{x^\nu}{x+\beta}\sin(x+\beta)J_\nu(x)\,dx = \frac{\pi}{2}\sec(\nu\pi)\beta^\nu J_{-\nu}(\beta) \qquad \left[|\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 340(8)

2. $$\int_0^\infty \frac{x^\nu}{x+\beta}\cos(x+\beta)J_\nu(x)\,dx = -\frac{\pi}{2}\sec(\nu\pi)\beta^\nu N_{-\nu}(\beta) \qquad \left[|\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 340(9)

6.716

1. $$\int_0^a x^\lambda \sin(a-x)J_\nu(x)\,dx = 2a^{\lambda+1}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(\nu-\lambda+2n)\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\Gamma(\nu+\lambda+3+2n)}(\nu+2n+1)J_{\nu+2n+1}(a)$$
$$[\operatorname{Re}(\lambda+\nu) > -1].$$

ET II 335(16)

2. $$\int_0^a x^\lambda \cos(a-x)J_\nu(x)\,dx = \frac{a^{\lambda+1}J_\nu(a)}{\lambda+\nu+1} + 2a^{\lambda+1} \times$$
$$\times\sum_{n=1}^\infty \frac{(-1)^n\Gamma(\nu-\lambda+2n-1)\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\Gamma(\nu+\lambda+2n+2)}(\nu+2n)J_{\nu+2n}(a)$$
$$[\operatorname{Re}(\lambda+\nu) > -1].$$

ET II 335(26)

6.717

$$\int_{-\infty}^\infty \frac{\sin[a(x+\beta)]}{x^\nu(x+\beta)}J_{\nu+2n}(x)\,dx = \pi\beta^{-\nu}J_{\nu+2n}(\beta) \qquad \left[1 \le a < \infty, \quad n = 0, 1, 2, \ldots; \quad \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 345(44)

6.718

1. $$\int_0^\infty \frac{x^\nu}{x^2+\beta^2}\sin(\alpha x)J_\nu(\gamma x)\,dx = \beta^{\nu-1}\operatorname{sh}(\alpha\beta)K_\nu(\beta\gamma)$$
$$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \frac{3}{2}\right].$$

2. $$\int_0^\infty \frac{x^{\nu+1}}{x^2+\beta^2} \cos(\alpha x) J_\nu(\gamma x)\,dx = \beta^\nu \operatorname{ch}(\alpha\beta) K_\nu(\beta\gamma)$$
$$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 37(33)

3. $$\int_0^\infty \frac{x^{1-\nu}}{x^2+\beta^2} \sin(\alpha x) J_\nu(\gamma x)\,dx = \frac{\pi}{2}\beta^{-\nu} e^{-\alpha\beta} I_\nu(\beta\gamma) \qquad \left[0 < \gamma \le \alpha, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 33(9)

4. $$\int_0^\infty \frac{x^{-\nu}}{x^2+\beta^2} \cos(\alpha x) J_\nu(\gamma x)\,dx = \frac{\pi}{2}\beta^{-\nu-1} e^{-\alpha\beta} I_\nu(\beta\gamma) \qquad \left[0 < \gamma \le \alpha,\ \operatorname{Re}\beta > 0,\ \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 37(34)

6.719

1.[6] $$\int_0^\alpha \frac{\sin(\beta x)}{\sqrt{\alpha^2-x^2}} J_\nu(x)\,dx = \pi\sum_{n=0}^\infty (-1)^n J_{2n+1}(\alpha\beta) J_{\frac{1}{2}\nu+n+\frac{1}{2}}\left(\frac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n-\frac{1}{2}}\left(\frac{1}{2}\alpha\right) \qquad [\operatorname{Re}\nu > -2].$$

ET II 335(17)

2. $$\int_0^\alpha \frac{\cos(\beta x)}{\sqrt{\alpha^2-x^2}} J_\nu(x)\,dx = \frac{\pi}{2} J_0(\alpha\beta)\left[J_{\frac{1}{2}\nu}\left(\frac{1}{2}\alpha\right)\right]^2 + \pi\sum_{n=1}^\infty (-1)^n J_{2n}(\alpha\beta) J_{\frac{1}{2}\nu+n}\left(\frac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n}\left(\frac{1}{2}\alpha\right).$$
$$[\operatorname{Re}\nu > -1].$$

ET II 336(27)

6.721

1. $$\int_0^\infty \sqrt{x} J_{\frac{1}{4}}(a^2x^2)\sin(bx)\,dx = 2^{-\frac{3}{2}} a^{-2}\sqrt{\pi b} J_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \qquad [b > 0].$$

ET I 108(1)

2. $$\int_0^\infty \sqrt{x} J_{-\frac{1}{4}}(a^2x^2)\cos(bx)\,dx = 2^{-\frac{3}{2}} a^{-2}\sqrt{\pi b} J_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \qquad [b > 0].$$

ET I 51(1)

3. $$\int_0^\infty \sqrt{x} N_{\frac{1}{4}}(a^2x^2)\sin(bx)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi b} a^{-2} H_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right).$$

4. $$\int_0^\infty \sqrt{x} N_{-\frac{1}{4}}(a^2x^2)\cos(bx)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi} b a^{-2} H_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right).$$

ET I 52(7)

5. $$\int_0^\infty \sqrt{x} K_{\frac{1}{4}}(a^2x^2)\sin(bx)\,dx = 2^{-\frac{5}{2}}\sqrt{\pi^3 b} a^{-2}\left[I_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - L_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right] \qquad \left[|\arg a| < \frac{\pi}{4},\ b > 0\right].$$

ET I 109(11)

6. $$\int_0^\infty \sqrt{x} K_{-\frac{1}{4}}(a^2x^2)\cos(bx)\,dx = 2^{-\frac{5}{2}}\sqrt{\pi^3 b} a^{-2}\left[I_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - L_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right] \qquad [b > 0].$$

ET I 52(10)

6.722

1. $$\int_0^\infty \sqrt{x} K_{\frac{1}{8}+\nu}(a^2x^2) I_{\frac{1}{8}-\nu}(a^2x^2)\sin(bx)\,dx = \sqrt{2\pi} b^{-\frac{3}{2}} \frac{\Gamma\left(\frac{5}{8}-\nu\right)}{\Gamma\left(\frac{5}{4}\right)} W_{\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{5}{8},\ |\arg a| < \frac{\pi}{4},\quad b > 0\right].$$

ET I 109(13)

2. $$\int_0^\infty \sqrt{x} J_{-\frac{1}{8}-\nu}(a^2x^2) J_{-\frac{1}{8}+\nu}(a^2x^2)\cos(bx)\,dx =$$
$$= \sqrt{\frac{2}{\pi}} b^{-\frac{3}{2}}\left[e^{-\frac{i\pi}{8}} W_{\nu,-\frac{1}{8}}\left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2}\right) W_{-\nu,-\frac{1}{8}}\left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2}\right) + \right.$$
$$\left. + e^{\frac{i\pi}{8}} W_{\nu,-\frac{1}{8}}\left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2}\right) W_{-\nu,-\frac{1}{8}}\left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2}\right)\right] \qquad [b > 0].$$

ET I 52(6)

3. $$\int_0^\infty \sqrt{x} J_{\frac{1}{8}-\nu}(a^2x^2) J_{\frac{1}{8}+\nu}(a^2x^2)\sin(bx)\,dx =$$
$$= \sqrt{\frac{2}{\pi}} b^{-\frac{3}{2}}\left[e^{\frac{\pi i}{8}} W_{\nu,\frac{1}{8}}\left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2}\right) + e^{-\frac{i\pi}{8}} W_{\nu,\frac{1}{8}}\left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2}\right)\right]$$
$$[b > 0].$$

4. $$\int_0^\infty \sqrt{x} K_{\frac{1}{8}-\nu}(a^2x^2) I_{-\frac{1}{8}-\nu}(a^2x^2)\, \cos(bx)\, dx = \sqrt{2\pi} b^{-\frac{3}{2}} \frac{\Gamma\left(\frac{3}{8}-\nu\right)}{\Gamma\left(\frac{3}{4}\right)} W_{\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{3}{8}, \quad b > 0\right].$$

ET I 52(12)

6.723

$$\int_0^\infty x J_\nu(x^2) \left[\sin(\nu\pi) J_\nu(x^2) - \cos(\nu\pi) N_\nu(x^2)\right] J_{4\nu}(4ax)\, dx = \frac{1}{4} J_\nu(a^2) J_{-\nu}(a^2)$$
$$[a > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 375(20)

6.724

1. $$\int_0^\infty x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \sin(bx)\, dx =$$
$$= \frac{\sqrt{\pi} a^{2\nu} \Gamma(\lambda-\nu+1) b^{2\nu-2\lambda-1}}{4^{2\nu-\lambda}\Gamma(2\nu+1)\Gamma\left(\nu-\lambda+\frac{1}{2}\right)}\, {}_0F_3\left(2\nu+1,\ \nu-\lambda,\ \nu-\lambda+\frac{1}{2};\ \frac{a^2b^2}{16}\right) +$$
$$+ \frac{a^{2\lambda+2}\Gamma(\nu-\lambda-1) b}{2^{2\lambda+3}\Gamma(\nu+\lambda+2)}\, {}_0F_3\left(\frac{3}{2},\ \lambda-\nu+2,\ \lambda+\nu+2;\ \frac{a^2b^2}{16}\right)$$
$$\left[-\frac{5}{4} < \operatorname{Re}\lambda < \operatorname{Re}\nu, \quad a > 0, \quad b > 0\right].$$

ET I 109(15)

2. $$\int_0^\infty x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \cos(bx)\, dx =$$
$$= 4^{\lambda-2\nu}\sqrt{\pi} a^{2\nu} b^{2\nu-2\lambda-1} \frac{\Gamma\left(\lambda-\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)\Gamma(\nu-\lambda)}\, {}_0F_3\left(2\nu+1,\ \nu-\lambda+\frac{1}{2},\ \nu-\lambda;\ \frac{a^2b^2}{16}\right) +$$
$$+ 4^{-\lambda-1} a^{2\lambda+1} \frac{\Gamma\left(\nu-\lambda-\frac{1}{2}\right)}{\Gamma\left(\nu+\lambda+\frac{3}{2}\right)}\, {}_0F_3\left(\frac{1}{2},\ \lambda-\nu+\frac{3}{2},\ \nu+\lambda+\frac{3}{2};\ \frac{a^2b^2}{16}\right)$$
$$\left[-\frac{3}{4} < \operatorname{Re}\lambda < \operatorname{Re}\nu - \frac{1}{2}, \quad a > 0, \quad b > 0\right].$$

ET I 53(14)

6.725

1. $$\int_0^\infty \frac{\sin(bx)}{\sqrt{x}} J_\nu(a\sqrt{x})\, dx = -\sqrt{\frac{\pi}{b}} \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{\nu}{2}}\left(\frac{a^2}{8b}\right)$$
$$[\operatorname{Re}\ \nu > -3, \quad a > 0, \quad b > 0].$$

2. $$\int_0^\infty \frac{\cos(bx)}{\sqrt{x}} J_\nu(a\sqrt{x})\,dx = \sqrt{\frac{\pi}{b}} \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{1}{2}\nu}\left(\frac{a^2}{8b}\right)$$
$$[\mathrm{Re}\ \nu > -1, \quad a > 0, \quad b > 0]\ .$$

ET I 54(25)

3. $$\int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \sin(bx)\,dx = 2^{-\nu} a^\nu b^{-\nu-1} \cos\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \qquad \left[-2 < \mathrm{Re}\,\nu < \frac{1}{2},\ a > 0,\ b > 0\right]\ .$$

ET I 110(28)

4. $$\int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \cos(bx)\,dx = 2^{-\nu} b^{-\nu-1} a^\nu \sin\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \qquad \left[-1 < \mathrm{Re}\,\nu < \frac{1}{2},\ a > 0,\ b > 0\right]\ .$$

ET I 54(26)

6.726

1. $$\int_0^\infty x(x^2+b^2)^{-\frac{1}{2}\nu} J_\nu(a\sqrt{x^2+b^2}) \sin(cx)\,dx =$$
$$= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{3}{2}} c (a^2-c^2)^{\frac{1}{2}\nu-\frac{3}{4}} J_{\nu-\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0 < c < a, \quad \mathrm{Re}\,\nu > \frac{1}{2}\right];$$
$$= 0 \qquad \left[0 < a < c, \quad \mathrm{Re}\,\nu > \frac{1}{2}\right].$$

ET I 111(37)

2. $$\int_0^\infty (x^2+b^2)^{-\frac{1}{2}\nu} J_\nu\left(a\sqrt{x^2+b^2}\right) \cos(cx)\,dx =$$
$$= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} (a^2-c^2)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0 < c < a, \quad b > 0, \quad \mathrm{Re}\,\nu > -\frac{1}{2}\right];$$
$$= 0 \qquad \left[0 < a < c, \quad b > 0, \quad \mathrm{Re}\,\nu > -\frac{1}{2}\right].$$

ET I 55(37)

3. $$\int_0^\infty x(x^2+b^2)^{\frac{1}{2}\nu} K_{\pm\nu}\left(a\sqrt{x^2+b^2}\right) \sin(cx)\,dx = \sqrt{\frac{\pi}{2}} a^\nu b^{\nu+\frac{3}{2}} c (a^2+c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} K_{-\nu-\frac{3}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$$[\mathrm{Re}\,a > 0, \quad \mathrm{Re}\,b > 0, \quad c > 0]\ .$$

4. $$\int_0^\infty (x^2+b^2)^{\mp\frac{1}{2}\nu} K_\nu\left(a\sqrt{x^2+b^2}\right)\cos(cx)\,dx = \sqrt{\frac{\pi}{2}}a^{\mp\nu}b^{\frac{1}{2}\mp\nu}(a^2+c^2)^{\pm\frac{1}{2}\nu-\frac{1}{4}}K_{\pm\nu-\frac{1}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0, \quad c>0].$$

ET I 56(45)

5. $$\int_0^\infty (x^2+a^2)^{-\frac{1}{2}\nu} N_\nu\left(b\sqrt{x^2+a^2}\right)\cos(cx)\,dx =$$
$$= \sqrt{\frac{a\pi}{2}}(ab)^{-\nu}(b^2-c^2)^{\frac{1}{2}\nu-\frac{1}{4}}N_{\nu-\frac{1}{2}}\left(a\sqrt{b^2-c^2}\right) \qquad \left[0<c<b, \quad a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right];$$
$$= -\sqrt{\frac{2a}{\pi}}(ab)^{-\nu}(c^2-b^2)^{\frac{1}{2}\nu-\frac{1}{4}}K_{\nu-\frac{1}{2}}\left(a\sqrt{c^2-b^2}\right) \qquad \left[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET I 56(41)

6.727

1. $$\int_0^a \frac{\sin(cx)}{\sqrt{a^2-x^2}}J_\nu(b\sqrt{a^2-x^2})\,dx = \frac{\pi}{2}J_{\frac{1}{2}\nu}\left[\frac{a}{2}(\sqrt{b^2+c^2}-c)\right]J_{\frac{1}{2}\nu}\left[\frac{a}{2}(\sqrt{b^2+c^2}+c)\right]$$
$$[\operatorname{Re}\nu>-1, \quad c>0, \quad a>0].$$

ET I 113(48)

2. $$\int_a^\infty \frac{\sin(cx)}{\sqrt{x^2-a^2}}J_\nu\left(b\sqrt{x^2-a^2}\right)\,dx = \frac{\pi}{2}J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2+b^2}\right)\right]J_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2+b^2}\right)\right]$$
$$[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-1].$$

ET I 113(49)

3. $$\int_a^\infty \frac{\cos(cx)}{\sqrt{x^2-a^2}}J_\nu\left(b\sqrt{x^2-a^2}\right)\,dx = -\frac{\pi}{2}J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2-b^2}\right)\right]N_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2-b^2}\right)\right]$$
$$[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-1].$$

ET I 58(54)

4. $$\int_0^a (a^2-x^2)^{\frac{1}{2}\nu}\cos I_\nu(\sqrt{a^2-x^2})\,dx = \frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)} \qquad \left[\operatorname{Re}\nu>-\frac{1}{2}\right].$$

WA 409(2)

6.728

1. $$\int_0^\infty x\sin(ax^2)J_\nu(bx)\,dx =$$
$$= \frac{\sqrt{\pi}b}{8a^{\frac{3}{2}}}\left[\cos\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right)-\sin\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-4].$$

2. $$\int_0^\infty x\cos(ax^2)J_\nu(bx)\,dx = \frac{\sqrt{\pi}b}{8a^{\frac{3}{2}}}\left[\cos\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right)+\sin\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-2]\,.$$

ET II 38(39)

3. $$\int_0^\infty J_0(\beta x)\sin(\alpha x^2)x\,dx = \frac{1}{2\alpha}\cos\frac{\beta^2}{4\alpha} \qquad [\alpha>0, \quad \beta>0]\,.$$

MO 47

4. $$\int_0^\infty J_0(\beta x)\cos(\alpha x^2)x\,dx = \frac{1}{2\alpha}\sin\frac{\beta^2}{4\alpha} \qquad [\alpha>0, \quad \beta>0]\,.$$

MO 47

5. $$\int_0^\infty x^{\nu+1}\sin(ax^2)J_\nu(bx)\,dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}}\cos\left(\frac{b^2}{4a}-\frac{\nu\pi}{2}\right) \qquad \left[a>0, \quad b>0, \quad -2<\operatorname{Re}\nu<\frac{1}{2}\right]\,.$$

ET II 34(15)

6. $$\int_0^\infty x^{\nu+1}\cos(ax^2)J_\nu(bx)\,dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}}\sin\left(\frac{b^2}{4a}-\frac{\nu\pi}{2}\right) \qquad \left[a>0, \quad b>0, \quad -1<\operatorname{Re}\nu<\frac{1}{2}\right]\,.$$

ET II 38(40)

6.729

1. $$\int_0^\infty x\sin(ax^2)J_\nu(bx)J_\nu(cx)\,dx = \frac{1}{2a}\cos\left(\frac{b^2+c^2}{4a}-\frac{\nu\pi}{2}\right)J_\nu\left(\frac{bc}{2a}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad \operatorname{Re}\nu>-2]\,.$$

ET II 51(26)

2. $$\int_0^\infty x\cos(ax^2)J_\nu(bx)J_\nu(cx)\,dx = \frac{1}{2a}\sin\left(\frac{b^2+c^2}{4a}-\frac{\nu\pi}{2}\right)J_\nu\left(\frac{bc}{2a}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad \operatorname{Re}\,\nu>-1]\,.$$

ET II 51(27)

6.731

1. $$\int_0^\infty x \sin(ax^2) J_\nu(bx^2) J_{2\nu}(2cx)\, dx =$$
$$= \frac{1}{2\sqrt{b^2-a^2}} \sin\left(\frac{ac^2}{b^2-a^2}\right) J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad [0 < a < b, \quad \operatorname{Re}\nu > -1]\ ;$$
$$= \frac{1}{2\sqrt{a^2-b^2}} \cos\left(\frac{ac^2}{a^2-b^2}\right) J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad [0 < b < a, \quad \operatorname{Re}\nu > -1]\ .$$

ET II 356(41)a

2.* $$\int_0^\infty x \cos(ax^2) J_\nu(bx^2) J_{2\nu}(2cx)\, dx =$$
$$= \frac{1}{2\sqrt{b^2-a^2}} \cos\left(\frac{ac^2}{b^2-a^2}\right) J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]\ ;$$
$$= \frac{1}{2\sqrt{a^2-b^2}} \sin\left(\frac{ac^2}{a^2-b^2}\right) J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]\ .$$

ET II 356(42)a

6.732

$$\int_0^\infty x^2 \cos\left(\frac{x^2}{2a}\right) N_1(x) K_1(x)\, dx = -a^3 K_0(a) \qquad [a > 0]\ .$$

ET II 371(52)

6.733

1. $$\int_0^\infty \sin\left(\frac{a}{2x}\right) [\sin x J_0(x) + \cos x N_0(x)]\ \frac{dx}{x} = \pi J_0(\sqrt{a}) N_0(\sqrt{a}) \qquad [a > 0]\ .$$

ET II 346(51)

2. $$\int_0^\infty \cos\left(\frac{a}{2x}\right) [\sin x N_0(x) - \cos x J_0(x)]\ \frac{dx}{x} = \pi J_0(\sqrt{a}) N_0(\sqrt{a}) \qquad [a > 0]\ .$$

ET II 347(52)

3. $$\int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x)\, dx = \frac{\pi a}{2} J_1(\sqrt{a}) K_1(\sqrt{a}) \qquad [a > 0]\ .$$

ET II 368(34)

4. $$\int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x)\, dx = -\frac{\pi a}{2} N_1(\sqrt{a}) K_1(\sqrt{a}) \qquad [a > 0]\ .$$

ET II 369(35)

6.734

$$\int_0^\infty \cos(a\sqrt{x})K_\nu(bx)\frac{dx}{\sqrt{x}} = \\ = \frac{\pi}{2\sqrt{b}}\sec(\nu\pi)\left[D_{\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right)D_{-\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right)+D_{\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right)D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right)\right] \\ \left[\operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 132(27)

6.735

1. $$\int_0^\infty x^{\frac{1}{4}}\sin(2a\sqrt{x})J_{-\frac{1}{4}}(x)\,dx = \sqrt{\pi}a^{\frac{3}{2}}J_{\frac{3}{4}}(a^2) \qquad [a > 0].$$

ET II 341(10)

2. $$\int_0^\infty x^{\frac{1}{4}}\cos(2a\sqrt{x})J_{\frac{1}{4}}(x)\,dx = \sqrt{\pi}a^{\frac{3}{2}}J_{-\frac{3}{4}}(a^2) \qquad [a > 0].$$

ET II 341(12)

3. $$\int_0^\infty x^{\frac{1}{4}}\sin(2a\sqrt{x})J_{\frac{3}{4}}(x)\,dx = \sqrt{\pi}a^{\frac{3}{2}}J_{-\frac{1}{4}}(a^2) \qquad [a > 0].$$

ET II 341(11)

4. $$\int_0^\infty x^{\frac{1}{4}}\cos(2a\sqrt{x})J_{-\frac{3}{4}}(x)\,dx = \sqrt{\pi}a^{\frac{3}{2}}J_{\frac{1}{4}}(a^2) \qquad [a > 0].$$

ET II 341(13)

6.736

1. $$\int_0^\infty x^{-\frac{1}{2}}\sin\cos(4a\sqrt{x})J_0(x)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi}\left[\cos\left(a^2-\frac{\pi}{4}\right)J_0\left(a^2\right)-\sin\left(a^2-\frac{\pi}{4}\right)N_0\left(a^2\right)\right] \\ [a > 0].$$

ET II 341(18)

2. $$\int_0^\infty x^{-\frac{1}{2}}\cos x\cos(4a\sqrt{x})J_0(x)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi}\left[\sin\left(a^2-\frac{\pi}{4}\right)J_0\left(a^2\right)+\cos\left(a^2-\frac{\pi}{4}\right)N_0\left(a^2\right)\right] \\ [a > 0].$$

3. $$\int_0^\infty x^{-\frac{1}{2}} \sin x \sin(4a\sqrt{x}) J_0(x)\,dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 + \frac{\pi}{4}\right) J_0(a^2) \qquad [a > 0]\,.$$

ET II 341(16)

4. $$\int_0^\infty x^{-\frac{1}{2}} \cos x \sin(4a\sqrt{x}) J_0(x)\,dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) \qquad [a > 0]\,.$$

ET II 342(20)

5. $$\int_0^\infty x^{-\frac{1}{2}} \sin x \cos(4a\sqrt{x}) N_0(x)\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}\left[3\sin\left(a^2 - \frac{\pi}{4}\right) J_0\left(a^2\right) - \cos\left(a^2 - \frac{\pi}{4}\right) N_0\left(a^2\right)\right]$$
$$[a > 0]\,.$$

ET II 347(55)

6. $$\int_0^\infty x^{-\frac{1}{2}} \cos x \cos(4a\sqrt{x}) N_0(x)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi}\left[3\cos\left(a^2 - \frac{\pi}{4}\right) J_0\left(a^2\right) + \sin\left(a^2 - \frac{\pi}{4}\right) N_0\left(a^2\right)\right]$$
$$[a > 0]\,.$$

ET II 347(56)

6.737

1. $$\int_0^\infty \frac{\sin\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} J_\nu(cx)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{b}{2}\left(a - \sqrt{a^2-c^2}\right)\right] J_{-\frac{1}{2}\nu}\left[\frac{b}{2}\left(a + \sqrt{a^2-c^2}\right)\right]$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re}\nu > -1]\,.$$

ET II 35(19)

2. $$\int_0^\infty \frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} J_\nu(cx)\,dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{b}{2}\left(a - \sqrt{a^2-c^2}\right)\right] N_{-\frac{1}{2}\nu}\left[\frac{b}{2}\left(a + \sqrt{a^2-c^2}\right)\right]$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re}\nu > -1]\,.$$

ET II 39(44)

3. $$\int_0^a \frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} J_\nu(cx)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2} - b\right)\right] J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2} + b\right)\right]$$
$$[c > 0, \quad \operatorname{Re}\nu > -1]\,.$$

4. $$\int_0^a x^{\nu+1} \frac{\cos\left(\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} I_\nu(x)\,dx = \frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)} \qquad [\operatorname{Re}\nu > -1]\,.$$

ET II 365(9)

5. $$\int_0^\infty x^{\nu+1} \frac{\sin\left(a\sqrt{b^2+x^2}\right)}{\sqrt{b^2+x^2}} J_\nu(cx)dx = \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2-c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} J_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right)$$
$$\left[0 < c < a,\ \operatorname{Re} b > 0,\ -1 < \operatorname{Re}\nu < \frac{1}{2}\right];$$
$$= 0 \qquad \left[0 < a < c,\ \operatorname{Re} b > 0,\ -1 < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 35(20)

6. $$\int_0^\infty x^{\nu+1} \frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} J_\nu(cx)\,dx = -\sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2-c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} N_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right)$$
$$\left[0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right];$$
$$= \sqrt{\frac{2}{\pi}} b^{\frac{1}{2}+\nu} c^\nu (c^2-a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} K_{\nu+\frac{1}{2}}\left(b\sqrt{c^2-a^2}\right)$$
$$\left[0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 39(45)

6.738

1. $$\int_0^a x^{\nu+1} \sin\left(b\sqrt{a^2-x^2}\right) J_\nu(x)\,dx = \sqrt{\frac{\pi}{2}} a^{\nu+\frac{3}{2}} b (1+b^2)^{-\frac{1}{2}\nu-\frac{3}{4}} J_{\nu+\frac{3}{2}}\left(a\sqrt{1+b^2}\right) \qquad [\operatorname{Re}\nu > -1]\,.$$

ET II 335(19)

2. $$\int_0^\infty x^{\nu+1} \cos\left(a\sqrt{x^2+b^2}\right) J_\nu(cx)\,dx =$$
$$= \sqrt{\frac{\pi}{2}} a b^{\nu+\frac{3}{2}} c^\nu (a^2-c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} \left[\cos(\pi\nu) J_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right) - \sin(\pi\nu) N_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right)\right]$$
$$\left[0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right];$$
$$= 0 \qquad \left[0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right].$$

ET II 39(43)

6.739

$$\int_0^t x^{-\frac{1}{2}} \frac{\cos(b\sqrt{t-x})}{\sqrt{t-x}} J_{2\nu}\left(a\sqrt{x}\right)\, dx = \pi J_\nu \left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}+b\right)\right] J_\nu \left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}-b\right)\right] \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

EH II 47(7)

6.741

1. $$\int_0^1 \frac{\cos(\mu \arccos x)}{\sqrt{1-x^2}} J_\nu(ax)\, dx = \frac{\pi}{2} J_{\frac{1}{2}(\mu+\nu)}\left(\frac{a}{2}\right) J_{\frac{1}{2}(\nu-\mu)}\left(\frac{a}{2}\right) \qquad [\operatorname{Re}(\mu+\nu) > -1, \quad a > 0]\,.$$

ET II 41(54)

2. $$\int_0^1 \frac{\cos\left[(\nu+1)\arccos x\right]}{\sqrt{1-x^2}} J_\nu(ax)\, dx = \sqrt{\frac{\pi}{a}} \cos\left(\frac{a}{2}\right) J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) \qquad [\operatorname{Re}\nu > -1, \quad a > 0]\,.$$

ET II 40(53)

3. $$\int_0^1 \frac{\cos\left[(\nu-1)\arccos x\right]}{\sqrt{1-x^2}} J_\nu(ax)\, dx = \sqrt{\frac{\pi}{a}} \sin\left(\frac{a}{2}\right) J_{\nu-\frac{1}{2}}\left(\frac{a}{2}\right) \qquad [\operatorname{Re}\nu > 0, \quad a > 0]\,.$$

ET II 40(52)a

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

6.751

1. $$\int_0^\infty e^{-\frac{1}{2}ax} \sin(bx) I_0\left(\frac{1}{2}ax\right)\, dx = \frac{1}{\sqrt{2b}} \frac{1}{\sqrt{b^2+a^2}} \sqrt{b+\sqrt{b^2+a^2}} \qquad [\operatorname{Re} a > 0, \quad b > 0]\,.$$

ET I 105(44)

2. $$\int_0^\infty e^{-\frac{1}{2}ax} \cos(bx) I_0\left(\frac{1}{2}ax\right)\, dx = \frac{a}{\sqrt{2b}} \frac{1}{\sqrt{a^2+b^2}\sqrt{b+\sqrt{a^2+b^2}}} \qquad [\operatorname{Re} a > 0, \quad b > 0]\,.$$

ET I 48(38)

3. $$\int_0^\infty e^{-bx} \cos(ax) J_0(cx)\, dx = \frac{\left[\sqrt{(b^2+c^2-a^2)^2+4a^2b^2}+b^2+c^2-a^2\right]^{\frac{1}{2}}}{\sqrt{2}\sqrt{(b^2+c^2-a^2)^2+4a^2b^2}} \qquad [c > 0]\,.$$

6.752

1. $$\int_0^\infty e^{-ax} J_0(bx) \sin(cx) \frac{dx}{x} = \arcsin\left(\frac{2c}{\sqrt{a^2+(c+b)^2}+\sqrt{a^2+(c-b)^2}}\right)$$
$$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0] \, .$$

ET I 101(17)

2. $$\int_0^\infty e^{-ax} J_1(cx) \sin(bx) \frac{dx}{x} = \frac{b}{c}(1-r), \qquad \left[b^2 = \frac{c^2}{1-r^2} - \frac{a^2}{r^2}, \quad c > 0\right] .$$

ET II 19(15)

6.753

1. $$\int_0^\infty \frac{\sin(xa\sin\psi)}{x} e^{-xa\cos\varphi\cos\psi} J_\nu(xa\sin\varphi)\, dx = \nu^{-1}\left(\operatorname{tg}\frac{\varphi}{2}\right)^\nu \sin(\nu\psi)$$
$$\left[\operatorname{Re}\nu > -1, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}\right] .$$

ET II 33(10)

2. $$\int_0^\infty \frac{\cos(xa\sin\psi)}{x} e^{-xa\cos\varphi\cos\psi} J_\nu(xa\sin\varphi)\, dx = \nu^{-1}\left(\operatorname{tg}\frac{\varphi}{2}\right)^\nu \cos(\nu\psi)$$
$$\left[\operatorname{Re}\nu > 0, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}\right] .$$

ET II 38(35)

3.* $$\int_0^\infty x^{\nu+1} e^{-sx} \sin(bx) I_\nu(ax) dx =$$
$$= -\frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma(\nu+3/2) R^{-2\nu-3} \left[b\cos(\nu+3/2)\varphi + s\sin(\nu+3/2)\varphi\right] ,$$
$$\left[\operatorname{Re}\nu > -3/2,\ \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\ R^4 = (s^2+a^2-b^2)^2 + 4b^2s^2,\ \varphi = \arg(s^2+a^2-b^2-2ibs)\right] .$$

4.* $$\int_0^\infty x^{\nu+1} e^{-sx} \cos(bx) J_\nu(ax) dx =$$
$$= \frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma(\nu+3/2) R^{-2\nu-3} \left[s\cos(\nu+3/2)Q - b\sin(\nu+3/2)\varphi\right] ,$$
$$\left[\operatorname{Re}\nu > -1,\ \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\ R^4 = (s^2+a^2-b^2)^2 + 4b^2s^2,\ Q = \arg(s^2+a^2-b^2-2ibs)\right] .$$

5. $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi}\sin(ax\sin\psi)J_\nu(ax\sin\varphi)\,dx =$$
$$= 2^\nu \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1}(\sin\varphi)^\nu(\cos^2\psi+\sin^2\psi\cos^2\varphi)^{-\nu-\frac{1}{2}}\sin\left[\left(\nu+\frac{3}{2}\right)\beta\right],$$
$$\operatorname{tg}\frac{\beta}{2} = \operatorname{tg}\psi\cos\varphi \qquad \left[a>0, \quad 0<\varphi, \quad \psi<\frac{\pi}{2}, \quad \operatorname{Re}\nu>-1\right].$$

ET II 34(12)

6. $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi}\cos(ax\sin\psi)J_\nu(ax\sin\varphi)\,dx =$$
$$= 2^\nu \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1}(\sin\varphi)^\nu(\cos^2\psi+\sin^2\psi\cos^2\varphi)^{-\nu-\frac{1}{2}}\cos\left[\left(\nu+\frac{1}{2}\right)\beta\right],$$
$$\operatorname{tg}\frac{\beta}{2} = \operatorname{tg}\psi\cos\varphi \qquad \left[a>0, \quad 0<\varphi, \quad \psi<\frac{\pi}{2}, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 38(37)

6.754

1. $$\int_0^\infty e^{-x^2}\sin(bx)I_0(x^2)\,dx = \frac{\sqrt{\pi}}{2^{\frac{3}{2}}}e^{-\frac{b^2}{8}}I_0\left(\frac{b^2}{8}\right) \qquad [b>0].$$

ET I 108(9)

2. $$\int_0^\infty e^{-ax}\cos(x^2)J_0(x^2)\,dx = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left[J_0\left(\frac{a^2}{16}\right)\cos\left(\frac{a^2}{16}-\frac{\pi}{4}\right) - N_0\left(\frac{a^2}{16}\right)\cos\left(\frac{a^2}{16}+\frac{\pi}{4}\right)\right]$$
$$[a>0].$$

MI 42

3. $$\int_0^\infty e^{-ax}\sin(x^2)J_0(x^2)\,dx = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left[J_0\left(\frac{a^2}{16}\right)\sin\left(\frac{a^2}{16}-\frac{\pi}{4}\right) - N_0\left(\frac{a^2}{16}\right)\sin\left(\frac{a^2}{16}+\frac{\pi}{4}\right)\right]$$
$$[a>0].$$

MI 42

6.755

1. $$\int_0^\infty x^{-\nu}e^{-x}\sin\left(4a\sqrt{x}\right)I_\nu(x)\,dx = (2^{\frac{3}{2}}a)^{\nu-1}e^{-a^2}W_{\frac{1}{2}-\frac{3}{2}\nu,\,\frac{1}{2}-\frac{1}{2}\nu}(2a^2) \qquad [a>0, \quad \operatorname{Re}\nu>0].$$

ET II 366(14)

2. $$\int_0^\infty x^{-\nu-\frac{1}{2}}e^{-x}\cos\left(4a\sqrt{x}\right)I_\nu(x)\,dx = 2^{\frac{3}{2}\nu-1}a^{\nu-1}e^{-a^2}W_{-\frac{3}{2}\nu,\,\frac{1}{2}\nu}(2a^2) \qquad \left[a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 366(16)

3. $$\int_0^\infty x^{-\nu} e^x \sin\left(4a\sqrt{x}\right) K_\nu(x)\,dx = (2^{\frac{3}{2}}a)^{\nu-1}\pi \frac{\Gamma\left(\frac{3}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} e^{a^2} W_{\frac{3}{2}\nu-\frac{1}{2},\,\frac{1}{2}-\frac{1}{2}\nu}(2a^2) \qquad \left[a>0, \quad 0<\operatorname{Re}\nu<\frac{3}{4}\right].$$

ET II 369(38)

4. $$\int_0^\infty x^{-\nu-\frac{1}{2}} e^x \cos\left(4a\sqrt{x}\right) K_\nu(x)\,dx = 2^{\frac{3}{2}\nu-1}\pi a^{\nu-1} \frac{\Gamma\left(\frac{1}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} e^{a^2} W_{\frac{3}{2}\nu,\,-\frac{1}{2}\nu}(2a^2) \qquad \left[a>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{4}\right].$$

ET II 369(42)

5. $$\int_0^\infty x^{\varrho-\frac{3}{2}} e^{-x} \sin\left(4a\sqrt{x}\right) K_\nu(x)\,dx = \frac{\sqrt{\pi}a\Gamma(\varrho+\nu)\Gamma(\varrho-\nu)}{2^{\varrho-2}\Gamma\left(\varrho+\frac{1}{2}\right)}\, {}_2F_2\left(\varrho+\nu,\ \varrho-\nu;\ \frac{3}{2},\ \varrho+\frac{1}{2};\ -2a^2\right) \qquad [\operatorname{Re}\varrho>|\operatorname{Re}\nu|].$$

ET II 369(39)

6. $$\int_0^\infty x^{\varrho-1} e^{-x} \cos\left(4a\sqrt{x}\right) K_\nu(x)\,dx = \frac{\sqrt{\pi}\Gamma(\varrho+\nu)\Gamma(\varrho-\nu)}{2^{\varrho}\Gamma\left(\varrho+\frac{1}{2}\right)}\, {}_2F_2\left(\varrho+\nu,\ \varrho-\nu;\ \frac{1}{2},\ \varrho+\frac{1}{2};\ -2a^2\right) \qquad [\operatorname{Re}\varrho>|\operatorname{Re}\nu|].$$

ET II 370(43)

7. $$\int_0^\infty x^{-\frac{1}{2}} e^{-x} \cos\left(4a\sqrt{x}\right) I_0(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-a^2} K_0(a^2) \qquad [a>0].$$

ET II 366(15)

ET II 369(40)

9. $\int_0^\infty x^{-\frac{1}{2}} e^{-x} \cos\left(4a\sqrt{x}\right) K_0(x)\, dx = \frac{1}{\sqrt{2}} \pi^{\frac{3}{2}} e^{-a^2} I_0(a^2).$

ET II 369(41)

6.756

1. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_\nu(bx)\, dx = \frac{i}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$ $$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1] .$$

ET II 34(17)

2. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_\nu(bx)\, dx = \frac{1}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$ $$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right] .$$

ET II 39(42)

3. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_0(bx)\, dx = \frac{1}{2b} a I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \qquad \left[|\arg a| < \frac{\pi}{4},\ b > 0\right] .$$

ET II 11(40)

4. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_0(bx)\, dx = \frac{a}{2b} I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \qquad \left[|\arg a| < \frac{\pi}{4},\ b > 0\right] .$$

ET II 12(49)

6.757

1. $$\int_0^\infty e^{-bx} \sin\left[a(1 - e^{-x})\right] J_\nu(ae^{-x})\, dx = 2\sum_{n=0}^\infty \frac{(-1)^n \Gamma(\nu - b + 2n + 1)\Gamma(\nu + b)}{\Gamma(\nu - b + 1)\Gamma(\nu + b + 2n + 2)} (\nu+2n-1) J_{\nu+2n+1}(a) \qquad [\operatorname{Re} b > -\operatorname{Re}\nu] .$$

2. $$\int_0^\infty e^{-bx} \cos\left[a(1-e^{-x})\right] J_\nu(ae^{-x})\,dx =$$
$$= \frac{J_\nu(a)}{\nu+b} + \sum_{n=0}^{\infty} 2(-1)^n \frac{\Gamma(\nu-b+2n)\Gamma(\nu+b)}{\Gamma(\nu-b+1)\Gamma(\nu+b+2n+1)} (\nu+2n) J_{\nu+2n}(a)$$
$$[\operatorname{Re} b > -\operatorname{Re}\nu]\,.$$

ET I 193(27)

6.758

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta} (\cos\theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu} J_{\nu+\mu}(\lambda z)\,d\theta = \pi(2az)^{-\mu}(2bz)^{-\nu} J_\mu(az) J_\nu(bz);$$
$$\lambda = \sqrt{2\cos\theta(a^2e^{i\theta}+b^2e^{-i\theta})} \qquad [\operatorname{Re}(\nu+\mu) > -1]\,.$$

EH II 48(12)

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

6.761

$$\int_0^\infty \operatorname{ch} x \cos(2a \operatorname{sh} x) J_\nu(be^x) J_\nu(be^{-x})\,dx = \frac{J_{2\nu}\left(2\sqrt{b^2-a^2}\right)}{2\sqrt{b^2-a^2}} \qquad [0 < a < b, \quad \operatorname{Re}\nu > -1]\,;$$
$$= 0 \qquad [0 < b < a, \quad \operatorname{Re}\nu > -1]\,.$$

ET II 359(10)

6.762

$$\int_0^\infty \operatorname{ch} x \sin(2a \operatorname{sh} x) \left[J_\nu(be^x)N_\nu(be^{-x}) - N_\nu(be^x)J_\nu(be^{-x})\right]\,dx =$$
$$= 0 \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]\,;$$
$$= -\frac{2}{\pi}\cos(\nu\pi)(a^2-b^2)^{-\frac{1}{2}} K_{2\nu}\left[2(a^2-b^2)^{\frac{1}{2}}\right] \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]\,.$$

ET II 360(12)

6.763

$$\int_0^\infty \operatorname{ch} x \cos(2a \operatorname{sh} x) N_\nu(be^x) N_\nu(be^{-x})\,dx =$$
$$= -\frac{1}{2}(b^2-a^2)^{-\frac{1}{2}} J_{2\nu}\left[2(b^2-a^2)^{\frac{1}{2}}\right] \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 1]\,;$$
$$= \frac{2}{\pi}\cos(\nu\pi)(a^2-b^2)^{-\frac{1}{2}} K_{2\nu}\left[2(a^2-b^2)^{\frac{1}{2}}\right] \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 1]\,.$$

ET II 360(11)

6.77 Combinations of Bessel functions and the logarithm, or arctangent

6.771

$$\int_0^\infty x^{\mu+\frac{1}{2}} \ln x J_\nu(ax)\,dx = \frac{2^{\mu-\frac{1}{2}}\Gamma\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right)a^{\mu+\frac{3}{2}}}\left[\psi\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)+\psi\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right)-\ln\frac{a^2}{4}\right]$$
$$\left[a>0, \quad -\operatorname{Re}\nu-\frac{3}{2}<\operatorname{Re}\mu<0\right].$$

ET II 32(25)

6.772

1. $$\int_0^\infty \ln x J_0(ax)\,dx = -\frac{1}{a}\left[\ln(2a)+\boldsymbol{C}\right].$$

WA 430(4)A, ET II 10(27)

2. $$\int_0^\infty \ln x J_1(ax)\,dx = -\frac{1}{a}\left[\ln\left(\frac{a}{2}\right)+\boldsymbol{C}\right].$$

ET II 19(11)

3. $$\int_0^\infty \ln(a^2+x^2)J_1(bx)\,dx = \frac{2}{b}\left[K_0(ab)+\ln a\right].$$

ET II 19(12)

4. $$\int_0^\infty J_1(t\,x)\ln\sqrt{1+t^4}\,dt = \frac{2}{x}\operatorname{ker} x.$$

MO 46

6.773

$$\int_0^\infty \frac{\ln\left(x+\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}J_0(bx)\,dx = \left[\frac{1}{2}K_0^2\left(\frac{ab}{2}\right)+\ln a I_0\left(\frac{ab}{2}\right)K_0\left(\frac{ab}{2}\right)\right] \qquad [a>0, \quad b>0].$$

ET II 10(28)

6.774

$$\int_0^\infty \ln \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} J_0(bx) \frac{dx}{\sqrt{x^2+a^2}} = K_0^2\left(\frac{ab}{2}\right) \qquad [\operatorname{Re} a > 0, \quad b > 0] \,.$$

ET II 10(29)

6.775

$$\int_0^\infty x\left[\ln\left(1+\sqrt{a^2+x^2}\right) - \ln x\right] J_0(bx)\,dx = \frac{1}{b^2}(1-e^{-ab}) \qquad [\operatorname{Re} a > 0, \quad b > 0] \,.$$

ET II 12(55)

6.776

$$\int_0^\infty x \ln\left(1+\frac{a^2}{x^2}\right) J_0(bx)\,dx = \frac{2}{b}\left[\frac{1}{b} - aK_1(ab)\right] \qquad [\operatorname{Re} a > 0, \quad b > 0] \,.$$

ET II 10(30)

6.777

$$\int_0^\infty J_1(t\,x) \operatorname{arctg} t^2\,dt = -\frac{2}{x} \operatorname{kei} x.$$

MO 46

6.78 Combinations of Bessel and other special functions

6.781

$$\int_0^\infty \operatorname{si}(ax) J_0(bx)\,dx = -\frac{1}{b} \arcsin\left(\frac{b}{a}\right) \qquad [0 < b < a] \,;$$
$$= 0 \qquad [0 < a < b] \,.$$

ET II 13(6)

6.782

1. $$\int_0^\infty \operatorname{Ei}(-x) J_0\left(2\sqrt{zx}\right)\,dx = \frac{e^{-z}-1}{z}.$$

NT 60(4)

2. $$\int_0^\infty \operatorname{si}(x) J_0\left(2\sqrt{zx}\right)\,dx = -\frac{\sin z}{z}.$$

3. $$\int_0^\infty \operatorname{ci}(x) J_0\left(2\sqrt{zx}\right)\, dx = \frac{\cos z - 1}{z}.$$

NT 60(5)

4. $$\int_0^\infty \operatorname{Ei}(-x) J_1\left(2\sqrt{zx}\right) \frac{dx}{\sqrt{x}} = \frac{\operatorname{Ei}(-z) - \boldsymbol{C} - \ln z}{\sqrt{z}}.$$

NT 60(7)

5. $$\int_0^\infty \operatorname{si}(x) J_1\left(2\sqrt{zx}\right) \frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \operatorname{si}(z)}{\sqrt{z}}.$$

NT 60(9)

783

6. $$\int_0^\infty \operatorname{ci}(z) J_1\left(2\sqrt{zx}\right) \frac{dx}{\sqrt{x}} = \frac{\operatorname{ci}(z) - \boldsymbol{C} - \ln z}{\sqrt{z}}.$$

NT 60(8)

7. $$\int_0^\infty \operatorname{Ei}(-x) N_0\left(2\sqrt{zx}\right)\, dx = \frac{\boldsymbol{C} + \ln z - e^2 \operatorname{Ei}(-z)}{\pi z}.$$

NT 63(5)

6.783

1. $$\int_0^\infty x \operatorname{si}(a^2x^2) J_0(bx)\, dx = -\frac{2}{b^2} \sin\left(\frac{b^2}{4a^2}\right) \qquad [a > 0]\,.$$

ET II 13(7)a

2. $$\int_0^\infty x \operatorname{ci}(a^2x^2) J_0(bx)\, dx = \frac{2}{b^2}\left[1 - \cos\left(\frac{b^2}{4a^2}\right)\right] \qquad [a > 0]\,.$$

ET II 13(8)a

3. $$\int_0^\infty \operatorname{ci}(a^2x^2) J_0(bx)\, dx = \frac{1}{b}\left[\operatorname{ci}\left(\frac{b^2}{4a^2}\right) + \ln\left(\frac{b^2}{4a^2}\right) + 2\boldsymbol{C}\right] \qquad [a > 0]\,.$$

4. $$\int_0^\infty \operatorname{si}(a^2x^2)J_1(bx)\,dx = \frac{1}{b}\left[-\operatorname{si}\left(\frac{b^2}{4a^2}\right) - \frac{\pi}{2}\right] \qquad [a>0].$$

ET II 20(25)a

6.784

1. $$\int_0^\infty x^{\nu+1}\left[1-\Phi(ax)\right] J_\nu(bx)\,dx = a^{-\nu}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{b^2\Gamma(\nu+2)}\exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu+\frac{1}{2},\,\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a|<\frac{\pi}{4},\quad b>0,\quad \operatorname{Re}\nu>-1\right].$$

ET II 92(22)

2. $$\int_0^\infty x^{\nu}\left[1-\Phi(ax)\right] J_\nu(bx)\,dx = \sqrt{\frac{2}{\pi}}\frac{a^{\frac{1}{2}-\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{b^{\frac{3}{2}}\Gamma\left(\nu+\frac{3}{2}\right)}\exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu-\frac{1}{4},\,\frac{1}{2}\nu+\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a|<\frac{\pi}{4},\quad \operatorname{Re}\nu>-\frac{1}{2},\quad b>0\right].$$

ET II 92(23)

6.785

$$\int_0^\infty \frac{\exp\left(\frac{a^2}{2x}-x\right)}{x}\left[1-\Phi\left(\frac{a}{\sqrt{2x}}\right)\right] K_\nu(x)\,dx = \frac{\pi^{\frac{5}{2}}}{4}\sec(\nu\pi)\left\{\left[J_\nu(a)\right]^2+\left[N_\nu(a)\right]^2\right\}$$
$$\left[\operatorname{Re}a>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET II 370(46)

6.786

$$\int_0^\infty x^{\nu-2\mu+2n+2}e^{x2}\Gamma(\mu,\,x^2)N_\nu(bx)\,dx =$$
$$= (-1)^n\frac{\Gamma\left(\frac{3}{2}-\mu+\nu+n\right)\Gamma\left(\frac{3}{2}-\mu+n\right)}{b\Gamma(1-\mu)}\exp\left(\frac{b^2}{8}\right)W_{\mu-\frac{1}{2}\nu-n-1,\,\frac{1}{2}\nu}\left(\frac{b^2}{4}\right)$$
$$\left[n - \text{an integer},\quad b>0,\quad \operatorname{Re}(\nu-\mu+n)>-\frac{3}{2},\quad \operatorname{Re}(-\mu+n)>-\frac{3}{2},\quad \operatorname{Re}\nu<\frac{1}{2}-2n\right].$$

ET II 108(2)

6.787

$$\int_0^\infty \frac{x^{\nu+2n-\frac{1}{2}}}{\mathrm{B}(a+x,\ a-x)} J_\nu(bx)\,dx = 0 \qquad \left[\pi \le b < \infty, \quad -1 < \operatorname{Re}\nu < 2a - 2n - \frac{7}{2}\right].$$

ET II 92(21)

6.79 Integration of Bessel functions with respect to the order

6.791

1. $$\int_{-\infty}^\infty K_{ix+iy}(a) K_{ix+iz}(b)\,dx = \pi K_{iy-iz}(a+b) \qquad [|\arg a| + |\arg b| < \pi].$$

ET II 382(21)

2. $$\int_{-\infty}^\infty J_{\nu-x}(a) J_{\mu+x}(a)\,dx = J_{\mu+\nu}(2a) \qquad [\operatorname{Re}(\mu+\nu) > 1].$$

ET II 379(1)

3. $$\int_{-\infty}^\infty J_{\{+x}(a) J_{\lambda-x}(a) J_{\mu+x}(a) J_{\nu-x}(a)\,dx =$$
$$= \frac{\Gamma(\{+\lambda+\mu+\nu+1)}{\Gamma(\{+\lambda+1)\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)\Gamma(\nu+\{+1)} \times$$
$$\times {}_4F_5\left(\frac{\{+\lambda+\mu+\nu+1}{2}, \frac{\{+\lambda+\mu+\nu+1}{2}, \frac{\{+\lambda+\mu+\nu}{2}+1, \frac{\{+\lambda+\mu+\nu}{2}+1;\right.$$
$$\left.\{+\lambda+\mu+\nu+1, \{+\lambda+1, \lambda+\mu+1, \mu+\nu+1, \nu+\{+1; -4a^2\right)$$
$$[\operatorname{Re}(\{+\lambda+\mu+\nu) > -1].$$

ET II 379(3)

6.792

1. $$\int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a) K_{ix+iz}(b)\,dx = \pi e^{-\pi z} K_{i(y-z)}(a-b) \qquad [a > b > 0].$$

ET II 382(22)

2. $$\int_{-\infty}^\infty e^{i\varrho x} K_{\nu+ix}(\alpha) K_{\nu-ix}(\beta)\,dx = \pi \left(\frac{\alpha e^\rho + \beta}{\alpha + \beta e^\rho}\right)^\nu K_{2\nu}\left(\sqrt{\alpha^2+\beta^2+2\alpha\beta\operatorname{ch}\varrho}\right)$$
$$[|\arg\alpha| + |\arg\beta| + |\operatorname{Im}\varrho| < \pi].$$

3. $$\int_{-\infty}^{\infty} e^{(\pi-\gamma)x} K_{ix+iy}(a) K_{ix+iz}(b)\, dx = \pi e^{-\beta y - \alpha z} K_{iy-iz}(c)$$
$[0 < \gamma < \pi, \quad a > 0, \quad b > 0, \quad c > 0, \quad \alpha,\, \beta,\, \gamma$ --- the angles of the triangle with sides $a,\, b,\, c\,]$.

ET II 382(24), EH II 55(44)a

785

4. $$\int_{-\infty}^{\infty} e^{-cxi} H^{(2)}_{\nu-ix}(a) H^{(2)}_{\nu+ix}(b)\, dx = 2i \left(\frac{h}{k}\right)^{2\nu} H^{(2)}_{2\nu}(hk),$$
$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \qquad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \qquad [a,\, b > 0, \quad \operatorname{Im} c = 0]\,.$$

ET II 380(11)

5. $$\int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{cxi} J_{\mu+x}(a) J_{\nu-x}(b)\, dx =$$
$$= \left[\frac{2\cos\left(\frac{c}{2}\right)}{a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}}\right]^{\frac{1}{2}\mu+\frac{1}{2}\nu} \exp\left[\frac{c}{2}(\nu-\mu)i\right] J_{\mu+\nu}\left\{\left[2\cos\left(\frac{c}{2}\right)\left(a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}\right)\right]^{\frac{1}{2}}\right\}$$
$$[b > 0, \quad a > 0, \quad |c| < \pi, \quad \operatorname{Re}(\mu+\nu) > 1]\;;$$
$$= 0 \qquad [a > 0, \quad b > 0, \quad |c| \geq \pi, \quad \operatorname{Re}(\mu+\nu) > 1]\;.$$

EH II 54(41), ET II 379(2)

6.793

1. $$\int_{-\infty}^{\infty} e^{-cxi} \left[J_{\nu-ix}(a) N_{\nu+ix}(b) + N_{\nu-ix}(a) J_{\nu+ix}(b)\right]\, dx = -2 \left(\frac{h}{k}\right)^{2\nu} J_{2\nu}(hk),$$
$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \qquad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \qquad [a,\, b > 0, \quad \operatorname{Im} c = 0]\,.$$

ET II 380(9)

2. $$\int_{-\infty}^{\infty} e^{-cxi} \left[J_{\nu-ix}(a) J_{\nu+ix}(b) - N_{\nu-ix}(a) N_{\nu+ix}(b)\right]\, dx = 2 \left(\frac{h}{k}\right)^{2\nu} N_{2\nu}(hk),$$
$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \qquad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \qquad [a,\, b > 0, \quad \operatorname{Im} c = 0]\,.$$

ET II 380(10)

3.* $$\int_{-\infty}^{\infty} e^{i\gamma x} \operatorname{sech}(\pi x) \left[J_{-ix}(\alpha) J_{ix}(\beta) - J_{ix}(\alpha) J_{-ix}(\beta)\right]\, dx = 2iH(\sigma)\operatorname{sgn}(\beta-\alpha) J_0(\sigma^{\frac{1}{2}})$$
$\left[\alpha,\, \beta,\, \gamma \in \mathsf{R}, \quad \alpha,\, \beta > 0, \quad \sigma = \alpha^2 + \beta^2 - 2\alpha\beta\cosh\gamma, \quad H(\sigma)\right.$ the Heaviside step function $\left.\right]$.

1. $$\int_0^\infty K_{ix}(a)K_{ix}(b)\,\operatorname{ch}\left[(\pi-\varphi)x\right]\,dx = \frac{\pi}{2}K_0\left(\sqrt{a^2+b^2-2ab\cos\varphi}\right).$$

EH II 55(42)

2. $$\int_0^\infty \operatorname{ch}\left(\frac{\pi}{2}x\right)K_{ix}(a)\,dx = \frac{\pi}{2} \qquad [a>0].$$

ET II 382(19)

3. $$\int_0^\infty \operatorname{ch}(\varrho x)K_{ix+\nu}(a)K_{-ix+\nu}(a)\,dx = \frac{\pi}{2}K_{2\nu}\left[2a\cos\left(\frac{\varrho}{2}\right)\right] \qquad [2|\arg a|+|\operatorname{Re}\varrho|<\pi].$$

ET II 383(28)

4. $$\int_{-\infty}^\infty \operatorname{sech}\left(\frac{\pi}{2}x\right)J_{ix}(a)\,dx = 2\sin a \qquad [a>0].$$

ET II 380(6)

5. $$\int_{-\infty}^\infty \operatorname{cosech}\left(\frac{\pi}{2}x\right)J_{ix}(a)\,dx = -2i\cos a \qquad [a>0].$$

ET II 380(7)

6. $$\int_0^\infty \operatorname{sech}(\pi x)\left\{\left[J_{ix}(a)\right]^2+\left[N_{ix}(a)\right]^2\right\}\,dx = -N_0(2a)-E_0(2a) \qquad [a>0].$$

ET II 380(12)

7. $$\int_0^\infty x\,\operatorname{sh}\left(\frac{\pi}{2}x\right)K_{ix}(a)\,dx = \frac{\pi a}{2} \qquad [a>0].$$

ET II 382(20)

8. $$\int_0^\infty x\,\theta(\pi x)K_{ix}(\beta)K_{ix}(\alpha)\,dx = \frac{\pi}{2}\sqrt{\alpha\beta}\frac{\exp\left(-\beta-\alpha\right)}{\alpha+\beta} \qquad [|\arg\beta|<\pi, \quad |\arg\alpha|<\pi].$$

ET II 175(4)

9. $$\int_0^\infty x\,\operatorname{sh}(\pi x)K_{2ix}(\alpha)K_{ix}(\beta)\,dx = \frac{\pi^{\frac{3}{2}}\alpha}{2^{\frac{5}{2}}\sqrt{\beta}}\exp\left(-\beta-\frac{\alpha^2}{8\beta}\right) \qquad \left[\beta>0, \quad |\arg\alpha|<\frac{\pi}{4}\right].$$

10. $$\int_0^\infty \frac{x\ \mathrm{sh}(\pi x)}{x^2+n^2} K_{ix}(\alpha) K_{ix}(\beta)\,dx = \frac{\pi^2}{2} I_n(\beta) K_n(\alpha) \qquad [0<\beta<\alpha; \quad n=0,\,1,\,2,\ldots]\ ;$$
$$= \frac{\pi^2}{2} I_n(\alpha) K_n(\beta) \qquad [0<\alpha<\beta; \quad n=0,\,1,\,2,\ldots]\ .$$

ET II 176(8)

11. $$\int_0^\infty x\ \mathrm{sh}(\pi x) K_{ix}(\alpha) K_{ix}(\beta) K_{ix}(\gamma)\,dx = \frac{\pi^2}{4}\ \exp\left[-\frac{\gamma}{2}\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{\alpha\beta}{\gamma^2}\right)\right]$$
$$\left[|\arg\alpha| + |\arg\beta| < \frac{\pi}{2}, \quad \gamma>0\right]$$

ET II 176(9)

12. $$\int_0^\infty x\ \mathrm{sh}\left(\frac{\pi}{2}x\right) K_{\frac{1}{2}ix}(\alpha) K_{\frac{1}{2}ix}(\beta) K_{ix}(\gamma)\,dx = \frac{\pi^2\gamma}{2\sqrt{\gamma^2+4\alpha\beta}} \exp\left[-\frac{(\alpha+\beta)\sqrt{\gamma^2+4\alpha\beta}}{2\sqrt{\alpha\beta}}\right]$$
$$[|\arg\alpha| + |\arg\beta| < \pi, \quad \gamma>0]\ .$$

ET II 176(10)

13. $$\int_0^\infty x\ \mathrm{sh}(\pi x) K_{\frac{1}{2}ix+\lambda}(\alpha) K_{\frac{1}{2}ix-\lambda}(\alpha) K_{ix}(\gamma)\,dx = 0 \qquad [0<\gamma<2\alpha]\ ;$$
$$= \frac{\pi^2\gamma}{2^{2\lambda+1}\alpha^{2\lambda} z} \qquad \left[(\gamma+z)^{2\lambda} + (\gamma-z)^{2\lambda}\right]\ ,$$
$$z=\sqrt{\gamma^2-4\alpha^2} \qquad [0<2\alpha<\gamma]\ .$$

ET II 176(11)

6.795

1. $$\int_0^\infty \cos(bx) K_{ix}(a)\,dx = \frac{\pi}{2} e^{-a\ \mathrm{ch}\, b} \qquad \left[|\operatorname{Im} b| < \frac{\pi}{2}, \quad a>0\right]\ .$$

EH II 55(46), ET II 175(2)

2. $$\int_0^\infty J_x(ax) J_{-x}(ax) \cos(\pi x)\,dx = \frac{1}{4}(1-a^2)^{-\frac{1}{2}} \qquad [|a|<1]\ .$$

ET II 380(4)

3. $$\int_0^\infty x \sin(ax) K_{ix}(b)\,dx = \frac{\pi b}{2}\ \mathrm{sh}\, a \exp(-b\ \mathrm{ch}\, a) \qquad \left[|\operatorname{Im} a| < \frac{\pi}{2}, \quad b>0\right]\ .$$

4. $$\int_{-\infty}^{\infty} \frac{\sin[(\nu+ix)\pi]}{n+\nu+ix} K_{\nu+ix}(a) K_{\nu-ix}(b)\, dx = \pi^2 I_n(a) K_{n+2\nu}(b) \qquad [0<a<b; \quad n=0,\, 1, \dots]\ ;$$
$$= \pi^2 K_{n+2\nu}(a) I_n(b) \qquad [0<b<a; \quad n=0,\, 1, \dots]\ .$$

ET II 382(25)

5. $$\int_0^{\infty} x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a) K_{ix}(b)\, dx = \frac{\pi^{\frac{3}{2}} b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right) \qquad \left[|\arg a| < \frac{\pi}{2}, \quad b>0\right].$$

ET II 175(6)

6.796

1. $$\int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}\pi x}\ \cos(bx)}{\operatorname{sh}(\pi x)} J_{ix}(a)\, dx = -i \exp(ia\ \operatorname{ch} b) \qquad [a>0, \quad b>0]\ .$$

ET II 380(8)

2. $$\int_0^{\infty} \cos(bx)\ \operatorname{ch}\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2}\cos(a\ \operatorname{sh} b).$$

EH II 55(47)

3. $$\int_0^{\infty} \sin(bx)\ \operatorname{sh}\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2}\sin(a\ \operatorname{sh} b).$$

EH II 55(48)

4. $$\int_0^{\infty} \cos(bx)\ \operatorname{ch}(\pi x) \left[K_{ix}(a)\right]^2\ dx = -\frac{\pi^2}{4} N_0\left[2a\ \operatorname{sh}\left(\frac{b}{2}\right)\right] \qquad [a>0, \quad b>0]\ .$$

ET II 383(27)

5. $$\int_0^{\infty} \sin(bx)\ \operatorname{sh}(\pi x) \left[K_{ix}(a)\right]^2\ dx = \frac{\pi^2}{4} J_0\left[2a\ \operatorname{sh}\left(\frac{b}{2}\right)\right] \qquad [a>0, \quad b>0]\ .$$

ET II 382(26)

6.797

1. $$\int_0^\infty xe^{\pi x}\,\operatorname{sh}(\pi x)\Gamma(\nu+ix)\Gamma(\nu-ix)H_{ix}^{(2)}(a)H_{ix}^{(2)}(b)\,dx =$$
$$= i2^\nu\sqrt{\pi}\Gamma\left(\frac{1}{2}+\nu\right)(ab)^\nu(a+b)^{-\nu}K_\nu(a+b) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\nu>0]\,.$$

ET II 381(14)

2. $$\int_0^\infty xe^{\pi x}\,\operatorname{sh}(\pi x)\,\operatorname{ch}(\pi x)\Gamma(\nu+ix)\Gamma(\nu-ix)H_{ix}^{(2)}(a)H_{ix}^{(2)}(b)\,dx = \frac{i\pi^{\frac{3}{2}}2^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}(b-a)^{-\nu}H_\nu^{(2)}(b-a)$$
$$\left[0<a<b, \quad 0<\operatorname{Re}\nu<\frac{1}{2}\right].$$

ET II 381(15)

3. $$\int_0^\infty xe^{\pi x}\,\operatorname{sh}(\pi x)\Gamma\left(\frac{\nu+ix}{2}\right)\Gamma\left(\frac{\nu-ix}{2}\right)H_{ix}^{(2)}(a)H_{ix}^{(2)}(b)\,dx =$$
$$= i\pi 2^{2-\nu}(ab)^\nu(a^2+b^2)^{-\frac{1}{2}\nu}H_\nu^{(2)}\left(\sqrt{a^2+b^2}\right) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\nu>0]\,.$$

ET II 381(16)

4. $$\int_0^\infty x\,\operatorname{sh}(\pi x)\Gamma(\lambda+ix)\Gamma(\lambda-ix)K_{ix}(a)K_{ix}(b)\,dx = 2^{\nu-1}\pi^{\frac{3}{2}}(ab)^\lambda(a+b)^{-\lambda}\Gamma\left(\lambda+\frac{1}{2}\right)K_\lambda(a+b)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\lambda>0, \quad b>0]\,.$$

ET II 176(12)

5. $$\int_0^\infty x\,\operatorname{sh}(2\pi x)\Gamma(\lambda+ix)\Gamma(\lambda-ix)K_{ix}(a)K_{ix}(b)\,dx = \frac{2^\lambda\pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2}-\lambda\right)}\left(\frac{ab}{|b-a|}\right)^\lambda K_\lambda(|b-a|)$$
$$\left[a>0, \quad 0<\operatorname{Re}\lambda<\frac{1}{2}, \quad b>0\right].$$

ET II 176(13)

6. $$\int_0^\infty x\,\operatorname{sh}(\pi x)\Gamma\left(\lambda+\frac{1}{2}ix\right)\Gamma\left(\lambda-\frac{1}{2}ix\right)K_{ix}(a)K_{ix}(b)\,dx = 2\pi^2\left(\frac{ab}{2\sqrt{a^2+b^2}}\right)K_{2\lambda}\left(\sqrt{a^2+b^2}\right)$$
$$\left[|\arg a|<\frac{\pi}{2}, \quad \operatorname{Re}\lambda>0, \quad b>0\right].$$

7. $$\int_0^\infty \frac{x\,\theta(\pi x) K_{ix}(a) K_{ix}(b)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}ix\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}ix\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi ab}{a^2+b^2}}\exp\left(-\sqrt{a^2+b^2}\right) \qquad \left[|\arg a| < \frac{\pi}{2}, \quad b>0\right],$$

(see also **7.335**).

7.335
ET II 177(15)

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.811

1. $$\int_0^\infty \mathbf{H}_\nu(bx)\,dx = -\frac{\operatorname{ctg}\left(\frac{\nu\pi}{2}\right)}{b} \qquad [-2 < \operatorname{Re}\nu < 0, \quad b>0].$$

ET II 158(1)

2. $$\int_0^\infty \mathbf{H}_\nu\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{J_{2\nu}\left(2a\sqrt{b}\right)}{b} \qquad \left[a>0, \quad b>0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 170(37)

3. $$\int_0^\infty \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\frac{dx}{x} = -\frac{1}{a\sqrt{b}}J_{2\nu-1}\left(2a\sqrt{b}\right)\left[a>0, \quad b>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 170(38)

6.812

1. $$\int_0^\infty \frac{\mathbf{H}_1(bx)\,dx}{x^2+a^2} = \frac{\pi}{2a}\left[I_1(ab) - \mathbf{L}_1(ab)\right] \qquad [\operatorname{Re}a>0, \quad b>0].$$

ET II 158(6)

2. $$\int_0^\infty \frac{\mathbf{H}_\nu(bx)}{x^2+a^2}\,dx = -\frac{\pi}{2a\sin\left(\frac{\nu\pi}{2}\right)}L_\nu(ab) + \frac{b\operatorname{ctg}\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\,{}_1F_2\left(1;\ \frac{3-\nu}{2};\ \frac{3+\nu}{2};\ \frac{a^2b^2}{2}\right)$$

$$[\operatorname{Re}a>0, \quad b>0, \quad |\operatorname{Re}\nu| < 2].$$

$$1.\quad \int_0^\infty x^{s-1}\mathbf{H}_\nu(ax)\,dx = \frac{2^{s-1}\Gamma\left(\frac{s+\nu}{2}\right)}{a^s\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}\ \operatorname{tg}\left(\frac{s+\nu}{2}\pi\right)$$
$$\left[a>0,\quad -1-\operatorname{Re}\nu<\operatorname{Re}s<\min\left(\frac{3}{2},\,1-\operatorname{Re}\,\nu\right)\right].$$

WA 429(2), ET I 335(52)

$$2.\quad \int_0^\infty x^{-\nu-1}\mathbf{H}_\nu(x)\,dx = \frac{2^{-\nu-1}\pi}{\Gamma(\nu+1)}\qquad \left[\operatorname{Re}\nu>-\frac{3}{2}\right].$$

ET II 383(2)

$$3.\quad \int_0^\infty x^{-\mu-\nu}\mathbf{H}_\mu(x)\mathbf{H}_\nu(x)\,dx = \frac{2^{-\mu-\nu}\sqrt{\pi}\Gamma(\mu+\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\qquad [\operatorname{Re}(\mu+\nu)>0].$$

WA 435(2), ET II 384(8)

$$4.\quad \int_0^1 x^{\nu+1}\mathbf{H}_\nu(ax)\,dx = \frac{1}{a}\mathbf{H}_{\nu+1}(a)\qquad \left[a>0,\quad \operatorname{Re}\nu>-\frac{3}{2}\right].$$

ET II 158(2)a

$$5.\quad \int_0^1 x^{1-\nu}\mathbf{H}_\nu(ax)\,dx = \frac{a^{\nu-1}}{2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}-\frac{1}{a}\mathbf{H}_{\nu-1}(a)\qquad [a>0].$$

ET II 158(3)a

6.814

$$1.\quad \int_0^\infty \frac{x^\lambda\mathbf{H}_\nu(bx)}{(x^2+a^2)^{1-\mu}}\,dx = \frac{1}{\sqrt{2b}}\frac{a^{\lambda+2\mu-\frac{3}{2}}}{\Gamma(1-\mu)}G_{24}^{22}\left(\frac{a^2b^2}{4}\middle|\begin{matrix}l,m\\ l,m-\mu,h,k\end{matrix}\right),$$
$$h=\frac{1}{4}+\frac{\nu}{2},\quad k=\frac{1}{4}-\frac{\nu}{2},\quad l=\frac{3}{4}+\frac{\nu}{2},\quad m=\frac{3}{4}-\frac{\lambda}{2}$$
$$\left[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}(\lambda+\nu)>-2,\quad \operatorname{Re}(\lambda+2\mu)<\frac{5}{2},\quad \operatorname{Re}(\lambda+2\mu+\nu)<2\right].$$

ET II 159(10)

$$2.\quad \int_0^\infty \frac{x^{\nu+1}\mathbf{H}_\nu(bx)}{(x^2+a^2)^{1-\mu}}\,dx = \frac{2^{\mu-1}\pi a^{\mu+\nu}b^{-\mu}}{\Gamma(1-\mu)\cos[(\mu+\nu)\pi]}\left[I_{-\mu-\nu}(ab)-\mathbf{L}_{\mu+\nu}(ab)\right]$$
$$\left[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}\nu>-\frac{3}{2},\quad \operatorname{Re}(\mu+\nu)<\frac{1}{2},\quad \operatorname{Re}(2\mu+\nu)<\frac{3}{2}\right].$$

6.815

1. $$\int_0^1 x^{\frac{1}{2}\nu}(1-x)^{\mu-1}\mathbf{H}_\nu(a\sqrt{x})\,dx = 2^\mu a^{-\mu}\Gamma(\mu)\mathbf{H}_{\mu+\nu}(a) \qquad \left[\operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET II 199(88)a

2. $$\int_0^1 x^{\lambda-\frac{1}{2}\nu-\frac{3}{2}}(1-x)^{\mu-1}\mathbf{H}_\nu(a\sqrt{x})\,dx = \frac{\mathrm{B}(\lambda,\,\mu)a^{\nu+1}}{2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_2F_3\left(1,\ \lambda;\ \frac{3}{2},\ \nu+\frac{3}{2},\ \lambda+\mu;\ -\frac{a^2}{4}\right)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

ET II 199(89)a

6.82 Combinations of Struve functions, exponentials, and powers

6.821

1.[6] $$\int_0^\infty e^{-\alpha x}\mathbf{H}_{-n-\frac{1}{2}}(\beta x)\,dx = (-1)^n\beta^{n+\frac{1}{2}}(\alpha+\sqrt{\alpha^2+\beta^2})^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

ET I 206(6)

2.[6] $$\int_0^\infty e^{-\alpha x}\mathbf{L}_{-n-\frac{1}{2}}(\beta x)\,dx = \beta^{n+\frac{1}{2}}(\alpha+\sqrt{\alpha^2-\beta^2})^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2-\beta^2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Re}\beta|].$$

ET I 208(26)

3. $$\int_0^\infty e^{-\alpha x}\mathbf{H}_0(\beta x)\,dx = \frac{2}{\pi}\frac{\ln\left(\dfrac{\sqrt{\alpha^2+\beta^2}+\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Im}\beta|].$$

ET II 205(1)

4. $$\int_0^\infty e^{-\alpha x}\mathbf{L}_0(\beta x)\,dx = \frac{2}{\pi}\frac{\arcsin\left(\dfrac{\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\alpha > |\operatorname{Re}\beta|].$$

ET II 207(18)

6.822

$$\int_0^\infty e^{(\nu+1)x}\mathbf{H}_\nu(a\ \mathrm{sh}\, x)\,dx = \sqrt{\frac{\pi}{a}}\operatorname{cosec}(\nu\pi)\left[\mathrm{sh}\left(\frac{a}{2}\right)I_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) - \mathrm{ch}\left(\frac{a}{2}\right)I_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$

$$[\operatorname{Re} a > 0, \quad -2 < \operatorname{Re}\nu < 0]\ .$$

ET II 385(11)

6.823

1. $$\int_0^\infty x^\lambda e^{-\alpha x}\mathbf{H}_\nu(bx)\,dx = \frac{b^{\nu+1}\Gamma(\lambda+\nu+2)}{2^\nu a^{\lambda+\nu+2}\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_3F_2\left(1,\ \frac{\lambda+\nu}{2}+1,\ \frac{\lambda+\nu+3}{2};\ \frac{3}{2},\ \nu+\frac{3}{2};\ -\frac{b^2}{a^2}\right)$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda+\nu) > -2]\ .$$

ET II 161(19)

2. $$\int_0^\infty x^\nu e^{-\alpha x}\mathbf{L}_\nu(\beta x)\,dx = \frac{(2\beta)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\sqrt{\alpha^2-\beta^2}\right)^{2\nu+1}} - \frac{\Gamma(2\nu+1)\left(\frac{\beta}{\alpha}\right)^\nu}{\sqrt{\frac{\pi}{2}}\alpha(\beta^2-\alpha^2)^{\frac{1}{2}\nu+\frac{1}{4}}}P\nu{-}\frac{1}{2}\nu{-}\frac{1}{2}\left(\frac{\beta}{\alpha}\right)$$

$$\left[\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]\ .$$

ET I 209(35)a

791

6.824

1. $$\int_0^\infty t^\nu e^{-at}\mathbf{L}_{2\nu}\left(2\sqrt{t}\right)\,dt = \frac{1}{a^{2\nu+1}}e^{\frac{1}{a}}\,\Phi\left(\frac{1}{\sqrt{a}}\right).$$

MI 51

2. $$\int_0^\infty t^\nu e^{-at}\mathbf{L}_{-2\nu}\left(\sqrt{t}\right)\,dt = \frac{1}{\Gamma\left(\frac{1}{2}-2\nu\right)a^{2\nu+1}}e^{\frac{1}{a}}\gamma\left(\frac{1}{2}-2\nu,\ \frac{1}{a}\right).$$

MI 51

6.825

$$\int_0^\infty x^{s-1}e^{-\alpha^2x^2}\mathbf{H}_\nu(\beta x)\,dx = \frac{\beta^{\nu+1}\Gamma\left(\frac{1}{2}+\frac{s}{2}+\frac{\nu}{2}\right)}{2^{\nu+1}\sqrt{\pi}\alpha^{\nu+s+1}\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_2F_2\left(1,\ \frac{\nu+s+1}{2};\ \frac{3}{2},\ \nu+\frac{3}{2};\ -\frac{\beta^2}{4\alpha^2}\right)$$

$$\left[\operatorname{Re} s > -\operatorname{Re}\nu - 1, \quad |\arg\alpha| < \frac{\pi}{4}\right]\ .$$

6.83 Combinations of Struve and trigonometric functions

6.831

$$\int_0^\infty x^{-\nu} \sin(ax)\mathbf{H}_\nu(bx)\,dx = 0 \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\frac{1}{2}\right];$$

$$= \sqrt{\pi}2^{-\nu}b^{-\nu}\frac{\left(b^2-a^2\right)^{\nu-\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 162(21)

6.832

$$\int_0^\infty \sqrt{x}\sin(ax)\mathbf{H}_{\frac{1}{4}}(b^2x^2)\,dx = -2^{-\frac{3}{2}}\sqrt{\pi}\frac{\sqrt{a}}{b^2}N_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right) \qquad [a > 0].$$

ET I 109(14)

6.84- 6.85 Combinations of Struve and Bessel functions

6.841

$$\int_0^\infty \mathbf{H}_{\nu-1}(ax)N_\nu(bx)\,dx = -a^{\nu-1}b^{-\nu} \qquad \left[0 < b < a, \quad |\operatorname{Re}\,\nu| < \frac{1}{2}\right];$$

$$= 0 \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 114(36)

6.842

$$\int_0^\infty \left[\mathbf{H}_0(ax) - N_0(ax)\right] J_0(bx)\,dx = \frac{4}{\pi(a+b)}\boldsymbol{K}\left[\frac{|a-b|}{a+b}\right] \qquad [a > 0, \quad b > 0].$$

ET II 15(22)

6.843

1. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{1}{b}N_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < \frac{5}{4}\right].$$

ET II 164(10)

2. $$\int_0^\infty K_{2\nu}\left(2a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{2^\nu}{\pi b}\Gamma(\nu+1)S_{-\nu-1,\,\nu}\left(\frac{a^2}{b}\right) \qquad [\operatorname{Re}a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1].$$

6.844

$$\int_0^\infty \left[\cos\left(\frac{\mu-\nu}{2}\pi\right) J_\mu\left(a\sqrt{x}\right) - \sin\left(\frac{\mu-\nu}{2}\pi\right) N_\mu\left(a\sqrt{x}\right)\right] K_\mu\left(a\sqrt{x}\right) \mathbf{H}_\nu(bx)\,dx =$$
$$= \frac{1}{a^2} W_{\frac{1}{2}\nu,\,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) W_{-\frac{1}{2}\nu,\,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) \qquad \left[|\arg a| < \frac{\pi}{4}, \quad b>0, \quad \operatorname{Re}\nu > |\operatorname{Re}\mu| - 2\right].$$

ET II 169(35)

6.845

1. $$\int_0^\infty \left[\mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - N_{-\nu}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx = \frac{4}{\pi b}\cos(\nu\pi) K_{2\nu}\left(2\sqrt{ab}\right)$$
$$\left[|\arg a| < \pi, \quad b>0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 73(7)

2. $$\int_0^\infty \left[J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{x}\right)\right] \mathbf{H}_\nu(bx)\,dx = \frac{1}{b}\left[\frac{2}{\pi}K_{2\nu}(2a\sqrt{b}) - N_{2\nu}(2a\sqrt{b})\right]$$
$$\left[a>0, \quad b>0, \quad -\frac{3}{2} < \operatorname{Re}\nu < 0\right].$$

ET II 170(39)

6.846

$$\int_0^\infty \left[\frac{2}{\pi}K_{2\nu}\left(2a\sqrt{x}\right) + N_{2\nu}\left(2a\sqrt{x}\right)\right] \mathbf{H}_\nu(bx)\,dx = \frac{1}{b} J_\nu\left(\frac{a^2}{b}\right) \qquad \left[a>0, \quad b>0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 169(30)

6.847

$$\int_0^\infty \left[\cos\frac{\nu\pi}{2} J_\nu(ax) + \sin\frac{\nu\pi}{2}\mathbf{H}_\nu(ax)\right] \frac{dx}{x^2+k^2} = \frac{\pi}{2k}\left[I_\nu(ak) - \mathbf{L}_\nu(ak)\right]$$
$$\left[a>0, \quad \operatorname{Re}k>0, \quad -\frac{1}{2} < \operatorname{Re}\nu < 2\right].$$

ET II 384(5)A, WA 467(8)

6.848

1. $$\int_0^\infty x\left[I_\nu(ax) - \mathbf{L}_{-\nu}(ax)\right] J_\nu(bx)\,dx = \frac{2}{\pi}\left(\frac{a}{b}\right)^{\nu-1}\cos(\nu\pi)\frac{1}{a^2+b^2}$$
$$\left[\operatorname{Re}a>0, \quad b>0, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right].$$

2. $$\int_0^\infty x\left[\mathbf{H}_{-\nu}(ax) - N_{-\nu}(ax)\right] J_\nu(bx)\,dx = 2\frac{\cos(\nu\pi)}{a^\nu\pi} b^{\nu-1}\frac{1}{a+b}$$
$$\left[|\arg a| < \pi, \quad -\frac{1}{2} < \operatorname{Re}\nu, \quad b > 0\right].$$

ET II 73(5)

6.849

1. $$\int_0^\infty xK_\nu(ax)\mathbf{H}_\nu(bx)\,dx = a^{-\nu-1}b^{\nu+1}\frac{1}{a^2+b^2} \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 164(12)

2. $$\int_0^\infty x\left[K_\mu(ax)\right]^2 \mathbf{H}_0(bx)\,dx = -2^{-\mu-1}\pi a^{-2\mu}\frac{\left[(z+b)^{2\mu} + (z-b)^{2\mu}\right]}{bz}\sec(\mu\pi), \qquad z = \sqrt{4a^2+b^2}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < \frac{3}{2}\right].$$

ET II 166(18)

6.851

1. $$\int_0^\infty x\left\{\left[J_{\frac{1}{2}\nu}(ax)\right]^2 - \left[N_{\frac{1}{2}\nu}(ax)\right]^2\right\}\mathbf{H}_\nu(bx)dx = 0 \quad \left[0 < b < 2a, \ -\frac{3}{2} < \operatorname{Re}\nu < 0\right];$$
$$= \frac{4}{\pi b}\frac{1}{\sqrt{b^2-4a^2}} \quad \left[0 < 2a < b, \ -\frac{3}{2} < \operatorname{Re}\nu < 0\right].$$

ET II 164(7)

2. $$\int_0^\infty x^{\nu+1}\left\{\left[J_\nu(ax)\right]^2 - \left[N_\nu(ax)\right]^2\right\}\mathbf{H}_\nu(bx)\,dx =$$
$$= 0 \qquad \left[0 < b < 2a, \quad -\frac{3}{4} < \operatorname{Re}\nu < 0\right];$$
$$= \frac{2^{3\nu+2}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}(b^2-4a^2)^{-\nu-\frac{1}{2}} \qquad \left[0 < 2a < b, \quad -\frac{3}{4} < \operatorname{Re}\nu < 0\right].$$

ET II 163(6)

6.852

1. $$\int_0^\infty x^{1-\mu-\nu}J_\nu(x)\mathbf{H}_\mu(x)\,dx = \frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1)\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \qquad \left[\operatorname{Re}\nu > \frac{1}{2}, \ \operatorname{Re}(\mu+\nu) > 1\right].$$

2. $\int_0^\infty x^{\mu-\nu+1} N_\mu(ax)\mathbf{H}_\nu(bx)\,dx =$

$= 0 \qquad \left[0 < b < a, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re}\mu < \frac{1}{2}\right]$;

$= \frac{2^{1+\mu-\nu} a^\mu b^{-\nu}}{\Gamma(\nu-\mu)} (b^2-a^2)^{\nu-\mu-1} \qquad \left[0 < a < b, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re}\mu < \frac{1}{2}\right]$.

ET II 163(3)

3. $\int_0^\infty x^{\mu+\nu+1} K_\mu(ax)\mathbf{H}_\nu(bx)\,dx = \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}} \Gamma\left(\mu+\nu+\frac{3}{2}\right) F\left(1,\, \mu+\nu+\frac{3}{2};\, \frac{3}{2};\, -\frac{b^2}{a^2}\right)$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) > -\frac{3}{2}\right]$.

ET II 165(13)

794

6.853

1. $\int_0^\infty x^{1-\mu} \left[\sin(\mu\pi) J_{\mu+\nu}(ax) + \cos(\mu\pi) N_{\mu+\nu}(ax)\right] \mathbf{H}_\nu(bx)\,dx =$

$= 0 \qquad \left[0 < b < a, \quad 1 < \operatorname{Re}\mu < \frac{3}{2}, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \frac{1}{2}\right]$;

$= \frac{b^\nu (b^2-a^2)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)} \qquad \left[0 < a < b, \quad 1 < \operatorname{Re}\mu < \frac{3}{2}, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \frac{1}{2}\right]$.

ET II 163(4)

2. $\int_0^\infty x^{\lambda+\frac{1}{2}} \left[I_\mu(ax) - \mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\,dx =$

$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}} G_{33}^{22}\left(\frac{b^2}{a^2} \middle| \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu+\lambda) > -\frac{3}{2}, \quad -\operatorname{Re}\nu - \frac{5}{2} < \operatorname{Re}(\lambda-\mu) < 1\right]$.

ET II 76(21)

3. $\int_0^\infty x^{\lambda+\frac{1}{2}} \left[\mathbf{H}_\mu(ax) - N_\mu(ax)\right] J_\nu(bx)\,dx =$

$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}} G_{33}^{23}\left(\frac{b^2}{a^2} \middle| \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$

$\left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda+\mu) < 1, \quad \operatorname{Re}(\lambda+\nu) + \frac{3}{2} > |\operatorname{Re}\mu|\right]$.

4. $$\int_0^\infty \sqrt{x}\left[I_{\nu-\frac{1}{2}}(ax)-\mathbf{L}_{\nu-\frac{1}{2}}(ax)\right] J_\nu(bx)\,dx = \sqrt{\frac{2}{\pi}}a^{\nu-\frac{1}{2}}b^{-\nu}\frac{1}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET II 74(11)

5. $$\int_0^\infty x^{\mu-\nu+1}\left[I_\mu(ax)-\mathbf{L}_\mu(ax)\right] J_\nu(bx)\,dx = \frac{2^{\mu-\nu+1}a^{\mu-1}b^{\nu-2\mu-1}}{\sqrt{\pi}\Gamma\left(\nu-\mu+\frac{1}{2}\right)}F\left(1,\ \frac{1}{2};\ \nu-\mu+\frac{1}{2};\ -\frac{b^2}{a^2}\right)$$
$$\left[-1<2\operatorname{Re}\mu+1<\operatorname{Re}\nu+\frac{1}{2}, \quad \operatorname{Re} a>0, \quad b>0\right].$$

ET II 74(13)

6. $$\int_0^\infty x^{\mu-\nu+1}\left[I_\mu(ax)-\mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\,dx = \frac{2^{\mu-\nu+1}a^{-\mu-1}b^{\nu-1}}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\nu\right)}F\left(1,\ \frac{1}{2}+\mu;\ \frac{1}{2}+\nu;\ -\frac{b^2}{a^2}\right)$$
$$\left[\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}, \quad \operatorname{Re}\mu>-1, \quad b>0\right].$$

ET II 75(18)

6.854

1. $$\int_0^\infty x\mathbf{H}_{\frac{1}{2}\nu}(ax^2)K_\nu(bx)\,dx = \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{2^{1-\frac{1}{2}\nu}a\pi}S_{-\frac{1}{2}\nu-1,\,\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad [a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\nu>-2].$$

ET II 150(75)

2. $$\int_0^\infty x\mathbf{H}_{\frac{1}{2}\nu}(ax^2)J_\nu(bx)\,dx = -\frac{1}{2a}N_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad \left[a>0, \quad b>0, \quad -2<\operatorname{Re}\nu<\frac{3}{2}\right].$$

ET II 73(3)

6.855

1. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[I_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)-\mathbf{L}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx = 2^{\frac{3}{2}}\frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi}b^{\nu+1}}J_{2\nu+1}\left(\sqrt{2ab}\right)K_{2\nu+1}(\sqrt{2ab})$$
$$\left[\operatorname{Re} a>0, \quad b>0, \quad -1<\operatorname{Re}\ \nu<\frac{1}{2}\right].$$

2. $$\int_0^\infty \left[\mathbf{H}_{-\nu-1}\left(\frac{a}{x}\right) - N_{-\nu-1}\left(\frac{a}{x}\right)\right] J_\nu(bx)\frac{dx}{x} = -\frac{4}{\pi\sqrt{ab}}\cos(\nu\pi)K_{-2\nu-1}\left(2\sqrt{ab}\right)$$
$$\left[|\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 74(8)

3. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[\mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) - N_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx =$$
$$= -2^{\frac{5}{2}}\pi^{-\frac{3}{2}}a^{\nu+\frac{1}{2}}b^{-\nu-1}\sin(\nu\pi)K_{2\nu+1}\left(\sqrt{2ab}e^{\frac{1}{4}\pi i}\right)K_{2\nu+1}\left(\sqrt{2ab}e^{-\frac{1}{4}\pi i}\right)$$
$$\left[|\arg a| < \pi, \quad b > 0, \quad -1 < \operatorname{Re}\nu < -\frac{1}{6}\right].$$

ET II 74(9)

6.856

$$\int_0^\infty xN_\nu\left(a\sqrt{x}\right)K_\nu\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{1}{2b^2}\exp\left(-\frac{a^2}{2b}\right) \qquad \left[b > 0,\ |\arg a| < \frac{\pi}{4},\ \operatorname{Re}\nu > -\frac{3}{2}\right].$$

ET II 169(32)

796

6.857

1. $$\int_0^\infty x\exp\left(\frac{a^2x^2}{8}\right)K_{\frac{1}{2}\nu}\left(\frac{a^2x^2}{8}\right)\mathbf{H}_\nu(bx)\,dx =$$
$$= \frac{2}{\sqrt{\pi}}a^{-\frac{\nu}{2}-1}b^{\frac{\nu}{2}-1}\cos\left(\frac{\nu\pi}{2}\right)\Gamma\left(-\frac{1}{2}\nu\right)\exp\left(\frac{b^2}{2a^2}\right)W_{k,\,m}\left(\frac{b^2}{a^2}\right),$$
$$k = \frac{1}{4}\nu, \qquad m = \frac{1}{2} + \frac{1}{4}\nu \qquad \left[|\arg a| < \frac{3}{4}\pi, \quad b > 0, \quad -\frac{3}{2} < \operatorname{Re}\nu < 0\right].$$

ET II 167(24)

2. $$\int_0^\infty x^{\sigma-2}\exp\left(-\frac{1}{2}a^2x^2\right)K_\mu\left(\frac{1}{2}a^2x^2\right)\mathbf{H}_\nu(bx)\,dx =$$
$$= \frac{\sqrt{\pi}}{2^{\nu+2}}a^{-\nu-\sigma}b^{\nu+1}\frac{\Gamma\left(\dfrac{\nu+\sigma}{2}+\mu\right)\Gamma\left(\dfrac{\nu+\sigma}{2}-\mu\right)}{\Gamma\left(\dfrac{3}{2}\right)\Gamma\left(\nu+\dfrac{3}{2}\right)\Gamma\left(\dfrac{\nu+\sigma}{2}\right)} \times$$
$$\times\ {}_3F_3\left(1,\ \frac{\nu+\sigma}{2}+\mu,\ \frac{\nu+\sigma}{2}-\mu;\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{\nu+\sigma}{2};\ -\frac{b^2}{4a^2}\right)$$
$$\left[b > 0, \quad |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}(\sigma+\nu) > 2|\operatorname{Re}\mu|\right].$$

ET II 167(23)

6.86 Lommel functions

6.861

$$\int_0^\infty x^{\lambda-1} s_{\mu,\,\nu}(x)\,dx = \frac{\Gamma\left[\frac{1}{2}(1+\lambda+\mu)\right]\Gamma\left[\frac{1}{2}(1-\lambda-\mu)\right]\Gamma\left[\frac{1}{2}(1+\mu+\nu)\right]\Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]}{2^{2-\lambda-\mu}\Gamma\left[\frac{1}{2}(\nu-\lambda)+1\right]\Gamma\left[1-\frac{1}{2}(\lambda+\nu)\right]}$$

$$\left[-\operatorname{Re}\mu < \operatorname{Re}\lambda + 1 < \frac{5}{2}\right].$$

ET II 385(17)

6.862

1. $$\int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}}(u-x)^{\sigma-1} s_{\mu,\,\nu}\left(a\sqrt{x}\right)\,dx =$$
$$= \Gamma(\sigma)\frac{a^{\mu+1}u^{\lambda+\sigma}\Gamma(\lambda+1)}{(\mu-\nu+1)(\mu+\nu+1)\Gamma(\lambda+\sigma+1)} \times$$
$$\times\ {}_2F_3\left(1,\ 1+\lambda;\ \frac{\mu-\nu+3}{2},\ \frac{\mu+\nu+3}{2},\ \lambda+\sigma+1;\ -\frac{a^2u}{4}\right) \qquad [\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\sigma > 0].$$

ET II 199(92)

2. $$\int_u^\infty x^{\frac{1}{2}\nu}(x-u)^{\mu-1} S_{\lambda,\,\nu}(a\sqrt{x})\,dx = \frac{\mathrm{B}\left[\mu,\ \frac{1}{2}(1-\lambda-\nu)-\mu\right]u^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{a^\mu} S_{\lambda+\mu,\,\mu+\nu}\left(a\sqrt{u}\right)$$
$$\left[|\arg(a\sqrt{u})| < \pi, \quad 0 < 2\operatorname{Re}\mu < 1 - \operatorname{Re}(\lambda+\nu)\right].$$

ET II 211(71)

6.863

$$\int_0^\infty \sqrt{x}e^{-\alpha x} s_{\mu,\,\frac{1}{4}}\left(\frac{x^2}{2}\right)dx = 2^{-2\mu-1}\sqrt{\alpha}\Gamma\left(2\mu+\frac{3}{2}\right)S_{-\mu-1,\,\frac{1}{4}}\left(\frac{\alpha^2}{2}\right) \qquad \left[\operatorname{Re}\alpha > 0,\ \operatorname{Re}\mu > -\frac{3}{4}\right].$$

ET I 209(38)

6.864

$$\int_0^\infty \exp\left[(\mu+1)x\right] s_{\mu,\,\nu}(a\ \operatorname{sh} x)\,dx = 2^{\mu-2}\pi\operatorname{cosec}(\mu\pi)\Gamma(\varrho)\Gamma(\sigma) \times$$
$$\times\left[I_\varrho\left(\frac{a}{2}\right)I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right)I_{-\sigma}\left(\frac{a}{2}\right)\right],$$
$$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1 \qquad [a > 0, \quad -2 < \operatorname{Re}\mu < 0].$$

$$\int_0^\infty \sqrt{\operatorname{sh} x}\ \operatorname{ch}(\nu x) S_{\mu,\,\frac{1}{2}}(a\ \operatorname{ch} x)\,dx = \frac{\mathrm{B}\left(\frac{1}{4}-\frac{\mu+\nu}{2},\ \frac{1}{4}-\frac{\mu-\nu}{2}\right)}{\sqrt{a}2^{\mu+\frac{3}{2}}} S_{\mu+\frac{1}{2},\,\nu}(a)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu + |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

ET II 388(31)

6.866

1. $$\int_0^\infty x^{-\mu-1}\cos(ax) s_{\mu,\,\nu}(x)\,dx =$$
$$= 0 \qquad [a > 1]\,;$$
$$= 2^{\mu-\frac{1}{2}}\sqrt{\pi}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)(1-a^2)^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}(a) \qquad [0 < a < 1]\,.$$

ET II 386(18)

2. $$\int_0^\infty x^{-\mu}\sin(ax) S_{\mu,\,\nu}(x)\,dx = 2^{-\mu-\frac{1}{2}}\sqrt{\pi}\Gamma\left(1-\frac{\mu+\nu}{2}\right)\Gamma\left(1-\frac{\mu-\nu}{2}\right)(a^2-1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a)$$
$$[a > 1, \quad \operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|]\,.$$

ET II 387(23)

6.867

1. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x) S_{2\mu-1,\,2\nu}(a\cos x)\,dx =$$
$$= \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1-\mu-\nu)\Gamma(1-\mu+\nu)}\left[J_{\mu+\nu}\left(\frac{a}{2}\right) N_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) N_{\mu+\nu}\left(\frac{a}{2}\right)\right]$$
$$[\operatorname{Re}\mu > -2, \quad |\operatorname{Re}\nu| < 1]\,.$$

ET II 388(29)

2. $$\int_0^{\frac{\pi}{2}} \cos\left[(\mu+1)x\right] s_{\mu,\,\nu}(a\cos x)\,dx = 2^{\mu-2}\pi\Gamma(\varrho)\Gamma(\sigma) J_\varrho\left(\frac{a}{2}\right) J_\sigma\left(\frac{a}{2}\right),$$
$$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1 \qquad [\operatorname{Re}\mu > -2]\,.$$

ET II 386(21)

6.868

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2\mu x)}{\cos x} S_{2\mu,\,2\nu}(a\sec x)\,dx = \frac{\pi 2^{2\mu-1}}{a} W_{\mu,\,\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\mu,\,\nu}\left(ae^{-i\frac{\pi}{2}}\right) \qquad [|\arg a| < \pi, \quad \operatorname{Re}\mu < 1]\,.$$

6.869

1. $$\int_0^\infty x^{1-\mu-\nu} J_\nu(ax) S_{\mu,\,-\mu-2\nu}(x)\,dx = \frac{\sqrt{\pi}a^{\nu-1}\Gamma(1-\mu-\nu)}{2^{\mu+2\nu}\Gamma\left(\nu+\frac{1}{2}\right)}(a^2-1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$$
$$\left[a > 1, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) < 1\right].$$

ET II 388(28)

2. $$\int_0^\infty x^{-\mu} J_\nu(ax) s_{\nu+\mu,\,-\nu+\mu+1}(x)\,dx =$$
$$= 2^{\nu-1}\Gamma(\nu)a^{-\nu}(1-a^2)^\mu \qquad \left[0 < a < 1, \quad \operatorname{Re}\mu > -1, \quad -1 < \operatorname{Re}\nu < \frac{3}{2}\right];$$
$$= 0 \qquad \left[1 < a, \quad \operatorname{Re}\mu > -1, \quad -1 < \operatorname{Re}\nu < \frac{3}{2}\right].$$

ET II 92(24)

3. $$\int_0^\infty x K_\nu(bx) s_{\mu,\,\frac{1}{2}\nu}(ax^2)\,dx = \frac{1}{4a}\Gamma\left(\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\mu-\frac{1}{2}\nu+1\right) S_{-\mu-1,\,\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$$\left[\operatorname{Re}\mu > \frac{1}{2}|\operatorname{Re}\nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0\right].$$

ET II 151(78)

6.87 Thomson functions

6.871

1. $$\int_0^\infty e^{-\beta x} \operatorname{ber} x\,dx = \frac{\left(\sqrt{\beta^4+1}+\beta^2\right)^{\frac{1}{2}}}{\sqrt{2(\beta^4+1)}}.$$

ME 40

2. $$\int_0^\infty e^{-\beta x} \operatorname{bei} x\,dx = \frac{\left(\sqrt{\beta^4+1}-\beta^2\right)^{\frac{1}{2}}}{\sqrt{2(\beta^4+1)}}.$$

ME 40

6.872

1. $$\int_0^\infty e^{-\beta x} \operatorname{ber}_\nu \left(2\sqrt{x}\right) dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu\pi}{4}\right) - - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right].$$

MI 49

2. $$\int_0^\infty e^{-\beta x} \operatorname{bei}_\nu \left(2\sqrt{x}\right) dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu}{4}\pi\right) - - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right].$$

MI 49

3. $$\int_0^\infty e^{-\beta x} \operatorname{ber}\left(2\sqrt{x}\right) dx = \frac{1}{\beta}\cos\frac{1}{\beta}.$$

ME 40

4. $$\int_0^\infty e^{-\beta x} \operatorname{bei}\left(2\sqrt{x}\right) dx = \frac{1}{\beta}\sin\frac{1}{\beta}.$$

ME 40

5. $$\int_0^\infty e^{-\beta x} \operatorname{ker}\left(2\sqrt{x}\right) dx = -\frac{1}{2\beta}\left[\cos\frac{1}{\beta}\operatorname{ci}\frac{1}{\beta}+\sin\frac{1}{\beta}\operatorname{si}\frac{1}{\beta}\right].$$

MI 50

6. $$\int_0^\infty e^{-\beta x} \operatorname{kei}\left(2\sqrt{x}\right) dx = -\frac{1}{2\beta}\left[\sin\frac{1}{\beta}\operatorname{ci}\frac{1}{\beta}-\cos\frac{1}{\beta}\operatorname{si}\frac{1}{\beta}\right].$$

MI 50

7. $$\int_0^\infty e^{-\beta x} \operatorname{ber}_\nu \left(2\sqrt{x}\right) \operatorname{bei}_\nu \left(2\sqrt{x}\right) dx = \frac{1}{2\beta}J_\nu\left(\frac{2}{\beta}\right)\sin\left(\frac{2}{\beta}+\frac{3\nu\pi}{2}\right) \qquad [\operatorname{Re}\nu > -1].$$

MI 49

6.873

ME 40

6.874

1. $$\int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{ber}_{2\nu}\left(2\sqrt{2x}\right) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

MI 49

2. $$\int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{bei}_{2\nu}\left(2\sqrt{2x}\right) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

MI 49

3. $$\int_0^\infty x^{\frac{\nu}{2}} \operatorname{ber}_\nu\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \cos\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \qquad [\operatorname{Re}\nu > -1].$$

ME 40

4. $$\int_0^\infty x^{\frac{\nu}{2}} \operatorname{bei}_\nu\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \sin\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \qquad [\operatorname{Re}\nu > -1].$$

ME 40

6.875

1. $$\int_0^\infty e^{-\beta x}\left[\operatorname{ker}\left(2\sqrt{x}\right) - \frac{1}{2}\ln x \operatorname{ber}\left(2\sqrt{x}\right)\right] dx = \frac{1}{\beta}\left[\ln\beta\cos\frac{1}{\beta} + \frac{\pi}{4}\sin\frac{1}{\beta}\right].$$

MI 50

2. $$\int_0^\infty e^{-\beta x}\left[\operatorname{kei}\left(2\sqrt{x}\right) - \frac{1}{2}\ln x \operatorname{bei}\left(2\sqrt{x}\right)\right] dx = \frac{1}{\beta}\left[\ln\beta\sin\frac{1}{\beta} - \frac{\pi}{4}\cos\frac{1}{\beta}\right].$$

MI 50

6.876

1. $$\int_0^\infty x \operatorname{kei} x J_1(ax)\, dx = -\frac{1}{2a} \operatorname{arctg} a^2 \qquad [a > 0].$$

ET II 21(32)

ET II 21(32)

2. $\int_0^\infty x \ker x J_1(ax)\,dx = \frac{1}{2a}\ln(1+a^4)^{\frac{1}{2}} \qquad [a>0]\,.$

ET II 21(33)

6.9 Mathieu Functions

Notation: $k^2 = q$. For definition of the coefficients $A_p^{(m)}$ and $B_p^{(m)}$ see 8.6

6.91 Mathieu functions

6.911

1. $\int_0^{2\pi} \mathrm{ce}_m\,(z,\,q)\,\mathrm{ce}_p\,(z,\,q)\,dz = 0 \qquad [m \neq p]\,.$

MA

2. $\int_0^{2\pi} \left[\mathrm{ce}_{2n}\,(z,\,q)\right]^2\,dz = 2\pi\left[A_0^{(2n)}\right]^2 + \pi\sum_{r=1}^{\infty}\left[A_{2r}^{(2n)}\right]^2 = \pi.$

MA

3. $\int_0^{2\pi} \left[\mathrm{ce}_{2n+1}\,(z,\,q)\right]^2\,dz = \pi\sum_{r=0}^{\infty}\left[A_{2r+1}^{(2n+1)}\right]^2 = \pi.$

MA

4. $\int_0^{2\pi} \mathrm{se}_m\,(z,\,q)\,\mathrm{se}_p\,(z,\,q)\,dz = 0 \qquad [m \neq p]\,.$

MA

5. $\int_0^{2\pi} \left[\mathrm{se}_{2n+1}\,(z,\,q)\right]^2\,dz = \pi\sum_{r=0}^{\infty}\left[B_{2r+1}^{(2n+1)}\right]^2 = \pi.$

MA

MA

7. $\int_0^{2\pi} \mathrm{se}_m(z, q)\, \mathrm{ce}_p(z, q)\, dz = 0 \qquad [m = 1, 2, \dots; \quad p = 1, 2, \dots].$

MA

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.921

1. $\int_0^{\pi} \mathrm{ch}(2k \cos u\ \mathrm{sh}\, z)\, \mathrm{ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}\left(\frac{\pi}{2}, q\right)} (-1)^n\ \mathrm{Ce}_{2n}(z, -q) \qquad [q > 0].$

MA

2. $\int_0^{\pi} \mathrm{ch}(2k \sin u\ \mathrm{ch}\, z)\, \mathrm{ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}(0, q)} (-1)^n\ \mathrm{Ce}_{2n}(z, -q) \qquad [q > 0].$

MA

3. $\int_0^{\pi} \mathrm{sh}(2k \sin u\ \mathrm{ch}\, z)\ \mathrm{se}_{2n+1}(u, q)\, du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0, q)} (-1)^n\, \mathrm{Ce}_{2n+1}(z, -q) \qquad [q > 0].$

MA

4. $\int_0^{\pi} \mathrm{sh}(2k \cos u\ \mathrm{sh}\, z)\ \mathrm{ce}_{2n+1}(u, q)\, du = \frac{\pi k A_1^{(2n+1)}}{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} (-1)^{n+1}\ \mathrm{Se}_{2n+1}(z, -q) \qquad [q > 0].$

MA

5. $\int_0^{\pi} \mathrm{sh}(2k \sin u \sin z)\ \mathrm{se}_{2n+1}(u, q)\, du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0, q)}\ \mathrm{se}_{2n+1}(z, q) \qquad [q > 0].$

MA

6.922

MA

2. $$\int_0^{\pi} \sin u \operatorname{sh} z \cos(2k \cos u \operatorname{ch} z) \operatorname{se}_{2n+1}(u, q)\, du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+1}(z, q) \qquad [q > 0].$$

MA

3. $$\int_0^{\pi} \sin u \operatorname{sh} z \sin(2k \cos u \operatorname{ch} z) \operatorname{se}_{2n+2}(u, q)\, du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+2}(z, q) \qquad [q > 0].$$

MA

4. $$\int_0^{\pi} \cos u \operatorname{ch} z \sin(2k \sin u \operatorname{sh} z) \operatorname{se}_{2n+2}(u, q)\, du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} \operatorname{Se}_{2n+2}(z, q) \qquad [q > 0].$$

MA

5. $$\int_0^{\pi} \sin u \operatorname{ch} z \operatorname{ch}(2k \cos u \operatorname{sh} z) \operatorname{se}_{2n+1}(u, q)\, du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q) \qquad [q > 0].$$

MA

6. $$\int_0^{\pi} \cos u \operatorname{sh} z \operatorname{ch}(2k \sin u \operatorname{ch} z) \operatorname{ce}_{2n+1}(u, q)\, du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} (-1)^n \operatorname{Se}_{2n+1}(z, -q) \qquad [q > 0].$$

MA

7. $$\int_0^{\pi} \sin u \operatorname{ch} z \operatorname{sh}(2k \cos u \operatorname{sh} z) \operatorname{se}_{2n+2}(u, q)\, du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} (-1)^{n+1} \operatorname{Se}_{2n+2}(z, -q) \qquad [q > 0].$$

MA

8. $$\int_0^{\pi} \cos u \operatorname{sh} z \operatorname{sh}(2k \sin u \operatorname{ch} z) \operatorname{se}_{2n+2}(u, q)\, du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} (-1)^n \operatorname{Se}_{2n+2}(z, -q) \qquad [q > 0].$$

6.923

1. $$\int_0^\infty \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+1}(u, q)\, du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+1}(z, q) \qquad [q > 0] .$$

MA

2. $$\int_0^\infty \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+1}(u, q)\, du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Gey}_{2n+1}(z, q) \qquad [q > 0] .$$

MA

3. $$\int_0^\infty \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+2}(u, q)\, du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Gey}_{2n+2}(z, q) \qquad [q > 0] .$$

MA

4. $$\int_0^\infty \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+2}(u, q)\, du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+2}(z, q) \qquad [q > 0] .$$

MA

5. $$\int_0^\infty \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \operatorname{Ce}_{2n}(z, q) \qquad [q > 0] .$$

MA

6. $$\int_0^\infty \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n}(u, q)\, du = -\frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{Fey}_{2n}(z, q) \qquad [q > 0] .$$

MA

7. $$\int_0^\infty \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n+1}(u, q)\, du = \frac{k\pi A_1^{(2n+1)}}{2 \operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Fey}_{2n+1}(z, q) \qquad [q > 0] .$$

8. $$\int_0^\infty \cos(2k \text{ ch } z \text{ ch } u) \text{ Ce}_{2n+1}(u, q)\, du = \frac{k\pi A_1^{(2n+1)}}{2 \text{ ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \text{ Ce}_{2n+1}(z, q) \qquad [q > 0].$$

MA

6.924

1. $$\int_0^\pi \cos(2k \cos u \cos z) \text{ ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{ ce}_{2n}(z, q) \qquad [q > 0].$$

MA

2. $$\int_0^\pi \sin(2k \cos u \cos z) \text{ ce}_{2n+1}(u, q)\, du = -\frac{\pi k A_1^{(2n+1)}}{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \text{ ce}_{2n+1}(z, q) \qquad [q > 0].$$

MA

3. $$\int_0^\pi \cos(2k \cos u \text{ ch } z) \text{ ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{ Ce}_{2n}(z, q) \qquad [q > 0].$$

MA

4. $$\int_0^\pi \cos(2k \sin u \text{ sh } z) \text{ ce}_{2n}(u, q)\, du = \frac{\pi A_0^{(2n)}}{\text{ce}_{2n}(0, q)} \text{ Ce}_{2n}(z, q) \qquad [q > 0].$$

MA

5. $$\int_0^\pi \sin(2k \cos u \text{ ch } z) \text{ ce}_{2n+1}(u, q)\, du = -\frac{\pi k A_1^{(2n+1)}}{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \text{ Ce}_{2n+1}(z, q) \qquad [q > 0].$$

MA

6. $$\int_0^\pi \sin(2k \sin u \text{ sh } z) \text{ se}_{2n+1}(u, q)\, du = \frac{\pi k B_1^{(2n+1)}}{\text{se}'_{2n+1}(0, q)} \text{ Se}_{2n+1}(z, q) \qquad [q > 0].$$

MA

6.925

Notation: $z_1 = 2k\sqrt{\mathrm{ch}^2\ \xi - \sin^2\ \eta}, \mathrm{tg}\,\alpha = \theta\xi\ \mathrm{tg}\,\eta$

1. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \mathrm{ce}_{2n}\left(\theta,\ q\right) d\theta = 0.$$

MA

2. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right]\ \mathrm{ce}_{2n}\left(\theta,\ q\right) d\theta = \frac{2\pi A_0^{(2n)}}{\mathrm{ce}_{2n}\left(0,\ q\right)\ \mathrm{ce}_{2n}\ \left(\frac{\pi}{2},\ q\right)}\ \mathrm{Ce}_{2n}(\xi,\ q)\ \mathrm{ce}_{2n}\left(\eta,\ q\right).$$

MA

3. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \mathrm{ce}_{2n+1}(\theta, q)\, d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\mathrm{ce}_{2n+1}(0, q)\,\mathrm{ce}_{2n+1}'\left(\frac{\pi}{2}, q\right)}\ \mathrm{Ce}_{2n+1}(\xi, q)\,\mathrm{ce}_{2n+1}(\eta, q).$$

MA

4. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right]\ \mathrm{ce}_{2n+1}\left(\theta,\ q\right) d\theta = 0.$$

MA

5. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right]\ \mathrm{se}_{2n+1}\left(\theta,\ q\right) d\theta = \frac{2\pi k B_1^{(2n+1)}}{\mathrm{se}_{2n+1}\left(0,\ q\right)\ \mathrm{se}_{2n+1}\ \left(\frac{\pi}{2},\ q\right)}\ \mathrm{Se}_{2n+1}\left(\xi,\ q\right) \mathrm{se}_{2n+1}\left(\eta,\ q\right).$$

MA

6. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right]\ \mathrm{se}_{2n+1}\left(\theta,\ q\right) d\theta = 0.$$

MA

7. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right]\ \mathrm{se}_{2n+2}\left(\theta,\ q\right) d\theta = 0.$$

MA

$$\int_0^{\pi} \sin u \, \sin z \, \sin(2k \, \cos u \, \cos z) \, \mathrm{se}_{2n+2}\,(u,\, q)\, du = -\frac{\pi k B_2^{(2n+2)}}{2\, \mathrm{se}'_{2n+2}\, \left(\frac{\pi}{2},\, q\right)}\, \mathrm{se}_{2n+2}\,(z,\, q) \qquad [q > 0].$$

MA

6.93 Combinations of Mathieu and Bessel functions

6.931

1. $$\int_0^{\pi} J_0\{k[2(\cos 2u + \cos 2z)]^{\frac{1}{2}}\}\, \mathrm{ce}_{2n}\,(u,\, q)\, du = \frac{\pi[A_0^{(2n)}]^2}{\mathrm{ce}_{2n}\,(0,\, q)\, \mathrm{ce}_{2n}\, \left(\frac{\pi}{2},\, q\right)}\, \mathrm{ce}_{2n}\,(z,\, q).$$

MA

2. $$\int_0^{2\pi} N_0\{k[2(\cos 2u + \mathrm{ch}\, 2z)]^{\frac{1}{2}}\}\, \mathrm{ce}_{2n}\,(u,\, q)\, du = \frac{2\pi[A_0^{(2n)}]^2}{\mathrm{ce}_{2n}\,(0,\, q)\, \mathrm{ce}_{2n}\, \left(\frac{\pi}{2},\, q\right)}\, \mathrm{Fey}_{2n}\,(z,\, q).$$

MA

7.1- 7.2 Associated Legendre Functions

7.11 Associated Legendre functions

7.111

$$\int_{\cos\varphi}^{1} \mathrm{P}_{\nu}(x)\, dx = \sin\varphi\, \mathrm{P}_{\nu}^{-1}(\cos\varphi).$$

MO 90

7.112

1. $$\int_{-1}^{1} \mathrm{P}_n^m(x) \mathrm{P}_k^m(x)\, dx = 0 \qquad [n \neq k];$$
$$= \frac{2}{2n+1}\, \frac{(n+m)!}{(n-m)!} \qquad [n = k].$$

2. $$\int_{-1}^{1} \mathrm{Q}_n^m(x)\mathrm{P}_k^m(x)\,dx = (-1)^m \frac{1-(-1)^{n+k}(n+m)!}{(k-n)(k+n+1)(n-m)!}.$$

EH I 171(18)

3. $$\int_{-1}^{1} \mathrm{P}_\nu(x)\mathrm{P}_\sigma(x)\,dx = \frac{2\pi\,\sin\pi(\sigma-\nu)+4\,\sin(\pi\nu)\,\sin(\pi\sigma)[\psi(\nu+1)-\psi(\sigma+1)]}{\pi^2(\sigma-\nu)(\sigma+\nu+1)} \qquad [\sigma+\nu+1\neq 0];$$

$$= \frac{\pi^2 - 2(\sin\pi\nu)^2\psi'(\nu+1)}{\pi^2\left(\nu+\frac{1}{2}\right)} \qquad [\sigma=\nu].$$

EH I 170(9)a
EH I 170(7)

805

4. $$\int_{-1}^{1} \mathrm{Q}_\nu(x)\mathrm{Q}_\sigma(x)\,dx =$$

$$= \frac{[\psi(\nu+1)-\psi(\sigma+1)][1+\cos(\pi\sigma)\,\cos(\nu\pi)] - \frac{\pi}{2}\,\sin\pi(\nu-\sigma)}{(\sigma-\nu)(\sigma+\nu+1)} \qquad [\sigma+\nu+1\neq 0; \quad \nu, \quad \sigma\neq -1,\,-2,\,-3,\dots];$$

$$= \frac{\frac{1}{2}\pi^2 - \psi'(\nu+1)[1+(\cos\nu\pi)^2]}{2\nu+1} \qquad [\nu=\sigma, \quad \nu\neq -1,\,-2,\,-3,\dots].$$

EH I 170(12)
EH I 170(11)

5. $$\int_{-1}^{1} \mathrm{P}_\nu(x)\mathrm{Q}_\sigma(x)\,dx =$$

$$= \frac{1-\cos\pi(\sigma-\nu) - 2\pi^{-1}\,\sin(\pi\nu)\,\cos(\pi\sigma)[\psi(\nu+1)-\psi(\sigma+1)]}{(\nu-\sigma)(\nu+\sigma+1)} \qquad [\operatorname{Re}\nu>0, \quad \operatorname{Re}\sigma>0, \quad \sigma\neq\nu];$$

$$= -\frac{\sin(2\nu\pi)\psi'(\nu+1)}{\pi(2\nu+1)} \qquad [\operatorname{Re}\nu>0, \quad \sigma=\nu].$$

EH I 171(14)
EH I 170(13)

7.113

Notation: $A = \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}$

1. $$\int_{0}^{1} \mathrm{P}_\nu(x)\mathrm{P}_\sigma(x)\,dx = \frac{A\,\sin\frac{\pi\sigma}{2}\,\cos\frac{\pi\nu}{2} - A^{-1}\,\sin\frac{\pi\nu}{2}\,\cos\frac{\pi\sigma}{2}}{\frac{1}{2}\pi(\sigma-\nu)(\sigma+\nu+1)}.$$

EH I 171(15)

2. $$\int_0^1 Q_\nu(x) Q_\sigma(x)\,dx = \\ = \frac{\psi(\nu+1) - \psi(\sigma+1) - \frac{\pi}{2}\left[(A - A^{-1})\, \sin\frac{\pi(\sigma+\nu)}{2}(A + A^{-1})\, \sin\frac{\pi(\sigma-\nu)}{2}\right]}{(\sigma-\nu)(\sigma+\nu+1)} \\ [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0].$$

EH I 171(16)

3. $$\int_0^1 P_\nu(x) Q_\sigma(x)\,dx = \frac{A^{-1}\,\cos\frac{\pi(\nu-\sigma)}{2} - 1}{(\sigma-\nu)(\sigma+\nu+1)} \qquad [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0].$$

EH I 171(17)

7.114

1. $$\int_1^\infty P_\nu(x) Q_\sigma(x)\,dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)} \qquad [\operatorname{Re}(\sigma-\nu) > 0, \quad \operatorname{Re}(\sigma+\nu) > -1].$$

ET II 324(19)

806

2. $$\int_1^\infty Q_\nu(x) Q_\sigma(x)\,dx = \frac{\psi(\sigma+1) - \psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)} \qquad [\operatorname{Re}(\nu+\sigma) > -1; \quad \sigma, \nu \neq -1, -2, -3, \dots].$$

EH I 170(5)

3. $$\int_1^\infty [Q_\nu(x)]^2\,dx = \frac{\psi'(\nu+1)}{2\nu+1} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

EH I 170(6)

7.115

$$\int_1^\infty Q_\nu(x)\,dx = \frac{1}{\nu(\nu+1)} \qquad [\operatorname{Re}\nu > 0].$$

ET II 324(18)

7.12- 7.13 Combinations of associated Legendre functions and powers

7.121

$$\int_{\cos\varphi}^{1} x\mathrm{P}_\nu(x)\,dx = \frac{-\sin\varphi}{(\nu-1)(\nu+2)}[\sin\varphi\mathrm{P}_\nu(\cos\varphi)+\cos\varphi\mathrm{P}_\nu^1(\cos\varphi)].$$

MO 90

7.122

1. $$\int_0^1 \frac{[\mathrm{P}_n^m(x)]^2}{1-x^2}\,dx = \frac{1}{2m}\,\frac{(n+m)!}{(n-m)!} \qquad [0 < m \le n].$$

MO 74

2. $$\int_0^1 [\mathrm{P}_\nu^\mu(x)]^2 \frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu\Gamma(1-\mu+\nu)} \qquad [\operatorname{Re}\mu < 0, \quad \nu+\mu\text{—a positive integer}].$$

EH I 172(26)

3. $$\int_0^1 [\mathrm{P}_\nu^{n-\nu}(x)]^2 \frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu)\Gamma(1-n+2\nu)} \qquad [n=0,\,1,\,2,\dots; \quad \operatorname{Re}\nu > n].$$

ET II 315(9)

7.123

$$\int_{-1}^1 \mathrm{P}_n^m(x)\mathrm{P}_n^k(x)\frac{dx}{1-x^2} = 0 \qquad [0 \le m \le n, \quad 0 \le k \le n; \quad m \ne k].$$

MO 74

7.124

$$\int_{-1}^1 x^k (z-x)^{-1}(1-x^2)^{\frac{1}{2}m}\mathrm{P}_n^m(x)\,dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z)\cdot z^k$$

[$m \le n; k = 0,\,1,\dots,n-m; z$ in the complex plane with a cut along the interval $(-1,\,1)$ on the real axis].

7.125

$$\int_{-1}^1 (1-x^2)^{\frac{1}{2}m}\mathrm{P}_k^m(x)\mathrm{P}_l^m(x)\mathrm{P}_n^m(x)\,dx =$$
$$= (-1)^m \pi^{-\frac{3}{2}} \frac{(k+m)!(l+m)!(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!}\times$$
$$\times \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(t-k+\frac{1}{2}\right)\Gamma\left(t-l+\frac{1}{2}\right)\Gamma\left(t-n+\frac{1}{2}\right)}{\dots}$$

7.126

1. $$\int_0^1 \mathrm{P}_\nu(x) x^\sigma\, dx = \frac{\sqrt{\pi}2^{-\sigma-1}\Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)} \qquad [\operatorname{Re}\sigma > -1].$$

EH I 171(23)

2. $$\int_0^1 x^\sigma \mathrm{P}_\nu^m(x)\, dx = \frac{(-1)^m \pi^{\frac{1}{2}} 2^{-2m-1}\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma(1+m+\nu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}m\right)\Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}\right)\Gamma(1-m+\nu)} \times$$
$$\times {}_3F_2\left(\frac{m+\nu+1}{2},\ \frac{m-\nu}{2},\ \frac{m}{2}+1;\ m+1,\ \frac{3+\sigma+m}{2};\ 1\right)$$
$$[\operatorname{Re}\sigma > -1; \quad m = 0,\ 1,\ 2, \ldots].$$

ET II 313(2)

3. $$\int_0^1 x^\sigma \mathrm{P}_\nu^\mu(x)\, dx = \frac{\pi^{\frac{1}{2}} 2^{2\mu-1}\Gamma\left(\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{3+\sigma-\mu}{2}\right)} \times$$
$$\times {}_3F_2\left(\frac{\nu-\mu+1}{2},\ -\frac{\mu+\nu}{2},\ 1-\frac{\mu}{2};\ 1-\mu,\ \frac{3+\sigma-\mu}{2};\ 1\right)$$
$$[\operatorname{Re}\sigma > -1, \quad \operatorname{Re}\mu < 2].$$

ET II 313(3)

4. $$\int_1^\infty x^{\mu-1} Q_\nu(ax)\, dx = e^{\mu\pi i}\Gamma(\mu) a^{-\mu}(a^2-1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a)$$
$$[|\arg(a-1)| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}(\nu-\mu) > -1].$$

ET II 325(26)

7.127

$$\int_{-1}^1 (1+x)^\sigma \mathrm{P}_\nu(x)\, dx = \frac{2^{\sigma+1}[\Gamma(\sigma+1)]^2}{\Gamma(\sigma+\nu+2)\Gamma(1+\sigma-\nu)} \qquad [\operatorname{Re}\sigma > -1].$$

ET II 316(15)

7.128

1. $$\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{3}{2}} \mathrm{P}_\nu^\mu(x)\, dx =$$
$$= -\frac{\Gamma\left(\mu-\frac{1}{2}\right)(z-1)^{\mu-\frac{1}{2}}(z+1)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}} e^{2\mu\pi i}\Gamma(\mu+\nu)\Gamma(\mu-\nu-1)} \times$$

2. $$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{1}{2}}\mathrm{P}_{\nu}^{\mu}(x)\,dx =$$
$$= \frac{2e^{-2\mu\pi i}\Gamma\left(\frac{1}{2}+\mu\right)}{\pi^{\frac{1}{2}}\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)}(z-1)^{\mu}Q_{\nu}^{\mu}\left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}}\right]Q_{-\nu-1}^{\mu}\left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}}\right]$$
$\left[-\frac{1}{2} < \operatorname{Re}\mu < 1, \quad z -\right.$ in the complex plane with a cut along the interval (-1, 1) of the real axis].

ET II 316(18)

7.129

$$\int_{-1}^{1}\mathrm{P}_{\nu}(x)\mathrm{P}_{\lambda}(x)(1+x)^{\lambda+\nu}\,dx = \frac{2^{\lambda+\nu+1}[\Gamma(\lambda+\nu+1)]^4}{[\Gamma(\lambda+1)\Gamma(\nu+1)]^2\Gamma(2\lambda+2\nu+2)} \qquad [\operatorname{Re}(\nu+\lambda+1) > 0].$$

EH I 172(30)

7.131

1. $$\int_{1}^{\infty}(x-1)^{-\frac{1}{2}\mu}(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{1}{2}}P_{\nu}^{\mu}(x)\,dx =$$
$$= \pi^{\frac{1}{2}}\frac{\Gamma(-\mu-\nu)\Gamma(1-\mu+\nu)}{\Gamma\left(\frac{1}{2}-\mu\right)}(z-1)^{\mu}\left\{P_{\nu}^{\mu}\left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}}\right]\right\}^2$$
$[\operatorname{Re}(\mu+\nu) < 0, \quad \operatorname{Re}(\mu-\nu) < 1, \quad |\arg(z+1)| < \pi]$.

ET II 321(6)

2. $$\int_{1}^{\infty}(x-1)^{-\frac{1}{2}\mu}(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{3}{2}}P_{\nu}^{\mu}(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(1-\mu-\nu)\Gamma(2-\mu+\nu)(z-1)^{\mu-\frac{1}{2}}(z+1)^{-\frac{1}{2}}}{\Gamma\left(\frac{3}{2}-\mu\right)}P_{\nu}^{\mu}\left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}}\right]P_{\nu}^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}}\right]$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu+\nu) < 1, \quad \operatorname{Re}(\mu-\nu) < 2, \quad \arg(1+z)| < \pi]$.

ET II 321(7)

7.132

1. $$\int_{-1}^{1}(1-x^2)^{\lambda-1}\mathrm{P}_{\nu}^{\mu}(x)\,dx =$$
$$= \frac{\pi 2^{\mu}\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{\Gamma\left(\lambda+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(-\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$

2. $$\int_1^\infty (x^2-1)^{\lambda-1} P_n^\mu(x)\,dx = \frac{2^{\mu-1}\Gamma\left(\lambda-\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\lambda-\frac{1}{2}\mu\right)}$$
$$[\operatorname{Re}\lambda > \operatorname{Re}\mu, \quad \operatorname{Re}(1-2\lambda-\nu) > 0, \quad \operatorname{Re}(2-2\lambda+\nu) > 0].$$

ET II 320(2)

3. $$\int_1^\infty (x^2-1)^{\lambda-1} Q_\nu^\mu(x)\,dx = e^{\mu\pi i}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{2^{2\lambda-\mu}\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\lambda+\frac{1}{2}\nu\right)}$$
$$[|\operatorname{Re}\mu| < 2\operatorname{Re}\lambda < \operatorname{Re}\nu+2].$$

ET II 324(23)

4. $$\int_0^1 x^\sigma (1-x^2)^{-\frac{1}{2}\mu} \mathrm{P}_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2}\right)}$$
$$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\sigma > -1].$$

EH I 172(24)

5. $$\int_0^1 x^\sigma (1-x^2)^{\frac{1}{2}m} \mathrm{P}_\nu^m(x)\,dx = \frac{(-1)^m 2^{-m-1}\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)\Gamma(1+m+\nu)}{\Gamma(1-m+\nu)\Gamma\left(1+\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}m+\frac{1}{2}\nu\right)}$$
$$[\operatorname{Re}\sigma > -1, \quad m \text{ is a positive integer}].$$

EH I 172(25), ET II 313(4)

6. $$\int_0^1 (1-x^2)^\eta \mathrm{P}_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\Gamma\left(1+\eta-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)}{\Gamma(1-\mu)\Gamma\left(\frac{3}{2}+\eta+\frac{1}{2}\sigma-\frac{1}{2}\mu\right)} \times$$
$$\times {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1+\eta-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}+\eta; 1\right)$$
$$\left[\operatorname{Re}\left(\eta-\frac{1}{2}\mu\right) > -1, \quad \operatorname{Re}\sigma > -1\right].$$

7. $$\int_1^\infty x^{-\varrho}(x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{\ldots\Gamma\left(\frac{\ldots}{2}\right)\Gamma\left(\frac{\ldots}{2}\right)}{\sqrt{\pi}\Gamma(\varrho)}$$
$$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\varrho+\mu+\nu) > 0, \quad \operatorname{Re}(\varrho+\mu-\nu) > 1].$$

7.133

1. $$\int_u^\infty Q_\nu(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}(u^2-1)^{\frac{1}{2}\mu}Q_\nu^{-\mu}(u)$$
$$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1 + \operatorname{Re}\nu].$$

MO 90a

2. $$\int_u^\infty (x^2-1)^{\frac{1}{2}\lambda}Q_\nu^{-\lambda}(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}(u^2-1)^{\frac{1}{2}\lambda+\frac{1}{2}\mu}Q_\nu^{-\lambda-\mu}(u)$$
$$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1 + \operatorname{Re}(\nu-\lambda)].$$

ET II 204(30)

7.134

1. $$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu}P_\nu^\mu(x)\,dx = \frac{2^{\lambda+\mu}\Gamma(\lambda)\Gamma(-\lambda-\mu-\nu)\Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)\Gamma(1-\lambda-\mu)}$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu+\nu) < 0, \quad \operatorname{Re}(\lambda+\mu-\nu) < 1].$$

ET II 321(4)

2. $$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu}P_\nu^\mu(x)\,dx = -\frac{2^{\lambda-\mu}\,\sin\pi\nu\Gamma(\lambda-\mu)\Gamma(-\lambda+\mu-\nu)\Gamma(1-\lambda+\mu+\nu)}{\pi\Gamma(1-\lambda)}$$
$$[\operatorname{Re}(\lambda-\mu) > 0, \quad \operatorname{Re}(\mu-\lambda-\nu) > 0, \quad \operatorname{Re}(\mu-\lambda+\nu) > -1].$$

ET II 321(5)

7.135

1. $$\int_{-1}^1 (1-x^2)^{-\frac{1}{2}\mu}(z-x)^{-1}\mathrm{P}_{\mu+n}^\mu(x)\,dx = 2e^{-i\mu\pi}(z^2-1)^{-\frac{1}{2}\mu}Q_{\mu+n}^\mu(z)$$

[$n = 0, 1, 2, \ldots, \operatorname{Re}\mu + n > -1$, z---in the complex plane with a cut along the interval $(-1, 1)$ of the real axis].

2. $$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{\mu/2}(x+z)^{-\rho}P_\nu^\mu(x)\,dx =$$
$$= \frac{2^{\lambda+\mu-\rho}\Gamma(\lambda-\rho)\Gamma(\rho-\lambda-\mu-\nu)\Gamma(\rho-\lambda-\mu+\nu+1)}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)\Gamma(1+\rho-\lambda-\mu)}\times$$
$$\times\, {}_3F_2\left(\rho,\ \rho-\lambda-\mu-\nu,\ \rho-\lambda-\mu+\nu+1;\ \rho-\lambda+1,\ \rho-\lambda-\mu+1;\ \frac{1+z}{2}\right) +$$
$$+\frac{\Gamma(\rho-\lambda)\Gamma(\lambda)}{\Gamma(\rho)\Gamma(1-\mu)}2^\mu(z+1)^{\lambda-\rho}\,{}_3F_2\left(\lambda,\ -\mu-\nu,\ 1-\mu+\nu;\ 1-\mu,\ 1-\rho+\lambda;\ \frac{1+z}{2}\right)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\rho-\lambda-\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda-\mu+\nu+1) > 0$$
$$|\arg(z+1)| < \pi].$$

3. $$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{-\mu/2}(x+z)^{-\rho}P_\nu^\mu(x)\,dx =$$
$$= -\frac{\sin(\nu\pi)\Gamma(\lambda-\mu-\rho)\Gamma(\rho-\lambda+\mu-\nu)\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\Gamma(1+\rho-\lambda)}\times$$
$$\times\, {}_3F_2\left(\rho,\, \rho-\lambda+\mu-\nu,\, \rho-\lambda+\mu+\nu+1;\, 1+\rho-\lambda,\, 1+\rho-\lambda+\mu;\, \frac{1+z}{2}\right) \;+$$
$$+\,\frac{\Gamma(\lambda-\mu)\Gamma(\rho-\lambda+\mu)}{\Gamma(\rho)\Gamma(1-\mu)}(z+1)^{\lambda-\rho-\mu}\times$$
$$\times\, {}_3F_2\left(\lambda-\mu,\, -\nu,\, \nu+1;\, 1+\lambda-\mu-\rho,\, 1-\mu;\, \frac{1+z}{2}\right)$$
$[\operatorname{Re}(\lambda-\mu)>0, \quad \operatorname{Re}(\rho-\lambda+\mu-\nu)>0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1)>0,$
$|\arg(z+1)|<\pi].$

ET II 322(10)

1. $$\int_{-1}^1 (1-x^2)^{\lambda-1}(1-a^2x^2)^{\mu/2}P_\nu(ax)\,dx = \frac{\pi 2^\mu\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\times$$
$$\times\, {}_2F_1\left(-\frac{\mu+\nu}{2},\, \frac{1-\mu+\nu}{2};\, \frac{1}{2}+\lambda;\, a^2\right)$$
$[\operatorname{Re}\lambda>0, \quad -1<a<1].$

ET II 318(31)

2. $$\int_1^\infty (x^2-1)^{\lambda-1}(a^2x^2-1)^{\mu/2}P_\nu^\mu(ax)\,dx =$$
$$= \frac{\Gamma(\lambda)\Gamma\left(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma(1-\lambda-\mu)}\times$$
$$\times\, 2^{\mu-1}a^{\mu-\nu-1}\, {}_2F_1\left(\frac{1-\mu+\nu}{2},\, 1-\lambda-\frac{\mu-\nu}{2};\, 1-\lambda-\mu;\, 1-\frac{1}{a^2}\right)$$
$[\operatorname{Re}a>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(\nu-\mu-2\lambda)>-2, \quad \operatorname{Re}(2\lambda+\mu+\nu)<1].$

ET II 325(25)

3. $$\int_1^\infty (x^2-1)^{\lambda-1}(a^2x^2-1)^{-\frac{1}{2}\mu}Q_\nu^\mu(ax)\,dx =$$
$$= \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma(\lambda)\Gamma\left(1-\lambda+\frac{\mu+\nu}{2}\right)2^{\mu-2}e^{\mu\pi i}a^{-\mu-\nu-1}}{\Gamma\left(\nu+\frac{3}{2}\right)}\times$$
$$\times\, {}_2F_1\left(\frac{\mu+\nu+1}{2},\, 1-\lambda+\frac{\mu+\nu}{2};\, \nu+\frac{3}{2};\, a^{-2}\right)$$
$[|\arg(a-1)|<\pi, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(2\lambda-\mu-\nu)<2].$

1. $$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}Q_\nu^\mu(1+2ax)\,dx = \pi^{-\frac{1}{2}}e^{-\mu\pi i}\Gamma\left(\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\{Q_\nu^\mu[(1+a)^{\frac{1}{2}}]\}^2$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > -1\right].$$

ET II 325(28)

2. $$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}Q_\nu^\mu(1+2ax)\,dx =$$
$$= -\pi^{-\frac{1}{2}}e^{-\mu\pi i}\Gamma\left(-\mu-\frac{1}{2}\right)a^{\frac{1}{2}\mu+\frac{1}{2}}(1+a^2)^{-\frac{1}{2}}Q_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]Q_\nu^\mu[(1+a)^{\frac{1}{2}}]$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu < -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu+2) > 0\right].$$

ET II 326(29)

3. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}P_\nu^\mu(1+2ax)\,dx = \pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\{P_\nu^\mu[(1+a)^{\frac{1}{2}}]\}^2$$
$$\left[\operatorname{Re}\mu < \frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(32)

4. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}P_\nu^\mu(1+2ax)\,dx =$$
$$= \pi^{\frac{1}{2}}\Gamma\left(-\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu+\frac{1}{2}}P_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]P_\nu^\mu[(1+a)^2] \qquad \left[\operatorname{Re}\mu < -\frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(33)

5. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu}P_\nu^\mu(1+2ax)\,dx = \pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\mu\right)a^{-\frac{1}{2}\mu}P_\nu^\mu[(1+a)^{\frac{1}{2}}]P_\nu^{-\mu}[(1+a)^{\frac{1}{2}}]$$
$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(34)

6. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}(1+ax)^{-\frac{1}{2}\mu}P_\nu^\mu(1+2ax)\,dx =$$
$$= \frac{1}{2}\pi^{\frac{1}{2}}\Gamma\left(\mu-\frac{1}{2}\right)a^{\frac{1}{2}-\frac{1}{2}\mu}(1+a)^{-\frac{1}{2}}\{P_\nu^{1-\mu}[(1+a)^{\frac{1}{2}}]P_\nu^\mu[(1+a)^{\frac{1}{2}}]+$$
$$+(\mu+\nu)(1-\mu+\nu)P_\nu^{-\mu}[(1+a)^{\frac{1}{2}}]P_\nu^\mu[(1+a)^{\frac{1}{2}}]\} \qquad \left[\operatorname{Re}\mu > \frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(35)

ET II 320(38)

8. $$\int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}Q_\nu^\mu(1+2ax)\,dx =$$
$$= \frac{1}{2}\pi^{\frac{1}{2}}\Gamma\left(-\mu-\frac{1}{2}\right)(1+a)^{-\frac{1}{2}}a^{\frac{1}{2}\mu+\frac{1}{2}} \times$$
$$\times\left\{P_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]Q_\nu^\mu[(1+a)^{\frac{1}{2}}] + P_\nu^\mu[(1+a)^{\frac{1}{2}}]Q_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]\right\}$$
$$\left[\operatorname{Re}\mu < -\frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 320(39)

813

9. $$\int_0^y (y-x)^{\mu-1}\left[x\left(1+\frac{1}{2}\gamma x\right)\right]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x)\,dx =$$
$$= \Gamma(\mu)\left(\frac{2}{\gamma}\right)^{\frac{1}{2}\mu}\left[y\left(1+\frac{1}{2}\gamma y\right)\right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1+\gamma y) \qquad [\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0, \quad \arg\gamma y| < \pi].$$

ET II 193(52)

10. $$\int_0^y (y-x)^{\mu-1}x^{\sigma+\frac{1}{2}\lambda-1}\left(1+\frac{1}{2}\gamma x\right)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x)\,dx =$$
$$= \frac{\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}\lambda}\Gamma(\sigma)\Gamma(\mu)y^{\sigma+\mu-1}}{\Gamma(1-\lambda)\Gamma(\sigma+\mu)}\,{}_3F_2\left(-\nu,\,1+\nu,\,\sigma;\,1-\lambda,\,\sigma+\mu;\,-\frac{1}{2}\gamma y\right)$$
$$[\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad |\gamma y| < 1].$$

ET II 193(53)

11. $$\int_0^y (y-x)^{\mu-1}[x(1-x)]^{-\frac{1}{2}\lambda}\mathrm{P}_\nu^\lambda(1-2x)\,dx = \Gamma(\mu)[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}\mathrm{P}_\nu^{\lambda-\mu}(1-2y)$$
$$[\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1].$$

ET II 193(54)

12. $$\int_0^y (y-x)^{\mu-1}x^{\sigma+\frac{1}{2}\lambda-1}(1-x)^{-\frac{1}{2}\lambda}\mathrm{P}_\nu^\lambda(1-2x)\,dx =$$
$$= \frac{\Gamma(\mu)\Gamma(\sigma)y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\Gamma(1-\lambda)}\,{}_3F_2(-\nu,\,1+\nu,\,\sigma;\,1-\lambda,\,\sigma+\mu;\,y)$$
$$[\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1].$$

$$\int_0^\infty (a+x)^{-\mu-\nu-2} P_\mu\left(\frac{a-x}{a+x}\right) P_\nu\left(\frac{a-x}{a+x}\right) dx = \frac{a^{-\mu-\nu-1}[\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1)\Gamma(\nu+1)]^2\Gamma(2\mu+2\nu+2)}$$
$$[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\nu) > -1].$$

ET II 326(3)

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

1. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu} \mathrm{P}_\nu^\mu(x)\, dx = \frac{a^{-\lambda-\mu}e^{-a}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G_{23}^{31}\left(2a\middle|\begin{matrix}1+\mu, 1\\ \lambda+\mu, -\nu, 1+\nu\end{matrix}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0].$$

ET II 323(13)

2. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu} Q_\nu^\mu(x)\, dx = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\Gamma(\nu-\mu+1)} a^{-\lambda-\mu}e^{-a} G_{23}^{22}\left(2a\middle|\begin{matrix}1+\mu, 1\\ \lambda+\mu, \nu+1, -\nu\end{matrix}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0].$$

ET II 325(24)

3. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\, dx = -\pi^{-1}\sin(\nu\pi) a^{\mu-\lambda}e^{-a} G_{23}^{31}\left(2a\middle|\begin{matrix}1, 1-\mu\\ \lambda-\mu,\ 1+\nu,\ -\nu\end{matrix}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda-\mu) > 0].$$

ET II 323(15)

4. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(x)\, dx = \frac{1}{2}e^{\mu\pi i}a^{\mu-\lambda}e^{-a} G_{23}^{22}\left(\middle|2a\begin{matrix}1-\mu, 1\\ \lambda-\mu, \nu+1, -\nu\end{matrix}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu) > 0].$$

ET II 323(14)

5. $$\int_1^\infty e^{-ax}(x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\, dx = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}a^{\mu-\frac{1}{2}}K_{\nu+\frac{1}{2}}(a) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\mu < 1].$$

ET II 323(11), MO 90

7.142

$$\int_1^\infty e^{-\frac{1}{2}ax}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, dx = \frac{2}{a}W_{\mu,\nu}(a) \qquad \left[\operatorname{Re}\mu < 1, \quad \nu - \frac{1}{2} \neq 0,\ \pm 1,\ \pm 2, \dots\right].$$

1. $$\int_0^\infty [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x)\, dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta} K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 179(1)

2. $$\int_0^\infty \left(1+\frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x)\, dx = \frac{e^{\frac{1}{2}\beta}}{\beta} W_{\mu,\, \nu+\frac{1}{2}}(\beta) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 179(2)

7.144

1. $$\int_0^\infty e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x)\, dx =$$
$$= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left\{ \frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu}\, \sin(\mu\pi)} E(-\nu,\, \nu+1,\, \lambda+\mu;\, \mu+1\colon 2\beta) \right. -$$
$$\left. - \frac{\sin[(\mu+\nu)\pi]}{2^{1-\mu}\beta^\lambda\, \sin(\mu\pi)} E(\nu-\mu+1,\, -\nu-\mu,\, \lambda\colon 1-\mu\colon 2\beta) \right\}$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0].$$

ET I 181(16)

2. $$\int_0^\infty e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x)\, dx =$$
$$= -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu}\, \sin(\mu\pi)} E(-\nu,\, \nu+1,\, \lambda-\mu\colon 1-\mu\colon 2\beta) -$$
$$- \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu}\beta^\lambda\, \sin(\mu\pi)} E(\mu+\nu+1,\, \mu-\nu,\, \lambda\colon 1+\mu\colon 2\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu)] > 0.$$

ET I 181(17)

7.145

1. $$\int_0^\infty \frac{e^{-\beta x}}{1+x} \mathrm{P}_\nu \left[\frac{1}{(1+x)^2} - 1\right] dx = \frac{e^\beta}{\beta} W_{\nu+\frac{1}{2},\, 0}(\beta) W_{-\nu-\frac{1}{2},\, 0}(\beta) \qquad [\operatorname{Re}\beta > 0].$$

ET I 180(6)

2. $$\int_0^\infty x^{-1} e^{-\beta x} Q_{-\frac{1}{2}}(1+2x^{-2})\, dx = \frac{\pi^2}{8} \left\{ \left[J_0\left(\frac{1}{2}\beta\right)\right]^2 + \left[N_0\left(\frac{1}{2}\beta\right)\right]^2 \right\} \qquad [\operatorname{Re}\beta > 0].$$

3. $$\int_0^\infty x^{-1} e^{-ax} Q_\nu(1+2x^{-2})\,dx = \frac{1}{2}[\Gamma(\nu+1)]^2 a^{-1} W_{-\nu-\frac{1}{2},\,0}(ai) W_{-\nu-\frac{1}{2},\,0}(-ai)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 327(6)

7.146

1. $$\int_0^\infty x^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(\sqrt{1+x})\,dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{5}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4},\,\frac{1}{2}\nu+\frac{1}{4}}(\beta) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0].$$

ET I 180(7)

2. $$\int_0^\infty x^{-\frac{1}{2}\mu} \frac{e^{-\beta x}}{\sqrt{1+x}} P_\nu^\mu(\sqrt{1+x})\,dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{3}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(\beta) \qquad [\operatorname{Re}\mu < 1,\ \operatorname{Re}\beta > 0].$$

ET I 180(8)a

3. $$\int_0^\infty \sqrt{x} e^{-\beta x} P_\nu^{\frac{1}{4}}(\sqrt{1+x^2}) P_\nu^{-\frac{1}{4}}(\sqrt{1+x^2})\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}} H_{\nu+\frac{1}{2}}^{(1)}\left(\frac{1}{2}\beta\right) H_{\nu+\frac{1}{2}}^{(2)}\left(\frac{1}{2}\beta\right) \qquad [\operatorname{Re}\beta > 0].$$

ET I 180(9)

7.147

$$\int_0^\infty x^{\lambda-1}(x^2+a^2)^{\frac{1}{2}\nu} e^{-\beta x} \mathrm{P}_\nu^\mu\left[\frac{x}{(x^2+a^2)^{\frac{1}{2}}}\right]\,dx =$$
$$= \frac{2^{-\nu-2} a^{\lambda+\nu}}{\pi\Gamma(-\mu-\nu)} G_{24}^{32}\left(\frac{a^2\beta^2}{4}\middle| 1-\frac{\lambda}{2},\ \frac{1-\lambda}{2} 0,\ \frac{1}{2},\ -\frac{\lambda+\mu+\nu}{2},\ -\frac{\lambda-\mu+\nu}{2}\right)$$
$$[a > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0].$$

ET II 327(7)

7.148

$$\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu+\nu-1} \exp\left(-\frac{1-x}{1+x}y\right) \mathrm{P}_\nu^\mu(x)\,dx = 2^\nu y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2},\,\frac{1}{2}\mu}(y)$$
$$[\operatorname{Re} y > 0].$$

ET II 317(21)

7.149

$$\int_1^\infty (\alpha^2+\beta^2+2\alpha\beta x)^{-\frac{1}{2}} \exp[-(\alpha^2+\beta^2+2\alpha\beta x)^{\frac{1}{2}}] P_\nu(x)\,dx = 2\pi^{-1}(\alpha\beta)^{-\frac{1}{2}} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0].$$

7.15 Combinations of associated Legendre and hyperbolic functions

7.151

1. $$\int_0^\infty (\operatorname{sh} x)^{\alpha-1} P_\nu^{-\mu}(\operatorname{ch} x)\,dx = \frac{2^{-1-\mu}\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\alpha+1\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1+\frac{1}{2}\mu-\frac{1}{2}\alpha\right)}$$
$$[\operatorname{Re}(\alpha+\mu)>0, \quad \operatorname{Re}(\nu-\alpha+2)>0, \quad \operatorname{Re}(1-\alpha-\nu)>0].$$

EH I 172(28)

2. $$\int_0^\infty (\operatorname{sh} x)^{\alpha-1} Q_\nu^{\mu}(\operatorname{ch} x)\,dx = \frac{e^{i\mu\pi}2^{\mu-\alpha}\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\alpha\right)}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha\right)}\times$$
$$\times\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\mu\right)$$
$$[\operatorname{Re}(\alpha\pm\mu)>0, \quad \operatorname{Re}(\nu-\alpha+2)>0].$$

EH I 172(29)

7.152

$$\int_0^\infty e^{-\alpha x}\ \operatorname{sh}^{2\mu}\left(\frac{1}{2}x\right) P_{2n}^{-2\mu}\left[\operatorname{ch}\left(\frac{1}{2}x\right)\right]\,dx = \frac{\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\alpha-n-\mu)\Gamma\left(\alpha+n-\mu+\frac{1}{2}\right)}{4^\mu\sqrt{\pi}\Gamma(\alpha+n+\mu+1)\Gamma\left(\alpha-n+\mu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\alpha>n+\operatorname{Re}\mu, \quad \operatorname{Re}\mu>-\frac{1}{4}\right].$$

ET I 181(15)

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.161

1. $$\int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{1}{2}\mu}\ \sin(ax)\mathrm{P}_\nu^\mu(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}2^{\mu-\lambda-1}\Gamma(\lambda+1)a}{\Gamma\left(1+\frac{\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{3+\lambda-\mu+\nu}{2}\right)}\times$$
$$\times\ {}_2F_3\left(\frac{1+\lambda}{2}, 1+\frac{\lambda}{2}; \frac{3}{2}, 1+\frac{\lambda-\mu-\nu}{2}, \frac{3+\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right)$$
$$[\operatorname{Re}\lambda>-1, \quad \operatorname{Re}\mu<1].$$

2. $$\int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{1}{2}\mu}\cos(ax)\mathrm{P}_\nu^\mu(x)\,dx = $$
$$= \frac{\pi^{\frac{1}{2}}2^{\mu-\lambda}\Gamma(\lambda)}{\Gamma\left(1+\frac{\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)}\times$$
$$\times {}_2F_3\left(\frac{\lambda}{2},\ \frac{\lambda+1}{2};\ \frac{1}{2},\ \frac{1+\lambda-\mu-\nu}{2},\ 1+\frac{\lambda-\mu+\nu}{2};\ -\frac{a^2}{4}\right) \quad [\operatorname{Re}\lambda>0,\ \operatorname{Re}\mu<1].$$

ET II 314(8)

3. $$\int_0^\infty (x^2-1)^{\frac{1}{2}\mu}\sin(ax)P_\nu^\mu(x)\,dx = \frac{2^\mu\pi^{\frac{1}{2}}a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}S_{\mu+\frac{1}{2},\,\nu+\frac{1}{2}}(a)$$
$$\left[a>0,\quad \operatorname{Re}\mu<\frac{3}{2},\quad \operatorname{Re}(\mu+\nu)<1\right].$$

ET II 320(1)

1. $$\int_a^\infty P_\nu(2x^2a^{-2}-1)\sin(bx)\,dx = -\frac{\pi a}{4\cos(\nu\pi)}\left\{\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2-\left[J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2\right\}$$
$$[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<0].$$

ET II 326(1)

2. $$\int_a^\infty P_\nu(2x^2a^{-2}-1)\cos(bx)\,dx = -\frac{\pi}{4}a\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)-N_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)N_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]$$
$$[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<0].$$

ET II 326(2)

3. $$\int_0^\infty (x^2+2)^{-\frac{1}{2}}\sin(ax)P_\nu^{-1}(x^2+1)\,dx = 2^{-\frac{1}{2}}\pi^{-1}a\sin(\nu\pi)[K_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a)]^2$$
$$[a>0,\quad -2<\operatorname{Re}\nu<1].$$

ET I 98(22)

4. $$\int_0^\infty (x^2+2)^{-\frac{1}{2}}\sin(ax)Q_\nu^1(x^2+1)\,dx = -2^{-\frac{3}{2}}\pi a K_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a)I_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a)$$
$$\left[a>0,\quad \operatorname{Re}\nu>-\frac{3}{2}\right].$$

5. $$\int_0^\infty \cos(ax) P_\nu(1+x^2)\,dx = -\frac{\sqrt{2}}{\pi}\sin(\nu\pi)\left[K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right)\right]^2 \qquad [a>0, \quad -1<\operatorname{Re}\nu<0].$$

ET I 42(23)

6. $$\int_0^\infty \cos(ax) Q_\nu(1+x^2)\,dx = \frac{\pi}{\sqrt{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right) I_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right) \qquad [a>0, \quad \operatorname{Re}\nu>-1].$$

ET I 42(24)

7. $$\int_0^1 \cos(ax) \mathrm{P}_\nu(2x^2-1)\,dx = \frac{\pi}{2} J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right) \qquad [a>0].$$

ET I 42(25)

7.163

1. $$\int_a^\infty (x^2-a^2)^{\frac{1}{2}\nu-\frac{1}{4}} \sin(bx)\, \mathrm{P}_0^{\frac{1}{2}-\nu}(ax^{-1})\,dx = b^{-\nu-\frac{1}{2}} \cos\left(ab - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) \qquad \left[a>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right].$$

ET I 98(24)

2. $$\int_0^1 x^{-1} \cos(ax) P_\nu(2x^{-2}-1)\,dx = -\frac{1}{2}\pi \operatorname{cosec}(\nu\pi)\ {}_1F_1(\nu+1\colon 1\colon ai)\ {}_1F_1(\nu+1\colon 1\colon -ai)$$
$$[a>0, \quad -1<\operatorname{Re}\nu<0].$$

ET II 327(4)

7.164

1. $$\int_0^\infty x^{\frac{1}{2}} \sin(bx)[P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2})]^2\,dx = \frac{\sqrt{\frac{2}{\pi}}a^{-1}b^{-\frac{1}{2}}}{\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{5}{4}<\operatorname{Re}\nu<\frac{1}{4}\right].$$

ET II 327(8)

2. $$\int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) Q_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2})\,dx =$$
$$= \frac{\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4}\pi i}\Gamma\left(\nu+\frac{5}{4}\right)}{ab^{\frac{1}{2}}\Gamma\left(\nu+\frac{3}{4}\right)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \qquad \left[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-\frac{5}{4}\right].$$

3. $$\int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) P_{\nu-1}^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} =$$
$$= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{5}{4}-\nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re}\nu < \frac{5}{4}\right].$$

ET II 328(10)

4. $$\int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) P_\nu^{-\frac{3}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} =$$
$$= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{7}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2 \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{7}{4} < \operatorname{Re}\nu < \frac{3}{4}\right].$$

ET II 328(11)

5. $$\int_0^\infty x^{\frac{1}{2}} \cos(bx) [P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2})]^2 \, dx = \frac{a^{-1}\left(\frac{\pi b}{2}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(-\frac{1}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{3}{4} < \operatorname{Re}\nu < -\frac{1}{4}\right].$$

ET II 328(12)

6. $$\int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) Q_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) \, dx = \frac{\sqrt{\frac{\pi}{2}} e^{\frac{1}{4}\pi i} \Gamma\left(\nu+\frac{3}{4}\right)}{ab^{\frac{1}{2}}\Gamma\left(\nu+\frac{5}{4}\right)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{4}\right].$$

ET II 328(13)

7. $$\int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) P_\nu^{\frac{3}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} =$$
$$= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2 \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re}\nu < \frac{1}{4}\right].$$

ET II 328(14)

8. $$\int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) P_{\nu-1}^{\frac{1}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} = \\ = \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{3}{4}\right].$$

ET II 329(15)

7.165

$$\int_0^\infty \cos(ax) P_\nu(\operatorname{ch} x)\, dx = -\frac{\sin(\nu\pi)}{4\pi^2}\Gamma\left(\frac{1+\nu+i\alpha}{2}\right)\Gamma\left(\frac{1+\nu-i\alpha}{2}\right)\Gamma\left(-\frac{\nu+i\alpha}{2}\right)\Gamma\left(-\frac{\nu-i\alpha}{2}\right) \\ [a > 0, \quad -1 < \operatorname{Re}\nu < 0].$$

ET II 329(18)

7.166

$$\int_0^\pi \mathrm{P}_\nu^{-\mu}(\cos\varphi)\, \sin^{\alpha-1}\varphi\, d\varphi = \\ = \frac{2^{-\mu}\pi\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)} \\ [\operatorname{Re}(\alpha \pm \mu) > 0].$$

MO 90, EH I 172(27)

7.167

$$\int_0^a \mathrm{P}_\nu^{-\mu}(\cos x)\mathrm{P}_\nu^{-\eta}[\cos(a-x)]\left[\frac{\sin(a-x)}{\sin x}\right]^\eta \frac{dx}{\sin x} = \frac{2^\eta\Gamma(\mu-\eta)\Gamma\left(\eta+\frac{1}{2}\right)(\sin a)^\eta}{\sqrt{\pi}\Gamma(\eta+\mu+1)}\mathrm{P}_\nu^{-\mu}(\cos a) \\ \left[\operatorname{Re}\mu > \operatorname{Re}\eta > -\frac{1}{2}\right].$$

ET II 329(16)

7.17 A combination of an associated Legendre function and the probability integral

7.171

$$\int_1^\infty (x^2-1)^{-\frac{1}{2}\mu} \exp(a^2x^2)[1-\Phi(ax)] P_\nu^\mu(x)\, dx = \\ = \pi^{-1}2^{\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right) a^{\mu-\frac{3}{2}} e^{\frac{a^2}{2}} W_{\frac{1}{4}-\frac{1}{2}\mu,\, \frac{1}{4}+\frac{1}{2}\nu}(a^2) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu+\nu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0].$$

7.181

1. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{\frac{1}{2}} N_\nu(ax)\,dx = 2^{-\frac{1}{2}} a^{-1} \left[\cos\left(\frac{1}{2}a\right) J_\nu\left(\frac{1}{2}a\right) - \sin\left(\frac{1}{2}a\right) N_\nu\left(\frac{1}{2}a\right)\right]$$
$$\left[a > 0, \quad \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 108(3)a

2. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{\frac{1}{2}} J_\nu(ax)\,dx = -\frac{1}{\sqrt{2a}} \left[\cos\left(\frac{1}{2}a\right) N_\nu\left(\frac{1}{2}a\right) + \sin\left(\frac{1}{2}a\right) J_\nu\left(\frac{1}{2}a\right)\right] \qquad \left[|\operatorname{Re}\nu| < \frac{1}{2}\right].$$

ET II 344(36)a

7.182

1. $$\int_1^\infty x^\nu (x^2-1)^{\frac{1}{2}\lambda-\frac{1}{2}} P_\lambda^{\lambda-1}(x) J_\nu(ax)\,dx = \frac{2^{\lambda+\nu} a^{-\lambda} \Gamma\left(\frac{1}{2}+\nu\right)}{\pi^{\frac{1}{2}} \Gamma(1-\lambda)} S_{\lambda-\nu,\,\lambda+\nu}(a)$$
$$\left[a > 0, \quad \operatorname{Re}\nu < \frac{5}{2}, \quad \operatorname{Re}(2\lambda+\nu) < \frac{3}{2}\right].$$

ET II 345(38)a

2. $$\int_1^\infty x^{\frac{1}{2}-\mu} (x^2-1)^{-\frac{1}{2}\mu} P^\mu_{\nu-\frac{1}{2}}(x) J_\nu(ax)\,dx = -2^{-\frac{3}{2}} \pi^{\frac{1}{2}} a^{\mu-\frac{1}{2}} \left[J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) N_\nu\left(\frac{a}{2}\right) + N_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) J_\nu\left(\frac{a}{2}\right)\right]$$
$$\left[-\frac{1}{4} < \operatorname{Re}\mu < 1, \quad a > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2} + 2\operatorname{Re}\mu\right].$$

ET II 344(37)a

3. $$\int_1^\infty x^{\frac{1}{2}-\mu} (x^2-1)^{-\frac{1}{2}\mu} P^\mu_{\nu-\frac{1}{2}}(x) N_\nu(ax)\,dx = 2^{-\frac{3}{2}} \pi^{\frac{1}{2}} a^{\mu-\frac{1}{2}} \left[J_\nu\left(\frac{a}{2}\right) J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - N_\nu\left(\frac{a}{2}\right) N_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$$\left[-\frac{1}{4} < \operatorname{Re}\mu < 1, \quad a > 0, \quad \operatorname{Re}(2\mu-\nu) > -\frac{1}{2}\right].$$

ET II 349(67)a

4. $$\int_0^1 x^{\frac{1}{2}-\mu} (1-x^2)^{-\frac{1}{2}\mu} \mathrm{P}^\mu_\nu(x) J_{\nu+\frac{1}{2}}(ax)\,dx = \sqrt{\frac{\pi}{2}} a^{\mu-\frac{1}{2}} J_{\frac{1}{2}-\mu}\left(\frac{1}{2}a\right) J_{\nu+\frac{1}{2}}\left(\frac{1}{2}a\right)$$
$$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu-\nu) < 2].$$

5. $$\int_1^\infty x^{\frac{1}{2}-\mu}(x^2-1)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)K_\nu(ax)\,dx = (2\pi)^{-\frac{1}{2}}a^{\mu-\frac{1}{2}}K_\nu\left(\frac{1}{2}a\right)K_{\mu-\frac{1}{2}}\left(\frac{1}{2}a\right)$$
$$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0].$$

ET II 135(5)a

6. $$\int_1^\infty x^{\mu+\frac{1}{2}}(x^2-1)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-\frac{3}{2}}e^{-\frac{1}{2}a}W_{\mu,\,\nu}(a) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0].$$

ET II 135(3)a

7. $$\int_1^\infty x^{\mu-\frac{3}{2}}(x^2-1)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-\frac{1}{2}}e^{-\frac{1}{2}a}W_{\mu-1,\,\nu}(a) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0].$$

ET II 135(4)a

8. $$\int_1^\infty x^{\mu-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{3}{2}}(x)K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-1}e^{-\frac{1}{2}a}W_{\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(a) \qquad [\operatorname{Re}\mu < 1].$$

ET II 135(6)a

9. $$\int_1^\infty x^{\frac{1}{2}}(x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}P^{\frac{1}{2}-\nu}_\mu(2x^2-1)K_\nu(ax)\,dx = \pi^{-\frac{1}{2}}a^{-\nu}2^{\nu-1}\left[K_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}a > 0\right].$$

ET II 136(11)a

10. $$\int_1^\infty x^{\frac{1}{2}}(x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}P^{\frac{1}{2}-\nu}_\mu(2x^2-1)N_\nu(ax)\,dx =$$
$$= \pi^{\frac{1}{2}}2^{\nu-2}a^{-\nu}\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - N_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)N_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu + |2\operatorname{Re}\mu + 1| < \frac{3}{2}\right].$$

ET II 108(5)a

11. $$\int_1^\infty x^{\frac{1}{2}}(x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}P^{\frac{1}{2}-\nu}_\mu(2x^2-1)J_\nu(ax)\,dx =$$
$$= -2^{\nu-2}a^{-\nu}\pi^{\frac{1}{2}}\sec(\mu\pi)\left\{\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 - \left[J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu - \frac{3}{2} < 2\operatorname{Re}\mu < \frac{1}{2} - \operatorname{Re}\nu\right].$$

12. $$\int_1^\infty x(x^2-1)^{-\frac{1}{2}\nu} P_\mu^\nu(2x^2-1) K_\nu(ax)\,dx = 2^{-\nu} a^{\nu-1} K_{\mu+1}(a) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\nu < 1].$$

ET II 136(10)a

13. $$\int_0^\infty x(x^2+a^2)^{\frac{1}{2}\nu} P_\mu^\nu(1+2x^2a^{-2}) K_\nu(xy)\,dx = 2^{-\nu} a y^{-\nu-1} S_{2\nu,\,2\mu+1}(ay)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1].$$

ET II 135(7)

14. $$\int_0^\infty x(x^2+a^2)^{\frac{1}{2}\nu}[(\mu-\nu)P_\mu^\nu(1+2x^2a^{-2})+(\mu+\nu)P_{-\mu}^\nu(1+2x^2a^{-2})] K_\nu(xy)\,dx =$$
$$= 2^{1-\nu}\mu y^{-\nu-2} S_{2\nu+1,\,2\mu}(ay) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1].$$

ET II 136(8)

15. $$\int_0^\infty x(x^2+a^2)^{\frac{1}{2}\nu-1}[P_\mu^\nu(1+2x^2a^{-2})+P_{-\mu}^\nu(1+2x^2a^{-2})] K_\nu(xy)\,dx = 2^{1-\nu} y^{-\nu} S_{2\nu-1,\,2\mu}(ay)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1].$$

ET II 136(9)

16. $$\int_0^\infty x^{\frac{1}{2}}(x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}(x^2+1) J_\nu(xy)\,dx = \frac{y^{-\frac{1}{2}} 2^{\frac{1}{2}-\nu} \pi^{-\frac{1}{2}} [K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y)]^2}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}$$
$$\left[-\frac{3}{2} - \operatorname{Re}\nu < \operatorname{Re}\mu < \operatorname{Re}\nu + \frac{1}{2}, \quad y > 0\right].$$

ET II 44(1)

17. $$\int_0^\infty x^{\frac{1}{2}}(x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} Q_\mu^{\nu+\frac{1}{2}}(x^2+1) J_\nu(xy)\,dx = 2^{-\nu-\frac{1}{2}} \pi^{\frac{1}{2}} e^{(\nu+\frac{1}{2})\pi i} y^\nu K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y) I_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y)$$
$$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -\frac{5}{2}, \quad y > 0\right].$$

ET II 46(12)

7.183

$$\int_0^\infty x^{1-\mu}(1+a^2x^2)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax) J_\nu(xy)\,dx =$$
$$= i(2\pi)^{\frac{1}{2}} e^{i\pi(\mu\mp\frac{1}{2}\nu\mp\frac{1}{4})} a^{-1} y^{\mu-1} I_\nu\left(\frac{1}{2}a^{-1}y\right) K_\mu\left(\frac{1}{2}a^{-1}y\right)$$
$$\left[-\frac{3}{4} - \frac{1}{2}\operatorname{Re}\nu < \operatorname{Re}\mu < 1 + \operatorname{Re}\nu, \quad y > 0, \quad \operatorname{Re} a > 0\right].$$

1. $$\int_1^\infty x^{\frac{1}{2}}(x^2-1)^{\frac{1}{2}\mu-\frac{1}{4}}P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}(x^{-1})J_\nu(xa)\,dx = 2^{\frac{1}{2}}a^{-1-\mu}\pi^{-\frac{1}{2}}\cos\left[a+\frac{1}{2}(\nu-\mu)\pi\right]$$
$$\left[|\operatorname{Re}\mu|<\frac{1}{2},\quad \operatorname{Re}\nu>-1,\quad a>0\right].$$

ET II 44(2)a

2. $$\int_1^\infty x^{-\nu}(x^2-1)^{\frac{1}{4}-\frac{1}{2}\nu}P_\mu^{\nu-\frac{1}{2}}(2x^{-2}-1)K_\nu(ax)\,dx = \pi^{\frac{1}{2}}2^{-\nu}a^{-2+\nu}W_{\mu+\frac{1}{2},\,\nu-\frac{1}{2}}(a)W_{-\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\nu<\frac{3}{2},\quad a>0\right].$$

ET II 370(45)a

3. $$\int_0^\infty x^\nu(1+x^2)^{\frac{1}{4}+\frac{\nu}{2}}Q_\mu^{\nu+\frac{1}{2}}\left(1+\frac{2}{x^2}\right)J_\nu(ax)\,dx =$$
$$= -ie^{i\pi\nu}\pi^{-\frac{1}{2}}2^\nu a^{-\nu-2}\left[\Gamma\left(\frac{3}{2}+\mu+\nu\right)\right]^2\Gamma\left(\frac{1}{2}+\nu-\mu\right)\times$$
$$\times W_{-\mu-\frac{1}{2},\,\nu+\frac{1}{2}}(a)\left[\frac{\cos(\mu\pi)}{\Gamma(2+2\nu)}M_{\mu+\frac{1}{2},\,\nu+\frac{1}{2}}(a)+\frac{\sin(\nu\pi)}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)}W_{\mu+\frac{1}{2},\,\nu+\frac{1}{2}}(a)\right]$$
$$\left[a>0,\quad \operatorname{Re}(\mu+\nu)>-\frac{3}{2},\quad \operatorname{Re}(\mu-\nu)<\frac{1}{2}\right].$$

ET II 46(14)

4. $$\int_0^1 x^\nu(1-x^2)^{\frac{1}{2}\nu+\frac{1}{4}}P_\mu^{-\nu-\frac{1}{2}}(2x^{-2}-1)J_\nu(xy)\,dx =$$
$$= 2^{\nu+\frac{1}{2}}y^\nu\frac{\Gamma\left(\frac{3}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}{(2\pi)^{\frac{1}{2}}\left[\Gamma\left(\frac{3}{2}+\nu\right)\right]^2}\times$$
$$\times\ {}_1F_1\left(\nu+\mu+\frac{3}{2};\,2\nu+2;\,iy\right)\ {}_1F_1\left(\nu+\mu+\frac{3}{2};\,2\nu+2;\,-iy\right)$$
$$\left[y>0,\quad -\frac{3}{2}-\operatorname{Re}\nu<\operatorname{Re}\mu<\operatorname{Re}\nu+\frac{1}{2}\right].$$

ET II 45(3)

5. $$\int_0^\infty x^{-\nu}(x^2+a^2)^{\frac{1}{4}-\frac{1}{2}\nu}Q_\mu^{\frac{1}{2}-\nu}(1+2a^2x^{-2})K_\nu(xy)\,dx =$$
$$= ie^{-i\pi\nu}\pi^{\frac{1}{2}}2^{-\nu-1}a^{-\nu-\frac{1}{2}}y^{\nu-2}\left[\Gamma\left(\frac{3}{2}+\mu-\nu\right)\right]^2W_{-\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(iay)W_{-\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(-iay)$$
$$\left[\operatorname{Re}a>0,\quad \operatorname{Re}y>0,\quad \operatorname{Re}\mu>-\frac{3}{2},\quad \operatorname{Re}(\mu-\nu)>-\frac{3}{2}\right].$$

6. $$\int_0^\infty x^{-\nu}(x^2+1)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}(1+2x^{-2}) J_\nu(ax)\,dx =$$
$$= 2^{-\nu} a^{-\nu-2} \frac{ie^{-i\nu\pi}\pi^{\frac{1}{2}}\Gamma\left(\frac{3}{2}+\mu-\nu\right)}{\Gamma(2\nu)} M_{\mu+\frac{1}{2},\,\nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(a)$$
$$\left[a>0, \quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu+\frac{3}{2}\right].$$

ET II 47(15)a

7. $$\int_0^\infty x^{-\nu}(x^2+a^2)^{\frac{1}{4}-\frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu}(1+2a^2x^{-2}) K_\nu(xy)\,dx =$$
$$= ie^{-i\pi\nu}\pi^{\frac{3}{2}} 2^{-\nu-3} a^{\frac{1}{2}-\nu} y^{\nu-1}[\Gamma(1-\nu)]^2 \times \left\{\left[J_{\nu-\frac{1}{2}}\left(\tfrac{ay}{2}\right)\right]^2 + \left[N_{\nu-\frac{1}{2}}\left(\tfrac{ay}{2}\right)\right]^2\right\}$$
$$[\operatorname{Re}a>0, \quad \operatorname{Re}y>0, \quad \operatorname{Re}\nu<1].$$

ET II 136(12)

7.185

$$\int_0^\infty x^{\frac{1}{2}} Q_{\nu-\frac{1}{2}}[(a^2+x^2)x^{-1}] J_\nu(xy)\,dx = 2^{-\frac{1}{2}}\pi y^{-1} \exp\left[-\left(a^2-\frac{1}{4}\right)^{\frac{1}{2}} y\right] J_\nu\left(\frac{1}{2}y\right)$$
$$\left[\operatorname{Re}\nu>-\frac{1}{2}, \quad y>0\right].$$

ET II 46(10)

7.186

$$\int_0^\infty x(1+x^2)^{-\nu-1} \mathrm{P}_\nu\left(\frac{1-x^2}{1+x^2}\right) J_0(xy)\,dx = y^{2\nu}[2^\nu\Gamma(\nu+1)]^{-2} K_0(y) \qquad [\operatorname{Re}\nu>0].$$

ET II 13(10)

7.187

1. $$\int_0^\infty x P_\mu^\nu(\sqrt{1+x^2}) K_\nu(xy)\,dx = y^{-\frac{3}{2}} S_{\nu+\frac{1}{2},\,\mu+\frac{1}{2}}(y) \qquad [\operatorname{Re}\nu<1, \quad \operatorname{Re}y>0].$$

ET II 137(14)

2. $$\int_0^\infty x[P_{\lambda-\frac{1}{2}}(\sqrt{1+a^2x^2})]^2 J_0(xy)\,dx = 2\pi^{-2}y^{-1}a^{-1}\cos(\lambda\pi)\left[K_\lambda\left(\frac{y}{2a}\right)\right]^2$$
$$\left[\operatorname{Re}a>0, \quad |\operatorname{Re}\lambda|<\frac{1}{4}, \quad y>0\right].$$

3. $\int_0^\infty x(1+x^2)^{-\frac{1}{2}} P_\mu^\nu(\sqrt{1+x^2}) K_\nu(xy)\, dx = y^{-\frac{1}{2}} S_{\nu-\frac{1}{2},\, \mu+\frac{1}{2}}(y) \qquad [\operatorname{Re}\nu < 1, \quad \operatorname{Re} y > 0].$

ET II 137(15)

4. $$\int_0^\infty x P_\mu^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) Q_\mu^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_\nu(xy)\, dx =$$
$$= \frac{y^{-1} e^{-\frac{1}{2}\nu\pi i} \Gamma\left(1+\mu+\frac{1}{2}\nu\right)}{a\Gamma\left(1+\mu-\frac{1}{2}\nu\right)} I_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\mu > -\frac{3}{4}, \quad \operatorname{Re}\nu > -1\right].$$

ET II 47(16)

5. $$\int_0^\infty x P^\mu_{\sigma-\frac{1}{2}}(\sqrt{1+a^2x^2}) Q^\mu_{\sigma-\frac{1}{2}}(\sqrt{1+a^2x^2}) J_0(xy) dx = y^{-2} e^{\mu\pi i} \frac{\Gamma\left(\frac{1}{2}+\sigma-\mu\right)}{\Gamma(1+2\sigma)} W_{\mu,\,\sigma}\left(\frac{y}{a}\right) M_{-\mu,\,\sigma}\left(\frac{y}{a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\sigma > -\frac{1}{4}, \quad \operatorname{Re}\mu < 1\right].$$

ET II 14(15)

6. $$\int_0^\infty x P^\mu_{\sigma-\frac{1}{2}}(\sqrt{1+a^2x^2}) P^{-\mu}_{\sigma-\frac{1}{2}}(\sqrt{1+a^2x^2}) J_0(xy)\, dx = 2\pi^{-1} y^{-2} \cos(\sigma\pi) W_{\mu,\,\sigma}\left(\frac{y}{a}\right) W_{-\mu,\,\sigma}\left(\frac{y}{a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re}\sigma| < \frac{1}{4}\right].$$

ET II 14(14)

7. $$\int_0^\infty x \{P^\mu_{\sigma-\frac{1}{2}}(\sqrt{1+a^2x^2})\}^2 J_0(xy)\, dx = -i\pi^{-1} y^{-2} W_{\mu,\,\sigma}\left(\frac{y}{a}\right) \left[W_{\mu,\,\sigma}\left(e^{\pi i}\frac{y}{a}\right) - W_{\mu,\,\sigma}\left(e^{-\pi i}\frac{y}{a}\right)\right]$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re}\sigma| < \frac{1}{4}, \quad \operatorname{Re}\mu < 1\right].$$

ET II 14(13)

8. $$\int_0^\infty x(1+a^2x^2)^{-\frac{1}{2}} P_\mu^{-\frac{1}{2}-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) P_\mu^{\frac{1}{2}-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_\nu(xy)\, dx =$$
$$= \frac{\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2}{\pi a^2 \Gamma\left(\frac{\nu}{2}+\mu+\frac{3}{2}\right)\Gamma\left(\frac{\nu}{2}-\mu+\frac{1}{2}\right)} \qquad \left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{5}{4} < \operatorname{Re}\mu < \frac{1}{4}\right].$$

9. $$\int_0^\infty x\{P_\mu^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2})\}^2 J_\nu(xy)\,dx = \frac{2\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2 y^{-1}}{\pi a\Gamma\left(1+\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)}$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{3}{4} < \operatorname{Re}\mu < -\frac{1}{4}, \quad \operatorname{Re}\nu > -1\right].$$

ET II 45(7)

10. $$\int_0^\infty x(1+a^2x^2)^{-\frac{1}{2}} P_\mu^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) P_{\mu+1}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_\nu(xy)\,dx =$$
$$= \frac{K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{3}{2}}\left(\frac{y}{2a}\right)}{\pi a^2\Gamma\left(2+\frac{1}{2}\nu+\mu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)} \qquad \left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{7}{4} < \operatorname{Re}\mu < -\frac{1}{4}\right].$$

ET II 45(8)

7.188

1. $$\int_0^\infty x(a^2+x^2)^{-\frac{1}{2}\mu} \mathrm{P}_{\mu-1}^{-\nu}\left[\frac{a}{\sqrt{a^2+x^2}}\right] J_\nu(xy)\,dx = \frac{y^{\mu-2}e^{-ay}}{\Gamma(\mu+\nu)}$$
$$\left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > \frac{1}{2}\right].$$

ET II 45(4)

825

2. $$\int_0^\infty x^{\nu+1}(x^2+a^2)^{\frac{1}{2}\nu} P_\nu\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{\nu+1}y^{-\nu-1}}{\pi\Gamma(-\nu)}\left[K_{\nu+\frac{1}{2}}\left(\frac{ya}{2}\right)\right]^2$$
$$[\operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < 0, \quad y > 0].$$

ET II 45(5)

3. $$\int_0^\infty x^{1-\nu}(x^2+a^2)^{-\frac{1}{2}\nu} P_{\nu-1}\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{1-\nu}y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) K_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)$$
$$[\operatorname{Re} a > 0, \quad y > 0, \quad 0 < \operatorname{Re}\nu < 1].$$

ET II 45(6)

7.189

1. $$\int_0^\infty (a+x)^\mu e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = 0$$
$$\left[-\frac{1}{2} < \operatorname{Re}\mu < 0, \quad -\frac{1}{2} + \operatorname{Re}\mu < \operatorname{Re}\nu < -\frac{1}{2} - \operatorname{Re}\mu\right].$$

2. $$\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = $$
$$= \frac{2^{\mu-1}\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu-\frac{1}{2}\right)e^a}{\pi^{\frac{1}{2}}\Gamma(2\mu+\nu+1)\Gamma(2\mu-\nu)} W_{\frac{1}{2}-\mu,\,\frac{1}{2}+\nu}(2a)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu > \left|\operatorname{Re}\nu + \frac{1}{2}\right|\right].$$

ET II 367(19)

3. $$\int_0^\infty x^{-\mu} e^{x} P_\nu^{2\mu}\left(1+\frac{2x}{a}\right) K_\mu(x+a)\,dx = $$
$$= \pi^{-\frac{1}{2}} 2^{\mu-1} \cos(\mu\pi)\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right) W_{\frac{1}{2}-\mu,\,\frac{1}{2}+\nu}(2a)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu > \left|\operatorname{Re}\nu + \frac{1}{2}\right|\right].$$

ET II 373(11)

4. $$\int_0^\infty x^{-\frac{1}{2}\mu}(x+a)^{-\frac{1}{2}} e^{-x} \mathrm{P}_{\nu-\frac{1}{2}}^{\mu}\left(\frac{a-x}{a+x}\right) K_\nu(a+x)\,dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}\mu}\Gamma(\mu,\, 2a) \qquad [a>0, \quad \operatorname{Re}\mu < 1].$$

ET II 374(12)

5. $$\int_0^\infty (\operatorname{sh} x)^{\mu+1}(\operatorname{ch} x)^{-2\mu-\frac{3}{2}} P_\nu^{-\mu}[\operatorname{ch}(2x)] I_{\mu-\frac{1}{2}}(a \operatorname{sech} x)\,dx = $$
$$= \frac{2^{\mu-\frac{1}{2}}\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)}{\pi^{\frac{1}{2}} a^{\mu+\frac{3}{2}}[\Gamma(\mu+1)]^2} M_{\nu+\frac{1}{2},\,\mu}(a) M_{-\nu-\frac{1}{2},\,\mu}(a)$$
$$[\operatorname{Re}\mu > \operatorname{Re}\nu, \quad \operatorname{Re}\mu > -\operatorname{Re}\nu - 1].$$

ET II 378(44)

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.191

1. $$\int_a^\infty x^{\frac{1}{2}}(x^2-a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}}(2x^2a^{-2}-1)[\mathbf{H}_\nu(x) - N_\nu(x)]\,dx = $$
$$= 2^{-\nu-2}\pi^{\frac{1}{2}} a \operatorname{cosec}(\mu\pi)\cos(\nu\pi)\left\{\left[N_\nu\left(\frac{1}{2}a\right)\right]^2 - \left[J_\nu\left(\frac{1}{2}a\right)\right]^2\right\}$$
$$\left[-1 < \operatorname{Re}\mu < 0, \quad \operatorname{Re}\nu < \frac{1}{2}\right].$$

2. $$\int_0^\infty x^{1/2}(x^2-a^2)^{-1/4-\nu/2}P_\mu^{\nu+1/2}(2x^2a^{-2}-1)[I_{-\nu}(x)-\mathbf{L}_\nu(x)]\,dx =$$
$$= 2^{-\nu-1}\pi^{1/2}a\ \operatorname{cosec}(2\mu\pi)\ \cos(\nu\pi)\left\{\left[I_\nu\left(\frac{1}{2}a\right)\right]^2-\left[I_{-\nu}\left(\frac{1}{2}a\right)\right]^2\right\}$$
$$\left[-1<\operatorname{Re}\mu<0,\quad \operatorname{Re}\nu<\frac{1}{2}\right].$$

ET II 385(15)

7.192

1. $$\int_0^1 x^{(\nu-\mu-1)/2}(1-x^2)^{(\nu-\mu-2)/4}P_{\nu-1/2}^{(\mu-\nu+2)/2}(x)S_{\mu,\,\nu}(ax)\,dx =$$
$$= 2^{\mu-3/2}\pi^{1/2}a^{-(\nu-\mu-1)/2}\Gamma\left(\frac{\mu+\nu+3}{4}\right)\Gamma\left(\frac{\mu-3\nu+3}{4}\right)\cos\left(\frac{\mu-\nu}{2}\pi\right)\times$$
$$\times\left[J_\nu\left(\frac{1}{2}a\right)N_{-(\mu-\nu+1)/2}\left(\frac{1}{2}a\right)-N_\nu\left(\frac{1}{2}a\right)J_{-(\mu-\nu+1)/2}\left(\frac{1}{2}a\right)\right]$$
$$[\operatorname{Re}(\mu-\nu)<0,\quad a>0,\quad |\operatorname{Re}(\mu+\nu)|<1,\quad \operatorname{Re}(\mu-3\nu)<1].$$

ET II 387(24)a

2. $$\int_1^\infty x^{1/2}(x^2-1)^{-\beta/2}P_\nu^\beta(x)S_{\mu,\,1/2}(ax)\,dx =$$
$$= \frac{2^{-3/2+\beta-\mu}a^{\beta-1}\Gamma\left(\frac{\beta-\mu+\nu}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\beta-\mu-\nu}{2}-\frac{1}{4}\right)}{\pi^{1/2}\Gamma\left(\frac{1}{2}-\mu\right)}S_{\mu-\beta+1,\,\nu+1/2}(a)$$
$$\left[\operatorname{Re}\beta<1,\quad a>0,\quad \operatorname{Re}(\mu+\nu-\beta)<-\frac{1}{2},\quad \operatorname{Re}(\mu-\nu-\beta)<\frac{1}{2}\right].$$

ET II 387(25)a

7.193

1. $$\int_1^\infty x^{-\nu}(x^2-1)^{1/4-\nu/2}P_{\mu/2-\nu/2}^{\nu-1/2}(2x^{-2}-1)S_{\mu,\,\nu}(ax)\,dx =$$
$$= \frac{2^{\mu-\nu}a^{\nu-2}\pi^{1/2}\Gamma\left(\frac{3\nu-\mu-1}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}W_{\rho,\,\sigma}(ae^{i\pi/2})W_{\rho,\,\sigma}(ae^{-i\pi/2});$$
$$\rho=\frac{1}{2}(\mu+1-\nu),\qquad \sigma=\nu-\frac{1}{2}$$
$$\left[\operatorname{Re}(\mu-\nu)<0,\quad a>0,\quad \operatorname{Re}\nu<\frac{3}{2},\quad \operatorname{Re}(3\nu-\mu)>1\right].$$

ET II 387(27)a

ET II 387(26)a

7.21 Integration of associated Legendre functions with respect to the order

7.211

1. $$\int_0^\infty \mathrm{P}_{-x-\frac{1}{2}}(\cos\theta)\,dx = \frac{1}{2}\ \operatorname{cosec}\left(\frac{1}{2}\theta\right) \qquad [0<\theta<\pi].$$

ET II 329(19)

2. $$\int_{-\infty}^\infty \mathrm{P}_x(\cos\theta)\,dx = \operatorname{cosec}\left(\frac{1}{2}\theta\right) \qquad [0<\theta<\pi].$$

ET II 329(20)

7.212

$$\int_0^\infty x^{-1}\,\theta(\pi x) P_{-\frac{1}{2}+ix}(\operatorname{ch} a)\,dx = 2e^{-\frac{1}{2}a}\boldsymbol{K}(e^{-a}) \qquad [a>0].$$

ET II 330(22)

7.213

$$\int_0^\infty \frac{x\,\theta(\pi x)}{a^2+x^2} P_{-\frac{1}{2}+ix}(\operatorname{ch} b)\,dx = Q_{a-\frac{1}{2}}(\operatorname{ch} b) \qquad [\operatorname{Re} a>0].$$

ET II 387(23)

7.214

$$\int_0^\infty \operatorname{sh}(\pi x)\ \cos(ax) P_{-\frac{1}{2}+ix}(b)\,dx = \frac{1}{\sqrt{2(b+\operatorname{ch} a)}} \qquad [a>0, \quad |b|<1].$$

ET I 42(27)

7.215

$$\int_0^\infty \cos(bx)\,P^{\mu}_{-\frac{1}{2}+ix}(\operatorname{ch} a)\,dx = 0 \qquad [0<a<b];$$

$$= \frac{\sqrt{\frac{\pi}{2}}(\operatorname{sh} a)^{\mu}}{\Gamma\left(\frac{1}{2}-\mu\right)(\operatorname{ch} a-\operatorname{ch} b)^{\mu+\frac{1}{2}}} \qquad [0<b<a].$$

$$\int_0^\infty \cos(bx)\Gamma(\mu+ix)\Gamma(\mu-ix)P^{\frac{1}{2}-\mu}_{-\frac{1}{2}+ix}(\operatorname{ch} a)\,dx = \frac{\sqrt{\frac{\pi}{2}}\Gamma(\mu)(\operatorname{sh} a)^{\mu-\frac{1}{2}}}{(\operatorname{ch} a+\operatorname{ch} b)^{\mu}}$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0].$$

ET II 330(24)

7.217

1. $$\int_{-\infty}^\infty \left(\nu-\frac{1}{2}+ix\right)\Gamma\left(\frac{1}{2}-ix\right)\Gamma\left(2\nu-\frac{1}{2}+ix\right)P^{\frac{1}{2}-\nu}_{\nu+ix-1}(\cos\theta)I_{\nu-\frac{1}{2}+ix}(a)K_{\nu-\frac{1}{2}+ix}(b)\,dx =$$
$$= \sqrt{2\pi}(\sin\theta)^{\nu-\frac{1}{2}}\left(\frac{ab}{\omega}\right)^{\nu}K_\nu(\omega); \qquad \omega=(a^2+b^2+2ab\,\cos\theta)^{\frac{1}{2}}.$$

ET II 383(29)

2. $$\int_0^\infty xe^{\pi x}\,\theta(\pi x)P_{-\frac{1}{2}+ix}(-\cos\theta)H^{(2)}_{ix}(ka)H^{(2)}_{ix}(kb)\,dx = -\frac{2(ab)^{\frac{1}{2}}}{\pi R}e^{-ikR};$$
$$R=(a^2+b^2-2ab\,\cos\theta)^{\frac{1}{2}} \qquad [a>0, \quad b>0, \quad 0<\theta<\pi, \quad \operatorname{Im}\,k\le 0].$$

ET II 381(17)

3. $$\int_0^\infty xe^{\pi x}\,\operatorname{sh}(\pi x)\Gamma(\nu+ix)\Gamma(\nu-ix)P^{\frac{1}{2}-\nu}_{-\frac{1}{2}+ix}(-\cos\theta)H^{(2)}_{ix}(a)H^{(2)}_{ix}(b)\,dx =$$
$$= i(2\pi)^{\frac{1}{2}}(\sin\theta)^{\nu-\frac{1}{2}}\left(\frac{ab}{R}\right)^{\nu}H^{(2)}_{\nu}(R); \qquad R=(a^2+b^2-2ab\,\cos\theta)^{\frac{1}{2}}$$
$$[a>0, \quad b>0, \quad 0<\theta<\pi, \quad \operatorname{Re}\nu>0].$$

ET II 381(18)

4. $$\int_0^\infty x\,\operatorname{sh}(\pi x)\Gamma(\lambda+ix)\Gamma(\lambda-ix)K_{ix}(a)K_{ix}(b)P^{\frac{1}{2}-\lambda}_{-\frac{1}{2}+ix}(\beta)\,dx = \frac{\pi^{\frac{1}{2}}}{\sqrt{2}}\left(\frac{ab}{z}\right)^{\lambda}(\beta^2-1)^{\frac{1}{2}\lambda-\frac{1}{4}}K_\lambda(z);$$
$$z=\sqrt{a^2+b^2+2ab\beta} \qquad \left[|\arg a|<\frac{\pi}{2}, \quad |\arg(\beta-1)|<\pi, \quad \operatorname{Re}\lambda>0\right].$$

ET II 177(16)

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.221

1. $$\int_{-1}^1 P_n(x)P_m(x)\,dx = 0 \qquad [m\ne n]$$
$$= \frac{2}{2n+1} \qquad [m=n].$$

2.[6] $$\int_0^1 P_n(x)P_m(x)\,dx = \frac{1}{2n+1} \qquad [m=n];$$
$$= 0 \qquad [n-m \text{ is even}, \quad m \neq n];$$
$$= \frac{(-1)^{\frac{1}{2}(m+n-1)} m!n!}{2^{m+n-1}(m-n)(n+m+1)\left[\left(\frac{n}{2}\right)!\left(\frac{m-1}{2}\right)!\right]^2} \qquad [n\text{−even}, m\text{−odd}].$$

WH

3. $$\int_0^{2\pi} P_{2n}(\cos\varphi)\,d\varphi = 2\pi\left[\binom{2n}{n}2^{-2n}\right]^2.$$

MO 70, EH II 183(50)

829

7.222

1. $$\int_{-1}^1 x^m P_n(x)\,dx = 0 \qquad [m<n].$$

2. $$\int_{-1}^1 (1+x)^{m+n} P_m(x)P_n(x)\,dx = \frac{2^{m+n+1}[(m+n)!]^4}{(m!n!)^2(2m+2n+1)!}.$$

ET II 277(15)

3. $$\int_{-1}^1 (1+x)^{m-n-1} P_m(x)P_n(x)\,dx = 0 \qquad [m>n].$$

ET II 278(16)

4. $$\int_{-1}^1 (1-x^2)^n P_{2m}(x)\,dx = \frac{2n^2}{(n-m)(2m+2n+1)}\int_{-1}^1 (1-x^2)^{n-1} P_{2m}(x)\,dx \qquad [m<n].$$

WH

5. $$\int_0^1 x^2 P_{n+1}(x)P_{n-1}(x)\,dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

WH

7.223

7.224

[z belongs to the complex plane with a discontinuity along the interval from -1 to $+1$].

1. $$\int_{-1}^{1} (z-x)^{-1} P_n(x)\,dx = 2Q_n(z).$$

ET II 277(7)

2. $$\int_{-1}^{1} x(z-x)^{-1} P_0(x)\,dx = 2Q_1(z).$$

ET II 277(8)

3. $$\int_{-1}^{1} x^{n+1}(z-x)^{-1} P_n(x)\,dx = 2z^{n+1}Q_n(z) - \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

ET II 277(9)

4. $$\int_{-1}^{1} x^m(z-x)^{-1} P_n(x)\,dx = 2z^m Q_n(z) \qquad [m \le n].$$

ET II 277(10)a

5. $$\int_{-1}^{1} (z-x)^{-1} P_m(x)P_n(x)\,dx = 2P_m(z)Q_n(z) \qquad [m \le n].$$

ET II 278(18)a

6. $$\int_{-1}^{1} (z-x)^{-1} P_n(x)P_{n+1}(x)\,dx = 2P_{n+1}(z)Q_n(z) - \frac{2}{n+1}.$$

ET II 278(19)

7. $$\int_{-1}^{1} x(z-x)^{-1} P_m(x)P_n(x)\,dx = 2zP_m(z)Q_n(z) \qquad [m < n].$$

ET II 278(21)

ET II 278(20)

7.225

1. $$\int_{-1}^{x} (x-t)^{-\frac{1}{2}} P_n(t)\, dt = \left(n+\frac{1}{2}\right)^{-1} (1+x)^{-\frac{1}{2}} [T_n(x) + T_{n+1}(x)].$$

EH II 187(43)

2. $$\int_{x}^{1} (t-x)^{-\frac{1}{2}} P^{-\frac{1}{2}} P_n(t)\, dt = \left(n+\frac{1}{2}\right)^{-1} (1-x)^{-\frac{1}{2}} [T_n(x) - T_{n+1}(x)].$$

EH II 187(44)

3. $$\int_{-1}^{1} (1-x)^{-1/2} P_n(x)\, dx = \frac{2^{3/2}}{2n+1}.$$

EH II 183(49)

4. $$\int_{-1}^{1} (\operatorname{ch} 2p - x)^{-1/2} P_n(x)\, dx = \frac{2\sqrt{2}}{2n+1} \exp[-(2n+1)p] \qquad [p>0].$$

WH

7.226

1. $$\int_{-1}^{1} (1-x^2)^{-1/2} P_{2m}(x)\, dx = \left[\frac{\Gamma\left(\frac{1}{2}+m\right)}{m!}\right]^2.$$

ET II 276(4)

2. $$\int_{-1}^{1} x(1-x^2)^{-1/2} P_{2m+1}(x)\, dx = \frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}$$

ET II 276(5)

3. $$\int_{-1}^{1} (1+px^2)^{-m-3/2} P_{2m}(x)\, dx = \frac{2}{2m+1}(-p)^m (1+p)^{-m-1/2} \qquad [|p|<1].$$

7.227

$$\int_0^1 x(a^2+x^2)^{-1/2} P_n(1-2x^2)\,dx = \frac{[a+(a^2+1)^{1/2}]^{-2n-1}}{2n+1} \qquad [\operatorname{Re} a > 0].$$

ET II 278(23)

7.228[6]

$$\frac{1}{2}\Gamma(1+\mu)\int_{-1}^1 P_l(x)(z-x)^{-\mu-1}\,dx = (z^2-1)^{-\mu/2} e^{-i\pi\mu} Q_l^\mu(z).$$

$$[l = 0,\ 1,\ 2, \dots, \quad |\arg(z-1)| < \pi]$$

7.23 Combinations of Legendre polynomials and powers

7.231

1. $$\int_0^1 x^\lambda P_{2m}(x)\,dx = \frac{(-1)^m \Gamma\left(m-\frac{1}{2}\lambda\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\lambda\right)}{2\Gamma\left(-\frac{1}{2}\lambda\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2}\lambda\right)} \qquad [\operatorname{Re}\lambda > -1].$$

EH II 183(51)

2.[6] $$\int_0^1 x^\lambda P_{2m+1}(x)\,dx = \frac{(-1)^m \Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)} \qquad [\operatorname{Re}\lambda > -2].$$

EH II 183(52)

7.232

1. $$\int_{-1}^1 (1-x)^{a-1} P_m(x) P_n(x)\,dx = \\ = \frac{2^a \Gamma(a)\Gamma(n-a+1)}{\Gamma(1-a)\Gamma(n+a+1)}\, {}_4F_3(-m,\ m{+}1,\ a,\ a;\ 1,\ a{+}n{+}1,\ a{-}n;\ 1) \qquad [\operatorname{Re} a > 0].$$

ET II 278(17)

2. $$\int_{-1}^1 (1-x)^{a-1}(1+x)^{b-1} P_n(x)\,dx = \frac{2^{a+b-1}\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\, {}_3F_2(-n,\ 1{+}n,\ a;\ 1,\ a{+}b;\ 1)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

3. $$\int_0^1 (1-x)^{\mu-1} P_n(1-\gamma x)\,dx = \frac{\Gamma(\mu) n!}{\Gamma(\mu+n+1)} P_n^{(\mu,\,-\mu)}(1-\gamma) \qquad [\operatorname{Re}\mu > 0].$$

ET II 190(37)a

4. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} P_n(1-\gamma x)\,dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\, {}_3F_2\left(-n,\ n+1,\ \nu;\ 1,\ \mu+\nu;\ \frac{1}{2}\gamma\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 190(38)

7.233

$$\int_0^1 x^{2\mu-1} P_n(1-2x^2)\,dx = \frac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n+1)\Gamma(\mu-n)} \qquad [\operatorname{Re}\mu > 0].$$

ET II 278(22)

7.24 Combinations of Legendre polynomials and other elementary functions

7.241

$$\int_0^\infty P_n(1-x) e^{-ax}\,dx = e^{-a} a^n \left(\frac{1}{a}\frac{d}{da}\right)^n \left(\frac{e^a}{a}\right);$$
$$= a^n \left(1 + \frac{1}{2}\frac{d}{da}\right)^n \left(\frac{1}{a^{n+1}}\right) \qquad [\operatorname{Re} a > 0].$$

ET I 171(2)

7.242

$$\int_0^\infty P_n(e^{-x}) e^{-ax}\,dx = \frac{(a-1)(a-2)\dots(a-n+1)}{(a+n)(a+n-2)\dots(a-n+2)} \qquad [n \geq 2, \quad \operatorname{Re} a > 0].$$

ET I 171(3)

7.243

1. $$\int_0^\infty P_{2n}(\operatorname{ch} x) e^{-ax}\,dx = \frac{(a^2-1^2)(a^2-3^2)\dots[a^2-(2n-1)^2]}{a(a^2-2^2)(a^2-4^2)\dots[a^2-(2n)^2]} \qquad [\operatorname{Re} a > 2n].$$

ET I 171(6)

2. $$\int_0^\infty P_{2n+1}(\operatorname{ch} x) e^{-ax}\,dx = \frac{a(a^2-2^2)(a^2-4^2)\dots[a^2-(2n)^2]}{(a^2-1)(a^2-3^2)\dots[a^2-(2n+1)^2]} \qquad [\operatorname{Re} a > 2n+1].$$

3. $$\int_0^\infty P_{2n}(\cos x)e^{-ax}\,dx = \frac{(a^2+1^2)(a^2+3^2)\dots[a^2+(2n-1)^2]}{a(a^2+2^2)(a^2+4^2)\dots[a^2+(2n)^2]} \qquad [\operatorname{Re} a > 0].$$

ET I 171(4)

4. $$\int_0^\infty P_{2n+1}(\cos x)e^{-ax}\,dx = \frac{a(a^2+2^2)(a^2+4^2)\dots[a^2+(2n)^2]}{(a^2+1^2)(a^2+3^2)\dots[a^2+(2n+1)^2]} \qquad [\operatorname{Re} a > 0].$$

ET I 171(5)

7.244

1. $$\int_0^1 P_n(1-2x^2)\,\sin ax\,dx = \frac{\pi}{2}\left[J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 \qquad [a>0].$$

ET I 94(2)

2. $$\int_0^1 P_n(1-2x^2)\,\cos ax\,dx = \frac{\pi}{2}(-1)^n J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right) \qquad [a>0].$$

ET I 38(1)

7.245

1. $$\int_0^{2\pi} P_{2m+1}(\cos\theta)\,\cos\theta\,d\theta = \frac{\pi}{2^{4m+1}}\binom{2m}{m}\binom{2m+2}{m+1}.$$

MO 70, EH II 183(5)

2. $$\int_0^\pi P_m(\cos\theta)\,\sin n\theta\,d\theta =$$
$$= \frac{2(n-m+1)(n-m+3)\dots(n+m-1)}{(n-m)(n-m+2)\dots(n+m)} \qquad [n>m, \quad n+m \text{ is odd}];$$
$$= 0 \qquad [n \le m \quad \text{or} \quad n+m \text{ is even}].$$

MO 71

7.246

$$\int_0^\pi P_n(1-2\,\sin^2\,x\,\sin^2\,\theta)\,\sin x\,dx = \frac{2\,\sin(2n+1)\theta}{(2n+1)\,\sin\theta}.$$

$$\int_0^1 P_{2n+1}(x)\,\sin ax \frac{dx}{\sqrt{x}} = (-1)^{n+1}\sqrt{\frac{\pi}{2a}}J_{2n+\frac{3}{2}}(a) \qquad [a > 0].$$

ET I 94(1)

7.248

1. $$\int_{-1}^1 (a^2+b^2-2abx)^{-\frac{1}{2}}\,\sin[\lambda(a^2+b^2-2abx)^{\frac{1}{2}}]P_n(x)\,dx = \pi(ab)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(a\lambda)J_{n+\frac{1}{2}}(b\lambda) \qquad [a > 0, \quad b > 0].$$

ET II 277(11)

2. $$\int_{-1}^1 (a^2+b^2-2abx)^{-\frac{1}{2}}\,\cos[\lambda(a^2+b^2-2abx)^{\frac{1}{2}}]P_n(x)\,dx = -\pi(ab)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(a\lambda)N_{n+\frac{1}{2}}(b\lambda) \qquad [0 \le a \le b].$$

ET II 277(12)

7.249

1. $$\int_{-1}^1 P_n(x)\,\arcsin x\,dx = 0 \qquad [n - \text{even}];$$

$$= \pi\left\{\frac{(n-2)!!}{2^{\frac{1}{2}(n+1)}\left(\frac{n+1}{2}\right)!}\right\}^2 \qquad [n - \text{odd}].$$

WH

* I. J. Good, Proc. Camb. Philos. Soc. *51*(1955), 385-388.

2. $$P_n(x) = \frac{1}{t}\sum_{t=0}^{t-1}\left(x + \sqrt{x^2-1}\,\cos\frac{2\pi r}{t}\right)^n \qquad [t > n].^*$$

7.25 Combinations of Legendre polynomials and Bessel functions

7.251

1. $$\int_0^1 xP_n(1-2x^2)N_\nu(xy)\,dx = \pi^{-1}y^{-1}[S_{2n+1}(y) + \pi N_{2n+1}(y)] \qquad [n = 0,\ 1, \dots; \quad y > 0, \quad \nu > 0].$$

ET II 108(1)

2. $$\int_0^1 xP_n(1-2x^2)K_0(xy)\,dx = y^{-1}\left[(-1)^{n+1}K_{2n+1}(y) + \frac{i}{2}S_{2n+1}(iy)\right] \qquad [y > 0].$$

ET II 134(1)

3. $$\int_0^1 xP_n(1-2x^2)J_0(xy)\,dx = y^{-1}J_{2n+1}(y) \qquad [y > 0].$$

ET II 13(1)

4. $$\int_0^1 xP_n(1-2x^2)[J_0(ax)]^2\,dx = \frac{1}{2(2n+1)}\{[J_n(a)]^2 + [J_{n+1}(a)]^2\}.$$

ET II 338(39)a

5. $$\int_0^1 xP_n(1-2x^2)J_0(ax)N_0(ax)\,dx = \frac{1}{2(2n+1)}[J_n(a)N_n(a)+J_{n+1}(a)N_{n+1}(a)].$$

ET II 339(48)a

6. $$\int_0^1 x^2P_n(1-2x^2)J_1(xy)\,dx = y^{-1}(2n+1)^{-1}[(n+1)J_{2n+2}(y)-nJ_{2n}(y)] \qquad [y > 0].$$

ET II 20(23)

7. $$\int_0^1 x^{\mu-1}P_n(2x^2-1)J_\nu(ax)\,dx = \frac{2^{-\nu-1}a^\nu\left[\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu\right)\right]^2}{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+n+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-n\right)}\times$$
$$\times\ {}_2F_3\left(\frac{\mu+\nu}{2},\ \frac{\mu+\nu}{2};\ \nu+1,\ \frac{\mu+\nu}{2}+n+1,\ \frac{\mu+\nu}{2}-n;\ -\frac{a^2}{4}\right)$$
$$[a > 0, \quad \operatorname{Re}(\mu+\nu) > 0].$$

ET II 337(32)a

7.252

$$\int_0^1 e^{-ax}P_n(1-2x)I_0(ax)\,dx = \frac{e^{-a}}{2n+1}[I_n(a)+I_{n+1}(a)] \qquad [a > 0].$$

$$\int_0^{\frac{\pi}{2}} \sin(2x) P_n(\cos 2x) J_0(a \sin x)\, dx = a^{-1} J_{2n+1}(a).$$

ET II 361(20)

7.254

$$\int_0^1 x P_n(1-2x^2)[I_0(ax) - L_0(ax)]\, dx = (-1)^n [I_{2n+1}(a) - \mathbf{L}_{2n+1}(a)] \qquad [a > 0].$$

ET II 385(14)a

7.3- 7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and powers

7.311

1. $$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)\, dx = 0 \qquad \left[n > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 280(1)

2. $$\int_0^1 x^{n+2\varrho}(1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)\, dx = \frac{\Gamma(2\nu+n)\Gamma(2\varrho+n+1)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\varrho+\frac{1}{2}\right)}{2^{n+1}\Gamma(2\nu)\Gamma(2\varrho+1) n! \Gamma(n+\nu+\varrho+1)}$$
$$\left[\operatorname{Re} \varrho > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 280(2)

3. $$\int_{-1}^1 (1-x)^{\nu-\frac{1}{2}}(1+x)^{\beta} C_n^{\nu}(x)\, dx = \frac{2^{\beta+\nu+\frac{1}{2}}\Gamma(\beta+1)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)\Gamma\left(\beta-\nu+\frac{3}{2}\right)}{n!\Gamma(2\nu)\Gamma\left(\beta-\nu-n+\frac{3}{2}\right)\Gamma\left(\beta+\nu+n+\frac{3}{2}\right)}$$
$$\left[\operatorname{Re} \beta > -1, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 280(3)

4. $$\int_{-1}^1 (1-x)^{\alpha}(1+x)^{\beta} C_n^{\nu}(x)\, dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+2\nu)}{n!\Gamma(2\nu)\Gamma(\alpha+\beta+2)} \times$$
$$\times\ {}_3F_2\left(-n,\, n+2\nu,\, \alpha+1;\, \nu+\frac{1}{2},\, \alpha+\beta+2;\, 1\right)$$
$$[\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]$$

7.312

In the following integrals, z belongs to the complex plane with a cut along the interval of the real axis from -1 to 1.

1. $$\int_{-1}^{1} x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x)\, dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \qquad \left[m \le n, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 281(5)

2. $$\int_{-1}^{1} x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x)\, dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) - \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu)\Gamma(\nu+n+1)} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 281(6)

3.[6] $$\int_{-1}^{1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x)\, dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z) Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \qquad \left[m \le n, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 283(17)

7.313

1. $$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x)\, dx = 0 \qquad \left[m \ne n, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 282(12), MO 98a, EH I 177(16)

2. $$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} \left[C_n^\nu(x)\right]^2 dx = \frac{\pi 2^{1-2\nu}\Gamma(2\nu+n)}{n!(n+\nu)[\Gamma(\nu)]^2} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 281(8), MO 98a, EH I 177(17)

7.314

1. $$\int_{-1}^{1} (1-x)^{\nu-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} \left[C_n^\nu(x)\right]^2 dx = \frac{\pi^{\frac{1}{2}} \Gamma\left(\nu-\frac{1}{2}\right) \Gamma(2\nu+n)}{n!\Gamma(\nu)\Gamma(2\nu)} \qquad \left[\operatorname{Re}\nu > \frac{1}{2}\right].$$

2. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{2\nu-1}[C_n^\nu(x)]^2\,dx=\frac{2^{3\nu-\frac{1}{2}}[\Gamma(2\nu+n)]^2\Gamma\left(2n+\nu+\dfrac{1}{2}\right)}{(n!)^2\Gamma(2\nu)\Gamma\left(3\nu+2n+\dfrac{1}{2}\right)}\qquad[\operatorname{Re}\nu>0].$$

ET II 282(10)

3. $$\int_{-1}^{1}(1-x)^{3\nu+2n-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}[C_n^\nu(x)]^2\,dx=$$
$$=\frac{\pi^{\frac{1}{2}}\left[\Gamma\left(\nu+\dfrac{1}{2}\right)\right]^2\Gamma\left(\nu+2n+\dfrac{1}{2}\right)\Gamma(2\nu+2n)\Gamma\left(3\nu+2n-\dfrac{1}{2}\right)}{2^{2\nu+2n}\left[n!\Gamma\left(\nu+n+\dfrac{1}{2}\right)\Gamma\left(2\nu\right)\right]^2\Gamma\left(2\nu+2n+\dfrac{1}{2}\right)}\qquad\left[\operatorname{Re}\nu>\frac{1}{6}\right].$$

ET II 282(11)

4. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{\nu+m-n-\frac{3}{2}}C_m^\nu(x)C_n^\nu(x)\,dx=$$
$$=(-1)^m\frac{2^{2-2\nu-m+n}\pi^{\frac{3}{2}}\Gamma(2\nu+n)}{m!(n-m)![\Gamma(\nu)]^2\Gamma\left(\dfrac{1}{2}+\nu+m\right)}\frac{\Gamma\left(\nu-\dfrac{1}{2}+m-n\right)\Gamma\left(\dfrac{1}{2}-\nu+m-n\right)}{\Gamma\left(\dfrac{1}{2}-\nu-n\right)\Gamma\left(\dfrac{1}{2}+m-n\right)}$$
$$\left[\operatorname{Re}\nu>-\frac{1}{2};\quad n\geq m\right].$$

ET II 282(13)a

5. $$\int_{-1}^{1}(1-x)^{2\nu-1}(1+x)^{\nu-\frac{1}{2}}C_m^\nu(x)C_n^\nu(x)\,dx=$$
$$=\frac{2^{3\nu-\frac{1}{2}}\Gamma\left(\nu+\dfrac{1}{2}\right)\Gamma(2\nu+m)\Gamma(2\nu+n)}{m!n!\Gamma(2\nu)\Gamma\left(\dfrac{1}{2}-\nu\right)}\frac{\Gamma\left(\nu+\dfrac{1}{2}+m+n\right)\Gamma\left(\dfrac{1}{2}-\nu+n-m\right)}{\Gamma\left(\nu+\frac{1}{2}+n-m\right)\Gamma\left(3\nu+\frac{1}{2}+m+n\right)}$$
$$[\operatorname{Re}\nu>0].$$

ET II 282(14)

6. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{3\nu+m+n-\frac{3}{2}}C_m^\nu(x)C_n^\nu(x)\,dx=$$
$$=\frac{2^{4\nu+m+n-1}\left[\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+m+n\right)\right]^2}{\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+n+\frac{1}{2}\right)\Gamma(2\nu+m)}\frac{\Gamma\left(\nu+m+n+\frac{1}{2}\right)\Gamma\left(3\nu+m+n-\frac{1}{2}\right)}{\Gamma(2\nu+n)\Gamma(4\nu+2m+2n)}$$
$$\left[\operatorname{Re}\nu>\frac{1}{6}\right].$$

7. $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\nu-\frac{1}{2}}C_m^{\mu}(x)C_n^{\nu}(x)\,dx =$$
$$= \frac{2^{\alpha+\nu+\frac{1}{2}}\Gamma(\alpha+1)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n-\frac{1}{2}\right)}{m!n!\Gamma\left(\nu-\alpha-\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n+\frac{3}{2}\right)}\frac{\Gamma(2\mu+m)\Gamma(2\nu+n)}{\Gamma(2\mu)\Gamma(2\nu)}\times$$
$$\times\ {}_4F_3\left(-m,\ m+2\mu,\ \alpha+1,\ \alpha-\nu+\frac{3}{2};\ \mu+\frac{1}{2},\ \nu+\alpha+n+\frac{3}{2},\ \alpha-\nu-n+\frac{3}{2};\ 1\right)$$
$$\left[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 283(16)

7.315

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}\nu-1}C_{2n}^{\nu}(ax)\,dx = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)}C_n^{\frac{1}{2}\nu}(2a^2-1) \qquad [\operatorname{Re}\nu > 0].$$

ET II 283(19)

7.316

$$\int_{-1}^{1}(1-x^2)^{\nu-1}C_n^{\nu}(\cos\alpha\,\cos\beta+x\,\sin\alpha\,\sin\beta)\,dx = \frac{2^{2\nu-1}n![\Gamma(\nu)]^2}{\Gamma(2\nu+n)}C_n^{\nu}(\cos\alpha)C_n^{\nu}(\cos\beta)$$
$$[\operatorname{Re}\nu > 0].$$

ET II 283(20)

7.317

1. $$\int_{0}^{1}(1-x)^{\mu-1}x^{\lambda-\frac{1}{2}}C_n^{\lambda}(1-\gamma x)\,dx = \frac{\Gamma(2\lambda+n)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(\mu)}{\Gamma(2\lambda)\Gamma\left(\lambda+\mu+n+\frac{1}{2}\right)}P_n^{(\alpha,\,\beta)}(1-\gamma)$$
$$\alpha=\lambda+\mu-\frac{1}{2} \qquad \beta=\lambda-\mu-\frac{1}{2} \qquad \left[\operatorname{Re}\lambda > -1, \quad \lambda\neq 0, \quad -\frac{1}{2}, \quad \operatorname{Re}\mu > 0\right].$$

ET II 190(39)a

2. $$\int_{0}^{1}(1-x)^{\mu-1}x^{\nu-1}C_n^{\lambda}(1-\gamma x)\,dx = \frac{\Gamma(2\lambda+n)\Gamma(\mu)\Gamma(\nu)}{n!\Gamma(2\lambda)\Gamma(\mu+\nu)}\times$$
$$\times\ {}_3F_2\left(-n,\ n+2\lambda,\ \nu;\ \lambda+\frac{1}{2},\ \mu+\nu;\ \frac{\gamma}{2}\right)$$
$$[2\lambda\neq 0,\ -1,\ -2,\ldots, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

$$\int_0^1 x^{2\nu}(1-x^2)^{\sigma-1} C_n^\nu(1-x^2y)\,dx = \frac{\Gamma(2\nu+n)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\sigma)}{2\Gamma(2\nu)\Gamma\left(n+\nu+\sigma+\frac{1}{2}\right)} P_n^{(\alpha,\,\beta)}(1-y),$$

$$\alpha = \nu+\sigma-\frac{1}{2}, \qquad \beta = \nu-\sigma-\frac{1}{2} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\sigma > 0\right].$$

ET II 283(21)

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{\frac{1}{2}})\,dx = (-1)^n \frac{\Gamma(\lambda+n)\Gamma(\mu)\Gamma(\nu)}{n!\Gamma(\lambda)\Gamma(\mu+\nu)}\ {}_3F_2\left(-n,\, n+\lambda,\, \nu;\ \frac{1}{2},\, \mu+\nu;\ \gamma^2\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{\frac{1}{2}})\,dx = \frac{(-1)^n 2\gamma\Gamma(\mu)\Gamma(\lambda+n+1)\Gamma\left(\nu+\frac{1}{2}\right)}{n!\Gamma(\lambda)\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times$$
$$\times {}_3F_2\left(-n,\, n+\lambda+1,\, \nu+\frac{1}{2};\ \frac{3}{2},\, \mu+\nu+\frac{1}{2};\ \gamma^2\right)$$
$$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and some elementary functions

7.321

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x)\,dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n!\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 281(7), MO 99a

7.322

$$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx}\,dx = (-1)^n \frac{\pi\Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab)$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi = 0 \qquad [n = 1,\ 2,\ 3, \dots];$$
$$= 2^{-2\nu}\pi\Gamma(2\nu+1)[\Gamma(1+\nu)]^{-2} \qquad [n = 0].$$

EH I 177(18)

2. $$\int_0^{\pi} C_n^{\nu}(\cos\psi\ \cos\psi' + sin\,\psi\ \sin\psi'\ \cos\varphi)(\sin\varphi)^{2\nu-1}\,d\varphi =$$
$$= 2^{2\nu-1}n![\Gamma(\nu)]^2 C_n^{\nu}(\cos\psi)C_n^{\nu}(\cos\psi')[\Gamma(2\nu+n)]^{-1} \qquad [\operatorname{Re}\nu > 0].$$

EH I 177(20)

7.324

1. $$\int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n+1}^{\nu}(x)\,\sin ax\,dx = (-1)^n\pi\frac{\Gamma(2n+2\nu+1)J_{2n+\nu+1}(a)}{(2n+1)!\Gamma(\nu)(2a)^{\nu}}$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0\right].$$

ET I 94(4)

2. $$\int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n}^{\nu}(x)\,\cos ax\,dx = \frac{(-1)^n\pi\Gamma(2n+2\nu)J_{\nu+2n}(a)}{(2n)!\Gamma(\nu)(2a)^{\nu}} \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0\right].$$

ET I 38(3)a

7.33 Combinations of the polynomials $C_n^{\nu}(x)$ and Bessel functions. Integration of Gegenbauer functions with respect to the index

7.331

1. $$\int_1^{\infty} x^{2n+1-\nu}(x^2-1)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n}\left(\frac{1}{x}\right) J_{\nu}(xy)\,dx =$$
$$= (-1)^n 2^{2n-\nu+1} y^{-\nu+2n-1}[(2n)!]^{-1}\Gamma(2\nu-2n)[\Gamma(\nu-2n)]^{-1}\ \cos y$$
$$\left[y > 0, \quad 2n - \frac{1}{2} < \operatorname{Re}\nu < 2n + \frac{1}{2}\right].$$

ET II 44(10)a

2. $$\int_1^{\infty} x^{2n-\nu+2}(x^2-1)^{\nu-2n-\frac{3}{2}} C_{2n+1}^{\nu-2n-1}\left(\frac{1}{x}\right) J_{\nu}(xy)\,dx =$$
$$= (-1)^n 2^{2n-\nu+2} y^{-\nu+2n}\Gamma(2\nu-2n-1)[(2n+1)!\Gamma(\nu-2n-1)]^{-1}\ \sin y$$
$$\left[y > 0, \quad 2n + \frac{1}{2} < \operatorname{Re}\nu < 2n + \frac{3}{2}\right].$$

1. $$\int_0^\infty x^{\nu+1}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}}[(x^2+\beta^2)^{-\frac{1}{2}}\beta] J_{\nu+\frac{3}{2}+2n}[(x^2+\beta^2)^{\frac{1}{2}}a] J_\nu(xy)\,dx =$$
$$= (-1)^n 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^\nu (a^2-y^2)^{-\frac{1}{2}} \sin[\beta(a^2-y^2)^{\frac{1}{2}}] C_{2n+1}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right] \qquad [0<y<a];$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1].$$

ET II 59(23)

2. $$\int_0^\infty x^{\nu+1}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}}[\beta(x^2+\beta^2)^{-\frac{1}{2}}] J_{\nu+\frac{1}{2}+2n}[(x^2+\beta^2)^{\frac{1}{2}}a] J_\nu(xy)\,dx =$$
$$= (-1)^n 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^\nu (a^2-y^2)^{-\frac{1}{2}} \cos[\beta(a^2-y^2)^{\frac{1}{2}}] C_{2n}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right] \qquad [0<y<a];$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1].$$

ET II 59(24)

7.333

1. $$\int_0^\pi (\sin x)^{\nu+1} \cos(a\cos\theta\cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a\sin\theta\sin x)\,dx =$$
$$= (-1)^{\frac{n}{2}} \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} (\sin\theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos\theta) J_{\nu+\frac{1}{2}+n}(a) \qquad [n=0,\,2,\,4,\dots];$$
$$= 0 \qquad [n=1,\,3,\,5,\dots] \qquad [\operatorname{Re}\nu>-1].$$

WA 414(2)a

2. $$\int_0^\pi (\sin x)^{\nu+1} \sin(a\cos\theta\cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a\sin\theta\sin x)\,dx =$$
$$= 0 \qquad [n=0,\,2,\,4,\dots];$$
$$= (-1)^{\frac{n-1}{2}} \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} (\sin\theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos\theta) J_{\nu+\frac{1}{2}+n}(a) \qquad [n=1,\,3,\,5,\dots] \qquad [\operatorname{Re}\nu>-1].$$

WA 414(3)a

7.334

1. $$\int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{J_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\Gamma(2\nu+n)}{2^{\nu-1}n!\Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{J_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = (\alpha^2+\beta^2-2\alpha\beta\cos x)^{\frac{1}{2}} \qquad \left[n=0,\,1,\,2,\dots; \quad \operatorname{Re}\nu>-\frac{1}{2}\right].$$

ET II 362(29)

ET II 362(29)

2. $$\int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{N_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\Gamma(2\nu+n)}{2^{\nu-1}n!\Gamma(\nu)}\,\frac{J_{\nu+n}(\alpha)}{\alpha^\nu}\,\frac{N_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = (\alpha^2+\beta^2-2\alpha\beta\,\cos x)^{\frac{1}{2}} \qquad \left[|\alpha|<|\beta|, \quad \operatorname{Re}\nu-\frac{1}{2}\right].$$

ET II 362(30)

Integration of gegenbauer functions with respect to the index

7.335

$$\int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1} t^\alpha C_\alpha^\nu(z)\,d\alpha = -2i(1+2tz+t^2)^{-\nu}$$
$$[-2<\operatorname{Re}\nu<c<0, \quad |\arg(z\pm 1)|<\pi].$$

EH I 178(25)

7.336

$$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x)\left(\nu-\frac{1}{2}+ix\right) K_{\nu-\frac{1}{2}+ix}(a) I_{\nu-\frac{1}{2}+ix}(b) C^\nu_{-\frac{1}{2}+ix}(-\cos\varphi)\,dx = \frac{2^{-\nu+1}(ab)^\nu}{\Gamma(\nu)}\omega^{-\nu}K_\nu(\omega),$$
$$\omega=\sqrt{a^2+b^2-2ab\,\cos\varphi}.$$

EH II 55(45)

7.34 Combinations of Chebyshev polynomials and powers

7.341

$$\int_{-1}^1 [T_n(x)]^2\,dx = 1-(4n^2-1)^{-1}.$$

ET II 271(6)

7.342

$$\int_{-1}^1 U_n[x(1-y^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}+yz]\,dx = \frac{2}{n+1}U_n(y)U_n(z) \qquad [|y|<1, \quad |z|<1].$$

ET II 275(34)

7.343

1. $$\int_{-1}^1 T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = 0 \qquad [m\neq n];$$
$$= \frac{\pi}{2} \qquad [m=n\neq 0];$$
$$= \pi \qquad [m=n=0].$$

$$2.\quad \int_{-1}^{1} \sqrt{1-x^2}\, U_n(x) U_m(x)\, dx = 0 \qquad [m \neq n];$$
$$= \frac{\pi}{2} \qquad [m = n].$$

ET II 274(28)
ET II 274(27), MO 105a

7.344

$$1.\quad \int_{-1}^{1} (y-x)^{-1} (1-y^2)^{-\frac{1}{2}} T_n(y)\, dy = \pi U_{n-1}(x) \qquad [n = 1, 2, \dots].$$

EH II 187(47)

$$2.\quad \int_{-1}^{1} (y-x)^{-1} (1-y^2)^{\frac{1}{2}} U_{n-1}(y)\, dy = -\pi T_n(x) \qquad [n = 1, 2, \dots].$$

EH II 187(48)

7.345

$$1.\quad \int_{-1}^{1} (1-x)^{-\frac{1}{2}} (1+x)^{m-n-\frac{3}{2}} T_m(x) T_n(x)\, dx = 0 \qquad [m > n].$$

ET II 272(10)

$$2.\quad \int_{-1}^{1} (1-x)^{-\frac{1}{2}} (1+x)^{m+n-\frac{3}{2}} T_m(x) T_n(x)\, dx = \frac{\pi (2m+2n-2)!}{2^{m+n} (2m-1)!(2n-1)!} \qquad [m+n \neq 0].$$

ET II 272(11)

$$3.\quad \int_{-1}^{1} (1-x)^{\frac{1}{2}} (1+x)^{m+n+\frac{3}{2}} U_m(x) U_n(x)\, dx = \frac{\pi (2m+2n+2)!}{2^{m+n+2} (2m+1)!(2n+1)!}.$$

ET II 274(31)

$$4.\quad \int_{-1}^{1} (1-x)^{\frac{1}{2}} (1+x)^{m-n-\frac{1}{2}} U_m(x) U_n(x)\, dx = 0 \qquad [m > n].$$

5. $$\int_{-1}^{1}(1-x)(1+x)^{\frac{1}{2}}U_m(x)U_n(x)\,dx = \frac{2^{\frac{5}{2}}(m+1)(n+1)}{\left(m+n+\frac{3}{2}\right)\left(m+n+\frac{5}{2}\right)\left[1-4(m-n)^2\right]}.$$

ET II 274(29)

6. $$\int_{-1}^{1}(1+x)^{-\frac{1}{2}}(1-x)^{\alpha-1}T_m(x)T_n(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}2^{\alpha-\frac{1}{2}}\Gamma(\alpha)\Gamma\left(n-\alpha+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\alpha+n+\frac{1}{2}\right)}\,{}_4F_3\left(-m,\, m,\, \alpha,\, \alpha+\frac{1}{2};\, \frac{1}{2},\, \alpha+n+\frac{1}{2},\, \alpha-n+\frac{1}{2};\, 1\right)$$
$$[\operatorname{Re}\alpha > 0].$$

ET II 272(12)

7. $$\int_{-1}^{1}(1+x)^{\frac{1}{2}}(1-x)^{\alpha-1}U_m(x)U_n(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}2^{\alpha-\frac{1}{2}}(m+1)(n+1)\Gamma(\alpha)\Gamma\left(n-\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\alpha\right)\Gamma\left(\frac{3}{2}+\alpha+n\right)} \times$$
$$\times\, {}_4F_3\left(-m,\, m+2,\, \alpha,\, \alpha-\frac{1}{2};\, \frac{3}{2},\, \alpha+n+\frac{3}{2},\, \alpha-n-\frac{1}{2};\, 1\right) \qquad [\operatorname{Re}\alpha > 0].$$

ET II 275(32)

841

7.346

$$\int_0^1 x^{s-1}T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s2^s\mathrm{B}\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n,\ \frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)} \qquad [\operatorname{Re}s > 0].$$

ET II 324(2)

7.347

1. $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}T_n(x)\,dx = \frac{2^{\alpha+\beta+2n+1}(n!)^2\Gamma(\alpha+1)\Gamma(\beta+1)}{(2n)!\Gamma(\alpha+\beta+2)} \times$$
$$\times\ {}_3F_2\left(-n,\, n,\, \alpha+1;\, \frac{1}{2},\, \alpha+\beta+2;\, 1\right)$$
$$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1].$$

ET II 271(2)

ET II 273(22)

7.348

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} U_{2n}(xz)\, dx = \pi P_n(2z^2-1) \qquad [|z|<1].$$

ET II 275(33)

7.349

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(1-x^2y)\, dx = \frac{1}{2}\pi[P_n(1-y)+P_{n-1}(1-y)].$$

ET II 222(14)

7.35 Combinations of Chebyshev polynomials and some elementary functions

7.351

$$\int_{0}^{1} x^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} e^{-\frac{2a}{x}} T_n(x)\, dx = \pi^{\frac{1}{2}} D_{n-\frac{1}{2}}(2a^{\frac{1}{2}}) D_{-n-\frac{1}{2}}(2a^{\frac{1}{2}}) \qquad [\operatorname{Re} a>0].$$

ET II 272(13)

7.352

1. $$\int_0^\infty \frac{xU_n[a(a^2+x^2)^{-\frac{1}{2}}]}{(a^2+x^2)^{\frac{1}{2}n+1}(e^{\pi x}+1)}\, dx = \frac{a^{-n}}{2n} - 2^{-n-1}\zeta\left(n+1,\, \frac{a+1}{2}\right) \qquad [\operatorname{Re} a>0].$$

ET II 275(39)

2. $$\int_0^\infty \frac{xU_n[a(a^2+x^2)^{-\frac{1}{2}}]}{(a^2+x^2)^{\frac{1}{2}n+1}(e^{2\pi x}-1)}\, dx = \frac{1}{2}\zeta(n+1,\, a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n} \qquad [\operatorname{Re} a>0].$$

ET II 276(40)

7.353

1. $$\int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} \operatorname{sech}\left(\frac{1}{2}\pi x\right) T_n[a(a^2+x^2)^{-\frac{1}{2}}]\, dx$$
$$= 2^{1-2n}\left[\zeta\left(n,\, \frac{a+1}{4}\right) - \zeta\left(n,\, \frac{a+3}{4}\right)\right] =$$
$$= 2^{1-n}\Phi\left(-1,\, n,\, \frac{a+1}{2}\right) \qquad [\operatorname{Re} a>0].$$

2. $$\int_0^\infty (a^2+x^2)^{-\frac{1}{2}n}\left[\operatorname{ch}\left(\frac{1}{2}\pi x\right)\right]^{-2} T_n[a(a^2+x^2)^{-\frac{1}{2}}]\,dx = \pi^{-1} n 2^{1-n}\zeta\left(n+1,\ \frac{a+1}{2}\right) \qquad [\operatorname{Re} a > 0].$$

ET II 273(20)

7.354

1. $$\int_{-1}^1 \sin(xyz)\ \cos[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z]T_{2n+1}(x)\,dx = (-1)^n\pi T_{2n+1}(y)J_{2n+1}(x).$$

ET II 271(4)

2. $$\int_{-1}^1 \sin(xyz)\ \sin[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z]U_{2n+1}(x)\,dx = (-1)^n\pi(1-y^2)^{\frac{1}{2}}U_{2n+1}(y)J_{2n+2}(z).$$

ET II 274(25)

3. $$\int_{-1}^1 \cos(xyz)\ \cos[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z]T_{2n}(x)\,dx = (-1)^n\pi T_{2n}(y)J_{2n}(z).$$

ET II 271(5)

4. $$\int_{-1}^1 \cos(xyz)\ \sin[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z]U_{2n}(x)\,dx = (-1)^n\pi(1-y^2)^{\frac{1}{2}}U_{2n}(y)J_{2n+1}(z).$$

ET II 274(24)

7.355

1. $$\int_0^1 T_{2n+1}(x)\ \sin ax\frac{dx}{\sqrt{1-x^2}} = (-1)^n\frac{\pi}{2}J_{2n+1}(a) \qquad [a > 0].$$

ET I 94(3)a

2. $$\int_0^1 T_{2n}(x)\ \cos ax\frac{dx}{\sqrt{1-x^2}} = (-1)^n\frac{\pi}{2}J_{2n}(a) \qquad [a > 0].$$

ET I 38(2)a

7.36 Combinations of Chebyshev polynomials and Bessel functions

7.361

$$\int_0^1 (1-x^2)^{-\frac{1}{2}}T_n(x)J_\nu(xy)\,dx = \frac{1}{2}\pi J_{\frac{1}{2}(\nu+n)}\left(\frac{1}{2}y\right)J_{\frac{1}{2}(\nu-n)}\left(\frac{1}{2}y\right) \qquad [y > 0, \quad \operatorname{Re}\nu > -n-1].$$

7.362

$$\int_1^\infty (x^2-1)^{-\frac{1}{2}} T_n\left(\frac{1}{x}\right) K_{2\mu}(ax)\,dx = \frac{\pi}{2a} W_{\frac{1}{2}n,\,\mu}(a) W_{-\frac{1}{2}n,\,\mu}(a) \qquad [\operatorname{Re} a > 0].$$

ET II 366(17)a

7.37- 7.38 Hermite polynomials

7.371

$$\int_0^x H_n(y)\,dy = [2(n+1)]^{-1}[H_{n+1}(x) - H_{n+1}(0)].$$

EH II 194(27)

7.372

$$\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} H_{2n}(\sqrt{x}t)\,dx = \frac{(-1)^n \pi^{\frac{1}{2}} (2n)! \Gamma\left(\alpha+\frac{1}{2}\right) L_n^\alpha(x)}{\Gamma(n+\alpha+1)} \qquad \left[\operatorname{Re} a > -\frac{1}{2}\right].$$

EH II 195(34)

7.373

1. $$\int_0^x e^{-y^2} H_n(y)\,dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x). \qquad (\text{see} \quad 8.956).$$

8.956

EH II 194(26)

2. $$\int_{-\infty}^\infty e^{-x^2} H_{2m}(xy)\,dx = \sqrt{\pi}\frac{(2m)!}{m!}(y^2-1)^m.$$

EH II 195(28)

7.374

1. $$\int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x)\,dx = 0 \qquad [m \neq n];$$
$$= 2^n \cdot n! \sqrt{\pi} \qquad [m = n].$$

2. $$\int_{-\infty}^{\infty} e^{-2x^2} H_m(x) H_n(x)\, dx = (-1)^{\frac{1}{2}(m+n)} 2^{\frac{m+n-1}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \qquad [m+n \text{ is even}].$$

ET II 289(10)a

3. $$\int_{-\infty}^{\infty} e^{-x^2} H_m(ax) H_n(x)\, dx = 0 \qquad [m < n].$$

ET II 290(20)a

4. $$\int_{-\infty}^{\infty} e^{-x^2} H_{2m+n}(ax) H_n(x)\, dx = \sqrt{\pi} 2^n \frac{(2m+n)!}{m!} (a^2-1)^m a^n.$$

ET II 291(21)a

5. $$\int_{-\infty}^{\infty} e^{-2\alpha^2 x^2} H_m(x) H_n(x)\, dx = 2^{\frac{m+n-1}{2}} \alpha^{-m-n-1} (1-2\alpha^2)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \times$$
$$\times\ {}_2F_1\left(-m,\ -n;\ \frac{1-m-n}{2};\ \frac{\alpha^2}{2\alpha^2-1}\right)$$
$$[\operatorname{Re}\alpha^2 > 0, \quad \alpha^2 \neq 1/2, \quad m+n \text{ is even}].$$

ET II 289(12)a

6. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(x)\, dx = \pi^{\frac{1}{2}} y^n 2^n.$$

ET II 288(2)A, EH II 195(31)

7. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x)\, dx = 2^n \pi^{\frac{1}{2}} m! y^{n-m} L_m^{n-m}(-2y^2) \qquad [m \le n].$$

BU 148(15), ET II 289(13)a

8. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x)\, dx = \pi^{\frac{1}{2}} (1-\alpha^2)^{\frac{n}{2}} H_n\left[\frac{\alpha y}{(1-\alpha^2)^{\frac{1}{2}}}\right].$$

ET II 290(17)a

9. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x)\, dx =$$
$$= \pi^{\frac{1}{2}} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} (1-\alpha^2)^{\frac{m+n}{2}-k} H_{m+n-2k}\left[\frac{\alpha y}{(1-\alpha^2)^{\frac{1}{2}}}\right].$$

10. $$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x)\,dx = (2\pi u)^{\frac{1}{2}}(1-2u)^{\frac{n}{2}} H_n[y(1-2u)^{-\frac{1}{2}}] \qquad \left[0 \le u < \frac{1}{2}\right].$$

EH II 195(30)

7.375

1. $$\int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x)\,dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)} \Gamma(s-k)\Gamma(s-m)\Gamma(s-n)$$
$$2s = k+m+n+1 \qquad [k+m+n \text{ is even}].$$

ET II 290(14)a

2. $$\int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x)\,dx = \frac{2^{\frac{m+n+k}{2}} \pi^{\frac{1}{2}} k!m!n!}{(s-k)!(s-m)!(s-n)!},$$
$$2s = m+n+k \qquad [k+m+n \text{ is even}].$$

ET II 290(15)a

7.376

1. $$\int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x)\,dx = (2\pi)^{\frac{1}{2}} e^{-\frac{y^2}{2}} H_n(y) i^n.$$

MO 165a

2. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n}(x)\,dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}(\nu+1)}} F\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1].$$

BU 150(18a)

3. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n+1}(x)\,dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}\nu+1}} F\left(-n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -2].$$

BU 150(18b)

7.377

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z)\,dx = 2^n \pi^{\frac{1}{2}} m! z^{n-m} L_m^{n-m}(-2yx) \qquad [m \le n].$$

$$\int_0^\infty x^{\alpha-1}e^{-\beta x}H_n(x)\,dx = 2^n\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{n!\Gamma(\alpha+n-2m)}{m!(n-2m)!}(-1)^m 2^{-2m}\beta^{2m-\alpha-n}$$
$$[\operatorname{Re}\alpha>0,\quad \text{if } n \text{ is even};\quad \operatorname{Re}\alpha>-1,\quad \text{if } n \text{ is odd};\quad \operatorname{Re}\beta>0].$$

ET I 172(11)a

7.379

1. $$\int_{-\infty}^\infty xe^{-x^2}H_{2m+1}(xy)\,dx = \pi^{\frac{1}{2}}\frac{(2m+1)!}{m!}y(y^2-1)^m.$$

EH II 195(28)

2. $$\int_{-\infty}^\infty x^n e^{-x^2}H_n(xy)\,dx = \pi^{\frac{1}{2}}n!P_n(y).$$

EH II 195(29)

7.381

$$\int_{-\infty}^\infty (x\pm ic)^\nu e^{-x^2}H_n(x)\,dx = 2^{n-1-\nu}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma(-\nu)}\exp\left[\pm\frac{1}{2}\pi(\nu+n)i\right]\qquad [c>0].$$

ET II 288(3)a

845

7.382

$$\int_0^\infty x^{-1}(x^2+a^2)^{-1}e^{-x^2}H_{2n+1}(x)\,dx = (-2)^n(\pi)^{\frac{1}{2}}a^{-2}[2^\nu n!-(2n+1)!e^{\frac{1}{2}a^2}D_{-2n-2}(a\sqrt{2})].$$

ET II 288(4)a

7.383

1. $$\int_0^\infty e^{-xp}H_{2n+1}(\sqrt{x})\,dx = (-1)^n 2^n(2n+1)!!\pi^{\frac{1}{2}}(p-1)^n p^{-n-\frac{3}{2}}\qquad [\operatorname{Re}p>0].$$

EF 151(261)A, ET I 172(12)a

2. $$\int_0^\infty e^{-(b-\beta x)}H_{2n+1}(\sqrt{(\alpha-\beta)x})\,dx = (-1)^n\sqrt{\pi}\sqrt{\alpha-\beta}\frac{(2n+1)!}{n!}\frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}}$$
$$[\operatorname{Re}(b-\beta)>0].$$

3. $$\int_0^\infty \frac{1}{\sqrt{x}} e^{-(b-\beta)x} H_{2n}(\sqrt{(\alpha-\beta)x})\,dx = (-1)^n \sqrt{\pi} \frac{(2n)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}} \qquad [\operatorname{Re}(b-\beta) > 0].$$

ET I 172(16)a

4. $$\int_0^\infty x^{a-\frac{1}{2}n-1} e^{-bx} H_n(\sqrt{x})\,dx = 2^n \Gamma(a) b^{-a}\, {}_2F_1\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1-a; b\right)$$

$$\left[\operatorname{Re} a > \frac{1}{2}n,\ \text{if } n \text{ is even; } \operatorname{Re} a > \frac{1}{2}n - \frac{1}{2},\ \text{if } n \text{ is odd; } \operatorname{Re} b > 0.\right.$$

$$\left.\text{If } a \text{ is even, only the first } 1 + \binom{n}{2} \text{ terms are kept in the series for } {}_2F_1\right].$$

ET I 172(14)a

5. $$\int_0^\infty x^{-\frac{1}{2}} e^{-px} H_{2n}(\sqrt{x})\,dx = (-1)^n 2^n (2n-1)!!\pi^{\frac{1}{2}} (p-1)^n p^{-n-\frac{1}{2}}.$$

MO 177a

7.384

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-bx} \left[H_n\left(\frac{\alpha+\sqrt{x}}{\lambda}\right) + H_n\left(\frac{a-\sqrt{x}}{\lambda}\right)\right] dx = \sqrt{\frac{2\pi}{b}} (1-\lambda^{-2}b^{-1})^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\lambda^2-\frac{1}{b}}}\right)$$

$$[\operatorname{Re} b > 0].$$

ET I 173(17)a

7.385

1. $$\int_0^\infty \frac{e^{-bx}}{\sqrt{e^x-1}} H_{2n}[\sqrt{s(1-e^{-x})}]\,dx = (-1)^n 2^{2n} \sqrt{\pi} \frac{(2n)!\Gamma\left(b+\frac{1}{2}\right)}{\Gamma(n+b+1)} L_n^n(s) \qquad \left[\operatorname{Re} b > -\frac{1}{2}\right].$$

ET I 174(23)a

2. $$\int_0^\infty e^{-bx} H_{2n+1}[\sqrt{s}\sqrt{1-e^{-x}}]\,dx = (-1)^n 2^{2n} \sqrt{\pi s} \frac{(2n+1)!\Gamma(b)}{\Gamma\left(n+b+\frac{3}{2}\right)} L_n^b(s) \qquad [\operatorname{Re} b > 0].$$

ET I 174(24)a

7.386

$$\int_0^\infty x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}} H_n\left(\frac{q}{2\sqrt{x}}\right) e^{-px}\,dx = 2^n \pi^{\frac{1}{2}} p^{\frac{n-1}{2}} e^{-q\sqrt{p}}.$$

EF 129(117)

7.387

1. $$\int_0^\infty e^{-x^2}\ \mathrm{sh}(\sqrt{2}\beta x) H_{2n+1}(x)\,dx = 2^{n-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{2n+1} e^{\frac{1}{2}\beta^2}.$$

ET II 289(7)a

2. $$\int_0^\infty e^{-x^2}\ \mathrm{ch}(\sqrt{2}\beta x) H_{2n}(x)\,dx = 2^{n-1} \pi^{\frac{1}{2}} \beta^{2n} e^{\frac{1}{2}\beta^2}.$$

ET II 289(8)a

7.388

1. $$\int_0^\infty e^{-x^2}\ \sin(\sqrt{2}\beta x) H_{2n+1}(x)\,dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{2n+1} e^{-\frac{1}{2}\beta^2}.$$

ET II 288(5)a

2. $$\int_0^\infty e^{-x^2}\ \sin(\sqrt{2}\beta x) H_{2n+1}(ax)\,dx = (-1)^n 2^{-1} \pi^{\frac{1}{2}} (a^2-1)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1}\left(\frac{a\beta}{\sqrt{2}(a^2-1)^{\frac{1}{2}}}\right).$$

ET II 290(18)a

3. $$\int_0^\infty e^{-x^2}\ \cos(\sqrt{2}\beta x) H_{2n}(x)\,dx = (-1)^n 2^{n-1} \pi^{\frac{1}{2}} \beta^{2n} e^{-\frac{1}{2}\beta^2}.$$

ET II 289(6)a

4. $$\int_0^\infty e^{-x^2}\ \cos(\sqrt{2}\beta x) H_{2n}(ax)\,dx = 2^{-1} \pi^{\frac{1}{2}} (1-a^2)^n e^{-\frac{1}{2}\beta^2} H_{2n}\left[\frac{a\beta}{\sqrt{2}(a^2-1)^{\frac{1}{2}}}\right].$$

ET II 290(19)a

5. $$\int_0^\infty e^{-y^2} [H_n(y)]^2\ \cos(\sqrt{2}\beta y)\,dy = \pi^{\frac{1}{2}} 2^{n-1} n! e^{-\frac{\beta^2}{2}} L_n(\beta^2).$$

6. $$\int_0^\infty e^{-x^2} \sin(bx) H_n(x) H_{n+2m+1}(x)\,dx = 2^n(-1)^m\sqrt{\frac{\pi}{2}}n!b^{2m+1}e^{-\frac{b^2}{4}}L_n^{2m+1}\left(\frac{b^2}{2}\right) \qquad [b>0].$$

ET I 39(11)a

7. $$\int_0^\infty e^{-x^2} \cos(bx) H_n(x) H_{n+2m}(x)\,dx = 2^{n-\frac{1}{2}}\sqrt{\frac{\pi}{2}}n!(-1)^m b^{2m}e^{-\frac{b^2}{4}}L_n^{2m}\left(\frac{b^2}{2}\right) \qquad [b>0].$$

ET I 39(11)a

7.389

$$\int_0^\pi (\cos x)^n H_{2n}[a(1-\sec x)^{\frac{1}{2}}]\,dx = 2^{-n}(-1)^n\pi\frac{(2n)!}{(n!)^2}[H_n(a)]^2.$$

ET II 292(31)

7.39 Jacobi polynomials

7.391

1. $$\int_{-1}^1 (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\,\beta)}(x)P_m^{(\alpha,\,\beta)}(x)\,dx =$$
$$= 0 \qquad [m\neq n, \quad \operatorname{Re}\alpha>-1, \quad \operatorname{Re}\beta>-1];$$
$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+1+2n)\Gamma(\alpha+\beta+n+1)} \qquad [m=n, \quad \operatorname{Re}\alpha>-1, \quad \operatorname{Re}\beta>-1].$$

ET II 285(5, 9)

847

2. $$\int_{-1}^1 (1-x)^\varrho(1+x)^\sigma P_n^{(\alpha,\,\beta)}(x)\,dx = \frac{2^{\varrho+\sigma+1}\Gamma(\varrho+1)\Gamma(\sigma+1)\Gamma(n+1+\alpha)}{n!\Gamma(\varrho+\sigma+2)\Gamma(1+\alpha)}\times$$
$$\times\ {}_3F_2(-n,\ \alpha+\beta+n+1,\ \varrho+1;\ \alpha+1,\ \varrho+\sigma+2;\ 1)$$
$$[\operatorname{Re}\varrho>-1, \quad \operatorname{Re}\sigma>-1].$$

ET II 284(3)

3.[6] $$\int_{-1}^1 (1-x)^\alpha(1+x)^\sigma P_n^{(\alpha,\,\beta)}(x)\,dx = \frac{2^{\alpha+\sigma+1}\Gamma(\sigma+1)\Gamma(\alpha+1)\Gamma(\sigma-\beta+1)}{n!\Gamma(\sigma-\beta-n+1)\Gamma(\alpha+\sigma+n+2)}$$
$$[\operatorname{Re}\alpha>-1, \quad \operatorname{Re}\sigma>-1].$$

4. $$\int_{-1}^{1}(1-x)^{\varrho}(1+x)^{\beta}P_n^{(\alpha,\,\beta)}(x)\,dx = \frac{2^{\beta+\varrho+1}\Gamma(\varrho+1)\Gamma(\beta+n+1)\Gamma(\alpha-\varrho+n)}{n!\Gamma(\alpha-\varrho)\Gamma(\beta+\varrho+n+2)}$$
$$[\operatorname{Re}\varrho > -1, \quad \operatorname{Re}\beta > -1].$$

ET II 284(2)

5. $$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta}[P_n^{(\alpha,\,\beta)}(x)]^2\,dx = \frac{2^{\alpha+\beta}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\alpha\Gamma(\alpha+\beta+n+1)}$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > -1].$$

ET II 285(6)

6. $$\int_{-1}^{1}(1-x)^{2\alpha}(1+x)^{\beta}[P_n^{(\alpha,\,\beta)}(x)]^2\,dx = \frac{2^{4\alpha+\beta+1}\Gamma\left(\alpha+\frac{1}{2}\right)[\Gamma(\alpha+n+1)]^2\Gamma(\beta+2n+1)}{\sqrt{\pi}(n!)^2\Gamma(\alpha+1)\Gamma(2\alpha+\beta+2n+2)}$$
$$\left[\operatorname{Re}\alpha > -\frac{1}{2}, \quad \operatorname{Re}\beta > -1\right].$$

ET II 285(7)

7. $$\int_{-1}^{1}(1-x)^{\varrho}(1+x)^{\beta}P_n^{(\alpha,\,\beta)}(x)P_n^{(\varrho,\,\beta)}(x)\,dx =$$
$$= \frac{2^{\varrho+\beta+1}\Gamma(\varrho+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+2n+1)}{n!\Gamma(\beta+\varrho+2n+2)\Gamma(\alpha+\beta+n+1)} \qquad [\operatorname{Re}\varrho > -1, \quad \operatorname{Re}\beta > -1].$$

ET II285(10)

8. $$\int_{-1}^{1}(1-x)^{\varrho-1}(1+x)^{\beta}P_n^{(\alpha,\,\beta)}(x)P_n^{(\varrho,\,\beta)}(x)\,dx = \frac{2^{\varrho+\beta}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\varrho)}{n!\Gamma(\alpha+1)\Gamma(\varrho+\beta+n+1)}$$
$$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\varrho > 0].$$

ET II 286(11)

9.[7] $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}P_n^{(\alpha,\,\beta)}(x)P_m^{(\alpha,\,\sigma)}(x)\,dx =$$
$$= \frac{2^{\alpha+\sigma+1}\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+m+n+1)\Gamma(\sigma+m+1)\Gamma(\sigma-\beta+1)}{m!(n-m)!\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\sigma+m+n+2)\Gamma(\sigma-\beta+m-n+1)}$$
$$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\sigma > -1].$$

ET II 286(12)

10.[6] $$\int_{-1}^{1}(1-x)^{\varrho}(1+x)^{\beta}P_n^{(\alpha,\,\beta)}(x)P_m^{(\varrho,\,\beta)}(x)\,dx =$$
$$= \frac{2^{\beta+\varrho+1}\Gamma(\alpha+\beta+m+n+1)\Gamma(\beta+n+1)\Gamma(\varrho+m+1)}{m!(n-m)!\Gamma(\alpha+\beta+n+1)\Gamma(\beta+\varrho+m+n+2)}\,\frac{\Gamma(\alpha-\varrho-m+n)}{\Gamma(\alpha-\varrho)}$$
$$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\varrho > -1].$$

11. $$\int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha,\,\beta)}(y)\,dy = \frac{1}{2n}[P_{n-1}^{(\alpha+1,\,\beta+1)}(0) - (1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\,\beta+1)}(x)].$$

EH II 173(38)

7.392

1. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} P_n^{(\alpha,\,\beta)}(1-\gamma x)\,dx = \\ = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)}\ {}_3F_2\left(-n,\ n+\alpha+\beta+1,\ \lambda;\ \alpha+1,\ \lambda+\mu;\ \frac{1}{2}\gamma\right)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

ET II 192(46)a

2. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} P_n^{(\alpha,\,\beta)}(\gamma x-1)\,dx = \\ = (-1)^n \frac{\Gamma(\beta+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\beta+1)\Gamma(\lambda+\mu)}\ {}_3F_2\left(-n,\ n+\alpha+\beta+1,\ \lambda;\ \beta+1,\ \lambda+\mu;\ \frac{1}{2}\gamma\right)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

ET II 192(47)a

3. $$\int_0^1 x^{\alpha}(1-x)^{\mu-1} P_n^{(\alpha,\,\beta)}(1-\gamma x)\,dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} P_n^{(\alpha+\mu,\,\beta-\mu)}(1-\gamma)$$
$$[\operatorname{Re} a > -1, \quad \operatorname{Re}\mu > 0].$$

ET II 191(43)a

4. $$\int_0^1 x^{\beta}(1-x)^{\mu-1} P_n^{(\alpha,\,\beta)}(\gamma x-1)\,dx = \frac{\Gamma(\beta+n+1)\Gamma(\mu)}{\Gamma(\beta+\mu+n+1)} P_n^{(\alpha-\mu,\,\beta+\mu)}(\gamma-1)$$
$$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\mu > 0].$$

ET II 191(44)a

7.393

1. $$\int_0^1 (1-x^2)^\nu\ \sin bx P_{2n+1}^{(\nu,\,\nu)}(x)\,dx = \frac{(-1)^n\sqrt{\pi}\Gamma(2n+\nu+2) J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu}(2n+1)!b^{\nu+\frac{1}{2}}}$$
$$[b > 0, \quad \operatorname{Re}\nu > -1].$$

ET I 94(5)

ET I 38(4)

7.41- 7.42 Laguerre polynomials

7.411

1. $$\int_0^t L_n(x)\,dx = L_n(t) - L_{n+1}(t)/(n+1).$$

MO 110

2. $$\int_0^t L_n^\alpha(x)\,dx = L_n^\alpha(t) - L_{n+1}^\alpha(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}.$$

EH II 189(16)a

849

3. $$\int_0^t L_{n-1}^{\alpha+1}(x)\,dx = -L_n^\alpha(t) + \binom{n+\alpha}{n}.$$

EH II 189(15)a

4. $$\int_0^t L_m(x)L_n(t-x)\,dx = L_{m+n}(t) - L_{m+n+1}(t).$$

EH II 191(31)

5. $$\sum_{k=0}^\infty \left[\int_0^t (L_k(x)/k!)\,dx\right]^2 = e^t - 1 \qquad [t \geq 0].$$

MO 110

7.412

1. $$\int_0^1 (1-x)^{\mu-1} x^\alpha L_n^\alpha(ax)\,dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} L_n^{\alpha+\mu}(a) \qquad [\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\mu > 0].$$

EH II 191(30)A, BU 129(14c)

2. $$\int_0^1 (1-x)^{\mu-1} x^{\lambda-1} L_n^\alpha(\beta x)\,dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)}\ {}_2F_2(-n,\ \lambda;\ \alpha+1,\ \lambda+\mu:\ \beta)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

7.413

$$\int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta[(1-x)y]\,dx = \frac{(m+n)!\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{m!n!\Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y)$$
$$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1].$$

ET II 293(7)

7.414

1. $$\int_0^\infty e^{-x} L_n^\alpha(x)\,dx = e^{-y}[L_n^\alpha(y) - L_{n-1}^\alpha(y)].$$

EH II 191(29)

2. $$\int_0^\infty e^{-bx} L_n(\lambda x) L_n(\mu x)\,dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n\left[\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right] \qquad [\operatorname{Re} b > 0].$$

ET I 175(34)

3. $$\int_0^\infty e^{-x\alpha} L_n^\alpha(x) L_m^\alpha(x)\,dx = 0 \qquad [m \neq n, \quad \operatorname{Re}\alpha > -1];$$
$$= \frac{\Gamma(\alpha+n+1)}{n!} \qquad [m = n, \quad \operatorname{Re}\alpha > 0].$$

BU 115(8), ET II 292(2)
BU 115(8), ET II 293(3)

4. $$\int_0^\infty e^{-bx} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha(\mu x)\,dx = \frac{\Gamma(m+n+\alpha+1)}{m!n!} \frac{(b-\lambda)^n(b-\mu)^m}{b^{m+n+\alpha+1}} \times$$
$$\times F\left[-m, -n; -m-n-\alpha, \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right]$$
$$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re} b > 0].$$

ET I 175(35)

4.* (1) $$\int_0^\infty e^{-x} x^{\alpha+1/2} L_n^\alpha(x) L_m^\alpha(x)\,dx = \frac{\Gamma(\alpha+n+1)^2\Gamma(\alpha+m+1)\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(m-\frac{1}{2}\right)}{n!m!\Gamma(\alpha+1)\Gamma\left(-\frac{1}{2}\right)} \times$$
$$\times\ {}_3F_2\left(-n, \alpha+\frac{3}{2}, \frac{3}{2}; \alpha+1, \frac{3}{2}-m; 1\right).$$

5. $$\int_0^\infty e^{-bx} L_n^a(x)\,dx = \sum_{m=0}^{n} \binom{a+m-1}{m} \frac{(b-1)^{n-m}}{b^{n-m+1}} \qquad [\operatorname{Re} b > 0].$$

6. $$\int_0^\infty e^{-bx} L_n(x)\,dx = (b-1)^n b^{-n-1} \qquad [\operatorname{Re} b > 0].$$

ET I 174(25)

7. $$\int_0^\infty e^{-st} t^\beta L_n^\alpha(t)\,dt = \frac{\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} s^{-\beta-1} F\left(-n,\ \beta+1;\ \alpha+1;\ \frac{1}{s}\right)$$
$$[\operatorname{Re}\beta > -1, \quad \operatorname{Re} s > 0].$$

BU 119(4b), EH II 191(133)

8. $$\int_0^\infty e^{-st} t^\alpha L_n^\alpha(t)\,dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n! s^{\alpha+n+1}} \qquad [\operatorname{Re}\alpha > -1, \quad \operatorname{Re} s > 0].$$

EH II 191(32), MO 176a

9. $$\int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x)\,dx = (-1)^{m+n}(\alpha+\beta)! \binom{\alpha+m}{n}\binom{\beta+n}{m} \qquad [\operatorname{Re}(\alpha+\beta) > -1].$$

ET II 293(4)

10.[6] $$\int_0^\infty e^{-bx} x^{2a} [L_n^a(x)]^2\,dx = \frac{2^{2a}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\pi(n!)^2 b^{2a+1}} \times$$
$$\times F\left(-n,\ a+\frac{1}{2};\ \frac{1}{2}-n;\ \left(1-\frac{2}{b}\right)^2\right)\Gamma(a+n+1)$$
$$\left[\operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re} b > 0\right].$$

ET I 174(30)

11. $$\int_0^\infty e^{-x} x^{\gamma-1} L_n^\mu(x)\,dx = \frac{\Gamma(\gamma)\Gamma(1+\mu+n-\gamma)}{n!\Gamma(1+\mu-\gamma)} \qquad [\operatorname{Re}\gamma > 0].$$

BU 120(4b)

12. $$\int_0^\infty e^{-x\left(s+\frac{a_1+a_2}{2}\right)} x^{\mu+\beta} L_k^\mu(a_1 x) L_k^\mu(a_2 x)\,dx =$$
$$= \frac{\Gamma(1+\mu+\beta)\Gamma(1+\mu+k)}{k!k!\Gamma(1+\mu)} \left\{ \frac{d^k}{dh^k} \left[\frac{F\left(\frac{1+\mu+\beta}{2},\ 1+\frac{\mu+\beta}{2};\ 1+\mu;\ \frac{A^2}{B^2}\right)}{(1-h)^{1+\mu} B^{1+\mu+\beta}} \right] \right\}_{h=0},$$
$$A^2 = \frac{4a_1 a_2 h}{(1-h)^2}; \qquad B = s + \frac{a_1+a_2}{2}\,\frac{1+h}{1-h}$$

13. $$\int_0^\infty e^{-x\left(s+\frac{a_1+a_2}{2}\right)} x^\mu L_k^\mu(a_1x) L_k^\mu(a_2x)\,dx = \frac{\Gamma(1+\mu+k)}{b_0^{1+\mu+k}} \cdot \frac{b_2^k}{k!} \cdot P_k^{(\mu,\,0)}\left(\frac{b_1^2}{b_0 b_2}\right),$$
$$b_0 = s+\frac{a_1+a_2}{2}, \quad b_1^2 = b_0b_2+2a_1a_2, \quad b_2 = s-\frac{a_1+a_2}{2}$$
$$\left[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\left(s+\frac{a_1+a_2}{2}\right) > 0\right].$$

BU 144(22)

7.415

$$\int_0^1 (1-x)^{\mu-1} x^{\lambda-1} e^{-\beta x} L_n^\alpha(\beta x)\,dx = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} \mathrm{B}(\lambda,\,\mu)\ {}_2F_2(\alpha+n+1,\,\lambda;\,\alpha+1,\,\lambda+\mu;\,-\beta)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0].$$

ET II 193(51)a

7.416

$$\int_{-\infty}^\infty x^{m-n} \exp\left[-\frac{1}{2}(x-y)^2\right] L_n^{m-n}(x^2)\,dx = \frac{(2\pi)^{\frac{1}{2}}}{n!} i^{n-m} 2^{-\frac{n+m}{2}} H_n\left(\frac{iy}{\sqrt{2}}\right) H_m\left(\frac{iy}{\sqrt{2}}\right).$$

BU 149(15b), ET II 293(8)a

7.417

1. $$\int_0^\infty x^{\nu-2n-1} e^{-ax} \sin(bx) L_{2n}^{\nu-2n-1}(ax)\,dx = (-1)^n i\Gamma(\nu) \frac{b^{2n}[(a-ib)^{-\nu}-(a+ib)^{-\nu}]}{2(2n)!}$$
$$[b>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\nu > 2n].$$

ET I 95(12)

2. $$\int_0^\infty x^{\nu-2n-2} e^{-ax} \sin(bx) L_{2n+1}^{\nu-2n-2}(ax)\,dx = (-1)^{n+1}\Gamma(\nu) \frac{b^{2n+1}[(a+ib)^{-\nu}+(a-ib)^{-\nu}]}{2(2n+1)!}$$
$$[b>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\nu > 2n+1].$$

ET I 95(13)

3. $$\int_0^\infty x^{\nu-2n} e^{-ax} \cos(bx) L^{\nu-2n}{}_{2n-1}(ax)\,dx = i(-1)^{n+1}\Gamma(\nu) \frac{b^{2n-1}[(a-ib)^{-\nu}-(a+ib)^{-\nu}]}{2(2n-1)!}$$
$$[b>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\nu > 2n-1].$$

ET I 39(12)

4. $$\int_0^\infty x^{\nu-2n-1} e^{-ax} \cos(bx) L_{2n}^{\nu-2n-1}(ax)\,dx = (-1)^n\Gamma(\nu) \frac{b^{2n}[(a+ib)^{-\nu}+(a-ib)^{-\nu}]}{2(2n)!}$$
$$[b>0, \quad \operatorname{Re}\nu > 2n, \quad \operatorname{Re}a>0].$$

ET I 39(13)

7.418

1. $$\int_0^\infty e^{-\frac{1}{2}x^2} \sin(bx) L_n(x^2)\, dx = (-1)^n \frac{i}{2} n! \frac{1}{\sqrt{2\pi}} \{[D_{-n-1}(ib)]^2 - [D_{-n-1}(-ib)]^2\} \qquad [b > 0].$$

ET I 95(14)

2. $$\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) L_n(x^2)\, dx = \sqrt{\frac{\pi}{2}} (n!)^{-1} e^{-\frac{1}{2}b^2} 2^{-n} \left[H_n\left(\frac{b}{\sqrt{2}}\right) \right]^2 \qquad [b > 0].$$

ET I 39(14)

3. $$\int_0^\infty x^{2n+1} e^{-\frac{1}{2}x^2} \sin(bx) L_n^{n+\frac{1}{2}}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}}\left(\frac{b^2}{2}\right) \qquad [b > 0].$$

ET I 95(15)

4. $$\int_0^\infty x^{2n} e^{-\frac{1}{2}x^2} \cos(bx) L_n^{n-\frac{1}{2}}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}}\left(\frac{1}{2}b^2\right) \qquad [b > 0].$$

ET I 39(16)

5. $$\int_0^\infty x e^{-\frac{1}{2}x^2} L_n^{\alpha}\left(\frac{1}{2}x^2\right) L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right) \sin(xy)\, dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} y e^{-\frac{1}{2}y^2} L_n^{\alpha}\left(\frac{1}{2}y^2\right) L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}y^2\right).$$

ET II 294(11)

6. $$\int_0^\infty e^{-\frac{1}{2}x^2} L_n^{\alpha}\left(\frac{1}{2}x^2\right) L_n^{-\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right) \cos(xy)\, dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{\alpha}\left(\frac{1}{2}y^2\right) L_n^{-\alpha-\frac{1}{2}}\left(\frac{1}{2}y^2\right).$$

ET II 294(12)

7.419

$$\int_0^\infty x^{n+2\nu-\frac{1}{2}} \exp[-(1+a)x] L_n^{2\nu}(ax) K_\nu(x)\, dx =$$
$$= \frac{\pi^{\frac{1}{2}} \Gamma\left(n+\nu+\frac{1}{2}\right) \Gamma\left(n+3\nu+\frac{1}{2}\right)}{2^{n+2\nu+\frac{1}{2}} n! \Gamma(2\nu+1)} F\left(n+\nu+\frac{1}{2},\, n+3\nu+\frac{1}{2};\, 2\nu+1;\, -\frac{1}{2}a\right)$$
$$\left[\operatorname{Re} a > -2, \quad \operatorname{Re}(n+\nu) > -\frac{1}{2}, \quad \operatorname{Re}(n+3\nu) > -\frac{1}{2}\right].$$

1. $$\int_0^\infty xe^{-\frac{1}{2}\alpha x^2} L_n\left(\frac{1}{2}\beta x^2\right) J_0(xy)\,dx = \frac{(\alpha-\beta)^n}{\alpha^{n+1}} e^{-\frac{1}{2\alpha}y^2} L_n\left[\frac{\beta y^2}{2\alpha(\beta-\alpha)}\right] \qquad [y>0, \quad \operatorname{Re}\alpha>0].$$

ET II 13(4)a

2. $$\int_0^\infty xe^{-x^2} L_n(x^2) J_0(xy)\,dx = \frac{2^{-2n-1}}{n!} y^{2n} e^{-\frac{1}{4}y^2}.$$

ET II 13(5)

3. $$\int_0^\infty x^{2n+\nu+1} e^{-\frac{1}{2}x^2} L_n^{\nu+n}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = y^{2n+\nu} e^{-\frac{1}{2}y^2} L_n^{\nu+n}\left(\frac{1}{2}y^2\right) \qquad [y>0, \quad \operatorname{Re}\nu>-1].$$

MO 183

4. $$\int_0^\infty x^{\nu+1} e^{-\beta x^2} L_n^\nu(\alpha x^2) J_\nu(xy)\,dx = 2^{-\nu-1}\beta^{-\nu-n-1}(\beta-\alpha)^n y^\nu e^{-\frac{y^2}{4\beta}} L_n^\nu\left[\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right].$$

ET II 43(5)

5. $$\int_0^\infty e^{-\frac{1}{2q}x^2} x^{\nu+1} L_n^\nu\left[\frac{x^2}{2q(1-q)}\right] J_\nu(xy)\,dx = \frac{q^{n+\nu+1}}{(q-1)^n} e^{-\frac{qy^2}{2}} y^\nu L_n^\nu\left(\frac{y^2}{2}\right) \qquad [\nu>0].$$

MO 183

7.422

1. $$\int_0^\infty x^{\nu+1} e^{-\beta x^2} [L_n^{\frac{1}{2}\nu}(\alpha x^2)]^2 J_\nu(xy)\,dx =$$
$$= \frac{y^\nu}{\pi n!}\Gamma\left(n+1+\frac{1}{2}\nu\right)(2\beta)^{-\nu-1} e^{-\frac{y^2}{4\beta}} \times$$
$$\times \sum_{l=0}^{n} \frac{(-1)^l \Gamma\left(n-l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l+1+\frac{1}{2}\nu\right)(n-l)!} \left(\frac{2\alpha-\beta}{\beta}\right)^{2l} L_{2l}^\nu\left[\frac{\alpha y^2}{2\beta(2\alpha-\beta)}\right]$$
$$[y>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1].$$

ET II 43(7)

2.[7] $$\int_0^\infty x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}(\alpha x^2) L_n^\sigma(\alpha x^2) J_\nu(xy)\,dx =$$
$$= (-1)^{m+n}(2\alpha)^{-\nu-1} y^\nu e^{-\frac{y^2}{4\alpha}} L_n^{\sigma+m-n}\left(\frac{y^2}{4\alpha}\right) L_m^{\nu-\sigma+m-n}\left(\frac{y^2}{4\alpha}\right)$$
$$[y>0, \quad \operatorname{Re}\alpha>0, \quad \operatorname{Re}\nu>-1]$$

1. $$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n+1}\left(\frac{x}{2\sqrt{2}}\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n+1}\left(\frac{y}{2\sqrt{2}}\right).$$

ET II 294(13)a

2. $$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n}\left(\frac{x}{2\sqrt{2}}\right)\cos(xy)\,dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n}\left(\frac{y}{2\sqrt{2}}\right).$$

ET II 294(14)a

7.5 Hypergeometric Functions

7.51 Combinations of hypergeometric functions and powers

7.511

$$\int_0^\infty F(a,\,b;\,c;\,-z)z^{-s-1}\,dx = \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c)\Gamma(-s)}{\Gamma(a)\Gamma(b)\Gamma(c+s)}$$
$$[c\neq 0,\,-1,\,-2,\ldots,\quad \operatorname{Re} s<0,\quad \operatorname{Re}(a+s)>0,\quad \operatorname{Re}(b+s)>0].$$

EH I 79(4)

7.512

1. $$\int_0^1 x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}F(\alpha,\,\beta;\,\gamma;\,x)\,dx =$$
$$= \frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)\Gamma\left(\gamma-\frac{\alpha}{2}-\beta\right)}{\Gamma(1+\alpha)\Gamma\left(1+\frac{\alpha}{2}-\beta\right)\Gamma\left(\gamma-\frac{\alpha}{2}\right)}$$
$$\left[\operatorname{Re}\alpha+1>\operatorname{Re}\gamma>\operatorname{Re}\beta,\quad \operatorname{Re}\left(\gamma-\frac{\alpha}{2}-\beta\right)>0\right].$$

ET II 398(1)

2. $$\int_0^1 x^{\varrho-1}(1-x)^{\beta-\gamma-n}F(-n,\,\beta;\,\gamma;\,x)\,dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\varrho+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\varrho)\Gamma(\beta-\gamma+\varrho+1)}$$
$$[n=0,\,1,\,2\ldots;\quad \operatorname{Re}\varrho>0,\quad \operatorname{Re}(\beta-\gamma)>n-1].$$

ET II 398(2)

3. $$\int_0^1 x^{\varrho-1}(1-x)^{\beta-\varrho-1}F(\alpha,\,\beta;\,\gamma;\,x)\,dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\beta-\varrho)\Gamma(\gamma-\alpha-\varrho)}{\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\varrho)}$$
$$[\operatorname{Re}\varrho>0,\quad \operatorname{Re}(\beta-\varrho)>0,\quad \operatorname{Re}(\gamma-\alpha-\varrho)>0].$$

4. $$\int_0^1 x^{\gamma-1}(1-x)^{\varrho-1}F(\alpha,\,\beta;\,\gamma;\,x)\,dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma+\varrho-\alpha-\beta)}{\Gamma(\gamma+\varrho-\alpha)\Gamma(\gamma+\varrho-\beta)}$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\varrho>0,\quad \operatorname{Re}(\gamma+\varrho-\alpha-\beta)>0].$$

ET II 399(4)

5. $$\int_0^1 x^{\varrho-1}(1-x)^{\sigma-1}F(\alpha,\,\beta;\,\gamma;\,x)\,dx = \frac{\Gamma(\varrho)\Gamma(\sigma)}{\Gamma(\varrho+\sigma)}\,{}_3F_2(\alpha,\,\beta,\,\varrho;\,\gamma,\,\varrho+\sigma;\,1)$$
$$[\operatorname{Re}\varrho>0,\quad \operatorname{Re}\sigma>0,\quad \operatorname{Re}(\gamma+\sigma-\alpha-\beta)>0].$$

ET II 399(5)

6.* $$\int_0^1 x^{\lambda-1}(1-x)^{\beta-\lambda-1}F\left(\alpha,\,\beta;\,\lambda;\,\frac{zx}{b}\right)\,dx = \mathrm{B}(\lambda,\,\beta-\lambda)(1-z/b)^{-\alpha}.$$

BU 9

7. $$\int_0^1 x^{\gamma-1}(1-x)^{\delta-\gamma-1}F(\alpha,\,\beta;\,\gamma;\,xz)F(\delta-\alpha,\,\delta-\beta;\,\delta-\gamma;\,(1-x)\zeta)\,dx =$$
$$= \frac{\Gamma(\gamma)\Gamma(\delta-\gamma)}{\Gamma(\delta)}(1-\zeta)^{2\alpha-\delta}F(\alpha,\,\beta;\,\delta;\,z+\zeta-z\zeta)$$
$$[0<\operatorname{Re}\gamma<\operatorname{Re}\delta,\quad |\arg(1-z)|<\pi,\quad |\arg(1-\zeta)|<\pi].$$

ET II 400(11)

8. $$\int_0^1 x^{\gamma-1}(1-x)^{\epsilon-1}(1-xz)^{-\delta}F(\alpha,\,\beta;\,\gamma;\,xz)F\left[\delta,\,\beta-\gamma;\,\epsilon;\,\frac{(1-x)z}{(1-xz)}\right]\,dx =$$
$$= \frac{\Gamma(\gamma)\Gamma(\epsilon)}{\Gamma(\gamma+\epsilon)}F(\alpha+\delta,\,\beta;\,\gamma+\epsilon;\,z)$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\epsilon>0,\quad |\arg(z-1)|<\pi].$$

ET II 400(12), Eh I 78(3)

9. $$\int_0^1 x^{\gamma-1}(1-x)^{\varrho-1}(1-zx)^{-\sigma}F(\alpha,\,\beta;\,\gamma;\,x)\,dx =$$
$$= \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma+\varrho-\alpha-\beta)}{\Gamma(\gamma+\varrho-\alpha)\Gamma(\gamma+\varrho-\beta)}(1-z)^{-\sigma}\times$$
$$\times\ {}_3F_2\left(\varrho,\,\sigma,\,\gamma+\varrho-\alpha-\beta;\,\gamma+\varrho-\alpha,\,\gamma+\varrho-\beta;\,\frac{z}{z-1}\right)$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\varrho>0,\quad \operatorname{Re}(\gamma+\varrho-\alpha-\beta)>0,\quad |\arg(1-z)|<\pi].$$

10. $$\int_0^\infty x^{\gamma-1}(x+z)^{-\sigma} F(\alpha,\, \beta;\, \gamma;\, -x)\, dx = \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\Gamma(\alpha+\beta-\gamma+\sigma)} \times \\ \times F(\alpha-\gamma+\sigma,\, \beta-\gamma+\sigma;\, \alpha+\beta-\gamma+\sigma;\, 1-z)$$
$[\operatorname{Re}\gamma > 0, \quad \operatorname{Re}(\alpha-\gamma+\sigma) > 0, \quad \operatorname{Re}(\beta-\gamma+\sigma) > 0, \quad |\arg\, z| < \pi].$

11. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, {}_pF_q(a_1, \dots, a_p;\, \nu,\, b_2, \dots, b_q;\, ax)\, dx = \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\, {}_pF_q(a_1, \dots, a_p;\, \mu+\nu,\, b_2, \dots, b_q;\, a)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad p \le q+1; \quad \text{if} \quad p = q+1, \quad \text{then} \quad |a| < 1].$

ET II 200(94)

12. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, {}_pF_q(a_1, \dots, a_p;\, b_1, \dots, b_q;\, ax)\, dx = \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\, {}_{p+1}F_{q+1}(\nu,\, a_1, \dots, a_p;\, \mu+\nu,\, b_1, \dots, b_q;\, a)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad p \le q+1, \quad \text{if} \quad p = q+1, \quad \text{then} \quad |a| < 1].$

ET II 200(95)

855

7.513

$$\int_0^1 x^{s-1}(1-x^2)^\nu F(-n,\, a;\, b;\, x^2)\, dx = \frac{1}{2}\mathrm{B}\left(\nu+1,\, \frac{s}{2}\right)\, {}_3F_2\left(-n,\, a,\, \frac{s}{2};\, b,\, \nu+1+\frac{s}{2};\, 1\right)$$
$[\operatorname{Re} s > 0, \quad \operatorname{Re}\nu > -1].$

ET I 336(4)

7.52 Combinations of hypergeometric functions and exponentials

7.521

$$\int_0^\infty e^{-st}\, {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q, t)\, dt = \frac{1}{s}\, {}_{p+1}F_q(1, a_1, \dots, a_p; b_1, \dots, b_q, s^{-1}) \quad [p \le q].$$

EH I 192

7.522

1. $$\int_0^\infty e^{-\lambda x} x^{\gamma-1}\, {}_2F_1(\alpha,\, \beta;\, \delta;\, -x)\, dx = \frac{\Gamma(\delta)\lambda^{-\gamma}}{\Gamma(\alpha)\Gamma(\beta)} E(\alpha,\, \beta,\, \gamma;\, \delta : \lambda)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\gamma > 0].$

EH I 205(10)

2.[6] $$\int_0^\infty e^{-bx} x^{a-1} F\left(\frac{1}{2}+\nu,\, \frac{1}{2}-\nu;\, a;\, -\frac{x}{2}\right)\, dx = 2^a e^b \frac{1}{\sqrt{\pi}} \Gamma(a)(2b)^{\frac{1}{2}-a} K_\nu(b)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$

3. $$\int_0^\infty e^{-bx}x^{\gamma-1}F(2\alpha,\,2\beta;\,\gamma;\,-\lambda x)\,dx = \Gamma(\gamma)b^{-\gamma}\left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}} e^{\frac{b}{2\lambda}}W_{\frac{1}{2}-\alpha-\beta,\,\alpha-\beta}\left(\frac{b}{\lambda}\right)$$
$$[\operatorname{Re} b > 0, \quad \operatorname{Re}\gamma > 0, \quad |\arg\lambda| < \pi].$$

BU 78(30), ET I 212(4)

4.[6] $$\int_0^\infty e^{-xt}t^{b-1}F(a,\,a-c+1;\,b;\,-t)\,dt = x^{a-b}\Gamma(b)\Psi(a,\,c;\,x) \qquad [\operatorname{Re} b > 0, \quad \operatorname{Re} x > 0].$$

EH I 273(11)

5. $$\int_0^\infty e^{-x}x^{s-1}\,{}_pF_q(a_1,\dots,a_p,\,b_1,\dots,b_q;\,ax)\,dx = \Gamma(s)\,{}_{p+1}F_q(s,\,a_1,\dots,a_p;\,b_1,\dots,b_q;\,a)$$
$$[p < q, \quad \operatorname{Re} s > 0].$$

ET I 337(11)

6. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\,{}_2F_2(-n,\,n+1;\,1,\,\beta;\,x)\,dx = \Gamma(\beta)\mu^{-\beta}P_n\left(1-\frac{2}{\mu}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0].$$

ET I 218(6)

7. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\,{}_2F_2\left(-n,\,n;\,\beta,\,\frac{1}{2};\,x\right)\,dx = \Gamma(\beta)\mu^{-\beta}\,\cos\left[2n\,\arcsin\left(\frac{1}{\sqrt{\mu}}\right)\right]$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0].$$

ET I 218(7)

8. $$\int_0^\infty x^{\varrho_n-1}e^{-\mu x}\,{}_mF_n(a_1,\dots,a_m;\varrho_1,\dots,\varrho_n;\lambda x)dx = \Gamma(\varrho_n)\mu^{-\varrho_n}\,{}_mF_{n-1}\left(a_1,\dots,a_m;\varrho_1,\dots,\varrho_{n-1};\frac{\lambda}{\mu}\right)$$
$$[m \le n; \quad \operatorname{Re}\varrho_n > 0, \quad \operatorname{Re}\mu > 0, \quad \text{if} \quad m < n; \operatorname{Re}\mu > \operatorname{Re}\lambda, \quad \text{if} \quad m = n].$$

ET I 219(16)a

9. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\,{}_mF_n(a_1,\dots,a_m;\varrho_1,\dots,\varrho_n;\lambda x)dx = \Gamma(\sigma)\mu^{-\sigma}\,{}_{m+1}F_n\left(a_1,\dots,a_m,\sigma;\varrho_1,\dots,\varrho_n;\frac{\lambda}{\mu}\right)$$
$$[m \le n, \quad \operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad \text{if} \quad m < n; \operatorname{Re}\mu > \operatorname{Re}\lambda, \quad \text{if} \quad m = n].$$

ET I 219(17)

7.523

$$\int_0^1 x^{\gamma-1}(1-x)^{\varrho-1}e^{-xz}F(\alpha,\,\beta;\,\gamma;\,x)\,dx =$$
$$= \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma+\varrho-\alpha-\beta)}{\Gamma(\gamma+\varrho-\alpha)\Gamma(\gamma+\varrho-\beta)}e^{-z}\,{}_2F_2(\varrho,\,\gamma+\varrho-\alpha-\beta;\,\gamma+\varrho-\alpha,\,\gamma+\varrho-\beta;\,z)$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\varrho>0,\quad \operatorname{Re}(\gamma+\varrho-\alpha-\beta)>0].$$

ET II 400(8)

7.524

1. $$\int_0^\infty e^{-\lambda x}F\left(\alpha,\,\beta;\,\frac{1}{2};\,-x^2\right)\,dx = \lambda^{\alpha+\beta-1}S_{1-\alpha-\beta,\,\alpha-\beta}(\lambda) \qquad [\operatorname{Re}\lambda>0].$$

ET II 401(13)

2. $$\int_0^\infty e^{-st}\,{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;t^2)dx = s^{-1}\,{}_{p+2}F_q\left(a_1,\dots,a_p,1,\frac{1}{2};b_1,\dots,b_q;\frac{4}{s^2}\right) \quad [p<q].$$

MO 176

3. $$\int_0^\infty e^{-st}\,{}_0F_q\left(\frac{1}{q},\,\frac{2}{q},\dots,\frac{q-1}{q},\,1;\,\frac{t^q}{q^q}\right)\,dt = s^{-1}\,\exp(s^{-q}).$$

MO 176

7.525

1. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\,{}_mF_n\left[a_1,\dots,a_m;\,\varrho_1,\dots,\varrho_n;\,(\lambda x)^k\right]\,dx =$$
$$=\Gamma(\sigma)\mu^{-\sigma}\,{}_{m+k}F_n\left[a_1,\dots,a_m,\,\frac{\sigma}{k},\,\frac{\sigma+1}{k},\dots,\frac{\sigma+k-1}{k};\,\varrho_1,\dots,\varrho_n;\,\left(\frac{k\lambda}{\mu}\right)^k\right]$$
$$\left[m+k\le n+1,\quad \operatorname{Re}\sigma>0;\quad \operatorname{Re}\mu>0,\quad \text{if}\quad m+k\le n;\right.$$
$$\left.\operatorname{Re}\left(\mu+k\lambda e^{\frac{2\pi i}{k}}\right)>0;\quad r=0,\,1,\dots,k-1\quad \text{for}\quad m+k=n+1\right].$$

1. $\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}s^{-b}F\left(a,\, b;\, a+b-c+1;\, 1-\frac{1}{s}\right)\, dx = 2\pi i\frac{\Gamma(a+b-c+1)}{\Gamma(b)\Gamma(b-c+1)}t^{b-1}\Psi(a;\, c;\, t)$

$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}(b-c) > -1, \quad \gamma > \frac{1}{2}\right].$

EH I 273(12)

2. $\int_0^\infty e^{-t}t^{\gamma-1}(x+t)^{-\alpha}(y+t)^{-a'}F\left[a,\, a';\, \gamma;\, \frac{t(x+y+t)}{(x+t)(y+t)}\right]\, dt = \Gamma(\gamma)\Psi(a,\, c;\, x)\Psi(a',\, c;\, y),$

$\gamma = a + a' - c + 1 \qquad [\operatorname{Re}\gamma > 0, \quad xy \neq 0].$

EH I 287(21)

3. $\int_0^\infty x^{\gamma-1}(x+y)^{-\alpha}(x+z)^{-\beta}e^{-x}F\left[\alpha,\, \beta;\, \gamma;\, \frac{x(x+y+z)}{(x+y)(x+z)}\right]\, dx =$

$= \Gamma(\gamma)(zy)^{-\frac{1}{2}-\mu}e^{\frac{y+z}{2}}W_{\nu,\,\mu}(y)W_{\lambda,\,\mu}(z),$

$2\nu = 1-\alpha+\beta-\gamma; \quad 2\lambda = 1+\alpha-\beta-\gamma; \quad 2\mu = \alpha+\beta-\gamma$

$[\operatorname{Re}\gamma > 0, \quad |\arg y| < \pi, \quad |\arg z| < \pi].$

ET II 401(15)

7.527

1. $\int_0^\infty (1-e^{-x})^{\lambda-1}e^{-\mu x}F(\alpha,\, \beta;\, \gamma;\, \delta e^{-x})\, dx = \mathrm{B}(\mu,\, \lambda)\, {}_3F_2(\alpha,\, \beta,\, \mu;\, \gamma,\, \mu+\lambda;\, \delta)$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg(1-\delta)| < \pi].$

ET I 213(9)

2. $\int_0^\infty (1-e^{-x})^{\mu}e^{-\alpha x}F(-n,\, \mu+\beta+n;\, \beta;\, e^{-x})\, dx = \frac{\mathrm{B}(\alpha,\, \mu+n+1)\mathrm{B}(\alpha,\, \beta+n-\alpha)}{\mathrm{B}(\alpha,\, \beta-\alpha)}$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -1].$

ET I 213(10)

3. $\int_0^\infty (1-e^{-x})^{\gamma-1}e^{-\mu x}F(\alpha,\, \beta;\, \gamma;\, 1-e^{-x})\, dx = \frac{\Gamma(\mu)\Gamma(\gamma-\alpha-\beta+\mu)\Gamma(\gamma)}{\Gamma(\gamma-\alpha+\mu)\Gamma(\gamma-\beta+\mu)}$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\mu > \operatorname{Re}(\alpha+\beta-\gamma), \quad \operatorname{Re}\gamma > 0].$

ET I 213(11)

4. $\int_0^\infty (1-e^{-x})^{\gamma-1}e^{-\mu x}F[\alpha,\, \beta;\, \gamma;\, \delta(1-e^{-x})]\, dx = \mathrm{B}(\mu,\, \gamma)F(\alpha,\, \beta;\, \mu+\gamma;\, \delta)$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\gamma > 0, \quad |\arg(1-\delta)| < \pi].$

ET I 213(12)

7.53 Hypergeometric and trigonometric functions

7.531

1. $$\int_0^\infty x \sin\mu x F\left(\alpha,\, \beta;\, \frac{3}{2};\, -c^2x^2\right)\, dx = 2^{-\alpha-\beta+1}\pi c^{-\alpha-\beta}\mu^{\alpha+\beta-2}\frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\left[\mu > 0, \quad \operatorname{Re}\alpha > \frac{1}{2}, \quad \operatorname{Re}\beta > \frac{1}{2}\right].$$

ET I 115(6)

2. $$\int_0^\infty \cos\mu x F\left(\alpha,\, \beta;\, \frac{1}{2};\, -c^2x^2\right)\, dx = 2^{-\alpha-\beta+1}\pi c^{-\alpha-\beta}\mu^{\alpha+\beta-1}\frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$[\mu > 0, \quad \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad c > 0].$$

ET I 61(9)

7.54 Combinations of hypergeometric and Bessel functions

7.541

$$\int_0^\infty x^{\alpha+\beta-2\nu-1}(x+1)^{-\nu}e^{xz}K_\nu[(x+1)z]F(\alpha,\, \beta;\, \alpha+\beta-2\nu;\, -x)\, dx =$$
$$= \pi^{-\frac{1}{2}}\cos(\nu\pi)\Gamma\left(\frac{1}{2}-\alpha+\nu\right)\Gamma\left(\frac{1}{2}-\beta+\nu\right)\Gamma(\gamma)\times$$
$$\times (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma}W_{\frac{1}{2}\gamma,\, \frac{1}{2}(\beta-\alpha)}(2z), \qquad \gamma = \alpha+\beta-2\nu$$
$$\left[\operatorname{Re}(\alpha+\beta-2\nu) > 0, \operatorname{Re}\left(\frac{1}{2}-\alpha+\nu\right) > 0, \operatorname{Re}\left(\frac{1}{2}-\beta+\nu\right) > 0, |\arg z| < \frac{3\pi}{2}\right].$$

ET II 401(16)

7.542

1. $$\int_0^\infty x^{\sigma-1}\,{}_pF_{p-1}\left(a_1, \dots, a_p;\, b_1, \dots, b_{p-1};\, -\lambda x^2\right) N_\nu(xy)\, dx =$$
$$= \frac{\Gamma(b_1)\dots\Gamma(b_{p-1})}{2\lambda^{\frac{1}{2}\sigma}\Gamma(a_1)\dots\Gamma(a_p)}G^{p+2,\,1}_{p+2,\,p+3}\left(\frac{y^2}{4\lambda}\middle|\begin{matrix} b_0^*, \dots, b_{p+1}^*, l \\ h,\, k,\, a_1^*, \dots, a_p^*, l\end{matrix}\right),$$
$$a_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \dots, p; \quad b_0^* = 1 - \frac{\sigma}{2}; \quad b_j^* = b_j - \frac{\sigma}{2},$$
$$j = 1, \dots, p-1;\; h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$$
$$\left[|\arg\lambda| < \pi, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|, \quad \operatorname{Re}a_j > \frac{1}{2}\operatorname{Re}\sigma - \frac{3}{4}, \quad y > 0\right].$$

2. $$\int_0^\infty x^{\sigma-1} {}_pF_p\left(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2\right) N_\nu(xy)\,dx =$$
$$= \frac{\Gamma(b_1)\dots\Gamma(b_p)}{2\lambda^{\frac{1}{2}\sigma}\Gamma(a_1)\dots\Gamma(a_p)} G^{p+2,\,1}_{p+2,\,p+3}\left(\frac{y^2}{4\lambda}\middle| \begin{matrix} b_0^*, \dots, b_p^*, l \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix}\right),$$
$$b_0^* = 1 - \frac{\sigma}{2}; \quad a_j^* = a_j - \frac{\sigma}{2}, \quad b_j^* = b_j - \frac{\sigma}{2}; \quad j = 1, \dots, p; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$$
$$\left[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|, \quad \operatorname{Re}a_j > \frac{1}{2}\operatorname{Re}\sigma - \frac{3}{4}, \quad y > 0\right].$$

ET II 119(54)

3. $$\int_0^\infty x^{\sigma-1} {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2\right) N_\nu(xy)\,dx =$$
$$= -\pi^{-1}2^{\sigma-1}y^{-\sigma} \cos\left[\frac{\pi}{2}(\sigma-\nu)\right]\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right) \times$$
$$\times {}_{p+2}F_q\left(a_1, \dots, a_p, \frac{\sigma+\nu}{2}, \frac{\sigma-\nu}{2}; b_1, \dots, b_q; -\frac{4\lambda}{y^2}\right)$$
$$[y > 0, \quad p \le q-1, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|].$$

ET II 119(55)

4. $$\int_0^\infty x^{\sigma-1} {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2\right) K_\nu(xy)\,dx =$$
$$= 2^{\sigma-2}y^{-\sigma}\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right) {}_{p+2}F_q\left(a_1, \dots, a_p, \frac{\sigma+\nu}{2}, \frac{\sigma-\nu}{2}; b_1, \dots, b_q; \frac{4\lambda}{y^2}\right)$$
$$[\operatorname{Re}y > 0, \quad p \le q-1, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|].$$

ET II 153(88)

5. $$\int_0^\infty x^{2\varrho} {}_pF_p\left(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{2^{2\varrho}\Gamma(b_1)\dots\Gamma(b_p)}{y^{2\varrho+1}\Gamma(a_1)\dots\Gamma(a_p)} G^{p+1,\,1}_{p+1,\,p+2}\left(\frac{y^2}{4\lambda}\middle| \begin{matrix} 1, & b_1, \dots, b_p & \\ h, & a_1, \dots, a_p, & k \end{matrix}\right),$$
$$h = \frac{1}{2} + \varrho + \frac{1}{2}\nu, \qquad k = \frac{1}{2} + \varrho - \frac{1}{2}\nu,$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -1 - \operatorname{Re}\nu < 2\operatorname{Re}\varrho < \frac{1}{2} + 2\operatorname{Re}a_r, \quad r = 1, \dots, p\right].$$

ET II 91(18)

6. $$\int_0^\infty x^{2\varrho} {}_{m+1}F_m\left(a_1, \dots, a_{m+1}; b_1, \dots, b_m; -\lambda^2 x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{2^{2\varrho}\Gamma(b_1)\dots\Gamma(b_m)y^{-2\varrho-1}}{\Gamma(a_1)\dots\Gamma(a_{m+1})} G^{m+2,\,1}_{m+1,\,m+3}\left(\frac{y^2}{4\lambda^2}\middle| \begin{matrix} 1, & b_1, \dots, b_m & \\ h, & a_1, \dots, a_{m+1}, & k \end{matrix}\right),$$
$$h = \frac{1}{2} + \varrho + \frac{1}{2}\nu, \qquad k = \frac{1}{2} + \varrho - \frac{1}{2}\nu,$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(2\varrho + \nu) > -1, \quad \operatorname{Re}(\varrho - a_r) < \frac{1}{4}; \quad r = 1, \dots, m+1\right].$$

7. $$\int_0^\infty x^\delta F(\alpha,\ \beta;\ \gamma;\ -\lambda^2 x^2) J_\nu(xy)\, dx =$$
$$= \frac{2^\delta \Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} y^{-\delta-1} G_{24}^{22}\left(\frac{y^2}{4\lambda^2}\left|\begin{matrix}1-\alpha, & 1-\beta & & \\ \frac{1+\delta+\nu}{2}, & 0, & 1-\gamma, & \frac{1+\delta-\nu}{2}\end{matrix}\right.\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -1 - \operatorname{Re}\nu - 2\ \min(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) < \operatorname{Re}\delta < -\frac{1}{2}\right].$$

ET II 82(9)

8. $$\int_0^\infty x^\delta F(\alpha,\ \beta;\ \gamma;\ -\lambda^2 x^2) J_\nu(xy)\, dx = \frac{2^\delta y^{-\delta-1}\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} G_{24}^{31}\left(\frac{y^2}{4\lambda^2}\left|\begin{matrix}1, & \gamma & & \\ \frac{1+\delta+\nu}{2}, & \alpha, & \beta, & \frac{1+\delta-\nu}{2}\end{matrix}\right.\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -\operatorname{Re}\nu - 1 < \operatorname{Re}\delta < 2\ \max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \frac{1}{2}\right].$$

ET II 81(6)

9. $$\int_0^\infty x^{\nu+1} F(\alpha,\ \beta;\ \gamma;\ -\lambda^2 x^2) J_\nu(xy)\, dx = \frac{2^{\nu+1}\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} y^{-\nu-2} G_{13}^{30}\left(\frac{y^2}{4\lambda^2}\left|\begin{matrix}\gamma & & \\ \nu+1, & \alpha, & \beta\end{matrix}\right.\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -1 < \operatorname{Re}\nu < 2\ \max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \frac{3}{2}\right].$$

ET II 81(5)

10. $$\int_0^\infty x^{\nu+1} F(\alpha,\ \beta;\ \nu+1;\ -\lambda^2 x^2) J_\nu(xy)\, dx = \frac{2^{\nu-\alpha-\beta+2}\Gamma(\nu+1)}{\lambda^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)} y^{\alpha+\beta-\nu-2} K_{\alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -1 < \operatorname{Re}\nu < 2\ \max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \frac{3}{2}\right].$$

ET II 81(3)

11. $$\int_0^\infty x^{\nu+1} F(\alpha,\ \beta;\ \nu+1;\ -\lambda^2 x^2) K_\nu(xy)\, dx = 2^{\nu+1}\lambda^{-\alpha-\beta} y^{\alpha+\beta-\nu-2}\Gamma(\nu+1) S_{1-\alpha-\beta,\, \alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$[\operatorname{Re} y > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 152(86)

12. $$\int_0^\infty x^{\nu+1} F\left(\alpha, \beta; \frac{\beta+\nu}{2}+1; -\lambda^2 x^2\right) J_\nu(xy) dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right) y^{\beta-1}\lambda^{-\nu-\beta-1}}{\pi^{\frac{1}{2}}\Gamma(\alpha)\Gamma(\beta) 2^{\beta-1}} \left[K_{\frac{1}{2}(\nu-\beta+1)}\left(\frac{y}{2\lambda}\right)\right]^2$$
$$\left[y > 0, \quad -1 < \operatorname{Re}\nu < 2\ \max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \frac{3}{2}\right].$$

ET II 81(4)

13. $$\int_0^\infty x^{\sigma+\frac{1}{2}} F(\alpha,\ \beta;\ \gamma;\ -\lambda^2 x^2) N_\nu(xy)\, dx = \frac{\lambda^{-\sigma-1} y^{-\frac{1}{2}}\Gamma(\gamma)}{\sqrt{2}\Gamma(\alpha)\Gamma(\beta)} G_{35}^{41}\left(\frac{y^2}{4\lambda^2}\left|\begin{matrix}1-p, & \gamma-p, & l & & \\ h, & k, & \alpha-p, & \beta-p, & l\end{matrix}\right.\right),$$
$$h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu, \quad p = \frac{1}{2} + \frac{1}{2}\sigma$$

ET II 118(52)

14. $$\int_0^\infty x^{\nu+2} F\left(\frac{1}{2},\, \frac{1}{2}-\nu;\, \frac{3}{2};\, -\lambda^2 x^2\right) N_\nu(xy)\, dx = \frac{2^\nu y^{-\nu-1}}{\pi^{\frac{1}{2}} \lambda^2 \Gamma\left(\frac{1}{2}-\nu\right)} K_\nu\left(\frac{y}{2\lambda}\right) K_{\nu+1}\left(\frac{y}{2\lambda}\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -\frac{3}{2} < \operatorname{Re}\nu < -\frac{1}{2}\right].$$

ET II 117(49)

15. $$\int_0^\infty x^{\nu+2} F\left(1,\, 2\nu+\frac{3}{2};\, \nu+2;\, -\lambda^2 x^2\right) N_\nu(xy)\, dx = \pi^{-\frac{1}{2}} 2^{-\nu} \lambda^{-2\nu-3} \frac{\Gamma(\nu+2)}{\Gamma\left(2\nu+\frac{3}{2}\right)} \left[K_\nu\left(\frac{y}{2\lambda}\right)\right]^2$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 117(50)

16. $$\int_0^\infty x^{\nu+2} F\left(1,\, \mu+\nu+\frac{3}{2};\, \frac{3}{2};\, -\lambda^2 x^2\right) N_\nu(xy)\, dx = \frac{\pi^{\frac{1}{2}} 2^{-\mu-\nu-1} \lambda^{-\mu-2\nu-3} y^{\mu+\nu}}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)} K_\mu\left(\frac{y}{\lambda}\right)$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad -\frac{3}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -\frac{3}{2}\right].$$

ET II 118(51)

17. $$\int_0^\infty x^{2\alpha+\nu} F\left(\alpha-\nu-\frac{1}{2},\, \alpha;\, 2\alpha;\, -\lambda^2 x^2\right) J_\nu(xy)\, dx =$$
$$= \frac{i\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\alpha+\nu\right)}{\pi 2^{1-\nu-2\alpha} \lambda^{2\alpha-1} y^{\nu+2}} W_{\frac{1}{2}-\alpha,\, -\frac{1}{2}-\nu}\left(\frac{y}{\lambda}\right) \left[W_{\frac{1}{2}-\alpha,\, -\frac{1}{2}-\nu}\left(e^{-i\pi}\frac{y}{\lambda}\right) - W_{\frac{1}{2}-\alpha,\, -\frac{1}{2}-\nu}\left(e^{i\pi}\frac{y}{\lambda}\right)\right]$$
$$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu < -\frac{1}{2}, \quad \operatorname{Re}(\alpha+\nu) > -\frac{1}{2}\right].$$

ET II 80(1)

18. $$\int_0^\infty x^{2\alpha-\nu} F\left(\nu+\alpha-\frac{1}{2},\, \alpha;\, 2\alpha;\, -\lambda^2 x^2\right) J_\nu(xy)\, dx =$$
$$= \frac{2^{2\alpha-\nu}\Gamma\left(\frac{1}{2}+\alpha\right) y^{\nu-2}}{\lambda^{2\alpha-1}\Gamma(2\nu)} M_{\alpha-\frac{1}{2},\, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right) W_{\frac{1}{2}-\alpha,\, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right).$$

ET II 80(2)

7.543

1. $$\int_0^\infty x^{-2\alpha-1} F\left(\frac{1}{2}+\alpha,\, 1+\alpha;\, 1+2\alpha;\, -\frac{4\lambda^2}{x^2}\right) J_\nu(xy)\, dx = \lambda^{-2\alpha} I_{\frac{1}{2}\nu+\alpha}(\lambda y) K_{\frac{1}{2}\nu-\alpha}(\lambda y)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\alpha>-\frac{1}{2}\right].$$

ET II 81(7)

2. $$\int_0^\infty x^{\nu+1-4\alpha} F\left(\alpha,\, \alpha+\frac{1}{2};\, \nu+1;\, -\frac{\lambda^2}{x^2}\right) J_\nu(xy)\, dx =$$
$$= \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1} I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\alpha-1<\operatorname{Re}\nu<4\operatorname{Re}\alpha-\frac{3}{2}\right].$$

ET II 81(8)

7.544

$$\int_0^\infty x^{\nu+1}(1+x)^{-2\alpha} F\left[\alpha,\, \nu+\frac{1}{2};\, 2\nu+1;\, \frac{4x}{(1+x)^2}\right] J_\nu(xy)\, dx =$$
$$= \frac{\Gamma(\nu+1)\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)} J_\nu(y) \qquad \left[y>0, \quad -1<\operatorname{Re}\nu<2\operatorname{Re}\alpha-\frac{3}{2}\right].$$

ET II 82(10)

7.6 Confluent Hypergeometric Functions

7.61 Combinations of confluent hypergeometric functions and powers

7.611

1. $$\int_0^\infty x^{-1} W_{k,\,\mu}(x)\, dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma\left(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}\mu\right)} \qquad \left[|\operatorname{Re}\mu|<\frac{1}{2}\right].$$

ET II 406(22)

2. $$\int_0^\infty x^{-1} M_{k,\,\mu}(x) W_{\lambda,\,\mu}(x)\, dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\Gamma\left(\frac{1}{2}+\mu-\lambda\right)} \qquad \left[\operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}(k-\lambda)>0\right].$$

3. $\displaystyle\int_0^\infty x^{-1} W_{k,\mu}(x) W_{\lambda,\,\mu}(x)\,dx =$

$$= \frac{1}{(k-\lambda)\,\sin(2\mu\pi)} \left[\frac{1}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-\lambda-\mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2}-k-\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}\right]$$

$$\left[|\operatorname{Re}\mu| < \frac{1}{2}\right].$$

BU 116(12), ET II 409(40)

4. $$\int_0^\infty \{W_{\{,\,\mu}(z)\}^2 \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi\left(\frac{1}{2}+\mu-\{\right) - \psi\left(\frac{1}{2}-\mu-\{\right)}{\Gamma\left(\frac{1}{2}+\mu-\{\right)\Gamma\left(\frac{1}{2}-\mu-\{\right)} \qquad \left[|\operatorname{Re}\mu| < \frac{1}{2}\right].$$

BU 117(12a)

5. $$\int_0^\infty \frac{1}{z}[W_{\{,\,0}(z)]^2\,dx = \frac{\psi'\left(\frac{1}{2}-\{\right)}{\left[\Gamma\left(\frac{1}{2}-\{\right)\right]^2}.$$

BU 117(12b)

6. $$\int_0^\infty x^{\varrho-1} W_{k,\,\mu}(x) W_{-k,\,\mu}(x)\,dx = \frac{\Gamma(\varrho+1)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}-\mu\right)}{2\Gamma\left(1+\frac{1}{2}\varrho+k\right)\Gamma\left(1+\frac{1}{2}\varrho-k\right)}$$

$$[\operatorname{Re}\varrho > 2|\operatorname{Re}\mu| - 1].$$

ET II 409(41)

7. $\displaystyle\int_0^\infty x^{\varrho-1} W_{k,\,\mu}(x) W_{\lambda,\,\nu}(x)\,dx =$

$$= \frac{\Gamma(1+\mu+\nu+\varrho)\Gamma(1-\mu+\nu+\varrho)\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\lambda-\nu\right)\Gamma\left(\frac{3}{2}-k+\nu+\varrho\right)} \times$$

$$\times\ {}_3F_2\left(1+\mu+\nu+\varrho,\ 1-\mu+\nu+\varrho,\ \frac{1}{2}-\lambda-\nu;\ 1+2\nu,\ \frac{3}{2}-k+\nu+\varrho;\ 1\right) \ +$$

$$+ \frac{\Gamma(1+\mu-\nu+\varrho)\Gamma(1-\mu-\nu+\varrho)\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right)\Gamma\left(\frac{3}{2}-k-\nu+\varrho\right)} \times$$

$$\times\ {}_3F_2\left(1+\mu-\nu+\varrho,\ 1-\mu-\nu+\varrho,\ \frac{1}{2}-\lambda-\nu;\ 1-2\nu,\ \frac{3}{2}-k-\nu+\varrho;\ 1\right)$$

1. $$\int_0^\infty t^{b-1}\,{}_1F_1(a;\,c;\,-t)\,dt = \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \qquad [0 < \operatorname{Re} b < \operatorname{Re} a].$$

2. $$\int_0^\infty t^{b-1}\Psi(a,\,c;\,t)\,dt = \frac{\Gamma(b)\Gamma(a-b)\Gamma(b-c+1)}{\Gamma(a)\Gamma(a-c+1)} \qquad [0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b+1].$$

EH I 285(11)

7.613

1. $$\int_0^t x^{\gamma-1}(t-x)^{c-\gamma-1}\,{}_1F_1(a;\,\gamma;\,x)\,dx = t^{c-1}\frac{\Gamma(\gamma)\Gamma(c-\gamma)}{\Gamma(c)}\,{}_1F_1(a;\,c;\,t) \qquad [\operatorname{Re} c > \operatorname{Re}\gamma > 0].$$

BU 9(16)A, EH I 271(16)

2. $$\int_0^t x^{\beta-1}(t-x)^{\gamma-1}\,{}_1F_1(t;\,\beta;\,x)\,dx = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}t^{\beta+\gamma-1}\,{}_1F_1(t;\,\beta+\gamma;\,t)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0].$$

ET II 401(1)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{2\mu-\lambda}\,{}_1F_1\left(\frac{1}{2}+\mu-\nu;\,\lambda;\,xz\right)\,dx = \mathrm{B}(\lambda,\,1+2\mu-\lambda)e^{\frac{1}{2}z}z^{-\frac{1}{2}-\mu}M_{\nu,\,\mu}(z)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(2\mu-\lambda) > -1].$$

BU 14(14)

4. $$\int_0^t x^{\beta-1}(t-x)^{\delta-1}\,{}_1F_1(t;\,\beta;\,x)\,{}_1F_1(\gamma;\,\delta;\,t-x)\,dx = \frac{\Gamma(\beta)\Gamma(\delta)}{\Gamma(\beta+\delta)}t^{\beta+\delta-1}\,{}_1F_1(t+\gamma;\,\beta+\delta;\,t)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\delta > 0].$$

ET II 402(2), EH I 271(15)

5. $$\int_0^t x^{\mu-\frac{1}{2}}(t-x)^{\nu-\frac{1}{2}}M_{k,\,\mu}(x)M_{\lambda,\,\nu}(t-x)\,dx = \frac{\Gamma(2\mu+1)\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)}t^{\mu+\nu}M_{k+\lambda,\,\mu+\nu+\frac{1}{2}}(t)$$
$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

BU 128(14), ET II 402(7)

6. $$\int_0^1 x^{\beta-1}(1-x)^{\sigma-\beta-1}\,{}_1F_1(\alpha;\,\beta;\,\lambda x)\,{}_1F_1[\sigma-\alpha;\,\sigma-\beta;\,\mu(1-x)]\,dx =$$
$$= \frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)}e^{\lambda}\,{}_1F_1(\alpha;\,\sigma;\,\mu-\lambda) \qquad [0 < \operatorname{Re}\beta < \operatorname{Re}\sigma].$$

7.62- 7.63 Combinations of confluent hypergeometric functions and exponentials

7.621

1. $$\int_0^\infty e^{-st} t^\alpha M_{\mu,\,\nu}(t)\,dt = \frac{\Gamma\left(\alpha+\nu+\frac{3}{2}\right)}{\left(\frac{1}{2}+s\right)^{\alpha+\nu+\frac{3}{2}}} F\left(\alpha+\nu+\frac{3}{2},\ -\mu+\nu+\frac{1}{2};\ 2\nu+1;\ \frac{2}{2s+1}\right)$$

$$\left[\operatorname{Re}\left(\alpha+\mu+\frac{3}{2}\right)>0, \quad \operatorname{Re} s > \frac{1}{2}\right].$$

BU 118(1), MO 176a, EH I 270(12)a

2. $$\int_0^\infty e^{-st} t^{\mu-\frac{1}{2}} M_{\lambda,\,\mu}(qt)\,dt = q^{\mu+\frac{1}{2}}\Gamma(2\mu+1)\left(s-\frac{1}{2}q\right)^{\lambda-\mu-\frac{1}{2}}\left(s+\frac{1}{2}q\right)^{-\lambda-\mu-\frac{1}{2}}$$

$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re} s > \frac{|\operatorname{Re} q|}{2}\right].$$

BU 119(4c), MO 176a, EH I 271(13)a

3. $$\int_0^\infty e^{-st} t^\alpha W_{\lambda,\,\mu}(qt)\,dt =$$

$$= \frac{\Gamma\left(\alpha+\mu+\frac{3}{2}\right)\Gamma\left(\alpha-\mu+\frac{3}{2}\right) q^{\mu+\frac{1}{2}}}{\Gamma(\alpha-\lambda+2)}\left(s+\frac{1}{2}q\right)^{-\alpha-\mu-\frac{3}{2}} \times$$

$$\times F\left(\alpha+\mu+\frac{3}{2},\ \mu-\lambda+\frac{1}{2};\ \alpha-\lambda+2;\ \frac{2s-q}{2s+q}\right)$$

$$\left[\operatorname{Re}\left(\alpha\pm\mu+\frac{3}{2}\right)>0, \quad \operatorname{Re} s > -\frac{q}{2}, \quad q>0\right].$$

EH I 271(14)A, BU 121(6), MO 176

4. $$\int_0^\infty e^{-st} t^{b-1}\,{}_1F_1(a;\ c;\ kt)\,dt = \Gamma(b)s^{-b}F(a,\ b;\ c;\ ks^{-1}) \qquad [|s|>|k|];$$

$$= \Gamma(b)(s-k)^{-b}F\left(c-a,\ b;\ c;\ \frac{k}{k-s}\right) \qquad [|s-k|>|k|];$$

$$[\operatorname{Re} b > 0, \quad \operatorname{Re} s > \max(0, \operatorname{Re} k)].$$

EH I 269(5)

5. $$\int_0^\infty t^{c-1}\,{}_1F_1(a;\ c;\ t)e^{-st}\,dt = \Gamma(c)s^{-c}(1-s^{-1})^{-a} \qquad [\operatorname{Re} c > 0, \quad \operatorname{Re} s > 1].$$

6. $\int_0^\infty t^{b-1}\Psi(a,\, c;\, t)e^{-st}\,dt =$

$$= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)} F(b,\, b-c+1;\, a+b-c+1;\, 1-s)$$

$$[\operatorname{Re} b > 0, \quad \operatorname{Re} c < \operatorname{Re} b + 1, \quad |1-s| < 1];$$

$$= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)} s^{-b} F(a,\, b;\, a+b-c+1;\, 1-s^{-1}) \qquad \left[\operatorname{Re} s > \frac{1}{2}\right].$$

EH I 270(7)

7. $$\int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\{,\,\mu}(bx)\,dx = \frac{\Gamma(1+2\mu)\Gamma(\{-\nu)\Gamma\left(\frac{1}{2}+\mu+\nu\right)}{\Gamma\left(\frac{1}{2}+\mu+\{\right)\,\Gamma\left(\frac{1}{2}+\mu-\nu\right)} b^\nu$$

$$\left[\operatorname{Re}\left(\nu+\frac{1}{2}+\mu\right) > 0, \quad \operatorname{Re}\left(\{-\nu\right) > 0\right].$$

BU 119(3)A, ET I 215(11)a

8. $$\int_0^\infty e^{-sx} M_{\{,\mu}(x)\frac{dx}{x} = \frac{2\Gamma(1+2\mu)e^{-i\pi\{}}{\Gamma\left(\frac{1}{2}+\mu+\{\right)} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\{}{2}} Q^{\{}_{\mu-\frac{1}{2}}(2s) \qquad \left[\operatorname{Re}\left(\frac{1}{2}+\mu\right) > 0,\ \operatorname{Re} s > \frac{1}{2}\right].$$

BU 119(4a)

9. $$\int_0^\infty e^{-sx} W_{\{,\,\mu}(x)\frac{dx}{x} = \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\{}{2}} P^{\{}_{\mu-\frac{1}{2}}(2s)$$

$$\left[\operatorname{Re}\left(\frac{1}{2}\pm\mu\right) > 0, \quad \operatorname{Re} s > -\frac{1}{2}\right].$$

BU 121(7)

10. $$\int_0^\infty x^{k+2\mu-1} e^{-\frac{3}{2}x} W_{k,\,\mu}(x)\,dx = \frac{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2k+6\mu+5)\right]}{\left(k+3\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2\mu-2k+3)\right]}$$

$$\left[\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k+3\mu) > -\frac{1}{2}\right].$$

11. $$\int_0^\infty e^{-\frac{1}{2}x} x^{\nu-1} W_{\{,\,\mu}(x)\,dx = \frac{\Gamma\left(\nu+\frac{1}{2}-\mu\right)\Gamma\left(\nu+\frac{1}{2}+\mu\right)}{\Gamma(\nu-\{+1)} \quad \left[\operatorname{Re}\left(\nu+\frac{1}{2}\pm\mu\right)>0\right].$$

BU 122(8b)

12. $$\int_0^\infty e^{\frac{1}{2}x} x^{\nu-1} W_{\{,\,\mu}(x)\,dx = \Gamma(-\{-\mu)\frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\mu-\{\right)\Gamma\left(\frac{1}{2}+\mu-\{\right)}$$
$$\left[\operatorname{Re}\left(\nu+\frac{1}{2}\pm\mu\right)>0, \quad \operatorname{Re}(\{+\nu)<0\right].$$

BU 122(8c)a

7.622

1. $$\int_0^\infty e^{-st} t^{c-1}\,{}_1F_1(a;\,c;\,t)\,{}_1F_1(\alpha;\,c;\,\lambda t)\,dt =$$
$$= \Gamma(c)(s-1)^{-a}(s-\lambda)^{-\alpha}s^{a+\alpha-c}F[a,\,\alpha;\,c;\,\lambda(s-1)^{-1}(s-\lambda)^{-1}]$$
$$[\operatorname{Re} c>0, \quad \operatorname{Re} s > \operatorname{Re}\lambda+1].$$

EH I 287(22)

2. $$\int_0^\infty e^{-t} t^{\varrho}\,{}_1F_1(a;\,c;\,t)\Psi(a';\,c';\,\lambda t)\,dt = C\frac{\Gamma(c)\Gamma(\beta)}{\Gamma(\gamma)}\lambda^{\sigma}F(c-a,\,\beta;\,\gamma;\,1-\lambda^{-1}),$$

$$\varrho = c-1, \quad \sigma=-c, \quad \beta = c-c'+1, \quad \gamma = c-a+a'-c'+1, \quad C=\frac{\Gamma(a'-a)}{\Gamma(a')},$$

or

$$\varrho = c+c'-2, \quad \sigma = 1-c-c', \quad \beta = c+c'-1, \quad \gamma = a'-a+c, \quad C = \frac{\Gamma(a'-a-c'+1)}{\Gamma(a'-c'+1)}.$$

EH I 287(24)

866

3. $$\int_0^\infty x^{\nu-1}e^{-bx}M_{\lambda_1,\,\mu_1-\frac{1}{2}}(a_1x)\dots M_{\lambda_n,\,\mu_n-\frac{1}{2}}(a_nx)\,dx =$$
$$= a_1^{\mu_1}\dots a_n^{\mu_n}(b+A)^{-\nu-M}\Gamma(\nu+M)\times$$
$$\times F_A\left(\nu+M;\,\mu_1-\lambda_1,\dots,\mu_n-\lambda_n;\,2\mu_1,\dots,2\mu_n\colon\,\frac{a_1}{b+A},\dots,\frac{a_n}{b+A}\right),$$
$$M=\mu_1+\dots+\mu_n, \qquad A=\frac{1}{2}(a_1+\dots+a_n)$$

1. $$\int_0^\infty e^{-x} x^{c+n-1} (x+y)^{-1} \, {}_1F_1(a;\, c;\, x)\, dx = (-1)^n \Gamma(c)\Gamma(1-a) y^{c+n-1} \Psi(c-a,\, c;\, y)$$
$$[-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \quad n = 0,\, 1,\, 2, \dots, \quad |\arg y| < \pi].$$

EH I 285(16)

2. $$\int_0^t x^{-1} (t-x)^{k-1} e^{\frac{1}{2}(t-x)} M_{k,\,\mu}(x)\, dx = \frac{\Gamma(k)\Gamma(2\mu+1)}{\Gamma\left(k+\mu+\frac{1}{2}\right)} \pi^{\frac{1}{2}} t^{k-\frac{1}{2}} l_\mu\left(\frac{1}{2}t\right)$$
$$\left[\operatorname{Re} k > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right].$$

ET II 402(5)

3. $$\int_0^t x^{k-1} (t-x)^{\lambda-1} e^{\frac{1}{2}(t-x)} M_{k+\lambda,\,\mu}(x)\, dx = \frac{\Gamma(\lambda)\Gamma\left(k+\mu+\frac{1}{2}\right) t^{k+\lambda-1}}{\Gamma\left(k+\lambda+\mu+\frac{1}{2}\right)} M_{k,\,\mu}(t)$$
$$\left[\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re} \lambda > 0\right].$$

ET II 402(6)

4. $$\int_0^t x^{-k-\lambda-1} (t-x)^{\lambda-1} e^{\frac{1}{2}x} W_{k,\mu}(x)\, dx = \frac{\Gamma(\lambda)\Gamma\left(\frac{1}{2}-k-\lambda+\mu\right)\Gamma\left(\frac{1}{2}-k-\lambda-\mu\right)}{t^{k+1}\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} W_{k+\lambda,\,\mu}(t)$$
$$\left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\lambda) < \frac{1}{2} - |\operatorname{Re} \mu|\right].$$

ET II 405(21)

5. $$\int_1^\infty (x-1)^{\mu-1} x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{k,\,\lambda}(ax)\, dx = \frac{\Gamma(\mu)\Gamma\left(\frac{1}{2}-k-\lambda-\mu\right)}{\Gamma\left(\frac{1}{2}-k-\lambda\right)} a^{-\frac{1}{2}\mu} e^{\frac{1}{2}a} W_{k+\frac{1}{2}\mu,\, \lambda+\frac{1}{2}\mu}(a)$$
$$\left[|\arg(a)| < \frac{3}{2}\pi, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re}(k+\lambda)\right].$$

ET II 211(72)a

ET II 211(74)a

7. $\displaystyle\int_1^\infty (x-1)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\,\lambda}(ax)\,dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{k-\mu,\,\lambda}(a) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re} a > 0].$

ET II 211(73)a

8. $$\int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\,\lambda}(ax)\,dx =$$
$$= \Gamma(\mu) e^{-\frac{1}{2}a} \sec\left[(k-\mu-\lambda)\pi\right] \times$$
$$\times \left\{ \sin(\mu\pi) \frac{\Gamma\left(k-\mu+\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda+1)} M_{k-\mu,\,\lambda}(a) + \cos[(k-\lambda)\pi] W_{k-\mu,\,\lambda}(a) \right\}$$
$$\left[0 < \operatorname{Re}\mu < \operatorname{Re} k - |\operatorname{Re}\lambda| + \frac{1}{2}\right].$$

ET II 200(93)a

7.624

1. $$\int_0^\infty x^{\varrho-1} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\,\mu}(x)\,dx =$$
$$= \frac{-\sigma\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{23}\left(a \,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1-k+\varrho \\ \frac{1}{2}+\mu+\varrho, -\sigma, \sigma, \frac{1}{2}-\mu+\varrho \end{matrix}\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\varrho) > -\frac{1}{2}, \quad \operatorname{Re}(k-\varrho-\sigma) > 0\right].$$

ET II 403(8)

2. $$\int_0^\infty x^{\varrho-1} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\,\mu}(x)\,dx =$$
$$= -\pi^{-\frac{1}{2}} \sigma a^\sigma G_{34}^{32}\left(a \,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1-k+\varrho \\ \frac{1}{2}+\mu+\varrho, \frac{1}{2}-\mu+\varrho, -\sigma, \sigma \end{matrix}\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\varrho > |\operatorname{Re}\mu| - \frac{1}{2}\right].$$

ET II 406(24)

ET II 406(25)

4. $$\int_0^\infty x^{\varrho-1}(a+x)^{-\frac{1}{2}}[x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}]^{2\sigma}e^{-\frac{1}{2}x}M_{k,\,\mu}(x)\,dx =$$
$$= \frac{\Gamma(2\mu+1)a^\sigma}{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{23}\left(a\left|\begin{matrix}0,\ \frac{1}{2},\ \frac{1}{2}-k-\varrho\\ -\sigma,\ \varrho+\mu,\ \varrho-\mu,\ \sigma\end{matrix}\right.\right)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}(\varrho+\mu)>-\frac{1}{2}, \quad \operatorname{Re}(k-\varrho-\sigma)>-\frac{1}{2}\right].$$

ET II 403(9)

5. $$\int_0^\infty x^{\varrho-1}(a+x)^{-\frac{1}{2}}[x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}]^{2\sigma}e^{-\frac{1}{2}x}W_{k,\,\mu}(x)\,dx =$$
$$= \frac{\pi^{-\frac{1}{2}}a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{34}^{33}\left(a\left|\begin{matrix}0,\ \frac{1}{2},\ \frac{1}{2}+k+\varrho\\ -\sigma,\ \varrho+\mu,\ \varrho-\mu,\ \sigma\end{matrix}\right.\right)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\varrho>|\operatorname{Re}\mu|-\frac{1}{2}, \quad \operatorname{Re}(k+\varrho+\sigma)<\frac{1}{2}\right].$$

ET II 406(26)

6. $$\int_0^\infty x^{\varrho-1}(a+x)^{-\frac{1}{2}}[x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}]^{2\sigma}e^{-\frac{1}{2}x}W_{k,\,\mu}(x)\,dx =$$
$$= \pi^{-\frac{1}{2}}a^\sigma G_{34}^{32}\left(a\left|\begin{matrix}0,\ \frac{1}{2},\ \frac{1}{2}-k+\varrho\\ -\sigma,\ \varrho+\mu,\ \varrho-\mu,\ \sigma\end{matrix}\right.\right) \qquad \left[|\arg a|<\pi, \quad \operatorname{Re}\varrho>|\operatorname{Re}\mu|-\frac{1}{2}\right].$$

ET II 406(27)

7.625

1. $$\int_0^\infty x^{\varrho-1}\exp\left[-\frac{1}{2}(\alpha+\beta)x\right]M_{k,\,\mu}(\alpha x)W_{\lambda,\,\nu}(\beta x)\,dx =$$
$$= \frac{\Gamma(1+\mu+\nu+\varrho)\Gamma(1+\mu-\nu+\varrho)}{\Gamma\left(\frac{3}{2}-\lambda+\mu+\varrho\right)}\alpha^{\mu+\frac{1}{2}}\beta^{-\mu-\varrho-\frac{1}{2}} \times$$
$$\times\ {}_3F_2\left(\frac{1}{2}+k+\mu,\ 1+\mu+\nu+\varrho,\ 1+\mu-\nu+\varrho;\ 2\mu+1,\ \frac{3}{2}-\lambda+\mu+\varrho;\ -\frac{\alpha}{\beta}\right)$$
$$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}(\varrho+\mu)>|\operatorname{Re}\nu|-1].$$

ET II 410(43)

2. $\int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x)\,dx =$
$= \beta^{-\varrho}\left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\nu\right)\right]^{-1} \times$
$\times G_{33}^{33}\left(\frac{\beta}{\alpha}\left|\begin{matrix}\frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ 1+\lambda+\varrho\\ \frac{1}{2}+\nu+\varrho,\ \frac{1}{2}-\nu+\varrho,\ -k\end{matrix}\right.\right)$
$[|\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\varrho + 1, \quad \operatorname{Re}(k+\lambda+\varrho) < 0].$

ET II 410(44)a

3. $\int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x)\,dx = \beta^{-\varrho} G_{33}^{22}\left(\frac{\beta}{\alpha}\left|\begin{matrix}\frac{1}{2}+\mu,\ \frac{1}{2}-\nu,\ 1-\lambda+\varrho\\ \frac{1}{2}+\nu+\varrho,\ \frac{1}{2}-\nu+\varrho,\ k\end{matrix}\right.\right)$
$[\operatorname{Re}(\alpha+\beta) > 0, \quad |\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\varrho + 1].$

ET II 411(46)

4. $\int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha-\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x)\,dx =$
$= \beta^{-\varrho}\left[\Gamma\left(\frac{1}{2}-\lambda+\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\nu\right)\right]^{-1} G_{33}^{23}\left(\frac{\beta}{\alpha}\left|\begin{matrix}\frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ 1+\lambda+\varrho\\ \frac{1}{2}+\nu+\varrho,\ \frac{1}{2}-\nu+\varrho,\ k\end{matrix}\right.\right)$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\varrho + 1].$

ET II 411(45)

7.626

1. $\int_0^1 \left[\frac{k}{x} - \frac{1}{4}(\xi+\eta)\right] \exp\left[-\frac{1}{2}(\xi+\eta)x\right] x^c \times$
$= \times\, {}_1F_1(a;\,c;\,\xi x)\, {}_1F_1(a;\,c;\,\eta x)\,dx = 0 \qquad [\xi \neq \eta, \quad \operatorname{Re} c > 0];$
$= \frac{a}{\xi} e^{-\xi} [\,{}_1F_1(a+1;\,c;\,\xi)]^2 \qquad [\xi = \eta, \quad \operatorname{Re} c > 0]$
[where ξ and η are any two zeros of the function ${}_1F_1(a;c;x)$].

EH I 285

2. $\int_1^\infty \left[\frac{k}{x} - \frac{1}{4}(\xi+\eta)\right] e^{-\frac{1}{2}(\xi+\eta)x} x^c \Psi(a,\,c;\,\xi x)\Psi(a,\,c;\,\eta x)\,dx =$
$= 0 \qquad [\xi \neq \eta];$
$= -\xi^{-1} e^{-\xi} [\Psi(a-1,\,c;\,\xi)]^2 \qquad [\xi = \eta]$
[where ξ and η are any two zeros of the function $\Psi(a,\,c;\,x)$].

7.627

1. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{\frac{1}{2}x}W_{k,\,\mu}(a+x)\,dx = \frac{\Gamma(2\lambda)\Gamma\left(\frac{1}{2}-k+\mu-2\lambda\right)}{\Gamma\left(\frac{1}{2}-k+\mu\right)}a^{\lambda-\mu-\frac{1}{2}}W_{k+\lambda,\,\mu-\lambda}(a)$$
$$\left[|\arg a|<\pi, \quad 0<2\operatorname{Re}\lambda<\frac{1}{2}-\operatorname{Re}(k+\mu)\right].$$

ET II 411(50)

2. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}M_{k,\,\mu}^{-\frac{1}{2}x}(a+x)\,dx =$$
$$= \frac{\Gamma(2\lambda)\Gamma(2\mu+1)\Gamma\left(k+\mu-2\lambda+\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(1-2\lambda+2\mu)}a^{\lambda-\mu-\frac{1}{2}}M_{k-\lambda,\,\mu-\lambda}(a)$$
$$\left[\operatorname{Re}\lambda>0, \quad \operatorname{Re}(k+\mu-2\lambda)>-\frac{1}{2}\right].$$

ET II 405(20)

3. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}W_{k,\,\mu}(a+x)\,dx = \Gamma(2\lambda)a^{\lambda-\mu-\frac{1}{2}}W_{k-\lambda,\,\mu-\lambda}(a)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\lambda>0].$$

ET II 411(47)

4. $$\int_0^\infty x^{\lambda-1}(a+x)^{k-\lambda-1}e^{-\frac{1}{2}x}W_{k,\,\mu}(a+x)\,dx = \Gamma(\lambda)a^{k-1}W_{k-\lambda,\,\mu}(a) \qquad [|\arg a|<\pi, \quad \operatorname{Re}\lambda>0].$$

ET II 411(48)

5. $$\int_0^\infty x^{\varrho-1}(a+x)^{-\sigma}e^{-\frac{1}{2}x}W_{k,\,\mu}(a+x)\,dx = \Gamma(\varrho)a^{\varrho}e^{\frac{1}{2}a}G_{23}^{30}\left(a\,\middle|\,\begin{matrix}0,\,1-k-\sigma\\ -\varrho,\,\frac{1}{2}+\mu-\sigma,\,\frac{1}{2}-\mu-\sigma\end{matrix}\right)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\varrho>0].$$

ET II 411(49)

6. $$\int_0^\infty x^{\varrho-1}(a+x)^{-\sigma}e^{\frac{1}{2}x}W_{k,\,\mu}(a+x)\,dx =$$
$$= \frac{\Gamma(\varrho)a^{\varrho}e^{-\frac{1}{2}a}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)}G_{23}^{31}\left(a\,\middle|\,\begin{matrix}k-\sigma+1,\,0\\ -\varrho,\,\frac{1}{2}+\mu-\sigma,\frac{1}{2}-\mu-\sigma\end{matrix}\right)$$
$$[|\arg a|<\pi, \quad 0<\operatorname{Re}\varrho<\operatorname{Re}(\sigma-k)].$$

7. $$\int_0^\infty e^{-\frac{1}{2}(a+x)} \frac{(a+x)^{2\{-1}}{(ax)^{\{}} W_{\{,\,\mu}(x) \frac{dx}{x} = \frac{\Gamma\left(\frac{1}{2}-\mu-\{\right)\, \Gamma\left(\frac{1}{2}+\mu-\{\right)}{a\Gamma(1-2\{)} W_{\{,\,\mu}(a)$$
$$\left[\operatorname{Re}\left(\frac{1}{2}\pm\mu-\{\right) > 0\right].$$

BU 126(7a)

871

8. $$\int_0^\infty e^{-\frac{1}{2}x} x^{\gamma+\alpha-1} M_{\{,\,\mu}(x) \frac{dx}{(x+a)^\alpha} =$$
$$= \frac{\Gamma(1+2\mu)\Gamma\left(\frac{1}{2}+\mu+\gamma\right)\Gamma(\{-\gamma)}{\Gamma\left(\frac{1}{2}+\mu-\gamma\right)\Gamma\left(\frac{1}{2}+\mu+\{\right)} {}_2F_2\left(\alpha,\ \{-\gamma;\ \frac{1}{2}+\mu-\gamma,\ \frac{1}{2}-\mu-\gamma;\ a\right) +$$
$$+ \frac{\Gamma\left(\alpha+\gamma+\frac{1}{2}+\mu\right)\Gamma\left(-\gamma-\frac{1}{2}-\mu\right)}{\Gamma(\alpha)} a^{\gamma+\frac{1}{2}+\mu} \times$$
$$\times\ {}_2F_2\left(\alpha+\gamma+\mu+\frac{1}{2},\ \{+\mu+\frac{1}{2};\ 1+2\mu,\ \frac{3}{2}+\mu+\gamma;\ a\right)$$
$$\left[\operatorname{Re}\left(\gamma+\alpha+\frac{1}{2}+\mu\right) > 0, \quad \operatorname{Re}(\gamma-\{) < 0\right].$$

BU 126(8)a

9. $$\int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}} M_{\{,\,\mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a} \Gamma(1+2\mu)\Gamma\left(\frac{1}{2}-\mu+\{\right) W_{-\{,\,\mu}(a)$$
$$\left[n=0,\ 1,\ 2,\dots, \quad \operatorname{Re}\left(\mu+1+\frac{n}{2}\right) > 0, \quad \operatorname{Re}\left(\{-\mu-\frac{1}{2}\right) < n, \quad |\arg a| < \pi\right].$$

BU 127(10a)a

7.628

1. $$\int_0^\infty e^{-st} e^{-t^2} t^{2c-2}\, {}_1F_1(a;\ c;\ t^2)\, dt = 2^{1-2c}\Gamma(2c-1)\Psi\left(c-\frac{1}{2},\ a+\frac{1}{2};\ \frac{1}{4}s^2\right)$$
$$\left[\operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re} s > 0\right].$$

EH I 270(11)

2. $$\int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{-3\nu,\,\nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}} \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8} K_{2\nu}\left(\frac{as^2}{8}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}, \quad \operatorname{Re} s > 0\right].$$

3. $$\int_0^\infty t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{\lambda,\,\mu}\left(\frac{t^2}{a}\right) dt = 2^{-3\mu-\lambda}\Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}} W_{-\frac{1}{2}(\lambda+3\mu),\,\frac{1}{2}(\lambda-\mu)}\left(\frac{as^2}{4}\right)$$

$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re}\mu > -\frac{1}{4}, \quad \operatorname{Re} s > 0\right].$$

ET I 215(13)

7.629

1.[8] $$\int_0^\infty t^k \exp\left(\frac{a}{2t}\right) e^{-st} W_{k,\,\mu}\left(\frac{a}{t}\right) dt = 2^{1-2k}\sqrt{a}\, s^{-k-\frac{1}{2}} S_{2k,\,2\mu}\left(2\sqrt{a}\, s\right)$$

$$\left[|\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\frac{1}{2}, \quad \operatorname{Re} s > 0\right].$$

ET I 217(21)

2. $$\int_0^\infty t^{-k} \exp\left(-\frac{a}{2t}\right) e^{-st} W_{k,\,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{a} s^{k-\frac{1}{2}} K_{2\mu}(2\sqrt{as}) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0].$$

ET I 217(22)

7.631

1. $$\int_0^\infty x^{\varrho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\,\mu}(\alpha^{-1}x) W_{\lambda,\,\nu}(\beta x^{-1})\, dx =$$
$$= \beta^\varrho \left[\Gamma\left(\frac{1}{2} - k + \mu\right)\Gamma\left(\frac{1}{2} - k - \mu\right)\right]^{-1} \times$$
$$\times G_{24}^{41}\left(\frac{\beta}{\alpha}\middle|\begin{matrix} 1+k, \quad 1-\lambda-\varrho \\ \frac{1}{2}+\mu, \quad \frac{1}{2}-\mu, \quad \frac{1}{2}+\nu-\varrho, \quad \frac{1}{2}-\nu-\varrho \end{matrix}\right)$$
$$\left[|\arg\alpha| < \frac{3}{2}\pi, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(k+\varrho) < -|\operatorname{Re}\nu| - \frac{1}{2}\right].$$

ET II 412(55)

2. $$\int_0^\infty x^{\varrho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\,\mu}(\alpha^{-1}x) W_{\lambda,\,\nu}(\beta x^{-1})\, dx =$$
$$= \beta^\varrho \left[\Gamma\left(\frac{1}{2} - k + \mu\right)\Gamma\left(\frac{1}{2} - k - \mu\right)\Gamma\left(\frac{1}{2} - \lambda + \nu\right)\Gamma\left(\frac{1}{2} - \lambda - \nu\right)\right]^{-1} \times$$
$$\times G_{24}^{42}\left(\frac{\beta}{\alpha}\middle|\begin{matrix} 1+k, \quad 1+\lambda-\varrho \\ \frac{1}{2}+\mu, \quad \frac{1}{2}-\mu, \quad \frac{1}{2}+\nu-\varrho, \quad \frac{1}{2}-\nu-\varrho \end{matrix}\right)$$
$$\left[|\arg\alpha < \frac{3}{2}\pi, \; |\arg\beta| < \frac{3}{2}\pi, \; \operatorname{Re}(\lambda-\varrho) < \frac{1}{2} - |\operatorname{Re}\mu|, \; \operatorname{Re}(k+\varrho) < \frac{1}{2} - |\operatorname{Re}\nu|\right].$$

$$= \beta^{\varrho} G_{24}^{40}\left(\frac{\beta}{\alpha}\middle|\begin{matrix} & 1-k, & 1-\lambda-\varrho & \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\varrho, & \frac{1}{2}-\nu-\varrho \end{matrix}\right) \qquad [\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0].$$

7.632

$$\int_0^\infty e^{-st}(e^t-1)^{\mu-\frac{1}{2}} \exp\left(-\frac{1}{2}\lambda e^t\right) M_{k,\,\mu}(\lambda e^t-\lambda)\,dt = \frac{\Gamma(2\mu+1)\Gamma\left(\frac{1}{2}+k-\mu+s\right)}{\Gamma(s+1)} W_{-k-\frac{1}{2}s,\,\mu-\frac{1}{2}s}(\lambda)$$
$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}s > \operatorname{Re}(\mu-k)-\frac{1}{2}\right].$$

ET I 216(15)

873

7.64 Combinations of confluent hypergeometric and trigonometric functions

7.641

$$\int_0^\infty \cos(ax)\,{}_1F_1(\nu+1;\,1;\,ix)\,{}_1F_1(\nu+1;\,1;\,-ix)\,dx = -a^{-1}\sin(\nu\pi)P_\nu(2a^{-2}-1) \qquad [0<a<1];$$
$$= 0 \qquad [1<a<\infty] \qquad [-1<\operatorname{Re}\nu<0].$$

ET II 402(4)

7.642

$$\int_0^\infty \cos(2xy)\,{}_1F_1(a;\,c;\,-x^2)\,dx = \frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma(c)}{\Gamma(a)}y^{2\alpha-1}e^{-y^2}\Psi\left(c-\frac{1}{2},\,a+\frac{1}{2};\,y^2\right).$$

EH I 285(12)

7.643

1. $$\int_0^\infty x^{4\nu}e^{-\frac{1}{2}x^2}\sin(bx)\,{}_1F_1\left(\frac{1}{2}-2\nu;\,2\nu+1;\,\frac{1}{2}x^2\right)dx = \sqrt{\frac{\pi}{2}}b^{4\nu}c^{-\frac{1}{2}b^2}\,{}_1F_1\left(\frac{1}{2}-2\nu;\,1+2\nu;\,\frac{1}{2}b^2\right)$$
$$\left[b>0, \quad \operatorname{Re}\nu > -\frac{1}{4}\right].$$

ET I 115(5)

2. $$\int_0^\infty x^{2\nu-1}e^{-\frac{1}{4}x^2}\sin(bx)M_{3\nu,\,\nu}\left(\frac{1}{2}x^2\right)dx = \sqrt{\frac{\pi}{2}}b^{2\nu-1}e^{-\frac{1}{4}b^2}M_{3\nu,\,\nu}\left(\frac{1}{2}b^2\right)$$
$$\left[b>0, \quad \operatorname{Re}\nu > -\frac{1}{4}\right].$$

ET I 116(10)

ET I 61(7)

4. $$\int_0^\infty x^{-2\nu} e^{\frac{1}{4}x^2} \sin(bx) W_{3\nu-1,\,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu} e^{\frac{1}{4}b^2} W_{3\nu-1,\,\nu}\left(\frac{1}{2}b^2\right) \qquad \left[\operatorname{Re}\nu < \frac{1}{2},\ b > 0\right].$$

ET I 116(9)

7.644

1. $$\int^\infty x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} \sin(2ax^{\frac{1}{2}}) M_{k,\,\mu}(x)\, dx = \pi^{\frac{1}{2}} a^{k+\mu-1} \frac{\Gamma(3-2\mu)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} \exp\left(-\frac{a^2}{2}\right) W_{\varrho,\,\sigma}(a^2),$$
$$2\varrho = k - 3\mu + 1, \qquad 2\sigma = k + \mu - 1 \qquad [a > 0, \quad \operatorname{Re}(k+\mu) > 0].$$

ET II 403(10)

2. $$\int_0^\infty x^{\varrho-1} \sin(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{k,\,\mu}(x)\, dx = \frac{c\Gamma(1+\mu+\varrho)\Gamma(1-\mu+\varrho)}{\Gamma\left(\frac{3}{2}-k+\varrho\right)} \times$$
$$\times\ {}_2F_2\left(1+\mu+\varrho,\ 1-\mu+\varrho;\ \frac{3}{2}, \frac{3}{2}-k+\varrho;\ -\frac{c^2}{4}\right)$$
$$[\operatorname{Re}\varrho > |\operatorname{Re}\mu| - 1].$$

ET II 407(28)

874

3. $$\int_0^\infty x^{\varrho-1} \sin(cx^{\frac{1}{2}}) e^{\frac{1}{2}x} W_{k,\,\mu}(x)\, dx =$$
$$= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{23}^{22}\left(\frac{c^2}{4} \middle| \begin{matrix} \frac{1}{2}+\mu-\varrho,\ \frac{1}{2}-\mu-\varrho \\ \frac{1}{2},\ -k-\varrho,\ 0 \end{matrix}\right)$$
$$\left[c > 0, \quad \operatorname{Re}\varrho > |\operatorname{Re}\mu| - 1, \quad \operatorname{Re}(k+\varrho) < \frac{1}{2}\right].$$

ET II 407(29)

4. $$\int_0^\infty x^{\varrho-1} \cos(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{k,\,\mu}(x)\, dx = \frac{\Gamma\left(\frac{1}{2}+\mu+\varrho\right)\Gamma\left(\frac{1}{2}-\mu+\varrho\right)}{\Gamma(1-k+\varrho)} \times$$
$$\times\ {}_2F_2\left(\frac{1}{2}+\mu+\varrho,\ \frac{1}{2}-\mu+\varrho;\ \frac{1}{2},\ 1-k+\varrho;\ -\frac{c^2}{4}\right)$$
$$\left[\operatorname{Re}\varrho > |\operatorname{Re}\mu| - \frac{1}{2}\right].$$

5. $$\int_0^\infty x^{\varrho-1}\, \cos(cx^{\frac{1}{2}})e^{\frac{1}{2}x}W_{k,\,\mu}(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G^{22}_{23}\left(\frac{c^2}{4}\left|\begin{matrix}\frac{1}{2}+\mu-\varrho,\ \frac{1}{2}-\mu-\varrho\\ 0,\ -k-\varrho,\ \frac{1}{2}\end{matrix}\right.\right)$$
$$\left[c>0, \quad \operatorname{Re}\varrho > |\operatorname{Re}\mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\varrho) < \frac{1}{2}\right].$$

ET II 407(31)

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.651

1. $$\int_0^\infty J_\nu(xy)M_{-\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ax)W_{\frac{1}{2}\mu,\,\frac{1}{2}\nu}(ax)\,dx =$$
$$= ay^{-\mu-1}\frac{\Gamma(\nu+1)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}[a+(a^2+y^2)^{\frac{1}{2}}]^\mu(a^2+y^2)^{-\frac{1}{2}}$$
$$\left[y>0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re}a>0\right].$$

ET II 85(19)

875

2. $$\int_0^\infty M_{k,\,\frac{1}{2}\nu}(-iax)M_{-k,\,\frac{1}{2}\nu}(-iax)J_\nu(xy)\,dx =$$
$$= \frac{ae^{-\frac{1}{2}(\nu+1)\pi i}[\Gamma(1+\nu)]^2}{\Gamma\left(\frac{1}{2}+k+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-k+\frac{1}{2}\nu\right)}y^{-1-2k}\times$$
$$\times (a^2-y^2)^{-\frac{1}{2}}\{[a+(a^2-y^2)^{\frac{1}{2}}]^{2k}+[a-(a^2-y^2)^{\frac{1}{2}}]^{2k}\} \qquad [0<y<a];$$
$$= 0 \qquad [a<y<\infty] \qquad \left[a>0, \quad \operatorname{Re}\nu>-1, \quad |\operatorname{Re}k| < \frac{1}{4}\right].$$

ET II 85(18)

7.652

$$\int_0^\infty M_{-\mu,\,\frac{1}{2}\nu}\{a[(b^2+x^2)^{\frac{1}{2}}-b]\}W_{\mu,\,\frac{1}{2}\nu}\{a[(b^2+x^2)^{\frac{1}{2}}+b]\}J_\nu(xy)\,dx =$$
$$= \frac{ay^{-2\mu-1}\Gamma(1+\nu)[(a^2+y^2)^{\frac{1}{2}}+a]^{2\mu}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\mu\right)(A^2+Y^2)^{\frac{1}{2}}}\ \exp[-b(a^2+y^2)^{\frac{1}{2}}]$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu<\frac{1}{4}, \quad \operatorname{Re}a>0, \quad \operatorname{Re}b>0\right].$$

ET II 87(29)

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

1. $$\int_0^\infty x^{-1} W_{k,\,\mu}(ax) M_{-k,\,\mu}(ax) J_0(xy)\,dx =$$
$$= e^{-ik\pi} \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu+k\right)} P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right] Q^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re} k<\frac{3}{4}\right].$$

ET II 18(44)

2. $$\int_0^\infty x^{-1} W_{k,\,\mu}(ax) W_{-k,\,\mu}(ax) J_0(xy)\,dx = \frac{1}{2}\pi\,\cos(\mu\pi) P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right] P^{-k}_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad |\operatorname{Re}\mu|<\frac{1}{2}\right].$$

ET II 18(45)

3. $$\int_0^\infty x^{2\mu-\nu} W_{k,\,\mu}(ax) M_{-k,\,\mu}(ax) J_\nu(xy)\,dx =$$
$$= 2^{2\mu-\nu+2k} a^{2k} y^{\nu-2\mu-2k-1} \frac{\Gamma(2\mu+1)}{\Gamma\left(\nu-k-\mu+\frac{1}{2}\right)} \times$$
$$\times\ {}_3F_2\left(\frac{1}{2}-k,\, 1-k,\, \frac{1}{2}-k+\mu;\, 1-2k,\, \frac{1}{2}-k-\mu+\nu;\, -\frac{y^2}{a^2}\right)$$
$$\left[y>0, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re} a>0, \quad \operatorname{Re}(2\mu+2k-\nu)<\frac{1}{2}\right].$$

ET II 85(20)

4. $$\int_0^\infty x^{2\varrho-\nu} W_{k,\,\mu}(iax) W_{k,\,\mu}(-iax) J_\nu(xy)\,dx =$$
$$= 2^{2\varrho-\nu} y^{\nu-2\varrho-1} \pi^{-\frac{1}{2}} \left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} G^{24}_{44}\left(\frac{y^2}{a^2}\,\middle|\, \begin{matrix} \frac{1}{2},\, 0,\, \frac{1}{2}-\mu,\, \frac{1}{2}+\mu \\ \varrho+\frac{1}{2},\, -k,\, k,\, \varrho-\nu+\frac{1}{2} \end{matrix}\right)$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\varrho>|\operatorname{Re}\mu|-1, \quad \operatorname{Re}(2\varrho+2k-\nu)<\frac{1}{2}\right].$$

5. $$\int_0^\infty x^{2\varrho-\nu} W_{k,\,\mu}(ax) M_{-k,\,\mu}(ax) J_\nu(xy)\,dx =$$
$$= \frac{2^{2\varrho-\nu}\Gamma(2\mu+1)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}-k+\mu\right)} y^{\nu-2\varrho-1} G_{44}^{23}\left(\frac{y^2}{a^2}\,\middle|\,\begin{matrix}\frac{1}{2},\ 0,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\mu \\ \varrho+\frac{1}{2},\ -k,\ k,\ \varrho-\nu+\frac{1}{2}\end{matrix}\right)$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\varrho>-1, \quad \operatorname{Re}(\varrho+\mu)>-1, \quad \operatorname{Re}(2e+2k+\nu)<\frac{1}{2}\right].$$

ET II 86(21)a

6. $$\int_0^\infty x^{2\varrho-\nu} W_{k,\,\mu}(ax) W_{-k,\,\mu}(ax) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma(\varrho+1+\mu)\Gamma(\varrho+1-\mu)\Gamma(2\varrho+2)}{\Gamma\left(\frac{3}{2}+k+\varrho\right)\Gamma\left(\frac{3}{2}-k+\varrho\right)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\varrho-1} \times$$
$$\times\ {}_4F_3\left(\varrho+1,\ \varrho+\frac{3}{2},\ \varrho+1+\mu,\ \varrho+1-\mu;\ \frac{3}{2}+k+\varrho,\ \frac{3}{2}-k+\varrho,\ 1+\nu;\ -\frac{y^2}{a^2}\right)$$
$$[y>0, \quad \operatorname{Re}\varrho>|\operatorname{Re}\mu|-1, \quad \operatorname{Re} a>0].$$

ET II 86(22)a

7.662

1. $$\int_0^\infty x^{-1} M_{-\mu,\,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) W_{\mu,\,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\nu-\mu\right)} I_{\frac{1}{4}\nu-\mu}\left(\frac{1}{4}y^2\right) K_{\frac{1}{4}\nu+\mu}\left(\frac{1}{4}y^2\right)$$
$$[y>0, \quad \operatorname{Re}\nu>-1].$$

ET II 86(24)

2. $$\int_0^\infty x^{-1} M_{\alpha-\beta,\,\frac{1}{4}\nu-\gamma}\left(\frac{1}{2}x^2\right) W_{\alpha+\beta,\,\frac{1}{4}\nu+\gamma}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma\left(1+\frac{1}{2}\nu-2\gamma\right)}{\Gamma\left(1+\frac{1}{2}\nu-2\beta\right)} y^{-2} M_{\alpha-\gamma,\,\frac{1}{4}\nu-\beta}\left(\frac{1}{2}y^2\right) W_{\alpha+\gamma,\,\frac{1}{4}\nu+\beta}\left(\frac{1}{2}y^2\right)$$
$$\left[y>0, \quad \operatorname{Re}\beta<\frac{1}{8}, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-4\gamma)>-2\right].$$

ET II 86(25)

3. $$\int_0^\infty x^{-1} M_{k,\,0}(iax^2) M_{k,\,0}(-iax^2) K_0(xy)\,dx = \frac{\pi}{16}\left\{\left[J_k\left(\frac{y^2}{8a}\right)\right]^2 + \left[N_k\left(\frac{y^2}{8a}\right)\right]^2\right\} \qquad [a>0].$$

4. $$\int_0^\infty x^{-1} M_{k,\mu}(iax^2) M_{k,\mu}(-iax^2) K_0(xy)\,dx = ay^{-2}[\Gamma(2\mu+1)]^2 W_{-\mu,k}\left(\frac{iy^2}{4a}\right) W_{-\mu,k}\left(-\frac{iy^2}{4a}\right)$$
$$\left[a>0, \quad \operatorname{Re} y>0, \quad \operatorname{Re}\mu>-\frac{1}{2}\right].$$

ET II 152(84)

7.663

1. $$\int_0^\infty x^{2\varrho}\,{}_1F_1(a;\,b;\,-\lambda x^2) J_\nu(xy)\,dx = \frac{2^{2\varrho}\Gamma(b)}{\Gamma(a) y^{2\varrho+1}} G_{23}^{21}\left(\frac{y^2}{4\lambda}\,\middle|\,\begin{matrix} 1,\, b \\ \frac{1}{2}+\varrho+\frac{1}{2}\nu,\, a,\, \frac{1}{2}+\varrho-\frac{1}{2}\nu \end{matrix}\right)$$
$$\left[y>0, \quad -1-\operatorname{Re}\nu < 2\operatorname{Re}\varrho < \frac{1}{2}+2\operatorname{Re} a, \quad \operatorname{Re}\lambda>0\right].$$

ET II 88(6)

2. $$\int_0^\infty x^{\nu+1}\,{}_1F_1\left(2a-\nu;\, a+1;\, -\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{2^{\nu-a+\frac{1}{2}}\Gamma(a+1)}{\pi^{\frac{1}{2}}\Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} K_{a-\nu-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(4a-3\nu)>\frac{1}{2}\right].$$

ET II 87(1)

3. $$\int_0^\infty x^a\,{}_1F_1\left(a;\, \frac{1+a+\nu}{2};\, -\frac{1}{2}x^2\right) J_\nu(xy)\,dx = y^{a-1}\,{}_1F_1\left(a;\, \frac{1+a+\nu}{2};\, -\frac{y^2}{2}\right)$$
$$\left[y>0, \quad \operatorname{Re} a>-\frac{1}{2}, \quad \operatorname{Re}(a+\nu)>-1\right].$$

ET II 87(2)

4. $$\int_0^\infty x^{\nu+1-2a}\,{}_1F_1\left(a;\, 1+\nu-a;\, -\frac{1}{2}x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(1+\nu-a)}{\Gamma(a)} 2^{-2a+\nu+\frac{1}{2}} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} I_{a-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$$\left[y>0, \quad \operatorname{Re} a-1<\operatorname{Re}\nu<4\operatorname{Re} a-\frac{1}{2}\right].$$

ET II 87(3)

5. $$\int_0^\infty x\,{}_1F_1(\lambda;\, 1;\, -x^2) J_0(xy)\,dx = [2^{2\lambda-1}\Gamma(\lambda)]^{-1} y^{2\lambda-2} e^{-\frac{1}{4}y^2} \qquad [y>0, \quad \operatorname{Re}\lambda>0].$$

ET II 18(46)

ET II 88(4)

7. $$\int_0^\infty x^{2b-\nu-1}\,{}_1F_1(a;\,b;\,-\lambda x^2)J_\nu(xy)\,dx$$
$$= \frac{2^{2b-2a-\nu-1}\Gamma(b)}{\Gamma(a-b+\nu+1)}\lambda^{-a}y^{2a-2b+\nu}\times$$
$$\times\,{}_1F_1\left(a;\,1+a-b+\nu;\,-\frac{y^2}{4\lambda}\right)$$
$$\left[y>0,\quad 0<\operatorname{Re}b<\frac{3}{4}+\operatorname{Re}\left(a+\frac{1}{2}\nu\right),\quad \operatorname{Re}\lambda>0\right].$$

ET II 88(5)

7.664

1. $$\int_0^\infty xW_{\frac{1}{2}\nu,\,\mu}\left(\frac{a}{x}\right)W_{-\frac{1}{2}\nu,\,\mu}\left(\frac{a}{x}\right)K_\nu(xy)\,dx = 2ay^{-1}K_{2\mu}[(2ay)^{\frac{1}{2}}e^{\frac{1}{4}i\pi}]K_{2\mu}[(2ay)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}]$$
$$[\operatorname{Re}y>0,\quad \operatorname{Re}a>0].$$

ET II 152(85)

2. $$\int_0^\infty xW_{\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)W_{-\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)J_\nu(xy)\,dx =$$
$$= -4y^{-1}\left\{\sin\left[\left(\mu-\frac{1}{2}\nu\right)\pi\right]J_{2\mu}\left(2y^{\frac{1}{2}}\right)+\cos\left[\left(\mu-\frac{1}{2}\nu\right)\pi\right]N_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}K_{2\mu}(2y^{\frac{1}{2}})$$
$$[y>0,\quad \operatorname{Re}(\nu\pm 2\mu)>-1].$$

ET II 87(27)

3. $$\int_0^\infty xW_{\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)W_{-\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)N_\nu(xy)\,dx =$$
$$= 4y^{-1}\left\{\left\{\cos\left[\left(\mu-\frac{1}{2}\nu\right)\pi\right]J_{2\mu}\left(2y^{\frac{1}{2}}\right)-\sin\left[\left(\mu-\frac{1}{2}\nu\right)\pi\right]N_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}K_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}$$
$$\left[y>0,\quad |\operatorname{Re}\mu|<\frac{1}{4}\right].$$

ET II 117(48)

4. $$\int_0^\infty xW_{-\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)M_{\frac{1}{2}\nu,\,\mu}\left(\frac{2}{x}\right)J_\nu(xy)\,dx = \frac{4\Gamma(1+2\mu)y^{-1}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\mu\right)}J_{2\mu}(2y^{\frac{1}{2}})K_{2\mu}(2y^{\frac{1}{2}})$$
$$\left[y>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>-\frac{1}{4}\right].$$

5. $$\int_0^\infty x W_{-\frac{1}{2}\nu,\,\mu}\left(\frac{ia}{x}\right) W_{-\frac{1}{2}\nu,\,\mu}\left(-\frac{ia}{x}\right) J_\nu(xy)\,dx =$$
$$= 4ay^{-1}\left[\Gamma\left(\frac{1}{2}+\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\mu+\frac{1}{2}\nu\right)\right]^{-1} K_\mu[(2iay)^{\frac{1}{2}}]K_\mu[(-2iay)^{\frac{1}{2}}]$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad |\operatorname{Re}\mu|<\frac{1}{2}, \quad \operatorname{Re}\nu>-1\right].$$

ET II 87(28)

7.665

1. $$\int_0^\infty x^{-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}}) K_{\frac{1}{2}\nu-\mu}\left(\frac{1}{2}x\right) M_{k,\,\mu}(x)\,dx =$$
$$= \frac{\Gamma(2\mu+1)}{a\Gamma\left(k+\frac{1}{2}\nu+1\right)} W_{\frac{1}{2}(k-\mu),\,\frac{1}{2}k-\frac{1}{4}\nu}\left(\frac{a^2}{2}\right) M_{\frac{1}{2}(k+\mu),\,\frac{1}{2}k+\frac{1}{4}\nu}\left(\frac{a^2}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re} k>-\frac{1}{4}, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}\nu>-1\right].$$

ET II 405(18)

2. $$\int_0^\infty x^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi(a,\,c;\,x)\,{}_1F_1(a';\,c';\,-x) J_{c+c'-2}[2(xy)^{\frac{1}{2}}]\,dx =$$
$$= \frac{\Gamma(c')}{\Gamma(a+a')} y^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi(c'-a',\,c+c'-a-a';\,y)\,{}_1F_1(a';\,a+a';\,-y)$$
$$\left[\operatorname{Re} c'>0, \quad 1<\operatorname{Re}(c+c')<2\ \operatorname{Re}(a+a')+\frac{1}{2}\right].$$

EH I 287(23)

7.666

$$\int_0^\infty x^{\frac{1}{2}c-\frac{1}{2}}\,{}_1F_1(a;\,c;\,-2x^{\frac{1}{2}})\Psi(a,\,c;\,2x^{\frac{1}{2}}) J_{c-1}[2(xy)^{\frac{1}{2}}]\,dx =$$
$$= 2^{-c}\frac{\Gamma(c)}{\Gamma(a)} y^{a-\frac{1}{2}c-\frac{1}{2}}[1+(1+y)^{\frac{1}{2}}]^{c-2a}(1+y)^{-\frac{1}{2}} \qquad \left[\operatorname{Re} c>2, \quad \operatorname{Re}(c-2a)<\frac{1}{2}\right].$$

EH I 285(13)

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

1. $$\int_0^\infty x^{k-\frac{3}{2}}\exp\left[-\frac{1}{2}(a+1)x\right] K_\nu\left(\frac{1}{2}ax\right) M_{k,\,\nu}(x)\,dx =$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(k)\Gamma(k+2\nu)}{a^{k+\nu}\Gamma\left(k+\nu+\frac{1}{2}\right)}\,{}_2F_1(k,\,k+2\nu;\,2\nu+1;\,-a^{-1})$$

2. $$\int_0^\infty x^{-k-\frac{3}{2}} \exp\left[-\frac{1}{2}(a-1)x\right] K_\mu\left(\frac{1}{2}ax\right) W_{k,\,\mu}(x)\,dx =$$
$$= \frac{\pi\Gamma(-k)\Gamma(2\mu-k)\Gamma(-2\mu-k)}{\Gamma\left(\frac{1}{2}-k\right)\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} 2^{2k+1} a^{k-\nu}\,{}_2F_1(-k,\,2\mu-k;\,-2k;\,1-a^{-1})$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} k < 2\operatorname{Re}\mu < -\operatorname{Re} k].$$

ET II 408(36)

7.672

1. $$\int_0^\infty x^{2\varrho} e^{-\frac{1}{2}ax^2} M_{k,\,\mu}(ax^2) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} 2^{2\varrho} y^{-2\varrho-1} G_{23}^{21}\left(\frac{y^2}{4a}\,\middle|\,\begin{matrix}\frac{1}{2}-\mu,\ \frac{1}{2}+\mu \\ \frac{1}{2}+\varrho+\frac{1}{2}\nu,\ k,\ \frac{1}{2}+\varrho-\frac{1}{2}\nu\end{matrix}\right)$$
$$\left[y > 0, \quad -1-\operatorname{Re}\left(\frac{1}{2}\nu+\mu\right) < \operatorname{Re}\varrho < \operatorname{Re} k - \frac{1}{4}, \quad \operatorname{Re} a > 0\right].$$

ET II 83(10)

2. $$\int_0^\infty x^{2\varrho} e^{-\frac{1}{2}ax^2} W_{k,\,\mu}(ax^2) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma\left(1+\mu+\frac{1}{2}\nu+\varrho\right)\Gamma\left(1-\mu+\frac{1}{2}\nu+\varrho\right) 2^{-\nu-1}}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-k+\frac{1}{2}\nu+\varrho\right)} a^{-\frac{1}{2}\nu-\varrho-\frac{1}{2}} y^\nu \times$$
$$\times\ {}_2F_2\left(\lambda+\mu,\,\lambda-\mu;\,\nu+1,\,\frac{1}{2}-k+\lambda;\,-\frac{y^2}{4a}\right), \qquad \lambda = 1+\frac{1}{2}\nu+\varrho$$
$$\left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\left(\varrho \pm \mu + \frac{1}{2}\nu\right) > -1\right].$$

ET II 85(16)

3. $$\int_0^\infty x^{2\varrho} e^{\frac{1}{2}ax^2} W_{k,\,\mu}(ax^2) J_\nu(xy)\,dx = \frac{2^{2\varrho} y^{-2\varrho-1}}{\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} \times$$
$$\times\, G_{23}^{22}\left(\frac{y^2}{4a}\,\middle|\,\begin{matrix}\frac{1}{2}-\mu, \quad \frac{1}{2}+\mu \\ \frac{1}{2}+\varrho+\frac{1}{2}\nu, \quad -k, \quad \frac{1}{2}+\varrho-\frac{1}{2}\nu\end{matrix}\right)$$
$$\left[y > 0, \quad |\arg a| < \pi, \quad -1-\operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) < \operatorname{Re}\varrho < -\frac{1}{4} - \operatorname{Re} k\right].$$

4. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\,\mu}\left(\frac{1}{2}x^2\right) N_\nu(xy)\,dx =$$
$$= \frac{2^\lambda y^{-\frac{1}{2}}\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{31}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & l & \\ h, & \{, & k-\lambda-\frac{1}{2}, & l\end{matrix}\right),$$
$$h = \frac{1}{4}+\frac{1}{2}\nu, \quad \{ = \frac{1}{4}-\frac{1}{2}\nu, \quad l = -\frac{1}{4}-\frac{1}{2}\nu$$
$$\left[y>0, \quad \operatorname{Re}(k-\lambda)>0, \quad \operatorname{Re}(2\lambda+2\mu\pm\nu) > -\frac{5}{2}\right].$$

ET II 116(45)

5. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\,\mu}\left(\frac{1}{2}x^2\right) N_\nu(xy)\,dx$$
$$= 2^\lambda \left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} \times$$
$$\times\, G_{34}^{32}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & l & \\ h, & \{, & -\frac{1}{2}-k-\lambda, & l\end{matrix}\right) y^{-\frac{1}{2}},$$
$$h = \frac{1}{4}+\frac{1}{2}\nu, \quad \{ = \frac{1}{4}-\frac{1}{2}\nu, \quad l = -\frac{1}{4}-\frac{1}{2}\nu$$
$$\left[y>0, \quad \operatorname{Re}(k+\lambda)<0, \quad \operatorname{Re}(2\lambda\pm2\mu\pm\nu) > -\frac{5}{2}\right].$$

ET II 117(47)

6. $$\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4},\,\frac{1}{2}\nu+\frac{1}{4}}(x^2) J_\nu(xy)\,dx = (2\nu+1)2^{-\nu}y^{\nu-1}\left[1-\Phi\left(\frac{1}{2}y\right)\right]$$
$$\left[y>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 82(1)

7. $$\int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}(x^2) J_\nu(xy)dx = \frac{\Gamma(\nu+2)y^\nu}{\Gamma\left(\nu+\frac{3}{2}\right)2^\nu}\left[1-\Phi\left(\frac{1}{2}y\right)\right] \qquad [y>0,\ \operatorname{Re}\nu>-1].$$

ET II 82(2)

8. $$\int_0^\infty e^{-\frac{1}{4}x^2} M_{k,\,\frac{1}{2}\nu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{2^{-k}\Gamma(\nu+1)}{\Gamma\left(k+\frac{1}{2}\nu+\frac{1}{2}\right)} y^{2k-1} e^{-\frac{1}{2}y^2}$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}k<\frac{1}{2}\right].$$

ET II 83(7)

ET II 83(9)

10. $$\int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2} W_{k,\,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = \frac{1}{2} + k + \nu - 3\mu, \quad 2\beta = \frac{1}{2} - k + \nu - \mu$$
$$[y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1].$$

ET II 84(14)

11. $$\int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} W_{k,\,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{\Gamma(1+\nu-2\mu)}{\Gamma\left(\frac{1}{2}+\mu-k\right)} 2^{\frac{1}{2}\left(\frac{1}{2}+k-3\mu+\nu\right)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2} W_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = k + 3\mu - \nu - \frac{1}{2}, \qquad 2\beta = k - \mu + \nu + \frac{1}{2}$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1, \quad \operatorname{Re}\left(k - \mu + \frac{1}{2}\nu\right) < -\frac{1}{4}\right].$$

ET II 84(15)

12. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\,\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx =$$
$$= \frac{\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k-\mu+\nu\right)} 2^{\frac{1}{2}\left(\frac{1}{2}-k+3\mu-\nu\right)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = \frac{1}{2} + k + 3\mu - \nu, \qquad 2\beta = -\frac{1}{2} + k - \mu + \nu$$
$$\left[y > 0, \quad -\frac{1}{2} < \operatorname{Re}\mu < \operatorname{Re}\left(k + \frac{1}{2}\nu\right) - \frac{1}{4}\right].$$

ET II 83(8)

13. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\,\mu}\left(\frac{1}{2}x^2\right) N_\nu(xy)\,dx =$$
$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} \Gamma(2\mu+1) \times$$
$$\times \Gamma\left(\frac{1}{2} - k - \mu\right) \left\{\cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)} M_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right) - \right.$$
$$\left. - \sin[(\nu+k-\mu)\pi] W_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right)\right\}, \qquad 2\alpha = 3\mu - \nu + k + \frac{1}{2}, \qquad 2\beta = \mu - \nu - k + \frac{1}{2}$$
$$\left[y > 0, \quad -1 < 2\operatorname{Re}\mu < \operatorname{Re}(2k+\nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu - \nu) > -1\right].$$

14. $\int_0 x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\,\mu}\left(\frac{1}{2}x^2\right) N_\nu(xy)\, dx =$

$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} \Gamma(2\mu+1) \times$$

$$\times \Gamma\left(\frac{1}{2} - \mu - k\right) e^{-\frac{1}{4}y^2} \left\{ \cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma\left(\mu+\nu-k+\frac{3}{2}\right)} M_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right) + \right.$$

$$\left. + \sin[(\mu-k)\pi] W_{\alpha,\,\beta}\left(\frac{1}{2}y^2\right) \right\}, \qquad 2\alpha = 3\mu+\nu+k+\frac{1}{2}, \qquad 2\beta = \mu+\nu-k+\frac{1}{2}$$

$$\left[y > 0, \quad -1 < 2 \operatorname{Re}\mu < \operatorname{Re}(2k-\nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1\right].$$

ET II 116(43)

883

15. $\int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\,\mu}(ax^2) K_\nu(xy)\, dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}} \times$

$$\times \Gamma(2\mu+1)\Gamma(2\mu+\nu+1) \exp\left(\frac{y^2}{8a}\right) W_{\{,\,m}\left(\frac{y^2}{4a}\right),$$

$$2\{ = -3\mu-\nu-k-\frac{1}{2}, \qquad 2m = \mu+\nu-k+\frac{1}{2}$$

$$\left[\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1\right].$$

ET II 152(82)

7.673

1.* $\int_0^\infty e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\{,\,\frac{1}{2}\mu}(ax) J_\nu(2\sqrt{bx})\, dx =$

$$= \left(\frac{b}{a}\right)^{\frac{\{-1}{2}-\frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)} \Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma\left(1+\frac{\{+\nu}{2}-\frac{1+\mu}{4}\right)} \times$$

$$\times M_{\frac{1}{2}(\{-\nu-1)+\frac{3}{4}(1+\mu),\; \frac{\{+\nu}{2}-\frac{1+\mu}{4}}\left(\frac{b}{a}\right)$$

$$\left[\operatorname{Re}(1+\mu) > 0, \quad \operatorname{Re}\left(\{+\frac{\nu-\mu}{2}\right) > -\frac{3}{4}, \quad \operatorname{Im}\, b = 0\right].$$

BU 128(12)a

2. $\int_0^\infty e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\{,\,\frac{1}{2}\mu}(ax) J_\nu(2\sqrt{bx})\, dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu) e^{\frac{b}{2a}}}{\Gamma\left(\frac{1\pm\mu}{2}-\{\right)} \left(\frac{a}{b}\right)^{\frac{1}{2}(\{+1)+\frac{1}{4}(1\mp\nu)} \times$

$$\times W_{\frac{1}{2}(\{+1-\nu)-\frac{3}{4}(1\mp\mu),\; \frac{1}{2}(\{+\nu)+\frac{1}{4}(1\mp\mu)}\left(\frac{b}{a}\right)$$

$$\left[\operatorname{Re}\left(\frac{\nu\mp\mu}{2}+\{\right) < \frac{3}{4}, \quad \operatorname{Re}\nu > -1\right].$$

BU 128(13)

7.674

ET II 409(37)

2. $$\int_0^\infty x^{\varrho-1} e^{-\frac{1}{2}\{} I_{\lambda+\nu}(ax^{\frac{1}{2}}) K_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{k,\mu}(x)\, dx = \frac{\pi^{-\frac{1}{2}}}{2} G_{45}^{24}\left(a^2 \middle| \begin{matrix} 0,\ \frac{1}{2},\ \frac{1}{2}+\mu-\varrho,\ \frac{1}{2}-\mu-\varrho \\ \lambda,\ \nu,\ -\lambda,\ -\nu,\ k-\varrho \end{matrix}\right)$$
$$\left[|\operatorname{Re}\mu| < \operatorname{Re}(\lambda+\varrho)+\frac{1}{2}, \quad |\operatorname{Re}\mu| < \operatorname{Re}(\nu+\varrho)+\frac{1}{2}\right].$$

ET II 409(38)

Combinations of Struve functions and confluent hypergeometric functions

7.675

1. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) H_\nu(xy)\, dx = \frac{2^{-\lambda}\Gamma(2\mu+1)}{y^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{22}\left(\frac{y^2}{2} \middle| \begin{matrix} l,\ -\mu-\lambda,\ \mu-\lambda \\ l,\ k-\lambda-\frac{1}{2},\ h,\ \{ \end{matrix}\right),$$
$$h = \frac{1}{4}+\frac{1}{2}\nu, \quad \{ = \frac{1}{4}-\frac{1}{2}\nu, \quad l = \frac{3}{4}+\frac{1}{2}\nu$$
$$\left[\operatorname{Re}(2\lambda+2\mu+\nu) > -\frac{7}{2},\ \operatorname{Re}(k-\lambda) > 0,\ y > 0,\ \operatorname{Re}(2\lambda-2k+\nu) < -\frac{1}{2}\right].$$

ET II 171(42)

2. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) H_\nu(xy)\, dx =$$
$$= 2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu}\pi^{-\frac{1}{2}} y^{\nu+1} \frac{\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda+\mu\right)\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda-\mu\right)}{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{9}{4}+\lambda-k-\frac{1}{2}\nu\right)} \times$$
$$\times\ {}_3F_3\left(1,\ \frac{7}{4}+\frac{\nu}{2}+\lambda+\mu,\ \frac{7}{4}+\frac{\nu}{2}+\lambda-\mu;\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{9}{4}+\lambda-k+\frac{\nu}{2};\ -\frac{y^2}{2}\right)$$
$$\left[\operatorname{Re}(2\lambda+\nu) > 2|\operatorname{Re}\mu| - \frac{7}{4}, \quad y > 0\right].$$

ET II 171(43)

3. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) H_\nu(xy)\, dx =$$
$$= \left[2^\lambda\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} y^{-\frac{1}{2}} G_{34}^{23}\left(\frac{y^2}{2} \middle| \begin{matrix} l,\ -\mu-\lambda,\ \mu-\lambda \\ l,\ -k-\lambda-\frac{1}{2},\ h,\ \{ \end{matrix}\right),$$
$$h = \frac{1}{4}+\frac{1}{2}\nu, \quad \{ = \frac{1}{4}-\frac{1}{2}\nu, \quad l = \frac{3}{4}+\frac{1}{2}\nu$$
$$\left[y > 0, \quad \operatorname{Re}(2\lambda+\nu) > 2|\operatorname{Re}\mu| - \frac{7}{2},\ \operatorname{Re}(2k+2\lambda+\nu) < -\frac{1}{2}, \quad \operatorname{Re}(k+\lambda) < 0\right].$$

4. $$\int_0^\infty e^{\frac{1}{2}x^2} W_{-\frac{1}{2}\nu-\frac{1}{2},\,\frac{1}{2}\nu}(x^2) H_\nu(xy)\,dx = 2^{-\nu-1} y^\nu \pi e^{\frac{1}{4}y^2}\left[1-\Phi\left(\frac{y}{2}\right)\right] \qquad [y>0, \quad \operatorname{Re}\nu > -1].$$

ET II 171(44)

7.68 Combinations of confluent hypergeometric functions and other special functions

Combinations of confluent hypergeometric functions and associated Legendre functions

7.681

1. $$\int_0^\infty x^{-\frac{1}{2}}(a+x)^\mu e^{-\frac{1}{2}x} P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right) M_{k,\,\mu}(x)\,dx =$$
$$= -\frac{\sin(\nu\pi)}{\pi\Gamma(k)}\Gamma(2\mu+1)\Gamma\left(k-\mu+\nu+\frac{1}{2}\right)\Gamma\left(k-\mu-\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}W_{\varrho,\,\sigma}(a),$$
$$\varrho = \frac{1}{2}-k+\mu, \qquad \sigma = \frac{1}{2}+\nu$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\mu) > \left|\operatorname{Re}\nu+\frac{1}{2}\right|\right].$$

ET II 403(11)

2. $$\int_0^\infty x^{-\frac{1}{2}}(a+x)^{-\mu} e^{-\frac{1}{2}x} P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right) M_{k,\,\mu}(x)\,dx =$$
$$= \frac{\Gamma(2\mu+1)\Gamma\left(k+\mu+\nu+\frac{1}{2}\right)\Gamma\left(k+\mu-\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(2\mu+\nu+1)\Gamma(2\mu-\nu)} W_{\frac{1}{2}-k-\mu,\,\frac{1}{2}+\nu}(a)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu) > \left|\operatorname{Re}\nu+\frac{1}{2}\right|\right].$$

ET II 403(12)

3. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} P^\mu_{k+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right) W_{k,\,\nu}(x)\,dx =$$
$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\,\sigma}(a), \quad 2\varrho = \frac{1}{2}+2\mu+\nu-k, \quad 2\sigma = k+3\nu-\frac{3}{2}$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\mu<1, \quad \operatorname{Re}(\mu+2\nu)<1].$$

ET II 407(32)

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P^\mu_{k+\mu+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right) W_{k,\,\nu}(x)\,dx =$$
$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\,\sigma}(a), \qquad 2\varrho = \frac{1}{2}-k+\nu, \qquad 2\sigma = k+2\mu+3\nu-\frac{3}{2}$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\mu<1, \quad \operatorname{Re}(\mu+2\nu)<1].$$

5. $$\int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}}(a+x)^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}Q^{\nu}_{\mu-k+\frac{3}{2}}\left(1+2\frac{x}{a}\right)M_{k,\,\nu}(x)\,dx =$$
$$= \frac{e^{\nu\pi i}\Gamma(1+2\mu-\nu)\Gamma(1+2\mu)\Gamma\left(\frac{5}{2}-k+\mu+\nu\right)}{2\Gamma\left(\frac{1}{2}+k+\mu\right)}a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)}e^{\frac{1}{2}a}W_{\varrho,\,\sigma}(a),$$
$$2\varrho=\frac{1}{2}-k-\mu+2\nu, \qquad 2\sigma=k-3\mu-\frac{3}{2}$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}(2\mu-\nu)>-1\right].$$

ET II 404(14)

7.682

1. $$\int_0^\infty x^{-\frac{1}{2}}e^{-\frac{1}{2}x}P^{-2\mu}_{\nu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\,\mu}(x)\,dx =$$
$$= \frac{\Gamma(2\mu+1)\Gamma\left(k+\frac{1}{2}\nu\right)\Gamma\left(k-\frac{1}{2}\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{2^{2\mu}a^{\frac{1}{4}}\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{2}\nu\right)}W_{\frac{3}{4}-k,\,\frac{1}{4}+\frac{1}{2}\nu}(a)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}k>\frac{1}{2}\operatorname{Re}\nu-\frac{1}{2}, \quad \operatorname{Re}k>-\frac{1}{2}\operatorname{Re}\nu\right].$$

ET II 404(13)

2. $$\int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1}(a+x)^{-\frac{1}{2}}e^{-\frac{1}{2}x}Q^{1-k+\mu-\nu}_{k-\mu-\nu-1}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\,\mu}(x)\,dx =$$
$$= e^{(1-k+\mu-\nu)\pi i}2^{\mu-k-\nu}a^{\frac{1}{2}(k+\mu-1)}\frac{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+2\mu)\Gamma(k+\mu+\nu)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}e^{\frac{1}{2}a}W_{\varrho,\,\sigma}(a),$$
$$\varrho=\frac{1}{2}-k-\frac{1}{2}\nu, \qquad \sigma=\mu+\frac{1}{2}\nu$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}(k+\mu+\nu)>0\right].$$

ET II 404(15)

3. $$\int_0^\infty x^{\nu-\frac{1}{2}}e^{-\frac{1}{2}x}Q^{2\mu-2\nu}_{2k-2\nu-3}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\,\mu}(x)\,dx =$$
$$= e^{2(\mu-\nu)\pi i}2^{2\mu-2\nu-1}a^{\frac{1}{2}(k+\mu-1)}e^{\frac{1}{2}a}\frac{\Gamma(2\mu+1)\Gamma(\nu+1)\Gamma\left(k+\mu-2\nu-\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}W_{\varrho,\,\sigma}(a),$$
$$2\varrho=1-k+\mu-2\nu, \qquad 2\sigma=k-\mu-2\nu-2$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(k+\mu-2\nu)>\frac{1}{2}\right].$$

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P^{\mu}_{2k+\mu+2\nu-3}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right] W_{k,\,\nu}(x)\,dx =$$
$$= \frac{2^{\mu}\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\sigma}(a),$$
$$2\varrho = 1-k+\mu+\nu, \qquad 2\sigma = k+\mu+3\nu-2$$
$$[|\arg a| < \pi, \quad \operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1].$$

ET II 408(34)

5.* $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} P^{\mu}_{2k+\mu+2\nu-2}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right] W_{k,\,\nu}(x)\,dx =$$
$$= \frac{2^{\mu}\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\,\sigma}(a),$$
$$2\varrho = \mu+\nu-k, \qquad 2\sigma = k+\mu+3\nu-1 \qquad [|\arg a| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 408(35)

A combination of confluent hypergeometric functions and orthogonal polynomials

7.683[7]

$$\int_0^1 e^{-\frac{1}{2}ax} x^{\alpha} (1-x)^{\frac{\mu-\alpha}{2}-1} L_n^{\alpha}(ax) M_{\{-\frac{1+\alpha}{2},\, \frac{\mu-\alpha-1}{1}}\,[a(1-x)]\,dx =$$
$$= \frac{\Gamma(\mu-\alpha)}{\Gamma(1+\mu)}\,\frac{\Gamma(1+n+\alpha)}{n!} a^{-\frac{1+\alpha}{2}} M_{\{+n,\,\frac{\mu}{2}}\,(a)$$
$$[\operatorname{Re} a > -1, \quad \operatorname{Re}(\mu-\alpha) > 0, \quad n = 0,\,1,\,2,\ldots].$$

BU 129(14b)

A combination of hypergeometric and confluent hypergeometric functions

7.684

$$\int_0^\infty x^{\varrho-1} e^{-\frac{1}{2}x} M_{\gamma+\varrho,\,\beta+\varrho+\frac{1}{2}}(x)\,{}_2F_1\left(\alpha,\,\beta;\,\gamma;\,-\frac{\lambda}{x}\right)\,dx =$$
$$= \frac{\Gamma(\alpha+\beta+2\varrho)\Gamma(2\beta+2\varrho)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta+\gamma+2\varrho)} \lambda^{\frac{1}{2}\beta+\varrho-\frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k,\,\mu}(\lambda);$$
$$k = \frac{1}{2}-\alpha-\frac{1}{2}\beta-\varrho, \qquad \mu = \frac{1}{2}\beta+\varrho$$
$$[|\arg\lambda| < \pi, \quad \operatorname{Re}(\beta+\varrho) > 0, \quad \operatorname{Re}(\alpha+\beta+2\varrho) > 0, \quad \operatorname{Re}\gamma > 0].$$

ET II 405(19)

7.69 Integration of confluent hypergeometric functions with respect to the index

7.691

$$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x) W_{ix,\,0}(\alpha) W_{-ix,\,0}(\beta)\, dx = 2\frac{(a\beta)^{\frac{1}{2}}}{\alpha+\beta} \exp\left[-\frac{1}{2}(\alpha+\beta)\right].$$

ET II 414(61)

7.692

$$\int_{-i\infty}^{i\infty} \Gamma(-a)\Gamma(c-a)\Psi(a,\, c;\, x)\Psi(c-a,\, c;\, y)\, da = 2\pi i \Gamma(c)\Psi(c,\, 2c;\, x+y).$$

EH I 285(15)

7.693

1. $$\int_{-\infty}^{\infty} \Gamma(ix)\Gamma(2k+ix) W_{k+ix,\, k-\frac{1}{2}}(\alpha) W_{-k-ix,\, k-\frac{1}{2}}(\beta)\, dx = \\ = 2\pi^{\frac{1}{2}}\Gamma(2k)(a\beta)^k(\alpha+\beta)^{\frac{1}{2}-2k} K_{2k-\frac{1}{2}}\left(\frac{a+\beta}{2}\right).$$

ET II 414(62)

2. $$\int_{-i\infty}^{i\infty} \Gamma\left(\frac{1}{2}+\nu+\mu+x\right)\Gamma\left(\frac{1}{2}+\nu+\mu-x\right) \times \\ = \times\Gamma\left(\frac{1}{2}+\nu-\mu+x\right)\Gamma\left(\frac{1}{2}+\nu-\mu-x\right) M_{\mu+ix,\,\nu}(\alpha) M_{\mu-ix,\,\nu}(\beta)\, dx = \\ = \frac{2\pi(a\beta)^{\nu+\frac{1}{2}}[\Gamma(2\nu+1)]^2\Gamma(2\nu+2\mu+1)\Gamma(2\nu-2\mu+1)}{(\alpha+\beta)^{2\nu+1}\Gamma(4\nu+2)} M_{2\mu,\, 2\nu+\frac{1}{2}}(\alpha+\beta) \\ \left[\operatorname{Re}\nu > |\operatorname{Re}\mu| - \frac{1}{2}\right].$$

ET II 413(59)

7.694

$$\int_{-\infty}^{\infty} e^{-2\varrho xi}\Gamma\left(\frac{1}{2}+\nu+ix\right)\Gamma\left(\frac{1}{2}+\nu-ix\right) M_{ix,\,\nu}(\alpha) M_{ix,\,\nu}(\beta)\, dx = \\ = \frac{2\pi(\alpha\beta)^{\frac{1}{2}}}{\operatorname{ch}\varrho} \exp[-(\alpha+\beta)\,\theta\varrho] J_{2\nu}\left(\frac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\operatorname{ch}\varrho}\right) \left[|\operatorname{Im}\ \varrho| < \frac{1}{2}\pi, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 414(60)

7.7 Parabolic-Cylinder Functions* See Whitaker, E. T., & Watson, G. N., *Modern Analysis*, Cambridge University Press 1952, page 437 for definition.

7.71 Parabolic-cylinder functions

7.711

1. $$\int_{-\infty}^{\infty} D_n(x) D_m(x)\,dx = 0 \qquad [m \neq n];$$
$$= n!(2\pi)^{\frac{1}{2}} \qquad [m = n].$$

WH

2. $$\int_0^{\infty} D_\mu(\pm t) D_\nu(t)\,dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu - \nu}\left[\frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu\right)\Gamma\left(-\frac{1}{2}\nu\right)} \mp \frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu\right)}\right]$$
[when the lower sign is taken, $\operatorname{Re}\mu > \operatorname{Re}\nu$].

BU 11 117(13a), EH II 122(21)

3. $$\int_0^{\infty} [D_\nu(t)]^2\,dt = \pi^{\frac{1}{2}} 2^{-\frac{3}{2}} \frac{\psi\left(\frac{1}{2}-\frac{1}{2}\nu\right) - \psi\left(-\frac{1}{2}\nu\right)}{\Gamma(-\nu)}.$$

BU 117(13b)A, EH II 122(22)a

7.72 Combinations of parabolic-cylinder functions, powers, and exponentials

7.721

1. $$\int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2}(x-z)^{-1} D_n(x)\,dx = \pm i e^{\mp n\pi i}(2\pi)^{\frac{1}{2}} n! e^{-\frac{1}{4}z^2} D_{-n-1}(\mp iz)$$

2. $$\int_1^\infty x^\nu (x-1)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp\left[-\frac{(x-1)^2a^2}{4}\right] D_\mu(ax)\,dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2}-\frac{\nu}{2}-1}\Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$$
$$[\operatorname{Re}(\mu-\nu) > 0].$$

ET II 395(4)a
WH

7.722

1. $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu+1}(x)\,dx = 2^{-\frac{1}{2}-\frac{1}{2}\nu}\Gamma(\nu+1)\sin\frac{1}{4}(1-\nu)\pi \qquad [\operatorname{Re}\nu > -1].$$

WH

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\mu-1} D_{-\nu}(x)\,dx = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{2}\mu-\frac{1}{2}\nu}\Gamma(\mu)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)} \qquad [\operatorname{Re}\mu > 0].$$

EH II 122(20)

3. $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu-1}(x)\,dx = 2^{-\frac{1}{2}\nu-1}\Gamma(\nu)\sin\frac{1}{4}\pi\nu \qquad [\operatorname{Re}\nu > -1].$$

ET II 395(2)

7.723

1. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^\nu (x^2+y^2)^{-1} D_\nu(x)\,dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Gamma(\nu+1)y^{\nu-1}e^{\frac{1}{4}y^2}D_{-\nu-1}(y) \qquad [\operatorname{Re}y > 0, \quad \operatorname{Re}\nu > -1].$$

EH II 121(18)A, ET II 396(6)a

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\nu-1} (x^2+y^2)^{-\frac{1}{2}} D_\nu(x)\,dx = y^{\nu-1}\Gamma(\nu)e^{\frac{1}{4}y^2}D_{-\nu}(y) \qquad [\operatorname{Re}y > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 396(7)

3. $$\int_0^1 x^{2\nu-1}(1-x^2)^{\lambda-1}e^{\frac{a^2x^2}{4}} D_{-2\lambda-2\nu}(ax)\,dx = \frac{\Gamma(\lambda)\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)}2^{\lambda-1}e^{\frac{a^2}{4}}D_{-2\nu}(a)$$
$$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 395(3)a

7.724

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2} D_\nu(x)\,dx = (2\pi\mu)^{\frac{1}{2}}(1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}} D_\nu[y(1-\mu)^{-\frac{1}{2}}] \qquad [0 < \operatorname{Re}\mu < 1].$$

EH II 121(15)

7.725

1. $$\int_0^{\infty} e^{-pt}(2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu-2}(\sqrt{2t})\,dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(\sqrt{p+1}-1)^{\nu+1}}{(\nu+1)p^{\nu+1}} \qquad [\operatorname{Re}\nu > -1].$$

MO 175

2. $$\int_0^{\infty} e^{-pt}(2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu}(\sqrt{2t})\,dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(\sqrt{p+1}-1)^{\nu}}{p^{\nu}\sqrt{p+1}} \qquad [\operatorname{Re}\nu > -1].$$

MO 175

3. $$\int_0^{\infty} e^{-bx} D_{2n+1}(\sqrt{2x})\,dx = (-2)^n \Gamma\left(n+\frac{3}{2}\right)\left(b-\frac{1}{2}\right)^n \left(b+\frac{1}{2}\right)^{-n-\frac{3}{2}} \qquad \left[\operatorname{Re} b > -\frac{1}{2}\right].$$

ET I 210(3)

4. $$\int_0^{\infty} (\sqrt{x})^{-1} e^{-bx} D_{2n}(\sqrt{2x})\,dx = (-2)^n \Gamma\left(n+\frac{1}{2}\right)\left(b-\frac{1}{2}\right)^n \left(b+\frac{1}{2}\right)^{-n-\frac{1}{2}} \qquad \left[\operatorname{Re} b > -\frac{1}{2}\right].$$

ET I 210(5)

5. $$\int_0^{\infty} x^{-\frac{1}{2}(\nu+1)} e^{-sx} D_\nu(\sqrt{x})\,dx = \sqrt{\pi}\left(1+\sqrt{\frac{1}{2}+2s}\right)^{\nu} \frac{1}{\sqrt{\frac{1}{4}+s}} \qquad \left[\operatorname{Re} s > -\frac{1}{4}, \quad \operatorname{Re}\nu < 1\right].$$

ET I 210(7)

6. $$\int_0^{\infty} e^{-zt} t^{-1+\frac{\beta}{2}} D_{-\nu}[2(kt)^{\frac{1}{2}}]\,dt = \frac{2^{1-\beta-\frac{\nu}{2}}\pi^{\frac{1}{2}}\Gamma(\beta)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}\right)}(z+k)^{-\frac{\beta}{2}} F\left(\frac{\nu}{2}, \frac{\beta}{2}; \frac{\nu+\beta+1}{2}; \frac{z-k}{z+k}\right)$$
$$\left[\operatorname{Re}(z+k) > 0, \quad \operatorname{Re}\frac{z}{k} > 0\right].$$

$$\int_{-\infty}^{\infty} e^{ixy-\frac{(1+\lambda)x^2}{4}} D_\nu[x(1-\lambda)^{\frac{1}{2}}]\,dx = (2\pi)^{\frac{1}{2}}\lambda^{\frac{1}{2}\nu} e^{-\frac{(1+\lambda)y^2}{4\lambda}} D_\nu[i(\lambda^{-1}-1)^{\frac{1}{2}}y] \qquad [\operatorname{Re}\lambda > 0].$$

EH II 121(16)

7.727

$$\int_0^\infty \frac{e^{\frac{1}{2}x}e^{-bx}}{(e^x-1)^{\mu+\frac{1}{2}}} \exp\left(-\frac{a}{1-e^{-x}}\right) D_{2\mu}\left(\frac{2\sqrt{a}}{\sqrt{1-e^{-x}}}\right) dx = e^{-a}2^{b+\mu}\Gamma(b+\mu)D_{-2b}(2\sqrt{a})$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > -\operatorname{Re}\mu].$$

ET I 211(13)

7.728

$$\int_0^\infty (2t)^{-\frac{\nu}{2}} e^{-pt} e^{-\frac{q^2}{8t}} D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} p^{\frac{1}{2}\nu-1} e^{-q\sqrt{p}}.$$

MO 175

7.73 Combinations of parabolic-cylinder and hyperbolic functions

7.731

1. $$\int_0^\infty \operatorname{ch}(2\mu x)\, \exp[-(a \operatorname{sh} x)^2] D_{2k}(2a \operatorname{ch} x)\,dx = 2^{k-\frac{3}{2}}\pi^{\frac{1}{2}}a^{-1}W_{k,\,\mu}(2a^2) \qquad [\operatorname{Re} a^2 > 0].$$

ET II 398(20)

2. $$\int_0^\infty \operatorname{ch}(2\mu x)\, \exp[(a \operatorname{sh} x)^2] D_{2k}(2a \operatorname{ch} x)\,dx = \frac{\Gamma(\mu-k)\Gamma(-\mu-k)}{2^{k+\frac{5}{2}}a\Gamma(-2k)} W_{k+\frac{1}{2},\,\mu}(2a^2)$$
$$\left[|\arg a| < \frac{3\pi}{4}, \quad \operatorname{Re} k + |\operatorname{Re}\mu| < 0\right].$$

ET II 398(21)

7.74 Combinations of parabolic-cylinder and trigonometric functions

7.741

1. $$\int_0^\infty \sin(bx)\{[D_{-n-1}(ix)]^2-[D_{-n-1}(-ix)]^2\}\,dx = (-1)^{n+1}\frac{i}{n!}\pi\sqrt{2\pi}e^{-\frac{1}{2}b^2}L_n(b^2) \qquad [b > 0].$$

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) D_{2n+1}(x)\,dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} \qquad [b > 0].$$

ET I 115(1)

3. $$\int_0^\infty e^{-\frac{1}{4}x^2} \cos(bx) D_{2n}(x)\,dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} \qquad [b > 0].$$

ET I 60(2)

4. $$\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx)[D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x)]\,dx = \sqrt{2\pi} \sin\left[\left(\nu - \frac{1}{4}\right)\pi\right] b^{2\nu-\frac{1}{2}} e^{-\frac{1}{2}b^2}$$
$$\left[\operatorname{Re}\nu > \frac{1}{4}, \quad b > 0\right].$$

ET I 115(2)

5. $$\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx)[D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x)]\,dx = \frac{2^{\frac{1}{4}-2\nu}\sqrt{\pi} b^{2\nu-\frac{1}{2}} e^{-\frac{1}{4}b^2}}{\operatorname{cosec}\left[\left(\nu + \frac{1}{4}\right)\pi\right]} \qquad \left[\operatorname{Re}\nu > \frac{1}{4}, \quad b > 0\right].$$

ET I 61(4)

7.742

1. $$\int_0^\infty x^{2\varrho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x)\,dx = 2^{\nu-\varrho-\frac{1}{2}} \pi^{\frac{1}{2}} a \frac{\Gamma(2\varrho+1)}{\Gamma(\varrho-\nu+1)} \times$$
$$\times {}_2F_2\left(\varrho + \frac{1}{2}, \varrho + 1; \frac{3}{2}, \varrho - \nu + 1; -\frac{a^2}{2}\right) \qquad \left[\operatorname{Re}\varrho > -\frac{1}{2}\right].$$

ET II 396(8)

2. $$\int_0^\infty x^{2\varrho-1} \sin(ax) e^{\frac{x^2}{4}} D_{2\nu}(x)\,dx = \frac{2^{\varrho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22}\left(\frac{a^2}{2}\middle|\begin{matrix}\frac{1}{2}-\varrho, 1-\varrho \\ -\varrho-\nu, \frac{1}{2}, 0\end{matrix}\right)$$
$$\left[a > 0, \quad \operatorname{Re}\varrho > -\frac{1}{2}, \quad \operatorname{Re}(\varrho+\nu) < \frac{1}{2}\right].$$

ET II 396(9)

3. $$\int_0^\infty x^{2\varrho-1} \cos(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x)\,dx = \frac{2^{\nu-\varrho}\Gamma(2\varrho)\pi^{\frac{1}{2}}}{\Gamma\left(\varrho-\nu+\frac{1}{2}\right)} {}_2F_2\left(\varrho, \varrho + \frac{1}{2}; \frac{1}{2}, \varrho - \nu + \frac{1}{2}; -\frac{a^2}{2}\right)$$
$$[\operatorname{Re}\varrho > 0].$$

4. $$\int_0^\infty x^{2\varrho-1}\cos(ax)e^{\frac{x^2}{4}}D_{2\nu}(x)\,dx = \frac{2^{\varrho-\nu-2}}{\Gamma(-2\nu)}G_{23}^{22}\left(\frac{a^2}{2}\middle|\begin{matrix}\frac{1}{2}-\varrho,\ 1-\varrho\\ -\varrho-\nu,\ 0,\ \frac{1}{2}\end{matrix}\right)$$
$$\left[a>0,\quad \operatorname{Re}\varrho>0,\quad \operatorname{Re}(\varrho+\nu)<\frac{1}{2}\right].$$

ET II 396(11)

7.743

$$\int_0^{\frac{\pi}{2}}(\cos x)^{-\mu-2}(\sin x)^{-\nu}D_\nu(a\sin x)D_\mu(a\cos x)\,dx = -\left(\frac{1}{2}\pi\right)^{\frac{1}{2}}(1+\mu)^{-1}D_{\mu+\nu+1}(a)$$
$$[\operatorname{Re}\nu<1,\quad \operatorname{Re}\mu<-1].$$

ET II 397(19)

7.744

1. $$\int_0^\infty \sin(bx)[D_{-\nu-\frac{1}{2}}(\sqrt{2x})-D_{-\nu-\frac{1}{2}}(-\sqrt{2x})]D_{\nu-\frac{1}{2}}(\sqrt{2x})\,dx =$$
$$= -\sqrt{2\pi}\sin\left[\left(\frac{1}{4}+\frac{1}{2}\nu\right)\pi\right]b^{-\nu-\frac{1}{2}}\frac{(1+\sqrt{1+b^2})^\nu}{\sqrt{1+b^2}}\qquad [b>0].$$

ET I 115(4)

2. $$\int_0^\infty \cos(bx)[D_{-2\nu-\frac{1}{2}}(\sqrt{2x})+D_{-2\nu-\frac{1}{2}}(-\sqrt{2x})]D_{2\nu-\frac{1}{2}}(\sqrt{2x})\,dx =$$
$$= -\frac{\sqrt{\pi}\sin\left[\left(\nu-\frac{1}{4}\right)\pi\right](1+\sqrt{1+b^2})^{2\nu}}{\sqrt{1+b^2}\,b^{2\nu+\frac{1}{2}}}\qquad [b>0].$$

ET I 60(3)

7.75 Combinations of parabolic-cylinder and Bessel functions

7.751

1. $$\int_0^\infty [D_n(ax)]^2J_1(xy)\,dx = (-1)^{n-1}y^{-1}\left[D_n\left(\frac{y}{a}\right)\right]^2\qquad [y>0].$$

ET II 20(24)

2. $$\int_0^\infty J_0(xy)D_n(ax)D_{n+1}(ax)\,dx = (-1)^ny^{-1}D_n\left(\frac{y}{a}\right)D_{n+1}\left(\frac{y}{a}\right)\qquad \left[y>0,\quad |\arg a|<\frac{1}{4}\pi\right]$$

3. $$\int_0^\infty J_0(xy)D_\nu(x)D_{\nu+1}(x)\,dx = 2^{-1}y^{-1}[D_\nu(-y)D_{\nu+1}(y) - D_{\nu+1}(-y)D_\nu(y)].$$

ET II 397(17)a

7.752

1. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu-1}(x)J_\nu(xy)\,dx = -\frac{1}{2}\sec(\nu\pi)y^{\nu-1}e^{-\frac{1}{4}y^2}[D_{2\nu-1}(y) - D_{2\nu-1}(-y)]$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 76(1), MO 183

2. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{2\nu-1}(x)J_\nu(xy)\,dx = 2^{\frac{1}{2}-\nu}\pi\sin(\nu\pi)y^{-\nu}\Gamma(2\nu)e^{\frac{1}{4}y^2}K_\nu\left(\frac{1}{4}y^2\right)$$
$$\left[y > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right].$$

ET II 77(4)

3. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu}(x)J_\nu(xy)\,dx = \frac{1}{2}\sec(\nu\pi)y^{\nu-1}e^{-\frac{1}{4}y^2}[D_{2\nu+1}(y) - D_{2\nu+1}(-y)]$$
$$[y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 78(13)

4. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu+1}(x)J_\nu(xy)\,dx = \frac{1}{2}\sec(\nu\pi)e^{-\frac{1}{4}y^2}y^\nu[D_{2\nu}(y) + D_{2\nu}(-y)]$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 77(5)

5. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu+2}(x)J_\nu(xy)\,dx = -\frac{1}{2}\sec(\nu\pi)y^\nu e^{-\frac{1}{4}y^2}[D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$$
$$[\operatorname{Re}\nu > -1, \quad y > 0].$$

ET II 78(16)

6. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{2\nu+2}(x)J_\nu(xy)\,dx = \pi^{-1}\sin(\nu\pi)\Gamma(2\nu+3)y^{-\nu-2}e^{\frac{1}{4}y^2}K_{\nu+1}\left(\frac{1}{4}y^2\right)$$
$$\left[y > 0, \quad -1 < \operatorname{Re}\nu < -\frac{5}{6}\right].$$

7. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy)\,dx = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{-\nu} e^{-\frac{1}{4}y^2} I_\nu\left(\frac{1}{4}y^2\right) \qquad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 77(8)

8. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy)\,dx = y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu}(y) \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}, \quad y > 0\right].$$

ET II 77(9), EH II 121(17)

9. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu-2}(x) J_\nu(xy)\,dx = (2\nu+1)^{-1} y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-1}(y) \qquad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 77(10)

10. $$\int_0^\infty x^\nu e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy)\,dx = \frac{2^{\mu-\frac{1}{2}} \Gamma\left(\nu+\frac{1}{2}\right) y^\nu}{\Gamma(\nu-\mu+1) a^{1+2\nu}} {}_1F_1\left(\nu+\frac{1}{2}; \nu-\mu+1; -\frac{y^2}{2a^2}\right)$$
$$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 77(11)

11. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy)\,dx = \frac{\Gamma\left(\frac{1}{2}+\nu\right) a^{2k} 2^{m+\mu}}{\Gamma\left(\frac{1}{2}-\mu\right) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{4a^2}\right),$$
$$2k = \frac{1}{2}+\mu-\nu, \quad 2m = \frac{1}{2}+\mu+\nu$$
$$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re}\nu < \operatorname{Re}\left(\frac{1}{2}-2\mu\right)\right].$$

ET II 78(12)

12. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy)\,dx = \frac{2^\mu \Gamma\left(\nu+\frac{3}{2}\right) y^\nu}{\Gamma\left(\nu-\mu+\frac{3}{2}\right) a^{2\nu+2}} {}_1F_1\left(\nu+\frac{3}{2}; \nu-\mu+\frac{3}{2}; -\frac{y^2}{2a^2}\right)$$
$$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re}\nu > -1\right].$$

13. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy)\, dx \frac{\Gamma\left(\frac{3}{2}+\nu\right) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu) y^{\mu+2}} e^{\frac{y^2}{4a^2}} W_{k,\,m}\left(\frac{y^2}{2a^2}\right),$$
$$2k = \mu - \nu - 1, \quad 2m = \mu + \nu + 1$$
$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2} - 2\operatorname{Re}\mu\right].$$

ET II 79(24)

14. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2x^2} D_\mu(ax) J_\nu(xy)\, dx = \frac{2^{\lambda-\frac{1}{2}\mu}\pi^{-\frac{1}{2}}}{\Gamma(-\mu) y^{\lambda+\frac{3}{2}}} G_{23}^{22}\left(\frac{y^2}{2a^2}\middle|\begin{matrix}\frac{1}{2}, 1\\ \frac{3}{4}+\frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2}\end{matrix}\right)$$
$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad \operatorname{Re}\mu < -\operatorname{Re}\lambda < \operatorname{Re}\nu + \frac{3}{2}\right].$$

ET II 80(26)

15. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-1}(x) J_\nu(xy)\, dx = (2\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu-2}(y) \qquad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 79(20)

16. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy)\, dx = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{-\nu-2} e^{-\frac{1}{4}y^2} I_{\nu+1}\left(\frac{1}{4}y^2\right) \qquad [y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 79(21)

17. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy)\, dx = y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-3}(y) \qquad [y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 79(22)

18. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{\frac{1}{2}\nu-\frac{1}{2}}(ax) N_\nu(xy)\, dx = -\pi^{-1} 2^{\frac{3}{4}\nu+\frac{3}{4}} a^{-\nu} y^{-1} \Gamma(\nu+1) e^{\frac{y^2}{4a^2}} W_{-\frac{1}{2}\nu-\frac{1}{2},\,\frac{1}{2}\nu}\left(\frac{y^2}{2a^2}\right)$$
$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{2}{3}\right].$$

ET II 115(39)

7.753

1. $$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2} I_{\nu-\frac{1}{2}}(2ax) D_\nu(2x)\, dx = \frac{1}{2}\pi^{-\frac{1}{2}} \Gamma(\nu) a^{\nu-\frac{1}{2}} D_{-\nu}(2a) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 0].$$

ET II 397(12)

ET II 397(13)

7.754

1. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu-1}(x) - D_{2\nu-1}(-x)\}J_\nu(xy)\,dx =$$
$$= \pm y^{\nu-1}e^{-\frac{1}{4}y^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu-1}(y) - D_{2\nu-1}(-y)\} \qquad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 76(2, 3)

2. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu+1}(x) - D_{2\nu+1}(-x)\}J_\nu(xy)\,dx =$$
$$= \mp y^{\nu}e^{-\frac{1}{4}y^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu}(y) + D_{2\nu}(-y)\} \qquad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 77(6, 7)

3. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2}\{[1 \pm 2\,\cos(\nu\pi)]D_{2\nu}(x) + D_{2\nu}(-x)\}J_\nu(xy)\,dx =$$
$$= \pm y^{\nu-1}e^{-\frac{1}{4}y^2}\{[1 \pm 2\,\cos(\nu\pi)]D_{2\nu+1}(y) - D_{2\nu+1}(-y)\} \qquad [y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 78(14, 15)

4. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu+2}(x) + D_{2\nu+2}(-x)\}J_\nu(xy)\,dx =$$
$$= \pm y^{\nu}e^{-\frac{1}{4}y^2}\{[1 \mp 2\,\cos(\nu\pi)]D_{2\nu+2}(y) + D_{2\nu+2}(-y)\} \qquad [y > 0, \quad \operatorname{Re}\nu > -1].$$

ET II 78(17, 18)

7.755

1. $$\int_0^\infty x^{-\frac{1}{2}} D_\nu(a^{\frac{1}{2}}x^{\frac{1}{2}})D_{-\nu-1}(a^{\frac{1}{2}}x^{\frac{1}{2}})J_0(xy)\,dx =$$
$$= 2^{-\frac{3}{2}}\pi a^{-\frac{1}{2}} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}}\left[\left(1+\frac{4y^2}{a^2}\right)^{\frac{1}{2}}\right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}}\left[\left(1+\frac{4y^2}{a^2}\right)^{\frac{1}{2}}\right] \qquad [y > 0,\ \operatorname{Re}a > 0].$$

ET II 17(43)

ET II 80(27)

3. $$\int_0^\infty D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i}x^{-\frac{1}{2}})D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i}x^{-\frac{1}{2}})J_\nu(xy)\,dx = 2^{\frac{1}{2}}\pi^{\frac{1}{2}}y^{-1}\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^{-1}\exp[-a(2y)^{\frac{1}{2}}]$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right].$$

ET II 80(28)a

4. $$\int_0^\infty x^{\frac{1}{2}}D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})N_\nu(xy)\,dx = y^{-\frac{3}{2}}\exp(-ay^{\frac{1}{2}})\sin\left[ay^{\frac{1}{2}}-\frac{1}{2}\left(\nu-\frac{1}{2}\right)\pi\right].$$
$$\left[y>0, \quad |\arg a|<\frac{1}{4}\pi\right].$$

ET II 115(40)

5. $$\int_0^\infty x^{\frac{1}{2}}D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})K_\nu(xy)\,dx = 2^{-1}y^{-\frac{3}{2}}\pi\,\exp[-a(2y)^{\frac{1}{2}}]$$
$$[\operatorname{Re} y>0, \quad |\arg a|<\frac{1}{4}\pi].$$

ET II 151(81)

Combinations of parabolic-cylinder and Struve functions

7.756

$$\int_0^\infty x^{-\nu}e^{-\frac{1}{4}x^2}[D_\mu(x)-D_\mu(-x)]\mathbf{H}_\nu(xy)\,dx =$$
$$= \frac{2^{\frac{3}{2}}\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu+\nu+1\right)}y^{\mu+\nu}\sin\left(\frac{1}{2}\mu\pi\right)\,{}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2};\ \frac{1}{2}\mu+\nu+1;\ -\frac{1}{2}y^2\right)$$
$$\left[y>0, \quad \operatorname{Re}(\mu+\nu)>-\frac{3}{2}, \quad \operatorname{Re}\mu>-1\right].$$

ET II 171(41)

7.76 Combinations of parabolic-cylinder functions and degenerate hypergeometric functions

7.761

1. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1} D_{-\nu}(t)\, {}_1F_1\left(a;\, c;\, -\frac{1}{2}pt^2\right)\, dt =$$
$$= \frac{\pi^{\frac{1}{2}}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c)\Gamma\left(\frac{1}{2}\nu - c + a\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}+\frac{1}{2}\nu\right)} F\left(a,\, c+\frac{1}{2};\, a+\frac{1}{2}+\frac{1}{2}\nu;\, 1-p\right)$$
$$[|1-p| < 1, \quad \operatorname{Re} c > 0, \quad \operatorname{Re}\nu > 2\operatorname{Re}(c-a)].$$

EH II 121(12)

2. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2} D_{-\nu}(t)\, {}_1F_1\left(a;\, c;\, -\frac{1}{2}pt^2\right)\, dt =$$
$$= \frac{\pi^{\frac{1}{2}}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1)\Gamma\left(\frac{1}{2}\nu + \frac{1}{2} - c + a\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}\nu\right)} F\left(a,\, c-\frac{1}{2};\, a+\frac{1}{2}\nu;\, 1-p\right)$$
$$\left[|1-p| < 1, \quad \operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re}\nu > 2\operatorname{Re}(c-a) - 1\right].$$

EH II 121(13)

7.77 Integration of a parabolic-cylinder function with respect to the index

7.771

$$\int_0^\infty \cos(ax) D_{x-\frac{1}{2}}(\beta) D_{-x-\frac{1}{2}}(\beta)\, dx = \frac{1}{2}\left(\frac{\pi}{\cos a}\right)^{\frac{1}{2}} \exp\left(-\frac{\beta^2\cos a}{2}\right) \qquad \left[|a| < \frac{1}{2}\pi\right];$$
$$= 0 \qquad \left[|a| > \frac{1}{2}\pi\right].$$

ET II 298(22)

7.772

1. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left[\frac{\left(\operatorname{tg}\frac{1}{2}\varphi\right)^\nu}{\cos\frac{1}{2}\varphi} D_\nu(-e^{\frac{1}{4}i\pi}\xi) D_{-\nu-1}(e^{\frac{1}{4}i\pi}\eta) + \right.$$
$$\left. + \frac{\left(\operatorname{ctg}\frac{1}{2}\varphi\right)^\nu}{\sin\frac{1}{2}\varphi} D_{-\nu-1}(e^{\frac{1}{4}i\pi}\xi) D_\nu(-e^{\frac{1}{4}i\pi}\eta)\right] \frac{d\nu}{\sin\nu\pi}$$
$$= -2i(2\pi)^{\frac{1}{2}} \exp\left[-\frac{1}{4}i(\xi^2-\eta^2)\cos\varphi - \frac{1}{2}i\xi\eta\sin\varphi\right].$$

2. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\left(\operatorname{tg}\frac{1}{2}\varphi\right)^{\nu}}{\cos\frac{1}{2}\varphi} D_{\nu}(-e^{\frac{1}{4}i\pi}\zeta) D_{-\nu-1}(e^{\frac{1}{4}i\pi}\eta) \frac{d\nu}{\sin\nu\pi} =$$
$$= -2iD_0\left[e^{\frac{1}{4}i\pi}\left(\zeta\,\cos\frac{1}{2}\varphi + \eta\,\sin\frac{1}{2}\,\varphi\right)\right] D_{-1}\left[e^{\frac{1}{4}i\pi}\left(\eta\,\cos\frac{1}{2}\varphi - \zeta\,\sin\frac{1}{2}\varphi\right)\right].$$

EH II 125(8)

7.773

1. $$\int_{c-i\infty}^{c+i\infty} D_{\nu}(z) t^{\nu} \Gamma(-\nu)\, d\nu = 2\pi i e^{-\frac{1}{4}z^2 - zt - \frac{1}{2}t^2} \qquad \left[c < 0, \quad |\arg t| < \frac{\pi}{4}\right].$$

EH II 126(10)

2. $$\int_{c-i\infty}^{c+i\infty} \left[D_{\nu}(x) D_{-\nu-1}(iy) + D_{\nu}(-x) D_{-\nu-1}(iy)\right] \frac{t^{-\nu-1}\, d\nu}{\sin(-\nu\pi)} =$$
$$= \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}} (1+t^2)^{-\frac{1}{2}} \exp\left[\frac{1}{4}\frac{1-t^2}{1+t^2}(x^2+y^2) + i\frac{txy}{1+t^2}\right] \qquad \left[-1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi\right].$$

EH II 126(11)

7.774

$$\int_{c-i\infty}^{c+i\infty} D_{\nu}\left[k^{\frac{1}{2}}(1+i)\xi\right] D_{-\nu-1}\left[k^{\frac{1}{2}}(1+i)\eta\right] \Gamma\left(-\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right) d\nu =$$
$$= 2^{\frac{1}{2}}\pi^2 H_0^{(2)}\left[\frac{1}{2}k(\xi^2+\eta^2)\right] \qquad [-1 < c < 0, \quad \operatorname{Re} ik \geq 0].$$

EH II 125(9)

7.8 Meijer's and MacRobert's Functions(*G* and *E*)

7.81 Combinations of the functions G and E and the elementary functions

7.811

1. $$\int_0^{\infty} G_{p,\,q}^{m,\,n}\left(\eta x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) G_{\sigma,\,\tau}^{\mu,\,\nu}\left(\omega x \,\middle|\, \begin{matrix} c_1, \dots, c_{\sigma} \\ d, \dots, d_{\tau} \end{matrix}\right) dx =$$
$$= \frac{1}{\eta} G_{q+\sigma,\,p+\tau}^{n+\mu,\,m+\nu}\left(\frac{\omega}{\eta} \,\middle|\, \begin{matrix} -b_1, \dots, -b_m,\ c_1, \dots, c_{\sigma},\ -b_{m+1}, \dots, -b_q \\ -a_1, \dots, -a_n,\ d_1, \dots, d_{\tau},\ -a_{n+1}, \dots, -a_p \end{matrix}\right)$$
$[m,\ n,\ p,\ q,\ \mu,\ \nu,\ \sigma,\ \tau$—are integers; $1 \leq n \leq p < q < p + \tau - \sigma$,
$\frac{1}{2}p + \frac{1}{2}q - n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma + \frac{1}{2}\tau - \nu < \mu \leq \tau;$
$\operatorname{Re}(b_j + d_k) > -1 \qquad (j = 1, \dots, m; k = 1, \dots, \mu),$
$\operatorname{Re}(a_j + c_k) < 1 \qquad (j = 1, \dots, n; k = 1, \dots, \tau);$

$$\begin{array}{ll} b_j - b_k & (j = 1, \ldots, m; k = 1, \ldots, m; j \neq k), \\ a_j - a_k & (j = 1, \ldots, n; k = 1, \ldots, n; j \neq k), \\ d_j - d_k & (j = 1, \ldots, \mu; k = 1, \ldots, \mu; j \neq k), \\ a_j + d_k & (j = 1, \ldots, n; k = 1, \ldots, n); \end{array}$$

must not be positive integers:

$$\begin{array}{ll} a_j - b_k & (j = 1, \ldots, n; k = 1, \ldots, m), \\ c_j - d_k & (j = 1, \ldots, \nu; k = 1, \ldots, \mu); \end{array}$$
$$\omega \neq 0, \quad \eta \neq 0, \quad |\arg \eta| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi, \quad |\arg \omega| < \left(\mu + \nu - \frac{1}{2}\sigma - \frac{1}{2}\tau\right)\pi \Bigg].$$

Formula 7.811 1 also holds for four sets of restrictions. See C. S. Meijer, Neue Integraldarstellungen für Whittakersche Funktionen, Nederl. Akad. Wetensch. Proc. **44** (1941), 82-92.

899

2. $$\int_0^1 x^{\varrho-1}(1-x)^{\sigma-1} G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right.\right) dx = \Gamma(\sigma) G_{p+1,\, q+1}^{m,\, n+1}\left(\alpha \left| \begin{matrix} 1-\varrho,\, a_1, \ldots, a_p \\ b_1, \ldots, b_q,\, 1-\varrho-\sigma \end{matrix} \right.\right)$$
$$\Bigg[(p+q) < 2(m+n), \quad |\arg a| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\varrho + b_j) > 0; \quad j = 1, \ldots, m; \quad \operatorname{Re}\sigma > 0,$$

ET II 422(14)

either

$$p + q \leq 2(m+n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}\varrho - \frac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\varrho + b_j) > 0; \quad j = 1, \ldots, m; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (p-q)\left(\varrho - \frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$p < q \quad (\text{or } p \leq q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(p + b_j) > 0; \quad j = 1, \ldots, m; \quad \operatorname{Re}\sigma > 0].$$

ET II 417(1)

3. $$\int_1^\infty x^{-\varrho}(x-1)^{\sigma-1} G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right.\right) dx = \Gamma(\sigma) G_{p+1,\, q+1}^{m+1,\, n}\left(\alpha \left| \begin{matrix} a_1, \ldots, a_p,\, \varrho \\ \varrho-\sigma,\, b_1, \ldots, b_q \end{matrix} \right.\right)$$
$$\Bigg[p + q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\varrho - \sigma - a_j) > -1; \quad j = 1, \ldots, n; \quad \operatorname{Re}\sigma > 0,$$

either

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\varrho-\sigma-a_j) > -1; \quad j=1,\dots,n; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (q-p)\left(\varrho-\sigma+\frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$q < p \quad (\text{or } q \le p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\varrho-\sigma-a_j) > -1; \quad j=1,\dots,n; \quad \operatorname{Re}\sigma > 0].$$

ET II 417(2)

900

4. $$\int_0^\infty x^{\varrho-1} G_{pq}^{mn}\left(\alpha x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \frac{\prod_{j=1}^{m}\Gamma(b_j+\varrho)\prod_{j=1}^{n}\Gamma(1-a_j-\varrho)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\varrho)\prod_{j=n+1}^{p}\Gamma(a_j+\varrho)}\alpha^{-\varrho}$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\right.$$
$$\left. -\min_{1\le j\le m} \operatorname{Re} b_j < \operatorname{Re}\varrho < 1 - \max_{1\le j\le n} \operatorname{Re} a_j\right].$$

ET II 418(3)A, ET I 337(14)

5. $$\int_0^\infty x^{\varrho-1}(x+\beta)^{-\sigma} G_{pq}^{mn}\left(\alpha x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \frac{\beta^{\varrho-\sigma}}{\Gamma(\sigma)} G_{p+1,\,q+1}^{m+1,\,n+1}\left(\alpha\beta \middle| \begin{matrix} 1-\varrho,\, a_1,\dots,a_p \\ \sigma-\varrho,\, b_1,\dots,b_q \end{matrix}\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg\beta| < \pi\right.$$
$$\operatorname{Re}(\varrho+b_j) > 0, \quad j=1,\dots,m, \quad \operatorname{Re}(\varrho-\sigma+a_j) < 1, \quad j=1,\dots,n,$$

either

$$p \le q, \quad p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg\beta| < \pi$$
$$\operatorname{Re}(\varrho = b_j) > 0, \quad j=1,\dots,m, \quad \operatorname{Re}(\varrho-\sigma+a_j) < 1, \quad j=1,\dots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j - (q-p)\left(\varrho-\sigma-\frac{1}{2}\right)\right] > 1,$$

$$p \geq q, \quad p+q \leq 2(m+n), \quad |\arg \alpha| \leq \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg \beta| < \pi,$$
$$\operatorname{Re}(\varrho+b_j) > 0, \quad j=1,\dots,m, \quad \operatorname{Re}(\varrho-\sigma+a_j) < 1, \quad j=1,\dots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (p-q)\left(\varrho-\frac{1}{2}\right)\right] > 1\Bigg].$$

ET II 418(4)

7.812

1. $$\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1} E\left(a_1,\dots,a_p\colon \varrho_1,\dots,\varrho_q;\ \frac{z}{x^m}\right) dx =$$
$$= \Gamma(\gamma-\beta) m^{\beta-\gamma} E(a_1,\dots,a_{p+m}\colon \varrho_1,\dots,\varrho_{q+m}\colon z),$$
$$a_{p+k} = \frac{\beta+k-1}{m}, \quad \varrho_{q+k} = \frac{\gamma+k-1}{m}, \qquad k=1,\dots,m$$
$$[\operatorname{Re}\gamma > \operatorname{Re}\beta > 0, \quad m=1,2,\dots].$$

ET II 414(2)

2. $$\int_0^\infty x^{\varrho-1}(1+x)^{-\sigma} E[a_1,\dots,a_p\colon \varrho_1,\dots,\varrho_q\colon (1+x)z]\,dx =$$
$$= \Gamma(\varrho) E(a_1,\dots,a_p,\ \sigma-\varrho\colon \varrho_1,\dots,\varrho_q,\ \sigma\colon z) \qquad [\operatorname{Re}\sigma > \operatorname{Re}\varrho > 0].$$

ET II 415(3)

3. $$\int_0^\infty (1+x)^{-\beta} x^{s-1} G_{p,q}^{m,n}\left(\frac{ax}{1+x}\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \Gamma(\beta-s) G_{p+1,q+1}^{m,n+1}\left(a\middle|\begin{matrix}1-s,\ a_1,\dots,a_p\\ b_1,\dots,b_q,\ 1-\beta\end{matrix}\right)$$
$$\Bigg[-\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re}\beta, \quad 1 \leq k \leq m;\ (p+q) < 2(m+n),$$
$$|\arg a| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi\Bigg].$$

ET I 338(19)

7.813

1. $$\int_0^\infty x^{-\varrho} e^{-\beta x} G_{pq}^{mn}\left(\alpha x\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \beta^{\varrho-1} G_{p+1,q}^{m,n+1}\left(\frac{\alpha}{\beta}\,|\varrho,\ a_1,\dots,a_p b_1,\dots,b_q\right)$$
$$\Bigg[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,$$
$$|\arg\beta| < \frac{1}{2}\pi, \quad \operatorname{Re}(b_j-\varrho) > -1, \quad j=1,\dots,m\Bigg].$$

ET II 419(5)

ET II 419(6)

7.814

1. $$\int_0^\infty x^{\beta-1} e^{-x} E(a_1, \ldots, a_p\colon \varrho_1, \ldots, \varrho_q\colon xz)\, dx =$$
$$= \pi \operatorname{cosec}(\beta\pi) \left[E(a_1, \ldots, a_p\colon 1-\beta,\ \varrho_1, \ldots, \varrho_q\colon e^{\pm i\pi} z)\ - \right.$$
$$\left. - z^{-\beta} E(a_1+\beta, \ldots, a_p+\beta\colon 1+\beta,\ \varrho_1+\beta, \ldots, \varrho_l+\beta\colon e^{\pm i\pi} z)\right]$$

[$p \geq q+1$, $\operatorname{Re}(a_r+\beta) > 0$, $r = 1, \ldots, p$, $|\arg z| < \pi$. The formula holds also for $p < q+1$, provided the integral converges].

2. $$\int_0^\infty x^{\beta-1} e^{-x} E(a_1, \ldots, a_p\colon \varrho_1, \ldots, \varrho_q\colon x^{-m} z) dx = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}} E(a_1, \ldots, a_{p+m}\colon \varrho_1, \ldots, \varrho_q\colon m^{-m} z)$$
$$\left[\operatorname{Re}\beta > 0, \quad a_{p+k} = \frac{\beta+k-1}{m}, \quad k = 1, \ldots, m;\ m = 1, 2, \ldots\right].$$

ET II 415(5)
ET II 415(4)

7.815

1. $$\int_0^\infty \sin(cx) G_{pq}^{mn}\left(\alpha x^2 \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx = pi^{\frac{1}{2}} c^{-1} G_{p+2,\, q}^{m,\, n+1}\left(\frac{4\alpha}{c^2} \middle| \begin{matrix} 0,\ a_1, \ldots, a_p,\ \frac{1}{2} \\ b_1, \ldots, b_q \end{matrix}\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\right.$$
$$\left. c > 0, \quad \operatorname{Re} b_j > -1, \quad j = 1, 2, \ldots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \ldots, n\right].$$

ET II 420(7)

2. $$\int_0^\infty \cos(cx) G_{pq}^{mn}\left(\alpha x^2 \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx = \pi^{\frac{1}{2}} c^{-1} G_{p+2,\, q}^{m,\, n+1}\left(\frac{4\alpha}{c^2} \middle| \begin{matrix} \frac{1}{2},\ a_1, \ldots, a_p,\ 0 \\ b_1, \ldots, b_q \end{matrix}\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\right.$$
$$\left. c > 0, \quad \operatorname{Re} b_j > -\frac{1}{2}, \quad j = 1, \ldots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \ldots, n\right].$$

ET II 420(8)

7.82 Combinations of the functions G and E and Bessel functions

7.821

$$1.\quad \int_0^\infty x^{-\varrho} J_\nu(2\sqrt{x}) G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx = G_{p+2,\,q}^{m,\,n+1}\left(\alpha \,\middle|\, \begin{matrix} \varrho - \frac{1}{2}\nu,\ a_1, \dots, a_p,\ \varrho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix}\right)$$

$$\left[p + q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi\ ,\right.$$

$$\left. -\frac{3}{4} + \max_{1 \le j \le n} \operatorname{Re} a_j < \operatorname{Re} \varrho < 1 + \frac{1}{2} \operatorname{Re} \nu + \min_{1 \le j \le m} \operatorname{Re} b_j \right].$$

ET II 420(9)

$$2.\quad \int_0^\infty x^{-\varrho} N_\nu(2\sqrt{x}) G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx = G_{p+3,q+1}^{m,n+2}\left(\alpha \,\middle|\, \begin{matrix} \varrho - \frac{1}{2}\nu, \varrho + \frac{1}{2}\nu, a_1, \dots, a_p, \varrho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q,\ \varrho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix}\right)$$

$$\left[p + q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,\right.$$

$$\left. -\ \frac{3}{4} + \max_{1 \le j \le n} \operatorname{Re} a_j < \operatorname{Re} \varrho < \min_{1 \le j \le m} \operatorname{Re} b_j + \frac{1}{2}|\operatorname{Re} \nu| + 1 \right]$$

ET II 420(10)

$$3.\quad \int_0^\infty x^{-\varrho} K_\nu(2\sqrt{x}) G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx = \frac{1}{2} G_{p+2,\,q}^{m,\,n+2}\left(\alpha \,\middle|\, \begin{matrix} \varrho - \frac{1}{2}\nu,\ \varrho + \frac{1}{2}\nu,\ a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$$

$$\left[p + q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,\right.$$

$$\left. \operatorname{Re} \varrho < 1 - \frac{1}{2}|\operatorname{Re} \nu| + \min_{1 \le j \le m} \operatorname{Re} b_j \right].$$

ET II 421(11)

7.822

$$1.\quad \int_0^\infty x^{2\varrho} J_\nu(xy) G_{pq}^{mn}\left(\lambda x^2 \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx =$$

$$= \frac{2^{2\varrho}}{y^{2\varrho+1}} G_{p+2,\,q}^{m,\,n+1}\left(\frac{4\lambda}{y^2} \,\middle|\, \begin{matrix} h,\ a_1, \dots, a_p,\ k \\ b_1, \dots, b_q \end{matrix}\right), \qquad h = \frac{1}{2} - \varrho - \frac{1}{2}\nu, \quad k = \frac{1}{2} - \varrho + \frac{1}{2}\nu$$

$$\left[p + q < 2(m+n), \quad |\arg \lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi, \quad \operatorname{Re}\left(b_j + \varrho + \frac{1}{2}\nu\right) > -\frac{1}{2}\ ,\right.$$

$$\left. j = 1, 2, \dots, m, \quad \operatorname{Re}(a_j + \varrho) < \frac{3}{4}, \quad j = 1, \dots, n, \quad y > 0 \right].$$

2. $$\int_0^\infty x^{\frac{1}{2}} N_\nu(xy) G_{pq}^{mn}\left(\lambda x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx =$$
$$= (2\lambda)^{-\frac{1}{2}} y^{-\frac{1}{2}} G_{q+1,\, p+3}^{n+2,\, m}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2}-b_1, \dots, \frac{1}{2}-b_q,\, l \\ h,\, k,\, \frac{1}{2}-a_1, \dots, \frac{1}{2}-a_p,\, l \end{matrix}\right)$$
$$h = \frac{1}{4}+\frac{1}{2}\nu, \quad k = \frac{1}{4}-\frac{1}{2}\nu, \quad l = -\frac{1}{4}-\frac{1}{2}\nu$$
$$\left[p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad y>0, \right.$$
$$\left. \operatorname{Re} a_j < 1, \quad j=1,\dots,n, \quad \operatorname{Re}\left(b_j \pm \frac{1}{2}\nu\right) > -\frac{3}{4}, \quad j=1,\dots,m\right].$$

ET II 119(56)

3. $$\int_0^\infty x^{\frac{1}{2}} K_\nu(xy) G_{pq}^{mn}\left(\lambda x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx =$$
$$= 2^{-\frac{3}{2}} \lambda^{-\frac{1}{2}} y^{-\frac{1}{2}} G_{q,\, p+2}^{n+2,\, m}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2}-b_1, \dots, \frac{1}{2}-b_q \\ h,\, k,\, \frac{1}{2}-a_1, \dots, \frac{1}{2}-a_p \end{matrix}\right), \qquad h = \frac{1}{4}+\frac{1}{2}\nu, \quad k = \frac{1}{4}-\frac{1}{2}\nu$$
$$\left[\operatorname{Re} y > 0, \quad p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi \;, \right.$$
$$\left. \operatorname{Re} b_j > \frac{1}{2}|\operatorname{Re}\nu| - \frac{3}{4}, \quad j=1,\dots,m\right].$$

ET II 153(90)

7.823

1. $$\int_0^\infty x^{\beta-1} J_\nu(x) E(a_1, \dots, a_p\colon \varrho_1, \dots, \varrho_q\colon x^{-2m} z)\, dx =$$
$$= (2\pi)^{-m}(2m)^{\beta-1}\left\{\exp\left[\frac{1}{2}\pi(\beta-\nu-1)i\right] E\left[a_1, \dots, a_{p+2m}\colon \varrho_1, \dots, \varrho_q\colon (2m)^{-2m} z e^{-m\pi i}\right] + \right.$$
$$\left. + \exp\left[-\frac{1}{2}\pi(\beta-\nu-1)i\right] E\left[a_1, \dots, a_{p+2m}\colon \varrho_1, \dots, \varrho_q\colon (2m)^{-2m} z e^{m\pi i}\right]\right\},$$
$$a_{p+k} = \frac{\beta+\nu+2k-2}{2m}, \quad a_{p+m+k} = \frac{\beta-\nu+2k-2}{2m}, \qquad m = 1,\, 2, \dots,; \quad k = 1, \dots, m$$
$$\left[\operatorname{Re}(\beta+\nu) > 0, \quad \operatorname{Re}(2a_r m - \beta) > -\frac{3}{2}, \quad r = 1, \dots, p\right]$$

ET II 415(7)

2. $$\int_0^\infty x^{\beta-1} K_\nu(x) E(a_1, \dots, a_p\colon \varrho_1, \dots, \varrho_q\colon x^{-2m} z)\, dx = (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1} \times$$
$$\times E[a_1, \dots, a_{p+2m}\colon \varrho_1, \dots, \varrho_q\colon (2m)^{-2m} z],$$
$$a_{p+k} = \frac{\beta+\nu+2k-2}{2m}, \quad a_{p+m+k} = \frac{\beta-\nu+2k-2}{2m}, \qquad k = 1,\, 2, \dots, m$$
$$[\operatorname{Re}\beta > |\operatorname{Re}\nu|, \quad m = 1,\, 2, \dots].$$

$$1.\quad \int_0^\infty x^{\frac{1}{2}} \mathbf{H}_\nu(xy) G_{pq}^{mn}\left(\lambda x^2 \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx = (2\lambda y)^{-\frac{1}{2}} G_{q+1,\, p+3}^{n+1,\, m+1}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} l,\ \frac{1}{2} - b_1, \ldots, \frac{1}{2} - b_q \\ l,\ \frac{1}{2} - a_1, \ldots, \frac{1}{2} - a_p,\ h,\ k \end{matrix}\right)$$

$$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$$

$$\left[p + q < 2(m+n), \quad |\arg\lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi, \quad y > 0,\right.$$
$$\left.\operatorname{Re} a_j < \min\left(1, \frac{3}{4} - \frac{1}{2}\nu\right), \quad j = 1, \ldots, n, \quad \operatorname{Re}(2b_j + \nu) > -\frac{5}{2}, \quad j = 1, \ldots, m\right].$$

ET II 172(47)

$$2.\quad \int_0^\infty x^{-\varrho} \mathbf{H}_\nu(2\sqrt{x}) G_{pq}^{mn}\left(\alpha x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx = G_{p+3,q+1}^{m+1,n+1}\left(\alpha \middle| \begin{matrix} \varrho - \frac{1}{2} - \frac{1}{2}\nu, a_1, \ldots, a_p, \varrho + \frac{1}{2}\nu, \varrho - \frac{1}{2}\nu \\ \varrho - \frac{1}{2} - \frac{1}{2}\nu, b_1, \ldots, b_q \end{matrix}\right)$$

$$\left[p + q < 2(m+n), \ |\arg\alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi\ ,\right.$$
$$\left.\max\left(-\frac{3}{4}, \operatorname{Re}\frac{\nu - 1}{2}\right) + \max_{1 \le j \le n} \operatorname{Re} a_j < \operatorname{Re}\varrho < \min_{1 \le j \le m} \operatorname{Re} b_j + \frac{1}{2}\operatorname{Re}\nu + \frac{3}{2}\right]$$

ET II 421(12)

7.83 Combinations of the functions G and E and other special functions

7.831

$$\int_1^\infty x^{-\varrho}(x-1)^{\sigma-1} F(k + \sigma - \varrho, \lambda + \sigma - \varrho; \sigma; 1 - x) G_{pq}^{mn}\left(\alpha x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx =$$
$$= \Gamma(\sigma) G_{p+2,\, q+2}^{m+2,\, n}\left(\alpha \middle| \begin{matrix} a_1, \ldots, a_p,\ k + \lambda + \sigma - \varrho,\ \varrho \\ k,\ \lambda,\ b_1, \ldots, b_q \end{matrix}\right)$$
$$\left[p + q < 2(m+n), \quad |\arg\alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,\right.$$
$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re}k \ge \operatorname{Re}\lambda > \operatorname{Re}a_j - 1, \quad j = 1, \ldots, n,$$

or

$$p + q \le 2(m+n), \quad |\arg\alpha| \le \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,$$
$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re}k \ge \operatorname{Re}\lambda > \operatorname{Re}a_j - 1, \quad j = 1, \ldots, n,$$
$$\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)\left(k + \frac{1}{2}\right)\right] > -\frac{1}{2},$$
$$\left.\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)\left(\lambda + \frac{1}{2}\right)\right] > -\frac{1}{2}\right].$$

$$\int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\{,\mu}(x) E(a_1, \dots, a_p \colon \varrho_1, \dots, \varrho_q \colon x^{-m} z)\, dx =$$
$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta+\{-\frac{1}{2}} E(a_1, \dots, a_{p+2m} \colon \varrho_1, \dots, \varrho_{q+m} \colon m^{-m} z),$$
$$a_{p+k} = \frac{\beta + k + \mu - \frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta - \mu + k - \frac{1}{2}}{m}, \quad \varrho_{q+k} = \frac{\beta - \{ + k}{m}, \qquad k = 1, \dots, m$$
$$\left[\operatorname{Re}\beta > |\operatorname{Re}\mu| - \frac{1}{2}, \quad m = 1, 2, \dots\right].$$

ET II 416(10)

8.-9. Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.110

1. Every integral of the form $\int R(x, \sqrt{P(x)})\, dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx, \quad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form*. The results of this reduction for the more frequently encountered integrals are given in formulas 3.13-3.17. The number k is called the *modulus** of these integrals, the number $k' = \sqrt{1-k^2}$ is called the complementary modulus, and the number n is called the parameter of the integral of the third kind.

BY (110.04)}

* The quantity k is sometimes called the *module* of the functions.

2. By means of the substitution $x = \sin\varphi$, elliptic integrals can be reduced to the normal trigonometric form

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \quad \int \sqrt{1-k^2\sin^2\varphi}\, d\varphi, \quad \int \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}.$$

BY (110.04)

The results of reducing integrals of trigonometric functions to normal form are given in 2.58-2.62.

3. Elliptic integrals from 0 to $\frac{\pi}{2}$ are called *complete elliptic integrals*.

8.111

Notations:

$$1. \quad \Delta\varphi = \sqrt{1 - k^2 \sin^2 \varphi}; \quad k' = \sqrt{1 - k^2}; \quad k^2 < 1.$$

2. The elliptic integral of the first kind:

$$F(\varphi,\, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

908

3. The elliptic integral of the second kind:

$$E(\varphi,\, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha = \int_0^{\sin\varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}}\, dx.$$

FI II 135

4. The elliptic integral of the third kind:

$$\Pi(\varphi,\, n,\, k) = \int_0^{\varphi} \frac{d\alpha}{(1 - n \sin^2 \alpha)\sqrt{1 - k^2 \sin^2 \alpha}} = \frac{\int_0^{\sin\varphi} dx}{(1 - nx^2)\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

BY (110.04)

$$5. \quad D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2} = \int_0^{\varphi} \frac{\sin^2 \alpha\, d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin\varphi} \frac{x^2\, dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

8.112

Complete elliptic integrals

In writing complete elliptic integrals, the modulus k, which acts as an independent variable, is often omitted and we write

$$\boldsymbol{K}(\equiv \boldsymbol{K}(k)), \quad \boldsymbol{K}'(\equiv \boldsymbol{K}'(k)), \quad \boldsymbol{E}(\equiv \boldsymbol{E}(k)), \quad \boldsymbol{E}'(\equiv \boldsymbol{E}'(k)).$$

Series representations

8.113

1. $$\boldsymbol{K} = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \ldots\right\} = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

FI II 487, WH

2. $$\boldsymbol{K} = \frac{\pi}{1+k'}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-k'}{1+k'}\right)^4 + \ldots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \ldots\right\}.$$

DW

3. $$\boldsymbol{K} = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2}\right)k'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)k'^4 + \\ + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{2}{5\cdot 6}\right)k'^6 + \ldots$$

DW

See also 8.197 1., 8.197 2.

8.114

1.[6] $$\boldsymbol{E} = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2\cdot 3}{2^2\cdot 4^2}k^4 - \ldots - \left[\frac{(2n-1)!!}{2^n n!}\right]^2\frac{k^{2n}}{2n-1} - \ldots\right\} = \frac{\pi}{2}F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

2. $\boldsymbol{E} = \frac{(1+k')\pi}{4}\left\{1 + \frac{1}{2^2}\left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^2\cdot 4^2}\left(\frac{1-k'}{1+k'}\right)^4 + \ldots + \left[\frac{(2n-3)!!}{2^n n!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \ldots\right\}$

DW

3. $$\boldsymbol{E} = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{1\cdot 2}\right)k'^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)k'^4 + \\ + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)k'^6 + \ldots$$

DW

8.115

$$\boldsymbol{D} = \pi\left\{\frac{1}{1}\left(\frac{1}{2}\right)^2 + \frac{2}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \ldots + \frac{n}{2n-1}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2(n-1)} + \ldots\right\}.$$

ZH 43(158)

8.116

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2\sin^2\varphi}}{1-n^2\sin^2\varphi}\,d\varphi = \sqrt{n'^2-k'^2}\left(\frac{\arccos\frac{1}{n'}}{n'\sqrt{n'^2-1}} + \boldsymbol{R}\right),$$

where

$$\boldsymbol{R} = \frac{k'^2}{2}\left(p+\frac{1}{2}\right)\frac{1}{n'^3} + \frac{k'^4}{16}\left[-1 + \left(p+\frac{1}{4}\right)\frac{1}{n'^3}\left(1+\frac{6}{n'^2}\right)\right] + \\ + \frac{k'^6}{16}\left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p+\frac{1}{6}\right)\frac{1}{n'^3}\left(\frac{3}{8}+\frac{1}{n'^2}+\frac{5}{n'^4}\right)\right] + \\ + \frac{15k'^8}{256}\left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p+\frac{1}{8}\right)\frac{1}{n'^3}\left(\frac{5}{24}+\frac{9}{20n'^2}+\frac{1}{n'^4}+\frac{14}{3n'^6}\right)\right] + \ldots, \\ p = \ln\frac{4}{k'}, \quad k' = 4e^{-p}, \quad k'^2 = 1-k^2, \quad n'^2 = 1-n^2.$$

ZH 44(163)

Trigonometric series

8.117

For *small* values of k and φ, we may use the series

1. $F(\varphi, k) = \frac{2}{\pi}\boldsymbol{K}\varphi - \sin\varphi\,\cos\varphi\left(a_0 + \frac{2}{3}a_1\sin^2\varphi + \frac{2\cdot 4}{3\cdot 5}a_2\sin^4\varphi + \ldots\right),$

where

$$a_0 = \frac{2}{\pi}\boldsymbol{K} - 1; \quad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}.$$

ZH 10(19)

2. $$E(\varphi,\, k) = \frac{2}{\pi}\boldsymbol{E}\varphi + \sin\varphi\,\cos\varphi\left(b_0 + \frac{2}{3}b_1\sin^2\,\varphi + \frac{2\cdot 4}{3\cdot 5}b_2\sin^4\,\varphi + \ldots\right),$$

where

$$b_0 = 1 - \frac{2}{\pi}\boldsymbol{E}, \quad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}.$$

ZH 27(86)

8.118*

For k close to 1, we may use the series

1. $$F(\varphi,\, k) = \frac{2}{\pi}\boldsymbol{K}'\,\ln\operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{4}\right) - \frac{\operatorname{tg}\varphi}{\cos\varphi}\left(a'_0 - \frac{2}{3}a'_1\operatorname{tg}^2\,\varphi + \frac{2\cdot 4}{3\cdot 5}a'_2\operatorname{tg}^4\,\varphi - \ldots\right),$$

where

$$a'_0 = \frac{2}{\pi}\boldsymbol{K}' - 1; \quad a'_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k'^{2n}.$$

ZH 10(23)

2. $$E(\varphi,\, k) = \frac{2}{\pi}(\boldsymbol{K}' - \boldsymbol{E}')\ln\operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{2}\right) + \\ + \frac{\operatorname{tg}\varphi}{\cos\varphi}\left(b'_1 - \frac{2}{3}b'_2\operatorname{tg}^2\,\varphi + \frac{2\cdot 4}{3\cdot 5}b'_3\operatorname{tg}^4\,\varphi - \ldots\right) + \frac{1}{\sin\varphi}[1 - \cos\varphi\sqrt{1-k^2\sin\varphi}],$$

$$b'_0 = \frac{2}{\pi}(\boldsymbol{K}' - \boldsymbol{E}'),\ b'_n = b'_{n-1} - \left[\frac{(2n-3)!!}{2^{n-1}(n-1)!}\right]^2 \left(\frac{2n-1}{2n}\right) k'^{2n}$$

ZH 27(90)

For the expansion of complete elliptic integrals in Legendre polynomials, see 8.928.

8.119

Representation in the form of an infinite product:

1. $\boldsymbol{K}(k) = \frac{\pi}{2}\prod_{n=1}^{\infty}(1 + k_n),$

where

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}; \qquad k_0 = k.$$

FI II 166

See also 8.197.

8.12 Functional relations between elliptic integrals

8.121

1. $F(-\varphi, k) = -F(\varphi, k).$

JA

2. $E(-\varphi, k) = -E(\varphi, k).$

JA

3. $F(n\pi \pm \varphi, k) = 2n\boldsymbol{K}(k) \pm F(\varphi, k).$

JA

4. $E(n\pi \pm \varphi,\, k) = 2n\boldsymbol{E}(k) \pm E(\varphi,\, k).$

JA

8.122

$$\boldsymbol{E}(k)\boldsymbol{K}'(k) + \boldsymbol{E}'(k)\boldsymbol{K}(k) - \boldsymbol{K}(k)\boldsymbol{K}'(k) = \frac{\pi}{2}.$$

FI II 691, 791

8.123

1. $$\frac{\partial F}{\partial k} = \frac{1}{k'^2}\left(\frac{E - k'^2 F}{k} - \frac{k \sin\varphi \cos\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\right).$$

MO 138, BY (710.07)

2. $$\frac{d\boldsymbol{K}(k)}{dk} = \frac{\boldsymbol{E}(k)}{kk'^2} - \frac{\boldsymbol{K}(k)}{k}.$$

FI II 691

3. $$\frac{\partial E}{\partial k} = \frac{E - F}{k}.$$

MO 138

4. $$\frac{d\boldsymbol{E}(k)}{dk} = \frac{\boldsymbol{E}(k) - \boldsymbol{K}(k)}{k.}$$

FI II 690

8.124

1. The functions K and K′ satisfy the equation

$$\frac{d}{dk}\left\{kk'^2 \frac{du}{dk}\right\} - ku = 0.$$

WH

2. The functions E and E′ - K′ satisfy the equation

$$k'^2 \frac{d}{dk}\left(k\frac{du}{dk}\right) + ku = 0.$$

WH

8.125

$$\left.\begin{aligned} &1. \quad F\left(\psi,\ \frac{1-k'}{1+k'}\right) = (1+k')\,F\,(\varphi,\ k) \\ &2. \quad E\left(\psi,\ \frac{1-k'}{1+k'}\right) = \frac{2}{1+k'}[E(\varphi,\ k) + k'F(\varphi,\ k)] - \frac{1-k'}{1+k'}\sin\psi \end{aligned}\right\} \quad [\operatorname{tg}(\psi-\varphi) = k'\ \operatorname{tg}\varphi]$$

MO 131

MO 130

912

$$\left.\begin{aligned} &3. \quad F\left(\psi,\ \frac{2\sqrt{k}}{1+k}\right) = (1+k)\,F\,(\varphi,\ k)\,. \\ &4. \quad E\left(\psi,\ \frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}[2E(\varphi,\ k) - k'^2F(\varphi,\ k) + \\ &\qquad\qquad + 2k\frac{\sin\varphi\ \cos\varphi}{1+k\ \sin^2\ \varphi}\sqrt{1-k^2\sin^2\ \varphi}\Big] \end{aligned}\right\} \quad \left[\sin\psi = \frac{(1+k)\ \sin\varphi}{1+k\ \sin^2\ \varphi}\right].$$

MO 131

8.126

In particular,

$$1. \quad \boldsymbol{K}\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}\boldsymbol{K}(k).$$

MO 130

$$2. \quad \boldsymbol{E}\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k'}[\boldsymbol{E}(k) + k'\boldsymbol{K}(k)].$$

MO 130

$$3. \quad \boldsymbol{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\boldsymbol{K}(k).$$

4. $\boldsymbol{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}[2\boldsymbol{E}(k) - k'^2\boldsymbol{K}(k)].$

MO 130

8.127

k_1	$\sin\varphi_1$	$\cos\varphi_1$	$F(\varphi_1, k_1)$	$E(\varphi_1, k_1)$
$i\frac{k}{k'}$	$k'\frac{\sin\varphi}{\Delta\varphi}$	$\frac{\cos\varphi}{\Delta\varphi}$	$k'F(\varphi,\, k)$	$\frac{1}{k'}\left[E(\varphi,\, k) - \frac{k^2\,\sin\varphi\,\cos\varphi}{\Delta\varphi}\right]$
k'	$-i\,\operatorname{tg}\varphi$	$\sec\varphi$	$-iF(\varphi,\, k)$	$i[E(\varphi,\, k) - F(\varphi,\, k) - \Delta\varphi\,\operatorname{tg}\varphi]$
$\frac{1}{k}$	$k\,\sin\varphi$	$\Delta\varphi$	$kF(\varphi,\, k)$	$\frac{1}{k}[E(\varphi,\, k) - k'^2F(\varphi, k)]$
$\frac{1}{k'}$	$-k'\,\operatorname{tg}\varphi$	$\frac{\Delta\varphi}{\cos\varphi}$	$-ik'F(\varphi, k)$	$\frac{i}{k'}[E(\varphi,\, k) - k'^2F(\varphi,\, k) - \Delta\varphi\,\operatorname{tg}\varphi]$
$\frac{k'}{ik}$	$\frac{-ik\,\sin\varphi}{\Delta\varphi}$	$\frac{1}{\Delta\varphi}$	$-ikF(\varphi,\, k)$	$\frac{i}{k}\left[E(\varphi,\, k) - F(\varphi,\, k) - \frac{k^2\sin\varphi\,\cos\varphi}{\Delta\varphi}\right]$

8.111
MO 131

8.128

In particular,

1. $\boldsymbol{K}\left(i\frac{k}{k'}\right) = k'\boldsymbol{K}(k)[\operatorname{Im}(k) < 0].$

MO 130

913

2. $\boldsymbol{K}'\left(i\frac{k}{k'}\right) = k'[\boldsymbol{K}(k') - i\boldsymbol{K}(k)][\operatorname{Im}(k) < 0].$

MO 130

3. $\boldsymbol{K}\left(\frac{1}{k}\right) = k[\boldsymbol{K}(k) + i\boldsymbol{K}'(k)][\operatorname{Im}(k) < 0].$

MO 130

For integrals of elliptic integrals, see 6.14-6.15. For indefinite integrals of complete elliptic integrals, see 5.11.

8.129

Special values:

1. $\boldsymbol{K}\left(\sin\frac{\pi}{4}\right) = \boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) = \boldsymbol{K}'\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

MO 130

2. $\boldsymbol{K}'(\sqrt{2}-1) = \sqrt{2}\boldsymbol{K}(\sqrt{2}-1).$

MO 130

3. $\boldsymbol{K}'\left(\sin\frac{\pi}{12}\right) = \sqrt{3}\boldsymbol{K}\left(\sin\frac{\pi}{12}\right).$

MO 130

4. $\boldsymbol{K}'\left(\operatorname{tg}^2\frac{\pi}{8}\right) = \boldsymbol{K}'\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2\boldsymbol{K}\left(\operatorname{tg}^2\frac{\pi}{8}\right).$

MO 130

8.13 Elliptic functions

8.130

Definition and general properties.

1. A single-valued function $f(z)$ of a complex variable, which is not a constant, is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$, that is

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \quad [m,\, n \text{integers}].$$

The ratio of the periods of an analytic function cannot be a real number. For an elliptic function $f(z)$, the z-plane can be partitioned

$f(z)$ z

into parallelograms--the period parallelograms--the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$. At corresponding points of these parallelograms, the function $f(z)$ has the same value.

ZH 117, SI 299

2. Suppose that α is the angle between the sides a and b of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b}e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi \sin\alpha}\left[\cos\left(\frac{a}{b}\pi \cos\alpha\right) + i \sin\left(\frac{a}{b}\pi \cos\alpha\right)\right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods.

SM III 598

4. A nonconstant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points a_1, $a_2, \ldots, a_n$ and that their orders are α_1, $\alpha_2, \ldots, \alpha_n$. Suppose that the zeros of an analytic function that occur in a single parallelogram are b_1, $b_2, \ldots, b_m$ and that the orders of the zeros are β_1, $\beta_2, \ldots, \beta_m$, respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \beta_1 + \beta_2 + \ldots + \beta_m.$$

ZH 118

The number γ representing this sum is called the *order* of the elliptic function.

914

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.

6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.

7. Every two elliptic functions with the same periods are related by an algebraic relationship.

GO II 151

8.[7] A nonconstant single-valued function cannot have more than two periods.

GO II 147

9. An elliptic function of order γ assumes *an arbitrary value* γ times in a period parallelogram.

SM 601, SI 301

8.14 Jacobian elliptic functions

8.141

Consider the upper limit φ of the integral

$$u = \int_0^{\varphi} \frac{da}{\sqrt{1 - k^2 \sin^2 \ \alpha}}$$

as a function of u. Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude.* The quantity u is called the *argument,* and its dependence on φ is written

$$u = \arg \varphi.$$

8.142

The amplitude is an *infinitely-many-valued* function of u and has a period of $4\boldsymbol{K}i$. The *branch points* of the amplitude correspond to the values of the argument

$$u = 2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i,$$

ZH 67- 69

where m and n are arbitrary integers (see also 8.151).

8.143

The first two of the following functions

$$\operatorname{sn} u = \sin \varphi = \sin \operatorname{am} \ u, \quad \operatorname{cn} u = \cos \varphi = \cos \operatorname{am} \ u,$$
$$\operatorname{dn} u = \Delta\varphi = \sqrt{1 - k^2 \sin^2 \ \varphi} = \frac{d\varphi}{du}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the

delta amplitude. All these elliptic functions were exhibited by Jacobi and they bear his name.

SI 16

The Jacobian elliptic functions are *doubly-periodic* functions and have *two simple poles* in a period parallelogram.

ZH 69

915

8.144

$$\left.\begin{aligned} &1. \quad u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &2. \quad u = \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}. \\ &3. \quad u = \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}. \end{aligned}\right\}$$

SI 21(23)

8.145

Power series representations:

$$1. \quad \operatorname{sn} u = u - \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \frac{1+135k^2+135k^2+k^6}{7!}u^7 + \\ + \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}u^9 - \dots \qquad [|u| < |\boldsymbol{K}'|].$$

ZH 81(97)

$$2. \quad \operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \\ + \frac{1+408k^2+912k^4+64k^6}{8!}u^8 - \dots \qquad [|u| < |\boldsymbol{K}'|].$$

ZH 81(98)

$$3. \quad \operatorname{dn} u = 1 - \frac{k^2}{2!}u^2 + \frac{k^2(4+k^2)}{4!}u^4 - \frac{k^2(16+44k^2+k^4)}{6!}u^6 + \\ + \frac{k^2(64+912k^2+408k^4+k^6)}{8!}u^8 - \dots \qquad [|u| < |\boldsymbol{K}'|].$$

4. $\operatorname{am} u = u - \frac{k^2}{3!}u^3 + \frac{k^2(4+k^2)}{5!}u^5 - \frac{k^2(16+44k^2+k^4)}{7!}u^7 + {} + \frac{k^2(64+912k^2+408k^4+k^6)}{9!}u^9 - \dots \qquad [|u| < |\boldsymbol{K}'|].$

LA 380(4)

8.146

Representation as a trigonometric series or a product $(q = e^{-\frac{\pi K'}{K}})$** The expansions 1-22 are valid in every strip of the form $\left|\operatorname{Im} \frac{\pi u}{2\boldsymbol{K}}\right| < \frac{1}{2}\pi \operatorname{Im}\tau$. The expansions 23-25 are valid in an arbitrary bounded portion of u.

:

1. $\operatorname{sn} u = \frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}.$

WH, ZH 84(108)

2. $\operatorname{cn} u = \frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}.$

WH, ZH 84(109)

3. $\operatorname{dn} u = \frac{\pi}{2\boldsymbol{K}} + \frac{2\pi}{\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{\boldsymbol{K}}.$

WH, ZH 84(110)

4. $amu = \frac{\pi u}{2\boldsymbol{K}} + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\frac{n\pi u}{\boldsymbol{K}}.$

WH

5. $\frac{1}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right].$

LA 369(3)

LA 369(3)

7. $$\frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[1 + 4\sum_{n=1}^{\infty}(-1)^n \frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{\boldsymbol{K}}\right].$$

LA 369(3)

8. $$\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[\operatorname{tg}\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}(-1)^n \frac{q^{2n}}{1+q^{2n}}\sin\frac{n\pi u}{\boldsymbol{K}}\right].$$

LA 369(4)

9. $$\frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'\boldsymbol{K}}\sum_{n=1}^{\infty}(-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}.$$

LA 369(4)

10. $$\frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\operatorname{ctg}\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin\frac{\pi n u}{\boldsymbol{K}}\right].$$

LA 369(5)

11. $$\frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}(-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}.$$

LA 369(5)

12. $$\frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right].$$

LA 369(6)

13. $$\frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\cos\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}(-1)^n \frac{q^{2n-1}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right].$$

LA 369(6)

14. $$\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\operatorname{ctg}\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right].$$

15. $$\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left\{\operatorname{tg}\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}\frac{q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right\}.$$

LA 369(7)

917

16. $$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}.$$

LA 369(7)

17. $$\frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2(1-k^2)\boldsymbol{K}}\left[\operatorname{tg}\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1-q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right].$$

LA 369(8)

18. $$\frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\operatorname{ctg}\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right].$$

LA 369(8)

19. $$\frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}\right].$$

LA 369(8)

20. $$\ln snu = \ln\frac{2\boldsymbol{K}}{\pi} + \ln\sin\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}.$$

LA 369(2)

21. $$\ln\operatorname{cn} u = \ln\cos\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}.$$

LA 369(2)

LA 369(2)

23. $$\operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin\frac{\pi u}{2\boldsymbol{K}} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n}}{1 - 2q^{2n-1}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n-2}}.$$

ZH 86(145)

24. $$\operatorname{cn} u = \frac{2\sqrt{k'}\,\sqrt[4]{q}}{\sqrt{k}} \cos\frac{\pi u}{2\boldsymbol{K}} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n}}{1 - 2q^{2n-1}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n-2}}.$$

ZH 86(146)

25. $$\operatorname{dn} u = \sqrt{k'} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n-1}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n-2}}{1 - 2q^{2n-1}\cos\dfrac{\pi u}{\boldsymbol{K}} + q^{4n-2}}.$$

ZH 86(147)

26. $$\operatorname{sn}^3 u = \sum_{n=0}^{\infty} \left[\frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3}\frac{\pi^2}{4\boldsymbol{K}^2}\right] \frac{2\pi q^{n+\frac{1}{2}} \sin(2n+1)\dfrac{\pi u}{2\boldsymbol{K}}}{\boldsymbol{K}(1-q^{2n+1})} \qquad \left[\left|\operatorname{Im}\frac{u}{2\boldsymbol{K}}\right| < \operatorname{Im}\tau\right].$$

MO 147

27. $$\frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4\boldsymbol{K}^2}\operatorname{cosec}^2\frac{\pi u}{2\boldsymbol{K}} + \frac{\boldsymbol{K}-\boldsymbol{E}}{\boldsymbol{K}} - \frac{2\pi^2}{\boldsymbol{K}^2}\sum_{n=1}^{\infty}\frac{nq^{2n}\cos\dfrac{n\pi u}{\boldsymbol{K}}}{1-q^{2n}} \qquad \left[\left|\operatorname{Im}\frac{u}{2\boldsymbol{K}}\right| < \frac{1}{2}\operatorname{Im}\tau\right].$$

MO 148

8.147

1. $$\operatorname{sn} u = \frac{\pi}{2k\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{1}{\sin\dfrac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}.$$

MO 149

2. $$\operatorname{cn} u = \frac{\pi i}{2k\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin\dfrac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}.$$

3. $\operatorname{dn} u = \frac{\pi i}{2\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\operatorname{tg} \frac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}.$

MO150

8.148

The Weierstrass expansions of the functions $\operatorname{sn}\ u$, $\operatorname{cn}\ u$, $\operatorname{dn}\ u$:

$$\operatorname{sn} u = \frac{B}{A}, \quad \operatorname{cn} u = \frac{C}{A}, \quad \operatorname{dn} u = \frac{D}{A},$$

where

$$A = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!}$$
$$[a_2 = 2k^2, \quad a_3 = 8(k^2+k^4), \quad a_4 = 32(k^2+k^6)+68k^4, \quad a_5 = 128(k^2+k^8)$$
$$+480(k^4+k^6), \quad a_6 = 512(k^2+k^{10})+3008(k^4+k^8)+5400k^6, \dots]$$
$$B = \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!}$$
$$[b_0 = 1, \quad b_1 = 1+k^2, \quad b_2 = 1+k^4+4k^2, \quad b_3 = 1+k^6+9(k^2+k^4),$$
$$b_4 = 1+k^8+16(k^2+k^6)-6k^4, \quad b_5 = 1+k^{10}+25(k^2+k^8)-494(k^4+k^6),$$
$$b_6 = 1+k^{12}+36(k^2+k^{10})-5781(k^4+k^8)-12184k^6, \dots].$$
$$C = \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!}$$
$$[c_0 = 1, \quad c_1 = 1, \quad c_2 = 1+2k^2, \quad c_3 = 1+6k^2+8k^4, \quad c_4 = 1+12k^2+60k^4+32k^6,$$
$$c_5 = 1+20k^2+348k^4+448k^6+128k^8, \quad c_6 = 1+30k^2+2372k^4+4600k^6+2880k^8+512k^{10}, \dots].$$
$$D = \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!}$$
$$[d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2+k^4, \quad d_3 = 8k^2+6k^4+k^6, \quad d_4 = 32k^2+60k^4+12k^4+k^8,$$
$$d_5 = 128k^2+448k^4+348k^6+20k^8+k^{10},$$
$$d_6 = 512k^2+2880k^4+4600k^6+2372k^8+30k^{10}+k^{12}, \dots].$$

ZH 82-83(105,106,107)

8.15 Properties of Jacobian elliptic functions and functional relationships between

them

8.151

The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
$\mathrm{sn}\, u$	$4m\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{k} + (2n+1)\boldsymbol{K}'i$	$(-1)^m \frac{1}{k}$
$\mathrm{cn}\, u$	$4m\boldsymbol{K} + 2n(\boldsymbol{K} + \boldsymbol{K}'i)$	$(2m+1)\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$(-1)^{m-1} \frac{i}{k}$
$\mathrm{dn}\, u$	$2m\boldsymbol{K} + 4n\boldsymbol{K}'i$	$(2m+1)\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$(-1)^{n-1} i$

SM 630, ZH 69-72}

2.

$u^* = u + \boldsymbol{K}$	$u + i\boldsymbol{K}$	$u + \boldsymbol{K} + i\boldsymbol{K}'$	$u + 2\boldsymbol{K}$	$u + 2i\boldsymbol{K}'$	$u + 2\boldsymbol{K} + 2i\boldsymbol{K}'$
$\mathrm{sn}\, u^* = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}$	$\frac{1}{k\, \mathrm{sn}\, u}$	$\frac{1}{k} \frac{\mathrm{dn}\, u}{\mathrm{cn}\, u}$	$-\mathrm{sn}\, u$	$\mathrm{sn}\, u$	$-\mathrm{sn}\, u$
$\mathrm{cn}\, u^* = -k' \frac{\mathrm{sn}\, u}{\mathrm{dn}\, u}$	$-\frac{i}{k} \frac{\mathrm{dn}\, u}{\mathrm{sn}\, u}$	$-\frac{ik'}{k\, \mathrm{cn}\, u}$	$-\mathrm{cn}\, u$	$-\mathrm{cn}\, u$	$\mathrm{cn}\, u$
$\mathrm{dn}\, u^* = k' \frac{1}{\mathrm{dn}\, u}$	$-i \frac{\mathrm{cn}\, u}{\mathrm{sn}\, u}$	$ik' \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u}$	$\mathrm{dn}\, u$	$-\mathrm{dn}\, u$	$-\mathrm{dn}\, u$

SM 630

3.

$u^* = 0$	$-u$	$\frac{1}{2}\boldsymbol{K}$	$\frac{1}{2}(\boldsymbol{K} + i\boldsymbol{K}')$	$\frac{1}{2}i\boldsymbol{K}'$	$u + 2m\boldsymbol{K} + 2n\boldsymbol{K}'i$
$\mathrm{sn}\, u^* = 0$	$-\mathrm{sn}\, u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k}+i\sqrt{1-k}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m\, \mathrm{sn}\, u$
$\mathrm{cn}\, u^* = 1$	$\mathrm{cn}\, u$	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n}\, \mathrm{cn}\, u$
$\mathrm{dn}\, u^* = 1$	$\mathrm{dn}\, u$	$\sqrt{k'}$	$\frac{\sqrt{k'}(\sqrt{1+k'}-i\sqrt{1-k'})}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n\, \mathrm{dn}\, u$

SI 19, SI 18(13), WH

920

8.152

Transformation formulas

u_1	k_1	$n(u_1, k_1)$	$\mathrm{cn}(u_1, k)$	$\ln(u_1, k$
ku	$\frac{1}{k}$	$k\, \mathrm{sn}(u, k)$	$\mathrm{dn}(u, k)$	$\mathrm{cn}(u, k$
iu	k'	$i \frac{\mathrm{sn}(u,\, k)}{\mathrm{cn}(u,\, k)}$	$\frac{1}{\mathrm{cn}(u,\, k)}$	$\frac{\mathrm{dn}(u,\, k}{\mathrm{cn}(u,\, k}$
$k'u$	$i\frac{k}{k'}$	$k' \frac{\mathrm{sn}(u,\, k)}{\mathrm{dn}(u,\, k)}$	$\frac{\mathrm{cn}(u,\, k)}{\mathrm{dn}(u,\, k)}$	$\frac{1}{\mathrm{dn}(u,\, k}$
iku	$i\frac{k'}{k}$	$ik \frac{\mathrm{sn}(u,\, k)}{\mathrm{dn}(u,\, k)}$	$\frac{1}{\mathrm{dn}(u,\, k)}$	$\frac{\mathrm{cn}(u, k}{\mathrm{dn}(u, k}$
$ik'u$	$\frac{1}{k'}$	$ik' \frac{\mathrm{sn}(u,\, k)}{\mathrm{cn}(u,\, k)}$	$\frac{\mathrm{dn}(u,\, k)}{\mathrm{cn}(u,\, k)}$	$\frac{1}{\mathrm{cn}(u,\, k}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k)\, \mathrm{sn}(u,\, k)}{1+k\, \mathrm{sn}^2(u,\, k)}$	$\frac{\mathrm{cn}(u,\, k)\, \mathrm{dn}(u,\, k)}{1+k\, \mathrm{sn}^2(u,\, k)}$	$\frac{1-k\, \mathrm{sn}^2(}{1+k\, \mathrm{sn}^2(}$
	$1-k'$	$\mathrm{sn}(u,\, k)\, \mathrm{cn}(u,\, k)$	$1-(1+k')\, \mathrm{sn}^2(u,\, k)$	$1-(1-k')\, \mathrm{sn}$

1. $\operatorname{sn}(iu,\, k) = i\dfrac{\operatorname{sn}(u,\, k')}{\operatorname{cn}(u,\, k')}.$

2. $\operatorname{cn}(iu,\, k) = \dfrac{1}{\operatorname{cn}(u,\, k')}.$

SI 50(65)

3. $\operatorname{dn}(iu,\, k) = \dfrac{\operatorname{dn}(u,\, k')}{\operatorname{cn}(u,\, k')}.$

SI 50(65)

4. $\operatorname{sn}(u,\, k) = k^{-1}\operatorname{sn}(ku,\, k^{-1}).$

5. $\operatorname{cn}(u,\, k) = \operatorname{dn}(ku,\, k^{-1}).$

6. $\operatorname{dn}(u,\, k) = \operatorname{cn}(ku,\, k^{-1}).$

7. $\operatorname{sn}(u,\, ik) = \dfrac{1}{\sqrt{1+k^2}}\dfrac{\operatorname{sn}(u\sqrt{1+k^2},\, k(1+k^2)^{-\frac{1}{2}})}{\operatorname{dn}(u\sqrt{1+k^2},\, k(1+k^2)^{-\frac{1}{2}})}.$

8. $\operatorname{cn}(u,\, ik) = \dfrac{\operatorname{sn}(u(1+k^2)^{\frac{1}{2}},\, k(1+k^2)^{-\frac{1}{2}})}{\operatorname{dn}(u(1+k^2)^{\frac{1}{2}},\, k(1+k^2)^{-\frac{1}{2}})}.$

9. $\operatorname{dn}(u,\, ik) = \dfrac{1}{\operatorname{dn}(u(1+k^2)^{\frac{1}{2}},\, k(1+k^2)^{-\frac{1}{2}})}.$

Functional relations

8.154

1. $\operatorname{sn}^2 u = \dfrac{1-\operatorname{cn} 2u}{1+\operatorname{dn} 2u}.$

MO 146

MO 146

3. $\operatorname{dn}^2 u = \dfrac{\operatorname{dn} 2u + k^2 \operatorname{cn} 2u + k'^2}{1 + \operatorname{dn} 2u}.$

MO 146

4. $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1.$

SI 16(9)

5. $\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1.$

SI 16(9)

8.155

1. $\dfrac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} = k^2 \dfrac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn}^2 u}.$

MO 146

2. $\dfrac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u} = \dfrac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{\operatorname{cn}^2 u}.$

MO 146

8.156

1. $\operatorname{sn}(u \pm v) = \dfrac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$

SI 46(56)

2. $\operatorname{cn}(u \pm v) = \dfrac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$

SI 46(57)

3. $\operatorname{dn}(u \pm v) = \dfrac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$

1. $\operatorname{sn}\frac{u}{2} = \pm\frac{1}{k}\sqrt{\frac{1-\operatorname{dn}u}{1+\operatorname{cn}u}} = \pm\sqrt{\frac{1-\operatorname{cn}u}{1+\operatorname{dn}u}}.$

SI 47(61), SU 67(15)

2. $\operatorname{cn}\frac{u}{2} = \pm\sqrt{\frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{dn}u}} = \pm\frac{k'}{k}\sqrt{\frac{1-\operatorname{dn}u}{\operatorname{dn}u-\operatorname{cn}u}}.$

SI 48(62), SI 67(16)

3. $\operatorname{dn}\frac{u}{2} = \pm\sqrt{\frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{cn}u}} = \pm k'\sqrt{\frac{1-\operatorname{cn}u}{\operatorname{dn}u+\operatorname{cn}u}}.$

SI 48(63), SI 67(17)

8.158

$$\left.\begin{array}{ll} 1. & \frac{d}{du}\operatorname{sn}u = \operatorname{cn}u\ \operatorname{dn}u. \\ 2. & \frac{d}{du}\operatorname{cn}u = -\operatorname{sn}u\ \operatorname{dn}u \\ 3.^{8} & \frac{d}{du}\operatorname{dn}u = -k^2\operatorname{dn}u\ \operatorname{cn}u. \end{array}\right\}$$

SI 21(21)

8.159

Jacobian elliptic functions are solutions of the following differential equations:

$$\left.\begin{array}{ll} 1. & \frac{d}{du}\operatorname{sn}u = \sqrt{(1-\operatorname{sn}^2\ u)(1-k^2\operatorname{sn}^2\ u)}. \\ 2. & \frac{d}{du}\operatorname{cn}u = -\sqrt{(1-\operatorname{cn}^2\ u)(k'^2+k^2\operatorname{cn}^2\ u)}, \\ 3. & \frac{d}{du}\operatorname{dn}u = -\sqrt{(1-\operatorname{dn}^2\ u)(\operatorname{dn}^2\ u-k'^2)}. \end{array}\right\}$$

SI 21(22)

For the indefinite integrals of Jacobi's elliptic functions, see 5.13.

8.16 The Weierstrass function $\wp(u)$

8.160

The Weierstrass elliptic function $\wp(u)$ is defined by

1. $\wp(u) = \frac{1}{u^2} + \sum_{m,n}{}' \left\{\frac{1}{(u-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2}\right\},$

SI 307(6)

where the symbol $\sum'$ means that the summation is made over all combinations of integers m and n except for the combination $m = n = 0$; $2\omega_1$ and $2\omega_2$ are the periods of the function $\wp(u)$. Obviously,

$$2. \quad \wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \quad \text{and} \quad \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0,$$

923

$$3. \quad \frac{d}{du}\,\wp(u) = -2\sum_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3},$$

where the summation is made over all integral values of m and n.

The series 8.160 1. and 8.160 3. converge everywhere except at the poles, that is, at the points $2m\omega_1 + 2n\omega_2$ (where m and n are integers).

4. The function $\wp(u)$ is a *second-order periodic function* and has *one second-order pole* in a period parallelogram.

SI 306

8.161

The function $\wp(u)$ satisfies the differential equation

$$1. \quad \left[\frac{d}{du}\wp(u)\right]^2 = 4\wp^3(u) - g_2\wp(u) - g_3,$$

SI 142, 310, WH

where

$$2. \quad g_2 = 60\sum_{m,n}{}' (m\omega_1 + n\omega_2)^{-4}; \quad g_3 = 140\sum_{m,n}{}' (m\omega_1 + n\omega_2)^{-6}.$$

WH, SI 310

The functions g_2 and g_3 are called the *invariants* of the function $\wp(u)$.

8.162

$$u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}},$$

where e_1, e_2, and e_3 are the roots of the equation $4z^3 - g_2 z - g_3 = 0$; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4}, \quad e_1 e_2 e_3 = \frac{g_3}{4}.$$

SI 142, 143, 144

8.163

$\wp(\omega_1) = e_1$, $\wp(\omega_1 + \omega_2) = e_2$, $\wp(\omega_2) = e_3$. Here, it is assumed that if e_1, e_2, and e_3 lie on a straight line in the complex plane, e_2 lies between e_1 and e_3.

8.164

The number $\Delta = g_2^3 - 27g_3^2$ is called the *discriminant* of the function $\wp(u)$. If $\Delta > 0$, all roots e_1, e_2, and e_3 of the equation $4z^3 - g_2 z - g_3 = 0$ (where g_2 and g_3 are real numbers) are *real*. In this case, the roots e_1, e_2, and e_3 are numbered in such a way that $e_1 > e_2 > e_3$.

1. If $\Delta > 0$, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \qquad \omega_2 = i\int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2 z - 4z^3}},$$

where ω_1 is real and ω_2 is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that ω_1 and $\frac{\omega_2}{i}$ will be positive.

2. If $\Delta < 0$, the root e_2 of the equation $4z^3 - g_2 z - g_3 = 0$ is *real* and the remaining two roots (e_1 and e_3) are *complex conjugates*. Suppose that $e_1 = \alpha = i\beta$, and $e_3 = \alpha - i\beta$. In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} \quad \text{and} \quad \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower-half plane.

SI 151(21, 22)

8.165

Series representation:

1. $\wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4 \cdot 5} + \frac{g_3 u^4}{4 \cdot 7} + \frac{g_2^2 u^6}{2^4 \cdot 3 \cdot 5^2} + \frac{3 g_2 g_3 u^8}{2^4 \cdot 5 \cdot 7 \cdot 11} + \dots$

WH

8.166

Functional relations

1. $\wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u).$

2. $\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2.$

SI 163(32)

8.167

$\wp(u; g_2, g_3) = \mu^2 \wp\left(\mu u;\, \frac{g_2}{\mu^4}, \frac{g_3}{\mu^6}\right)$ (the formula for homogeneity).

SI 149(13)

The special case: $\mu = i$.

1. $\wp(u;\ g_2,\ g_3) = -\wp(iu;\ g_2,\ -g_3).$

8.168

An arbitrary elliptic function can be expressed in terms of the elliptic function $\wp(u)$ having the same periods as the original function and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

$\wp(u)$

and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

8.169

A connection with the Jacobian elliptic functions. For $\Delta > 0$ (see 8.164 1.).

$$1. \quad \wp\left(\frac{u}{\sqrt{e_1 - e_2}}\right) = e_1 + (e_1 - e_3)\frac{\operatorname{cn}^2(u;\, k)}{\operatorname{sn}^2(u;\, k)};$$
$$= e_2 + (e_1 - e_3)\frac{\operatorname{dn}^2(u;\, k)}{\operatorname{sn}^2(u;\, k)};$$
$$= e_3 + (e_1 - e_3)\frac{1}{\operatorname{sn}^2(u,\, k)};$$

SI 145(5), ZH 120(197-199)a

$$2. \quad \omega_1 = \frac{\boldsymbol{K}}{\sqrt{e_1 - e_3}}, \qquad \omega_2 = \frac{i\boldsymbol{K}'}{\sqrt{e_1 - e_3}},$$

SI 154(29)

where

$$3. \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \qquad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}.$$

SI 145(7)

For $\Delta < 0$ (see 8.164 2.)

$$4. \quad \wp\left(\frac{u}{\sqrt[4]{9\alpha^2 + \beta^2}}\right) = e_2 + \sqrt{9\alpha^2 + \beta^2}\,\frac{1 + \operatorname{cn}(2u;\, k)}{1 - \operatorname{cn}(2u;\, k)};$$

SI 147(12)

$$5. \quad \omega' = \frac{\boldsymbol{K} - i\boldsymbol{K}'}{2\sqrt{9\alpha^2 + \beta^2}}, \qquad \omega'' = \frac{\boldsymbol{K} + i\boldsymbol{K}'}{\sqrt[4]{9\alpha^2 + \beta^2}},$$

SI 153(28)

where

$$6.\quad k = \sqrt{\frac{1}{2} - \frac{3e_2}{\sqrt{9\alpha^2 + \beta^2}}}; \qquad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{\sqrt{9\alpha^2 + \beta^2}}}.$$

SI 147

For $\Delta = 0$, all the roots e_1, e_2, and e_3 are real and if $g_2 g_3 \neq 0$, two of them are equal to each other. If $e_1 = e_2 \neq e_3$, then

$$7.\quad \wp(u) = \frac{3g_3}{g_2} - \frac{9g_3}{2g_2}\operatorname{cth}^2\left(u\sqrt{-\frac{9g_3}{2g_2}}\right).$$

SI 148

If $e_1 \neq e_2 = e_3$, then

$$8.\quad \wp(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2}\frac{1}{\sin^2\left(u\sqrt{\frac{9g_3}{2g_2}}\right)}.$$

SI 149

If $g_2 = g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and

$$9.\quad \wp(u) = \frac{1}{u^2}.$$

SI 149

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.171

Definitions:

$$1.\quad \zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2}\right)\,dz.$$

SI 181(45)

$$2.\quad \sigma(u) = u\,\exp\left\{\int_0^u \left(\wp(z) - \frac{1}{z^2}\right)\,dz\right\}.$$

8.172

Series and infinite-product representation

1. $\zeta(u) = \frac{1}{u} + \sum' \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 - 2n\omega_2)^2} \right).$

SI 307(8)

2. $\sigma(u) = u \prod' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2} \right) \exp \left\{ \frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2} \right\}.$

SI 308(9)

8.173

1. $\zeta(u) = u - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \frac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \frac{3 g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \cdots$

SI 181(49)

2. $\zeta(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^9 \cdot 3^2 \cdot 5 \cdot 7} - \frac{3 g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \cdots$

SI 181(49)

8.174

$$\zeta(u) = \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{\pi}{2\omega_1} \sum_{n=1}^{\infty} \left\{ \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} + n\pi \frac{\omega_2}{\omega_1} \right) + \right.$$
$$\left. + \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} - n\pi \frac{\omega_2}{\omega_1} \right) \right\};$$
$$= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin \frac{\pi n u}{\omega_1}$$

MO 155

MO 154

Functional relations and properties

8.175

SI 181

8.176

1. $\zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1).$

SI 184(57)

2. $\zeta(u + 2\omega_2) = \zeta(u) + 2\zeta(\omega_2).$

SI 184(57)

3. $\sigma(u + 2\omega_1) = -\sigma(u)\ \exp\{2(u + \omega_1)\zeta(\omega_1)\}.$

SI 185(60)

4. $\sigma(u + 2\omega_2) = -\sigma(u)\ \exp\{2(u + \omega_2)\zeta(\omega_2)\}.$

SI 185(60)

5. $\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{\pi}{2}i.$

SI 186(62)

8.177

1. $\zeta(u + v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}.$

SI 182(53)

2. $\wp(u) - \wp(v) = -\frac{\sigma(u - v)\sigma(u + v)}{\sigma^2(u)\sigma^2(v)}.$

SI 183(54)

3. $\zeta(u - v) + \zeta(u + v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}.$

SI 182(51)

8.178

1. $\zeta(u;\,\omega_1,\,\omega_2) = t\zeta(tu;\,t\omega_1,\,t\omega_2).$

MO 154

2.[7] $\sigma(u;\,\omega_1,\,\omega_2) = t\sigma(tu;\,t\omega_1,\,t\omega_2).$

MO 156

For the (indefinite) integrals of Weierstrass elliptic functions, see 5.14.

8.18- 8.19 Theta functions

8.180

Theta functions are defined as the sums (for $|q| < 1$) of the following series:

1. $$\vartheta_4(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nu.$$

WH

2. $$\vartheta_1(u) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2\sum_{n=1}^{\infty}(-1)^{n+1} q^{(n-\ \frac{1}{2})^2} \sin(2n{-}1)u.$$

WH

3. $$\vartheta_2(u) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2\sum_{n=1}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n-1)u.$$

WH

4. $$\vartheta_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nu.$$

WH

The notations $\vartheta(u,\, q)$, $\vartheta(u|\tau)$, where τ and q are related by $q = e^{i\pi\tau}$, are also used. q is called the *nome* of the theta function and τ its *parameter.*

8.181

Representation of theta functions in terms of infinite products

$$1.\quad \vartheta_4(u) = \prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2u + q^{2(2n-1)})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$2.\quad \vartheta_3(u) = \prod_{n=1}^{\infty}(1 + 2q^{2n-1}\cos 2u + q^{2(2n-1)})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$3.\quad \vartheta_1(u) = 2\sqrt[4]{q}\,\sin u\prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2u + q^{4n})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$4.\quad \vartheta_2(u) = 2\sqrt{2}\cos u\prod_{n=1}^{\infty}(1 + 2q^{2n}\cos 2u + q^{4n})(1 - q^{2n}).$$

SI 200(0), ZH 90(9)

Functional relations and properties

8.182

Quasiperiodicity.

Suppose that $q = e^{\pi\tau i}(\operatorname{Im}\tau > 0)$. Then, theta functions that are periodic functions of u are called *quasiperiodic functions* of τ and u. This property follows from the equations

$$1.\quad \vartheta_4(u + \pi) = \vartheta_4(u).$$

2. $\vartheta_4(u+\tau\pi) = -\frac{1}{q}e^{-2iu}\vartheta_4(u).$

SI 200(10)

3. $\vartheta_1(u+\pi) = -\vartheta_1(u).$

SI 200(10)

4. $\vartheta_1(u+\tau\pi) = -\frac{1}{q}e^{-2iu}\vartheta_1(u).$

SI 200(10)

5. $\vartheta_2(u+\pi) = -\vartheta_2(u).$

SI 200(10)

6. $\vartheta_2(u+\tau\pi) = \frac{1}{q}e^{-2iu}\vartheta_2(u).$

SI 200(10)

7. $\vartheta_3(u+\pi) = \vartheta_3(u).$

SI 200(10)

8. $\vartheta_3(u+\tau\pi) = \frac{1}{q}e^{-2iu}\vartheta_3(u).$

SI 200(10)

$9^*.\vartheta_4(u+\pi) = \vartheta_4(u).$

LW 6(1.3.5)

928

8.183

1. $\vartheta_4\left(u+\frac{1}{2}\pi\right) = \vartheta_3(u).WH$

2. $\vartheta_1\left(u+\frac{1}{2}\pi\right)=\vartheta_2(u).$

WH

3. $\vartheta_2\left(u+\frac{1}{2}\pi\right)=-\vartheta_1(u).$

WH

4. $\vartheta_3\left(u+\frac{1}{2}\pi\right)=\vartheta_4(u).$

WH

5. $\vartheta_4\left(u+\frac{1}{2}\pi\tau\right)=iq^{-\frac{1}{4}}e^{-iu}\vartheta_1(u).$

WH

6. $\vartheta_1\left(u+\frac{1}{2}\pi\tau\right)=iq^{-\frac{1}{4}}e^{-iu}\vartheta_4(u).$

WH

7. $\vartheta_2\left(u+\frac{1}{2}\pi\tau\right)=q^{-\frac{1}{4}}e^{-iu}\vartheta_3(u).$

WH

8. $\vartheta_3\left(u+\frac{1}{2}\pi\tau\right)=q^{-\frac{1}{4}}e^{-iu}\vartheta_2(u).$

WH

8.184

Even and odd theta functions

1. $\vartheta_1(-u)=-\vartheta_1(u)$

WH

WH

3. $\vartheta_3(-u) = \vartheta_3(u)$

WH

4. $\vartheta_4(-u) = \vartheta_4(u)$.

WH

8.185

$$\vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u).$$

WH

8.186[7]

Considering the theta functions as functions of two independent variables u and τ, we have

$$\pi i \frac{\partial^2 \vartheta_k(u|\tau)}{\partial u^2} + 4\frac{\partial \vartheta_k(u|\tau)}{\partial \tau} = 0 \qquad [k = 1,\ 2,\ 3,\ 4].$$

WH

8.187

We denote the partial derivatives of the theta functions with respect to u by a prime and consider them as functions of the single argument u. Then,

1. $\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$.

WH

2. $\dfrac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \dfrac{\vartheta_2''(0)}{\vartheta_2(0)} + \dfrac{\vartheta_3''(0)}{\vartheta_3(0)} + \dfrac{\vartheta_4''(0)}{\vartheta_4(0)}$.

WH

8.188

$$\vartheta_1(u)\vartheta_2(u)\vartheta_3(u)\vartheta_4(0) = \frac{1}{2}\vartheta_1(2u)\vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$$

8.189

The zeros of the theta functions:

$$1.^{8}\quad \vartheta_4(u) = 0 \quad \text{for} \quad u = 2m\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2}.$$

SI 201

$$2.^{*}\quad \vartheta_1(u) = 0 \quad \text{for} \quad u = 2m\frac{\pi}{2} + 2n\frac{\pi\tau}{2}.$$

SI 201

$$3.\quad \vartheta_2(u) = 0 \quad \text{for} \quad u = (2m-1)\frac{\pi}{2} + 2n\frac{\pi\tau}{2}.$$

SI 201

$$4.\quad \vartheta_3(u) = 0 \quad \text{for} \quad u = (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2} \qquad [m \text{ and } n-\text{integers or zero}].$$

SI 201

For integrals of theta functions, see 6.16.

8.191

Connections with the Jacobian elliptic functions:

$$\text{For} \quad \tau = i\frac{\boldsymbol{K}'}{\boldsymbol{K}}, \quad \text{i.e.} \quad \text{for} \quad q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right),$$

$$1.\quad \operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}.$$

SI 206(22), SI 209(35)

$$2.\quad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}.$$

3. $\operatorname{dn} u = \sqrt{k'} \dfrac{\vartheta_3\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{k'} \dfrac{\Theta_1(u)}{\Theta(u)}.$

SI 207(24), SI 209(35)

8.192

Series representation of the functions $H,\ H_1,\ \Theta,\ \Theta_1$.

1. $\Theta(u) = \vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos \dfrac{n\pi u}{\boldsymbol{K}}.$

SI 207(25), SI 212(42)

2. $H(u) = \vartheta_1\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[4]{q^{(2n+1)^2}} \sin(2n-1)\dfrac{\pi u}{2\boldsymbol{K}}.$

SI 207(25), SI 212(43)

3. $\Theta_1(u) = \vartheta_3\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos \dfrac{n\pi u}{\boldsymbol{K}}.$

SI 207(25), SI 212(45)

4. $H_1(u) = \vartheta_2\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 2\sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1)\dfrac{\pi u}{2\boldsymbol{K}}.$

SI 207(25), SI 212(44)

In formulas 8.192 $q = \exp\left(-\pi \dfrac{\boldsymbol{K}'}{\boldsymbol{K}}\right)$.

930

8.193

Connections with the Weierstrass elliptic functions

1. $\wp(u) = e_1 + \left[\dfrac{H_1(u\sqrt{\lambda})H'(0)}{H_1(0)H(u\sqrt{\lambda})}\right]^2 \lambda = e_2 + \left[\dfrac{\Theta_1(u\sqrt{\lambda})H'(0)}{\Theta_1(0)H(u\sqrt{\lambda})}\right]^2 \lambda = e_3 + \left[\dfrac{\Theta(u\sqrt{\lambda})H'(0)}{\Theta(0)H(u\sqrt{\lambda})}\right]^2 \lambda.$

2. $\zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda}\frac{H'(u\sqrt{\lambda})}{H(u\sqrt{\lambda})}.$

SI 234(73)

3. $\sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H(u\sqrt{\lambda})}{H'(0)},$

SI 234(72)

where

$$\lambda = e_1 - e_3; \quad \eta_1 = \zeta(\omega_1) = -\frac{\omega_1 \lambda}{3} \frac{H'''(0)}{H'(0)}.$$

SI 236

8.194

The connection with elliptic integrals:

1. $E(u,\, k) = u - u\frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)}.$

SI 228(65)

2. $\Pi(u,\, -k^2 \sin^2\, a,\, k) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2\, a\, \operatorname{sn}^2\, \varphi} = u + \frac{\operatorname{sn} a}{\operatorname{cn} a\, \operatorname{dn} a}\left[\frac{\Theta'(a)}{\Theta(a)} u + \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)}\right].$

SI 232(69)

$$q-\text{series and products} \quad \left[q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right)\right]$$

8.195

$$\frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2 = \boldsymbol{K} = \frac{\pi}{2}\Theta^2(\boldsymbol{K}) \qquad (\text{cf.}\,\mathbf{8}.1971.).$$

8.196

$$\boldsymbol{E} = \boldsymbol{K} - \boldsymbol{K}\frac{\Theta''(0)}{\Theta(0)} = \boldsymbol{K} - \frac{2\pi^2}{\boldsymbol{K}}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{n^2}}{1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}}.$$

SI 230(67)

8.197

1. $$1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2\boldsymbol{K}}{\pi}} = \vartheta_3(0) \qquad (\text{c.f.}\,\mathbf{8}.195).$$

8.195
WH

2. $$\sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{k\boldsymbol{K}}{2\pi}} = \frac{1}{2}\vartheta_2(0).$$

WH

931

3. $$4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4 = k.$$

SI 206(17, 18)

4. $$\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 = k'$$

SI 206(19, 20)

5. $$2\sqrt[4]{q}\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right)^2 = 2\sqrt{k}\frac{\boldsymbol{K}}{\pi}.$$

WH

WH

8.198

1. $$\lambda = \frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1+2\sum_{n=1}^{\infty} q^{4n^2}} \qquad \left[\text{for } 0<k<1, \text{ we have } 0<\lambda<\frac{1}{2}\right]$$

WH

The series

2. $q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \ldots$ is used to determine q from the given modulus k.

WH

8.199

* Identities involving products of theta functions

1. $$\theta_1(x, q)\theta_1(y, q) = \theta_3(x+y, q^2)\theta_2(x-y, q^2) - \theta_2(x+y, q^2)\theta_3(x-y, q^2).$$

LW 7(1.4.7)

2. $$\theta_1(x, q)\theta_2(y, q) = \theta_1(x+y, q^2)\theta_4(x-y, q^2) + \theta_4(x+y, q^2)\theta_1(x-y, q^2).$$

LW 8(1.4.8)

3. $$\theta_2(x, q)\theta_2(y, q) = \theta_2(x+y, q^2)\theta_3(x-y, q^2) + \theta_3(x+y, q^2)\theta_2(x-y, q^2).$$

LW 8(1.4.9)

4. $$\theta_3(x, q)\theta_3(y, q) = \theta_3(x+y, q^2)\theta_3(x-y, q^2) + \theta_2(x+y, q^2)\theta_2(x-y, q^2).$$

LW 8(1.4.10)

5. $$\theta_3(x, q)\theta_4(y, q) = \theta_4(x+y, q^2)\theta_4(x-y, q^2) - \theta_1(x+y, q^2)\theta_1(x-y, q^2).$$

6. $\theta_4(x,\,q)\theta_4(y,\,q) = \theta_3(x+y,\,q^2)\theta_3(x-y,\,q^2) - \theta_2(x+y,\,q^2)\theta_2(x-y,\,q^2).$

LW 8(1.4.12)

7. $\theta_1(x+y)\theta_1(x-y)\theta_4^2(0) = \theta_3^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_3^2(y) = \theta_1^2(x)\theta_4^2(y) - \theta_4^2(x)\theta_1^2(y).$

LW 8(1.4.16)

8. $\theta_2(x+y)\theta_2(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_2^2(y) - \theta_1^2(x)\theta_3^2(y) = \theta_2^2(x)\theta_4^2(y) - \theta_3^2(x)\theta_1^2(y).$

LW 8(1.4.17)

9. $\theta_3(x+y)\theta_3(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_3^2(y) - \theta_1^2(x)\theta_2^2(y) = \theta_3^2(x)\theta_4^2(y) - \theta_2^2(x)\theta_1^2(y).$

LW 8(1.4.18)

10. $\theta_4(x+y)\theta_4(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_4^2(y) - \theta_1^2(x)\theta_1^2(y).$

LW 8(1.4.15)

11. $\theta_4(x+y)\theta_4(x-y)\theta_4^2(0) = \theta_3^2(x)\theta_3^2(y) - \theta_2^2(x)\theta_2^2(y) = \theta_4^2(x)\theta_4^2(y) - \theta_1^2(x)\theta_1^2(y).$

LW 9(1.4.19)

12. $\theta_1(x+y)\theta_1(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_1^2(y) = \theta_4^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_4^2(y).$

LW 9(1.4.23)

13. $\theta_2(x+y)\theta_2(x-y)\theta_3^2(0) = \theta_2^2(x)\theta_3^2(y) - \theta_4^2(x)\theta_1^2(y) = \theta_3^2(x)\theta_2^2(y) - \theta_1^2(x)\theta_4^2(y).$

LW 9(1.4.24)

14. $\theta_3(x+y)\theta_3(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_1^2(y) + \theta_3^2(x)\theta_3^2(y) = \theta_2^2(x)\theta_2^2(y) + \theta_4^2(x)\theta_4^2(y).$

15. $\theta_4(x+y)\theta_4(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_2^2(y)+\theta_3^2(x)\theta_4^2(y) = \theta_2^2(x)\theta_1^2(y)+\theta_4^2(x)\theta_3^2(y)$.

LW 9(1.4.26)

16. $\theta_1(x+y)\theta_1(x-y)\theta_2^2(0) = \theta_1^2(x)\theta_2^2(y)-\theta_2^2(x)\theta_1^2(y) = \theta_4^2(x)\theta_3^2(y)-\theta_3^2(x)\theta_4^2(y)$.

LW 9(1.4.30)

17. $\theta_2(x+y)\theta_2(x-y)\theta_2^2(0) = \theta_2^2(x)\theta_2^2(y)-\theta_1^2(x)\theta_1^2(y) = \theta_3^2(x)\theta_3^2(y)-\theta_4^2(x)\theta_4^2(y)$.

LW 10(1.4.31)

18. $\theta_3(x+y)\theta_3(x-y)\theta_2^2(0) = \theta_3^2(x)\theta_2^2(y)+\theta_4^2(x)\theta_1^2(y) = \theta_2^2(x)\theta_3^2(y)+\theta_1^2(x)\theta_4^2(y)$.

LW 10(1.4.32)

19. $\theta_4(x+y)\theta_4(x-y)\theta_2^2(0) = \theta_4^2(x)\theta_2^2(y)+\theta_3^2(x)\theta_1^2(y) = \theta_1^2(x)\theta_3^2(y)+\theta_2^2(x)\theta_4^2(y)$.

LW 10(1.4.33)

20. $\theta_3^2(x)\theta_3^2(0) = \theta_4^2(x)\theta_4^2(0) + \theta_2^2(x)\theta_2^2(0)$.

LW 11(1.4.49)

21. $\theta_4^2(x)\theta_3^2(0) = \theta_1^2(x)\theta_2^2(0) + \theta_3^2(x)\theta_4^2(0)$.

LW 11(1.4.50)

22. $\theta_4^2(x)\theta_2^2(0) = \theta_1^2(x)\theta_3^2(0) + \theta_2^2(x)\theta_4^2(0)$.

LW 11(1.4.51)

23. $\theta_3^2(x)\theta_2^2(0) = \theta_1^2(x)\theta_4^2(0) + \theta_2^2(x)\theta_3^2(0)$.

24. $\theta_3^4(x)\theta_2^4(0) + \theta_4^4(0)$.

LW 11(1.4.53)

8.199(1)*

Theta functions as infinite products

1. $$\theta_1(z) = 2q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2z + q^{4n}).$$

LW 15(1.6.23)

2. $$\theta_2(z) = 2q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2z + q^{4n}).$$

LW 15(1.5.24)

3. $$\theta_3(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2z + q^{4n-2}).$$

LW 15(1.6.25)

4. $$\theta_4(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos 2z + q^{4n-2}).$$

LW 15(1.6.26)

8.199(2)*

Derivatives of ratios of theta functions

1. $$\frac{d}{dx}(\theta_1/\theta_4) = \theta_4^2(0)\theta_2(x)\theta_3(x)/\theta_4^2(x).$$

LW 19(1.9.3)

2. $$\frac{d}{dx}(\theta_2/\theta_4) = -\theta_3^2(0)\theta_1(x)\theta_3(x)/\theta_4^2(x).$$

3. $\dfrac{d}{dx}(\theta_3/\theta_4) = -\theta_2^2(0)\theta_1(x)\theta_2(x)/\theta_4^2(x).$

LW 19(1.9.7)

4. $\dfrac{d}{dx}(\theta_1/\theta_3) = \theta_3^2(0)\theta_2(x)\theta_4(x)/\theta_3^2(x).$

LW 19(1.9.8)

5. $\dfrac{d}{dx}(\theta_2/\theta_3) = -\theta_4^2(0)\theta_1(x)\theta_4(x)/\theta_3^2(x).$

LW 19(1.9.9)

6. $\dfrac{d}{dx}(\theta_1/\theta_2) = \theta_2^2(0)\theta_3(x)\theta_4(x)/\theta_2^2(x).$

LW 19(1.9.10)

7. $\dfrac{d}{dx}(\theta_4/\theta_1) = -\theta_4^2(0)\theta_2(x)\theta_3(x)/\theta_1^2(x).$

LW 19(1.9.11)

8. $\dfrac{d}{dx}(\theta_4/\theta_2) = \theta_3^2(0)\theta_1(x)\theta_3(x)/\theta_2^2(x).$

LW 20(1.9.12)

9. $\dfrac{d}{dx}(\theta_4/\theta_3) = \theta_2^2(0)\theta_1(x)\theta_2(x)/\theta_3^2(x).$

LW 20(1.9.13)

10. $\dfrac{d}{dx}(\theta_3/\theta_1) = -\theta_3^2(0)\theta_2(x)\theta_4(x)/\theta_1^2(x).$

LW 20(1.9.14)

11. $\dfrac{d}{dx}(\theta_3/\theta_2) = \theta_4^2(0)\theta_1(x)\theta_4(x)/\theta_2^2(x).$

12. $\frac{d}{dx}(\theta_2/\theta_1) = -\theta_2^2(0)\theta_3(x)\theta_4(x)/\theta_1^2(x).$

LW 20(1.9.16)

8.199(3)*

Derivatives of theta functions

1. $\frac{d}{du} \ln \theta_1(u) = \cot u + 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - 2q^{2n} \cos 2u + q^{4n}}.$

2. $\frac{d}{du} \ln \theta_2(u) = -\tan u - 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + 2q^{2n} \cos 2u + q^{4n}}.$

3. $\frac{d}{du} \ln \theta_3(u) = -4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 + 2q^{2n} \cos 2u + q^{4n-2}}.$

4. $\frac{d}{du} \ln \theta_4(u) = 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - 2q^{2n} \cos 2u + q^{4n-2}}.$

5. $\frac{d^2}{du^2} \ln \theta_2(u) = -\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 \{i(u + n\pi\tau)\}.$

8.2 The Exponential-Integral Function and Functions Generated by It

8.21 The exponential-integral function Ei(x)

8.211

1. $\operatorname{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \int_{-\infty}^{x} \frac{e^{t}}{t}\, dt = \operatorname{li}(e^x) \qquad [x < 0].$

$$2.^{8}\quad \operatorname{Ei}(x) = -\lim_{e\to+0}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}\,dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}\,dt\right] pv\int_{-\infty}^{x}\frac{e^{t}}{t}dt \qquad [x>0].$$

$$3.^{7}\ \operatorname{Ei}(x) = \frac{1}{2}\{\operatorname{Ei}(x+i0)+\operatorname{Ei}(x-i0)\} \qquad [x>0].$$

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8.212

$$1.\quad \operatorname{Ei}(-x) = \boldsymbol{C} = \ln x + \int_0^x \frac{e^{-t}-1}{t}\,dt \qquad [x>0];$$
$$= \boldsymbol{C} + e^{-x}\ln x + \int_0^x e^{-t}\ln t\,dt \qquad [x>0].$$

NT 11(10)
NT 11(1)

$$2.^{7}\quad \operatorname{Ei}(x) = e^{x}\left[\frac{1}{x}+\int_0^\infty \frac{e^{-t}dt}{(x-t)^2}\right] \qquad [\text{integral is divergent}].$$

8.211

$$3.\quad \operatorname{Ei}(-x) = e^{-x}\left[-\frac{1}{x}+\int_0^\infty \frac{e^{-t}dt}{(x+t)^2}\right] \qquad [x>0] \qquad (\text{cf. } \mathbf{8}.211\ 1.).$$

8.211
LA 281(28)

$$4.\quad \operatorname{Ei}(\pm x) = \pm e^{\pm x}\int_0^1 \frac{dt}{x\pm\ln t} \qquad [x>0] \qquad (\text{cf. } \mathbf{8}.211\ \ 1.).$$

8.211

$$5.\quad \operatorname{Ei}(\pm xy) = \pm e^{\pm xy}\int_0^\infty \frac{e^{-xt}}{y\mp t}\,dt \qquad [\operatorname{Re} y>0,\quad x>0].$$

NT 19(11)

6. $\mathrm{Ei}(\pm x) = -e^{\pm x} \int_0^\infty \frac{e^{-it}}{t \pm ix}\, dt \qquad [x > 0].$

NT 23(2, 3)

$$7.^{8} \quad \mathrm{Ei}(xy) = e^{xy} \int_0^1 \frac{t^{y-1}}{x + \ln t}\, dt;$$

LA 283(46)a
LA 282(44)a

$$8. \quad \mathrm{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t}\, dt;$$
$$= x^{-1} e^{-xy} \left[\int_0^1 \frac{t^{x-1}}{(y - \ln t)^2}\, dt - y^{-1} \right] \qquad [x > 0, \quad y > 0].$$

LA 283(47)a
LA 282(45)a

9. $\mathrm{Ei}(x) = e^x \int_1^\infty \frac{1}{x - \ln t} \frac{dt}{t^2} \qquad [x > 0].$

LA 283(48)

10. $\mathrm{Ei}(-x) = -e^{-x} \int_1^\infty \frac{1}{x + \ln t} \frac{dt}{t^2} \qquad [x > 0].$

LA 283(48)

11. $\mathrm{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t + x \sin t}{t^2 + x^2}\, dt \qquad [x > 0].$

NT 23(6)

12. $\mathrm{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t - x \sin t}{t^2 + x^2}\, dt \qquad [x < 0].$

NT 23(6)

13. $\mathrm{Ei}(-x) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \operatorname{arctg} \frac{t}{x}\, dt \qquad [\mathrm{Re}\, x > 0].$

14. $\operatorname{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^\infty \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t \, dt \qquad [x > 0].$

NT 26(7)

935

15. $\operatorname{Ei}(x) = 2 \ln x - \frac{2e^{x}}{\pi} \int_0^\infty \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t \, dt \qquad [x > 0].$

NT 27(8)

16. $\operatorname{Ei}(-x) = -x \int_1^\infty e^{-tx} \ln t \, dt \qquad [x > 0].$

NT 32(12)

See also 3.327, 3.881 8., 3.916 2. and 3., 4.326 1., 4.326 2., 4.331 2., 4.351 3., 4.425 3., 4.581. For integrals of the exponential-integral function, see 6.22-6.23, 6.78.

Series and asymptotic representations

8.213

1. $\operatorname{li}(x) = \boldsymbol{C} + \ln(-\ln x) + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \qquad [0 < x < 1].$

NT 3(9)

2. $\operatorname{li}(x) = \boldsymbol{C} + \ln \ln x + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \qquad [x > 1].$

NT 3(10)

8.214

1. $\operatorname{Ei}(x) = \boldsymbol{C} + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} \qquad [x < 0].$

NT 39(13)

8.215

$$\mathrm{Ei}(z) = \frac{e^z}{z}\left[\sum_{k=0}^{n}\frac{k!}{z^k} + R_n(z)\right], |R_n(z)| = O(|z|^{-n-1}),$$
$$[z\to\infty, |\arg(-z)| \le \pi-\delta; \quad \delta > 0 \text{ small}]. \quad |R_n(z)| \le (n+1)!|z|^{-n-1}[\mathrm{Re}\, z \le 0].$$

where

$$|R_n| < \frac{n!}{|x|^{n+1}\cos\frac{\varphi}{2}}, \qquad x = |x|e^{i\varphi}, \quad \varphi^2 < \pi^2.$$

NT 37(9)

8.216[7]

$$\mathrm{Ei}(nx) - \mathrm{Ei}(-nx) = e^{nx'}\left(\frac{1}{nx} + \frac{1}{n^2x^2} + \frac{k_n}{n^3x^3}\right),$$

where

$$x' = x \text{ sign}\,\mathrm{Re}(x),\ k_n = O(1),\ \text{and}\ n\to\infty.$$

NT 39(15)

8.217

Functional relations:

$$1.\quad e^{x'}\ \mathrm{Ei}(-x') - e^{-x'}\ \mathrm{Ei}(x') = -2\int_0^\infty \frac{x'\ \sin t}{t^2+x^2}\,dt =$$
$$= \frac{4}{\pi}\int_0^\infty \frac{x'\ \cos t}{t^2+x^2}\ln t\,dt - 2e^{-x'}\ln x' \qquad [x' = x \text{ sign}\,\mathrm{Re}\, x].$$

NT 27(9)
NT 24(11)

NT 24(10), NT 27(10)

3. $\mathrm{Ei}(-x) - \mathrm{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi}\int_0^\infty \frac{\cos t}{t} \operatorname{arctg} \frac{t\left(x - \frac{1}{x}\right)}{1+t^2}\,dt \qquad [\operatorname{Re} x > 0].$

NT 25(14)

4. $\mathrm{Ei}(-\alpha x)\ \mathrm{Ei}(-\beta x) - \ln(\alpha\beta)\ \mathrm{Ei}[-(\alpha+\beta)x] = e^{-(\alpha+\beta)x}\int_0^\infty \frac{e^{-tx}\ln[(\alpha+t)(\beta+t)]}{t+\alpha+\beta}\,dt.$

NT 32(9)

See also 3.723 1. and 5., 3.742 2. and 4., 3.824 4., 4.573 2.

For a connection with a degenerate hypergeometric function, see 9.237.

For integrals of the exponential-integral function, see 5.21, 5.22, 5.23, and 6.22-6.23.

8.218

Two numerical values:

1. $\mathrm{Ei}(-1) = -0.219\quad 383\quad 934\quad 395\quad 520\quad 273\quad 665\ldots$

NT 89

2. $\mathrm{Ei}(1) = 1.895\quad 117\quad 816\quad 355\quad 936\quad 755\quad 478\ldots$

NT 89

8.22 The hyperbolic-sine-integral shi x and the hyperbolic-cosine-integral chi x

8.221

1. $\operatorname{shi} x = \int_0^x \frac{\operatorname{sh} t}{t}\,dt = -i\left[\frac{\pi}{2} + \operatorname{si}(ix)\right] \qquad (\text{see } \mathbf{8}.2301.).$

8.230

EH II 146(17)

2. $\operatorname{chi} x = \boldsymbol{C} + \ln x + \int_0^x \frac{\operatorname{ch} t - 1}{t}\, dt.$

EH II 146(18)

8.23 The sine integral and the cosine integral: si(x) and ci(x)

8.230

1.* $\operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt = -\frac{\pi}{2} + \operatorname{Si}(x), \text{where } \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt.$

NT 11(3)

2.* $\operatorname{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt = \boldsymbol{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\, dt \qquad [\operatorname{ci}(x) \text{is also written } \operatorname{Ci}(x)].$

NT 11(2)

8.231

1. $\operatorname{si}(xy) = -\int_x^\infty \frac{\sin ty}{t}\, dt.$

NT 18(7)

2. $\operatorname{ci}(xy) = -\int_x^\infty \frac{\cos ty}{t}\, dt.$

NT 18(6)

937

3. $\operatorname{si}(x) = -\int_0^{\frac{\pi}{2}} e^{-x \cos t} \cos(x \sin t)\, dt.$

NT 13(26)

8.232

1. $\operatorname{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^\infty \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!}.$

2.[7] $\operatorname{ci}(x) = \boldsymbol{C} + \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!}.$

NT 7(3)

8.233

1. $\operatorname{ci}(x) \pm i \operatorname{si}(x) = \operatorname{Ei}(\pm ix).$

NT 6a

2. $\operatorname{ci}(x) - \operatorname{ci}(xe^{\pm\pi i}) = \mp\pi i.$

NT 7(5)

3. $\operatorname{si}(x) + \operatorname{si}(-x) = -\pi.$

NT 7(7)

8.234

1.[7] $\operatorname{Ei}(-x) - \operatorname{ci}(x) = \int_0^{\frac{\pi}{2}} e^{-x\cos\varphi} \sin(s \sin\varphi)\, d\varphi.$

NT 13(27)

2. $[\operatorname{ci}(x)]^2 + [\operatorname{si}(x)]^2 = -2\int_0^{\frac{\pi}{2}} \frac{\exp(-x \operatorname{tg}\varphi) \ln\cos\varphi}{\sin\varphi \cos\varphi}\, d\varphi \qquad [\operatorname{Re} x > 0] \qquad \text{(see also 4.366)}.$

4.366
NT 32(11)

See also 3.341, 3.351 1. and 2., 3.354 1 and 2., 3.721 2. and 3., 3.722 1., 3., 5. and 7., 3.723 8. and 11., 4.338 1., 4.366 1.

8.235

1. $\lim_{x\to+\infty} (x^\varrho \operatorname{si}(x)) = 0, \quad \lim_{x\to+\infty} (x^\varrho \operatorname{ci}(x)) = 0 \qquad [\varrho < 1].$

NT 38(5)

NT 38(6)

For integrals of the sine integral and cosine integral, see 6.24-6.26, 6.781, 6.782, and 6.783. For indefinite integrals of the sine-integral and cosine-integral, see 5.3.

8.24 The logarithm-integral li(x)

8.240

$$1.\quad \operatorname{li}(x) = \int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x) \qquad [x < 1].$$

JA

$$2.\quad \operatorname{li}(x) = \lim_{\varepsilon \to 0} \left[\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right] = \operatorname{Ei}(\ln x) \qquad [x > 1].$$

JA

$$3.\quad \operatorname{li}\{\exp(-xe^{\pm\pi i})\} = \operatorname{Ei}(-xe^{\pm i\pi}) = \operatorname{Ei}(x \mp i0) = \operatorname{Ei}(x) \pm i\pi = \operatorname{li}(e^x) \pm i\pi \qquad [x > 0].$$

JA, NT 2(6)

8.241

$$1.\quad \operatorname{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t}\,dt = x\,\ln\ln\frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t}\,\ln t\,dt \qquad [x < 1].$$

LA 281(33)

$$\begin{aligned} 2.\quad \operatorname{li}(x) &= x \int_0^1 \frac{dt}{\ln x + \ln t}; \\ &= \frac{x}{\ln x} + x \int_0^1 \frac{dt}{(\ln x + \ln t)^2}; \\ &= x \int_1^\infty \frac{1}{\ln x - \ln t}\frac{dt}{t^2} \qquad [x < 1]. \end{aligned}$$

3. $\operatorname{li}(a^x) = \frac{1}{\ln a}\int_{-\infty}^{x} \frac{a^t}{t}\,dt \qquad [x > 0].$

For integrals of the logarithm integral, see 6.21

8.25 The probability integral Φ(x), the Fresnel integrals S(x), C(x), the error function erf(x) and the complementary error function erfc(x)

8.250

Definition:

1. $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad \text{(also called the error function and denoted by erf } (x)).$

2. $S(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \sin t^2\,dt.$

3. $C(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \cos t^2\,dt.$

4.* $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \Phi(x) \qquad \text{(called the complementary error function)}\ .$

Integral representations

8.251

1. $\Phi(x) = \frac{1}{\sqrt{\pi}}\int_0^{x^2} \frac{e^{-t}}{\sqrt{t}}\,dt \qquad (\text{see also } \mathbf{3}.3611.).$

3.361

2. $S(x) = \frac{1}{\sqrt{2\pi}}\int_0^{x^2} \frac{\sin t}{\sqrt{t}}\,dt.$

3. $C(x) = \frac{1}{\sqrt{2\pi}}\int_0^{x^2} \frac{\cos t}{\sqrt{t}}\,dt.$

8.252

1. $\Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2y^2}\, dt.$

2. $S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(t^2y^2)\, dt.$

3. $C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(t^2y^2)\, dt.$

4. $$\left.\begin{aligned} \Phi(xy) &= 1 - \frac{2}{\sqrt{\pi}} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2}\, ty\, dt}{\sqrt{t^2+x^2}} \\ &= 1 - \frac{2x}{\pi} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2}\, dt}{t^2+x^2} \end{aligned}\right\} \qquad [\operatorname{Re} y^2 > 0]$$

NT 19(13)a
NT 19(11)a

5.[7] $\Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2y^2} \sin(ty)\, dt \qquad [\operatorname{Re} x^2 > 0].$

NT 28(3)a

6. $\Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} x e^{-\frac{y^2}{4x^2}} \int_0^\infty e^{-t^2x^2 - ty}\, dt \qquad [\operatorname{Re} x^2 > 0].$

NT 27(1)a

See also 3.322, 3.362 2., 3.363, 3.468, 3.897, 6.511 4. and 5.

8.253

Series representations:

1. $$\begin{aligned} \Phi(x) &= \frac{2}{\sqrt{\pi}} \sum_{k=1}^\infty (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!}; \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^\infty \frac{2^k x^{2k+1}}{(2k+1)!!}. \end{aligned}$$

NT 7(9)a

$$2.\quad S(x) = \frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)};$$
$$= \frac{2}{\sqrt{2\pi}}\left\{\sin x^2\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!}\right\}.$$

NT 8(14)a
NT 10(13)a

$$3.\quad C(x)\frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)};$$
$$= \frac{2}{\sqrt{2\pi}}\left\{\sin x^2\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!}\right\}$$

NT 10(12)a
NT 8(13)a

For the expansions in Bessel functions, see 8.515 2., 8.515 3.

Asymptotic representations

8.254[7]

$$\Phi(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi}z}\left[\sum_{k=0}^{n}(-1)^k\frac{(2k-1)!!}{(2z^z)k} + O(|z|^{-2n-2})\right], [z\to\infty, |\arg -z)| \le \pi - \delta; \delta > 0 \text{ small}]$$

940

where

$$|R_n| < \frac{\Gamma(n+\frac{1}{2})}{|x|^{n+\frac{1}{2}}\cos\frac{\varphi}{2}}, \quad x = |x|e^{i\varphi} \quad \text{and} \quad \varphi^2 < \pi^2.$$

NT 37(10)a

8.255

$$1.\quad S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}x}\cos x^2 + O\left(\frac{1}{x^2}\right) \qquad [x\to\infty].$$

2. $C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi x}} \sin x^2 + O\left(\frac{1}{x^2}\right) \qquad [x \to \infty].$

MO 127a

8.256

Functional relations:

1. $C(z) + iS(z) = \sqrt{\frac{i}{2}}\Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}}\int_0^z e^{it^2}\,dt.$

2. $C(z) - iS(z) = \frac{1}{\sqrt{2i}}\Phi(z\sqrt{i}) = \frac{2}{\sqrt{2\pi}}\int_0^z e^{-it^2}\,dt.$

3. $[\cos u^2 C(u) + \sin u^2 S(u)] = \frac{1}{2}[\cos u^2 + \sin u^2] + \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2ut}\sin t^2\,dt \qquad [\operatorname{Re} u \geq 0].$

NT 28(6)a

4. $[\cos u^2 S(u) - \sin u^2 C(u)] = \frac{1}{2}[\cos u^2 - \sin u^2] - \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2ut}\cos t^2\,dt \qquad [\operatorname{Re} u \geq 0].$

NT 28(5)a

5. $\left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\exp(-x^2 \operatorname{tg}\varphi)\,\sin\frac{\varphi}{2}\sqrt{\cos\varphi}}{\sin 2\varphi}\,d\varphi. \qquad \text{see also } 6.322.$

6.322
NT 33(18)a

For a connection with a degenerate hypergeometric function, see 9.236.

For a connection with a parabolic-cylinder function, see 9.254.

8.257

NT 38(11)

2. $\lim\limits_{x\to+\infty}\left(x^{\varrho}\left[C(x)-\frac{1}{2}\right]\right)=0 \qquad [\varrho<1].$

NT 38(11)

3. $\lim\limits_{x\to+\infty} S(x)=\frac{1}{2}.$

NT 38(12)a

941

4. $\lim\limits_{x\to+\infty} C(x)=\frac{1}{2}.$

NT 38(12)a

For integrals of the probability integral, see 6.28-6.31.

For integrals of Fresnel's sine-integral and cosine-integral, see 6.32.

8.258*

Integrals involving the complementary error function

1. $$\int_0^\infty \operatorname{erfc}^2(x)e^{-\beta x^2}\,dx=\frac{1}{\sqrt{\beta\pi}}\left(-\arccos\left(\frac{1}{1+\beta}\right)+2\ \operatorname{arctg}\left(\sqrt{\beta}\right)\right) \qquad [\beta>0].$$

2. $$\int_0^\infty x\ \operatorname{erfc}^2(x)e^{-\beta x^2}\,dx=\frac{1}{2\beta}\left(1-\frac{4}{\pi}\frac{\operatorname{arctg}(\sqrt{1+\beta})}{\sqrt{1+\beta}}\right) \qquad [\beta>0].$$

3. $$\int_0^\infty x^3\ \operatorname{erfc}^2(x)e^{-\beta x^2}\,dx=\frac{1}{2\beta^2}\left(1-\frac{4}{\pi}\frac{\operatorname{arctg}(\sqrt{1+\beta})}{\sqrt{1+\beta}}\right)+\frac{1}{\beta\pi}\left(\frac{1}{(1+\beta)(\beta^2+2\beta+2)}-\frac{\operatorname{arctg}(\sqrt{1+\beta})}{(1+\beta)^{\frac{3}{2}}}\right) \qquad [\beta>0].$$

8.26 Lobachevskiy's function L(x)

8.260

Definition:

$$L(x) = -\int_0^x \ln\cos\, t\, dt.$$

LO III 184(10)

For integral representations of the function $L(x)$, see also 3.531 8., 3.532 2., 3.533, and 4.224.

8.261

Representation in the form of a series:

$$L(x) = x\, \ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin 2kx}{k^2}.$$

LO III 185(11)

8.262

Functional relationships:

1. $L(-x) = -L(x) \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right].$

LO III 185(13)

2. $L(\pi - x) = \pi\, \ln 2 - L(x).$

LO III 286

3. $L(\pi + x) = \pi\, \ln 2 + L(x).$

LO III 286

LO III 186(14)

8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

8.31 The gamma function (Euler's integral of the second kind): Γ(z)

8.310[7]

Definition:

$$1.\quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad [\operatorname{Re} z > 0]. \qquad (\text{Euler}).$$

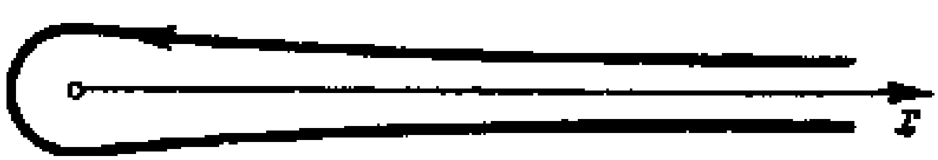
FI II 777(6)

Generalization:

$$2.\quad \Gamma(z) = -\frac{1}{2i\, \sin \pi z} \int_C (-t)^{z-1} e^{-t}\, dt$$

for z not an integer.

The contour C is shown in the drawing.



$\Gamma(z)$ is an analytic function z with simple poles at the points $z = -l$ (for $l = 0$, 1, 2,. . .) which correspond to residues $\dfrac{(-1)^l}{l!}$.

WH, MO 1

8.311

$$\Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_{\infty}^{(0+)} e^{-t} t^{z-1}\, dt.$$

MO 2
WH

8.312

1. $$\Gamma(z) = \int_0^1 \left(\ln \frac{1}{t}\right)^{z-1} dt \qquad [\operatorname{Re} z > 0].$$

FI II 778

2. $$\Gamma(z) = x^z \int_0^{\infty} e^{-xt} t^{z-1}\, dt \qquad [\operatorname{Re} z > 0, \quad \operatorname{Re} x > 0].$$

FI II 779(8)

3. $$\Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^{\infty} e^{-at^2} (1+t^2)^{z-\frac{1}{2}} \cos[2at+(2z-1) \operatorname{arctg} t]\, dt \qquad [a > 0].$$

WH

4. $$\Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^{\infty} e^{-t^2} t^{z-1} (1+t^2)^{\frac{z}{2}} \{3 \sin[t+z \operatorname{arcctg}(-t)]+\sin[t+(z-2) \operatorname{arcctg}(-t)]\}\, dt$$
$$[\operatorname{arcctg} \text{ denotes an obtuse angle}].$$

WH

5. $$\Gamma(y) = x^y e^{-i\beta y} \int_0^{\infty} t^{y-1} \exp(-xte^{-i\beta})\, dt \qquad \left[x,\, y,\, \beta \text{ real}, \quad x > 0, \quad y > 0, \quad |\beta| < \frac{\pi}{2}\right].$$

MO 8

6. $$\Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^{\infty} e^{bti} (it)^{z-1}\, dt \qquad [b > 0, \quad 0 < \operatorname{Re} z < 1].$$

NH 154(3)

7. $$\Gamma(z) = \frac{(\sqrt{a^2+b^2})^z}{\cos\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^\infty e^{-at} \cos(bt) t^{z-1}\, dt; \quad = \frac{(\sqrt{a^2+b^2})^z}{\sin\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^\infty e^{-at} \sin(bt) t^{z-1}\, dt \qquad [a > 0, \quad b \geq 0, \quad \operatorname{Re} z > 0].$$

NH 152(1)a
NH 152(2)

8. $$\Gamma(z) = \frac{b^z}{\cos\frac{\pi z}{2}} \int_0^\infty \cos(bt) t^{z-1}\, dt; \quad = \frac{b^z}{\sin\frac{\pi z}{2}} \int_0^\infty \sin(bt) t^{z-1}\, dt \qquad [b > 0, \quad 0 < \operatorname{Re} z < 1].$$

NH 152(5)

9. $$\Gamma(z) = \int_0^\infty e^{-t}(t-z) t^{z-1} \ln t\, dt;$$

10. $$\Gamma(z) = \int_{-\infty}^\infty \exp(zt - e^t)\, dt \qquad [\operatorname{Re} z > 0].$$

NH 145(14)
NH 173(7)

11. $$\Gamma(z) \cos \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \cos(\lambda t \sin\alpha)\, dt;$$

12. $$\Gamma(x) \sin \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \sin(\lambda t \sin\alpha)\, dt \qquad \left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right].$$

WH

13. $$\Gamma(-z) = \int_0^\infty \left[\frac{e^{-t} - \sum_{k=0}^{n} (-1)^k \frac{t^k}{k!}}{t^{z+1}}\right] dt \qquad [n = [\operatorname{Re} z]].$$

MO 2

8.313

JA, MO 7a

8.315

1.[7] $$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t}\, dt \qquad (\text{For } C \text{ see } 8.310.2)$$

8.310

944

2.[8] $$\int_{-\infty}^{\infty} \frac{e^{bti}}{(a+it)^z}\, dt = \frac{2\pi e^{-ab} b^{z-1}}{\Gamma(z)}$$
$$\int_{-\infty}^{\infty} \frac{e^{-bti}}{(a+it)^z}\, dt = 0 \qquad \left[\operatorname{Re} a > 0, b > 0, \operatorname{Re} z > 0, |\arg(a+it)| < \frac{1}{2}\pi\right].$$

NH 155(8), MO 7
NH 155(8), MO 7

3. $$\frac{1}{\Gamma(z)} = a^{1-z} \frac{e^a}{\pi} \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg}\theta - z\theta) \cos^{z-2}\theta\, d\theta \qquad [\operatorname{Re} z > 1].$$

NH 157(14)

See also 3.324 2., 3.326, 3.328, 3.381 4., 3.382 2., 3.389 2., 3.433, 3.434, 3.478 1., 3.551 1., 2., 3.827 1., 4.267 7., 4.272, 4.353 1., 4.369 1., 6.214, 6.223, 6.246, 6.281.

8.32 Representation of the gamma function as series and products

8.321

Representation in the form of a series:

1.[6] $$\Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$
$$\left[c_0 = 1,\ c_{n+1} = \frac{\displaystyle\sum_{k=0}^{n} (-1)^{k+1} s_{k+1} c_{n-k}}{n+1};\ s_1 = \boldsymbol{C},\ s_n = \zeta(n) \quad \text{for} \quad n \ge 2,\ |z| < 1\right].$$

2. $$\frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$
$$\left[d_0 = 1,\ d_{n+1} = \frac{\sum_{k=0}^{n} (-1)^k s_{k+1} d_{n-k}}{n+1};\ s_1 = \boldsymbol{C},\ s_n = \zeta(n) \quad \text{for} \quad n \geq 2\right].$$

NH 41(4, 6)

Infinite-product representation

8.322

$$\Gamma(z) = e^{-\boldsymbol{C}z} \frac{1}{z} \prod_{k=1}^{\infty} \frac{e^{\frac{z}{k}}}{1 + \frac{z}{k}} \qquad [\operatorname{Re} z > 0];$$
$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}} \qquad [\operatorname{Re} z > 0];$$
$$= \lim_{n \to \infty} \frac{n^z}{z} \prod_{k=1}^{n} \frac{k}{z+k} \qquad [\operatorname{Re} z > 0].$$

SM 267(130)
WH
SM 269

8.323[7]

$$\Gamma(z) = 2z^z e^{-z} \prod_{k=1}^{\infty} \sqrt[2^k]{\mathrm{B}\left(2^{k-1} z,\ \frac{1}{2}\right)}.$$

NH 98(12)

8.324[7]

$$\Gamma(1+z) = 4^z \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right)}{\sqrt{\pi}}.$$

MO 3

8.325

1. $$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right)\left(1 - \frac{\gamma}{\beta+k}\right)\right].$$

NH 62(2)

2. $$\frac{e^{\boldsymbol{C}x}\Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty} \left[\left(1 - \frac{x}{z+k}\right) e^{\frac{x}{k}}\right] \qquad [z \neq 0,\ -1,\ -2, \ldots;\quad \operatorname{Re} z > 0, \quad \operatorname{Re}(z-x) > 0].$$

3.[7] $$\frac{\sqrt{\pi}}{\Gamma\left(1+\frac{z}{2}\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)} = \prod_{k=1}^{\infty}\left(1-\frac{z}{2k-1}\right)\left(1+\frac{z}{2k}\right).$$

MO 2

8.326

1. $$\frac{\frac{[\Gamma(x)]^2}{\Gamma(2x)}}{\mathrm{B}(x+iy,\, x-iy)} = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2 = \prod_{k=0}^{\infty}\left(1+\frac{y^2}{(x+k)^2}\right) \qquad [x,\, y \text{ real}, \quad x \neq 0, \quad -1,\, -2, \ldots].$$

LO V, NH 63(4)

2. $$\frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{xe^{-i\boldsymbol{C}y}}{x+iy}\prod_{n=1}^{\infty}\frac{\exp\left(\frac{iy}{n}\right)}{1+\frac{iy}{x+n}} \qquad [x,\, r \text{ real}, \quad x \neq 0, \quad -1,\, -2, \ldots].$$

MO 2

8.327

Asymptotic representation for large values of $|z|$:

$$\Gamma(z) = z^{z-\frac{1}{2}}e^{-z}\sqrt{2\pi}\left\{1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+O(z^{-5})\right\} \qquad [|\arg z| < \pi].$$

WH

For z real and positive, the remainder of the series is less than the last term that is retained.

8.328

1. $$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi} \qquad [x \text{ and } y \text{ are real}].$$

MO 6

2. $$\lim_{|z|\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\ \ln z} = 1.$$

MO 6

8.33 Functional relations involving the gamma function

8.331

$$\Gamma(x+1) = x\Gamma(x).$$

8.332

$$\left.\begin{aligned} &1.\quad |\Gamma(iy)|^2 = \frac{\pi}{y\,\mathrm{sh}\,\pi y} \\ &2.\quad \left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\mathrm{ch}\,\pi y} \end{aligned}\right\} \qquad [y \text{ is real}].$$

$$3.\quad \Gamma(1+ix)\Gamma(1-ix) = \frac{\pi x}{\mathrm{sh}\,x\pi} \qquad [x \text{ is real}].$$

$$4.\quad \Gamma(1+x+iy)\Gamma(1-x+iy)\Gamma(1+x-iy)\Gamma(1-x-iy) = \frac{2\pi^2(x^2+y^2)}{\mathrm{ch}\,2y\pi - \cos 2x\pi} \qquad [x \text{ and } y \text{ are real}].$$

LO V
LO V
MO 3

8.333

$$[\Gamma(n+1)]^n = G(n+1)\prod_{k=1}^{n} k^k,$$

where n is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left[-\frac{z(z+1)}{2} - \frac{\boldsymbol{C}}{2}z^2\right] \prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n \exp\left(-z+\frac{z^2}{2n}\right)\right\}.$$

WH

8.334

$$1.\quad \prod_{k=1}^{n} \frac{1}{\Gamma\left(-z\,\exp\dfrac{2\pi k i}{n}\right)} = -z^n \prod_{k=1}^{\infty}\left[1-\left(\frac{z}{k}\right)^n\right] \qquad [n = 2,\, 3,\, 4\ldots].$$

MO 2

FI II 430

Special cases

8.335[7]

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \qquad [\text{product theorem}]\,.$$

FI II 782a, WH

1. $$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right) \qquad [\text{doubling formula}]\,.$$

2. $$\Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x) \Gamma\left(x + \frac{1}{3}\right) \Gamma\left(x + \frac{2}{3}\right).$$

3. $$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \Gamma\left(1 - \frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}.$$

WH

947

4*. $$\sum_{n=0}^{\infty} \frac{\Gamma\left(n - \frac{1}{2}\right)^2}{4(n!)^2 \Gamma\left(-\frac{1}{2}\right)^2} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{1024} + \frac{25}{65536} + \ldots = \frac{1}{\pi}.$$

8.336

$$\Gamma\left(-\frac{yz + xi}{2y}\right) \Gamma(1 - z) = (2i)^{z+1} y \Gamma\left(1 + \frac{yz - xi}{2y}\right) \int_0^\infty e^{-tx} \sin^z(ty)\, dt$$
$$[\operatorname{Re}(yi) > 0, \quad \operatorname{Re}(x - yzi) > 0].$$

NH 133(10)

For a connection with the psi function, 8.334 1.

For a connection with the beta function, see 8.384 1.

For integrals of the gamma function, see 8.412 4., 8.414, 9.223, 9.242 3., 9.242 4.

8.337

1. $[\Gamma'(x)]^2 < \Gamma(x)\Gamma''(x) \qquad [x > 0].$

2. For $x > 0$, $\min \Gamma(1+x) = 0.88560\ldots$ is attained when $x = 0.46163$. JA

Particular values

8.338

1. $\Gamma(1) = \Gamma(2) = 1.$

MO 1

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$

4. $$\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2[(4k+1)^2-1]}{[(4k-1)^2-1](4k+1)^2}.$$

MO 1a

5. $$\prod_{k=1}^{8} \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3.$$

WH

8.339

For n a natural number

1. $\Gamma(n) = (n-1)!$

2. $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!$

3. $\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}.$

4. $\frac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)} = \frac{(4p^2-1^2)(4p^2-3^2)\dots[4p^2-(2n-1)^2]}{2^{2n}}.$

WA 221

948

8.34 The logarithm of the gamma function

8.341

Integral representation:

1. $\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}\,dt \qquad [\operatorname{Re} z > 0].$

WH

2.[7] $\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + 2\int_0^\infty \frac{\operatorname{arctg}\frac{t}{z}}{e^{2\pi t}-1}\,dt$
$\left[\operatorname{Re} z > 0 \text{ and } \operatorname{arctg} w = \int_0^w \frac{du}{1+u^2} \text{ is taken over a rectangular path in the } w\text{-plane}\right]$

WH

3. $\ln\Gamma(z) = \int_0^\infty \left\{\frac{e^{-zt}-e^{-t}}{1-e^{-t}} + (z-1)e^{-t}\right\}\frac{dt}{t} \qquad [\operatorname{Re} z > 0].$

4. $\ln\Gamma(z) = \int_0^\infty \left\{(z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)}\right\} \frac{dt}{t} \qquad [\mathrm{Re}\, z > 0].$

WH

5. $\ln\Gamma(x) = \frac{\ln\pi - \ln\sin\pi x}{2} + \frac{1}{2}\int_0^\infty \left\{\frac{\mathrm{sh}\left(\frac{1}{2} - x\right) t}{\mathrm{sh}\, \frac{t}{2}} - (1-2x)e^{-t}\right\} \frac{dt}{t} \qquad [0 < x < 1].$

WH

6. $\ln\Gamma(z) = \int_0^1 \left\{\frac{t^z - t}{t-1} - t(z-1)\right\} \frac{dt}{t\, \ln t} \qquad [\mathrm{Re}\, z > 0].$

WH

7. $\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}}\right] \frac{dt}{t} \qquad [\mathrm{Re}\, z > 0].$

NH 187(7)

See also 3.427 9., 3.554 5.

8.342

Series representations:

1. $$\ln\Gamma(z+1) = \frac{1}{2}\left[\ln\left(\frac{\pi 2}{\sin\pi z}\right) - \ln\frac{1+z}{1-z}\right] + (1-\boldsymbol{C})z + \\ + \sum_{k=1}^\infty \frac{1 - \zeta(2k+1)}{2k+1} z^{2k+1} = \\ = -\boldsymbol{C}z + \sum_{k=2}^\infty (-1)^k \frac{z^k}{k}\zeta(k) \qquad [|z| < 1].$$

NH 38(16, 12)

2. $\ln\Gamma(1+x) = \frac{1}{2}\ln\frac{\pi x}{\sin\pi x} - \boldsymbol{C}x - \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1}\{\zeta(2n+1)\} \qquad [|x| < 1].$

1. $\ln\Gamma(x) = \ln\sqrt{2\pi} + \sum_{n=1}^{\infty}\left\{\frac{1}{2n}\cos 2n\pi x + \frac{1}{n\pi}(\boldsymbol{C} + \ln 2n\pi)\sin 2n\pi x\right\} \qquad [0 < x < 1].$

FI III 558

2. $\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}} \qquad [|\arg z| < \pi].$

MO 9

8.344[7]

Asymptotic expansion for large values of $|z|$:

$$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \sum_{k=1}^{n-1}\frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1}\cos^{2n-1}\left(\frac{1}{2}\arg z\right)}.$$

MO5

For integrals of $\ln\Gamma(x)$, see 6.44.

8.35 The incomplete gamma functions

8.350

Definition:

1. $\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}\,dt \qquad [\operatorname{Re}\alpha > 0].$

EH II 133(1), NH 1(1)

2.[7] $\Gamma(\alpha, x) = \int_x^{\infty} e^{-t}t^{\alpha-1}\,dt.$

EH II 133(2), NH 2(2), LE 339

8.351

1. $\gamma^*(\alpha, x) = \frac{x^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha, x)$ is an analytic function with respect to α and x.

EH II 133(5)

2. Another definition of $\gamma(\alpha, x)$, that is also suitable for the case Re $\alpha \le 0$:

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha} e^{-x} \Phi(1, \quad 1+\alpha; \quad x) = \frac{x^\alpha}{\alpha} \Phi(a, 1+a; -x).$$

EH II 133(3)

3. For fixed x, $\Gamma(\alpha, x)$ is an entire function of α. For nonintegral α, $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$.

4. A second definition of $\Gamma(\alpha, x)$:

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \Psi(1, 1+\alpha; x) = e^{-x} \Psi(1-\alpha, 1-\alpha; x).$$

EH II 133(4)

8.352

Special cases:

1. $$\gamma(1+n, x) = n! \left[1 - e^{-x} \left(\sum_{m=0}^{n} \frac{x^m}{m!}\right)\right] \qquad [n = 0, 1, \ldots].$$

EH II 136(17, 16), NH 6(11)

2. $$\Gamma(1+n, x) = n! e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!} \qquad [n = 0, 1, \ldots].$$

EH II 136(16, 18)

3.[7] $$\Gamma(-n, x) = \frac{(-1)^n}{n!} \left[-\operatorname{Ei}(-x) - e^{-x} \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}}\right] \qquad [n = 1, 2, \ldots].$$

Integral representations:

1. $\gamma(\alpha, x) = x^{\alpha} \operatorname{cosec} \pi\alpha \int_0^{\pi} e^{x} \cos\theta \cos(\alpha\theta + x \sin\theta)\, d\theta \qquad [x \neq 0, \quad \operatorname{Re}\alpha > 0, \quad \alpha \neq 1, 2, \dots].$

EH II 137(2)

2. $\gamma(\alpha, x) = x^{\frac{1}{2}\alpha} \int_0^{\infty} e^{-t} t^{\frac{1}{2}\alpha - 1} J_{\alpha}(2\sqrt{xt})\, dt \qquad [\operatorname{Re}\alpha > 0].$

EH II 138(4)

3. $\Gamma(\alpha, x) = \frac{\rho^{-x} x^{\alpha}}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{e^{-t} t^{-\alpha}}{x + t}\, dt \qquad [\operatorname{Re}\alpha < 1, \quad x > 0].$

EH II 137(3), NH 19(12)

4. $\Gamma(\alpha, x) = \frac{2x^{\frac{1}{2}\alpha} e^{-x}}{\Gamma(1-\alpha)} \int_0^{\infty} e^{-t} t^{-\frac{1}{2}\alpha} K_{\alpha}[2\sqrt{xt}]\, dt \qquad [\operatorname{Re}\alpha < 1].$

EH II 138(5)

5. $\Gamma(\alpha, xy) = y^{\alpha} e^{-xy} \int_0^{\infty} e^{-ty} (t+x)^{\alpha-1}\, dt \qquad [\operatorname{Re} y > 0, \quad x > 0, \quad \operatorname{Re}\alpha > 1].$

(See also 3.936 5.– 3.944 1.–4.)

3.944

3.936
NH 19(10)

For integrals of the gamma function, see 6.45.

8.354

Series representations:

1. $\gamma(\alpha, x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}.$

2. $\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \qquad [\alpha \neq 0, -1, -2, \dots].$

EH II 135(5), LE 340(2)

3. $$\Gamma(\alpha, x) - \Gamma(\alpha, x+y)\gamma(\alpha, x+y) - \gamma(\alpha, x) = \\ = e^{-x} x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - e^{-y} e_k(y)] \Gamma(1-\alpha+k)}{x^k \Gamma(1-\alpha)}, \\ e_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!} \qquad [|y| < |x|].$$

EH II 139(2)

4. $\gamma(\alpha, x) = \Gamma(\alpha) e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} x^{\frac{1}{2}n} I_{n+\alpha}(2\sqrt{x}) \sum_{m=0}^{n} \frac{(-1)^m}{m!} \qquad [x \neq 0, \quad \alpha \neq 0, \quad -1, -2, \dots].$

EH II 139(3)

5. $\Gamma(\alpha, x) = e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \qquad [x > 0].$

EH II 140(5)

8.355

$$\Gamma(\alpha, x)\gamma(\alpha, y) = e^{-x-y} (xy)^{\alpha} \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha)}{(n+1)\Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) \\ [y > 0, x \geq y, \alpha \neq 0, -1, \dots].$$

EH II 139(4)

8.356

Functional relations:

1. $\nu(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^{\alpha} e^{-x}.$

EH II 134(2)

EH II 134(3)

3. $\Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha).$

EH II 134(1)

4. $\frac{d\gamma(\alpha, x)}{dx} = -\frac{d\Gamma(\alpha, x)}{dx} = x^{\alpha-1}e^{-x}.$

EH II 135(8)

5. $\frac{\Gamma(\alpha+n, x)}{\Gamma(\alpha+n)} = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} + e^{-x}\sum_{s=0}^{n-1}\frac{x^{\alpha+s}}{\Gamma(\alpha+s+1)}.$

NH 4(3)

6. $\Gamma(\alpha)\Gamma(\alpha+n, x) - \Gamma(\alpha+n)\Gamma(\alpha, x) = \Gamma(\alpha+n)\gamma(\alpha, x) - \Gamma(\alpha)\Gamma(\alpha+n, x).$

NH 5

8.357

Asymptotic representation for large values of $|x|$:

$$\Gamma(\alpha, x) = x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(-1)^m\Gamma(1-\alpha+m)}{x^m\Gamma(1-\alpha)} + O(|x|^{-M})\right]$$
$$\left[|x|\to\infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, M = 1, 2, \ldots\right].$$

EH II 135(6), NH 37(7), LE 340(3)

8.358

Representation as a continued fraction:

$$\Gamma(\alpha, x) = \cfrac{e^{-x}x^{\alpha}}{x + \cfrac{1-\alpha}{1 + \cfrac{1}{x + \cfrac{2-\alpha}{1 + \cfrac{2}{x + \cfrac{3-\alpha}{1+\ldots}}}}}}$$

EH II 136(13), NH 42(9)

8.359

Relationships with other functions:

1. $\Gamma(0,\, x) = -\operatorname{Ei}(-x).$

EH II 143(1)

2. $\Gamma\left(0,\, \ln\frac{1}{x}\right) = -\operatorname{li}(x).$

EH II 143(2)

3. $\Gamma\left(\frac{1}{2},\, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\Phi(x).$

EH II 147(2)

4. $\gamma\left(\frac{1}{2},\, x^2\right) = \sqrt{\pi}\Phi(x).$

EH II 147(1)

8.36 The psi function ψ(x)

8.360

Definition:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x).$$

8.361

Integral representations:

1.[8] $\psi(z) = \frac{d\,\ln\Gamma(z)}{dz} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt \qquad [\operatorname{Re} z > 0].$

NH 183(1), WH

NH 184(7), WH

3. $\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty \frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)} \qquad [\operatorname{Re} z > 0].$

WH

4. $\psi(z) = \int_0^1 \left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t}\right) dt \qquad [\operatorname{Re} z > 0].$

WH

$$\left.\begin{aligned} &5.\quad \psi(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1-e^{-t}}\,dt - \boldsymbol{C}, \\ &6.\quad \psi(z) = \int_0^\infty \{(1+t)^{-1} - (1+t)^{-z}\}\frac{dt}{t} - \boldsymbol{C}, \\ &7.\quad \psi(z) = \int_0^1 \frac{t^{z-1}-1}{t-1}\,dt - \boldsymbol{C} \end{aligned}\right\} \qquad [\operatorname{Re} z > 0].$$

WH

FI II 796, WH

8. $\psi(z) = \ln z + \int_0^\infty e^{-tz}\left[\frac{1}{t} - \frac{1}{1-e^{-t}}\right] dt \qquad [\operatorname{Re} z > 0].$

MO 4

See also 3.244 3., 3.311 6., 3.317 1., 3.457, 3.458 2., 3.471 14., 4.253 1. and 6., 4.275 2., 4.281 4., 4.282 5.

For integrals of the psi function, see 6.46-6.47.

Series representation

8.362

1. $\psi(x) = -\boldsymbol{C} - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right);$

$$= -\boldsymbol{C} - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(x+k)}.$$

2. $\psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln\left(1 + \frac{1}{x+k}\right) \right].$

MO 4

953

3. $\psi(x) = -\boldsymbol{C} + \frac{\pi^2}{6}(x-1) - (x-1)\sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}.$

NH 54(12)

8.363

1. $\psi(x+1) = -\boldsymbol{C} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1}.$

NH 37(5)

2. $\psi(x+1) = \frac{1}{2x} - \frac{\pi}{2} \operatorname{ctg} \pi x - \frac{x^2}{1-x^2} - \boldsymbol{C} + \sum_{k=1}^{\infty} [1 - \zeta(2k+1)] x^{2k}.$

NH 38(10)

3. $\psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left(\frac{1}{y+k} - \frac{1}{x+k} \right)$

(see also 3.219, 3.231 5., 3.311 7., 3.688 20., 4.253 1., 4.295 37.) .

4.295

4.253

3.688

3.311

3.231

3.219
NH 99(3)

4. $\psi(x+iy) - \psi(x-iy) = \sum_{k=0}^{\infty} \frac{2yi}{y^2+(x+k)^2}.$

5. $\psi\left(\frac{p}{q}\right) = -\boldsymbol{C} + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{q}{p+kq}\right)$ (see also 3.244 3.).

3.244
NH 29(1)

6. $\psi\left(\frac{p}{q}\right) = -\boldsymbol{C} - \ln(2q) - \frac{\pi}{2}\operatorname{ctg}\frac{p\pi}{q} + 2\sum_{k=1}^{\left[\frac{q+1}{2}\right]-1}\left[\cos\frac{2kp\pi}{q}\ln\sin\frac{k\pi}{q}\right]$
$[q = 2, 3, \dots, p, = 1, 2, \dots, q-1].$

MO 4, EH I 19(29)

7. $\psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+kq)^n - 1}.$

NH 59(3)

8.[7] $\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, x).$

NH 37(1)

Infinite-product representation

8.364

1. $e^{\psi(x)} = x\prod_{k=0}^{\infty}\left(1+\frac{1}{x+k}\right)e^{-\frac{1}{x+k}}.$

NH 65(12)

NH 65(11)

See also 8.37.

For a connection with Riemann's zeta function, see 9.533 2.

For a connection with the gamma function, see 4.325 12. and 4.352 1.

For a connection with the beta function, see 4.253 1.

For series of psi functions, see 8.403 2., 8.446, and 8.447 3. (Bessel functions) 8.761 (derivatives of associated Legendre functions with respect to the degree), 9.153, 9.154 (hypergeometric function), 9.237 (confluent hypergeometric function).

For integrals containing psi functions, see 6.46-6.47.

954

8.365

Functional relations:

1. $$\psi(x+1) = \psi(x) + \frac{1}{x}.$$

JA

2. $$\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\beta(x) \qquad \text{(cf. 8.37 0)}.$$

8.37

3. $$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}.$$

GA 154(64)a

4. $$\psi(n+1) = -\boldsymbol{C} + \sum_{k=1}^{n} \frac{1}{k}.$$

5. $\lim_{n\to\infty}[\psi(z+n) - \ln n] = 0.$

MO 3

6. $\psi(nz) = \frac{1}{n}\sum_{k=0}^{n-1}\psi\left(z+\frac{k}{n}\right) + \ln n \qquad [n = 2,\ 3,\ 4, \dots].$

MO 3

7. $\psi(x-n) = \psi(x) - \sum_{k=1}^{n}\frac{1}{x-k}.$

8. $\psi(1-z) = \psi(z) + \pi\ \operatorname{ctg}\pi z.$

GA 155(68)a

9. $\psi\left(\frac{1}{2}+z\right) = \psi\left(\frac{1}{2}-z\right) + \pi\operatorname{tg}\pi z.$

JA

10. $\psi\left(\frac{3}{4}-n\right) = \psi\left(\frac{1}{4}+n\right) + \pi \qquad [n = 0, \quad \pm 1, \quad \pm 2, \dots].$

8.366

Particular values

1. $\psi(1) = -\boldsymbol{C} \qquad (\text{cf. } 8.367\ 1.).$

8.367

2. $\psi\left(\frac{1}{2}\right) = -\boldsymbol{C} - 2\ \ln 2 = -1.963\,510\,026\dots$

3. $\psi\left(\frac{1}{2} \pm n\right) = -\boldsymbol{C} + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right].$

JA

4. $\psi\left(\frac{1}{4}\right) = -\boldsymbol{C} - \frac{\pi}{2} - 3\ \ln 2.$

GA 157a

5. $\psi\left(\frac{3}{4}\right) = -\boldsymbol{C} + \frac{\pi}{2} - 3\ \ln 2.$

GA 157a

6. $\psi\left(\frac{1}{3}\right) = -\boldsymbol{C} - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ \ln 3.$

GA 157a

7. $\psi\left(\frac{2}{3}\right) = -\boldsymbol{C} + \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ \ln 3.$

GA 157a

955

8. $\psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\ldots.$

JA

9. $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\ldots.$

JA

10. $\psi'(-n) = \infty$

11. $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$

12. $\psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$

13. $\psi'\left(\frac{1}{2} - n\right) = \frac{\pi^2}{2} + 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$

[n – a natural number].

1. $\boldsymbol{C} = -\psi(1) = 0.577\,215\,664\,90\ldots$.

FI II 319, 795

2. $\boldsymbol{C} = \lim_{n\to\infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n\right]$.

FI II 801a

3. $\boldsymbol{C} = \lim_{x\to 1+0} \left[\zeta(x) - \frac{1}{x-1}\right]$.

FI II 804

Integral representations:

4. $\boldsymbol{C} = -\int_0^\infty e^{-t}\, \ln t\, dt.$

FI II 807

5. $\boldsymbol{C} = -\int_0^1 \ln\left(\ln\frac{1}{t}\right)\, dt.$

FI II 807

6. $\boldsymbol{C} = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t}\right]\, dt.$

DW

7. $\boldsymbol{C} = -\int_0^\infty \left[\cos t - \frac{1}{1+t}\right] \frac{dt}{t}.$

MO 10

8. $\boldsymbol{C} = 1 - \int_0^\infty \left[\frac{\sin t}{t} - \frac{1}{1+t}\right] \frac{dt}{t}.$

9. $\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t}\right] \frac{dt}{t}.$

FI II 795, 802

10. $\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t^2}\right] \frac{dt}{t}.$

DW, MO 10

11. $\boldsymbol{C} = \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{te^t}\right] dt.$

DW

956

12. $\boldsymbol{C} = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t}\, dt.$

FI II 802

See also 8.361 5.- 8.361 7., 3.311 6., 3.435 3. and 4., 3.476 2., 3.481 1. and 2., 3.951 10., 4.283 9., 4.331 1., 4.421 1., 4.424 1., 4.553 , 4.572, 6.234, 6.264 1., 6.468.

13. Asymptotic expansions

$$\boldsymbol{C} = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$
$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)} \frac{\theta}{n^{2r+2}} \qquad [0 < \theta < 1].$$

FI II 827

8.37 The function β(x)

Definition:

8.370

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right].$$

8.371

Integral representations:

$$1.^{3}\ \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt \qquad [\operatorname{Re} x > 0].$$

WH

$$2.\quad \beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt \qquad [\operatorname{Re} x > 0].$$

MO 4

$$3.\quad \beta\left(\frac{x+1}{2}\right) = \int_0^\infty \frac{e^{-xt}}{\operatorname{ch} t}\,dt \qquad [\operatorname{Re} x > -1] \qquad\qquad (\text{cf } 8.371\ 1.).$$

8.371

See also 3.241 1., 3.251 7., 3.522 2. and 4., 3.623 2. and 3., 4.282 2., 4.389 3., 4.532 1. and 3.

Series representation

8.372

$$1.^{7}\quad \beta(x) = \sum_{k=0}^\infty \frac{(-1)^k}{x+k} \qquad [-x \notin \mathrm{N}].$$

NH 37, 101(1)

$$2.^{7}\quad \beta(x) = \sum_{k=0}^\infty \frac{1}{(x+2k)(x+2k+1)} \qquad [-x \notin \mathrm{N}].$$

NH 101(2)

$$3.^{7}\quad \beta(x) = \frac{1}{2}\sum_{k=0}^\infty \frac{k!}{x(x+1)\dots(x+k)}\,\frac{1}{2^k} \qquad [-x \notin \mathrm{N}].$$

NH 246(7)

8.373

$$1.^{6}\ \beta(x+1) = \ln 2 + \sum_{k=1}^{\infty}(-1)^k(1-2^{-k})\zeta(k+1)x^k \qquad [|x|<1].$$

NH 37(5)

$$2.^{6}\ \beta(x+1) = \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2\sin\pi x} + \frac{1}{1-x^2} - $$
$$- \sum_{k=1}^{\infty}[1-(1-2^{-2k})\zeta(2k+1)]x^{2k}\ [0<|x|<2;\ x\neq\pm 1].$$

NH 38(11)

8.374[7]

$$\beta^{(n)}(x) = (-1)^n n! \sum_{k=0}^{\infty}\frac{(-1)^k}{(x+k)^{n+1}} \qquad [-x \notin \mathrm{N}].$$

NH 37(2)

8.375

Representation in the form of a finite sum:

$$1^{6}.\quad \beta\left(\frac{p}{q}\right) = \frac{\pi}{2\sin\frac{p\pi}{q}} - \sum_{k=0}^{\left[\frac{q-1}{2}\right]} \cos\frac{p(2k+1)\pi}{q}\ \ln\sin\frac{(2k+1)\pi}{2q}$$
$$[q=2,\,3,\dots,p=1,\,2,\,3,\dots,q-1] \qquad (\text{see also } 8.362\ 5.–7.).$$

8.362
NH 23(9)

$$2.\quad \beta(n) = (-1)^{n+1}\ln 2 + \sum_{k=1}^{n-1}\frac{(-1)^{k+n+1}}{k}.$$

Functional relations

8.376

$$\sum_{k=0}^{2n}(-1)^k\beta\left(\frac{x+k}{2n+1}\right)=(2n+1)\beta(x).$$

NH 19

8.377

$$\sum_{k=1}^{n}\beta(2^k x)=\psi(2^n x)-\psi(x)-n\ln 2.$$

NH 20(10)

8.38 The beta function (Euler's integral of the first kind): B(x,y)

Integral representation

8.380

1. $$\mathrm{B}\,(x,\,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt^*);$$
$$=2\int_0^1 t^{2x-1}(1-t^2)^{y-1}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0].$$

* This equation is used as the definition of the function B

$(x,\,y)$.

FI II 774(1)

2. $$\mathrm{B}\,(x,\,y)=2\int_0^{\frac{\pi}{2}}\sin^{2x-1}\,\varphi\,\cos^{2y-1}\,\varphi\,d\varphi \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0].$$

KU 10

3. $$\mathrm{B}\,(x,\,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt=2\int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0].$$

$$4.\quad \mathrm{B}\,(x,\,y) = 2^{2-y-x}\int_{-1}^{1}\frac{(1+t)^{2x-1}(1-t)^{2y-1}}{(1+t^2)^{x+y}}\,dt \qquad [\operatorname{Re} x>0,\quad \operatorname{Re} y>0].$$

MO 7

$$\left.\begin{aligned}
&5.\quad \mathrm{B}\,(x,\,y) = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt = \int_1^\infty \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt\\
&6.\quad \mathrm{B}\,(x,\,y) = \frac{1}{2^{x+y-1}}\int_0^1 [(1+t)^{x-1}(1-t)^{y-1}+(1+t)^{y-1}(1-t)^{x-1}]\,dt
\end{aligned}\right\}$$

$$[\operatorname{Re} x>0,\quad \operatorname{Re} y>0].$$

BI ((1))(15)

$$\left.\begin{aligned}
&7.\quad \mathrm{B}\,(x,\,y) = z^y(1+z)^x\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\\
&8.\quad \mathrm{B}\,(x,\,y) = z^y(1+z)^x\int_0^{\frac{\pi}{2}} \frac{\cos^{2x-1}\ \varphi\ \sin^{2y-1}\ \varphi}{(z+\cos^2\ \varphi)^{x+y}}\,d\varphi
\end{aligned}\right\},$$

$$[\operatorname{Re} x>0,\quad \operatorname{Re} y>0,\quad 0>z>-1,\quad \operatorname{Re}(x+y)<1].$$

NH 163(8)

See also 3.196 3., 3.198, 3.199, 3.215, 3.238 3., 3.251 1.-;3., 11., 3.253, 3.312 1., 3.512 1. and 2., 3.541 1., 3.542 1., 3.621 5., 3.623 1., 3.631 1., 8., 9., 3.632 2., 3.633 1., 4., 3.634 1., 2., 3.637, 3.642 1., 3.667 8., 3.681 2.

$$9.\quad \mathrm{B}\,(x,\,x) = \frac{1}{2^{2x-2}}\int_0^1 (1-t^2)^{x-1}\,dt = \frac{1}{2^{2x-1}}\int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}}\,dt.$$

See 8.384 4., 8.382 3., and also 3.621 1., 3.642 2., 3.665 1., 3.821 6., 3.839 6.

$$10.\quad \mathrm{B}\,(x+y,\,x-y) = 4^{1-x}\int_0^\infty \frac{\operatorname{ch} 2yt}{\operatorname{ch}^{2x} t}\,dt \qquad [\operatorname{Re} x>|\operatorname{Re} y|,\quad \operatorname{Re} x>0].$$

MO 9

$$11.\quad \mathrm{B}\,\left(x,\,\frac{y}{z}\right) = z\int_0^1 (1-t^z)^{x-1}t^{y-1}\,dt \qquad \left[\operatorname{Re} z>0,\quad \operatorname{Re}\frac{y}{z}>0,\quad \operatorname{Re} x>0\right].$$

FI II 787a

8.381

MO 7

3. $\mathrm{B}\,(x+iy,\ x-iy) = 2^{1-2x}\alpha e^{-2i\gamma y}\int_{-\infty}^{\infty}\frac{e^{2i\alpha yt}\,dt}{\operatorname{ch}^{2x}(\alpha t-\gamma)}\qquad [y,\ \alpha,\ \gamma \text{ are real},\quad \alpha>0;\quad \operatorname{Re}x>0].$

MI 8a

For an integral representation of ln B$(x,\ y)$, see 3.428 7.

4.
$$\frac{1}{\mathrm{B}\,(x,\ y)} = \frac{2^{x+y-1}(x+y-1)}{\pi}\int_0^{\frac{\pi}{2}}\cos[(x-y)t]\ \cos^{x+y-2}\ t\,dt;$$
$$= \frac{2^{x+y-2}(x+y-1)}{\pi\ \cos\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi}\cos[(x-y)t]\ \sin^{x+y-2}\ t\,dt;$$
$$= \frac{2^{x+y-2}(x+y-1)}{\pi\ \sin\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi}\sin[(x-y)t]\ \sin^{x+y-2}\ t\,dt.$$

NH 159(9)a
NH 159(8)a
NH 158(5)a

Series representation

8.382

1. $\mathrm{B}\,(x,\ y) = \frac{1}{y}\sum_{n=0}^{\infty}(-1)^n y\frac{(y-1)\dots(y-n)}{n!(x+n)}\qquad [y>0].$

WH

2.
$$\ln \mathrm{B}\left(\frac{1+x}{2},\ \frac{1}{2}\right) = \ln\sqrt{2\pi} + \frac{1}{2}\left[\ln\left(\frac{\operatorname{tg}\frac{\pi x}{2}}{x}\right) - \ln\left(\frac{1+x}{1-x}\right)\right] +$$
$$+\sum_{k=0}^{\infty}\frac{1-(1-2^{-2k})\zeta(2k+1)}{2k+1}\,x^{2k+1}\qquad [|x|<2].$$

NH 39(17)

3. $\mathrm{B}\left(z,\ \frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k k!}\frac{1}{z+k} + \frac{1}{z}\qquad (\text{see also 8.384 and 8.380 9. }).$

8.380

8.384
WH

8.383

Infinite-product representation:

$$(x+y+1)\,\mathrm{B}\,(x+1,\,y+1) = \prod_{k=1}^{\infty} \frac{k(x+y+k)}{(x+k)(y+k)} \qquad [x, \quad y \neq -1, \quad -2, \ldots].$$

MO 2

8.384

Functional relations involving the beta function:

1. $\mathrm{B}\,(x,\,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \mathrm{B}\,(y,\,x)$

FI II 779

2. $\mathrm{B}\,(x,\,y)\,\mathrm{B}\,(x+y,\,z) = \mathrm{B}\,(y,\,z)\,\mathrm{B}\,(y+z,\,x).$

MO 6

3. $\displaystyle\sum_{k=0}^{\infty} \mathrm{B}\,(x,\,y+k) = \mathrm{B}\,(x-1,\,y).$

WH

4. $\mathrm{B}\,(x,\,x) = 2^{1-2x}\mathrm{B}\left(\dfrac{1}{2},\,x\right) \qquad$ (see also 8.380 9. and 8.382 3.)

8.382

8.380
FI II 784

6. $$\frac{1}{\mathrm{B}\,(n,\ m)} = m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1} \qquad [m \text{ and } n\text{—natural numbers}].$$

For a connection with the psi function, see 4.253 1.

8.39 The incomplete beta function $B_x(p, q)$

8.391[7]

$$\mathrm{B}_x(p,\ q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{x^p}{p}\,{}_2F_1(p,\ 1-q;\ p+1;\ x).$$

ET I 373

8.392

$$I_x(p,\ q) = \frac{\mathrm{B}_x(p,\ q)}{\mathrm{B}\,(p,\ q)}.$$

ET II 429

8.4- 8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.401

Bessel functions $Z_\nu(z)$ are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{dZ_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0.$$

KU 37(1)

Special types of Bessel functions are what are called Bessel functions of the first kind $J_\nu(z)$, Bessel functions of the second kind $N_\nu(z)$ (also called Neumann functions and often written $Y_\nu(z)$), and Bessel functions of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (also

$J_\nu(z)$

$N_\nu(z)$ $Y_\nu(z)$ $H_\nu^{(1)}(z)$ $H_\nu^{(2)}(z)$

called Hankel's functions).

8.402

$$J_\nu(z) = \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{2^{2k}k!\Gamma(\nu+k+1)} \qquad [|\arg z| < \pi].$$

KU 55(1)

8.403

1. $$N_\nu(z) = \frac{1}{\sin\nu\pi}[\cos\nu\pi J_\nu(z) - J_{-\nu}(z)] \qquad [\text{for nonintegral } \nu, \quad |\arg z| < \pi].$$

KU 41(3)

2. $$\pi N_n(z) = 2J_n(z)\ \ln\frac{z}{2} - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} - \\ - \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!(k+n)!}\left(\frac{z}{2}\right)^{n+2k}[\psi(k+1)+\psi(k+n+1)]; \\ = 2J_n(z)\left(\ln\frac{z}{2}+\boldsymbol{C}\right) - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} - \\ - \left(\frac{z}{2}\right)^n\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{n+2k}}{k!(k+n)!}\left[\sum_{m=1}^{n+k}\frac{1}{m}+\sum_{m=1}^{k}\frac{1}{m}\right] \\ [n+1- \text{ a natural number}, \ |\arg z| < \pi].$$

KU 43(10)
KU 44, WA 75(3)a

961

8.404

$$\left.\begin{aligned} &1.\quad N_{-n}(z) = (-1)^n N_n(z) \\ &2.\quad J_{-n}(z) = (-1)^n J_n(z) \end{aligned}\right\} \qquad [n\text{—a natural number}].$$

KU 41(2)

8.405[7]

$$\left.\begin{aligned} &1.\quad H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z). \\ &2.\quad H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z). \end{aligned}\right\}$$

$J_\nu(z)$ $N_\nu(z)$ $H_\nu^{(1)}(z)$ $H_\nu^{(2)}(z)$

we shall write simply the letter Z instead of the letters $J,\ N,\ H^{(1)}$, and $H^{(2)}$.

Modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$

8.406

1. $I_\nu(z) = e^{-\frac{\pi}{2}\nu i} J_\nu\left(e^{\frac{\pi}{2}i} z\right) \qquad \left[-\pi < \arg z \le \frac{\pi}{2}\right].$

WA 92

2. $I_\nu(z) = e^{\frac{3}{2}\pi\nu i} J_\nu(e^{-\frac{3}{2}\pi i} z) \qquad \left[\frac{\pi}{2} < \arg z \le \pi\right].$

WA 92

For integral ν,

3. $I_n(z) = i^{-n} J_n(iz).$

KU 46(1)

8.407

1. $K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}(iz).$

2. $K_\nu(z) = \frac{\pi i}{2} e^{-\frac{\pi}{2}\nu i} H_{-\nu}^{(1)}(iz).$

WA 92(8)

For the differential equation defining these functions, see 8.494.

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

1.[7] $J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta}\, d\theta;$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta \qquad [n = 0, 1, 2, \dots].$$

2. $$J_{2n}(z) = \frac{1}{\pi}\int_0^{\pi} \cos 2n\theta \, \cos(z \, \sin\theta)\, d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos 2n\theta \, \cos(z \, \sin\theta)\, d\theta \qquad [n\text{—an integer}].$$

WA 30(7)

3. $$J_{2n+1}(z) = \frac{1}{\pi}\int_0^{\pi} \sin(2n+1)\theta \, \sin(z \, \sin\theta)\, d\theta = \\ = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(2n+1)\theta \, \sin(z \, \sin\theta)\, d\theta \qquad [n - \text{an integer}].$$

WA 30(6)

4. $$J_\nu(z) = 2\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \sin^{2\nu}\theta \, \cos(z\cos\theta)\, d\theta \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WH

5. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} \sin^{2\nu}\theta \, \cos(z\cos\theta)\, d\theta \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

6. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(z\sin\theta)\, \cos^{2\nu}\theta \, d\theta \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

KU 65(5), WA 35(4)a

7. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} e^{\pm iz\cos\varphi} \sin^{2\nu}\varphi \, d\varphi \qquad \left[\operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right].$$

WH

8. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \cos zt \, dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

9. $$J_\nu(x) = 2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^\infty \frac{\sin xt}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2},\ x > 0\right].$$

MO 37

10. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^1 e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 34(3)

963

11. $$J_\nu(x) = \frac{2}{\pi}\int_0^\infty \sin\left(x\operatorname{ch}t - \frac{\nu\pi}{2}\right)\operatorname{ch}\nu t\,dt.$$

WA 199(12)

12. $$J_\nu(z) = \frac{2^{\nu+1}z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}}\theta\,\sin\left(z-\nu\theta+\frac{1}{2}\theta\right)}{\sin^{2\nu+1}\theta}e^{-2z\operatorname{ctg}\theta}\,d\theta$$
$$\left[|\arg z| < \frac{\pi}{2},\ \operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right].$$

WH

13.[8] $$J_\nu(z)\frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu\theta-z\operatorname{sh}\theta}\,d\theta$$
$$[\operatorname{Re}z > 0].$$

WA 195(4)

14. $$J_\nu(z) = \frac{e^{\pm\nu\pi i}}{\pi}\left[\int_0^\pi \cos(\nu\theta + z\sin\theta)\,d\theta - \sin\nu\pi\int_0^\infty e^{-\nu\theta+z\operatorname{sh}\theta}\,d\theta\right]$$
$$\left[\text{for } \frac{\pi}{2} < |\arg z| < \pi,\ \text{with the upper sign taken for } \arg z > \frac{\pi}{2} \text{ and the lower sign taken for } \arg z < -\frac{\pi}{2}\right].$$

WH

8.412

1. $$J_\nu(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]dt \qquad \left[|\arg z| < \frac{\pi}{2}\right].$$

2. $J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) dt.$

WA 195(1)

3. $J_\nu(z) = \frac{z}{2^{\nu+1}\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!} \int_{-\infty}^{(0+)} e^t t^{-\nu-k-1} \, dt.$

WA 195(1)

4. $J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)} \left(\frac{x}{2}\right)^{\nu+2t} dt \qquad [\operatorname{Re}\nu \geq 0, \quad x > 0].$

WA 214(7)

5.[7] $J_\nu(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right)\left(\frac{z}{2}\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \int_A^{(1+,-1-)} (t^2-1)^{\nu-\frac{1}{2}} \cos(zt) \, dt$

$\left[\nu \neq \frac{1}{2}, \frac{3}{2}, \ldots \text{ The point } A \text{ falls to the right of the point } t = 1, \text{ and } \arg(t-1) = \arg(t+1) = 0 \text{ at the point } A\right].$

WH

6.[7] $J_\nu(z) = \frac{1}{2\pi} \int_{-\pi+\infty i}^{\pi+\infty i} e^{-iz\sin\theta + i\nu\theta} \, d\theta \qquad [\operatorname{Re} z > 0].$

The path of integration is taken around the semi-infinite strip $y \geq 0, -\pi \leq x \leq \pi$.

8.413

$$\frac{J_\nu(\sqrt{z^2+\zeta^2})}{(z^2-\zeta^2)^{\frac{\nu}{2}}} = = \frac{1}{\pi(z+\zeta)^\nu} \left\{ \int_0^\infty e^{\zeta\cos t} \cos(z\sin t - \nu t) \, dt - \right.$$

$$\left. - \sin\nu\pi \int_0^\infty \exp(-z \operatorname{sh} t - \zeta \operatorname{ch} t - \nu t) \, dt \right\} \qquad [\operatorname{Re}(z+\zeta) > 0].$$

$$\int_{2x}^{\infty} \frac{J_0(t)}{t}\,dt = \frac{1}{4\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-t)}{t\Gamma(1+t)} x^{2t}\,dt \qquad [x>0]$$

MO 41

See 3.715 2., 9., 10., 13., 14., 19.- 21., 3.865 1., 2., 4., 3.996 4. For an integral representation of $J_0(z)$ see 3.714 2., 3.753 2., 3., and 4.124. For an integral representation of $J_1(z)$ see 3.697, 3.711, 3.752 2., and 3.753 5.

8.415

1. $$N_0(x) = \frac{4}{\pi^2}\int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}}\,\sin(xt)\,dt - \frac{4}{\pi^2}\int_1^{\infty} \frac{\ln(t+\sqrt{t^2-1})}{\sqrt{t^2-1}}\,\sin(xt)\,dt \qquad [x>0].$$

MO 37

2. $$N_\nu(x) = -2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^{\infty} \frac{\cos xt}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x>0\right].$$

KU 89(28)A, MO 38

3. $$N_\nu(x) = -\frac{2}{\pi}\int_0^{\infty} \cos\left(x \operatorname{ch} t - \frac{\nu\pi}{2}\right) \operatorname{ch}\nu t\,dt \qquad [-1 < \operatorname{Re}\nu < 1, \quad x>0].$$

WA 199(13)

4. $$N_\nu(z) = \frac{1}{\pi}\int_0^{\infty} \sin(z\sin\theta - \nu\theta)\,d\theta - \frac{1}{\pi}\int_0^{\infty} (e^{\nu t} + e^{-\nu t}\cos\nu\pi) e^{-z \operatorname{sh} t}\,dt \qquad [\operatorname{Re} z > 0].$$

WA 197(1)

5. $$N_\nu(z) = \frac{2\left(\frac{z}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left[\int_0^{\frac{\pi}{2}} \sin(z\sin\theta)\cos^{2\nu}\theta\,d\theta - \int_0^{\infty} e^{-z \operatorname{sh}\theta} \operatorname{ch}^{2\nu}\theta\,d\theta\right]$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0\right].$$

WA 181(5)a

WA 186(8)

For an integral representation of $N_0(z)$, see 3.714 3., 3.753 4., 3.864. See also 3.865 3.

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

8.421

$$\left.\begin{aligned} 1.\quad H_\nu^{(1)}(x) &= \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^{\infty} e^{ix\,\operatorname{ch} t-\nu t}\,dt = \\ &= \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_0^{\infty} e^{ix\,\operatorname{ch} t}\,\operatorname{ch}\nu t\,dt \\ 2.\quad H_\nu^{(2)}(x) &= -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^{\infty} e^{-ix\,\operatorname{ch} t-\nu t}\,dt = \\ &= -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i}\int_0^{\infty} e^{-ix\,\operatorname{ch} t}\,\operatorname{ch}\nu t\,dt \end{aligned}\right\}\qquad [-1 < \operatorname{Re}\nu < 1,\quad x > 0].$$

WA 199(11)
WA 199(10)

$$3.\quad H_\nu^{(1)}(z) = -\frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} t\, e^{i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t}\exp(-2z\,\operatorname{ctg} t)\,dt$$

$$\left[\operatorname{Re}\nu > -\frac{1}{2},\quad \operatorname{Re} z > 0\right].$$

WA 186(5)

$$4.\quad H_\nu^{(2)}(z) = \frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} t\, e^{-i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t}\exp(-2z\,\operatorname{ctg} t)\,dt$$

$$\left[\operatorname{Re}\nu > -\frac{1}{2},\quad \operatorname{Re} z > 0\right].$$

WA 186(6)

$$5.\quad H_\nu^{(1)}(x) = -\frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^{\infty}\frac{e^{ixt}}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2},\quad x > 0\right].$$

6. $$H_\nu^{(2)}(x) = \frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^\infty \frac{e^{-ixt}}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x>0\right].$$

WA 187(2)

7. $$H_\nu^{(1)}(z) = -\frac{i}{\pi}e^{-\frac{1}{2}i\nu\pi}\int_0^\infty \exp\left[\frac{1}{2}iz\left(t+\frac{1}{t}\right)\right]t^{-\nu-1}\,dt$$
$$[0<\arg z<\pi;\ \text{or}\arg z=0\ \text{and}\ -1<\operatorname{Re}\nu<1].$$

MO 38

8. $$H_\nu^{(1)}(xz) = -\frac{i}{\pi}e^{-\frac{1}{2}i\nu\pi}\,z^\nu\int_0^\infty \exp\left[\frac{1}{2}ix\left(t+\frac{z^2}{t}\right)\right]t^{-\nu-1}\,dt$$
$$\left[0<\arg z<\frac{\pi}{2},\ x>0,\ \operatorname{Re}\nu>-1;\ \text{or}\ \arg z=\frac{\pi}{2},\ x>0\ \text{and}\ -1<\operatorname{Re}\nu<1\right].$$

MO 38

9. $$H_\nu^{(1)}(xz) = \sqrt{\frac{2}{\pi z}}\frac{x^\nu\ \exp\left[i\left(xz-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\right]}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty\left(1+\frac{it}{2z}\right)^{\nu-\frac{1}{2}}t^{\nu-\frac{1}{2}}e^{-xt}\,dt$$
$$\left[\operatorname{Re}\nu>-\frac{1}{2},\ -\frac{\pi}{2}<\arg z<\frac{3}{2}\pi,\ x>0\right].$$

MO 39

10. $$H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi}\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{iz\ \operatorname{ch}t}\operatorname{sh}^{2\nu}t\,dt$$
$$\left[0<\arg z<\pi,\ \operatorname{Re}\nu>-\frac{1}{2}\ \text{or}\ \arg z=0\ \text{and}\ -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}\right].$$

MO 38

11. $$H_0^{(1)}(x) = -\frac{i}{\pi}\int_{-\infty}^\infty\frac{\exp(i\sqrt{x^2+t^2})}{\sqrt{x^2+t^2}}\,dt \qquad [x>0].$$

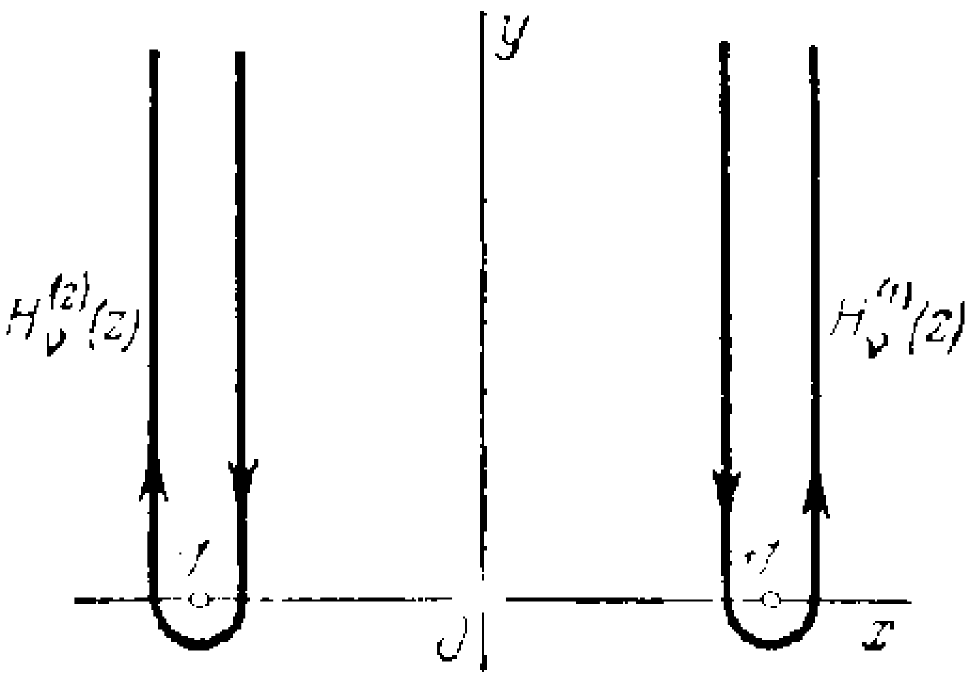
MO 38

8.422

1. $$H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i\Gamma\left(\frac{1}{2}\right)}\int_{1+\infty i}^{(1+)}e^{izt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \qquad [-\pi<\arg z<2\pi].$$

2. $$H_\nu^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i \Gamma\left(\frac{1}{2}\right)} \times \int_{-1+\infty i}^{(-1-)} e^{izt}(t^2-1)^{\nu-\frac{1}{2}}\,dt$$
$[-2\pi < \arg z < \pi]$.

The paths of integration are shown in the drawing.



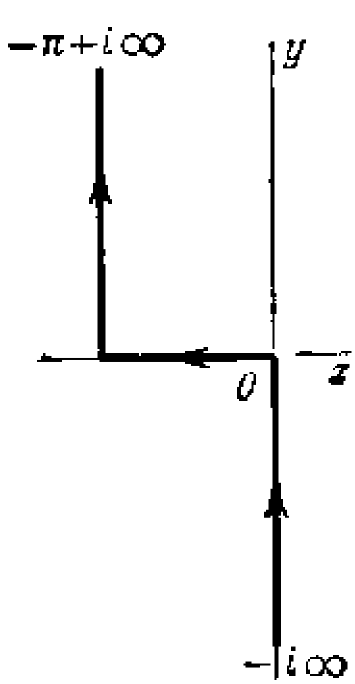
8.423

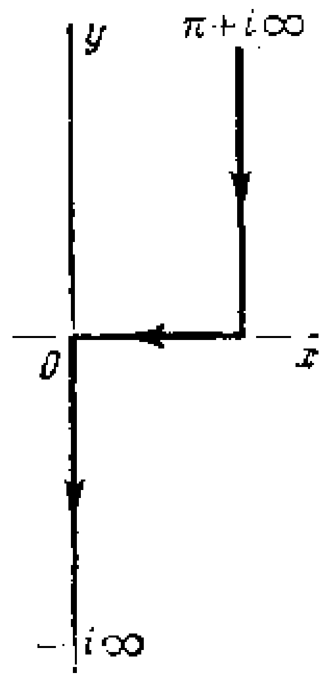
1. $$H_\nu^{(1)}(z) = -\frac{1}{\pi}\int_{-\infty i}^{-\pi+\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0].$$

WA 197(2)a

2. $$H_\nu^{(2)}(z) = -\frac{1}{\pi}\int_{\pi+\infty i}^{-\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0].$$

The path of integration for formula 1 is shown in the left hand drawing and for formula 2 in the right hand drawing.





8.424

$$\left.\begin{aligned} 1.\quad & H_\nu^{(1)}(z)J_\nu(\zeta) = \frac{1}{\pi i}\int_0^{\gamma+i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t} \\ 2.\quad & H_\nu^{(2)}(z)J_\nu(\zeta) = \frac{i}{\pi}\int_0^{\gamma-i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t} \end{aligned}\right\}$$

$$[\gamma > 0, \quad \operatorname{Re}\nu > -1, \quad |\zeta| < |z|].$$

MO 45

968

8.43 Integral representations of the function $I_\nu(z)$ and $K_\nu(z)$

The function $I_\nu(Z)$

8.431

$$1.\quad I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{\nu-\frac{1}{2}}e^{\pm zt}\,dt$$

$$2.\quad I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{\nu-\frac{1}{2}}\operatorname{ch} zt\,dt$$

5. $I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\nu\theta\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\operatorname{ch}t-\nu t}\,dt \qquad \left[|\arg z| \le \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right].$

WA 201(4)

See also 3.383 2., 3.387 1., 3.471 6., 3.714 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see 3.366 1., 3.534 3.856 6.

The function $K_\nu(Z)$

8.432

1. $K_\nu(z) = \int_0^\infty e^{-z\operatorname{ch}t}\operatorname{ch}\nu t\,dt \qquad \left[|\arg z| < \frac{\pi}{2}\text{or}\operatorname{Re}z = 0\text{and}\nu = 0\right].$

MO 39

2. $K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-z\operatorname{ch}t}\operatorname{sh}^{2\nu}t\,dt$

$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}z > 0; \quad \text{or}\operatorname{Re}z = 0\text{and} -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right].$

WA 190(5), WH

3. $K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt$

$\left[\operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \quad \text{or}\operatorname{Re}z = 0\text{and}\nu = 0\right].$

WA 190(4)

4. $K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty \cos(x\operatorname{sh}t)\operatorname{ch}\nu t\,dt \qquad [x > 0, \quad -1 < \operatorname{Re}\nu < 1].$

WA 202(13)

WA 191(1)

6.[7] $$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-\frac{z^2}{4t}}\,dt}{t^{\nu+1}} \qquad \left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0\right].$$

WA 203(15)

7.[7] $$K_\nu(xz) = \frac{z^\nu}{2}\int_0^\infty \exp\left[-\frac{x}{2}\left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1}\,dt \qquad \left[|\arg z| < \frac{\pi}{4} \text{or} |\arg z| = \frac{\pi}{4} \text{and} \operatorname{Re}\nu < 1\right].$$

MO 39

8. $$K_\nu(xz) = \sqrt{\frac{\pi}{2z}}\frac{x^\nu e^{-xz}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{\nu-\frac{1}{2}}\left(1+\frac{t}{2z}\right)^{\nu-\frac{1}{2}}\,dt \qquad \left[|\arg z| < \pi,\ \operatorname{Re}\nu > -\frac{1}{2},\ x > 0\right].$$

MO 39

9. $$K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2z}\right)^\nu \int_0^\infty \frac{\exp(-x\sqrt{t^2+z^2})}{\sqrt{t^2+z^2}}\,t^{2\nu}\,dt$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re}\sqrt{t^2+z^2} > 0, \quad x > 0\right].$$

MO 39

See also 3.337 4., 3.383 3., 3.387 3., 6., 3.388 2., 3.389 4., 3.391, 3.395 1., 3.471 9., 3.483, 3.547 2., 3.856, 3.871 3., 4., 7.141 5.

8.433

$$K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}}\int_0^\infty \cos(t^3 + xt)\,dt.$$

KU 98(31), WA 211(2)

For an integral representation of $K_0(z)$, see 3.754 2., 3.864, 4.343, 4.356, 4.367.

8.44 Series representation

8.440

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k} \; [|\arg z| < \pi].$$

8.441

Special cases:

1. $$J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k}(k!)^2}.$$

2. $$J_1(z) = -J_0'(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!(k+1)!}.$$

3. $$J_{\frac{1}{3}}(z) = \frac{\sqrt[3]{\frac{z}{2}}}{\Gamma\left(\frac{4}{3}\right)} \sum_{k=0}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3k+1)}.$$

4. $$J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)} \sqrt[3]{\frac{2}{z}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3k-1)} \right\}.$$

For the expansion of $J_\nu(z)$ in Laguerre polynomials, see 8.975 3.

8.442

1.[7] $$J_\nu(z) J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{\mu+\nu+2m} (\mu+\nu+m+1)_m}{m!\Gamma(\mu+m+1)\Gamma(\nu+m+1)},$$

2.[8] $$J_\nu(az) J_\mu(bz) = \frac{\left(\frac{az}{2}\right)^\nu \left(\frac{bz}{2}\right)^\mu}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu+1; \frac{b^2}{a^2}\right)}{k!\Gamma(\nu+k+1)}.$$

The function $N_\nu(Z)$

8.443

$$N_\nu(z) = \frac{1}{\sin \nu\pi} \left\{ \cos \nu\pi \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} - \right.$$
$$\left. - \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(k - \nu + 1)} \right\} \qquad [\nu \neq \text{an integer}] \qquad (\text{cf. } \mathbf{8}.4031.).$$

8.403

For $\nu + 1$ a natural number, see 8.403 2.; for ν a negative integer see 8.404 1

971

8.444

Special cases:

$$1. \quad \pi N_0(z) = 2J_0(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^{k} \frac{1}{m}.$$

KU 44

$$2. \quad \pi N_1(z) = 2J_1(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - \frac{2}{z} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2}\right)^{2k-1}}{k!(k-1)!} \left\{ \frac{1}{k} + 2\sum_{m=1}^{k-1} \frac{1}{m} \right\}.$$

The functions $I_\nu(z)$ and $K_n(z)$

8.445

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{\nu + 2k}.$$

WH

8.446[7]

$$K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k! \left(\frac{z}{2}\right)^{n-2k}} +$$
$$+ (-1)^{n+1} \sum^{\infty} \left(\frac{z}{2}\right)^{n+2k} \left[\ln \frac{z}{2} - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(n+k+1)\right]$$

WA 95(15)

8.447

Special cases:

1. $$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}.$$

2. $$I_1(z) = I_0'(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!}.$$

3. $$K_0(z) = -\ln\frac{z}{2} I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\psi(k+1).$$

WA 95(14)

8.45 Asymptotic expansions of Bessel functions

8.451

For large values of |z|* An estimate of the remainders in formulas 8.451 is given in 8.451 7. and 8.451 8.

1. $$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma\left(\nu + 2k + \frac{1}{2}\right)}{(2k)!\Gamma\left(\nu - 2k + \frac{1}{2}\right)} + R_1 \right] - \right.$$
$$\left. -\sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma\left(\nu + 2k + \frac{3}{2}\right)}{(2k+1)!\Gamma\left(\nu - 2k - \frac{1}{2}\right)} + R_2 \right] \right\}$$
$$[|\arg z| < \pi] (\text{see } 8.339\ 4.).$$

8.339
WA 222(1, 3)

$$2.\quad N_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}}\left\{\sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma\left(\nu+2k+\frac{1}{2}\right)}{(2k)!\Gamma\left(\nu-2k+\frac{1}{2}\right)} + R_1\right] + \right.$$

$$\left. + \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma\left(\nu+2k+\frac{3}{2}\right)}{(2k+1)!\Gamma\left(\nu-2k-\frac{1}{2}\right)} + R_2\right]\right\}$$

$$[|\arg z| < \pi] (\text{see } 8.339\ 4.).$$

8.339
WA 222(2, 4, 5)

$$3.\quad H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\Gamma\left(\nu-k+\frac{1}{2}\right)} + \theta_1\frac{(-1)^n}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$

$$\left[\operatorname{Re}\nu > -\frac{1}{2}, |\arg z| < \pi\right] (\text{see } 8.339\ 4.).$$

8.339
WA 221(5)

$$4.\quad H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}e^{-i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{1}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\Gamma\left(\nu-k+\frac{1}{2}\right)} + \theta_2\frac{1}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$

$$\left[\operatorname{Re}\nu > -\frac{1}{2}, |\arg z| < \pi\right] (\text{see } 8.339\ 4.).$$

8.339
WA 221(6)

For indices of the form $\nu = \frac{2n-1}{2}$ (where n is a natural number), the series 8.451 terminate. In this case, the closed formulas 8.46 are valid for all values.

[The sign $+$ is taken for $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, the sign $-$ for $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}{}^*$)] (see 8.339 4.).* The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

$$6. \quad K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[\sum_{k=0}^{n-1} \frac{1}{(2z)^k} \frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k!\Gamma\left(\nu - k + \frac{1}{2}\right)} + \theta_3 \frac{\Gamma\left(\nu + n + \frac{1}{2}\right)}{(2z)^n \, n!\Gamma\left(\nu - n + \frac{1}{2}\right)} \right] \text{(see 8.339 4.)}.$$

8.339
WA 231, 245(9)
WA 226(2,3)

An estimate of the remainders of the asymptotic series in formulas 8.451:

$$7. \quad |R_1| < \left| \frac{\Gamma\left(\nu + 2n + \frac{1}{2}\right)}{(2z)^{2n}(2n)!\Gamma\left(\nu - 2n + \frac{1}{2}\right)} \right| \left[n > \frac{\nu}{2} - \frac{1}{4} \right].$$

WA 231

$$8. \quad |R_2| < \left| \frac{\Gamma\left(\nu + 2n + \frac{3}{2}\right)}{(2z)^{2n+1}(2n+1)!\Gamma\left(\nu - 2n - \frac{1}{2}\right)} \right| \left[n \geq \frac{\nu}{2} - \frac{3}{4} \right].$$

WA 231

For $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, ν real, and $n + \frac{1}{2} > |\nu|$

$$|\theta_1| < 1, \text{ if } \operatorname{Im} z \geq 0; \quad |\theta_1| < |\sec(\arg z)|, \text{ if } \operatorname{Im} z \leq 0.$$

WA 245

For $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$, ν real, and $n + \frac{1}{2} > |\nu|$

$$|\theta_2| < 1, \text{ if } \operatorname{Im} z \leq 0; \quad |\theta_2| < |\sec(\arg z)|, \text{ if } \operatorname{Im} z \geq 0.$$

WA 246

974

For ν real,

$$|\theta_3| < 1 \text{ and } \operatorname{Re}\theta_3 \geq 0, \text{if} \operatorname{Re} z \geq 0;$$
$$|\theta_3| < |\operatorname{cosec}(\arg z)|, \text{if} \operatorname{Re} z < 0.$$

WA 245

For ν and z real and $n \geq \nu - \frac{1}{2}$,

$$0 \leq |\theta_3| \leq 1.$$

WA 231

In particular, it follows from 8.451 7. and 8.451 8. that for real positive values of z and ν, the errors $|R_1|$ and $|R_2|$ are less than the absolute value of the first discarded term. For values of $|\arg z|$ close to π, the series 8.451 1. and 8.451 2. may not be suitable for calculations. In particular, the error for $|\arg z| > \pi$ can be greater in absolute value than the first discarded term.

"Approximation by tangents"

8.452

For large values of the index (where the argument is less than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \operatorname{ch}\alpha$. Then, for large values of ν, the following expansions are valid:

$$1. \quad J_\nu\left(\frac{\nu}{\operatorname{ch}\alpha}\right) \sim \frac{\exp(\nu\theta\alpha - \nu\alpha)}{\sqrt{2\nu\pi\theta\alpha}}\left\{1 + \frac{1}{\nu}\left(\frac{1}{8}\operatorname{cth}\alpha - \frac{5}{24}\operatorname{cth}^3\alpha\right) + \right.$$
$$\left. + \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{cth}^2\alpha - \frac{231}{576}\operatorname{cth}^4\alpha + \frac{1155}{3456}\operatorname{cth}^6\alpha\right) + \dots\right\}.$$

2. $$N_\nu\left(\frac{\nu}{\operatorname{ch}\alpha}\right) \sim -\frac{\exp(\nu\alpha - \nu\theta\alpha)}{\sqrt{\frac{\pi}{2}\nu\theta\alpha}}\left\{1 - \frac{1}{\nu}\left(\frac{1}{8}\operatorname{cth}\alpha - \frac{5}{24}\operatorname{cth}^3\alpha\right) + \right.$$
$$\left. + \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{cth}^2\alpha - \frac{231}{576}\operatorname{cth}^4\alpha + \frac{1155}{3456}\operatorname{cth}^6\alpha\right) + \dots\right\}.$$

WA 270(5)

8.453

For large values of the index (where the argument is greater than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cos\beta$. Then, for large values of ν, the following expansions are valid:

1. $$J_\nu(\nu\sec\beta) \sim \sqrt{\frac{2}{\nu\pi\operatorname{tg}\beta}}\left\{\left[1 - \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{ctg}^2\beta + \frac{231}{576}\operatorname{ctg}^4\beta\right.\right.\right.$$
$$\left.\left. + \frac{1155}{3456}\operatorname{ctg}^6\beta\right) + \dots\right]\cos\left(\nu\operatorname{tg}\beta - \nu\beta - \frac{\pi}{4}\right) +$$
$$\left. + \left[\frac{1}{\nu}\left(\frac{1}{8}\operatorname{ctg}\beta + \frac{5}{24}\operatorname{ctg}^3\beta\right) - \dots\right]\sin\left(\nu\operatorname{tg}\beta - \nu\beta - \frac{\pi}{4}\right)\right\}.$$

WA 271(4)

2. $$N_\nu(\nu\sec\beta) \sim \sqrt{\frac{2}{\nu\pi\operatorname{tg}\beta}}\left\{\left[1 - \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{ctg}^2\beta + \frac{231}{576}\operatorname{ctg}^4\beta\right.\right.\right. +$$
$$\left.\left. + \frac{1155}{3456}\operatorname{ctg}^6\beta\right) + \dots\right]\sin\left(\nu\operatorname{tg}\beta - \nu\beta - \frac{\pi}{4}\right) -$$
$$\left. - \left[\frac{1}{\nu}\left(\frac{1}{8}\operatorname{ctg}\beta + \frac{5}{24}\operatorname{ctg}^3\beta\right) - \dots\right]\cos\left(\nu\operatorname{tg}\beta - \nu\beta - \frac{\pi}{4}\right)\right\}.$$

WA 271(5)

3. $$H_\nu^{(1)}(\nu\sec\beta) \sim \frac{\exp\left[\nu i(\operatorname{tg}\beta - \beta) - \frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\operatorname{tg}\beta}}\left\{1 - \frac{i}{\nu}\left(\frac{1}{8}\operatorname{ctg}\beta + \frac{5}{24}\operatorname{ctg}^3\beta\right) - \right.$$
$$\left. - \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{ctg}^2\beta + \frac{231}{576}\operatorname{ctg}^4\beta + \frac{1155}{3456}\operatorname{ctg}^6\beta\right) + \dots\right\}.$$

WA 271(1)

4. $$H_\nu^{(2)}(\nu\sec\beta) \sim \frac{\exp\left[-\nu i(\operatorname{tg}\beta - \beta) + \frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\operatorname{tg}\beta}}\left\{1 + \frac{i}{\nu}\left(\frac{1}{8}\operatorname{ctg}\beta + \frac{5}{24}\operatorname{ctg}^3\beta\right) - \right.$$
$$\left. - \frac{1}{\nu^2}\left(\frac{9}{128}\operatorname{ctg}^2\beta + \frac{231}{576}\operatorname{ctg}^4\beta + \frac{1155}{3456}\operatorname{ctg}^6\beta\right) + \dots\right\}.$$

$|x-\nu|$ $x^{\frac{1}{3}}$ $|x-\nu|$

may use the following formulas:

8.454

Suppose that $x>0$ and $\nu>0$, we set

$$w=\sqrt{\frac{x^2}{\nu^2}-1};$$

Then,

$$1.\quad H_\nu^{(1)}(x)=\frac{w}{\sqrt{3}}\exp\left\{\left[\frac{\pi}{6}+\nu\left(w-\frac{w^3}{3}-\operatorname{arctg} w\right)\right]i\right\}H_{\frac{1}{3}}^{(1)}\left(\frac{\nu}{3}w^3\right)+O\left(\frac{1}{|\nu|}\right).$$

$$2.\quad H_\nu^{(2)}(x)=\frac{w}{\sqrt{3}}\exp\left\{\left[-\frac{\pi}{6}-\nu\left(w-\frac{w^3}{3}-\operatorname{arctg} w\right)\right]i\right\}H_{\frac{1}{3}}^{(2)}\left(\frac{\nu}{3}w^3\right)+O\left(\frac{1}{|\nu|}\right).$$

MO 34

The absolute value of the error $O\left(\frac{1}{|\nu|}\right)$ is then less than $24\sqrt{2}\left|\frac{1}{\nu}\right|$.

976

8.455

For x real and ν a natural number $(\nu=n)$, if $n\gg 1$, the following approximations are valid:

$$1.^{7}\quad J_n(x)\ \text{t}\frac{1}{\pi}\sqrt{\frac{2(n-x)}{3x}}K_{\frac{1}{3}}\left\{\frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}}\right\}[n>x]\qquad(\text{see also } 8.433);$$

$$\text{t}\frac{1}{2}e^{\frac{2}{3}\pi i}\sqrt{\frac{2(n-x)}{3x}}H_{\frac{1}{3}}^{(1)}\left\{\frac{i}{3}\frac{[2(n-x)]^{\frac{3}{2}}}{\sqrt{x}}\right\}[n>x];$$

$$\text{t}\frac{1}{\sqrt{3}}\sqrt{\frac{2(x-n)}{3x}}\left\{J_{\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]+J_{-\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]\right\}\qquad[x>n].$$

$$(\text{see also } 8.441\ 3.,\ 8.441\ 4.).$$

8.441

8.441

8.433
WA 276(2)
MO 34
WA 276(1)

$$2. \quad N_n(x) \ \text{t} \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] - J_{\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} [x > n].$$

WA 276(3)

An estimate of the error in formulas 8.455 has not yet been achieved.

8.456

$$J_\nu^2(z) + N_\nu^2(z) \ \text{t} \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k!\Gamma\left(\nu - k + \frac{1}{2}\right)} [|\arg z| < \pi] \qquad \text{(see also 8.479 1.).}$$

8.479
WA 250(5)

8.457

$$J_\nu^2(x) + J_{\nu+1}^2(x) \ \text{t} \frac{2}{\pi x} [x \gg |\nu|].$$

WA 223

8.46 Bessel functions of order equal to an integer plus one-half

The function $J_\nu(Z)$

8.461

$$1. \quad J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2} n\right) \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!} (2z)^{2k} + \right.$$

$$\left. + \cos\left(z - \frac{\pi}{2} n\right) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \right\}$$

$$[n+1 - \text{is a natural number}] \qquad \text{(cf. 8.451 1.).}$$

2. $J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} - \right.$

$$\left. - \sin\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \right\}$$

$[n+1 -$ is a natural number$]$ (cf. 8.451 1.).

8.451
KU 58(7), WA 67(5)

8.462

1. $J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^{n} \frac{i^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^{n} \frac{(-i)^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k} \right\}$

$[n+1 -$ is a natural number$]$.

KU 59(6), WA 66(1)

2. $J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^{n} \frac{i^{n+k}(n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^{n} \frac{(-i)^{n+k}(n+k)!}{k!(n-k)!(2z)^k} \right\}$

$[n+1 -$ is a natural number$]$.

KU 59(7), WA 67(4)

8.463

1. $J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\sin z}{z}\right).$

KU 58(4)

2. $J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left(\frac{\cos z}{z}\right).$

KU 58(5)

8.464

Special cases:

1. $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$

2. $J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z.$

DW

3. $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$

DW

4. $J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\sin z - \frac{\cos z}{z} \right).$

DW

5.[8] $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left(\frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}.$

DW

6. $J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left(\frac{3}{z^2} - 1 \right) \cos z \right\}.$

DW

The function $N_{N+12}(Z)$

8.465

1. $N_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z).$

JA

2. $N_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z).$

JA

8.466

$$1. \quad H^{(1)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k} \qquad \text{(cf. 8.451 3.)}.$$

8.451

$$2. \quad H^{(2)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{n} e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k} \qquad \text{(cf. 8.451 4.)}.$$

8.451

8.467

$$I_{\pm\left(n+\frac{1}{2}\right)}(z) = \frac{1}{\sqrt{2\pi z}} \left[e^{z} \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k} \right] \qquad \text{(cf. 8.451 5.)}.$$

8.451
KU 60a

8.468

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k} \qquad \text{(cf. 8.451 6.)}.$$

8.451
KU 60

8.469

Special cases:

$$1. \quad N_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z.$$

2. $N_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$

3. $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$

WA 95(13)

4. $H^{(1)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}.$

MO 27

5. $H^{(2)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{-iz}}{-i}.$

MO 27

6. $H^{(1)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}.$

MO 27

7. $H^{(2)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}.$

MO 27

8.47- 8.48 Functional relations

8.471[8]

Recursion formulas:

1. $zZ_{\nu-1}(z) + zZ_{\nu+1}(z) = 2\nu Z_\nu(z).$

KU 56(13), WA 56(1), WA 79(1), WA 88(3)

2. $Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2\frac{d}{dz} Z_\nu(z).$

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of z that satisfy the recursion relations 8.471. Z denotes J, N, $H^{(1)}$, $H^{(2)}$ or any linear combination of these functions, the coefficients of which are independent of z and ν.

8.472

Consequences of the recursion formulas for Z defined as above:

1. $z\frac{d}{dz}Z_\nu(z) + \nu Z_\nu(z) = zZ_{\nu-1}(z).$

KU 56(11), WA 56(3), WA 79(3), WA 88(5)

2. $z\frac{d}{dz}Z_\nu(z) - \nu Z_\nu(z) = -zZ_{\nu+1}(z).$

KU 56(10), WA 56(4), WA 79(4), WA 88(6)

3. $\left(\frac{d}{zdz}\right)^m (z^\nu Z_\nu(z)) = z^{\mu-m} Z_{\nu-m}(z).$

KU 56(8), WA 57(5), WA 89(9)

4. $\left(\frac{d}{zdz}\right)^m (z^{-\nu} Z_\nu(z)) = (-1)^m z^{-\nu-m} Z_{\nu+m}(z).$

WA 89(10), Ku 55(5), WA 57(6)

5. $Z_{-n}(z) = (-1)^n Z_n(z)[n \text{ is a natural number}]. \qquad (\text{cf. } 8.404).$

8.404

8.473

Special cases:

1. $J_2(z) = \frac{2}{z}J_1(z) - J_0(z).$

2. $N_2(z) = \frac{2}{z} N_1(z) - N_0(z).$

3. $H_2^{(1,2)}(z) = \frac{2}{z} H_1^{(1,2)}(z) - H_0^{(1,2)}(z).$

4. $\frac{d}{dz} J_0(z) = -J_1(z).$

5. $\frac{d}{dz} N_0(z) = -N_1(z).$

6. $\frac{d}{dz} H_0^{(1,2)}(z) = -H_1^{(1,2)}(z).$

8.474[8]

Each of the pairs of functions $J_\nu(z)$ and $J_{-\nu}(z)$ (for $\nu \neq 0, \pm 1, \pm 2, \ldots$), $J_\nu(z)$ and $N_\nu(z)$, and $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, which are solutions of equation 8.401, and also the pair $I_\nu(z)$ and $K_\nu(z)$ is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z} \sin \nu\pi, \frac{2}{\pi z}, -\frac{4i}{\pi z}, -\frac{1}{z}.$$

KU 52(10, 11, 12), WA 90(1, 4)

980

8.475[6]

The functions $J_\nu(z)$, and $N_\nu(z)$, $H_\nu^{(1,2)}(z)$, $I_\nu(z)$, $K_\nu(z)$ with the exception of $J_n(z)$ and $I_n(z)$ for n an integer are *non-single-valued*: $z = 0$ is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut ($-\infty$, 0) are connected by the relations

8.476

1. $J_\nu(e^{m\pi i} z) = e^{m\nu\pi i} J_\nu(z).$

WA 90(1)

WA 90(3)

3. $N_{-\nu}(e^{m\pi i}z) = e^{-m\nu\pi i}N_{-\nu}(z) + 2i\sin m\nu\pi \operatorname{cosec}\nu\pi J_\nu(z).$

WA 90(4)

4. $I_\nu(e^{m\pi i}z) = e^{m\nu\pi i}I_\nu(z).$

WA 95(17)

5. $K_\nu(e^{m\pi i}z) = e^{-m\nu\pi i}K_\nu(z) - i\pi\frac{\sin m\nu\pi}{\sin\nu\pi}I_\nu(z)[\nu \text{ not an integer}].$

WA 95(18)

6. $$H_\nu^{(1)}(e^{m\pi i}z) = e^{-m\nu\pi i}H_\nu^{(1)}(z) - 2e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}J_\nu(z) = \\ = \frac{\sin(1-m)\nu\pi}{\sin\nu\pi}H_\nu^{(1)}(z) - e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}H_\nu^{(2)}(z).$$

WA 95(5)

7. $$H_\nu^{(2)}(e^{m\pi i}z) = e^{-m\nu\pi i}H_\nu^{(2)}(z) + 2e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}J_\nu(z) = \\ = \frac{\sin(1+m)\nu\pi}{\sin\nu\pi}H_\nu^{(2)}(z) + e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}H_\nu^{(1)}(z) \quad [m\text{—an integer}].$$

WA 90(6)

8. $H_\nu^{(1)}(e^{i\pi}z) = -H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu}H_\nu^{(2)}(z).$

MO 26

9. $H_\nu^{(2)}(e^{-i\pi}z) = -H_{-\nu}^{(1)}(z) = -e^{i\pi\nu}H_\nu^{(1)}(z).$

MO 26

10. $\overline{H_\nu^{(2)}}(z) = H_\nu^{(1)}(\overline{z}).$

1. $J_\nu(z)N_{\nu+1}(z) - J_{\nu+1}(z)N_\nu(z) = -\frac{2}{\pi z}.$

WA 91(12)

2. $I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z}.$

WA 95(20)

See also 3.863.

For a connection with Legendre functions, see 8.722.

For a connection with the polynomials $C_n^\lambda(t)$, see 8.936 4.

For a connection with a confluent hypergeometric function, see 9.235.

8.478

For $\nu > 0$ and $x > 0$, the product

$$x[J_\nu^2(x) + N_\nu^2(x)],$$

considered as a function of x, decreases monotonically, if $\nu > \frac{1}{2}$ and increases monotonically if $0 < \nu < \frac{1}{2}$.

981

8.479

1. $\frac{1}{\sqrt{x^2-\nu^2}} > \frac{\pi}{2}[J_\nu^2(x) + N_\nu^2(x)] \geq \frac{1}{x}\left[x \geq \nu \geq \frac{1}{2}\right]$ (see also 6.518, 6.664 4., 8.456).

8.456

6.664

6.518

MO 35

MO 35

Relations between bessel functions of the first, second, and third kinds

8.481

$$J_\nu(z) = \frac{N_{-\nu}(z) - N_\nu(z)\cos\nu\pi}{\sin\nu\pi} = H_\nu^{(1)}(z) - iN_\nu(z) = \\ = H_\nu^{(2)}(z) + iN_\nu(z) = \frac{1}{2}(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)) \qquad (\text{cf. } 8.403\ 1.,\ 8.405\,).$$

8.405

8.403
WA 89(1), JA

8.482

$$N_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} = iJ_\nu(z) - iH_\nu^{(1)}(z) = \\ = iH_\nu^{(2)}(z) - iJ_\nu(z) = \frac{i}{2}(H_\nu^{(2)}(z) - H_\nu^{(1)}(z)) \qquad (\text{cf. } 8.403\ 1.,\ 8.405\,).$$

8.405

8.403
WA 89(3), JA

8.483

1. $$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi} = \frac{N_{-\nu}(z) - e^{-\nu\pi i}N_\nu(z)}{\sin\nu\pi} = J_\nu(z) + iN_\nu(z).$$

WA 89(5)

2. $$H_\nu^{(2)}(z) = \frac{e^{\nu\pi i}J_\nu(z) - J_{-\nu}(z)}{i\sin\nu\pi} = \frac{N_{-\nu}(z) - e^{\nu\pi i}N_\nu(z)}{\sin\nu\pi} = J_\nu(z) - iN_\nu(z) \qquad (\text{cf. } 8.405).$$

8.405
WA 89(6)

8.484

1. $H^{(1)}_{-\nu}(z) = e^{\nu\pi i} H^{(1)}_{\nu}(z).$

WA 89(7)

2. $H^{(2)}_{-\nu}(z) = e^{-\nu\pi i} H^{(2)}_{\nu}(z).$

WA 89(7)

8.485[7]

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \qquad [\nu \text{ not an integer}] \qquad (\text{see also } 8.407)$$

8.407
WA 92(6)

8.486

Recursion formulas for the functions $I_\nu(z)$ and $K_\nu(z)$ and their consequences:

1. $zI_{\nu-1}(z) - zI_{\nu+1}(z) = 2\nu I_\nu(z).$

WA 93(1)

2. $I_{\nu-1}(z) + I_{\nu+1}(z) = 2\frac{d}{dz} I_\nu(z).$

WA 93(2)

3. $z\frac{d}{dz} I_\nu(z) + \nu I_\nu(z) = zI_{\nu-1}(z).$

WA 93(3)

4. $z\frac{d}{dz} I_\nu(z) - \nu I_\nu(z) = zI_{\nu+1}(z).$

WA 93(4)

5. $$\left(\frac{d}{zdz}\right)^m \{z^\nu I_\nu(z)\} = z^{\nu-m} I_{\nu-m}(z).$$

WA 93(5)

6. $$\left(\frac{d}{zdz}\right)^m \{z^{-\nu} I_\nu(z)\} = z^{-\nu-m} I_{\nu+m}(z).$$

WA 93(6)

7. $$I_{-n}(z) = I_n(z) \qquad [n - \text{a natural number}].$$

WA 93(8)

8. $$I_2(z) = -\frac{2}{z} I_1(z) + I_0(z).$$

9. $$\frac{d}{dz} I_0(z) = I_1(z).$$

WA 93(7)

10. $$zK_{\nu-1}(z) - zK_{\nu+1}(z) = -2\nu K_\nu(z).$$

WA 93(1)

11. $$K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz} K_\nu(z).$$

WA 93(2)

12. $$z\frac{d}{dz} K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z).$$

WA 93(3)

13. $$z\frac{d}{dz} K_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z).$$

14. $$\left(\frac{d}{zdz}\right)^m \{z^\nu K_\nu(z)\} = (-1)^m z^{\nu-m} K_{\nu-m}(z).$$

WA 93(5)

15. $$\left(\frac{d}{zdz}\right)^m \{z^{-\nu} K_\nu(z)\} = (-1)^m z^{-\nu-m} K_{\nu+m}(z).$$

WA 93(6)

16. $$K_{-\nu}(z) = K_\nu(z).$$

WA 93(8)

17. $$K_2(z) = \frac{2}{z} K_1(z) + K_0(z).$$

18. $$\frac{d}{dz} K_0(z) = -K_1(z).$$

WA 93(7)

8.486(1)[7]

Differentiation with respect to order

1. $$\frac{\partial J_\nu(z)}{\partial \nu} = J_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\Gamma(\nu+k+1)}.$$

MS 3.1.3

2. $$\frac{\partial J_{-\nu}(z)}{\partial \nu} = -J_{-\nu}(z) \ln\left(\frac{1}{2}z\right) + \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{-\nu+2k} \frac{\psi(-\nu+k+1)}{k!\Gamma(-\nu+k+1)}$$

MS 3.1.3

3. $$\frac{\partial N_\nu(z)}{\partial \nu} = \operatorname{ctg} \pi\nu \frac{\partial J_\nu(z)}{\partial \nu} - \operatorname{cosec} \pi\nu \frac{\partial J_{-\nu}(z)}{\partial \nu} - \pi \operatorname{cosec} \pi\nu N(z).$$

4. $$\frac{\partial I_\nu(z)}{\partial \nu} = I_\nu(z)\ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty}\left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\Gamma(\nu+k+1)}$$

MS 3.1.3

5. $$\frac{\partial K_\nu(z)}{\partial \nu} = -\pi \operatorname{ctg} \pi\nu K_\nu(z) + \frac{1}{2}\pi \operatorname{cosec} \pi\nu \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right].$$

MS 3.1.3

6. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \frac{1}{2}\pi(\pm 1)^n N_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} J_k(z)}{k!(n-k)} \qquad [n = 0, 1, \dots]$$

MS 3.2.3

7. $$\left[\frac{\partial N_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = -\frac{1}{2}\pi(\pm 1)^n J_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} N_k(z)}{k!(n-k)} \qquad [n = 0, 1, \dots]$$

MS 3.2.3

8. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm (-1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)} \qquad [n = 0, 1, \dots]$$

MS 3.2.3

9. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \pm\frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)} \qquad [n = 0, 1, \dots]$$

MS 3.2.3

10. $$(-1)^n \left[\frac{\partial}{\partial \nu} I_\nu(z)\right]_{\nu=n} = -K_n(z) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)} \qquad [n = 0, 1, \dots]$$

AS 9.6.44

11. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=n} = \frac{1}{2}n!\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)}. \qquad [n=0,1,\ldots]$$

AS 9.6.45

Special cases

12. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=0} = \frac{1}{2}\pi N_0(z).$$

MS 3.2.3

13. $$\left[\frac{\partial N_\nu(z)}{\partial \nu}\right]_{\nu=0} = -\frac{1}{2}\pi J_0(z).$$

MS 3.2.3

14. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=0} = -K_0(z).$$

MS 3.2.3

15. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=0} = 0.$$

MS 3.2.3

16. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} [\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)].$$

MS 3.3.3

17. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} [\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)].$$

18. $$\left[\frac{\partial N_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} \{\cos x\,\mathrm{Ci}(2x) + \sin x[\mathrm{Si}(2x) - \pi]\}.$$

MS 3.3.3

19. $$\left[\frac{\partial N_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = -\left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} \{\sin x\,\mathrm{Ci}(2x) - \cos x[\mathrm{Si}(2x) - \pi]\}.$$

MS 3.3.3

20. $$\left[\frac{\partial I_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-\frac{1}{2}}[e^x\,\mathrm{Ei}(-2x) \mp e^{-x}\mathrm{Ei}(2x)].$$

MS 3.3.3

21. $$\left[\frac{\partial K_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = \mp\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^x\,\mathrm{Ei}(-2x).$$

MS 3.3.3

8.487

Continuity with respect to the order* The continuity of the functions $J_\nu(z)$ and $I_\nu(z)$ follows directly from the series representations of these functions.

:

$$\left.\begin{aligned} &1. \quad \lim_{\nu\to n} N_\nu(z) = N_n(z) \\ &2. \quad \lim_{\nu\to n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z) \\ &3. \quad \lim_{\nu\to n} K_\nu(z) = K_n(z) \end{aligned}\right\} \quad [n - \text{an integer}].$$

WA 183
WA 76

8.49 Differential equations leading to Bessel functions

See also 8.401

8.491

1. $$\frac{1}{z}\frac{d}{dz}(zu') + \left(\beta^2 - \frac{\nu^2}{z^2}\right)u = 0, \qquad u = Z_\nu(\beta z).$$

2. $$\frac{1}{z}\frac{d}{dz}(zu') + \left[(\beta\gamma z^{\gamma-1})^2 - \left(\frac{\nu\gamma}{z}\right)^2\right] u = 0, \qquad u = Z_\nu(\beta z^\gamma).$$

JA

3. $$u'' + \frac{1-2\alpha}{z}u' + \left[(\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2\gamma^2}{z^2}\right] u = 0, \qquad u = z^\alpha Z_\nu(\beta z^\gamma).$$

JA

4. $$u'' + \left[(\beta\gamma z^{\gamma-1})^2 - \frac{4\nu^2\gamma^2 - 1}{4z^2}\right] u = 0, \qquad u = \sqrt{z} Z_\nu(\beta z^\gamma).$$

JA

5. $$u'' + \left(\beta^2 - \frac{4\nu^2 - 1}{4z^2}\right) u = 0, \qquad u = \sqrt{z} Z_\nu(\beta z).$$

JA

6. $$u'' + \frac{1-2\alpha}{z}u' + \left(\beta^2 + \frac{\alpha^2 - \nu^2}{z^2}\right) u = 0, \qquad u = z^\alpha Z_\nu(\beta z).$$

JA

7. $$u'' + bz^m u = 0, \qquad u = \sqrt{z} Z \frac{1}{m+2} \left(\frac{2\sqrt{b}}{m+2} z \frac{m+2}{2}\right).$$

JA 111(5)

8. $$u'' + \frac{1}{z}u' + 4\left(z^2 - \frac{\nu^2}{z^2}\right) u = 0, \qquad u = Z_\nu(z^2).$$

WA 111(6)

9. $$u'' + \frac{1}{z}u' + \frac{1}{4z}\left(1 - \frac{\nu^2}{z}\right) u = 0, \qquad u = Z_\nu(\sqrt{z}).$$

10. $u'' + \frac{1-\nu}{z}u' + \frac{1}{4}\frac{u}{z} = 0, \qquad u = z\frac{\nu}{2}Z_\nu(\sqrt{z}).$

WA 111(9)a

11. $u'' + \beta^2\gamma^2 z^{2\beta-2}u = 0, \qquad u = z^{\frac{1}{2}} Z_{\frac{1}{2\beta}}(\gamma z^\beta).$

WA 110(3)

12. $z^2u'' + (2\alpha - 2\beta\nu + 1)zu' + [\beta^2\gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta\nu)]u = 0, \qquad u = z^{\beta\nu-\alpha} Z_\nu(\gamma z^\beta).$

WA 112(21)

8.492

1. $u'' + (e^{2z} - \nu^2)u = 0, \qquad u = Z_\nu(e^z).$

WA 112(22)

2. $u'' + \frac{e^{\frac{2}{z}} - \nu^2}{z^4}u = 0, \qquad u = zZ_\nu\left(e^{\frac{1}{z}}\right).$

WA 112(22)

8.493

1. $u'' + \left(\frac{1}{z} - 2\operatorname{tg} z\right)u' - \left(\frac{\nu^2}{z^2} + \frac{\operatorname{tg} z}{z}\right)u = 0, \qquad u = \sec z Z_\nu(z).$

JA

2. $u'' + \left(\frac{1}{z} + 2\operatorname{ctg} z\right)u' - \left(\frac{\nu^2}{z^2} - \frac{\operatorname{ctg} z}{z}\right)u = 0, \qquad u = \operatorname{cosec} z Z_\nu(z).$

JA

8.494

1. $u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right)u = 0, \qquad u = Z_\nu(iz) = C_1 I_\nu(z) + C_2 K_\nu(z).$

2. $u'' + \frac{1}{z}u' - \left[\frac{1}{z} + \left(\frac{\nu}{2z}\right)^2\right] u = 0, \qquad u = Z_\nu(2i\sqrt{z}).$

JA

3. $u'' + u' + \frac{1}{z^2}\left(\frac{1}{4} - \nu^2\right) u = 0, \qquad u = \sqrt{z}e^{-\frac{z}{2}} Z_\nu\left(\frac{iz}{2}\right).$

JA

4. $u'' + \left(\frac{2\nu+1}{z} - k\right) u' - \frac{2\nu+1}{2z}ku = 0, \qquad u = z^{-\nu}e^{\frac{1}{2}k^2} Z_\nu\left(\frac{ikz}{2}\right).$

JA

5. $u'' + \frac{1-\nu}{z}u' - \frac{1}{4}\frac{u}{z} = 0, \qquad u = z^{\frac{\nu}{2}} Z_\nu(i\sqrt{z}).$

WA 111(8)

6. $u'' \pm \frac{u}{\sqrt{z}} = 0, \qquad u = \sqrt{z}Z_{\frac{2}{3}}\left(\frac{4}{3}z^{\frac{3}{4}}\right), \qquad \sqrt{z}Z_{\frac{2}{3}}\left(\frac{4}{3}iz^{\frac{3}{4}}\right).$

WA 111(10)

7. $u'' \pm zu = 0, \qquad u = \sqrt{z}Z_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right), \qquad \sqrt{z}Z_{\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right).$

WA 111(10)

8. $u'' - \left(c^2 + \frac{\nu(\nu+1)}{z^2}\right) u = 0, \qquad u = \sqrt{z}Z_{\nu+\frac{1}{2}}(icz).$

WA 108(1)

9. $u'' - \frac{2\nu}{z}u' - c^2u = 0, \qquad u = z^{\nu+\frac{1}{2}} Z_{\nu+\frac{1}{2}}(icz).$

WA 109(3, 4)

10. $u'' - c^2z^{2\nu-2}u = 0, \qquad u = \sqrt{z}Z_{\frac{1}{2\nu}}\left(i\frac{c}{\nu}z^\nu\right).$

1. $u'' + \frac{1}{z}u' + \left(i - \frac{\nu^2}{z^2}\right)u = 0, \qquad u = Z_\nu(z\sqrt{i}).$

JA

2. $u'' + \left(\frac{1}{z} \mp 2i\right)u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right)u = 0, \qquad u = e^{\pm iz}Z_\nu(z).$

JA

3. $u'' + \frac{1}{z}u' + se^{i\alpha}u = 0, \qquad u = Z_0\left(\sqrt{s}ze^{\frac{i}{2}\alpha}\right).$

JA

4. $u'' + \left(se^{i\alpha} + \frac{1}{4z^2}\right)u = 0, \qquad u = \sqrt{z}Z_0\left(\sqrt{s}ze^{\frac{i}{2}\alpha}\right).$

JA

8.496

1. $\frac{d^2}{dz^2}\left(z^4\frac{d^2u}{dz^2}\right) - z^2u = 0, \qquad u = \frac{1}{z}\{Z_2(2\sqrt{z}) + \overline{Z_2(2i\sqrt{z})}\}.$

WA 122(7)

2. $\frac{d^2}{dz^2}\left(z^{\frac{16}{5}}\frac{d^2u}{dz^2}\right) - z^{\frac{8}{5}}u = 0, \qquad u = z^{-\frac{7}{10}}\left\{Z_{\frac{5}{6}}\left(\frac{5}{3}z^{\frac{3}{5}}\right) + \overline{Z_{\frac{5}{6}}\left(\frac{5}{3}iz^{\frac{3}{5}}\right)}\right\}.$

WA 122(8)

3. $\frac{d^2}{dz^2}\left(z^{12}\frac{d^2u}{dz^2}\right) - z^6u = 0, \qquad u = z^{-4}\{Z_{10}(2z^{-\frac{1}{2}}) + \overline{Z_{10}(2iz^{-\frac{1}{2}})}\}.$

WA 122(9)

4. $\frac{d^4u}{dz^4} + \frac{2}{z}\frac{d^3u}{dz^3} - \frac{2\nu^2+1}{z^2}\frac{d^2u}{dz^2} + \frac{2\nu^2+1}{z^3}\frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1\right)u = 0,$
$u = A_1J_\nu(z) + A_2N_\nu(z) + A_3I_\nu(z) + A_4K_\nu(z),$ where A_1, A_2, A_3, A_4 are constants

8.51- 8.52 Series of Bessel functions

8.511

Generating function for Bessel functions:

$$1.\quad \exp\frac{1}{2}\left(t-\frac{1}{t}\right)z = J_0(z)+\sum_{k=1}^{\infty}[t^k+(-t)^{-k}]J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z)t^k \qquad [|z|<|t|].$$

KU 119(12)

$$2.\quad \exp\left(t-\frac{1}{t}\right)z = \left\{\sum_{k=-\infty}^{\infty} t^k J_k(z)\right\}\left\{\sum_{m=-\infty}^{\infty} t^m J_m(z)\right\}.$$

WA 40

$$3.\quad \exp(\pm iz\sin\varphi) = J_0(z)+2\sum_{k=1}^{\infty} J_{2k}(z)\cos 2k\varphi \pm 2i\sum_{k=0}^{\infty} J_{2k+1}(z)\sin(2k+1)\varphi.$$

KU 120(13)

$$4.\quad \exp(iz\cos\varphi) = \sqrt{\frac{\pi}{2z}}\sum_{k=0}^{\infty}(2k+1)i^k J_{k+\frac{1}{2}}(z)P_k(\cos\varphi);$$

$$= \sum_{k=-\infty}^{\infty} i^k J_k(z)e^{ik\varphi};$$

$$= J_0(z)+2\sum_{k=1}^{\infty} i^k J_k(z)\cos k\varphi.$$

MO 27

MO 27

WA 401(1)

$$5.\quad \sqrt{\frac{i}{\pi}}e^{iz\cos 2\varphi}\int_{-\infty}^{\sqrt{2z}\cos\varphi} e^{-it^2}\,dt = \frac{1}{2}J_0(z)+\sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i}J_{\frac{k}{2}}(z)\cos k\varphi.$$

MO 28

The series $\sum J_k(z)$

8.512

1. $J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) = 1.$

WA 44

2. $\sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \qquad [n = 1, 2, \dots].$

WA 45

3. $\sum_{k=0}^{\infty} \frac{(4k+1)(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) = \sqrt{2z/\pi}.$

8.513

1. $\sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^{p} Q_{2k}^{(2p)} z^{2k} \qquad [p = 1, 2, 3, \dots].$

WA 46(1)

2. $\sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^{p} Q_{2k+1}^{(2p+1)} z^{2k+1} \qquad [p = 0, 1, 2, 3, \dots].$

WA 46(2)

$$\left[\text{In formulas } \mathbf{8}.513 \quad Q_k^{(p)} = \sum_{m=0}^{\left[\frac{k-1}{2}\right]} \frac{(-1)^m \binom{k}{m} (k-2m)^p}{2^k k!}\right].$$

8.513

In particular:

3. $\sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2}(z + z^3).$

4. $\sum_{k=1}^{\infty}(2k)^2 J_{2k}(z) = \frac{1}{2}z^2.$

WA 47(4)

5. $\sum_{k=1}^{\infty}2k(2k+1)(2k+2)J_{2k+1}(z) = \frac{1}{2}z^3.$

WA 47(4)

8.514

1. $\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(z) = \frac{\sin z}{2}.$

WH

2. $J_0(z) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(z) = \cos z.$

WH

3. $\sum_{k=1}^{\infty}(-1)^{k+1}(2k)^2 J_{2k}(z) = \frac{z \sin z}{2}.$

WA 32(9)

4. $\sum_{k=0}^{\infty}(-1)^k (2k+1)^2 J_{2k+1}(z) = \frac{z \cos z}{2}.$

WA 32(10)

5. $J_0(z) + 2\sum_{k=1}^{\infty}J_{2k}(z)\cos 2k\theta = \cos(z \sin\theta).$

KU 120(14), WA 32

6. $\sum_{k=0}^{\infty}J_{2k+1}(z)\sin(2k+1)\theta = \frac{\sin(z\sin\theta)}{2}.$

7. $$\sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2}\int_0^x J_0(t)\,dt \qquad [x \quad \text{real}].$$

WA 638

8.515

1. $$\sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \left(\frac{2z+t}{2z}\right)^k J_{\nu+k}(z) = \left(\frac{z}{z+t}\right)^{\nu} J_{\nu}(z+t).$$

AD (9140)

2. $$\sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}(x^2) = S(x).$$

MO 127a

3. $$\sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}(x^2) = C(x).$$

MO 127a

989

8.516

$$\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z\sin\theta) = (z\sin\theta)^{2n}$$

WA 47

The series $\sum a_k J_k(kx)$ and $\sum a_k J_k'(kx)$

8.517

1. $$\sum_{k=1}^{\infty} J_k(kz) = \frac{z}{2(1-z)}$$

2. $$\sum_{k=1}^{\infty} (-1)^k J_k(kz) = -\frac{z}{2(1+z)}$$

3. $$\sum_{k=1}^{\infty} J_{2k}(2kz) = \frac{z^2}{2(1-z^2)}$$

$$\left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1\right].$$

1. $\sum_{k=1}^{\infty} \frac{J_k'(kx)}{k} = \frac{1}{2} + \frac{x}{4} \qquad [0 \le x < 1].$

MO 58

2. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{J_k'(kx)}{k} = \frac{1}{2} - \frac{x}{4} \qquad [0 \le x < 1].$

MO 58

3. $\sum_{k=1}^{\infty} k J_k'(kx) = \frac{1}{2(1-x)^2} \qquad [0 \le x < 1].$

MO 58

4. $\sum_{k=1}^{\infty} (-1)^{k-1} J_k'(kx) k = \frac{1}{2(1+x)^2} \qquad [0 \le x < 1].$

MO 58

The series $\sum A_K J_0(Kx)$

8.519

If, on the interval $[0 \le x \le \pi]$, a function $f(x)$ possesses a continuous derivative with respect to x that is of bounded variation, then

1. $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \qquad [0 < x < \pi],$

where

2. $a_0 = 2f(0) + \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin\varphi) d\varphi.$

3. $a_n = \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin\varphi) \cos nu d\varphi.$

WH

8.521

Examples:

1. $$\sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2\sum_{m=1}^{n} \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \qquad [2n\pi < x < 2(n+1)\pi].$$

MO 59

2. $$\sum_{k=1}^{\infty} (-1)^{k+1} J_0(kx) = \frac{1}{2} \qquad [0 < x < \pi].$$

KU 124(12)

990

3. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\{(2k-1)x\} \frac{\pi^2}{8} - \frac{|x|}{2} \qquad [-\pi < x < \pi];$$
$$= \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos \frac{\pi}{x} [\pi < x < 2\pi].$$

MO 59
KU 124

4. $$\sum_{k=1}^{\infty} e^{-kz} J_0(k\sqrt{x^2+y^2}) = \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{(2ki\pi + z)^2 + x^2 + y^2}} - \frac{1}{\sqrt{(2ki\pi - z)^2 + x^2 + y^2}} \right\};$$
$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1}\left(\frac{z}{r}\right) \qquad [0 < r < 2\pi],$$

MO 59

where $r = \sqrt{x^2 + y^2 + z^2}$ and where the radical indicates the square root with a positive real part. In formula 8.521 4., the first equation holds when x and y are real and Re $z > 0$; the second equation holds when x, y, and z are all real.

The series $\sum a_k Z_0(kx) \sin kx$ and $\sum a_k Z_0(kx) \cos kx$

8.522

1. $$\sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^{m} \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^{n} \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}.$$

$$2.\quad \sum_{k=1}^{\infty} J_0(kx)\sin kxt = \frac{1}{2\pi}\left\{\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{l}\right\} + \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l+tx)^2 - x^2}} - \frac{1}{2\pi l}\right\} - \\ - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l}\right\}.$$

MO 59

$$3.\quad \sum_{k=1}^{\infty} N_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{1}{2\pi}\left\{\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{n}\frac{1}{l}\right\} - \\ - \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l+tx)^2 - x^2}} - \frac{1}{2\pi l}\right\} - \\ - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l}\right\}.$$

MO 60

In formulas 8.522, $x > 0, 0 \le t < 1, 2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.523

$$1.\quad \sum_{k=1}^{\infty}(-1)^k J_0(kx)\cos kxt = -\frac{1}{2} + \sum_{l=1}^{m}\frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} + \sum_{l=1}^{n}\frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}.$$

MO 60

$$2.\quad \sum_{k=1}^{\infty}(-1)^k J_0(kx)\sin kxt = \frac{1}{2\pi}\left\{\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{l}\right\} + \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \\ - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}.$$

MO 60

$$3.\quad \sum_{k=1}^{\infty}(-1)^k N_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{1}{2\pi}\left\{\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{n}\frac{1}{l}\right\} - \\ - \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \\ - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}.$$

$x > 0, 0 \le t < 1$ $(2m-1)\pi < x(1-t) < (2m+1)\pi$ $(2n-1)\pi < x(1+t) < (2n+1)\pi$ m n

8.524[6]

1. $$\sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^{n} \frac{1}{\sqrt{x^2 - (2l\pi - tx)^2}}.$$

MO 60

2. $$\sum_{k=1}^{\infty} J_0(kx) \sin kxt = \sum_{l=0}^{m} \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} + \frac{1}{2\pi} \sum_{l=1}^{n} \frac{1}{l}.$$

MO 60

3.[6] $$\sum_{k=1}^{\infty} N_0(kx) \cos kxt = -\frac{1}{\pi} \left(\boldsymbol{C} + \ln \frac{x}{4\pi} \right) - \sum_{l=0}^{m} \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} + \\ + \frac{1}{2\pi} \sum_{l=1}^{n} \frac{1}{l} - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\}.$$

MO 61

In formulas 8.524, $x > 0, t > 1, 2m\pi < x(t-1) < 2(m+1)\pi$, $2n\pi < x(t+1) < 2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.525

1. $$\sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^{n} \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}.$$

MO 61

2. $$\sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = = \sum_{l=1}^{m} \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^{n} \frac{1}{l} + \\ + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}.$$

$$3.\quad \sum_{k=1}^{\infty}(-1)^k N_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)+\frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} - \\ -\sum_{l=1}^{m}\frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}} - \\ -\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi+tx]^2-x^2}}-\frac{1}{2l\pi}\right\} - \\ -\sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}}-\frac{1}{2l\pi}\right\}.cr$$

MO 61

In formulas 8.525, $x > 0, t > 1$, $(2m-1)\pi < x(t-1) < (2m+1)\pi$, $(2n-1)\pi < x(t+1) < (2n+1)\pi$, m and n are natural numbers.

8.526

$$1.\quad \sum_{k=1}^{\infty}K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)+\frac{\pi}{2x\sqrt{1+t^2}}+\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi-tx)^2}}-\frac{1}{2l\pi}\right\}+ \\ +\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi+tx)^2}}-\frac{1}{2l\pi}\right\}.$$

MO 61

$$2.\quad \sum_{k=1}^{\infty}(-1)^k K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)+\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+[(2l-1)\pi-xt]^2}}-\frac{1}{2l\pi}\right\}+ \\ +\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+[(2l-1)\pi+xt]^2}}-\frac{1}{2l\pi}\right\} \qquad [x>0, \quad t \text{ real}], \\ (\text{see also } 8.66)\,.$$

8.66
MO 62

8.53 Expansion in products of Bessel functions

8.530

Suppose that $r > 0, \varrho > 0, \varphi > 0$, and $R = \sqrt{r^2+\varrho^2-2r\varrho\cos\varphi}$; that is, suppose that r, ϱ, and R are the sides of a triangle such that the angle between the sides r and ϱ is equal to φ. Suppose also that

$\varrho < r$ and that ψ is the angle opposite the side ϱ, so that

$$1.\quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}.$$

When these conditions are satisfied, we have the "summation theorem" for Bessel functions:

$$2.\quad e^{i\nu\psi} Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho) Z_{\nu+k}(mr) e^{ik\varphi} \qquad [m\text{—an arbitrary complex number }].$$

WA 394(6)

For $Z_\nu = J_\nu$ and ν an integer, the restriction $\varrho < r$ is superfluous.

8.531

Special cases:

$$1.\quad J_0(mR) = J_0(m\varrho) J_0(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho) J_k(mr) \cos k\varphi.$$

WA 391(1)
MO 31

$$2.\quad H_0^{(1,2)}(mR) = J_0(m\varrho) H_0^{(1,2)}(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho) H_k^{(1,2)}(mr) \cos k\varphi.$$

MO 31

$$3.\quad J_0(z \sin\alpha) = J_0^2\left(\frac{z}{2}\right) + 2\sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right) \cos 2k\alpha;$$

$$= \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) P_{2k}(\cos\alpha).$$

MO 31

8.532

The term "summation theorem" is also applied to the formula

$$1. \quad \frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu}\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)\frac{J_{\nu+k}(m\varrho)}{\varrho^\nu}\frac{Z_{\nu+k}(mr)}{r^\nu}C_k^\nu(\cos\varphi).$$

[$\nu \neq -1, -2, -3, \dots$; the conditions on r, ϱ, R, φ, and m are the same as in formula 8.530; for $Z_\nu = J_\nu$ and ν an integer, formula 8.532 1 is valid for arbitrary r, ϱ, and φ].

8.533

Special cases:

$$1. \quad \frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}}\sum_{k=0}^{\infty}(2k+1)J_{k+\frac{1}{2}}(m\varrho)H^{(1)}_{k+\frac{1}{2}}(mr)P_k(\cos\varphi).$$

MO 31
WA 398(4)

$$2. \quad \frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}}\sum_{k=0}^{\infty}(2k+1)J_{k+\frac{1}{2}}(m\varrho)H^{(2)}_{k+\frac{1}{2}}(mr)P_k(\cos\varphi).$$

MO 31

8.534

A degenerate addition theorem ($r \to \infty$):

$$e^{im\varrho\cos\varphi} = \sqrt{\frac{\pi}{2m\varrho}}\sum_{k=0}^{\infty}i^k(2k+1)J_{k+\frac{1}{2}}(m\varrho)P_k(\cos\varphi);$$
$$= 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)i^k(m\varrho)^{-\nu}J_{\nu+k}(m\varrho)C_k^\nu(\cos\varphi) \qquad [\nu \neq 0, -1, -2, \dots].$$

WA 401(2)
WA 401(1)

8.535

The term "product theorem" is also applied to the formula

For $Z_\nu = J_\nu$, it is valid for all values of λ and z.

8.536

1. $$\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0].$$

WA 47(1)
MO 32

2. $$2\sum_{k=n}^{\infty} \frac{k\Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0].$$

WA 47(2)

3. $$J_0^2(z) + 2\sum_{k=1}^{\infty} J_k^2(z) = 1.$$

WA 41(3)

8.537

$$\sum_{k=-\infty}^{\infty} Z_{\nu-k}(t) J_k(z) = Z_\nu(z+t) \qquad [|z| < |t|].$$

WA 158(2)

In particular:

$$\sum_{k=-\infty}^{\infty} J_k(z) J_{n-k}(z) = J_n(2z).$$

WA 41

8.538

1. $$\sum_{k=-\infty}^{\infty} (-1)^k J_{-\nu+k}(t) J_k(z) = J_{-\nu}(z+t) \qquad [|z| < |t|].$$

2. $$\sum_{k=-\infty}^{\infty} Z_{\nu+k}(t) J_k(z) = Z_\nu(t-z) \qquad [|z| < |t|].$$

WA 159(5)

8.54 The zeros of Bessel functions

8.541

For arbitrary real ν, the function $J_\nu(z)$ has infinitely many real zeros. For $\nu > -1$, all its zeros are real.

A Bessel function $Z_\nu(z)$ has no multiple zeros except possibly the coordinate origin.

WA 528

8.542

All zeros of the function $N_0(z)$ with positive real parts are real.

WA 531

8.543

If $-(2s+2) < \nu < -(2s+1)$, where s is a natural number or 0, then $J_\nu(z)$ has exactly $4s+2$ complex roots, two of which are purely imaginary. If $-(2s+1) < \nu < -2s$, where s is a natural number, then the function $J_\nu(z)$ has exactly $4s$ complex zeros none of which are purely imaginary.

WA 532

8.544

If x_ν and x'_ν are, respectively, the smallest positive zeros of the functions $J_\nu(z)$ and $J'_\nu(z)$ for $\nu > 0$, then $x_\nu > \nu$ and $x'_\nu > \nu$. Suppose also that y_ν is the smallest positive zero of the function $N_\nu(z)$. Then, $x_\nu < y_\nu < x'_\nu$.

WA 534, 536

Suppose that $z_{\nu,m}$ (for $m = 1, 2, 3, \ldots$) are the zeros of the function $z^{-\nu} J_\nu(z)$, numbered in order of the absolute value of their real parts. Here, we assume that $\nu \neq -1, -2, -3, \ldots$. Then, for arbitrary z

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right).$$

WA 550

WA 526, 530

8.545

The number of zeros of the function $z^{-\nu} J_\nu(z)$ that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + O\left(\frac{1}{m}\right)\right)\pi + \left[\frac{1}{2}\operatorname{Re}\nu + \frac{1}{4}\right]\pi,$$

WA 548

is $m + O\left(\frac{1}{m}\right)$, for large values of m.

8.546

For $\nu \geq 0$, the number of zeros of the function $K_\nu(z)$ that occur in the region $\operatorname{Re} z < 0$, $|\arg z| < \pi$ is equal to the even number closest to $\nu - \frac{1}{2}$.

8.547

Large zeros of the functions $J_\nu(z)\cos\alpha - N_\nu(z)\sin\alpha$, where ν and α are real numbers, are given by the asymptotic expansion

$$x_{\nu,m} \sim \left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]} - \\ - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]^3} - \dots$$

KU 109(24), WA 558
WA 562

8.548

In particular, large zeros of the function $J_0(z)$ are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m - 1) + \frac{1}{2\pi(4m - 1)} - \frac{31}{6\pi^3(4m - 1)^3} + \frac{3779}{15\pi^5(4m - 1)^5} - \dots$$

KU 109(25), WA 556

This series is suitable for calculating all (except the smallest x_{01}) zeros of the function $J_0(z)$ correctly to at least five digits.

8.549

To calculate the roots $x_{\nu,m}$ of the function $J_\nu(z)$ of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}.$$

KU 112(27)a
WA 554

8.55 Struve functions

8.550

Definitions:

1. $$\mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\nu+m+\frac{3}{2}\right)}.$$

WA 358(2)

2. $$\mathbf{L}_\nu(z) = -ie^{-i\nu\frac{\pi}{2}}\mathbf{H}_\nu\left(ze^{i\frac{\pi}{2}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\nu+m+\frac{3}{2}\right)}.$$

WA 360(11)

8.551

Integral representations:

1. $$\mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin zt\,dt = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \sin(z\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi$$
$$\left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

WA 358(1)

2. $$\mathbf{L}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \operatorname{sh}(z\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi \left[\operatorname{Re}\nu > -\frac{1}{2}\right].$$

8.552

Special cases:

1.[6] $$\mathbf{H}_n(z) = \frac{1}{\pi}\sum_{m=0}^{[\frac{n-1}{2}]} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma\left(n+\frac{1}{2}-m\right)} - \mathbf{E}_n(z) \qquad [n=1,2,\dots].$$

EH II 40(66), WA 337(1)

2.[6] $$\mathbf{H}_{-n}(z) = (-1)^{n+1}\frac{1}{\pi}\sum_{m=0}^{[\frac{n-1}{2}]} \frac{\Gamma\left(n-m-\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)} - \mathbf{E}_{-n}(z) [n=1,2,\dots].$$

EH II 40(67), WA 337(2)

3. $$\mathbf{H}_{n+\frac{1}{2}}(z) = N_{n+\frac{1}{2}}(z) + \frac{1}{\pi}\sum_{m=0}^{n} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)} [n=0,1,\dots].$$

EH II 39(64)

4. $$\mathbf{H}_{-(n+\frac{1}{2})}(z) = (-1)^n J_{n+\frac{1}{2}}(z) [n=0,1,\dots].$$

EH II 39(65)

5. $$\mathbf{L}_{-(n+\frac{1}{2})}(z) = I_{n+\frac{1}{2}}(z) [n=0,1,\dots].$$

EH II 39(65)

6. $$\mathbf{H}_{\frac{1}{2}}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}}(1-\cos z).$$

EH II 39, WA 364(3)

7. $$\mathbf{H}_{\frac{3}{2}}(z) = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\left(1+\frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\sin z + \frac{\cos z}{z}\right).$$

WA 364(3)

8.553

Functional relations:

1. $\mathbf{H}_\nu(ze^{im\pi}) = e^{i\pi(\nu+1)m}\mathbf{H}_\nu(z)[m = 1, 2, 3, \dots].$

WA 362(5)

2. $\frac{d}{dz}[z^\nu \mathbf{H}_\nu(z)] = z^\nu \mathbf{H}_{\nu-1}(z).$

WA 358

3. $\frac{d}{dz}[z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu}\pi^{-\frac{1}{2}} \left[\Gamma\left(\nu + \frac{3}{2}\right)\right]^{-1} - z^{-\nu}\mathbf{H}_{\nu+1}(z).$

WA 359

4. $\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1}\mathbf{H}_\nu(z) + \pi^{-\frac{1}{2}} \left(\frac{z}{2}\right)^\nu \left[\Gamma\left(\nu + \frac{3}{2}\right)\right]^{-1}.$

WA 359(5)

5. $\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}'_\nu(z) - \pi^{-\frac{1}{2}} \left(\frac{z}{2}\right)^\nu \left[\Gamma\left(\nu + \frac{3}{2}\right)\right]^{-1}.$

WA 359(6)

8.554

Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = N_\nu(\xi) + \frac{1}{\pi}\sum_{m=0}^{p-1} \frac{\Gamma\left(m + \frac{1}{2}\right)\left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu + \frac{1}{2} - m\right)} + O(|\xi|^{\nu-2p-1})[|\arg\xi| < \pi].$$

EH II 39(63), WA 363(2)

For the asymptotic representation of $N_\nu(\xi)$, see 8.451 2.

8.555

The differential equation for Struve functions:

$$z^2y'' + zy' + (z^2 - \nu^2)y = \frac{1}{\sqrt{\pi}}\frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\Gamma\left(\nu + \frac{1}{2}\right)}.$$

WA 359(10)

8.56 Thomson functions and their generalizations: $\mathrm{ber}_\nu(z)$, $\mathrm{bei}_\nu(z)$, $\mathrm{her}_\nu(z)$, $\mathrm{hei}_\nu(z)$, ker(z), kei(z)

8.561

$$\left.\begin{array}{ll} 1. & \mathrm{ber}_\nu(z) + i\,\mathrm{bei}_\nu(z) = J_\nu(ze^{\frac{3}{4}\pi i}). \\ 2. & \mathrm{ber}_\nu(z) - i\,\mathrm{bei}_\nu(z) = J_\nu(ze^{-\frac{3}{4}\pi i}). \end{array}\right\}$$

WA 96(6)

8.562

$$\left.\begin{array}{ll} 1. & \mathrm{her}_\nu(z) + i\,\mathrm{hei}_\nu(z) = H_\nu^{(1)}(ze^{\frac{3}{4}\pi i}) \\ 2. & \mathrm{her}_\nu(z) - i\,\mathrm{hei}_\nu(z) = H_\nu^{(1)}(ze^{-\frac{3}{4}\pi i}) \end{array}\right\} \qquad \text{(see also } \mathbf{8}.567\text{)}.$$

8.567

WA 96(7)

8.563

$$\left.\begin{array}{ll} 1. & \mathrm{ber}_0(z) \equiv \mathrm{ber}(z); \quad \mathrm{bei}_0(z) \equiv \mathrm{bei}(z). \\ 2. & \mathrm{ker}(z) \equiv -\frac{\pi}{2}\mathrm{hei}_0(z); \quad \mathrm{kei}(z) \equiv \frac{\pi}{2}\mathrm{her}_0(z). \end{array}\right\}$$

WA 96(8)

For integral representations, see 6.251, 6.536, 6.537, 6.772 4., 6.777.

Series representation

8.564

1. $\operatorname{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k}[(2k)!]^2}.$

WA 96(3)

2. $\operatorname{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2}[(2k+1)!]^2}.$

WA 96(4)

3. $\operatorname{ker}(z) = \left(\ln \frac{2}{z} - \boldsymbol{C}\right) \operatorname{ber}(z) + \frac{\pi}{4} \operatorname{bei}(z) + \sum_{k=1}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k}[(2k)!]^2} \sum_{m=1}^{2k} \frac{1}{m}.$

WA 96(9)A, DW

4. $\operatorname{kei}(z) = \left(\ln \frac{2}{z} - \boldsymbol{C}\right) \operatorname{bei}(z) - \frac{\pi}{4} \operatorname{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2}[(2k+1)!]^2} \sum_{m=1}^{2k+1} \frac{1}{m}.$

WA 96(10)A, DW

8.565

$$\operatorname{ber}_\nu^2(z) + \operatorname{bei}_\nu^2(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2\nu+4k}}{k!\Gamma(\nu+k+1)\Gamma(\nu+2k+1)}.$$

WA 163(6)

Asymptotic representation

8.566

1. $\operatorname{ber}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos \beta(z) \qquad \left[|\arg z| < \frac{\pi}{4}\right].$

WA 227(1)

WA 227(1)

$$2.\quad \operatorname{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin\beta(z) \qquad \left[|\arg z| < \frac{\pi}{4}\right].$$

WA 227(1)

$$3.\quad \ker(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos\beta(-z) \qquad \left[|\arg z| < \frac{5}{4}\pi\right].$$

WA 227(2)

$$4.\quad \operatorname{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin\beta(-z) \qquad \left[|\arg z| < \frac{5}{4}\pi\right],$$

WA 227(2)

where

$$\alpha(z) \sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots,$$
$$\beta(z) \sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots.$$

8.567

Functional relations

$$\left.\begin{array}{ll} 1. & \ker(z) + i\operatorname{kei}(z) = K_0(z\sqrt{i}) \\ 2. & \ker(z) - i\operatorname{kei}(z) = K_0(z\sqrt{-i}) \end{array}\right\} \qquad (\text{see } \mathbf{8}.562).$$

8.562
WA 96(5), DW

For integrals of Thomson's functions, see 6.87.

8.57 Lommel functions

8.570

Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$:

1. $$s_{\mu,\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]\dots[(\mu+2m+1)^2-\nu^2]};$$
$$= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)}$$
$[\mu \pm \nu \text{ is not a negative odd integer }]$.

EH II 40(69), WA 377(2)

1000

2. $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + \left[2^{\mu-1}\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)\right] \times$$
$$\times \frac{\cos\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu+\nu)\pi\right] J_{\nu}(z)}{\sin \nu\pi}$$
$[\mu \pm \nu \text{ is a positive odd integer, } \nu \text{ is an odd integer }]$;
$$= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right) \times$$
$$\times \left\{\sin\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{\nu}(z) - \cos\left[\frac{1}{2}(\mu-\nu)\pi\right] N_{\nu}(z)\right\}$$
$[\mu \pm \nu \text{ is a positive odd integer, } \nu \text{ is an integer }]$.

EH II 41(71), WA 379(3)
EH II 40(71), WA 379(2)

Integral representations

8.571

$$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[N_{\nu}(z)\int_0^z z^{\mu} J_{\nu}(z)dz - J_{\nu}(z)\int_0^z z^{\mu} N_{\nu}(z)dz\right].$$

WA 378(9)

8.572

$$s_{\mu,\nu}(z) = 2^{\mu}\left(\frac{z}{2}\right)^{\frac{1}{2}(1+\nu+\mu)} \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu\right) \times \int_0^{\frac{\pi}{2}} J_{\frac{1}{2}(1+\mu-\nu)}(z\sin\theta)(\sin\theta)^{\frac{1}{2}(1+\nu-\mu)}(\cos\theta)^{\nu+\mu}d\theta$$
$[\mathrm{Re}(\nu+\mu+1) > 0]$.

1. $S_{1,2n}(z) = zO_{2n}(z).$

WA 382(1)

2. $S_{0,2n+1}(z) = \frac{z}{2n+1}O_{2n+1}(z).$

WA 382(1)

3. $S_{-1,2n}(z) = \frac{1}{4n}S_{2n}(z).$

WA 382(2)

4. $S_{0,2n+1}(z) = \frac{1}{2}S_{2n+1}(z).$

WA 382(2)

5. $s_{\nu,\nu}(z) = \Gamma\left(\nu + \frac{1}{2}\right)\sqrt{\pi}2^{\nu-1}\mathbf{H}_\nu(z).$

EH II 42(84)

6. $S_{\nu,\nu}(z) = [\mathbf{H}_\nu(z) - N_\nu(z)]2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right).$

EH II 42(84)

8.574

Connections with other special functions:

1. $\mathbf{J}_\nu(z) = \frac{1}{\pi}\sin(\nu\pi)[s_{0,\nu}(z) - \nu s_{-1,\nu}(z)].$

EH II 41(82)

2. $\mathbf{E}_\nu(z) = -\frac{1}{\pi}[(1+\cos\nu\pi)s_{0,\nu}(z) + \nu(1-\cos\nu\pi)s_{-1,\nu}(z)].$

A connection with a hypergeometric function

3. $$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right).$$

EH II 40(69), WA 378(10)

8.575

Functional relations:

1. $$s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu+1)^2 - \nu^2]s_{\mu,\nu}(z).$$

EH II 41(73), WA 380(1)

2. $$s_{\mu,\nu}(z) + \left(\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z).$$

EH II 41(74), WA 380(2)

3. $$s'_{\mu,\nu}(z) - \left(\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu-\nu-1)s_{\mu-1,\nu+1}(z).$$

EH II 41(75), WA 380(3)

4. $$\left(2\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)s_{\mu-1,\nu+1}(z).$$

EH II 41(76), WA 380(4)

5. $$2s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)s_{\mu-1,\nu+1}(z).$$

EH II 41(77), WA 380(5)

In formulas 8.575 1.- 5., $s_{\mu,\nu}(z)$ can be replaced with $S_{\mu,\nu}(z)$.

8.576

Asymptotic expansion of $S_{\mu,\nu}(z)$. In the case in which $\mu \pm \nu$ is not a positive odd integer, the following asymptotic expansion is valid for $S_{\mu,\nu}(z)$:

WA 385

8.577

Lommel functions satisfy the following differential equation:

$$z^2 w'' + z w' + (z^2 - \nu^2) w = z^{\mu+1}.$$

WA 377(1), EH II 40(68)

8.578

Lommel functions of two variables $U_\nu(w,z), V_\nu(w,z)$:

Definition

1. $$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z).$$

EH II 42(87), WA 591(5)

2. $$V_\nu(w,z) = \cos\left[\frac{1}{2}\left(w + \frac{z^2}{w} + \nu\pi\right)\right] + U_{-\nu+2}(w,z).$$

EH II 42(88), WA 591(6)

Particular values:

3. $$U_0(z,z) = V_0(z,z) = \frac{1}{2}\{J_0(z) + \cos z\}.$$

WA 591(9)

4. $$U_1(z,z) = -V_1(z,z) = \frac{1}{2}\sin z.$$

WA 591(10)

5. $$U_{2n}(z,z) = V_{2n}(z,z) = \\ = \frac{(-1)^n}{2}\left\{\cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z)\right\} \qquad [n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

6. $U_{2n+1}(z,z) - V_{2n+1}(z,z) =$
$$= \frac{(-1)^n}{2}\left\{\sin z - \sum_{m=0}^{n-1}(-1)^m \varepsilon_{2m+1} J_{2m+1}(z)\right\}, \qquad [n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

WA 591(12)

7. $$V_n(w,z) = (-1)^n U_n\left(\frac{z^2}{w}, z\right).$$

8. $$U_\nu(w,0) = \frac{\left(\frac{w}{2}\right)^{\frac{1}{2}}}{\Gamma(\nu-1)} s_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right).$$

WA 593(9)

9. $$V_{-\nu+2}(w,0) = \frac{\left(\frac{w}{2}\right)^{\frac{1}{2}}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right).$$

WA 593(10)

8.579

Functional relations:

1. $$2\frac{\partial}{\partial w}U_\nu(w,z) = U_{\nu-1}(w,z) + \left(\frac{z}{w}\right)^2 U_{\nu+1}(w,z).$$

WA 593(2)

2. $$2\frac{\partial}{\partial w}V_\nu(w,z) = V_{\nu+1}(w,z) + \left(\frac{z}{w}\right)^2 V_{\nu-1}(w,z).$$

WA 593(4)

3. The function $U_\nu(w,z)$ is a particular solution of the differential equation

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z}\frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z).$$

$V_\nu(w,z)$

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z}\frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z).$$

WA 592(3)

8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.580

Definitions:

1. The Anger function $\mathbf{J}_\nu(z)$:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,d\theta.$$

WA 336(1), EH II 35(32)

2. The Weber function $\mathbf{E}_\nu(z)$:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)\,d\theta.$$

WA 336(2), EH II 35(32)

8.581

Series representations:

$$1.\quad \mathbf{J}_\nu(z) = \cos\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} + \sin\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}.$$

EH II 36(36), WA 337(3)

EH II 36(37), WA 338(4)

8.582

Functional relations:

$$1.^{6}\quad 2\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z).$$

EH II 36(40), WA 340(2)

$$2.^{6}\quad 2\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z).$$

EH II 36(41), WA 340(6)

$$3.^{6}\quad \mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1}\mathbf{J}_\nu(z) - 2(\pi z)^{-1}\sin(\nu\pi).$$

EH II 36(42), WA 340(1)

$$4.^{6}\quad \mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1}\mathbf{E}_\nu(z) - 2(\pi z)^{-1}(1 - \cos\nu\pi).$$

EH II 36(43), WA 340(5)

8.583

Asymptotic expansions:

$$1.^{6}\quad \mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin\nu\pi}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)}\frac{\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n} + \right.$$
$$\left. + O(|z|^{-2p}) - \nu\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1} + \nu O(|z|^{-2p-1})\right]$$
$$[|\arg z| < \pi].$$

EH II 37(47), WA 344(1)

$$2.\quad \mathbf{E}_\nu(z) = -N_\nu(z) -$$
$$-\frac{1+\cos(\nu\pi)}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n} + O(|z|^{-2p})\right] -$$
$$-\frac{\nu(1-\cos\nu\pi)}{\ldots}\times$$

For the asymptotic expansion of $J_\nu(z)$ and $N_\nu(z)$, see 8.451.

8.584

The Anger and Weber functions satisfy the differential equation

$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right) y = f(\nu, z),$$

where $f(\nu, z) = \frac{z-\nu}{\pi z^2} \sin \nu\pi$ for $\mathbf{J}_\nu(z)$

WA 341(9), EH II 37(44)

and $f(\nu, z) = -\frac{1}{\pi z^2}[z + \nu + (z - \nu) \cos \nu\pi]$ for $\mathbf{E}_\nu(z)$

EH II 37(45), WA 341(10)

8.59 Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$

8.590

Definition of Neumann's polynomials

1. $$O_n(z) = \frac{1}{4} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n-1} \quad [n \geq 1].$$

WA 299(2), EH II 33(6)

2. $$O_{-n}(z) = (-1)^n O_n(z) \qquad [n \geq 1].$$

WA 303(8)

3. $$O_0(z) = \frac{1}{z}.$$

WA 299(3), EH II 33(7)

4. $$O_1(z) = \frac{1}{z^2}.$$

EH II 33(7)

EH II 33(7)

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$.

8.591

Functional relations:

1. $O_0'(z) = -O_1(z)$.

EH II 33(9), WA 301(3)

2. $2O_n'(z) = O_{n-1}(z) - O_{n+1}(z) \qquad [n \geq 1]$.

EH II 33(10), WA 301(2)

3. $(n-1)O_{n+1}(z)+(n+1)O_{n-1}(z)-2z^{-1}(n^2-1)O_n(z) = 2nz^{-1}\left(\sin n\frac{\pi}{2}\right)^2 \qquad [n \geq 1]$.

EH II 33(11), WA 301(1)

4. $nzO_{n-1}(z) - (n^2-1)O_n(z) = (n-1)zO_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2$.

EH II 33(12), WA 303(4)

5. $nzO_{n+1}(z) - (n^2-1)O_n(z) = -(n+1)zO_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2$.

EH II 33(13), WA 303(5)a

8.592

The generating function:

$$\frac{1}{z-\xi} = J_0(\xi)z^{-1} + 2\sum_{n=1}^{\infty} J_n(\xi)O_n(z) \qquad [|\xi| < |z|].$$

EH II 32(1), WA 298(1)

8.593

The integral representation:

$$O_n(z) = \int_0^\infty \frac{[u + \sqrt{u^2+z^2}]^n + [u - \sqrt{u^2+z^2}]^n}{2z^{n+1}} e^{-u} du.$$

See also 3.547 6., 8., 3.549 1., 2.

EH II 32(3), WA 305(1)

8.594

The inequality

$$|O_n(z)| \leq 2^{n-1} n! |z|^{-n-1} e^{\frac{1}{4}|z|^2} \qquad [n > 1].$$

EH II 33(8), WA 300(8)

8.595

Neumann's polynomial $O_n(z)$ satisfies the differential equation

$$z^2 \frac{d^2 y}{dz^2} + 3z \frac{dy}{dz} + (z^2 + 1 - n^2) y = z \left(\cos n \frac{\pi}{2}\right)^2 + n \left(\sin n \frac{\pi}{2}\right)^2 .$$

EH II 33(14), WA 303(1)

8.596

Schläfli's polynomials $S_n(z)$. These are the functions that satisfy the formulas

1. $S_0(z) = 0.$

EH II 34(18), WA 312(2)

2. $$S_n(z) = \frac{1}{n}\left[2zO_n(z) - 2\left(\cos n\frac{\pi}{2}\right)^2\right] \; [n \geq 1];$$
$$= \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n} \qquad [n \geq 1].$$

EH II 34(18)

EH II 34(19), WA 312(3)

3. $S_{-n}(z) = (-1)^{n+1} S_n(z).$

WA 313(6)

1006
8.597

Functional relations:

1. $S_{n-1}(z) + S_{n+1}(z) = 4O_n(z).$

WA 313(7)

Other functional relations may be obtained from 8.591 by replacing $O_n(z)$ with the expression for $S_n(z)$ given by 8.596 2.

8.6 Mathieu Functions

8.60 Mathieu's equation

$$\frac{d^2y}{dz^2} + (a - 2k^2 \cos 2z)y = 0, \quad k^2 = q.$$

MA

8.61 Periodic Mathieu functions

8.610

In general, Mathieu's equation 8.60 does not have periodic solutions. If k is a real number, there exist infinitely many *eigenvalues a*, not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z),$$

If k is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions.*

8.611

Mathieu's equation has four series of distinct periodic solutions:

1. $\mathrm{ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz.$

MA

2. $\mathrm{ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z.$

MA

3. $\mathrm{se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z.$

MA

4. $\mathrm{se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z.$

MA

5. The coefficients A and B depend on q. The eigenvalues a of the functions ce_{2n}, ce_{2n+1}, se_{2n}, se_{2n+1} are denoted by a_{2n}, a_{2n+1}, b_{2n}, b_{2n+1}.

8.612

The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2 dx = \pi.$$

MO 65

8.613

1. $\lim_{q \to 0} \mathrm{ce}_0(x) = \frac{1}{\sqrt{2}}.$

2. $\lim_{q\to 0} \mathrm{ce}_n(x) = \cos nx \qquad [n \neq 0]$

3. $\lim_{q\to 0} \mathrm{se}_n(x) = \sin nx.$

MO 65

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$

8.621

1. $aA_0^{(2n)} - qA_2^{(2n)} = 0.$

MA

2. $(a-4)A_2^{(2n)} - q(A_4^{(2n)} + 2A_0^{(2n)}) = 0.$

MA

3. $(a-4r^2)A_{2r}^{(2n)} - q(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}) = 0 \qquad [r \geq 2].$

MA

8.622

1. $(a-1-q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0.$

MA

2. $[a-(2r+1)^2]A_{2r+1}^{(2n+1)} - q(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)}) = 0 \qquad [r \geq 1].$

MA

8.623

1. $(a-1+q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0.$

MA

MA

8.624

1. $(a-4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0.$

MA

2. $(a-4r^2)B_{2r}^{(2n+2)} - q(B_{2r+2}^{(2n+2)} - B_{2r-2}^{(2n+2)}) = 0 \qquad [r \geq 2].$

MA

8.625

We can determine the coefficients A and B from equations 8.612, 8.613 and 8.621-8.624 provided a is known. Suppose, for example, that we need to determine the coefficients $A_{2r}^{(2n)}$ for the function $\mathrm{ce}_{2n}(z,q)$. From the recursion formulas, we have

1. $$\begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a-4 & -q & 0 & 0 & \dots \\ 0 & -q & a-16 & -q & 0 & \dots \\ 0 & 0 & -q & a-36 & -q & \dots \\ 0 & 0 & 0 & -q & a-64 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

ST

For given q in equation 8.625 1., we may determine the eigenvalues

2. $a = a_0, a_2, a_4, \dots [|a_0| \leq |a_2| \leq |a_4| \leq \dots].$

If we now set $a = a_{2n}$, we can determine the coefficients $A_{2r}^{(2n)}$ from the recursion formulas 8.621 up to a proportionality coefficient. This coefficient is determined from the formula

3. $2[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty}[A_{2r}^{(2n)}]^2 = 1,$

MA

which follows from the conditions of normalization.

1008

8.63 Mathieu functions with a purely imaginary argument

8.630

If, in equation 8.60, we replace z with iz, we arrive at the differential equation

$$1.\quad \frac{d^2y}{dz^2} + (-a + 2q\operatorname{ch} 2x)y = 0.$$

We can find the solutions of this equation if we replace the argument z with iz in the functions $\operatorname{ce}_n(z,q)$ and $\operatorname{se}_n(z,q)$. The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

$$2.\quad \operatorname{Ce}_{2n}(z,q),\quad \operatorname{Ce}_{2n+1}(z,q),\quad \operatorname{Se}_{2n+1}(z,q),\quad \operatorname{Se}_{2n+2}(z,q).$$

8.631

$$1.\quad \operatorname{Ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \operatorname{ch} 2rz.$$

MA

$$2.\quad \operatorname{Ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \operatorname{ch}(2r+1)z.$$

MA

$$3.\quad \operatorname{Se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \operatorname{sh}(2r+1)z.$$

MA

$$4.\quad \operatorname{Se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \operatorname{sh}(2r+2)z.$$

Along with each periodic solution of equation 8.60, there exists a second nonperiodic solution that is linearly independent. The nonperiodic solutions are denoted as follows:

$$\mathrm{fe}_{2n}(z,q), \quad \mathrm{fe}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+2}(z,q).$$

Analogously, the second solutions of equation 8.630 1. are denoted by

$$\mathrm{Fe}_{2n}(z,q), \quad \mathrm{Fe}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+2}(z,q).$$

8.65 Mathieu functions for negative q

8.651

If we replace the argument z in equation 8.60 with $\pm\left(\frac{\pi}{2} \pm z\right)$, we get the equation

$$\frac{d^2y}{dz^2} + (a + 2q\cos 2z)y = 0.$$

MA

This equation has the following solutions:

1009

8.652

1. $$\mathrm{ce}_{2n}(z,-q) = (-1)^n \,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi - z, q\right).$$

MA

2. $$\mathrm{ce}_{2n+1}(z,-q) = (-1)^n \,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

MA

3. $$\mathrm{se}_{2n+1}(z,-q) = (-1)^n \,\mathrm{ce}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

4. $\mathrm{se}_{2n+2}(z,-q) = (-1)^n \, \mathrm{se}_{2n+2}\left(\frac{1}{2}\pi - z, q\right).$

MA

5. $\mathrm{fe}_{2n}(z,-q) = (-1)^{n+1} \, \mathrm{fe}_{2n}\left(\frac{1}{2}\pi - z, q\right).$

MA

6. $\mathrm{fe}_{2n+1}(z,-q) = (-1)^n \, \mathrm{ge}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$

MA

7. $\mathrm{ge}_{2n+1}(z,-q) = (-1)^n \, \mathrm{fe}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$

MA

8. $\mathrm{ge}_{2n+2}(z,-q) = (-1)^n \, \mathrm{ge}_{2n+2}\left(\frac{1}{2}\pi - z, q\right).$

MA

8.653

Analogously, if we replace z with $\frac{\pi}{2}i + z$ in equation 8.630 1. we get the equation

$$\frac{d^2y}{dz^2} - (a + 2q \operatorname{ch} z)y = 0.$$

It has the following solutions:

8.654

1. $\mathrm{Ce}_{2n}(z,-q) = (-1)^n \, \mathrm{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right).$

MA

2. $\mathrm{Ce}_{2n+1}(z,-q) = (-1)^{n+1} i \, \mathrm{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right).$

MA

3. $\operatorname{Se}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ce}_{2n+1}\left(\frac{\pi}{2} i + z, q\right).$

MA

4. $\operatorname{Se}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Se}_{2n+2}\left(\frac{\pi}{2} i + z, q\right).$

MA

5. $\operatorname{Fe}_{2n}(z, -q) = (-1)^{n} \operatorname{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right).$

MA

6. $\operatorname{Fe}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ge}_{2n+1}\left(\frac{\pi}{2} i + z, q\right).$

MA

7. $\operatorname{Ge}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Fe}_{2n+1}\left(\frac{\pi}{2} i + z, q\right).$

MA

8. $\operatorname{Ge}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Ge}_{2n+2}\left(\frac{\pi}{2} i + z, q\right).$

MA

8.66 Representation of Mathieu functions as series of Bessel functions

8.661

1. $$\operatorname{ce}_{2n}(z, q) = \frac{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_{2r}(2k \cos z);$$
$$= \frac{\operatorname{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(2k \sin z).$$

2. $$\mathrm{ce}_{2n+1}(z,q) = -\frac{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{kA_1^{(2n+1)}}\sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)} J_{2r+1}(2k\cos z);$$
$$= \frac{\mathrm{ce}_{2n+1}(0,q)}{kA_1^{(2n+1)}}\operatorname{ctg} z\sum_{r=0}^{\infty}(-1)^r(2r+1)A_{2r+1}^{(2n+1)} I_{2r+1}(2k\sin z).$$

MA
MA

3. $$\mathrm{se}_{2n+1}(z,q) = \frac{\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{kB_1^{(2n+1)}}\operatorname{tg} z\sum_{r=0}^{\infty}(-1)^r(2r+1)B_{2r+1}^{(2n+1)} J_{2r+1}(2k\cos z);$$
$$= \frac{\mathrm{se}'_{2n+1}(0,q)}{kB_1^{(2n+1)}}\sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(2k\sin z).$$

MA
MA

4. $$\mathrm{se}_{2n+2}(z,q) = \frac{-\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{k^2B_2^{(2n+2)}}\operatorname{tg} z\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}^{(2n+2)} J_{2r+2}(2k\cos z);$$
$$= \frac{\mathrm{se}'_{2n+2}(0,q)}{k^2B_2^{(2n+2)}}\operatorname{ctg} z\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}^{(2n+2)} I_{2r+2}(2k\sin z).$$

MA
MA

8.662

1. $$\mathrm{fe}_{2n}(z,q) = -\frac{\pi\,\mathrm{fe}'_{2n}(0,q)}{2\,\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}\,\mathrm{Im}[J_r(ke^{iz})N_r(ke^{-iz})].$$

MA

2. $$\mathrm{fe}_{2n+1}(z,q) = \frac{\pi k\,\mathrm{fe}'_{2n+1}(0,q)}{2\,\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}\times$$
$$\times\sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\,\mathrm{Im}[J_r(ke^{iz})N_{r+1}(ke^{-iz})+J_{r+1}(ke^{iz})N_r(ke^{-iz})].$$

MA

3. $$\mathrm{ge}_{2n+1}(z,q) = -\frac{\pi k\,\mathrm{ge}_{2n+1}(0,q)}{2\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}\times$$
$$\times\sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}\,\mathrm{Re}[J_r(ke^{iz})N_{r+1}(ke^{-iz})-J_{r+1}(ke^{iz})N_r(ke^{-iz})].$$

4. $$\mathrm{ge}_{2n+2}(z,q) = -\frac{\pi k^2 \,\mathrm{ge}_{2n+2}(0,q)}{2\,\mathrm{se}'_{2n+2}\left(\frac{1}{2}\pi, q\right)} \times$$
$$\times \sum_{r=0}^{\infty} (-1)^r \operatorname{Re}[J_k(ke^{iz})N_{r+2}(ke^{-iz}) - J_{r+2}(ke^{iz})N_r(ke^{-iz})].$$

MA

The expansions of the functions Fe_n and Ge_n as series of the functions N_ν are denoted, respectively, by Fey_n and Gey_n and the expansions of these functions as series of the functions K_ν are denoted, respectively, by Fek_n and Gek_n.

8.663

1. $$\mathrm{Fey}_{2n}(z,q) = \frac{\mathrm{ce}_{2n}(0,q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} N_{2r}(2k \operatorname{sh} z), \quad k^2 = q[|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0];$$
$$= \frac{\mathrm{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} N_{2r}(2k \operatorname{ch} z) \qquad [|\operatorname{ch} z| > 1];$$
$$= \frac{\mathrm{ce}_{2n}(0,q)\,\mathrm{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{[A_0^{(2n)}]^2} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(ke^{-z}) N_r(ke^z).$$

MA
MA
MA

2. $$\mathrm{Fey}_{2n+1}(z,q) = \frac{\mathrm{ce}_{2n+1}(0,q)\operatorname{cth} z}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(2n+1)} N_{2r+1}(2k \operatorname{sh} z), \quad k^2 = q$$
$$[|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0];$$
$$= -\frac{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} N_{2r+1}(2k \operatorname{ch} z) \qquad [|\operatorname{ch} z| > 1];$$
$$= -\frac{\mathrm{ce}_{2n+1}(0,q)\,\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k[A_1^{(2n+1)}]^2} \times$$
$$\times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} [J_r(ke^{-z}) N_{r+1}(ke^z) + J_{r+1}(ke^{-z}) N_r(ke^z)].$$

MA
MA
MA

3. $$\mathrm{Gey}_{2n+1}(z,q) = \frac{\mathrm{se}'_{2n+1}(0,q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} N_{2r+1}(2k \operatorname{sh} z) \qquad [|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0];$$
$$= \frac{\mathrm{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \theta z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} N_{2r+1}(2k \operatorname{ch} z) \qquad [|\operatorname{ch} z| > 1];$$
$$= \frac{\mathrm{se}_{2n+1}(0,q)\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k[B_1^{(2n+1)}]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} [J_r(ke^{-z}) N_{r+1}(ke^z) - J_{r+1}(ke^{-z}) N_r(ke^z)].$$

MA
MA
MA

4. $$\mathrm{Gey}_{2n+2}(z,q) = \frac{\mathrm{se}'_{2n+2}(0,q)}{k^2 B_2^{(2n+2)}} \operatorname{cth} z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} N_{2r+2}(2k \operatorname{sh} z) \qquad [|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0];$$
$$= -\frac{\mathrm{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \theta z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} N_{2r+2}(2k \operatorname{ch} z) \qquad [|\operatorname{ch} z| > 1];$$
$$= \frac{\mathrm{se}'_{2n+2}(0,q)\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2[B_2^{(2n+2)}]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} [J_r(ke^{-z}) N_{r+2}(ke^z) - J_{r+2}(ke^{-z}) N_r(ke^z)].$$

MA
MA
MA

8.664

1. $$\mathrm{Fek}_{2n}(z,q) = \frac{\mathrm{ce}_{2n}(0,q)}{\pi A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} K_{2r}(-2ik \operatorname{sh} z), \quad k^2 = q \qquad [|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0].$$

MA

2. $$\mathrm{Fek}_{2n+1}(z,q) = \frac{\mathrm{ce}_{2n+1}(0,q)}{\pi k A_1^{(2n+1)}} \operatorname{cth} z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \operatorname{sh} z), \quad k^2 = q \qquad [|\operatorname{sh} z| > 1, \quad \operatorname{Re} z > 0]$$

MA

MA

4. $\mathrm{Gek}_{2n+2}(z,q) = \dfrac{\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{\pi k^2 B_2^{(2n+2)}} \theta z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} K_{2r+2}(-2ik\,\mathrm{ch}\,z).$

MA

8.67 The general theory

If $i\mu$ is not an integer, the general solution of equation 8.60 can be found in the form

1013

8.671

1. $y = Ae^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} + Be^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rzi}.$

MA

The coefficients c_{2r} can be determined from the homogeneous system of linear algebraic equations

2. $c_{2r} + \xi_{2r}(c_{2r+2} + c_{2r-2}) = 0, r = \dots, -2, -1, 0, 1, 2, 2\dots,$

MA

where

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}.$$

The condition that this system be compatible yields an equation that μ must satisfy:

3.[7] $\Delta(i\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$

4. $\operatorname{ch}\mu\pi = 1 - 2\Delta(0)\sin^2\left(\frac{\pi\sqrt{a}}{2}\right)$, where $\Delta(0)$ is the value that is assumed by the determinant of the preceding article if we set $\mu = 0$ in the expressions for ξ_{2r}.

5. If the pair (a, q) is such that $|\operatorname{ch}\mu\pi| < 1$, then $\mu = i\beta, \operatorname{Im}\beta = 0$, and the solution 8.671 1. is bounded on the real axis.

6. If $|\operatorname{ch}\mu\pi| > 1$, μ may be real or complex and the solution 8.671 1. will not be bounded on the real axis.

7. If $\operatorname{ch}\mu\pi = \pm 1$, $i\mu$ will be an integer. In this case, one of the solutions will be of period π or 2π (depending on whether n is even or odd). The second solution is nonperiodic (see 8.61 and 8.64).

8.7- 8.8 Associated Legendre Functions

8.70 Introduction

8.700

An *associated Legendre function* is a solution of the differential equation

$$1. \quad (1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]u = 0,$$

in which ν and μ are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see 9.151). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

1014

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable z that lie in the interval $[-1, 1]$ and, on the other hand, in solutions corresponding to an arbitrary complex number z such that $\operatorname{Re} z > 1$. These are multiple-valued in the z-plane. To separate these functions into single-valued branches, we make a cut along the real axis from

$-\infty$ to $+1$. We are also interested in those solutions of equation 8.700 1. for which ν or μ or both are integers. Of especial significance is the case in which $\mu = 0$.

8.701

In connection with this, we shall use the following notations:

The letter z will denote *an arbitrary complex variable*; the letter x will denote a *real* variable that varies over the interval $[-1, +1]$. We shall sometimes set $x = \cos\varphi$, where φ is a real number.

We shall use the symbols $P_\nu^\mu(z), Q_\nu^\mu(z)$ to denote those solutions of equation 8.700 1., that are single-valued and regular for $|z| < 1$ and, in particular, uniquely determined for $z = x$.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation 8.700 1. that are single-valued and regular for Re $z > 1$. When these functions cannot be unrestrictedly extended without violating their single-valuedness we make a cut along the real axis to the left of the point $z = 1$. The values of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ on the upper and lower boundaries of that portion of the cuts lying between the points -1 and $+1$ are denoted respectively by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters n and m denote natural numbers or zero. The letters ν and μ denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z).$$

The *linearly independent* functions

8.702

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1-\mu; \quad \frac{1-z}{2}\right)$$
$$\left[\arg\frac{z+1}{z-1} = 0, \text{if } z \text{ is real and greater than } 1\right] \text{and}$$

$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}(z^2-1)^{\frac{\mu}{2}}z^{-\nu-\mu-1}F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$[\arg(z^2-1)=0$ when z is real and greater than 1; $\arg z = 0$ when z is real and greater than zero] which are solutions of the differential equation 8.700 1., are called *associated Legendre functions* (or *spherical functions*) of *the first* and *second kinds* respectively. They are uniquely defined, respectively, in the intervals $|1-z|<2$ and $|z|>1$ with the portion of the real axis that lies between $-\infty$ and $+1$ excluded. They can be extended by means of hypergeometric series to the entire z-plane where the above-mentioned cut was made. These expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ lose their meaning when $1-\mu$ and $\nu+\frac{3}{2}$ are nonpositive integers respectively.

MO 80

1015

When z is a real number lying on the interval $[-1,+1]$, so that $(z=x=\cos\varphi)$, we take the following functions as linearly independent Solutions of the equation

8.704

$$P_\nu^\mu(x) = \frac{1}{2}\left[e^{\frac{1}{2}\mu\pi i}P_\nu^\mu(\cos\varphi+i0) + e^{-\frac{1}{2}\mu\pi i}P_\nu^\mu(\cos\varphi-i0)\right] ;$$
$$= \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right) .$$

EH I 143(6)
EH I 143(1)

8.705

$$Q_\nu^\mu(x) = \frac{1}{2}e^{-\mu\pi i}\left[e^{-\frac{1}{2}\mu\pi i}Q_\nu^\mu(x+i0) + e^{\frac{1}{2}\mu\pi i}Q_\nu^\mu(x-i0)\right] ;$$
$$= \frac{\pi}{2\sin\mu\pi}\left[P_\nu^\mu(x)\cos\mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(x)\right] \qquad \text{(cf. 8.732 5.)}$$

8.732
EH I 143(2)

If $\mu = \pm m$ is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

8.706

1. $$Q_\nu^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_\nu(x) \qquad (\text{cf. } 8.752\ 1).$$

8.752
EH I 149(7)

2. $$Q_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} Q_\nu^m(x).$$

EH I 144(18)

The functions $Q_\nu^\mu(z)$ are not defined when $\nu+\mu$ is equal to a negative integer. Therefore, we must exclude the cases when $\nu+\mu = -1, -2, -3, \ldots$ for these formulas.

The functions

$$P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z).$$

are *linearly independent solutions* of the differential equation for $\nu+\mu \neq 0, \pm 1, \pm 2, \ldots$.

8.707

Nonetheless, two linearly independent solutions can always be found. Specifically, for $\nu \pm \mu$ not an integer, the differential equation 8.700 1. has the following solutions:

1. $$P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

respectively, for $z = x = \cos\varphi$,

2. $$P_\nu^{\pm\mu}(\pm x), \quad Q_\nu^{\pm\mu}(\pm x), \quad P_{-\nu-1}^{\pm\mu}(\pm x), \quad Q_{-\nu-1}^{\pm\mu}(\pm x).$$

If $\nu \pm \mu$ is not an integer, the solutions

$\nu \pm \mu$

$$3.\quad P_\nu^\mu(z),\quad Q_\nu^\mu(z),\ \text{respectively, and}\ P_\nu^\mu(x),\quad Q_\nu^\mu(x)$$

are linearly independent. If $\nu \pm \mu$ is an integer but μ itself is not an integer, the following functions are linearly independent solutions of equation 8.700 1.:

$$4.\quad P_\nu^\mu(z),\quad P_\nu^{-\mu}(z),\ \text{respectively, and}\ P_\nu^\mu(x),\quad P_\nu^{-\mu}(x).$$

If $\mu = \pm m, \nu = n$, or $\nu = -n-1$, the following functions are linearly independent solutions of equation 8.700 1. for $n \geq m$:

1016

$$5.\quad P_n^m(z),\quad Q_n^m(z),\ \text{respectively, and}\ P_n^m(x),\quad Q_n^m(x),$$

and for $n < m$, the following functions will be linearly independent solutions

$$6.\quad P_n^{-m}(z),\quad Q_n^m(z),\ \text{respectively, and}\ P_n^{-m}(x),\quad Q_n^m(x).$$

8.71 Integral representations

8.711

$$1.\quad P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\frac{\mu}{2}}}{2^\mu\sqrt{\pi}\Gamma\left(\mu+\frac{1}{2}\right)}\int_{-1}^{1}\frac{(1-t^2)^{\mu-\frac{1}{2}}}{(z+t\sqrt{z^2-1})^{\mu-\nu}}dt \qquad \left[\operatorname{Re}\mu > -\frac{1}{2},\quad |\arg(z\pm 1)| < \pi\right].$$

MO 88

$$2.\quad P_\nu^m(z) = \frac{(\nu+1)(\nu+2)\dots(\nu+m)}{\pi}\int_0^\pi\left[z+\sqrt{z^2-1}\cos\varphi\right]^\nu\cos m\varphi d\varphi;$$
$$= (-1)^m\frac{\nu(\nu-1)\dots(\nu-m+1)}{\pi}\int_0^\pi\frac{\cos m\varphi d\varphi}{\left[z+\sqrt{z^2-1}\cos\varphi\right]^{\nu+1}}$$
$$\left[|\arg z| < \frac{\pi}{2},\quad \arg(z+\sqrt{z^2-1}\cos\varphi) = \arg z \quad \text{for} \quad \varphi = \frac{\pi}{2}\right] \quad (\text{cf. } 8.822\ 1.).$$

8.822
SM 483(15), WH

$$3.\quad Q^{\mu}_{\nu}(z)=\sqrt{\pi}\frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)}{2^{\mu}\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu+1)}(z^2-1)^{\frac{\mu}{2}}\int_0^{\infty}\frac{\operatorname{sh}^{2\mu}t\,dt}{(z+\sqrt{z^2-1}\operatorname{ch}t)^{\nu+\mu+1}}$$

$$[\operatorname{Re}(\nu\pm\mu)>-1,\quad |\arg(z\pm 1)<\pi]\qquad (\text{cf. } 8.822\ 2.).$$

8.822
MO 88

$$4.\quad Q^{\mu}_{\nu}(z)=\frac{e^{\mu\pi i}\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^{\infty}\frac{\operatorname{ch}\mu t\,dt}{(z+\sqrt{z^2-1}\operatorname{ch}t)^{\nu+1}}$$

$$[\operatorname{Re}(\nu+\mu)>-1,\nu\neq-1,-2,-3,\ldots,\quad |\arg(z\pm 1)|<\pi]\ .$$

WH, MO 88

$$5.\quad \int_{-1}^{1}P^2_l(x)P^0_l(x)dx=-\frac{l!}{(l-2)!}\frac{1}{2l+1}=-\frac{l(l-1)}{2l+1}.$$

8.712

$$Q^{\mu}_{\nu}(z)=\frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+1)}(z^2-1)^{-\frac{\mu}{2}}\int_{-1}^{1}(1-t^2)^{\nu}(z-t)^{-\nu-\mu-1}dt$$

$$[\operatorname{Re}(\nu+\mu)>-1,\quad \operatorname{Re}\mu>-1,\quad |\arg(z\pm 1)|<\pi]\qquad (\text{cf. } 8.821\ 2.).$$

8.821
MO 88a, EH I 155(5)a

8.713

$$1.\quad Q^{\mu}_{\nu}(z)=\frac{e^{\mu\pi i}\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2\pi}}(z^2-1)^{\frac{\mu}{2}}\left\{\int_0^{\pi}\frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(z-\cos t)^{\mu+\frac{1}{2}}}-\cos\nu\pi\int_0^{\infty}\frac{e^{-(\nu+\frac{1}{2})t}dt}{(z+\operatorname{ch}t)^{\mu+\frac{1}{2}}}\right\}$$

$$\left[\operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}(\nu+\mu)>-1,\quad |\arg(z\pm 1)|<\pi\right]\ .$$

2. $P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\frac{\mu}{2}}}{2^\nu\Gamma(\mu-\nu)\Gamma(\nu+1)}\int_0^\infty \frac{\operatorname{sh}^{2\nu+1} t}{(z+\operatorname{ch} t)^{\nu+\mu+1}}dt$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+1) > 0, \quad \operatorname{Re}(\mu-\nu) > 0]$.

MO 89

3. $P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)(z^2-1)^{\frac{\mu}{2}}}{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}\int_0^\infty \frac{\operatorname{ch}\left(\nu+\frac{1}{2}\right)t dt}{(z+\operatorname{ch} t)^{\mu+\frac{1}{2}}}$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$.

MO 89

8.714

1. $P_\nu^{\mu}(\cos\varphi) = \sqrt{\frac{2}{\pi}}\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\varphi \frac{\cos\left(\nu+\frac{1}{2}\right)t dt}{(\cos t-\cos\varphi)^{\mu+\frac{1}{2}}} \qquad \left[0<\varphi<\pi, \operatorname{Re}\mu < \frac{1}{2}\right]; \quad \text{(cf. 8.823)}$

8.823
MO 87

2. $P_\nu^{-\mu}(\cos\varphi) = \frac{\Gamma(2\mu+1)\sin^\mu\varphi}{2^\mu\Gamma(\mu+1)\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}\int_0^\infty \frac{t^{\nu+\mu}dt}{(1+2t\cos\varphi+t^2)^{\mu+\frac{1}{2}}}$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$.

MO 89

3. $Q_\nu^{\mu}(\cos\varphi) = \frac{1}{2^{\mu+1}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\times$
$\times\int_0^\infty\left[\frac{\operatorname{sh}^{2\mu} t}{(\cos\varphi+i\sin\varphi\operatorname{ch} t)^{\nu+\mu+1}}+\frac{\operatorname{sh}^{2\mu} t}{(\cos\varphi-i\sin\varphi\operatorname{ch} t)^{\nu+\mu+1}}\right]dt$
$\left[\operatorname{Re}(\nu+\mu+1) > 0, \quad \operatorname{Re}(\nu-\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$.

MO 89

1018

4. $P_\nu^{\mu}(\cos\varphi) = \frac{i}{2^{\mu}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\times$
$\times\int_0^\infty\left[\frac{\operatorname{sh}^{2\mu} t}{(\cos\varphi+i\sin\varphi\operatorname{ch} t)^{\nu+\mu+1}}-\frac{\operatorname{sh}^{2\mu} t}{(\cos\varphi-i\sin\varphi\operatorname{ch} t)^{\nu+\mu+1}}\right]dt$
$\left[\operatorname{Re}(\nu\pm\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$.

1. $P_\nu^\mu(\operatorname{ch}\alpha) = \frac{\sqrt{2}\operatorname{sh}^\mu\alpha}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\alpha \frac{\operatorname{ch}\left(\nu+\frac{1}{2}\right)t\,dt}{(\operatorname{ch}\alpha - \operatorname{ch}t)^{\mu+\frac{1}{2}}} \qquad \left[\alpha > 0, \quad \operatorname{Re}\mu < \frac{1}{2}\right].$

MO 87

2. $Q_\nu^\mu(\operatorname{ch}\alpha) = \sqrt{\frac{\pi}{2}}\frac{e^{\mu\pi i}\operatorname{sh}^\mu\alpha}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\alpha^\infty \frac{e^{-\left(\nu+\frac{1}{2}\right)t}dt}{(\operatorname{ch}t - \operatorname{ch}\alpha)^{\mu+\frac{1}{2}}} \qquad \left[\alpha > 0, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re}(\nu+\mu) > -1\right].$

MO 87

See also 3.277 1., 4., 5., 7., 3.318, 3.516 3., 3.518 1., 2., 3.542 2., 3.663 1., 3.894, 3.988 3., 6.622 3., 6.628 1., 4.-7., and also 8.742.

8.72 Asymptotic series for large values of |ν|

8.721

[6] For real values of $\mu, |\nu| \gg 1, |\nu| \gg |\mu|, |\arg\nu| < \pi$, we have:

1. $P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}}\Gamma(\nu+\mu+1)\sum_{k=0}^\infty \frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)} \frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi + \frac{\pi}{4}(2k-1) + \frac{\mu\pi}{2}\right]}{k!\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$

$\left[\nu+\mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \frac{7}{2}\dots; \quad \text{for} \quad \frac{\pi}{6} < \varphi < \frac{5\pi}{6}\right.$

this series also converges for complex values of ν and μ. In the remaining cases, it is an asymptotic expansion

$\left.\text{for} \quad |\nu| \gg |\mu|, |\nu| \gg 1, \quad \text{if} \quad \nu > 0, \mu > 0 \quad \text{and} \quad 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon\right].$

MO 92

2.[6] $Q_\nu^\mu(\cos\varphi) = \sqrt{\pi}\Gamma(\nu+\mu+1) \times$

$\times \sum_{k=0}^\infty (-1)^k \frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)} \frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi - \frac{\pi}{4}(2k-1) + \frac{\mu\pi}{2}\right]}{k!\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$

$\left[\nu+\mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \quad \text{for} \quad \frac{\pi}{6} < \varphi < \frac{5}{6}\pi\right.$

$\nu \quad \mu$

$$\text{for} \quad |\nu| \gg |\mu|, \quad |\nu| \gg 1, \quad \text{if} \quad \nu > 0, \quad \mu > 0, \quad 0 < \varepsilon \le \varphi \le \pi - \varphi] \ .$$

EH I 147(6), MO 92

$$3. \quad P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} \frac{\cos\left[\left(\nu+\frac{1}{2}\right)\varphi - \frac{\pi}{4} + \frac{\mu\pi}{2}\right]}{\sqrt{2\sin\varphi}} \left[1 + O\left(\frac{1}{\nu}\right)\right]$$

$$\left[0 < \varepsilon \le \varphi \le \pi - \varepsilon, |\nu| \gg \frac{1}{\varepsilon}\right] .$$

MO 92

For $\nu > 0, \mu > 0$ and $\nu > \mu$, it follows from formulas 8.721 1. and 8.721 2. that

$$4. \quad \nu^{-\mu} P_\nu^\mu(\cos\varphi) = \sqrt{\frac{2}{\nu\pi\sin\varphi}} \cos\left[\left(\nu+\frac{1}{2}\right)\varphi - \frac{\pi}{4} + \frac{\mu\pi}{2}\right] + O\left(\frac{1}{\sqrt{\nu^3}}\right) .$$

$$5. \quad \nu^{-\mu} Q_\nu^\mu(\cos\varphi) = \sqrt{\frac{\pi}{2\nu\sin\varphi}} \cos\left[\left(\nu+\frac{1}{2}\right)\varphi + \frac{\pi}{4} + \frac{\mu\pi}{2}\right] O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

$$\left[0 < \varepsilon \le \varphi \le \pi - \varepsilon; \quad \nu \gg \frac{1}{\varepsilon}\right] .$$

MO 92

8.722

If φ is sufficiently close to 0 or π that $\nu\varphi$ or $\nu(\pi - \varphi)$ is small in comparison with 1, the asymptotic formulas 8.721 become unsuitable. In this case, the following asymptotic representation is applicable for $\mu \le 0, \nu \gg 1$, and *small* values of φ:

$$1. \quad \left[\left(\nu+\frac{1}{2}\right)\cos\frac{\varphi}{2}\right]^\mu P_\nu^{-\mu}(\cos\varphi) = J_\mu(\eta) + \sin^2\frac{\varphi}{2}\left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6} J_{\mu+3}(\eta)\right] + O\left(\sin^4\frac{\varphi}{2}\right),$$

where $\eta = (2\nu+1)\sin\frac{\varphi}{2}$. In particular, it follows that

$$2. \quad \lim_{\nu\to\infty} \nu^\mu P_\nu^{-\mu}\left(\cos\frac{x}{\nu}\right) = J_\mu(x)\,[x \ge 0, \mu \ge 0] .$$

MO 93

8.723

We can see how the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ behave for large $|\nu|$and real values of $z > \frac{3}{2\sqrt{2}}$:

$$1.\quad P_\nu^\mu(\operatorname{ch}\alpha) = \frac{2^\mu}{\sqrt{\pi}}\left\{\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}\frac{e^{(\mu-\nu)\alpha}\operatorname{sh}^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}}F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};\nu+\frac{3}{2};\frac{1}{1-e^{2\alpha}}\right)\right. +$$
$$\left. + \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{e^{(\nu+\mu+1)\alpha}\operatorname{sh}^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}}F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};-\nu+\frac{1}{2};\frac{1}{1-e^{2\alpha}}\right)\right\}$$
$$\left[\nu\neq\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\dots;\quad a>\frac{1}{2}\ln 2\right]$$

MO 94

$$2.\quad Q_\nu^\mu(\operatorname{ch}\alpha) = e^{\mu\pi i}2^\mu\sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{e^{-(\nu+\mu+1)\alpha}}{(1-e^{-2\alpha})^{\mu+\frac{1}{2}}}\operatorname{sh}^\mu\alpha\times$$
$$\times F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};\nu+\frac{3}{2};\frac{1}{1-e^{2\alpha}}\right)$$
$$\left[\mu+\nu+1\neq 0,\quad -1,-2,\dots;\quad \alpha>\frac{1}{2}\ln 2\right].$$

MO 94

See also 8.776

8.724

The inequalities

$$1.\quad |P_\nu^{\pm\mu}(\cos\varphi)| < \sqrt{\frac{8}{\nu\pi}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi},$$
$$2.\quad |Q_\nu^{\pm\mu}(\cos\varphi)| < \sqrt{\frac{2\pi}{\nu}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi},$$
$$3.\quad |P_\nu^{\pm m}(\cos\varphi)| < \frac{2}{\sqrt{\nu\pi}}\frac{\Gamma(\nu\pm m+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{m+\frac{1}{2}}\varphi},$$
$$4.\quad |Q_\nu^{\pm m}(\cos\varphi)| < \sqrt{\frac{\pi}{\nu}}\frac{\Gamma(\nu\pm m+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{m+\frac{1}{2}}\varphi}$$

[ν and μ are arbitrary real numbers satisfying the inequalities $\nu\geq 1$, $\nu-\mu+$ $+1>0$, $\mu\geq 0$].

$$5.^{8}\quad \left|\sqrt{\sin\theta}P_n^m(\cos\theta)\right| < \frac{\Gamma(n+1/2)}{\Gamma(n-m+1)}2^{(m+n)^2/n}x\operatorname{Sup}_{o<t<\infty}\left|\sqrt{t}J_m(t)\right| \qquad [\text{uniformly } 0\leq m\leq n]$$

1. $$P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}{\pi\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\sin\mu\pi\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\ - \right.$$
$$\left. -\ \frac{\sin\nu\pi}{\sin\mu\pi}e^{\mp i\mu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$

AS 8.10.1

2. $$Q_\nu^\mu(z) = \frac{1}{2}e^{i\mu\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\Gamma(\mu-\nu)\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\ - \right.$$
$$\left. -\ e^{\mp i\nu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$

AS 8.10.2

3. $$Q_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi}\csc\left[\pi(\nu-\mu)\right]}{2\pi\Gamma(1+\mu)}\left[e^{\mp i\nu\pi}\left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\ - \right.$$
$$\left. -\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\nu+1;\quad 1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\right]$$

AS 8.10.3

8.73- 8.74 Functional relations

8.731

1. $$(z^2-1)\frac{dP_\nu^\mu(z)}{dz} = (\nu-\mu+1)P_{\nu+1}^\mu(z)-(\nu+1)zP_\nu^\mu(z) \qquad \text{(cf. 8.832 1., 8.914 2.).}$$

8.914

8.832

EH I 161(10), MO 81

1(1).* $$(z^2-1)\frac{dP_\nu^\mu(z)}{dz} = \nu zP_\nu^\mu(z)-(\nu+\mu)P_{\nu-1}^\mu(z).$$

AS 8.5.4

1(2).* $$(z^2-1)\frac{dP_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)(z^2-1)P_\nu^{\mu-1}(z)-\mu zP_\nu^\mu(z).$$

2. $(2\nu+1)zP_\nu^\mu(z) = (\nu-\mu+1)P_{\nu+1}^\mu(z)+(\nu+\mu)P_{\nu-1}^\mu(z)$ (cf. 8.832 2., 8.914 1.).

8.914

8.832
EH I 160(2), MO 81

3. $P_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}P_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)P_\nu^\mu(z).$

MO 82, EH I 160(1)

3(1).* $P_\nu^{\mu+1}(z) = (z^2-1)^{-\frac{1}{2}}\left[(\nu-\mu)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)\right]$.

AS 8.5.1

4. $P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu+1)\sqrt{z^2-1}P_\nu^{\mu-1}(z).$

EH I 160(3), MO 82

4(1).* $(\nu-\mu+1)P_{\nu+1}^\mu(z) = (2\nu+1)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z).$

AS 334(8.5.3)

4(2).* $P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{\frac{1}{2}}P_\nu^{\mu-1}(z).$

AS 334(8.5.5)

5. $P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$ (cf. 8.820, 8.832 4.).

8.832

8.820
EH I 140(1), MO 82

8.832
MO 82

2.* $(2\nu+1)zQ_\nu^\mu(z) = (\nu-\mu+1)Q_{\nu+1}^\mu(z) + (\nu+\mu)Q_{\nu-1}^\mu(z) \qquad \text{(cf. 8.832 4.)}.$

8.832
MO 82

3. $Q_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}Q_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)Q_\nu^\mu(z).$

MO 82

4. $Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu+1)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z).$

MO 82a

5. $e^{-\mu\pi i}Q_\nu^\mu(x\pm i0) = e^{\pm\frac{1}{2}\mu\pi i}\left[Q_\nu^\mu(x) \mp i\frac{\pi}{2}P_\nu^\mu(x)\right].$

MO 83

8.733

1. $$\begin{aligned}(1-x^2)\frac{dP_\nu^\mu(x)}{dx} &= (\nu+1)xP_\nu^\mu(x) - (\nu-\mu+1)P_{\nu+1}^\mu(x) \qquad \text{(cf. 8.731 1.)};\\ &= -\nu xP_\nu^\mu(x) + (\nu+\mu)P_{\nu-1}^\mu(x);\\ &= -\sqrt{1-x^2}\,P_\nu^{\mu+1}(x) - \mu xP_\nu^\mu(x);\\ &= (\nu-\mu+1)(\nu+\mu)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x) + \mu xP_\nu^\mu(x).\end{aligned}$$

8.731
MO 82

2. $(2\nu+1)xP_\nu^\mu(x) = (\nu-\mu+1)P_{\nu+1}^\mu(x) + (\nu+\mu)P_{\nu-1}^\mu(x) \qquad \text{(cf. 8.731 2.)}.$

3. $P_\nu^{\mu+2}(x)+2(\mu+1)\frac{x}{\sqrt{1-x^2}}P_\nu^{\mu+1}(x)+(\nu-\mu)(\nu+\mu+1)P_n^\mu(x)=0$ (cf. 8.731 3.).

8.731
MO 82

4. $P_{\nu-1}^\mu(x)-P_{\nu+1}^\mu(x)=(2\nu+1)\sqrt{1-x^2}P_\nu^{\mu-1}(x)$ (cf. 8.731 4.).

8.731
MO 82

5. $P_{-\nu-1}^\mu(x)=P_\nu^\mu(x)$ (cf.8.731 5.).

8.731

8.734

1. $(\nu+\mu+1)zQ_\nu^\mu(z)+\sqrt{z^2-1}Q_\nu^{\mu+1}(z)=(\nu-\mu+1)Q_{\nu+1}^\mu(z).$

MO 82

2. $(\nu+\mu)Q_{\nu-1}^\mu(z)+\sqrt{z^2-1}Q_\nu^{\mu+1}(z)=(\nu-\mu)zQ_\nu^\mu(z).$

MO 82

3. $Q_{\nu-1}^\mu(z)-zQ_\nu^\mu(z)=-(\nu-\mu+1)\sqrt{z^2-1}Q_\nu^{\mu-1}(z).$

MO 82

4. $zQ_\nu^\mu(z)-Q_{\nu+1}^\mu(z)=-(\nu+\mu)\sqrt{z^2-1}Q_\nu^{\mu-1}(z).$

MO 82

5. $(\nu+\mu)(\nu+\mu+1)Q_{\nu-1}^\mu(z)+(2\nu+1)\sqrt{z^2-1}Q_\nu^{\mu+1}(z)=(\nu-\mu)(\nu-\mu+1)Q_{\nu+1}^\mu(z).$

8.735

1. $(\nu+\mu+1)xP_\nu^\mu(x)+\sqrt{1-x^2}P_\nu^{\mu+1}(x)=(\nu-\mu+1)P_{\nu+1}^\mu(x).$

MO 83

2. $(\nu-\mu)xP_\nu^\mu(x)-(\nu+\mu)P_{\nu-1}^\mu(x)=\sqrt{1-x^2}P_\nu^{\mu+1}(x).$

MO 83

3. $P_{\nu-1}^\mu(x)-xP_\nu^\mu(x)=(\nu-\mu+1)\sqrt{1-x^2}P_\nu^{\mu-1}(x).$

MO 83

4. $xP_\nu^\mu(x)-P_{\nu+1}^\mu(x)=(\nu+\mu)\sqrt{1-x^2}P_\nu^{\mu-1}(x).$

MO 83

5. $(\nu-\mu)(\nu-\mu+1)P_{\nu+1}^\mu(x)=(\nu+\mu)(\nu+\mu+1)P_{\nu-1}^\mu(x)+(2\nu+1)\sqrt{1-x^2}P_\nu^{\mu+1}(x).$

MO 83

8.736

1. $P_\nu^{-\mu}(z)=\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z)-\frac{2}{\pi}e^{-\mu\pi i}\sin\mu\pi Q_\nu^\mu(z)\right].$

MO 83

1023

2. $P_\nu^\mu(-z)=e^{\nu\pi i}P_\nu^\mu(z)-\frac{2}{\pi}\sin\left[(\nu+\mu)\pi\right]e^{-\mu\pi i}Q_\nu^\mu(z)\qquad[\operatorname{Im} z<0]\qquad$ (cf. 8.833 1.).

8.833
MO 83

3. $P_\nu^\mu(-z)=e^{-\nu\pi i}P_\nu^\mu(z)-\frac{2}{\pi}\sin\left[(\nu+\mu)\pi\right]e^{-\mu\pi i}Q_\nu^\mu(z)\qquad[\operatorname{Im} z>0]\qquad$ (cf. 8.833 2.).

MO 83

4. $Q_\nu^{-\mu}(z) = e^{-2\mu\pi i} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} Q_\nu^\mu(z).$

MO 82

5. $Q_\nu^\mu(-z) = -e^{-\nu\pi i} Q_\nu^\mu(z)\,[\operatorname{Im} z < 0] \qquad \text{(cf. 8.833 3.)}.$

8.833
MO 82

6. $Q_\nu^\mu(-z) = -e^{\nu\pi i} Q_\nu^\mu(z)\,[\operatorname{Im} z > 0] \qquad \text{(cf. 8.833 4.)}.$

8.833
MO 82

7.[6] $Q_\nu^\mu(z) \sin[(\nu+\mu)\pi] - Q_{-\nu-1}^\mu(z) \sin[(\nu-\mu)\pi] = \pi e^{\mu\pi i} \cos\nu\pi P_\nu^\mu(z).$

MO 83

8.737

1. $P_\nu^{-\mu}(x) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \left[\cos\mu\pi P_\nu^\mu(x) - \frac{2}{\pi}\sin(\mu\pi) Q_\nu^\mu(x)\right].$

MO 84

2. $P_\nu^\mu(-x) = \cos[(\nu+\mu)\pi]\, P_\nu^\mu(x) - \frac{2}{\pi}\sin[(\nu+\mu)\pi]\, Q_\nu^\mu(x).$

MO 84

3. $Q_\nu^\mu(-x) = -\cos[(\nu+\mu)\pi]\, Q_\nu^\mu(x) - \frac{\pi}{2}\sin[(\nu+\mu)\pi]\, P_\nu^\mu(x).$

MO 83, EH I 144(15)

4. $Q_{-\nu-1}^\mu(x) = \frac{\sin[(\nu+\mu)\pi]}{\sin[(\nu-\mu)\pi]} Q_\nu^\mu(x) - \frac{\pi\cos\nu\pi\cos\mu\pi}{\sin[(\nu-\mu)\pi]} P_\nu^\mu(x).$

8.738

1.[6] $$Q_\nu^\mu(i\operatorname{ctg}\varphi) = \exp\left[i\pi\left(\mu-\frac{1}{4}\right)\right]\sqrt{\pi}\Gamma(\nu+\mu+1)\sqrt{\frac{1}{2}\sin\varphi}P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi) \qquad \left[0<\varphi<\frac{\pi}{2}\right].$$

MO 83

2.[6] $$P_\nu^\mu(i\operatorname{ctg}\varphi) = \sqrt{\frac{2}{\pi}}\exp\left[i\pi\left(\nu+\frac{1}{4}\right)\right]\frac{\sqrt{\sin\varphi}}{\Gamma(-\nu-\mu)}Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi - i0) \qquad \left[0<\varphi<\frac{\pi}{2}\right].$$

MO 83

8.739

$$e^{-\mu\pi i}Q_\nu^\mu(\operatorname{ch}\alpha) = \frac{\sqrt{\pi}\Gamma(\nu+\mu+1)}{\sqrt{2\operatorname{sh}\alpha}}P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\operatorname{cth}\alpha) \qquad [\operatorname{Re}(\operatorname{ch}\alpha)>0].$$

MO 83

8.741

1. $$P_\nu^{-\mu}(x)\frac{dP_\nu^\mu(x)}{dx} - P_\nu^\mu(x)\frac{dP_\nu^{-\mu}(x)}{dx} = \frac{2\sin\mu\pi}{\pi(1-x^2)}.$$

MO 83

2. $$P_\nu^\mu(x)\frac{dQ_\nu^\mu(x)}{dx} - Q_\nu^\mu(x)\frac{dP_\nu^\mu(x)}{dx} = \frac{2^{2\mu}}{1-x^2}\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)}.$$

MO 83

8.742

1. $$\frac{\Gamma(\nu-\mu-1)}{\Gamma(\nu+\mu+1)}\left\{\cos\mu\pi P_\nu^\mu(\cos\varphi) - \frac{2}{\pi}\sin\mu\pi Q_\nu^\mu(\cos\varphi)\right\} =$$
$$= \sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\varphi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(\cos t-\cos\varphi)^{\frac{1}{2}-\mu}} \qquad \left[\operatorname{Re}\mu>-\frac{1}{2}\right].$$

2. $$\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left\{\cos\nu\pi P_\nu^\mu(\cos\varphi)-\frac{2}{\pi}\sin\nu\pi Q_\nu^\mu(\cos\varphi)\right\}=$$
$$=\sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_\varphi^\pi\frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]dt}{(\cos\varphi-\cos t)^{\frac{1}{2}-\mu}}\left[\operatorname{Re}\mu>-\frac{1}{2}\right].$$

MO 88

3. $$P_\nu^\mu(\cos\varphi)\cos(\nu+\mu)\pi-\frac{2}{\pi}Q_\nu^\mu(\cos\varphi)\sin(\nu+\mu)\pi=$$
$$=\sqrt{\frac{2}{\pi}}\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\varphi^\pi\frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]dt}{(\cos\varphi-\cos t)^{\mu+\frac{1}{2}}}\qquad\left[\operatorname{Re}\mu<\frac{1}{2}\right].$$

MO 88

4. $$\cos\mu\pi P_\nu^\mu(\cos\varphi)-\frac{2}{\pi}\sin\mu\pi Q_\nu^\mu(\cos\varphi)=$$
$$=\frac{1}{2^\mu\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\pi\frac{\sin^{2\mu}t\,dt}{(\cos\varphi\pm i\sin\varphi\cos t)^{\nu-\mu}}$$
$$\left[\operatorname{Re}\mu>-\frac{1}{2},\quad 0<\varphi<\pi\right].$$

MO 38

For integrals of Legendre functions, see 7.11-7.21.

8.75 Special cases and particular values

8.751

1. $$P_\nu^m(x)=(-1)^m\frac{\Gamma(\nu+m+1)(1-x^2)^{\frac{m}{2}}}{2^m\Gamma(\nu-m+1)m!}F\left(m-\nu,m+\nu+1;m+1;\frac{1-x}{2}\right).$$

MO 84

2. $$P_\nu^m(z)=\frac{\Gamma(\nu+m+1)(z^2-1)^{\frac{m}{2}}}{2^m m!\Gamma(\nu-m+1)}F\left(m-\nu,m+\nu+1;m+1;\frac{1-z}{2}\right).$$

MO 84

3.* $$Q_{n-\frac{3}{2}}^\mu(z)=\frac{e^{\mu\pi i}\Gamma\left(\mu+n+\frac{3}{2}\right)}{2^{n+\frac{3}{2}}(n+1)!}(z^2-1)^{\frac{\mu}{2}}\pi^{\frac{1}{2}}z^{-n-\mu-3/2}F\left(\frac{\mu+n+\frac{5}{2}}{2},\frac{\mu+n+\frac{3}{2}}{2};n+2;\frac{1}{z^2}\right).$$

1. $P_\nu^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\nu(x).$

WH, MO 84, EH I 148(6)

2. $P_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(x) = (1-x^2)^{-\frac{m}{2}} \int_x^1 \dots \int_x^1 P_\nu(x)(dx)^m \qquad [m \geq 1]\,.$

HO 99a, MO 85, EH I 149(10)a

3. $P_\nu^{-m}(z) = (z^2-1)^{-\frac{m}{2}} \int_1^z \dots \int_1^z P_\nu(z)(dz)^m \qquad [m \geq 1]\,.$

MO 85, EH I 149(8)

4. $Q_\nu^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_\nu(z).$

WH, MO 85, EH I 148(5)

5. $Q_\nu^{-m}(z) = (-1)^m (z^2-1)^{-\frac{m}{2}} \int_z^\infty \dots \int_z^\infty Q_\nu(z)(dz)^m \qquad [m \geq 1]\,.$

MO 85, EH I 149(9)

Special values of the indices

8.753

1. $P_0^\mu(\cos\varphi) = \frac{1}{\Gamma(1-\mu)} \operatorname{ctg}^\mu \frac{\varphi}{2}.$

MO 84

2. $P_\nu^{-1}(\cos\varphi) = -\frac{1}{\nu(\nu+1)} \frac{dP_\nu(\cos\varphi)}{d\varphi}.$

MO 84

3. $P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0 \quad \text{for} \quad m > n.$

8.754

1. $P^{\frac{1}{2}}_{\nu-\frac{1}{2}}(\operatorname{ch}\alpha) = \sqrt{\dfrac{2}{\pi\operatorname{sh}\alpha}}\operatorname{ch}\nu\alpha.$

MO 85

2. $P^{\frac{1}{2}}_{\nu-\frac{1}{2}}(\cos\varphi) = \sqrt{\dfrac{2}{\pi\sin\varphi}}\cos\nu\varphi.$

MO 85

3. $P^{-\frac{1}{2}}_{\nu-\frac{1}{2}}(\cos\varphi) = \sqrt{\dfrac{2}{\pi\sin\varphi}}\dfrac{\sin\nu\varphi}{\nu}.$

MO 85

4. $Q^{\frac{1}{2}}_{\nu-\frac{1}{2}}(\operatorname{ch}\alpha) = i\sqrt{\dfrac{\pi}{2\operatorname{sh}\alpha}}e^{-\nu\alpha}.$

MO 85

1026

8.755

1. $P^{-\nu}_{\nu}(\cos\varphi) = \dfrac{1}{\Gamma(1+\nu)}\left(\dfrac{\sin\varphi}{2}\right)^{\nu}.$

MO 85

2. $P^{-\nu}_{\nu}(\operatorname{ch}\alpha) = \dfrac{1}{\Gamma(1+\nu)}\left(\dfrac{\operatorname{sh}\alpha}{2}\right)^{\nu}.$

MO 85

Special values of legendre functions

8.756

1. $P^{\mu}_{\nu}(0) = \dfrac{2^{\mu}\sqrt{\pi}}{\Gamma\left(\dfrac{\nu-\mu}{2}+1\right)\Gamma\left(\dfrac{-\nu-\mu+1}{2}\right)}.$

2. $$\frac{dP_\nu^\mu(0)}{dx} = \frac{2^{\mu+1}\sin\frac{1}{2}(\nu+\mu)\pi\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-\mu+1}{2}\right)}.$$

MO 84

3. $$Q_\nu^\mu(0) = -2^{\mu-1}\sqrt{\pi}\sin\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}.$$

MO84

4. $$\frac{dQ_\nu^\mu(0)}{dx} = 2^\mu\sqrt{\pi}\cos\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}.$$

MO 84

8.76 Derivatives with respect to the order

8.761

$$q\frac{\partial P_\nu^{-\mu}(x)}{\partial\nu} = \frac{1}{\Gamma(\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}}\sum_{n=1}^{\infty}\frac{(-\nu)(1-\nu)\ldots(n-1-\nu)(\nu+1)(\nu+2)\ldots(\nu+n)}{(\mu+1)(\mu+2)\ldots(\mu+n)1\cdot 2\ldots n}\times$$
$$\times[\psi(\nu+n+1)-\psi(\nu-n+1)]\left(\frac{1-x}{2}\right)^n \qquad [\nu\neq 0,\ \pm1,\pm2,\ldots;\ \operatorname{Re}\mu>-1].$$

MO 94

8.762

1. $$\left[\frac{\partial P_\nu(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = 2\ln\cos\frac{\varphi}{2}.$$

MO 94

2. $$\left[\frac{\partial P_\nu^{-1}(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = -\operatorname{tg}\frac{\varphi}{2} - 2\operatorname{ctg}\frac{\varphi}{2}\ln\cos\frac{\varphi}{2}.$$

MO 94

3. $$\left[\frac{\partial P_\nu^{-1}(\cos\varphi)}{\partial\nu}\right]_{\nu=1} = -\frac{1}{2}\operatorname{tg}\frac{\varphi}{2}\sin^2\frac{\varphi}{2} + \sin\varphi\ln\cos\frac{\varphi}{2}.$$

MO 94

For a connection with the polynomials $C_n^\lambda(x)$, see 8.936.

For a connection with a hypergeometric function, see 8.77.

8.77 Series representation

For a representation in the form of a series, see 8.721. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

8.771

1. $$P_\nu^\mu(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right).$$

MO 15

2. $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)(z^2-1)^{\frac{\mu}{2}}}{2^{\nu+\mu+1}} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right).$$

MO 15

See also 8.702, 8.703, 8.704, 8.723, 8.751, 8.772.

The analytic continuation for $|Z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see 9.154 and 9.155):

8.772

1. $$P_\nu^\mu(z) = \frac{\sin(\nu+\mu)\pi\Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi}\cos\nu\pi\Gamma\left(\nu+\frac{3}{2}\right)}(z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) + \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu-\mu+1)}(z^2-1)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$
$$[2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |z| > 1; \quad |\arg(z \pm 1)| < \pi].$$

2. $$P_\nu^\mu(z) = \frac{\Gamma\left(-\nu - \frac{1}{2}\right)(z^2-1)^{-\frac{\nu+1}{2}}}{2^{\nu+1}\sqrt{\pi}\Gamma(-\nu-\mu)} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right) + \\ + \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu-\mu+1)}(z^2-1)^{\frac{\nu}{2}} F\left(\frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}; \frac{1}{2}-\nu; \frac{1}{1-z^2}\right) \\ \left[2\nu \neq \pm 1, \pm 3, \pm 5; \dots; \quad |1-z^2|>1; \quad |\arg(z\pm 1)|<\pi\right].$$

MO 85

1028

3. $$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}}\left(\frac{z+1}{2}\right)^\nu F\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right) \quad \left[\left|\frac{z-1}{z+1}\right|<1\right]$$

MO 86

8.773

1. $$Q_\nu^\mu(z) = e^{\mu\pi i}\frac{\sqrt{\pi}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}(z^2-1)^{-\frac{\nu+1}{2}} F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right) \\ \left[\nu+\mu \neq -1, -2, -3, \dots; \quad |\arg(z\pm 1)|<\pi; \quad |1-z^2|>1\right].$$

MO 86

2. $$Q_\nu^\mu(z)\frac{1}{2}e^{\mu\pi i}\left\{\Gamma(\mu)\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) + \right. \\ \left. + \frac{\Gamma(-\mu)\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1+\mu; \quad \frac{1-z}{2}\right)\right\} \\ [|\arg(z\pm 1)|<\pi, \quad |1-z|<2].$$

MO 86

8.774

$$P_\nu^\mu(i \operatorname{ctg}\varphi) = \sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}e^{-i(\nu+1)\frac{\pi}{2}}\left(\operatorname{tg}\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \nu+\frac{3}{2}; \sin^2\frac{\varphi}{2}\right) + \\ + \sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}e^{i\nu\frac{\pi}{2}}\left(\operatorname{ctg}\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{1}{2}-\nu; \sin^2\frac{\varphi}{2}\right) \\ \left[2\nu \neq \pm 1, \pm 3, \pm 5, \dots, 0<\varphi<\frac{\pi}{2}\right].$$

1.[6] $$P_\nu^\mu(x) = \frac{2^\mu \cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-\mu}{2}+1\right)}(1-x^2)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right) + \\ + \frac{2^{\mu+1}}{\sqrt{\pi}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \times \\ \times x(1-x^2)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu}{2}+1,\frac{-\nu+\mu+1}{2};\frac{3}{2};x^2\right).$$

MO 87

1029

2.[6] $$Q_\nu^\mu(x) = -\frac{\sqrt{\pi}}{2^{1-\mu}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}(1-x^2)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right) + \\ + 2^\mu\sqrt{\pi}\frac{\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \times \\ \times x(1-x^2)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu}{2}+1,\frac{\mu-\nu+1}{2};\frac{3}{2};x^2\right)$$

MO 87

8.776

For $|z| \gg 1$

1. $$P_\nu^\mu(z) = \left\{\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\nu-\mu+1\right)}z^\nu + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\Gamma\left(-\nu-\mu\right)}z^{-\nu-1}\right\}\left(1+O\left(\frac{1}{z^2}\right)\right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \ldots, |\arg z| < \pi] .$$

MO 87

2. $$Q_\nu^\mu(z) = \sqrt{\pi}\frac{e^{\mu\pi i}}{2^{\nu+1}}\frac{\Gamma(\mu+\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}\left(1+O\left(\frac{1}{z^2}\right)\right) [2\nu \neq -3, -5, -7, \ldots; \quad |\arg z| < \pi] .$$

MO 87

8.777

Set $\zeta = z + \sqrt{z^2-1}$. The variable ζ is uniquely defined by this equation on the entire z-plane in which a cut is made from $-\infty$ to

$+1$. Here, we are considering that branch of the variable ζ for which values of ζ exceeding 1 correspond to real values of z exceeding 1. In this case,

$$1.\quad P_\nu^\mu(z) = \frac{2^\mu\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(-\nu-\mu)}\frac{(z^2-1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}}F\left(\frac{1}{2}+\mu, \nu+\mu+1; \nu+\frac{3}{2}; \frac{1}{\zeta^2}\right) + \\ + \frac{2^\mu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{(z^2-1)^{\frac{\mu}{2}}}{\zeta^{\mu-\nu}}F\left(\frac{1}{2}+\mu, \mu-\nu; \frac{1}{2}-\nu; \frac{1}{\zeta^2}\right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |\arg(z-1)| < \pi]\ .$$

MO 86

$$2.\quad Q_\nu^\mu(z) = 2^\mu e^{\mu\pi i}\sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{(z^2-1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}}F\left(\frac{1}{2}+\mu, \nu+\mu+1; \nu+\frac{3}{2}; \frac{1}{\zeta^2}\right) \\ [|\arg(z-1)| < \pi]\ .$$

MO 86

1030

8.78 The zeros of associated Legendre functions

8.781

The function $P_\nu^{-\mu}(\cos\varphi)$, considered as a function of ν has infinitely many zeros for $\mu \geq 0$. These are all simple and real. If a number ν_0 is a zero of the function $P_\nu^{-\mu}(\cos\varphi)$, the number $-\nu_0 - 1$ is also a zero of this function.

8.782

If ν and μ are both real and $\mu \leq 0$, or if ν and μ are integers, the function $P_\nu^\mu(t)$ has no *real* zeros exceeding 1. If ν and μ are both real with $\nu < \mu < 0$, the function $P_\nu^\mu(t)$ has no real zeros exceeding 1 when $\sin\mu\pi\sin(\mu-\nu)\pi > 0$, but does have one such zero when $\sin\mu\pi\sin(\mu-\nu)\pi < 0$. Finally, if $\mu \leq \nu$, the function $P_\nu^\mu(t)$ has no zeros exceeding 1 for $[\mu]$ even but does have one zero for $[\mu]$ odd.

8.783

If $\nu > -\frac{3}{2}$ and $\nu+\mu+1 > 0$, the function $Q_\nu^\mu(t)$ has no real zeros exceeding 1. MO 91

8.784

The function $P_{-\frac{1}{2}+i\lambda}(z)$ has infinitely many zeros for real λ. All these zeros are *real* and *greater than unity*.

8.785

For n a natural number, the function $P_n(x)$ has exactly n real zeros which lie in the closed interval $-1, +1$.

8.786

The function $Q_n(z)$ has no zeros for which $|\arg(z-1)| < \pi$ if n is a natural number. The function $Q_n(\cos\varphi)$ has exactly $n+1$ zeros in the interval $0 \le \varphi \le \pi$.

MO 91

8.787

The following approximate formula can be used to calculate the values of ν for which the equation $P_\nu^{-\mu}(\cos\varphi) = 0$ holds for given small values of φ:

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2\sin\frac{\varphi}{2}}\left\{1 - \frac{\sin^2\frac{\varphi}{2}}{6}\left(1 - \frac{4\mu^2-1}{j_\mu^2}\right) + O\left(\sin^4\frac{\varphi}{2}\right)\right\};$$

MO 93
MO 91

Here, j_μ denotes an arbitrary nonzero root of the equation $J_\mu(z) = 0$ (for $\mu \ge 0$). If φ is close to π then, instead of this formula, we can use the following formulas:

1. $$\nu \ \mathsf{t} \mu + k + \frac{\Gamma(2\mu+k+1)}{\Gamma(\mu)\Gamma(\mu+1)\Gamma(k+1)}\left(\frac{\pi-\varphi}{3}\right)^{2\mu} \qquad [\mu > 0; \quad k = 0, 1, 2, \dots] .$$

MO 93

2. $$\nu \ \mathsf{t} k + \frac{1}{2\ln\left(\frac{2}{\pi-\varphi}\right)} \qquad [\mu = 0, \quad k = 0, 1, 2, \dots] .$$

MO 93

8.79 Series of associated Legendre functions

8.791

1. $\dfrac{1}{z-t} = \sum_{k=0}^{\infty}(2k+1)P_k(t)Q_k(z) \qquad \left[|t+\sqrt{t^2-1}| < |z+\sqrt{z^2-1}|\right]$;

1031

Here, t must lie inside an ellipse passing through the point z with foci at the points ± 1.

2. $\dfrac{1}{\sqrt{1-2tz+t^2}} \ln \dfrac{z-t+\sqrt{1-2tz+t^2}}{\sqrt{z^2-1}} = \sum_{k=0}^{\infty} t^k Q_k(z) \qquad [\operatorname{Re} z > 1, \quad |t| < 1]$.

MO 78

8.792

$$P_\nu^{-\alpha}(\cos\varphi)P_\nu^{-\beta}(\cos\psi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right]P_k^{-\alpha}(\cos\varphi)P_k^{-\beta}(\cos\psi)$$
$$[a \ge 0, \quad \beta \ge 0, \quad \nu \text{ real}, \ -\pi < \varphi \pm \psi < \pi] \ .$$

MO 94

8.793

$$P_\nu^{-\mu}(\cos\varphi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right)P_k^{-\mu}(\cos\varphi) \qquad [\mu \ge 0, \quad 0 < \varphi < \pi] \ .$$

MO 94

Addition theorems

8.794

1. $P_\nu(\cos\psi_1\cos\psi_2 + \sin\psi_1\sin\psi_2\cos\varphi) =$

$$= P_\nu(\cos\psi_1)P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)P_\nu^k(\cos\psi_2)\cos k\varphi;$$
$$= P_\nu(\cos\psi_1)P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}P_\nu^k(\cos\psi_1)P_\nu^k(\cos\psi_2)\cos k\varphi$$

$[0 \le \psi_1 < \pi, \quad 0 \le \psi_2 < \pi, \quad \psi_1+\psi_2 < \pi; \quad \varphi \text{ real}\ ] \qquad$ (cf. 8.814, 8.844 1.).

8.844

8.814
MO 90

2. $Q_\nu(\cos\psi_1\cos\psi_2+\sin\psi_1\sin\psi_2\cos\varphi)=$
$$=P_\nu(\cos\psi_1)Q_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)Q_\nu^k(\cos\psi_2)\cos k\varphi$$
$\left[0<\psi_1<\frac{\pi}{2},\quad 0<\psi_2<\pi,\quad 0<\psi_1+\psi_2<\pi;\quad \varphi \text{ real}\right]$ (cf. 8.844 3.).

8.844
MO 90

8.795

1. $$P_\nu\left(z_1z_2-\sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\varphi\right)=P_\nu(z_1)P_\nu(z_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^k(z_1)P_\nu^{-k}(z_2)\cos k\varphi$$
$[\operatorname{Re} z_1>0,\quad \operatorname{Re} z_2>0,\quad |\arg(z_1-1)|<\pi,\quad |\arg(z_2-1)|<\pi]$.

MO 91

2. $$Q_\nu\left(x_1x_2-\sqrt{x_1^2-1}\sqrt{x_2^2-1}\cos\varphi\right)=P_\nu(x_1)Q_\nu(x_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(x_1)Q_\nu^k(x_2)\cos k\varphi$$
$[1<x_1<x_2,\quad \nu\neq-1,-2,-3,\dots;\quad \varphi \text{ real}]$

MO 91

3. $$Q_n\left(x_1x_2+\sqrt{x_1^2+1}\sqrt{x_2^2+1}\operatorname{ch}\alpha\right)=\sum_{k=n+1}^{\infty}\frac{1}{(k-n-1)!(k+n)!}Q_n^k(ix_1)Q_n^k(ix_2)e^{-k\alpha}$$
$[x_1>0,\quad x_2>0;\quad \alpha>0]$.

MO 91

1032
8.796

$$P_\nu(-\cos\psi_1\cos\psi_2-\sin\psi_1\sin\psi_2\cos\varphi)=P_\nu(-\cos\psi_1)P_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)}\times$$
$$\times P_\nu^{-k}(-\cos\psi_1)P_\nu^{-k}(\cos\psi_2)\cos k\varphi$$
$[0<\psi_2<\psi_1<\pi;\quad \varphi \text{ real}]$ (cf. 8.844 2.).

See also 8.934 3.

8.81 Associated Legendre functions with integral indices

8.810

For *integral* values of ν and μ, the differential equation 8.700 1. (with $|\nu| > |\mu|$) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions $P_n^m(x)$ are called *associated Legendre functions* (or *spherical functions*) *of the first kind*. The number n is called the *degree* and the number m is called the *order* of the function $P_n^m(x)$. The functions

$$\cos m\vartheta P_n^m(\cos\varphi), \quad \sin m\vartheta P_n^m(\cos\varphi),$$

which depend on the angles φ and ϑ, are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for $m < n$ and *sectoral harmonics* for $m = n$. These last functions are periodic with respect to the angles φ and ϑ. Their periods are respectively π and 2π. They are single-valued and continuous everywhere on the surface of the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (where $x_1 = \sin\varphi\cos\vartheta, x_2 = \sin\varphi\sin\vartheta, x_3 = \cos\varphi$) and they are solutions of the differential equation

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\vartheta^2} + n(n+1)Y = 0.$$

8.811[7]

The integral representation

$$P_n^m(\cos\varphi) = \frac{(-1)^m (n+m)!}{\Gamma\left(m+\frac{1}{2}\right)(n-m)!}\sqrt{\frac{2}{\pi}}\sin^{-m}\varphi\int_0^\varphi (\cos t - \cos\varphi)^{m-\frac{1}{2}} \cos\left(n+\frac{1}{2}\right) t dt.$$

MO 75

8.812

The series representation:

$$P_n^m(x) = \frac{(-1)^m(n+m)!}{2^m m!(n-m)!}(1-x^2)^{\frac{m}{2}}\left\{1 - \frac{(n-m)(m+n+1)}{1!(m+1)}\frac{1-x}{2} + \right.$$
$$\left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)}\left(\frac{1-x}{2}\right)^2 - \dots\right\};$$
$$= \frac{(-1)^m(2n-1)!!}{(n-m)!}(1-x^2)^{\frac{m}{2}}\left\{x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}x^{n-m-2} + \right.$$
$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-m-4} - \dots\right\};$$
$$= \frac{(-1)^m(2n-1)!!}{(n-m)!}(1-x^2)^{\frac{m}{2}}x^{n-m}F\left(\frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right).$$

MO 73

8.813

Special cases:

1. $P_1^1(x) = -(1-x^2)^{\frac{1}{2}} = -\sin\varphi.$

MO 73

2. $P_2^1(x) = -3(1-x^2)^{\frac{1}{2}}x = -\frac{3}{2}\sin 2\varphi.$

MO 73

3. $P_2^2(x) = 3(1-x^2) = \frac{3}{2}(1-\cos 2\varphi).$

MO 73

4. $P_3^1(x) = -\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1) = -\frac{3}{8}(\sin\varphi + 5\sin 3\varphi).$

MO 73

5. $P_3^2(x) = 15(1-x^2)x = \frac{15}{4}(\cos\varphi - \cos 3\varphi).$

6. $P_3^3(x) = -15(1-x^2)^{\frac{3}{2}} = -\frac{15}{4}(3\sin\varphi - \sin 3\varphi).$

MO 73

Functional relations

For recursion formulas, see 8.731.

8.814

$$P_n(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2\cos\Theta) =$$
$$= P_n(\cos\varphi_1)P_n(\cos\varphi_2) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\varphi_1)P_n^m(\cos\varphi_2)\cos m\Theta$$
$$[0 \le \varphi_1 \le \pi, \quad 0 \le \varphi_2 \le \pi] \quad (\text{“addition theorem”}).$$

MO 74

8.815

If

$$Y_{n_1}(\varphi,\vartheta) = a_0 P_{n_1}(\cos\varphi) + \sum_{m=1}^{n_1}(a_m\cos m\vartheta + b_m\sin m\vartheta)P_{n_1}^m(\cos\varphi),$$

$$Z_{n_2}(\varphi,\vartheta) = \alpha_0 P_{n_2}(\cos\varphi) + \sum_{m=1}^{n_2}(\alpha_m\cos m\vartheta + \beta_m\sin m\vartheta)P_{n_2}^m(\cos\varphi),$$

then

$$\int_0^{2\pi} d\vartheta \int_0^{\pi} \sin\varphi d\varphi Y_{n_1}(\varphi,\vartheta)Z_{n_2}(\varphi,\vartheta) = 0,$$

$$\int_0^{2\pi} d\vartheta \int_0^{\pi} \sin\varphi d\varphi Y_n(\varphi,\vartheta)P_n\left[\cos\varphi\cos\psi + \sin\varphi\sin\psi\cos(\vartheta-\theta)\right] = \frac{4\pi}{2n+1}Y_n(\psi,\theta).$$

$$(\cos\varphi + i\sin\varphi\cos\vartheta)^n = P_n(\cos\varphi) + 2\sum_{m=1}^{n}(-1)^m \frac{n!}{(n+m)!}\cos m\vartheta P_n^m(\cos\varphi).$$

MO 75

For integrals of the functions, $P_n^m(x)$, see 7.112 1., 7.122 1.

8.82- 8.83 Legendre functions

8.820

The differential equation

$$\frac{d}{dz}\left[(1-z^2)\frac{du}{dz}\right] + \nu(\nu+1)u = 0 \qquad \text{(cf. 8.700 1.)},$$

8.700

where the parameter ν can be an arbitrary number, has the following two linearly independent solutions:

$$1. \quad P_\nu(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right).$$

$$2. \quad Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{z^2}\right).$$

SM 518(137)

The functions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first* and *second kind* respectively. If ν is not an integer, the function $P_\nu(z)$ has *singularities* at $z=-1$ and $z=\infty$. However, if $\nu = n = 0, 1, 2, \ldots$, the function $P_\nu(z)$ becomes the *Legendre polynomial* $P_n(z)$ (see 8.91). For $\nu = -n = -1, -2, \ldots$, we have

$$P_{-n-1}(z) = P_n(z).$$

3. If $\nu \neq 0, 1, 2, \dots$, the function $Q_\nu(z)$ has singularities at the points $z = \pm 1$ and $z = \infty$. These points are branch points of the function. On the other hand, if $\nu = n = 0, 1, 2, \dots$, the function $Q_n(z)$ is single-valued for $|z| > 1$ and regular for $z = \infty$.

1035

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \qquad [\operatorname{Re} z > 0] .$$

5. The function $P_\nu(z)$ is uniquely determined by equations 8.820 1. and 8.820 4. within a circle of radius 2 with its center at the point $z = 1$ in the right half-plane.

For $z = x = \cos\varphi$, a solution of equation 8.820 is the function

$$6. \quad P_\nu(x) = P_\nu(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\frac{\varphi}{2}\right) ;$$

In general,

$$7. \quad P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x), \quad \text{for } z = x.$$

8. The function $Q_\nu(z)$ for $|z| > 1$ is uniquely determined by equation 8.820 2. everywhere in the z-plane in which a cut is made from the point $z = -\infty$ to the point $z = 1$. By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut $(-1 \le x \le +1)$ of the real axis, the function $Q_\nu(x)$ is determined by the equation

$$9. \quad Q_\nu(x) = \frac{1}{2}\left[Q_\nu(x+i0) + Q_\nu(x-i0)\right] .$$

HO 52(53), WH

Integral representations

8.821

Here, A is a point on the real axis to the right of the point $t = 1$ and to the right of z if z is real. At the point A, we set

$$\arg(t-1) = \arg(t+1) = 0 \quad \text{and} \qquad [|\arg(t-z)| < \pi] \; .$$

WH

2. $$Q_\nu(z) = \frac{1}{4i \sin \nu\pi} \int_A^{(1-,1+)} \frac{(t^2-1)^\nu}{2^\nu (z-t)^{\nu+1}} dt.$$

[ν is not an integer; the point A is at the end of the major axis of an ellipse to the right of $t = 1$ drawn in the t-plane with foci at the points ± 1 and with a minor axis sufficiently small that the point z lies outside it. The contour begins at the point A, follows the path $(1-, -1+)$ and returns to A; $|\arg z| \leq \pi$ and $|\arg(z-t)| \to \arg z$ as $t \to 0$ on the contour; $\arg(t+1) = \arg(t-1) = 0$ at the point A; z does not lie on the real axis between -1 and 1.]

For $\nu = n$ an integer,

3. $$Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n (z-t)^{-n-1} dt.$$

SM 517(134), WH

1036

8.822

1. $$P_\nu(z) = \frac{1}{\pi} \int_0^\pi \frac{d\varphi}{(z + \sqrt{z^2-1} \cos\varphi)^{\nu+1}} = \frac{1}{\pi} \int_0^\pi (z + \sqrt{z^2-1} \cos\varphi)^\nu d\varphi$$
$$\left[\operatorname{Re} z > 0 \quad \text{and} \qquad \arg\left\{z + \sqrt{z^2-1}\cos\varphi\right\} = \arg z \quad \text{for } \varphi = \frac{\pi}{2}\right] .$$

WH

2. $$Q_\nu(z) = \int_0^\infty \frac{d\varphi}{(z + \sqrt{z^2-1} \operatorname{ch}\varphi)^{\nu+1}},$$
$$\left[\operatorname{Re}\nu > -1; \quad \text{if } \nu \text{ is not an integer}\,, \left\{(z + \sqrt{z^2-1}) \operatorname{ch}\varphi\right\} \text{ for } \varphi = 0 \text{ has its principal value}\right] .$$

WH

8.823

$$P_\nu(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos\left(\nu+\frac{1}{2}\right)\varphi}{\sqrt{2(\cos\varphi-\cos\theta)}}d\varphi.$$

WH

8.824

$$Q_n(z) = 2^n n!\int_z^\infty \ldots \int_z^\infty \frac{(dz)^{n+1}}{(z^2-1)^{n+1}} = 2^n\int_z^\infty \frac{(t-z)^n}{(t^2-1)^{n+1}}dt;$$
$$= \frac{(-1)^n}{(2n-1)!!}\frac{d^n}{dz^n}\left[(z^2-1)^n\int_z^\infty \frac{dt}{(t^2-1)^{n+1}}\right] \qquad [\operatorname{Re} z > 1]\,.$$

WH, MO 78

8.825

$$Q_n(z) = \frac{1}{2}\int_{-1}^1 \frac{P_n(t)}{z-t}dt \qquad [|\arg(z-1)| < \pi]\,.$$

WH, MO 78

See also 6.622 3., 8.842.

8.826

Fourier series:

1. $$P_n(\cos\varphi) = \frac{2^{n+2}}{\pi}\frac{n!}{(2n+1)!!}\left[\sin(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\sin(n+3)\varphi + \right.$$
$$\left. + \frac{1\cdot 3(n+1)(n+2)}{1\cdot 2(2n+3)(2n+5)}\sin(n+5)\varphi + \ldots\right] \qquad [0 < \varphi < \pi]\,.$$

MO 79

2. $$Q_n(\cos\varphi) = 2^{n+1}\frac{n!}{(2n+1)!!}\left[\cos(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\cos(n+3)\varphi + \right.$$
$$\left. + \frac{1\cdot 3}{1\cdot 2}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\cos(n+5)\varphi + \ldots\right] \qquad [0 < \varphi < \pi]\,.$$

MO 79

The expressions for Legendre functions in terms of a hypergeometric function (see 8.820) provide other series representations of these functions.

Special cases and particular values

8.827

1. $Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{Arth} x.$

JA

2. $Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1.$

JA

3. $Q_2(x) = \frac{1}{4}(3x^2 - 1) \ln \frac{1+x}{1-x} - \frac{3}{2}x.$

JA

4. $Q_3(x) = \frac{1}{4}(5x^3 - 3x) \ln \frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}.$

JA

5. $Q_4(x) = \frac{1}{16}(35x^4 - 30x^2 + 3) \ln \frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x.$

JA

6. $Q_5(x) = \frac{1}{16}(63x^5 - 70x^3 + 15x) \ln \frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}.$

JA

8.828

1. $P_\nu(1) = 1.$

2. $P_\nu(0) = -\frac{1}{2}\frac{\sin\nu\pi}{\sqrt{\pi^3}}\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right).$

MO 79

8.829

$$Q_\nu(0) = \frac{1}{4\sqrt{\pi}}(1-\cos\nu\pi)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right).$$

MO 79

Functional relationships

8.831

1. $Q_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left[\cos\nu\pi P_\nu(x) - P_\nu(-x)\right] \qquad [\nu\neq 0, \quad \pm 1, \pm 2, \dots].$

MO 76

2. $Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - W_{n-1}(x) \qquad [n = 0, 1, 2, \dots],$

where

3. $W_{n-1}(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{2(n-2k)-1}{(2k+1)(n-k)}P_{n-2k-1}(x) = \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)P_{n-k}(x)$

and

$W_{-1}(x) \equiv 0 \qquad \text{(see also 8.839)}.$

8.839
SM 516(131), MO 76

4. $\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right)P_k(\cos\varphi) = \frac{\pi}{\sin\nu\pi}P_\nu(\cos\varphi)$

$[\nu \text{ not an integer}; \quad 0 \le \varphi < \pi].$

5. $$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k}-\frac{1}{\nu+k+1}\right)P_k(\cos\varphi)P_k(\cos\psi)=\frac{\pi}{\sin\nu\pi}P_\nu(\cos\varphi)P_\nu(\cos\psi)$$
$$[\nu \text{ not an integer}, \quad -\pi<\varphi+\psi<\pi, \quad -\pi<\varphi-\psi<\pi] .$$

MO 77

See also 8.521 4.

8.832

1. $$(z^2-1)\frac{d}{dz}P_\nu(z)=(\nu+1)\left[P_{\nu+1}(z)-zP_\nu(z)\right] .$$

WH

2. $$(2\nu+1)zP_\nu(z)=(\nu+1)P_{\nu+1}(z)+\nu P_{\nu-1}(z).$$

WH

3. $$(z^2-1)\frac{d}{dz}Q_\nu(z)=(\nu+1)\left[Q_{\nu+1}(z)-zQ_\nu(z)\right] .$$

WH

4. $$(2\nu+1)zQ_\nu(z)=(\nu+1)Q_{\nu+1}(z)+\nu Q_{\nu-1}(z).$$

WH

8.833

1. $$P_\nu(-z)=e^{\nu\pi i}P_\nu(z)-\frac{2}{\pi}\sin\nu\pi Q_\nu(z) \qquad [\operatorname{Im} z<0] .$$

MO 77

2. $$P_\nu(-z)=e^{-\nu\pi i}P_\nu(z)-\frac{2}{\pi}\sin\nu\pi Q_\nu(z) \qquad [\operatorname{Im} z>0] .$$

MO 77

3. $$Q_\nu(-z)=-e^{-\nu\pi i}Q_\nu(z) \qquad [\operatorname{Im} z<0] .$$

4. $Q_\nu(-z) = -e^{\nu\pi i} Q_\nu(z) \qquad [\operatorname{Im} z > 0]$.

MO 77

8.834

1. $Q_\nu(x \pm i0) = Q_\nu(x) \mp \frac{\pi i}{2} P_\nu(x)$.

MO 77

2. $Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z) \qquad$ (see 8.831 3.).

8.831
MO 77

8.835

1. $Q_\nu(z) - Q_{-\nu-1}(z) = \pi \operatorname{ctg} \nu\pi P_\nu(z) \qquad [\sin \nu\pi \neq 0]$.

MO 77

2. $Q_{-\nu-1}(\cos\varphi) = Q_\nu(\cos\varphi) - \pi \operatorname{ctg} \nu\pi P_\nu(\cos\varphi) \qquad [\sin \nu\pi \neq 0]$.

MO 77

3. $Q_\nu(-\cos\varphi) = -\cos \nu\pi Q_\nu(\cos\varphi) - \frac{\pi}{2} \sin \nu\pi P_\nu(\cos\varphi)$.

MO 77

8.836

1. $Q_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2-1)^n \ln \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1}$.

MO 79

2. $Q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n \ln \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x}$.

1. $P_\nu(x) = P_\nu(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\frac{\varphi}{2}\right) \qquad \text{(cf. 8.820 6.)}.$

8.820
MO 76

2. $$P_\nu(z) = \; = \frac{\operatorname{tg}\nu\pi}{2^{\nu+1}\sqrt{\pi}} \frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1} F\left(\frac{\nu}{2}+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) + \\ + \frac{2^\nu}{\sqrt{\pi}} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} z^\nu F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right).$$

MO 78

See also 8.820.

For integrals of Legendre functions, see 7.1-7.2.

8.838

Inequalities:

1. $|P_\nu(\cos\varphi) - P_{\nu+2}(\cos\varphi)| \le 2C_0\sqrt{\frac{1}{\nu\pi}}.$

MO 78

2. $|Q_\nu(\cos\varphi) - Q_{\nu+2}(\cos\varphi)| < C_0\sqrt{\frac{\pi}{\nu}}.$

MO 78

[$0 \le \varphi \le \pi, \nu > 1, C_0$ is a number that does not depend on the values of ν or φ].

With regard to the zeros of Legendre functions of the second kind, see 8.784, 8.785, and 8.786. For the expansion of Legendre functions in series of associated Legendre functions, see 8.794, 8.795, and 8.796.

8.839

A differential equation leading to the functions $W_{n-1}(x)$ (see 8.831 3.):

$$(1-x^2)\frac{d^2W_{n-1}}{dx^2} - 2x\frac{dW_{n-1}}{dx} + (n+1)nW_{n-1} = 2\frac{dP_n(x)}{dx}.$$

MO 76

8.84 Conical functions

8.840

Let us set

$$\nu = -\frac{1}{2} + i\lambda,$$

where λ is a real parameter, in the defining differential equation 8.700 1. for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x), \quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class--the class of conical functions. The most important of these peculiarities is the following

8.841

The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos\varphi) = 1 + \frac{4\lambda^2+1^2}{2^2}\sin^2\frac{\varphi}{2} + \frac{(4\lambda^2+1^2)(4\lambda^2+3^2)}{2^2 4^2}\sin^4\frac{\varphi}{2} + \dots$$

1040

are real for real values of φ. Also,

$$P_{-\frac{1}{2}+i\lambda}(x) \equiv P_{-\frac{1}{2}-i\lambda}(x).$$

8.842

Integral representations:

1. $$P_{-\frac{1}{2}+i\lambda}(\cos\varphi) = \frac{2}{\pi}\int_0^{\varphi} \frac{\operatorname{ch}\lambda u du}{\sqrt{2(\cos u - \cos\varphi)}} = \frac{2}{\pi}\operatorname{ch}\lambda\pi\int_0^{\infty} \frac{\cos\lambda u du}{\sqrt{2(\cos\varphi + \operatorname{ch} u)}}.$$ MO 95

2.[6] $$Q_{-\frac{1}{2}\mp\lambda i}(\cos\varphi) = \pm i \operatorname{sh}\lambda\pi\int_0^{\infty} \frac{\cos\lambda u du}{\sqrt{2(\operatorname{ch} u + \cos\varphi)}} + \int_0^{\infty} \frac{\cos\lambda u du}{\sqrt{2(\operatorname{ch} u - \cos\varphi)}}.$$ MO 95

Functional relations (see also 8.73)

8.843

$$P_{-\frac{1}{2}+i\lambda}(-\cos\varphi) = \frac{\operatorname{ch}\lambda\pi}{\pi}\left[Q_{-\frac{1}{2}+i\lambda}(\cos\varphi) + Q_{-\frac{1}{2}-i\lambda}(\cos\varphi)\right].$$ MO 95

8.844

1. $$\begin{aligned} &P_{-\frac{1}{2}+i\lambda}(\cos\psi\cos\vartheta + \sin\psi\sin\vartheta\cos\varphi) = \\ &= P_{-\frac{1}{2}+i\lambda}(\cos\psi)P_{-\frac{1}{2}+i\lambda}(\cos\vartheta) + 2\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k} P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)P^k_{-\frac{1}{2}+i\lambda}(\cos\vartheta)\cos k\varphi}{(4\lambda^2+1^2)(4\lambda^2+3^2)\dots[4\lambda^2+(2k-1)^2]} \\ &\left[0 < \vartheta < \frac{\pi}{2}, \quad 0 < \psi < \pi, \quad 0 < \psi + \vartheta < \pi\right] \quad \text{(cf. 8.794 1.).} \end{aligned}$$ 8.794 MO 95

2. $$\begin{aligned} &P_{-\frac{1}{2}+i\lambda}(-\cos\psi\cos\vartheta - \sin\psi\sin\vartheta\cos\varphi) = \\ &= P_{-\frac{1}{2}+i\lambda}(\cos\psi)P_{-\frac{1}{2}+i\lambda}(-\cos\vartheta) + 2\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k} P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)P^k_{-\frac{1}{2}+i\lambda}(-\cos\vartheta)\cos k\varphi}{(4\lambda^2+1)(4\lambda^2+3^2)\dots[4\lambda^2+(2k-1)^2]} \\ &\left[0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi\right] \quad \text{(cf. 8.796).} \end{aligned}$$

3. $Q_{-\frac{1}{2}+i\lambda}(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi)=$

$$=P_{-\frac{1}{2}+i\lambda}(\cos\psi)Q_{-\frac{1}{2}+i\lambda}(\cos\vartheta)+2\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k}P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)Q^k_{-\frac{1}{2}+i\lambda}(\cos\vartheta)\cos k\varphi}{(4\lambda^2+1)(4\lambda^2+3^2)\dots[4\lambda^2+(2k-1)^2]}$$

$$\left[0<\psi<\frac{\pi}{2}<\vartheta,\quad \psi+\vartheta<\pi\right]\quad \text{(cf. 8.794 2.)}.$$

8.794
MO 96

Regarding the zeros of conical functions, see 8.784.

8.85 Toroidal functions* Sometimes called torus functions.

8.850

Solutions of the differential equation

$$1.\quad \frac{d^2u}{d\eta^2}+\frac{\operatorname{ch}\eta}{\operatorname{sh}\eta}\frac{du}{d\eta}-\left(n^2-\frac{1}{4}+\frac{m^2}{\operatorname{sh}^2\eta}\right)u=0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions

$$P^m_{n-\frac{1}{2}}(\operatorname{ch}\eta),\quad Q^m_{n-\frac{1}{2}}(\operatorname{sh}\eta).$$

MO 96

are solutions of equation 8.850 1.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

8.851

Integral representations:

$$1.\quad P^m_{n-\frac{1}{2}}(\operatorname{ch}\eta)=\frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\frac{(\operatorname{sh}\eta)^m}{2^m\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)}\int_0^{\pi}\frac{\sin^{2m}\varphi\,d\varphi}{(\operatorname{ch}\eta+\operatorname{sh}\eta\cos\varphi)^{n+m+\frac{1}{2}}}=$$

$$(-1)^m \quad \Gamma\left(n+\frac{1}{2}\right) \quad \int^{2\pi} \quad \cos m\varphi d\varphi$$

2. $$Q^m_{n-\frac{1}{2}}(\operatorname{ch}\eta) = (-1)^m \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)} \int_0^\infty \frac{\operatorname{ch} mt\,dt}{(\operatorname{ch}\eta + \operatorname{sh}\eta \operatorname{ch} t)^{n+\frac{1}{2}}}; \qquad [n \geq m]$$
$$= (-1)^m \frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{\ln \operatorname{cth}\frac{\eta}{2}} (\operatorname{ch}\eta - \operatorname{sh}\eta \operatorname{ch} t)^{n-\frac{1}{2}} \operatorname{ch} mt\,dt.$$

MO 96

8.852

Functional relations:

1. $$Q^m_{n-\frac{1}{2}}(\operatorname{ch}\eta) = (-1)^m \frac{2^m \Gamma\left(n+m+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)} \operatorname{sh}^m \eta e^{-(n+m+\frac{1}{2})\eta} \times$$
$$\times F\left(m+\frac{1}{2}, n+m+\frac{1}{2}; n+1; e^{-2\eta}\right).$$

MO 96

2. $$P^{-m}_{n-\frac{1}{2}}(\operatorname{ch}\eta) = \frac{2^{-2m}}{\Gamma(m+1)} (1-e^{-2\eta})^m e^{-(n+\frac{1}{2})\eta} F\left(m+\frac{1}{2}, n+m+\frac{1}{2}; 2m+1; 1-e^{-2\eta}\right).$$

MO 96

8.853

An asymptotic representation $P_{n-\frac{1}{2}}(\operatorname{ch}\eta)$ for large values of n:

$$P_{n-\frac{1}{2}}(\operatorname{ch}\eta) = \frac{\Gamma(n) e^{(n-\frac{1}{2})\eta}}{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)} \times$$
$$\times \left[\frac{2\Gamma^2\left(n+\frac{1}{2}\right)}{\pi n! \Gamma(n)} \ln(4e^\eta) e^{-2n\eta} F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; e^{-2\eta}\right) + A + B\right],$$

where

Here,

$$u_r = \sum_{s=1}^{r} \frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^{r} \frac{2}{2s-1} \qquad [r - \text{a natural number}] .$$

MO 97

8.9 Orthogonal Polynomials

8.90 Introduction

8.901

Suppose that $w(x)$ is a nonnegative real function of a real variable x. Let (a, b) be a fixed interval on the x-axis. Let us suppose further that, for $n = 0, 1, 2, \ldots$, the integral

$$\int_a^b x^n w(x) dx$$

exists and that the integral

$$\int_a^b w(x) dx$$

is positive. In this case, there exists a sequence of polynomials $p_0(x), p_1(x), \ldots , p_n(x), \ldots$, that is uniquely

determined by the following conditions:

1. $p_n(x)$ is a polynomial of degree n and the coefficient of x^n in this polynomial is positive.

2. The polynomials $p_0(x), p_1(x), \ldots$ are orthonormal; that is,

We say that the polynomials $p_n(x)$ constitute *a system of orthogonal polynomials on the interval* (a, b) *with the weight function* $w(x)$.

8.902

If q_n is the coefficient of x^n in the polynomial $p_n(x)$, then

$$1.\quad \sum_{k=0}^{n} p_k(x) p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \qquad \text{(Darboux-Christoffel formula)}$$

EH II 159(10)

$$2.\quad \sum_{k=0}^{n} [p_k(x)]^2 = \frac{q_n}{q_{n+1}} \left[p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)\right] .$$

EH II 159(11)

8.903

Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n) p_{n-1}(x) - C_n p_{n-2}(x) \qquad [n = 2, 3, 4, \dots] .$$

In this formula, A_n, B_n, and C_n are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2}.$$

MO 102

8.904

Examples of normalized systems of orthogonal polynomials:

Notation and name	Interval	Weight
$\left(n+\frac{1}{2}\right)^{\frac{1}{2}} P_n(x)$, see 8.91	$(-1,+1)$	1
$2^\lambda \Gamma(\lambda) \left[\frac{(n+\lambda)n!}{2\pi\Gamma(2\lambda+n)}\right]^{\frac{1}{2}} C_n^\lambda(x)$, see 8.93	$(-1,+1)$	$(1-x^2)^{\lambda-\frac{1}{2}}$
$\sqrt{\frac{\varepsilon_n}{\pi}} T_n(x), \quad \varepsilon_0 = 1, \varepsilon_n = 2$ for $n = 1, 2, 3, \dots$, see 8.94	$(-1,+1)$	$(1-x^2)^{-\frac{1}{2}}$
$2^{-\frac{n}{2}} \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} H_n(x)$, see 8.95	$(-\infty,\infty)$	e^{-x^2}
$\left[\frac{\Gamma(n+1)\Gamma(\alpha+\beta+1+n)(\alpha+\beta+1+2n)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)2^{\alpha+\beta+1}}\right]^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x)$, see 8.96	$(-1,+1)$	$(1-x)^\alpha(1+x)^\beta$

8.95

8.94

8.93

8.91

Cf. 7.221 1., 7.313, 7.343, 7.374 1., 7.391 1., 7.414 3.

8.91 Legendre polynomials

8.910

Definition. The Legendre polynomials $P_n(z)$ are polynomials satisfying equation 8.700 1. withT $\mu = 0$ and $\nu = n$: that is, they satisfy the equation

$$1. \quad (1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + n(n+1)u = 0.$$

This equation has a polynomial solution if, and only if, n is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree n are of the form

$$2. \quad P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n.$$

8.911

Legendre polynomials written in expanded form:

$$1. \quad P_n(z) = \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}z^{n-2k} =$$
$$= \frac{(2n)!}{2^n(n!)^2}\left(z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}z^{n-4} - \dots\right);$$
$$= \frac{(2n-1)!!}{n!}z^n F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}-n; \frac{1}{z^2}\right).$$

2. $$P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n n!} \left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \ldots\right); \\ = (-1)^n \frac{(2n-1)!!}{2^n n!} F\left(-n, n+\frac{1}{2}; \frac{1}{2}; z^2\right).$$

AD (9002), MO 69

3. $$P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n n!} \left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \ldots\right); \\ = (-1)^n \frac{(2n+1)!!}{2^n n!} zF\left(-n, n+\frac{3}{2}; \frac{3}{2}; z^2\right).$$

AD (9002), MO 69

4. $$P_n(\cos\varphi) = \frac{(2n-1)!!}{2^n n!} \left(\cos n\varphi + \frac{1}{1}\frac{n}{2n-1}\cos(n-2)\varphi + \right. \\ + \frac{1\cdot 3}{1\cdot 2}\frac{n(n-1)}{(2n-1)(2n-3)}\cos(n-4)\varphi + \\ \left. + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos(n-6)\varphi - \ldots\right).$$

WH

5. $$P_{2n}(\cos\varphi) = (-1)^n \frac{(2n-1)!!}{2^n n!} \times \\ \times \left\{\sin^{2n}\varphi - \frac{(2n)^2}{2!}\sin^{2n-2}\varphi\cos^2\varphi + \ldots + (-1)^n \frac{2^n n!}{(2n-1)!!}\cos^{2n}\varphi\right\}.$$

AD (9011)

6. $$P_{2n+1}(\cos\varphi) = (-1)^n \frac{(2n+1)!!}{2^n n!}\cos\varphi \times \\ \times \left\{\sin^{2n}\varphi - \frac{(2n)^2}{3!}\sin^{2n-2}\varphi\cos^2\varphi + \ldots + (-1)^n \frac{2^n n!}{(2n+1)!!}\cos^{2n}\varphi\right\}.$$

AD (9012)

7. $$P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{(n-k)!(k!)^2 2^{k+1}} \left[(1-z)^k + (-1)^n (1+z)^k\right].$$

WH

8.912

Special cases:

1. $P_0(x) = 1.$

JA

2. $P_1(x) = x = \cos\varphi.$

JA

3. $P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3\cos 2\varphi + 1).$

JA

4. $P_3(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{8}(5\cos 3\varphi + 3\cos\varphi).$

JA

5. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = \frac{1}{64}(35\cos 4\varphi + 20\cos 2\varphi + 9).$

JA

6. $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) = \frac{1}{128}(63\cos 5\varphi + 35\cos 3\varphi + 30\cos\varphi).$

JA

7.* $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) = \frac{1}{512}(231\cos 6\varphi + 126\cos 4\varphi + 105\cos 2\varphi + 50).$

8.* $$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) = \\ = \frac{1}{1024}(429\cos 7\varphi + 231\cos 5\varphi + 189\cos 3\varphi + 175\cos\varphi).$$

9.* $P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
$= \frac{1}{16384}(6435\cos 8\varphi - 3432\cos 6\varphi + 2772\cos 4\varphi - 2520\cos 2\varphi + 1225).$

8.913

Integral representations:

1. $$P_n(\cos\varphi) = \frac{2}{\pi}\int_{\varphi}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{\sqrt{2(\cos\varphi - \cos t)}}\,dt.$$

WH

See also 3.611 3., 3.661 3., 4.

2.[7] Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{(t^2 - 1)^n}{2^n(t - z)^{n+1}}\,dt,$$

with C a closed contour containing z.

1046

3.* Laplace integral formula:

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \left[x + (x^2 - 1)^{1/2}\cos\varphi\right]^n d\varphi, \quad |x| \le 1$$

SA 180(19)
SA 175(9)

Functional relations

8.914

Recurrence formulas:

1. $$(n + 1)P_{n+1}(z) - (2n + 1)zP_n(z) + nP_{n-1}(z) = 0$$

WH

WH

8.915

1.* $$\sum_{k=0}^{n}(2k+1)P_k(x)P_k(y) = (n+1)\frac{P_n(x)P_{n+1}(y) - P_n(y)P_{n+1}(x)}{y-x}.$$

(Christoffel summation formula)

MO 70

1(1).* $$(y-x)\sum_{k=0}^{n}(2k+1)P_k(x)Q_k(y) = 1-(n+1)\left[P_{n+1}(x)Q_n(y) - P_n(x)Q_{n+1}(y)\right] .$$

AS 335(8.9.2)

2.[7] $$\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(2n-4k-1)P_{n-2k-1}(z) = P_n'(z) \qquad \text{(summation theorem)}$$

MO 70

3.[7] $$\sum_{k=0}^{\left[\frac{n-2}{2}\right]}(2n-4k-3)P_{n-2k-2}(z) = zP_n'(z) - nP_n(z)$$

SM 491(42), WH

4.* $$\sum_{k=1}^{\left[\frac{n}{2}\right]}(2n-4k+1)\left[k(2n-2k+1)-2\right]\, P_{n-2k}(z) = z^2P_n''(z) - n(n-1)P_n(z).$$

WH

5.* $$\sum_{k=0}^{m}\frac{a_{m-k}a_ka_{n-k}}{a_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right)P_{n+m-2k}(z) = P_n(z)P_m(z)$$

$$\left[a_k = \frac{(2k-1)!!}{k!}, \quad m \le n\right].$$

AD (9036)

8.916

1. $$P_n(\cos\varphi) = \frac{(2n-1)!!}{2^n n!}e^{\mp in\varphi}F\left(\frac{1}{2}, -n; \frac{1}{2}-n; e^{\pm 2i\varphi}\right).$$

2. $P_n(\cos\varphi) = F\left(n+1, -n; 1; \sin^2\frac{\varphi}{2}\right).$

MO 69

3. $P_n(\cos\varphi) = (-1)^n F\left(n+1, -n; 1; \cos^2\frac{\varphi}{2}\right).$

WH

4. $P_n(\cos\varphi) = \cos^n\varphi F\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1; -\operatorname{tg}^2\varphi\right).$

HO 23

5. $P_n(\cos\varphi) = \cos^{2n}\frac{\varphi}{2} F\left(-n, -n; 1; -\operatorname{tg}^2\frac{\varphi}{2}\right).$

HO 23, 29, WH

See also 8.911 1., 8.911 2., 8.911 3. For a connection with other functions, see 8.936 3., 8.836, 8.962 2. For integrals of Legendre polynomials, see 7.22-7.25. For the zeros of Legendre polynomials, see 8.785.

8.917

Inequalities:

1. For $x > 1 P_0(x) < P_1(x) < P_2(x) < \ldots < P_n(x) < \ldots$

MO 71

2. For $x > -1 P_0(x) + P_1(x) + \ldots + P_n(x) > 0.$

MO 71

3. $[P_n(\cos\varphi)]^2 > \frac{\sin(2n+1)\varphi}{(2n+1)\sin\varphi} \qquad [0 < \varphi < \pi].$

4. $\sqrt{n \sin \varphi}|P_n(\cos \varphi)| \leq 1.$

MO 71

5. $|P_n(\cos \varphi)| \leq 1.$

WH

6.* Let $n \geq 2$. The successive relative maxima of $|P_n(x)|$, when x decreases from 1 to 0, form a decreasing sequence. More precisely, if $\mu_1, \mu_2, \ldots, \mu_{[n/2]}$ denote these maxima corresponding to decreasing values of x, we have

$$1 > \mu_1 > \mu_2 > \ldots > \mu_{[n/2]}.$$

SZ 162(7.3.1)

7.* Let $n \geq 2$. The successive relative maxima of $(\sin \theta)^{\frac{1}{2}}|P_n(\cos \theta)|$ when θ increases from 0 to $\pi/2$, form an increasing sequence.

8.* We have

$$(\sin \theta)^{\frac{1}{2}}|P_n(\cos \theta)| < (2/\pi)^{\frac{1}{2}} n^{-\frac{1}{2}}, \qquad 0 \leq \theta \leq \pi.$$

SZ 163(7.3.8)
SZ 163(7.3.2)

Here the constant $(2/\pi)^{\frac{1}{2}}$ cannot be replaced by a smaller one.

9.* $\max_{0 \leq \theta \leq \pi} (\sin \theta)^{\frac{1}{2}}|P_n(\cos \theta)| \cong (2/\pi)^{\frac{1}{2}} n^{-\frac{1}{2}}, \qquad n \to \infty$.

10.* Stieltjes' first theorem:

$$|P_n(\cos \theta)| \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{4}{\sqrt{n \sin \theta}}, \qquad n = 1, 2, \ldots, 0 < \theta < \pi.$$

SA 197(8)
SZ 164(7.3.12)

11.* Stieltjes' second theorem:

SA 199(15)

12.* $\left|\frac{dP_n(x)}{dx}\right| < \frac{2}{\sqrt{\pi}}\frac{\sqrt{n}}{1-x^2}, \qquad |x|<1, \quad n=1,2,\ldots$

SA 201(18)

13.* $|P_{n+1}(x)+P_n(x)| < 6\left(\frac{2}{\pi n}\right)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}, \qquad |x|<1, \quad n=0,1,\ldots$

SA 201(19)

1048
8.918*

Asymptotic approximations:

1. $P_n(\cos\theta) = \left(\frac{2}{\pi n\sin\varphi}\right)^{\frac{1}{2}}\cos\left[\left(n+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]+O(n^{-\frac{3}{2}}), \qquad \varepsilon\le\theta\le\pi-\varepsilon, \quad 0<\varepsilon<\pi/2m$
(Laplace's formula).

SA 208(1)

2. $$P_n(\cos\theta) = \left(\frac{2}{\pi n\sin\theta}\right)^{\frac{1}{2}}\left\{\left(1-\frac{1}{4n}\right)\cos\left[\left(n+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]+\frac{1}{8n}\cos\theta\sin\left[\left(n+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]\right\}+ \\ +O(n^{-\frac{5}{2}}), \qquad \varepsilon\le\theta\le\pi-\varepsilon, \quad 0<\varepsilon<\pi/2$$
(Bonnet-Heine formula).

SA 208(2)

8.92 Series of Legendre polynomials

8.921

The generating function:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty}t^kP_k(z) \qquad \left[|t|<\min|z\pm\sqrt{z^2-1}|\right]; \\ = \sum_{k=0}^{\infty}\frac{1}{t^{k+1}}P_k(z) \qquad \left[|t|>\max|z\pm\sqrt{z^2-1}|\right].$$

1. $$z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z).$$

MO 72

2. $$z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z).$$

MO 72

3. $$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 P_{2k}(x) \qquad [|x| < 1, \quad (-1)!! \equiv 1] \ .$$

MO 72, LA 385(15)

4. $$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!(2k+1)!!}{2^{2k+1} k!(k+1)!} P_{2k+1}(x) \qquad [|x| < 1, \quad (-1)!! \equiv 1] \ .$$

LA 385(17)

5. $$\sqrt{1-x^2} = \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!(2k-1)!!}{2^{2k+1} k!(k+1)!} P_{2k}(x) \right\} \qquad [|x| < 1, \quad (-1)!! \equiv 1] \ .$$

LA 385(18)

6.* $$\sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)} P_n(x) \qquad [-1 \le x \le 1] \ .$$

7.* $$\frac{1-\rho^2}{(1-2\rho x+\rho^2)^{\frac{1}{2}}} = 1 + \sum_{n=0}^{\infty} (2n+1) \rho^n P_n(x), \quad |\rho| < 1, \quad |x| \le 1.$$

SA 170(4)

$$\arcsin x = \frac{\pi}{2} \sum_{k=1}^{\infty} \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 [P_{2k+1}(x) - P_{2k-1}(x)] + \pi x/2 \qquad [|x| < 1, \quad (-1)!! \equiv 1] \ .$$

8.924

1. $$-\frac{1+\cos n\pi}{2(n^2-1)}P_0(\cos\theta)-\frac{1+\cos n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2(n^2-2^2)\dots\left[n^2-(2k)^2\right]}{(n^2-1^2)(n^2-3^2)\dots[n^2-(2k+3)^2]}P_{2k+2}(\cos\theta)-$$
$$-\frac{3(1-\cos n\pi)}{2(n^2-2^2)}P_1(\cos\theta)-$$
$$-\frac{1-\cos n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)(n^2-1^2)\dots\left[n^2-(2k-1)^2\right]}{(n^2-2^2)(n^2-4^2)\dots[n^2-(2k+2)^2]}P_{2k+1}(\cos\theta)=\cos n\theta.$$

AD (9060.1)

2. $$\frac{-\sin n\pi}{2(n^2-1)}P_0(\cos\theta)-\frac{\sin n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2(n^2-2^2)\dots\left[n^2-(2k)^2\right]}{(n^2-1^2)(n^2-3^2)\dots[n^2-(2k+3)^2]}P_{2k+2}(\cos\theta)+$$
$$+\frac{3\sin n\pi}{2(n^2-2^2)}P_1(\cos\theta)+$$
$$+\frac{\sin n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)(n^2-1^2)(n^2-3^2)\dots\left[n^2-(2k-1)^2\right]}{(n^2-2^2)(n^2-4^2)\dots[n^2-(2k+2)^2]}P_{2k+1}(\cos\theta)=\sin n\theta.$$

AD (9060.2)

3.[3] $$\frac{2^{n-1}n!}{(2n-1)!!}P_n(\cos\theta)+$$
$$-n\sum_{k=1}^{[n/2]}(2n-4k+1)\frac{2^{n-2k-1}(n-k-1)!(2k-3)!!}{(2n-2k+1)!!k!}P_{n-2k}(\cos\theta)=\cos n\theta.$$

AD (9061.1)

4. $$\frac{(2n-1)!!P_{n-1}(\cos\theta)}{2^{n-1}(n-1)!}-$$
$$-\frac{n}{2^{n+1}}\sum_{k=0}^{\infty}\frac{(2n+2k-1)!!(2k-1)!!(2n+4k+3)}{2^{2k}(n+k+1)!(k+1)!}P_{n+2k+1}(\cos\theta)=\frac{4\sin n\theta}{\pi}.$$

AD (9061.2)

8.925

1. $$\sum_{k=1}^{\infty}\frac{4k-1}{2^{2k}(2k-1)^2}\left[\frac{(2k-1)!!}{k!}\right]^2P_{2k-1}(\cos\theta)=1-\frac{2\theta}{\pi}.$$

2. $$\sum_{k=1}^{\infty}\frac{4k+1}{2^{2k+1}(2k-1)(k+1)}\left[\frac{(2k-1)!!}{k!}\right]^2P_{2k}(\cos\theta)=\frac{1}{2}-\frac{2\sin\theta}{\pi}.$$

3. $$\sum_{k=1}^{\infty} \frac{k(4k-1)}{2^{2k-1}(2k-1)} \left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k-1}(\cos\theta) = \frac{2\operatorname{ctg}\theta}{\pi}.$$

AD (9062.3)

4. $$\sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k}(\cos\theta) = \frac{2}{\pi\sin\theta} - 1.$$

AD (9062.4)

8.926

1. $$\sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos\theta) = \ln \frac{2\operatorname{tg}\dfrac{\pi-\theta}{4}}{\sin\theta} = -\ln\sin\frac{\theta}{2} - \ln\left(1+\sin\frac{\theta}{2}\right).$$

AD (9063.2)

2. $$\sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos\theta) = \ln \frac{1+\sin\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} - 1.$$

AD (9063.1)

8.927

$$\sum_{k=0}^{\infty} \cos\left(k+\frac{1}{2}\right)\beta P_k(\cos\varphi) = \frac{1}{\sqrt{2(\cos\beta-\cos\varphi)}} \qquad [0 \le \beta < \varphi < \pi];$$
$$= 0 \qquad [0 < \varphi < \beta < \pi].$$

MO 72

8.928

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n(4k+1)\left[(2n-1)!!\right]^3}{2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\boldsymbol{K}(\sin\theta)}{\pi^2} - 1.$$

AD (9064.1)

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1)\left[(2n-1)!!\right]^3}{(2n-1)(2n+2)2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\boldsymbol{E}(\sin\theta)}{\pi^2} - \frac{1}{2}.$$

AD (9064.2)

For series of products of Bessel functions and Legendre polynomials, see 8.511 4., 8.531 3., 8.533 1., 8.533 2.[6], and 8.534.

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.930

Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$(1-2t\alpha+\alpha^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t)\alpha^n.$$

WH

Thus, the polynomials $C_n^\lambda(t)$ are a *generalization of the Legendre polynomials*.

1.* $C_0^\lambda(t) = 1.$

2.* $C_1^\lambda(t) = 2\lambda t.$

3.* $C_2^\lambda(t) = 2\lambda(\lambda+1)t^2 - \lambda.$

4.* $C_3^\lambda(t) = \frac{1}{3}\lambda(4\lambda^2+12\lambda+8)t^3 - 2\lambda(\lambda+1)t.$

5.* $C_4^\lambda(t) = \frac{2}{3}\lambda(\lambda^3+6\lambda^2+11\lambda+6)t^4 - 2\lambda(\lambda^3+3\lambda+2)t^2 + \frac{1}{2}\lambda(\lambda+1).$

6.* $C_5^\lambda(t) = \; = \frac{1}{15}\lambda(4\lambda^4+40\lambda^3+140\lambda^2+200\lambda+96)t^5 - - \frac{1}{3}\lambda(4\lambda^3+24\lambda^2+44\lambda+24)t^3 + \lambda(\lambda^2+3\lambda+2)t.$

7.* $C_6^\lambda(t) = \frac{1}{45}\lambda(\lambda^5 + 60\lambda^4 + 340\lambda^3 + 900\lambda^2 + 1096\lambda + 480)t^6 -$
$$- \frac{1}{3}\lambda(2\lambda^4 + 20\lambda^3 + 70\lambda^2 + 100\lambda + 48)t^4 +$$
$$+ \lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^2 + \frac{1}{6}\lambda(\lambda^2 + 3\lambda + 2).$$

8.931

Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\lambda+n)}{n!\Gamma(2\lambda)} \frac{\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(\lambda)} \int_0^\pi (t + \sqrt{t^2-1}\cos\varphi)^n \sin^{2\lambda-1}\varphi d\varphi.$$

MO 99

See also 3.252 11., 3.663 2., 3.664 4.

Functional relations

8.932

Expressions in terms of hypergeometric functions:

$$1. \quad C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(2\lambda)} F\left(2\lambda+n, -n; \lambda+\frac{1}{2}; \frac{1-t}{2}\right)^*;$$
$$= \frac{2^n\Gamma(\lambda+n)}{n!\Gamma(\lambda)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-\lambda-n; \frac{1}{t^2}\right).$$

* This equation is used for defining the generalized functions $C_n^\lambda(t)$, where the subscript n can be an arbitrary number.

MO 99
MO 97

$$2. \quad C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda+n)B(\lambda, n+1)} F\left(-n, n+\lambda; \frac{1}{2}; t^2\right).$$

MO 99

$$3. \quad C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{B(\lambda, n+1)} F\left(-n, n+\lambda+1; \frac{3}{2}; t^2\right).$$

Recursion formulas:

1. $(n+2)C_{n+2}^{\lambda}(t) = 2(\lambda+n+1)tC_{n+1}^{\lambda}(t) - (2\lambda+n)C_n^{\lambda}(t).$

Mo 98

2. $nC_n^{\lambda}(t) = 2\lambda\left[tC_{n-1}^{\lambda+1}(t) - C_{n-2}^{\lambda+1}(t)\right].$

WH

3. $(2\lambda+n)C_n^{\lambda}(t) = 2\lambda\left[C_n^{\lambda+1}(t) - tC_{n-1}^{\lambda+1}(t)\right].$

WH

4. $nC_n^{\lambda}(t) = (2\lambda+n-1)tC_{n-1}^{\lambda}(t) - 2\lambda(1-t^2)C_{n-2}^{\lambda+1}(t).$

WH

8.934

1. $$C_n^{\lambda}(t) = \frac{(-1)^n}{2^n}\frac{\Gamma(2\lambda+n)\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(2\lambda)\Gamma\left(\frac{2\lambda+1}{2}+n\right)}\frac{(1-t^2)^{\frac{1}{2}-\lambda}}{n!}\frac{d^n}{dt^n}\left[(1-t^2)^{\lambda+n-\frac{1}{2}}\right].$$

WH

2. $$C_n^{\lambda}(\cos\varphi) = \sum_{\substack{k,l=0\\k+l=n}}^{n}\frac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{k!l!\left[\Gamma(\lambda)\right]^2}\cos(k-l)\varphi.$$

MO 99

3. $$C_n^{\lambda}(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi) =$$
$$= \frac{\Gamma(2\lambda-1)}{\left[\Gamma(\lambda)\right]^2}\sum_{k=0}^{n}\frac{2^{2k}(n-k)!\left[\Gamma(\lambda+k)\right]^2}{\Gamma(2\lambda+n+k)}(2\lambda+2k-1)\sin^k\psi\sin^k\vartheta\times$$
$$\times C_{n-k}^{\lambda+k}(\cos\psi)C_{n-k}^{\lambda+k}(\cos\vartheta)C_k^{\lambda-\frac{1}{2}}(\cos\varphi)$$
$\left[\psi,\vartheta,\varphi \text{ real};\ \lambda\neq\frac{1}{2}\right]$ ["summation theorem"] (see also 8.794–8.796).

8.796
WH

4. $\lim_{\lambda\to 0}\Gamma(\lambda)C_n^\lambda(\cos\varphi) = \frac{2\cos n\varphi}{n}.$

MO 98

For orthogonality, see 8.904, 7.313.

8.935

Derivatives:

1. $\frac{d^k}{dt^k}C_n^\lambda(t) = 2^k\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}C_{n-k}^{\lambda+k}(t).$

MO 99

In particular,

2. $\frac{dC_n^\lambda(t)}{dt} = 2\lambda C_{n-1}^{\lambda+1}(t).$

WH

For integrals of the polynomials $C_n^\lambda(x)$ see 7.31-7.33.

8.936

Connections with other functions:

1. $C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\Gamma(n+1)}\left\{\frac{1}{4}(t^2-1)\right\}^{\frac{1}{4}-\frac{\lambda}{2}}P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t).$

MO 98

2. $C_{n-m}^{m+\frac{1}{2}}(t) = \frac{1}{(2m-1)!!}\frac{d^mP_n(t)}{dt^m} = (-1)^m\frac{(1-t^2)^{-\frac{m}{2}}m!2^m}{(2m)!}P_n^m(t)$
$[m+1 \text{ a natural number}]\,.$

3. $C_n^{\frac{1}{2}}(t) = P_n(t).$

4. $J_{\lambda-\frac{1}{2}}(r\sin\vartheta\sin\alpha)(r\sin\vartheta\sin\alpha)^{-\lambda+\frac{1}{2}}e^{-ir\cos\vartheta\cos\alpha} =$

$$= \sqrt{2}\frac{\Gamma(\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\sum_{k=0}^{\infty}(\lambda+k)i^{-k}\frac{\boldsymbol{J}_{\lambda+k}(r)C_k^{\lambda}(\cos\vartheta)C_k^{\lambda}(\cos\alpha)}{r^{\lambda}C_k^{\lambda}(1)}.$$

MO 99

5. $\lim_{\lambda\to\infty}\lambda^{-\frac{n}{2}}C_n^{\frac{\lambda}{2}}\left(t\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-\frac{n}{2}}}{n!}H_n(t).$

MO 99a

See also 8.932.

8.937

Special cases and particular values:

1. $C_n^1(\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}.$

MO 99

2. $C_0^0(\cos\varphi) = 1.$

MO 98

3. $C_0^{\lambda}(t) \equiv 1.$

MO 98

4. $C_n^{\lambda}(1) \equiv \binom{2\lambda+n-1}{n}.$

MO 98

8.938

A differential equation leading to the polynomials $C_n^\lambda(t)$:

$$y'' + \frac{(2\lambda+1)t}{t^2-1}y' - \frac{n(2\lambda+n)}{t^2-1}y = 0 \qquad \text{(cf. 9.174)}.$$

9.174
WH

For series of products of Bessel functions and the polynomials $C_n^\lambda(x)$, see 8.532, 8.534.

8.939*

Differentiation and Rodrigues' formulas and orthogonality relation

1. $\frac{d}{dt}C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t).$

MS 5.3.2

2. $\frac{d^m}{dt^m}C_n^\lambda(t) = 2^m\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+m-1)C_{n-m}^{\lambda+m}(t).$

MS 5.3.2

3. $\frac{d}{dt}C_{n-1}^\lambda(t) = t\frac{d}{dt}C_n^\lambda(t) - nC_n^\lambda(t).$

MS 5.3.2

4. $\frac{d}{dt}C_{n+1}^\lambda(t) = t\frac{d}{dt}C_n^\lambda(t) + (2\lambda+n)C_n^\lambda(t).$

MS 5.3.2

5. $(1-t^2)\frac{d}{dt}C_n^\lambda(t) = \; = (n+2\lambda-1)C_{n-1}^\lambda(t) - ntC_n^\lambda(t) = (n+2\lambda)tC_n^\lambda(t) - (n+1)C_{n+1}^\lambda(t) =$
$= 2\lambda(1-t^2)C_{n-1}^{\lambda+1}(t).$

MS 5.3.2

6. $\frac{d}{dt}\left[C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)\right] = 2(n+\lambda)C_n^\lambda(t).$

7. $$C_n^\lambda(t) = \frac{(-1)^n 2\lambda(2\lambda+1)(2\lambda+2)\dots(2\lambda+n-1)(1-t^2)^{\frac{1}{2}-\lambda}}{2^n n! \left(\lambda+\frac{1}{2}\right)\left(\lambda+\frac{3}{2}\right)\dots\left(\lambda+n-\frac{1}{2}\right)} \frac{d^n}{dt^n}\left[(1-t^2)^{n+\lambda-\frac{1}{2}}\right] =$$
$$= \frac{(-1)^n \Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(n+2\lambda)(1-t^2)^{\frac{1}{2}-\lambda}}{2^n n! \Gamma(2\lambda)\Gamma\left(n+\lambda+\frac{1}{2}\right)} \frac{d^n}{dt^n}\left[(1-t^2)^{n+\lambda-\frac{1}{2}}\right].$$

[Rodrigues' formula]

MS 5.3.2

8. $$\int_{-1}^{1} C_n^\lambda(t) C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt = \begin{cases} 0, & n \neq m \\ \dfrac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(\lambda+n)\left[\Gamma(\lambda)\right]^2}, & n = m. \end{cases} \qquad (\lambda \neq 0)$$

[Orthogonality relation]

MS 5.3.2

8.94 The Chebyshev polynomials $T_n(x)$ *and* $U_n(x)$

8.940

Definition

1. Chebyshev's polynomials of the first kind

$$T_n(x) = \cos(n \arccos x) = \frac{1}{2}\left[(x+i\sqrt{1-x^2})^n + (x-i\sqrt{1-x^2})^n\right] =$$
$$= x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \binom{n}{6} x^{n-6}(1-x^2)^6 + \dots$$

NA 66, 71

2. Chebyshev's polynomials of the second kind:

$$U_n(x) = \frac{\sin\left[(n+1)\arccos x\right]}{\sin\left[\arccos x\right]} = \frac{1}{2i\sqrt{1-x^2}}\left[(x+i\sqrt{1-x^2})^{n+1} - (x-i\sqrt{1-x^2})^{n+1}\right] =$$
$$= \binom{n+1}{1} x^n - \binom{n+1}{3} x^{n-2}(1-x^2) + \binom{n+1}{5} x^{n-4}(1-x^2)^2 - \dots$$

Recursion formulas:

1. $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$

NA 358

2. $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0.$

3. $T_n(x) = U_n(x) - xU_{n-1}(x).$

EH II 184(3)

4. $(1-x^2)U_{n-1}(x) = xT_n(x) - T_{n+1}(x).$

EH II 184(4)

For the orthogonality, see 7.343 and 8.904.

1055

8.942

Relations with other functions:

1. $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right).$

MO 104

2. $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}.$

MO 104

3. $U_n(x) = \frac{(-1)^n(n+1)}{\sqrt{1-x^2}(2n+1)!!} \frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}.$

EH II 185(15)

See also 8.962 3.

8.943*

Special cases

1. $T_0(x) = 1.$

2. $T_1(x) = x.$

3. $T_2(x) = 2x^2 - 1.$

4. $T_3(x) = 4x^3 - 3x.$

5. $T_4(x) = 8x^4 - 8x^2 + 1.$

6. $T_5(x) = 16x^5 - 20x^3 + 5x.$

7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$

8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x.$

9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1.$

10. $U_0(x) = 1.$

11. $U_1(x) = 2x.$

12. $U_2(x) = 4x^2 - 1.$

13. $U_3(x) = 8x^3 - 4x.$

14. $U_4(x) = 16x^4 - 12x^2 + 1.$

15. $U_5(x) = 32x^5 - 32x^3 + 6x.$

16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.$

17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x.$

18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1.$

8.944

Particular values:

1. $T_n(1) = 1.$
2. $T_n(-1) = (-1)^n.$
3. $T_{2n}(0) = (-1)^n.$
4. $T_{2n+1}(0) = 0.$
5. $U_{2n+1}(0) = 0.$
6. $U_{2n}(0) = (-1)^n.$

1056

8.945

The generating function:

1. $$\frac{1 - t^2}{1 - 2tx + t^2} = T_0(x) + 2\sum_{k=1}^{\infty} T_k(x) t^k.$$

2. $$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x) t^k.$$

MO 104a, EH II 186(31)

8.946

Zeros.

The polynomials $T_n(x)$ and $U_n(x)$ only have real simple zeros. All these zeros lie in the interval $(-1, +1)$.

8.947

The functions $T_n(x)$ and $\sqrt{1-x^2}U_{n-1}(x)$ are two linearly independent solutions of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0.$$

NA 69(58)

8.948

Of all polynomials of degree n with leading coefficient equal to 1, the one that deviates the least from zero on the interval $[-1, +1]$ is the polynomial $2^{-n+1}T_n(x)$.

8.949*

Differentiation and Rodrigues' formulas and orthogonality relations

1. $$\frac{d}{dx}T_n(x) = nU_{n-1}(x).$$

MS 5.7.2

2. $$\frac{d^m}{dx^m}T_n(x) = 2^{m-1}\Gamma(m)nC_{n-m}^{m}(x).$$

MS 5.7.2

3. $$(1-x^2)\frac{d}{dx}T_n(x) = n\left[T_{n-1}(x) - xT_n(x)\right] = n\left[xT_n(x) - T_{n+1}(x)\right].$$

MS 5.7.2

4. $$\frac{d}{dx}U_n(x) = 2C_{n-1}^{2}(x).$$

MS 5.7.2

5. $\dfrac{d^m}{dx^m}U_n(x) = 2^m m! C_{n-m}^{m+1}(x).$

MS 5.7.2

6. $(1-x^2)\dfrac{d}{dx}U_n(x) = (n+1)U_{n-1}(x) - nxU_n(x) = (n+2)xU_n(x) - (n+1)U_{n+1}(x).$

MS 5.7.2

7. $T_n(x) = \dfrac{(-1)^n \pi^{\frac{1}{2}}(1-x^2)^{c\frac{1}{2}}}{2^{n+1}\Gamma\left(n+\frac{1}{2}\right)} \dfrac{d^n}{dx^n}\left[(1-x^2)^{n-\frac{1}{2}}\right].$ [Rodrigues' formula]

MS 5.7.2

8. $U_n(x) = \dfrac{(-1)^n \pi^{\frac{1}{2}}(n+1)(1-x^2)^{-\frac{1}{2}}}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)} \dfrac{d^n}{dx^n}\left[(1-x^2)^{n+\frac{1}{2}}\right].$ [Rodrigues' formula]

MS 5.7.2

9. $\displaystyle\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-\frac{1}{2}}\,dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0. \end{cases}$ [Orthogonality relation]

MS 5.7.2

10. $\displaystyle\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{-\frac{1}{2}}\,dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n. \end{cases}$ [Orthogonality relation]

MS 5.7.2

8.95 The Hermite polynomials $H_n(x)$

8.950

Definition

SM 567(14)

or

2. $H_n(x) = 2^n x^n - 2^{n-1}\binom{n}{2}x^{n-2} + 2^{n-2}\cdot 1\cdot 3\cdot\binom{n}{4}x^{n-4} - 2^{n-3}\cdot 1\cdot 3\cdot 5\cdot\binom{n}{6}x^{n-6} + \dots$

MO 105a

3.* $H_0(x) = 1.$

4.* $H_1(x) = 2x.$

5.* $H_2(x) = 4x^2 - 2.$

6.* $H_3(x) = 8x^3 - 12x.$

7.* $H_4(x) = 16x^4 - 48x^2 + 12.$

8.* $H_5(x) = 32x^5 - 160x^3 + 120x.$

9.* $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$

10.* $H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x.$

11.* $H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680.$

8.951

The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2} \, dt.$$

MO 106a

Functional relations

8.952

Recursion formulas:

1. $\dfrac{dH_n(x)}{dx} = 2nH_{n-1}(x).$

SM 569(22)

2. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$

SM 570(23)

For the orthogonality, see 7.374 1. and 8.904.

3.* $nH_n(x) = -nH'_{n-1}(x) + xH'_n(x).$

MS 5.6.2

4.* $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x).$

MS 5.6.2

8.953

The connection with other functions:

1. $H_{2n}(x) = (-1)^n \dfrac{(2n)!}{n!} \Phi\left(-n, \dfrac{1}{2}; x^2\right).$

2. $H_{2n+1}(x) = (-1)^n 2 \frac{(2n+1)!}{n!} x \Phi\left(-n, \frac{3}{2}; x^2\right).$

MO 106a

For a connection with the polynomials $C_n^\lambda(x)$, see 8.936 5.

For a connection with the Laguerre polynomials, see 8.972 2. and 8.972 3.

For a connection with functions of a parabolic cylinder, see 9.253.

8.954

Inequalities:

1.* $|H_n(x)| \leq 2^{\frac{n}{2}-[n/2]} \frac{n!}{[[n/2]]!} e^{2x\sqrt{[n/2]}} \qquad [x > 0].$

MO 106a

2.* $|H_n(x)| < k\sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k = 1 \cdot 086435.$

SA 324

8.955

Asymptotic representation:

1. $H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[\cos(\sqrt{4n+1}x) + O\left(\frac{1}{\sqrt[4]{n}}\right)\right].$

SM 579

2. $H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1} e^{x^2/2} \left[\sin(\sqrt{4n+3}x) + O\left(\frac{1}{\sqrt[4]{n}}\right)\right].$

SM 579

8.956

Special cases and particular values:

SM 570(24)

7. $H_{2n+1}(0) = 0.$

Series of hermite polynomials

8.957

The generating function:

1. $\exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x).$

SM 569(21)

2. $\frac{1}{e} \operatorname{sh} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x).$

MO 106a

3. $\frac{1}{e} \operatorname{ch} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x).$

MO 106a

4. $e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x).$

MO 106a

5. $e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x).$

"The summation theorem":

1. $$\frac{\left(\sum_{k=1}^{r} a_k^2\right)^{\frac{n}{2}}}{n!} H_n \left(\frac{\sum_{k=1}^{r} a_k x_k}{\sqrt{\sum_{k=1}^{r} a_k^2}} \right) = \sum_{m_1+m_2+\ldots+m_r=n} \prod_{k=1}^{r} \left\{ \frac{a_k^m k}{m_k!} H_{m_k}(x_k) \right\}.$$

MO 106a

2. A special case:

1. $$2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}).$$

MO 107a

8.959

Hermite polynomials satisfy the differential equation

1. $$\frac{d^2 u_n}{dx^2} - 2x \frac{du_n}{dx} + 2n u_n = 0;$$

SM 566(9)

A second solution of this differential equation is provided by the functions:

2. $$u_{2n} = (-1)^n A x \Phi\left(\frac{1}{2} - n; \frac{3}{2}; x^2\right),$$

3. $$u_{2n+1} = (-1)^n B \Phi\left(-\frac{1}{2} - n; \frac{1}{2}; x^2\right) \qquad [A \text{ and } B\text{-arbitrary constants}].$$

MO 107

8.959(1)*

Rodrigues' formula and orthogonality relation

MS 5.6.2

2. $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0, & m \neq n \\ \pi^{1/2} 2^n n!, & m = n. \end{cases}$$

MS 5.6.2

8.96 Jacobi's polynomials

8.960

Definition

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] ;$$
$$= \frac{1}{2^n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m}(x+1)^m.$$

EH II 169(2)
EH II 169(10), CO

8.961

Functional relations:

1. $$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

EH II 169(13)

2. $$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) =$$
$$= (2n+\alpha+\beta+1)\left[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2\right] \; P_n^{(\alpha,\beta)}(x) -$$
$$- 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x).$$

EH II 169(11)

3. $$(2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = n\left[(\alpha-\beta) - (2n+\alpha+\beta)x\right] \; P_n^{(\alpha,\beta)}(x) +$$
$$+ 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x).$$

EH II 170(15)

4. $$\frac{d^m}{dx^m}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{2^m}\frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-m}^{(\alpha+m,\beta+m)}(x) \qquad [m=1,2,\ldots,n].$$

EH II 170(17)

5. $$\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)P_n^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x).$$

EH II 173(32)

6. $$\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1+x)P_n^{(\alpha,\beta+1)}(x) = (n+\beta+1)P_n^{(\alpha,\beta)}(x)+(n+1)P_{n+1}^{(\alpha,\beta)}(x).$$

EH II 173(33)

7. $$(1-x)P_n^{(\alpha+1,\beta)}(x)+(1+x)P_n^{(\alpha,\beta+1)}(x) = 2P_n^{(\alpha,\beta)}(x).$$

EH II 173(34)

8. $$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)-(n+\beta)P_{n-1}^{(\alpha,\beta)}(x).$$

EH II 173(35)

9. $$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)+(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x).$$

EH II 173(36)

10. $$P_n^{(\alpha,\beta-1)}(x)-P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x).$$

EH II 173(37)

8.962

Connections with other functions:

1. $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\Gamma(n+1+\beta)}{n!\Gamma(1+\beta)}F\left(n+\alpha+\beta+1,-n;1+\beta;\frac{1+x}{2}\right);$$
$$= \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}F\left(n+\alpha+\beta+1,-n;1+\alpha;\frac{1-x}{2}\right);$$
$$= \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}\left(\frac{1+x}{2}\right)^n F\left(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}\right);$$
$$\Gamma(n+1+\beta)\ \left(x-1\right)^n F\left(\qquad \beta+1 \quad x+1\right)$$

2. $P_n(x) = P_n^{(0,0)}(x).$

CO, EH II 179(3)

3. $T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x).$

CO, EH II 184(5)a

1061

4. $C_n^{\nu}(x) = \frac{\Gamma(n+2\nu)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{(\nu-1/2,\nu-1/2)}(x).$

MO 108a, EH II 174(4)

8.963

The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta},$$

$$R = \sqrt{1-2xz+z^2} \qquad [|z|<1]\,.$$

EH II 172(29)

8.964

The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]\; y' + n(n+\alpha+\beta+1)y = 0.$$

EH II 169(14)

8.965

Asymptotic representation

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos\left\{\left[n+\frac{1}{2}(\alpha+\beta+1)\right]\theta - \left(\frac{1}{2}\alpha+\frac{1}{4}\right)\pi\right\}}{\sqrt{\pi n}\left(\sin\frac{1}{2}\theta\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{1}{2}\theta\right)^{\beta+\frac{1}{2}}} + O(n^{-\frac{3}{2}})$$

$$[\operatorname{Im}\alpha = \operatorname{Im}\beta = 0, \quad 0<\theta<\pi]\,.$$

8.966

A limit relationship:

$$\lim_{n\to\infty}\left[n^{-\alpha}P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)\right]=\left(\frac{z}{2}\right)^{-\alpha}J_\alpha(z).$$

EH II 173(41)

8.967

If $\alpha>-1$ and $\beta>-1$, all the zeros of the polynomial $P_n^{(\alpha,\beta)}(x)$ are simple and they lie in the interval $(-1,1)$.

8.97 The Laguerre polynomials

8.970

Definition.

1. $$L_n^\alpha(x)=\frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha});\qquad \text{[Rodrigues' formula]}$$
$$=\sum_{m=0}^{n}(-1)^m\binom{n+\alpha}{n-m}\frac{x^m}{m!}.$$

MO 109, EH II 188(7)
EH II 188(5), MO 108

2. $L_n^0(x)=L_n(x).$

ET I 369

3.* $L_0^\alpha(x)=1.$

4.* $L_1^\alpha(x)=-x+\alpha+1.$

5.* $L_2^\alpha(x)=\frac{1}{2}\left[x^2-2(\alpha+2)x+(\alpha+1)(\alpha+2)\right]$.

6.* $L_3^\alpha(x) = -\frac{1}{6}\left[x^3 - 3(\alpha+3)x^2 + 3(\alpha+2)(\alpha+3)x - (\alpha+1)(\alpha+2)(\alpha+3)\right]$.

7.* $L_4^\alpha(x) = \frac{1}{24}\left[x^4 - 4(\alpha+4)x^3 + 6(\alpha+3)(\alpha+4)x^2 - 4(\alpha+2)(\alpha+3)(\alpha+4)x + \right.$
$\left. + (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\right]$.

8.* $L_5^\alpha(x) = -\frac{1}{120}\left[x^5 - 5(\alpha+5)x^4 + 10(\alpha+4)(\alpha+5)x^3 - 10(\alpha+3)(\alpha+4)(\alpha+5)x^2\right.$
$\left. + 5(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)x - (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)\right]$.

8.971

Functional relations:

1. $\frac{d}{dx}\left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] = L_n^\alpha(x).$

EH II 189(16)

2. $\frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)}{x}.$

EH II 189(15), SM 575(42)a

3. $x\frac{d}{dx}L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x);$
$= (n+1)L_{n+1}^\alpha(x) - (n+\alpha+1-x)L_n^\alpha(x).$

EH II 189(12), MO 109

4. $xL_n^{\alpha+1}(x) = (n+\alpha+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x);$
$= (n+\alpha)L_{n-1}^\alpha(x) - (n-x)L_n^\alpha(x).$

SM 575(43)A, EH II 190(23)

5. $L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x).$

SM 575(44)A, EH II 190(24)

6. $(n+1)L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0 \qquad [n = 1, 2, \ldots]$.

7.* $(n+\alpha)L_n^{\alpha-1}(x) = (n+1)L_{n+1}^{\alpha}(x) - (n+1-x)L_n^{\alpha}(x)$

MS 5.5.2

8.* $nL_n^{\alpha}(x) = (2n+\alpha-1-x)L_{n-1}^{\alpha}(x) - (n+\alpha-1)L_{n-2}^{\alpha}(x)n = 2, 3, \ldots$

MS 5.5.2

8.972

Connections with other functions:

1. $L_n^{\alpha}(x) = \binom{n+\alpha}{n} \Phi(-n, \alpha+1; x).$

MO 109, FI II 189(14)

2. $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(x^2).$

EH II 193(2), SM 576(47)

3. $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2).$

EH II 193(3), SM 577(48)

8.973

Special cases:

1. $L_0^{\alpha}(x) = 1$

EH II 188(6)

2. $L_1^{\alpha}(x) = \alpha + 1 - x.$

EH II 188(6)

EH II 189(13)

4. $L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}.$

MO 109

5. $L_1(x) = 1 - x.$

6. $L_2(x) = 1 - 2x + \frac{x^2}{2}.$

MO 109

8.974

Finite sums:

1. $$\sum_{m=0}^{n} \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x) L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} \left[L_n^\alpha(x) L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x) L_n^\alpha(y)\right]$$

EH II 188(9)

2. $$\sum_{m=0}^{n} \frac{\Gamma(\alpha-\beta+m)}{\Gamma(\alpha-\beta)m!} L_{n-m}^\beta(x) = L_n^\alpha(x).$$

MO 110, EH II 192(39)

3. $$\sum_{m=0}^{n} L_m^\alpha(x) = L_n^{\alpha+1}(x).$$

EH II 192(38)

4. $$\sum_{m=0}^{n} L_m^\alpha(x) L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y).$$

EH II 192(41)

8.975

Arbitrary functions:

1. $$(1-z)^{-\alpha-1}\exp\frac{xz}{z-1}=\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n \qquad [|z|<1]\,.$$

EH II 189(17), MO 109

2. $$e^{-xz}(1+z)^{\alpha}=\sum_{n=0}^{\infty}L_n^{\alpha-n}(x)z^n \qquad [|z|<1]\,.$$

MO 110, EH II 189(19)

3. $$J_{\alpha}(2\sqrt{xz})e^{z}(xz)^{-\frac{1}{2}\alpha}=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+\alpha+1)}L_n^{\alpha}(x) \qquad [\alpha>-1]\,.$$

EH II 189(18), MO 109

8.976

Other series of Laguerre polynomials:

1. $$\sum_{n=0}^{\infty}n!\frac{L_n^{\alpha}(x)L_n^{\alpha}(y)z^n}{\Gamma(n+\alpha+1)}=\frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z}\exp\left(-z\frac{x+y}{1-z}\right)I_{\alpha}\left(2\frac{\sqrt{xyz}}{1-z}\right) \qquad [|z|<1]\,.$$

EH II 189(20)

2. $$\sum_{n=0}^{\infty}\frac{L_n^{\alpha}(x)}{n+1}=e^{x}x^{-\alpha}\Gamma(\alpha,x) \qquad [\alpha>-1, \quad x>0]\,.$$

EH II 215(19)

3.[6] $$[L_n^{\alpha}(x)]^2=\frac{\Gamma(n+\alpha+1)}{2^{2n}n!}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\frac{(2k)!}{k!}\frac{1}{\Gamma(\alpha+k+1)}L_{2k}^{2\alpha}(2x).$$

MO 110

4.[6] $$L_n^{\alpha}(x)L_n^{\alpha}(y)=\frac{\Gamma(1+\alpha+n)}{n!}\sum_{k=0}^{n}\frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)}\frac{(xy)^k}{k!}.$$

8.977

Summation theorems:

1. $L_n^{\alpha_1+\alpha_2+\ldots+\alpha_k+k-1}(x_1+x_2+\ldots+x_k) = \sum_{(i_1+i_2+\ldots+i_k=n)} L_{i_1}^{\alpha_1}(x_1) L_{i_2}{}^{\alpha_2}(x_2)\ldots L_{i_k}^{\alpha_k}(x_k).$

MO 110

2. $L_n^{\alpha}(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L_n^{\alpha+k}(x).$

MO 110

8.978

Limit relations and asymptotic behavior:

1. $L_n^{\alpha}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right).$

EH II 191(35)

2. $\lim_{n\to\infty}\left[n^{-\alpha} L_n^{\alpha}\left(\frac{x}{n}\right)\right] = x^{-\frac{1}{2}\alpha} J_{\alpha}(2\sqrt{x}).$

EH II 191(36)

3. $L_n^{\alpha}(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha-\frac{1}{4}} n^{\frac{1}{2}\alpha-\frac{1}{4}} \cos\left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O(n^{\frac{1}{2}\alpha-\frac{3}{4}}) \qquad [\operatorname{Im}\alpha = 0, \quad x>0].$

EH II 199(1)

8.979

Laguerre polynomials satisfy the following differential equation:

$$x\frac{d^2u}{dx^2} + (\alpha - x + 1)\frac{du}{dx} + nu = 0.$$

EH II 188(10), SM 574(34)

8.980*

Orthogonality relation

$$\int_0^\infty e^{-x}x^\alpha L_m^\alpha(x)L_n^\alpha(x)dx = \begin{cases} 0, & m \neq n \\ \Gamma(1+\alpha)\binom{n+\alpha}{n}, & m = n \end{cases}$$

MS 5.5.2

8.981*

Behavior of relative maxima of $|L_n^\alpha(x)|$

1. *Let α be arbitrary and real. The sequence formed by the relative maxima of $|L_n^\alpha(x)|$ and by the value of this function at $x = 0$, is decreasing for $x < \alpha + \frac{1}{2}$, and increasing for $x > \alpha + \frac{1}{2}$. The successive relative maxima of $|H_n(x)|$ form a decreasing sequence for $x \leq 0$, and an increasing sequence for $x \geq 0$.*

SZ 174(7.6.1)

2. *Let α be an arbitrary real number. The successive relative maxima of*

$$(7.6.3) \qquad e^{-x/2}x^{(\alpha+1)/2}|L_n^\alpha(x)| \quad and \quad e^{-x/2}x^{\alpha/2+\frac{1}{4}}|L_n^\alpha(x)|$$

form an increasing sequence provided $x > x_0$. In the first case

$$(7.6.4) \qquad x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \dfrac{\alpha^2-1}{2n+\alpha+1} & \text{if } \alpha^2 > 1. \end{cases}$$

In the second case

$$(7.6.5) \qquad x_0 = \begin{cases} 0 & if\ \alpha^2 \leq \dfrac{1}{4}, \\ \left(\alpha^2 - \dfrac{1}{4}\right)^{\frac{1}{2}} & if\ \alpha^2 > \dfrac{1}{4}. \end{cases}$$

SZ 174(7.6.2)

In the first case we take n *so large that* $2n+\alpha+1>0$.

8.982*

Asymptotic and limiting behavior of $L_n^\alpha(x)$

1. *Let* α *be arbitrary and real,* c *and* w *fixed positive constants, and let* $n \to \infty$. *Then*

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}} O(n^{\alpha/2-\frac{1}{4}}) & \text{if } cn^{-1} \le x \le \omega, \\ O(n^\alpha) & \text{if } 0 \le x \le cn^{-1}. \end{cases}$$

These bounds are precise as regards their orders in n. *For* $\alpha \ge -\frac{1}{2}$, *both bounds hold in both intervals, that is,*

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}} O(n^{\alpha/2-\frac{1}{4}}), \\ O(n^\alpha), \end{cases} 0 < x \le \omega, \alpha \ge -\frac{1}{2}.$$

SZ 175(7.6.4)

2. *Let* α *be arbitrary and real. Then for an arbitrary complex* z

$$\lim_{n\to\infty} n^{-\alpha} L_n(z/n) = z^{-\alpha/2} J_\alpha(2z^{\frac{1}{2}}),$$

SZ 191(8.1.3)

uniformly if z *is bounded.*

9.1 Hypergeometric Functions

9.10 Definition

9.100[7]

A *hypergeometric series*, also called a Gaussian hypergeometric function, is a series of the form

$$F(\alpha,\, \beta;\, \gamma;\, z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 1\cdot 2\cdot 3} z^3 + \dots$$

A hypergeometric series terminates if α or β is equal to a negative integer or to zero. For $\gamma = -n (n = 0, 1, 2, \ldots)$, the hypergeometric series is indeterminate if neither α nor β is equal to $-m$ (where $m < n$ and m is a natural number). However,

$$1. \quad \lim_{\gamma \to -n} \frac{F(\alpha,\ \beta;\ \gamma;\ z)}{\Gamma(\gamma)} = \frac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{(n+1)!} \times \\ \times z^{n+1} F(\alpha+n+1, \beta+n+1; n+2; z).$$

EH I 62(16)

9.102

If we exclude these values of the parameters α, β, γ, a hypergeometric series converges in the unit circle $|z| < 1$. F then has a branch point at $z = 1$. Then we have the following conditions for convergence on the unit circle:

1. $1 > \operatorname{Re}(\alpha + \beta - \gamma) \geq 0$. The series converges throughout the entire unit circle except at the point $z = 1$.

2. $\operatorname{Re}(\alpha + \beta - \gamma) < 0$. The series converges (absolutely) throughout the entire unit circle.

3. $\operatorname{Re}(\alpha + \beta - \gamma) \geq 1$. The series diverges on the entire unit circle.

FI II 410, WH

9.11 Integral representations

9.111

$$F(\alpha,\ \beta;\ \gamma;\ z) = \frac{1}{B(\beta,\ \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt \qquad [\operatorname{Re}\,\gamma > \operatorname{Re}\,\beta > 0]\,.$$

WH

9.112[8]

$$F(p,\ n+p;\ n+1;\ z^2) = \frac{z^{-n}}{2\pi} \frac{\Gamma(p)n!}{\Gamma(p+n)} \int_0^{2\pi} \frac{\cos\ nt\,dt}{(1 - 2z\cos t + z^2)^p} \\ [n = 0,\ 1,\ 2, \ldots; \quad p \neq 0,\ -1,\ -2, \ldots;\ |z| < 1].$$

WH, MO 16

9.113

$$F(\alpha,\ \beta;\ \gamma;\ z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t\,dt,$$

Here, $|\arg(-z)| < \pi$ and the path of integration is chosen in such a way that the poles of the functions $\Gamma(\alpha+t)$ and $\Gamma(\beta+t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of it.

9.114

$$F\left(-m, -\frac{p+m}{2}; 1-\frac{p+m}{2}; -1\right) = \frac{(-2)^m(p+m)}{\sin p\pi}\int_0^\pi \cos^m\varphi\cos p\varphi\,d\varphi$$
$$[m+1 \text{ a natural number}; p \neq 0, \quad \pm 1, \ldots].$$

EH I 80(8), MO 16

See also 3.194 1., 2., 5., 3.196 1., 3.197 6., 9., 3.259 3, 3.312 3., 3.518 4.-6., 3.665 2., 3.671 1., 2., 3.681 1., 3.984 7.

9.12 The representation of elementary functions in terms of a hypergeometric function

9.121

1. $F(-n, \beta; \beta; -z) = (1+z)^n \qquad [\beta \text{ arbitrary}]$

EH I 101(4), GA 127 Ia

2. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n + (t-z)^n}{2t^n}.$

GA 127 II

3. $\lim\limits_{\omega\to\infty} F\left(-n, \omega; 2\omega; -\frac{z}{t}\right) = \left(1+\frac{z}{2t}\right)^n.$

GA 127 IIIa

4. $F\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}.$

GA 127 IV

5. $F\left(1-n, 1; 2; -\frac{z}{t}\right) = \frac{(t+z)^n - t^n}{nzt^{n-1}}.$

6. $F(1,\ 1;\ 2;\ -z) = \dfrac{\ln(1+z)}{z}.$

GA 127 VI

7. $F\left(\dfrac{1}{2},\ 1;\ \dfrac{3}{2};\ z^2\right) = \dfrac{\ln\frac{1+z}{1-z}}{2z}.$

GA 127 VII

8. $$\lim_{k\to\infty} F\left(1,\ k;\ 1;\ \frac{z}{k}\right) = 1 + z \lim_{k\to\infty} F\left(1,\ k;\ 2;\ \frac{z}{k}\right) = = 1 + z + \frac{z^2}{2} \lim_{k\to\infty} F\left(1,\ k;\ 3;\ \frac{z}{k}\right) = \ldots = e^z.$$

GA 127 VIII

9. $$\lim_{\substack{k\to\infty \\ k'\to\infty}} F\left(k,\ k';\ \frac{1}{2};\ \frac{z^2}{4kk'}\right) = \frac{e^z + e^{-z}}{2} = \operatorname{ch}\ z.$$

GA 127 IX

10. $$\lim_{\substack{k\to\infty \\ k'\to\infty}} F\left(k,\ k';\ \frac{3}{2};\ \frac{z^2}{4kk'}\right) = \frac{e^z - e^{-z}}{2z} = \frac{\operatorname{sh}\ z}{z}.$$

GA 127 X

11. $$\lim_{\substack{k\to\infty \\ k'\to\infty}} F\left(k,\ k';\ \frac{3}{2};\ -\frac{z^2}{4kk'}\right) = \frac{\sin\ z}{z}.$$

GA 127 XI

12. $$\lim_{\substack{k\to\infty \\ k'\to\infty}} F\left(k,\ k';\ \frac{1}{2};\ -\frac{z^2}{4kk'}\right) = \cos\ z.$$

GA 127 XII

13. $$F\left(\frac{1}{2},\ \frac{1}{2};\ \frac{3}{2}; \sin^2\ z\right) = \frac{z}{\sin\ z}.$$

GA 127 XIII

GA 127 XIV

15. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{z}{\operatorname{tg} z}.$

GA 127 XV

16. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z}.$

GA 127 XVI

17. $F\left(\frac{n+2}{2}, -\frac{n-2}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z \cos z}.$

GA 127 XVII

18. $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}.$

GA 127 XVIII

19. $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}.$

GA 127 XIX

20. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz.$

EH I 101(11), GA 127 XX

21. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}.$

EH I 101(11), GA 127 XXI

22. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \frac{\cos nz}{\cos^n z}.$

23. $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \cos nz \cos^n z.$

GA 127 XXIII

24. $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z} \qquad \left[|z| \leq \frac{1}{2}; \quad |z(1-z)| \leq \frac{1}{4}\right].$

25. $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z.$

26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$ (cf. 9.121 13.).

9.121

27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arctg} z}{z}$ (cf. 9.121 15.).

9.121

28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{Arsh} z}{z}$ (cf. 9.121 26.).

9.121

29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$ (cf. 9.121 16.).

9.121

30. $F\left(1+\frac{n}{2}, 1-\frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$ (cf. 9.121 17.).

9.121

31. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z) \qquad (\text{cf. } 9.121\ 20.).$

9.121

32. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}. \qquad (\text{cf. } 9.121\ 21.).$

9.121

The representation of special functions in terms of a hypergeometric function

For complete elliptic integrals, see 8.113 1. and 8.114 1.;

for integrals of Bessel functions, see 6.574 1., 3., 6.576 2.-5., 6.621 1.-3.;

for Legendre polynomials, 8.911 and 8.916. (All these hypergeometric series terminate; that is, these series are finite sums);

for Legendre functions, see 8.820 and 8.837;

for associated Legendre functions, see 8.702, 8.703, 8.751, 8.77, 8.852, and 8.853;

for Chebyshev polynomials, see 8.942 1.;

for Jacobi's polynomials, see 8.962;

for Gegenbauer polynomials $C_n^{\prime\lambda}(x)$, see 8.932;

for integrals of parabolic-cylinder functions, see 7.725 6.

9.122

Particular values:

GA 147(48), FI II 793

1069

$$2.\quad F(\alpha,\,\beta;\,\gamma;\,1) = F(-\alpha,\,-\beta;\,\gamma-\alpha-\beta;\,1), \mathrm{Re}\;\gamma > \mathrm{Re}(\alpha+\beta)s$$
$$= \frac{1}{F(-\alpha,\,\beta;\,\gamma-\alpha;\,1)}, \qquad \mathrm{Re}\;\gamma > \mathrm{Re}\,(\alpha+\beta)$$
$$= \frac{1}{F(\alpha,\,-\beta;\,\gamma-\beta;\,1)}, \qquad \mathrm{Re}\,\gamma > \mathrm{Re}\,(\alpha+\beta)$$

GA 148(51)
GA 148(50)
GA 148(49)

$$3.\quad F\left(1,\,1;\,\frac{3}{2};\,\frac{1}{2}\right) = \frac{\pi}{2}(\text{cf. } 9.121\ 14).$$

9.121

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.130

The series $F(\alpha,\beta;\gamma;z)$ defines an analytic function that, speaking generally, has singularities at the points $z=0,1$, and ∞ (In the general case, there are branch points). We make a cut in the z-plane along the real axis from $z=1$ to $z=\infty$; that is, we require that $|\arg(-z)|<\pi$ for $|z|\geq 1$. Then, the series $f(\alpha,\beta;\gamma;z)$ will, in the cut plane, yield a single-valued analytic continuation which we can obtain by means of the formulas below (provided $\gamma+1$ is not a natural number and $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers). These formulas make it possible to calculate the values of F in the given region even in the case in which $|z|>1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α,β,γ.

Transformation formulas

9.131

$$1.\quad F(\alpha,\,\beta;\,\gamma;\,z) = (1-z)^{-\alpha}F\left(\alpha,\,\gamma-\beta;\,\gamma;\,\frac{z}{z-1}\right);$$
$$= (1-z)^{-\beta}F\left(\beta,\,\gamma-\alpha;\,\gamma;\,\frac{z}{z-1}\right);$$
$$= (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha;\,\gamma-\beta;\,\gamma;\,z).$$

2. $F(\alpha,\, \beta;\, \gamma;\, z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\, \beta;\, \alpha+\beta-\gamma+1;\, 1-z) +$
$+ (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma-\alpha,\, \gamma-\beta;\, \gamma-\alpha-\beta+1;\, 1-z).$

EH I 94, MO 13

9.132

1. $F(\alpha,\, \beta;\, \gamma;\, z) = \frac{(1-z)^{-\alpha}\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} F\left(\alpha,\, \gamma-\beta;\, \alpha-\beta+1;\, \frac{1}{1-z}\right) +$
$+ (1-z)^{-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} F\left(\beta,\, \gamma-\alpha;\, \beta-\alpha+1;\, \frac{1}{1-z}\right).$

MO 13

2.[8] $F(\alpha,\, \beta;\, \gamma;\, z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-z)^{-\alpha} z^{-\alpha} F\left(\alpha,\, \alpha+1-\gamma;\, \alpha+1-\beta;\, \frac{1}{z}\right) +$
$+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-z)^{-\beta} z^{-\beta} F\left(\beta,\, \beta+1-\gamma;\, \beta+1-\alpha;\, \frac{1}{z}\right).$

GA 220(93)

9.133

$$F\left(2\alpha,\, 2\beta;\, \alpha+\beta+\frac{1}{2};\, z\right) = F\left[\alpha,\, \beta;\, \alpha+\beta+\frac{1}{2};\, 4z(1-z)\right] \quad \left[|z| \le \frac{1}{2},\, |z(1-z)| \le \frac{1}{4}\right].$$

WH

9.134

1. $F(\alpha,\, \beta;\, 2\beta;\, z) = \left(1-\frac{z}{2}\right)^{-\alpha} F\left[\frac{\alpha}{2},\, \frac{\alpha+1}{2};\, \beta+\frac{1}{2};\, \left(\frac{z}{2-z}\right)^2\right].$

MO 13, EH I 111(4)

2. $F(2\alpha,\, 2\alpha+1-\gamma;\, \gamma;\, z) = (1+z)^{-2\alpha} F\left(\alpha,\, \alpha+\frac{1}{2};\, \gamma;\, \frac{4z}{(1+z)^2}\right).$

GA 225(100)

GA 225(101)

9.135

$$F\left(\alpha,\ \beta;\ \alpha+\beta+\frac{1}{2};\sin^2\ \varphi\right)\ = F\left(2\alpha,\ 2\beta;\ \alpha+\beta+\frac{1}{2};\sin^2\frac{\varphi}{2}\right)$$

$$\left[x=\sin^2\ \frac{\varphi}{2}\ \text{real};\qquad \frac{1-\sqrt{2}}{2}<x<\frac{1}{2}\right].$$

MO 13

9.136

We set

$$A=\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)},\qquad B=\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)};$$

then

1. $$F\left(2\alpha,\ 2\beta;\ \alpha+\beta+\frac{1}{2};\ \frac{1-\sqrt{z}}{2}\right)\ = AF\left(\alpha,\ \beta;\ \frac{1}{2};\ z\right)+B\sqrt{z}F\left(\alpha+\frac{1}{2},\ \beta+\frac{1}{2};\ \frac{3}{2};\ z\right)$$

GA 227(106)

2. $$F\left(2\alpha,\ 2\beta;\ \alpha+\beta+\frac{1}{2};\ \frac{1+\sqrt{z}}{2}\right)\ = AF\left(\alpha,\ \beta;\ \frac{1}{2};\ z\right)-B\sqrt{z}F\left(\alpha+\frac{1}{2},\ \beta+\frac{1}{2};\ \frac{3}{2};\ z\right)$$

GA 227(107)

3. $$\frac{\left(\alpha-\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)}{\alpha+\beta-\frac{1}{2}}A\sqrt{z}F\left(\alpha,\ \beta;\ \frac{3}{2};\ z\right)=F\left(2\alpha-1,\ 2\beta-1;\ \alpha+\beta-\frac{1}{2};\ \frac{1+\sqrt{z}}{2}\right)\ -$$

$$-\ F\left(2\alpha-1,\ 2\beta-1;\ \alpha+\beta-\frac{1}{2};\ \frac{1-\sqrt{z}}{2}\right).$$

GA 229(110)

9.137[8]

Gauss' recursion formulas:

1. $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z]F(\alpha,\, \beta;\, \gamma;\, z) +$
$+ (\gamma - \alpha)(\gamma - \beta)zF(\alpha,\, \beta;\, \gamma + 1;\, z) + \gamma(\gamma - 1)(z - 1)F(\alpha,\, \beta;\, \gamma - 1;\, z) = 0.$
2. $(2\alpha - \gamma - \alpha z + \beta z)F(\alpha,\, \beta;\, \gamma;\, z) + (\gamma - \alpha)F(\alpha - 1,\, \beta;\, \gamma;\, z) + \alpha(z - 1)F(\alpha + 1,\, \beta;\, \gamma;\, z) = 0.$
3. $(2\beta - \gamma - \beta z + \alpha z)F(\alpha,\, \beta;\, \gamma;\, z) + (\gamma - \beta)F(\alpha,\, \beta - 1;\, \gamma;\, z) + \beta(z - 1)F(\alpha,\, \beta + 1;\, \gamma;\, z) = 0.$
4. $\gamma F(\alpha,\, \beta - 1;\, \gamma;\, z) - \gamma F(\alpha - 1,\, \beta;\, \gamma;\, z) + (\alpha - \beta)zF(\alpha,\, \beta;\, \gamma + 1;\, z) = 0.$
5. $\gamma(\alpha - \beta)F(\alpha,\, \beta;\, \gamma;\, z) - \alpha(\gamma - \beta)F(\alpha + 1,\, \beta;\, \gamma + 1;\, z) + \beta(\gamma - \alpha)F(\alpha,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
6. $\gamma(\gamma + 1)F(\alpha,\, \beta;\, \gamma;\, z) - \gamma(\gamma + 1)F(\alpha,\, \beta;\, \gamma + 1;\, z) - \alpha\beta zF(\alpha + 1,\, \beta + 1;\, \gamma + 2;\, z) = 0.$
7. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) - (\gamma - \alpha)F(\alpha,\, \beta + 1;\, \gamma + 1;\, z) - \alpha(1 - z)F(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
8. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) + (\beta - \gamma)F(\alpha + 1,\, \beta;\, \gamma + 1;\, z) - \beta(1 - z)F(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
9. $\gamma(\gamma - \beta z - \alpha)F(\alpha,\, \beta;\, \gamma;\, z) - \gamma(\gamma - \alpha)F(\alpha - 1,\, \beta;\, \gamma;\, z) +$
$+ \alpha\beta z(1 - z)F(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
10. $\gamma(\gamma - \alpha z - \beta)F(\alpha,\, \beta;\, \gamma;\, z) - \gamma(\gamma - \beta)F(\alpha,\, \beta - 1;\, \gamma;\, z) +$
$+ \alpha\beta z(1 - z)F(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
11. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) - \gamma F(\alpha,\, \beta + 1;\, \gamma;\, z) + \alpha zF(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
12. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) - \gamma F(\alpha + 1,\, \beta;\, \gamma;\, z) + \beta zF(\alpha + 1,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
13. $\gamma[\alpha - (\gamma - \beta)z]F(\alpha,\, \beta;\, \gamma;\, z) - \alpha\gamma(1 - z)F(\alpha + 1,\, \beta;\, \gamma;\, z) +$
$+ (\gamma - \alpha)(\gamma - \beta)zF(\alpha,\, \beta;\, \gamma + 1;\, z) = 0.$
14. $\gamma[\beta - (\gamma - \alpha)z]F(\alpha,\, \beta;\, \gamma;\, z) - \beta\gamma(1 - z)F(\alpha,\, \beta + 1;\, \gamma;\, z) +$
$+ (\gamma - \alpha)(\gamma - \beta)zF(\alpha,\, \beta;\, \gamma + 1;\, z) = 0.$
15. $\gamma(\gamma + 1)F(\alpha,\, \beta;\, \gamma;\, z) - \gamma(\gamma + 1)F(\alpha,\, \beta + 1;\, \gamma + 1;\, z) +$
$+ \alpha(\gamma - \beta)zF(\alpha + 1,\, \beta + 1;\, \gamma + 2;\, z) = 0.$
16. $\gamma(\gamma + 1)F(\alpha,\, \beta;\, \gamma;\, z) - \gamma(\gamma + 1)F(\alpha + 1,\, \beta;\, \gamma + 1;\, z) +$
$+ \beta(\gamma - \alpha)zF(\alpha + 1,\, \beta + 1;\, \gamma + 2;\, z) = 0.$
17. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) - (\gamma - \beta)F(\alpha,\, \beta;\, \gamma + 1;\, z) - \beta F(\alpha,\, \beta + 1;\, \gamma + 1;\, z) = 0.$
18. $\gamma F(\alpha,\, \beta;\, \gamma;\, z) - (\gamma - \alpha)F(\alpha,\, \beta;\, \gamma + 1;\, z) - \alpha F(\alpha + 1,\, \beta;\, \gamma + 1;\, z) = 0.$

MO 13-14

9.14 A generalized hypergeometric series

The series

1. $${}_pF_q(\alpha_1,\, \alpha_2, \ldots, \alpha_p;\, \beta_1,\, \beta_2, \ldots, \beta_q;\, z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k} \frac{z^k}{k!}$$

is called a *generalized hypergeometric series* (see also 9.210).

MO 14

MO 15

For integral representations, see 3.254 2., 3.259 2., and 3.478 3.

1072

9.15 The hypergeometric differential equation

9.151

A hypergeometric series is one of the solutions of the differential equation

$$z(1-z)\frac{d^2u}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{du}{dz}-\alpha\beta u=0,$$

WH

which is called the *hypergeometric equation*.

The solution of the hypergeometric differential equation

9.152

The hypergeometric differential equation 9.151 possesses *two linearly independent solutions*. These solutions have analytic continuations to the entire z-plane except possibly for the three points 0, 1, and ∞. Generally speaking, the points $z=0,1,\infty$ are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio $w(z)$ of two linearly independent solutions satisfies the differential equation

$$2\frac{w'''}{w'}-3\left(\frac{w''}{w'}\right)^2=\frac{1-a_1^2}{z^2}+\frac{1-a_2^2}{(z-1)^2}+\frac{a_1^2+a_2^2-a_3^2-1}{z(z-1)},$$

where

$$a_1^2=(1-\gamma)^2,\quad a_2^2=(\gamma-\alpha-\beta)^2,\quad a_3^2=(\alpha-\beta)^2.$$

If α,β,γ are real, the function $w(z)$ maps the upper (Im $z>0$) or the lower (Im $z<0$) half-plane onto a curvilinear triangle whose

α, β, γ $w(z)$ $z > 0$ $z < 0$

angles are $\pi a_1, \pi a_2, \pi a_3$. The vertices of this triangle are the images of the points $z = 0, z = 1$, and $z = \infty$.

9.153

Within the unit circle $|z| < 1$, the linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeo- metric differential equation are given by the following formulas:

1. If γ is not an integer,

$$u_1 = F(\alpha,\ \beta;\ \gamma;\ z),$$

$$u_2 = z^{1-\gamma} F(\alpha - \gamma + 1,\ \beta - \gamma + 1;\ 2 - \gamma;\ z).$$

2. If $\gamma = 1$, then

$$u_1 = F(\alpha,\ \beta;\ 1;\ z),$$
$$u_2 = F(\alpha,\ \beta;\ 1;\ z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(k!)^2} \times$$
$$\times \{\psi(\alpha + k) - \psi(\alpha) + \psi(\beta + k) - \psi(\beta) - 2\psi(k + 1) + 2\psi(1)\}$$
$$\text{(see 9.14 2.)}.$$

9.14

1073

3. If $\gamma = m + 1$ (where m is a natural number), and if neither α nor β is a positive number not exceeding m, then

$$u_1 = F(\alpha,\ \beta;\ m + 1;\ z),$$
$$u_2 = F(\alpha,\ \beta;\ m{+}1;\ z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(1 + m)_k} \{h(k) - h(0)\} - \sum_{k=1}^{m} \frac{(k-1)!(-m)_k}{(1-\alpha)_k (1-\beta)_k} z^{-k}$$
$$\text{(see 9.14 2.)},$$

9.14

where

$$h(n) = \psi(\alpha+n)+\psi(\beta+n)-\psi(m+1+n)-\psi(n+1) \qquad [n+1 \text{ a natural number}].$$

4.[7] Suppose that $\gamma = m+1$ (where m is a natural number) and that α or β is equal to $m'+1$, where $0 \le m' < m$. Then, for example, for $\alpha = m'+1$, we obtain

$$u_1 = F(1+m',\, \beta;\, 1+m;\, z),$$

$$u_2 = z^{-m} F(1+m' - m,\, \beta - m;\, 1-m;\, z).$$

In this case, u_2 is a polynomial in z^{-1}.

5. If $\gamma = 1-m$ (where m is a natural number) and if α and β are both different from the numbers $0, -1, -2, \ldots, 1-m$, then

$$u_1 = z^m F(\alpha+m,\, \beta+m;\, 1+m;\, z),$$
$$u_2 = z^m F(\alpha+m,\, \beta+m;\, 1+m;\, z)\ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha+m)_k(\beta+m)_k}{(1+m)_k k!}\left\{h^*(k)-h^*(0)\right\} -$$
$$-\sum_{k=1}^{\infty}\frac{(k-1)!(-m)_k}{(1-\alpha-m)_k(1-\beta-m)_k} z^{m-n} \qquad (\text{see } 9.14\ 2.),$$

9.14

where

$$h^*(n) = \psi(\alpha+m+n)+\psi(\beta+m+n)-\psi(1+m+n)-\psi(1+n).$$

We note that

$$\psi(\alpha+n)-\psi(\alpha) = \frac{1}{\alpha}+\frac{1}{\alpha+1}+\ldots+\frac{1}{\alpha+n-1} \qquad (\text{cf. } 8.365\ 3.)$$

and that, for $\alpha = -\lambda$, where λ is a natural number or zero and $n = \lambda + 1, \lambda + 2, \ldots$ the expression

$$(\alpha)_k[\psi(\alpha + n) - \psi(\alpha)]$$

in formulas 9.153 2.-5. should be replaced with the expression

$$(-1)^\lambda \lambda!(n - \lambda - 1)!.$$

1074

6. Suppose that $\gamma = 1 - m$ (where m is a natural number) and that α or β is an integer $(-m')$, where m' is one of the following numbers: $0, 1, \ldots, m - 1$. Suppose, for example, that $\alpha = -m'$. Then,

$$u_1 = F(-m', \beta; 1 - m; z),$$

$$u_2 = F(-m' + m, \beta + m; 1 + m; z).$$

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7. For $\gamma = \frac{1}{2}(\alpha + \beta + 1)$

$$u_1 = F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); z\right),$$

$$u_2 = F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); 1 - z\right)$$

are two linearly independent solutions of the hypergeometric differential equation provided α, β and γ are not zero or negative integers.

The analytic continuation of a solution that is regular at the point $z = 0$

9.154

Formulas 9.153 make possible the analytic continuation, by means of the hypergeometric series, of the function $F(\alpha, \beta; \gamma; z)$ defined inside the circle $|z| < 1$ to the region $|z| > 1, |\arg(-z)| < \pi$. Here, it is assumed that $\alpha - \beta$ is not an integer. In the event that $\alpha - \beta$ is an integer (for example, if $\beta = \alpha + m$, where m is a natural number), then, for $|z| > 1, |\arg(-z)| < \pi$ we have:

$$1.\quad \frac{\Gamma(\alpha)\Gamma(\alpha+m)}{\Gamma(\gamma)} F(\alpha,\, \alpha+m;\, \gamma;\, z) = \\ = \frac{\sin\pi(\gamma-\alpha)}{\pi} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha+k)\Gamma(1-\gamma+\alpha+k)\Gamma(m-k)}{k!} (-z)^{-\alpha-k} \right. + \\ \left. + \, (-z)^{-\alpha-m} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+m+k)\Gamma(1-\gamma+\alpha+m+k)}{k!(k+m)!} g(k) z^{-k} \right\},$$

where

$$2.\quad g(n) = \ln(-z) + \pi \operatorname{ctg} \pi(\gamma-\alpha) + \psi(n+1) + \psi(n+m+1) - \\ - \psi(\alpha+m+n) - \psi(1-\gamma+\alpha+m+n).$$

For $m = 0$, we should set $\sum\limits_{k=0}^{m-1} = 0$.

9.155

This formula loses its meaning when α, γ, or $\alpha - \gamma + 1$ is equal to one of the numbers $0, -1, -2, \ldots$. In this last case, we have

1. If α is a nonpositive integer and γ is not an integer, $F(\alpha, \alpha+m; \gamma; z)$ is a polynomial in z.

2. Suppose that γ is a nonpositive integer and that α is not an integer. We then set $\gamma = -\lambda$, where $\lambda = 0, 1, 2, \ldots$. Then,

$$\frac{\Gamma(\alpha+\lambda+1)\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)} z^{\lambda+1} F(\alpha+\lambda+1,\, \alpha+\lambda+m+1;\, \lambda+2;\, z)$$

$$\frac{\Gamma(\alpha+\lambda+1)\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)} z^{\lambda+1} F(\alpha+\lambda+1,\ \alpha+\lambda+m+1;\ \lambda+2;\ z)$$

is a solution of the hypergeometric equation that is regular at the point $z=0$. This solution is equal to the right hand member of formula 9.154 1 if we replace γ with λ in this equation and in formula 9.154 2.

3. If $\alpha-\gamma+1$ is a nonpositive integer and if α and γ are not themselves integers, we may use the formula

$$F(\alpha,\ \alpha+m;\ \gamma;\ z) = (1-z)^{\gamma-2\alpha-m} F(\gamma-\alpha-m,\ \gamma-\alpha;\ \gamma;\ z)$$

and apply formula 9.154 1. to its right hand member provided $\gamma-\alpha-m>0$. However, if $\alpha-\gamma-m\leq 0$, the right member of this expression is a polynomial taken to the $(1-z)$th power.

4.[7] If α, β, and γ are integers, the hypergeometric differential equation always has a solution that is regular for $z=0$ and that is of the form

$$R_1(z) + \ln(1-z) R_2(z),$$

where $R_1(z)$ and $R_2(z)$ are rational functions of z. To get a solution of this form, we need to apply formulas 9.137 1.-9.137 3. to the function $F(\alpha,\beta;\gamma;z)$. However, if $\gamma=-\lambda$, where $\lambda+1$ is a natural number, formulas 9.137 1. and 9.137 2. should be applied not to $F(\alpha,\beta;\gamma;z)$ but to the function $z^{\lambda+1}F(\alpha+\lambda+1,\beta+\lambda+1;\lambda+2,z)$.

By successive applications of these formulas, we can reduce the positive values of the parameters to one or zero. Furthermore, we can obtain the desired form of the solution from the formulas

$$F(1,\ 1;\ 2;\ z) = -z^{-1}\ \ln(1-z),$$

$$F(0,\ \beta;\ \gamma;\ z) = F(\alpha,\ 0;\ \gamma;\ z) = 1.$$

MO 19-20

9.16 Riemann's differential equation

9.160

The hypergeometric differential equation is a particular case of Riemann's differential equation

WH

The coefficients of this equation have poles at the points a, b, and c, and the numbers $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are called the indices corresponding to these poles. The indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are related by the following equation:

$$\alpha + \alpha' + \beta - \beta' + \gamma + \gamma' - 1 = 0.$$

WH

1076

2. The differential equations 9.160 1. are written diagramatically as follows:

$$3. \quad u = P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\}.$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column.

WH

9.161

The two following transformation formulas are valid for Riemann's P-equation:

$$1. \quad \left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\} = P \left\{ \begin{matrix} a & b & c & \\ \alpha + k & \beta - k - 1 & \gamma + l & z \\ \alpha' + k & \beta' - k - l & \gamma' + l & \end{matrix} \right\}.$$

WH

$$2.^7 \quad P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\} = P \left\{ \begin{matrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\}.$$

WH

The first of these formulas means that if

$$u = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\},$$

the function

$$u_1 = \left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l u$$

satisfies a second-order differential equation having the same singular points as equation 9.161 2. and indices equal to $\alpha + k, \alpha' + k; \beta - k - l, \beta' - k - l; \gamma + l, \gamma' + l$. The second transformation formula converts a differential equation with singularities at the points a, b, and c, indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$, and an independent variable z into a differential equation with the same indices, singular points a_1, b_1, and c_1, and independent variable z_1. The variable z_1 is connected with the variable z by the bilinear transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \qquad [AD - BC \neq 0].$$

The same transformation connects the points a_1, b_1, and c_1 with the points a, b, and c.

WH, MO 20

9.162

By the successive application of the two transformation formulas 9.161 1. and 9.161 2., we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

1077

For $k = -\alpha, l = -\gamma$, and $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$, we have

$$1. \quad u = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P\left\{\begin{matrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{matrix}\right\} =$$
$$= \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{matrix}\right\}.$$

2. $u = \left(\frac{z-a}{z-b}\right)^{\alpha} \left(\frac{z-c}{z-b}\right)^{\gamma} F\left(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\alpha-\alpha';\ \frac{(z-a)(c-b)}{(z-b)(c-a)}\right)$.

If the constants $a, b, c; \alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences $\alpha-\alpha', \beta-\beta', \gamma-\gamma'$ are integers):

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9.163

1. $u_1 = \left(\frac{z-a}{z-b}\right)^{\alpha} \left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\alpha-\alpha';\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$.

2. $u_2 = \left(\frac{z-a}{z-b}\right)^{\alpha'} \left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha'+\beta+\gamma,\ \alpha'+\beta'+\gamma;\ 1+\alpha'-\alpha;\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$.

3. $u_3 = \left(\frac{z-a}{z-b}\right)^{\alpha} \left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma';\ 1+\alpha-\alpha';\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$.

4. $u_4 = \left(\frac{z-a}{z-b}\right)^{\alpha'} \left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha'+\beta+\gamma',\ \alpha'+\beta'+\gamma;\ 1+\alpha'-\alpha;\ \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$.

9.164

1.* $u_5 = \left(\frac{z-b}{z-c}\right)^{\beta} \left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta+\gamma+\alpha,\ \beta+\gamma'+\alpha;\ 1+\beta-\beta';\ \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$.

2. $u_6 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta'+\gamma+\alpha,\ \beta'+\gamma'+\alpha;\ 1+\beta'-\beta;\ \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$.

3. $u_7 = \left(\frac{z-b}{z-c}\right)^{\beta} \left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta+\gamma+\alpha',\ \beta+\gamma'+\alpha';\ 1+\beta-\beta';\ \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$.

4. $u_8 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta'+\gamma+\alpha',\ \beta'+\alpha'+\gamma';\ 1+\beta'-\beta;\ \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$.

9.165

1. $u_9 = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma+\alpha+\beta,\ \gamma+\alpha'+\beta;\ 1+\gamma-\gamma';\ \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$.

2. $u_{10} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma'+\alpha+\beta,\ \gamma'+\alpha'+\beta;\ 1+\gamma'-\gamma;\ \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$.

3. $u_{11} = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma+\alpha+\beta',\ \gamma+\alpha'+\beta';\ 1+\gamma-\gamma';\ \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$.

4. $u_{12} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma'+\alpha+\beta',\ \gamma'+\alpha'+\beta';\ 1+\gamma'-\gamma;\ \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$.

9.166

1. $u_{13} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha+\gamma+\beta,\ \alpha+\gamma'+\beta;\ 1+\alpha-\alpha';\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$.

2. $u_{14} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha'+\gamma+\beta,\ \alpha'+\gamma'+\beta;\ 1+\alpha'-\alpha;\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$.

3. $u_{15} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha+\gamma+\beta',\ \alpha+\gamma'+\beta';\ 1+\alpha-\alpha';\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$.

4. $u_{16} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha'+\gamma+\beta',\ \alpha'+\gamma'+\beta';\ 1+\alpha'-\alpha;\ \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$.

1. $u_{17} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma+\beta+\alpha,\, \gamma+\beta'+\alpha;\, 1+\gamma-\gamma';\, \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}.$

2. $u_{18} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma'+\beta+\alpha,\, \gamma'+\beta'+\alpha;\, 1+\gamma'-\gamma;\, \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}.$

3. $u_{19} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma+\beta+\alpha',\, \gamma+\beta'+\alpha';\, 1+\gamma-\gamma';\, \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}.$

4. $u_{20} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma'+\beta+\alpha',\, \gamma'+\beta'+\alpha';\, 1+\gamma'-\gamma;\, \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}.$

9.168

1. $u_{21} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta+\alpha+\gamma,\, \beta+\alpha'+\gamma;\, 1+\beta-\beta';\, \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$

2. $u_{22} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta'+\alpha+\gamma,\, \beta'+\alpha'+\gamma;\, 1+\beta'-\beta;\, \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$

3. $u_{23} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta+\alpha+\gamma',\, \beta+\alpha'+\gamma';\, 1+\beta-\beta';\, \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$

4. $u_{24} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta'+\alpha+\gamma',\, \beta'+\alpha'+\gamma';\, 1+\beta'-\beta;\, \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}.$

WH

9.17 Representation of certain second-order differential equations by means of a Reimann scheme

9.171

The hypergeometric equation (see 9.151):

$$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & z \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{matrix}\right\}.$$

WH

9.172

The associated Legendre's equation defining the functions $P_n^m(z)$ for n and m integers (see 8.700 1.):

$$1. \quad u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & \frac{1-z}{2} \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}.$$

WH

$$2. \quad u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ -\frac{1}{2}n & \frac{1}{2}m & 0 & \frac{1}{1-z}^2 \\ \frac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} & \end{matrix}\right\}.$$

WH

9.173

The function $P_n^m\left(1-\frac{z^2}{2n^2}\right)$ satisfies the equation

$$u = P\left\{\begin{matrix} 4n^2 & \infty & 0 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & z^2 \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}.$$

WH

The function $J_m(z)$ satisfies the limiting form of this equation obtained as $n \to \infty$.

9.174

The equation defining the polynomials $C_n^\lambda(z)$ (see 8.938):

$$u = P\begin{Bmatrix} -1 & \infty & 1 & \\ \frac{1}{2}-\lambda & n+2\lambda & \frac{1}{2}-\lambda & z \\ 0 & -n & 0 & \end{Bmatrix}.$$

WH

1080
9.175

Bessel's equation (see 8.401) is the limiting form of the equations:

$$1.\quad u = P\begin{Bmatrix} 0 & \infty & c & \\ n & ic & \frac{1}{2}+ic & z \\ -n & -ic & \frac{1}{2}-ic & \end{Bmatrix},$$

WH

$$2.\quad u = e^{iz}P\begin{Bmatrix} 0 & \infty & c & \\ n & \frac{1}{2} & 0 & z \\ -n & \frac{3}{2}-2ic & 2ic-1 & \end{Bmatrix},$$

WH

$$3.\quad u = P\begin{Bmatrix} 0 & \infty & c^2 & \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 & z^2 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 & \end{Bmatrix},$$

WH

as $c \to \infty$.

9.18 Hypergeometric functions of two variables

9.180

$$1.\quad F_1(\alpha,\, \beta,\, \beta',\, \gamma;\, x,\, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!} x^m y^n.$$

Region of convergence

$$|x| < 1, \quad |y| < 1.$$

AK 16

2. $$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} x^m y^n.$$

EH I 224(7), AK 14(12)

Region of convergence

$$|x| + |y| < 1.$$

AK 17

3. $$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} x^m y^n.$$

EH I 224(8), AK 14(13)

Region of convergence

$$|x| < 1, \quad |y| < 1.$$

AK 17

4. $$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!n!} x^m y^n.$$

EH I 224(9), AK 14(14)

Region of convergence

$$|\sqrt{x}| + |\sqrt{y}| < 1.$$

9.181

The functions F_1, F_2, F_3, and F_4 satisfy the following systems of partial differential equations for z:

1. System of equations for $z = F_1$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha+\beta+1)x]\frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 233(9)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x(1-y)\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha+\beta'+1)y]\frac{\partial z}{\partial x} - \beta' x \frac{\partial z}{\partial x} - \alpha\beta' z = 0.$$

2. System of equations for $z = F_2$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha+\beta+1)x]\frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(10)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma' - (\alpha+\beta'+1)y]\frac{\partial z}{\partial y} - \beta' x \frac{\partial z}{\partial x} - \alpha\beta' z = 0.$$

3. System of equations for $z = F_3$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha+\beta+1)x]\frac{\partial z}{\partial x} - \alpha\beta z = 0,$$

EH I 234(11)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha'+\beta'+1)y]\frac{\partial z}{\partial y} - \alpha'\beta' z = 0.$$

4. System of equations for $z = F_4$:

EH I 234(12)

$$(1-y)\frac{\partial^2 z}{\partial y^2}-x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+[\gamma'-(\alpha+\beta+1)y]\frac{\partial z}{\partial y}-(\alpha+\beta+1)x\frac{\partial z}{\partial x}-\alpha\beta z=0.$$

AK 44

9.182

For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

1. $$F_1(\alpha,\,\beta,\,\beta',\,\beta+\beta';\,x,\,y)=(1-y)^{-\alpha}F\left(\alpha,\,\beta;\,\beta+\beta';\,\frac{x-y}{1-y}\right).$$

EH I 238(1), AK 24(28)

2. $$F_2(\alpha,\,\beta,\,\beta',\,\beta,\,\gamma';\,x,\,y)=(1-x)^{-\alpha}F\left(\alpha,\,\beta';\,\gamma';\,\frac{y}{1-x}\right).$$

EH I 238(2), AK 23

3. $$F_2(\alpha,\,\beta,\,\beta',\,\alpha,\,\alpha;\,x,\,y)=(1-x)^{-\beta}(1-y)^{-\beta'}F\left[\beta,\,\beta';\,\alpha;\,\frac{x\,y}{(1-x)(1-y)}\right].$$

EH I 238(3)

4. $$F_3(\alpha,\,\gamma-\alpha,\,\beta,\,\gamma-\beta,\,\gamma;\,x,\,y)=(1-y)^{\alpha+\beta-\gamma}F(\alpha,\,\beta;\,\gamma;\,x+y-xy).$$

EH I 238(4), AK 25(35)

5. $$F_4[\alpha,\,\gamma+\gamma'-\alpha-1,\,\gamma,\,\gamma';\,x(1-y),\,y(1-x)]=$$
$$=F(\alpha,\,\gamma+\gamma'-\alpha-1;\,\gamma;\,x)F(\alpha,\,\gamma+\gamma'-\alpha-1;\,\gamma';\,y).$$

EH I 238(5)

6. $$F_4\left[\alpha,\,\beta,\,\alpha,\,\beta;\,-\frac{x}{(1-x)(1-y)},\quad\frac{-y}{(1-x)(1-y)}\right]=\frac{(1-x)^{\beta}(1-y)^{\alpha}}{(1-xy)}.$$

7. $F_4\left[\alpha,\,\beta,\,\beta,\,\beta;\,-\frac{x}{(1-x)(1-y)},\quad -\frac{y}{(1-x)(1-y)}\right]=$
$$=(1-x)^{\alpha}(1-y)^{\alpha}F(\alpha,\,1+\alpha-\beta;\,\beta;\,x\,y).$$

EH I 238(7)

8. $F_4\left[\alpha,\,\beta,\,1+\alpha-\beta,\,\beta;\,-\frac{x}{(1-x)(1-y)},\quad -\frac{y}{(1-x)(1-y)}\right]=$
$$=(1-y)^{\alpha}F\left[\alpha,\,\beta;\,1+\alpha-\beta;\,-\frac{x(1-y)}{1-x}\right].$$

EH I 238(8)

9. $F_4\left(\alpha,\,\alpha+\frac{1}{2},\,\gamma,\,\frac{1}{2};\,x,\,y\right)=\frac{1}{2}(1+\sqrt{y})^{-2\alpha}F\left(\alpha,\,\alpha+\frac{1}{2};\,\gamma;\,\frac{x}{(1+\sqrt{y})^2}\right)+$
$$+\frac{1}{2}(1-\sqrt{y})^{-2\alpha}F\left(\alpha,\,\alpha+\frac{1}{2};\,\gamma;\,\frac{x}{(1-\sqrt{y})^2}\right).$$

AK 23

10. $F_1(\alpha,\,\beta,\,\beta',\,\gamma;\,x,\,1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}F(\alpha,\,\beta\text{:}\,\gamma-\beta';\,x).$

EH I 239(10), AK 22(23)

11. $F_1(\alpha,\,\beta,\,\beta',\,\gamma;\,x,\,x)=F(\alpha,\,\beta+\beta';\,\gamma;\,x).$

EH I 239(11), AK 23(25)

9.183

Functional relations between hypergeometric functions of two variables:

1. $F_1(\alpha,\,\beta,\,\beta',\,\gamma;\,x,\,y)=(1-x)^{-\beta}(1-y)^{-\beta}F_1\left(\gamma-\alpha,\,\beta,\,\beta',\,\gamma;\,\frac{x}{x-1},\,\frac{y}{y-1}\right);$
$$=(1-x)^{-\alpha}F_1\left(\alpha,\,\gamma-\beta-\beta',\,\beta',\,\gamma;\,\frac{x}{x-1},\,\frac{y-x}{1-x}\right);$$
$$=(1-y)^{-\alpha}F_1\left(\alpha,\,\beta,\,\gamma-\beta-\beta',\,\gamma;\,\frac{y-x}{y-1},\,\frac{y}{y-1}\right);$$
$$=(1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha,\,\gamma-\beta-\beta',\,\beta',\,\gamma;\,x,\,\frac{x-y}{1-y}\right);$$
$$=(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1\left(\gamma-\alpha,\,\beta,\,\gamma-\beta-\beta',\,\gamma;\,\frac{x-y}{x-1},\,y\right).$$

2. $F_2(\alpha,\,\beta,\,\beta',\,\gamma,\,\gamma';\,x,\,y) = (1-x)^{-\alpha} F_2\left(\alpha,\,\gamma-\beta,\,\beta',\,\gamma,\,\gamma';\,\frac{x}{x-1},\,\frac{y}{1-x}\right);$

$$= (1-y)^{-\alpha} F_2\left(\alpha,\,\beta,\,\gamma'-\beta',\,\gamma,\,\gamma';\,\frac{x}{1-y},\,\frac{y}{y-1}\right);$$

$$= (1-x-y)^{-\alpha} F_2\left(\alpha,\,\gamma-\beta,\,\gamma'-\beta',\,\gamma,\,\gamma';\,\frac{x}{x+y-1},\,\frac{y}{x+y-1}\right).$$

EH I 240(8), AK 32(6)
EH I 240(7)
EH I 240(6)

3.[7] $F_4(\alpha,\,\beta,\,\gamma,\,\gamma';x,\,y) = \frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}(-y)^{-\alpha} F_4\left(\alpha,\,\alpha+1-\gamma',\,\gamma,\,\alpha+1-\beta;\,\frac{x}{y},\,\frac{1}{y}\right) +$

$$+\frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}(-y)^{-\beta} F_4\left(\beta+1-\gamma',\,\beta,\,\gamma,\,\beta+1-\alpha;\,\frac{x}{y},\,\frac{1}{y}\right).$$

EH I 240(9), AK 26(37)

9.184

Integral representations:

Double integrals of the Euler type

1. $F_1(\alpha,\,\beta,\,\beta',\,\gamma;\,x,\,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\times$

$$\times \iint\limits_{\left(\substack{u\geq 0,\,v\geq 0\\ u+v\leq 1}\right)} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-\alpha}\,du\,dv$$

$$[\mathrm{Re}\;\beta>0,\quad \mathrm{Re}\;\beta'>0,\quad \mathrm{Re}\,(\gamma-\beta-\beta')>0].$$

EH I 230(1), AK 28(1)

2. $F_2(\alpha,\,\beta,\,\beta',\,\gamma,\,\gamma';\,x,\,y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}\times$

$$\times \int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}\,du\,dv$$

$$[\mathrm{Re}\;\beta>0,\quad \mathrm{Re}\;\beta'>0,\quad \mathrm{Re}\,(\gamma-\beta)>0,\quad \mathrm{Re}\,(\gamma'-\beta')>0].$$

EH I 230(2), AK 28(2)

3. $F_3(\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma;\,x,\,y) =$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\times$$

$$\times \iint_{\left(\substack{u\geq 0,\,v\geq 0\\ u+v\leq 1}\right)} u^{\beta-1}v^{\beta'-1}(1-u-v)^{-\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}\,du\,dv$$

$$[\mathrm{Re}\;\beta>0,\quad \mathrm{Re}\;\beta'>0,\quad \mathrm{Re}\,(\gamma-\beta-\beta')>0].$$

4. $F_4[\alpha,\,\beta,\,\gamma,\,\gamma';\,x(1-y),\,y(1-x)] =$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} \times$$
$$\times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1}\, du\, dv$$
$$[\mathrm{Re}\ \alpha > 0, \quad \mathrm{Re}\ \beta > 0, \quad \mathrm{Re}\,(\gamma-\alpha) > 0, \quad \mathrm{Re}\,(\gamma'-\beta) > 0]\ .$$

EH I 230(4)

Integrals of the Mellin-Barnes type

9.185

The functions F_1, F_2, F_3 and F_4 can be represented by means of double integrals of the following form:

$$F(x,\,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s,\,t)\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t\, ds\, dt.$$

$\Psi(s,\,t)$	$F(x,\,y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\beta')\Gamma(\gamma+s+t)}$	$F_1(\alpha,\,\beta,\,\beta',\,\gamma;\,x,\,y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s)\Gamma(\beta'+t)\Gamma(\gamma')}{\Gamma(\beta')\Gamma(\gamma+s)\Gamma(\gamma'+t)}$	$F_2(\alpha,\,\beta,\,\beta',\,\gamma,\,\gamma';\,x,\,y)$
$\frac{\Gamma(\alpha+s)\Gamma(\alpha'+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\alpha')\Gamma(\beta')\Gamma(\gamma+s+t)}$	$F_3(\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma;\,x,\,y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s+t)\Gamma(\gamma')}{\Gamma(\gamma+s)\Gamma(\gamma'+t)}$	$F_4(\alpha,\,\beta,\,\gamma,\,\gamma';\,x,\,y)$

[$\alpha, \alpha', \beta, \beta'$ may not be negative integers].

EH I 232(9-13), AK 41(33)

9.19 A hypergeometric function of several variables

$$F_A(\alpha;\,\beta_1,\ldots,\beta_n;\,\gamma_1,\ldots,\gamma_n;\,z_1,\ldots,z_n) =$$
$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n} (\beta_1)_{m_1} \ldots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \ldots (\gamma_n)_{m_n} m_1! \ldots m_n!} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}.$$

ET I 385

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.201*

A *confluent hypergeometric function* is obtained by taking the limit as $c \to \infty$ in the solution of Riemann's differential equation

$$P \left\{ \begin{matrix} 0 & \infty & c & \\ \frac{1}{2} + \mu & -c & c - \lambda & z \\ \frac{1}{2} - \mu & 0 & \lambda & \end{matrix} \right\}.$$

WH

1085

9.202

The equation obtained by means of this limiting process is of the form

$$1. \quad \frac{d^2u}{dz^2} + \frac{du}{dz} + \left(\frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0.$$

WH

Equation 9.202 1. has the following two linearly independent solutions:

$$2. \quad z^{\frac{1}{2}+\mu} e^{-z} \Phi \left(\frac{1}{2} + \mu - \lambda,\ 2\mu + 1;\ z \right),$$

$$3. \quad z^{\frac{1}{2}-\mu} e^{-z} \Phi \left(\frac{1}{2} - \mu - \lambda,\ -2\mu + 1;\ z \right),$$

which are defined for all values of $\mu \neq \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \ldots$

MO 111

9.21 The functions Φ(α, γ; z) and Φ(α, γ;z)

9.210*

The series

1. $$\Phi(\alpha,\,\gamma;\,z)=1+\frac{\alpha}{\gamma}\frac{z}{1!}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!}+\dots$$

is also called a *confluent hypergeometric function.*

A second notation: $\Phi(\alpha,\gamma;z)={}_1F_1(\alpha;\gamma;z)$.

2. $$\Psi(\alpha,\,\gamma;\,z)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha,\,\gamma;\,z)+\frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(\alpha-\gamma+1,\,2-\gamma;\,z).$$

EH I 257(7)

3. Bateman's function $k_\nu(x)$ is defined by

$$k_\nu(x)=\frac{2}{\pi}\int_0^{\pi/2}\cos(x\tan\theta-\nu\theta)\,d\theta\qquad[x,\,\nu\text{ real}].$$

EH I 267

9.211

Integral representation:

1. $$\Phi(\alpha,\,\gamma;\,z)=\frac{2^{1-\gamma}e^{\frac{1}{2}z}}{B(\alpha,\,\gamma-\alpha)}\int_{-1}^{1}(1-t)^{\gamma-\alpha-1}(1+t)^{\alpha-1}e^{\frac{1}{2}zt}\,dt\qquad[0<\operatorname{Re}\,\alpha<\operatorname{Re}\,\gamma].$$

MO 114

2. $$\Phi(\alpha,\,\gamma;\,z)=\frac{1}{B(\alpha,\,\gamma-\alpha)}z^{1-\gamma}\int_0^z e^t t^{\alpha-1}(z-t)^{\gamma-\alpha-1\,dt}\qquad[0<\operatorname{Re}\,\alpha<\operatorname{Re}\,\gamma].$$

MO 114

3. $$\Phi(-\nu,\,\alpha+1;\,z)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}e^z z^{-\frac{\alpha}{2}}\int_0^\infty e^{-t}t^{\nu+\frac{\alpha}{2}}J_\alpha(2\sqrt{zt})\,dt$$
$$\left[\operatorname{Re}\,(\alpha+\nu+1)>0,\quad|\arg\,z|<\frac{\pi}{2}\right].$$

4.[8] $\Psi(\alpha,\, \gamma;\, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1}\, dt \qquad [\operatorname{Re}\alpha > 0, \operatorname{Re} z > 0].$

EH I 255(2)

9.212

1. $\Phi(\alpha,\, \gamma;\, z) = e^z \Phi(\gamma - \alpha,\, \gamma;\, -z).$

MO 112

2. $\frac{z}{\gamma} \Phi(\alpha + 1,\, \gamma + 1;\, z) = \Phi(\alpha + 1,\, \gamma;\, z) - \Phi(\alpha,\, \gamma;\, z).$

MO 112

3. $\alpha \Phi(\alpha + 1,\, \gamma + 1;\, z) = (\alpha - \gamma) \Phi(\alpha,\, \gamma + 1;\, z) + \gamma \Phi(\alpha,\, \gamma;\, z).$

MO 112

4. $\alpha \Phi(\alpha + 1,\, \gamma;\, z) = (z + 2a - \gamma) \Phi(\alpha,\, \gamma;\, z) + (\gamma - \alpha) \Phi(\alpha - 1,\, \gamma;\, z).$

MO 112

9.213

$$\frac{d\Phi}{dz} = \frac{\alpha}{\gamma} \Phi(\alpha + 1,\, \gamma + 1;\, z).$$

MO 112

9.214

$$\lim_{\gamma \to -n} \frac{1}{\Gamma(\gamma)} \Phi(\alpha,\, \gamma;\, z) = z^{n+1} \binom{\alpha + n}{n + 1} \Phi(\alpha{+}n{+}1,\, n{+}2;\, z) \qquad [n = 0,\, 1,\, 2, \ldots].$$

MO 112

9.215*

1. $\Phi(\alpha,\, \alpha;\, z) = e^z.$

2. $\Phi(\alpha,\, 2\alpha;\, 2z) = 2^{\alpha-\frac{1}{2}}\, \exp\left[\frac{1}{4}(1-2\alpha)\pi i\right] \Gamma\left(\alpha+\frac{1}{2}\right) e^{z} z^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}\left(ze^{\frac{\pi}{2}i}\right).$

MO 112

3. $\Phi\left(p+\frac{1}{2},\, 2p+1;\, 2iz\right) = \Gamma(p+1)\left(\frac{z}{2}\right)^{-p} e^{iz} J_p(z).$

MO 15

For a representation of special functions in terms of a confluent hypergeometric function $\Phi(\alpha, \gamma; z)$, see:

for the probability integral, 9.236;

for integrals of Bessel functions, 6.631 1.;

for Hermite polynomials, 8.953 and 8.959;

for Laguerre polynomials, 8.972 1.;

for parabolic-cylinder functions, 9.240;

for the functions $M_{\lambda,\mu}(z)$, 9.220 2. and 9.220 3.

9.216

The function $\Phi(\alpha, \gamma; z)$ is a solution of the differential equation

1. $z\frac{d^2F}{dz^2} + (\gamma - z)\frac{dF}{dz} - \alpha F = 0.$

MO 111

This equation has two linearly independent solutions:

2. $\Phi(\alpha,\, \gamma;\, z)$

3. $z^{1-\gamma}\Phi(\alpha-\gamma+1,\, 2-\gamma;\, z)$

9.22- 9.23 The Whittaker functions $M_{\lambda,\mu}$(z) and $W_{\lambda,\mu}$(x)

9.220

If we make the change of variable $u = e^{-\frac{z}{2}} W$ in equation 9.202 1., we obtain the equation

$$1. \quad \frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) W = 0.$$

MO 115

1087

Equation 9.220 1. has the following two linearly independent solutions:

$$2. \quad M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} \Phi\left(\mu - \lambda + \frac{1}{2},\, 2\mu + 1;\, z\right).$$

$$3. \quad M_{\lambda,-\mu}(z) = z^{-\mu+\frac{1}{2}} e^{-z/2} \Phi\left(-\mu - \lambda + \frac{1}{2},\, 2\mu + 1;\, z\right).$$

MO 115

To obtain solutions that are also suitable for $2\mu = \pm 1, \pm 2, \ldots$, we introduce Whittaker's function

$$4. \quad W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda,-\mu}(z),$$

WH

which, for 2μ approaching an integer, is also a solution of equation 9.220 1.

For the functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z), z = 0$ is a branch point and $z = \infty$ is an essential singular point. Therefore, we shall examine these functions only for $|\arg z| < \pi$.

These functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are linearly independent solutions of equation 9.220 1.

9.221

$$M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu} B\left(\mu+\lambda+\frac{1}{2},\, \mu-\lambda+\frac{1}{2}\right)} \int_{-1}^{1} (1+t)^{\mu-\lambda-\frac{1}{2}} (1-t)^{\mu+\lambda-\frac{1}{2}} e^{\frac{1}{2}zt}\, dt,$$

WH

if the integral converges. See also 6.631 1. and 7.623 3.

9.222

1. $$W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} \int_0^\infty e^{-zt} t^{\mu-\lambda-\frac{1}{2}} (1+t)^{\mu-\lambda-\frac{1}{2}}\, dt.$$

$$\left[\operatorname{Re}(\mu-\lambda) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right].$$

MO 118

2. $$W_{\lambda,\mu}(z) = \frac{z^{\lambda} e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1+\frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt.$$

$$\left[\operatorname{Re}(\mu-\lambda) > -\frac{1}{2}, \quad |\arg z| < \pi\right].$$

WH

9.223

$$W_{\lambda,\mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(u-\lambda)\Gamma\left(-u-\mu+\frac{1}{2}\right)\Gamma\left(-u+\mu+\frac{1}{2}\right)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right)\Gamma\left(-\lambda-\mu+\frac{1}{2}\right)} z^u\, du$$

[the path of integration is chosen in such a way that the poles of the function $\Gamma(u-\lambda)$ are separated from the poles of the functions $\Gamma\left(-u-\mu+\frac{1}{2}\right)$ and $\Gamma\left(-u+\mu+\frac{1}{2}\right)$]. See also 7.142.

MO 118

9.224

$$W_{\mu,\frac{1}{2}+\mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^\infty (1+t)^{2\mu} e^{-zt}\, dt = z^{-\mu} e^{\frac{1}{2}z} \int_z^\infty t^{2\mu} e^{-t}\, dt \qquad [\mathrm{Re}\ z > 0].$$

WH

9.225

1. $$W_{\lambda,\mu}(x) W_{-\lambda,\mu}(x) = -x \int_0^\infty \mathrm{th}^{\,2\lambda} \frac{t}{2} \{ J_{2\mu}(x \ \mathrm{sh}\ t) \sin(\mu-\lambda)\pi + {} + N_{2\mu}(x \ \mathrm{sh}\ t) \cos(\mu-\lambda)\pi \}\ dt \qquad \left[|\,\mathrm{Re}\ \mu| - \mathrm{Re}\ \lambda < \frac{1}{2}; \quad x > 0 \right].$$

MO 119

2. $$W_{\{,\mu}(z_1) W_{\lambda,\mu}(z_2) = \frac{(z_1 z_2)^{\mu+\frac{1}{2}} \exp\left[-\frac{1}{2}(z_1+z_2)\right]}{\Gamma(1-\{-\lambda)} \times \times \int_0^\infty e^{-t} t^{-\{-\lambda} (z_1+t)^{-\frac{1}{2}+\{-\mu} (z_2+t)^{-\frac{1}{2}+\lambda-\mu} \times \times F\left(\frac{1}{2} - \{ + \mu,\ \frac{1}{2} - \lambda + \mu;\ 1 - \{ - \lambda;\ \Theta \right)\ dt,$$

$$\Theta = \frac{t(z_1+z_2+t)}{(z_1+t)(z_2+t)} \qquad [z_1 \neq 0, \quad z_2 \neq 0, \quad |\arg\ z_1| < \pi, \quad |\arg\ z_2| < \pi, \quad \mathrm{Re}\,(\{ + \lambda) < 1]$$

MO 119

See also 3.334, 3.381 6., 3.382 3., 3.383 4., 8., 3.384 3., 3.471 2.

9.226

Series representations

$$M_{0,\mu}(z) = z^{\frac{1}{2}+\mu} \left\{ 1 + \sum_{k=1}^\infty \frac{z^{2k}}{2^{4k} k! (\mu+1)(\mu+2)\dots(\mu+k)} \right\}.$$

WH

Asymptotic representations

9.227[7]

For large values of $|z|$

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^{\lambda} \left(1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right]\left[\mu^2 - \left(\lambda - \frac{3}{2}\right)^2\right] \dots \left[\mu^2 - \left(\lambda - k + \frac{1}{2}\right)^2\right]}{k! z^k} \right)$$

$$[|\arg\ z| \le \pi - \alpha < \pi] \, .$$

WH

1089

9.228

For large values of $|\lambda|$

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu+1) \lambda^{-\mu-\frac{1}{4}} z^{\frac{1}{4}} \cos\left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi\right) .$$

MO 118

9.229

1. $$W_{\lambda,\mu} \sim -\left(\frac{4z}{\lambda}\right)^{\frac{1}{4}} e^{-\lambda + \lambda \ln \lambda} \sin\left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4}\right) .$$

MO 118

2. $$W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda}\right)^{\frac{1}{4}} e^{\lambda - \lambda\ \ln\ \lambda - 2\sqrt{\lambda z}}$$

MO 118

Formulas 9.228 and 9.229 are applicable for

$|\lambda| \gg 1, |\lambda| \gg |z|, |\lambda| \gg |\mu|, z \neq 0, |\arg\sqrt{z}| < \frac{3\pi}{4}$ and $|\arg\lambda| < \frac{\pi}{2}$.

MO 118

Functional relations

9.231

MO 117

2. $z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z) \qquad [2\mu \neq -1, -2, -3, \ldots].$

WH

9.232

1. $W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z).$

MO 116

2. $$W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu+\lambda\right)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu+\lambda\right)} M_{-\lambda,-\mu}(-z) \quad \left[|\arg(-z)| < \frac{3}{2}\pi\right].$$

WH

9.233

1. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{i\pi\lambda} W_{-\lambda,\mu}(e^{i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda,\mu}(z) \quad \left[-\frac{3}{2}\pi < \arg\, z < \frac{\pi}{2}; \quad 2\mu \neq -1, -2, \ldots\right].$$

MO 117

2. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{-i\pi\lambda} W_{-\lambda,\mu}(e^{-i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[-i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda,\mu}(z) \quad \left[-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi; \quad 2\mu \neq -1, -2, \ldots\right].$$

MO 117

9.234

Recursion formulas

WH

2. $W_{\mu,\lambda}(z) = \sqrt{z}W_{\mu-\frac{1}{2},\,\lambda-\frac{1}{2}}(z) + \left(\frac{1}{2} - \lambda - \mu\right) W_{\mu-1,\lambda}(z).$

WH

3. $z\frac{d}{dz}W_{\lambda,\mu}(z) = \left(\lambda - \frac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z).$

WH

4. $$\left[\left(\mu + \frac{1-z}{2}\right) W_{\lambda,\mu}\left(z\right) - z\frac{d}{dz}W_{\lambda,\mu}(z)\right] \left(\mu + \frac{1}{2} + \lambda\right) = \\ = \left[\left(\mu + \frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z\frac{d}{dz}W_{\lambda,\mu+1}(z)\right] \left(\mu + \frac{1}{2} - \lambda\right)$$

MO 117

5. $$\left(\frac{3}{2} + \lambda + \mu\right) \left(\frac{1}{2} + \lambda + \mu\right) zW_{\lambda,\mu}(z) = z(z + 2\mu + 1)\frac{d}{dz}W_{\lambda+1,\mu+1}(z) + \\ + \left[\frac{1}{2}z^2 + \left(\mu - \lambda - \frac{1}{2}\right) z + 2\mu^2 + 2\mu + \frac{1}{2}\right] W_{\lambda+1,\mu+1}(z).$$

MO 117

Connections with other functions

9.235

1. $M_{0,\mu}(z) = 2^{2\mu}\Gamma(\mu+1)\sqrt{z}I_\mu\left(\frac{z}{2}\right).$

MO 125a

2. $W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}K_\mu\left(\frac{z}{2}\right).$

MO 125

9.236

1. $\Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}}W_{-\frac{1}{4},\frac{1}{4}}(x^2) = \frac{2x}{\sqrt{\pi}}\Phi\left(\frac{1}{2},\ \frac{3}{2};\ -x^2\right).$

2. $\operatorname{li}(z) = -\dfrac{\sqrt{z}}{\sqrt{\ln \dfrac{1}{2}}} W_{-\frac{1}{2},0}(-\ln z).$

WH

1091

3. $\Gamma(\alpha, x) = e^{-x}\Psi(1-\alpha, 1-\alpha; x).$

EH I 266(21)

4. $\gamma(\alpha, x) = \dfrac{x^\alpha}{\alpha}\Phi(\alpha, \alpha+1; -x).$

EH I 266(22)

9.237

$$W_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)\Gamma\left(\frac{1}{2}+\mu-\lambda\right)} \times$$
$$\times \left\{ \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+k-\lambda+\frac{1}{2}\right)}{k!\,(2\mu+k)!} z^k \left[\psi(k+1)+\psi(2\mu+k+1)-\psi\left(\mu+k-\lambda+\frac{1}{2}\right)-\ln z\right] + \right.$$
$$\left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu-k)\Gamma\left(k-\mu-\lambda+\frac{1}{2}\right)}{k!}(-z)^k \right\}^*$$
$$\left[|\arg z| < \frac{3\pi}{2}; \quad 2\mu+1 \text{ is a natural number}\right].$$

MO 116

2. Set $\lambda - \mu - \frac{1}{2} = l$, where $l+1$ is a natural number. Then

$$W_{l+\mu+\frac{1}{2},\mu}(z) = (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}(2\mu+1)(2\mu+2)\dots(2\mu+l)\Phi(-l, 2\mu+1; z) =$$
$$= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} L_l^{2\mu}(z).$$

MO 116

9.238

1. $$J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix} \Phi\left(\frac{1}{2}+\nu,\, 1+2\nu;\, 2ix\right).$$

EH I 265(9)

2. $$I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x} \Phi\left(\frac{1}{2}+\nu,\, 1+2\nu;\, 2x\right).$$

EH I 265(10)

3. $$K_\nu(x) = \sqrt{\pi} e^{-x} (2x)^\nu \Psi\left(\frac{1}{2}+\nu,\, 1+2\nu;\, 2x\right).$$

EH I 265(13)

1092

9.24- 9.25 Parabolic cylinder functions $D_p(z)$

9.240

$$D_p(z) = 2^{\frac{1}{4}+\frac{p}{2}} W_{\frac{1}{4}+\frac{p}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-\frac{1}{2}} =$$
$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(-\frac{p}{2},\, \frac{1}{2};\, \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2},\, \frac{3}{2};\, \frac{z^2}{2}\right) \right\}$$

MO 120a

are called *parabolic cylinder functions.*

Integral representations

9.241

1. $$D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz}\, dx \qquad [\operatorname{Re} p > -1; \quad \text{for } x<0 \arg x^p = p\pi i]$$

MO 122

3.462
MO 122

9.242

1.* $D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_\infty^{(0+)} e^{-zt-\frac{1}{2}t^2} (-t)^{-p-1}\, dt \qquad [|\arg(-t)| \le \pi].$

WH

2. $$D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)}\, dt$$
$$\left[|\arg z| < \frac{\pi}{4}; \quad |\arg(1+t)| \le \pi\right].$$

WH

3. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t\, dt$$
$$\left[|\arg z| < \frac{3}{4}\pi; \quad p \text{ is not a positive integer}\right].$$

WH

4. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_\infty^{(0-)} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t\, dt$$

[for all values of $\arg z$; also, the contours encircle the poles of the function $\Gamma(-t)$ but they do not encircle the poles of the function $\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)$].

WH

1093
9.243

1. $$D_n(z) = (-1)^\mu \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} (\sqrt{n})^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} (zt\sqrt{n})\, dt + \right.$$
$$\left. + \int_0^\infty [e^{\frac{1}{2}n(1-t^2)} t^n - e^{-n(t-1)^2}] \begin{matrix}\cos\\ \sin\end{matrix} (zt\sqrt{n})\, dt - \int_{-\infty}^0 e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} (zt\sqrt{n})\, dt \right\}$$
$$[n \text{ is a natural number}].$$

2. $D_n(z) = (-1)^\mu 2^{n+2} (2\pi)^{-\frac{1}{2}} e^{\frac{1}{4}z^2} \int_0^\infty t^n e^{-2t^2} \begin{matrix}\cos\\ \sin\end{matrix} (2zt)\, dt$

[n is a natural number, $\mu = \left[\frac{n}{2}\right]$, and the cosine or sine is chosen according as n is even or odd].

WH

9.244

1. $D_{-p-1}[(1+i)z] = \dfrac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}}\Gamma\left(\dfrac{p+1}{2}\right)} \int_0^\infty \dfrac{e^{-ix^2z^2} x^p}{(1+x^2)^{1+\frac{p}{2}}}\, dx \qquad [\mathrm{Re}\ p > -1, \quad \mathrm{Re}\ iz^2 \geq 0].$

MO 122

2. $D_p[(1+i)z] = \dfrac{2^{\frac{p+1}{2}}}{\Gamma\left(-\dfrac{p}{2}\right)} \int_1^\infty e^{-\frac{i}{2}z^2x} \dfrac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}}\, dx \qquad [\mathrm{Re}\ p < 0; \quad \mathrm{Re}\ iz^2 \geq 0].$

MO 122

See also 3.383 6., 7., 3.384 2., 6., 3.966 5., 6.

9.245

1.* $D_p(x) D_{-p-1}(x) = -\dfrac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{cth}^{p+\frac{1}{2}}\, \dfrac{t}{2} \dfrac{1}{\sqrt{\mathrm{sh}\ t}} \sin \dfrac{x^2\ \mathrm{sh}\ t + p\pi}{2}\, dt \qquad [x \text{ is real, } Re\, p < 0].$

MO 122

2. $D_p\left(ze^{\frac{\pi}{4}i}\right) D_p\left(ze^{-\frac{\pi}{4}i}\right) = \dfrac{1}{\Gamma(-p)} \int_0^\infty \mathrm{cth}^p\ t \exp\left(-\dfrac{z^2}{2}\ \mathrm{sh}\ 2t\right) \dfrac{dt}{\mathrm{sh}\ t} \qquad \left[|\arg\ z| < \dfrac{\pi}{4}; \quad \mathrm{Re}\ p < 0\right]$

MO 122

See also 6.613.

9.246

Asymptotic expansions. If $|z| \gg 1, |z| \gg |p|$, then

1. $D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \dfrac{p(p-1)}{2z^2} + \dfrac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$

$\left[|\arg\ z| < \dfrac{3}{4}\pi\right]$

2. $$D_p(z) \sim e^{-z2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) - \\ - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \right. \\ \left. + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right) \qquad \left[\frac{\pi}{4} < \arg\, z < \frac{5}{4}\pi\right].$$

MO 121

3. $$D_p(z) \sim e^{-z2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) - \\ - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \right. \\ \left. + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right) \qquad \left[-\frac{\pi}{4} > \arg\, z > \frac{5}{4}\pi\right].$$

MO 121

Functional relations

9.247

Recursion formulas:

1. $$D_{p+1}(z) - zD_p(z) + pD_{p-1}(z) = 0.$$

WH

2. $$\frac{d}{dz} D_p(z) + \frac{1}{2} z D_p(z) - pD_{p-1}(z) = 0.$$

WH

3. $$\frac{d}{dz} D_p(z) - \frac{1}{2} z D_p(z) + D_{p+1}(z) = 0.$$

MO 121

9.248

Linear relations:

MO 121

9.249

$$D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp\left[-\frac{i}{2}\left(x^2 + p\frac{\pi}{2}\right)\right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it2/4}\,dt$$

$$[x \text{ real}; \ -1 < \operatorname{Re}\, p < 0].$$

MO 122

9.251

$$D_n(z) = (-1)^n e^{z2/4} \frac{d^n}{dz^n}(e^{-z2/2}) \qquad [n = 0,\ 1,\ 2, \dots].$$

WH

1095

9.252

$$D_p(ax+by) = \exp\frac{(bx-ay)^2}{4}\left(\frac{a}{\sqrt{a^2+b^2}}\right)^p \sum_{k=0}^{\infty}\binom{p}{k} D_{p-k}(\sqrt{a^2+b^2}x) D_k(\sqrt{a^2+b^2}y)\left(\frac{b}{a}\right)^k$$

$$[a > b > 0, \quad x > 0, \quad y > 0, \quad \operatorname{Re}\, p \geq 0] \qquad [\text{“summation theorem”}].$$

MO 124

Connections with other functions

9.253*

$$D_n(z) = -2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right).$$

MO 123a

9.254

1. $D_{-1}(z) = e^{\frac{z^2}{4}}\sqrt{\frac{\pi}{2}}\left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right].$

MO 123

2.[3] $D_{-2}(z) = e^{\frac{z^2}{4}}\sqrt{\frac{\pi}{2}}\left\{\sqrt{\frac{\pi}{2}}e^{-\frac{z^2}{2}} - z\left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right]\right\}.$

Differential equations leading to parabolic cylinder functions:

1. $$\frac{d^2u}{dz^2} + \left(p + \frac{1}{2} - \frac{z^2}{4}\right) u = 0,$$
$$u = D_p(z),\ D_p(-z),\ D_{-p-1}(iz),\ D_{-p-1}(-iz)$$

(These four solutions are linearly dependent. See 9.248)

2. $$\frac{d^2u}{dz^2} + (z^2 + \lambda)u = 0, \qquad u = D_{-\frac{1+i\lambda}{2}}[\pm(1+i)z].$$

EH II 118(12,13)A, MO 123

3.[7] $$\frac{d^2u}{dz^2} + z\frac{du}{dz} + (p+1)u = 0, \qquad u = e^{-\frac{z^2}{4}} D_p(z).$$

MO 123

9.26 Confluent hypergeometric series of two variables

9.261

1.[6] $$\Phi_1(\alpha,\ \beta,\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n} m!n!} x^m y^n \qquad [|x| < 1].$$

EH I 225(20)

2. $$\Phi_2(\beta,\ \beta',\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_m}{(\gamma)_{m+n} m!n!} x^m y^n.$$

EH I 225(21)A, ET I 385

3. $$\Phi_3(\beta,\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n} m!n!} x^m y^n.$$

EH I 225(22)

The functions Φ_1, Φ_2, Φ_3 satisfy the following systems of partial differential equations:

9.262

1. $z = \Phi_1(\alpha,\, \beta,\, \gamma,\, x,\, y)$

$$\begin{cases} x(1-x)\dfrac{\partial^2 z}{\partial x^2} + y(1-x)\dfrac{\partial^2 z}{\partial x\, \partial y} + [\gamma - (\alpha+\beta+1)x]\dfrac{\partial z}{\partial x} - \beta y\dfrac{\partial z}{\partial y} - \alpha\beta z = 0, \\ y\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial^2 z}{\partial x\, \partial y} + (\gamma - y)\dfrac{\partial z}{\partial y} - x\dfrac{\partial z}{\partial x} - \alpha z = 0. \end{cases}$$

EH I 235(23)

1096

2. $$z = \Phi_2(\beta,\, \beta',\, \gamma, x,\, y)\begin{cases} x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x \partial y} + (\gamma - x)\dfrac{\partial z}{\partial x} - \beta z = 0, \\ y\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial^2 z}{\partial x \partial y} + (\gamma - y)\dfrac{\partial z}{\partial y} - \beta' z = 0. \end{cases}$$

EH I 235(24)

3. $$z = \Phi_3(\beta,\, \gamma,\, x,\, y)\begin{cases} x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x \partial y} + (\gamma - x)\dfrac{\partial z}{\partial x} - \beta z = 0, \\ y\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial^2 z}{\partial x \partial y} + \gamma\dfrac{\partial z}{\partial y} - z = 0. \end{cases}$$

EH I 235(25)

9.3 Meijer's *G*-Function

9.30 Definition

9.301[7]

$$\mathbf{G}^{m,n}_{p,q}\left(x \,\middle|\, \begin{smallmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{smallmatrix}\right) = \tfrac{1}{2\pi i}\int \frac{\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-s)} x^s\, ds$$

[$0 \le m \le q, 0 \le n \le p$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_k + s)$ for any j and k (where $j = 1, \dots, m; k = 1, \dots, n$)]. Besides 9.301, the following notations are also used

$$G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right), \quad G^{mn}_{pq}(x), \quad G(x).$$

9.302

Three types of integration paths L in the right member of 9.301. can be exhibited:

1 The path L runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1-a_k+s)$ lie to the left, and the poles of the functions $\Gamma(b_j-s)$ lie to the right of L (for $j=1,2,\ldots,m$ and $k=1,2,\ldots,n$). In this case, the conditions under which the integral 9.301 converges are of the form

$$p+q<2(m+n), \quad |\arg\, x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi.$$

EH I 207(2)

2) L is a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j-s)$ (for $j=1,2,\ldots,m$) once in the negative direction. All the poles of the functions $\Gamma(1-a_k+s)$ must remain outside this loop. Then, the conditions under which the integral 9.301 converges are:

$$q\geq 1 \text{ and either } p<q \text{ or } p=q \text{ and } |x|<1.$$

EH I 207(3)

1097

3) L is a loop, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1-a_k+s)$ (for $k=1,2,\ldots,n$) once in the positive direction. All the poles of the functions $\Gamma(b_j-s)$ (for $j=1,2,\ldots,m$) must remain outside this loop. The conditions under which the integral in 9.301 converges are

$$p\geq 1 \text{ and either } p>q \text{ or } p=q \text{ and } |x|>1.$$

EH I 207(4)

The function $G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right)$ is analytic with respect to x; it is symmetric with respect to the parameters $a_1,\ldots,a_n$ and also with respect to $a_{n+1},\ldots,a_p$; $b_1,\ldots,b_m$; $b_{m+1},\ldots,b_q$.

EH I 208

9.303[7]

If no two b_j (for $j = 1, 2, \dots, n$) differ by an integer, then, under the conditions that either $p < q$ or $p = q$ and $|x| < 1$,

$$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{n=1}^{m} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - b_h) \prod\limits_{j=1}^{n} \Gamma(1 + b_h - a_j)}{\prod\limits_{j=m+1}^{q} \Gamma(1 + b_h - b_j) \prod\limits_{j=n+1}^{p} \Gamma(a_j - b_h)} x^{b_h} \times$$
$$\times {}_pF_{q-1}[1+b_h-a_1, \dots, 1+b_h-a_p; \quad 1+b_h-b_1, \dots$$
$$\dots, , \dots, 1 + b_h - b_q; \quad (-1)^{p-m-n}x]^*).$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when $j = h$. The asterisk under the symbol for the function ${}_pF_{q-1}$ denotes the omission of the hth parameter.

9.304[7]

If no two a_k (for $k = 1, 2, \dots, n$) differ by an integer, then, under the conditions that $q < p$ or $q = p$ and $|x| > 1$,

$$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{h=1}^{n} \frac{\prod\limits_{j=1}^{n} \Gamma(a_h - a_j) \prod\limits_{j=1}^{m} \Gamma(b_j - a_h + 1)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - a_h + 1) \prod\limits_{j=m+1}^{q} \Gamma(a_h - b_j)} x^{a_h - 1} \times$$
$$\times {}_qF_{p-1}[1+b_1-a_h, \dots, 1+b_q-a_h; \quad 1+a_1-a_h, \dots$$
$$\dots, , \dots, 1 + a_p - a_h; \quad (-1)^{q-m-n}x^{-1}]^*).$$

EH I 208(6)

9.31 Functional relations

If one of the parameters a_j (for $j = 1, 2, \dots, n$) coincides with one of the parameters b_j (for $j = m+1, m+2, \dots, q$), the order of the G-function decreases. For example,

1. $$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, a_1 \end{matrix}\right) = G_{p-1,q-1}^{m,n-1}\left(x \middle| \begin{matrix} a_2, \dots, a_p \\ b_1, \dots, b_{q-1} \end{matrix}\right) \qquad [n, p, q \geq 1].$$

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \dots, m$) coincides with one of the a_j (for $j = n+1, \dots, p$). In this case, it is m and not n that decreases by one unit.

EH I 209(7)

The G-function with $p > q$ can be transformed into the G-function with $p < q$ by means of the relationships:

2. $$G_{pq}^{mn}\left(x^{-1}\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) = G_{qp}^{nm}\left(x\,\middle|\,\begin{matrix}1-b_s\\ 1-a_r\end{matrix}\right).$$

EH I 209(9)

3. $$x\frac{d}{dx}G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) = G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_1-1, a_2, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right) + (a_1-1)G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) \qquad [n \geq 1].$$

EH I 210(13)

9.32 A differential equation for the G-function

$G_{pq}^{mn}\left(x\,\middle|\,\begin{smallmatrix}a_r\\ b_s\end{smallmatrix}\right)$ satisfies the following linear qth-order differential equation

$$\left[(-1)^{p-m-n}x\prod_{j=1}^{p}\left(x\frac{d}{dx}-a_j+1\right) - \prod_{j=1}^{q}\left(x\frac{d}{dx}-b_j\right)\right]y = 0 \qquad [p \leq q].$$

EH I 210(1)

9.33 Series of G-functions

$$G_{pq}^{mn}\left(\lambda x\,\middle|\,\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right) = \lambda^{b_1}\sum_{r=0}^{\infty}\frac{1}{r!}(1-\lambda)^r G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_1, \dots, a_p\\ b_1+r, b_2, \dots, b_q\end{matrix}\right)$$

$$[|\lambda-1| < 1, \quad m \geq 1, \quad \text{if } m=1 \text{ and } p<q, \lambda \text{ may be arbitrary}];$$

$$= \lambda^{b_q}\sum_{r=0}^{\infty}\frac{1}{r!}(\lambda-1)^r G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_{q-1}, b_q+r\end{matrix}\right)$$

$$[m < q, \quad |\lambda-1| < 1];$$

$$= \lambda^{a_1-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\lambda-\frac{1}{\lambda}\right)^r G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_1-r, a_2, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right)$$

$$\left[n \geq 1, \quad \operatorname{Re}\lambda > \frac{1}{2}(\text{if } n=1 \text{ and } p>q, \text{ then } \lambda \text{ may be arbitrary})\right];$$

$$= \lambda^{a_p-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{1}{\lambda}-1\right)^r G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_1, \dots, a_{p-1}, a_p-r\\ b_1, \dots, b_q\end{matrix}\right)$$

$$\left[n < p, \quad \operatorname{Re}\gamma > \frac{1}{2}\right].$$

1. $J_\nu(x)x^\mu = 2^\mu G_{02}^{10}\left(\frac{1}{4}x^2 \,\middle|\, \frac{1}{2}\nu + \frac{1}{2}\mu,\ \frac{1}{2}\mu - \frac{1}{2}\nu\right).$

EH I 219(44)

2. $N_\nu(x)x^\mu = 2^\mu G_{13}^{20}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \\ \frac{1}{2}\mu - \frac{1}{2}\nu,\ \frac{1}{2}\mu + \frac{1}{2}\nu,\ \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \end{matrix}\right).$

EH I 219(46)

3. $K_\nu(x)x^\mu = 2^{\mu-1} G_{02}^{20}\left(\frac{1}{4}x^2 \,\middle|\, \frac{1}{2}\mu + \frac{1}{2}\nu,\ \frac{1}{2}\mu - \frac{1}{2}\nu\right).$

EH I 219(47)

4. $K_\nu(x) = e^x\sqrt{\pi}\, G_{12}^{20}\left(2x \,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu,\ -\nu \end{matrix}\right).$

EH I 219(49)

5. $\mathbf{H}_\nu(x)x^\mu = 2^\mu G_{13}^{11}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu \\ \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu,\ \frac{1}{2}\mu - \frac{1}{2}\nu,\ \frac{1}{2}\mu + \frac{1}{2}\nu \end{matrix}\right).$

EH I 220(51)

6. $S_{\mu,\nu}(x) = 2^{\mu-1}\frac{1}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)} G_{13}^{31}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2} + \frac{1}{2}\mu \\ \frac{1}{2} + \frac{1}{2}\mu,\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu \end{matrix}\right).$

EH I 220(55)

7.[7] ${}_2F_1(a,\, b;\, c;\, -x) = \frac{\Gamma(c)x}{\Gamma(a)\Gamma(b)} G_{22}^{12}\left(x \,\middle|\, \begin{matrix} -a, -b \\ -1, -c \end{matrix}\right).$

EH I 222(74)a

8. $$ {}_pF_q(a_1, \ldots, a_p;\, b_1, \ldots, b_q;\, x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} G_{p,q+1}^{1,p}\left(-x \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0,\ 1-b_1, \ldots, 1-b_q \end{matrix}\right); $$

$$ = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} G_{q+1,p}^{p,1}\left(-\frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right). $$

9. $W_{k,m}(x) = \frac{2^k \sqrt{x} e^{\frac{1}{2}x}}{\sqrt{2\pi}} G_{24}^{40}\left(\frac{x^2}{4} \middle| \begin{matrix} \frac{1}{4} - \frac{1}{2}k,\ \frac{3}{4} - \frac{1}{2}k \\ \frac{1}{2} + \frac{1}{2}m,\ \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{matrix} \right).$

EH I 221(70)

9.4 MacRobert's *E*-Function

9.41 Representation by means of multiple integrals

$$E(p;\ \alpha_r\colon\ q;\ \varrho_s\colon\ x) = \frac{\Gamma(\alpha_{q+1})}{\Gamma(\varrho_1 - \alpha_1)\Gamma(\varrho_2 - \alpha_2)\dots\Gamma(\varrho_q - \alpha_q)} \times$$
$$\times \prod_{\mu=1}^{q} \int_0^\infty \lambda_\mu^{\varrho_\mu - \alpha_\mu - 1}(1 - \lambda_\mu)^{-\varrho_\mu} d\lambda_\mu \prod_{\nu=2}^{p-q-1} \int_0^\infty e^{-\lambda_{q+\nu}} \lambda_{q+\nu}^{\alpha_{q+\nu}-1} d\lambda_{q+\nu} \times$$
$$\times \int_0^\infty e^{-\lambda_p} \lambda_p^{\alpha_p - 1} \left[1 + \frac{\lambda_{q+2}\lambda_{q+3}\dots\lambda_p}{(1+\lambda_1)\dots(1+\lambda_q)x}\right]^{-\alpha_{q+1}} d\lambda_p$$

[$|\arg x| < \pi$, $p \ge q+1$, α_r and ϱ_s are bounded by the condition that the integrals on the right be convergent.]

EH I 204(3)

9.42 Functional relations

1. $\alpha_1 x E(\alpha_1, \dots, \alpha_p\colon\ \varrho_1, \dots, \varrho_q\colon\ x) = xE(\alpha_1 + 1,\ \alpha_2, \dots, \alpha_p\colon\ \varrho_1, \dots, \varrho_q\colon\ x) + E(\alpha_1 + 1,\ \alpha_2 + 1, \dots, \alpha_p + 1\colon\ \varrho_1 + 1, \dots, \varrho_q + 1\colon\ x).$

EH I 205(7)

2. $(\varrho_1 - 1)xE(\alpha_1, \dots, \alpha_p\colon\ \varrho_1, \dots, \varrho_q\colon\ x) = xE(\alpha_1, \dots, \alpha_p\colon\ \varrho_1 - 1,\ \varrho_2, \dots, \varrho_q\colon\ x) + E(\alpha_1 + 1, \dots, \alpha_p + 1\colon\ \varrho_1 + 1, \dots, \varrho_q + 1\colon\ x).$

EH I 205(9)

3. $\frac{d}{dx}E(\alpha_1, \dots, \alpha_p\colon\ \varrho_1, \dots, \varrho_q\colon\ x) = x^{-2}E(\alpha_1 + 1, \dots, \alpha_p + 1\colon\ \varrho_1 + 1, \dots, \varrho_q + 1\colon\ x).$

9.5 Riemann's Zeta Functions $\zeta(z,q)$, and $\zeta(z)$, and the Functions $\Phi(z,s,v)$ and $\xi(s)$

9.51 Definition and integral representations

9.511

$$\zeta(z,\,q)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}e^{-qt}}{1-e^{-t}}\,dt;$$
$$=\frac{1}{2}q^{-z}+\frac{q^{1-z}}{z-1}+2\int_0^\infty(q^2+t^2)^{-\frac{z}{2}}\left[\sin\left(z\ \mathrm{arctg}\ \frac{t}{q}\right)\right]\frac{dt}{e^{2\pi t}-1}$$
$$[0<q<1,\quad \mathrm{Re}\ z>1]\ .$$

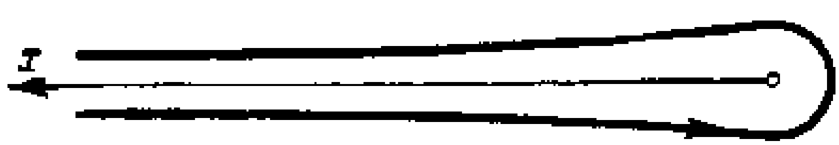
WH

9.512

$$\zeta(z,\,q)=-\frac{\Gamma(1-z)}{2\pi i}\int_\infty^{(0+)}\frac{(-\theta)^{z-1}e^{-q\theta}}{1-e^{-\theta}}\,d\theta.$$

1101

This equation is valid for all values of z except for $z=1,2,3,\ldots$. It is assumed that the path of integration (see drawing below) does not pass through the points $2n\pi i$ (where n is a natural number).



See also 4.251 4., 4.271 1., 4., 8., 4.272 9., 12., 4.294 11.

9.513

$$1.\quad \zeta(z)=\frac{1}{(1-2^{1-z})\Gamma(z)}\int_0^\infty\frac{t^{z-1}}{e^t+1}\,dt\qquad[\mathrm{Re}\ z>0].$$

WH

WH

3. $\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left[\frac{1}{z(z-1)} + \int_1^\infty \left(t^{\frac{1-z}{2}} + t^{\frac{z}{2}} \right) t^{-1} \sum_{k=1}^\infty e^{-k^2 \pi i} \, dt \right].$

WH

4. $\zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty (1+t^2)^{-\frac{z}{2}} \sin(z \text{ arctg } t) \frac{dt}{e^{\pi t}+1}.$

WH

5. $\zeta(z) = \frac{2^{z-1}}{2^z-1} \frac{z}{z-1} + \frac{2}{2^z-1} \int_0^\infty \left(\frac{1}{4} + t^2 \right)^{-z/2} \sin(z \text{ arctg } 2t) \frac{dt}{e^{2\pi t}-1}.$

WH

See also 3.411 1., 3.523 1., 3.527 1., 3., 4.271 8.

9.52 Representation as a series or as an infinite product

9.521

1. $\zeta(z, q) = \sum_{n=0}^\infty \frac{1}{(q+n)^z} \qquad [\text{Re } z > 1, \quad q \neq 0, \quad -1, -2, \ldots].$

WH

2. $\zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin \frac{z\pi}{2} \sum_{n=1}^\infty \frac{\cos 2\pi q n}{n^{1-z}} + \cos \frac{z\pi}{2} \sum_{n=1}^\infty \frac{\sin 2\pi q n}{n^{1-z}} \right]$
$[\text{Re } z < 0, \quad 0 < q \leq 1].$

WH

3.[8] $\zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^\infty F_n(z),$

where

3.[8] $\zeta(z, q) = \sum_{n=0}^{N} \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$

where

$$F_n(z) = \frac{1}{1-z}\left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}}\right) - \frac{1}{(n+1+q)^z} = $$
$$= z\int_n^{n+1} \frac{(t-n)\,dt}{(t+q)^{z+1}} \qquad [\operatorname{Re} z > 1].$$

WH

1102

9.522

1. $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad [\operatorname{Re} z > 1].$

WH

2. $\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \qquad [\operatorname{Re} z > 0].$

WH

9.523

1.[7] $\zeta(z) = \prod \frac{1}{1-p^{-z}},$

2. $\ln \zeta(z) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{kz}}.$

The product and the summation are taken over all primes p. $[\operatorname{Re} z > 1].$

WH

9.524[7]

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z}, \qquad [\operatorname{Re} z > 1]$$

where $\Lambda(k) = 0$ when k is not a power of a prime and $\Lambda(k) = \ln p$ when k is a power of a prime p. [for Re $z > 1$].

WH

9.53 Functional relations

9.531

$$\zeta(-n,\, q) = -\frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1} \qquad [n \text{ is a nonnegative integer}].$$

[see EH I 27(11).]

WH

9.532

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k,\, q) = \ln\, \frac{e^{-\boldsymbol{C}z}\Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)} \qquad [|z| < q].$$

WH

9.533

1. $$\lim_{z\to 1} \frac{\zeta(z,\, q)}{\Gamma(1-z)} = -1.$$

WH

2. $$\lim_{z\to 1} \left\{\zeta(z,\, q) - \frac{1}{z-1}\right\} = -\psi(q).$$

WH

3. $$\left\{\frac{d}{dz}\zeta(z,\, q)\right\}_{z=0} = \ln\, \Gamma(q) - \frac{1}{2}\ln 2\pi.$$

WH

9.534

$$\zeta(z,\, 1) = \zeta(z).$$

9.535

1. $$\zeta(z) = \frac{1}{2^z - 1}\zeta\left(z,\, \frac{1}{2}\right) \qquad [\mathrm{Re}\; z > 1].$$

WH

2. $2^z\Gamma(1-z)\zeta(1-z)\sin\frac{z\pi}{2}\pi^{1-z}\zeta(z).$

WH

3. $2^{1-z}\Gamma(z)\zeta(z)\cos\frac{z\pi}{2} = \pi^z\zeta(1-z).$

WH

4. $\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{\frac{z-1}{2}}\zeta(1-z).$

WH

9.536

$$\lim_{z\to 1}\left\{\zeta(z) - \frac{1}{z-1}\right\} = \boldsymbol{C}.$$

9.537

$$\text{Set } z = \frac{1}{2} + it; \quad \text{Then, } \Xi(t) = \frac{(z-1)\Gamma\left(\frac{z}{2}+1\right)}{\sqrt{\pi^z}}\zeta(z) = \Xi(-t)$$

is an even function of t with real coefficients in its expansion in powers of t^2.

JA

9.54 Singular points and zeros

9.541[7]

1. $z = 1$ is the only singular point of the function $\zeta(z)$.

WH

2. The function $\zeta(z)$ has simple zeros at the points $-2n$, where n is a natural number. All other zeros of the function $\zeta(z)$ lie in the strip $0 < \operatorname{Re} z < 1$.

3.[8] Riemann's hypothesis: All zeros of the function $\zeta(z)$ lie on the straight line $\operatorname{Re} z = \frac{1}{2}$. It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first $1, 500, 000, 001$ zeros lying in $0 < \operatorname{Im} z < 545, 439, 823.215$ are known to have $\operatorname{Re} z = 1/20$

WH

9.542

Particular values:

$$\left.\begin{array}{ll} 1. & \zeta(2m) = \dfrac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!}, \\ 2. & \zeta(1-2m) = -\dfrac{B_{2m}}{2m}, \\ 3. & \zeta(-2m) = 0 \end{array}\right\} \qquad [m \text{ is a natural number}].$$

$$4. \quad \zeta'(0) = -\frac{1}{2}\ln 2\pi.$$

WH

9.55 The Lerch function $\Phi(z, s, v)$

9.550

Definition:

$$\Phi(z,\ s,\ v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \qquad [|z| < 1, \quad v \neq 0, \quad -1, \ldots].$$

EH I 27(1)

Functional relations

9.551

$$\Phi(z,\ s,\ v) = z^m\Phi(z,\ s,\ m+v) + \sum_{n=0}^{m-1}(v+n)^{-s}z^n$$
$$[m = 1,\ 2,\ 3, \ldots,\ v \neq 0,\ -1,\ -2, \ldots]\ .$$

EH I 27(1)

9.552

9.552

$$\Phi(z,\, s,\, v) = iz^{-v}(2\pi)^{s-1}\Gamma(1-s)\left[e^{-i\pi\frac{s}{2}}\Phi\left(e^{-2\pi iv},\, 1-s,\, \frac{\ln\, z}{2\pi i}\right) - \right.$$
$$\left. - \, e^{i\pi\left(\frac{s}{2}-2v\right)}\Phi\left(e^{2\pi iv},\, 1-s,\, 1-\frac{\ln\, z}{2\pi i}\right)\right].$$

EH I 29(7)

Series representation

9.553

$$\Phi(z,\, s,\, v) = z^{-v}\Gamma(1-s)\sum_{n=-\infty}^{\infty}(-\ln\, z + 2\pi ni)^{s-1}e^{2\pi nvi}$$
$$[0 < v \le 1, \quad \mathrm{Re}\; s < 0, \quad |\arg\,(-\ln\, z + 2\pi ni)| \le \pi].$$

EH I 28(6)

9.554

$$\Phi(z,\, m,\, v) = z^{-v}\left\{\sum_{n=0}^{\infty}\zeta(m-n,\, v)\frac{(\ln\, z)^n}{n!} + \frac{(\ln\, z)^{m-1}}{(m-1)!}\left[\psi(m) - \psi(v) - \ln\left(\ln\, \frac{1}{z}\right)\right]\right\}^*$$
$$[m = 2,\, 3,\, 4, \ldots,\, |\ln\, z| < 2\pi, \quad v \ne 0, \quad -1,\, -2,\, \ldots].$$

EH I 30(9)

* The prime on the symbol $\sum$ means that the term corresponding to $n = m - 1$ is omitted.

9.555

$$\Phi(z,\, -m,\, v) = \frac{m!}{z^v}\left(\ln\, \frac{1}{z}\right)^{-m-1} - \frac{1}{z^v}\sum_{r=0}^{\infty}\frac{B_{m+r+1}(v)(\ln\, z)^r}{r!(m+r+1)} \qquad [|\ln\, z| < 2\pi].$$

EH I 30(11)

Integral representation

9.556

$$\Phi(z,\, s,\, v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}\,dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}\,dt}{e^t - z}$$
$$[\mathrm{Re}\; v > 0, \quad \text{or} \quad |z| \le 1, \quad z \ne 1, \quad \mathrm{Re}\; s > 0, \quad \text{or} \quad z = 1, \quad \mathrm{Re}\; s > 1].$$

Limit relationships

9.557

$$\lim_{z\to 1}(1-z)^{1-s}\Phi(z,\, s,\, v) = \Gamma(1-s) \qquad [\mathrm{Re}\ s < 1].$$

EH I 30(12)

9.558

$$\lim_{z\to 1}\frac{\Phi(z,\, 1,\, v)}{-\ln(1-z)} = 1.$$

EH I 30(13)

A connection with a hypergeometric function

9.559

$$\Phi(z,\, 1,\, v) = v^{-1}{}_2F_1(1,\, v;\, 1+v;\, z) \qquad [|z| < 1].$$

EH I 30(10)

9.56 The function ξ(s)

9.561

$$\xi(s) = \frac{1}{2}s(s-1)\frac{\Gamma\left(\frac{1}{2}s\right)}{\pi^{\frac{1}{2}s}}\zeta(s).$$

EH III 190(10)

9.562

$$\xi(1-s) = \xi(s).$$

EH III 190(11)

9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions

$\nu(x), \nu(x,\alpha), \mu(x,\beta), \mu(x,\beta,\alpha), \lambda(x,y)$ and Euler Polynomials

9.61 Bernoulli numbers

9.610

The numbers B_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad [0 < |t| < 2\pi],$$

are called *Bernoulli* numbers. Thus, the function $\frac{t}{e^t - 1}$ is a generating function for the Bernoulli numbers.

GE 48(57), FI II 520

9.611

Integral representations

1. $$B_{2n} = (-1)^{n-1} 4n \int_0^\infty \frac{x^{2n-1}}{e^{2\pi x} - 1}\, dx \qquad [n = 1, 2, \ldots] \qquad \text{(cf. 3.411 2., 4.)}.$$
2. $$B_{2n} = (-1)^{n-1} \pi^{-2n} \int_0^\infty \frac{x^{2n}}{\operatorname{sh}^2 x}\, dx \qquad [n = 1, 2, \ldots].$$
3. $$B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^\infty x^{2n-2} \ln(1 - e^{-2\pi x})\, dx \qquad [n = 1, 2, \ldots].$$

3.411
FI II 721a

See also 3.523 2., 4.271 3.

Properties and functional relations

9.612[8]

A symbolic notation:

$$(B + \alpha)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} B_k \alpha^{n-k}, \qquad [n \geq z];$$

$$B_n = (B+1)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad [n \geq z];$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!}, \qquad [n \geq z].$$

9.613

All the Bernoulli numbers are rational numbers.

1106
9.614

Every number B_n can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where C_n is an integer and the sum is taken over all $k > 0$ such that $k+1$ is a prime and k is a divisor of n.

GE 64

9.615[8]

All the Bernoulli numbers with odd index are equal to zero except that $B_1 = -\frac{1}{2}$; that is, $B_{2n+1} = 0$ for n a natural number.

GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1}^{n-1} \frac{2n(2n-1)\dots(2n-k+2)}{(2k)!} B^{2k} \qquad [n \geq 1].$$

9.616

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \qquad [n \geq 0]. \qquad (\text{cf. } 9.542)$$

GE 56(79), FI II 721a

7

9.617

$$B_{2n}(-1)^{n-1} = \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{\prod\limits_{p}\left(1 - \frac{1}{p^{2n}}\right)} \qquad [n \geq 1] \qquad (\text{cf. } 9.523)$$

9.523

(where the product is taken over all primes p).

For a connection with Riemann's zeta function, see 9.542.

For a connection with the Euler numbers, see 9.635.

For a table of values of the Bernoulli numbers, see 9.71

9.618[6]

$$\text{Symbolic notation: } (B+\alpha)^{[n]} \equiv \sum_{k=0}^{n} \binom{n}{k} B_k \alpha^{n-k} \qquad [n = 0,\, 1, \ldots].$$

CE 337

9.619

An inequality $|(B-\theta)^{[n]}| \leq |B_n| \qquad [0 < \theta < 1]$.

9.62 Bernoulli polynomials

9.620

The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

GE 51(62)

or symbolically, $B_n(x) = (B+x)^{[n]}$.

GE 52(68)

9.621

The generating function

$$\frac{e^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^{n-1}}{n!} \qquad [0 < |t| < 2\pi] \qquad \text{(cf. 1.213)}$$

GE 65(89)a

9.622

Series representation

$$1.^7 \quad B_n(x) = -2\frac{n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos\left(2\pi kx - \frac{1}{2}\pi n\right)}{k^n},\ [n > 1,\ 1 \ge x \ge 0,\ n = 1,\ 1 > x > 0].$$

AS 805(23.1.16)

1107

$$2.^7 \quad B_{2n-1}(x) = 2\frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n-1}},\ [n > 1,\ 1 \ge x \ge 0,\ n = 1,\ 1 > x > 0].$$

AS 805(23.1.17)

$$3.^* \quad B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}} \qquad [0 \le x \le 1, \quad n = 1, 2, \ldots].$$

GE 71

9.623

Functional relations and properties:

$$1. \quad B_{m+1}(n) = B_{m+1} + (m+1)\sum_{k=1}^{n-1} k^m \qquad [n \text{ and } m \text{ are natural numbers}] \qquad \text{(see also 0.121)}$$

GE 51(65)

$$2. \quad B_n(x+1) - B_n(x) = nx^{n-1}.$$

3. $B_n'(x) = nB_{n-1}(x) \qquad [n = 1, 2, \ldots].$

GE 66

4. $B_n(1-x) = (-1)^n B_n(x).$

GE 66

5.* $(-1)^n B_n(-x) = B_n(x) + nx^{n-1} \qquad [n = 0, 1, \ldots]$

AS 804(23.1.9)

9.624[7]

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \qquad [m = 1, 2, \ldots ; n = 0, 1, \ldots ; \quad \text{“summation theorem”}].$$

GE 67

9.625

For n odd, the differences

$$B_n(x) - B_n$$

vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1. They change sign at the point $x = \frac{1}{2}$. For n even, these differences vanish at the end points of the interval $[0, 1]$. Within this interval, they do not change sign and their greatest absolute value occurs at the point $x = \frac{1}{2}$.

9.626

The polynomials

$$B_{2n}(x) - B_{2n} \text{ and } B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval $(0, 1)$.

GE 87

9.627

Special cases:

$$\left.\begin{array}{ll} 1. & B_1(x) = x - \frac{1}{2}. \\ 2. & B_2(x) = x^2 - x + \frac{1}{6}. \\ 3. & B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2x}. \\ 4. & B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}. \\ 5. & B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \end{array}\right\}$$

GE 70

1108
9.628

Particular values:

$$1. \quad B_n(0) = B_n.$$

$$2. \quad B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n \qquad [n \neq 1].$$

GE 76

9.63 Euler numbers

9.630

The numbers E_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{1}{\text{ch } t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \left[|t| < \frac{\pi}{2}\right],$$

are known as the *Euler numbers*. Thus, the function $\frac{1}{\text{ch } t}$ is a generating function for the Euler numbers.

CE 330

9.631

A recursion formula

$$(E+1)^{[n]} + (E-1)^{[n]} = 0 \qquad [n \geq 1], \qquad E_0 = 1.$$

CE 329

Properties of the euler numbers

9.632

The Euler numbers are integers.

9.633

The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0.$$

CE 329

9.634

If $\alpha, \beta\gamma, \ldots$ are the divisors of the number $n-m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers

$2\alpha+1, 2\beta+1, 2\gamma+1, \ldots$, that are primes.

9.635

A connection with the Bernoulli numbers (symbolic notation):

$$1.^6 \quad E_{n-1} + 4(-1)^n(3^{n-1}-1)B_1 = \frac{(4B-1)^{[n]}(4B-3)^{[n]}}{2n} + 4(-1)^{n+1}(3^{n-1}-1)B_1.$$

CE 330

$$2. \quad B_n = \frac{n(E+1)^{[n-1]}}{2^n(2^n-1)} \qquad [n \geq 2].$$

CE 330

$$3.^6 \quad \left(B + \frac{1}{4}\right)^{[2n+1]} = -4^{-2n-1}(2n+1)E_{2n} \qquad [n \geq 0].$$

For a table of values of the Euler numbers, see 9.72.

9.64 The functions $\nu(x), \nu(x,\alpha), \mu(x,\beta), \mu(x,\beta,\alpha), \lambda(x,y)$

9.640

1. $$\nu(x) = \int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}.$$

EH III 217(1)

2. $$\nu(x,\,\alpha) = \int_0^\infty \frac{x^{\alpha+t}\,dt}{\Gamma(\alpha+t+1)}.$$

EH III 217(1)

3. $$\mu(x,\,\beta) = \int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\Gamma(t+1)}.$$

EH III 217(2)

4. $$\mu(x,\,\beta,\,\alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\beta\,dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)}.$$

EH III 217(2)

5. $$\lambda(x,\,y) = \int_0^y \frac{\Gamma(u+1)\,du}{x^u}.$$

MI 9

9.65 Euler polynomials

9.650

The Euler polynomials are defined by

AS 804 (23.1.7)

9.651

The generating function:

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

AS 804 (23.1.1)

9.652

Series representation:

1. $$E_n(x) = 4\frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)\pi x - \frac{1}{2}\pi n\right)}{(2k+1)^{n+1}}. \qquad [n>0, \quad 1 \geq x \geq 0, \quad n=1, \quad 1>x>0].$$

AS 804 (23.1.16)

2. $$E_{2n-1}(x) = \frac{(-1)4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}}. \qquad [n=1,\,2,\ldots, \quad 1 \geq x \geq 0].$$

AS 804 (23.1.17)

3. $$E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}}. \qquad [n>0, \quad 1 \geq x \geq 0, \quad n=0, \quad 1>x>0].$$

AS 804 (23.1.18)

9.653

Functional relations and properties:

1. $$E_m(n+1) = 2\sum_{k=1}^{n} (-1)^{n-k} k^m + (-1)^{n+1} E_m(0), \qquad [m,\,n \text{ natural numbers}].$$

AS 804 (23.1.4)

As 804 (23.1.5)

3. $E_n(x+1) + E_n(x) = 2x^n. \qquad [n = 0,\ 1, \dots]$

As 804 (23.1.6)

4.[8] $E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x + \frac{k}{m}\right). \qquad [n = 0,\ 1, \dots, m = 1,\ 3, \dots]$

As 804 (23.1.10)

5. $E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right) \qquad [n = 0,\ 1, \dots, m = 2,\ 4, \dots].$

AS 804 (23.1.10)

9.654

Special cases:

1. $E_1(x) = x - \frac{1}{2}.$

2. $E_2(x) = x^2 - x.$

3. $E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$

4. $E_4(x) = x^4 - 2x^3 + x.$

5. $E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}.$

9.655

Particular values:

1. $E_{2n+1} = 0. \qquad [n = 0,\, 1, \ldots]$

As 805 (23.1.19)

2. $E_n = (0) = -E_n(1) = -2(n+1)^{-1}(2^{n+1} - 1)B_{n+1}. \qquad [n = 1,\, 2, \ldots]$

As 805 (23.1.20)

3. $E_n\left(\frac{1}{2}\right) = 2^{-n}E_n. \qquad [n = 0,\, 1, \ldots]$

As 805 (23.1.21)

4. $E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -(2n)^{-1}(1-3^{1-2n})(2^{2n}-1)B_{2n}. \qquad [n = 1,\, 2, \ldots]$

As 806 (23.1.22)

9.7 Constants

9.71 Bernoulli numbers

$$
\begin{array}{ll}
B_0 = 1, & \\
B_1 = -\frac{1}{2}, & B_{18} = \frac{43\,867}{798}, \\
B_2 = \frac{1}{6}, & B_{20} = -\frac{174\,611}{330}, \\
B_4 = -\frac{1}{30}, & B_{22} = \frac{854\,513}{138}, \\
B_6 = \frac{1}{42}, & B_{24} = -\frac{236\,364\,091}{2730}, \\
B_8 = -\frac{1}{30}, & B_{26} = \frac{8\,553\,103}{6}, \\
B_{10} = \frac{5}{66}, & B_{28} = -\frac{23\,749\,461\,029}{870}, \\
B_{12} = -\frac{691}{2730}, & B_{30} = \frac{8\,615\,841\,276\,005}{14\,322}, \\
B_{14} = \frac{7}{6}, & B_{32} = -\frac{7\,709\,321\,041\,217}{510}, \\
B_{16} = -\frac{3617}{510}, & B_{34} = \frac{2\,577\,687\,858\,367}{6}.
\end{array}
$$

9.72 Euler numbers

The Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

9.73 Euler's and Catalan's constants

Euler's constant

$$\boldsymbol{C} = 0.577\,215\,664\,901\,532\,860\,606\,512\ldots$$

Catalan's constant

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\,965\,594\ldots$$

9.74 Stirling numbers

9.740

The **Stirling number of the first kind** $S_n^{(m)}$ is defined by the requirement that $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles.

AS 824 (23.1.3)

9.741

Generating functions:

1. $$x(x-1)\ldots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m.$$

AS 824 (24.1.3)

2. $$\{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \qquad [|x| < 1]$$

Recurrence relations:

1.[8] $S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)};\ S_n^{(0)} = \delta_{on};\ S_n^{(1)} = (-1)^{n-1}(n-1)!;\ S_n^{(n)} = 1. \qquad [n \geq m \geq 1]$

AS 824 (24.1.3)

2. $\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)} \qquad [n \geq m \geq r]$

AS 824 (24.1.3)

9.743

Functional relations and properties

1. $x(x-h)(x-2h)\dots(x-mh+h) = \dfrac{h^m \Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}-m+1\right)} = h^m \sum_{k=1}^{m} \left(\frac{x}{h}\right)^k S_k^{(m)}.$

1113

2. $[(x+1)(x+2)\dots(x+m)]^{-1} = \left[\binom{x+m}{m} m!\right]^{-1} = \left[\sum_{k=1}^{p} (x+m)^k S_k^{(m)}\right]^{-1}.$

3. $[(x+h)(x+2h)\dots(x+mh)]^{-1} = \dfrac{\Gamma\left(\frac{x}{h}+1\right)}{h^m \Gamma\left(\frac{x}{h}+m+1\right)} = \left[h^m \sum_{k=1}^{m} \left(\frac{x}{h}+m\right)^k S_k^{(m)}\right]^{-1}.$

9.744[7]

The Stirling number of the second kind $\mathsf{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

9.745

Generating functions:

1. $x^n = \sum_{m=0}^{n} \mathsf{S}_n^{(m)} x(x-1)\dots(x-m+1).$

2. $(e^x - 1)^m = m! \sum_{n=m}^{\infty} \mathsf{S}_n^{(m)} \frac{x^n}{n!}$.

As 824 (24.1.4)

3. $[(1-x)(1-2x)\dots(1-mx)]^{-1} = \sum_{n=m}^{\infty} \mathsf{S}_n^{(m)} x^{n-m} \qquad [|x| < m^{-1}]$

As 824 (24.1.4)

9.746

Closed form:

$$\delta_n^{(m)} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n.$$

AS 824 (24.1.4)

9.747

Recurrence relations:

1.[8] $\mathsf{S}_{n+1}^{(m)} = m\mathsf{S}_n^{(m)} + \mathsf{S}_n^{(m-1)}; \delta_n^{(0)} = \delta_{on}; \delta_n^{(1)} = \delta_n^{(n)} = 1. \qquad [n \geq m \geq 1]$

AS 825(24.1.4)

2. $\binom{m}{r} \mathsf{S}_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} \mathsf{S}_{n-k}^{(r)} \mathsf{S}_k^{(m-r)}. \qquad [n \geq m \geq r]$

AS 825 (24.1.4)

3. $S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathsf{S}_{n-m+k}^{(k)}$.

AS 824 (24.1.3)

9.748[8]

Particular values:

Stirling numbers of the first kind $S_n^{(m)}$

m	$S_1^{(m)}$	$S_2^{(m)}$	$S_3^{(m)}$	$S_4^{(m)}$	$S_5^{(m)}$	$S_6^{(m)}$	$S_7^{(m)}$	$S_8^{(m)}$	$S_9^{(m)}$
1	1	−1	2	−6	24	−120	720	−5040	40320
2		1	−3	11	−50	274	−1764	13068	−109584
3			1	−6	35	−225	1624	−13132	118124

Stirling numbers of the second kind $\mathsf{S}_n^{(m)}$

m	$\mathsf{S}_1^{(m)}$	$\mathsf{S}_2^{(m)}$	$\mathsf{S}_3^{(m)}$	$\mathsf{S}_4^{(m)}$	$\mathsf{S}_5^{(m)}$	$\mathsf{S}_6^{(m)}$	$\mathsf{S}_7^{(m)}$	$\mathsf{S}_8^{(m)}$	$\mathsf{S}_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

9.749[8]

Relationship between Stirling numbers of the first kind and derivatives of $(\ln x)^{-m}$:

$$1. \quad \frac{d^n}{dx^n}\left(\frac{1}{\ln^m n}\right) = \frac{1}{\ln^m x}\sum_{k=1}^{n}\frac{(-1)^k (m)_k S_n^{(x)}}{x^n \ln^k x},$$

where

$$(m)_k = \Gamma(m+k)/\Gamma(m) \qquad [m,\, n \text{ positive integers}]$$

10. Vector Field Theory

10.1- 10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_2)$ be arbitrary vectors, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the set of orthogonal unit vectors in terms of which the components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written $\mathbf{a} \cdot \mathbf{b}$, and the vector product, written either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$. Their properties are as follows:

$$1. \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(scalar product)}.$$

2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ (vector product).

3. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (triple scalar product).

4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ (triple vector product).

10.12 Properties of scalar product

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative).

2. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}$.

Note: $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is also written $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; thus (2) may also be written

3. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$.

10.13 Properties of vector product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative).
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$.
3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

10.14 Differentiation of vectors

If $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$, $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, $\phi(t)$ is a scalar and all functions of t are

$\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$ $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$ $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$ $\phi(t)$ t

differentiable, then

1. $\dfrac{d\mathbf{a}}{dt} = \dfrac{da_1}{dt}\,\mathbf{i} + \dfrac{da_2}{dt}\,\mathbf{j} + \dfrac{da_3}{dt}\,\mathbf{k}.$

2. $\dfrac{d}{dt}(\mathbf{a} + \mathbf{b}) = \dfrac{d\mathbf{a}}{dt} + \dfrac{d\mathbf{b}}{dt}.$

3. $\dfrac{d}{dt}(\phi\mathbf{a}) = \dfrac{d\phi}{dt}\,\mathbf{a} + \phi\,\dfrac{d\mathbf{a}}{dt}.$

4. $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \dfrac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \dfrac{d\mathbf{b}}{dt}.$

5. $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \dfrac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \dfrac{d\mathbf{b}}{dt}.$

6. $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \dfrac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \dfrac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \dfrac{d\mathbf{c}}{dt}.$

7. $\dfrac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \dfrac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\dfrac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \dfrac{d\mathbf{c}}{dt}\right).$

10.21 Operators grad, div, and curl

In cartesian coordinates $O\{x_1, x_2, x_3\}$, in which system it is convenient to denote the triad of unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the vector operator $\boldsymbol{\nabla}$, called either ``del'' or ``nabla'', has the form

1. $\boldsymbol{\nabla} \equiv \mathbf{e}_1\,\dfrac{\partial}{\partial x_1} + \mathbf{e}_2\,\dfrac{\partial}{\partial x_2} + \mathbf{e}_3\,\dfrac{\partial}{\partial x_3}.$

If $\Phi(x, y, z)$ is any differentiable scalar function, the gradient of Φ, written grad Φ, is

$$2. \quad \text{grad } \Phi \equiv \boldsymbol{\nabla}\Phi = \frac{\partial\Phi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial\Phi}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial\Phi}{\partial x_3}\,\mathbf{e}_3.$$

The divergence of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written div **f**, is

$$3. \quad \text{div } f \equiv \boldsymbol{\nabla}\cdot\mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

The curl, or rotation, of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written either curl **f** or rot **f**, is

$$4. \quad \text{curl } \mathbf{f} \equiv \text{rot } \mathbf{f} \equiv \boldsymbol{\nabla}\times\mathbf{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{e}_3,$$

or equivalently,

$$\text{curl } \mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

10.31 Properties of the operator ∇

Let $\Phi(x_1, x_2, x_3)$, $\Psi(x_1, x_2, x_3)$ be any two differentiable scalar functions, $\mathbf{f}(x_1, x_2, x_3)$, $\mathbf{g}(x_1, x_2, x_3)$ any two differentiable vector functions, and $\mathbf{a}$ an arbitrary vector. Define the scalar operator ∇^2, called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Then, in terms of the operator $\boldsymbol{\nabla}$, we have the following:

1. $\boldsymbol{\nabla}(\Phi + \Psi) = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\Psi.$
2. $\boldsymbol{\nabla}(\Phi\Psi) = \Phi\boldsymbol{\nabla}\Psi + \Psi\boldsymbol{\nabla}\Phi.$
3. $\boldsymbol{\nabla}(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\boldsymbol{\nabla})\mathbf{g} + (\mathbf{g}\cdot\boldsymbol{\nabla})\mathbf{f} + \mathbf{f}\times(\boldsymbol{\nabla}\times\mathbf{g}) + \mathbf{g}\times(\boldsymbol{\nabla}\times\mathbf{f}).$
4. $\boldsymbol{\nabla}\cdot(\Phi\mathbf{f}) = \Phi(\boldsymbol{\nabla}\cdot\mathbf{f}) + \mathbf{f}\cdot\boldsymbol{\nabla}\Phi.$
5. $\boldsymbol{\nabla}\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\boldsymbol{\nabla}\times\mathbf{f}) - \mathbf{f}\cdot(\boldsymbol{\nabla}\times\mathbf{g}).$

1117

The equivalent results in terms of grad, div, and curl are as follows:

1′. $\operatorname{grad}(\Phi + \Psi) = \operatorname{grad}\Phi + \operatorname{grad}\Psi$.
2′. $\operatorname{grad}(\Phi\Psi) = \Phi \operatorname{grad}\Psi + \Psi \operatorname{grad}\Phi$.
3′. $\operatorname{grad}(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\operatorname{grad})\mathbf{g} + (\mathbf{g}\cdot\operatorname{grad})\mathbf{f} + \mathbf{f}\times\operatorname{curl}\mathbf{g} + \mathbf{g}\times\operatorname{curl}\mathbf{f}$.
4′. $\operatorname{div}(\Phi\mathbf{f}) = \Phi\operatorname{div}\mathbf{f} + \mathbf{f}\cdot\operatorname{grad}\Phi$.
5′. $\operatorname{div}(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot\operatorname{curl}\mathbf{f} - \mathbf{f}\cdot\operatorname{curl}\mathbf{g}$.
6′. $\operatorname{curl}(\Phi\mathbf{f}) = \Phi\operatorname{curl}\mathbf{f} + \operatorname{grad}\Phi\times\mathbf{f}$.
7′. $\operatorname{curl}(\mathbf{f}\times\mathbf{g}) = \mathbf{f}\operatorname{div}\mathbf{g} - \mathbf{g}\operatorname{div}\mathbf{f} + (\mathbf{g}\cdot\operatorname{grad})\mathbf{f} - (\mathbf{f}\cdot\operatorname{grad})\mathbf{g}$.
8′. $\operatorname{curl}(\operatorname{curl}\mathbf{f}) = \operatorname{grad}(\operatorname{div}\mathbf{f}) - \boldsymbol{\nabla}^2\mathbf{f}$.
9′. $\operatorname{curl}(\operatorname{grad}\Phi) \equiv \mathbf{0}$.
10′. $\operatorname{div}(\operatorname{curl}\mathbf{f}) \equiv 0$.
11′. $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2\operatorname{grad}\Phi\cdot\operatorname{grad}\Psi + \Psi\nabla^2\Phi$.

The expression $(\mathbf{a}\cdot\boldsymbol{\nabla})$ or, equivalently $(\mathbf{a}\cdot\operatorname{grad})$, defined by

$$(\mathbf{a}\cdot\boldsymbol{\nabla}) \equiv a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector $\mathbf{a}$.

10.41 Solenoidal fields

A vector field $\mathbf{f}$ is said to be solenoidal if $\operatorname{div}\mathbf{f} \equiv 0$. We have the following representation.

10.411

Representation theorem for vector Helmholtz equation. If u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and $\mathbf{m}$ is a constant unit vector, then the vectors

$$\mathbf{X} = \operatorname{curl}(\mathbf{m}u), \qquad \mathbf{Y} = \frac{1}{\lambda}\operatorname{curl}\mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector $\mathbf{H}$. The general solution of the equation is

$$\mathbf{H} = \operatorname{curl}(\mathbf{m}u) + \frac{1}{\lambda} \operatorname{curl} \operatorname{curl}(\mathbf{m}u).$$

10.51- 10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates $O\{x_1, x_2, x_3\}$ to the general orthogonal curvilinear coordinates $O\{u_1, u_2, u_3\}$:

$$x_1 = x_1(u_1,\, u_2,\, u_3), \qquad x_2 = x_2(u_1,\, u_2,\, u_3), \qquad x_3 = x_3(u_1,\, u_2,\, u_3).$$

Then,

1. $$dx_i = \frac{\partial x_i}{\partial u_1}\, du_1 + \frac{\partial x_i}{\partial u_2}\, du_2 + \frac{\partial x_i}{\partial u_3}\, du_3 \qquad (i = 1,\, 2,\, 3),$$

and the length element dl may be determined from

2. $$dl^2 = g_{11}\, du_1^2 + g_{22}\, du_2^2 + g_{33}\, du_3^2 + 2g_{23}\, du_2\, du_3 + 2g_{31}\, du_3\, du_1 + 2g_{12}\, du_1\, du_2,$$

where

3.[3] $$g_{ij} = \frac{\partial x_1}{\partial u_i}\,\frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i}\,\frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i}\,\frac{\partial x_3}{\partial u_j} = g_{ji}, \qquad g_{ij} = 0, \quad i \neq j.$$

provided the Jacobian of the transformation

$$4. \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

does not vanish (see 4.313).

Define the metrical coefficients

$$5. \quad h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

then the volume element dV in orthogonal curvilinear coordinates is

$$6. \quad dV = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3,$$

and the surface elements of area ds_i on the surfaces $u_i = \text{const}.$, for $i = 1, 2, 3$, are

$$7. \quad ds_1 = h_2 h_3 \, du_2 \, du_3, \quad ds_2 = h_1 h_3 \, du_1 \, du_3, \quad ds_3 = h_1 h_2 \, du_1 \, du_2.$$

Denote by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ the triad of orthogonal unit vectors that are tangent to the u_1, u_2, and u_3 coordinate lines through any given point P, and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components f_{u_1}, f_{u_2}, and f_{u_3} of $\mathbf{f}$ along the coordinate line,

$$8. \quad \mathbf{f} = f_{u_1}\mathbf{e}_1 + f_{u_2}\mathbf{e}_2 + f_{u_3}\mathbf{e}_3.$$

MF I 115

10.611

$\text{grad}\,\Phi$, $\text{div}\,\mathbf{f}$, $\text{curl}\,\mathbf{f}$, and ∇^2 *in general orthogonal curvilinear coordinates.*

$$1. \quad \text{grad}\,\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial \Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial \Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial \Phi}{\partial u_3}.$$

2.[3] $$\operatorname{div}\mathbf{f} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3f_{u_1}) + \frac{\partial}{\partial u_2}(h_3h_1f_{u_2}) + \frac{\partial}{\partial u_3}(h_1h_2f_{u_3})\right) .$$

3. $$\operatorname{curl}\mathbf{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1f_{u_1} & h_2f_{u_2} & h_3f_{u_3} \end{vmatrix} .$$

4. $$\nabla^2 \equiv \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right) .$$

MF I 21-31

10.612

Cylindrical polar coordinates. In terms of the coordinates $O\{r, \phi, z\}$, that is, $u_1 = r, u_2 = \phi, u_3 = z$, where $x_1 = r\cos\phi, x_2 = r\sin\phi, x_3 = z$ for $-\pi < \phi \le \pi$, it follows that

1. $$h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

and

2. $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z,$$

3. $$\operatorname{div}\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_z}{\partial z},$$

4. $$\operatorname{curl}\ \mathbf{f} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ f_r & rf_\phi & f_z \end{vmatrix},$$

5. $\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$

MF I 116

10.613

Spherical polar coordinates. In terms of the coordinates $O\{r, \theta, \phi\}$, that is, $u_1 = r, u_2 = \theta, u_3 = \phi$, where $x_1 = r$ sin θ cos $\phi, x_2 = r$ sin θ sin $\phi, x_3 = r$ cos θ, for $0 \leq \theta \leq \pi, -\pi < \phi \leq \pi$, we have

1. $h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta,$

and

2. $\operatorname{grad} \Phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r \sin \theta}\frac{\partial \Phi}{\partial \phi}\mathbf{e}_\phi,$

3. $\operatorname{div} \mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(f_\theta \sin \theta) + \frac{1}{r \sin \theta}\frac{\partial f_\phi}{\partial \phi},$

4. $\operatorname{curl} \mathbf{f} = \frac{1}{r^2 \sin \theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_r & r f_\theta & r \sin \theta f_\phi \end{vmatrix},$

5. $\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2}{\partial \phi^2}.$

MF I 116

10.614

Elliptic cylinder coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = u_1 u_2, \qquad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \qquad x_3 = u_3.$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \qquad h_2 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = 1.$$

MF I 657

10.615

Parabolic cylinder coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \frac{1}{2}(u_1^2 - u_2^2), \qquad x_2 = u_1 u_2, \qquad x_3 = u_3.$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = 1.$$

MF I 658

10.616

Conical coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \frac{u_1}{a}\sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \qquad x_2 = \frac{u_1}{b}\sqrt{(b^2 + u_2^2)(b^2 - u_3^2)},$$
$$x_3 = \frac{u_1 u_2 u_3}{ab} \qquad \text{with} \quad a^2 + b^2 = 1.$$

$$2. \quad h_1 = 1, \qquad h_2 = u_1\sqrt{\frac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \qquad h_3 = u_1\sqrt{\frac{u_2^2 + u_3^2}{(a^2 + u_3^2)(b^2 - u_3^2)}}.$$

10.617

Rotational parabolic coordinates $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_1 u_2 u_3, \qquad x_2 = u_1 u_2 \sqrt{1-u_3^2}, \qquad x_3 = \frac{1}{2}(u_1^2 - u_2^2).$

2. $h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = \frac{u_1 u_2}{\sqrt{1-u_3^2}}.$

MF I 660

10.618

Rotational prolate spheroidal coordinates $O\{u_1, u_2, u_3\}$.

1. $x_1 = \sqrt{(u_1^2 - a^2)(1-u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 - a^2)(1-u_2^2)(1-u_3^2)}, \qquad x_3 = u_1 u_2.$

2. $h_1 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \qquad h_2 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{1-u_2^2}}, \qquad h_3 = \sqrt{\frac{(u_1^2 - a^2)(1-u_2^2)}{1-u_3^2}}.$

MF I 661

10.619

Rotational oblate spheroidal coordinates $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_3 \sqrt{(u_1^2 + a^2)(1-u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 + a^2)(1-u_2^2)(1-u_3^2)}, \qquad x_3 = u_1 u_2.$

2. $h_1 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \qquad h_2 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{1-u_2^2}}, \qquad h_3 = \sqrt{\frac{(u_1^2 + a^2)(1-u_2^2)}{1-u_3^2}}.$

MF I 662

10.620

Ellipsoidal coordinates $O\{u_1, u_2, u_3\}$.

$$1.\quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \qquad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}},$$
$$x_3 = u_1 u_2 u_3 / ab.$$

1121

$$2.\quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \qquad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}.$$

MF I 663

10.621

Paraboloidal coordinates $O\{u_1, u_2, u_3\}$.

$$1.\quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \qquad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}},$$
$$x_3 = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2).$$

$$2.\quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = u_2\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \qquad h_3 = u_3\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}.$$

MF I 664

10.622

Bispherical coordinates $O\{u_1, u_2, u_3\}$.

$$1.\quad x_1 = a u_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \qquad x_2 = a\frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \qquad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2}.$$

2. $h_1 = \dfrac{a}{(u_1 - u_2)\sqrt{u_1^2 - 1}}, \qquad h_2 = \dfrac{a}{(u_1 - u_2)\sqrt{1 - u_2^2}}, \qquad h_3 = \left(\dfrac{a}{u_1 - u_2}\right)\sqrt{\dfrac{1 - u_2^2}{1 - u_3^2}}.$

MF I 665

10.71- 10.72 Vector integral theorems

10.711

Gauss's divergence theorem. Let V be a volume bounded by a simple closed surface S and let $\mathbf{f}$ be a continuously differentiable vector field defined in V and on S. Then, if $d\mathbf{S}$ is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f} \, dV.$$

KE 39

10.712

Green's theorems. Let Φ and Ψ be scalar fields which, together with $\nabla^2\Phi$ and $\nabla^2\Psi$, are defined both in a volume V and on its surface S, which we assume to be simple and closed. Then, if $\partial/\partial n$ denotes differentiation along the outward drawn normal to S, we have

10.713

Green's first theorem.

$$\int_S \Phi \frac{\partial \Psi}{\partial n} \, dS = \int_V (\Phi \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi) \, dV.$$

KE 212

10.714

Green's second theorem.

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \int_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV.$$

10.715

Special cases.

1. $$\int_S (\Phi \text{ grad } \Phi) \cdot d\mathbf{S} = \int_V (\Phi \nabla^2 \Phi + (\text{grad } \Phi)^2) dV.$$

2. $$\int_S \frac{\partial \Phi}{\partial n} \, dS = \int_V \nabla^2 \Phi \, dV.$$

MV 81

10.716

Green's reciprocal theorem. If Φ and Ψ are harmonic, so that $\nabla^2 \Phi = \nabla^2 \Psi = 0$, then

3. $$\int_S \Phi \frac{\partial \Psi}{\partial n} \, dS = \int_S \Psi \frac{\partial \Phi}{\partial n} \, dS.$$

MM 105

10.717

Green's representation theorem. If Φ and $\nabla^2 \Phi$ are defined within a volume V bounded by a simple closed surface S, and P is an interior point of V, then in three dimensions

4. $$\Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \Phi \, dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} \, dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS.$$

KE 219

If Φ is harmonic within V, so that $\nabla^2 \Phi = 0$, then the previous result becomes

5. $$\Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} \, dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS.$$

In the case of two dimensions, result (4) takes the form

$$6. \quad \Phi(p) = \frac{1}{2\pi} \int_S \nabla^2 \Phi(q) \ln |p-q| \, dS + \\ + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p-q| \, dq - \frac{1}{2\pi} \int \ln |p-q| \frac{\partial}{\partial n_q} \Phi(q) \, dq,$$

MM 116

where C is the boundary of the planar region S, and result (5) the form

$$7. \quad \Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p-q| \, dq - \frac{1}{2\pi} \int_C \ln |p-q| \frac{\partial}{\partial n_q} \Phi(q) \, dq.$$

VL 280

10.718

Green's representation theorem in R^n. If Φ is twice differentiable within a region Ω in R^n bounded by the surface Σ with outward drawn unit normal $\mathbf{n}$, then for $p \notin \Sigma$ and $n > 3$

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_\Omega \frac{\nabla^2 \Phi(q)}{|p-q|^{n-2}} \, d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_\Sigma \left(\frac{1}{|p-q|^{n-2}} \frac{\partial \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p-q|^{n-2}} \right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

VL 279

is the area of the unit sphere in R^n.

1123
10.719

Green's theorem of the arithmetic mean. If Φ is harmonic in a sphere, then the value of Φ at the center of the sphere is the arithmetic mean of its value on the surface. KE 223

10.720

Poisson's integral in three dimensions. If Φ is harmonic in the interior of a spherical volume V of radius R and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates (r, θ, ϕ), it satisfies the boundary condition $\Phi(R, \theta, \phi) = f(\theta, \phi)$, then

$$\Phi(r,\, \theta,\, \phi) = \frac{R(R^2 - r^2)}{4\pi} \int_0^{\pi} \int_{-\pi}^{\pi} \frac{f(\theta',\, \phi')\, \sin\, \theta'\, d\theta'\, d\phi'}{(r^2 + R^2 - 2rR\, \cos\, \gamma)^{3/2}},$$

where

$$\cos\, \gamma = \cos\, \theta\, \cos\, \theta' + \sin\, \theta\, \sin\, \theta'\, \cos\, (\phi - \phi').$$

KE 241

10.721

Poisson's integral in two dimensions. If Φ is harmonic in the interior of a circular disk S of radius R and is continuous on the boundary of the disk on which, in terms of the polar coordinates (r, θ), it satisfies the boundary condition $\Phi(R, \theta) = f(\theta)$, then

$$\Phi(r,\, \theta) = \frac{(R^2 - r^2)}{2\pi} \int_{-\pi}^{\pi} \frac{f(\phi)\, d\phi}{r^2 + R^2 - 2rR\, \cos\, (\theta - \phi)}.$$

10.722

Stokes's theorem. Let a simple closed curve C be spanned by a surface S. Define the positive normal $\mathbf{n}$ to S, and the positive sense of description of the curve C with line element $d\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $d\mathbf{S} = \mathbf{n} dS$,

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \int_S \text{curl}\ \mathbf{f} \cdot d\mathbf{S},$$

MM 143

where the line integral around C is taken in the positive sense.

10.723

Planar case of Stokes's theorem. If a region R in the (x, y)-plane is bounded by a simple closed curve C, and $f_1(x, y), f_2(x, y)$ are any two functions having continuous first derivatives in R and on C, then

$$\oint_C (f_1\, dx + f_2\, dy) = \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy,$$

MM 143

where the line integral is taken in the anticlockwise sense.

10.81 Integral rate of change theorems

10.811

Rate of change of volume integral bounded by a moving closed surface. Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a

simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt} \int_{V(t)} f\, dV = \int_{V(t)} \frac{\partial f}{\partial t}\, dV + \int_{S(t)} f\mathbf{v} \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad}\, .$$

By virtue of Gauss's theorem this also takes the form

$$\frac{D}{Dt} \int_{V(t)} f\, dV = \int_{V(t)} \left(\frac{Df}{Dt} + f\ \text{div}\ \mathbf{v} \right) dV.$$

MV 88

10.812

Rate of change of flux through a surface. Let $\mathbf{q}$ be a vector function that may also depend on the time t, and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \int_S \mathbf{q} \cdot \mathbf{n} \, dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \ \mathrm{div} \ \mathbf{q} + \mathrm{curl} \, (\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n} \, dS.$$

MV 90

10.813

Rate of change of the circulation around a given moving curve. Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \int_C \mathbf{q} \cdot d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathrm{curl} \ \mathbf{q}) \times \mathbf{v} \right) \cdot d\mathbf{r}.$$

MV 94

11. Algebraic Inequalities

11.1-11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.111

Lagrange's identity. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2.$$

BB 3

11.112

Cauchy-Schwarz-Buniakowsky inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional. MT 30

11.113

Minkowski's inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers and let $p > 1$; then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional. MT 55

11.114

Hölder's inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers, and let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$; then

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \ge \sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, the sequences $a_1^p, a_2^p, \ldots, a_n^p$ and $b_1^q, b_2^q, \ldots, b_n^q$ are proportional. MT 50

1126

11.115

Chebyshev's inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers such that either

$a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, or $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$; then

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \leq \frac{1}{n}\sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

11.116

Arithmetic-geometric inequality. Let $a_1, a_2, \ldots, a_n$ be any set of positive numbers, with arith- metic mean

$$A_n = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

and geometric mean

$$G_n = (a_1 a_2 \ldots a_n)^{1/n};$$

then $A_n \geq G_n$ or, equivalently,

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq (a_1 a_2 \ldots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers a_i are equal. BB 4

11.117

Carleman's inequality. If $a_1, a_2, \ldots, a_n$ is any set of positive numbers, then the geometric and arithmetic means satisfy the inequality

$$a_1, a_2, \ldots, a_n$$

$$\sum_{r=1}^{n} G_r \leq eA_n$$

or, equivalently,

$$\sum_{r=1}^{n} (a_1 a_2 \ldots a_r)^{1/r} \leq e \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right),$$

where e is the best possible constant in this inequality.

MT 131

11.118

An inequality involving absolute values. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^{n} \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \qquad 0 < p \leq 2.$$

1127

11.21 Algebraic inequalities involving complex numbers

If α, β are any two real numbers, the complex number $z = \alpha + i\beta$ with real part α and imaginary part β has for its modulus $|z|$ the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude) arg z the angle arg $z = \theta$ such that

$$\cos\,\theta = \frac{\alpha}{|z|} \qquad \text{and} \qquad \sin\,\theta = \frac{\beta}{|z|},$$

where $-\pi < \theta \leq \pi$. The complex number $\overline{z} = \alpha - i\beta$ is said to be the **complex conjugate** of $z = \alpha + i\beta$.

$$\text{If } z = re^{i\theta} = r(\cos\,\theta + i\,\sin\,\theta),$$

then

$$z^n = r^n e^{in\theta} = r^n(\cos\, n\theta + i\, \sin\, n\theta),$$

and setting $r = 1$ we have **de Moivre's theorem**

$$(\cos\, \theta + i\, \sin\, \theta)^n = \cos\, n\theta + i\, \sin\, n\theta.$$

It follows directly that, if $z = e^{i\theta}$, then

$$\cos\, \theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\, \alpha = -\frac{i}{2}\left(z - \frac{1}{z}\right),$$

and

$$\cos\, r\theta = \frac{1}{2}\left(z^r + \frac{1}{z^r}\right), \qquad \sin\, r\theta = -\frac{i}{2}\left(z^r - \frac{1}{z^r}\right).$$

If $w = z^{p/q}$ with p, q integral, and $z = re^{i\theta}$, then the q roots of $w_0, w_1, \ldots, w_{q-1}$ of z are

$$w_k = r^{p/q}\left[\cos\, \left(\frac{p\theta + 2k\pi}{q}\right) + i\, \sin\, \left(\frac{p\theta + 2k\pi}{q}\right)\right],$$

with $k = 0, 1, 2, \ldots, q - 1.$

11.211

Simple properties and inequalities involving the modulus and the complex conjugate. If the real part of z is denoted by Re z and the imaginary part by Im z, then

$$\begin{aligned}
z + \overline{z} &= 2\ \mathrm{Re}\ z = 2\alpha,\\
z - \overline{z} &= 2i\ \mathrm{Im}\ z = 2i\beta,\\
z &= \overline{(\overline{z})},\\
\frac{1}{\overline{z}} &= \overline{\left(\frac{1}{z}\right)},\\
\overline{(z^n)} &= (\overline{z})^n,\\
\left|\frac{\overline{z}_1}{\overline{z}_2}\right| &= \frac{|\overline{z}_1|}{|\overline{z}_2|},\\
\overline{(z_1 + z_2 + \cdots + z_n)} &= \overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_n,
\end{aligned}$$

a, b

(i) $|a+b| \le |a| + |b|$ (triangle inequality),

(ii) $|a-b| \ge ||a| - |b||$.

11.31 Inequalities for sets of complex numbers

11.311

Complex Cauchy-Schwarz-Buniakowsky inequality. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers; then

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \le \left(\sum_{k=1}^{n} |a_k|^2\right) \left(\sum_{k=1}^{n} |b_k|^2\right).$$

The equality holds if, and only if, the sequences $\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n$ and $b_1, b_2, \dots, b_n$ are proportional. MT 42

11.312

Complex Minkowski inequality. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers, and let the real number p be such that $p > 1$; then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}.$$

MT 56

11.313

Complex Hölder inequality. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be any two arbitrary sets of complex numbers, and let the real numbers p, q be such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/p} \ge \left|\sum_{k=1}^{n} a_k b_k\right|.$$

The equality holds if, and only if, the sequences

$$|a_1|^p, \quad |a_2|^p, \dots, |a_n|^p \quad \text{and} \quad |b_1|^p, \quad |b_2|^p, \dots, |b_n|^p$$

are proportional and arg $a_k b_k$ is independent of k for $k = 1, 2, \dots, n$. MT 53

12. Integral Inequalities

12.1- 12.5 Properties of Integrals and Integral Inequalities

12.11 Mean value theorems

12.111

First mean value theorem. Let $f(x)$ and $g(x)$ be two bounded functions integrable in $[a, b]$ and let $g(x)$ be of one sign in this interval. Then

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx,$$

CA 105

with

$a \le \xi \le b$.

12.112

Second mean value theorem.

(i) Let $f(x)$ be a bounded, monotonic decreasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx,$$

with $a \le \xi \le b$.

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx,$$

$a \le \xi \le b$

(ii) Let $f(x)$ be a bounded, monotonic increasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(b)\int_{\eta}^{b} g(x)\,dx,$$

with $a \le \eta \le b$.

(iii) Let $f(x)$ be bounded and monotonic in $[a, b]$, and let $g(x)$ be a bounded integrable function which experiences only a finite number of sign changes in $[a, b]$. Then,

$$\int_a^b f(x)g(x)\,dx = f(a+0)\int_a^{\xi} g(x)\,dx + f(b-0)\int_{\xi}^{b} g(x)\,dx,$$

CA 107

with $a \le \xi \le b$.

1130
12.113

First mean value theorem for infinite integrals. Let $f(x)$ be bounded for $x \ge a$, and integrable in the arbitrary interval $[a, b]$, and let $g(x)$ be of one sign in $x \ge a$ and such that $\int_a^{\infty} g(x)dx$ is finite. Then,

$$\int_a^{\infty} f(x)g(x)\,dx = \mu\int_a^{\infty} g(x)\,dx,$$

CA 123

where $m \le \mu \le M$ and m, M are, respectively, the lower and upper bounds of $f(x)$ for $x \ge a$.

12.114[6]

Second mean value theorem for infinite integrals. Let $f(x)$ be bounded and monotonic when $x \ge a$, and $g(x)$ be bounded and

$f(x)$ $x \geq a$ $g(x)$

integrable in the arbitrary interval $[a, b]$ in which it experiences only a finite number of changes of sign. Then, provided $\int_a^\infty g(x)dx$ is finite,

$$\int_a^\infty f(x)g(x)\,dx = f(a+0)\int_a^\xi g(x)\,dx + f(\infty)\int_\xi^\infty g(x)\,dx,$$

CA 123

with $a \leq \xi \leq \infty$.

12.21 Differentiation of definite integral containing a parameter

12.211

Differentiation when limits are finite. Let $\phi(\alpha)$ and $\psi(\alpha)$ be twice differentiable functions in some interval $c \leq \alpha \leq d$, and let $f(x, \alpha)$ be both integrable with respect to x over the interval $\phi(\alpha) \leq x \leq \psi(\alpha)$ and differentiable with respect to α. Then,

$$\frac{d}{d\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\,\alpha)\,dx = \left(\frac{d\psi}{d\alpha}\right) f(\psi(\alpha),\,\alpha) - \left(\frac{d\phi}{d\alpha}\right) f(\phi(\alpha),\,\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial\alpha}\,dx.$$

FI II 680

12.212

Differentiation when a limit is infinite. Let $f(x, \alpha)$ and $\partial f/\partial\alpha$ both be integrable with respect to x over the semi-infinite region $x \geq a, b \leq \alpha < c$. Then, if the integral

$$f(\alpha) = \int_a^\infty f(x,\,\alpha)\,dx$$

exists for all $b \leq \alpha \leq c$, and if

$$\int_a^\infty \frac{\partial f}{\partial\alpha}\,dx$$

is uniformly convergent for α in $[b, c]$, it follows that

$$\frac{d}{d\alpha}\int_a^\infty f(x,\,\alpha)\,dx = \int_a^\infty \frac{\partial f}{\partial\alpha}\,dx.$$

12.31 Integral inequalities

1131

12.311

Cauchy-Schwarz-Buniakowsky inequality for integrals. Let $f(x)$ and $g(x)$ be any two real integrable functions on $[a, b]$. Then,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right),$$

and the equality will hold if, and only if, $f(x) = kg(x)$, with k real.

BB 21

12.312

Hölder's inequality for integrals. Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^q$ are integrable on $[a, b]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\int_a^b f(x)g(x)\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(\int_a^b |g(x)|^q\,dx\right)^{1/q}.$$

The equality holds if, and only if, $\alpha|f(x)|^p = \beta|g(x)|^q$, where α and β are positive constants. BB 21

12.313

Minkowski's inequality for integrals. Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^p$ are integrable on $[a, b]$ for $p > 0$; then

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

The equality holds if, and only if, $f(x) = kg(x)$ for some real $k \ge 0$.

BB 21

12.314

Chebyshev's inequality for integrals. Let $f_1, f_2, \ldots, f_n$ be nonnegative integrable functions on $[a, b]$ which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x)\,dx \int_a^b f_2(x)\,dx \ldots \int_a^b f_n(x)\,dx \le (b-a)^{n-1} \int_a^b f_1(x) f_2(x) \ldots f_n(x)\,dx.$$

MT 39

12.315

Young's inequality for integrals. Let $f(x)$ be a real-valued continuous strictly monotonic increasing function on the interval $[0, a]$, with $f(0) = 0$ and $b \le f(a)$. Then

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

where $f^{-1}(y)$ denotes the function inverse to $f(x)$. The equality holds if, and only if, $b = f(a)$. BB 15

12.316

Steffensen's inequality for integrals. Let $f(x)$ be nonnegative and monotonic decreasing in $[a, b]$ and $g(x)$ be such that $0 \le g(x) \le 1$ in $[a, b]$. Then

$$\int_{b-k}^b f(x)\,dx \le \int_a^b f(x) g(x)\,dx \le \int_a^{a+k} f(x)\,dx,$$

where

$$k = \int_a^b g(x)\,dx.$$

MT 107

12.317

Gram's inequality for integrals. Let $f_1(x), f_2(x), \dots, f_n(x)$ be real square integrable functions on $[a,b]$; then

$$\begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x)f_2(x)\,dx & \cdots & \int_a^b f_1(x)f_n(x)\,dx \\ \int_a^b f_2(x)f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x)f_n(x)\,dx \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \int_a^b f_n(x)f_1(x)\,dx & \int_a^b f_n(x)f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix} \geq 0.$$

MT 47

12.318

Ostrowski's inequality for integrals. Let $f(x)$ be a monotonic function integrable on $[a,b]$, and let $f(a)f(b) \geq 0, |f(a)| \geq |f(b)|$. Then, if g is a real function integrable on $[a,b]$,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq |f(a)| \max_{a \leq \xi \leq b} \left|\int_a^\xi g(x)\,dx\right|.$$

12.41 Convexity and Jensen's inequality

A function $f(x)$ is said to be **convex** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}.$$

A function $f(x)$ is said to be **concave** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$ the function $-f(x)$ is convex in that interval.

If the function $f(x)$ possesses a second derivative in the interval $[a,b]$, then a necessary and sufficient condition for it to be convex on that interval is that $f''(x) > 0$ for all x in $[a,b]$.

A function $f(x)$ is said to be **logarithmically convex** on the interval $[a,b]$ if $f > 0$ and $\log f(x)$ is concave on $[a,b]$.

If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a, b]$, then the functions $f(x) + g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a, b]$. MT 17

12.411

Jensen's inequality. Let $f(x), p(x)$ be two functions defined for $a \leq x \leq b$ such that $\alpha \leq f(x) \leq \beta$

1133

and $p(x) \geq 0$, with $p(x) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then

$$\phi\left(\frac{\int_a^b f(x)p(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{\int_a^b \phi(f)p(x)\,dx}{\int_a^b p(x)\,dx}.$$

HL 151

12.51 Fourier series and related inequalities

The trigonometric **Fourier series** representation of the function $f(x)$ integrable on $[-\pi, \pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients** a_n and b_n of $f(x)$ are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx, b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

(See 0.320-0.328 for convergence of Fourier series on $(-l, l)$.) TF 1

12.511

Riemann-Lebesgue lemma. If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t\to\infty} \int_{-\pi}^{\pi} f(x) \sin tx\,dx \to 0$$

$$\lim_{t\to\infty} \int_{-\pi}^{\pi} f(x)\, \cos\, tx\, dx \to 0.$$

TF 11

12.512

Dirichlet lemma.

$$\int_0^{\pi} \frac{\sin\, \left(n + \frac{1}{2}\right) x}{2\, \sin\, \frac{1}{2} x}\, dx = \frac{\pi}{2},$$

in which $\sin \left(n + \frac{1}{2}\right) x \Big/ 2 \sin \frac{1}{2} x$ is called the **Dirichlet kernel**.

ZY 21

12.513

Parseval's theorem for trigonometric Fourier series. If $f(x)$ is square integrable on $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx.$$

ZY 10

12.514

Integral representation of nth *partial*

sum. If $f(x)$ is integrable on $[-\pi, \pi]$, then the nth partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^{n} (a_r\, \cos\, rx + b_r\, \sin\, rx)$$

has the following integral representation in terms of the Dirichlet kernel,

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^{n} (a_r \cos rx + b_r \sin rx)$$

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2}t} \, dt.$$

ZY 20

12.515

Generalized Fourier series. Let the set of functions $\{\phi_n\}_{n=0}^{\infty}$ form an **orthonormal set** over $[a, b]$, so that

$$\int_a^b \phi_m(x)\phi_n(x)\, dx = \begin{cases} 1 & \text{for} \quad m = n, \\ 0 & \text{for} \quad m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function $f(x)$ on $[a, b]$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x),$$

where the generalized Fourier coefficients of $f(x)$ are given by

$$c_n = \int_a^b f(x)\phi_n(x)\, dx.$$

12.516

Bessel's inequality for generalized Fourier series. For any square integrable function defined on $[a, b]$,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x)\, dx,$$

where the c_n are the generalized Fourier coefficients of $f(x)$.

12.517

Parseval's theorem for generalized Fourier series. If $f(x)$ is a square integrable function defined on $[a, b]$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ is a

complete orthonormal set of continuous functions defined on $[a, b]$, then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x)\, dx,$$

where the c_n are generalized Fourier coefficients of $f(x)$.

13. Matrices and Related Results

13.1- 13.4 Matrices and Quadratic Forms

13.11- 13.12 Special matrices

13.111

Diagonal matrix. A square matrix **A** of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

13.112

Identity matrix and null matrix. The **identity matrix** is a diagonal matrix **I** in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

13.113

Reducible and irreducible matrices. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is said to be **reducible**, if the indices $1,\ 2, \dots,\ n$ can be divided into two disjoint nonempty sets $i_1,\ i_2, \dots,\ i_\mu;\ j_1,\ j_2, \dots,\ j_\nu (\mu + \nu = n)$, such that

$$a_{i_\alpha j_\beta} = 0 \qquad (\alpha = 1,\ 2, \dots,\ \mu; \quad \beta = 1,\ 2, \dots,\ \nu).$$

Otherwise $\mathbf{A}$ will be said to be irreducible.

13.114

Equivalent matrices. An $m \times n$ matrix $\mathbf{A}$ is **equivalent** to an $m \times n$ matrix $\mathbf{B}$ if, and only if, $\mathbf{B} = \mathbf{P}AQ$ for suitable nonsingular $m \times m$ and $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$, respectively.

13.115

Transpose of a matrix. If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix with element a_{ij} in the ith row and the jth column, then the transpose $\mathbf{A}^T$ of $\mathbf{A}$ is the $n \times m$ matrix

$$\mathbf{A}^T = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from $\mathbf{A}$ by interchanging rows and columns.

1136

13.116

Adjoint matrix. If $\mathbf{A}$ is an $n \times n$ matrix, then its **adjoint**, denoted by adj $\mathbf{A}$, is the transpose of the matrix of cofactors A_{ij} of $\mathbf{A}$, so that

$$adj\mathbf{A} = [A_{ij}]^T \qquad \text{(see 14.13)}.$$

14.13

13.117

Inverse matrix. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with a nonsingular determinant $|\mathbf{A}|$, then its **inverse** $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|.}$$

13.118

Trace of a matrix. The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, written tr $\mathbf{A}$, is defined to be the sum of the terms on the leading diagonal, so that

$$\text{tr}\,\mathbf{A} = a_{11} + a_{22} + \ldots + a_{nn}.$$

13.119

Symmetric matrix. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **symmetric** if $a_{ij} = a_{ji}$ for $i,\ j = 1,\ 2, \dots, n$.

13.120

Skew-symmetric matrix. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for $i,\ j = 1,\ 2, \dots, n$.

13.121

Triangular matrices. An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is of **upper triangular type** if $a_{ij} = 0$ for $i > j$ and of **lower triangular type** if $a_{ij} = 0$ for $j > i$.

13.122

Orthogonal matrices. A real $n \times n$ matrix $\mathbf{A}$ is **orthogonal** if, and only if, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

13.123

Hermitian transpose of a matrix. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with complex elements, then its **hermitian transpose** $\mathbf{A}^H$ is defined to be

$$\mathbf{A}^H = [\overline{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

13.124

Hermitian matrix. An $n \times n$ matrix $\mathbf{A}$ is **hermitian** if $\mathbf{A} = \mathbf{A}^H$, or equivalently, if $\mathbf{A} = \overline{\mathbf{A}}^T$, with the bar denoting the complex conjugate operation.

13.125

Unitary matrix. An $n \times n$ matrix $\mathbf{A}$ is **unitary** if $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A} = \mathbf{I}$.

13.126

Eigenvalues and eigenvectors. If $\mathbf{A}$ is an $n \times n$ matrix, each eigenvector $\mathbf{X}$ corresponding to λ satisfies the equation

$$\mathbf{A}X = \lambda\mathbf{X},$$

while the **eigenvalues** λ satisfy the **characteristic equation**

1137

13.127*

Nilpotent matrix. An $n \times n$ matrix $\mathbf{A}$ is **nilpotent** if $\mathbf{A}^k = \mathbf{0}$ for some k.

13.128*

Idempotent matrix. An $n \times n$ matrix $\mathbf{A}$ is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

13.129*

Positive definite. An $n \times n$ matrix $\mathbf{A}$ is **positive definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.130*

Non-negative definite. An $n \times n$ matrix $\mathbf{A}$ is **non-negative definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.131*

Diagonally dominant. An $n \times n$ matrix $\mathbf{A}$ is **diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i.

13.21 Quadratic forms

A **quadratic form** involving the n real variables $x_1,\ x_2, \ldots ,x_n$ that are associated with the real $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar expression

$$Q(x_1,\ x_2, \ldots ,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j.$$

In terms of matrix notation, if $\mathbf{x}$ is the $n \times 1$ column vector with real elements $x_1,\ x_2, \ldots ,x_n$, and $\mathbf{x}^T$ is the transpose of $\mathbf{x}$, then

$$Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x},\ \mathbf{A}\mathbf{x}).$$

If the $n \times n$ matrix $\mathbf{A}$ is hermitian, so that $\overline{\mathbf{A}}^T = \mathbf{A}$, where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix $\mathbf{A}$ and the vector $\mathbf{x}$ which may have complex elements is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x},\ \mathbf{A}\mathbf{x}).$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x},\ \mathbf{A}\mathbf{x}),$$

where $\mathbf{A} = [a_{ij}]$ is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \qquad \text{for } i = 1,\ 2, \ldots , n$$

1138

and

$$a_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) \qquad \text{for } i,\ j = 1,\ 2, \ldots , n \quad \text{and} \quad i \neq j.$$

13.211

Sylvester's law of inertia. When a quadratic form Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2,$$

the number p of positive squares appearing in the reduction is an invariant of the quadratic form Q, and does not depend on the

$$Q = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2,$$

p Q

method of reduction itself. ML 377

13.212

Rank. The **rank** of the quadratic form Q in the above canonical form is the total number r of squared terms (both positive and negative) appearing in its reduced form.

ML 360

13.213

Signature. The **signature** of the quadratic form Q above is the number s of positive squared terms appearing in its reduced form. It is sometimes also defined to be $2s - r$.

ML 378

13.214

Positive definite and semidefinite quadratic form. The quadratic form $Q(\mathbf{x}) = (\mathbf{x},\ \mathbf{A}\mathbf{x})$ is said to be **positive definite** when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. It is said to be **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq 0$.

ML 394

13.215

Basic theorems on quadratic forms.

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.

2. A real quadratic form in n variables is positive definite if, and only if, its canonical form is

$$Q = z_1^2 + z_2^2 + \ldots + z_n^2.$$

3. A real symmetric matrix $\mathbf{A}$ is positive definite if, and only if, there exists a real nonsingular matrix $\mathbf{M}$ such that $\mathbf{A} = \mathbf{M}\,\mathbf{M}^T$.

4. Any real quadratic form in n variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \ldots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form $Q = (\mathbf{x},\ \mathbf{Ax})$ is positive definite if, and only if, every eigenvalue of $\mathbf{A}$ is positive; it is positive semidefinite if, and only if, all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is indefinite if the eigenvalues of $\mathbf{A}$ are of both signs.

6. The necessary conditions for an hermitian matrix $\mathbf{A}$ to be positive definite are

(i) $a_{ii} > 0$ for all i,

(ii) $a_{ii}a_{ij} > |a_{ij}|^2$ for $i \neq j$,

(iii) the element of largest modulus must lie on the leading diagonal,

(iv) $|\mathbf{A}| > 0$.

1139

7. The quadratic form $Q = (\mathbf{x},\ \mathbf{Ax})$ with $\mathbf{A}$ hermitian will be positive definite if all the principal minors in the top left-hand corner of $\mathbf{A}$ are positive, so that

$$a_{11} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots .$$

ML 353-379

13.31 Differentiation of matrices

If the $n \times m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have elements that are differentiable functions of t, so that

$$\mathbf{A}(t) = [a_{ij}(t)], \qquad \mathbf{B}(t) = [b_{ij}(t)],$$

then

1. $\displaystyle \frac{d}{dt}\mathbf{A}(t) = \left[\frac{d}{dt}a_{ij}(t)\right];$

13.41 The matrix exponential

If $\mathbf{A}$ is a square matrix, and z is any complex number, then the matrix exponential $e^{\mathbf{A}z}$ is defined to be

$$e^{\mathbf{A}z} = \mathbf{I} + \mathbf{A}z + \ldots + \frac{\mathbf{A}^n z^n}{n!} + \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}\, \mathbf{A}^r z^r.$$

1140
13.411

Basic properties.

1. $e^0 = \mathbf{I}, \quad e^{\mathbf{I}z} = \mathbf{I}e^z, \quad e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2}, \quad e^{-\mathbf{A}z} = (e^{\mathbf{A}z})^{-1},$
   $e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$ when $\mathbf{A}+\mathbf{B}$ is defined and $\mathbf{AB} = \mathbf{BA}$

2. $\dfrac{d^r}{dz^r}(e^{\mathbf{A}z}) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z}\mathbf{A}^r.$ ML 340

3. If the square matrix $\mathbf{A}$ can be expressed in the form
   $\mathbf{A} = \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{pmatrix},$

with $\mathbf{B}$ and $\mathbf{C}$ square matrices, then

$$e^{\mathbf{A}z} = \begin{pmatrix} e^{\mathbf{B}z} & 0 \\ 0 & e^{\mathbf{C}z} \end{pmatrix}.$$

14. Determinants

14.1- 14.3 Properties of Determinants

14.11 Expansion of second-and third-order determinants

1. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$

2. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$

14.12 Basic properties

Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices, then the following statements are true.

1. If any two rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.

2. If any two rows (or columns) of a determinant are identical, the determinant is zero.

3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).

4. $|k\mathbf{A}| = k^n|\mathbf{A}|$ for any scalar k.

5. $|\mathbf{A}^T| = |\mathbf{A}|$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

6. $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

7. $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$.

8. If the elements a_{ij} of $\mathbf{A}$ are functions of x, then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,\,j=1}^{n} \frac{da_{ij}}{dx}\, A_{ij} \qquad \text{(see 14.13)}.$$

14.13 Minors and cofactors of a determinant

The **minor** M_{ij} of the element a_{ij} in the nth-order determinant $|\mathbf{A}|$ associated with the square $n \times n$ matrix $\mathbf{A}$ is the $(n-1)$th-order determinant derived from $\mathbf{A}$ by deletion of the ith row and jth column. The cofactor A_{ij} of the element a_{ij} is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

ML 20

1142

14.14 Principal minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of $\mathbf{A}$. ML 197

14.15 Laplace expansion of a determinant

The nth-order determinant denoted by $|\mathbf{A}|$, or det $\mathbf{A}$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ may be expanded either by elements of the ith row as

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

or by elements of the jth column as

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{ij} A_{ij},$$

where A_{ij} is the cofactor of element a_{ij}. The cofactors A_{ij} satisfy the following n linear equations:

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \qquad \sum_{i=1}^{n} a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$
$$\text{for } i,\ j,\ k = 1,\ 2, \ldots, n \text{ and, } \delta_{ij} = 1 \text{ for } i = j \text{ and } 0 \text{ for } i \neq j.$$

14.16 Jacobi's theorem

Let M_r be an r-rowed minor of the nth-order determinant $|\mathbf{A}|$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, in which the rows $i_1,\ i_2, \ldots, i_r$ are represented together with the columns $k_1,\ k_2, \ldots, k_r$.

Define the **complementary minor** to M_r to be the $(n-k)$-rowed minor obtained from $|\mathbf{A}|$ by deleting all the rows and columns associated with M_r, and the **signed complementary minor** $M^{(r)}$ to M_r to be

$$M^{(r)} = (-1)^{i_1+i_2+\ldots+i_r+k_1+k_2+\ldots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if Δ is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and M_r and M'_r are corresponding r-rowed minors of $|\mathbf{A}|$ and Δ, it follows that

$$M'_r = |\mathbf{A}|^{r-1} M^{(r)}.$$

ML 25

1143

Corollary. If $|\mathbf{A}| = 0$, then

$$A_{pk} A_{nq} = A_{nk} A_{pq}.$$

14.17 Hadamard's theorem

If $|\mathbf{A}|$ is an $n \times n$ determinant with elements a_{ij} that may be complex, then $|\mathbf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ij}|.$$

14.18 Hadamard's inequality

Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $n \times n$ nonsingular matrix with real elements and determinant $|\mathbf{A}|$. Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}^2 \right).$$

This result is also true when $\mathbf{A}$ is hermitian. ML 418

Deductions.

1. If $M = \max |a_{ij}|$, then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$

ML 419

2. If the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is positive definite, then

$$|\mathbf{A}| \leq a_{11} a_{22} \ldots a_{nn}.$$

BL 126

3. If the real $n \times n$ matrix $\mathbf{A}$ is diagonally dominant, so that $\sum_{j\neq 1}^{n} |a_{ij}| < |a_{ii}|$ for $i = 1,\, 2, \ldots, n$, then $|\mathbf{A}| \neq 0$.

14.21 Cramer's rule

If n linear equations

If n linear equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad . \qquad . \qquad \cdots \quad . \qquad . \\ \quad . \qquad . \qquad \cdots \quad . \qquad . \\ \quad . \qquad . \qquad \cdots \quad . \qquad . \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \end{array}$$

1144

have a nonsingular coefficient matrix $\mathbf{A} = [a_{ij}]$, so that $|\mathbf{A}| \neq 0$, then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \ldots + A_{nj}b_j}{|A|}$$

for $j = 1,\ 2, \ldots\ , n$, where A_{ij} is the cofactor of element a_{ij} in the coefficient matrix $\mathbf{A}$.

ML 134

14.31 Some special determinants

14.311

Vandermonde's determinant (alternant).

Third order.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the nth-order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i),$$

$$\begin{vmatrix} \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}n(n-1)$ pairs of numbers taken from $x_1,\ x_2, \ldots\ , x_n$, with the order of the differences taken in the reverse order of the suffixes that are involved.

ML 17

14.312

Circulants.

Second order.

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

Third order.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3),$$

where ω and ω^2 are the complex cube roots of 1.

1145

In general, the nth-order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1} (x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1}),$$

where ω_j is an nth root of 1.

The eigenvalues λ (see 15.61) of an $n \times n$ circulant matrix are

$$\lambda_j = x_1 + x_2\omega_j + x_3\omega_j^2 + \ldots + x_n\omega_j^{n-1},$$

ω_j n

14.313

Jacobian determinant. If $f_1,\ f_2,\dots,f_n$ are n real-valued functions which are differentiable with respect to $x_1,\ x_2,\dots,x_n$, then the Jacobian $J_f(x)$ of the f_i with respect to the x_j is the determinant

$$J_f(x)=\begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

The notation

$$\frac{\partial(f_1,\ f_2,\dots,f_n)}{\partial(x_1,\ x_2,\dots,x_n)}$$

is also used to denote the Jacobian $J_f(x)$.

14.314

Hessian determinants. The Jacobian of the derivatives $\frac{\partial\phi}{\partial x_1},\ \frac{\partial\phi}{\partial x_2},\dots,\frac{\partial\phi}{\partial x_n}$ of a function

1146

$\phi(x_1,\ x_2,\dots,x_n)$ with respect to $x_1,\ x_2,\dots,x_n$ is called the Hessian H of ϕ, so that

$$H=\begin{vmatrix} \dfrac{\partial^2\phi}{\partial x_1^2} & \dfrac{\partial^2\phi}{\partial x_1\partial x_2} & \dfrac{\partial^2\phi}{\partial x_1\partial x_3} & \cdots & \dfrac{\partial^2\phi}{\partial x_1\partial x_n} \\ \dfrac{\partial^2\phi}{\partial x_2\partial x_1} & \dfrac{\partial^2\phi}{\partial x_2^2} & \dfrac{\partial^2\phi}{\partial x_2\partial x_3} & \cdots & \dfrac{\partial^2\phi}{\partial x_2\partial x_n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial^2\phi}{\partial x_n\partial x_1} & \dfrac{\partial^2\phi}{\partial x_n\partial x_2} & \dfrac{\partial^2\phi}{\partial x_n\partial x_3} & \cdots & \dfrac{\partial^2\phi}{\partial x_n^2} \end{vmatrix}.$$

14.315

Wronskian determinants.

$f_1,\ f_2, \ldots\ , f_n$ $\quad n \quad\quad\quad\quad n \quad\quad\quad\quad\quad\quad\quad\quad x$

$(a,\ b)$. Then the Wronskian $W(x)$ of $f_1,\ f_2, \ldots\ , f_n$ is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix},$$

where $f_i^{(r)} = \frac{d^r f_i}{dx^r}$.

14.316

Properties.

1. $\dfrac{dW}{dx}$ follows from $W(x)$ by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}$, $f_2^{(n)}, \ldots\ , f_n^{(n)}$.

2. If constants $k_1,\ k_2, \ldots\ , k_n$ exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \ldots + k_n f_n = 0$$

for all x in $(a,\ b)$, then $W(x) = 0$ for all x in $(a,\ b)$.

3. The vanishing of the Jacobian throughout $(a,\ b)$ is necessary, but not sufficient, for the linear dependence of $f_1,\ f_2, \ldots\ , f_n$.

14.317*

Gram-Kowalewski theorem on linear dependence. A necessary and sufficient condition for n functions $f_1,\ f_2, \ldots\ , f_n$ square integrable over $a \leq n \leq b$ to be linearly dependent in this interval is the

vanishing of the Gram determinant

$$G(f_1,\ f_2, \ldots\ , f_n) = \begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x) f_2(x)\,dx & \cdots & \int_a^b f_1(x) f_n(x)\,dx \\ \int_a^b f_2(x) f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x) f_n(x)\,dx \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \int_a^b f_n(x) f_1(x)\,dx & \int_a^b f_n(x) f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix}.$$

n $f_1,\ f_2, \dots , f_n$ $a \leq n \leq b$

$$G(f_1,\ f_2, \dots , f_n) \geq 0,$$

and the equality sign holds only when the functions are linearly dependent in $a \leq n \leq b$.

SA 4 (Corollary 1)

14.319*

The rank of the matrix corresponding to the Gram determinant $G(f_1,\ f_2, \dots , f_n)$ gives the maximum number of linearly independent functions $f_1,\ f_2, \dots , f_n$ in $a \leq x \leq b$. If the rank is r, then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

SA 3 (Theorem 4)

15. Norms

15.1- 15.9 Vector Norms

15.11 General properties

The **vector norm** $\|\mathbf{x}\|$ of an $n \times 1$ column vector $\mathbf{x}$ is a nonnegative number having the property that

(a) $\|\mathbf{x}\| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x}\| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;

(b) $\|k\mathbf{x}\| = |k|\ \|\mathbf{x}\|$ for any scalar k;

(c) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

15.21 Principal vector norms

15.211

The norm $\|\mathbf{x}\|_1$.If $\mathbf{x}$ is a vector with complex components $x_1,\ x_2, \dots , x_n$, then

$$\|\mathbf{x}\|_1 = \sum_{r=1}^{n} |x_r|.$$

VA 15

15.212

The norm $\|\boldsymbol{x}\|_2$ *(euclidean or* L_2 *norm).* If $\mathbf{x}$ is a vector with complex components $x_1,\ x_2, \ldots, x_n$, then

$$\|\mathbf{x}\|_2 = \left(\sum_{r=1}^{n} |x_r|^2\right)^{\frac{1}{2}}.$$

VA8

15.213

The norm $\|\boldsymbol{x}\|_\infty$.If $\mathbf{x}$ is a vector with complex components $x_1,\ x_2, \ldots, x_n$, then

$$\|\mathbf{x}\|_\infty = \max_i |x_i|.$$

VA 15

15.31 Matrix norms

15.311

General properties. The **matrix norm** $\|\mathbf{A}\|$ of a square matrix $\mathbf{A}$ is a nonnegative number associated with $\mathbf{A}$ having the property that

(a) $\|\mathbf{A}\| > 0$ when $\mathbf{A} \neq \mathbf{0}$ and $\|\mathbf{A}\| = 0$ if, and only if, $\mathbf{A} = \mathbf{0}$;

(b) $\|k\mathbf{A}\| = |k|\ \|\mathbf{A}\|$ for any scalar k;

(c) $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$;

(d) $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\ \|\mathbf{B}\|$.

VA 9

The matrix norm $\|\mathbf{A}\|$ associated with $\mathbf{A} = [a_{ij}]$, and the vector norm $\|\mathbf{x}\|$ associated with the column vector $\mathbf{x}$ for which the matrix product $\mathbf{Ax}$ is defined, are said to be **compatible** if

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\|\ \|\mathbf{x}\|.$$

15.312

Induced norms. When a vector $\mathbf{z}$ with norm $\|\mathbf{z}\|$ exists such that the maximum is attained in the expression

$$\|\mathbf{A}\| = \max_{\|z\|=1} \|\mathbf{Az}\|,$$

then $\|\mathbf{A}\|$ is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm $\|\mathbf{z}\|$.

NO 428

15.313

Natural norm of unit matrix. If $\mathbf{I}$ is the unit matrix, then for any natural norm

$$\|\mathbf{I}\| = 1.$$

NO 429

15.41 Principal natural norms

The natural matrix norms induced on matrix $\mathbf{A} = [a_{ij}]$ by the 1, 2, and ∞ vector norms are as follows:

15.411

Maximum absolute column sum norm.

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|.$$

NO 429

15.412

Spectral norm. If $\mathbf{A}^H$ denotes the hermitian transpose of the square matrix $\mathbf{A} = [a_{ij}]$, so that $\mathbf{A}^H = [\overline{a}_{ji}]$ with a bar denoting the complex conjugate operation, then

$$\|\mathbf{A}\|_2 = \{\max \text{ eigenvalue of } \mathbf{A}^H\mathbf{A}\}^{\frac{1}{2}},$$

or, equivalently,

$$\|\mathbf{A}\|_2 = \{\max \ \text{eigenvalue of } \mathbf{A}^H\mathbf{A}\}^{\frac{1}{2}},$$

$$\|\mathbf{A}\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}.$$

NO 429

15.413

Maximum absolute row sum norm.

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

NO 429

15.51 Spectral radius of a square matrix

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with elements that may be complex, and with eigenvalues $\lambda_1,\ \lambda_2, \ldots, \lambda_n$. Then the **spectral radius** $\rho(\mathbf{A})$ of $\mathbf{A}$ is the number

$$\rho(\mathbf{A}) = \max_{1 \leq i \leq n} |\lambda_i|.$$

VA 9

1150
15.511

Inequalities concerning matrix norms and the spectral radius.

1. $\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_\infty.$

NO 431

2. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, and the $n \times n$ matrix $\mathbf{U}$ is unitary, so that $\mathbf{U}^H = \mathbf{U}^{-1}$, with ^{H} denoting the hermitian transpose of $\mathbf{A}$ (see 13.123), then

$$\|\mathbf{AU}\| = \|\mathbf{UA}\| = \|\mathbf{A}\|.$$

3. If $\mathbf{A}$ is any nonsingular $n \times n$ matrix with elements that may be complex with eigenvalues λ_1, λ_2, λ_n, then

$$\frac{1}{\|\mathbf{A}^{-1}\|} \leq |\lambda| \leq \|\mathbf{A}\|.$$

VA 16

4. For any square matrix $\mathbf{A}$ with spectral radius $\rho(\mathbf{A})$ and any natural norm $\|\mathbf{A}\|$,

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|.$$

NO 430

5. If the square matrix $\mathbf{A}$ is hermitian, then

$$\rho(\mathbf{A}) = \|\mathbf{A}\|.$$

6. If the square matrix $\mathbf{A}$ is hermitian and $P_m(x)$ is any polynomial of degree m with real coefficients, then

$$\|P_m(\mathbf{A})\| = \rho(P_m(\mathbf{A})).$$

7. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, then the sequence of matrices $\mathbf{A},\ \mathbf{A}^2,\ \mathbf{A}^3, \ldots$ converges to the null matrix as $n \to \infty$ if, and only if, $\rho(\mathbf{A}) < 1$.

NO 303

15.512

Deductions from Gerschgorin's theorem (*see* 15.814).

1. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex; then

$$\rho(\mathbf{A}) \leq \min\left(\max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|,\ \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}|\right).$$

VA 17

2. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex, and $x_1,\ x_2, \ldots, x_n$ be any set of n positive numbers; then

$$\rho(\mathbf{A}) \leq \min\left(\max_{1\leq i\leq n}\left(\frac{\sum_{j=1}^{n}|a_{ij}|x_j}{x_i}\right), \max_{1\leq j\leq n}\left(x_j\sum_{i=1}^{n}\frac{|a_{ij}|}{x_i}\right)\right).$$

VA 18

15.61 Inequalities involving eigenvalues of matrices

The **eigenvalues** (**characteristic values** or **latent roots**) λ of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

When expanded, the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the **characteristic polynomial** and it has the form

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of $\mathbf{A}$. In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \qquad \text{(sum of all principal minors of } |\mathbf{A}| \text{ of order } r).$$

It then follows that

$$\begin{aligned} b_{n-1} &= (-1)^n(a_{11} + a_{22} + \cdots + a_{nn}), \\ b_{n-2} &= (-1)^n\sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}), \\ b_0 &= |\mathbf{A}|. \end{aligned}$$

$\mathbf{A}$ $\mathbf{A}$ $\mathbf{A}$

$b_{n-1} = (-1)^n \text{ tr } \mathbf{A}$

15.611

Cayley-Hamilton theorem. Every square matrix $\mathbf{A}$ satisfies its characteristic equation, so that

$$(-1)^n \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + c_{n-2}\mathbf{A}^{n-2} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}.$$

ML 206

15.612

Corollaries.

1. If $\mathbf{A}$ is nonsingular then its adjoint, denoted by $\operatorname{adj} \mathbf{A}$, is

$$\operatorname{adj} \mathbf{A} = -[(-1)^n \mathbf{A}^{n-1} + c_{n-1}\mathbf{A}^{n-2} + c_{n-2}\mathbf{A}^{n-3} + \cdots + c_2\mathbf{A} + c_1\mathbf{I}].$$

2. If $\mathbf{A}$ is nonsingular, then the characteristic polynomial of $\mathbf{A}^{-1}$ is

$$(-1)^n \left(\lambda^n + \frac{c_1}{|\mathbf{A}|}\lambda^{n-1} + \frac{c_2}{|\mathbf{A}|}\lambda^{n-2} + \cdots + \frac{(-1)^n}{|\mathbf{A}|} \right).$$

15.71 Inequalities for the characteristic polynomial

The first group of inequalities that follow, which relate to the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ whose elements may be complex, refer directly to the coefficients of the polynomial when written in the form

$$P(\lambda) \equiv |\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n,$$

and only implicitly to the coefficients a_{ij} of $\mathbf{A}$ that give rise to the b_i.

1152

15.711

Named and unnamed inequalities. The first group of inequalities relating to the eigenvalues λ satisfying $P(\lambda) = 0$ are unnamed and are as follows:

1. all the eigenvalues λ lie within or on the circle $|z| \leq r$, where r is the positive root of

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0;$$

MG 122

2. all the eigenvalues λ lie within the circle

$$|z| < 1 + \max_i |b_i|;$$

MG 123

3. when $b_n \neq 0$ the eigenvalue λ of smallest modulus lies in the annulus $R \leq |z| \leq \frac{R}{2^{1/n}-1}$, where R is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0;$$

MG 126

4. all the eigenvalues λ lie on or outside the circle

$$|z| = \min_k \left[\frac{|b_n|}{(|b_n| + |b_k|)} \right];$$

MG 126

5. if the eigenvalues λ are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > 1 \geq |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|,$$

then

$$|z_1 z_2 \ldots z_p| \leq N, \qquad |z_p| \leq N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2;$$

6. all the eigenvalues λ lie in or on the circle

$$|z| \leq \sum_{j=1}^{n} |b_j|^{\frac{1}{j}};$$

MG 126

7. all the eigenvalues λ lie on the disk

$$\left|z + \frac{b_1}{2}\right| \leq \left|\frac{b_1}{2}\right| + |b_2|^{\frac{1}{2}} + |b_3|^{\frac{1}{3}} + \cdots + |b_n|^{\frac{1}{n}};$$

MG 145

1153

8. all the eigenvalues λ lie in the annulus $m \leq |z| \leq M,$ where

$$m^2 = \max\left\{0, \min_{1 \leq j \leq n-1} [1 - |b_j|, |b_n|^2]\right\}$$

and

$$M^2 = \max\left\{1 + |b_j|, |b_n|^2 + 2\sum_{j=1}^{n-1} |b_j|^2\right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial $P(\lambda)$.

MG 145

15.712

Parodi's theorem. The eigenvalues λ satisfying $P(\lambda) = 0$ lie in the union of the disks

$$|z| \leq 1, \qquad |z + b_1| \leq \sum_{j=1}^{n} |b_j|.$$

MG 143

15.713

Corollary of Brauer's theorem. If

$$|b_1| > 1 + \sum_{j=2}^{n} |b_j|,$$

then one and only one eigenvalue satisfying $P(\lambda) = 0$ lies on the disk

$$|z + b_1| \le \sum_{j=2}^{n} |b_j|.$$

MG 141

15.714

Ballieu's theorem. For any set $\mu = (\mu_1,\ \mu_2, \dots, \mu_n)$ of positive numbers, let $\mu_0 = 0$ and

$$M_\mu = \max_{0 \le k \le n-1} \left[\frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}} \right].$$

Then all the eigenvalues satisfying $P(\lambda) = 0$ lie on the disk $|z| \le M_\mu$.

MG 144

15.715

Routh-Hurwitz theorem.

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \lambda + b_n = 0$$

determining the n eigenvalues λ of the real $n \times n$ matrix $\mathbf{A}$. Then the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \dots, \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \dots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & \cdot & \dots & b_k \end{vmatrix}.$$

GM 230

15.81- 15.82 Named theorems on eigenvalues

In the following theorems involving eigenvalue inequalities the elements a_{ij} of matrix $\mathbf{A}$ enter directly, and not in the form of the coefficients of the characteristic polynomial.

15.811

Schur's inequalities. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with elements that may be complex, and eigenvalues $\lambda_1,\ \lambda_2, \dots ,\lambda_n$, then

$$1.\quad \sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2,$$

$$2.\quad \sum_{i=1}^{n} |\operatorname{Re}\ \lambda_i|^2 \leq \sum_{i,j=1}^{n} \left| \frac{a_{ij} + \overline{a}_{ji}}{2} \right|^2,$$

$$3.\quad \sum_{i=1}^{n} |\operatorname{Im}\ \lambda_i|^2 \leq \sum_{i,j=1}^{n} \left| \frac{a_{ij} - \overline{a}_{ji}}{2} \right|^2.$$

ML 309

15.812

Sturmian separation theorem. Let $\mathbf{A}_r = [a_{ij}]$ with $i, j = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, N$ be a sequence of N symmetric matrices of increasing order. Then if $\lambda_k(\mathbf{A}_r)$ for $k = 1, 2, \ldots, r$ denotes the $k\theta$ eigenvalue of A_r, where the ordering is such that

$$\lambda_1(A_r) \geq \lambda_2(A_r) \geq \cdots \geq \lambda_r(A_r),$$

it follows that

$$\lambda_{k+1}(A_{i+1}) \leq \lambda_k(A_i) \leq \lambda_k(A_{i+1}).$$

BL 115

15.813

Poincare's separation theorem. Let $\{\mathbf{y}^k\}$, with $k = 1, 2, \ldots, K$, be a set of orthonormal vectors so that the inner product $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Set

$$\mathbf{x} = \sum_{k=1}^{K} u_k \mathbf{y}^k,$$

so that for any square matrix $\mathbf{A}$ for which the product $\mathbf{Ax}$ is defined, the quadratic form

$$(\mathbf{x}, \mathbf{Ax}) = \sum_{k,l=1}^{K} u_k u_l (\mathbf{y}^k, \mathbf{Ay}^l).$$

1155

Then if

$$\mathbf{B}_K = (\mathbf{y}^k, \mathbf{Ay}^l) \qquad \text{for} \quad k, l = 1, 2, \ldots, K,$$

it follows that

$$\begin{aligned} \lambda_i(\mathbf{B}_K) \leq \lambda_i(\mathbf{A}) \qquad &\text{for} \quad i = 1, 2, \ldots, K, \\ \lambda_{K-j}(\mathbf{B}_K) \geq \lambda_{N-j}(\mathbf{A}) \qquad &\text{for} \quad j = 0, 1, 2, \ldots, K-1, \end{aligned}$$

Gerschgorin's theorem. Let $\mathbf{A} = [a_{ij}]$ be any arbitrary $n \times n$ matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{j=1, i \neq j}^{n} |a_{ij}| \qquad \text{for} \quad i = 1,\ 2, \ldots, n.$$

Then all of the eigenvalues λ_i of $\mathbf{A}$ lie in the union of the n disks Γ_i, where

$$\Gamma_i\colon \quad |z - a_{ii}| \leq \Lambda_i \qquad \text{for} \quad i = 1,\ 2, \ldots, n.$$

VA 16

15.815

Brauer's theorem. If in Gerschgorin's theorem for a given m

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all $j \neq m$, then one and only one eigenvalue of $\mathbf{A}$ lies in the disk Γ_m.

MG 141

15.816

Perron's theorem. If $\boldsymbol{\mu} = (\mu_1,\ \mu_2, \ldots, \mu_n)$ is an arbitrary set of positive numbers, then all the eigenvalues λ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ lie on the disk $|z| \leq \mathbf{M}_\mu$, where

$$\mathbf{M}_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i} |a_{ij}|.$$

MG 141

15.817

Frobenius theorem. If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients, so that $a_{ij} > 0$ for all $i,\ j = 1,\ 2, \ldots, n$, then $\mathbf{A}$ has a positive eigenvalue λ_0 and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0.$$

MG 142

15.818

Perron-Frobenius theorem. If all elements a_{ij} of an irreducible matrix $\mathbf{A}$ are nonnegative, then $R = \min\, M_\lambda$ is a simple eigenvalue of $\mathbf{A}$ and all the eigenvalues of $\mathbf{A}$ lie on the disk

$$|z| \le R,$$

where, if $\boldsymbol{\lambda} = (\lambda_1,\ \lambda_2, \ldots\ , \lambda_n)$ is a set of nonnegative numbers, not all zero,

$$M_\lambda = \inf\left\{\mu\colon\ \mu\lambda_i > \sum_{j=1}^{n} |a_{ij}|\lambda_j,\ 1 \le i \le n\right\}$$

and $R = \min M_\lambda$.

1156

Furthermore, if $\mathbf{A}$ has exactly p eigenvalues $(p \le n)$ on the circle $|z| = R$, then the set of all its eigenvalues is invariant under rotations $2\pi/p$ about the origin.

GM 69

15.819

Wielandt's theorem. If the $n \times n$ matrix $\mathbf{A}$ satisfies the conditions of the Perron-Frobenius theorem and if in the $n \times n$ matrix $\mathbf{C} = [c_{ij}]$

$$|c_{ij}| \le a_{ij}, \qquad i,\ j = 1,\ 2, \ldots\ , n,$$

then any eigenvalue λ_0 of $\mathbf{C}$ satisfies the inequality $|\lambda_0| \le R$. The equality sign holds only when there exists an $n \times n$ matrix $\mathbf{D} = [\pm\delta_{ij}]$ such that $\delta_{ii} = 1$ for all i, $\delta_{ij} = 0$ for all $i{\ne}j$, and

$$\mathbf{C} = (\lambda_0/R)\,\mathbf{D}AD^{-1}.$$

GM 69

15.820

Ostrowski's theorem. If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients and λ_0 is the positive eigenvalue in Frobenius' theorem, then the $n-1$ eigenvalues $\lambda_j \neq \lambda_0$ satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \qquad m = \min a_{ij} \qquad \text{for} \quad i,\, j = 1,\, 2, \ldots, n.$$

MG 145

15.821

First theorem due to Lyapunov. In order that all the eigenvalues of the real $n \times n$ matrix $\mathbf{A}$ have negative real parts, it is necessary and sufficient that if V is an $n \times n$ matrix, the equation

$$\mathbf{A}^T\mathbf{V} + \mathbf{V}\mathbf{A} = -\mathbf{I}$$

has as a solution the matrix of coefficients $\mathbf{V}$ of some positive-definite quadratic form $(\mathbf{x},\, \mathbf{V}\mathbf{x})$ (see 13.21).

GM 224

15.822

Second theorem due to Lyapunov.

If all the eigenvalues of the real matrix $\mathbf{A}$ have negative real parts, then to an arbitrary negative- definite quadratic form $(\mathbf{x},\, \mathbf{W}\mathbf{x})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive-definite quadratic form $(\mathbf{x},\, \mathbf{V}\mathbf{x})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x},$$

then $(\mathbf{x},\, \mathbf{V}\mathbf{x})$ and $(\mathbf{x},\, \mathbf{W}\mathbf{x})$ satisfy

$$\frac{d}{dt}(\mathbf{x},\, \mathbf{V}\mathbf{x}) = (\mathbf{x},\, \mathbf{W}\mathbf{x}).$$

Conversely, if for some negative-definite form $(\mathbf{x},\ \mathbf{Wx})$ there exists a positive-definite form $(\mathbf{x},\ \mathbf{Vx})$ connected to $(\mathbf{x},\ \mathbf{Wx})$ by the preceding two equations, then all the eigenvalues of $\mathbf{A}$ have negative real parts (see 13.21, 13.31).

GM 222

1157
15.823

Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov.

1. The off-diagonal hermitian matrix $\mathbf{A}$ of rank n whose elements are given by

$$a_{jk} = (1 - \delta_{jk}) \left\{ 1 + i \text{ ctg } \left[\frac{(j-k)\pi}{n} \right] \right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \qquad \text{for} \quad s = 1,\ 2, \dots ,n,$$

and the corresponding eigenvectors $v^{(s)}$ have the components

$$v_j^{(s)} = \exp\left(-\frac{2\pi i s j}{n} \right) \qquad \text{for} \quad j = 1,\ 2, \dots ,n.$$

2. The two off-diagonal hermitian matrices $\mathbf{B}$ and $\mathbf{C}$ whose elements are defined by the formulas

$$b_{jk} = (1 - \delta_{jk}) \sin^{-2} \left[\frac{(j-k)\pi}{n} \right],$$
$$c_{jk} = (1 - \delta_{jk}) \sin^{-4} \left[\frac{(j-k)\pi}{n} \right],$$

are related to the matrix $\mathbf{A}$ in (1) by the equations

$$\mathbf{B} = \frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}),$$
$$\mathbf{C} = -\frac{1}{6}(\mathbf{B}^2 - 2(2 + \sigma_n^{(1)})\mathbf{B} - \sigma_n^{(2)}\mathbf{I}),$$

$$\mathbf{B} = \frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}),$$
$$\mathbf{C} = -\frac{1}{6}(\mathbf{B}^2 - 2(2 + \sigma_n^{(1)})\mathbf{B} - \sigma_n^{(2)}\mathbf{I}),$$

where $\mathbf{I}$ is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3}(n^2 - 1), \qquad \sigma_n^{(2)} = \frac{1}{45}(n^2 - 1)(n^2 + 11).$$

The eigenvalues of $\mathbf{B}$ and $\mathbf{C}$ corresponding to the eigenvector $v_j^{(s)}$ in (1) have the form

$$\lambda_s^{(b)} = \sigma_n^{(1)} - 2s(n-s) \qquad \text{for} \quad s = 1,\, 2, \dots, n,$$
$$\lambda_s^{(c)} = \sigma_n^{(2)} - 2s(n-s)\frac{s(n-s)+2}{3} \qquad \text{for} \quad s = 1,\, 2, \dots, n.$$

1158

3. Together, the above two results imply the following diophantine summation rules:

$$\text{(a)} \quad \sum_{k=1}^{n-1} \operatorname{ctg}\left(\frac{k\pi}{n}\right) \sin\left(\frac{2sk\pi}{n}\right) = n - 2s \qquad \text{for} \quad s = 1,\, 2, \dots, n-1,$$
$$\text{(b)} \quad \sum_{k=1}^{n-1} \sin^{-2}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = b_s \qquad \text{for} \quad s = 1,\, 2, \dots, n-1,$$
$$\text{(c)} \quad \sum_{k=1}^{n-1} \sin^{-4}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = c_s \qquad \text{for} \quad s = 1,\, 2, \dots, n-1,$$
$$\text{(d)} \quad \sum_{k=1}^{n-1} \sin^{-2p}\left(\frac{k\pi}{n}\right) = \sigma_n^{(p)},$$

with $b_s,\ c_s,\ \sigma_n^{(1)}$ and $\sigma_n^{(2)}$ as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)} \frac{2n^4 + 23n^2 + 191}{315},$$
$$\sigma_n^{(4)} = \sigma_n^{(2)} \frac{3n^4 + 10n^2 + 227}{315}.$$

15.91 Variational principles

15.911

Rayleigh quotient.

If $\mathbf{A}$ is an hermitian matrix, the Rayleigh quotient $\rho(\mathbf{x})$ is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x},\ \mathbf{Ax})}{(\mathbf{x},\ \mathbf{x})}.$$

NO 407

15.912

Basic theorems.

1. If the $n \times n$ matrix A is hermitian and has eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

$$\lambda_1 \le \rho \le \lambda_n,$$

where ρ is the Rayleigh quotient for any $\mathbf{x} \neq \mathbf{0}$, and

$$\lambda_1 = \min_{x \neq 0} \frac{(\mathbf{x},\ \mathbf{Ax})}{(\mathbf{x},\ \mathbf{x})} \qquad \text{and} \qquad \lambda_n = \max_{x \neq 0} \frac{(\mathbf{x},\ \mathbf{Ax})}{(\mathbf{x},\ \mathbf{x})}.$$

NO 407

2. If the $n \times n$ matrix $\mathbf{A}$ is hermitian and has eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ corresponding to the eigenvectors $\mathbf{x}_1,\ \mathbf{x}_2, \ldots,\ \mathbf{x}_n$, respectively, and $\mathbf{x} \neq \mathbf{0}$ is such that

$$(\mathbf{x},\ \mathbf{x}_1) = (\mathbf{x},\ \mathbf{x}_2) = \cdots = (\mathbf{x},\ \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_{x} \frac{(\mathbf{x},\ \mathbf{Ax})}{(\mathbf{x},\ \mathbf{x})},$$

and

NO 410

3. If the $n \times n$ matrix $\mathbf{A}$ is hermitian, then the eigenvalue

$$\lambda_r = \max\left(\min \frac{(\mathbf{x},\ \mathbf{A}\mathbf{x})}{(\mathbf{x},\ \mathbf{x})}\right),$$

where first the minimum over $\mathbf{x}$ is taken subject to $(\mathbf{b}_i,\ \mathbf{x}) = 0,\ \ i = 1,\ 2, \ldots, r-1$, with the $\mathbf{b}_i$ regarded as fixed vectors, and then the maximum over all possible $\mathbf{b}_i$. Also, the eigenvalue

$$\lambda_r = \min\left(\max \frac{(\mathbf{x},\ \mathbf{A}\mathbf{x})}{(\mathbf{x},\ \mathbf{x})}\right),$$

where now the maximum over $\mathbf{x}$ is taken first subject to $(\mathbf{b}_i,\ \mathbf{x}) = 0,\ \ i = r+1,\ r+2, \ldots, n$ for fixed $\mathbf{b}_i$, and then the minimum over all possible $\mathbf{b}_i$.

NO 414

4. The $(n-1)$ eigenvalues $\lambda'_1,\ \lambda'_2, \ldots, \lambda'_{n-1}$ obtained from the $(n-1) \times (n-1)$ matrix derived from an hermitian matrix $\mathbf{A}$ from which the last row and column have been omitted separate the n eigenvalues of $\mathbf{A}$, so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \cdots < \lambda'_{n-1} < \lambda_n \qquad (\text{see } 15.812).$$

15.812

16. Ordinary Differential Equations

16.1- 16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-order equations

16.111

Solution of a first-order equation.

Consider the real function $f(t,\ x)$ that is defined and continuous in an open set $D \subset R^2$. Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t,\ x)$$

in the open interval $I \subset R$ is a real function $u(t)$ that is defined and is both continuous and differentiable in I, with the property that

$$\begin{array}{lll} \text{(i)} & (t, u(t)) \in D & \text{for } t \in I, \\ \text{(ii)} & \frac{du}{dt} = f(t, u(t)) & \text{for } t \in I. \end{array}$$

16.112

Cauchy problem.

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t,\ x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where $(t_0,\ u(t_0)) \in D$, the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^{t} f(\tau,\ u(\tau))\, d\tau \qquad \text{(see 16.316).}$$

16.316

16.113

Approximate solution to an equation.

The real function $\phi(t)$ is said to be an **approximate solution**, to within the error ϵ, of the differential equation

$$\frac{dx}{dt} = f(t,\, x)$$

1161

if ϕ' is piecewise continuous, and for a given $\epsilon > 0$ and an open interval $I \subset R$,

$$|\phi'(t) - f(t,\, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative.

HU 3

16.114

Lipschitz continuity of a function.

The real function $f(t,\, x)$ defined and continuous in some open set $D \subset R^2$ is said to be

Lipschitz continuous with respect to x for some constant $k > 0$ if, for all points $(t,\, x_1)$ and $(t,\, x_2)$ belonging to D

$$|f(t,\, x_1) - f(t,\, x_2)| \leq k|x_1 - x_2|.$$

HU 5

16.21 Fundamental inequalities and related results

16.211

Gronwall's lemma.

Let the three piecewise continuous, nonnegative functions $u,\, v$, and w be defined in the interval $[0,\, a]$ and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau) w(\tau)\, d\tau,$$

$$w(t) \leq u(t) + \int_0^t v(\tau) w(\tau) \, d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau) v(\tau) \, \exp \, \left(\int_\tau^t v(\sigma) \, d\sigma \right) \, d\tau.$$

BB 135

16.212

Comparison of approximate solutions of a differential equation.

Let f be a real function that is defined in an open set $D \subset R^2$, in which it is both continuous and Lipschitz continuous. In addition, let u_1 and u_2 be two approximate solutions of

$$\frac{dx}{dt} = f(t, \, x)$$

in an open set $I \subset R$ in the sense already defined, with

$$|u_1'(t) - f(t, \, u_1(t))| \leq \epsilon_1, |u_2'(t) - f(t, \, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous.

Then, if for all $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \, \exp \, \{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k} \right) [\exp \, \{k|t - t_0|\} - 1].$$

HU 6

16.31 First-order systems

16.311

Solution of a system of equations.

The **system** of n first-order differential equations

$$\frac{dx_1}{dt} = f_1(t,\, x_1,\, x_2, \ldots ,x_n),$$

$$\frac{dx_2}{dt} = f_2(t,\, x_1,\, x_2, \ldots ,x_n),$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(t,\, x_1,\, x_2, \ldots ,x_n),$$

in which the functions $f_1,\, f_2, \ldots ,f_n$ are real and continuous in an open set $D \subset R^{n+1}$ may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t,\, \mathbf{x}),$$

where $\mathbf{x}$ and **f** are $n \times 1$ column vectors. Its solution in the open interval $I \subset R$ is the vector $\mathbf{u}(t)$ with elements

$u_1(t),\, u_2(t), \ldots ,u_n(t)$ with the property that

$$\text{(i)} \quad (t,\, \mathbf{u}(t)) \in D \quad \text{for} \quad t \in I,$$
$$\text{(ii)} \quad \frac{d\mathbf{u}}{dt} = \mathbf{f}(t,\, \mathbf{u}(t)) \qquad \text{for} \quad t \in I.$$

Cauchy problem for a system.

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t,\, \mathbf{x})$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector** condition

$$\mathbf{u}(t_0) = \mathbf{x}_0,$$

where $(t_0,\ \mathbf{u}(t_0) \in D$, the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau,\, \mathbf{u}(\tau))\, d\tau.$$

16.313

Approximate solution to a system.

The real vector $\Phi(t)$ is said to be an **approximate vector solution**, to within the order ϵ, of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t,\, \mathbf{x}),$$

if the elements of Φ' are piecewise continuous, and for a given $\epsilon > 0$ and open interval $I \subset R$,

$$\|\phi'(t) - \mathbf{f}(t,\, \phi(t))\| \leq \epsilon,$$

except at points of discontinuity of the derivative, where $\|\mathbf{w}\|$ denotes the supremum norm

$$\|\mathbf{w}\| = \sup\,(|w_1|,\, |w_2|, \ldots ,|w_n|).$$

16.314

Lipschitz continuity of a vector.

The real vector $\mathbf{f}(t,\, x)$ defined and continuous in some open set $D \subset R^n$ is said to be

Lipschitz continuous with respect to x for some constant $k > 0$ if, for all points $(t,\, \mathbf{x}_1)$, $(t,\, \mathbf{x}_2)$ belonging to D,

$$\|\mathbf{f}(t,\, \mathbf{x}_1) - \mathbf{f}(t,\, \mathbf{x}_2)\| \leq k\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

HU 26

16.315

Comparison of approximate solutions of a system.

Let **f** be a real vector defined in an open set $D \subset R \times R^n$ in which it is both continuous and Lipschitz continuous. In addition, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t,\, \mathbf{x})$$

in an open set $I \subset R$ in the sense already defined, with

$$\|\mathbf{u}_1'(t) - \mathbf{f}(t,\, \mathbf{u}_1(t))\| \leq \epsilon_1, \qquad \|\mathbf{u}_2'(t) - \mathbf{f}(t,\, \mathbf{u}_2(t))\| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$\|\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)\| \leq \delta,$$

it follows that

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \leq \delta\, \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)[\exp\{k|t - t_0|\} - 1].$$

HU 27

16.316

First-order linear differential equation.

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t)\,dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)}\left(\mu(t_0)y_0 + \int_{t_0}^{t} \mu(\xi)Q(\xi)\,d\xi\right),$$

where $y_0 = y(t_0)$.

1164

16.317

Linear systems of differential equations.

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where $\mathbf{x}$ is an $n \times 1$ column vector and $\mathbf{A}(t)$ an $n \times n$ matrix. Then a **fundamental system** of solutions of this system is a set of n linearly independent solution vectors $\Phi_1(t),\ \Phi_2(t), \ldots, \Phi_n(t)$,

The square matrix $\mathbf{K}(t)$ whose columns comprise the vectors $\Phi_1(t),\ \Phi_2(t), \ldots, \Phi_n(t)$ is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}(t_0)|\ \exp\left(\int_{t_0}^{t} \operatorname{tr}\ \mathbf{A}(\tau)\,d\tau\right).$$

Using the fundamental matrix $\mathbf{K}(t)$ defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, is

$$\phi(t) = \mathbf{K}(t)\mathbf{K}(t_0)^{-1}\mathbf{x}_0 + \mathbf{K}(t)\int_{t_0}^{t} \mathbf{K}(\tau)^{-1}\mathbf{b}(\tau)\, d\tau,$$

HU 43

where $\mathbf{b}(t)$ is an $n \times 1$ column vector.

CL 69

16.41 Some special types of elementary differential equations

16.411

Variables separable.

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y)\, dx + R(x)S(y)\, dy = 0.$$

It may then be written in the form

$$M(x)\, dx - \frac{1}{N(y)}\, dy = 0,$$

or

$$M(x)\,dx - \frac{1}{N(y)}\,dy = 0,$$

$$\frac{P(x)}{R(x)}\,dx + \frac{S(y)}{Q(y)}\,dy = 0,$$

provided $R(x)Q(y) \neq 0$.

1165
16.412

Exact differential equations.

A differential equation

$$M(x,\, y)\,dx + N(x,\, y)\,dy = 0$$

is said to be **exact** if there exists a function $h(x,\, y)$ such that

$$d[h(x,\, y)] = M(x,\, y)\,dx + N(x,\, y)\,dy.$$

IN 16

16.413

Conditions for an exact equation.

A necessary and sufficient condition that an equation of this form is exact is that the functions $M(x,\, y)$ and $N(x,\, y)$ together with their partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

IN 16

16.414

Homogeneous differential equations.

A differential equation

$$M(x,\, y)\,dx + N(x,\, y)\,dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary k,

$$\frac{M(kx,\ ky)}{N(kx,\ ky)} = \frac{M(x,\ y)}{N(x,\ y)}.$$

Setting $y = sx$, it may then be expressed in the form

$$[M(1,\ s) + sN(1,\ s)]\, dx + xN(1)\, dx = 0,$$

in which the variables s and x are separable.

IN 18

16.51 Second-order equations

16.511

Adjoint and self-adjoint equations.

The linear second-order differential equation

$$L(u) \equiv a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0,$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2}[a(x)v] - \frac{d}{dx}[b(x)v] + c(x)v = 0.$$

The equation $L(u) = 0$ is said to be **self-adjoint** if $L(u) \equiv M(u)$.

A linear self-adjoint second-order differential equation defined on $[\alpha,\ \beta]$ can always be expressed in the form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

where $p(x)$ and $q(x)$ are continuous on $[\alpha,\ \beta]$ and $p(x) > 0$. The general equation $L(u) = 0$ can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right],$$

when

$$p(x) = \exp\int\frac{b(x)}{a(x)}\,dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right].$$

In general, if

$$L(u) = p_0\,\frac{d^n u}{dx^n} + p_1\,\frac{d^{n-1}u}{dx^{n-1}} + \ldots + p_{n-1}\,\frac{du}{dx} + p_n u, i$$

then its adjoint is

$$M(v) = (-1)^n\,\frac{d^n}{dx^n}[p_0 v] + (-1)^{n-1}\,\frac{d^{n-1}}{dx^{n-1}}[p_1 v] + \ldots - \frac{d}{dx}[p_{n-1}v] + p_n v.$$

HI 391

16.512

Abel's identity.

If $p(x)$ and $q(x)$ are continuous in $[\alpha,\ \beta]$ in which $p(x) > 0$, and $u(x)$ and $v(x)$ are suitably differentiable with

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

then the result

$$p(x)\left(u\,\frac{dv}{dx} - v\,\frac{du}{dx}\right) \equiv \text{const.}$$

More generally, if we consider the linear nth-order equation

$$p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_{n-1} \frac{du}{dx} + p_n = 0,$$

and Δ is the Wronskian of a (fundamental) set of linearly independent solutions $u_1,\ u_2, \ldots\ , u_n$, the Abel identity takes the form

$$\Delta = \Delta_0\ \exp\left(-\int_{x_0}^{x} \frac{p_1(x)}{p_0(x)}\, dx\right),$$

where Δ_0 is the value of Δ at $x = x_0$.

IN 119

1167
16.513

Lagrange identity.

If the linear nth-order equation $L(u) = 0$ is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx}\{P(u,\ v)\},$$

where $M(v)$ is the adjoint of $L(u)$, is called the **Lagrange identity**. The expression $P(u,\ v)$, which is linear and homogeneous in

$$u,\ \frac{du}{dx}, \ldots\ , \frac{d^{n-1}u}{dx^{n-1}} \quad \text{and} \quad v,\ \frac{dv}{dx}, \ldots\ , \frac{d^{n-1}v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**.

In the case of the second-order equation

$$L(u) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0,$$

with adjoint $M(v)$, the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx}\left(a(x)v\frac{du}{dx} - \frac{d}{dx}(a(x)v)u + b(x)uv\right) \ .$$

IN 124

16.514

The Riccati equation.

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x)z + b(x)z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{\left(p(x)\frac{du}{dx}\right)}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0.$$

1168

The further substitution $v = u\left(\exp\,\int_{\alpha}^{x} a(x)\,dx\right)$ in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

$$v = u \left(\exp \int_{\alpha} a(x)\,dx\right)$$

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

with

$$r(x) = b(x)\,\exp\left(-\int_{\alpha}^{x} a(x)\,dx\right) \quad \text{and} \quad s(x) = c(x)\,\exp\left(\int_{\alpha}^{x} a(x)\,dx\right).$$

HI 273

16.515

Solutions of the Riccati equation.

If in the Riccati equation

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0$$

$r(x) \neq 0$, while $r(x)$ and $s(x)$ are continuous on the interval $[\alpha,\ \beta]$, then every solution $v(x)$ may be expressed in the form

$$\frac{1}{r(x)}\,\frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with $A,\ B$ arbitrary constants, not both zero, and the prime denoting differentiation, while u and v are linearly independent solutions of

$$\frac{d}{dx}\left(\frac{1}{r(x)}\,\frac{dz}{dx}\right) + s(x)z = 0.$$

Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions of this last equation and A and B are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)}\,\frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever $Au(x) = Bv(x) \neq 0$.

IN 24

16.516

Solution of a second-order linear differential equation.

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} = \phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t),$$

1169

when the solution to the inhomogeneous equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t),$$

may be written

$$x(t) = \int_{t_0}^{t} \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)}\, f(\xi)\, d\xi.$$

The linear combination $c_1\phi_1(t) + c_2\phi_2(t)$ is known as the **complementary function** where c_1 and c_2 are arbitrary constants.

16.61- 16.62 Oscillation and nonoscillation theorems for second-order equations

Equations whose solutions possess an infinite number of zeros in the interval $(0, \infty)$ are said to have **oscillatory** solutions. The following theorems relate to such properties.

16.611

First basic comparison theorem.

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if $\psi(x) \geq \phi(x)$ and some solutions v are nonoscillatory, then so also must some solutions u be nonoscillatory.

BS 119

16.622

Second basic comparison theorem.

If all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0$$

are oscillatory as $x \to \infty$, and if

$$q_2(x) \geq q_1(x),$$
$$p_2(x) \geq p_1(x) > 0,$$

1170

then all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0$$

BS 120

16.623

Interlacing of zeros.

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and suppose that $y_1(x)$ has at least two zeros in the interval $(a,\ b)$. Then if x_1 and x_2 are two consecutive zeros of $y_1(x)$, the function $y_2(x)$ has one, and only one, zero in the interval $(x_1,\ x_2)$.

HI 374

16.624

Sturm separation theorem.

Let $u(x)$ and $v(x)$ be two linearly independent solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) = 0,$$

in which $p(x) > 0$ and $p(x),\ q(x)$ are continuous on $[a,\ b]$. Then, between any two consecutive zeros of $u(x)$ there will be one, and only one, zero of $v(x)$.

IN 224

16.625

Sturm comparison theorem.

Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions in the differential equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0,$$

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0.$$

Then between any two zeros of a nontrivial solution $u(x)$ of the first equation there will be at least one zero of every nontrivial solution $v(x)$ of the second equation.

IN 228

16.626

Szegö's comparison theorem.

Suppose, under the conditions of the Sturm comparison theorem, that $p_1(x) \equiv p_2(x),\ q_1(x) \not\equiv q_2(x)$, and $u(x) > 0,\ v(x) > 0$ for $a < x < b$, together with

$$\lim_{x \to a} p_1(x) \left(\frac{du}{dx}\, v - \frac{dv}{dx}\, u \right) = 0.$$

Then, if $u(b) = 0$, there is a point ξ in $(a,\ b)$ such that $v(\xi) = 0.$

HI 379

1171
16.627

Picone's identity.

Consider the equations

$$\begin{aligned} \frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u &= 0, \\ \frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v &= 0, \end{aligned}$$

with $p_1,\ p_2,\ q_1$, and q_2 positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then with $a < \alpha < \beta < b$, Picone's identity is

$$\left(\frac{u}{v} \left(p_1\, \frac{du}{dx}\, v - p_2\, \frac{dv}{dx}\, u \right) \right)_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (q_2 - q_1) u^2\, ds + \int_{\alpha}^{\beta} (p_1 - p_2) \left(\frac{du}{ds} \right)^2 ds + \int_{\alpha}^{\beta} \frac{p_2}{v^2} \left(v\, \frac{du}{ds} - u\, \frac{dv}{ds} \right)^2 ds.$$

IN 226

16.628

Sturm-Picone theorem.

Consider the self-adjoint equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0$$

and

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0.$$

Let $p_1,\ p_2,\ q_1$, and q_2 be positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then, if x_1 and x_2 is a pair of consecutive zeros of $u(x)$ in $(a,\ b),\ v(x)$ has at least one zero in the open interval $(a,\ b)$.

IN 225

16.629

Oscillation on the half line.

Consider the self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0.$$

We then have the following results.

(i) Let $p(x) > 0$ and $p,\ q$ be continuous on $[0,\ \infty)$. If the two improper integrals

$$\int_1^\infty \frac{dx}{p(x)} \quad \text{and} \quad \int_1^\infty q(x)\,dx$$

diverge, then every solution $u(x)$ has infinitely many zeros on the interval $[1,\ \infty)$. Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)} = +\infty \quad \text{and} \quad \int_0^1 q(x)\,dx = +\infty,$$

then every solution $u(x)$ has infinitely many zeros on the interval (0, 1).

1172

(ii) (Moore's theorem). Every nontrivial solution $u(x)$ has at most a finite number of zeros on the interval $[a,\ \infty)$ if the improper integral

$$\int_a^\infty \frac{dx}{p(x)}$$

converges, and if

$$\left| \int_a^x q(s)\, ds \right| < M \quad \text{for} \quad a \le x < \infty$$

with $M > 0$ a finite constant.

16.71 Two related comparison theorems

16.711

Theorem 1.

Consider the equations in the Sturm comparison theorem with the same assumptions on $p(x)$ and $q(x)$, and let $u(x),\ v(x)$ be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if $u(x)$ is increasing in $[x_1,\ x_2]$ and reaches a maximum at x_2, the function $v(x)$ reaches a maximum at some point x_3 such that $x_1 < x_3 < x_2$.

HI 376

16.712

Theorem 2.

Consider the equation

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

in which $F(x)$ is continuous in $(a,\ b)$ and such that

$$0 < m \leq F(x) \leq M.$$

Then, if the solution $y(x)$ has two successive zeros $x_1,\ x_2$, it follows that

$$\pi M^{-\frac{1}{2}} \leq x_2 - x_1 \leq \pi m^{-\frac{1}{2}}.$$

16.81- 16.82 Nonoscillatory solutions

The real solution $y(x)$ of

$$\frac{d^2y}{dx^2} + F(x)y = 0$$

is said to be **nonoscillatory** in the wide sense in $(0,\ \infty)$ if there exists a finite number c such that the solution has no zeros in $[c,\ \infty)$.
HI 376

1173
16.811

Kneser's nonoscillation theorem.

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup\,[x^2 F(x)] = \gamma^*,$$

$$\liminf\,[x^2F(x)] = \gamma_*.$$

Then the solution $y(x)$ is nonoscillatory if $\gamma^* < \frac{1}{4},$ oscillatory if $\frac{1}{4} < \gamma_*$ and no conclusion can be drawn if either γ^* or γ_* equals $\dfrac{1}{4}$.

HI 461

16.822

Comparison theorem for nonoscillation.

Consider the differential equations

$$\frac{d^2y}{dx^2} + F(x)y = 0, \quad f(x) = x\int_x^\infty F(s)\,ds,$$

$$\frac{d^2y}{dx^2} + G(x)y = 0, \quad g(x) = x\int_x^\infty G(s)\,ds,$$

where $0 < g(x) < f(x)$. Then if the first equation is nonoscillatory in the wide sense, so also is the second.

HI 460

16.823

Necessary and sufficient conditions for nonoscillation.

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x\to\infty}\,\sup\left(x\int_x^\infty F(s)\,ds\right) = F^*,$$

$$\lim_{x\to\infty} \inf \left(x \int_x^\infty F(s)\, ds \right) = F_*,$$

it follows that:

(i) a necessary condition that the solution $y(x)$ be nonoscillatory is that $F_* \le \frac{1}{4}$ and $F^* \le 1$;

(ii) a sufficient condition that the solution $y(x)$ be nonoscillatory is that $F^* < \frac{1}{4}$.

1174

16.91 Some growth estimates for solutions of second-order equations

16.911

Strictly increasing and decreasing solutions.

Suppose that $G(x) > 0$ be continuous in $(-\infty, \infty)$ and such that $xG(x) \notin L(0, \infty)$. Then the equation

$$\frac{d^2 y}{dx^2} - G(x) y = 0$$

has one, and only one, solution $y_+(x)$ passing through the point (0, 1) which is positive and strictly monotonic decreasing for all x, and one and only one solution $y_-(x)$ through the point (0, 1) which is positive and strictly increasing for all x. The solution $y_+(x)$ has the property that

$$[G(x)]^{\frac{1}{2}} y_+(x) \in L_2(0, \infty) \quad \text{and} \quad \frac{dy_+(x)}{dx} \in L_2(0, \infty).$$

If, in addition,

$$0 < \alpha^2 \le G(x) \le \beta^2 < \infty,$$

then

$$e^{-\beta x} \le y_+(x) \le e^{-\alpha x} \quad \text{for} \quad x > 0.$$

16.912

General result on dominant and subdominant solutions.

Consider the equations

$$\frac{d^2y}{dx^2} - g(x)y = 0, \quad \frac{d^2Y}{dx^2} - G(x)Y = 0,$$

where g and G are continuous on $(0,\, \infty)$ with $0 < g(x) < G(x)$, and $xg(x) \notin L(0,\, \infty)$. In addition, let y_α and Y_α be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \quad y'_\alpha(0) = Y'_\alpha(0) = \alpha \quad \text{for} \quad -\infty < \alpha < \infty.$$

Let y_ω and Y_ω be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \quad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let y_+ and Y_+ be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while $[y'_+(x)]^2,\ g(x)[y_+(x)]^2,\ [Y'_+(x)]^2$, and $G(x)[Y'_+(x)]^2$ belong to $L(0,\, \infty)$. Then, if β and γ are such that $y_{-\beta} = y_+$ and $Y_{-\gamma} = Y_+$, it follows that $\beta < \gamma$ and

$$\begin{aligned} &y_\alpha(x) < Y_\alpha(x), \quad 0 < x < \infty, \quad -\gamma \le \alpha, \\ &y_\omega(x) < Y_\omega(x), \\ &y_+(x) > Y_+(x). \end{aligned}$$

HI 440

1175
16.913

Estimate of dominant solution.

Let $G(x)$ be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G'(x) < \frac{5}{4}[G'(x)]^2.$$

Then there exists a **dominant solution** $y(x)$ of the fundamental solutions $Y_0(x)$ and $Y_1(x)$ of

$$\frac{d^2y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$\begin{aligned} Y_0(0) &= 0, \quad Y_1(0) = 1, \\ Y_0'(0) &= 1, \quad Y_1'(0) = 0, \end{aligned}$$

such that

$$y(x) < [G(x)]^{-\frac{1}{4}} \exp\left(\int_0^x [G(\xi)]^{\frac{1}{2}}\, d\xi\right),$$

and a positive constant C such that the normalized subdominant solution $y_+(x)$, for which $y_+(0) = 1$ and $[y_+'(x)]^2 \in L(0, \infty)$, $G(x)[y_+(x)]^2 \in L(0, \infty)$, satisfies

$$y_+(x) > C[G(x)^{-\frac{1}{4}} \exp\left(-\int_0^x [G(\xi)]^{\frac{1}{2}}\, d\xi\right).$$

HI 443

16.914

A theorem due to Lyapunov.

Let $y(x)$ be any solution of

$$\frac{d^2y}{dx^2} - G(x)y = 0$$

with $G(x)$ positive and continuous in $(0,\ \infty)$ with $xG(x) \in L(0,\ \infty)$. Then

$$C\ \exp\left(-\int_0^x [G(\xi)+1]\,d\xi\right) < [y(x)]^2 + [y'(x)]^2 < C\ \exp\left(\int_0^x [G(\xi)+1]\,d\xi\right),$$

HI 446

where $C = [y(0)]^2 + [y'(0)]^2$.

16.92 Boundedness theorems

16.921[6]

All solutions of the equation

$$\frac{d^2u}{dx^2} + (1+\phi(x)+\psi(x))u = 0$$

are bounded, provided that

$$\text{(i)} \quad \int^\infty |\phi(x)|\,dx < \infty,$$

$$\text{(ii)} \quad \int^\infty |\psi(x)|\,dx < \infty \quad \text{and} \quad \psi(x) \to 0 \quad \text{as} \quad x \to \infty.$$

BS 112

16.922

If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x)+b(x))u = 0$$

are also bounded if

$$\int^{\infty} |b(x)|\,dx < \infty.$$

BS 112

16.923

If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded as $x \to \infty$.

BS 113

16.924

Consider the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

in which

$$\int^{\infty} x|a(x)|\,dx < \infty.$$

Then $\lim_{x\to\infty}\left(\frac{du}{dx}\right)$ exists, and the general solution is asymptotic to $d_0 + d_1 x$ as $x \to \infty$, where d_0 and d_1 may be zero, but not simultaneously.

BS 114

*16.93**

Growth of maxima of $|y|$. Sonin's theorem generalized by Pólya may be stated as follows. *Let y(x) satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

where $k(x) > 0,\ \phi(x) > 0$, and both functions $k(x),\ \phi(x)$ have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\phi(x)$ is decreasing or increasing.

SZ 164

17. Fourier, Laplace and Mellin Transforms

17.1- 17.4 Integral Transforms

17.11 Laplace transform

The **Laplace transform** of the function $f(x)$, denoted by $F(s)$, is defined by the integral

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx, \qquad \operatorname{Re} s > 0.$$

The functions $f(x)$ and $F(s)$ are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^\infty |f(x)|e^{-c|x|}\,dx,$$

is finite, then the Laplace transform exists when $s = \sigma + i\tau$ is such that $\sigma \geq c$.

Setting

$$F(s) = \mathcal{L}[f(x);\ s],$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}[F(s);\ x].$$

The inversion of the Laplace transform is accomplished for analytic functions $F(s)$ of order $O(s^{-k})$ with $k > 1$ by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{sx}\,ds,$$

where γ is a real constant that exceeds the real part of all the singularities of $F(s)$.

SN 30

17.12 Basic properties of the Laplace transform

1.[8] For a and b arbitrary constants,

$$\mathcal{L}[af(x) + bg(x)] = aF(s) + bG(s) \qquad \text{(linearity)}.$$

2. If $n > 0$ is an integer and $\lim\limits_{x\to\infty} f(x)e^{-sx} = 0$, then for $x > 0$,

$$\mathcal{L}[f^{(n)}(x);\, s] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0) \quad \text{(transform of a derivative)}.$$

SN 32

3. If $\lim_{x\to\infty}\left(e^{-sx}\int_0^x f(\xi)\,d\zeta\right) = 0$, then

$$\mathcal{L}\left[\int_0^x f(\xi)\,d\xi;\, s\right] = \frac{1}{s}F(s) \qquad \text{(transform of an integral)}.$$

SN 37

4. $\mathcal{L}[e^{-ax}f(x);\, s] = F(s+a)$ (shift theorem).

SU 143

5. The **Laplace convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \int_0^x f(x-\xi)g(\xi)\,d\xi,$$

and it has the property that $f * g = g * f$ and $f * (g * h) = (f * g) * h$. In terms of the convolution operation

$$\mathcal{L}[f * g(x);\, s] = F(s)G(s) \text{ (convolution (Faltung) theorem).}$$

SN 30

17.13* Table of Laplace transform pairs

	$f(x)$	$F(s)$	
1.	1	$1/s$	
2.	$x^n,\ n = 0, 1, 2, \ldots$	$n!/s^{n+1}, \quad \operatorname{Re} s > 0$	ET I 133(3)
3.	$x^\nu,\ \operatorname{Re}\nu > -1$	$\Gamma(\nu+1)/s^{\nu+1}, \quad \operatorname{Re} s > 0$	ET I 137(1)
4.[8]	$x^{n-\frac{1}{2}}$	$\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{n-1}{2}\right)/s^{n+\frac{1}{2}} = 1^7\left(n+\frac{1}{2}\right)/s^{n+\frac{1}{2}}, \quad \operatorname{Re} s > 0$	ET I 135(17)
5.	$x^{-\frac{1}{2}}(x+a)^{-1},\ \lvert\arg a\rvert < \pi$	$\pi a^{-\frac{1}{2}} e^{as} \operatorname{erfc}(a^{\frac{1}{2}} s^{\frac{1}{2}}), \quad \operatorname{Re} s \ge 0 = \Gamma\left(n+\frac{1}{2}\right)/s^{n+\frac{1}{2}}$	ET I 136(25)
6.	x for $0 < x < 1$ 1 for $x > 1$	$(1-e^{-s})/s^2, \quad \operatorname{Re} s > 0$	ET I 142(14)
7.	e^{-ax}	$1/(s+a), \quad \operatorname{Re} s > -\operatorname{Re} a$	ET I 143(1)
8.	xe^{-ax}	$1/(s+a)^2, \quad \operatorname{Re} s > -\operatorname{Re} a$	ET I 144(2)
9.	$(e^{-ax} - e^{-bx})/(b-a)$	$(s+a)^{-1}(s+b)^{-1}, \quad \operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$	AS 1022(29.3.12)
10.	$(ae^{-ax} - be^{-bx})/(b-a)$	$s(s+a)^{-1}(s+b)^{-1}, \quad \operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$	AS 1022(29.3.13)
11.	$(e^{ax} - 1)/a$	$s^{-1}(s-a)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a$	
12.	$(e^{ax} - ax - 1)/a^2$	$s^{-2}(s-a)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a$	
13.	$\left(e^{ax} - \frac{1}{2}a^2x^2 - ax - 1\right)/a^3$	$s^{-3}(s-a)^{-1}, \quad \operatorname{Re} s > \operatorname{Re} a$	
14.	$(1+ax)e^{ax}$	$s(s-a)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a$	
15.	$[1+(ax-1)e^{ax}]/a^2$	$s^{-1}(s-a)^{-2}, \quad \operatorname{Re} ps > \operatorname{Re} a$	
16.	$[2+ax+(ax-2)e^{ax}]/a^3$	$s^{-2}(s-a)^{-2}, \quad \operatorname{Re} s > \operatorname{Re} a$	
17.	$x^n e^{ax},\ n = 0, 1, 2, \ldots$	$n!(s-a)^{-(n+1)}, \quad \operatorname{Re} s > \operatorname{Re} a$	
18.	$\left(x + \frac{1}{2}ax^2\right)e^{ax}$	$s(s-a)^{-3}, \quad \operatorname{Re} s > \operatorname{Re} a$	
19.	$\left(1 + 2ax + \frac{1}{2}a^2x^2\right)e^{ax}$	$s^2(s-a)^{-3}, \quad \operatorname{Re} s > \operatorname{Re} a$	
20.	$\frac{1}{6}x^3e^{ax}$	$(s-a)^{-4}, \quad \operatorname{Re} s > \operatorname{Re} a$	

ET I 133(3)
ET I 137(1)
ET I 135(17)
ET I 136(25)
ET I 142(14)
ET I 143(1)
ET I 144(2)
AS 1022(29.3.12)

17.13 (continued)

	$f(x)$	$F(s)$
21.	$\left(\frac{1}{2}x^2 + \frac{1}{6}ax^3\right)e^{ax}$	$s(s-a)^{-4}, \quad \operatorname{Re} s > \operatorname{Re} a$
22.	$\left(x + ax^2 + \frac{1}{6}a^2x^3\right)e^{ax}$	$s^2(s-a)^{-4}, \quad \operatorname{Re} s > \operatorname{Re} a$
23.	$\left(1 + 3ax + \frac{3}{2}a^2x^2 + \frac{1}{6}a^3x^3\right)e^{ax}$	$s^3(s-a)^{-4}, \quad \operatorname{Re} s > \operatorname{Re} a$
24.	$(ae^{ax} - be^{bx})/(a-b)$	$s(s-a)^{-1}(s-b)^{-1}, \quad \operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$

ET I 147(37)
ET I 146(26)
ET I 146(27)
ET I 146(28)
ET I 150(1)
ET I 154(43)
ET I 150(2)
ET I 155(44)
AS 1022(29.3.19)
AS 1022(29.3.20)
AS 1022(29.3.21)

17.13 (continued)

	$f(x)$	$F(s)$	
39.	$[x \sin p(ax)]/(2a)$	$s(s^2+a^2)^{-2}, \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	ET I 152(14)
40.	$[\sin p(ax) + ax \cos p(ax)]/(2a)$	$s^2(s^2+a^2)^{-2}, \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	AS 1023(29.3.23)
41.	$x \cos p(ax)$	$(s^2-a^2)(s^2+a^2)^{-2}, \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	ET I 157(57)
42.	$[\cos p(ax) - \cos p(bx)]/(b^2-a^2)$	$s(s^2+a^2)^{-1}(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \{\lvert \operatorname{Im} a \rvert, \lvert \operatorname{Im} b \rvert\}$	AS 1023(29.3.25)
43.	$\left[\frac{1}{2} a^2x^2 - 1 + \cos p(ax)\right]/a^4$	$s^{-3}(s^2+a^2)^{-1}, \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	
44.	$\left[1 - \cos p(ax) - \frac{1}{2} ax \sin p(ax)\right]/a^4$	$s^{-1}(s^2+a^2)^{-2}, \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	
45.	$\left[\frac{1}{b} \sin p(bx) - \frac{1}{a} \sin p(ax)\right]/(a^2-b^2)$	$(s^2+a^2)^{-1}(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \{\lvert \operatorname{Im} a \rvert, \lvert \operatorname{Im} b \rvert\}$	
46.	$\left[1 - \cos p(ax) + \frac{1}{2} ax \sin p(ax)\right]/a^2$	$s^{-1}(s^2+a^2)^{-2}(25^2+a^2), \quad \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	
47.	$[a \sin p(ax) - b \sin p(bx)]/(a^2-b^2)$	$s^2(s^2+a^2)^{-1}(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \{\lvert \operatorname{Im} a \rvert, \lvert \operatorname{Im} b \rvert\}$	
48.	$\sin p(a+bx)$	$(s \sin pa + b \cos pa)(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \lvert \operatorname{Im} b \rvert$	
49.	$\cos p(a+bx)$	$(s \cos pa - b \sin pa)(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \lvert \operatorname{Im} b \rvert$	
50.	$\left[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sin p(bx)\right]/(a^2+b^2)$	$(s^2-a^2)^{-1}(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \{\lvert \operatorname{Re} a \rvert, \lvert \operatorname{Im} b \rvert\}$	

ET I 152(14)
AS 1023(29.3.23)
ET I 157(57)
AS 1023(29.3.25)

17.13 (continued)

	$f(x)$	$F(s)$
51.	$[\cosh(ax) - \cos p(bx)]/(a^2+b^2)$	$s(s^2-a^2)^{-1}(s^2+b^2)^{-1}, \quad \operatorname{Re} s > \{\lvert \operatorname{Re} a \rvert, \lvert \operatorname{Im} b \rvert\}$
52.	$[a \sinh(ax) + b$	$s^2(s^2-a^2)^{-1}(s^2+b^2)^{-1},$

ET I 152(16)
ET I 157(59)
ET I 162(1)
ET I 162(2)
ET I 164(18)

17.13 (continued)

	$f(x)$	$F(s)$	
66.	$x^{\nu-1} \cosh(ax), \quad \operatorname{Re}\nu > 0$	$\frac{1}{2}\Gamma(\nu)[(s-a)^{-\nu} + (s+a)^{-\nu}], \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	ET I 164(19)
67.	$x \sinh(ax)$	$2as(s^2-a^2)^{-2}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	
68.	$x \cosh(ax)$	$(s^2+a^2)(s^2-a^2)^{-2}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	
69.	$\sinh(ax) - \sin p(ax)$	$2a^3(s^4-a^4)^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Im}\ a\rvert\}$	AS 1023(29.3.31)
70.	$\cosh(ax) - \cos p(ax)$	$2a^2s(s^4-a^4)^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Im}\ a\rvert\}$	AS 1023(29.3.32)
71.	$\sinh(ax) + ax\cosh(ax)$	$2as^2(a^2-s^2)^{-2}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	
72.	$ax\cosh(ax) - \sinh(ax)$	$2a^3(a^2-s^2)^{-2}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	
73.	$x\sinh(ax) - \cosh(ax)$	$s(a^2+2a-s^2)(a^2-s^2)^{-2}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$	
74.	$\left[\frac{1}{a}\sinh(ax) - \frac{1}{b}\sinh(bx)\right]/(a^2-b^2)$	$(a^2-s^2)^{-1}(b^2-s^2)^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$	
75.	$[\cosh(ax) - \cosh(bx)]/(a^2-b^2)$	$s(s^2-a^2)^{-1}(s^2-b^2)^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$	
76.	$[a\sinh(ax) - b\sinh(bx)]/(a^2-b^2)$	$s^2(s^2-a^2)^{-1}(s^2-b^2)^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$	
77.	$sinh\,(a+bx)$	$(b\cosh\ a + s\sinh\ a)(s^2-b^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}b\rvert$	
78.	$cosh\,(a+bx)$	$(s\cosh\ a + b\sinh\ a)(s^2-b^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}b\rvert$	
79.	$\sinh(ax)\sinh(bx)$	$2abs[s^2-(a+b)^2]^{-1}[s^2-(a-b)^2]^{-1}, \quad \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$	

AS 1022(29.3.13)
ET I 164(19)
AS 1023(29.3.31)
AS 1023(29.3.32)

17.13 (continued)

	$f(x)$	$F(s)$
80.[8]	$\cosh(ax)\cosh(bx)$	$s(s^2-a^2-b^2)[s^2-(a+b)^2]^{-1}[s^2-(a-b)^2]^{-1}, \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$
81.	$\sinh(ax)\cosh(bx)$	$a(s^2-a^2+b^2)[s^2-(a+b)^2]^{-1}[s^2-(a-b)^2]^{-1}, \operatorname{Re}s > \{\lvert\operatorname{Re}a\rvert, \lvert\operatorname{Re}b\rvert\}$
82.	$\sinh^2(ax)$	$2a^2s^{-1}(s^2-4a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$
83.	$\cosh^2(ax)$	$(s^2-2a^2)s^{-1}(s^2-4a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$
84.	$\sinh(ax)\cosh(ax)$	$a(s^2-4a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$
85.	$[\cosh(ax) - 1]/a^2$	$s^{-1}(s^2-a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$
86.	$[\sinh(ax) - ax]/a^3$	$s^{-2}(s^2-a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$
87.	$[\cosh(ax)$	$s^{-3}(s^2-a^2)^{-1}, \quad \operatorname{Re}s > \lvert\operatorname{Re}a\rvert$

17.13 (continued)

$f(x)$	$F(s)$	
97. $\mathrm{Si}(x) \equiv \int_0^x \frac{\sin p\xi}{\xi}\, d\xi \equiv \frac{1}{2}\pi + \mathrm{si}\,(x)$	s^{-1} arccot s, $\operatorname{Re} s > 0$	ET I 177(17)
98. $\mathrm{Ci}(x) \equiv \mathrm{ci}(x) \equiv -\int_x^\infty \frac{\cos p\xi}{\xi}\, d\xi$	$-\frac{1}{2}\, s^{-1} \ln(1+s^2)$, $\operatorname{Re} s > 0$	ET I 178(19)
99.[8] $\mathrm{erf}\left(\frac{x}{2a}\right)$	$s^{-1} e^{a^2 s^2}$ erfc (as), $\operatorname{Re} s > 0$, $\lvert \arg a \rvert < \pi/4$	ET I 176(2)
100. $\mathrm{erf}(a\sqrt{x})$	$as^{-1}(s+a^2)^{-\frac{1}{2}}$, $\operatorname{Re} s > \{0, -\operatorname{Re} a^2\}$	ET I 176(4)
101. erfc $(a\sqrt{x})$	$s^{-1}(s+a^2)^{-\frac{1}{2}}[(s+a^2)^{\frac{1}{2}} - a]$, $\operatorname{Re} s > 0$	ET I 177(9)
102.[8] erfc $(a/\sqrt{x})$	$s^{-1} e^{-2a\sqrt{s}}$, $\operatorname{Re} s > 0$, $\operatorname{Re} a > 0$	ET I 177(11)
103. $J_\nu(ax)$	$a^\nu[s+(s^2+a^2)^{\frac{1}{2}}]^{-\nu}(s^2+a^2)^{-\frac{1}{2}}$, $\operatorname{Re} s > \lvert \operatorname{Im} a \rvert, \operatorname{Re} \nu > -1$	ET I 182(1)
104. $xJ_\nu(ax)$	$a^\nu[s+\nu(s^2+a^2)^{\frac{1}{2}}][s+(s^2+a^2)^{\frac{1}{2}}]^{-\nu}$ $(s^2+a^2)^{-\frac{3}{2}}, \operatorname{Re} s > \lvert \operatorname{Im} a \rvert$, $\operatorname{Re} \nu > -2$	ET I 182(2)
105. $x^{-1} J_\nu(ax)$	$a^\nu \nu^{-1}[s+(s^2+a^2)^{\frac{1}{2}}]^{-\nu}$, $\operatorname{Re} s \geq \lvert \operatorname{Im} a \rvert$	ET I 182(5)
106. $x^n J_n(ax)$	$1 \cdot 3 \cdot 5 \ldots (2n-1) a^n (s^2+a^2)^{-(n+\frac{1}{2})}$, $\operatorname{Re} s > \lvert \operatorname{Im} a \rvert$	ET I 182(4)
107. $x^\nu J_\nu(ax)$	$2^\nu \pi^{-\frac{1}{2}} \Gamma\left(\nu+\frac{1}{2}\right) a^\nu (s^2+a^2)^{-(\nu+\frac{1}{2})}$, $\operatorname{Re} s > \lvert \operatorname{Im} a \rvert$, $\operatorname{Re} \nu > -\frac{1}{2}$	ET I 182(7)
108. $x^{\nu+1} J_\nu(ax)$	$2^{\nu+1} \pi^{-\frac{1}{2}} \Gamma\left(\nu+\frac{3}{2}\right) a^\nu s(s^2+a^2)^{-(\nu+\frac{3}{2})}$, $\operatorname{Re} s > \lvert \operatorname{Im} a \rvert$, $\operatorname{Re} \nu > -1$	ET I 182(8)
109. $I_\nu(ax)$	$a^\nu (s^2-a^2)^{-\frac{1}{2}}[s+(s^2-a^2)^{\frac{1}{2}}]^{-\nu}$, $\operatorname{Re} s > \lvert \operatorname{Re} a \rvert, \operatorname{Re} \nu > -1$	ET I 195(1)

ET I 177(17)
ET I 178(19)
ET I 176(2)
ET I 176(4)
ET I 177(9)
ET I 177(11)
ET I 182(1)
ET I 182(2)
ET I 182(5)
ET I 182(4)
ET I 182(7)
ET I 182(8)
ET I 195(1)

17.13 (continued)

$f(x)$	$F(s)$	
110. $x^\nu I_\nu(ax)$	$2^\nu \pi^{-\frac{1}{2}} \Gamma\left(\nu+\frac{1}{2}\right) a^\nu (s^2-a^2)^{-(\nu+\frac{1}{2})}$, $\operatorname{Re} s > \lvert \operatorname{Re} a \rvert$, $\operatorname{Re} \nu > -\frac{1}{2}$	ET I 195(6)
111. $x^{\nu+1} I_\nu(ax)$	$2^{\nu+1} \pi^{-\frac{1}{2}} \Gamma\left(\nu+\frac{3}{2}\right) a^\nu s(s^2-a^2)^{-(\nu+\frac{3}{2})}$, $\operatorname{Re} s > \lvert \operatorname{Re} a \rvert$, $\operatorname{Re} \nu > -1$	ET I 196(7)
112. $x^{-1} I_\nu(ax)$	$\nu^{-1} a^\nu [s+(s^2-a^2)^{\frac{1}{2}}]^{-\nu}$, $\operatorname{Re} s > \lvert \operatorname{Re} a \rvert$, $\operatorname{Re} \nu > 0$	ET I 195(4)
113. $\sin p(2a^{\frac{1}{2}} x^{\frac{1}{2}})$	$(\pi a)^{\frac{1}{2}} s^{-\frac{3}{2}} e^{-a/s}$, $\operatorname{Re} s > 0$	ET I 153(32)
114. $x^{-\frac{1}{2}} \cos p(2a^{\frac{1}{2}} x^{\frac{1}{2}})$	$\pi^{\frac{1}{2}} s^{-\frac{1}{2}} e^{-a/s}$, $\operatorname{Re} s > 0$	ET I 158(67)
115. $x^{-1} e^{-ax} I_1(ax)$	$[(s+2a)^{\frac{1}{2}} - s^{\frac{1}{2}}][(s+2a)^{\frac{1}{2}} + s^{\frac{1}{2}}]^{-1}$, $\operatorname{Re} s > \lvert \operatorname{Re} a \rvert$	AS 1024(29.3.52)
116. $x^{-1} J_k(ax)$	$k^{-1} a^{-k} [(s^2+a^2)^{\frac{1}{2}} - s]^k$, $\operatorname{Re} s > \lvert \operatorname{Im} a \rvert$, $k > -1$	AS 1025(29.3.58)

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function $f(x)$, denoted by $F(\xi)$, is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x}\, dx.$$

The functions $f(x)$ and $F(\xi)$ are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered.

Setting

$$F(\xi) = \mathcal{F}[f(x);\, \xi],$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{F}^{-1}[F(\xi);\, x].$$

The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x}\, d\xi.$$

17.22 Basic properties of the Fourier transform

1. For a and b arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \qquad \text{(linearity)}.$$

2. If $n > 0$ is an integer, and $\lim_{|x|\to\infty} f^{(r)}(x) = 0$ for $r = 0,\ 1, \ldots, n-1$ with $f^{(0)}(x) \equiv f(x)$, then

3. The **Fourier convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi,$$

and it has the property $f * g = g * f$, and $f * (g * h) = (f * g) * h$. In terms of the convolution operation.

$$\mathcal{F}[f * g(x);\, \xi] = F(\xi) G(\xi) \qquad \text{(convolution (Faltung) theorem).}$$

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17.23* Table of Fourier transform pairs

$f(x)$	$F(\xi)$	
1. 1	$(2\pi)^{\frac{1}{2}}\delta(\xi)$	SU 496
2.[7] $\frac{1}{x}$	$(\pi/2)^{\frac{1}{2}} i \operatorname{sgn} \xi$	SU 50
3. $\delta(x)$	$(2\pi)^{-\frac{1}{2}}$	SU 496
4.[8] $\delta(ax+b),\ a,\ b \in \mathrm{R},\ a \neq 0$	$(2\pi)^{-\frac{1}{2}}\, e^{-ib\xi/a}$	SU 517
5. $1, \quad \lvert x\rvert < a$; $0, \quad \lvert x\rvert > a,\ a > 0$	$(2/\pi)^{\frac{1}{2}} \xi^{-1} \sin(a\xi)$	
6.[8] $\frac{1}{\lvert x\rvert}$	divergent	
7. $\frac{1}{\lvert x\rvert^a}, \quad 0 < \operatorname{Re} a < 1$	$\dfrac{(2/\pi)^{\frac{1}{2}} \Gamma(1-a) \sin\left(\frac{1}{2}\, a\pi\right)}{\lvert\xi\rvert^{1-a}}$	SN 523
8.[8] $e^{iax}, \quad a \in \mathrm{R}$	$(2\pi)^{\frac{1}{2}}\delta(\xi + a)$	SU 517
9. $e^{-a\lvert x\rvert}, \quad a > 0$	$\frac{a(2/\pi)^{\frac{1}{2}}}{a^2+\xi^2}$	SU 517
10.[7] $xe^{-a\lvert x\rvert}, \quad a > 0$	$\frac{2ai\xi(2/\pi)^{\frac{1}{2}}}{(a^2+\xi^2)^2},$	SU 517
11. $\lvert x\rvert e^{-a\lvert x\rvert}, \quad a > 0$	$\frac{(2/\pi)^{\frac{1}{2}}(a^2-\xi^2)}{(a^2+\xi^2)^2}$	SU 517
12. $\frac{e^{-a\lvert x\rvert}}{\lvert x\rvert^{\frac{1}{2}}}, \quad a > 0$	$\frac{[a+(a^2+\xi^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{x(a^2+\xi^2)^{\frac{1}{2}}}$	SN 523
13. $e^{-a^2x^2}, \quad a > 0$	$(a\sqrt{2})^{-1} e^{-\xi^2/4a^2}$	SU 517
14. $\frac{1}{a^2+x^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} e^{-a\lvert\xi\rvert}}{a}$	SU 517
15.[7] $\frac{x}{a^2+x^2}, \quad \operatorname{Re} a > 0$	$i \operatorname{sgn} \xi (\pi/2)^{\frac{1}{2}} e^{-a\lvert\xi\rvert}$	SU 517
16. $\sin(ax^2)$	$\frac{1}{(2a)^{\frac{1}{2}}} \sin\left(\frac{\xi^2}{4a} + \frac{\pi}{4}\right)$	SN 523
17. $\cos(ax^2)$	$\frac{1}{(2a)^{\frac{1}{2}}} \cos\left(\frac{\xi^2}{4a} - \frac{\pi}{4}\right)$	SN 523
18. $e^{-a\lvert x\rvert} \cos(bx),$ $a > 0, \quad b > 0$	$a(2\pi)^{-\frac{1}{2}} \left[\frac{1}{a^2+(b+\xi)^2} + \frac{1}{a^2+(b-\xi)^2}\right]$	
19. $e^{-\frac{1}{2}ax^2} \sin(bx),$ $a > 0, \quad b > 0$	$\frac{1}{2}\, ia^{-\frac{1}{2}} \left\{\exp\left[-\frac{1}{2}(\xi-b)^2/a\right] - \exp\left[-\frac{1}{2}(\xi+b)^2/a\right]\right\}$	

	$-1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\operatorname{cth}(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121(26)
28.	$e^{-ax} \log(1+e^{-x})$, $-1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\csc(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121 (27)

SN 523
SU 123
SU 506
ET I 121(26)
ET I 121 (27)

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In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $1/(2\pi)^{\frac{1}{2}}$ employed in our definition of F has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

17.31 Fourier sine and cosine transforms

The **Fourier sine** and **cosine transforms** of the function $f(x)$, denoted by $F_s(\xi)$ and $F_c(\xi)$, respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x)\, dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\xi x)\, dx.$$

The functions $f(x)$ and $F_s(\xi)$ are called a **Fourier sine transform pair**, and the functions $f(x)$ and $F_c(\xi)$ a **Fourier cosine transform pair**, and knowledge of either $F_s(\xi)$ or $F_c(\xi)$ enables $f(x)$ to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s[f(x);\, \xi] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c[f(x);\, \xi],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s^{-1}[F_s(\xi);\, x] \quad \text{and} \quad f(x) = \mathcal{F}_c^{-1}[F_c(\xi);\, x].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\xi) \sin(\xi x)\, d\xi (x \geq 0),$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\xi) \cos(\xi x) \, d\xi (x \geq 0).$$

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17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s[af(x) + bg(x)] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c[af(x) + bg(x)] = aF_c(\xi) + bG_c(\xi) \qquad \text{(linearity)}.$$

2. If $\lim_{x\to\infty} f^{(r-1)}(x) = 0$ and $\lim_{x\to\infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by $F_s^{(r)}$ and $F_c^{(r)}$, respectively,

$$\begin{array}{rl}
\text{(i)} & F_c^{(r)}(\xi) = -a_{r-1} + \xi F_s^{(r-1)}, \\
\text{(ii)} & F_s^{(r)}(\xi) = -\xi F_c^{(r-1)}(\xi), \\
\text{(iii)} & F_c^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2n} F_c(\xi), \\
\text{(iv)} & F_c^{(2r+1)}(\xi) = -\sum_{n=0}^{r} (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi), \\
\text{(v)} & F_s^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s^{(r-2)}(\xi), \\
\text{(vi)}^6 & F_s^{(2r)}(\xi) = -\sum_{n=1}^{r} (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi), \\
\text{(vii)} & F_s^{(2r+1)}(\xi) = -\sum_{n=1}^{r} (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1} \xi^{2r+1} F_c(\xi).
\end{array}$$

SN 28

$$\begin{array}{ll}
3. \quad \text{(i)} & \int_0^\infty F_s(\xi) G_s(\xi) \cos(\xi x) \, d\xi = \frac{1}{2} \int_0^\infty g(s)[f(s+x) + f(s-x)] \, ds, \\
\text{(ii)} & \int_0^\infty F_c(\xi) G_c(\xi) \, \cos p(\xi x) \, d\xi = \frac{1}{2} \int_0^\infty g(s)[f(s+x) + f(|x-s|)] \, ds \\
& \qquad \text{(convolution (Faltung) theorem)}.
\end{array}$$

SN 24

4. (i) If $F_s(\xi)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $F_s(x)$ is $f(\xi)$.

(ii) If $F_c(\xi)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $F_c(x)$ is $f(\xi)$.

(iii) If $f(x)$ is an odd function in $(-\infty, \infty)$, then the Fourier sine transform of $f(x)$ in $(0, \infty)$ is $-iF(\xi)$.

(iv) If $f(x)$ is an even function in $(-\infty, \infty)$, then the Fourier cosine transform of $f(x)$ in $(0, \infty)$ is $F(\xi)$.

(v) The Fourier sine transform of $f(x/a)$ is $aF_s(a\xi)$.

(vi) The Fourier cosine transform of $f(x/a)$ is $aF_c(a\xi)$.

(vii) $\mathcal{F}_s[f(x);\, \xi] = F_s(|\xi|)\ \mathrm{sgn}\ \xi\,.$

SU 45

17.33 Table of Fourier sine transforms

	$f(x)$	$F_s(\xi)$	
1.	x^{-1}	$(\pi/2)^{\frac{1}{2}}, \quad \xi > 0$	ET I 64(3)
2.	$x^{-\nu}, \quad 0 < \mathrm{Re}\,\nu < 2$	$(2/\pi)^{\frac{1}{2}}\xi^{\nu-1}\Gamma(1-\nu)\cos(\nu\pi/2), \quad \xi > 0$	ET I 68(1)
3.	$x^{-\frac{1}{2}}$	$\xi^{-\frac{1}{2}}, \quad \xi > 0$	ET I 64(6)
4.	$x^{-\frac{3}{2}}$	$2\xi^{\frac{1}{2}}, \quad \xi > 0$	ET I 64(9)
5.	$1, \quad 0 < x < a$ $0, \quad x > a$	$(2/\pi)^{\frac{1}{2}}\xi^{-1}[1-\cos(a\xi)], \quad \xi > 0$	ET I 63(1)
6.	$x^{-1}, \quad 0 < x < a$ $0, \quad x > a$	$(2/\pi)^{\frac{1}{2}}\ \mathrm{Si}\,(a\xi), \quad \xi > 0$	ET I 64(4)
7.	$(a-x)^{-1}, \quad a > 0$	$(2/\pi)^{\frac{1}{2}}\left\{\sin(a\xi)\,\mathrm{Ci}(a\xi) - \cos(a\xi)\left[\tfrac{1}{2}\pi + \mathrm{Si}(a\xi)\right]\right\}, \quad \xi > 0$	ET I 64(11)
8.[7]	$(x^2+a^2)^{-1}, \quad a > 0$	$(2\pi)^{-\frac{1}{2}}a^{-1}[e^{-a\xi}\ \mathrm{Ei}(a\xi) - e^{a\xi}\ \mathrm{Ei}(-a\xi)], \quad \xi > 0$	ET I 65(14)
9.	$x(x^2+a^2)^{-\frac{3}{2}}, \quad \mathrm{Re}\,a > 0$	$(2/\pi)^{\frac{1}{2}}\xi K_0(a\xi), \quad \xi > 0$	ET I 66(27)
10.	$x^{-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}, \quad \mathrm{Re}\,a > 0$	$\xi^{\frac{1}{2}}I_{\frac{1}{4}}\left(\tfrac{1}{2}a\xi\right)K_{\frac{1}{4}}\left(\tfrac{1}{2}a\xi\right), \quad \xi > 0$	ET I 66(28)
11.[7]	$x(x^2+a^2)^{-\nu-\frac{3}{2}}$ $\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,a > 0$	$\frac{\xi^{\nu+1}}{\sqrt{2}(2a)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}K_\nu(a\xi)$	
12.	$\frac{x}{a^2+x^2}, \quad \mathrm{Re}\,a > 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\ e^{-a\xi}, \quad \xi > 0$	ET I 65(15)
13.	$\frac{x}{(a^2+x^2)^2}$	$\sqrt{\pi/8}a^{-1}\xi e^{-a\xi}, \quad \xi > 0$	
14.	$x^{-1}(x^2+a^2)^{-1}, \quad \mathrm{Re}\,a > 0$	$(\pi/2)^{\frac{1}{2}}a^{-2}(1-e^{-a\xi}), \quad \xi > 0$	ET I 65(20)
15.	$x^{-1}e^{-ax}, \quad \mathrm{Re}\,a > 0$	$(2/\pi)^{\frac{1}{2}}\ \tan^{-1}\left(\frac{\xi}{a}\right), \quad \xi > 0$	ET I 72(2)
16.	$x^{\nu-1}e^{-ax}, \quad \mathrm{Re}\,\nu > -1,$	$(2/\pi)^{\frac{1}{2}}\Gamma(\nu)(a^2+\xi^2)^{-\nu/2}\sin$	

17.33 (continued)

	$f(x)$	$F_s(\xi)$	
17.	$e^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}}\xi/(a^2+\xi^2), \quad \xi > 0$	ET I 72(1)
18.	$xe^{-ax}, \quad \operatorname{Re} a > 0$	$\frac{(2/\pi)^{\frac{1}{2}} 2a\xi}{(a^2+\xi^2)^2}, \quad \xi > 0$	ET I 72(3)
19.	$xe^{-ax^2}, \quad \lvert\arg a\rvert < \pi/2$	$(2a)^{-\frac{3}{2}}\xi \exp(-\xi^2/4a), \quad \xi > 0$	ET I 73(19)
20.	$\frac{\sin pax}{x}, \quad a > 0$	$\frac{1}{(2\pi)^{\frac{1}{2}}} \log\left\lvert\frac{\xi+a}{\xi-a}\right\rvert, \quad \xi > 0$	ET I 78(1)
21.	$\frac{\sin pax}{x^2}, a > 0$	$\xi\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$ for $0 < \xi < a$ $a\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$ for $a < \xi < \infty, \quad \xi > 0$	ET I 78(2)
22.	$\sin\left(\frac{a^2}{x}\right), \quad a > 0$	$a\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi^{-\frac{1}{2}} J_1(2a\xi\frac{1}{2}), \quad \xi > 0$	ET I 83(6)
23.	$x^{-1} \sin\left(\frac{a^2}{x}\right), \quad a > 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} N_0(2a\xi^{\frac{1}{2}}) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_0(2a\xi^{\frac{1}{2}})$	ET I 83(7)
24.	$x^{-2} \sin\left(\frac{a^2}{x}\right), \quad a > 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} a^{-1}\xi^{\frac{1}{2}} J_1(2a\xi^{\frac{1}{2}}), \quad \xi > 0$	ET I 83(8)
25.	$\operatorname{cosec}(ax), \quad \operatorname{Re} a > 0$	$(\pi/2)^{\frac{1}{2}} a^{-1} \tanh\left(\frac{1}{2}\pi a^{-1}\xi\right), \quad \xi > 0$	ET I 88(2)
26.	$\operatorname{ctnh}\left(\frac{1}{2}ax\right) - 1,$ $\operatorname{Re} a > 0$	$(2\pi)^{\frac{1}{2}} a^{-1} \operatorname{ctnh}(\pi a^{-1}\xi) - \xi, \quad \xi > 0$	ET I 88(3)
27.	$(1-x^2)^{-1} \sin p(\pi x)$	$(2/\pi)^{\frac{1}{2}} \sin p\xi, \quad 0 \le \xi \le \pi$ $0, \quad \pi < \xi$	ET I 78(4)
28.	$e^{-ax^2} \sin p(bx), \quad \operatorname{Re} a > 0$	$(2a)^{-\frac{1}{2}} \exp p[-(\xi^2+b^2)/(4a)] \sinh(b\xi/2a),$ $\xi > 0$	ET I 78(7)
29.	$x^{-1} \sin^2(ax), \quad a > 0$	$\pi^{\frac{1}{2}} 2^{-\frac{3}{2}}, \quad 0 < \xi < 2a$ $\pi^{\frac{1}{2}} 2^{-\frac{5}{2}}, \quad \xi = 2a$ $0, \quad 2a < \xi$	ET I 78(8)
30.	$\sin(ax^2), \quad a > 0$	$a^{-\frac{1}{2}}\{\cos(\xi^2/4a)C[(2\pi a)^{-\frac{1}{2}}\xi] +$ $+\sin(\xi^2/4a)S[(2\pi a)^{-\frac{1}{2}}\xi]\}, \quad \xi > 0$	ET I 82(1)

ET I 72(1)
ET I 72(3)
ET I 73(19)
ET I 78(1)
ET I 78(2)
ET I 83(6)
ET I 83(7)
ET I 83(8)
ET I 88(2)
ET I 88(3)
ET I 78(4)
ET I 78(7)
ET I 78(8)
ET I 82(1)

17.33 (continued)

	$f(x)$	$F_s(\xi)$	
31.	$\cos(ax^2), \quad a > 0$	$a^{-\frac{1}{2}}\{\sin(\xi^2/4a)C[(2\pi a)^{-\frac{1}{2}}\xi] -$ $-\cos(\xi^2/4a)S[(2\pi a)^{-\frac{1}{2}}\xi]\}, \quad \xi > 0$	ET I 83(3)
32.	$\operatorname{arctg}(x/a), \quad a > 0$	$(\pi/2)^{\frac{1}{2}}\xi^{-1}e^{-a\xi}, \quad \xi > 0$	ET I 87(3)
33.[7]	$\operatorname{arctg}(2a/x), \quad \operatorname{Re} a > 0$	$(2\pi)^{\frac{1}{2}}e^{-a\xi} \sinh(a\xi)/\xi, \quad \xi > 0$	ET I 87(8)
34.	$x^{-1} \ln px$	$-(\pi/2)^{\frac{1}{2}}(C + \ln p\xi), \quad \xi > 0$	ET I 76(2)
35.	$\ln\left\lvert\frac{x+a}{x-a}\right\rvert, \quad a > 0$	$(2\pi)^{\frac{1}{2}}\xi^{-1} \sin(a\xi), \quad \xi > 0$	ET I 77(11)
36.[7]	$x^{-1} \ln(1+a^2x^2), \quad a > 0$	$-(2\pi)^{\frac{1}{2}} \operatorname{Ei}(-\xi/a), \quad \xi > 0$	ET I 77(14)
37.	$J_0(ax), \quad a > 0$	$0, \quad 0 < \xi < a$	

ET I 99(3)
ET I 99(4)
ET I 100(12)
ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_s has not been used in those tables.

17.34 Table of Fourier cosine transforms

	$f(x)$	$F_c(\xi)$	
1.	$x^{-\nu}, \quad 0 < \operatorname{Re}\nu < 1$	$(\pi/2)^{\frac{1}{2}}[\Gamma(\nu)]^{-1} \sec\left(\frac{1}{2}\nu\pi\right)\xi^{\nu-1}, \quad \xi > 0$	ET I 10(1)
2.	$1, \quad 0 < x < a$ $0, \quad x > a$	$(2/\pi)^{\frac{1}{2}}\frac{\sin(a\xi)}{\xi}, \quad \xi > 0$	ET I 7(1)
3.	$0, \quad 0 < x < a$ $1/x, \quad x > a$	$-(2/\pi)^{\frac{1}{2}} \operatorname{Ci}(a\xi), \quad \xi > 0$	ET I 8(3)
4.	$x^{-\frac{1}{2}}, \quad 0 < x < a$ $0, \quad x > a$	$2\xi^{-\frac{1}{2}} C(a\xi), \quad \xi > 0$	ET I 8(5)
5.	$0, \quad 0 < x < a$ $x^{-\frac{1}{2}}, \quad x > a$	$2\xi^{-\frac{1}{2}}\left[\frac{1}{2} - C(a\xi)\right], \quad \xi > 0$	ET I 8(6)
6.	$x^{\nu-1}, \quad 0 < \operatorname{Re}\nu < 1$	$(2/\pi)^{\frac{1}{2}}\Gamma(\nu)\xi^{-\nu} \sin\left(\frac{1}{2}\nu\pi\right), \quad \xi > 0$	ET I 10(1)
7.	$(x^2 + a^2)^{-1}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} e^{-a\xi}}{a}, \quad \xi > 0$	ET I 8(11)
8.	$(x^2 + a^2)^{-2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{\frac{1}{2}}(1+a\xi)e^{-a\xi}}{2a^3}, \quad \xi > 0$	
9.	$(x^2 + a^2)^{-\nu-\frac{1}{2}}, \quad \operatorname{Re} a > 0,$ $\operatorname{Re}\nu > -\frac{1}{2}$	$\sqrt{2}\left(\frac{\xi}{2a}\right)^{\nu} \frac{K_\nu(a\xi)}{\Gamma(\nu+\frac{1}{2})}, \quad \xi > 0$	ET I 11(7)
10.	$(a^2 - x^2)^{\nu}, \quad 0 < x < a$ $0, \quad x > a$ $\operatorname{Re}\nu > -1$	$2^{\nu}\Gamma(\nu+1)(a/\xi)^{\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}(a\xi), \quad \xi > 0$	ET I 11(8)
11.	$0, \quad 0 < x < a$ $(x^2 - a^2)^{-\nu-\frac{1}{2}}, \quad x > a,$ $-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$	$-2^{-(\nu+\frac{1}{2})}\Gamma\left(\frac{1}{2} - \nu\right)(\xi/a)^{\nu} Y_\nu(a\xi), \quad \xi > 0$	ET I 11(9)
12.	$e^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}} a(a^2 + \xi^2)^{-1}, \quad \xi > 0$	ET I 14(1)

ET I 10(1)
ET I 7(1)
ET I 8(3)
ET I 8(5)
ET I 8(6)
ET I 10(1)
ET I 8(11)
ET I 11(7)

ET I 11(8)
ET I 11(9)
ET I 14(1)

17.34 (continued)

	$f(x)$	$F_c(\xi)$	
13.	$xe^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}}(a^2-\xi^2)(a^2+\xi^2)^{-2}, \quad \xi > 0$	ET I 14(5)
14.[7]	$x^{\nu-1}e^{-ax}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 0$	$(2/\pi)^{\frac{1}{2}}\Gamma(\nu)(a^2+\xi^2)^{-\nu/2} \cos\left[\nu \tan^{-1}\left(\frac{\xi}{a}\right)\right], \quad \xi > 0$	ET I 15(7)
15.	$x^{-\frac{1}{2}}e^{-ax}, \quad \operatorname{Re} a > 0$	$(a^2+\xi^2)^{-\frac{1}{2}}[(a^2+\xi^2)^{\frac{1}{2}}+a]^{\frac{1}{2}}, \quad \xi > 0$	ET I 14(4)
16.[7]	$e^{-a^2x^2}, \quad \operatorname{Re} a > 0$	$2^{-\frac{1}{2}}a^{-1}e^{-\xi^2/4a^2}, \quad \xi > 0$	ET I 15(11)
17.	$x^{-1}e^{-x} \sin px$	$(2\pi)^{-\frac{1}{2}} \tan^{-1}\left(\frac{2}{\xi^2}\right), \quad \xi > 0$	ET I 19(7)
18.	$\sin(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right) - \sin\left(\frac{\xi^2}{4a}\right)\right], \quad \xi > 0$	ET I 23(1)
19.	$\cos(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right) + \sin\left(\frac{\xi^2}{4a}\right)\right], \quad \xi > 0$	ET I 24(7)
20.	$x^{-1} \sin(ax), \quad a > 0$	$(\pi/2)^{\frac{1}{2}}, \quad \xi < a$ $\frac{1}{2}(\pi/2)^{\frac{1}{2}}, \quad \xi = a$ $0, \quad \xi > a$	ET I 18(1)
21.[7]	$x^{-2} \sin^2(ax), \quad a > 0$	$(\pi/2)^{\frac{1}{2}}\left(a - \frac{1}{2}\xi\right), \quad \xi < 2a$ $0, \quad 2a < \xi$	ET I 19(8)
22.[7]	$e^{-bx} \sin(ax), \quad a > 0, \quad \operatorname{Re} b > 0$	$(2\pi)^{-\frac{1}{2}}\left[\frac{a+\xi}{b^2+(a+\xi)^2} + \frac{a-\xi}{b^2+(a-\xi)^2}\right], \quad \xi > 0$	ET I 19(6)
23.	$(x^2+a^2)^{-2} \sin[b(x^2+a^2)^{\frac{1}{2}}], a > 0$	$(b/a)(\pi/2)^{\frac{1}{2}}e^{-a\xi}, \quad \xi > 0$	ET I 26(29)
24.	$(x^2+a^2)^{-\frac{1}{2}} \sin[b(x^2+a^2)^{\frac{1}{2}}], a > 0$	$(\pi/2)^{\frac{1}{2}}J_0[a(b^2-\xi^2)^{\frac{1}{2}}], \quad 0 < \xi < b$ $0, \quad b < \xi$	ET I 26(30)

ET I 14(5)
ET I 15(7)
ET I 14(4)
ET I 15(11)
ET I 19(7)
ET I 23(1)
ET I 24(7)
ET I 18(1)
ET I 19(8)
ET I 19(6)
ET I 26(29)

17.34 (continued)

	$f(x)$	$F_c(\xi)$	
25.	$x^{-2}[1-\cos(ax)], \quad a > 0$	$\pi/2)^{\frac{1}{2}}(a-\xi), \quad \xi < a$ $0, \quad a < \xi$	ET I 20(16)
26.	$e^{-ax^2} \sin(bx^2), \quad \operatorname{Re} a > \lvert \operatorname{Im} b \rvert$	$2^{-\frac{1}{2}}(a^2+b^2)^{-\frac{1}{4}} \exp\{-a\xi^2/[4(a^2+b^2)]\} \times$ $\times \sin\left[\frac{1}{2} \operatorname{arctg}(b/a) - \frac{1}{4}b\xi^2(a^2+b^2)^{-1}\right]$	

ET I 23(5)
ET I 24(6)
ET I 31(14)
ET I 31(12)
ET I 30(1)
ET I 32(19)
ET I 18(10)
ET I 18(12)
ET I 45(14)
ET I 45(15)

1193

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_c has not been used in those tables.

17.35 Relationships between transforms

The following relationships exist between transforms and they may be used to derive further transform pairs from among the results given in Sections 17.13-17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

17.351

Fourier cosine transform and Laplace transform relationship.

$$\mathcal{F}_c[f(x);\,\xi] = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f(x);\,i\xi] + \frac{1}{\sqrt{2\pi}}\mathcal{L}[f(x);\,-i\xi].$$

17.352

Fourier sine transform and Laplace transform relationship.

$$\mathcal{F}_s[f(x);\,\xi] = \frac{i}{\sqrt{2\pi}}\mathcal{L}[f(x);\,i\xi] - \frac{i}{\sqrt{2\pi}}\mathcal{L}[f(x);\,-i\xi].$$

17.353

Exponential Fourier transform and Laplace transform relationship.

$$\mathcal{F}[f(x);\,\xi] = \sqrt{2\pi}\mathcal{L}[f(x);\,-i\xi] + \sqrt{2\pi}\mathcal{L}[f(-x);\,i\xi].$$

17.41 Mellin transform

The **Mellin transform** of the function $f(x)$, denoted by $f^*(s)$, is defined by the integral

$$f^*(s) = \int_0^\infty f(x)x^{s-1}\,dx.$$

The functions $f(x)$ and $f^*(s)$ are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

1194

The transform exists provided the integral

$$\int_0^\infty |f(x)|x^{k-1}\,dx$$

is bounded for some $k > 0$, and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}\,ds,$$

where $c > k$.

Setting

$$f^*(s) = M[f(x);\, s]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = M^{-1}[f^*(s);\, x].$$

MS 397(6)

17.42 Basic properties of the Mellin transform

1. For a and b arbitrary constants,

$$\mathcal{M}[af(x)+bg(x)] = af^*(s)+bg^*(s) \qquad \text{(linearity)}.$$

2. If $\lim_{x\to 0}\; x^{s-r-1}f^{(r)}(x)=0, \quad r=0,\,1,\ldots\,,n-1\,$,

$$\text{(i)}\quad \mathcal{M}[f^{(n)}(x);\,s] = (-1)^n\frac{\Gamma(s)}{\Gamma(s-n)}f^*(s-n) \qquad \text{(transform of a derivative)}.$$

$$\text{(ii)}\quad \mathcal{M}[x^nf^{(n)}(x);\,s] = (-1)^n\frac{\Gamma(s+n)}{\Gamma(s)}f^*(s) \qquad \text{(transform of a derivative)}.$$

SU 267(4.2.5)
SU 267(4.2.3)

3. Denoting the nth repeated integral of $f(x)$ by $I_n\;f(x)$, where

$$I_nf(x) = \int_0^x I_{n-1}f(u)\,du,$$

$$\text{(i)}\quad \mathcal{M}[I_nf(x);\,s] = (-1)^n\frac{\Gamma(s)}{\Gamma(n+s)}f^*(s+n) \qquad \text{(transform of an integral)}.$$

$$\text{(ii)}\quad \mathcal{M}[I_n^\infty f(x);\,s] = \frac{\Gamma(s)}{\Gamma(s+n)}f^*(s+n),$$

SU 269(4.2.15)

where

$$I_n^\infty f(x) = \int_x^\infty I_{n-1}^\infty f(u)\,du \qquad \text{(transform of an integral)}.$$

SU 269(4.2.18)

$$4.\quad \mathcal{M}[f(x)g(x);\,s] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(u)g^*(s-u)\,du \qquad \text{(Mellin convolution theorem)}.$$

SU 275(4.4.1)

17.43 Table of Mellin transform pairs

	$f(x)$	$F^*(\xi)$	
1.	e^{-x}	$\Gamma(s), \quad \operatorname{Re} s > 0$	SU 521(M13)
2.	e^{-x^2}	$\frac{1}{2}\Gamma\left(\frac{1}{2}\, s\right), \quad \operatorname{Re} s > 0$	SU 521(M14)
3.	$\cos px$	$\Gamma(s)\cos\left(\frac{1}{2}\,\pi s\right), \quad 0 < \operatorname{Re} s < 1$	SU 521(M15)
4.	$\sin px$	$\Gamma(s)\sin\left(\frac{1}{2}\,\pi s\right), \quad 0 < \operatorname{Re} s < 1$	SU 521(M16)
5.	$(1-x)^{-1}$	$\pi\,\cot(\pi s), \quad 0 < \operatorname{Re} s < 1$	SU 521(M1)
6.	$(1+x)^{-1}$	$\pi\,\operatorname{cosec}(\pi s), \quad 0 < \operatorname{Re} s < 1$	SU 521(M2)
7.	$(1+x^a)^{-b}$	$\frac{\Gamma(s/a)\Gamma(b-s/a)}{a\Gamma(b)}, \quad 0 < \operatorname{Re} s < ab$	SU 521(M3)
8.	$\frac{T_n(x)H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{-s}\pi\Gamma(s)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\,s+\frac{1}{2}\,n\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\,s-\frac{1}{2}\,n\right)},$ $\operatorname{Re} s > 0$	SU 521(M4)
9.	$\frac{T_n(x^{-1})H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{s-2}\Gamma\left(\frac{1}{2}\,n+\frac{1}{2}\,s\right)\Gamma\left(\frac{1}{2}\,s-\frac{1}{2}\,n\right)}{\Gamma(s)},$ $\operatorname{Re} s > n$	SU 521(M5)
10.	$P_n(x)H(1-x)$	$\frac{\Gamma\left(\frac{1}{2}\,s\right)\Gamma\left(\frac{1}{2}\,s+\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}\,s-\frac{1}{2}\,n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s+\frac{1}{2}\,n+1\right)},$ $\operatorname{Re} s > 0$	SU 521(M6)
11.	$P_n(x^{-1})H(1-x)$	$\frac{2^{s-1}\Gamma\left(\frac{1}{2}\,s+\frac{1}{2}\,n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s-\frac{1}{2}\,n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(s+1)},$ $\operatorname{Re} s > n$	SU 521(M7)

SU 521(M13)
SU 521(M14)
SU 521(M15)
SU 521(M16)
SU 521(M1)
SU 521(M2)
SU 521(M3)
SU 521(M4)
SU 521(M5)
SU 521(M6)
SU 521(M7)

17.43 (continued)

	$f(x)$	$F^*(\xi)$	
12.	$\frac{1+x\,\cos p\phi}{1-2x\,\cos p\phi+x^2}$ $-\pi < \phi < \pi$	$\frac{\pi\,\cos(s\phi)}{\sin(s\pi)}, \quad 0 < \operatorname{Re} s < 1$	SU 521(M11)
13.	$\frac{x\,\sin p\phi}{1-2x\,\cos p\phi+x^2},$	$\frac{\pi\,\sin(s\phi)}{\sin(s\pi)}, \quad 0 < \operatorname{Re} s < 1$	SU 521(M12)

SU 521(M12)
SU 522(M17)
SU 522(M18)
SU 522(M19)
SU 522(M20)
SU 522(M21)
SU 522(M22)
MS 453

17.43 (continued)

$f(x)$	$F^*(\xi)$	
21. $(1+ax^h)^{-\nu}$	$h^{-1}a^{-s/h}\mathrm{B}(s/h, \quad \nu-(s/h))$,	MS 454
$h>0, \quad \lvert\arg a\rvert<\pi$	$0<\operatorname{Re} s<h \operatorname{Re}\nu$	
22. $(1-x^h)^{\nu-1}$		
for $\quad 0<x<1$		
for $\quad x>1$		
$h>0, \quad \operatorname{Re}\nu>0$	$h^{-1}\mathrm{B}(\nu,\, s/h)$	MS 454
23. $\ln(1+ax)$		
$\lvert\arg a\rvert<\pi$	$\pi s^{-1}a^{-s} \operatorname{cosec}(\pi s), \quad -1<\operatorname{Re} s<0$	MS 454
24. $\operatorname{arctg} px$	$-\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad -1<\operatorname{Re} s<0$	MS 454
25. $\operatorname{arccot} x$	$\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad 0<\operatorname{Re} s<1$	MS 454
26. $\operatorname{csch}(ax)$		
$\operatorname{Re} a>0$	$a^{-s}2(1-2^{-s})\Gamma(s)\zeta(s), \quad \operatorname{Re} s>1$	MS 454
27.[7] $\operatorname{sech}^2(ax)$		
$\operatorname{Re} a>0$	$4a^{-s}(1-2^{2-s})\Gamma(s)2^{-s}\zeta(s-1), \quad \operatorname{Re} s>2$	MS 454
28. $\operatorname{csch}^2(ax)$		
$\operatorname{Re} a>0$	$4a^{-s}\Gamma(s)2^{-s}\zeta(s-1), \quad \operatorname{Re} s>0$	MS 454
29. $(x^2+b^2)^{-\frac{1}{2}\nu}\times$		
$\times J_\nu[a(x^2+b^2)^{\frac{1}{2}}]$	$2^{\frac{1}{2}s-1}a^{-\frac{1}{2}s}b^{\frac{1}{2}s-\nu}\Gamma\left(\frac{1}{2}s\right)J_{\nu-\frac{1}{2}}(ab)$	
	$0<\operatorname{Re} s<\frac{3}{2}+\operatorname{Re}\nu$	MS 454
30. $(a^2-x^2)^{\frac{1}{2}\nu}\times$		
$\times J_\nu[a(b^2-x^2)^{\frac{1}{2}}]$	$2^{\frac{1}{2}s-1}\Gamma\left(\frac{1}{2}s\right)b^{-\frac{1}{2}s}a^{\nu+\frac{1}{2}s}J_{\nu+\frac{1}{2}s}(ab)$	MS 455
$0<x<a$	$\operatorname{Re} s>0$	
$0, \quad x>a$		
$\operatorname{Re}\nu>-1$		

MS 454
MS 455

17.43 (continued)

$f(x)$	$F^*(\xi)$	
31. $(a^2-x^2)^{-\frac{1}{2}\nu}\times$		
$\times J_\nu[b(a^2-x^2)^{\frac{1}{2}}]$		
$0<x<a$	$2^{1-\nu}[\Gamma(\nu)]^{-1}a^{\frac{1}{2}s-\nu}b^{-\frac{1}{2}\nu}s_{\nu-1+\frac{1}{2}s,\,\frac{1}{2}s-\nu}(ab)$	MS 455
$0 \qquad x>a$	$\operatorname{Re} s>0$	
32. $K_\nu(\alpha x)$	$\alpha^{-s}2^{s-2}\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)$,	
	$\operatorname{Re} s>\lvert\operatorname{Re}\nu\rvert$	MS 455

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The letters and numbers following equations [visible with the **refs** stylesheet selected from the View Main] refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography below. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources, and numbers in double parentheses denote the number of a table in the source.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example we may use the reference to equation 3.354-5:

$$\text{ET I 118 (1)}\,a.$$

The key below indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms.*

The Roman numerals denotes volume one of the work, 118 is the page on which the formula will be found, (1) refers to the number of the formula in this source, and the a indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases the editors have used Russian editions of works published in other languages, under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

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