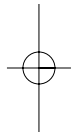
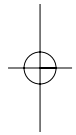
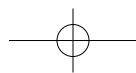


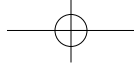
# A handbook of essential mathematical formulae



Alan Davies and Diane Crann

**University of Hertfordshire Press**



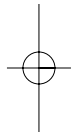
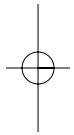


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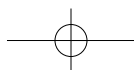
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# Contents

<b>1</b>	<b>Algebra, Trigonometry and Geometry</b>	<b>1</b>
1.1	Indices and logarithms . . . . .	1
1.2	Factors and roots of equations . . . . .	2
1.3	Partial fractions . . . . .	2
1.4	Permutations and combinations . . . . .	3
1.5	Trigonometric functions . . . . .	3
1.6	Hyperbolic functions . . . . .	5
1.7	Geometry . . . . .	5
1.8	Conic sections . . . . .	6
1.9	Mensuration . . . . .	7
1.10	Complex numbers . . . . .	7
1.11	Inequalities . . . . .	9
<b>2</b>	<b>Determinants and Matrices</b>	<b>10</b>
2.1	Determinants . . . . .	10
2.2	Matrices . . . . .	11
2.3	Systems of equations . . . . .	12
2.4	Eigenvalues and eigenvectors . . . . .	13
<b>3</b>	<b>Vector Algebra</b>	<b>14</b>
3.1	Vector addition . . . . .	14
3.2	Vector products . . . . .	14
3.3	Polar coordinates in two and three dimensions . . . . .	15
<b>4</b>	<b>Calculus</b>	<b>17</b>
4.1	Rules for manipulation of derivatives and integrals . . . . .	17
4.2	Standard derivatives and integrals . . . . .	19
4.3	Definite integrals . . . . .	21
4.4	Radius of curvature of a curve . . . . .	23
4.5	Stationary points . . . . .	23
4.6	Limits and series . . . . .	24
4.7	Multiple integration . . . . .	25
4.8	Applications of integration . . . . .	27

4.9	Kinematics and dynamics . . . . .	31
<b>5</b>	<b>Ordinary Differential Equations</b>	<b>32</b>
5.1	First order equations . . . . .	32
5.2	Second order equations . . . . .	33
5.3	Higher order equations . . . . .	34
<b>6</b>	<b>Fourier Series</b>	<b>35</b>
<b>7</b>	<b>Vector Calculus</b>	<b>37</b>
7.1	grad, div and curl . . . . .	37
7.2	Integral theorems of the vector calculus . . . . .	40
<b>8</b>	<b>Tables of Transforms</b>	<b>41</b>
8.1	The Laplace transform . . . . .	41
8.2	The Z-transform . . . . .	45
8.3	The Fourier transform . . . . .	46
8.4	Fourier sine and cosine transforms . . . . .	46
8.5	Some periodic functions: Laplace transforms and Fourier series	47
<b>9</b>	<b>Numerical Methods</b>	<b>50</b>
9.1	Interpolation . . . . .	50
9.2	Finite difference operators . . . . .	51
9.3	Non-linear algebraic equations . . . . .	52
9.4	Numerical integration . . . . .	52
9.5	Numerical solution of ordinary differential equations . . . . .	54
9.6	Systems of linear equations, $n \times n$ . . . . .	55
9.7	Chebyshev polynomials . . . . .	57
9.8	Numerical eigenvalues and eigenvectors . . . . .	57
9.9	Least squares approximation . . . . .	59
<b>10</b>	<b>Statistics</b>	<b>60</b>
10.1	Sample statistics . . . . .	60
10.2	Regression and correlation . . . . .	61
10.3	Distributions . . . . .	61
<b>11</b>	<b>S. I. Units (Système International d'Unités)</b>	<b>64</b>
11.1	Fundamental units . . . . .	64
11.2	S. I. Prefixes and multiplication factors . . . . .	65
11.3	Basic and derived units . . . . .	65
11.4	Values of some physical constants . . . . .	67
11.5	Useful masses . . . . .	67
11.6	Astronomical constants . . . . .	69
11.7	Mathematical constants . . . . .	69
11.8	The Greek alphabet . . . . .	70

Throughout the handbook the symbol  $j$  is used for the unit imaginary number *i.e.*  $j^2 = -1$  or  $j = \sqrt{-1}$ . Alternatively the symbol  $i$  is frequently used instead of  $j$ .



# Chapter 1

## Algebra, Trigonometry and Geometry

A *polynomial* of degree  $n$  is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x^1 + a_0$$

*Sigma*  $\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$

*Factorial*  $n! = n(n-1)(n-2)\dots$  3.2.1

*Finite series*

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^n k^3 = \left(\sum_{k=1}^n k\right)^2 = \frac{1}{4}n^2(n+1)^2$$

### 1.1 Indices and logarithms

If  $x > 0$  then we have the following properties:

$$x^0 = 1, \quad x^{-1} = 1/x, \quad x^{m+n} = x^m x^n, \quad x^{m-n} = x^m / x^n, \quad x^{mn} = (x^m)^n \quad x^{\frac{1}{n}} = \sqrt[n]{x}$$

If  $a > 0$  and  $a \neq 1$  then

$$\log_a 1 = 0, \quad \log_a(x^{-1}) = \log_a(1/x) = -\log_a x$$

$$\log_a x + \log_a y = \log_a(xy), \quad m \log_a x = \log_a(x^m)$$

$$\log_a x - \log_a y = \log_a(x/y)$$

The logarithm and power functions are *inverse functions*, i.e.

$$\text{if } x = \log_a y \text{ then } y = a^x$$

$$\text{and if } x = a^y \text{ then } y = \log_a x$$

*Change of base*  $\log_a x = \log_b x / \log_b a$

Logarithms to base  $e$ , i.e.  $\log_e x$ , are often written  $\ln x$ . Such logarithms are called *natural logarithms*.  $e$  is the exponential constant given by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = 2.71828\dots$$

## 1.2 Factors and roots of equations

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1})$$

*Quadratic equation*  $ax^2 + bx + c = 0$  with roots  $\alpha, \beta$

$$\alpha, \beta = \left[ -b \pm \sqrt{b^2 - 4ac} \right] / 2a$$

$$\alpha + \beta = -b/a, \quad \alpha\beta = c/a$$

*Cubic equation*  $ax^3 + bx^2 + cx + d = 0$  with roots  $\alpha, \beta, \gamma$

$$\alpha + \beta + \gamma = -b/a, \quad \alpha\beta + \beta\gamma + \gamma\alpha = c/a, \quad \alpha\beta\gamma = -d/a$$

$f(a) = 0$  if and only if  $(x - a)$  is a factor of  $f(x)$ .

### Remainder Theorem

Suppose  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a polynomial of degree  $n$  and that  $a$  is a root of the equation  $P(x) = 0$ . Then  $(x - a)$  is a factor of  $P(x)$ .

## 1.3 Partial fractions

If the degree of the polynomial  $f(x)$  is less than that of the denominator then:

$$\begin{aligned} \frac{f(x)}{(x+a)(x+b)\dots} &= \frac{A}{x+a} + \frac{B}{x+b} + \dots \\ \frac{f(x)}{(ax^2+bx+c)(dx+e)} &= \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e} \\ \frac{f(x)}{(ax+b)(cx+d)^2} &= \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} \end{aligned}$$



## 1.4 Permutations and combinations

The number of ways of selecting  $r$  objects from  $n$  objects with due regard to order is

$${}^n P_r = \frac{n!}{(n-r)!}$$

The number of ways of selecting  $r$  objects from  $n$  objects without regard to order is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \text{ (sometimes written } {}^n C_r \text{)}$$

### Properties

$$\binom{n}{r} = \binom{n}{n-r}, \quad \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

### Binomial Theorem

For any positive integer,  $n$ ,

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$$

## 1.5 Trigonometric functions

### Degrees and Radians

$$360^\circ = 2\pi \text{ rad} \quad 1^\circ = \frac{\pi}{180} \text{ rad} \quad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.296^\circ$$

$$\cos n\pi = (-1)^n, \quad \sin n\pi = 0$$

$$\cos [(2n+1)\pi/2] = 0, \quad \sin [(2n+1)\pi/2] = (-1)^n$$

$$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$\cos(\pi/3) = \sin(\pi/6) = 1/2$$

$$\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$$

$$\cos(\pi/2 - \theta) = \sin \theta, \quad \sin(\pi/2 - \theta) = \cos \theta$$

### Trigonometric identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x, \quad \tan^{-1} x = \pi/2 - \tan^{-1}(1/x)$$

$$\sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\end{aligned}$$

$$\begin{aligned}\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ \tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y}\end{aligned}$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \\ \sin 2x &= 2 \sin x \cos x, \quad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}\end{aligned}$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x, \quad \sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\begin{aligned}\sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \\ \sin x - \sin y &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}, \\ \cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \\ \cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}\end{aligned}$$

$$\begin{aligned}\cos x \cos y &= \frac{1}{2} [\cos(x+y) + \cos(x-y)] \\ \sin x \sin y &= \frac{1}{2} [\cos(x-y) - \cos(x+y)] \\ \sin x \cos y &= \frac{1}{2} [\sin(x+y) + \sin(x-y)]\end{aligned}$$

$$\cos^2 x = \frac{1}{2} (1 + \cos 2x), \quad \sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\begin{aligned}\sin^{-1} x \pm \sin^{-1} y &= \sin^{-1} \left( x \sqrt{1-y^2} \pm y \sqrt{1-x^2} \right) \\ \cos^{-1} x \pm \cos^{-1} y &= \cos^{-1} \left( xy \mp \sqrt{1-x^2} \sqrt{1-y^2} \right) \\ \tan^{-1} x \pm \tan^{-1} y &= \tan^{-1} [(x \pm y) / (1 \mp xy)]\end{aligned}$$

If  $t = \tan \frac{x}{2}$ , then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}$$

$$\cos x = (e^{jx} + e^{-jx}) / 2, \quad \sin x = (e^{jx} - e^{-jx}) / 2j$$

### De Moivre's Theorem

$$\begin{aligned}(\cos x + j \sin x)^n &= \cos nx + j \sin nx \\ e^{jx} &= \cos x + j \sin x, \quad e^{-jx} = \cos x - j \sin x\end{aligned}$$

## 1.6 Hyperbolic functions

$$\cosh x = (e^x + e^{-x})/2, \quad \sinh x = (e^x - e^{-x})/2$$
$$e^x = \cosh x + \sinh x, \quad e^{-x} = \cosh x - \sinh x$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

$$\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right)$$

$$\cosh^{-1} x = \pm \ln \left( x + \sqrt{x^2 - 1} \right)$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$$

### Relationship with trigonometric functions

$$\cos jx = \cosh x, \quad \cosh jx = \cos x$$
$$\sin jx = j \sinh x, \quad \sinh jx = j \sin x$$

### Osborne's rule

An identity involving hyperbolic functions may be obtained from the equivalent trigonometric identity by replacing the trigonometric functions with the corresponding hyperbolic functions and changing the sign of a product (or implied product) of *two* sines.

*e.g.*

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

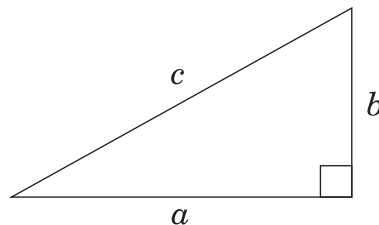
*N.B.* The Maclaurin series in powers of  $x$  for the hyperbolic functions may be found from the Maclaurin series for the corresponding trigonometric function by changing the sign of a product of two  $x$ 's

*e.g.*

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$$

$$\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \dots + (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{(2n+1)} + \dots$$

## 1.7 Geometry



## Pythagoras' theorem

$$a^2 + b^2 = c^2$$

### Theorems

The angle sum of a triangle is  $180^\circ$  or  $\pi$  rad.

The sum of the interior angles of an  $n$ -sided polygon is  $(2n - 4) \times 90^\circ$  or  $(n - 2)\pi$  rad.

The exterior angle of a triangle is equal to the sum of the interior opposite angles.

The angle subtended by a diameter at the circumference of a circle is  $90^\circ$ .

The angle subtended by a chord at the centre of a circle is twice the angle subtended by the same chord in the opposite segment.

The diagonals of a parallelogram bisect each other.

## 1.8 Conic sections

A conic section is the locus of a point that moves in a plane so that the ratio (the eccentricity) of its distance from a fixed point (the focus) in the plane, to its distance from a fixed line (the directrix), is a constant,  $\epsilon$ .

1. Parabola ( $\epsilon = 1$ ); focus at  $(a, 0)$ , directrix  $x = -a$   
Cartesian equation:  $y^2 = 4ax$   
Parametric equation:  $x = at^2, y = 2at$
2. Ellipse ( $\epsilon < 1$ ); foci at  $(\pm a\epsilon, 0)$ , directrices  $x = \pm a/\epsilon$   
Major axis of length  $2a$ , minor axis of length  $2b$   
Cartesian equation:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with  $b = a(1 - \epsilon^2)^{\frac{1}{2}}$   
Parametric equation:  $x = a \cos \theta, y = b \sin \theta$
3. Hyperbola ( $\epsilon > 1$ ); foci at  $(\pm a\epsilon, 0)$ , directrices at  $x = \pm a/\epsilon$   
Cartesian equation:  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  with  $b = a(\epsilon^2 - 1)^{\frac{1}{2}}$   
Parametric equation:  $x = a \sec \theta, y = b \tan \theta$   
Rectangular hyperbola referred to its asymptotes as axes:  $xy = c^2$
4. Circle ( $\epsilon = 0$ )  
Cartesian equation  $x^2 + y^2 = a^2$   
Parametric equation  $x = a \cos \theta, y = a \sin \theta$

The polar equation for these three conic sections with the pole at a focus is

$$\frac{1}{r} = 1 + \epsilon \cos \theta$$

### The general conic

The equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents:

a *circle* if  $a = b$  and  $h = 0$

a *pair of straight lines* if  $af^2 + bg^2 + ch^2 = 2fgh + abc$

a *parabola* if  $h^2 = ab$

an *ellipse* if  $h^2 < ab$

a *hyperbola* if  $h^2 > ab$

a *rectangular hyperbola* if  $a + b = 0$

## 1.9 Mensuration

*Circle*, radius  $r$ : perimeter is  $2\pi r$ , area is  $\pi r^2$

For a segment of angular width  $\theta$  (radians), arc length is  $r\theta$  and area is  $\frac{1}{2}r^2\theta$

*Ellipse*, axes  $2a$  and  $2b$ : perimeter is approximately  $2\pi\sqrt{(a^2 + b^2)/2}$ ,

area is  $\pi ab$

*Cylinder*, radius  $r$ , height  $h$ : surface area is  $2\pi r(h + r)$ , volume is  $\pi r^2 h$

*Cone*, base radius  $r$ , height  $h$ , slant height  $l$ :

curved surface area is  $\pi rl$ , volume is  $\pi r^2 h/3$

*Sphere*, radius  $r$ : area is  $4\pi r^2$ , volume is  $4\pi r^3/3$ ,

area cut off by parallel planes distance  $h$  apart is  $2\pi rh$

*Triangle*,  $ABC$  sides  $a, b, c$ :

area of triangle is

$$\Delta = \frac{1}{2}bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ (Heron's formula)}$$

where  $2s = a + b + c$

Radius of circumcircle is  $R = abc/4\Delta$

Radius of inscribed circle is  $r = \Delta/s$

$$\text{sine rule } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\text{cosine rule } a^2 = b^2 + c^2 - 2bc \cos A$$

$$\sin(A/2) = \sqrt{\{(s-b)(s-c)/bc\}}$$

$$\cos(A/2) = \sqrt{\{(s-a)/bc\}}$$

## 1.10 Complex numbers

The algebra of complex numbers follows the same rules as that for real numbers with the addition of the unit imaginary number  $j$  which has the

property  $j^2 = -1$  or  $j = \sqrt{-1}$ .

*Cartesian form*  $z = x + jy$  where  $x$  is called the *real part* of  $z$ ,  $\text{Re}(z)$ , and  $y$  is called the *imaginary part* of  $z$ ,  $\text{Im}(z)$ .

The *Argand diagram* is a geometric representation of the complex number  $z = x + jy$ , the point  $(x, y)$  represents  $z$ .

*Polar form*  $z = r(\cos \theta + j \sin \theta)$  where  $(r, \theta)$  are the polar coordinates of  $(x, y)$

*Exponential form*  $z = re^{j\theta}$

*Complex conjugate*  $\bar{z} = x - jy = r(\cos \theta - j \sin \theta) = re^{-j\theta}$

*Modulus*  $|z| = \sqrt{x^2 + y^2}$ ,  $z\bar{z} = r^2 = x^2 + y^2$

*Argument* (principal value)  $\arg z = \theta$ ,  $(-\pi < \theta \leq \pi)$ .

*De Moivre's theorem*  $(\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta$

*Euler's formulae*

$$\begin{aligned} e^{j\theta} &= \cos \theta + j \sin \theta, & e^{-j\theta} &= \cos \theta - j \sin \theta \\ \cos \theta &= \frac{1}{2}(e^{j\theta} + e^{-j\theta}), & \sin \theta &= \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \end{aligned}$$

*Complex roots* If  $z = re^{j\theta}$  then the  $n$  complex roots of  $z$  are given by

$$z^{1/n} = r^{1/n} \exp\left\{j \frac{\theta + 2k\pi}{n}\right\} \quad k = 0, 1, 2, \dots, n-1$$

These roots are equally spaced around the circle, radius  $r^{1/n}$ , centred on the origin.

### **Fundamental theorem of algebra**

A polynomial,  $P(z)$ , of degree  $n$  given by

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \quad (a_n \neq 0)$$

can be factorized into  $n$  complex factors:

$$P(z) = a_n (z - z_1)(z - z_2) \dots (z - z_n).$$

The numbers  $z_1, z_2, \dots, z_n$  are called the *roots* of the equation  $P(z) = 0$ . If the coefficients  $a_0, a_1, \dots, a_n$  are all real then the complex roots occur in conjugate pairs.

### Complex variable

If  $f(z) = u(x, y) + jv(x, y)$  is an *analytic function* of the complex variable  $z = x + jy$ , then  $u$  and  $v$  satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

### 1.11 Inequalities

Basic rules based on ‘greater than’ (the same rules apply to  $>$ ,  $<$ ,  $\geq$ ,  $\leq$ )

$$x > y \Leftrightarrow x + a > y + a$$

$$\text{if } p > 0 \text{ then } x > y \Leftrightarrow px > py$$

$$\text{if } n < 0 \text{ then } x > y \Leftrightarrow nx < ny$$

$$\text{if } a > b \text{ and } x > y \text{ then } a + x > b + y$$

$$\text{if } x > y \text{ and } y > z \text{ then } x > z$$

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$$

$$\text{if } x, y > 0 \text{ then } x < y \Leftrightarrow x^2 < y^2 \Leftrightarrow \sqrt{x} < \sqrt{y}$$

$$\text{if } x > 1 \text{ then } x^m > x^n \Leftrightarrow m > n$$

$$\text{if } 0 < x < 1 \text{ then } x^m > x^n \Leftrightarrow m < n$$

$$\text{if } x > 1 \text{ then } x^{1/m} > x^{1/n} \Leftrightarrow m > n$$

*Bernoulli's inequality*

$$\text{If } x > -1 \text{ then } (1+x)^n \geq 1+nx$$

*Arithmetic mean*  $\frac{1}{2}(x+y)$     *Geometric mean*  $\sqrt{xy}$

$$\frac{1}{2}(x+y) \geq \sqrt{xy}$$

*Triangle inequality*  $|x+y| \leq |x| + |y|$

$$|x| - |y| \leq ||x| - |y|| \leq |x - y|$$

*Cauchy Schwarz inequality*  $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$

*Minkowski inequality*  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

## Chapter 2

# Determinants and Matrices

### 2.1 Determinants

$$2 \times 2 : \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$3 \times 3 : \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$n \times n : \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The *minor*,  $\alpha_{ij}$ , of the element  $a_{ij}$  is the  $(n-1)$ th order determinant formed from  $\Delta$  by omitting the row and the column containing  $a_{ij}$ .

The *cofactor*,  $A_{ij}$ , of the element  $a_{ij}$  is given by  $A_{ij} = (-1)^{i+j} \alpha_{ij}$ .

The value of the  $n \times n$  determinant is

$$\begin{aligned} \Delta &= a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} \text{ (expansion by } i\text{th row)} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} \text{ (expansion by } j\text{th column)} \\ &\text{valid for } i, j = 1 \text{ to } n. \end{aligned}$$

#### Properties

1. The value of the determinant is unchanged if its rows and columns are interchanged.
2. The value of a determinant is unchanged if a multiple of any row (or column) is added to any other row (or column).
3. If the elements of two rows (or columns) are proportional, *i.e.* linearly dependent, then the value of the determinant is zero.



4. If the elements of two rows (or columns) are interchanged then the resulting determinant has the same numerical value but its sign is changed.
5. If the elements of a row (or column) are all multiplied by a constant  $k$ , then the value of the determinant is also multiplied by  $k$ .

**Cramer's rule** (efficient for only two equations in two unknowns or three equations in three unknowns).

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= h_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= h_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= h_3 \end{aligned} \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} h_1 & a_{12} & a_{13} \\ h_2 & a_{22} & a_{23} \\ h_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & h_1 & a_{13} \\ a_{21} & h_2 & a_{23} \\ a_{31} & h_3 & a_{33} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & h_1 \\ a_{21} & a_{22} & h_2 \\ a_{31} & a_{32} & h_3 \end{vmatrix}$$

The solution is

$$x_1 = \Delta_1/\Delta, \quad x_2 = \Delta_2/\Delta, \quad x_3 = \Delta_3/\Delta$$

## 2.2 Matrices

The  $m \times n$  matrix is written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or  $\mathbf{A} = [a_{ij}]$

*Transpose matrix*  $\mathbf{A}^T = [a_{ji}]$  is  $n \times m$

$\mathbf{A}$  is called a *symmetric matrix* if  $\mathbf{A}^T = \mathbf{A}$

If  $\mathbf{A}$  and  $\mathbf{B}$  are of the same order then  $\alpha\mathbf{A} + \beta\mathbf{B} = [\alpha a_{ij} + \beta b_{ij}]$

If  $\mathbf{A}$  is  $m \times r$  and  $\mathbf{B}$  is  $r \times n$  then

$$\mathbf{AB} = \left[ \sum_{k=1}^r a_{ik}b_{kj} \right] \text{ is } m \times n$$

*Unit matrix* or *identity matrix*  $\mathbf{I} = [\delta_{ij}]$  where the *Kronecker*  $\delta$  is given by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Null matrix or zero matrix  $\mathbf{0} = [0]$

For a square matrix  $\mathbf{A}$  if  $\det \mathbf{A} \neq 0$  then the *inverse matrix*,  $\mathbf{A}^{-1}$ , exists and is given by  $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A}$ , where  $\text{adj } \mathbf{A}$  is the transposed matrix of cofactors.

The inverse matrix has the property  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

If  $\det \mathbf{A} = 0$  then  $\mathbf{A}$  is said to be *singular*.

$\mathbf{A}$  is called an *orthogonal matrix* if  $\mathbf{A}^T = \mathbf{A}^{-1}$

For a  $2 \times 2$  matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For an  $n \times n$  matrix

$$\det \mathbf{A} = \det \mathbf{A}^T \text{ and } \det (k\mathbf{A}) = k^n \det \mathbf{A}$$

The *rank* of a matrix,  $\mathbf{A}$ , is the largest non-zero sub-determinant of  $\mathbf{A}$ .

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  for a finite dimensional space is said to be *linearly dependent* if there exist scalars  $\alpha_1, \alpha_2, \dots, \alpha_n$ , not all zero such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

otherwise the set is *linearly independent*.

Suppose that  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is an arbitrary basis for a finite dimensional inner product space. Then the orthogonal basis  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  may be obtained by the *Gram Schmidt process*

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2} \mathbf{w}_2, \dots, - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\|\mathbf{w}_{n-1}\|^2} \mathbf{w}_{n-1}$$

## 2.3 Systems of equations

For a system of  $m$  linear equations in  $n$  unknowns we write in matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

The explicit form is

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + & \cdots & + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + & \cdots & + a_{2n}x_n & = & b_2 \\ & \vdots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \cdots & + a_{mn}x_n & = & b_m \end{array}$$

1. If  $\text{rank } [\mathbf{A}, \mathbf{b}] > \text{rank } \mathbf{A}$  then there is no solution.
2. If  $\text{rank } [\mathbf{A}, \mathbf{b}] = \text{rank } \mathbf{A} = n$  then there is a unique solution.

In the case  $m = n$ , the solution is given by  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ .

3. If  $\text{rank} [\mathbf{A}, \mathbf{b}] = \text{rank} \mathbf{A} < n$  then there is an infinite number of solutions.

It follows from 2. and 3. that, for a square matrix  $\mathbf{A}$ , the homogeneous set of equations  $\mathbf{Ax} = \mathbf{0}$  has a non-trivial solution if and only if  $\det \mathbf{A} = 0$ .

## 2.4 Eigenvalues and eigenvectors

For a square matrix  $\mathbf{A}$ , the eigenvalues  $\lambda_i$  and corresponding eigenvectors  $\mathbf{x}_i$  are given by

$$\mathbf{Ax}_i = \lambda_i \mathbf{x}_i$$

Thus the eigenvalues satisfy  $\det [\mathbf{A} - \lambda \mathbf{I}] = 0$

### Properties

Matrix	$p\mathbf{A}$	$\mathbf{A}^T$	$\mathbf{A}^{-1}$	$\mathbf{A}^k$	$\mathbf{A} + q\mathbf{I}$	$[\mathbf{A} + p\mathbf{I}]^{-1}$
Eigenvalues	$p\lambda_i$	$\lambda_i$	$1/\lambda_i$	$\lambda_i^k$	$\lambda_i + q$	$1/(\lambda_i + p)$

In each case the corresponding eigenvectors are  $\mathbf{x}_i$ .

The *trace* of the matrix  $\mathbf{A}$ , is given by

$$\text{tr} \mathbf{A} = a_{11} + a_{22} + \dots + a_{nn}$$

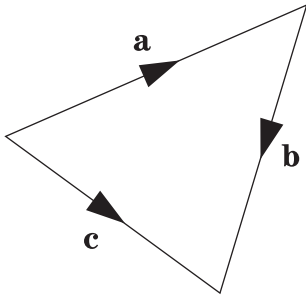
$$\sum \lambda_i = \text{tr} \mathbf{A}, \quad \prod \lambda_i = \det \mathbf{A}$$

Suppose that  $\lambda$  is an eigenvalue of  $\mathbf{A}$  then  $\lambda = 0$  iff  $\det \mathbf{A} = 0$ .

## Chapter 3

# Vector Algebra

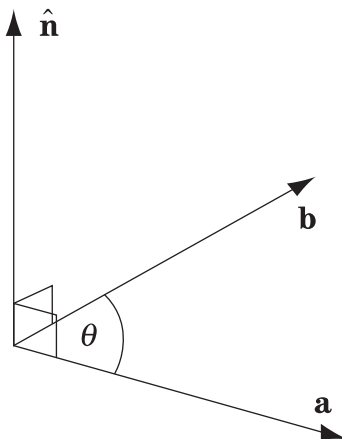
### 3.1 Vector addition



*Triangle law*  $\mathbf{a} + \mathbf{b} = \mathbf{c}$

*Component form*  $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{i}} + (a_2 + b_2)\hat{\mathbf{j}} + (a_3 + b_3)\hat{\mathbf{k}}$

### 3.2 Vector products



$\theta$  is the angle between the two vectors  $\mathbf{a}$  and  $\mathbf{b}$ ,  $\hat{\mathbf{n}}$  is the unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  in the sense that  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\hat{\mathbf{n}}$ , in that order, form a right-handed system.

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = \mathbf{b} \cdot \mathbf{a}, \quad \mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

In component form

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}, \quad \mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$$

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\mathbf{b} \times \mathbf{a}$$

Magnitude:  $|\mathbf{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector in the direction of the vector  $\mathbf{a}$  is  $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} =$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

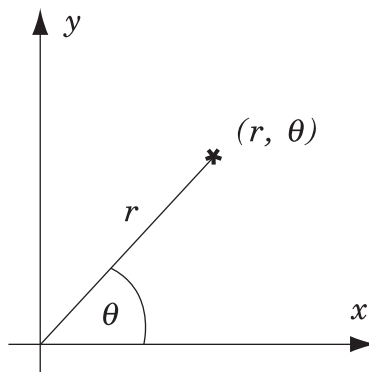
$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{d} - [\mathbf{a} \ \mathbf{b} \ \mathbf{d}]\mathbf{c}$$

### 3.3 Polar coordinates in two and three dimensions

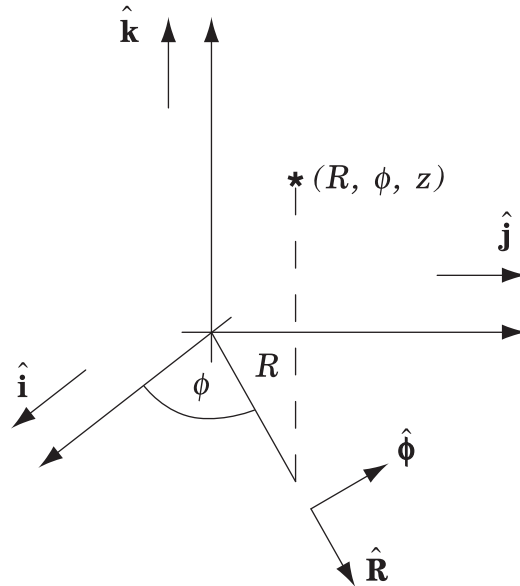
Plane polar coordinates  $(r, \theta)$

$$x = r \cos \theta, \quad y = r \sin \theta$$



**Cylindrical polar coordinates**  $(R, \phi, z)$

$$x = R \cos \phi, \quad y = R \sin \phi$$



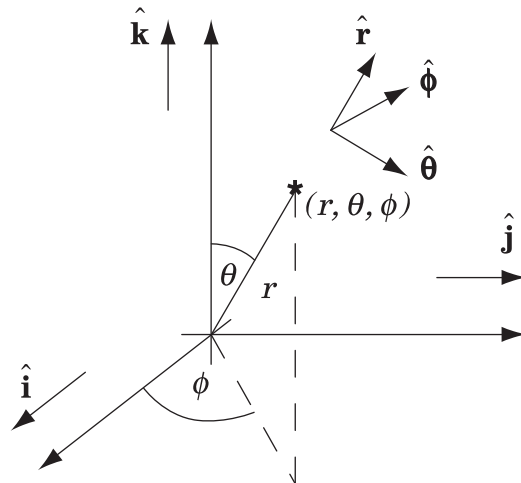
Unit vectors  $(\hat{\mathbf{R}}, \hat{\phi}, \hat{\mathbf{k}})$  form a right-handed system.

Relationships with Cartesian unit vectors:

$$\hat{\mathbf{R}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}, \quad \hat{\phi} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}$$

**Spherical polar coordinates**  $(r, \theta, \phi)$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$



Unit vectors  $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$  form a right-handed system.

Relationship with cylindrical unit vectors:

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{R}} + \cos \theta \hat{\mathbf{k}}, \quad \hat{\theta} = -\cos \theta \hat{\mathbf{R}} + \sin \theta \hat{\mathbf{k}}$$

# Chapter 4

## Calculus

### 4.1 Rules for manipulation of derivatives and integrals

#### Differentiation

*Linearity rule*  $\frac{d}{dx}(au + bv) = a\frac{du}{dx} + b\frac{dv}{dx}$

*Product rule*  $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

*Quotient rule*  $\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)/v^2$

*Leibniz rule*

$$\begin{aligned} D^n(uv) &= uD^n v + \binom{n}{1}(Du)D^{n-1}v + \binom{n}{2}(D^2u)D^{n-2}v + \dots + (D^n u)v \\ &= \sum_{r=0}^n \binom{n}{r}(D^r u)D^{n-r}v, \text{ where } D \equiv \frac{d}{dx} \end{aligned}$$

*Chain rule for ordinary differentiation* (function of a function rule)

*Leibniz notation*

$$y = y(x) \text{ and } x = x(u);$$

$$\frac{dy}{du} = \frac{dy}{dx} \frac{dx}{du}$$

*Function notation*

$$[F(g(x))] = F'(g(x))g'(x)$$

*Chain rule for partial differentiation*

$$F(u, v) = f(x, y) \text{ with } x = x(u, v), y = y(u, v)$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}; \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$$

Total differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

### Integration

Integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Integration by substitution

$$\int f(x) dx = \int F(u) \frac{dx}{du} du \text{ where } F(u) = f(x(u))$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)|$$

$$\int f'(x) [f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1}, \quad n \neq -1$$

### Differentiation of integrals

If  $I(x) = \int_{u(x)}^{v(x)} F(x, t) dt$ , then

$$\frac{dI}{dx} = F(x, v(x)) \frac{dv}{dx} - F(x, u(x)) \frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial F}{\partial x}(x, t) dt$$



## 4.2 Standard derivatives and integrals

Table of derivatives and integrals 1

$df/dx$	$f(x)$	$F(x) = \int f(x)dx$	
			(add arbitrary constant where necessary)
$nx^{n-1}$	$x^n$	$\frac{x^{n+1}}{n+1} (n \neq -1)$	(4.1)
$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln x $	(4.2)
$e^x$	$e^x$	$e^x$	(4.3)
$a^x \ln a$	$a^x$	$\frac{a^x}{\ln a}$	(4.4)
$\frac{1}{x}$	$\ln x$	$x(\ln x - 1)$	(4.5)
$\frac{1}{x \ln b}$	$\log_b x$	$x(\log_b x - \log_b e)$	(4.6)
$\cos x$	$\sin x$	$-\cos x$	(4.7)
$-\sin x$	$\cos x$	$\sin x$	(4.8)
$\sec^2 x$	$\tan x$	$\ln \sec x $	(4.9)
$\sec x \tan x$	$\sec x$	$\begin{cases} \ln \sec x + \tan x  \\ \ln \tan(x/2 + \pi/4)  \end{cases}$	(4.10)
$-\operatorname{cosec} x \cot x$	$\operatorname{cosec} x$	$\begin{cases} \ln \operatorname{cosec} x - \cot x  \\ \ln \tan(x/2)  \end{cases}$	(4.11)
$-\operatorname{cosec}^2 x$	$\cot x$	$\ln \sin x $	(4.12)
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$x \sin^{-1} x + \sqrt{1-x^2}$	(4.13)
$\frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$x \cos^{-1} x - \sqrt{1-x^2}$	(4.14)
$\frac{1}{1+x^2}$	$\tan^{-1} x$	$x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$	(4.15)
$\cosh x$	$\sinh x$	$\cosh x$	(4.16)
$\sinh x$	$\cosh x$	$\sinh x$	(4.17)
$\operatorname{sech}^2 x$	$\tanh x$	$\begin{cases} \ln \cosh x \\ \ln(e^x + e^{-x}) \end{cases}$	(4.18)

continued...

**Table of derivatives and integrals 2**

$df/dx$	$f(x)$	$F(x) = \int f(x)dx$	
		(add arbitrary constant where necessary)	
$-\operatorname{sech} x \tanh x$	$\operatorname{sech} x$	$\begin{cases} 2 \tan^{-1}(\tanh x/2) \\ 2 \tan^{-1} e^x \\ \tan^{-1}(\sinh x) \end{cases}$	(4.19)
$-\operatorname{cosech} x \coth x$	$\operatorname{cosech} x$	$\begin{cases} \ln  \tanh(x/2)  \\ -\ln  \operatorname{cosech} x + \coth x  \\ \ln  (e^x - 1)/(e^x + 1)  \end{cases}$	(4.20)
$-\operatorname{cosech}^2 x$	$\coth x$	$\begin{cases} \ln  \sinh x  \\ \ln  e^x - e^{-x}  \end{cases}$	(4.21)
$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1} x$	$x \sinh^{-1} x - \sqrt{1+x^2}$	(4.22)
$\frac{1}{\sqrt{x^2-1}}$	$\cosh^{-1} x$	$x \cosh^{-1} x - \sqrt{x^2-1}$	(4.23)
$\frac{1}{1-x^2}$	$\tanh^{-1} x$	$x \tanh^{-1} x - \frac{1}{2} \ln(1-x^2)$	(4.24)
$\frac{x}{(a^2-x^2)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\begin{cases} \sin^{-1} \frac{x}{a} \\ -\cos^{-1} \frac{x}{a} \end{cases}$	(4.25)
$\frac{-2x}{(x^2+a^2)^2}$	$\frac{1}{x^2+a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$	(4.26)
$\frac{a^2-2x^2}{x^2(x^2-a^2)^{\frac{3}{2}}}$	$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a} \sec^{-1} \frac{x}{a}$	(4.27)
$\frac{-x}{(a^2+x^2)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\begin{cases} \sinh^{-1} \frac{x}{a} \\ \ln  x + \sqrt{x^2+a^2}  \end{cases}$	(4.28)
$\frac{-x}{(x^2-a^2)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\begin{cases} \cosh^{-1} \frac{x}{a} \\ \ln  x + \sqrt{x^2-a^2}  \end{cases}$	(4.29)
$\frac{2x}{(a^2-x^2)^2}$	$\frac{1}{(a^2-x^2)}$	$\begin{cases} \frac{1}{a} \tanh^{-1} \frac{x}{a} \\ \frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  \end{cases}$	(4.30)

continued...

### Table of derivatives and integrals 3

$$df/dx \quad f(x) \quad F(x) = \int f(x)dx$$

(add arbitrary constant where necessary)

$$\frac{x}{\sqrt{x^2 \pm a^2}} \quad \sqrt{x^2 \pm a^2} \quad \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln |x + \sqrt{x^2 \pm a^2}| \quad (4.31)$$

$$\frac{-x}{\sqrt{a^2 - x^2}} \quad \sqrt{a^2 - x^2} \quad \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \quad (4.32)$$

$$e^{ax}(a \cos bx - b \sin bx) \quad e^{ax} \cos bx \quad \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) \quad (4.33)$$

$$e^{ax}(a \sin bx + b \cos bx) \quad e^{ax} \sin bx \quad \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) \quad (4.34)$$

### 4.3 Definite integrals

Wallis's formulae (reduction formulae also hold if upper limit is  $\pi$  or  $2\pi$ .)

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \frac{n-1}{n} S_{n-2}$$

$$C_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \frac{n-1}{n} C_{n-2}.$$

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{m-1}{m+n} I_{m-2,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

$$S_n = C_n = \frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)\dots} p;$$

$$I_{m,n} = \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} q;$$

where  $p = \begin{cases} \frac{\pi}{2} & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \quad q = \begin{cases} \frac{\pi}{2} & \text{both } m \text{ and } n \text{ even} \\ 1 & \text{otherwise} \end{cases}$

For all integer values of  $m$  and  $n$ :

$$\int_{-\pi}^{\pi} \sin m\theta \cos n\theta d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin m\theta \sin n\theta d\theta = \int_{-\pi}^{\pi} \cos m\theta \cos n\theta d\theta = \pi \delta_{mn}, \quad m, n \neq 0$$

( $\delta_{mn}$  is the Kronecker  $\delta$  see page 11)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}; \quad \int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \sqrt{\frac{\pi}{a}} \exp(b^2/4a) \quad a > 0$$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2} dx = \pi e^{-|a|}; \quad \int_{-\infty}^{\infty} \frac{\sin ax}{1+x^2} dx = 0$$

*Error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

*Complementary error function*

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

$$\operatorname{erf}(0) = \operatorname{erfc}(\infty) = 0, \quad \operatorname{erf}(\infty) = \operatorname{erfc}(0) = 1$$

For the *Chebyshev polynomials*  $T_n(x)$ ,  $T_m(x)$  (see page 57)

$$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \end{cases}$$

*Gamma function*

$$\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1} dx \quad t \neq 0, -1, -2, \dots$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

$$= (t-1)(t-2)(t-3) \dots \Gamma(t-[t])$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = n! \text{ when } n \text{ is an integer}$$

*Beta function*

$$B(s, t) = \int_0^1 x^{s-1}(1-x)^{t-1} dx = \Gamma(s)\Gamma(t)/\Gamma(s+t)$$

*Legendre polynomial*

$$P_n(x) = \frac{1}{2\pi} \int_0^{2\pi} [x + \sqrt{x^2-1} \cos \theta]^n d\theta = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2-1)/2, \quad P_3(x) = (5x^3-3x)/2, \dots$$

*Generating function*

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

$P_n(x)$  is a solution of *Legendre's equation*  $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ . A second, linearly independent, solution is the *Legendre function* of the second kind  $Q_n(x)$  which is not a polynomial.

*Bessel function*

$$J_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\Gamma(n + \frac{1}{2})} \int_0^{\frac{\pi}{2}} \cos(x \sin \theta) \cos^{2n} \theta d\theta.$$

$J_n(x)$  is a solution of *Bessel's equation*  $x^2y'' + xy' + (x^2 - n^2)y = 0$ . A second, linearly independent, solution,  $Y_n(x)$ , is the Bessel function of the second kind.

The *modified Bessel function*,  $I_n(x)$ , satisfies the equation  $x^2y'' + xy' - (x^2 + n^2)y = 0$  and a second linearly independent solution is  $K_n(x)$ , the modified Bessel function of the second kind.

#### 4.4 Radius of curvature of a curve

Intrinsic coordinates:  $\rho = \frac{ds}{d\psi}$ . Curvature:  $\kappa = \frac{1}{\rho}$ .

Cartesian coordinates:  $\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|}$ .

Parametric coordinates:  $\rho = \frac{((\dot{x})^2 + (\dot{y})^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}$

#### 4.5 Stationary points

##### Functions of one variable

The function  $f(x)$  has stationary points given by  $f'(x) = 0$ . The stationary point,  $(x_0, f(x_0))$ , is a local *maximum* if  $f''(x_0) < 0$  or a local *minimum* if  $f''(x_0) > 0$ . If  $f''(x_0) = 0$  then the second derivative test is inconclusive and we consider the sign of  $f'(x \pm \epsilon)$  where  $\epsilon$  is small and positive. If  $f''(x_0) = 0$  and  $f'(x_0 \pm \epsilon)$  are both positive or both negative then  $x_0$  is a *point of inflection*.

##### Functions of more than one variable

Stationary points occur when the first partial derivatives vanish. For a function of two variables,  $f(x, y)$ , stationary points are given by  $f_x = f_y = 0$ . The Hessian is given by

$$H(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

The stationary point is:  
a *maximum* if  $f_{xx} < 0, f_{yy} < 0$  and  $H > 0$ ,

a *minimum* if  $f_{xx} > 0, f_{yy} > 0$  and  $H > 0$ ,  
a *saddle point* if  $H < 0$ .

#### Lagrange multipliers

To find the stationary point of  $f(x, y)$  subject to the constraint  $g(x, y) = 0$ , find the stationary points of the function

$$\phi(x, y; \lambda) = f(x, y) + \lambda g(x, y).$$

## 4.6 Limits and series

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1, \quad \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\lim_{x \rightarrow 0} x^\alpha \ln x = 0, \quad \lim_{x \rightarrow \infty} x^{-\alpha} \ln x = 0, \quad \lim_{x \rightarrow \infty} x^\alpha e^{-x} = 0, \quad (\text{in all cases } \alpha > 0).$$

#### De L'Hôpital's rule

If  $f(a) = g(a) = 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ , provided  $g'(a) \neq 0$ .

#### Taylor's theorem

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \dots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \epsilon_n$$

$$\text{where } \epsilon_n = \frac{(x - a)^{n+1}}{(n + 1)!}f^{(n+1)}(c) \text{ for some } c \in (a, x).$$

**Taylor's series** for a function of one variable

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \dots + \frac{(x - a)^n}{n!}f^{(n)}(a),$$

or, writing  $x = a + h$ ,

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \dots$$

**Maclaurin series** (put  $a = 0, h = x$  in Taylor's series)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

#### Power series

If the power series  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots = \sum_{n=0}^{\infty} a_nx^n$  is convergent for  $|x| < R$  then  $R$  is called the *radius of convergence* of the series.

$$R = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

**Some useful series:**

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n} + \dots, \quad -1 < x \leq 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots - \frac{x^n}{n} - \dots, \quad -1 < x \leq 1$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad -1 < x < 1$$

[This is the *binomial series*. If  $n$  is a positive integer the series terminates after  $n+1$  terms, otherwise it converges if and only if  $|x| < 1$ .]

$$(1 \mp x)^{-1} = 1 \pm x + x^2 \pm x^3 + \dots + x^{2n} \pm x^{2n+1} + \dots, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots, \quad \text{all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots, \quad \text{all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots, \quad \text{all } x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^7}{315} + \dots, \quad |x| < \frac{\pi}{2}$$

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \dots + \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{(2n+1)} + \dots, \quad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)} + \dots, \quad |x| < 1$$

To obtain the series for the corresponding hyperbolic functions, see the note following Osborne's rule on page 5.

**Taylor's series** for a function of two variables

$$\begin{aligned} f(a+h, b+k) &= f(a, b) + \{hf_x(a, b) + kf_y(a, b)\} \\ &\quad + \frac{1}{2!}\{h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)\} + \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(a, b) \end{aligned}$$

## 4.7 Multiple integration

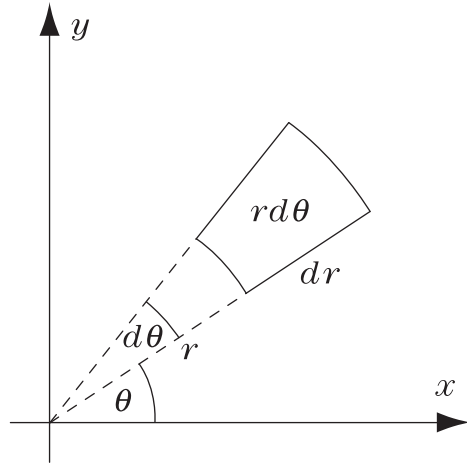
$$\int \int_R f(x, y) dx dy = \int \int_{R'} f(x(u, v), y(u, v)) \left| J \left( \frac{x, y}{u, v} \right) \right| du dv$$

where the region  $R$  in the  $xy$ -plane is mapped to the region  $R'$  in the  $uv$ -plane and the *Jacobian* of the transformation is given by

$$J \left( \frac{x, y}{u, v} \right) = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

*Plane polar coordinates*

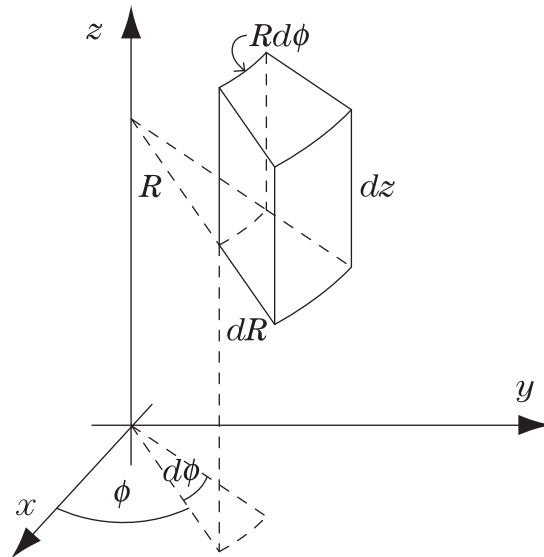
$$J = r, \text{ so that } dA = dx dy = r dr d\theta$$



**Volume and surface integrals**

*Cylindrical polar coordinates*

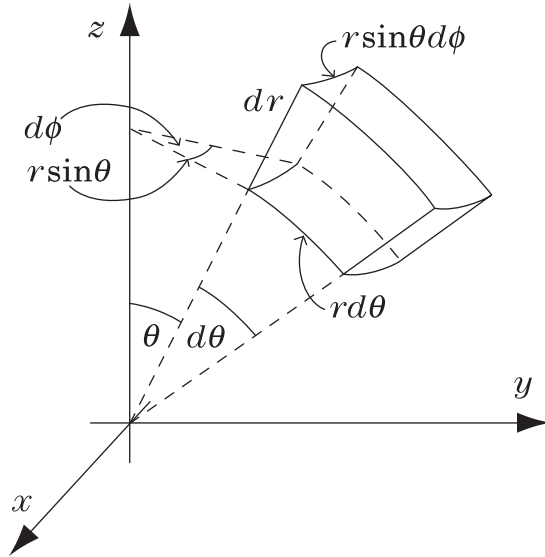
$$dS = R d\phi dz \text{ and } dV = R dR d\phi dz$$





Spherical polar coordinates

$$dS = r^2 \sin \theta d\theta d\phi \text{ and } dV = r^2 \sin \theta dr d\theta d\phi$$



### Line integrals

Green's theorem in the plane

$$\int_C (Pdx + Qdy) = \int \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

where  $C$  is the closed curve bounding the region  $R$ .

$\int_C (Pdx + Qdy)$  is independent of the path if and only if  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ .

The line integral of  $\mathbf{F}$  is independent of the path  $C$  joining  $A$  to  $B$ , if and only if there exists a potential function  $\phi$  such that  $\mathbf{F} = \text{grad}\phi$ , or equivalently, if and only if  $\text{curl}\mathbf{F} \equiv \mathbf{0}$ . In this case

$$\int_{C_{AB}} \mathbf{F} \cdot d\mathbf{s} = \int_A^B d\phi = \phi_B - \phi_A$$

## 4.8 Applications of integration

### Arc length of a curve

Cartesian form:  $\int_a^b \sqrt{1 + (y')^2} dx$

Polar form:  $\int_{\theta_1}^{\theta_2} \sqrt{r^2 + (dr/d\theta)^2} d\theta$

Parametric form:  $\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} dt$

**Solid of revolution**

The surface area of the solid of revolution obtained by one complete revolution of the curve  $y = f(x)$  about the  $x$ -axis is given by

$$2\pi \int_a^b y \sqrt{1 + (y')^2} dx$$

The volume of the solid of revolution obtained by one complete revolution of the curve  $y = f(x)$  about the  $x$ -axis is given by

$$\pi \int_a^b y^2 dx$$

**Centroid**

The centroid of the volume of the solid of revolution above is at  $(\bar{x}, 0)$  where

$$\bar{x} = \int_a^b xy^2 dx \bigg/ \int_a^b y^2 dx$$

The centroid of the area between the curve  $y = f(x)$ , the  $x$ -axis and the lines  $x = a, x = b$  is given by

$$\bar{x} = \int_a^b xy dx \bigg/ \int_a^b y dx, \quad \bar{y} = \int_a^b y^2 dx \bigg/ 2 \int_a^b y dx$$

**Mean value**

The mean value of  $f(x)$  between  $x = a, x = b$  is

$$\frac{1}{b-a} \int_a^b f(x) dx$$

**Root mean square value**

The *RMS* of the function  $f(x)$  between  $x = a, x = b$  is given by

$$\left( \frac{1}{b-a} \int_a^b [f(x)]^2 dx \right)^{\frac{1}{2}}$$

**Moments of area and inertia**

The first and second moments of a plane area  $A$  about an axis are given respectively by

$$\int \int_A r dA \text{ and } \int \int_A r^2 dA$$

where  $r$  is the distance from the axis of the element  $dA$ .

The moment of inertia,  $I$ , of a body, of density  $\rho$  and volume  $V$ , about an axis is given by

$$\int_V r^2 \rho dV$$

where  $r$  is the distance from the axis of the element  $dV$ .

**Parallel axes theorem**

If  $I_G$  is the moment of inertia about an axis through the centroid and  $I$  is the moment of inertia about a parallel axis distance  $d$  away, then

$$I = I_G + md^2$$

**Table of moments of inertia 1**

Uniform Body mass $m$	Axis	M.of I.
1. Bar length $2a$	Perpendicular to bar through one end	$\frac{ma^2}{3}$
	Perpendicular to bar through centroid	$\frac{4ma^2}{3}$
2. Rectangular lamina sides $2a$ and $2b$	Parallel to side $2b$ through centroid	$\frac{ma^2}{3}$
	Perpendicular to plane through centroid	$\frac{m(a^2 + b^2)}{3}$
3. Rectangular solid edges $2a$ and $2b$ , depth $2d$	Perpendicular to face $4ab$ through centroid	$\frac{m(a^2 + b^2)}{3}$
	Coinciding with edge $2d$	$\frac{4m(a^2 + b^2)}{3}$
4. Solid sphere radius $r$	Diameter	$\frac{2mr^2}{5}$
	Tangent	$\frac{7mr^2}{5}$
5. Spherical shell radii $R$ and $r$	Diameter	$\frac{2m(R^5 - r^5)}{3(R^3 - r^3)}$
	Tangent	$\frac{m(7R^5 - 5R^2r^3 - 2r^5)}{5(R^3 - r^3)}$

continued...

**Table of moments of inertia 2**

Uniform Body mass $m$	Axis	M.of I.
6. Disc radius $r$	Diameter	$\frac{mr^2}{4}$
	Perpendicular to disc through centroid	$\frac{mr^2}{2}$
7. Annular ring radii $R$ and $r$	Diameter	$\frac{m(R^2 + r^2)}{4}$
	Perpendicular to plane of ring through centroid	$\frac{m(R^2 + r^2)}{2}$
8. Solid cylinder radius $r$ length $2a$	Axis	$\frac{mr^2}{2}$
	Perpendicular to axis through centroid	$m \left( \frac{a^2}{3} + \frac{r^2}{4} \right)$
9. Cylindrical shell radii $R$ and $r$ length $2a$	Axis	$\frac{m(R^2 + r^2)}{2}$
	Perpendicular to axis through centroid	$m \left( \frac{a^2}{3} + \frac{R^2 + r^2}{4} \right)$

For entries 2, 6 and 7 the second moment of area  $J$  is given by  $J = IA/m$  where  $A$  is the area.

## 4.9 Kinematics and dynamics

### Kinematics

*Position vector*

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} && \text{(Cartesian coordinates)} \\ &= r\hat{\mathbf{r}} && \text{(Plane polar coordinates)}\end{aligned}$$

*Velocity vector*

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} && \text{(Cartesian coordinates)} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} && \text{(Plane polar coordinates)} \\ &= \dot{s}\hat{\mathbf{t}} && \text{(Intrinsic coordinates)}\end{aligned}$$

*Acceleration vector*

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} && \text{(Cartesian coordinates)} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}} && \text{(Plane polar coordinates)} \\ &= \ddot{s}\hat{\mathbf{t}} + \frac{\dot{s}^2}{\rho}\hat{\mathbf{n}} && \text{(Intrinsic coordinates)}\end{aligned}$$

For uniform motion with angular velocity  $\omega$  in a circle of radius  $a$ , speed is  $v = a\omega$  and acceleration is  $v^2/a = a\omega^2$  directed towards the centre.

### Dynamics

*Newton's second law* for a particle of mass  $m$  moving under the influence of an external force  $\mathbf{F}$

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

For a rigid body moving about an axis with angular velocity  $\omega$  and external torque  $\Gamma$  we have

$$\Gamma = I\dot{\omega}$$

where  $I$  is the moment of inertia about the axis.

## Chapter 5

# Ordinary Differential Equations

### 5.1 First order equations

$$\frac{dy}{dx} = F(x, y) \text{ or } y' = F(x, y)$$

**Variables separable**

$$\frac{dy}{dx} = f(x)g(y) \quad (g(y) \neq 0)$$

*General Solution* is

$$\int \frac{1}{g(y)} dy = \int f(x) dx + \text{constant}$$

**Linear equation** (*integrating factor method*)

$$\frac{dy}{dx} + P(x)y = Q(x)$$

*Integrating factor:*  $\mu(x) = \exp(\int P(x) dx)$

Then  $\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$  and the

*General Solution* is

$$\mu(x)y = \int \mu(x)Q(x) dx + \text{constant}$$

**Initial-value problem**

Find the general solution of

$$\frac{dy}{dx} = F(x, y) \quad \text{subject to } y(x_0) = y_0$$

*i.e.* solve the differential equation containing one arbitrary constant and determine the value of the constant to satisfy the *initial condition* that  $y = y_0$  when  $x = x_0$ .

## 5.2 Second order equations

The general linear equation with constant coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \text{ or } ay'' + by' + cy = f(x)$$

### Complementary function $y_c$

Solution of the associated homogeneous equation

$$ay'' + by' + cy = 0 \quad (a, b, c \text{ real})$$

*Auxiliary equation*  $a\lambda^2 + b\lambda + c = 0$

Three cases:

Two real distinct roots,  $\lambda_1, \lambda_2$

Linearly independent solutions  $e^{\lambda_1 x}$  and  $e^{\lambda_2 x}$ ;

$$y_c = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

One real repeated root,  $\lambda_1$

Linearly independent solutions  $e^{\lambda_1 x}$  and  $x e^{\lambda_1 x}$ ;

$$y_c = e^{\lambda_1 x} (C_1 + C_2 x)$$

Complex conjugate roots,  $\alpha \pm j\beta$

Linearly independent solutions  $e^{\alpha x} \cos \beta x$  and  $e^{\alpha x} \sin \beta x$ ;

$$y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

### Particular integral $y_p$

Any solution of the non-homogeneous equation

$$ay'' + by' + cy = f(x)$$

### Method of undetermined coefficients

Choose a suitable *trial function* for  $y_p$  according to the right-hand-side,  $f(x)$ .

$f(x)$	trial function
constant	constant
$px + q$	$lx + m$
$px^2 + qx + r$	$lx^2 + mx + n$
polynomial degree $n$	$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$
$e^{\omega x}$	$ke^{\omega x}$
$p \cos \omega x$	$m \cos \omega x + n \sin \omega x$
$q \sin \omega x$	$m \cos \omega x + n \sin \omega x$
$\text{Re}[(p - jq)e^{j\omega x}]$	$\text{Re}[ze^{j\omega x}]$

**Beware:** If the *rhs* includes one of the complementary function linearly independent solutions then choose  $x$  times the appropriate trial function.

**General Solution** The sum of the complementary function and a particular integral  $y = y_c + y_p$ .

**Initial-value problem**

$$ay'' + by' + cy = f(x) \text{ subject to } y(x) = y_0, \quad y'(x_0) = y'_0$$

Find the *general solution* containing two arbitrary constants and then use the initial conditions to determine the values of the constants.

### 5.3 Higher order equations

The general linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

The complementary function,  $y_c$ , is found in a similar manner to that on page 33 using the auxiliary equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

A particular integral,  $y_p$ , is also found in a similar manner to that on page 33 using suitable trial functions.

The general solution is  $y = y_c + y_p$ .





## Chapter 6

# Fourier Series

### Full-range series

$f(x)$  is defined for  $-l < x < l$ .

The Fourier series for  $f(x)$  is given by:

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx,$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx; \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$F(x)$  extends  $f(x)$  periodically with period  $2l$ .

$F(x) = f(x)$  at all points,  $-l < x < l$ , at which  $f$  is continuous. If  $f$  is discontinuous at  $x_0$ , then  $F(x_0) = \{f(x_0+) + f(x_0-)\}/2$ .

### Half-range series

$f(x)$  is defined for  $0 < x < l$ .

The Fourier half-range cosine series is:

$$F_c(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}; \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

$F_c(x)$  extends  $f(x)$  periodically as an even function with period  $2l$ .

The Fourier half-range sine series is

$$F_s(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}; \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

$F_s(x)$  extends  $f(x)$  periodically as an odd function with period  $2l$ .

The Fourier series of some frequently occurring functions are given on page 47.

**Complex Fourier series**  $f(x)$  is defined for  $-l < x < l$ .  
The complex Fourier series for  $f(x)$  is given by

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/l}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-jn\pi x/l} dx = \begin{cases} (a_n - jb_n)/2 & n > 0 \\ a_0/2 & n = 0 \\ (a_{-n} + jb_{-n})/2 & n < 0 \end{cases}$$

The constant term, *i.e.*  $\frac{1}{2}a_0$  or  $c_0$ , is the average value of  $f(x)$  over the interval  $(-l, l)$ .

**Parseval's theorem**

$$\frac{1}{2l} \int_{-l}^l [f(x)]^2 dx = \frac{1}{4} a_0^2 + \sum_{n=1}^{\infty} \frac{1}{2} (a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

## Chapter 7

# Vector Calculus

### 7.1 grad, div and curl

$$\text{Nabla operator } \nabla = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

$$\text{grad } \phi \equiv \nabla \phi = \frac{\partial \phi}{\partial x} \hat{\mathbf{i}} + \frac{\partial \phi}{\partial y} \hat{\mathbf{j}} + \frac{\partial \phi}{\partial z} \hat{\mathbf{k}}$$

$$\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\begin{aligned} \text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\mathbf{i}} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\mathbf{j}} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\mathbf{k}} \\ \text{grad } \phi &= \frac{\partial \phi}{\partial n} \hat{\mathbf{n}} \end{aligned}$$

where  $\hat{\mathbf{n}}$  is the unit vector normal to the surface  $\phi = \text{constant}$ .

*Directional derivative.* The rate of change of a scalar field,  $\phi$ , in the direction of the unit vector  $\hat{\mathbf{n}}$  is given by

$$\frac{\partial \phi}{\partial n} = \text{grad } \phi \cdot \hat{\mathbf{n}}$$

If  $\delta V$  is volume bounded by the surface  $\delta S$  then

$$\text{grad } \phi = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} \phi d\mathbf{S}$$

$$\text{div } \mathbf{A} = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} \mathbf{A} \cdot d\mathbf{S}$$

$$\text{curl}\mathbf{A} = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} d\mathbf{S} \times \mathbf{A}$$

*Laplacian operator*

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

### Vector identities

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

$$\nabla \cdot (\phi\mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{A}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\nabla\phi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla^2(\phi + \psi) = \nabla^2\phi + \nabla^2\psi$$

$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

### Cylindrical polar coordinates

$$\nabla\psi = \frac{\partial\psi}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{k}}$$

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R}\frac{\partial}{\partial R}(RA_R) + \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left\{ \frac{1}{R}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right\} \hat{\mathbf{R}} + \left\{ \frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right\} \hat{\boldsymbol{\phi}} + \frac{1}{R} \left\{ \frac{\partial}{\partial R}(RA_\phi) - \frac{\partial A_R}{\partial\phi} \right\} \hat{\mathbf{k}}$$

### Spherical polar coordinates

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} \\ \nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \\ \nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\ \nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right\}\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\left\{\frac{\partial A_r}{\partial\theta} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right\}\hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\hat{\boldsymbol{\phi}}\end{aligned}$$

### General orthogonal curvilinear coordinates

Transformation from  $\mathbf{x} = (x_1, x_2, x_3)$  to  $\mathbf{u} = (u_1, u_2, u_3)$

where  $x_i = x_i(u_1, u_2, u_3)$   $i = 1, 2, 3$

$$d\mathbf{x} = (du_1h_1\hat{\mathbf{e}}_1, du_2h_2\hat{\mathbf{e}}_2, du_3h_3\hat{\mathbf{e}}_3)$$

where  $h_i$  are scale factors (metric coefficients)

$\hat{\mathbf{e}}_i$  are unit vectors

$$\text{and } h_i\hat{\mathbf{e}}_i = \frac{\partial\mathbf{x}}{\partial u_i} \quad h_i = \left|\frac{\partial\mathbf{x}}{\partial u_i}\right| \quad i = 1, 2, 3$$

Element of arc  $dl = [h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2]^{1/2}$

Area element on  $u_1$ -coordinate surface  $dA = h_2h_3du_2du_3$

Volume element  $dV = h_1h_2h_3du_1du_2du_3$

### Vector calculus

$$\text{grad}\psi = \nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{\mathbf{e}}_3$$

$$\mathbf{A} = \mathbf{A}_1\hat{\mathbf{e}}_1 + \mathbf{A}_2\hat{\mathbf{e}}_2 + \mathbf{A}_3\hat{\mathbf{e}}_3$$

$$\text{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

$$\text{curl}\mathbf{A} = \nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left[\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right] + \frac{\partial}{\partial u_2}\left[\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right] + \frac{\partial}{\partial u_3}\left[\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right]\right]$$

## 7.2 Integral theorems of the vector calculus

$V$  is the volume bounded by the closed surface  $S$

**Gauss's divergence theorem**

$$\int_V \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

**Green's theorem**

First form:

$$\int_V (\phi \nabla^2 \psi + \operatorname{grad} \phi \cdot \operatorname{grad} \psi) dV = \int_S \phi \operatorname{grad} \psi \cdot d\mathbf{S}$$

Second form:

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \int_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\mathbf{S}$$

$C$  is the curve bounding the open surface  $S$

**Stokes's theorem**

$$\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

# Chapter 8

## Tables of Transforms

### 8.1 The Laplace transform

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt \quad \text{or} \quad \mathcal{L}[x(t)] = \int_0^{\infty} e^{-st} x(t) dt$$

$$= F(s) \qquad \qquad \qquad = \bar{x}(s)$$

$$f(t) = \mathcal{L}^{-1}[F(s)] \qquad \qquad x(t) = \mathcal{L}^{-1}[\bar{x}(s)]$$

**Table of Laplace transforms 1**

$f(t)$	$\mathcal{L}[f(t)]$	
$\mathcal{L}^{-1}[F(s)]$	$F(s)$	
1	$\frac{1}{s}$	(8.1)
$t$	$\frac{1}{s^2}$	(8.2)
$t^n$	$\frac{n!}{s^{n+1}}$	(8.3)
(n a positive integer)		
$t^\lambda (\lambda > 1)$	$\frac{\Gamma(\lambda + 1)}{s^{\lambda+1}}$	(8.4)
$e^{-at}$	$\frac{1}{s + a}$	(8.5)
$\sin bt$	$\frac{b}{s^2 + b^2}$	(8.6)
$\cos bt$	$\frac{s}{s^2 + b^2}$	(8.7)
$\sin(at + \phi)$	$\frac{s \sin \phi + a \cos \phi}{s^2 + a^2}$	(8.8)

continued...

**Table of Laplace transforms 2**

$f(t)$ $\mathcal{L}^{-1}[F(s)]$	$\mathcal{L}[f(t)]$ $F(s)$	
$\sinh bt$	$\frac{b}{s^2 - b^2}$	(8.9)
$\cosh bt$	$\frac{s}{s^2 - b^2}$	(8.10)
$t \sin bt$	$\frac{2bs}{(s^2 + b^2)^2}$	(8.11)
$t \cos bt$	$\frac{s^2 - b^2}{(s^2 + b^2)^2}$	(8.12)
$\sin bt - bt \cos bt$	$\frac{2b^3}{(s^2 + b^2)^2}$	(8.13)
$\sin bt + bt \cos bt$	$\frac{2bs^2}{(s^2 + b^2)^2}$	(8.14)
$e^{-at} t^n$	$\frac{n!}{(s + a)^{n+1}}$	(8.15)
( $n$ a positive integer)		
$e^{-at} \sin bt$	$\frac{b}{(s + a)^2 + b^2}$	(8.16)
$e^{-at} \cos bt$	$\frac{s + a}{(s + a)^2 + b^2}$	(8.17)
$e^{-at} \sinh bt$	$\frac{b}{(s + a)^2 - b^2}$	(8.18)
$e^{-at} \cosh bt$	$\frac{s + a}{(s + a)^2 - b^2}$	(8.19)
$H(t)$	$\frac{1}{s}$	(8.20)
$H(t - a)$	$\frac{e^{-as}}{s}$	(8.21)
$H(t) - H(t - a)$	$\frac{1 - e^{-as}}{s}$	(8.22)
$\delta(t)$	$1$	(8.23)
$\delta(t - a)$	$e^{-as}$	(8.24)
$[t]$	$\frac{e^{-as}}{s(1 - e^{-s})}$	(8.25)
$\frac{\sin at}{t}$	$\tan^{-1} \frac{a}{s}$	(8.26)

continued...



### Table of Laplace transforms 3

$f(t)$ $\mathcal{L}^{-1}[F(s)]$	$\mathcal{L}[f(t)]$ $F(s)$	
$\frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$	$\frac{e^{-a/s}}{\sqrt{s}}$	(8.27)
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$	(8.28)
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	(8.29)
$\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	(8.30)
$2\sqrt{\frac{t}{\pi}}e^{-a^2/4t} - a\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s^3}}$	(8.31)
$\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}$	$\operatorname{erfc}\left(\frac{s}{2a}\right)e^{s^2/4a^2}$	(8.32)
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	(8.33)
$H(t-a)f(t-a)$	$e^{-as}F(s)$	(8.34)
$e^{-at}f(t)$	$F(s+a)$	(8.35)
$f'(t)$	$sF(s) - f(0)$	(8.36)
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	(8.37)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$	(8.38)
$\int_0^t f(u)du$	$\frac{F(s)}{s}$	(8.39)
$-tf(t)$	$F'(s)$	(8.40)
$(-t)^n f(t)$	$F^{(n)}(s)$	(8.41)
$\frac{f(t)}{t}$	$\int_s^\infty F(u)du$	(8.42)
$\int_0^t f(u)g(t-u)du$ (convolution integral)	$F(s)G(s)$	(8.43)
$f(t+T) = f(t)$	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-su} f(u)du$	(8.44)
$\sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)} \exp(\alpha_k t)$	$P(s)/Q(s)$	(8.45)

$P$  polynomial of degree less than  $n$   $Q(s) = (s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_n)$   
 where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are all distinct

Limiting theorems  $\lim_{s \rightarrow \infty} F(s) = 0, \quad \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

**Inversion formula**

$$\frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} e^{st} F(s) ds = \begin{cases} f(t) & t > 0 \\ 0 & t < 0 \end{cases}$$

**Stehfest numerical inversion**

Given  $\bar{f}(s)$ , the Laplace transform of  $f(t)$ , seek the value  $f(T)$  for a specific value  $t = T$ .

Choose  $s_j = j \frac{\ln 2}{T} \quad j = 1, 2, \dots, M$  where  $M$  is even.

The approximate numerical inversion is given by

$$f(T) \approx \frac{\ln 2}{T} \sum_{j=1}^M w_j \bar{f}(s_j)$$

where the weights,  $w_j$ , are given by

$$w_j = (-1)^{\frac{M}{2}+j} \sum_{k=\lceil \frac{1}{2}(1+j) \rceil}^{\min(j, \frac{M}{2})} \frac{k^{\frac{M}{2}} (2k)!}{(\frac{M}{2} - k)! k! (k-1)! (j-k)! (2k-j)!}$$

**Stehfest's weights for  $M = 6, 8, 10, 12$  and  $14$**

$M = 6$	$M = 8$	$M = 10$	$M = 12$	$M = 14$
1	-1/3	1/12	-1/60	1/360
-49	145/3	-385/12	961/60	-461/72
366	-906	1279	-1247	18481/20
-858	16394/3	-46871/3	82663/3	-6227627/180
810	-43130/3	505465/6	-1579685/6	4862890/9
-270	18730	-473915/2	13241387/10	-131950391/30
	-35840/3	1127735/3	-58375583/15	189788326/9
	8960/3	-1020215/3	21159859/3	-2877521087/45
		328125/2	-16010673/2	2551951591/20
		-65625/2	11105661/2	-2041646257/12
			-10777536/5	4509824011/30
			1796256/5	-169184323/2
				824366543/30
				-117766649/30

## 8.2 The Z-transform

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n} = \bar{x}(z), \quad x(n) = Z^{-1}[\bar{x}(z)]$$

### Table of Z-transforms

The variable  $t$  is related to the sampling interval,  $T$ , by  $t = nT$ .

$x(n)$	$Z[x(n)]$	
$Z^{-1}[\bar{x}(z)]$	$\bar{x}(z)$	
$\delta(n)$	1	(8.46)
1 or $H(n)$	$\frac{z}{z-1}$	(8.47)
$n$	$\frac{Tz}{(z-1)^2}$	(8.48)
$n^2$	$\frac{T^2z(z+1)}{(z-1)^3}$	(8.49)
$\binom{n}{k}$	$\frac{z}{(z-1)^{k+1}}$	(8.50)
$a^n$	$\frac{z}{z-a}$	(8.51)
$\delta(n-m)$	$z^{-m}$	(8.52)
$e^{-an}$	$\frac{z}{z-e^{-aT}}$	(8.53)
$\sin(an)$	$\frac{z \sin(aT)}{z^2 - 2z \cos(aT) + 1}$	(8.54)
$ne^{-an}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$	(8.55)
$\cos(an)$	$\frac{z^2 - z \cos(aT)}{z^2 - 2z \cos(aT) + 1}$	(8.56)
$\alpha x_1(n) + \beta x_2(n)$	$\alpha \bar{x}_1(z) + \beta \bar{x}_2(z)$	(8.57)
$x(n-m)$	$z^{-m} \bar{x}(z)$	(8.58)
$e^{-an} x(n)$	$\bar{x}(ze^{aT})$	(8.59)
$a^{-n} x(n)$	$\bar{x}(az)$	(8.60)
$x(n+1)$	$z[\bar{x}(z) - x(0)]$	(8.61)
$x(n+2)$	$z^2[\bar{x}(z) - x(1)z^{-1} - x(0)]$	(8.62)
$\sum_{m=0}^n x(m)y(n-m)$	$\bar{x}(z)\bar{y}(z)$	(8.63)

*Limiting theorems*

$$\lim_{n \rightarrow \infty} x(n) = \lim_{z \rightarrow 1} \frac{z-1}{z} \bar{x}(z), \quad \lim_{n \rightarrow 0} x(n) = \lim_{z \rightarrow \infty} \bar{x}(z)$$

### 8.3 The Fourier transform

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-j\xi x} dx \quad \mathcal{F}^{-1}[F(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)e^{j\xi x} d\xi$$

$$= F(\xi) \quad = f(x)$$

Table of Fourier transforms

$f(x)$ $\mathcal{F}^{-1}[F(\xi)]$	$\mathcal{F}[f(x)]$ $F(\xi)$
$\frac{1}{ x }$	$\frac{1}{ \xi }$
$\frac{1}{a - jx}$	$\sqrt{2\pi}e^{-a\xi}H(\xi)$
$H( x ) - H( x  - a)$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\xi}{\xi}$
$e^{-x^2/a}$	$\sqrt{\frac{a}{2}}e^{-a\xi^2/4}$
$f'(x)$	$j\xi F(\xi)$
$f''(x)$	$-\xi^2 F(\xi)$

### 8.4 Fourier sine and cosine transforms

$$\mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \xi x dx \quad \mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \xi x dx$$

$$= F_s(\xi) \quad = F_c(\xi)$$

$$\mathcal{F}_s^{-1}[F_s(\xi)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\xi) \sin \xi x d\xi \quad \mathcal{F}_c^{-1}[F_c(\xi)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\xi) \cos \xi x d\xi$$

$$= f(x) \quad = f(x)$$

**Table of Fourier sine and cosine transforms**

$f(x)$	$\mathcal{F}_s(\xi)$	$f(x)$	$\mathcal{F}_c(\xi)$
$e^{-x}$	$\sqrt{\frac{2}{\pi}} \frac{1}{(1+\xi^2)}$	$e^{-x}$	$\sqrt{\frac{2}{\pi}} \frac{1}{(1+\xi^2)}$
$xe^{-x^2/2}$	$e^{-\xi^2/2}$	$e^{-x^2}$	$e^{-\xi^2}$
$\frac{\sin x}{x}$	$\frac{1}{\sqrt{2\pi}} \ln \left  \frac{1+\xi}{1-\xi} \right $	$H(x) - H(x-a)$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\xi}{\xi}$
$f'(x)$	$-\xi \mathcal{F}_c(\xi)$	$f'(x)$	$\xi \mathcal{F}_s(\xi) - \sqrt{\frac{2}{\pi}} f(0)$
$f''(x)$	$-\xi^2 \mathcal{F}_s(\xi) + \xi \sqrt{\frac{2}{\pi}} f(0)$	$f''(x)$	$-\xi^2 \mathcal{F}_c(\xi) - \sqrt{\frac{2}{\pi}} f'(0)$

### 8.5 Some periodic functions: Laplace transforms and Fourier series

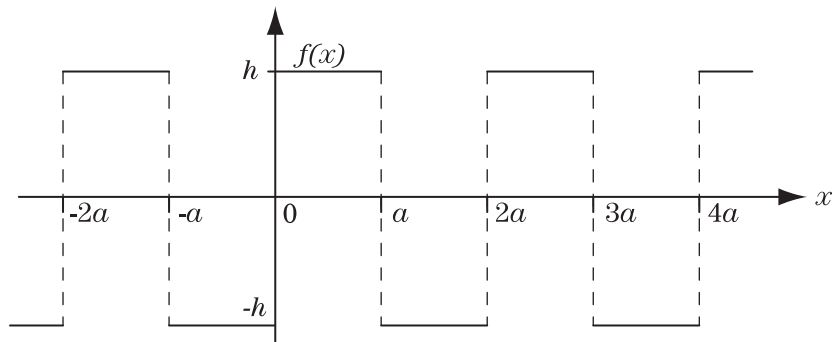
Graphs of the Laplace transform,  $F(s)$ , and the Fourier series,  $F_s(x), F_c(x)$  or  $F(x)$ :

For the Laplace transform it is assumed that  $f(x) \equiv 0$  for  $x < 0$ .

**1. Square wave (odd function) height  $h$ , period  $2a$**

$$f(x) = \begin{cases} -h & -a \leq x < 0 \\ h & 0 \leq x < a \end{cases}$$

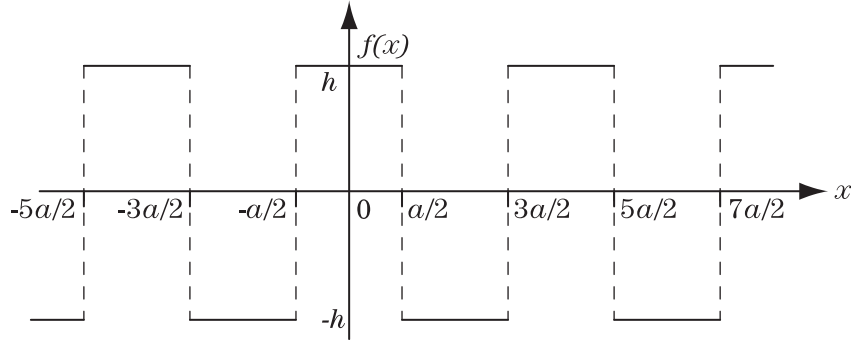
$$F(s) = \frac{h}{s} \tanh\left(\frac{as}{2}\right) \quad F_s(x) = \frac{4h}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left[\frac{(2n+1)\pi x}{a}\right]$$



2. Square wave (even function), period  $2a$

$$f(x) = \begin{cases} -h & -a \leq x < -a/2 \\ h & -a/2 \leq x < a/2 \\ -h & a/2 \leq x < a \end{cases}$$

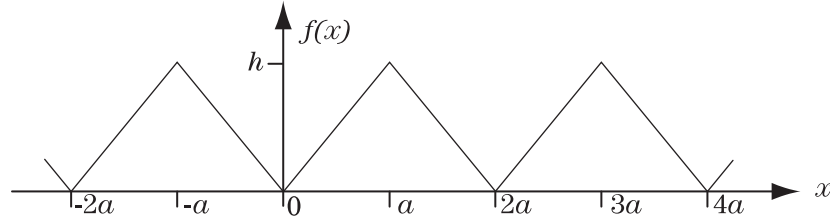
$$F(s) = \frac{h}{s} e^{as/2} \tanh\left(\frac{as}{2}\right) \quad F_c(x) = \frac{4h}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left[\frac{(2n+1)\pi x}{a}\right]$$



3. Triangular wave, period  $2a$

$$f(x) = \begin{cases} -hx/a & -a \leq x < 0 \\ hx/a & 0 \leq x < a \end{cases}$$

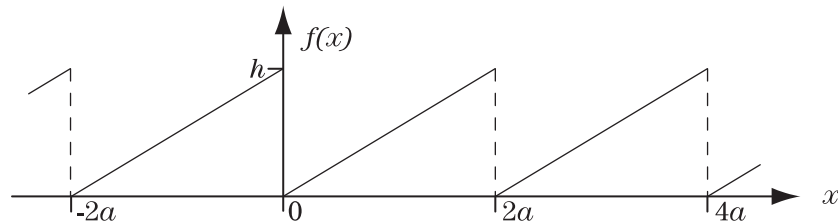
$$F(s) = \frac{h}{as^2} \tanh\left(\frac{as}{2}\right) \quad F_c(x) = \frac{h}{2} - \frac{4h}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left[\frac{(2n+1)\pi x}{a}\right]$$



4. Sawtooth wave, period  $2a$

$$f(x) = hx/2a \quad 0 \leq x < 2a$$

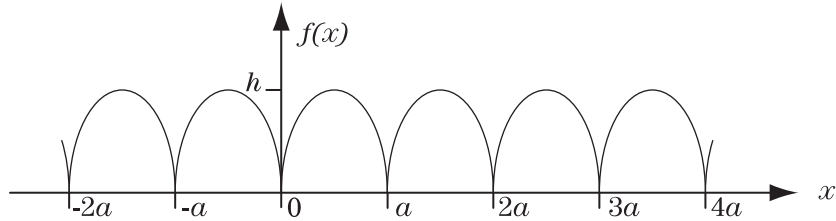
$$F(s) = \frac{h}{2as^2} - \frac{he^{-2as}}{s(1-e^{-2as})} \quad F_c(x) = \frac{h}{2} - \frac{4h}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left[\frac{n\pi x}{2a}\right]$$



5. Rectified sine wave, period  $a$

$$f(x) = h \sin(x/a) \quad 0 \leq x < a$$

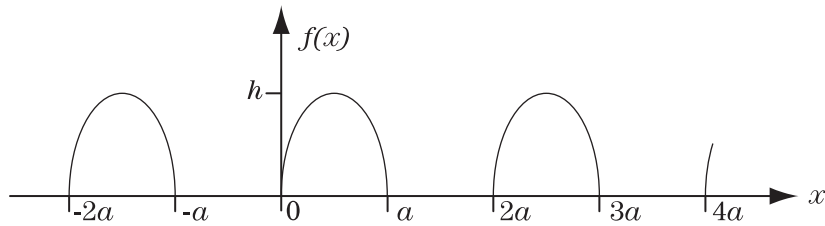
$$F(s) = \frac{\pi ah}{a^2 s^2 + \pi^2} \coth\left(\frac{as}{2}\right) \quad F_c(x) = \frac{2h}{\pi} - \frac{4h}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x/a)}{(4n^2 - 1)}$$



6. Half-rectified sine wave, period  $a$

$$f(x) = \begin{cases} h \sin(2x/a) & 0 \leq x < a/2 \\ 0 & a/2 \leq x < a \end{cases}$$

$$F(s) = \frac{\pi ah}{(a^2 s^2 + \pi^2)(1 - e^{-as})} \quad F_c(x) = \frac{h}{\pi} - \frac{2h}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x/a)}{(4n^2 - 1)} + \frac{h}{2} \sin\left(\frac{\pi x}{a}\right)$$



# Chapter 9

## Numerical Methods

### 9.1 Interpolation

*Polynomial interpolation:* Suppose that  $f$  is defined on the interval  $-1 \leq x \leq 1$  with  $n+1$  continuous derivatives. If the  $n$ -degree polynomial approximation at the  $n+1$  points  $x_0, x_1, \dots, x_n$  is given by  $p_n(x)$  then the error at any point  $x \in (-1, 1)$  is given by

$$f(x) - p_n(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_n)}{(n + 1)!} f^{(n+1)}(\xi) \text{ for some } \xi \in (-1, 1)$$

#### Lagrange interpolation polynomials

$$L_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

The  $n^{\text{th}}$  degree collocation polynomial through the points  $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$  is given by

$$P_n(x) = \sum_{i=0}^n L_i(x) y_i$$

#### Cubic splines

The cubic spline interpolating function,  $S(x)$ , through the points  $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$  has the properties  $S(x_i) = y_i$  i.e.  $S$  is continuous at  $(x_i, y_i)$ .

$S'(x)$  and  $S''(x)$  are continuous.

$S(x)$  is a cubic polynomial,  $S_i(x)$ , in each interval  $[x_i, x_{i+1}]$ .

$$S_i(x) = \frac{y_{i+1}(x - x_i)}{h_i} - \frac{y_i(x - x_{i+1})}{h_i} - \frac{M_i}{6} \left[ \frac{(x - x_{i+1})^3}{h_i} - h_i(x - x_{i+1}) \right] - \frac{M_{i+1}}{6} \left[ \frac{(x - x_i)^3}{h_i} - h_i(x - x_i) \right]$$

where  $M_i$  are found from



$$\begin{bmatrix} a_{00} & a_{01} & 0 & 0 & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \cdots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & \cdots & 0 & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & 0 & 0 & a_{nn-1} & a_{nn} \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix}$$

$$= 6 \begin{bmatrix} b_0 \\ \frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0} \\ \frac{y_3 - y_2}{h_2} - \frac{y_2 - y_1}{h_1} \\ \vdots \\ b_n \end{bmatrix}$$

where  $h_i = x_{i+1} - x_i$ .

Natural spline,  $M_0 = M_n = 0$  so that

$$a_{00} = a_{nn} = 1, \quad a_{01} = a_{nn-1} = b_0 = b_n = 0$$

If the derivative of the underlying function is known at  $x_0$  and  $y_0$  then

$$a_{00} = 2h_0, \quad a_{01} = h_0, \quad b_0 = (y_1 - y_0)/h_0 - f'(x_0)$$

$$a_{nn-1} = h_{n-1}, \quad a_{nn} = 2h_{n-1}, \quad b_n = f'(x_n) - (y_n - y_{n-1})/h_{n-1}$$

## 9.2 Finite difference operators

Forward	$\Delta y_i = y_{i+1} - y_i$	Backward	$\nabla y_i = y_i - y_{i-1}$
Central	$\delta y_{i+\frac{1}{2}} = y_{i+1} - y_i$	Shift	$E y_i = y_{i+1}$
Average	$\mu y_{i+\frac{1}{2}} = (y_i + y_{i+1})/2$	Differential	$D y_i = (dy/dx)_{x_i}$

### Relationship between the operators

	$E$	$\Delta$	$\delta, \mu$	$\nabla$	$hD$
$E$	$E$	$1 + \Delta$	$1 + \mu\delta + \delta^2/2$	$(1 - \nabla)^{-1}$	$e^{hD}$
$\Delta$	$E - 1$	$\Delta$	$\mu\delta + \delta^2/2$	$\nabla(1 - \nabla)^{-1}$	$e^{hD} - 1$
$\delta$	$E^{\frac{1}{2}} - E^{-\frac{1}{2}}$	$\Delta(1 + \Delta)^{\frac{1}{2}}$	$\delta$	$\nabla(1 - \nabla)^{-\frac{1}{2}}$	$2 \sinh(hD/2)$
$\nabla$	$1 - E^{-1}$	$\Delta(1 + \Delta)^{-1}$	$\mu\delta - \delta^2/2$	$\nabla$	$1 - e^{-hD}$
$hD$	$\ln E$	$\ln(1 + \Delta)$	$2 \sinh^{-1}(\delta/2)$	$-\ln(1 - \nabla)$	$hD$
$\mu$	$(E^{\frac{1}{2}} + E^{-\frac{1}{2}})/2$		$(1 + \delta^2/4)^{\frac{1}{2}}$		$\cosh(hD/2)$

### 9.3 Non-linear algebraic equations

**Formula iteration** method for  $x = F(x)$ .

An iterative scheme has the form  $x_{n+1} = F(x_n)$ , which converges to the root  $x = \alpha$ , only if  $|F'(\alpha)| < 1$ . The convergence is first order; *i.e.* if  $|F'(x)| \leq M$  for  $x$  near  $\alpha$ , then the error at each stage satisfies  $|\epsilon_{n+1}| \leq M|\epsilon_n|$ .

**Newton-Raphson** method for  $f(x) = 0$ .

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The method converges to the root  $x = \alpha$  if  $x_0$  is sufficiently close to  $\alpha$ . The convergence is second order *i.e.* if  $|f''(x)| \leq 2M$  for  $x$  near  $\alpha$  then the error at each stage satisfies  $|\epsilon_{n+1}| \leq M|\epsilon_n|^2$ .

**Newton's method** for a system of equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ , where  $\mathbf{f} = \{f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})\}$  and  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ .

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - \mathbf{J}_n^{-1} \mathbf{f}(\mathbf{x}^{(n)})$$

where  $\mathbf{J}_n$  is the Jacobian matrix given by

$$(\mathbf{J}_n)_{ij} = \left[ \frac{\partial f_i}{\partial x_j}(\mathbf{x}^{(n)}) \right]$$

### 9.4 Numerical integration

The region of integration is subdivided into  $n$  equal intervals of width  $h = (b - a)/n$ ,  $x_i = a + ih$  and  $f_i = f(x_i)$ .

**Trapezoidal rule**

$$\int_a^b f(x) dx \approx h \left\{ (f_0 + f_n)/2 + (f_1 + f_2 + \dots + f_{n-1}) \right\}$$

The error,  $\epsilon_n$ , is bounded as follows:

$$|\epsilon_n| \leq (b - a) \frac{h^2}{12} M_1, \text{ where } M_1 = \max_{x \in [a, b]} |f''(x)|$$

**Simpson's rule** ( $n$  must be even)

$$\int_a^b f(x) dx \approx \frac{h}{3} \left\{ f_0 + f_n + 4(f_1 + f_3 + \dots + f_{n-1}) + 2(f_2 + f_4 + \dots + f_{n-2}) \right\}$$

The error,  $\epsilon_n$ , is bounded as follows:

$$|\epsilon_n| \leq (b-a) \frac{h^4}{180} M_2, \text{ where } M_2 = \max_{x \in [a,b]} |f^{(iv)}(x)|$$

### Gauss quadrature

Whenever  $f(x)$  is known explicitly Gauss quadrature can be employed:

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n \omega_i f(x_i)$$

The error,  $\epsilon_n$ , is bounded as follows:

$$|\epsilon_n| \leq \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} M_3 \text{ where } M_3 = \max_{x \in [-1,1]} |f^{(2n)}(x)|$$

### Gauss quadrature: integration points and weights

$n$	$\pm x_i$	$\omega_i$
1	0.00000 00000 00000	2.00000 00000 00000
2	0.57735 02691 89626	1.00000 00000 00000
3	0.00000 00000 00000 0.77459 66692 41483	0.88888 88888 88889 0.55555 55555 55556
4	0.33998 10435 84856 0.86113 63115 94053	0.65214 51548 62546 0.34785 48451 37454
5	0.00000 00000 00000 0.53846 93101 05683 0.90617 98459 38664	0.56888 88888 88889 0.47862 86704 99366 0.23692 68850 56189
6	0.23861 91860 83197 0.66120 93864 66265 0.93246 95142 03152	0.46791 39345 72691 0.36076 15730 48139 0.17132 44923 79170

The  $x_i$  are the positive zeros of the Legendre polynomial  $P_n(x)$ . The formula integrates exactly all polynomials of degree  $2n - 1$ . Any interval  $a \leq X \leq b$  can be transformed to the interval  $-1 \leq x \leq 1$  by the change of variable

$$X = (a+b)/2 + (b-a)x/2$$

For double integrals:

$$\int_{-1}^1 \int_{-1}^1 f(x,y) dx dy \approx \sum_{i=1}^n \sum_{j=1}^n \omega_i \omega_j f(x_i, y_j)$$

## 9.5 Numerical solution of ordinary differential equations

### First order equations

$$y' = f(x, y) \text{ with } y(x_0) = y_0$$

The equation is solved successively at discrete points  $x_1, x_2, x_3, \dots$

where  $x_{i+1} - x_i = h$ .

$y_n$  is the exact value  $y(x_n)$ .

$Y_n$  is the approximation to  $y_n$ , used in the recurrence relation.

### Single-step methods

#### *Euler's method*

$$Y_{i+1} = Y_i + hf(x_i, Y_i), \quad Y_0 = y_0$$

The error is  $\frac{h^2}{2}y''(\xi)$  where  $x_i < \xi < x_{i+1}$ .

#### *Modified Euler method*

$$Y_{i+1} = Y_i + \frac{h}{2} \{f(x_i, Y_i) + f(x_{i+1}, Y_{i+1}^P)\}, \quad Y_0 = y_0$$

where  $Y_{i+1}^P = Y_i + hf(x_i, Y_i)$

The error is  $-\frac{h^3}{12}y'''(\xi)$  where  $x_i < \xi < x_{i+1}$ .

#### *Runge-Kutta (fourth order formulae)*

$$Y_{i+1} = Y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad Y_0 = y_0$$

where  $k_1 = hf(x_i, Y_i), \quad k_2 = hf(x_i + h/2, Y_i + k_1/2),$   
 $k_3 = hf(x_i + h/2, Y_i + k_2/2), \quad k_4 = hf(x_i + h, Y_i + k_3).$

The actual form of the local truncation error,  $\epsilon_{i+1}$ , is extremely complicated; it is, however, of the order  $h^5$ .

### Multi-step methods (Predictor-corrector)

#### *Milne-Simpson method*

$$\text{Predict } Y_{i+1}^P = Y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}]$$

$$\text{Correct } Y_{i+1}^C = Y_{i-1} + \frac{h}{3}[f_{i-1} + 4f_i + f_{i+1}^P]$$

The error in the predictor is  $\frac{14}{45}h^5y^{(v)}(\xi_1)$

and the error in the corrector is  $-\frac{h^5}{90}y^{(v)}(\xi_2)$

where  $x_{i-3} < \xi_1 < x_{i+1}, \quad x_{i-1} < \xi_2 < x_{i+1}.$

*Adams-Moulton method*

$$\text{Predict } Y_{i+1}^P = Y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$$

$$\text{Correct } Y_{i+1}^C = Y_i + \frac{h}{24}[f_{i-2} - 5f_{i-1}^P + 19f_i + 9f_{i+1}]$$

The error in the predictor is  $\frac{14}{45}h^5 y^{(v)}(\xi_1)$

and the error in the corrector is  $-\frac{h^5}{90}y^{(v)}(\xi_2)$

where  $x_i < \xi_1 < x_{i+1}$  and  $x_i < \xi_2 < x_{i+1}$ .

### Higher order equations

The methods given above are easily generalised for use with a system of first order equations, thus we consider only the first order equation in one dependent variable.

The  $n^{\text{th}}$  order equation  $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$  may be reduced to the system of first order equations

$$\begin{aligned} y_1' &= y, & y_2 &= y_1', & y_3 &= y_2', & \dots & y_{n-1} &= y_{n-2}' \\ y_n' &= f(x, y_1, y_2, \dots, y_n) \\ \text{i.e. } \mathbf{y}' &= \mathbf{f}(x, \mathbf{y}) \end{aligned}$$

The corresponding Euler recurrence relation is

$$\mathbf{Y}_{i+1} = \mathbf{Y}_i + hf(x_i, \mathbf{y}_i), \quad \mathbf{Y}_0 = \mathbf{y}_0$$

## 9.6 Systems of linear equations, $n \times n$

$$\begin{array}{cccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 & & & \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 & & & \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n & & & \end{array} \quad \text{in matrix form } \mathbf{Ax}=\mathbf{b}$$

### Direct method

*Gauss elimination*, with partial pivoting

Computational procedure, ( $k = 1 \dots (n-1)$ ):

1. Rearrange the equations so that

$$|a_{kk}| \geq |a_{ik}|, \quad i = (k+1) \dots n$$

2. Compute

$$\mu_{ik} = -a_{ik}/a_{kk}, \quad i = (k+1) \dots n$$

3. Compute

$$\left. \begin{aligned} a'_{ij} &= a_{ij} + \mu_{ik}a_{kj}, & j &= (k+1) \dots n \\ b'_i &= b_i + \mu_{ik}b_k \end{aligned} \right\}$$

4. Compute

$$x_n = b'_n / a'_{nn}$$

$$x_i = \left( b'_i - \sum_{j=i+1}^n a'_{ij} x_j \right) / a'_{ii}, \quad i = (n-1) \dots 1$$

### Indirect methods

*Jacobi*

An iterative scheme is given by

$$x_i^{(r+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{n'} a_{ij} x_j^{(r)} \right), \quad i = 1, 2, \dots, n$$

In matrix form

$$\mathbf{x}^{(r+1)} = \mathbf{b} - [\mathbf{L} + \mathbf{U}]\mathbf{x}^{(r)}$$

*Gauss-Seidel*

An iterative scheme is given by

$$x_i^{(r+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(r+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(r)} \right), \quad i = 1, 2, \dots, n.$$

A sufficient condition for convergence of both methods is that the matrix  $\mathbf{A}$  is *diagonally dominant*

*i.e.*

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad i = 1, 2, \dots, n$$

In matrix form

$$\mathbf{x}^{(r+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(r+1)} - \mathbf{U}\mathbf{x}^{(r)}$$

where

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{a_{21}}{a_{22}} & 0 & 0 & 0 & 0 \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 0 & 0 & 0 \\ \vdots & \vdots & & & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \dots & \frac{a_{nn-1}}{a_{nn}} & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \dots & \frac{a_{1n}}{a_{11}} \\ 0 & 0 & \frac{a_{23}}{a_{22}} & \dots & \frac{a_{2n}}{a_{22}} \\ & & & & \vdots \\ 0 & 0 & 0 & 0 & \frac{a_{n-1n}}{a_{n-1n-1}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and  $\mathbf{b} = [b_1/a_{11} \ b_2/a_{22} \ \dots \ b_n/a_{nn}]^T$ .

The Gauss-Seidel process converges if and only if all the eigenvalues of the matrix  $[\mathbf{I} + \mathbf{L}]^{-1}\mathbf{U}$  have modulus less than one.

*Successive over-relaxation (SOR)*

The *SOR* iterative scheme is

$$\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \omega(\mathbf{b} - \mathbf{L}\mathbf{x}^{(r+1)} - \mathbf{x}^{(r)} - \mathbf{U}\mathbf{x}^{(r)}),$$

where  $1 < \omega < 2$  for over-relaxation, and  $\omega = 1$  for Gauss-Seidel.

The *SOR* process converges if and only if all the eigenvalues of the matrix  $[\mathbf{I} + \omega\mathbf{L}]^{-1}[(1 - \omega)\mathbf{I} - \omega\mathbf{U}]$  have modulus less than one.

## 9.7 Chebyshev polynomials

Recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); \quad T_0(x) = 1, \quad T_1(x) = x$$

$$T_n(x) = \cos(n \cos^{-1} x) \quad (\text{coefficient of } x^n \text{ is } 2^{n-1})$$

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1,$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

The polynomial  $(1/2^{n-1})T_n(x)$  has a smaller upper bound to its magnitude over  $[-1, 1]$  than that of any other polynomial with leading term  $x^n$ .

## 9.8 Numerical eigenvalues and eigenvectors

The matrix eigenvalue problem is given by, see page 13,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

**Power method**

(for a non-repeated dominant eigenvalue  $\lambda_1$ , and eigenvector  $\mathbf{x}_1$ ).

Let  $\mathbf{z}_0$  be an arbitrary vector, but not  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ , then the iterative process  $\mathbf{z}_{i+1} = \frac{1}{k_i}\mathbf{A}\mathbf{z}_i$  (where  $k_i$  is the element with the largest absolute value in  $\mathbf{z}_i$ ) converges, with  $k_i \rightarrow \lambda_1$  and  $\mathbf{z}_i \rightarrow \mathbf{x}_1$ , provided that  $\mathbf{A}$  has  $n$  linearly independent eigenvectors. The convergence will be slow if  $|\lambda_2| \approx |\lambda_1|$ .

### Inverse iteration

(for the eigenvalue  $\lambda$  closest to  $p$  and eigenvector  $\mathbf{x}$ )

Let  $\mathbf{z}_0$  be an arbitrary vector, then the iterative process

$$\mathbf{z}_{i+1} = \frac{1}{k_i}[\mathbf{A} - p\mathbf{I}]^{-1}\mathbf{z}_i$$

converges with  $k_i \rightarrow \frac{1}{\lambda - p}$  and  $\mathbf{z}_i \rightarrow \mathbf{x}$ , where  $k_i$  is the element with the largest absolute value in  $\mathbf{z}_i$ .

### Jacobi method

(for symmetric matrices)

Suppose that  $\mathbf{A}$  is diagonalised by using a sequence of orthogonal transformations

$$\mathbf{D} = \mathbf{T}_k^t \mathbf{T}_{k-1}^t \dots \mathbf{T}_2^t \mathbf{T}_1^t \mathbf{A} \mathbf{T}_1 \mathbf{T}_2 \dots \mathbf{T}_{k-1} \mathbf{T}_k = \mathbf{M}^t \mathbf{A} \mathbf{M}, \text{ say,}$$

then the columns of  $\mathbf{M}$  are the eigenvectors and the diagonal of  $\mathbf{D}$  comprises the corresponding eigenvalues.

Computational procedure:

1. Locate largest off-diagonal element  $a_{pq}$ , say.
2. Compute  $\theta$ , where  $\tan 2\theta = 2a_{pq}/(a_{qq} - a_{pp})$ ,  $|\theta| \leq \pi/4$ .
3. Compute new elements in rows  $p$  and  $q$   
 $a'_{pp} = a_{pp} - (\tan \theta)a_{pq}$        $a'_{qq} = a_{qq} + (\tan \theta)a_{pq}$   
 $a'_{pq} = 0$   
 $a'_{pj} = (\cos \theta)a_{pj} - (\sin \theta)a_{qj}$      $a'_{qj} = (\sin \theta)a_{pj} + (\cos \theta)a_{qj}$
4. If any off-diagonal element is non-zero, return to 1.

### LR method

(for all the eigenvalues)

Form the sequence  $\mathbf{A}_0 = \mathbf{A}$ ,  $\mathbf{A}_{r+1} = \mathbf{U}_r \mathbf{L}_r$ , where  $\mathbf{A}_r = \mathbf{L}_r \mathbf{U}_r$  with  $\mathbf{L}_r$  lower triangular, all diagonal elements are equal to 1 and  $\mathbf{U}_r$  is upper triangular.

For suitable  $\mathbf{A}$  the sequence converges to an upper triangular matrix whose diagonal elements are the eigenvalues of  $\mathbf{A}$ , arranged in order of decreasing modulus.

If the method fails to converge try applying it again to  $\mathbf{A} + p\mathbf{I}$  where  $p$  is a suitable real number. Convergence in this case will be to  $\lambda_i + p$ .



## 9.9 Least squares approximation

Given the data set  $\{(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N)\}$ , the least squares best fit  $n^{\text{th}}$  degree polynomial ( $n < N - 1$ ) is

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

where the coefficients  $a_i$  are found from the normal equations

$$\begin{bmatrix} N & \sum x_i & \sum x_i^2 & \cdots & \sum x_i^n \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \cdots & \sum x_i^{n+1} \\ & \text{symmetric} & \sum x_i^4 & \cdots & \sum x_i^{n+2} \\ & & & \vdots & \\ & & & & \sum x_i^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \vdots \\ \sum x_i^n y_i \end{bmatrix}$$

# Chapter 10

## Statistics

### 10.1 Sample statistics

The following definitions are concerned with the data set  $x_1, x_2, \dots, x_n$ .  
Where appropriate, the data value  $x_i$  occurs with frequency  $f_i$ .

Sample Mean

(arithmetic)  
or Average

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{or} \quad \frac{\sum_{i=1}^k f_i x_i}{\sum_{i=1}^k f_i}$$

Population variance

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n}$$

or

$$\sigma^2 = \frac{1}{\sum_{i=1}^k f_i} \sum_{i=1}^k f_i (x_i - \bar{x})^2 = \frac{\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{\sum f_i}}{\sum f_i}$$

Standard deviation

$\sigma$

Sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{(n-1)}$$

Pooled variance

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Standard error of mean

$$\frac{\sigma}{\sqrt{n}}$$

## 10.2 Regression and correlation

Pearson's Product-moment  
Correlation Coefficient of  
 $n$  sample pairs  $(x_i, y_i)$

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{[\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2]}}$$

$$= \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sqrt{\left\{ \left[ \sum x_i^2 - \frac{(\sum x_i)^2}{n} \right] \left[ \sum y_i^2 - \frac{(\sum y_i)^2}{n} \right] \right\}}}$$

Least squares estimates  
 $a$  and  $b$  in the fitted  
regression line  $\hat{y} = a + bx$

$$b = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}$$

$$a = \bar{y} - b\bar{x}$$

Residual Variance  
in simple linear regression

$$s^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - a - bx_i)^2 = \frac{SSE}{n-2}$$

Spearman's rank correlation  
coefficient of  $n$  pairs  
of sample rankings  $(x_i, y_i)$

$$r_s = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)} \text{ where } d_i = x_i - y_i$$

## 10.3 Distributions

### Discrete uniform distribution

$k$ : the possible number of values

Probability function  $P(x) = \frac{1}{k}$   
 $x = x_1, x_2, \dots, x_k$

Mean  $\frac{k+1}{2}$

Variance  $\frac{k^2 - 1}{12}$

### Hypergeometric distribution

$N$ : population size

$n$ : sample size

$k$ : number of items in population labelled "success"

Probability function  $P(x) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$

Mean	$\frac{nk}{N}$
Variance	$n\frac{k}{N}\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$

**Binomial distribution**

$n$ : number of trials

$p$ : probability of “success” at each trial

Probability function	$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$
	$x = 0, 1, \dots, n$

Mean	$np$
Variance	$np(1-p)$

**Negative binomial distribution**

$r$ : no. of successes required

$x$ : no. of the trial when the  $r^{th}$  “success” occurs

or alternatively, if  $c$  is the number of “failures” preceding the  $r^{th}$  “success”

Probability function	$P(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$
	$x = r, r+1, \dots, \infty$

Mean	$\frac{r}{p}$
Variance	$\frac{r(1-p)}{p^2}$

Probability function	$P(c) = \binom{c+r-1}{c} p^r q^c$
	the general term of the binomial expansions of $p^r(1-q)^{-r}$

Mean	$\frac{r(1-p)}{p}$
Variance	$\frac{r(1-p)}{p^2}$

**Geometric distribution**

$p$ : probability of “success” at each trial

Probability function	$P(x) = p(1-p)^{x-1}$
	$x = 1, 2, \dots$

Mean	$\frac{1}{p}$
------	---------------

Variance	$\frac{1-p}{p^2}$
----------	-------------------

**Poisson distribution**

$m$ : average number of random events in a given interval

Probability function  $P(x) = \frac{e^{-m}m^x}{x!}$   
 $x = 0, 1, \dots$

Mean  $m$

Variance  $m$

**Continuous uniform distribution**

$\alpha$ : minimum value

$\beta$ : maximum value

Probability function  $f(x; \alpha, \beta) = \frac{1}{\beta - \alpha}$   
 $\alpha \leq x \leq \beta$

Mean  $\frac{\beta + \alpha}{2}$

Variance  $\frac{(\beta - \alpha)^2}{12}$

**Exponential distribution**

$m$ : average number of random events in a given interval

Probability function  $f(x; m) = me^{-mx} \quad x > 0$

Mean  $\frac{1}{m}$

Variance  $\frac{1}{m^2}$

**Normal distribution**

$\mu$ : mean

$\sigma^2$ : variance

Probability function  $f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right\}$   
 $-\infty < x < \infty$

Mean  $\mu$

Variance  $\sigma^2$

## Chapter 11

# S. I. Units (Système International d'Unités)

### 11.1 Fundamental units

The *kilogram* (kg) is defined as the mass of the international prototype platinum-iridium cylinder kept at Sèvres.

The *second* (s) is defined as the time taken by 9 192 631 770 periods of the radiation from the transition between the two hyperfine levels of the ground state of the atom Caesium-133.

The *metre* (m) is defined as the path length of light in a vacuum during an instant of  $1/(2.99792458 \times 10^8)$  seconds.

The *mole* (mol) is defined as the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilograms of the isotope Carbon-12.

The *ampère* (A) is defined as that constant current which, if maintained in each of two infinitely long straight parallel wires of negligible cross-section placed 1 metre apart in vacuo, would produce, between the wires, a force  $2 \times 10^{-7}$  newtons per metre length.

The *kelvin* (K) is defined as the fraction  $1/273.16$  of the thermodynamic temperature of the triple point of water.

The *candela* (cd) is defined as the luminous intensity from a source of monochromatic radiation of frequency  $540 \times 10^{12}$  Hz which has a radiant intensity of  $1/683$  Watts per steradian in a given direction.

## 11.2 S. I. Prefixes and multiplication factors

Factor	Prefix	Symbol
$10^{24}$	yotta	Y
$10^{21}$	zetta	Z
$10^{18}$	exa	E
$10^{15}$	peta	P
$10^{12}$	tera	T
$10^9$	giga	G
$10^6$	mega	M
$10^3$	kilo	k
$10^2$	hecto	h
$10^1$	deca	da
$10^{-1}$	deci	d
$10^{-2}$	centi	c
$10^{-3}$	milli	m
$10^{-6}$	micro	$\mu$
$10^{-9}$	nano	n
$10^{-12}$	pico	p
$10^{-15}$	femto	f
$10^{-18}$	atto	a
$10^{-21}$	zepto	z
$10^{-24}$	yocto	y

## 11.3 Basic and derived units

**Table of basic units**

Physical quantity	Dimensions	S. I. unit	Symbol
mass	$M$	kilogram	kg
length	$L$	metre	m
time	$T$	second	s
amount of substance	dimensionless	mole	mol
electric current	$I$	ampère	A
temperature	$\theta$	kelvin	K
luminous intensity	$C$	candela	cd

**Table of supplementary units**

Physical quantity	Dimensions	S. I. unit	Symbol
angle	dimensionless	radian degree	rad $^\circ$
solid angle	dimensionless	steradian	sr
$2\pi \text{ rad} = 360^\circ; \quad 1 \text{ rad} = 57.296^\circ$			

**Table of derived units**

Physical quantity	Dimensions	S. I. unit (symbol)
area	$L^2$	$m^2$
volume	$L^3$	$m^3$
frequency	$T^{-1}$	hertz(Hz)
speed	$LT^{-1}$	$ms^{-1}$
acceleration	$LT^{-2}$	$ms^{-2}$
angular speed	$T^{-1}$	$rads^{-1}$
angular acceleration	$T^{-2}$	$rads^{-2}$
density	$ML^{-3}$	$kgm^{-3}$
momentum	$MLT^{-1}$	$kgms^{-1}$
moment of inertia	$ML^2$	$kgm^2$
angular momentum	$ML^2T^{-1}$	$kgm^2s^{-1}$
force	$MLT^{-2}$	newton(N)
torque, moment of force	$ML^2T^{-2}$	Nm
energy, work	$ML^2T^{-2}$	joule(J)Nm
pressure, stress	$ML^{-1}T^{-2}$	pascal(Pa)Nm $^{-2}$
power, radiant flux	$ML^2T^{-3}$	watt(W)Js $^{-1}$
viscosity (dynamic)	$ML^{-1}T^{-1}$	Pas
surface tension	$MT^{-2}$	Nm $^{-1}$
electric charge	$TI$	coulomb(C)
electric potential	$ML^2T^{-3}I^{-1}$	volt(V)WA $^{-1}$
electric resistance	$ML^2T^{-3}I^{-2}$	ohm( $\Omega$ )VA $^{-1}$
electric conductance	$M^{-1}L^{-2}T^3I^2$	siemens(S)
electric field strength	$MLT^{-3}I^{-1}$	Vm $^{-1}$
electric charge density	$L^{-3}TI$	Cm $^{-3}$
electric flux density	$L^{-2}TI$	Cm $^{-2}$
capacitance	$M^{-1}L^{-2}T^4I^2$	farad(F)AsV $^{-1}$
magnetic flux	$ML^2T^{-2}I^{-1}$	weber(Wb)
magnetic flux density	$MT^{-2}I^{-1}$	tesla(T)Wbm $^{-2}$
inductance	$ML^2T^{-2}I^{-2}$	henry(H)WbA $^{-1}$
permittivity( $\epsilon$ )	$M^{-1}L^{-3}T^4I^2$	Fm $^{-1}$
permeability( $\mu$ )	$MLT^{-2}I^{-2}$	Hm $^{-1}$
quantity of heat	$ML^2T^{-2}$	joule(J)
heat capacity, entropy	$ML^2T^{-2}\theta^{-1}$	JK $^{-1}$
specific heat capacity, specific entropy	$L^2T^{-2}\theta^{-1}$	Jkg $^{-1}$ K $^{-1}$
radiant flux density	$MT^{-3}$	Wm $^{-2}$
thermal conductivity	$MLT^{-3}\theta^{-1}$	Wm $^{-1}$ K $^{-1}$
latent heat	$L^2T^{-2}$	Jkg $^{-1}$
molar energy	$ML^2T^{-2}$	Jmol $^{-1}$
molar entropy, molar heat capacity	$ML^2T^{-2}\theta^{-1}$	Jmol $^{-1}$ K $^{-1}$
luminous flux(1cd=1lm sr $^{-1}$ )	$ML^2T^{-3}$	lumen(lm)
illuminance	$MT^{-2}$	lux(lx)



## 11.4 Values of some physical constants

Quantity	Symbol	Value
speed of light in a vacuum	$c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$	$2.99792458 \times 10^8 \text{ms}^{-1}$
permeability of a vacuum	$\mu_0$	$4\pi \times 10^{-7} \text{Hm}^{-1}$
permittivity of a vacuum	$\epsilon_0$	$8.854187817 \times 10^{-12} \text{FM}^{-1}$
elementary charge	$e$	$1.60217733 \times 10^{-19} \text{C}$
Coulomb force constant	$k_e$	$8.9875 \times 10^9 \text{Nm}^2\text{C}^{-2}$
fine structure constant	$\alpha = e^2/(4\pi\epsilon_0\hbar c)$	1/137.0
gravitational constant	$G$	$6.67258 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$
atomic mass unit	u	$1.6605402 \times 10^{-27} \text{kg}$
energy equivalent of u		931.494MeV
rest mass of electron	$m_e$	$9.1093897 \times 10^{-31} \text{kg}$
		0.0005486u
electron rest energy	$m_e c^2$	0.5109991MeV
rest mass of proton	$m_p$	$1.6726231 \times 10^{-27} \text{kg}$
		1.007276u
proton rest energy	$m_p c^2$	938.27231MeV
rest mass of neutron	$m_n$	$1.6749286 \times 10^{-27} \text{kg}$
		1.008665u
neutron rest energy	$m_n c^2$	939.56563MeV
Planck's constant	$h$	$6.626075 \times 10^{-34} \text{Js}$
	$\hbar = h/(2\pi)$	$1.05457266 \times 10^{-34} \text{Js}$
Rydberg energy	$R_\infty = \alpha^2 m_e c^2 / 2$	13.61eV
Bohr radius	$a_0$	$5.29177249 \times 10^{-11} \text{m}$
Compton wavelength of electron	$\lambda_C$	$2.42631058 \times 10^{-12} \text{m}$
Avogadro constant	$N_A$	$6.0221367 \times 10^{23} \text{mol}^{-1}$
Boltzmann constant	$k$	$1.380658 \times 10^{-23} \text{JK}^{-1}$
Stefan-Boltzmann constant	$\sigma$	$5.67052 \times 10^{-8} \text{Wm}^{-2}\text{K}^{-4}$
Wien's (displacement law) constant		$2.8978 \times 10^{-3} \text{mK}$
Bohr constant	$\mu_B$	$9.2740154 \times 10^{-24} \text{JT}^{-1}$
nuclear magneton	$\mu_N$	$5.05057866 \times 10^{-27} \text{JT}^{-1}$
triple point temperature	$T_t$	273.16K
molar gas constant	$R$	$8.315 \text{JK}^{-1} \text{mol}^{-1}$
micron	$\mu\text{m}$	$10^{-6} \text{m}$
ångström	$\text{Å}$	$10^{-10} \text{m}$
femtometre or fermi	fm	$10^{-15} \text{m}$
barn	b	$10^{-28} \text{m}^2$

## 11.5 Useful masses

The atomic mass unit, u, is defined to be  $\frac{1}{12}$  of the mass of the Carbon-12 atom.

### Table of particle masses

particle/atom	mass
$p$	$m_p = 1.6726231 \times 10^{-27} \text{kg}$ $= 1.007276 \text{u}$ $= 938.272 \text{MeV}/c^2$
$e$	$m_e = 9.1093897 \times 10^{-31} \text{kg}$ $= 0.0005486 \text{u}$ $= 0.511 \text{MeV}/c^2$
$n$	$m_n = 1.6749286 \times 10^{-27} \text{kg}$ $= 1.008665 \text{u}$ $= 939.566 \text{MeV}/c^2$
${}^1_1\text{H}$	1.007825u
${}^2_1\text{H}$	2.014102u
${}^3_1\text{H}$	3.016049u
${}^3_2\text{He}$	3.016029u
${}^4_2\text{He}$	4.002603u
${}^6_3\text{Li}$	6.015121u
${}^7_3\text{Li}$	7.016003u
${}^{12}_5\text{B}$	12.014352u
${}^{12}_6\text{C}$	12.000000u
${}^{13}_6\text{C}$	13.003355u
${}^{14}_6\text{C}$	14.003242u
${}^{12}_7\text{N}$	12.018613u
${}^{13}_7\text{N}$	13.005738u
${}^{14}_7\text{N}$	14.003074u
${}^{92}_{36}\text{Kr}$	91.8973u
${}^{141}_{56}\text{Ba}$	140.9139u
${}^{210}_{84}\text{Po}$	209.982848u
${}^{218}_{84}\text{Po}$	218.008965u
${}^{220}_{86}\text{Rn}$	220.011369u
${}^{222}_{86}\text{Rn}$	222.017574u
${}^{224}_{88}\text{Ra}$	224.020187u
${}^{226}_{88}\text{Ra}$	226.025402u
${}^{227}_{90}\text{Th}$	227.027701u
${}^{228}_{90}\text{Th}$	228.028716u
${}^{230}_{90}\text{Th}$	230.033127u
${}^{231}_{91}\text{Pa}$	231.035880u
${}^{234}_{90}\text{Th}$	234.043593u
${}^{235}_{92}\text{U}$	235.043924u
${}^{238}_{92}\text{U}$	238.050784u
${}^{235}_{93}\text{Np}$	235.044057u

## 11.6 Astronomical constants

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Mass of Earth	$m_E$	$5.976 \times 10^{24} \text{kg}$
Radius of Earth (equatorial)	$R_E$	$6.378 \times 10^6 \text{m}$
Gravity at Earth's surface	$g$	$9.80665 \text{ms}^{-2}$
Mass of Sun	$M_\odot$	$1.989 \times 10^{30} \text{kg}$
Radius of Sun	$R_\odot$	$6.9599 \times 10^8 \text{m}$
Solar effective temperature	$T_e$	5800K
Luminosity of Sun	$L_\odot$	$3.826 \times 10^{26} \text{W}$
Astronomical unit	AU	$1.496 \times 10^{11} \text{m}$
Parsec	pc	$3.086 \times 10^{16} \text{m}$
Jansky	Jy	$10^{-26} \text{Wm}^{-2} \text{Hz}^{-1}$
Tropical year		$3.1557 \times 10^7 \text{s}$
Standard atmosphere	atm	101325Pa

## 11.7 Mathematical constants

---

pi (Archimedes' constant)	$\pi$	3.141 592 653 793
exponential constant	$e$	2.718 281 828 459
Apery's constant	$\zeta(3)$	1.202 056 903 160
Catalan's constant	$K$	0.915 965 594 177
Dottie's number	$r$	0.739 085 133 215
Euler's constant	$\gamma$	0.577 215 664 902
Feigenbaum's constant	$\alpha$	2.502 907 875 096
Feigenbaum's constant	$\delta$	4.669 201 609 102
Gelfond's constant	$e^\pi$	23.140 692 632 78
Gibb's constant	$G$	1.851 937 051 982
Golden mean	$\phi$	1.618 033 988 750
Khintchine's constant	$K$	2.685 452 001 065
omega constant	$\Omega$	0.567 143 290 410
parabolic constant	$P_2$	2.295 587 149 393
plastic constant	$P$	1.324 717 957 245
Sierpiński's constant	$K$	2.584 981 759 579
Trott's constant	$T_2$	0.108 410 151 223
Wallis's constant	$W$	2.094 551 481 542

## 11.8 The Greek alphabet

Letter	lowercase	uppercase
Alpha	$\alpha$	$A$
Beta	$\beta$	$B$
Gamma	$\gamma$	$\Gamma$
Delta	$\delta$	$\Delta$
Epsilon	$\epsilon$	$E$
Zeta	$\zeta$	$Z$
Eta	$\eta$	$H$
Theta	$\theta$	$\Theta$
Iota	$\iota$	$I$
Kappa	$\kappa$	$K$
Lambda	$\lambda$	$\Lambda$
Mu	$\mu$	$M$
Nu	$\nu$	$N$
Xi	$\xi$	$\Xi$
Omicron	$o$	$O$
Pi	$\pi$	$\Pi$
Rho	$\rho$	$P$
Sigma	$\sigma$	$\Sigma$
Tau	$\tau$	$T$
Upsilon	$\upsilon$	$Y$
Phi	$\phi, \varphi$	$\Phi$
Chi	$\chi$	$X$
Psi	$\psi$	$\Psi$
Omega	$\omega, \varpi$	$\Omega$

# Index

- Acceleration vector, 31
- Adams-Moulton method, 55
- Ampère, 64
- Analytic function, 9
- Apery's constant, 69
- Arc length of a curve, 27
- Archimedes' constant, 69
- Argand diagram, 8
- Argument, 8
- Arithmetic mean, 9
- Astronomical constants, 69
- Auxiliary equation, 33
- Average, 60
  
- Bernoulli's inequality, 9
- Bessel function, 23
- Bessel's equation, 23
- Beta function, 22
- Binomial series, 25
- Binomial theorem, 3
  
- Candela, 64
- Catalan's constant, 69
- Cauchy Schwarz inequality, 9
- Cauchy-Riemann equations, 9
- Centroid, 28
- Chain rule, 17
  - ordinary differentiation, 17
  - partial differentiation, 17
- Chebyshev polynomials, 22, 57
- Circle, 6, 7
- Circumcircle, 7
- Cofactor, 10
- Combinations, 3
- Complementary error function, 22
- Complementary function, 33
  
- Complex conjugate, 8
- Complex numbers, 7
- Complex roots, 8
- Complex variable, 9
- Cone, 7
- Conic sections, 6
- Convolution, 43
- Correlation, 61
- Cosine rule, 7
- Cramer's rule, 11
- Cubic equation, 2
- Cubic splines, 50
- Curl, 37
- Curvature, 23
- Cylinder, 7
  
- De L'Hôpital's rule, 24
- De Moivre's theorem, 4, 8
- Degrees and Radians, 3
- Determinant, 10
- Diagonally dominant, 56
- Differentiation, 17
  - chain rule, 17
  - differentiation of integrals, 18
  - Leibniz rule, 17
  - linearity rule, 17
  - product rule, 17
  - quotient rule, 17
  - standard forms, 19–21
  - total differential, 18
- Directional derivative, 37
- Directrix, 6
- Distributions, 61
  - binomial, 62
  - continuous uniform, 63
  - discrete uniform, 61

- exponential, 63
- geometric, 62
- hypergeometric, 61
- negative binomial, 62
- normal, 63
- Poisson, 63
- Div, 37
- Dottie's number, 69
- Dynamics, 31
- Eccentricity, 6
- Eigenvalues, 13
- Eigenvalues and eigenvectors, 13
  - inverse iteration, 58
  - Jacobi method, 58
  - LR method, 58
  - power method, 57
- Eigenvectors, 13
- Ellipse, 6, 7
- Error function, 22
- Euler's constant, 69
- Euler's formulae, 8
- Euler's method, 54
  - modified, 54
- Exponential constant, 2, 69
- Factorial, 1
- Factors and roots of equations, 2
- Feigenbaum's constant, 69
- Finite difference operators, 51
- Finite series, 1
- Focus, 6
- Formula iteration, 52
- Fourier series, 35, 47
  - complex series, 36
  - half-range series, 35
  - sine and cosine series, 35
- Fourier transform, 46
  - sine and cosine transform, 46
- Fundamental theorem of algebra, 8
- Fundamental units, 64
- Gamma function, 22
- Gauss elimination, 55
- Gauss quadrature, 53
  - double integrals, 53
- Gauss's divergence theorem, 40
- Gauss-Seidel iteration, 56, 57
- Gelfond's constant, 69
- General conic, 7
- General orthogonal curvilinear coordinates, 39
- Geometric mean, 9
- Geometry, 5
- Gibb's constant, 69
- Golden mean, 69
- Grad, 37
- Gram Schmidt process, 12
- Green's theorem, 40
- Green's theorem in the plane, 27
- Half-rectified sine wave, 49
- Heron's formula, 7
- Hessian, 23
- Hyperbola, 6, 7
- Hyperbolic functions, 5
- Imaginary part, 8
- Indices, 1
- Inequalities, 9
- Initial condition, 32
- Initial-value problem, 32, 34
- Inscribed circle, 7
- Integrating factor method, 32
- Integration, 18
  - by parts, 18
  - by substitution, 18
  - definite integrals, 21
  - differentiation of integrals, 18
  - standard forms, 19–21
- Interpolation, 50
- Intrinsic coordinates, 23, 31
- Jacobi iteration, 56
- Jacobian, 25
- Kelvin, 64
- Khinchine's constant, 69
- Kilogram, 64
- Kinematics, 31

Kronecker  $\delta$ , 11, 21  
 Lagrange interpolation poly., 50  
 Lagrange multipliers, 24  
 Laplace transform, 41, 44, 47  
     inversion formula, 44  
     standard forms, 41–43  
 Laplacian operator, 38  
 Least squares approximation, 59  
 Least squares regression, 61  
 Legendre function, 23  
 Legendre polynomial, 22  
 Legendre's equation, 23  
 Limiting theorems, 44  
 Limits, 24  
 Line integrals, 27  
 Linear equations, 12, 55  
 Linearly dependent, 12  
 Linearly independent, 12  
 Logarithms, 1  
  
 Maclaurin series, 5, 24  
 Mathematical constants, 69  
 Matrices, 11  
     identity matrix, 11  
     inverse matrix, 12  
     null matrix, 12  
     orthogonal matrix, 12  
     singular matrix, 12  
     symmetric matrix, 11  
     trace of a matrix, 13  
     transpose matrix, 11  
     unit matrix, 11  
     zero matrix, 12  
 Mean, 60  
 Mean value, 28  
 Metre, 64  
 Milne-Simpson method, 54  
 Minkowski inequality, 9  
 Minor, 10  
 Modified Bessel function, 23  
 Modulus, 8  
 Mole, 64  
 Moment of inertia, 28, 31  
     standard table, 29, 30  
 Multiple integration, 25  
 Nabla operator, 37  
 Natural logarithm, 2  
 Newton's second law, 31  
 Newton-Raphson method, 52  
     systems of equations, 52  
 Numerical integration, 52  
 Numerical odes, 54  
     first order equations, 54  
     higher order equations, 55  
  
 Omega constant, 69  
 Ordinary differential equations, 32  
     first order equations, 32  
     higher order equations, 34  
     second order equations, 33  
 Osborne's rule, 5, 25  
  
 Parabola, 6, 7  
 Parabolic constant, 69  
 Parallel axes theorem, 29  
 Parseval's theorem, 36  
 Partial fractions, 2  
 Particular integral, 33  
 Pearson's correlation coefficient, 61  
 Periodic functions, 47  
 Permutations, 3  
 Physical constants, 67  
 Plane polar coordinates, 31  
 Plastic constant, 69  
 Polar coordinates, 15  
     cylindrical, 16, 26, 38  
     plane, 15, 26, 31  
     spherical, 16, 27, 39  
 Polar equation, 7  
 Polynomial, 1, 2  
 Polynomial interpolation, 50  
 Pooled variance, 60  
 Population variance, 60  
 Position vector, 31  
 Power function, 2  
 Power series, 24  
 Predictor-corrector methods, 54

Probability function, 61  
 Pythagoras' theorem, 6

Quadratic equation, 2

Radius of convergence, 24  
 Radius of curvature, 23  
 Rank, 12  
 Real part, 8  
 Rectangular hyperbola, 6, 7  
 Rectified sine wave, 49  
 Regression, 61  
 Remainder theorem, 2  
 Residual variance, 61  
 Root mean square (*RMS*), 28  
 Roots of equation, 9  
 Runge-Kutta 4th order method, 54

Sample statistics, 60  
 Sample variance, 60  
 Sawtooth wave, 48  
 Second, 64  
 Second moment of area, 30  
 Series, 24  
 Sierpiński's constant, 69  
 Sigma notation, 1  
 Simpson's rule, 52  
 Sine rule, 7  
 Solid of revolution, 28
 

- surface area, 28
- volume, 28

Spearman's rank correlation coef., 61  
 Sphere, 7  
 Square wave, 47, 48  
 Standard deviation, 60  
 Standard error of mean, 60  
 Stationary points, 23
 

- more than one variable, 23
- one variable, 23

Stehfest method, 44  
 Stokes's theorem, 40  
 Straight lines, 7  
 Successive over-relaxation (*SOR*), 57  
 Surface integrals, 26  
 Système International d'Unités, 64

Basic and derived units, 65  
 S. I. Prefixes, 65

Taylor's series, 24
 

- one variable, 24
- two variables, 25

Taylor's theorem, 24  
 Trapezoidal rule, 52  
 Trial function, 33  
 Triangle, 7  
 Triangle inequality, 9  
 Triangle law for vectors, 14  
 Triangular wave, 48  
 Trigonometric identities, 3  
 Trott's constant, 69

Undetermined coefficients, 33  
 Unit vector, 15
 

- cartesian, 16
- cylindrical, 16

Variables separable, 32  
 Vector addition, 14  
 Vector calculus, 39  
 Vector identities, 38  
 Vector products, 14  
 Velocity vector, 31  
 Volume integrals, 26

Wallis's constant, 69  
 Wallis's formulae, 21  
 Z-transform, 45