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Rudrapatna V. Ramnath

Computation and Asymptotics

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ISSN 2191-5342
ISBN 978-3-642-25748-3
DOI 10.1007/978-3-642-25749-0
Springer Heidelberg Dordrecht London New York

e-ISSN 2191-5350
e-ISBN 978-3-642-25749-0

Library of Congress Control Number: 2011944257

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to the fond memory of my parents,
Sri R. Venkataramaiya and Smt. Seethamma

Foreword

The monograph *Computation and Asymptotics* is published in the series of SpringerBriefs in Applied Sciences and Technology—Computational Mechanics, which features cutting-edge research and practical applications presented and introduced by leading experts.

The author of this contribution, Professor Rudrapatna V. Ramnath, received a M.Sc. degree in Electrical Engineering (Flight Control) from Cranfield Institute of Technology (U.K.), M.S.E., M.A. and Ph.D. degrees in Aerospace and Mechanical Engineering from Princeton University. He generalized the multiple scales theory and significantly contributed to the theory and applications of asymptotic analysis. Using multiple scales theory, he solved a large class of problems in advanced aircraft and spacecraft dynamics and control including: hover-forward flight transitions of VTOL aircrafts, analytical theory of high angle-of-attack flight dynamics (for 1, 2, and 3 degree-of-freedom wing rock), re-entry vehicle dynamics, stability and parameter sensitivity of time-varying systems applied to the Generic Hypersonic Aerodynamic Model of NASA, etc. Furthermore, he developed a new handling quality theory of advanced aircrafts through variable flight conditions. In spacecraft he developed new and useful approaches for such as new attitude control design for single spin and dual spin satellites, dynamics, stability and control of large space structures with deformable reflectors, and mode shape determination for Heliogyro spacecraft, etc. Professor Ramnath's work also includes the development of rapid computational algorithms and reduced order models which were applied to NASA and Defense Department systems. He applied his theory to many aerospace systems at The Charles Stark Draper Laboratory and SPARTA, Inc., and also designed guidance laws for tactical missiles at Raytheon Missile Systems. He founded the company Vimanic Systems for performing research contracts for NASA centers including Dryden Flight Research Center and U.S. Department of Defense Laboratories.

For many years he taught undergraduate and graduate courses at Massachusetts Institute of Technology (MIT) on advanced aircraft and spacecraft dynamics and control and systems modeling. His scientific achievements are highly recognized through numerous technical papers and books on nonlinear control (published by

ASME), *Multiple Scales Theory and Aerospace Applications* (published by AIAA) and book chapters. In addition at MIT, Professor Ramnath conceived and developed the instrumentation, test methodology and procedures for performance evaluation of sports equipment in tennis and racket sports and golf clubs. He served as the Technical Editor of *Tennis* magazine and as Technical Advisor of ATP (Association of Tennis Professionals).

The monograph *Computation and Asymptotics* is a state-of-the-art presentation of the pioneering works of Professor Ramnath in applying and developing asymptotic analysis and the generalized multiple scales theory for efficient computations. The novelty of the approach is to introduce these two concepts in demanding computational problems in order to facilitate and to speed-up the solution procedure while ensuring the required accuracy. This work is well balanced between theoretical concepts and computation-intensive applications from the aerospace field. It serves as an excellent introduction to the power of combining asymptotic analysis and multiple scales approaches to complex computational problems to enable the aspiring engineer, interested in applied mathematical approaches, to achieve greater rewards in enhanced insight and computational efficiency in dynamic analysis, design and simulation. As such, I would highly recommend this work as a valuable addition to the technical library of a serious engineering analyst and practitioner.

December 2011

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Preface

In contrast to the traditional technical literature dealing with numerical computation and analytical asymptotic solution approaches, this book aims at presenting a methodology and philosophy that combine both aspects in a common synergistic combination. The purpose is to utilize the best features of each approach in an effort to facilitate efficient computation and glean insight into the nature of the system under study. In order to do this, it must be recognized that some degree of clever insight and understanding of the system behavior is needed in addition to implementing the optimal computational algorithms. In this sense the philosophy of this book marks a departure or a variation from the traditional and independent roles of computation and analytical solutions. The purpose is to combine the power of each approach and achieve a result that is greater than the sum of the parts. This is the view pursued and presented in this book.

This book is addressed to the dynamics and control systems analysts and designers who deal with computational issues of complex problems involving advanced scientific and engineering systems. In particular, engineers and scientists working with the computational aspects of sophisticated systems would find it very useful. Several important applications illustrating the benefits of a unified approach are presented. In addition, the techniques and methodology should be of interest to analysts and designers in mechanical, electrical and chemical engineering, and also some areas of physics, chemistry and biology.

The aim of this work is to present the powerful concepts and techniques of asymptotics and multiple scales approach in the context of computations of the dynamics of modern high performance scientific and engineering systems. Although the presentation is intended to be systematic and rigorous, the emphasis is on the concept, applicability and usefulness of these methods rather than on deep mathematical rigor. Reflecting my own interests, I have attempted to motivate the reader and appeal to a need for completeness, connectedness and philosophical abstraction in developing the theoretical framework. Recent research has culminated in a wealth of useful information and techniques which are generally unfamiliar to the practising engineer. I feel that substantial rewards are to be

gained by applying these techniques. The book intended to be used as a monograph reflecting the above spirit and philosophy.

Approximate solutions to complex physical and mathematical problems have been in use for a long time. This work concerns an area of approximations in applied mathematics known as asymptotic analysis and perturbation theory which deal with the study of applied mathematical systems in limiting cases. There are a number of good books on asymptotic analysis and perturbation theory, which form the basis upon which the multiple scales method rests. They are cited in the references and the reader would benefit by consulting them. In the technical literature the treatment has ranged from simple constructs to highly theoretical topics. However, the subject of multiple scales has been only briefly outlined in some books until recently. This powerful technique is relatively recent in its development, which is still continuing. It has led to a number of general and useful results which have been applied to a large number of diverse advanced engineering systems. Much of this body of knowledge, however, resides in research papers or is only partially treated in a small number of specialized mathematical books. A full and general development of the technique in available books appears to be lacking. Consequently, the power and usefulness of the technique is not well known among practising engineers. The value of this approach is being appreciated more and more with time as new applications are seen. A comprehensive development of this technique in its own right seems to be warranted by the rapid growth of the theory and the range of it applications covering a rather broad spectrum of engineering and scientific disciplines. It is in this spirit that this book is written. Engineering analysts and designers should derive much benefit from the simplicity of the concept and the general applicability of the method. The book is aimed at filling this gap and covering the middle ground between an entirely heuristic treatment and one of deep rigor and sophistication. It is intended as a bridge between esoteric mathematical theory and practical applications in the "real world".

The early chapters present the basic concept, foundations and the techniques of asymptotic analysis, perturbation theory, multiple scales and an outline of standard numerical methods. As the basic ideas and concepts of asymptotic analysis are essential to a proper development of the multiple scales theory, a brief discussion of asymptotic analysis is first presented. Next, elements of perturbation theory are discussed, mainly as relevant to multiple scaling. Deeper insight into perturbation theory may be gained by the interested reader by consulting many well known works on the subject cited in the references.

The computational advantages of asymptotics is illustrated through applications. First, a classical problem of computing the effective nuclear charge near a singularity in the Thomas-Fermi problem in atomic physics is solved by multiple scales approach (Chap. 6). Next the problem of computational speed-up is demonstrated in the case of the satellite attitude prediction (Chap. 7) with gravity gradient and geomagnetic torques for earth satellites. Chapter 8 presents the satellite attitude control problem for which the multiple scales approach facilitates the task of control design and computational efficiency by enabling the use of large

step sizes without loss of accuracy. All these problems utilize the multiple scales technique, which leads to solutions which are easily calculable and are accurate when compared to the conventional numerical solutions. Also included is an appendix on earth's environmental gravity gradient torque and geomagnetic torques and Floquet theory which is mainly of reference value.

The selection and presentation of the subjects reflects my own interests and experience. The material is an outgrowth of the lectures and courses taught by me at Princeton University and Massachusetts Institute of Technology and includes much of the research carried out by me with my students.

It is with pleasure that I record here that the original motivation, interest and inspiration came through meetings with Professor M.J. Lighthill first in India and later in the U.K. when I was a student. Later as a student at Princeton University, I was initiated into the subject by my Professors W.D. Hayes and M.D. Kruskal, my friend Dr. G.V. Ramanathan and by Dr. G. Sandri. I wish to record the interest and insight into engineering applications that I received from Professor D. Graham and Professor D. Seckel at Princeton and Dr. D.C. Fraser and Dr. R.H. Battin at the C.S. Draper Laboratory. Further, I wish to recognize the vigorous interaction and participation by my students in this research leading to many useful and important results.

Finally, I recognize with great appreciation the encouragement and support from my wife Vijaya, in motivating and helping me greatly in preparing many figures and the final manuscript. I also acknowledge the considerable joy that filled me from my wife and my children,—my son Venktesh, and daughters Seetha and Leela.

Lexington, MA, December 2011 Rudrapatna V. Ramnath

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Part I
Theory

Chapter 1
Introduction

Recognizing the important role of computation in scientific and engineering investigations, we will address, in this work, the topic of quantitative approaches required in the simulation and computation of the dynamics of engineering systems. For this purpose, we will consider a novel look at utilizing a class of classical methods of asymptotic analysis. Thus, we will make use of asymptotic solutions in order to calculate the dynamics of systems, as an alternative to their conventional use of representing the qualitative behavior of systems. This book represents a special viewpoint in which asymptotic solutions are sought for computational purposes. This is in contrast to the traditional approach in which asymptotic methods are used to generate analytical solutions describing the qualitative features of a physical phenomenon. However, a good case can be made to utilize asymptotic solutions for computational purposes. We note that in order to do this, a good deal of cleverness and intuition are required and are not, in general, as straightforward as numerical methods. A comparison of asymptotic and numerical approaches is presented later.

It is well known that mathematical analysis of scientific and engineering systems is usually in terms of differential equations which are derived by invoking Newton's laws. The next step is to try to solve them by some means. If they can be solved analytically, the computation becomes straightforward. Generally, the governing equations do not lend themselves to analytical solutions. The main recourse then, is to develop quasi-analytical approximate solutions or numerical solutions. Historically, these two approaches have been treated somewhat independently. Each approach has been developed in great breadth and depth leading to a large number of applications.

At the outset, it is of interest to consider each approach in its own right, with its own advantages and drawbacks. Later it may be worthwhile to combine the power of each approach for a common purpose. Approximations are usually generated by making some assumption in the mathematical model about the solution or by neglecting terms in the governing equation. The simpler model is then solved, resulting in partial information about the true solution. While numerical solutions are also a form of approximation, we make a clear distinction between purely numerical

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics,
DOI: 10.1007/978-3-642-25749-0_1, © The Author(s) 2012

solutions and other analytical approximations. For example, in a numerical approach the mathematical operation of differentiation is replaced by a finite difference model, thus bypassing the limiting process of the latter to yield the former. The numerical approaches are well suited to digital computer implementation and provide specific answers to specific questions.

Analytical approximations are usually obtained, not by brute force, but involve some degree of clever and subtle insight into the nature of the system dynamics. Such solutions are useful in qualitative studies of complex systems and enable us to delve into such issues as the trend and behavior of the system, instead of providing specific and definitive answers to facilitate quantitative calculation on particular mathematical models. The numerical and analytical approaches will be treated separately.

There are several kinds of analytical approximations. Of these, mention can be made of techniques such as Fourier series in periodic functions and generalizations in terms of the Fourier integral and orthonormal functions, asymptotic analysis and perturbation theory, and so on. In this book, we are particularly interested in asymptotic analysis and multiple scales approach, to be discussed later in detail.

Most of the problems commonly encountered in the mathematical analysis of dynamic systems are not amenable to exact solutions. Often the only recourse is to develop methods to approximate the behavior of the system under study. Indeed, it matters little whether an exact problem is solved approximately, or an approximate problem solved exactly. it is seldom that one finds a technique perfectly suited to a problem. Even on the rare occasion when there is an *exactly* solvable situation, the problem must somehow be brought into a form which is suitable to the methods of analysis. For a large class of systems such a modification is usually not easy. While particular methods are successful on specific problems, a unified approach is highly desirable.

Historically, numerical and asymptotic methods have been developed relatively independently. Traditionally, analytical solutions have enabled the analyst to predict the qualitative features of the phenomenon being studied. This is the classical approach, usually addressed by asymptotic solutions. On the other hand, numerical methods, by means of computer implementation, have been used to obtain solutions to specific problems which may be analytically intractable. All through history, there has been great interest in the development of computational methods. Many mathematicians have expended much effort on this class of problems. This underscores the fact that the lack of high-speed computational capability spurred mathematicians to resort to clever approximations in order to analyze the complex dynamics of physical systems. One great advantage is that these highly efficient methods, in conjunction with high-speed computers, lead to useful approaches for solving complex problems. In general, approximations yield qualitative insight and shed light on such questions as system stability, response, parameter sensitivity, etc. However, when one wishes to obtain a solution to a specific problem which is analytically intractable, then one usually resorts to the use of a computer. A great wealth of technical literature exists in both areas. The recent growth in high-speed computer technology suggests that a combination of these two approaches might be better than either one in enhancing the

power of computation. Ideally, the result might be the development of highly efficient computational capability and also provide great insight into system behavior.

The complexity of modern engineering systems necessitates the use of sophisticated techniques in order to solve the various problems. For example, in the field of flight vehicles, the development of high performance systems such as aircraft with large flight envelopes, vertical and short take-off and landing (V/STOL) vehicles and space vehicles requires elaborate and detailed study to meet the performance requirements. Physical intuition and experience as well as computer simulations usually provide answers to many of the questions. However, in order to glean physical insight and understand the system behavior, one needs to *simplify* the mathematical representation of the system so that conventional methods are applicable.

Note that in general, it is *impossible* to develop *exact* analytical solutions even for a first order nonlinear or a second order linear differential equation with variable coefficients. In contrast, it is generally agreed that linear differential equations with constant coefficients of any order are solvable. Actually, this only means that the problem of solving a linear differential equation with constant coefficients can be reduced to the problem of solving an algebraic equation of the polynomial type. In fact, in a strict mathematical sense, such an algebraic equation is not analytically solvable *exactly* in general, when the degree of the equation is greater than four. It is well known that such equations of degree no greater than four can be solved in general, in terms of radicals. However, a fact that is not well known is that the French mathematician, Hermite [1] solved *exactly* the generic algebraic equation of degree five [1]. Here there is no contradiction because Hermite's ingenious solution is not in terms of radicals, but requires modular functions. These issues are mainly of mathematical interest. In practical cases, there are well known numerical algorithms (eg. Newton-Raphson method, discussed later) which can be used to obtain accurate approximate numerical solutions to polynomial type algebraic equations of high degree. Therefore, it is generally agreed that linear differential equations with constant coefficients are solvable exactly. In order to settle the question of exact mathematical solvability of these equations of any order, one has to invoke the mathematical theories of Galois [2] and Abel [2] on the theory of groups. These issues are beyond the current scope.

Approximations are universally resorted to, though implicitly in many cases. However, a systematic use of rigorous approximation theory offers much greater rewards in general, than *ad hoc* approaches (which may sometimes be called *ad hoaxes*). A systematic approach to approximation theory can be found in the subject of *asymptotics*, -the branch of mathematics that deals with the study of applied mathematical systems in *limiting* cases. A substantial body of knowledge exists in this field, in both the rigorous and applied sense.

Asymptotic methods have been successfully applied to many problems in fluid mechanics, astrodynamics, physics etc. It is the purpose of this work to present the general concept and technique of asymptotic analysis in the context of simulation and computation on various problems of modern engineering. The subject of asymptotics has a vast literature, rich in results, both in the classical problems of mathematical physics and in the more applied problems of science and advanced engineering.

There are many examples of asymptotic phenomena which cover a wide range of problems, from the very practical to the highly theoretical analysis. We may now consider some examples of problems in the analysis of dynamic systems which are amenable to approximation theory. In linear time-invariant (LTI) systems an important problem is that of widely separated eigenvalues (i.e., *stiff systems*). The motion of flight vehicles is a combination of the different aspects of rigid body modes, actuator dynamics, structural modes, etc., all of which are well separated in frequency. Such a frequency separation could itself be the basis of approximation in an asymptotic sense. It would show at what level and to what extent the different modes would begin to contribute to the total motion.

The earliest systematic approach using asymptotic analysis was motivated by the motion of heavenly bodies. Indeed, Poincaré's *les Methodes Nouvelles de la Mecanique Celeste* [3] established the firm foundations of asymptotic analysis and demonstrated its success on problems in celestial mechanics. In astrodynamics, there are many dynamic phenomena which exhibit a mixture of rapid and slow variations. For example, the motion of an earth satellite under the influence of gravity of an oblate earth shows a fast variation of the coordinates around the orbit, while the orbit itself shows a slow rotation. Other perturbations such as gravity due to the moon and sun, atmospheric drag or geomagnetic field, occur at different rates. These effects must be included in computing the orbits accurately.

In the motion of conventional aircraft one can see that symmetric motions are characterized by the fast *short period* mode and the slower *phugoid* mode. In aerodynamics, the boundary layer theory of Prandtl [4] is a well-known example of multiple scale behavior. Fast changes in the fluid velocity occur inside the boundary layer, while slow changes occur outside it.

Recognition of the existence of different rates in a dynamic system is often useful in computation. For example, in the computation of highly elliptic orbits of an earth satellite, the integration step size has to be small enough to describe fast changes near the perigee, while larger steps would suffice at other parts of the orbit, leading to a nonuniform step size situation.

These are but a few examples of the presence of mixed dynamics at different rates. Most dynamic phenomena exhibit such a mixture and have been so recognized in many fields such as chemical reactor kinetics, economic systems, and so on. In these systems, it would be highly desirable to make use of the presence of fast and slow variations in order to separate them. The rewards to be gained are many, including enhanced insight and computational facility in addition to the development of constructive solutions for many important problems of analysis and design in engineering. The Multiple Time Scales (MTS) method is naturally suited to exploit the different time scales inherent in the system dynamics. In Chap. 5, the MTS method is extended into the Generalized Multiple Scales (GMS) method. It has been developed and applied to many problems involving linear time-varying and nonlinear differential equations in aerospace systems.

In this book a general approach to the analysis and computation of the dynamics of complex systems is presented. It incorporates the Method of Multiple Scales [5, 6], and related techniques of asymptotic analysis. In the Multiple Scales method, the

independent variable, usually *time*, is extended into a higher dimensional space. We consider new time scales, i.e., *fast time*, *slow time*, etc., as generated from the original time variable, i.e., real time. The different aspects of the dynamic phenomena are now described with respect to each of the new time scales, resulting in a composite description of the complex dynamics. The primary advantage is that the description with respect to each new time scale is obtained more easily than the complete description in the original time. However, the success of the approach is predicated on approximation theory and, in particular, on the concept of asymptotic analysis. It is, therefore, useful to present an introduction to the ideas and methods of asymptotic analysis. Such a brief outline is presented next.

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Chapter 2
Computation

2.1 Introduction

Very often, the ability to generate quantitative information on specific systems is central to the analysis and design in the investigation of science and engineering systems. Applied analysts and designers of complex systems rely heavily on numerical solutions to the mathematical models representing a given physical system under study. This leads to the task of computation, which must be accurate and efficient.

It is well known that there are many problems in the analysis of dynamic systems which are not amenable to exact solutions. Indeed, this is the usual case, and one seldom finds a real problem to be *exactly* solvable. Thus the only recourse is to develop methods to approximate the behavior of the system under study. Even this relatively modest goal is often found to be elusive.

The mathematical models are usually in terms of differential equations, and there is a great need to solve them in order to investigate the dynamic behavior of the systems. However, these equations are not, in general, *exactly* solvable, and therefore, it is necessary to resort to approximations. For example, the notion of a derivative of a function, which involves the concept of a mathematical limit, is replaced by a finite difference operation as an approximation. The differential equation is thus replaced by an equation involving finite differences. This latter model is solved with the expectation that its solution represents the solution of the original differential equation as the differential step size becomes smaller and smaller. While this expectation is not fulfilled with absolute rigor in all cases, this approach works well for the most part and forms the backbone of quantitative analysis of engineering systems. In particular, it does not matter whether an approximate problem is solved *exactly* or an exact problem is solved *approximately*.

As we go deeper into this task, we soon realize that the ease and success of computation is influenced strongly by the nature of the phenomenon being studied. In particular, the rate at which the system behavior unfolds has a crucial bearing on the ability to compute. There are many dynamic phenomena in which the system behavior is *nonuniform*, i.e., varies in different parts of the domain of interest.

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 9
DOI: 10.1007/978-3-642-25749-0_2, © The Author(s) 2012

For example, the velocity of a fluid changes rapidly near a solid boundary because of viscosity, resulting in a *boundary layer* and *slowly* in a region far away from the boundary. Another classic example is that of the motion of a smaller planet orbiting around a larger body in a highly elliptic orbit. The coordinates of the orbiting planet change rapidly near the periapsis (eg. the perigee) and slowly in the other parts of the orbit. In such cases, the computational step size needs to be small in regions of rapid changes and larger in other regions in order to describe the behavior accurately. Such a behavior is characterized as a *multiple scale* system or a *stiff* system. This type of behavior is also called an *asymptotic phenomenon* [1] and gives rise to incongruities and nonunformities in a simplistic desription of the phenomenon. Such systems require special methods and care in developing either numerical or analytical approximations to represent the system behavior accurately.

The philosophy and methodology discussed in this book are based on a general approach to the analysis of complex dynamic systems as embodied in the Method of Multiple Scales [2, 3]. In this approach, the independent variable, usually *time*, is extended into a higher dimensional space. Thus, we consider new time scales, i.e., *fast time*, *slow time*, etc., as generated from the original time variable, i.e., *real time*. The different aspects of the dynamic phenomena are now described with respect to each of the new time scales, resulting in a composite description of the complex dynamics. The primary advantage is that the description with respect to each new time scale is obtained more easily than the description in the original time. However, the success of the approach is predicated on approximation theory and, in particular, on the concept of asymptotic analysis. It is, therefore, necessary to present an introduction to the ideas and methods of asymptotic analysis.

Utilizing the great advances in high-speed computer technology, it is tempting to combine the ideas and techniques of asymptotic analysis with numerical methods to develop powerful methods for computation. With the advent of high-speed computers two aspects of systems analysis can be recognized. First, the exact analytical solutions, if available, will enable the analyst to predict the qualitative and quantitative features of the phenomenon being studied. On the other hand, exact solutions may be impossible, but accurate approximate solutions are often achievable. Thus, one may obtain numerical solutions to specific problems which may be analytically intractable.

Historically, there has been a great deal of interest in the computational aspects in physical applications. Many great mathematicians like Newton and Gauss were interested in computation and devoted considerable effort to the task of computation. Newton developed many numerical methods still in use today, and Gauss spent considerable time and effort in computing the trajectories of celestial bodies. In contrast, some of these tedious computations could be done readily by a personal computer today! This underscores the fact that until recently the lack of high-speed computational capability forced mathematicians and analysts to resort to clever approximation methods in order to analyze the complex dynamics of physical systems. These ingenious and highly efficient analytical methods, in conjunction with high-speed computers, lead to extremely useful approaches for solving complex problems. In general, approximations yield qualitative insight on such questions as

system stability, response, parameter sensitivity, etc. However, when one wishes to obtain a solution to a specific problem which is analytically intractable, then one usually resorts to the use of a computer. A great wealth of technical literature exists in both areas. In this work, an attempt is presented to combine asymptotic methods with numerical computation.

In the investigation of complex engineering systems the analyst and designer have found it necessary to employ sophisticated methods to solve the various problems. For example, in the field of flight vehicles, the development of high performance systems such as aircraft with large flight envelopes, vertical and short take-off and landing (V/STOL) vehicles and space vehicles requires elaborate and detailed study to meet the performance requirements. Physical intuition and experience as well as computer simulations usually provide answers to many of the questions. However, in order to glean physical insight and better understand the system behavior, one needs to *simplify* the mathematical representation of the system so that conventional methods are applicable. On the other hand, computers provide *specific* answers to *specific* questions.

Assuming that linear systems with constant coefficients can be solved exactly, the analyst and designer have relied on such models to investigate measures of system behavior and performance such as stability, response and design criteria. They have been successfully applied to many problems. Indeed, this approach is in common use, and is the first treatment of choice. The success of the simple approach rests on the validity of approximating a complex physical system by one which is tractable. However, as the systems become more complex in an effort to achieve greater performance capability, situations arise where simplistic representations are no longer adequate. More sophisticated methods then become necessary. Several such methods are available, and are constantly undergoing further development and refinement, but most of these are not well known. As the problems amenable to these techniques are widely prevalent, adequate dissemination and knowledge of such techniques would prove to be extremely useful.

In systems analysis, approximations are widely used, though implicitly in many cases. However, a systematic use of rigorous approximation theory offers much greater rewards in general, than *ad hoc* approaches. A systematic approach to approximation theory can be found in the subject of *asymptotics*, -the branch of mathematics that deals with the study of applied mathematical systems in *limiting* cases. A substantial body of knowledge exists in this field, in both the rigorous and applied senses.

Asymptotic methods have been successfully applied to many problems in many fields such as fluid mechanics, astrodynamics, and physics. However, the fields of dynamic systems analysis, control and computation have not largely made good use of these powerful techniques. It is the purpose of this work to develop a general concept and technique in applied mathematics and to discuss its wide applicability on various problems of analysis and computation in modern engineering systems. The subject of asymptotics traditionally concerns a class of phenomena in a somewhat unclear region of overlap between applied mathematics and physics. There is a vast literature, rich in techniques and results, both in the classical problems of mathematical physics and in the more applied problems of science and advanced engineering. The subject

is deep enough to satisfy the need for strict rigor of the most demanding analyst and yet is suitable for practical problem solving. In recent times, it is this latter feature that is assuming greater significance in the analysis of complex engineering systems.

We will now address some basic concepts in approximation theory.

2.2 Convergence and Computation

Asymptotic approximations usually involve power series. The following well-known classical examples may be considered in the context of computation.

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots \tag{2.1}$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \tag{2.2}$$

$$\frac{1}{(1-t)} = 1 + t + t^2 + t^3 + \cdots \qquad |t| < 1 \tag{2.3}$$

The sum of these series is finite and they converge to the function being represented. It is believed that the term "convergent series" was first introduced by J. Gregory in 1667. Rigorous general tests for convergence were not developed until much later, through the works of D'Alembert (1768), Cauchy (1821), Abel (1826), Raabe (1832), DeMorgan (1842), and Dirichlet (1862). These led to the practice of defining functions by series, still used today. In fact, solutions to differential equations are often developed in series form such as those using Frobenius' method. The series representations suggest the possibility of using the truncated series to calculate the function approximately. While this approach works rather well with some series, it is notoriously inadequate with others. The practical success of such a scheme depends critically on the *speed* or rate of convergence. For the above examples, for small values of $t\,(0 < t < 1)$, the convergence is quite rapid and the truncated series provides a good approximation of the function. For large values of t the truncated series does not lead to a good approximation and Eq. (2.3) is not even valid for $t > 1$. Further, consider the following representations.

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots) = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \tag{2.4}$$

$$ln\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tag{2.5}$$

$$\frac{1}{\sqrt{2}} = 1 - \frac{1}{2(1!)} + \frac{1(3)}{2^2(2!)} - \frac{1(3)(5)}{2^3(3!)} + \cdots \tag{2.6}$$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n^s}; \quad Re(s) > 1 \tag{2.7}$$

All these series are known to be convergent but their rate of convergence is so slow that they cannot be used directly to calculate the quantity approximately to good accuracy. For instance, Leibnitz's series for π as given by Eq. (2.4) requires 40 million terms for an accuracy of eight significant figures and the tenth partial sum is good only to one significant figure ! Similarly, the other series given above are computationally unsuitable. To compute $ln\,2$ with an error less than half the fifth significant figure, more than 100,000 terms are needed. The convergence of Eq. (2.6) is so slow that nearly 10^{10} terms are required to compute $1/\sqrt{2}$ to an accuracy of 10^{-5}. The famous Riemann Zeta function $\zeta(s)$, so useful in the analytic theory of numbers, converges for $Re(s) > 1$ and diverges for $Re(s) \leq 1$. Near $Re(s) = 1$, the convergence is extremely slow. To compute $\zeta(1.1)$ for instance, with an error less than 1%, more than 10^{20} terms are needed, an impossible task even with modern high speed computers!

In contrast, consider the example first discussed by Euler,

$$S(x) = 1 - 1!x + 2!x^2 - 3!x^3 + \cdots = \sum_{0}^{\infty} (-1)^n n! x^n \tag{2.8}$$

Clearly, the series diverges for all $x \neq 0$. Yet for small x (say of order 10^{-2}) the terms at first decrease rapidly. It can be shown that the first few terms of the series provide a very good approximation (for small x) of the function

$$f(x) = \int_0^{\infty} \frac{e^{-t}}{(1 + xt)} \tag{2.9}$$

Similarly the function

$$Ei(x) = \int_{-\infty}^{x} e^t t^{-1} dt; \quad x < 0 \tag{2.10}$$

can be represented by the series

$$Ei(x) = e^x x^{-1} (1 + \frac{1}{x} + \frac{2!}{x^2} + \cdots + \frac{m!}{x^m} + \cdots) \tag{2.11}$$

Clearly this series diverges for *all* finite values of x as m increases.

Finally, consider one of the earliest and celebrated examples of the Stirling series for the factorial function (n − 1)! = Γ (n) which is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt; \; Re(z) > 0 \tag{2.12}$$

Table 2.1 Comparison of Taylor and Stirling series

N	z=1; 0!	z=1; 0!	z=2; 1!	z=2; 1!	z=4; 3!	z=4; 3!
	Taylor Srs	Stirling Srs	Taylor Srs	Stirling Srs	Taylor Srs	Stirling Srs
1	1.0000000	0.9989817	0.5000000	0.9994814	0.2500000	5.9989717
2	0.631028	1.0021036	0.2307980	1.0003143	0.0755546	6.0002470
3	1.0053784	0.9997110	−1.0659140	0.9999927	−0.0347938	6.0000008
4	1.1372230	0.9994994	−0.6210392	0.9999789	−0.02532067	5.9999956
5	0.9561385	1.0002224	0.2689872	1.0000024	0.0076113	6.0000001
6	0.9963376	1.0002867	0.9819551	1.0000035	−0.0239235	6.0000002
7	1.0059817	0.9997401	−4.6896423	0.9999990	−0.0050138	5.9999999
	–	–	–	–	–	–
	–	–	–	–	–	–
∞	1.00000000		1.00000000		6.00000000	

This has the series representation

$$(n-1)! \sim \left(\sqrt{\frac{2\pi}{n}}\right)(n/e)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\cdots\right) \tag{2.13}$$

Although this series diverges for all n, it provides a very accurate approximation of $(n-1)!$ for large values of n. Indeed, it is remarkable that even when n is not large the approximation is still very good. For example, the first term of the Stirling series for 3! yields 5.9989 in comparison with the exact answer of 6. And yet the series was developed as an approximation for $n \to \infty$. The extreme accuracy of the Stirling series even for small values of $n!$ such as 1! and 3! is illustrated in the following table generated as a numerical experiment. Table 2.1 shows a term-by-term comparison of the rates of convergence of the partial sums of the Taylor series for $\frac{1}{\Gamma(z)}$ and the Stirling series for $\Gamma(z)$ for $z = 1, 2, 4$. Note that the Taylor series ultimately converges to the correct value, but it requires a very large number of terms (actually infinite terms)! The Stirling series, however, is far more useful in numerical computations. The usefulness of such a series was noticed by Stirling and others. The first rigorous foundation of these "asymptotic series" and their interpretation is due to H. Poincaré in 1886.

As already noted, the series representation for π given by Leibnitz converges so slowly that forty million terms are needed to compute π to an accuracy of eight significant figures! However, the series developed by the Indian mathematical genius Srinivasa Ramanujan converges so rapidly that π can be computed to eight significant figures by the very *first term* of the series! Indeed, every successive term of Ramanujan's series increases the computational accuracy by eight digits. This series is given by

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n!)(1103+26390n)}{(n!)^4 396^{4n}} \tag{2.14}$$

According to mathematicians, this is the fastest converging series for π. The incredible accuracy of the series depends crucially on the specific numbers in the series. For example, it has been proved that the number 1103 makes the series true to within one part in $10^{10,000000}$ [4]. It has been said that Ramanujan-type algorithm for approximating π can be shown to be very close to the best possible. It was not known for a long time how Ramanujan came up with the correct numbers as he did not give any proofs. It has since been established that the numbers were obtained as singular solutions of modular equations of which he was a master. The interested reader can refer to the sources cited.

Ramanujan's is a remarkable story. He was born in a poor family in Southern India in 1887. He did not complete high school as he did not pass the subject of English, which was required for matriculation in British-ruled India. However, he was a mathematical genius of the highest order, and worked entirely on his own, as he did not receive a formal education in mathematics. Therefore, much of his profound work is rediscovery. He sent a humble letter containing some results he had derived, to G.H.Hardy, the world famous number theorist at Cambridge University in England. Upon seeing this, Hardy recognized genius and arranged for Ramanujan to travel to England. They collaborated from 1914-1919 and these were years of the First World War. As recounted by Hardy, during the war time conditions, life was hard and Ramanujan became seriously ill and was hispitalized. Hardy said that he visited Ramanujan at the hospital and remarked that the taxi in which he rode had an uninteresting number. They were two number theorists conversing, and Ramanujan asked what the number was. Hardy replied that it was 1729 for which he did not see any interesting properties. Ramanujan said it was one of the most interesting numbers. When Hardy asked why, Ramanujan said, "Don't you see, it is the smallest number that can be expressed as the sum of two cubes in two different ways". That is, $1729 = 9^3 + 10^3 = 12^3 + 1^3$. Soon after, Ramanujan returned to India and died in 1920 at the age of thirty three!

In 1985, Ramanujan's formula was used to compute π to 17.5 million digits, in 1986, to 29 million digits, in 1987 to 134 million digits, and so on. Using this formula in the year 2002, the Japanese computed π on a Hitachi supercomputer, to 1.24 trillion decimal places ! One may wonder about the need for determining π to such a large number of decimal places. In fact, calculating π to the 39*th* digit is sufficient to compute the radius of the known universe to an accuracy of the order of the radius of the hydrogen atom ! One whimsical reason to compute π is "because it is there", like the need to climb a mountain. A more serious reason might be that an accurate knowledge of π is helpful in solving inverse problems by using π as a benchmark.

In fact, Ramanujan has given fourteen other representations for computing π. One of these is the infinite continued fraction

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}} \tag{2.15}$$

Ramanujan was one of the few mathematicians, along with Gauss, who were skilled in continued fractions. According to the mathematician Berndt, "In this area, Ramanujan is probably unsurpassed in all of mathematical history" [4, 5].

Stirling [6] was one of the first to use asymptotics in giving a representation of the factorial function. MacLaurin and Euler in the eighteenth century were also among the earliest in employing asymptotic series. In the nineteenth century, Poincaré [7, 8] was pre-eminent in establishing the rigorous mathematical foundations of asymptotic analysis. In current practice, following Poincaré, the term *asymptotic series* has replaced the older *semi-convergent* series used by Stieltjes [9]. There has been a tremendous growth in the development of asymptotics in the twentieth century, with a vast literature. A large number of techniques have been developed and the subject abounds with applications in diverse fields of investigation. It is, therefore, impossible to do justice to the large number of contributors. The prominent contributions are cited in the references.

In the field of simulation and computation of the dynamics of engineering systems, the use of asymptotic approximations has resulted in great success not only in describing the phenomenology but also in efficiency in computation. This is especially true of *stiff* systems which are chaacterized by the existence of different rates of change in the dynamics. In such problems the conventional numerical and analytical methods of approximation encounter considerable difficulty. Special approaches are then necessary. It is precisely this class of problems that is amenable to asymptotic methods such as the Multiple Scales approach, which will be discussed later. There are many examples of asymptotic phenomena which cover a wide range of problems, from the very practical to the highly theoretical applied analysis. We may now consider some examples of problems in the analysis of dynamic systems which are amenable to approximation theory. In linear time-invariant (LTI) systems an important problem is that of widely separated eigenvalues. The motion of flight vehicles is a combination of the different aspects of rigid body modes, actuator dynamics, structural modes, etc., all of which are well separated in frequency. Such a frequency separation could itself be the basis of approximation in an asymptotic sense. It would show at what level and to what extent the different modes would begin to contribute to the total motion.

The earliest systematic approach using asymptotic analysis was motivated by the motion of heavenly bodies. Indeed, Poincaré's *les Methodes Nouvelles de la Mecanique Celeste* [7] established the firm foundations of asymptotic analysis and demonstrated its success on problems in celestial mechanics. In astrodynamics, there are many dynamic phenomena which exhibit a mixture of rapid and slow variations. For example, the motion of an earth satellite under the influence of gravity of an oblate earth shows a fast variation of the coordinates around the orbit, while the orbit itself shows a slow rotation. Other perturbations such as gravity due to the moon and sun, atmospheric drag or geomagnetic field, occur at different rates. These effects must be included in computing the orbits accurately. In the motion of conventional aircraft one can see that symmetric motions are characterized by the fast *short period* mode and the slower *phugoid* mode. In aerodynamics, boundary layer theory of Prandtl

[1] is a well-known example of multiple scale behavior. Fast changes in the fluid velocity occur inside the boundary layer, while slow changes occur outside it.

Recognition of the existence of different rates in a dynamic system is often useful in computation. For example, in the computation of highly elliptic orbits of an earth satellite, the integration step size has to be small enough to describe fast changes near the perigee, while larger steps would suffice at other parts of the orbit, leading to a nonuniform step size situation.

These are but a few examples of the presence of mixed dynamics at different rates. Most dynamic phenomena exhibit such a mixture and have been so recognized in many fields such as chemical reactor kinetics, economic systems, and so on. In these systems, it would be highly desirable to make use of the presence of fast and slow variations in order to separate them. The rewards to be gained are many, including enhanced insight and computational facility in addition to the development of constructive solutions for many important problems of analysis and design in engineering. The Multiple Time Scales(MTS) method is naturally suited to exploit the different time scales inherent in the system dynamics. In Chap. 5 and later chapters, the MTS method is extended into the Generalized Multiple Scales (GMS) method. It has been developed and applied to many problems involving linear time-varying and nonlinear differential equations in advanced engineering systems (see references).

Another common problem is that involving linear time-varying (LTV) systems. For example, in the analysis of flight vehicle dynamics through variable flight conditions, the usual approach is to treat the time variation of the system parameters as "slow", and study the LTV system as a sequence of LTI systems. This approach is widely prevalent among dynamicists. For example, the motion of re-entry vehicles in the earth's atmosphere, the dynamics of VTOL vehicles during transitions from hover to forward flight, etc., have been so treated, in terms of a number of LTI systems. At steady flight, the equations of motion are indeed time-invariant, and useful results have been obtained in the study of airplane motion. However, such an approximation, while it can be useful, has a limited range of validity. Because the approximation is valid only for very short times, a number of such solutions must be patched. This opens up questions on the actual number of such patched solutions and the boundary conditions at the instants of patching, errors of the approximation, etc., besides the tedium of patching the different solutions. Then there is the rather fundamental question on the slowness of time variation,—i.e., "*how slow is slow* ?" The conclusions on stability of an LTV system from its LTI or *frozen* approximation can be quite erroneous. It can be shown that the frozen approximation is a rather simplistic asymptotic representation. For slowly varying systems a much more accurate asymptotic theory can be developed, in which the approximations are uniformly valid over long intervals [2]. LTI systems arise as a special case of this general result. A substantial amount of work exists on this subject. However, these powerful techniques are not widely known and not well exploited in engineering.

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Chapter 3
Outline of Numerical Methods

3.1 Introduction

Recalling that most of the commonly used computational approaches involve numerical methods, we will now address the task of generating numerical solutions of problems in engineering systems and computation of their dynamics. Of the large number of numerical methods which are used for computation of the solutions, this chapter will present a brief account of some of the well-known methods [1–4]. These can be generally considered in several categories. They include, for example, (1) numerical approximations to equations in one variable, (2) numerical differentiation, (3) numerical integration, (4) integration of ordinary differential equations, (5) partial differential equations, and so on.

We will briefly outline these topics and the popular solution approaches.

3.2 Equations with One Variable

This class includes such problems as finding the zeros of a nonlinear equation or the roots of a polynomial equation.The commonly used methods are, (1) Newton-Raphson method, (2) secant mathod, etc. One of the most widely used extrapolation methods is the Newton-Raphson method which is based on the slope of the function as the estimated root.

3.2.1 Newton-Raphson Method

In order to obtain a good approximation to a zero of a given function $f(x)$, we start with a guess x_i and a small step h, and expand the function $f(x)$ as a Taylor series,

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 19
DOI: 10.1007/978-3-642-25749-0_3, © The Author(s) 2012

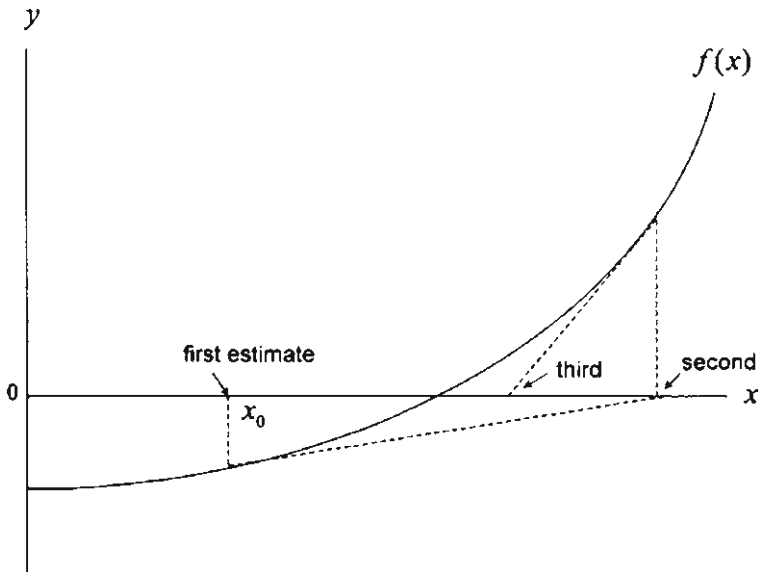

Fig. 3.1 Newton-Raphson method

$$f(x_{i+1}) = f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots \tag{3.1}$$

Neglecting terms of second and higher orders and assuming that x_{i+1} is a root, i.e., $f(x_i + h) = 0$, we can write

$$f(x_i) + hf'(x_i) = 0 \tag{3.2}$$

$$f(x_i) + (x_{i+1} - x_i) f'(x_i) = 0 \tag{3.3}$$

This equation can be rewritten as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = 0 \tag{3.4}$$

This is shown interpreted graphically in Fig. 3.1.

3.2.2 Numerical Differentiation

The approximation of the derivative of a function directly leads to *difference formulas*. These are obtained by truncating the Taylor series after the first derivative term. For example, given a function $f(x)$ and an interval h,

$$f(x \pm h) = f(x) \pm hf^i(x) + O(h^2); \quad f^i(x) \equiv \frac{df}{dx} \tag{3.5}$$

The following difference approximations can be written as:

Forward difference

$$f^i(x) = \frac{[f(x+h) - [f(x)]}{h} + O(h) \tag{3.6}$$

Backward difference

$$f^i(x) = \frac{[f(x) - f(x-h)]}{h} + O(h) \tag{3.7}$$

Central difference

$$f^i(x) = \frac{[f(x+h) - [f(x-h)]}{2h} + O(h^2) \tag{3.8}$$

3.2.3 Numerical Integration

Next, we will consider the numerical integration of a function of a single variable,

$$\int_a^b f(x)dx \tag{3.9}$$

The function to be integrated is first replaced by a simple polynomial $Q_{n-1}(x)$ of degree $n-1$ which coincides with $f(x)$ at n points x_i where $i = 1, 2, 3, \ldots n$. Thus,

$$\int_a^b f(x)dx = \int_a^b Q_{n-1}(x)dx \tag{3.10}$$

where

$$Q_{n-1}(x) = \sum_{i=0}^{n-1} a_i x^i \tag{3.11}$$

and a_i are constant coefficients.

In order to compute the the numerical integral of a function of a single variable, the standard *integration rule* is usually expressed as a summation of the form

$$\int_a^b Q_{n-1}(x)dx = \sum_{i=1}^{n} w_i Q_{n-1}(x_i) \tag{3.12}$$

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{3.13}$$

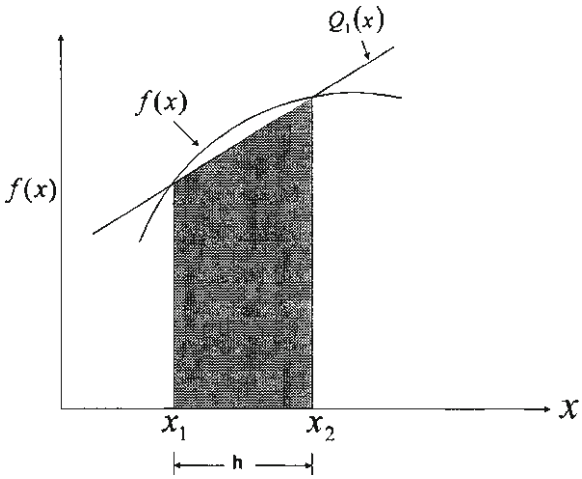

Fig. 3.2 Trapezoidal rule

The x_i are called *sampling points* where the function $f(x)$ is evaluated, and the w_i are constant *weighting coefficients*. This equation forms the basis of most of the usual numerical integration methods for a single variable. Note that the number and location of the sampling points and the corresponding weighting coefficients determine the different numerical integration methods. The Newton-Cotes rules involve equally spaced sampling points within the range of integration and include the limits of integration. On the other hand, if the sampling points are optimally placed, they lead to Gaussian rules and greater accuracy and efficiency. These will not discussed here. The interested reader can consult the standard references, eg., [1], etc.

Of the Newton-Cotes rules, we may consider two popular methods which are called: (1) Trapezoidal Rule, and (2) Simpson's Rule.

3.2.4 Trapezoidal Rule

In this method, the function $f(x)$ is approximated by a first order polynomial $Q_1(x)$ which coincides $f(x)$ at both limits of integration. The integral is approximated by the area of the trapezoid expressed as follows (Fig. 3.2).

3.2.5 Simpson's Rule

This is a well-known method in which the function $f(x)$ is approximated by a second order polynomial $Q_2(x)$ which coincides with $f(x)$ at three points, i.e., the limits of integration and at the middle of the range.

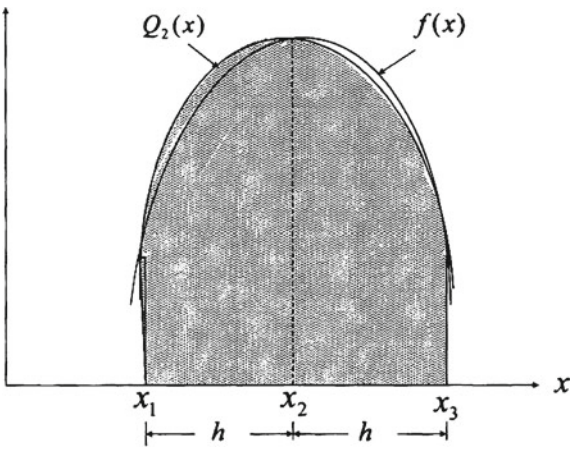

Fig. 3.3 Simpson's rule

In this case, the weighting coefficients of a three-point Newton-Cotes rule are derived by substituting the polynomial in the general form (Fig. 3.3),

$$\int_{x_1}^{x_3} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \tag{3.14}$$

where $x_2 = \frac{1}{2}(x_1 + x_3)$ and $x_3 - x_2 = x_2 - x_1 = h$. Note that Simpson's rule is exact if $f(x)$ is a polynomial of order 0,1 or 2. This will enable us to generate three equations in the unknowns w_1, w_2, w_3. It can be shown that $w_1 = w_3 = h/3$ and $w_2 = 4h/3$ [1]. The final result is

$$\int_{x_1}^{x_3} f(x)dx \approx \frac{1}{3} f(x_1) + \frac{4}{3} f(x_2) + \frac{1}{3} f(x_3) \tag{3.15}$$

More accurate results can be derived by considering higher order Newton-Cotes rules. For instance, the above approach can be used to derive a four-point rule which would fit a cubic to $f(x)$ and is exact for $f(x)$ of degree three or less.

Similarly, a five-point rule would fit a quartic to $f(x)$ and is exact for $f(x)$ of degree five or less, and so on.

$$\int_a^b f(x)dx \approx \frac{1}{2} h(f(x_1) + f(x_2)) \tag{3.16}$$

3.2.6 Accuracy of Newton-Cotes Rules

In order to estimate the accuracy of the Newton-Cotes rules formally, one can represent the function exactly as a Taylor series and consider the largest term in the series

that is neglected. For a small step size, this will yield a useful estimate of the error incurred by neglecting the term. If the range of integration is large, the accuracy of integration can be improved by either (1) fitting a higher order polynomial, or by (2) breaking up the range into strips and using lower order polynomials over each strip. The latter approach leads to repeated Newton-Cotes rules such as the repeated Trapezoidal or the repeated Simpson's rule. Usually, the frequently repeated low order methods are preferable to the high order methods. Further, the step size h should be *small*, but not so small as to require excessive computer time, or exceed computer word length limitations.

Some other techniques include, for example, (3) Romberg Integration, (4) Gaussian Quadrature, etc. For these and related ideas, one may consult the references.

3.3 Integration of Ordinary Differential Equations

This is an important area of numerical analysis and is widely applicable to many areas of science and engineering. The major techniques are: (1) Euler method (2) Modified Euler Method (3) Runge-Kutta Methods, (4) Adams-Bashforth Predictor-Corrector Method, (4) Milne-Simpson Predictor-Corrector Method, (6) Hamming Method, and so on.

A particularly difficult but important class of differential equations involve *stiff* systems. These are equations whose solutions exhibit the presence of a a mixture of time scales which are *fast* and *slow*. A stiff system can be defined as one in which the larger scale is of interest, but the smaller scale dictates the step size on the basis of system numerical stability. Such systems occur naturally in the study of multiple scale systems discussed in Chap. 2. A well-known numerical method for stiff systems is Gear's method [3].

A brief outline of the numerical methods for integrating differential equations will now be presented. We will mainly consider ordinary differential equations (ODE). The simplest type of differential equation can be written as

$$\frac{dy}{dx} = f(x) \tag{3.17}$$

The analytical solution of this equation can be written as

$$y(x) = \int f(x)dx + C \tag{3.18}$$

where C is an arbitrary constant. In order to generate numerical solution we will start with an initial-value problem

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \tag{3.19}$$

The numerical methods for solving this equation involve starting at the initial condition x_0, y_0 and stepping along the x-axis. At each step, a new value of y is estimated. As more steps are taken, an approximation to the solution $y(x)$ is obtained. Thus,

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} \frac{dy}{dx} \tag{3.20}$$

There are two main approaches to perform this integration:

(a) One-step methods, which use the information from only one preceding point (x_i, y_i) to approximate the next point (x_{i+1}, y_{i+1}).
(b) Multi-step methods which use the information about several previous points (x_i, y_i), $(x_{i-1}, y_{i-1}, \ldots$ etc. to approximate the next point (x_{i+1}, y_{i+1}). These methods are also known as *predictor-corrector* methods, because, each step makes use of two formulas: one that *predicts* the new solution y_{i+1}, and another that refines or *corrects* it.

One-step methods are self-starting, as they need only the initial condition given. However, multi-step methods need several consecutive initial values of x and y to get started. These additional initial values may be provided by a One-step method if not given in the initial data.

3.4 One-Step Methods

These are well suited to computer implementation as each step leads to the next one. There are several one-step methods in use, with the degree of accuracy depending on how much information is used at each step in comparison with the number of steps to span the range of interest. Of the many methods, mention may be made of (1) The Euler Method, (2) Modified Euler Method, (3) Runge-Kutta Methods, and so on.

3.4.1 (1) The Euler Method

This method is simple to implement, but requires small steps for reasonable accuracy. Starting with the equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \tag{3.21}$$

the Euler method approximates the new value of y using the relation

$$K_0 = hf(x_0, y_0) \tag{3.22}$$

$$y_1 = y_0 + K_0 \tag{3.23}$$

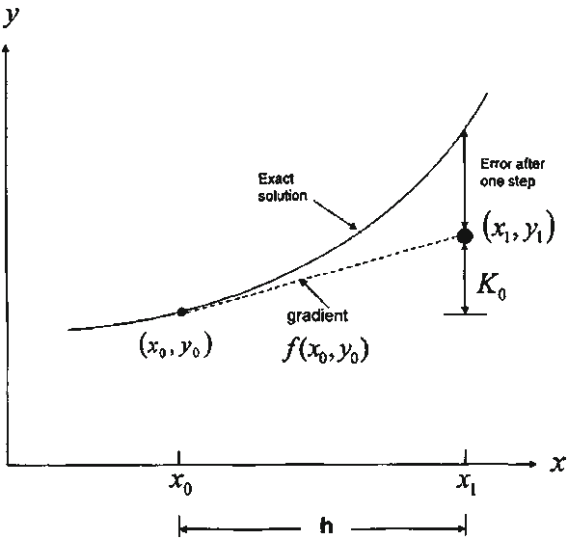

Fig. 3.4 The Euler method

where

$$h = x_1 - x_0 \tag{3.24}$$

This procedure is repeated using x_1, y_1 to generate y_2 and so on. Thus,

$$K_0 = hf(x_i, y_i) \tag{3.25}$$

$$y_{i+1} = y_i + K_0 \tag{3.26}$$

where

$$h = x_{i+1} - x_i \tag{3.27}$$

This procedure is illustrated in Fig. 3.4.

3.4.2 Modified Euler Method

This method uses a higher order integration to approximate the change in y. Using the Trapezoidal rule, on the equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \tag{3.28}$$

the new value of y is obtained by using

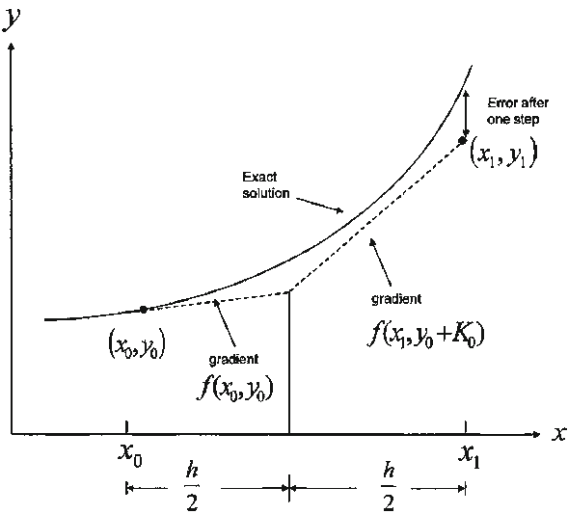

Fig. 3.5 Modified Euler method

$$K_0 = hf(x_0, y_0) \tag{3.29}$$

$$K_1 = hf(x_0 + h, y_0 + K)) \tag{3.30}$$

$$y_1 = y_0 + \frac{1}{2}(K_0 + K_1) \tag{3.31}$$

The procedure is shown in Fig. 3.5.

It is more accurate than the simple Euler method.

3.4.3 Runge-Kutta Methods

These include the previous two methods The general for the new approximation y_{i+1} is given by

$$y_{i+1} = y_i + \frac{\sum_{j=0}^{r-1} W_j K_j}{\sum_{j=0}^{r-1} W_j} \tag{3.32}$$

where the W_j are weighting coefficients and r is the order of the method, which indicates the highest power of h retained in the Taylor series expansion of f.

A frequently used fourth order Runge-Kutta method leads to the relations

$$K_0 = hf(x_0, y_0) \tag{3.33}$$

$$K_1 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_0}{2}) \tag{3.34}$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) \tag{3.35}$$

$$K_3 = hf(x_o + h, y_0 + K_2) \tag{3.36}$$

$$y_1 = y_0 + \frac{1}{6}(K_0 + 2K_1 + 2K_2 + K_3) \tag{3.37}$$

This is like Simpson's rule where the sampled points are in the ratio of 1:4:1 for approximating the change in f.

3.5 Predictor-Corrector Methods

These methods utilize several previous known points to compute the next point. However, these methods are not self-starting and may need to generate a few points by a one-step method to start. Note that while a fourth order Runge-Kutta approach uses four values of the function at each step, the Predictor-Corrector approach uses the information more efficiently to compute the next step. Two formulas are used in the Predictor-Corrector approach,—one to predict the next point by extrapolation, and one to correct by improving the next point.

Predictor formulas estimate the new value of y_{i+1} by integrating under curve of y" vs. x using sampling points at x_i, x_{i-1}, x_{i-2} etc. Any numerical integration formula which does not require a prior estimate of y'_{i+1} can be used as a predictor. The predicted value y_{i+1} is improved by the corrector formulas by again integrating under the curve of y' vs. x, but this time using sampling points x_{i+1}, x_i etc.

The same order of accuracy is used in both the prediction and correction parts of the algorithm in the commonly used Predictor-Corrector methods Fig. 3.6.

3.5.1 Milne-Simpson Method

For example, we may consider the in which the predictor formula is due to Milne, i.e.,

$$y^p_{i+1} = y_{i-3} + \frac{4h}{3}[2f(x_{i-2}, y_{i-2}) - f(x_{i-1}, y_{i-1}) + 2f(x_i, y_i)] + O(h^5) \tag{3.38}$$

and the corrector is the Simpson's rule,

$$y^c_{i+1} = y_{i-1} + \frac{h}{3}[f(x_{i-1}, y_{i-1}) + 4f(x_i, y_i) + f(x_{i+1}, y^p_{i+1})] - O(h^5) \tag{3.39}$$

A cautionary note is that the use of Simpson's rule for correction may result in the growth of the errors from stage to stage. This difficulty is obviated by the following method Fig. 3.7.

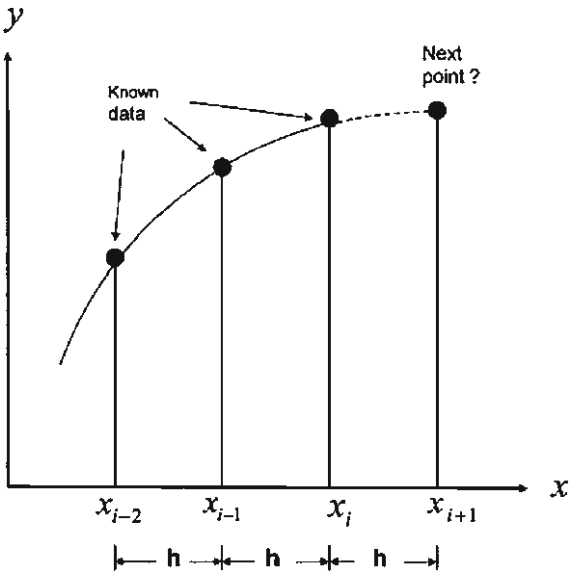

Fig. 3.6 Predictor-corrector methods

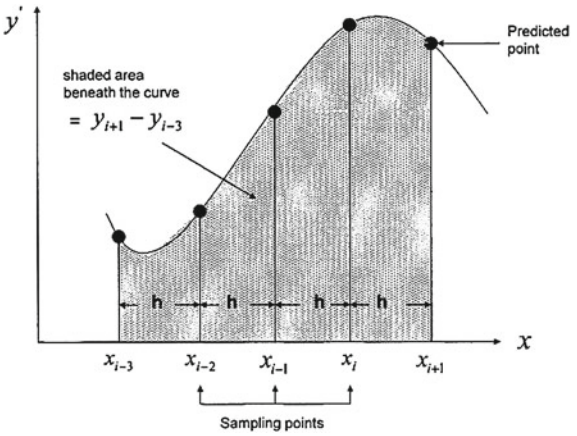

Fig. 3.7 Milne predictor

3.5.2 Adams-Bashforth-Moulton Method

This uses the Adams-Bashforth predictor and the Adams-Moulton corrector which results in an overall approach of fourth order in which the errors do not grow fast. In the Adams-Bashforth predictor (Fig. 3.8),

$$y_{i+1}^{p} = y_i + \frac{h}{24}[-9f(x_{i-3}, y_{i-3}) + 37f(x_{i-2}, y_{i-2}) - 59f(x_{i-1}, y_{i-1}) + 55f(X_i, y_i)] + O(h^5) \tag{3.40}$$

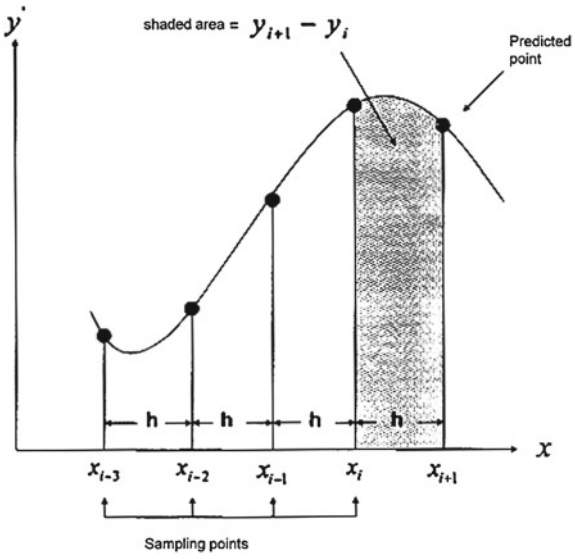

Fig. 3.8 Adams-Bashforth predictor

and the Adams-Moulton corrector (Fig. 3.8),

$$y_{i+1}^c = y_i + \frac{h}{24}[f(x_{i-2}, y_{i-2}) - 5f(x_{i-1}, y_{i-1}) + 19f(x_i, y_i) + 9f(x_{i+1}, y_{i+1}^p)] - O(h^5) \tag{3.41}$$

3.5.3 Stiff Equations

Some differential equations possess solutions which vary on widely different time scales. Such a system is characterized by the presence of small coefficients multiplying the highest derivatives indicating solutions which vary rapidly, as seen in *boundary layer* problems. In such cases, conventional numerical methods discussed so far, are not readily suitable to generate the numerical solutions and other approaches must be resorted to.These involve *multiple scale phenomena*, which will be discussed in greater detail later.

3.5.4 Other Methods

The field of numerical analysis has a vast range of technical literature with a large number of concepts and techniques. They include applications to partial differential equations and other advanced topics. For example, the finite element method proceeds

to solve a differential equation approximately over subregions of the original solution domain, by means of *finite elements* which are usually confined to simple shapes such as triangles and quadrilaterals for plane areas and tetrahedra for volumes. These and related techniques are beyond the current scope and will not be treated in this work. The interested reader can consult the references and bibliography for detailed expositions of the many methods (eg. [5, 6]).

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Chapter 4
Asymptotics and Perturbation

4.1 Introduction

Approximations of some kind are essential in the computation of the solutions of differential equations representing the system dynamics. Asymptotic solutions constitute a large class of such approximations which have been widely used in many areas such as the analysis of complex systems. They have been used, for example, for preliminary design purposes or for computing the response of a dynamic system, such as a linear or nonlinear (constant or variable) system. They are computationally efficient as evidenced by the classical examples in Chap. 2. They have the highly desirable property that the salient aspects of the system dynamics are preserved and can be calculated rapidly. This is generally because, a complex phenomenon is represented in terms of simpler descriptions as it is impossible, in general, to derive exact representations of the phenomena under study. The mathematical models are usually in the form of nonlinear or nonautonomous differential equations. Even simpler models such as linear time-invariant (LTI) systems, in certain situations present difficulties. This is the case, for instance, when the system has widely separated eigenvalues, or equivalently, a mixture of motions at different rates. Such systems are often called *stiff* systems. For many complex mathematical models, the only recourse is through approximations.

Approximate solutions to mathematical models may be obtained in many ways. A common feature of most approximation schemes is that the true behavior is represented by the approximation in a specific way. Inherent in any approximation is the error with respect to the exact solution. This error has to be *small* if Xthe approximation is to be useful. Such an error may simply be represented in terms of the difference or ratio, of the exact and approximate solutions or take on a more complicated form. For example, the mean square error is minimum in the case of the Fourier series representation. Numerical solutions of differential equations require the step size to be very small and have many measures of the errors depending on the method. Some commonly used numerical methods have been discussed in Chap. 3. In this chapter we will mainly consider asymptotic approximations.

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 33
DOI: 10.1007/978-3-642-25749-0_4, © The Author(s) 2012

The subject of asymptotics was initiated by Stieltjes [1] and Poincaré [2, 3] both in 1886. In his researches on celestial mechanics Poincaré suggested representing the unknown quantity,—the solution of a differential equation, as a series in powers of a small parameter ϵ. Grouping terms of like powers of ϵ, leads to approximations. Greater accuracy is achieved by considering terms of higher powers of ϵ. When it works, the approach is simple and accurate. Often, however, such a simple representation fails to yield a uniformly good approximation. This was recognized by Poincaré, who then gave prescriptions to render the results more useful. A rather remarkable feature of the asymptotic approximations in the form of series is that they *may not converge*. The usefulness of such divergent series was demonstrated by Poincaré. Heuristically, however, and following Friedrichs [4], we may simply call *asymptotic*, all phenomena which exhibit apparent discontinuities, quick transitions, nonuniformities and other incongruities resulting from an approximate description of the phenomena.

Asymptotic analysis deals with the limiting behavior of functions which usually arise as solutions of differential equations. Two aspects of the concept can be identified,—i.e., behavior with respect to the independent variable (eg. time), or a parameter ϵ. Of these two, we will be mainly concerned with the system behavior as the parameter ϵ becomes very small, i.e., $\epsilon \to 0$. Large values of a parameter can be treated by considering its inverse to be very small.

There are two broad approaches in the development of asymptotic analysis. The first is the classical and rigorous theory which deals with a precise analysis of the various special functions of mathematical physics, dealing with such issues as general asymptotic expansions to large orders, intricate questions on errors, analytic continuation into the complex plane, and so on. The other part of the subject is more applied in its flavor and more recent in its development. It is aimed at solving specific classes of problems of a practical nature. Only a few terms of the asymptotic expansion are sought, and usually no attempt is made to generate all terms of the expansion. Questions of rigor are examined only as necessary. The emphasis is on obtaining simple approximations for general classes of problems leading to insight and ease of calculation. This work is aimed at the latter aspect of asymptotics. Strict mathematical rigor is often replaced by heuristic reasoning in order to enhance physical insight.

Asymptotics is the branch of mathematics that deals with the behavior of mathematical functions in limiting cases. It is classical in its development and can be said to be the *science* of the study of systems in limiting cases. In applying asymptotic techniques to problems of practical interest, we often find a need for the *art* as well, in order to be successful. Thus the *science* of asymptotic analysis has to be combined with a certain degree of *art* in the process,—such as the particular technique to be used, the proper transformations of the variables that are required, etc. *Asymptotology* was defined by Kruskal [4] as the theory of approximation which includes the *art* and *science* of the analysis of applied mathematical systems in limiting cases. In an asymptotic approach we identify a "small" parameter ϵ (say $0 < \epsilon \ll 1$) and study the system in the limit $\epsilon \to 0$. Aapproximation of the unknown quantity, i.e., the solution $x(t, \epsilon)$, of a differential equation,—is obtained in powers

of ϵ as suggested by Poincaré [2]. Improved accuracy is achieved by including terms of higher order in the parameter ϵ.

In developing the approximations to represent complicated functions by simple mathematical functions, we can proceed in several ways. In asymptotics the approximation is good when an independent variable approaches a limit. Let us now consider the function $f(\epsilon)$ to be the quantity being approximated, and ϵ being the parameter approaching the limit $\epsilon \to 0$. The behavior of the function $f(\epsilon)$ can be described in different ways as follows. Apart from the simple questions of existence and boundedness of $f(\epsilon)$ as $\epsilon \to 0$, the comparison must include qualitative and quantitative information. With this motivation, the Landau order symbols are defined as follows.

4.2 Order Symbols

While comparing the order of magnitude of different mathematical functions, two order symbols are commonly used. Typically the magnitude of a complicated function $f(\epsilon)$ is described in terms of a simpler function $g(\epsilon)$. They are defined as:

- $f(\epsilon) = O(g(\epsilon))$ if $\lim_{\epsilon \to 0} [f(\epsilon)/g(\epsilon)] = constant$
- $f(\epsilon) = o(g(\epsilon))$ if $\lim_{\epsilon \to 0} [f(\epsilon)/g(\epsilon)] = 0$

Therefore in the first case, $f(\epsilon) \to 0$ precisely as fast as $g(\epsilon) \to 0$ so that the ratio $f(\epsilon)/g(\epsilon)$ has a finite, nonzero limit. In the second case $f(\epsilon) \to 0$ faster than $g(\epsilon) \to 0$, so that the ratio $f(\epsilon)/g(\epsilon)$ vanishes in the limit $\epsilon \to 0$. If $f(\epsilon) = O(g(\epsilon))$ we can write $f(\epsilon) \sim g(\epsilon)$; i.e.,

$$f(\epsilon) \sim g(\epsilon) = O(g(\epsilon)) + o(g(\epsilon)) \tag{4.1}$$

This is the *asymptotic representation* of the function. Examples:

- $\sin\epsilon = \epsilon - \epsilon^3/3! + \epsilon^5/5! - \cdots = O(\epsilon) = o(1)$
- $(1-\epsilon)^{\frac{1}{3}} = 1 - \epsilon/3 \cdots = O(1)$
- $\csc(\epsilon) = 1/sin\epsilon = O(1/\epsilon)$

A more precise description can be developed by adding terms to the representation, $f(\epsilon) \sim c_1 g_1(\epsilon) + c_2 g_2(\epsilon)$ as $\epsilon \to 0$, where $g_2(\epsilon)$ is much smaller than $g_1(\epsilon)$ in the limit, i.e., $g_2(\epsilon) = o(g_1(\epsilon))$. By a similar addition of terms we can arrive at the representation

$$f(\epsilon) = \sum_{i=1}^{n} c_i g_i(\epsilon) \ \ as \ \ \epsilon \to 0 \tag{4.2}$$

The functions $\{g_i(\epsilon)\}$ are known as gauge functions and Eq. (4.20) is called an *asymptotic expansion* (AE) of $f(\epsilon)$. If the gauge functions are of the form $\{\epsilon^i\}$,

$i = 1, 2, ...n$ then Eq. (4.20) is an *asymptotic power series* (APS). We must note that the gauge functions are not always positive integral powers of ϵ. Sometimes exponential and logarithmic terms are needed to represent the physical phenomena. For example, $ln(1/\epsilon) \to \infty$ as $\epsilon \to 0$, but does so slower than any finite power of ϵ. Likewise, as $\epsilon \to 0$, $exp(-1/\epsilon) \to 0$ faster than any power of ϵ. As needed, additional terms such as $ln\epsilon$, $\epsilon ln\epsilon$, $lnln\epsilon$ must be used in some cases in order to describe the correct asymptotic behavior. Some examples are:

$$arcsech(\epsilon) = O(ln\ \epsilon) \tag{4.3}$$

$$cosh(1/\epsilon) = O(e^{1/\epsilon}) \tag{4.4}$$

$$arccoshK_0(\epsilon) = O(ln\ ln\ (\frac{1}{\epsilon})) \tag{4.5}$$

where K_0 is the modified Bessel function of the zeroth order. Another point to note is that the gauge functions are not unique in that a particular function may have different-looking asymptotic expansions depending on the set of gauge functions chosen. For example,

$$cos\big(\frac{2\sqrt{\epsilon}}{1+\epsilon}\big) - 1 = O(\epsilon) = O(4\epsilon) = O\big(\frac{4\epsilon}{(1+\epsilon)^2}\big) \tag{4.6}$$

In asymptotic analysis the numerical coefficient within the argument of the O-symbol and its algebraic sign are ignored. That is,

$$sin(-2\epsilon) = O(\epsilon) = O(-2\epsilon) \tag{4.7}$$

The O and o symbols do not necessarily describe the precise rate of the limiting behavior, but describe only the upper bound. Formally,

$$cos\big(\frac{2\sqrt{\epsilon}}{1+\epsilon}\big) - 1 = O(\epsilon) = o(\epsilon^{1/3}) = o(\sqrt{\epsilon}) = o(1) \tag{4.8}$$

Manipulations of the order symbols follow simple rules. The O-symbol is more frequently used; the o-symbol is used only in case of insufficient knowledge to warrant the use of the O-relation. The simple rules for performing operations with the order symbols can be quite useful in problem solving. The following examples are to be understood to have the same limiting conditions.
If $f = O(g)$, then

- $|f|^n = O(|g|^n); n > 0$
- $\sum_{i=1}^{N} c_i f_i = O\big(\sum_{i=1}^{N} |c_i||g_i|\big)$; *where* c_i *are constants*
- $f_1 f_2 f_3 ... f_n = O(g_1 g_2 ... g_n)$
- $\int_t^b f(s)ds = O(\int_t^b |g(t)|dt); t \to b$
- $\int_a^b f(x, y)dy = O(\int_a^b |g(x, y)|dy); x \to x_0$

However, in general, differentiation of order relations with respect to the independent variable (or parameters) is not permitted. For example, consider

$$f(t) = e^{-t} sin(e^t) \tag{4.9}$$

We note that $f(t) = o(1)$ as $t \to \infty$, but

$$\frac{df}{dt} = cos(e^t) - e^{-t} sin(e^t) = O(1) \tag{4.10}$$

as $t \to \infty$. Therefore,

$$\frac{df}{dt} \neq O(\frac{dg}{dt}) \tag{4.11}$$

Some general rules of differentiation of order relations are available in the case of analytic functions of a complex variable. They are discussed in [5]. For some useful relations on the combinations of the order relations the interested reader can consult [5].

4.3 Asymptotic Sequences and Expansions

A successful method of generating approximations to the solutions of differential equations is by developing asymptotic expansions of the solutions. This generates a sequence of functions which are arranged in the order of decreasing magnitudes in the limit. A sequence $\{g_n\}$ is called an asymptotic sequence in the limit $t \to t_0$ is, for every n, $g_{n+1} = o(g_n)$. The formal series $\sum c_i f_i(t)$ is said to be an *asymptotic expansion* to n terms of a function $F(t)$ as $t \to t_0$ if

$$F(t) = \sum_{i=1}^{n} c_i f_i(t) + o(f_n) \quad as \quad t \to t_0 \tag{4.12}$$

It is often denoted by

$$F(t) \sim \sum_{i=1}^{n} c_i f_i(t) \quad as \quad t \to t_0 \tag{4.13}$$

An asymptotic expansion to one term is called an asymptotic representation and is denoted by

$$F(t) \sim c_1 f_1(t) \tag{4.14}$$

The coefficients of an asymptotic expansion to N terms may be calculated by the recurrence relation

$$c_n = lim_{t\to t_0}([F(t) - \sum_{i=1}^{n-1} c_i f_i(t)]/f_n(t)); n = 1, \cdots N \tag{4.15}$$

A few points regarding the representation of functions by asymptotic expansions must be noted. A certain function may be represented by using different asymptotic sequences, i.e., using different gauge functions. For instance, consider the following examples of asymptotic expansions [5].

$$\begin{aligned}
\frac{1}{1+t} &\sim \sum(-1)^{n-1}t^{-n}; && t\to\infty \\
&\sim \sum(t-1)t^{-2n}; && t\to\infty \\
&\sim \sum(-1)^{n-1}(t^2-t+1)^{-3n}; && t\to\infty \\
Sin(2\epsilon) &\sim 2\epsilon - \frac{4}{3}\epsilon^3 + \frac{4}{15}\epsilon^5 + \cdots \\
&\sim 2tan\epsilon - 2tan^3\epsilon - 2tan^5\epsilon + \cdots \\
&\sim 2log(1+\epsilon) + log(1+\epsilon^2) - 2log(1+\epsilon^3) + \cdots
\end{aligned}$$

The corresponding terms in the asymptotic expansions need not be of the same order; *the sequences are not equivalent*. But once the asymptotic sequence is prescribed, the asymptotic expansion of a function is unique. Also, different functions can have the same asymptotic expansion. Therefore, an asymptotic expansion does not have a unique "sum", $F(t)$. For example, the functions

$$\frac{1}{(1+t)}$$
$$\frac{1}{(1+t+e^{-t})}$$
$$\frac{1}{(1+t+e^{-t^2})}$$

all have the *same* asymptotic expansion $\sum(-1)^{n-1}t^{-n}$ as $t\to\infty$. As expressed by Erdelyi [5], an asymptotic series represents a class of asymptotically equivalent functions rather than a single function. As for other operations with asymptotic expansions, the following results are useful.

1. Summation of asymptotic expansions(AE) with the same asymptotic sequence generates an asymptotic expansion, i.e., if $f \sim \sum_1^n a_i\phi_i$ and $g \sim \sum_1^n b_i\phi_i$, then $\alpha f(t) + \beta g(t) \sim \sum_1^n(\alpha a_i + \beta b_i)\phi_i(t)$.
2. Integration of AE. Termwise integration is permitted and leads to the AE of the integral.
3. Differentiation of AE. This is much more difficult than the above problem. The task is to find the AE of the integrand, knowing the AE of the integral. In other words, given the AE of a function, what is the AE of the derivative of the function? It is important to note that differentiation an AE either with respect to a variable or

parameters is *not valid* in general. To illustrate, consider the example of a function $f(t) = e^{(-1/t)} sin(e^{(1/t)})$. For $t \to 0$, $f(t) \sim 0$. But the derivative is

$$f'(t) = t^{-2}[e^{-1/t} sin(e^{1/t}) + cos(e^{1/t})] \tag{4.16}$$

$lim_{t\to 0} f'(t)$ is not defined. Therefore, $f'(t)$ does not have an asymptotic power series. However, this is an unusual case. Termwise differentiation of the AE valid only on a single ray in the complex plane is not allowed. By suitably restricting the AE in different sectors of a complex t plane, a proper asymptotic representation of $f'(t)$ can be developed. Further, when $f(t)$ is an analytic function of a complex variable t, some general results on the differentiation of AE are available [5].

If $f(t)$ is differentiable and $f'(t)$ possesses an asymptotic power series expansion (APSE) and $f(t) \sim \sum_0^N a_i t^{-i}$ as $t \to \infty$, then

$$f'(t) \sim -a_1/t^2 - 2a_2/t^3 - 3a_3/t^4 \cdots \tag{4.17}$$

to $(N-1)$ terms as $t \to \infty$. If $f(t)$ is analytic, the requirement that $f'(t)$ possess an APSE may be relaxed; i.e., if $f(t)$ is regular and $f(t) \sim \sum_1^{N-1} a_i t^{-i}$ as $t \to \infty$ uniformly in arg t, then

$$f'(t) \sim \sum_1^{N-1} (ia_i/t^{i+1}) as\ t \to \infty \tag{4.18}$$

4. Multiplication of an asymptotic series does not always lead to an asymptotic series. The reason is that, in the formal product of $\sum a_m f_m$ and $\sum b_n g_n$, all products $f_m g_n$ occur. Rearrangement to get an AE is not possible in general. However, some general results are available on the multiplication of two asymptotic series [5]. For several classes of functions such as polynomials, formal multiplication of two asymptotic series leads to an asymptotic series. Similar results are available for the division of asymptotic expansions for polynomial and rational functions. Under certain circumstances, functions of functions can be similarly treated. Asymptotic expansions of functions such as $exp[P(f_1(t), f_2(t), \cdots f_n(t)]$ can be justified in this manner. A more detailed discussion of multiplication of asymptotic expansions is found in other works, for eg. Erdelyi [5].

By analogy with a convergent series whose sum is finite even with infinite number of terms, the "*sums*" of asymptotic series (even when they are *divergent*) have been considered. Such sums are those of Borel, Euler, Cesaro and Riesz. They will not be discussed here. The interested reader may consult the references.

The mathematical description of physical problems usually are in the form of differential equations involving one or more physical constants. The dependence of the solutions on these parameters is important in theoretical investigations as well as in applications. Historically, asymptotic methods in applied mathematics have been developed treating the parameters as *small* quantities. For example, boundary layer theory in fluid mechanics treats the kinematic viscosity to be small. In the three-body

problem of celestial mechanics, the ratio of the mass of the smallest body to the total mass of the system is taken as a small parameter. There are many other examples. The dependence of solutions on a parameter is not, in general, analytic, even if the coefficients of the differential equation are analytic in the parameter. This can be seen by noting that the product of a solution of a linear differential equation and any function of the parameter is also a solution. We will primarily deal with asymptotic expansions of solutions of differential equations with respect to a single parameter.

4.4 Convergence and Asymptoticity

It is useful to distinguish between the concepts of convergence and asymptoticity of the approximations in order to avoid any confusion in understanding the difference between them. Indeed, a series of approximations is said to "converge" to the solution, by which it is meant that the approximation approaches the true solution more and more closely by including additional terms, i.e., it is *asymptotic*.

On the other hand, the mathematical property of convergence of a series concerns the boundedness of an unlimited number of quantities. For example, the exponential function $e^{-\epsilon t}(|\epsilon| \ll 1)$ has the series representation

$$e^{-\epsilon t} = 1 - \epsilon t + \epsilon^2 t^2/2 - \cdots = \sum_0^\infty \frac{(-1)^n \epsilon^n t^n}{n!} \tag{4.19}$$

The exponential function converges for all values of t. in order to compute the value of the function from the series, the number of terms required for a specified accuracy depends on the value of t for which the calculation is made. For small $t \leq O(1)$, a small number of terms will suffice, as the series is asymptotic. For large values of t, the magnitude of the successive terms increases up to a point and then starts to decrease as the factorial term in the denominator becomes increasingly large, thereby reducing the magnitude of the terms. Therefore, a large number of terms is needed to compute the function for large values of t. Thus, the series is not uniformly asymptotic for all values of t, although it is convergent for all values of t.

Another example is that of the error function

$$\phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{(-s^2)} ds \tag{4.20}$$

which can be represented by an absolutely convergent series

$$\phi(t) = \frac{2}{\sqrt{\pi}} \sum_0^\infty \frac{(-t^2)^n}{n!(2n+1)} \tag{4.21}$$

The terms in the series do not start decreasing until $n \sim t^2$ and the partial sum does not approximate the function until much later, $(n \sim exp(t^2))$. More complicated

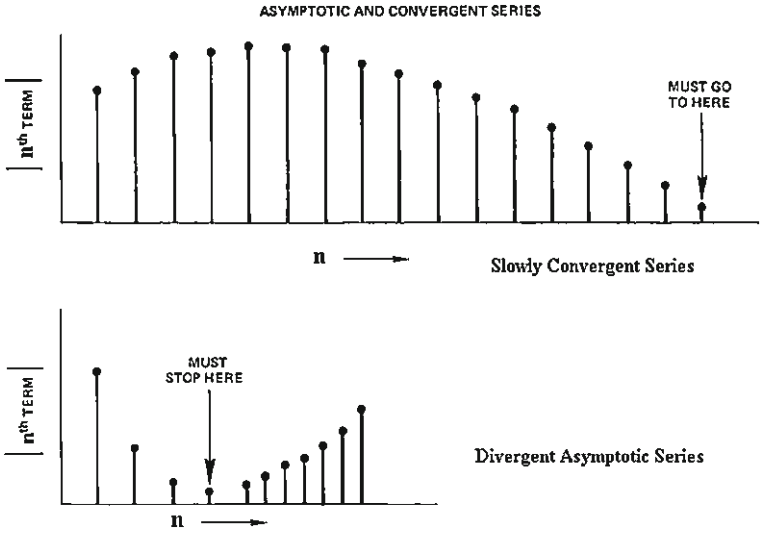

Fig. 4.1 Convergent and asymptotic series

functions and their series discussed in Sect. 4.2 make this matter much worse. On the other hand, it is possible that a series which diverges for all nontrivial values of the argument may yet provide a useful means of computing a function approximately.

As an example, we may consider

$$S(t) = 1 - 1!t + 2!t^2 - 3!t^3 + \cdots = \sum_0^\infty (-1)^n n! t^n \tag{4.22}$$

which diverges for all $t \neq 0$). Yet, for small $t(\sim 10^{-2})$, the magnitude of the terms at first decreases rapidly and an approximation to the function may be numerically computed. The series actually represents the Stieltjes integral

$$f(t) = \int_0^\infty \frac{e^{-x} dx}{(1 + tx)} \tag{4.23}$$

in an asymptotic sense as $t \to 0$. More dramatic examples can be constructed.

The distinction between a slowly convergent series and an asymptotic series is illustrated graphically in Fig. 4.1. Convergence of a series is a useful property. However, for computational purposes, an *asymptotic* series is more useful than a convergent series, i.e., the higher order terms rapidly decrease in magnitude. The accuracy is enhanced by evaluating successively larger number of terms. this property has been seen in the examples in Chap. 2. Therefore, the function can be calculated by a small number of terms.

In a slowly convergent (but not asymptotic) series, for example, Eq. (2.1) for $t = 10$, the magnitudes of the terms increase initially as more terms are taken and

will eventually begin to decrease when the number of terms becomes large enough. Thus a large number of terms may be needed in order to approximate the function accurately. On the other hand, in a divergent asymptotic series (for example, Eq. (2.11) for $x \to 0$, or Eq. 2.16 for $n \to \infty$), the magnitude of each successive term decreases (compared to the previous term) up to a point of minimum. However, if this process is continued, the terms begin to increase in magnitude. Before reaching the minimum, the accuracy of approximation can be sharply increased by including more terms of the series. The accuracy is maximum when the magnitude of the terms is minimum. This represents the ultimate achievable accuracy of an asymptotic series for a fixed *small* parameter ϵ. However, the error can be reduced by decreasing the magnitude of the parameter ϵ for a fixed number of terms n, although the error increases as n increases for a fixed ϵ. These properties are often useful in practical applications. The point at which the error starts to increase is not ususally reached in applications, as only the first few terms of the series are calculated and the parameter ϵ is small. Thus an asymptotic series provides a good approximation.

4.5 Error of Approximation

Asymptotic analysis inherently possesses an estimate of the error. As prescribed by Poincaré, at any stage of the asymptotic expansion the error is of the order of the first term that is neglected. The error associated with an n-term asymptotic expansion is of the order of the $(n+1)^{th}$ term. That is,

$$\left| f(t,\epsilon) - \sum_{0}^{n} \epsilon^i g_i(t) \right| = O(\epsilon^{n+1}) \tag{4.24}$$

This is very useful in computation, especially when ϵ is a *small* parameter ($|\epsilon| \ll 1$). In this case the error would indeed be very small. The magnitude of the terms of the expansion decreases rapidly, and a sufficiently accurate approximation can be thus obtained. However, a few questions arise. First, it is not clear that the initial favorable behavior of the various terms would continue at higher orders. Second, it is known that a change in the ordering of terms in an infinite series (even if it is convergent) can change the sum of the series. Therefore, if the perturbation methods lead to a rearrangement of the series to reflect the improving accuracy as one proceeds to higher orders, does the series lead to the correct *sum* and not *any sum*? These and other challenging questions of mathematical rigor are of interest to the mathematicians. Starts in this direction have been made through the theory of *terminants* using the concept of Borel sums [see references].

Without delving into such questions, the analysis of errors can be illustrated by the following example discussed by Whittaker and Watson [6]. Consider the function

$$y(t) = \int_t^\infty \frac{e^t e^{-s} ds}{s} \tag{4.25}$$

where the integral is along the real axis and $t \geq 0$. Integrating by parts repeatedly, we have

$$y(t) = \frac{1}{t} - \frac{1}{t^2} - \cdots + \frac{(-1)^n(n-1)}{t^n} + R_n \tag{4.26}$$

where

$$R_n = (-1)^n n! \int_0^\infty \frac{e^{t-s}ds}{s^{n+1}} \tag{4.27}$$

Consider the partial sums

$$S_n(t) = \sum_{k=0}^{n} \frac{(-1)^k k!}{t^{k+1}} \tag{4.28}$$

The convergence properties of this series as $k \to \infty$ can be examined by Cauchy's ratio test. The ratio of the k^{th} term to the $(k-1)^{th}$ term is $(\frac{k}{t})$ as $k \to \infty$. The series is, therefore, divergent for all values of t. However, for large values of t, it is an asymptotic series and can be used to calculate $y(t)$. For a *fixed* n, we have

$$y(t) - S_n(t) = R_n(t) = (-1)^{n+1}(n+1)! \int_t^\infty \frac{e^{t-s}ds}{s^{n+2}} \tag{4.29}$$

Noting that $e^{t-s} \leq 1$, we can bound the error as

$$|y(t) - S_n(t)| = (n+1)! \int_0^\infty \frac{e^{t-s}}{s^{n+2}} ds < (n+1) \int_t^\infty \frac{dt}{t^{n+2}} = \frac{n!}{t^{n+1}} \tag{4.30}$$

For large t, this difference is very small. For $t \geq 2n$,

$$|y(t) - S_n(t)| < \left(\frac{1}{2^{n+1}n^2}\right) \tag{4.31}$$

For large n the right hand side is very small. Therefore, for large values of t, the series can be used to compute $y(t)$ to great accuracy by taking an appropriate sum of terms. Further, we see that $|t^n\{y(t) - S_n(t)\}| < \frac{n!}{t}$ as $t \to \infty$. Therefore, $S_n(t)$ is an asymptotic expansion of $y(t)$ as it satisfies the definition stated in [5].

When applying asymptotics to engineering systems, some of the important considerations are, the comparison of the order of magnitude of the functions, the requirement of uniform validity, distinction between convergence and asymptoticity, uniqueness (or otherwise) of asymptotic expansions, and error analysis, etc. As noted by Poincaré, the asymptotic expansions provide an estimate of the approximation error. However, strict and sharp bounds on the error in the case of some well-known approximations, such as, for example, slowly varying linear differential equations of second order, are given by Olver [7].

4.6 Uniform Validity

We will now consider the important property of *uniform validity*. The concept of uniformity is important in many mathematical processes including continuity, summation and asymptoticity of series, and especially in approximating a function in terms of *simpler* representations. A precise discussion of this concept is given in [6]. Given a function $x(t, \epsilon)$ of arbitrary shape [and ϵ is a small parameter, $(|\epsilon| \ll 1)$, $x_0(t, \epsilon)$] is said to be a uniformly valid approximation of $x(t, \epsilon)$ to order ϵ in a specific interval, if and only if, for all t in that interval,

$$x(t, \epsilon) = x_0(t, \epsilon) + O(\epsilon) \tag{4.32}$$

More generally, we could have

$$x(t, \epsilon) = x_0(t, \epsilon) + o(1) \tag{4.33}$$

That is, the error between the function $x(t, \epsilon)$ and its approximation $x_0(t, \epsilon)$ is *uniformly* small during the interval of interest.

Recalling the order symbols already discussed, we write $f = O(g)$ if there is a constant C such that $|f(\epsilon)| \leq C|g(\epsilon)|$ for all ϵ in the interval. If the functions involved in an order relation depend on other quantities such as time and space variables, the constant C and the governing neighborhood also depend on these quantities. If the constant C and the neighborhoods are independent of the parameters, the order relation is said to hold *uniformly* in the parameters. Some examples of nonuniformities are

$f(t, \epsilon) = \epsilon t = O(\epsilon)$ for $t \leq 1$ but not for $t \geq 1/\epsilon$

$f(t, \epsilon) = \sqrt{\epsilon/t^3} = O(\sqrt{\epsilon})$ for $t \geq 1$ but not $t \leq \epsilon^{\frac{2}{3}}$. In these examples $\epsilon t = O(\epsilon)$ implies that $lim_{\epsilon \to 0} \frac{\epsilon t}{\epsilon} = A$ and $\epsilon^{\frac{1}{2}} t^{\frac{3}{2}} = O(\sqrt{\epsilon})$ implies that $lim_{\epsilon \to 0} \sqrt{\epsilon} t^{\frac{3}{2}} = B$. If A and B are independent of t, then the order relations hold uniformly. It is clear that the aim of any approximation scheme is to construct approximate solutions which are accurate and uniformly valid in the region of interest. A straightforward perturbation expansion may not lead to such a uniform approximation. Several uniformizing techniques such as Lighthill's [8] expansion of the independent variable, the averaging method of Bogoliubov [9], the method of matched asymptotic expansions (MAE)[10], and the method of multiple scales [11, 12] have found great success on many applications. The interested reader may consult the references for further details.

4.7 Uniqueness of Asymptotic Expansions

While dealing with asymptotic expansions, a question that occurs naturally is that of uniqueness of such expansions. One may wonder whether such uniqueness is true *bilaterally*, i.e.,

(i) A given function has a unique asymptotic expansion.
(ii) A given asymptotic series is the asymptotic expansion of a unique function.

Statement (i) is true if the gauge functions are specified. In other words, for a particular asymptotic sequence, a function has a unique asymptotic expansion and cannot be represented by more than one distinct asymptotic expansion. This can be shown as follows. Suppose that two series,
$\sum_0^\infty a_i/t^i$ and $\sum_0^\infty b_i/t^i$ are asymptotic expansions of a function $f(t)$, i.e., $f(t) \sim \sum_0^\infty a_i t^{-i}$ and $f(t) \sim \sum_0^\infty b_i t^{-i}$. Then

$$lim_{t\to\infty} t^n \left(\sum_{i=0}^{n} a_i t^{-i} - \sum_{i=0}^{\infty} b_i t^{-i} \right) = 0 \tag{4.34}$$

Therefore, $a_i = b_i$; $i = 0, 1, 2, \cdots$.

However, statement (ii) is not true. This is made clear by considering that there are functions $g(t)$ which have asymptotic representations whose terms are all zero, such that $lim_{t\to\infty} t^n g(t) = 0$ for all n.

An example of such a function is e^{-t}. Therefore, the asymptotic expansion of a function $f(t)$ for $t \to \infty$ is also the asymptotic expansion of $f(t) + g(t)$. This ambiguity occurs essentially because of Poincaŕe's definition of an asymptotic expansion, that $\sum a_i/t^i$ is an asymptotic power series expansion of a function $f(t)$ if

$$lim_{|t|\to\infty} t^n \left(f(t) - \sum_{0}^{n} a_i/t^i \right) = 0 \quad (n \ \ fixed) \tag{4.35}$$

for all zero and positive n even though

$$\lim_{n\to\infty} \left| t^n \left\{ f(t) - \sum_{0}^{n} \frac{a_i}{t^i} \right\} \right| = \infty \quad (t \ \ fixed) \tag{4.36}$$

When this is the case, we can make

$$\left| t^n \left[f(t) - \sum_{0}^{n} \frac{a_i}{t^i} \right] \right| \leq \epsilon \tag{4.37}$$

where ϵ is arbitrarily small by taking $|t|$ sufficiently large. Literal adherence to this definition requires one to abandon all exponentially small terms [such as the function $g(t)$ above] in an asymptotic expansion. Recent work in this field [see references] concerns the development of a complete asymptotic expansion of a function including exponentially small terms. A discussion of these ideas is beyond the current scope.

4.8 Perturbation Theory

In applications the asymptotic analysis is incorporated in perturbation theory. The mathematical models, which are usually in the form of algebraic and differential

equations contain a *small* parameter $\epsilon \ll 1$. For convenience the parameter is assumed to be positive and not zero. It is further assumed that the problem is well posed, i.e., has a meaningful solution $x_0(t)$ when $\epsilon = 0$. Then the solution $x(t, \epsilon)$ is perturbed in powers of the parameter ϵ in a power series in the form

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{4.38}$$

As discussed earlier, the subject of perturbation methods was placed on a firm foundation of asymptotic theory by Poincaré [2, 3]. The understanding of the nature of fluid flow, especially the effects of viscosity, provided a strong impetus to the development of perturbation theory from an applied standpoint. Poincaré developed the subject mainly in the context of celestial mechanics. In this section, the basic mechanism of perturbation methods is illustrated by means of simple examples occurring in physical problems.

Consider a physical problem whose mathematical model is given by a differential equation of the form

$$\mathcal{L}(x, t, \epsilon) = 0 \tag{4.39}$$

where $\mathcal{L}$ is a differential operator, $x(t, \epsilon)$ is the solution and ϵ is a small parameter $(0 < \epsilon \ll 1)$. Perturbation theory assumes the existence of a nominal solution $x(t, 0)$ which satisfies the equation

$$\mathcal{L}(x, t, 0) = 0 \tag{4.40}$$

That is, Eq. (4.39) has a meaningful limiting case as $\epsilon \to 0$ and $x(t, 0)$ is denoted by $x_0(t)$. The solution $x(t, \epsilon)$ can be represented as a series in powers of ϵ as

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{4.41}$$

We note that we usually seek Eq. 4.41 in the form of an asymptotic series rather than a convergent series. When ϵ is small (i.e., $0 < \epsilon \ll 1$), the deviation of the actual solution $x(t, \epsilon)$ from the approximation $x_0(t)$ will also be small. This is the rationale by which a nonlinear dynamic system

$$\aleph(x, t) = 0 \tag{4.42}$$

is represented by a linear system for small deviations from the nominal state.

Direct Perturbation

Consider the differential equation

$$\mathcal{L}(x, t, \epsilon) = 0; \quad 0 < \epsilon \ll 1 \tag{4.43}$$

Poincaré suggested that approximations to the solution $x(t, \epsilon)$ can be sought in the form

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \cdots + \epsilon^n x_n(t) + \cdots \tag{4.44}$$

Upon expanding the operator $\mathcal{L}$ in powers of ϵ, i.e.,

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots \tag{4.45}$$

Substitution of this series into Eq. (4.39) yields, upon simplification,

$$\mathcal{L}_0 x_0(t) + \epsilon\mathcal{L}_1[x_0(t), x_1(t)] + O(\epsilon^2) = 0 \tag{4.46}$$

As Eq. (4.46) is true for any ϵ, we may equate the coefficients of like powers of ϵ. We thus have a recursive set of equations

$$\mathcal{L}_0\{x_0(t)\} = 0 \tag{4.47}$$

$$\mathcal{L}_1[x_0(t), x_1(t)] = 0 \tag{4.48}$$

$$\vdots \tag{4.49}$$

$$\mathcal{L}_i[x_i(t), x_{i-1}(t)], \cdots x_0(t)] \tag{4.50}$$

Solving each of these equations in succession, we obtain the functions $x_0(t), x_1(t), \ldots$. Using Eq. (4.44) we can construct a straight perturbation expansion which is an approximation of the solution $x(t,\epsilon)$.

Nonuniformities in Direct Perturbation

When straight perturbation works, we have a process of systematically generating approximations order by order. If the approximate solution to a certain order is not found to be sufficiently accurate, then one simply proceeds to a higher degree of approximation by generating corrections of a higher order in the expansion parameter ϵ. However, such a direct expansion often leads to difficulties and straight perturbation solutions may fail in the major domain of interest. In such cases, the addition of higher order perturbation solutions may increase the accuracy in one part of the domain, while degrading the accuracy in another part of the domain of interest. This breakdown of the approximation is termed a *nonuniformity* of the perturbation expansion. Most of the sophisticated techniques of perturbation theory are aimed at solving this problem of nonuniformity, rendering the approximations uniformly valid.

We will first present a few basic examples of nonuniformities and their classification. Later some well-known methods of removing such nonuniformities will be outlined.

4.9 Types of Nonuniformities

Three basic types of nonuniformities are categorized according to their behavior.

(a) **secular** - nonuniformity occurs as $t \to \infty$.

(b) **singular** - nonuniformity occurs as $t \to 0$, or
(c) **boundary-layer** systems, resulting in a loss of boundary conditions.

Historically, each type has given rise to the development of different techniques for solving the particular problem.

In the context of computation, singular perturbations of the boundary layer type need to be carefully treated as they exhibit the presence of different scales on which the solution varies and therefore necessitate different rates for computation. Such problems are also called *stiff* systems, as stated earlier. For example, the boundary-layer model is given by

$$\epsilon x'' + x' + x = 0 \tag{4.51}$$

where the prime denotes differentiation with respect to t. In this case, separate expansions are made to represent the solution, so that each approximation is valid in a different region. The existence of different regions can be discerned by reexamining Eq. (4.51).

Now, we invoke the minimal simplification (or dominant balance) principle [13–15]. Note that each term of Eq. (4.51) contains information about the solution. Neglecting the first term in Eq. (4.51) in favor of the other two, we have the approximate subsystem

$$x' + x = 0 \tag{4.52}$$

which has the solution

$$x(t) \approx ce^{-t} \tag{4.53}$$

In this case the neglected term $\epsilon x''$ is indeed small in comparison with the terms retained,—i.e., either x' or X. Therefore, this is the correct asymptotic behavior of $x(t)$ for large values of t, i.e., as $t \to \infty$. It represents the *slow* behavior of the solution.

The other asymptotic behavior is obtained by assuming that the first term $\epsilon x''$ is *not* negligible. Indeed, if it is as large as the second term x', then, neglecting the third term X, we have the approximate subsystem

$$\epsilon x'' + x' = 0 \tag{4.54}$$

with the solution

$$x(t, \epsilon) = c_1 e^{-t/\epsilon} + c_2 \tag{4.55}$$

where c_1, c_2 are arbitrary constants. This is the correct asymptotic behavior for small values of t, i.e., as $t \to 0$. It represents the *fast* behavior of the solution which is valid within the boundary layer. It is easy to verify that the neglected term, X is indeed negligible when compared with the retained terms,—i.e., $\epsilon x''$ or x'.

Starting with Eq. (4.51)

$$\xi = \epsilon^m t \tag{4.56}$$

Equation 4.51 can now be written as

$$\epsilon^{(1+2m)} \frac{d^2x}{d\xi^2} + \epsilon^m \frac{dx}{d\xi} + x = 0 \tag{4.57}$$

The coefficients of the different terms are of the form ϵ^{a+bm}. Imposing *minimal simplification* leads to the choice of $m = 0$ and $m = -1$, which can be justified by plotting a and b along the ordinate and abscissa respectively on the Kruskal diagram [13, 15]. Two *lower convex support lines* are obtained and corresponds to $m = 0$ and $m = -1$. These lead to the correct change of the independent variable and to the two correct asymptotic behaviors, $-x \sim e^{-t}$ and $x \sim e^{-t/\epsilon}$.

Summarizing, the solution changes rapidly within the *boundary layer*, and slowly outside this region. Thus, inside the boundary layer, the solution has the dominant approximation given by $exp(-t/\epsilon)$, whereas outside the boundary layer, the solution is represented by the function $exp(-t)$. The sum of the two approximations is taken as the representation of the true solution. Generally, there are arbitrary constants associated with each of these approximations. They are chosen to satisfy the boundary conditions. The approximation valid within the boundary layer is also called the *inner* solution, and the slower approximation valid outside the boundary layer is called the *outer* solution. At this stage, a *matching principle* is invoked, which is stated as follows:

$$outer\ limit\ of\ the\ inner\ solution = inner\ limit\ of\ the\ outer\ solution \tag{4.58}$$

This principle enables us to determine the free constants present in the "inner" and "outer" approximations. This idea can be extended if there are several terms in the two approximations. A discussion of these is beyond the current scope and the interested reader may consult the references.

However, we prefer to treat these solutions in the context of multiple scales. Thus, the "inner" or boundary layer approximation represents the *fast* solution and the "outer" solution describes the *slow* behavior. With this perspective, the multiple scales technique leads to a systematic and correct development of the solutions [12]. The application of the multiple scales technique to nonuniform perturbation problems (including the boundary-layer case) is presented in the next chapter.

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Chapter 5
Stiff Systems and Multiple Scales

Most physical or engineering dynamic systems exhibit a mixture of fast and slow variations. Such systems are known as *multiple scale systems* or *stiff* systems. Some examples have already been considered. It is relatively rare that the dynamics of a physical system evolve on a single time scale. In general, an understanding of the behavior of these systems is facilitated by a separation of the rapid and slow aspects of the phenomenon. The method of multiple scales is designed to achieve such a systematic separation.

The Multiple Scales technique has its origin in the work of Poincaré [1] on the secular expansion in celestial mechanics and to the works of Krylov [2] and Bogoliubov and Mitropolsky [2] who allowed the constants arising in direct perturbation theory to be slowly varying functions. Cole, Kevorkian [3] and Nayfeh [4] applied it to several problems in the context of some nonlinear differential equations. Frieman [5] used it in the theory of irreversible processes and Sandri [6] applied it in statistical mechanics. However, an important point is that most of the previous applications treat the multiple scales approach merely as a useful transformation. However, Sandri [6] has also investigated a general development of the technique and its role in uniformization from the fundamental standpoint involving composition of mappings. Ramnath [7, 8] generalized and extended the theory through diverse applications. We will first present a heuristic and conceptual development of the basic idea of multiple scales.

5.1 Concept of Multiple Time Scales

Ordinarily, we are used to thinking of *time* as a monotonic, unidirectional and scalar quantity with one dimension. However, observations of natural phenomena and physical processes suggest that time can be conveniently thought of, not merely as a *scalar* quantity, but indeed as an entity having more than one dimension, i.e., as a *vector*. Many physical phenomena exhibit this feature. Of the many examples, the following may be considered. Dynamic phenomena in fields such as physics, astronomy, the motion aircraft and spacecraft, chemical reactions etc., show fast and slow variations.

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 51
DOI: 10.1007/978-3-642-25749-0_5, © The Author(s) 2012

Other examples include biology, economics, stock market fluctuations, and so on. In each of these, there are daily, weekly, monthly, seasonal and yearly variations. In astrodynamics, an earth satellite orbits the earth, the moon revolves around the earth, the earth rotates on its axis, the earth-moon system goes around the sun, and so on. Each of these motions occurs at a different rate and gives rise to the motion on different time scales.

In the above examples it is useful and convenient to to think of slowing down *time* to allow for the observation of fast occurring phenomena. In instances of slow phenomena, time can be speeded up to facilitate the clear observation of the slow events. We can extend this idea through a generalization in which the *fast* and *slow* variations are observed *simultaneously* by different observers using different clocks which count time at different rates.

The *multiple time scales* technique makes use of these inherent natural time scales. The basic idea is as follows. A complex dynamic phenomenon unfolds as a function of time as the independent variable. It may be a combination of rapid and slow variations. For example, it may be a mixture of motions at low and high frequencies, or very low and very high frequencies, etc. A conventional and direct approach with the use of *one* time scale would not be an efficient way to describe *all* the variations which occur at different rates. While the chosen time scale might be efficient in describing the motion at one frequency, (eg. low frequency), it may not be the best in representing high frequency motions. It would be highly attractive to be able to describe the motion at each frequency very efficiently. This is the crux of the multiple scales technique. By this means, each motion is described in terms of an appropriate time scale, which may be fast, slow, very fast, very slow, etc., according to the nature of the motion which is being represented. However, the fundamental idea is that all the different variations are described *simultaneously*.

5.2 Multiple Scales Extension

- $t \rightarrow \{\vec{\tau}\}$ is not single-valued. It is a multiple-valued mapping.
- Many independent variables are used. The different aspects of the phenomenon are observed *simultaneously* on different time scales.
- Nonperiodic and nonoscillatory phenomena can be treated.

5.3 Example

In general, a time history may display a mixture of fast and slow motions and therefore, the existence of a number of time scales. For example, there may be a fast oscillation, combined with a slow oscillation and a very slow amplitude decay [9]. In this case, all three motions are combined when viewed on the normal t time scale. However, when viewed on the different fast and slow time scales, τ_0, τ_1 and τ_2, the time evolution becomes transparent. Such a separation of the different motions can

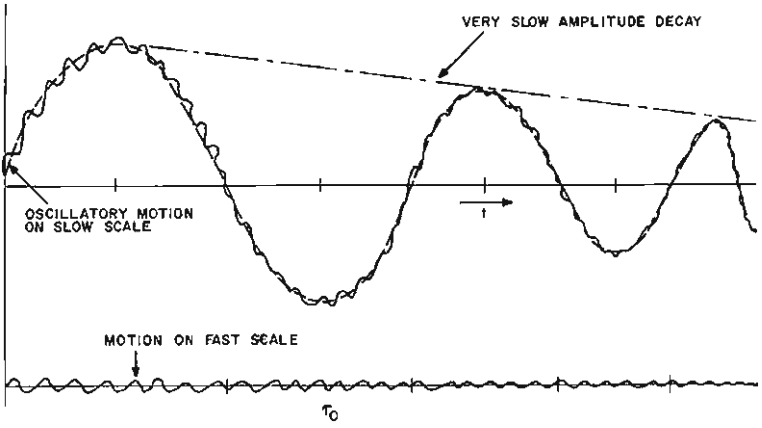

Fig. 5.1 Mixed motions

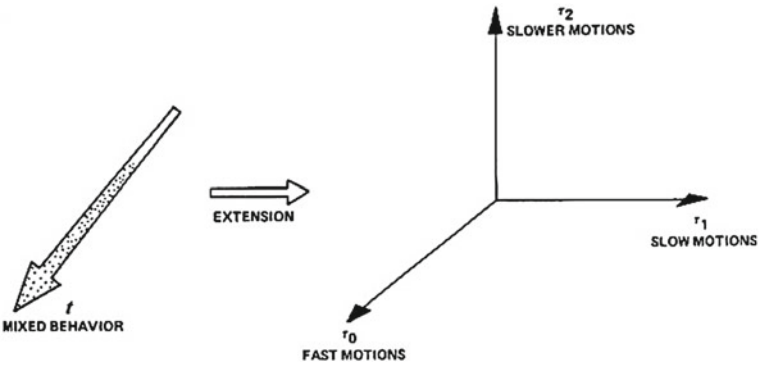

Fig. 5.2 GMS extension

be exploited and constitutes the essential aspect of the Multiple Time Scales (MTS) method. The multiple scales method postulates a number of *independent* observers, who *simultaneously* perform measurements of the phenomenon using different clocks like $\tau_0, \tau_1, \cdots$. That is, one observer uses a fast clock, a second observer uses a slow clock, and so on. Thus, each observer picks up each aspect of the dynamics very efficiently. Finally, the information obtained by *all* the observers is combined, thereby giving the complete description (Figs. 5.1 and 5.2).

A graphical interpretation can be given [9] by considering a three-dimensional space with orthogonal axes τ_0, τ_1 and X. Readings on the "fast" and "slow" clocks are represented, respectively, by points along τ_0 and τ_1 coordinates and X is defined to be the function $X(\tau_0, \tau_1) = exp(-\tau_1)$. Graphically, $X(\tau_0, \tau_1)$ is represented by a surface which is constant along τ_0, but decays exponentially along τ_1.To relate $X(\tau_0, \tau_1)$ to $x(t, \epsilon)$, we substitute $\tau_0 = t$ and $\tau_1 = \epsilon t$ into $X(\tau_0, \tau_1)$, giving $X(t, \epsilon t) = x(t, \epsilon)$. The function $X(\tau_0, \tau_1)$ is said to be an extension of $x(t, \epsilon)$. The concept and operation of extension are defined as follows:

Extension: Given a function $x(t)$, where t is, in general, an n-dimensional vector, and a function $X(\tau_0, \tau_1, \cdots \tau_{N-1})$ of the N independent variables $\tau_0, \tau_1, \cdots \tau_{N-1}$, each of which is an n-dimensional vector, X is said to be an *extension* of x if and only if there exists a set of $N \bullet n$ functions $\tau_i = \tau_i(t, \epsilon)$; $i = 0, 1, 2, \cdots N-1$, which, when inserted into X, gives, $X\big(\tau_0(t, \epsilon), \tau_1(t, \epsilon), \cdots \tau_{N-1}\big) = x(t, \epsilon)$.

The space of N-tuplets $\tau = \{\tau_0, \tau_1, \cdots \tau_{N-1}\}$ is called the *extension* of the domain, and the loci defined by the equations $\tau_i = \tau_I(t, \epsilon)$ are called the *trajectories* in the extended domain. The result of substituting the trajectory in the extended function is called the *restriction* of X and is denoted by $X(\tau_i)|_t$ (Fig. 5.4). Mathematically, this leads to the following process. The independent variable, time, is extended into higher dimensions, i.e., a vector *space of times* $t \rightarrow \{\overline{\tau}\}$.

Two degrees of freedom are available: the choice of the trajectories and the choice of the extension of x itself. Both are utilized in obtaining X with a smoother dependence on the parameter ϵ than that offered by x. Such a dependence should clearly facilitate the determination of uniformly valid approximations in the domain of interest.

The fundamental idea of the method of multiple scales is to enlarge the domain of the independent variable into a space of higher dimension. The extension of the dimension is achieved by introducing "clocks" (in the case of time) or scale functions. A complete reparameterization of the lowest order term in the perturbation expansion can thus be obtained. The clock variables, in general, will not be restricted to be *linear* or *real*. Indeed, in many problems, the clocks turn out to be nonlinear and complex quantities. The clocks are so chosen as to eliminate the nonuniformities of direct perturbation theory. In the extended domain, uniform approximations to the unknown functions can thus be obtained. The broad aim of extension is to facilitate the understanding of the intricate behavior of the phenomenon represented as a function of a small parameter. The general framework of mappings is useful in unifying several perturbation approaches in a fundamental way. It is also an important reason why this uniformization approach is different from other treatments of the subject. This general approach is useful not only for pedagogical reasons but also in providing an underlying unity of the concept and philosophy that transcends its mere applicability in specific problems.

5.4 A Simple Example

In order to illustrate the multiple scales method, reconsider the example of the slowly decaying exponential $exp(-\epsilon t)$ which satisfies the equation

$$\phi(t) \equiv \frac{dx}{dt} + \epsilon x = 0; \quad x(0) = 1 \tag{5.1}$$

We note that derivatives, and indeed, entire differential expressions can be treated as functions on t and can be extended using the definitions already given. Derivatives of x are, of course, functionals on x but functions on t. Let us consider Eq. (5.9) in the

context of multiple scales, and postulate two observers. One observes fast variations and the other follows slower changes. The variables are extended as follows:

$$t \to \{\overline{\tau}\};\ \tau_0 = t,\ \tau_1 = \epsilon t \tag{5.2}$$

$$x(t, \epsilon) \to x(\overline{\tau}) = x(\tau_0, \tau_1) \tag{5.3}$$

As a consequence, we have, using the chain rule of differentiation,

$$\frac{d}{dt} \to \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1} \tag{5.4}$$

An extension of $\phi(t)$, i.e., Eq. (5.1), corresponding to the trajectory Eq. (5.2) is

$$\phi(t) \to \Phi(\tau_0, \tau_1) \equiv \frac{\partial x}{\partial \tau_0} + \epsilon \frac{\partial x}{\partial \tau_1} + \epsilon x = 0 \tag{5.5}$$

The extended perturbation equations are:

$$O(1):\ \frac{\partial x}{\partial \tau_0} = 0 \tag{5.6}$$

$$O(\epsilon)\ \frac{\partial x}{\partial \tau_1} + x = 0 \tag{5.7}$$

Integrating Eq. (5.6), $x(\tau_0, \tau_1) = A(\tau_1)$. Substituting this into Eq. (5.7) yields

$$\frac{dA}{d\tau_1} + A = 0 \quad or \quad A(\tau_1) = cexp(-\tau_1) \tag{5.8}$$

where c is a constant. On the τ_1 scale, the fully developed exponential can be observed. On the τ_0 scale, however, it would be missed completely. The solution in the extended space is $x(\tau_0, \tau_1) = cexp(-\tau_1)$. Restriction to the original variables consists in substituting for τ_0, τ_1 in $X(\tau_0, \tau_1)$. Thus,

$$x(t, \epsilon) = x(\tau_0(t), \tau_1(t, \epsilon)) = ce^{-\epsilon t} \tag{5.9}$$

which is clearly the *exact* solution. Note that although we only wanted a uniformly valid perturbation approximation to first order in ϵ, the method yields the exact solution. Further, the fast and slow variations have been separated. To be sure, the problem is very simple, but it illustrates the mechanism of the multiple scales approach.

5.5 Generalization: Nonlinear Scales

In the above example, we note that the time scales are *linear*, i.e., τ_0 and τ_1 are *linear* functions of t. However, there are other kinds of systems where such linear scales are inadequate and more general representations are needed in order to describe the

dynamics. In the study of more complicated phenomena described by time-varying and nonlinear differential equations, the simple extension Eq. (5.2) with linear scales is not adequate and a more general choice of scale functions becomes necessary. For instance, time-varying systems are best treated with *nonlinear* scales, as shown by Ramnath [8, 7].

The choice of a *nonlinear* scale depends on the problem at hand, i.e., the form of the equation to be solved. This can be illustrated in the case of the most general first order linear differential equation. Indeed, the correct time scale turns out to be the integrating factor of the equation. Consider the equation

$$\frac{dx}{dt} + \epsilon\omega(t)x = 0 \tag{5.10}$$

We will first show that the extension with linear scales of Eq. (5.2) is inadequate. Using the trajectory Eq. (5.2) and the extension

$$\omega(t) \to \omega(\tau_0) \tag{5.11}$$

the extended perturbation equations are:

$$O(1): \quad \frac{\partial x}{\partial \tau_0} = 0 \tag{5.12}$$

$$O(\epsilon): \quad \frac{\partial x}{\partial \tau_1} + \omega(\tau_0)x = 0 \tag{5.13}$$

This simple and direct extension of Eq. (5.11) will be maintained in this chapter. Clearly, this extension can be used only if the variable coefficient depends on a single time scale. Generally, for a linear differential equation for $x(t)$ with nonconstant coefficients $\omega_i(t)$,

$$\mathcal{L}\big([\omega_i(t)], x\big) = 0 \tag{5.14}$$

where $\mathcal{L}$ is a linear differential operator. The freedom available in extension corresponds to the choice of the trajectory *and* to the extension of $\omega_i(t)$. From Eqs. (5.12) and (5.13), we have $x(\tau_0, \tau_1) = A(\tau_1)$ and

$$\frac{dA}{d\tau_1} + \omega(\tau_0)A = 0 \tag{5.15}$$

leading to a contradiction in that A is required to be independent of τ_0, but actually depends on τ_0 through $\omega(\tau_0)$. A uniform approximation to x cannot, therefore, be obtained with the *linear* scales of Eq. (5.2). Let the linearity of the scales be relaxed and the scales be allowed to be *nonlinear* functions.

$$t \to \{\tau_0, \tau_1\};\ \tau_0 = t,\ \tau_1 = \epsilon k(t) \tag{5.16}$$

where $k(t)$ is as yet an undetermined "clock" function. The time derivative operator is now extended as:

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1} \tag{5.17}$$

The extended perturbation equation now become

$$O(1): \quad \frac{\partial x}{\partial \tau_0} = 0 \tag{5.18}$$

$$O(\epsilon): \quad \dot{k} \frac{\partial x}{\partial \tau_1} + \omega(\tau_0) x = 0 \tag{5.19}$$

Upon integrating, we obtain, as before,

$$x(\tau_0, \tau_1) = A(\tau_1) \tag{5.20}$$

Substituting this, Eq. (5.19) becomes

$$\dot{k} \frac{\partial A}{\partial \tau_1} + \omega(\tau_0) A = 0 \tag{5.21}$$

This is an equation with two unknowns, $A(\tau_1)$ and $k(\tau_0)$. However, we can solve it by separating the variables, as

$$-\frac{1}{A} \frac{dA}{d\tau_1}(\tau_1) = \frac{\omega}{\dot{k}}(\tau_0) = s = constant \tag{5.22}$$

The resulting equations for $A(\tau_1)$ and $k(\tau_0)$ can be integrated as $A(\tau_1) = ce^{-s\tau_1}$, (where c is an arbitrary constant) and

$$\dot{k} = \omega(\tau_0); \quad or, \quad k(\tau_0) = (1/s) \int \omega(\tau_0) d\tau_0 \tag{5.23}$$

Therefore,

$$\tau_1 = \frac{\epsilon}{s} \int \omega(\tau_0) d\tau_0 \tag{5.24}$$

Clearly, the constant s can be set equal to unity without loss of generality. Upon restriction, the solution is given by

$$x(t, \epsilon) = C \, exp\Big(- \epsilon \int \omega(t) dt \Big) \tag{5.25}$$

which is the *exact* solution of Eq. (5.10).

We note that the method "simplifies" the problem by reducing the original variable coefficient equation to one with constant coefficients plus an explicit quadrature. The method yields a suitable change of variables to solve the equation exactly. It can also be viewed as leading to the integrating factor of the original equation rendering it solvable.

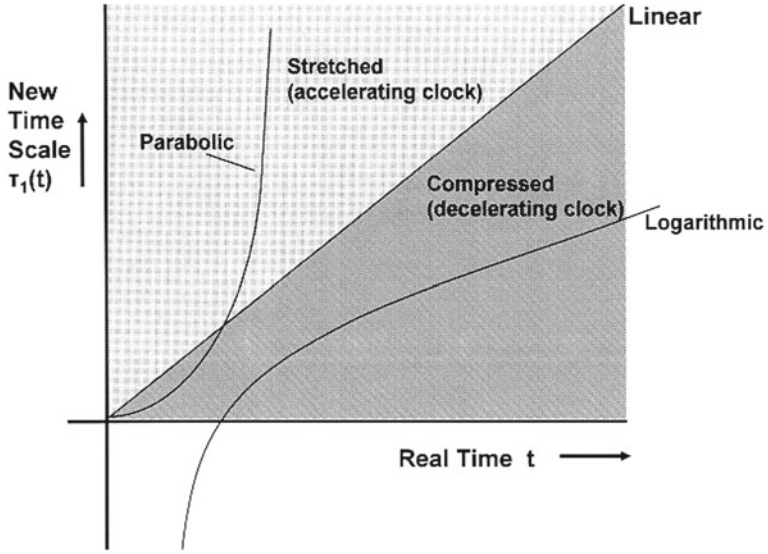

Fig. 5.3 Linear and nonlinear time scales

5.6 Generalization: Complex Scales

Another degree of generalization is that the time scales can be *complex* quantities (as different from *real* expressions). This form appears in other problems, for example, in higher order equations.

It is also instructive to verify that the trajectories represent the characteristics of the partial differential equation obtained by the extension of the original ordinary differential equation. The multiple scales technique can be further generalized to solve more complex problems such as those involving time-varying and nonlinear systems. In order to achieve this, we must relax the restrictions on the time scales and allow them to be *nonlinear* and *complex* quantities. As already seen, the nonlinear scale functions represent *accelerating* or *decelerating* clocks, which count time either faster and faster, or slower and slower. The concept of linear and nonlinear time scales is illustrated in Fig. 5.3 [9]. In some situations it is convenient to allow the scales to be *complex* quantities instead of real ones.

This generalized technique can be applied in a similar manner to other problems, where it can lead to uniform approximations capturing the salient aspects of a complex phenomenon. This idea has been developed in the context of slowly varying and other types of systems [7, 8, 10, 11], where illustrative examples of linear time-varying systems and nonlinear differential equations are discussed, showing the usefulness of the generalized method. As such, the approach will be referred to as the Generalized Multiple Scales (GMS) Method.

The general procedure of applying the GMS method is shown in Fig. 5.4. A given mathematical model, usually in terms of differential equations, is extended

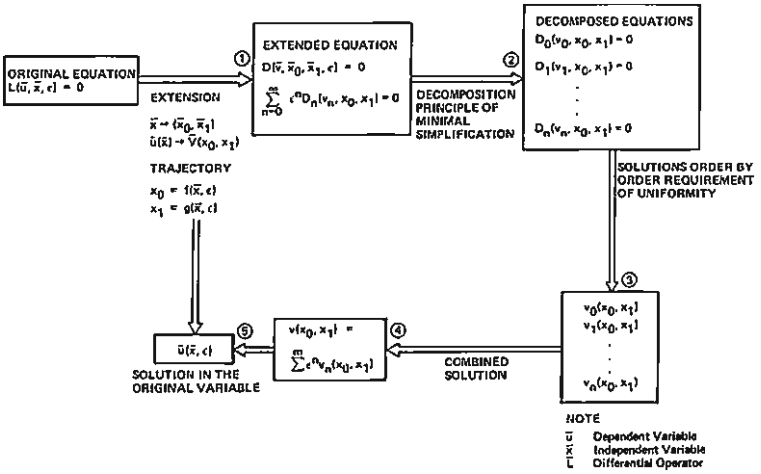

Fig. 5.4 GMS decomposition procedure

Table 5.1 A comparison of numerical and perturbation approaches

	Property	Numerical	Perturbation
1	Complicated geometry	Efficient	Inefficient
2	Results	Numbers	Analytical and numbers
3	Qualitative analysis	Inefficient	Efficient
4	Accuracy	Decrease stepsize	Add expansion terms
5	Singular perturbation	Inefficient	Efficient
6	Errors	Many sources	Single
7	Cost	Expensive	Inexpensive

into a set of partial differential equations by means of multiple scales and using a small parameter ϵ. By invoking asymptotological principles such as the *minimal simplification* [7, 12], the equations are decomposed into a set of asymptotically ordered subsystems. These are then solved order by order and combined to yield a composite solution in the extended variables. The final step is to restrict the extended solutions by means of time scales, resulting in a separable description of the complex dynamics.

5.7 A Comparison of Numerical and Perturbation Methods

For purposes of preliminary analysis and design of systems, it is often useful to utilize the computational efficiency of asymptotic solutions. They have the highly desirable property of preserving the salient aspects of the dynamics and ease of rapid computation.

A rather general observation of numerical and perturbation methods suggests the following features (Table 5.1).

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Part II
Applications

Chapter 6
Example

6.1 Thomas–Fermi Problem in Atomic Physics

Noting the existence of different scales in the dynamics of physical systems, we will now present some examples and applications demonstrating the computational advantages of asymptotic solutions. In this chapter, the example given is of second order, followed by applications of higher order in the next two chapters.

The Thomas–Fermi differential equation arises in the investigations of the periodic table in atomic physics. Thomas [1] and Fermi [2] developed a mathematical model in terms of a second order nonlinear differential equation. The model describes the effective nuclear charge in a heavy neutral atom. The electrons surrounding the nucleus are considered as a completely degenerate electron gas under the influence of the Coulomb field of the nucleus. The different states of the electron are neglected and the individual wave functions are replaced by a statistical average. Radial symmetry is assumed and Poisson's equation is used to obtain a description of the potential distribution for heavy neutral atoms by means of the Fermi-Dirac statistics. Based on this simple model, Thomas and Fermi were able to explain many properties of the periodic table. After suitable normalization, the effecctive nuclear charge ϕ was shown to satisfy the following nonlinear differential equation.

$$\frac{d^2\phi}{dx^2} = \sqrt{\frac{\phi^3}{x}} \tag{6.1}$$

with the boundary conditions. $\phi(0) = 1; \quad \phi(x) \to 0 \quad \text{as } x \to \infty$. This equation is not easy to integrate numerically as the initial condition is imposed at the singular point $x = 0$ and Fermi had to transform the equation first into a suitable form in order to calculate the numerical approximation. From graphical considerations Fermi [2] developed the approximation

$$\phi(x) = 1 - Bx + \frac{4}{3}x^{3/2}; \quad B \approx 1.58 \tag{6.2}$$

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 63
DOI: 10.1007/978-3-642-25749-0_6, © The Author(s) 2012

This approximation is only valid for small x. Baker [3] extended this result into the series

$$\phi(x) = 1 - Bx + \frac{4}{3}x^{3/2} - \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} - \frac{2}{15}Bx^4 + \frac{4}{63}\left(\frac{2}{3} + \frac{1}{16}B^3\right)x^{9/2} + \cdots \tag{6.3}$$

and

$$\phi' = -B + 2x^{1/2} - Bx^{3/2} + x^2 + \frac{3}{20}B^2x^{5/2} - \frac{8}{15}Bx^3 + \frac{2}{7}\left(\frac{2}{3} + \frac{1}{16}B^3\right)x^{7/2} + \cdots \tag{6.4}$$

where $B \approx 1.588558$, by a method of successive approximations, which are rapidly convergent near $\phi(0) = 1$. This approximation is good for small x only.

Using a known particular solution

$$\phi_p(x) = \frac{144}{x^3} \tag{6.5}$$

Sommerfeld [4] developed an interesting approximation

$$\phi(x) = \phi_p(x)\left[1 + \phi_p(x)^{\lambda_1/3}\right]^{\lambda_2/2} \tag{6.6}$$

where λ_1 is the positive root (≈ 0.772) and λ_2 is the negative root(≈ -7.772) of the equation

$$\lambda^2 - 7\lambda - 6 = 0 \tag{6.7}$$

This approximation is accurate for large x but underestimates the solution for small values of x.

Using multiple scales theory, Ramnath [5] developed an analytical approximation for $\phi(x)$, which is valid both for small x and large x. In addition his multiple scales solution uniformizes the Baker series and further recovers the known particular exact solution $\phi_p(x)$.

The calculation is rendered straightforward by means of an asymptotic approximation. This process is facilitated by first introducing a small parameter ϵ by the change of variables

$$x = \epsilon^{(2/3)}t \tag{6.8}$$

and converting the original equation into the form

$$\frac{d^2\phi}{dt^2} = \epsilon\sqrt{\frac{\phi^3}{t}} \tag{6.9}$$

We now invoke the GMS extension

$$t \rightarrow \{\tau_0, \tau_1\};\ \tau_0 = \tau,\ t = \epsilon \int_0^t k(t)dt \tag{6.10}$$

and

$$\phi(t, \epsilon) \rightarrow \Phi(\tau_0, \tau_1, \epsilon) \tag{6.11}$$

The extended perturbation equations are,

$$O(1) : \frac{\partial^2 \Phi}{\partial \tau_0^2} = 0 \tag{6.12}$$

$$O(\epsilon) : \dot{k} \frac{\partial \Phi}{\partial \tau_1} + 2k \frac{\partial^2 \Phi}{\partial \tau_o \partial \tau_1} = \sqrt{\frac{\Phi^3}{\tau_0}} \tag{6.13}$$

The $O(1)$ equation is solved as

$$B(\tau_1) + A(\tau_1)\tau_0 \tag{6.14}$$

We treat the two solutions as linearly independent in τ_0 and substitute them individually into the $O(\epsilon)$ equation to obtain the corrections separately. Thus,

$$\dot{k}_1(\tau_0) \frac{dB}{d\tau_1} = \tau_o^{-\frac{1}{2}} B^{\frac{3}{2}} \tag{6.15}$$

Note that this one equation, Eq. (6.15) has two unknowns, $k(\tau_0)$ and $B(\tau_1)$. However, by a separation of variables, we write

$$B^{-\frac{3}{2}} \frac{dB}{d\tau_1} = \frac{1}{\dot{k}\sqrt{\tau_0}} = constant = 1 \tag{6.16}$$

The left side is a function of τ_1 only and the right side is a function of τ_0 only. Each is equal to a constant which can be set equal to unity without loss of generality. These can be solved as

$$B(\tau_1) = 4(b_0 \pm \tau_1)^{-2} \tag{6.17}$$

and

$$k_1(\tau_0) = \frac{4}{3}\tau_0^{\frac{3}{2}} + b_1\tau_0 + b_2 \tag{6.18}$$

where b_i are arbitrary constants. The approximation after restriction is

$$\Phi \sim \tilde{\phi}_1(t, \epsilon) = \left(b_0 + b_1 t \pm \epsilon \frac{2}{3} t^{\frac{3}{2}}\right)^{-2} \tag{6.19}$$

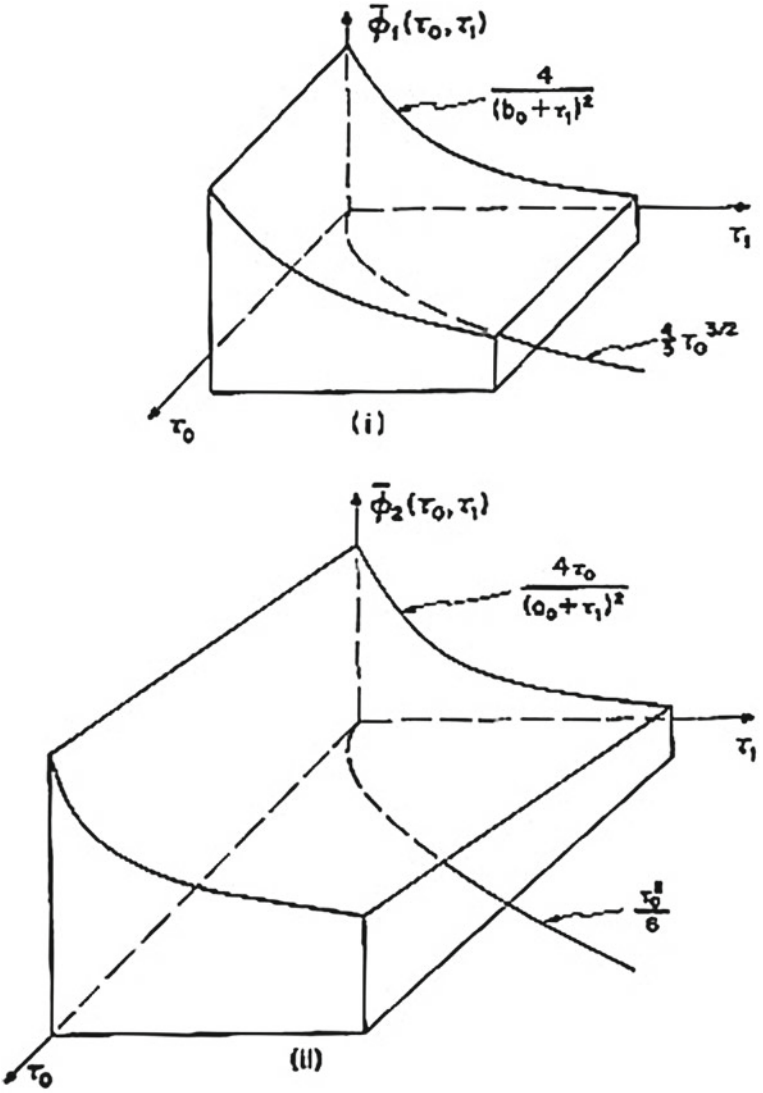

Fig. 6.1 Function surface in extended space

The other solution is obtained similarly by substituting $A(\tau_1)\tau_0$ as

$$\Phi \sim \tilde{\phi}_2(t, \epsilon) = 144t^3\Big(a_1 + a_0 t + \epsilon t^3\Big)^{-2} \tag{6.20}$$

It is interesting that, after transforming from t to x, the artificial parameter ϵ completely disappears from the solution. Imposing the boundary conditions, we have,

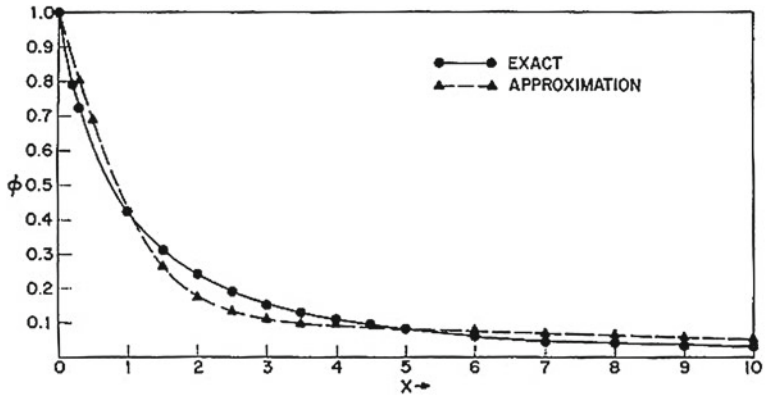

Fig. 6.2 Thomas–Fermi potential

$$\Phi \sim \tilde{\phi}(x) = \left(1 + b_1 x \pm \frac{2}{3} x^{\frac{3}{2}}\right)^{-2} + 144 x^3 \left(a_1 + a_0 x \pm x^3\right)^{-2} \tag{6.21}$$

We note that a particular solution

$$\phi_p = \frac{144}{x^3} \tag{6.22}$$

is obtained as a special case of our GMS solution. As discussed earlier, this is the particular solution used by Sommerfeld [4] to develop an approximation given by Eqs. (6.6) and (6.7). This approximation is accurate for large x but underestimates the solution for small x. Baker [3] developed an initially valid approximation. But the GMS approximation covers both regions (Figs. 6.1, 6.2).

We note that the Thomas–Fermi problem occurs in atomic physics and concerns the effective nuclear charge in the interior of hevy neutral atoms. It is interesting that a generalization of this type of equation occurs in a study of the temperature and density of stellar bodies [6–10]. Ramnath has developed solutions for this class of nonlinear differential equations by the multiple scales approach [6].

In the next two chapters, we will present two interesting applications of aerospace systems which cannot be readily handled by conventional methods. These difficult problems are rendered tractable by the GMS approach.

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Chapter 7
Satellite Attitude Prediction

7.1 Introduction

Attitude prediction of spacecraft is an important area in space flight requiring rapid and accurate methods of computation. We will now present an elaborate application in space flight to illustrate the computational benefits of asymptotic solutions. The application involves the attitude prediction of satellites orbiting the earth and is based on [1–3].

The usefulness of asymptotic approximations in computation is demonstrated by the following example of two configurations of earth satellites,—(1) rigid body satellite and (2) dual spin satellite. Asymptotic approximations are derived by the multiple time scales (MTS) technique as developed in [1–4]. Here we will not treat the development of the solutions in detail but mainly quote the result in order to illustrate their computational value in comparison with direct numerical solutions.

In space research an important problem is that of the attitude prediction of an earth satellite under the influence of environmental torques. Mathematically, it involves the integratiion a set of nonlinear differential equations with given initial conditions, so that the attitude motions are obtained as functions of time. But these equations are impossible to solve analytically exactly in general. The use of direct numerical methods to obtain solutions for long time intervals can be prohibitive because of the computational cost and the possible propagation of errors. From a computational point of view, a reasonable alternative is to use accurate asymptotic solutions. Our goal is to reduce the computational effort and develop rapid algorithms for attitude prediction for long intervals, at the cost of introducing small approximation errors. The approximate solution itself is numerically implemented. This leads to a fast computation and can handle a broad class of situations.

This chapter presents the problem of predicting the satellite attitudes under the influence of various disturbing torques. The presentation is based on the work of Tao and Ramnath [1–3, 5]. The main objective is to formulate a fast and accurate method of simulating the rotational dynamics for long time intervals, in terms of the angular velocities and Euler parameters as functions of time. The formulation has to be

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 69
DOI: 10.1007/978-3-642-25749-0_7, © The Author(s) 2012

general, able to handle any orbit, initial conditions, different mass distributions, etc. These goals are achieved by means of the Multiple Time Scales (MTS) technique, as already discussed in earlier chapters. Two different satellite configurations are considered,—a rigid body satellite and a dual spin satellite. In the latter case, in order to stabilize the attitude motion, the satellite contains a single flywheel mounted along one of the body principal axes. These models are considered to be typical of many classes of satellites currently in operation. We mainly consider the disturbing torques due to, (a) gravity gradient, and (b) geomagnetic interaction. For a high orbit earth satellite these two torques are at least a 100 times bigger than any other environmental torques. If needed, other torques could also be incorporated into the formulation.

The gravity gradient and geomagnetic torques depend on the position and the attitude of the satellite with respect to the earth. Therefore, the orbital and attitude motions are slowly mixed by the action of these disturbances. However, the attitude motion of the vehicle about its center of mass occurs at a much faster rate than the orbital motion of the vehicle around the earth. The effect of the orbit of the vehicle on the satellite attitude is much greater than the effect of the attitude on the orbit, which is therefore, neglected. Furthermore, this separation of the rates enables us to invoke the MTS method to develop a highly efficient attitude prediction algorithm as will be presently shown. In particular, the MTS method enables us to separate the fast attitude dynamics from the slow orbital motion. The secular effect as well as the orbit-attitude coupling is described by the slow time scale equation, while the non-biased oscillatory motion is given by the fast time scale equation. In some situations the slow equation for the secular effects can be useful in the design of an attitude control system. The control forces, which are naturally small, can be utilized to stabilize the long term secular motions.

A geometrical representation of the rigid body rotational motion was given by Poinsot [6]. The Euler–Poinsot problem, for the torque-free case, was first solved by Kirchhoff [7] in terms of Jacobian elliptic functions. Klein and Sommerfeld [8] solved the problem by formulating it in terms of singularity-free Euler symmetric parameters. Recently, Morton et al. [9] solved the equations for Euler symmetric parameters by introducing a set of complex orientation parameters.

7.2 Attitude Prediction

Euler's rotational equations together with the Euler symmetric parameters, embody the rigid body satellite attitude dynamics and are first formulated as an Encke problem, in which the torque-free case is considered as a nominal solution.

7.2.1 Euler's Rotational Equations

By invoking Newton's Second Law, the rigid body rotational motion in a moving reference frame can be described by Euler's equations

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = M_x \tag{7.1}$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = M_y \tag{7.2}$$

$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = M_z \tag{7.3}$$

In vector notation, they can be written in the form

$$I\dot{\vec{\omega}} + (\omega\times)I\vec{\omega} = \vec{M} \tag{7.4}$$

Euler's equations describe the angular velocity of a rigid body with respect to inertial space although this angular velocity is expressed in the instantaneous body-fixed principal axes [10].

7.2.2 Euler Symmetric Parameters

The role of Euler symmetric parameters is similar to that of Euler angles, which describe the relative orientation between two coordinate systems. It can be shown that a transformation matrix can be calculated from either set, and a vector can be transformed from one axis system to another by premultiplying by the transformation matrix. We will present computation of the rotational motion of a rigid body with respect to body axes. The orientation of the body axes with respect to inertial axes is described in terms of the Euler parameters by the equation

$$\frac{d\vec{\beta}}{dt} = A(t)\vec{\beta} \tag{7.5}$$

where $\vec{\beta}$ is a 4-vector of Euler parameters and $A(t)$ is a skew-symmetric 4×4 matrix function of angular velocities as:

$$A = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & -\omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{7.6}$$

Euler's equation and the Euler parameter equation constitute the main problem to be solved.

7.2.3 Euler–Poinsot Problem

Closed form solutions for the rotational motion of a rigid body with external torques are generally not possible except for a few special cases. One such solvable case is

that of zero external torque, named after Euler and Poinsot. This case is useful as the disturbing torques acting on a satellite are small, and the Euler–Poinsot problem is solved to determine the nominal solution. Kirchoff [7] was the first to derive the complete analytical solution $\vec{\omega}$ for Euler's equation in terms of time, using an elliptic integral of the first kind. In the following, we will utilize Kirchoff's solution for the angular velocities, $\vec{\omega}$, and the solution for Euler symmetric parameters, $\vec{\beta}$, by Morton et al. [9].

7.2.4 Solution for Euler Symmetric Parameters

The Euler symmetric parameters β_i satisfy a linear differential equation [1–3, 10]

$$\frac{d\vec{\beta}}{dt} = A(\vec{\omega})\vec{\beta} \tag{7.7}$$

with the constraint $\sum_i \beta_i^2 = 1$.

It can be shown that (see references) the general solution for $\vec{\beta}$ is

$$\begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{bmatrix} \begin{bmatrix} E_1 cos(p_1 + Rt) & 0 & -E_1 sin(p_1 + Rt) & 0 \\ 0 & E_2 cos(p_2 + Rt) & 0 & E_2 sin(p_2 + Rt) \\ E_1 sin(p_1 + Rt) & 0 & E_1 cos(p_1 + Rt) & 0 \\ 0 & -E_2 sin(p_2 + Rt) & 0 & E_2 cos(p_2 + Rt) \end{bmatrix} \times \begin{bmatrix} \beta_0(t_0) \\ \beta_1(t_0) \\ \beta_2(t_0) \\ \beta_3(t_0) \end{bmatrix} \tag{7.8}$$

7.2.5 Disturbing Torques on a Satellite

Among the many disturbing torques which act on an earth satellite, the gravity gradient torque (GGT) and the geomagnetic torque (GMT) are the most important. The orders of magnitude of the various environmental torques are illustrated in Fig. 7.1. We note that, except for very low orbits, the GGT and GMT are at least a 100 times as big as the others.

(a) Gravity Gradient Torque (GGT)

From orbital dynamics [11], the magnitude of the position vector R can be expressed in terms of the eccentricity e, semi-major axis a and true anomaly f as

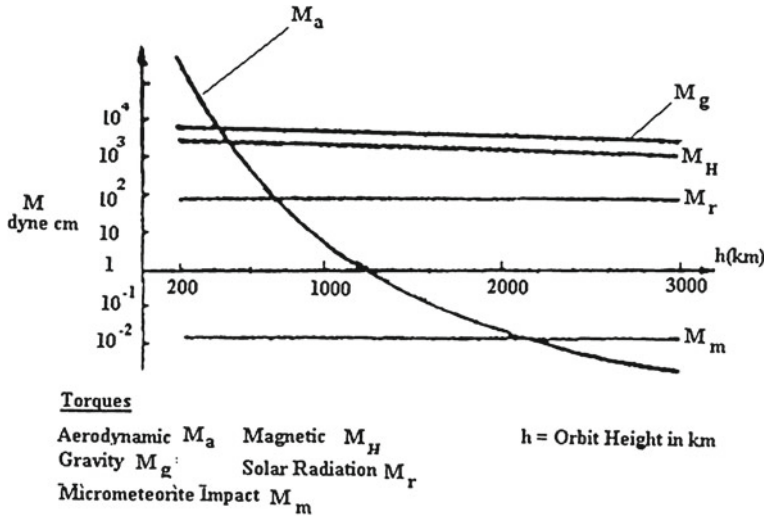

Fig. 7.1 Environmental torques on an earth satellite

$$R = \frac{a(1-e^2)}{(1+ecosf)} \tag{7.9}$$

The orbital period is

$$p = 2\pi/\omega_{orbit} = 2\pi\sqrt{a^3/\mu} \tag{7.10}$$

It can be shown that the GGT can be written as [5],

$$\vec{L}_G^b = \left[3\omega_{orbit}^2(1+ecosf)^3/(1-e^2)^3\right]\left(\frac{\vec{R}^b}{R}\times\right)I\left(\frac{\vec{R}^b}{R}\right) \tag{7.11}$$

$$= O(\omega_{orbit}^2) \tag{7.12}$$

if $e \ll 1$ and the inertia matrix I is not approximately an identity matrix.

In addition the GGT can be expressed as a product of attitude-dependent terms with a higher frequency and the orbit-dependent terms with the relatively lower frequency [5]. This will be useful later in the development of the solution.

(b) Geomagnetic Torque (GMT)

An earth satellite interacts with the geomagnetic field resulting in a torque

$$\vec{L}_M = \vec{V}_M \times \vec{B} \tag{7.13}$$

where $\vec{B}$ = geomagnetic field and $\vec{V}_M$ = magnetic moment of the spacecraft. The latter could arise from any current-carrying devices in the satellite payload or by

eddy currents in the metal structure, which cause undesirable torques. On the other hand, the vehicle magnetic moment could also be purposely generated by passing an electric current through an onboard coil to create a torque for attitude control. This will be discussed in the next chapter.

The geomagnetic field modeled as a dipole has the form

$$\vec{B} = (\mu_B/R^5)\left[R^2\vec{e}_B - 3(\vec{e}_B \cdot \vec{R})\vec{R}\right] \tag{7.14}$$

where $\vec{e}_B$ is a unit vector in the direction of the geomagnetic dipole axis, which is inclined about 11.5° from the geophysical polar axis. $\vec{R}$ is the satellite position vector and $\mu_B = 8.1 \times 10^{25}\text{gauss} - \text{cm}^3$. The GMT can be written in the form [5]

$$\vec{L}_M^b = \left[\vec{V}_M^b \times\right] C_{ib}(\mu_B/R^5)\left[R^2\vec{e}_B^i - 3(\vec{e}_B^i \cdot \vec{R}^i)\vec{R}^i\right] \tag{7.15}$$

Although neither the geomagnetic field nor the body magnetic moment can be precisely determined in general, modeling them as dipoles is sufficiently accurate for our purpose.

Note that GGT and GMT are of the order of ω_{orbit}^2, provided that the eccentricity is not too high and if the satellite mass distribution is not too nearly spherical. Further, both GGT and GMT can be expressed in a form separating the attitude and orbital frequencies.

7.3 Perturbed Attitude Equations

Euler's equation in vector form is

$$I\dot{\vec{\omega}} + \left[\vec{\omega}\times\right] I\vec{\omega} = \epsilon^2\vec{T}_1 + \epsilon^3\vec{T}_2 + \cdots \tag{7.16}$$

with the initial condition $\vec{\omega}(t_0) = \vec{\omega}_0$. $\epsilon^2 T_1 + \epsilon^3 T_2 + \cdots$ represent the disturbing torques and $\epsilon(\ll 1)$ is a small parameter defined as the ratio of the orbital and attitude frequencies. Let $\vec{\omega}_N(t)$ be the torque-free Kirchoff's solution which satisfies the given initial conditions. Thus

$$I\dot{\vec{\omega}}_N + \left[\vec{\omega}_N\times\right] I\vec{\omega}_N = 0 \tag{7.17}$$

with the initial condition $\vec{\omega}_N(t_0) = \vec{\omega}_0$. By Encke's approach [11], let

$$\vec{\omega}(t) = \vec{\omega}_N(t) + \epsilon\delta\vec{\omega}(t) \tag{7.18}$$

This leads to an equation for $\delta\vec{\omega}(t)$

$$I\delta\dot{\vec{\omega}} + \left[\vec{\omega}_N\times\right] I\delta\vec{\omega} + \left[\delta\vec{\omega}\times\right] I\vec{\omega}_N + \epsilon\left[\delta\vec{\omega}\times\right] I\delta\vec{\omega} = \epsilon T_1 + \epsilon^2 T_2 + \cdots \tag{7.19}$$

with $\delta\vec{\omega}(t_0) = \vec{0}$. For notational simplification, we define the operator A^* as

$$A^* = -I^{-1}\left[(\vec{\omega}_N\times)I - (I\vec{\omega}_N\times)\right] \tag{7.20}$$

The perturbed equation can be written as

$$\delta\dot{\vec{\omega}} - A^*(t)\delta\vec{\omega} + \epsilon I^{-1}(\delta\vec{\omega}\times)I\delta\vec{\omega} = \epsilon I^{-1}\vec{T}_1 + \epsilon^2 I^{-1}\vec{T}_2 + \cdots \tag{7.21}$$

We see that this is a weakly nonlinear equation, as the nonlinear terms are smaller than the linear terms by at least one order of magnitude. Further, by means of Floquet theory [12], it is known that a vector linear differential equation with a periodically varying coefficient can be reduced to an equation with constant coefficients as follows. Let the matrices R_A and $P_A^{-1}(t)$ be defined as

$$R_A = (1/T_\omega)\ln\left[\Phi_A(T_\omega, 0)\right] \tag{7.22}$$

$$P_A^{-1}(t) = \Phi_A(t, 0)exp(-R_A t) \tag{7.23}$$

$\Phi_A(t, 0)$ is the state transition matrix for $A(t)$. It can be proved that [13],

$$P_A^{-1}(t + T_\omega) = P_A^{-1}(t) \tag{7.24}$$

$$P_A(t)A(t)P_A^{-1}(t) + \frac{dP_A(t)}{dt}P_A^{-1}(t) = R_A = constant \tag{7.25}$$

Consider the transformation

$$\vec{u} = P_A(t)\delta\vec{\omega} \tag{7.26}$$

Then the perturbed equation takes the form

$$\dot{\vec{u}} = R_A\vec{u} - \epsilon P_A I^{-1}(P_A^{-1}\vec{u}\times)I(P_A^{-1}\vec{u}) + \epsilon P_A I^{-1}\vec{T}_1 + \epsilon^2 P_A I^{-1}\vec{T}_2 + \cdots \tag{7.27}$$

In the present case, the constant matrix $R_A \approx (1/T + \omega)$. Therefore, it can be considered to be $O(\epsilon)$. Also, for simplicity, the matrix R_A can be transformed into a diagonal (or Jordan) form by a similarity transformation. With the change of variables

$$\vec{v} = M^{-1}\vec{u} \tag{7.28}$$

Eq. (7.27) becomes

$$\dot{\vec{v}} = \epsilon\Lambda\vec{v} - \epsilon Q(I^{-1}Q^{-1}\vec{v}\times)Q_{-1}\vec{v} + Q(\epsilon\vec{T}_1 + \epsilon^2\vec{T}_2 + \cdots) \tag{7.29}$$

where

$$Q = M^{-1}P_A I^{-1} \tag{7.30}$$

7.4 MTS Solution of Attitude Equations

We will develop asymptotic solutions to the transformed attitude perturbation equations (7.29) by the multiple time scales (MTS) approach. Accordingly we extend the time domain t into a multi-dimensional space,

$$t \rightarrow \{\tau_0, \tau_1, \cdots\}$$

choosing $\tau_0, \tau_1, \cdots$ as:

$$\tau_0 = t, \ \tau_1 = \epsilon t, \ \cdots, \tau_n = \epsilon^n t$$

The time derivative operator is extended as

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1} + \epsilon^2 \frac{\partial}{\partial \tau_1}$$

The dependent variable $\vec{v}(t)$ is expanded into an asymptotic series in ϵ:

$$\vec{v}(t) = \vec{v}_0(\tau_0, \tau_1, \cdots) + \epsilon \vec{v}_1(\tau_0, \tau_1, \cdots) + \cdots \tag{7.31}$$

It can be shown that [1–3, 5]

$$\vec{v}_0 = \vec{v}_0(\tau_1, \tau_2, \cdots) \tag{7.32}$$

where $\vec{v}_0$ is independent of τ_0. The initial condition is $\vec{v}_0(0) = 0$.

Note that the zeroth order solution for Euler's equation with small disturbances is Kirchoff's solution. The $O(\epsilon)$ equation with initial condition $\vec{v}_1(0) = 0$ can be solved as,

$$\vec{v}_1 = \left(-\frac{\partial \vec{v}_0}{\partial \tau_1} + \Lambda \vec{v}_0\right) \tau_0 - \int_0^{\tau_0} Q(I - 1 Q^{-1} \vec{v}_0 \times) Q^{-1} \vec{v}_0 d\tau_0 + \int_0^{\tau_0} Q \vec{T}_1 d\tau_0 \tag{7.33}$$

The term $Q(I - 1 Q^{-1} \vec{v}_0 \times) Q^{-1} \vec{v}_0$ in the above equation can be written [5] as

$$Q(I - 1 Q^{-1} \vec{v}_0 \times) Q^{-1} \vec{v}_0 = \sum_{i=1}^{3} F_i(\tau_0) v_{0i} \vec{v}_0 \tag{7.34}$$

where F_i are periodic matrices with period $\vec{T}_\omega$, and $v_i; i = 1, 2, 3$ are the three components of the vector $\vec{v}_0$. Expanding F_i as a Fourier series,

$$Q(I - 1 Q^{-1} \vec{v}_0 \times) Q^{-1} \vec{v}_0 \approx \sum_{i=1}^{3} \sum_{j=0}^{n} \left[E_{ij} sin\,(2\pi j \tau_0 / T_\omega)\, F_{ij} cos\,(2\pi j \tau_0 / T_\omega)\right] v_{0i} \vec{v}_0 \tag{7.35}$$

E_{ij}, F_{ij} are constant matrices. Also, when the external torque $\vec{T}_1$ is not an explicit function of time, $Q\vec{T}_1$ can be transformed into the particular form

$$Q\vec{T}_1 = \vec{G}_1(\tau_1, \tau_2, \cdots) + Q_p(\tau_0)\vec{G}_2(\tau_1, \tau_2, \cdots) \tag{7.36}$$

where $\vec{G}_1, \vec{G}_2$ are functions of the slow time scales $\tau_1, \tau_2, \cdots$ only, and $Q_p(\tau_0)$ depends on the *fast* time. Substituting the above, we have,

$$\vec{v}_1 = \left[\frac{\partial \vec{v}_0}{\partial \tau_1} + \Lambda\vec{v}_0 - \sum_{i=1}^{3} F_{i0}v_{0i}\vec{v}_0 + \vec{G}_1(\tau_1)\right]\tau_0$$
$$- \int_0^{\tau_0} \sum_{i=1}^{3}\sum_{j=1}^{n}\left[E_{ij}sin\,(2\pi j\tau_0/T_\omega) + F_{ij}cos\,(2\pi j\tau_0/T_\omega)\right] v_{0i}\vec{v}_0 d\tau_0$$
$$+ \int_0^{\tau_0} Q_p(\tau_0)\vec{G}_2(\tau_1)d\tau_0 \tag{7.37}$$

For achieving a uniformly valid approximate solution, we impose the uniformity condition,– i.e., $\|\vec{v}_1(\tau_0, \tau_1)\|/\|\vec{v}_0(\tau_0, \tau_1)\|$ is bounded uniformly as $\tau_0 \to \infty$. Using this condition to $\vec{v}_0(\tau_0)$, we set the first bracketed term in Eq. (7.37) to zero, i.e.,

$$\frac{\partial \vec{v}_0}{\partial \tau_1} = \Lambda\vec{v}_0 - \sum_{i=1}^{3} F_{i0}v_{0i}\vec{v}_0 + \vec{G}_1(\tau_1) \tag{7.38}$$

with initial condition $\vec{v}_0(0) = 0$. This will eliminate a *secular* term which becomes unbounded as $\tau_0 \to \infty$. Thus,

$$\vec{v}_1(\tau_0, \tau_1) \approx - \int_0^{\tau_0} \sum_{i=1}^{3}\sum_{j}^{n}\left[E_{ij}sin\,(2\pi j\sigma/T_\omega) + F_{ij}cos\,(2\pi j\sigma/T_\omega)\right] v_{0i}\vec{v}_0 d\sigma$$
$$+ \int_0^{\tau_0} Q_p(\sigma)\vec{G}_2(\tau_1)d\sigma \tag{7.39}$$

Eq. (7.39) yields the first order MTS asymptotic representations of $\vec{v}(t)$. From this the asymptotic solutions of the attitude equations can be constructed as:

$$\vec{\omega}(t) = \vec{\omega}_N + \delta\vec{\omega} \tag{7.40}$$

$$= \vec{\omega}_N(r) + \epsilon P_A^{-1} M(\vec{v}_0 + \vec{v}_1) \tag{7.41}$$

The matrices P_A^{-1} and M are obtained from Eq. (7.23) and by diagonalizing R_A respectively.

The above approach gives us an alternative way of evaluating the angular velocity of a satellite $\vec{\omega}(t)$ besides direct integration of the attitude differential equations.

Since the MTS solution is in terms of the time scales $\tau_i(t, \epsilon)$, it allows us to use a large integration step, thereby saving computer time. Further, we note the fast oscillatory motion and the slower secular motion of the satellite angular velocity are systematically separated.

7.5 Euler Symmetric Parameters

The Euler symmetric parameters are related to the angular velocities through the linear differential equation

$$\dot{\vec{\beta}} = A(\omega)\vec{\beta} \tag{7.42}$$

From Eq. (7.18),

$$\vec{\omega}(t) = \vec{\omega}_N + \epsilon\delta\vec{\omega} \tag{7.43}$$

$$= \vec{\omega}_N + \epsilon P_A^{-1} M(\vec{v}_0 + \epsilon\vec{v}_1) \tag{7.44}$$

Hence

$$\dot{\vec{\beta}} = \frac{1}{2}\{[\vec{\omega_N}] + \epsilon[\delta\vec{\omega}]\}\vec{\beta}A(\omega_N + \epsilon\delta\omega)\vec{\beta} \tag{7.45}$$

Again, by the MTS method,

$$t \to \{\tau_0, \tau_1, \cdots\};\ \tau_i = \epsilon^i t \tag{7.46}$$

Further, we expand

$$\vec{\beta}(t) \to \vec{\beta}_0(\tau_0, \tau_1, \cdots) + \epsilon\vec{\beta}_1(\tau_0, \tau_1, \cdots) + \cdots \tag{7.47}$$

Substituting and equating like powers of ϵ, we proceed to solve the extended perturbation equations.

Let $\Phi_\beta(t, t_0)$ be the transition matrix for $\frac{1}{2}[\vec{\omega}_N]$; i.e.,

$$\frac{d}{dt}\Phi_\beta = \frac{1}{2}[\vec{\omega}_N]\Phi_\beta \tag{7.48}$$

The expression for $\Phi_\beta(t, t_0)$ is given by Eq. (7.48); that is, from the solution of β_i by Morton's approach [9]. Similarly $\Phi_\beta(t, t_0)$ can be obtained by invoking Floquet's theorem, although the latter is more numerically oriented and requires several transformations. The solution of the $O(\epsilon)$ equation is

$$\vec{\beta}_0(\tau_0, \tau_1) = \Phi_\beta(\tau_0, 0)\vec{\beta}_{0N}(\tau_1, \tau_2 \cdots) \tag{7.49}$$

with the initial conditions $\vec{\beta}_{0N}(0) = \vec{\beta}(0)$. We note that $\vec{\beta}_{0N}(\tau_1, \tau_2, \cdots)$ is yet to be determined. Upon substitution we note that (a) $\delta\vec{\omega}(\tau_0, \tau_1)$ is a function of τ_0 and τ_1, and (b) $\Phi_\beta(\tau_0, 0)$ is a function of τ_0 only. Further, we note that by re-grouping the terms, we can write

$$\Phi_\beta^{-1}[\delta\vec{\omega}]\Phi_\beta = R_1(\tau_1) + P_B(\tau_0)R_2(\tau_1) \tag{7.50}$$

Using these the solution is determined as

$$\vec{\beta}_1 = \Phi_\beta(\tau_0, 0)\left[\left(-\frac{\partial\vec{\beta}_{0N}}{\partial\tau_1} + \frac{1}{2}R_1(\tau_1)\vec{\beta}_{0N}(\tau_1)\right)\tau_0 + \frac{1}{2}\int_0^{\tau_0} P_B(\sigma)R_2(\tau_1)\vec{\beta}_{0N}(\tau_1)d\sigma\right] \tag{7.51}$$

In order that $\|\vec{\beta}_1\|/\|\vec{\beta}_0\|$ is bounded in τ_0, it is necessary that $\vec{\beta}_1$ should not increase with time faster that β_0. Therefore, the terms that increase with τ_0, i.e., *secular* terms must be set equal to zero. This leads to the equation

$$\frac{\partial\vec{\beta}_{0N}}{\partial\tau_1} + \frac{1}{2}R_1(\tau_1)\vec{\beta}_{0N}(\tau_1) = 0 \tag{7.52}$$

Therefore,

$$\vec{\beta}_1 = \Phi_\beta(\tau_0, 0]\left[+\frac{1}{2}\int_0^{\tau_0} P_B(\sigma)R_2(\tau_1)\vec{\beta}_{0N}(\tau_1)d\sigma\right] \tag{7.53}$$

Thus the asymptotic approximation to the Euler parameters is given by

$$\vec{\beta}(\tau_0, \tau_1) = \Phi_\beta(\tau_0, 0)\vec{\beta}_{0N}(\tau_1) + \epsilon\vec{\beta}(\tau_0, \tau_1) + \cdots \tag{7.54}$$

where $\vec{\beta}_{0N}(\tau_1)$ describes the secular variation of the perturbed motion and $\vec{\beta}_1(\tau_0)$ represents the non-biased oscillatory motions.

Equation (7.52) is integrated in τ_1, and correspond to large time steps ib t. The oscillatory motions can be analytically determined if the external torques $\vec{T}_1$ do not explicitly depend on time.

7.6 Attitude Prediction with Gravity Gradient Torque

Using the periodicity (with period T_ω) of the nominal solutions and their representation in terms of Fourier series, the equations can be rearranged into the form

$$Q\vec{L}_G^b = \vec{G}_1(\tau_1) + Q_p(\tau_0)\vec{G}_2(\tau_1) \tag{7.55}$$

Table 7.1 Satellite parameters

Moments of inertia ($slug.ft^2$)			-
$I_x = 39.4$	$I_y = 33.3$	$I_z = 10.3$	-
Orbit parameters			
Eccentricity, e = 0.16	Inclination, i = 0	Period, $T_{orbit} = 10,000$ s	-
Initial conditions			-
$\omega_x = 0.0246\ r/s$	$\omega_y = 0$	$\omega_z = 0$	-
Initial orientation parameters			
$\beta_0 = 0.7071$	$\beta_1 = 0$	$\beta_2 = 0$	$\beta_3 = 0.7071$

where

$$\vec{G}_1(\tau_1) = M\vec{G}(\tau_1) \tag{7.56}$$

$$\vec{G}_2(\tau_1) = \frac{3\omega_{orbit}^2(1+cosf)^3}{(1-e^2)^3}\begin{bmatrix} sin^2 f \\ cos^2 f \\ sinfcosf \end{bmatrix} \tag{7.57}$$

$$\begin{aligned} Q_p(\tau_0) \approx & \sum_{j=1}^{n}\left[M_{1j}cos(2\pi jt/T_\omega) + N_{1j}sin(2\pi jt/T_\omega\right] \\ & + \sum_{j=0}^{n}\left[M_{2j}cos(2\pi jt/T_\omega) + N_{2j}sin(2\pi jt/T_\omega\right] sin2Rt \\ & + \sum_{j=0}^{n}\left[M_{3j}cos(2\pi jt/T_\omega) + N_{3j}sin(2\pi jt/T_\omega)\right] cos(2Rt) \\ & + \cdots \end{aligned} \tag{7.58}$$

Therefore, $Q_p(\tau_0)\vec{G}_2(\tau_1)$ can be analytically integrated. (Table 7.1)

7.6.1 Example 1

For purposes of illustration, the MTS solution derived above is now applied to a class of rigid body satellites in an elliptic orbit. Typical parameters of a representative satellite are:

In this case the small parameter ϵ becomes

$$\epsilon = \frac{\omega_{orbit}}{\omega_{attitude}} \approx 0.03 \tag{7.59}$$

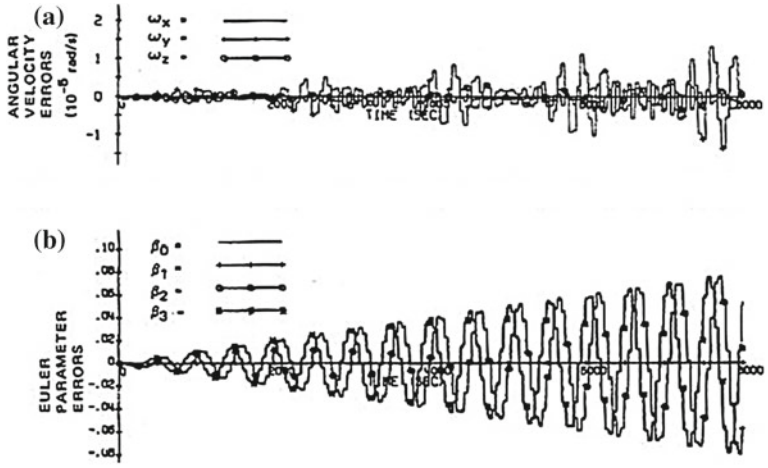

Fig. 7.2 Simulation errors: rigid satellite, gravity gradient torque. **a** Simulation errors for angular velocities (direct integration, $\Delta T = 50$ s). **b** Simulation errors for Euler parameters (direct integration, $\Delta T = 50$ s)

Table 7.2 Satellite parameters (example 3)

$I_x = 30\ slug - ft^2$	$I_y = 25\ slug - ft^2$	$I_z = 16\ slug - ft^2$	-
Orbit parameters			
Eccentricity, e=0.16	Inclination, i=0	Period, $T_{orbit} = 10,000$ s	-
Initial conditions			-
$\omega_x(0) = 0.03\ r/s$	$\omega_y(0) = 0.01 r/s$	$\omega_z(0) = 0.001 r/s$	-
Orientation parameters			
$\beta_0(0) = 0.7071$	$\beta_1(0) = 0.1031$	$\beta_2(0) = 0.1065$	$\beta_3(0) = 0.6913$

The accuracy of the MTS solution is investigated as follows. First, the satellite attitude dynamics are directly integrated using a fourth order Runge–Kutta method (see Chap. 3) with a small integration time step of 10 s for a total interval of 8000 s. This is considered to be very accurate and taken as a *reference case*. The other simulations are computed and compared to this reference case. A number of solutions are obtained, both by (a) the MTS approach and (b) by direct integration, using different integration step sizes in each case. The errors in each case,—i.e., the differences between each simulation and the reference solution,—are plotted against *time* and shown in Figs. 7.2, 7.3, 7.4 and 7.5. Note that Fig. 7.6 shows a plot of maximum numerical computational error as a function of step size. We see that with direct integration, the step size $\Delta T < 25$ s. On the other hand, the step size can be as large as 500 s, for comparable accuracy of the MTS solution. Since the MTS integration is done on a large time step (corresponding to a slow time scale), it requires a computer time which is approximately less than 10% of the time needed for direct integration.

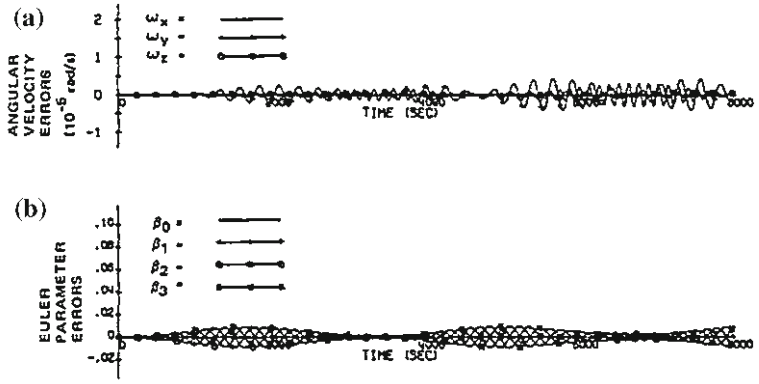

Fig. 7.3 Simulation errors: rigid satellite, gravity gradient torque. **a** Simulation errors for angular velocities (MTS solution, $\Delta T = 10$ s). **b** Simulation errors for Euler parameters (MTS solution, $\Delta T = 10$ s)

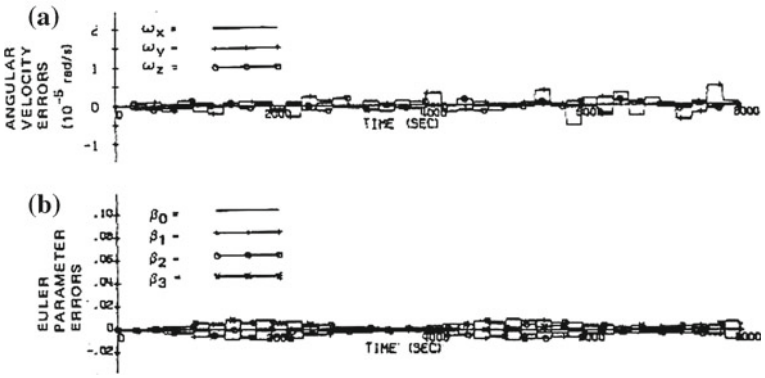

Fig. 7.4 Simulation errors: rigid satellite, gravity gradient torque. **a** MTS solution errors for angular velocities, $\Delta T = 200$ s. **b** MTS solution errors for Euler parameters, $\Delta T = 200$ s

7.7 Attitude Prediction with Geomagnetic Torque

As seen earlier, the geomagnetic torque can be written as

$$\vec{L}_M^b = \left[\vec{V}_M^b \times\right] C_{ib}(\mu_B/R^5)\left[R^2\vec{e}_B^i - 3(\vec{e}_B^i \cdot \vec{R}^i)\vec{R}^i\right] \tag{7.60}$$

where $\vec{V}_M^b$ is the magnetic moment of the satellite and $\vec{e}_B$ is a unit vector in the direction of the geomagnetic dipole axis, which is assumed, for simplicity, to coincide with the polar axis. Upon substitution into $\vec{L}_M^b$, we find that

$$Q\vec{L}_M^b = Q_p(\tau_0)\vec{G}_2(\tau_1) \tag{7.61}$$

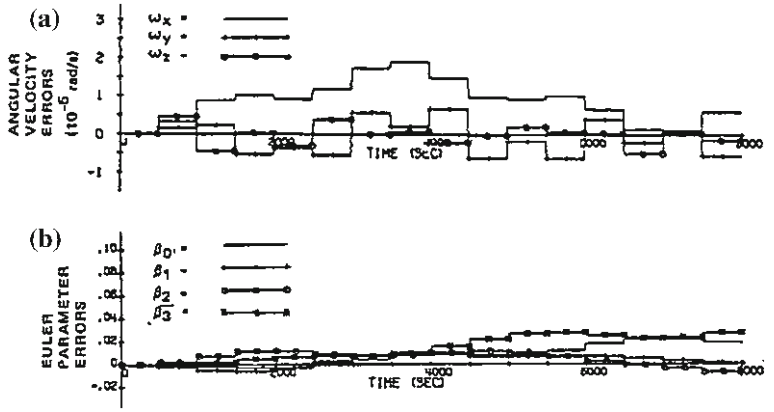

Fig. 7.5 Simulation errors: rigid satellite, gravity gradient torque. **a** MTS solution errors for angular velocities, $\Delta T = 500$ s. **b** MTS solution errors for Euler parameters, $\Delta T = 500$ s

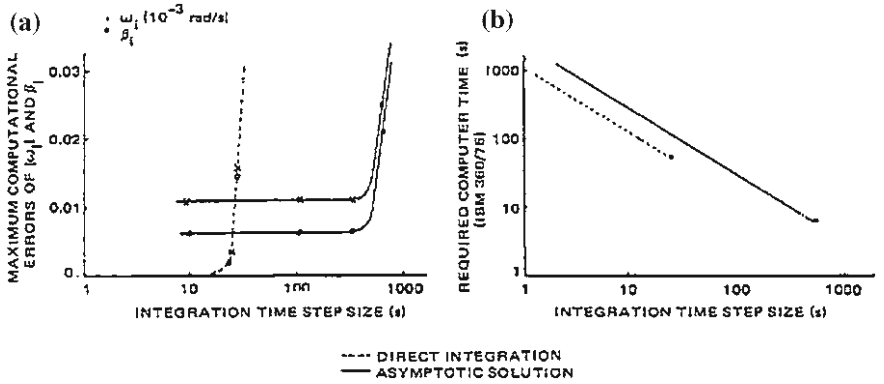

Fig. 7.6 Maximum simulation errors and computer time. **a** Max simulation errors. **b** Computer time

where;

$$Q_p(\tau_0, \tau_1) = Q(\vec{V}_M^b \times)\,[C_1 cos(2Rt) + C_2 sin(2Rt) + C_3] \tag{7.62}$$

$$\vec{G}_2(\tau_1) = (\mu_B/R^5)\left[R^2\vec{e}_B^i - 3(\vec{e}_B^i \cdot \vec{R}^i)\vec{R}^i\right] \tag{7.63}$$

By expanding C_1, C_2, C_3 into a Fourier series, we see that $Q\vec{L}_M^b$ can be analytically integrated in terms of τ_0 [5]. The MTS asymptotic solution can be easily applied.

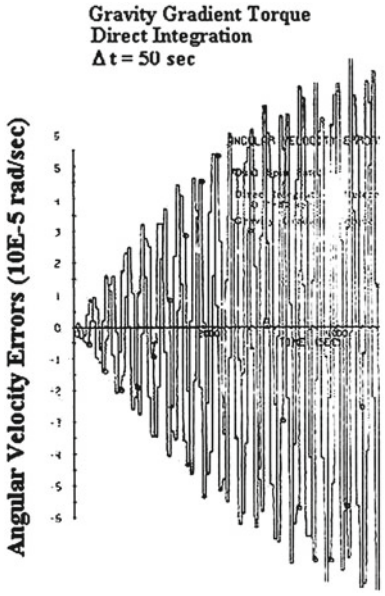

Fig. 7.7 Dual spin case: errors for ω; numerical integration, $\Delta T = 50\,s$

7.7.1 Example 2

As before, the theory is applied to a typical rigid body satellite perturbed by the geomagnetic torque. The parameters are the same as in Example 1. However, in addition, we assume that the satellite magnetic moment, $\vec{V}_M$, (modeled as a dipole), which is aligned with the body *x-axis*, i.e., $\vec{V}_M^b = (3, 0, 0) ft.A.s$. The geomagnetic field is assumed to be $\mu_B = 22.2 \times 10^{24} slug\,.ft^4/s^3.A$.

Again, these numbers are used to generate simulations by the MTS method and direct integration for different step sizes. Errors are calculated in each case by comparison with the reference case. The same conclusions are reached in this case also, just like in the case of gravity gradient torques (Figs. 7.2, 7.3, 7.4, 7.5).

7.8 Dual Spin Satellites

As already discussed, this is a class of satellites which are used in situations combining the inherent aerodynamic efficiency of a *prolate* launch vehicle placing the satellite in an orbit. It is well known that for stability of the spacecraft in the orbit, an *oblate* configuration is preferred. Further, if the spacecraft is utilizing spin stabilization, then any oriented sensors cannot be located on the spacecraft. In order to satisfy all these requirements, dual spin satellites are used. This configuration incorporates an on-board platform and the spinning spacecraft each with an independent degree-

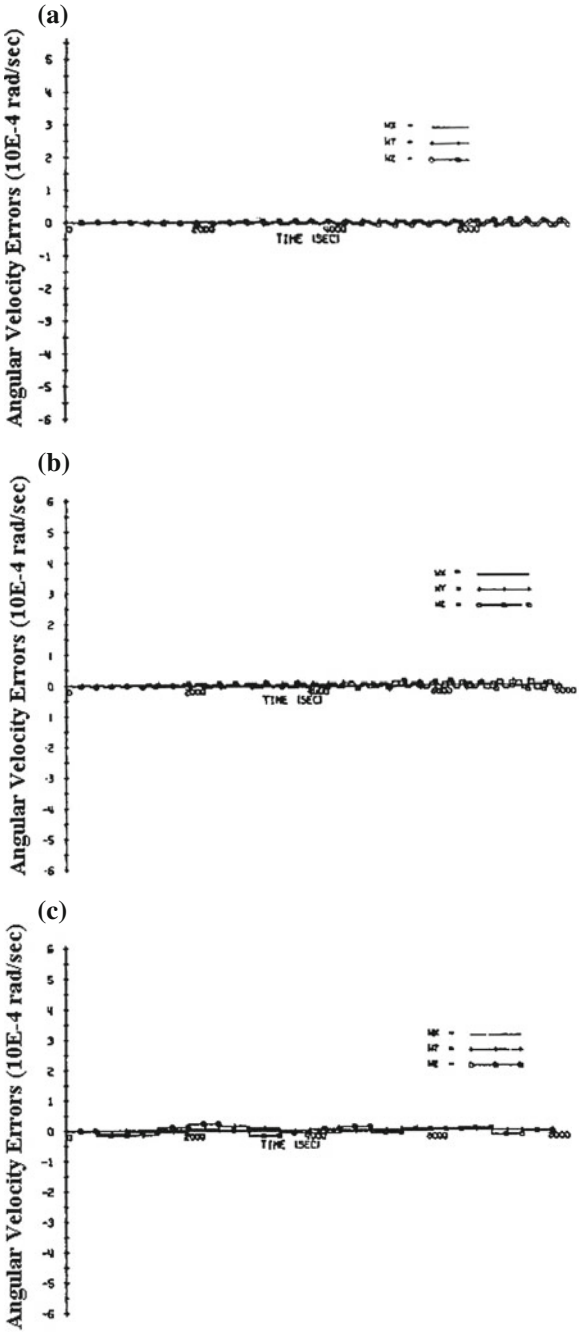

Fig. 7.8 **a** Dual spin case: MTS solution errors for ω; $\Delta t = 10$ s. **b** Dual spin case: MTS solution errors for ω; $\Delta t = 100$ s. **c** Dual spin case: MTS solution errors for ω; $\Delta t = 500$ s

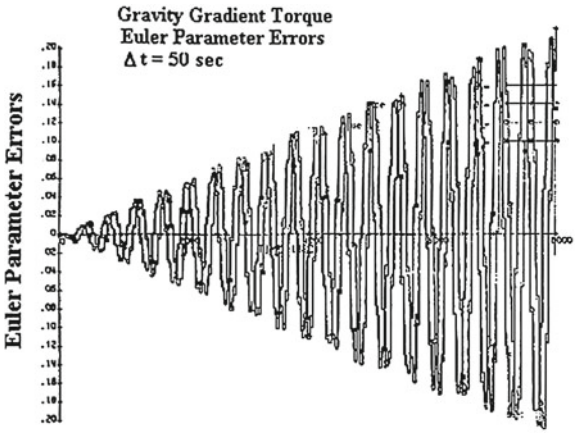

Fig. 7.9 Dual spin case: β errors for numerical integration, $\Delta t = 50$ s

of-freedom in rotation. Usually the platform carries the sensors and instruments. Communication satellites are mostly of this type. Often flywheels are mounted on board for control and stabilization. The GMS technique can be extended to solve this case also. Again, the presentation is based on [1–3, 5, 12].

7.8.1 Equations of Motion

Consider a satellite with one flywheel mounted along a body principal axis and moving in an elliptical orbit with arbitrary initial conditions the total angular momentum H_T can be resolved into

$$H_T = H + H_w \tag{7.64}$$

where:

H = Angular momentum of the whole system regarded as a rigid body.
H_w = Angular momentum of the fly-wheel with respect to the satellite.

In a body-fixed "b" frame,

$$I\dot{\omega}_b + \omega_b \times I\omega_b + \frac{d}{dt}H_{wb} + \omega_b \times H_{wb} = T \tag{7.65}$$

Assume the angular velocity of wheels with respect to the vehicle is constant.

$$\frac{d}{dt}(H_{wb}) = 0 \tag{7.66}$$

Therefore,

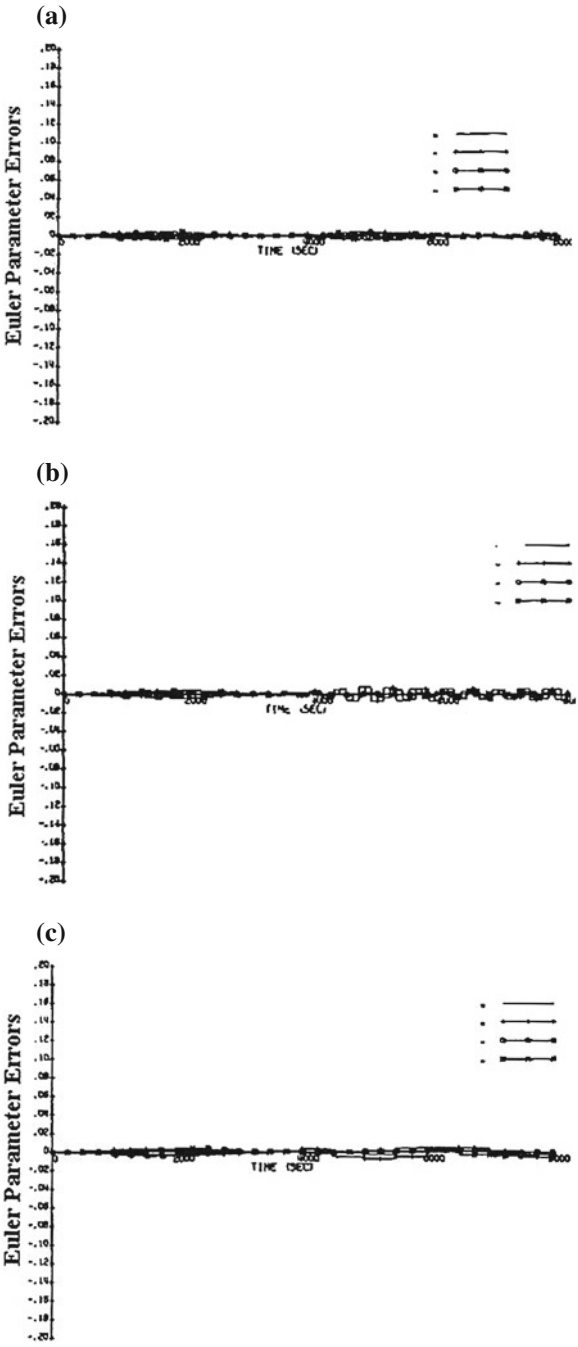

Fig. 7.10 **a** Dual spin: β errors for MTS solution, $\Delta t = 10$ s. **b** Dual spin: β errors for MTS solution, $\Delta t = 100$ s. **c** Dual spin: β errors for MTS solution, $\Delta t = 500$ s

$$I\dot{\omega}_b + (\omega_b \times)(I\omega_b + H_{wb}) = T \tag{7.67}$$

The orientation with respect to an inertial reference frame is given by

$$\frac{d\vec{\beta}}{dt} = \tilde{\omega}(t)\vec{\beta} \tag{7.68}$$

7.8.2 Torque-Free Solution

Assume that the flywheel is along the principal x-axis with angular momentum h. With zero external torque, Euler's equations are:

$$I_x\dot{\omega}_x = (I_y - I_z)\omega_y\omega_z \tag{7.69}$$

$$I_y\dot{\omega}_y = (I_z - I_x)\omega_z\omega_x - \omega_z h \tag{7.70}$$

$$I_z\dot{\omega}_z = (I_x - I_y)\omega_x\omega_y + \omega_y h \tag{7.71}$$

From these equations we obtain

$$I_y(I_y - I_z)\omega_y d\omega_y = \left[(I_z - I_x)I_x\omega_x - hI_x\right] d\omega_x \tag{7.72}$$

Therefore,

$$\omega_y = \pm\sqrt{\frac{(I_z - I_x)I_x\omega_x^2 - 2hI_x\omega_x}{I_y(I_y - I_z)} + C_y} \tag{7.73}$$

Similarly,

$$\omega_z = \pm\sqrt{\frac{(I_x - I_y)I_x\omega_x^2 + 2hI_x\omega_x}{I_z(I_y - I_z)} + C_z} \tag{7.74}$$

where C_y, C_z are constants. If the external torque $T=0$, the rotational kinetic energy E and the total angular momentum H_T are constant. Therefore, denoting ω_{wheel} = angular velocity of the flywheel,

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 + h\omega_{wheel} = 2E \tag{7.75}$$

and

$$(I_x\omega_x + h)^2 + I_y^2\omega_y^2 + I_z^2\omega^2 = H_T^2 \tag{7.76}$$

Let $\omega_y = \pm\sqrt{\frac{I_x}{I_y - I_z} P_y}$ and $\omega_z = \pm\sqrt{\frac{I_x}{I_z - I_y} P_z}$ where:

$$P_y = \frac{(I_z - I_x)}{I_y}\omega_x^2 - \frac{2h}{I_y}\omega_x + C_y\frac{(I_y - I_z)}{I_x} \tag{7.77}$$

$$P_z = \frac{(I_y - I_x)}{I_z}\omega_x^2 - \frac{2h}{I_z}\omega_x + C_y\frac{(I_z - I_y)}{I_x} \tag{7.78}$$

Now,

$$\dot{\omega}_x = \pm\sqrt{-P_y P_z} \tag{7.79}$$

Therefore,

$$t = \pm\int_{\omega_{x(0)}}^{\omega_x} \frac{d\omega_x}{\sqrt{-P_y P_z}} \tag{7.80}$$

This can be written in the standard form

$$t = m\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 sin^2\phi}} \tag{7.81}$$

which is an elliptic integral of the first kind [14, 15].

7.8.3 MTS Solution

The solution approach is similar to the discussion in Sect. 7.4. However, in this case, the attitude equation is written as:

$$I\dot{\vec{\omega}} + (\vec{\omega}\times)(I\vec{\omega} + \vec{H}_w) = \epsilon^2 T_1 + \cdots \tag{7.82}$$

As discussed earlier, the nominal equation is

$$I\dot{\vec{\omega}}_N + \left[\vec{\omega}_N\times\right] I\vec{\omega}_N + H_w = 0 \tag{7.83}$$

Proceeding as before, the operator A^* is given as

$$A^*(t) = -I^{-1}\left[(\omega_N\times)I - (I\omega_N + H_w)\times\right] \tag{7.84}$$

The rest of the procedure is the same as before, and leads to asymptotic solutions using the MTS technique.

7.8.4 Example 3

As an example, we consider a rigid body satellite in an elliptic orbit such that the fly-wheel is oriented along the body x-axis with angular momentum $0.2\ slug - ft^2/s$. The other parameters are as given in Table 7.2.

In this case the small parameter ϵ becomes

$$\epsilon = \frac{\omega_{orbit}}{\omega_{attitude}} = 0.027 \tag{7.85}$$

As before, the accuracy of the MTS solution is shown by displaying the errors of the MTS and numerical solutions for increasing step sizes, with respect to the reference solution (a high accuracy numerical solution). These are shown in Figs. 7.7, 7.8, 7.9 and 7.10. We see that the numerical solutions for angular velocities $\vec{\omega}$ and Euler parameters $\vec{\beta}$ show numerical instability for step size $\Delta t = 50$ s, which is unacceptable. The MTS solutions, on the other hand, show small and bounded errors for step sizes $\Delta t = 100, 200$ s, or even for $\Delta t = 500$ s. The maximum simulation errors and the computer time for the dual spin satellite, upon computation, follow almost exactly as in Fig. 7.6.

This is for the first order solution. Even greater accuracies can potentially be achieved by developing higher order approximations.

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Chapter 8
Satellite Attitude Control

8.1 Introduction

This chapter presents another example in space flight showing the computational advantages of asymptotic solutions. This application involves the attitude control of a dual spin satellite in an earth orbit by utilizing the interaction of an on-board electromagnet with the geomagnetic field. The presentation is based on [1, 2].

The interaction between the body magnetic moment of an earth satellite and the geomagnetic field produces a torque which can be utilized for controlling the attitude motion of the satellite. By installing one or several current-carrying coils on board, it is possible to generate an adjustable magnetic moment inside the vehicle and thus, a control torque on the satellite. As this magnetic control device needs no fuel and has no moving parts, it is likely to increase the satellite reliability.

The design of such a system, however, poses a difficult problem because the control torque is extremely small. The electric currents available to feed through the on-board coils are limited, and the magnetic torque generated in this way is not large enough to correct the satellite attitude motion in a short period of time. In fact, it is realized that one must depend on the interaction of the long-term accumulating control effort with the geomagnetic field in order to bring the vehicle into a desired orientation. For this reason, the system cannot be studied readily by classical control design techniques.

8.2 The Problem

The system consists of a dual-spin satellite moving in a circular orbit with its antenna required to point towards the center of the earth (Fig. 8.1).

A momentum wheel is assumed to be mounted along the satellite pitch axis to control the pitch motion. A small roll-yaw oscillation, called *nutation*, is possible if the vehicle angular momentum vector is not perfectly aligned with its angular velocity

R. V. Ramnath, *Computation and Asymptotics*, SpringerBriefs in Computational Mechanics, 93
DOI: 10.1007/978-3-642-25749-0_8, © The Author(s) 2012

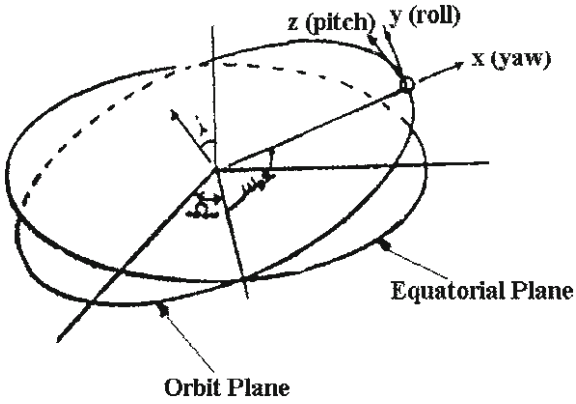

Fig. 8.1 Satellite geometry

vector due to external disturbances or initial misalignment, etc. A magnetic control device, using geomagnetic interaction, is to be designed to damp out the nutational oscillation and to keep the angular momentum of the satellite perpendicular to the orbit plane.

We consider a linear feedback control law involving (1) the angular velocity and (2) cross-feedback of the roll and yaw angles (yaw angle to the roll channel and roll angle to the yaw channel).

The desired feedback control torque $\vec{M}_d$ can be written as,

$$\vec{M}_d = -k_1 \begin{bmatrix} h\phi \\ -h\psi \\ 0 \end{bmatrix} - k_2 \begin{bmatrix} I_x \dot{\psi} \\ I_y \dot{\phi} \\ 0 \end{bmatrix} \tag{8.0}$$

where

k_1 = feedback control gain for angular error
k_2 = feedback control gain for angular rate error
I_x = Moment of inertia of vehicle about yaw axis in body axes
I_y = Moment of inertial of vehicle about roll axis in body axes
ψ = Yaw angle
ϕ = Roll angle

The pitch axis is stabilized by a momentum wheel and needs no magnetic control torque in that channel. The on-board magnetic attitude control device consists of a single electrical coil mounted along the satellite body pitch axis. The electric current feeding the coil can be monitored freely and will be used to produce the desired control torque.

$$\vec{M}_d = \vec{V}_d \times \vec{B} \tag{8.1}$$

where:

$\vec{V}_d =$ desired angular momentum of an on-board magnet.
$\vec{B} =$ geomagnetic field

The magnetic moment, $\vec{V}_d$, is not unique for a given $\vec{M}_d$. Considering a cross product of $\vec{M}_d$ with a vector $\vec{B}$,

$$\vec{B} \times \vec{M}_d = \vec{B} \times (\vec{V}_d \times \vec{B}) \tag{8.2}$$

$$i.e., \quad \vec{B} \times \vec{M}_d = |\vec{B}|^2 \vec{V}_d - \vec{B}(\vec{V}_d \cdot \vec{B}) \tag{8.3}$$

Since the smallest V_d for a desired $\vec{M}_d$ is the one perpendicular to both the geomagnetic field $\vec{B}$ and the desired torque $\vec{M}_d$, we set $\vec{V}_d \cdot \vec{B} = 0$, and obtain

$$\vec{V}_d = (\vec{B} \times \vec{M}_d)/|\vec{B}|^2 \tag{8.4}$$

As $\vec{V}_d$ is perpendicular to the geomagnetic field $\vec{B}$, it can be readily shown that it is the smallest moment required to generate a desired torque $\vec{M}_d$. Thus, the control is chosen to be a cross product with respect to the geomagnetic field. The control law developed here is different from the one in [3] in structure and implementation. However, in order to satisfy this law exactly, three on-board coils would be necessary. To save on hardware, we choose to have only one coil on board. In this case, $\vec{V}_d$ will not be exactly perpendicular to $\vec{B}$, but for high inclination orbits, the angle between them will be relatively close to 90°. For low inclination orbits, three coils may be required in some situations. The present approach to the problem is valid even for the three-coil case, although it requires greater algebraic manipulation. For this reason, we consider the design of the one coil case. Combining Eqs. (8.4) and (8.5), the smallest $\vec{V}_d$ is given by

$$\vec{V}_d^b = \frac{1}{|\vec{B}|^2} \begin{bmatrix} -B_z(-k_2 I_y \dot{\phi} + k_1 h \psi) \\ B_z(-k_2 I_x \dot{\psi} - k_1 h \phi) \\ B_x(-k_2 I_y \dot{\phi} + k_1 h \psi) - B_y(-k_2 I_x \dot{\psi} - k_1 h \phi) \end{bmatrix} \tag{8.5}$$

where $\vec{V}_d^b$ represents $\vec{V}_d$ in body coordinates.

A horizon sensor and a rate gyro are required to generate (ψ, ϕ) and $\dot{\psi}, \dot{\phi})$. These could also be provided by an inertial navigation system or an inertial measurement unit (IMU). Because our control device has the electrical coil only in the pitch axis, it is not possible to generate the body magnetic moment in the roll and yaw axes. By eliminating the first and second terms in Eq. (8.6), the control law is given by

$$\vec{V}_d^b = \frac{1}{|\vec{B}|^2} \begin{bmatrix} 0 \\ 0 \\ B_x(-k_2 I_y \dot{\phi} + k_1 h \psi) - B_y(-k_2 I_x \dot{\psi} - k_1 h \phi) \end{bmatrix} \tag{8.6}$$

Note that $\vec{V}_d^b$ as given in Eq. (8.7) is close to the best choice for a high-inclination orbit. However, even when the inclination is not high, this approach leads to useful results as shown by the application given later. For a rigorous determination of the optimal $\vec{V}_d^b$ with only one coil, optimization theory should be invoked.

8.3 System Dynamics

The angular motion of a dual spin body can be described by Euler's equations

$$I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z + h\omega_y = M_x \tag{8.7}$$

$$I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z - h\omega_x = M_y \tag{8.8}$$

$$I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y = M_z \tag{8.9}$$

Assuming the angles to be small, the angular velocity, $\vec{\omega}$, is related to the Euler angles as

$$\vec{\omega}^b = \begin{bmatrix} \dot{\psi} - \omega_0\phi \\ \dot{\phi} + \omega_0\psi \\ \dot{\theta} + \omega_0 \end{bmatrix} \tag{8.10}$$

where $\omega_0 =$ orbital angular velocity = a constant. The linearized equations of motion are,

$$I_x(\ddot{\psi} - \omega_0\dot{\phi}) + (I_z - I_y)(\dot{\phi} + \omega_0\psi)\left[\omega_0 + \frac{h}{I_z - I_y}\right] = M_x \tag{8.11}$$

$$I_y(\ddot{\phi} + \omega_0\dot{\psi}) + (I_x - I_z)(\dot{\psi} - \omega_0\phi)\left[\omega_0 - \frac{h}{I_x - I_z}\right] = M_y \tag{8.12}$$

$$I_z\ddot{\theta} = M_z \tag{8.13}$$

The geomagnetic field $\vec{B}$ is approximated by the magnetic dipole

$$\vec{B} = \mu_\beta / R^5\left[R^2\vec{e}_\beta - 3(\vec{e}_\beta \cdot \vec{R})\vec{R}\right] \tag{8.14}$$

or, in body coordinates,

$$\vec{B}^b = \begin{bmatrix} -2B_0 \sin\omega_0 t \\ B_0 \cos\omega_0 t \\ B_1 \end{bmatrix} \tag{8.15}$$

where:

$B_0 = (\omega_0^2\mu_\beta/\mu sin\, i)$
$B_1 = (\omega_0^2\mu_\beta/\mu cos\, i)$
$\vec{e}_\beta$ = unit vector along the magnetic dipole axis
μ_β = geomagnetic field constant
μ = gravitational constant

With the control law as given in Eq. (8.7), the equations of motion for the roll-yaw nutation are:

$$
\begin{aligned}
I_x\ddot{\psi} &+ \left[\frac{B_0^2}{|\vec{B}|^2}cos^2(\omega_0 t)k_2 I_x\right]\dot{\psi} \\
&+ \left[\frac{B_0^2}{|\vec{B}|^2}sin(2\omega_0 t)k_2 I_y - I_x\omega_0 + (I_z - I_y)\omega_0 + h\right]\dot{\phi} \\
&+ \left[-\frac{B_0^2}{|\vec{B}|^2}sin(2\omega_0 t)k_1 h + \omega_0^2(I_z - I_y) + h\omega_0\right]\psi \\
&+ \left[\frac{B_0^2}{|\vec{B}|^2}cos^2(\omega_0 t)k_1 h\right]\phi = 0
\end{aligned}
\tag{8.16}
$$

$$
\begin{aligned}
I_y\ddot{\phi} &+ \left[\frac{4B_0^2 k_2 I_y}{|\vec{B}|^2}sin^2(\omega_0 t)\right]\dot{\phi} \\
&+ \left[\frac{B_0^2}{|\vec{B}|^2}sin(2\omega_0 t)k_2 I_x + I_y\omega_0 + (I_x - I_z)\omega_0 - h\right]\dot{\psi} \\
&+ \left[\frac{B_0^2}{|\vec{B}|^2}sin(2\omega_0 t)k_1 h - \omega_0^2(I_x - I_z) + h\omega_0\right]\phi \\
&- \left[\frac{4B_0^2}{|\vec{B}|^2}sin^2(\omega_0 t)k_1 h\right]\psi = 0
\end{aligned}
\tag{8.17}
$$

The dynamics of the system exhibit two modes; one is the fast nutational oscillation, and the other is a relatively slow satellite orbital motion. A small parameter, ϵ, is defined as the ratio of the orbital frequency to the nutational frequency, i.e.,

$$
\epsilon \equiv \frac{\omega_0\sqrt{I_x I_y}}{h} \ll 1 \tag{8.18}
$$

Considering h and I_x, I_y to be of order unity leads to the ordering that ω_0, k_1 and k_2 are $O(\epsilon)$, and B_0, B_1 and $|\vec{B}|$ are $O(\epsilon^2)$. The problem can then be cast into a perturbation form by classifying each individual term with a proper ϵ^i attached to it to indicate the appropriate order of magnitude. Thus, the equations of the controlled motion are given as:

$$
I_x\ddot{\psi} + \epsilon C_\psi(t)\dot{\psi} + \left[h + \epsilon D_\psi(t)\right]\dot{\phi} + \left[\epsilon E_\psi(t) + O(\epsilon^2)\right]\psi + \epsilon F_\psi(t)\phi = 0 \tag{8.19}
$$

$$
I_y\ddot{\phi} + \epsilon C_\phi(t)\dot{\phi} + \left[-h + \epsilon D_\phi(t)\right] + \left[\epsilon E_\phi(t) + O(\epsilon^2)\right]\phi + \epsilon F_\phi(t)\psi = 0 \tag{8.20}
$$

where the coefficients $C_\psi(t)$, $C_\phi(t)$, etc., are defined by comparing them to the terms in Eqs. (8.17) and (8.18).

8.4 Asymptotic Solution

The problem is solved approximately by developing asymptotic solutions by means of the Multiple Scales Method. This technique has already been discussed in detail in the previous chapters. The independent variable, time, is extended into a higher dimensional space by means of the extension

$$t \to \{\tau_0, \tau_1, \cdots\}; \quad \tau_0 = t; \quad \tau_1 = \epsilon t$$

In general,

$$\tau_i = \tau_i(t, \epsilon)$$

and

$$\psi(t, \epsilon) \to \overline{\psi}(\tau_0, \tau_1, \cdots, \epsilon)$$

and

$$\phi(t, \epsilon) \to \overline{\phi}(\tau_0, \tau_1, \cdots, \epsilon)$$

The variables $\overline{\psi}$ and $\overline{\phi}$ are now expanded in an asymptotic series in the new scales

$$\overline{\psi} = \overline{\psi}_0(\tau_0, \tau_1, \cdots) + \epsilon \overline{\psi}_1(\tau_0, \tau_1, \cdots) + O(\epsilon^2) \tag{8.21}$$

$$\overline{\phi} = \overline{\phi}_0(\tau_0, \tau_1, \cdots) + \epsilon \overline{\phi}_1(\tau_0, \tau_1, \cdots) + O(\epsilon^2) \tag{8.22}$$

The terms of zeroth order in the time scales are written as:

$$I_x \frac{\partial^2 \overline{\psi}_0}{\partial \tau_0^2} + h \frac{\partial \overline{\phi}_0}{\partial \tau_0} = 0 \tag{8.23}$$

$$I_y \frac{\partial^2 \overline{\phi}_0}{\partial \tau_0^2} - h \frac{\partial \overline{\psi}_0}{\partial \tau_0} = 0 \tag{8.24}$$

The perturbation equations to the first order in ϵ are given by

$$I_x \frac{\partial^2 \overline{\psi}_1}{\partial \tau_0^2} + h \frac{\partial \overline{\phi}_1}{\partial \tau_0} = -2I_x \frac{\partial^2 \overline{\psi}_0}{\partial \tau_0 \partial \tau_1} - C_\psi \frac{\partial \overline{\psi}_0}{\partial \tau_o} - h \frac{\partial \overline{\phi}_0}{\partial \tau_1} - D_\psi \frac{\partial \overline{\phi}_0}{\partial \tau_0} - E_\psi \overline{\psi}_0 - F_\psi \overline{\phi}_0 \tag{8.25}$$

$$I_y \frac{\partial^2 \overline{\phi}_1}{\partial \tau_0^2} - h \frac{\partial \overline{\psi}_1}{\partial \tau_0} = -2I_y \frac{\partial^2 \overline{\phi}_0)}{\partial \tau_0 \partial \tau_1} - C_\phi \frac{\partial \overline{\phi}_0}{\partial \tau_o} + h \frac{\partial \overline{\psi}_0}{\partial \tau_1} - D_\phi \frac{\partial \overline{\psi}_0}{\partial \tau_0} - E_\phi \overline{\phi}_0 - F_\phi \overline{\psi}_0 \tag{8.26}$$

Terms of higher order are not considered in this analysis but can be developed systematically if needed. The solutions of the zeroth order equations (8.24) and (8.25) are

$$\overline{\psi}_0 = p(\tau_1, \tau_2, \cdots)exp(-i\frac{h\tau_0}{\sqrt{I_x I_y}}) + q(\tau_1, \tau_2, \cdots)exp(i\frac{h\tau_0}{\sqrt{I_x I_y}}) + r(\tau_1, \cdots) \tag{8.27}$$

$$\overline{\phi}_0 = i\frac{I_x p}{\sqrt{I_x I_y}}exp(-\frac{ih\tau_0}{\sqrt{I_x I_y}}) - i\frac{I_x q}{\sqrt{I_x I_y}}exp(i\frac{h\tau_0}{\sqrt{I_x I_y}}) + s(\tau_1, \cdots) \tag{8.28}$$

p, q, r and *s* are not functions of τ_0, but they can be functions of τ_1, τ_2, etc., and are yet to be determined.

With the above solutions of $\overline{\psi}_o$ and $\overline{\phi}_0$, the equation of the first order in ϵ leads to the solutions

$$\begin{bmatrix} \frac{\partial\overline{\psi}_1}{\partial\tau_0} \\ \frac{\partial\overline{\phi}_1}{\partial\tau_0} \end{bmatrix} = \frac{\Phi^*}{2(\tau_0, 0)} \begin{bmatrix} (-U_\phi + iU_\psi)\frac{exp(-2iN_u\tau_0)}{2N_u} - (iV_\psi + V_\phi)\frac{exp(2iN_u\tau_0)}{2N_u} \\ +(iW_\psi - W_\phi)\frac{exp(-iN_u\tau_0)}{N_u} - (W_\phi + iW_\psi)\frac{exp(iN_u\tau_0)}{N_u} \\ +[(V_\psi + U_\psi) + i(V_\phi - U_\phi)]\tau_0 \\ (-U_\psi + iU_\phi)\frac{exp(-2iN_u\tau_0)}{2N_u} - (V_\psi + iV_\phi)\frac{exp(2iN_u\tau_0)}{2N_u} \\ +(W_\psi + iW_\phi)\frac{exp(-iN_u\tau_0)}{N_u} - (W_\psi + iW_\phi)\frac{exp(iN_u\tau_0)}{N_u} \\ +[(V_\phi + U_\phi) + i(U_\psi - V_\psi)]\tau_0 \end{bmatrix} \tag{8.29}$$

where:

$$\begin{aligned} N_u &= \frac{h}{\sqrt{I_x I_y}} \\ U_\psi &= N_u i\{I_x\frac{\partial p}{\partial\tau_1} + [C_\psi - F_\psi\frac{I_x}{h} + \frac{D_\psi I_x i}{\sqrt{I_x I_y}} + E_\psi\frac{i}{N_u}]p\} \\ V_\psi &= -N_u i\{I_x\frac{\partial q}{\partial\tau_1} + C_\psi - F_\psi\frac{I_x}{h} - D_\psi I_x\frac{i}{\sqrt{I_x I_y}} - E_\psi\frac{i}{N_u}]q\} \\ W_\psi &= -h\frac{\partial s}{\partial\tau_1} - E_\psi r - F_\psi s \\ U_\phi &= -h\frac{\partial p}{\partial\tau_1} - [\frac{C_\phi h}{I_y} + F_\phi - D_\phi N_u i + E_\phi I_x\frac{i}{\sqrt{I_x I_y}}]p \\ V_\phi &= -h\frac{\partial q}{\partial\tau_1} - [C_\phi\frac{h}{I_y} + F_\phi + D_\phi N_u i - E_\phi I_x i\sqrt{I_x I_y}]q \\ W_\phi &= h\frac{\partial r}{\partial\tau_1} - E_\phi s - F_\phi r \end{aligned} \tag{8.30}$$

Φ^* is the transition matrix

$$\Phi^*(\tau_0, 0) = \begin{pmatrix} cosN_u\tau_0 & -sinN_u\tau_0 \\ sinN_u\tau_0 & cosN_u\tau_0 \end{pmatrix} \tag{8.31}$$

$$= \frac{1}{2}\begin{pmatrix} exp(iN_u\tau_0) + exp(-iN_u\tau_0) & iexp(iN_u\tau_0) - iexp(iN_u\tau_0) \\ -iexp(iN_u\tau_0) + iexp(-iN_u\tau_0) & exp(iN_u\tau_0) + exp(-iN_u\tau_0) \end{pmatrix} \tag{8.32}$$

The first order solution will diverge and the asymptotic property will be violated (i.e., *secular nonuniformity*), unless the terms $(V_\psi + U_\psi + iV_\phi - iU_\phi)\tau_0$ and $(V_\phi + U_\phi - iV_\psi + iU_\psi)\tau_0)$ are set to zero. This can be achieved by using the freedom in the selection of p and q. Therefore, for uniformity in τ_0, we require that

$$V_\psi + U_\psi + iV_\phi - iU_\phi = 0 \tag{8.33}$$

$$V_\phi + U_\phi - iV_\psi + iU_\psi = 0 \tag{8.34}$$

These are equivalent to

$$U_\phi + iU_\psi = 0 \tag{8.35}$$

$$V_\phi - iV_\psi = 0 \tag{8.36}$$

Similarly, by using the freedom in the choice of the bias terms $r(\tau_1, \cdots)$ and $s(\tau_1, \cdots)$, it is possible to force W_ψ and W_ϕ to zero. Because W_ψ and W_ϕ will lead to secular terms when multiplied by the transition matrix, we eliminate such nonuniformities by setting

$$W_\psi \equiv -h\frac{\partial s}{\partial \tau_1} - E_\psi r - F_\psi s = 0 \tag{8.37}$$

$$W_\phi \equiv -h\frac{\partial r}{\partial \tau_1} - E_\phi s - F_\phi r = 0 \tag{8.38}$$

We integrate the partial differential equations to first order in ψ and ϕ, and thereby obtain the solution with amplitude and phase to first order in ϵ. These solutions are given by

$$\overline{\psi} = p(\tau_1, \cdots)e^{(-iN_u\tau_0)} + q(\tau_1, \cdots)e^{(iN_u\tau_0)} + r(\tau_1, \cdots) \tag{8.39}$$

$$\overline{\phi} = i\frac{I_x}{\sqrt{I_x I_y}}p(\tau_1, \cdots)e^{(-iN_u\tau_0)} - i\frac{I_x}{\sqrt{I_x I_y}}q(\tau_1, \cdots)e^{(iN_u\tau_0)} + s(\tau_1, \cdots) \tag{8.40}$$

where the amplitudes p and q are governed by Eqs. (8.36) and (8.37). This is equivalent to

$$(h + N_u I_x)\frac{\partial p}{\partial \tau_1} + [(\frac{C_\phi h}{I_y} + F_\phi - iD_\phi N_u + i\frac{E_\phi I_x}{\sqrt{I_x I_y}}) + N_u(C_\psi - \frac{F_\psi I_x}{h} + i\frac{D_\psi I_x}{\sqrt{I_x I_y}} + i\frac{E_\psi}{N_u})]p = 0 \tag{8.41}$$

$$(h + N_u I_x)\frac{\partial q}{\partial \tau_1} + [(\frac{C_\phi h}{I_y} + F_\phi + iD_\phi N_u - i\frac{E_\phi I_x}{\sqrt{I_x I_y}}) + N_u(C_\psi - \frac{F_\psi I_x}{h} - i\frac{D_\psi I_x}{\sqrt{I_x I_y}} - i\frac{E_\psi}{N_u})]q = 0 \tag{8.42}$$

where $p(\tau_1)$ and $q(\tau_1)$ are complex conjugates. The bias terms $r(\tau_1)$ and $s(\tau_1)$ are governed by Eqs. (8.38) and (8.39).

8.5 Feedback Control Design

We note that in the original problem, the control torque is small and the system is time-varying. Therefore, the effect of the control on the system dynamics is not clear. However, the asymptotic solutions obtained by the MTS method render the control design problem tractable. The approximate solutions Eqs. (8.40) and (8.41), describe a constant frequency oscillating motion with slowly time-varying amplitudes *p, q* and the biases *r, s*. The variables *p, q, r, s* are functions of the control torque and are given by Eqs. (8.42)–(8.45). The problem can be considered as requiring the control torque to force the amplitudes p and q of the nutation oscillation and the slowly changing orientation errors r and s to go to zero. In this way, since the nutation oscillation has been separated out, the task of selecting the feedback control gains k_1 and k_2 in the slow Eqs. (8.40) and (8.41) becomes much easier. They can be solved in terms of quadratures, and the solutions for $p(\tau_1(\text{or}\, q(\tau_1))$ can be written as

$$p = p_0 exp\left[\left(\frac{-\frac{C_\psi}{I_y} - \frac{F_\phi}{h} - \frac{C_\psi}{\sqrt{I_x I_y}} + \frac{F_\psi I_x}{h\sqrt{I_x I_y}}}{1 + \frac{I_x}{\sqrt{I_x I_y}}}\right)\tau_1 + Imaginary\ Part\right] \tag{8.43}$$

Thus p and q will approach zero as $\tau_1 \to \infty$ if

$$-\frac{C_\psi}{I_y} - \frac{F_\phi}{h} - \frac{C_\psi}{\sqrt{I_x I_y}} + \frac{F_\psi I_x}{h\sqrt{I_x I_y}} < 0 \tag{8.44}$$

where $C_\phi, F_\phi, \cdots$ are as defined earlier by comparing Eqs. (8.17) and (8.18) with (Eqs. 8.20) and (8.21). Upon substitution, the condition for p (or q) to be stable is

$$4(k_1 - k_2)sin^2(\omega_0 t) + (k_1 - k_2)\frac{I_x}{\sqrt{I_x I_y}}cos^2(\omega_0 t) < 0 \tag{8.45}$$

Therefore, for p and q to be stable, we require

$$k_1 < k_2 \tag{8.46}$$

It can be shown that [1, 2] the time constant T_c varies inversely with $(k_1 - k_2)$. The torque for the control of the satellite orientation relating to k_1, will excite the nutational motion. The torque for the angular rate control relating to k_2, will damp out the nutational motion. The term $(k_1 - k_2)T_c$ depends on the inclination angle i and the moments of inertia I_x and I_y. The orbital period depends on the control gain k_1 and the orbital inclination i [1] depends on the yaw, roll moments of inertia.

First, the particular control device works better for a high inclination orbit than for an orbit of low inclination. However, the difference becomes less when the inclination angle becomes larger than 40°. Second, for control purposes, it is preferable to have the moment of inertia about the roll axis larger than that about the yaw axis. The slowly changing terms r and s satisfy the equation:

$$\begin{pmatrix} \frac{\partial s}{\partial \tau_1} \\ \frac{\partial r}{\partial \tau_1} \end{pmatrix} = \frac{1}{h} \begin{pmatrix} -F_\psi & -E_\psi \\ E_\phi & F_\phi \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix}$$
$$= \begin{pmatrix} [-B_0^2 k_1 cos^2(\omega_0 \tau_1)]/|\vec{B}|^2 & [k_1 B_0^2 sin(2\omega_0 \tau_1)]/|\vec{B}|^2 - \omega_0 \\ \omega_0 + [k_1 B_0^2 sin(2\omega_0 \tau_1)]/|\vec{B}|^2 & [-4 B_0^2 k_1 sin^2(\omega_0 \tau_1)/|\vec{B}|^2 \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix}$$
$$= A(\tau_1) \begin{pmatrix} s \\ r \end{pmatrix} \tag{8.48}$$

This is a second order linear differential equation with periodic coefficients. Invoking Floquet theory [4, 5], we see that the necessary and sufficient condition for the solutions to go to zero (i.e., to be *stable*) as $\tau_1 \to \infty$ is that the eigenvalues λ_i of the transition matrix $\Phi(\tau_1 + T, \tau_1)$ lie in a *unit disc* $|\lambda_i| < 1$ and T is the period of the coefficients. Note that the system matrix $A(\tau_1)$, and therefore the transition matrix $\Phi(\tau_1, 0)$, do not depend on the gain k_2. The eigenvalues λ_1 and λ_2 of $\Phi(\tau_1 + T, \tau_1)$ are shown in Fig. 8.1 as dependent on the control gain k_1, the orbital inclination i, and the orbital period. For small inclinations of the orbit (say 20°), there are two real eigenvalues λ_1, λ_2 (Fig. 8.1). However, as the angle of inclination increases (to say 80°) the eigenvalues become complex conjugates with magnitudes as shown in Fig. 8.1. It is found that the system is stable for all positive k_1. Further, as expected, the eigenvalues of $\Phi(\tau_1 + T, \tau_1)$ become smaller (an indication of more rapid convergence) as the orbital inclination increases.

8.6 Application

The above approach is now applied to a specific example. Consider a satellite with the parameters as follows:

- Moments of inertia about the principal axes:
  $I_x = 120\, slug\, ft^2$; $I_y = 100\, slug\, ft^2$; $I_z = 150\, slug\, ft^2$
- Angular momentum of the flywheel (along the pitch axis):
  $h = 4\, slug\, ft^2/s$
- Satellite orbit parameters:
  Eccentricity = $e = 0$, Inclination = $i = 20°$; Period = 10,000 s

The small parameter ϵ for this case can be calculated as

$$\epsilon = \frac{\omega_0}{h/\sqrt{I_x I_y}} \approx 0.017 \tag{8.49}$$

The value of the small parameter ϵ plays an important role in the asymptotic analysis. It provides an estimate of the accuracy of the approximate solution. The roll-yaw nutational motion can be approximated by

$$\overline{\psi}|_t = p\, exp(-i N_u t) + q\, exp(i N_u t) + r \tag{8.50}$$

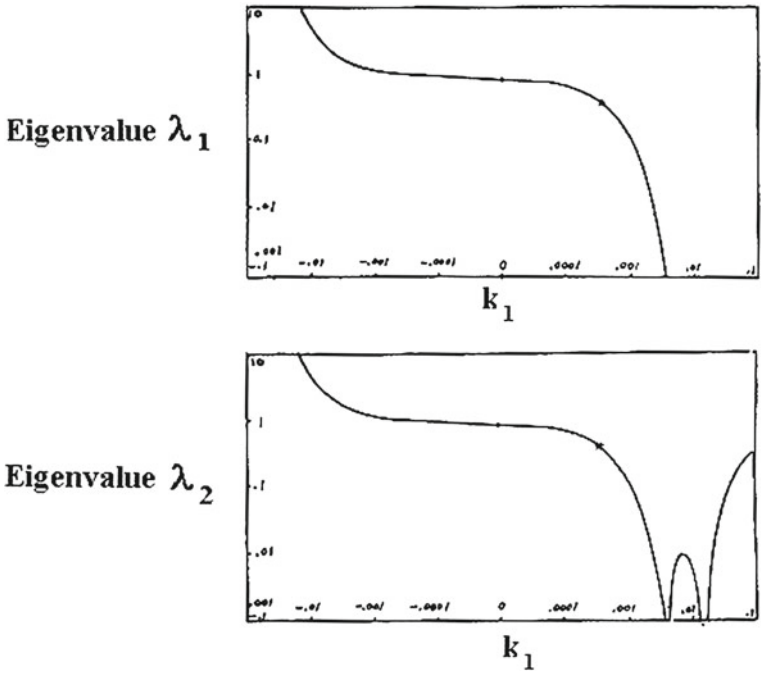

Fig. 8.2 Variation of system eigenvalues with control gain

$$\overline{\phi}|_t = (\frac{iN_u I_x p}{h})exp(-iN_u t) - (\frac{iN_u I_x q}{h})exp(iN_u t) + s \tag{8.51}$$

In order for the system to be stable, the amplitude terms p, q and the bias terms r, s must all decay. Therefore, recalling that k_1 and k_2 are the feedback control gains for angular error and angular rate error respectively,

- $k_1 < k_2$
- k_1 is chosen such that the eigenvalues of the transition matrix $\Phi(\tau_1 + T, \tau_1)$ for the Eq. (8.50) are less than unity.

The eigenvalues λ_1 and λ_2 are then plotted in terms of k_1 in Fig. 8.3. They show that if $0 < k_1 < 0.1$, then $\lambda_1 < 1$ and $\lambda_2 < 1$ and the bias terms r and s will be damped. Select $k_1 = 4 \times 10^{-4}$ and $k_2 = 8 \times 10^{-4}$. Implementation of the control law of Eq. (8.7) with these k_1 and k_2 values requires electric power of about 10 W [1]. The time constant for p and q, calculated from Eq. (8.50) is about 5 h.

In order to validate this synthesis, the Eqs. (8.17) and (8.18) constituting the dynamic system, are first numerically integrated with the control gains k_1 and k_2 and initial conditions, $\psi(0) = 0.3^\circ$ $\dot{\psi}(0) = 0.1^\circ/s$ $\phi(0) = 0.2^\circ$ and $\dot{\phi}(0) = 0.1^\circ/s$.

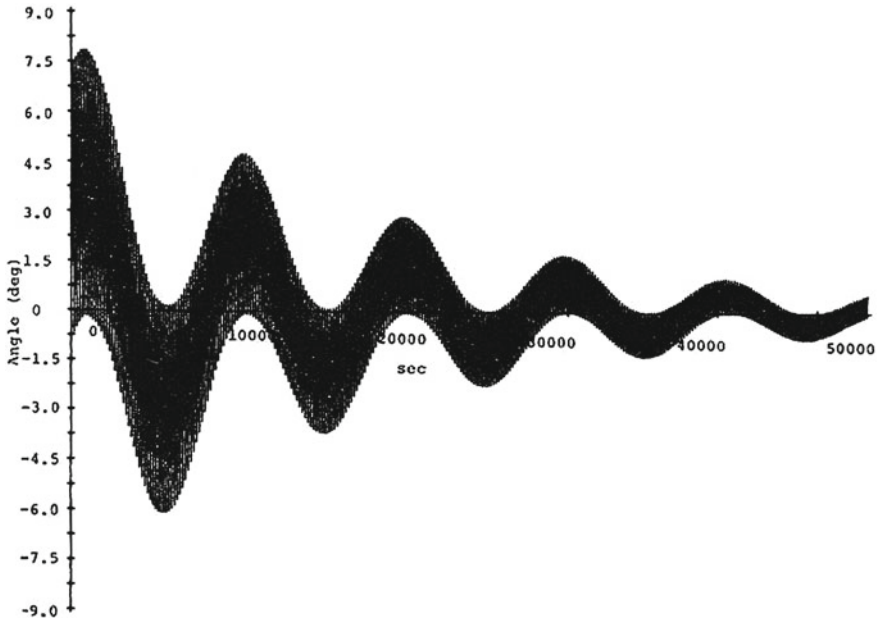

Fig. 8.3 Roll motion by direct integration: $\Delta t = 10\,s$

The roll and yaw oscillations by direct numerical integration are plotted in Figs. 8.2 and 8.7 respectively.

For the same initial conditions the solutions of Eqs. (8.17) and (8.18) are obtained by integrating Eqs. (8.42) and (8.43) with respect to τ_1. The asymptotic approximations by the GMS method (Fig. 8.4) are found to be extremely accurate when compared with the direct numerical solutions (Fig. 8.2) (with $\Delta t = 10\,s$) for the roll motion. Similarly for the yaw motion, a comparison of the GMS approximation (Fig. 8.10) with the direct numerical solution (Fig. 8.8 with $\Delta t = 10\,s$) shows very good accuracy. The GMS approximations are so accurate in fact, as to be indistinguishable from the numerical solutions on the plots for these cases. Further, the system is stable, and the variations of (p, q) and (r, s) are clearly identifiable.

The control of the satellite orientation and the damping of its nutation oscillation by an on-board magnetic device has been studied. We find that the orientation control torque could excite the nutational mode of the spacecraft. Therefore, for attitude stability, a larger damping torque is needed. Furthermore, the characteristics of the orientation control system can be given in terms of the orbit inclination angle, orbital period, and the control gain. It is demonstrated that orbits of large inclination angle and longer orbit period require less control torque. Finally, the control and damping characteristics of the system in terms of the orbital parameters and satellite inertia properties are given.

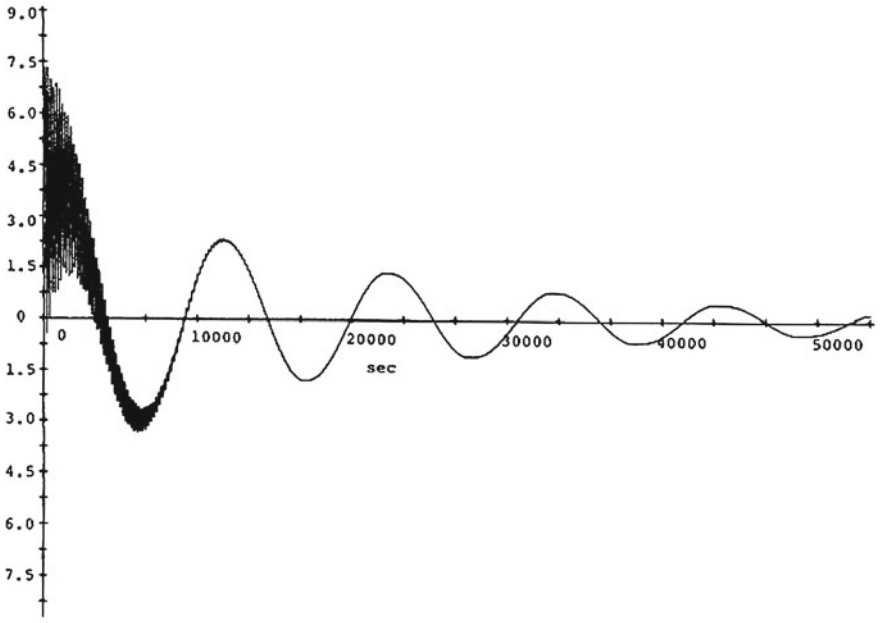

Fig. 8.4 Roll motion by direct integration: $\Delta t = 30\,s$

8.7 Computational Considerations

The time histories of the roll angle $\phi(t)$ and $\psi(t)$ are computed by (a) direct integration and (b) multiple scales technique. They show a combination of a high frequency oscillation and a slower oscillatory variation. The integration time step is progressively increased in each approach in an effort to speed up the computation. The results are shown in Figs. 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, and 8.12. However, this approach when applied to direct integration results in a highly *inaccurate* computation and a total misrepresentation of the solution due to the loss of the high frequency part of the dynamics. Specifically, this occurs (Figs. 8.4, and 8.9) when the integration step size is increased from 10 to 30 s!

For the numerical integration of the roll motion, when the step size is increased from $\Delta t = 10\,s$ (Fig. 8.3) to $\Delta t = 30\,s$ (Fig. 8.4), the high frequency part of the solution is lost quickly, leaving only the low frequency part of the solution. This is seen clearly in Fig. 8.4. In contrast, the multiple scales technique allows increases in step size to a much greater extent, from $\Delta t = 10\,s$ (Fig. 8.6) to $\Delta t = 2000\,s$ (Fig. 8.8) for the roll motion.

Similarly for the yaw motion, the numerical integration with $\Delta t = 30\,s$ loses the high frequency part (Fig. 8.9). However, the GMS solution allows increases in step size from $\Delta t = 10\,s$ (Fig. 8.10) to $\Delta t = 2000\,s$ (Fig. 8.12) for the yaw motion.

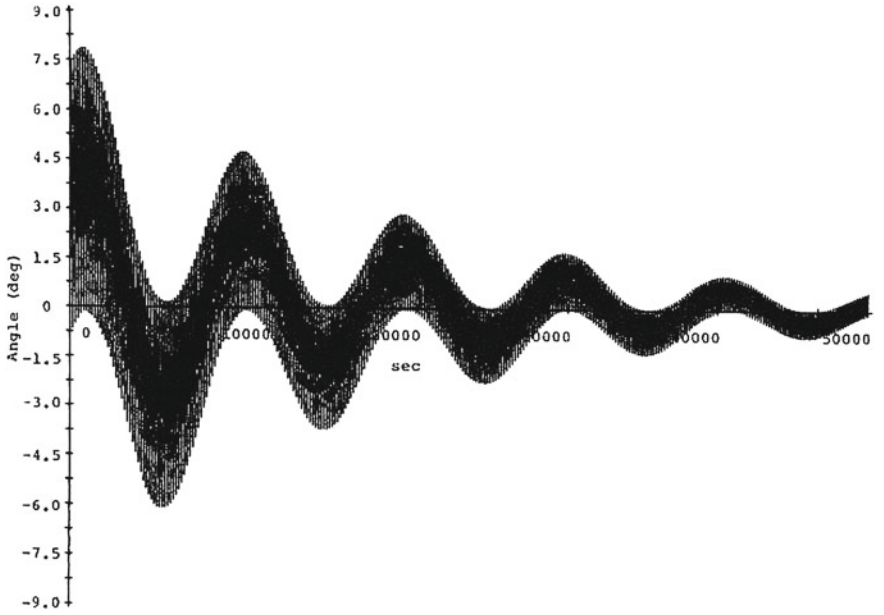

Fig. 8.5 Roll motion by GMS method: $\Delta t = 10\,s$

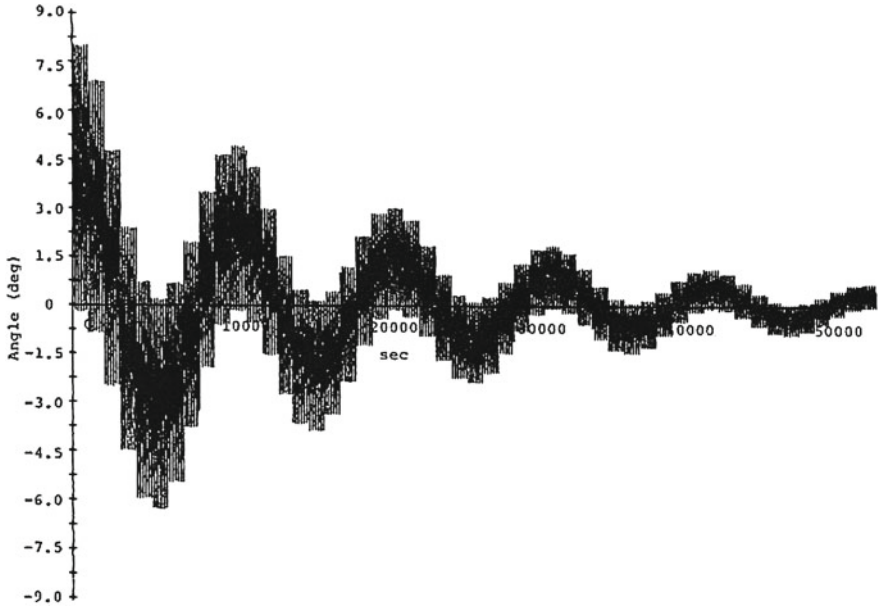

Fig. 8.6 Roll motion by GMS method: $\Delta t = 1000\,s$

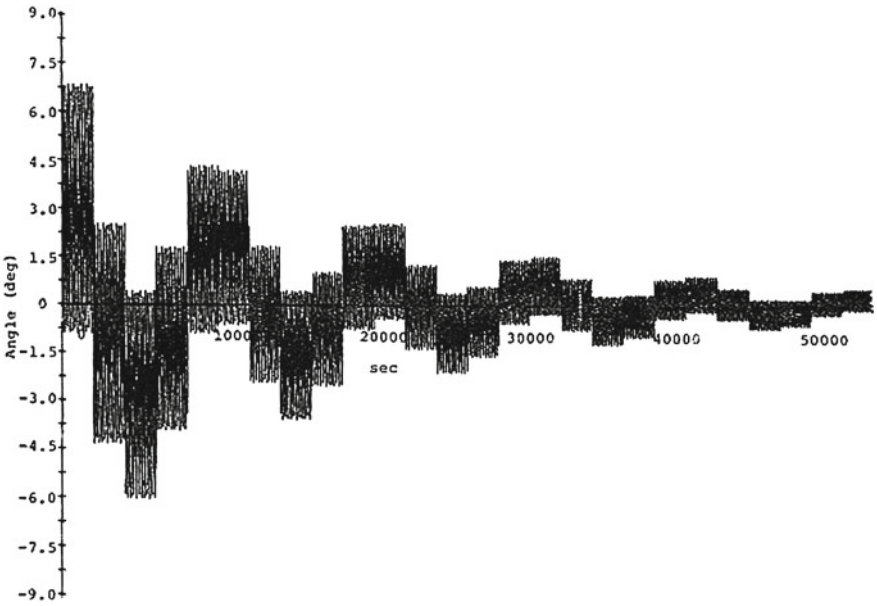

Fig. 8.7 Roll motion by GMS method: $\Delta t = 2000\,s$

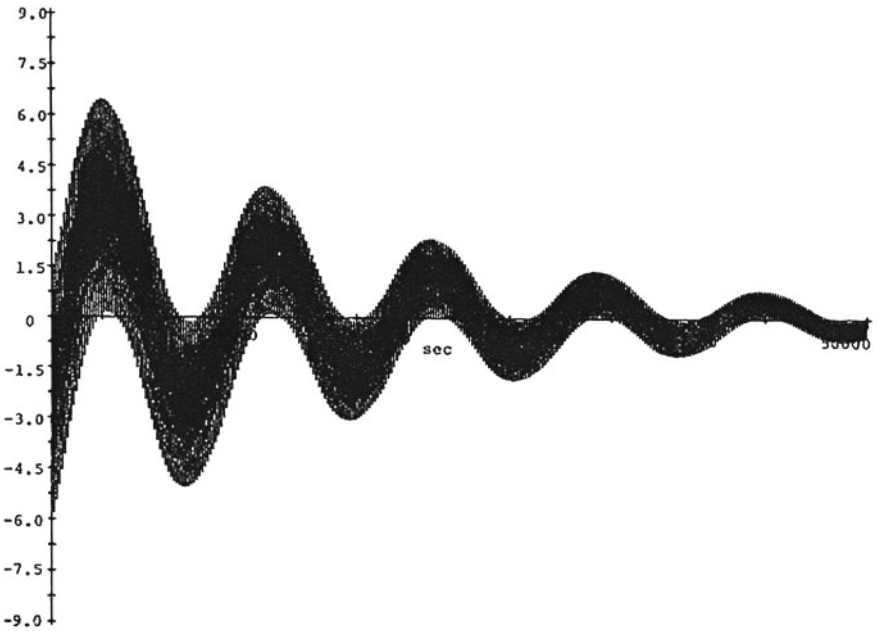

Fig. 8.8 Yaw motion by direct integration: $\Delta t = 10\,s$

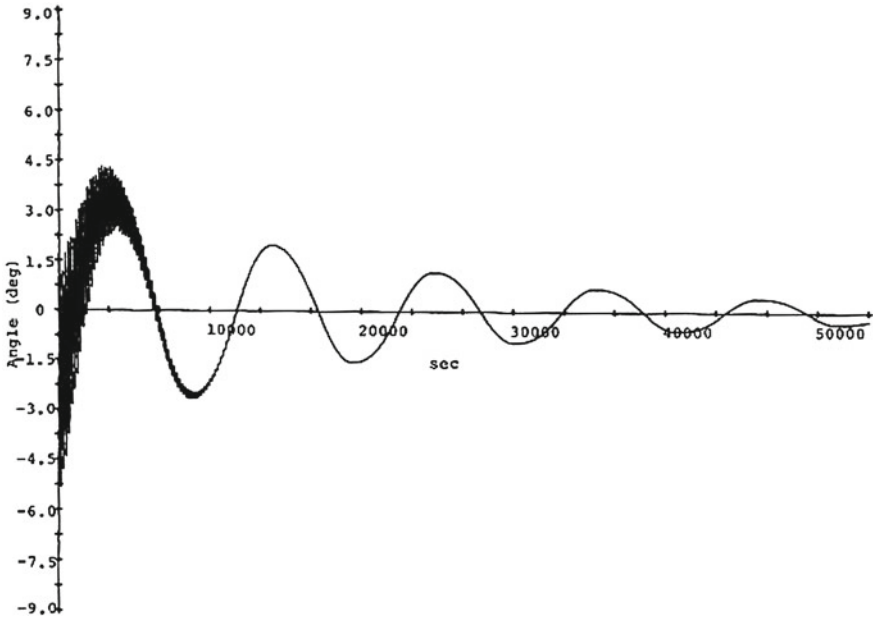

Fig. 8.9 Yaw motion by direct integration: $\Delta t = 30\,s$

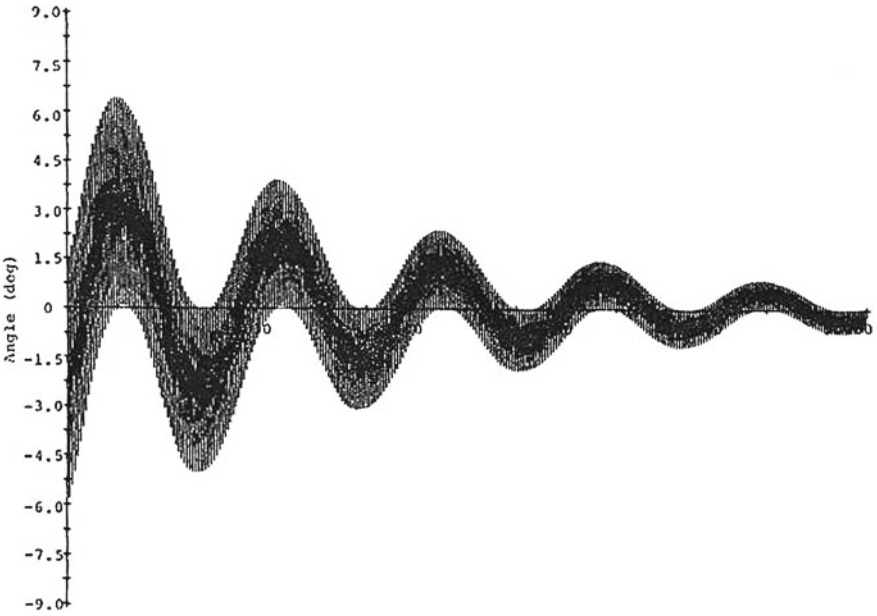

Fig. 8.10 Yaw motion by GMS method: $\Delta t = 10\,s$

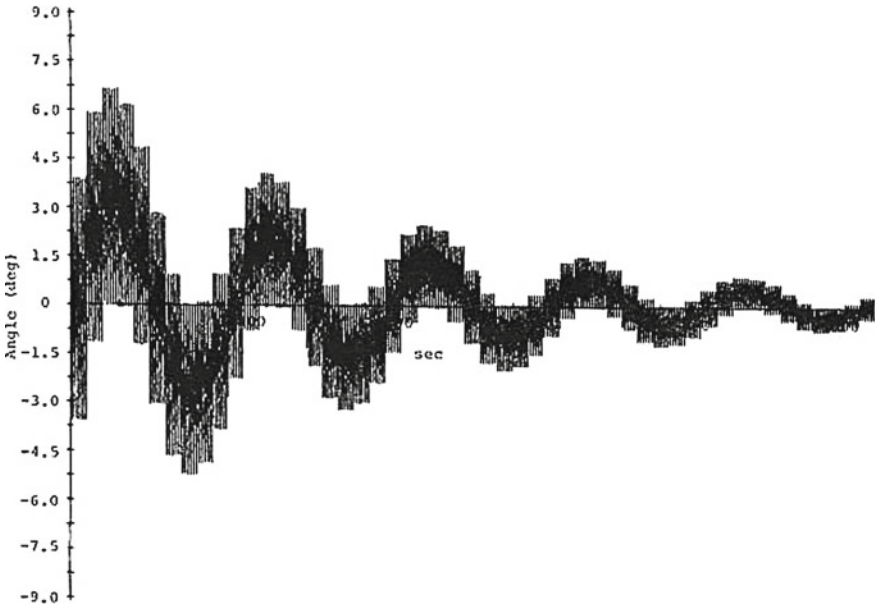

Fig. 8.11 Yaw motion by GMS method: $\Delta t = 1000\,s$

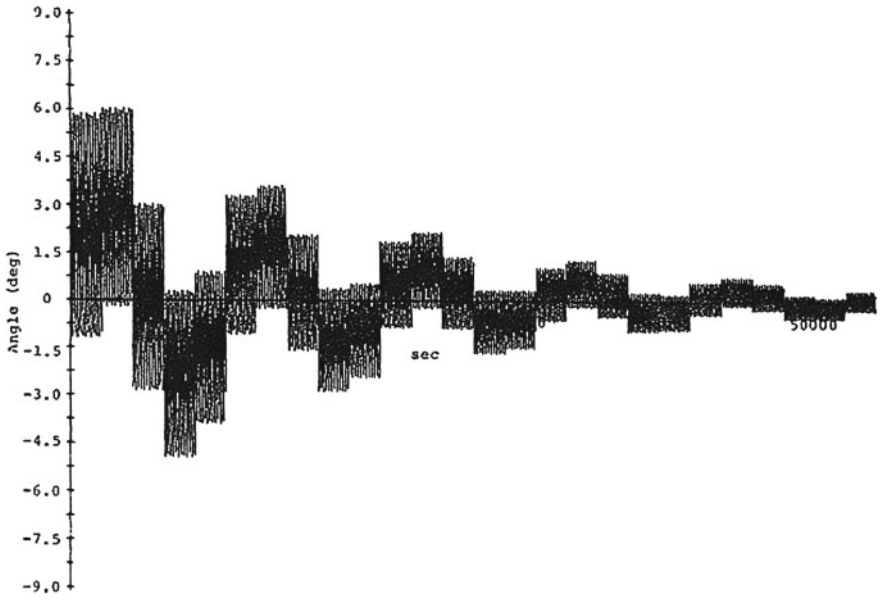

Fig. 8.12 Yaw motion by GMS method: $\Delta t = 2000\,s$

Table 8.1 Computation time for GMS and numerical integration

Case	Description	Computer time (s)
1	GMS Integration, $\Delta t = 1000\,s$	2.9
2	GMS Integration, $\Delta t = 2000\,s$	1.6
3	Direct integration, $\Delta t = 10\,s$	58

The general nature of the dynamics is represented well. Using larger step sizes in the GMS method, the roll and yaw oscillations can be computed without losing the high and low frequency aspects, or much loss in accuracy, although the peaks are flattened due to the increase in integration step size, as can be expected. Thus the computation is rendered efficient and results in a considerable saving of computer time. Further refinements using the nonlinear scales of the generalized (GMS) technique could lead to even more saving of computer time and greater efficiency. The relative computational times are shown in Table 8.1.

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Chapter 9
Summary and Conclusions

Having considered the different kinds of approximations, this book advances a relatively novel viewpoint of using asymptotics for computational purposes. This is in contrast to the classical approach in which asymptotic aolutions were used mainly for qualitative descriptions of physical phenomena and to provide insight into their behavior. However, as evidenced by many well-known examples presented in this book, asymptotic solutions have also been used in computation resulting in great efficiency.

It is hoped that this would lead to further work combining the two fields of computation and asymptotics in a quest of a combination retaining the significant advantages of both fields.

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DOI: 10.1007/978-3-642-25749-0_9, © The Author(s) 2012

Appendix A
Gravity Gradient Torque

We will now consider the gravity gradient torque on an earth satellite in a circular orbit. We assume that the earth is a perfect rigid sphere with a radially symmetric mass distribution and denote

$\vec{R} =$ vector from Earth center to the center of mass of the satellite.
$\vec{r} =$ vector from Earth center to the mass element *dm* of the satellite.
$\vec{\rho} =$ position vector of mass element *dm* with respect to the satellite center of mass.

Then, if $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the body axes *X*, *Y*, *Z* respectively, we have

$$\vec{\rho} = x\vec{i} + y\vec{j} + z\vec{k} \tag{A.1}$$
$$\vec{r} = \vec{R} + \vec{\rho} \tag{A.2}$$

The attractive force on the mass element *dm* is

$$d\vec{F} = -\mu \frac{dm}{r^3}\vec{r} \tag{A.3}$$

where: $\mu = GM =$ Earth's gravitational parameter, with $G =$ universal gravitational constant, and $M =$ total mass of the earth. The torque about the center of mass of the satellite caused by the force on *dm* is

$$\begin{aligned} d\vec{L}_g &= -\vec{\rho} \times \mu \frac{dm}{r^3}\vec{r} \\ &= -\mu \frac{dm}{r^3}\vec{\rho} \times (\vec{R} + \vec{\rho}) \\ &= -\mu \frac{dm}{r^3}\vec{\rho} \times \vec{R} \end{aligned} \tag{A.4}$$

Since

$$r^2 = (\vec{R} + \vec{\rho}) \cdot (\vec{R} + \vec{\rho}) = R^2 \left[1 + (\rho/R)^2 + 2\frac{2\vec{R} \cdot \vec{\rho}}{R^2} \right] \tag{A.5}$$

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r^{-3} can be written as

$$r^{-3} = \frac{1}{R^3}\left[1 + \frac{\rho^2}{R} + 2\frac{\vec{R}\cdot\vec{\rho}}{R^2}\right]^{-3/2}$$
$$\approx \frac{1}{R^3}\left[1 - 3\frac{\vec{R}\cdot\vec{\rho}}{R^2}\right] \tag{A.6}$$

Note that in the above expression $\rho \ll R$ and neglecting second and higher order terms in the binomial expansion. The total torque exerted on the satellite becomes

$$\vec{L}_g = -\frac{\mu}{R^3}\int_m \left[1 - 3\frac{\vec{R}\cot\vec{\rho}}{R^2}\right](\vec{\rho}\times\vec{R})dm$$
$$= -\frac{\mu}{R^3}\int_m \vec{\rho}dm \times \vec{R} + \frac{3\mu}{R^3}\int_m \frac{1}{R^2}(\vec{R}\cdot\vec{\rho})(\vec{\rho}\times\vec{R})dm \tag{A.7}$$

Since the origin of the body axes coincides with the center of mass of the satellite,

$$\int_m \rho dm = 0 \tag{A.8}$$

so that

$$\vec{L}_g = \frac{3\mu}{R^3}\int_m \frac{1}{R^2}(\vec{R}\cdot\vec{\rho})(\vec{\rho}\times\vec{R})dm \tag{A.9}$$

Now, choose the orbit reference axes with the origin at the center of mass of the satellite. Note that Z_0 is directed from the center of mass of the satellite to the center of the Earth, and X_0 is in the orbit plane along the forward direction normal to Z_0, and Y_0 is normal to the orbit plane.

The attitude of the satellite can be identified by three successive rotations from the orbit reference axes in the following sequence: ψ about Z_0, θ about the newly displaced Y_0 axis and ϕ about the final position of axis $X_0 \equiv X$. Therefore, $\vec{R}$ can be expressed as

$$\vec{R} = R(c_1\vec{i} + c_2\vec{j} + c_3\vec{k}) \tag{A.10}$$

where $c_1,\ c_2\, c_3$ denote:

$c_1 \equiv sin\theta$
$c_2 \equiv -cos\theta sin\phi$
$c_3 \equiv -cos\theta cos\phi$

Substituting into the torque equation and integrating, the components of the gravity gradient torque along the body axes can be written as $\left[Note: \Omega^2 = \mu/R^3\right]$

$$L_{g_x} = \frac{3\mu}{R^3}\int (c_1x + c_2y + c_3z)c_3y - c_2z)dm \tag{A.11}$$

$$= \frac{3\mu}{R^3}\left[\int (x^2 + y^2)dm - \int (x^2 + z^2)dm\right]c_2c_3 \tag{A.12}$$

$$= \frac{3}{2}\Omega^2(I_z - I_y)sin2\phi cos^2\theta \tag{A.13}$$

$$L_{g_y} = \frac{3\mu}{R^3}\int (c_1x + c_2y + c_3z)c_1z - c_3x)dm \tag{A.14}$$

$$= \frac{3\mu}{R^3}\left[\int (y^2 + z^2)dm - \int (x^2 + y^2)dm\right]c_1c_3 \tag{A.15}$$

$$= \frac{3}{2}\Omega^2(I_z - I_x)cos\phi sin2\theta \tag{A.16}$$

$$L_{gz} = \frac{3\mu}{R^3}\int (c_1x + c_2y + c_3z)c_3x - c_1y)dm \tag{A.17}$$

$$= \frac{3\mu}{R^3}\left[\int (x^2 + z^2)dm - \int (y^2 + z^2)dm\right]c_2c_3 \tag{A.18}$$

$$= \frac{3}{2}\Omega^2(I_x - I_y)sin\phi sin^2\theta \tag{A.19}$$

In summary, we can write

(a) Gravity Gradient Torque (GGT)

From orbital dynamics [1], the magnitude of the position vector R can be expressed in terms of the eccentricity e, w3mi-major axis a and true anomaly f as

$$R = \frac{a(1 - e^2)}{(1 + ecosf)} \tag{A.20}$$

The orbital period is

$$p = 2\pi/\omega_{orbit} = 2\pi\sqrt{a^3/\mu} \tag{A.21}$$

It can be shown that the GGT can be written as [2],

$$\vec{L}_G^b = \left[3\omega_{orbit}^2(1 + ecosf)^3/(1 - e^2)^3\right]\left(\frac{\vec{R}^b}{R}\times\right)I\left(\frac{\vec{R}^b}{R}\right) \tag{A.22}$$

$$= O(\omega_{orbit}^2) \tag{A.23}$$

if $e \ll 1$ and the inertia matrix I is not approximately an identity matrix.

It can be shown that the GGT can be expressed as a product of attitude-dependent terms with a higher frequency and the orbit-dependent terms with the relatively lower frequency [2]. This is useful later in the development of the solution.

Appendix B
Geomagnetic Torque (GMT) Torque

Similarly, the Geomagnetic Torque Torque (GMT) can be written as follows: An earth satellite interacts with the geomagnetic field resulting in a torque

$$\vec{L}_M = \vec{V}_M \times \vec{B} \tag{B.1}$$

where $\vec{B}$ = geomagnetic field and $\vec{V}_M$ = magnetic moment of the spacecraft. The latter could arise from any current-carrying devices in the satellite payload or by eddy currents in the metal structure, which cause undesirable torques. On the other hand, the vehicle magnetic moment could also be purposely generated by passing an electric current through an onboard coil to create a torque for attitude control. This has been discussed in Chap. 8.

The geomagnetic field modeled as a dipole has the form

$$\vec{B} = (\mu_B/R^5)\left[R^2\vec{e}_B - 3(\vec{e}_B \cdot \vec{R})\vec{R}\right] \tag{B.2}$$

where $\vec{e}_B$ is a unit vector in the direction of the geomagnetic dipole axis, which is inclined about 11.5 degrees from the geophysical polar axis. $\vec{R}$ is the satellite position vector and $\mu_B = 8.1 \times 10^{25} gauss - cm^3$. The GMT can be written in the form [2]

$$\vec{L}^b_M = \left[\vec{V}^b_M \times\right] C_{ib}(\mu_B/R^5)\left[R^2\vec{e}^i_B - 3(\vec{e}^i_B \cdot \vec{R}^i)\vec{R}^i\right] \tag{B.3}$$

Although neither the geomagnetic field nor the body magnetic moment can be precisely determined in general, modeling them as dipoles is sufficiently accurate for our purpose.

Two points are worthy of note. First, both GGT and GMT are of order of ω^2_{orbit}, provided that the eccentricity is not too high and if the satellite mass distribution is not too nearly spherical. Second, both GGT and GMT can be expressed in a form separating the attitude and orbital frequencies.

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Appendix C
Floquet's Theory

Floquet's theory [3] establishes the form and nature of the solutions of a linear differential equation of *any* order, with coefficients which are periodic functions of a fixed period T. The theory is sufficiently explained if we limit its application to a differential equation of second order.

Let $u_1(t)$ and $u_2(t)$ be any linearly independent solutions of the differential equation We can write the general solution as

$$U(t) = Au_1(t) + Bu_2(t) \tag{C.1}$$

where A and B are arbitrary constants. Note that the coefficients of the equation are periodic functions with period T. Therefore, both $u_1(t+T)$ and $u_2(t+T)$ are also solutions of the equation. Hence these functions can be expressed linearly in terms of the fundamental set as follows.

$$u_1(t+T) = a_1u_1(t) + a_2u_2(t); \quad u_2(t+T) = b_1u_1(t) + b_2u_2(t) \tag{C.2}$$

The general solution can be written as

$$U(t+T) = (Aa_1 + Bb_1)u_1(t) + (Aa_2 + Bb_2)u_2(t) \tag{C.3}$$

$$U(t+T) = kU(t) \tag{C.4}$$

A and B must satisfy the equations

$$Ak = Aa_1 + Bb_1 \tag{C.5}$$

$$Bk = Aa_2 + Bb_2 \tag{C.6}$$

These are homogenous equations in A and B. The necessary and sufficient conditions for the existence of nontrivial solutions is given by

$$det\begin{pmatrix} a_1 - k & b_1 \\ a_2 & b_2 - k \end{pmatrix} = 0 \tag{C.7}$$

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If k is one of the roots of this equation, then the general solution of the differential equation will satisfy Eq. C.4. Let us now write $k = e^{\lambda T}$ and define the function

$$W(t) = e^{-\lambda t} U(t) \tag{C.8}$$

We then have, from Eq. C.4,

$$W(t+T) = e^{-\lambda(t+T)} U(t+T) = e^{-\lambda t} U(t) = W(t) \tag{C.9}$$

Therefore, the differential equation has a solution of the form

$$U(t) = e^{\lambda t} W(t) \tag{C.10}$$

where $W(t)$ is a periodic function.

Proceeding in a similar manner, Floquet generalized the result to a linear differential equation of any order with coefficients that are periodic in one period. He showed that the solutions have the form of a product of an exponential function and a periodic function.

The principal difficulty concerns the determination of λ, which is called the *Floquet exponent*. If $\lambda = 0$, the solution is periodic. Otherwise, it is either stable, if $\lambda < 0$, and unstable if $\lambda > 0$. Usually, λ cannot be determined analytically. One has to resort to asymptotic approximations or numerical approaches to determine λ.

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