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Tractability of Multivariate Problems

Volume II: Standard Information for Functionals

European Mathematical Society

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For (für) my parents Esther and Heinz.

For (dla) my grandson Mateusz Woźniakowski.

Today, Mateusz is almost two years old.
But we hope that one day he will read
the book and join his older brother
Filip in solving some of the
remaining open problems.

Preface

This is the second volume of a three-volume set comprising a comprehensive study of the tractability of multivariate problems. The subjects treated in the three volumes can be briefly characterized as follows.

- Volume I [222]: we primarily studied multivariate problems specified by *linear operators* and algorithms that use arbitrary linear information given by arbitrary *linear functionals*.

- Volume II: we study multivariate problems specified by *linear functionals* and a few selected *nonlinear functionals*, and algorithms that use *standard* information given by *function values*.

- Volume III: we will study multivariate problems specified by *linear operators* and a few *nonlinear operators*, and algorithms that use *standard* information given by *function values*.

The problems studied in the three volumes are defined on spaces of d-variate functions. It often happens in computational practice that d is very large, perhaps even arbitrarily large. By tractability we mean that we can approximate the d-variate problem with error at most ε and with cost that is *not* exponential either in d or in ε^{-1}. Tractability has been studied since the 1990s, see [348], [349].

We study tractability in different settings. Each setting is specified by the definition of the error and the cost of an algorithm. We present tractability results in the worst case, average case, probabilistic and randomized settings. We do this for the absolute, normalized and relative error criteria.

There are many ways of measuring the lack of exponential dependence; therefore, we have various notions of tractability. Examples include polynomial tractability, T-tractability and weak tractability. The reader is referred to the first two chapters of Volume I for an overview and motivation of tractability studies.

Many multivariate problems specified by linear functionals suffer from the *curse of dimensionality*. This means that even the best possible algorithm must use exponentially many (in d) function values to have error at most ε. The *curse of dimensionality* is usually present for linear functionals defined over standard (unweighted) spaces. In this case all variables and groups of variables play the same role. The curse of dimensionality can often be vanquished if we switch to *weighted* spaces, in which we monitor the importance of all variables and groups of variables by sufficiently decaying weights. As in Volume I, we want to find necessary and sufficient conditions on weights to get various kinds of tractability.

Only standard information (or its analogues) makes sense for the approximation of linear functionals. Standard information was not systematically studied in Volume I.

Therefore the tractability results presented in Volume I for linear information are irrelevant for linear functionals.

The proof techniques for linear and standard information are quite different. For linear information and linear operators defined over Hilbert spaces, tractability depends on the singular values of a corresponding problem. For standard information the situation is much more complex and tractability results depend very much on the specific spaces and linear functionals. It is relatively easy to establish upper error bounds; however, many of these bounds are obtained by *non-constructive* arguments. It is especially hard to establish meaningful lower error bounds. Here, the concept of *decomposable* reproducing kernels is helpful, allowing us to find matching lower and upper error bounds for some linear functionals. We can then conclude tractability results from such error bounds.

Tractability results for linear functionals are very rich in possibilities, and almost anything can happen for linear functionals. They can be *trivial*, since there are Hilbert spaces of infinite dimension for which all linear functionals can be solved with arbitrary small error by using just one function value. They can be *very hard*, since there are Hilbert spaces for which all non-trivial linear functionals suffer from the curse of dimensionality. The last two properties hold for rather esoteric Hilbert spaces. For "typical" Hilbert spaces some linear functionals are easy and some are hard. One of the main challenges is to characterize which linear functionals are tractable and which are not.

Volume II consists of twelve chapters numbered from 9 to 20 since Volume I has the first eight chapters. We comment on their order and contents. We decided to start with a chapter on discrepancy and integration. The notion of discrepancy is simple and beautiful with a clear geometrical meaning. It is striking in how many areas of mathematics discrepancy plays an important role. In particular, discrepancy is intimately related to integration and we thoroughly explain these relations in Chapter 9. Many people would claim that integration is the most important multivariate problem among linear functionals since it appears in many applications such as finance, physics, chemistry, economic, and statistics. There is a tremendous need to compute high dimensional integrals with d in the hundreds and thousands, see Traub and Werschulz [306]. If so, discrepancy is also very important. Moreover, it connects us to many other mathematical areas. That is why we decided to start from discrepancy and explain its relation to integration from the very beginning. We hope the reader will appreciate our decision.

The order of the next chapters is as follows. We begin with the worst case setting. In Chapter 10 we study general linear functionals, whereas in Chapter 11 we study linear functionals specified by tensor products. In Chapter 11 we explain the idea of decomposable kernels and present lower error bounds. We use these lower bounds to obtain necessary conditions on tractability, as well as to show that the curse of dimensionality indeed occurs for many linear functionals. This also serves as a motivation for switching to weighted spaces in Chapter 12. We present a number of necessary tractability conditions in terms of the behavior of weights. As in Volume I, we concentrate on product and finite-order weights.

In Chapter 13 we analyze the average case setting. For linear functionals, there is a pleasing relation between the average case and worst case settings. Knowing this relation, we can translate all tractability results from the worst case setting to the average case setting. So there is no need to do additional work in the average case setting, and so this chapter is relatively short. We stress that such a relation is only present for linear functionals; we will not be so lucky in Volume III with linear operators.

In Chapter 14 we study the probabilistic setting. Here we also have a surprising and pleasing relation to the average case setting. Since the average case is related to the worst case, we conclude that the probabilistic setting is also related to the worst case setting. This means that, with some care, we can translate all tractability results from the worst case to the probabilistic setting. We also study the relative error. We show that only negative results hold for the relative error in the worst case and average case settings, and positive results are only possible in the probabilistic setting.

The relations between the average case, probabilistic and worst case settings mean that it is enough to study tractability in the worst case setting. That is why in the next two long chapters, Chapter 15 and Chapter 16, we return to the worst case setting. Our emphasis is on constructive results, since many tractability results presented so far have been based on non-constructive arguments. Chapter 15 is on the Smolyak/sparse grid algorithms for unweighted and weighted tensor product linear functionals. These algorithms are very popular. Many people have been analyzing error bounds of the Smolyak/sparse grid algorithms with the emphasis on the best order of convergence. The dependence of the error bounds on the number d of variables has been addressed only in a few papers. Of course, this dependence is crucial for tractability which is our emphasis in this chapter.

In Chapter 16 we return to multivariate integration. This problem was analyzed earlier in this book. However, this was usually done as an illustration or specification of general tractability results. We analyze multivariate integration for the Korobov spaces of smooth and periodic functions in the first part of Chapter 16. Our emphasis is on constructive lattice rules. We present the beautiful CBC (component-by-component) algorithm that efficiently computes a generating vector of the lattice rule. The history of this algorithm is reported in the introduction of Chapter 16. In the second part of Chapter 16 we exhibit relations between Korobov and Sobolev spaces. We show how the shifted lattice rules with the generating vectors computed by the CBC algorithm can be used for non-periodic functions from the Sobolev space.

In Chapter 17 we turn our attention to the randomized setting. We first report when the standard Monte Carlo algorithm for multivariate integration leads to tractability. It is well known that the rate of convergence of the standard Monte Carlo algorithm is independent of d. However, it is often overlooked that since the randomized error of the standard Monte Carlo algorithm depends on the variance of a function and the variance may depend on d, then tractability is not necessarily achieved. Indeed, this is the case for a number of standard spaces. Again, weighted spaces can help. Tractability conditions on weights for the standard Monte Carlo algorithm are usually more lenient than for the best algorithms in the worst case setting. We also discuss how importance sampling can help to relax tractability conditions or even guarantee tractability for unweighted spaces. In the final section, we discuss how we can approximate the local solution of the Laplace equation by algorithms based on a random walk.

In Chapter 18 we study tractability of a few selected nonlinear functionals in the worst case and randomized settings. We study a nonlinear integration problem where we integrate with respect to a partially known density, the local solution of Fredholm integral equations, global optimization, and computation of fixed points and computation of volumes, primarily of convex bodies.

Finally in Chapter 19, we briefly mention two generalizations of the material of the previous chapters. The first generalization is when we switch from d-variate problems with finite (and maybe arbitrarily large) d to problems for which $d = \infty$. We illustrate this point for path integration and integration over a Sobolev space of functions depending on infinitely many variables. The second generalization is when we switch from computations performed on a classical computer to computations performed on a (future) *quantum* computer and check how tractability study can be done in the quantum setting.

In Chapter 20 we present a summary of tractability results for multivariate integration defined over three standard weighted Sobolev spaces. We cover four settings:

- worst case,
- average case,
- probabilistic,
- randomized,

and three error criteria:

- absolute,
- normalized,
- relative.

Many specific results presented in this volume have been already published and we tried to carefully report the authors of these results in each chapter and additionally in the Notes and Remarks of each chapter. In fact, each chapter is based on a single paper or a few papers although in many cases we needed to generalize, synthesize or modify the existing results. There are also many new results. Again all this is described in the Notes and Remarks.

In the course of this volume we present a number of open problems. In Volume I we have 30 open problems, and so we started the count of new problems in Volume II from 31. The last open problem has the number 91 so there are 61 open problems in Volume II. The list of open problems is in Appendix D. We call it Appendix D since there are three appendices A, B and C in Volume I. We hope that the open problems will be of interest to a general audience of mathematicians and many of them will be solved soon. In this way, research on tractability will be further intensified.

We realize that this volume is very long since it has more than 650 pages. Despite this book's length, we did not cover some issues and tractability of linear functionals will be continued in Volume III. The reason for this is that there are interesting relations between linear functionals and some linear operators. These relations allow us to apply tractability results for linear operators to linear functionals. Since linear operators for standard information will be thoroughly studied in Volume III, we have to wait to present relations between linear functionals and operators, as well as the corresponding tractability results, till Volume III.

We decided to be repetitious in a number of places. This is the case with some notation as well as with some assumptions in the successive theorems or lemmas. This was done to help the reader who will not have to flip too many pages and look for the meaning of the specific notation or the specific assumption. We believe that our approach will be particularly useful after the first reading of the book when the reader wants to return to some specific result without remembering too much about the hidden assumptions and notation used in the book.

At the expense of some repetitions, we tried to write each chapter as much independent as possible of the other chapters. We hope that the reader may study Chapter n without knowing the previous $n-1$ chapters. We also think that even the last chapter with the summary of tractability results should be understood with only some knowledge of terminology already presented in Volume I.

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Contents

Chapter 9
Discrepancy and Integration

9.1 Introduction

The purpose of this introductory chapter is to show that discrepancy and multivariate integration over some classes of functions are ultimately related. There are many ways to define discrepancy and, as we shall see here, each of them corresponds to multivariate integration over a specific class. For standard discrepancies, these classes of functions are Sobolev spaces with smoothness measured by first mixed derivatives.

The notion of *discrepancy* goes back to the work of Weyl [344] in 1916 and van der Corput [34], [35] in the 1930s. Discrepancy is a quantitative measure of the uniformity of the distribution of points in d-dimensional Euclidean space. Today we have various notions of discrepancy, and there are literally thousands of papers studying different aspects of discrepancy. Research on discrepancy is very intensive, and the reader is referred to the books by Beck and Chen [9], Beck and Sós [11], Chazelle [23], Drmota and Tichy [61], Matoušek [184], Niederreiter [201], Sloan and Joe [273], Strauch and Porubský [290], Tezuka [301], and Triebel [312]. The reader is also referred to a recent book [27] *Analytic Number Theory: Essays in Honor of Klaus Roth* edited by Chen, Gowers, Halberstam, Schmidt and Vaughan [27], which has many surveys of discrepancy as well as to a recent book of Lemieux [170] *Monte Carlo and Quasi-Monte Carlo Sampling*, which covers randomized and deterministic algorithms for multivariate integration. We also want to add that Dick and Pillichshammer [53] are finishing their research monograph *Digital Nets and Sequences; Discrepancy and Quasi-Monte Carlo Integration*, where the reader may find the current state of the art for discrepancy and multivariate integration. Their book presents many constructions of points that nearly minimize the discrepancy or, equivalently, the error for multivariate integration for Quasi-Monte Carlo algorithms.

Various notions of discrepancy are widely used and studied in many areas of mathematics such as number theory, approximation, stochastic analysis, combinatorics, ergodic theory and numerical analysis. The notions of discrepancy are related to Sobolev spaces, Wiener measure, VC (Vapnik–Chervonenkis) dimension and Ramsey theory, see the books just mentioned, as well as papers by Frank and Heinrich [68], Larcher [165], Niederreiter and Xing [202] and [346].

We discuss first the L_2 discrepancy and turn later to the case of L_p discrepancy for general $p \in [1, \infty]$. The case $p = \infty$ is called the star discrepancy, and is usually the most challenging.

Many L_2 discrepancies are defined over the d-dimensional unit cube $[0, 1]^d$. Probably the most celebrated discrepancy in the L_2 norm is the discrepancy anchored at the origin or at 0. In this case, for $x = [x_1, \dots, x_d] \in [0, 1]^d$ we consider discrepancy of the sets

$$[0, x) = [0, x_1) \times [0, x_2) \times \cdots \times [0, x_d).$$

We know sharp lower and upper bounds for the minimal discrepancy anchored at 0. More precisely, if we use n points then it is of order $n^{-1}(\ln n)^{(d-1)/2}$. The lower bound was established by Roth [254] and the upper bound by Roth [255] and Frolov [71]. The original proofs of the upper bounds were not fully constructive, and we needed to wait until 2002 when Chen and Skriganov [30] presented a construction of n points satisfying the upper bound. The construction of points and sequences with L_2 discrepancy anchored at 0, (or more generally, with L_p discrepancy anchored at 0 for $p \in [1, \infty])$ of order $n^{-1}(\ln n)^{c(d-1)}$ for some positive c is still a challenging problem. There are beautiful theories explaining how to achieve this goal, see the books which we already mentioned above.

It is known that the minimal discrepancy anchored at 0 of n points is the same as the minimal worst case error of algorithms using n function values for approximating integrals from the unit ball of the Sobolev space anchored at zero (or one). For $d = 1$, the Sobolev space anchored at $\beta \in [0, 1]$ consists of absolutely continuous functions vanishing at β whose first derivatives are in the space $L_2([0, 1])$. For $d \geq 2$, the Sobolev space anchored at $\beta = [\beta_1, \dots, \beta_d] \in [0, 1]^d$ is the d-fold tensor product of the Sobolev spaces anchored at β_j. In particular, this space consists of functions vanishing at any d-dimensional vector x having one component equal to β_j for some $j \in [d] := \{1, 2, \dots, d\}$. All functions in this space are differentiable with respect to all variables and the resulting partial derivatives are in the space $L_2([0, 1]^d)$. The Sobolev space anchored at β is the Hilbert space with the reproducing kernel given by

$$K_d(x, y) = \prod_{j=1}^{d} \left[1 + \tfrac{1}{2}\big(|x_j - \beta_j| + |y_j - \beta_j| - |x_j - y_j|\big)\right].$$

We are not sure who first realized the equality between the minimal discrepancy anchored at 0 and the minimal worst case errors for multivariate integration for the Sobolev space anchored at 0 (or 1), but this result can be easily deduced from Hlawka's [135] and Zaremba's [362] identity, see also Section 3.1.5 of Volume I.

We achieve the same discrepancy and the worst case integration error if we use the points t_j for the discrepancy anchored at 0 and the same points for multivariate integration over the Sobolev space anchored at 1. For the Sobolev space anchored at 0 we should use the points $1 - t_j$, component-wise. That is why the construction of points minimizing the discrepancy anchored at 0 and the integration error for these Sobolev spaces is the same.

We also study the L_2 *discrepancy anchored at* α for $\alpha \in [0, 1]^d$. In this case, we consider the discrepancy of the sets

$$[\min(\alpha_1, x_1), \max(\alpha_1, x_1)) \times \cdots \times [\min(\alpha_d, x_d), \max(\alpha_d, x_d)).$$

Then the minimal discrepancy anchored at α is the same as the minimal worst case error of multivariate integration for the Sobolev space anchored at α. More precisely, we achieve the same error if we use the points t_j for discrepancy anchored at α and the points

$$\tau_j = (\alpha - t_j) \bmod 1$$

for multivariate integration. Here, the mod operation is applied component-wise. Furthermore, we show that if $\alpha \in (0,1)^d$ then the change of the points between discrepancy and multivariate integration is necessary.

Then we consider the L_2 *quadrant discrepancy anchored at* α whose analog in the L_∞ norm was studied by Hickernell, Sloan and Wasilkowski [123]. This is the discrepancy of the sets

$$[w_1(x), z_1(x)) \times \cdots \times [w_d(x), z_d(x))$$

with

$$[w_j(x), z_j(x)) = [0, x_j) \ \text{ if } x_j < \alpha_j, \quad \text{and} \quad [w_j(x), z_j(x)) = [x_j, 1) \ \text{ if } x_j \geq \alpha_j.$$

The minimal quadrant discrepancy anchored at α is the same as the minimal worst case error of multivariate integration for the Sobolev space anchored at α with no change of points.

The next example of L_2 discrepancy is the *extreme* or *unanchored discrepancy* proposed by Morokoff and Caflisch [191]. This is the discrepancy of the sets

$$[x, y) \quad \text{for } x \leq y, \text{ component-wise},$$

with $x, y \in [0,1]^d$. Then the minimal unanchored discrepancy is the same as the minimal worst case errors of multivariate integration for the Sobolev subspace anchored at 0 consisting of periodic functions with period 1. That is, $f(x) = 0$ if one component of x is either 0 or 1, see [221].

Having these relations between L_2 discrepancy and multivariate integration, it is natural to ask if such relations hold in general. The answer is *yes*, see [223], if we define the L_2 discrepancy for measurable sets $B(t)$ that are subsets of $\mathbb{R}^d$ for $t \in D \subseteq \mathbb{R}^{\tau(d)}$ for some integer $\tau(d)$. Here, we assume that t is distributed according to some density ϱ, so that $\int_D \varrho(t)\,\mathrm{d}y = 1$.

For specific choices of $B(t)$, we obtain the L_2 discrepancy anchored at 0 or at α, the L_2 quadrant discrepancy anchored at α or the unanchored discrepancy. We may also obtain the ball discrepancy and periodic ball discrepancy studied by Beck [8], Beck and Chen [10], Chen [26], Montgomery [190] and Travaglini [309]. The ball discrepancy may be defined over a bounded set or over the whole space $\mathbb{R}$.

The L_2 discrepancy for general sets $B(t)$ is called the *B-discrepancy*. Under natural assumptions on $B(t)$, see (9.44) and (9.48), we prove that the minimal B-discrepancy is the same as the minimal worst case error of multivariate integration for the unit ball of the Hilbert space $H(K_d)$ with the reproducing kernel K_d given by the formula

$$K_d(x, y) = \int_D 1_{B(t)}(x)\, 1_{B(t)}(y)\, \varrho(t)\,\mathrm{d}t \quad \text{for } x, y \in \mathbb{R}^d. \tag{9.1}$$

Furthermore, we obtain the same errors if we use the same points for both discrepancy and multivariate integration. Hence, if we take $B(t)$ corresponding to the discrepancy anchored at α for $\alpha \in (0,1)^d$ then we have a double relation to multivariate integration. The first relation is for the Sobolev space anchored at α with the need of the point change, and the second relation is for the space $H(K_d)$ which is similar to the Sobolev space of periodic functions anchored at 0 which may be, however, discontinuous at α with no need of the point change.

One can also ask the opposite question whether multivariate integration over reproducing kernel Hilbert spaces is always related to L_2 discrepancy for some sets $B(t)$. The answer is now *no*, which simply follows from the fact that not every reproducing kernel has the form (9.1). Note that if K_d is given by (9.1) then $K_d(x,y) \in [0,1]$. There are kernels that can take arbitrary values, including negative values as well. This holds, for example, for Korobov spaces that have been extensively studied in many papers, see also Volume I.

We indicated so far that the minimal L_2 discrepancy for sets $B(t)$ is related to the minimal *worst case errors* of multivariate integration for the unit ball of $H(K_d)$ with K_d given by (9.1). It is also possible to show that the minimal L_2 discrepancy for sets $B(t)$ is equal to the minimal *average case error* of multivariate integration for some normed space F_d that is equipped with a zero-mean probability measure μ_d. We should take the space F_d such that $L(f) = f(x)$ are well defined bounded linear functionals for all x, and the measure μ_d whose covariance function is K_d given by (9.1). For the L_2 discrepancy anchored at α, and for the quadrant and unanchored discrepancy, we may take F_d as the space of continuous functions with the max norm, and μ_d as the zero-mean Gaussian measure with the covariance function K_d. The first such relation to the average case setting was presented in [346]. More precisely, it was shown that the minimal discrepancy anchored at 0 of n points is the same as the minimal average case error of algorithms using n function values. These algorithms approximate the integrals for the space of continuous functions with the max norm and equipped with the Wiener sheet measure. In this case, we obtain the same errors if we use the points t_j for discrepancy, and the points $1 - t_j$ for multivariate integration.

In this chapter we also discuss bounds on the L_2 discrepancy anchored at 0, focusing on their dependence on d. The factors in the lower and upper bounds on the minimal discrepancy anchored at 0 depend on d, and the exact form of this dependence is *not* known. Even the asymptotic constants, for which $d \ge 2$ is fixed and n tends to infinity, are not known. The minimal discrepancy anchored at 0 is at most equal to $3^{-d/2}$ and this bound is sharp for $n = 0$. The case $n = 0$ corresponds to the *initial* discrepancy anchored at 0, which is exponentially small for large d. In terms of the corresponding integration problem, this means that the boundary conditions, $f(x) = 0$ if one component of x is zero (or one), for functions f from the unit ball of the Sobolev space anchored at 0 (or 1) imply that their integrals are at most $3^{-d/2}$. This may indicate that this L_2 discrepancy and multivariate integration are not properly normalized for large d. We can remove the boundary conditions and switch to the *weighted* discrepancy anchored at 0, and to multivariate integration for the *weighted* Sobolev space anchored at 0 (or 1). Again the minimal weighted discrepancy anchored at 0 of n points is the same as the minimal worst case error of algorithms using n function values for approximating the integrals for the unit ball of the weighted Sobolev space anchored at 0 (or 1).

We use weights to moderate the influence of all groups of variables, see Volume I and in particular see Section 5.3.2 for a discussion of weights. As we know from Volume I, *finite-order weights*, see [54], allow us to model functions that are sums of functions of at most ω variables, with ω independent of d. *Product weights*, see [277], allow us to model functions that depend on the successive variables in a diminishing way.

The weights are especially needed for large d, which occurs (as we already know) in many applications including financial mathematics, physics, chemistry and statistics. In such applications, d is often in the hundreds or thousands. Another type of application is path integration, where $d = \infty$ and a finite (but usually very large) d is obtained by approximation of a path integral.

The choice of weights is, in general, a delicate problem. We believe that the weights should be chosen such that the initial weighted discrepancy anchored at 0 is of order 1 or at most polynomially dependent on d. To see this point more clearly, let $\gamma_{d,\mathfrak{u}}$, where $\mathfrak{u} \subseteq [d]$, be the weight that moderates the behavior of the variables in $\mathfrak{u}$. Then the initial weighted discrepancy anchored at 0 is

$$\Big(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|} \Big)^{1/2}.$$

Suppose that $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| < d$ and $\gamma_{d,[d]} = 1$; we then obtain the previous discrepancy anchored at 0 with the initial discrepancy $3^{-d/2}$. Next, suppose that $\gamma_{d,\mathfrak{u}} = 1$ for all $\mathfrak{u} \subseteq [d]$. This corresponds to the case in which all groups of variables are equally important. Then the initial discrepancy anchored at 0 is $(4/3)^{d/2}$, which is now exponentially *large* in d. Neither choice is satisfactory. However, if we have finite-order weights, $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$ and $\gamma_{d,\mathfrak{u}} \in [0,1]$ for $|\mathfrak{u}| \le \omega$, then the initial weighted discrepancy is of order of the number of non-zero weights, which is of order d^{ω}. For product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_{d,j}$ for $\gamma_{d,j} \le \gamma_{d,j-1}$ and $\gamma_{d,j} \in [0,1]$, the initial discrepancy is $\prod_{j=1}^{d} (1 + \frac{1}{3}\gamma_{d,j})^{1/2}$. This is even uniformly bounded in d iff $\sup_d \sum_{j=1}^{d} \gamma_{d,j} < \infty$, and is polynomial in d iff $\sum_{j=1}^{d} \gamma_{d,j} = \mathcal{O}(\ln(d+1))$.

We present several estimates of the L_2 weighted discrepancy anchored at 0, from which we deduce *tractability* of the corresponding multivariate integration problem. More precisely, we study the minimal number of points $n = n(\varepsilon, d)$ for which the weighted L_2 discrepancy anchored at 0 in the d-dimensional case is at most ε, which corresponds to the *absolute error criterion*, or is at most ε times the initial discrepancy, which corresponds to the *normalized error criterion*. The minimal n means that we choose points t_j optimally. The coefficients in the discrepancy formula can be also chosen optimally, or we may fix them to be n^{-1}, as is done for the widely used QMC (Quasi-Monte Carlo) algorithms. As we know, tractability means that $n(\varepsilon, d)$ is *not* exponential in $\varepsilon^{-1} + d$. There are different ways of measuring the lack of exponential dependence. As in Volume I, we discuss

- *weak* tractability[1], in which $\lim_{\varepsilon^{-1}+d\to\infty} (\varepsilon^{-1} + d)^{-1} \ln n(\varepsilon, d) = 0$,
- *polynomial* and *strong polynomial* tractability, in which $n(\varepsilon, d)$ is bounded polynomially in ε^{-1} and d, or only polynomially in ε^{-1},
- T-tractability and *strong* T-tractability, in which $n(\varepsilon, d)$ is bounded by a multiple of some power of $T(\varepsilon^{-1}, d)$ or $T(\varepsilon^{-1}, 1)$. Here T is an increasing function of both arguments and T is *not exponential*, i.e.,

$$\lim_{\varepsilon^{-1}+d\to\infty} (\varepsilon^{-1} + d)^{-1} \ln T(\varepsilon^{-1}, d) = 0.$$

We present a number of tractability results. In particular, for finite-order weights we have polynomial tractability of QMC algorithms for both the absolute and normalized error criteria, see [275]. Furthermore, polynomial tractability can be achieved if we use Niederreiter, Halton, Sobol or shifted lattice rules points. For product weights, (strong) polynomial tractability for QMC algorithms is equivalent to (strong) polynomial tractability. Strong polynomial tractability holds iff

$$\limsup_{d} \sum_{j=1}^{d} \gamma_{d,j} < \infty,$$

and polynomial tractability holds iff

$$\limsup_{d} \sum_{j=1}^{d} \frac{\gamma_{d,j}}{\ln d} < \infty,$$

see [221], [277]. Weak tractability for QMC algorithms is also equivalent to weak tractability and holds, see [85], iff

$$\lim_{d} \sum_{j=1}^{d} \frac{\gamma_{d,j}}{d} = 0.$$

We also cite conditions on T-tractability and strong T-tractability from [85].

Although the main stress in this chapter is on the (weighted) L_2 discrepancy, we also briefly discuss (weighted) L_p discrepancy for $p \in [1, \infty]$ and its relation to multivariate integration. Section 9.8 mainly covers the case $p < \infty$. We also report various tractability results for the absolute and normalized error criteria. The case $p = \infty$ corresponds to the star discrepancy, which is probably the most challenging, see Section 9.9. In this case, the absolute and normalized error criteria coincide since

[1] In this book by ln we always mean the natural logarithm. It may happen that $n(\varepsilon, d) = 0$ for some ε and d, indicating that the problem is trivial for this pair. By convention, we define $\ln 0 = 0$.

the initial star discrepancy is 1. Surprisingly even the unweighted case is polynomially tractable. We also report tractability results for the weighted star discrepancy. In particular, we report a surprising result of Hinrichs, Pillichshammer and Schmid [132] and give a (more or less) complete proof.

As in Volume I, we will propose a number of open problems throughout the successive chapters. Since thirty open problems were presented in Volume I, and we want to number all our open problems consecutively, the open problems in this volume start with the number 31. In this chapter we will propose twelve open problems related to discrepancy and integration, which are numbered from 31 to 42. The reader is also referred to Heinrich [103] for more open problems concerning the star discrepancy.

9.2 L_2 Discrepancy Anchored at the Origin

In this section we discuss L_2 discrepancy for the d-dimensional unit cube $[0,1]^d$, the case that has been most extensively studied. Let $x = [x_1, \dots, x_d] \in [0,1]^d$. The box $[0,x)$ denotes the set $[0,x_1) \times \cdots \times [0,x_d)$ whose (Lebesgue) volume is clearly $x_1 \cdots x_d$. The boxes $[0,x)$ are anchored at the origin, which is why the corresponding concept of L_2 discrepancy is called the L_2 discrepancy anchored at the origin (or at 0). Later we consider more general sets than $[0,x)$ and that will lead to different notions of discrepancy.

For given points $t_1, t_2, \dots, t_n \in [0,1)^d$, we approximate the volume of $[0,x)$ by the fraction of the points t_j which are in the box $[0,x)$. The error of such an approximation is

$$x_1 \cdots x_d - \frac{1}{n} \sum_{j=1}^{n} 1_{[0,x)}(t_j),$$

where $1_{[0,x)}(t_j)$ is the indicator (characteristic) function, which is equal to 1 if $t_j \in [0,x)$, and to 0 otherwise.

Observe that we use equal coefficients n^{-1} in the previous approximation scheme. As we shall see, this corresponds to QMC (Quasi-Monte Carlo) algorithms for multivariate integration. It is a good idea to generalize this approach by allowing arbitrary real coefficients a_j instead of n^{-1}. That is, we approximate the volume of $[0,x)$ by the weighted sum of points in $[0,x)$, with the error given by the *discrepancy function*

$$\operatorname{disc}(x) := x_1 \cdots x_d - \sum_{j=1}^{n} a_j 1_{[0,x)}(t_j). \tag{9.2}$$

The L_2 *discrepancy anchored at the origin* for points $t_1, t_2, \dots, t_n$ and coefficients $a_1, a_2, \dots, a_n$ is just the L_2 norm of the error function (9.2), i.e.,

$$\operatorname{disc}_2(\{t_j\},\{a_j\}) = \left(\int_{[0,1]^d} \left(x_1 \cdots x_d - \sum_{j=1}^{n} a_j 1_{[0,x)}(t_j) \right)^2 \mathrm{d}x \right)^{1/2}. \tag{9.3}$$

We sometimes simply call $\mathrm{disc}_2(\{t_j\},\{a_j\})$ the L_2 *discrepancy* if it is clear from the context that we use the boxes $[0,x)$ as our test sets.

By direct integration, we have the explicit formula of the L_2 discrepancy for the points $t_j = [t_{j,1},\dots,t_{j,d}]$ and the coefficients a_j,

$$\begin{aligned}\mathrm{disc}_2^2(\{t_j\},\{a_j\}) \\ = \frac{1}{3^d} - \frac{1}{2^{d-1}} \sum_{j=1}^{n} a_j \prod_{k=1}^{d} (1-t_{j,k}^2) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \big(1-\max(t_{i,k},t_{j,k})\big)\,,\end{aligned} \tag{9.4}$$

this was first presented by Warnock in [323].

Hence, $\mathrm{disc}_2^2(\{t_i\},\{a_i\})$ can be computed using $O(dn^2)$ arithmetic operations. Faster algorithms for computing L_2 discrepancy for relatively small d have been found by Heinrich [98] and Frank and Heinrich [68].

The major problem for L_2 discrepancy is to find points t_j and coefficients a_j that minimize $\mathrm{disc}_2(\{t_j\},\{a_j\})$. Let

$$\overline{\mathrm{disc}}_2(n,d) = \inf_{t_1,t_2,\dots,t_n \in [0,1)^d} \mathrm{disc}_2(\{t_j\},\{n^{-1}\})$$

and

$$\mathrm{disc}_2(n,d) = \inf_{\substack{t_1,t_2,\dots,t_n \in [0,1)^d \\ a_1,a_2,\dots,a_n \in \mathbb{R}}} \mathrm{disc}_2(\{t_j\},\{a_j\})$$

denote the minimal L_2 discrepancy when we use n points in dimension d. For the minimal L_2 discrepancy $\overline{\mathrm{disc}}_2(n,d)$ we choose optimal t_j for coefficients $a_j = n^{-1}$ whereas for $\mathrm{disc}_2(n,d)$ we also choose optimal a_j.

Observe that for $n = 0$ we do not use any points t_j or coefficients a_j, and so the initial L_2 discrepancy satisfies

$$\overline{\mathrm{disc}}_2(0,d) = \mathrm{disc}_2(0,d) = \left(\int_{[0,1]^d} x_1^2 \cdots x_d^2 \,\mathrm{d}x \right)^{1/2} = 3^{-d/2}. \tag{9.5}$$

Hence, the initial L_2 discrepancy is exponentially small in d. This may suggest that for large d, the L_2 discrepancy is not properly normalized. We shall see later how we can cope with this problem.

9.2.1 Bounds for the L_2 Discrepancy

We briefly discuss bounds on the minimal L_2 discrepancy. For a fixed d, the asymptotic behavior of the minimal L_2 discrepancy as a function of n is known. There exist positive numbers c_d and C_d such that

$$c_d \,\frac{\ln^{(d-1)/2} n}{n} \le \mathrm{disc}_2(n,d) \le \overline{\mathrm{disc}}_2(n,d) \le C_d \,\frac{\ln^{(d-1)/2} n}{n} \tag{9.6}$$

for all $n \geq 2$. The lower bound is a celebrated result of Roth [254] proved in 1954 for $a_j = n^{-1}$. For arbitrary a_j, using essentially the same proof technique, the lower bound was proved by Chen [24], [25]. The upper bound was proved by Roth and Frolov in 1980 again for $a_j = n^{-1}$, see Roth [255] and Frolov [71]. The original proofs of the upper bounds were not fully constructive and based on a randomized argument. In fact, there are many papers along this line. We mention only one of them, where the upper bound is achieved by using randomized digital nets in a prime base, see Cristea, Dick and Pillichshammer [38]. A fully constructive proof was finally given in 2002 by Chen and Skriganov [30].

The essence of (9.6) is that (modulo a logarithmic factor) the L_2 discrepancy behaves asymptotically in n like n^{-1}, independently of d. The power of the logarithmic factor is $(d-1)/2$ and as long as d is not too large this factor is negligible. On the other hand, if d is large, say $d = 360$ as in some financial applications, the factor $\ln^{(d-1)/2} n$ is very important. Indeed, the function

$$\frac{\ln^{(d-1)/2} n}{n}$$

is increasing for $n \leq \exp((d-1)/2)$. The latter number for $d = 360$ is

$$\exp(179.5) \approx 9 \cdot 10^{77}.$$

Obviously, it is impossible to use such a large valued n. Hence when d is large, the good asymptotic behavior of $\mathrm{disc}_2(n,d)$ cannot be really utilized for practical purposes.

Obviously if d is large, then the numbers c_d and C_d from (9.6) are also very important. We do not know much about them. However, we know that the asymptotic constant

$$A_d = \limsup_{n\to\infty} \overline{\mathrm{disc}}_2(n,d)\, \frac{n}{\ln^{(d-1)/2} n}$$

is super-exponentially small in d.

For large d and a relatively small n, we need other estimates on $\mathrm{disc}_2(n,d)$. By a simple averaging argument of (9.4) with respect to t_j for $a_j = n^{-1}$, we have

$$\int_{[0,1]^{nd}} \mathrm{disc}_2^2(\{t_j\},\{n^{-1}\})\, \mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_n = \frac{1}{3^d} - \frac{2}{3^d} + \frac{1}{n2^d} + \frac{1-n^{-1}}{3^d} = \frac{2^{-d}-3^{-d}}{n}.$$

By the mean value theorem we conclude that there are points $t_1, t_2, \ldots, t_n$ for which the square of the L_2 discrepancy is at most $2^{-d}/n$. Therefore

$$\overline{\mathrm{disc}}_2(n,d) \leq \left(\int_{[0,1]^{nd}} \mathrm{disc}_2^2(\{t_j\},\{n^{-1}\})\, \mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_n\right)^{1/2} \leq \frac{2^{-d/2}}{n^{1/2}}. \tag{9.7}$$

The last estimate looks very promising since we have an exponentially small dependence on d through $2^{-d/2}$. However, we should keep in mind that even the initial L_2 discrepancy is $3^{-d/2}$ which is much smaller than $2^{-d/2}$ for large d. We can apply

Chebyshev's inequality and conclude from the last estimate that for any number $C > 1$, the set of sample points

$$A_c = \left\{(t_1, \dots, t_n) \mid \mathrm{disc}_2(\{t_j\}, \{n^{-1}\}) \le C\, 2^{-d/2} n^{-1/2} \right\}$$

has Lebesgue measure at least $1 - C^{-2}$. Hence, for $C = 10$ we have a set of points $t_1, t_2, \dots, t_n$ of measure at least 0.99 for which the L_2 discrepancy is at most $10 \cdot 2^{-d/2}\, n^{-1/2}$. Surprisingly enough, we still do *not* know how to construct such points. Of course, such points can be found computationally. Indeed, it is enough to take points $t_1, t_2, \dots, t_n$ randomly (as independent and uniformly distributed points over $[0, 1]^d$), and compute their discrepancy with $a_j = n^{-1}$. If their discrepancy is at most $10 \cdot 2^{-d/2} n^{-1/2}$ we are done. If not, we repeat random selection of points $t_1, t_2, \dots, t_n$. Then after a few such selections we will get the desired points since the failure of k trials is C^{-2k}, or 10^{-2k} for $C = 10$, which is exponentially small in k.

The bound (9.6) justifies the definition of *low discrepancy* sequences (and points which we do not cover here), for the coefficients a_j equal n^{-1}. Namely, the sequence $\{t_j\}$ is a *low discrepancy sequence* if there is a positive number C_d such that

$$\mathrm{disc}_2(\{t_j\}, \{n^{-1}\}) \le C_d\, \frac{\ln^d n}{n} \quad \text{for all } n \ge 2. \tag{9.8}$$

That is, the L_2 discrepancy of low discrepancy sequences enjoys almost the same asymptotic behavior as the minimal L_2 discrepancy with the only difference being in the power of the logarithmic factor. The search for low discrepancy sequences has been a very active research area, and many beautiful and deep constructions have been obtained. Such sequences usually bear the name of their finders. Today we know the low discrepancy sequences (and points) of Faure, Halton, Hammersley, Niederreiter, Sobol, and Tezuka, as well as (t, m, s) points and (t, m) nets, and lattice or shifted lattice points as their counterparts for the periodic case, see Beck and Chen [9], Beck and Sós [11], Drmota and Tichy [61], Matoušek [184], Niederreiter [201], Sloan and Joe [273] and Tezuka [301]. The reader interested in definitions and many old and new properties of such sequences and points is referred to a recent monograph of Dick and Pillichshammer [53].

There are also points and sequences satisfying (9.8) with more general coefficients a_j than n^{-1} in (9.2). An example is provided by hyperbolic points, see Temlyakov [297] and [329], although in this case we have $\ln^{1.5\,d} n$ instead of $\ln^d n$ in (9.8). Explicit bounds on the L_2 discrepancy for hyperbolic points can be found in [329]. In particular, hyperbolic points $t_1, t_2, \dots, t_n$ and coefficients $a_1, a_2, \dots, a_n$ were constructed such that $\mathrm{disc}_2(\{t_j\}, \{a_j\}) \le \varepsilon$ with

$$n \le 3.304 \left(1.77959 + 2.714 \frac{-1.12167 + \ln \varepsilon^{-1}}{d-1}\right)^{1.5(d-1)} \frac{1}{\varepsilon} \tag{9.9}$$

as well as

$$n \le 7.26 \left(\frac{1}{\varepsilon}\right)^{2.454}. \tag{9.10}$$

Observe an intriguing dependence on d in the bound (9.9). For a fixed large d, and ε tending to zero we have

$$n = \mathcal{O}\left(\frac{1}{\varepsilon}\,(\ln \varepsilon^{-1})^{1.5(d-1)}\right).$$

On the other hand, (9.10) shows that n can be bounded by a polynomial in ε^{-1} for *all* d. Again it looks more surprising than it really is since the initial L_2 discrepancy is exponentially small in d. As we shall see in the next subsection, the exponent 2.454 in (9.10) is not sharp. Clearly, it has to be at least one but its exact value is still not known.

9.2.2 The Exponent of the L_2 Discrepancy

The bound (9.10) of the previous subsection can be rewritten as

$$\mathrm{disc}_2(n,d) \le \frac{2.244}{n^{0.408}} \quad \text{for all } n, d,$$

and is obtained by hyperbolic points.

This bound suggests that we should try to find the smallest (or the infimum of) positive p for which there exists a positive C such that

$$\mathrm{disc}_2(n,d) \le C\, n^{-1/p} \quad \text{for all } n, d \in \mathbb{N}. \tag{9.11}$$

We stress that the last estimate holds for all n and d; hence neither C nor p depend on n and d. Such a minimal p is denoted by p^* and is called the *exponent* p^* of the L_2 discrepancy, see [331].

The bound $p^* \ge 1$ is obvious, since for $d = 1$ we have

$$\mathrm{disc}_2(n,1) = \Theta\left(\overline{\mathrm{disc}}_2(n,1)\right) = \Theta(n^{-1}).$$

For $a_j = n^{-1}$, it is proved by Matoušek [183] that p in (9.11) must be at least 1.0669. This means that the case of arbitrary d is harder than the univariate case, so that the presence of the logarithmic factors in (9.6) cannot be entirely neglected.

For general coefficients a_j, the upper bound $p^* \le 1.4779$ was proved in [331]. Recently it was improved in [336] to

$$p^* \le 1.41274. \tag{9.12}$$

The proof of this upper bound is *non-constructive*. The best constructive bound currently known is $p = 2.454$ from the estimate (9.10) for hyperbolic points, see [329]. It was proved by Plaskota [244] that for hyperbolic points or (more generally) for nested sparse grids points we have $p \ge 2.1933$. Hence, to obtain $p < 2.1933$ we must use points that do not form a sparse grid. This leads us to the first open problems in this volume.

Open Problem 31.

- Improve the bounds for the exponent p^* of the L_2 discrepancy anchored at 0, both for arbitrary coefficients a_j and for the QMC coefficients $a_j = n^{-1}$.

Open Problem 32.

- Construct points t_j for which (9.11) holds with $p < 2$. The cost of construction of $t_1, t_2, \dots, t_n$ must be polynomial in d and n. Of course, we prefer low degree polynomials.

9.2.3 Normalized L_2 Discrepancy

One way to omit the exponentially small initial L_2 discrepancy is to switch to the normalized case. By the *normalized* L_2 discrepancy, we mean

$$\frac{\mathrm{disc}_2(\{t_j\}, \{a_j\})}{\mathrm{disc}_2(0, d)}.$$

That is, we normalize the problem by the initial value of the L_2 discrepancy, which is $3^{-d/2}$. We now define

$$\bar{n}(\varepsilon, d) = \min\{n \mid \overline{\mathrm{disc}}_2(n, d) \le \varepsilon\, \mathrm{disc}_2(0, d)\}, \tag{9.13}$$

$$n(\varepsilon, d) = \min\{n \mid \mathrm{disc}_2(n, d) \le \varepsilon\, \mathrm{disc}_2(0, d)\} \tag{9.14}$$

as the minimal number of points necessary to reduce the initial discrepancy by a factor ε, either with the coefficients $a_j = n^{-1}$ or with optimally chosen a_j. We ask whether $\bar{n}(\varepsilon, d)$ and $n(\varepsilon, d)$ behave polynomially in ε^{-1} and d, or at least not exponentially in ε^{-1} and d. We stress that the polynomial bounds on the L_2 discrepancy that we have presented above are useless for the normalized case, since we now have to compare the minimal L_2 discrepancy to $\varepsilon\, 3^{-d/2}$ instead of ε.

The problem how $\bar{n}(\varepsilon, d)$ and $n(\varepsilon, d)$ depend on d has been partially solved and we now report its solution. First of all, notice that it directly follows from (9.5) and (9.7) that

$$\bar{n}(\varepsilon, d) \le 1.5^d\, \varepsilon^{-2}. \tag{9.15}$$

It was proved in [352], see also [277], that

$$\bar{n}(\varepsilon, d) \ge (1.125)^d\, (1 - \varepsilon^2). \tag{9.16}$$

The bound (9.16) is also valid if all coefficients a_j are non-negative. Hence, we have exponential dependence on d.

For arbitrary a_j, it was proved in [221], see also Chapter 11, that for any positive $\varepsilon_0 < 1$ there exists a positive c such that

$$c\, 1.0628^d \le n(\varepsilon, d) \le 1.5^d\, \varepsilon^{-2} \quad \text{for all } d \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{9.17}$$

Hence, $n(\varepsilon, d)$ goes to infinity exponentially fast in d.

The lower bound 1.0628^d can be slightly improved to 1.0833^d, as we will show in Section 11.5.4 of Chapter 11. Moreover, Plaskota, Wasilkowski and Zhao [248] showed that we can improve the upper bound in (9.17) by replacing 1.5^d by $(4/3)^d$. We will show this in Section 10.7.11 of Chapter 10. Therefore for some positive c and ε_0 we have

$$c\,1.0833^d \le n(\varepsilon, d) \le [4/3]^d\,\varepsilon^{-2} \quad \text{for all } d \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{9.18}$$

It is not known whether 9/8 in (9.16) and 1.0833 in (9.18) can be increased, and whether 1.5 in (9.15) and 4/3 in (9.18) can be decreased. This leads us to the next two open problems.

Open Problem 33.

- For the normalized L_2 discrepancy and $a_j = n^{-1}$, find the largest C_1 and the smallest C_2 for which for any positive ε_0 there exists a positive c such that

  $$c\,C_1^d \le \bar{n}(\varepsilon, d) \le C_2^d\,\varepsilon^{-2} \quad \text{for all } d \text{ and } \varepsilon \in (0, \varepsilon_0).$$

  Today, we know that $C_1 \ge 9/8$ and $C_2 \le 1.5$.

Open Problem 34.

- For the normalized L_2 discrepancy and optimally chosen a_j, find the largest C_1 and the smallest C_2 for which for any positive ε_0 there exists a positive c such that

  $$c\,C_1^d \le n(\varepsilon, d) \le C_2^d\,\varepsilon^{-2} \quad \text{for all } d \text{ and } \varepsilon \in (0, \varepsilon_0).$$

  Today, we know that $C_1 \ge 1.0833$ and $C_2 \le 4/3$.

9.3 Weighted L_2 Discrepancy

As already discussed in the introduction, the L_2 discrepancy anchored at 0 is related to multivariate integration for functions satisfying some boundary conditions that are probably not very common in computational practice. To remove these boundary conditions we need to consider a more general L_2 discrepancy. Furthermore, the integrands may depend differently on groups of variables when d is large. To address this property, we need to consider the *weighted* L_2 discrepancy, which is the subject of this section.

As always, by $[d] := \{1, 2, \dots, d\}$ we mean the set of the first d indices, and by $\mathfrak{u}$ we denote an arbitrary subset of $[d]$, and $|\mathfrak{u}|$ is its cardinality. We are given a sequence

$$\gamma = \{\gamma_{d,\mathfrak{u}}\}_{\mathfrak{u} \subseteq [d],\, d=1,2,\dots,}$$

of non-negative weights. For simplicity, we assume that $\gamma_{d,\mathfrak{u}} \in [0, 1]$.

As in Volume I, for a vector $x \in [0,1]^d$ let $x_{\mathfrak{u}}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ with the components of x whose indices are in $\mathfrak{u}$. For example, for $d = 7$ and $\mathfrak{u} = \{2,4,5,6\}$ we have $x_{\mathfrak{u}} = [x_2, x_4, x_5, x_6]$. We let $\mathrm{d}x_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \mathrm{d}x_j$. By $(x_{\mathfrak{u}}, 1)$, we mean the vector from $[0,1]^d$ with the same components as x for indices in $\mathfrak{u}$ and with the rest of components being replaced by 1. For our example, we have $(x_{\mathfrak{u}}, 1) = [1, x_2, 1, x_4, x_5, x_6, 1]$. Recall that for given points $t_1, t_2, \dots, t_n \in [0,1)^d$ and real coefficients $a_1, a_2, \dots, a_n$, the function $\mathrm{disc}(x_{\mathfrak{u}}, 1)$ is given by (9.2) and takes now the form

$$\mathrm{disc}(x_{\mathfrak{u}}, 1) = \prod_{k \in \mathfrak{u}} x_k - \sum_{j=1}^{n} a_j 1_{[0, x_{\mathfrak{u}})}\left((t_j)_{\mathfrak{u}}\right).$$

The *weighted* L_2 discrepancy anchored at the origin, or simply the L_2 weighted discrepancy, is then defined as

$$\mathrm{disc}_{2,\gamma}(\{t_j\}, \{a_j\}) = \left(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}^2(x_{\mathfrak{u}}, 1)\, \mathrm{d}x_{\mathfrak{u}} \right)^{1/2}. \tag{9.19}$$

Note that if $\gamma_{d,\mathfrak{u}} = 0$ for all $\mathfrak{u}$ with $|\mathfrak{u}| < d$, and $\gamma_{d,[d]} = 1$, then the L_2 weighted discrepancy reduces to the L_2 discrepancy studied before.

By using Warnock's formula (9.4), we obtain an explicit formula for the L_2 weighted discrepancy. Namely,

$$\mathrm{disc}^2_{2,\gamma}(\{t_j\}, \{a_j\}) =$$
$$\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left(\frac{1}{3^{|\mathfrak{u}|}} - \frac{1}{2^{|\mathfrak{u}|-1}} \sum_{j=1}^{n} a_j \prod_{k \in \mathfrak{u}} (1 - t_{j,k}^2) + \sum_{i,j=1}^{n} a_i a_j \prod_{k \in \mathfrak{u}} (1 - \max(t_{i,k}, t_{j,k})) \right).$$

The standard (unweighted) case corresponds to $\gamma = 1$, i.e., $\gamma_{d,\mathfrak{u}} = 1$ for all d and $\mathfrak{u} \subseteq [d]$. Since

$$\sum_{\mathfrak{u} \subseteq [d]} 3^{-|\mathfrak{u}|} = \sum_{k=0}^{d} \binom{d}{k} 3^{-k} = (1 + 3^{-1})^d = \left(\tfrac{4}{3}\right)^d$$

and

$$\sum_{\mathfrak{u} \subseteq [d]} \prod_{k \in \mathfrak{u}} \frac{1 - t_{j,k}^2}{2} = \prod_{k=1}^{d} \left(1 + \frac{1 - t_{j,k}^2}{2} \right),$$

we have

$$\mathrm{disc}^2_{2,\{1\}}(\{t_j\}, \{a_j\}) =$$
$$\left(\frac{4}{3}\right)^d - 2 \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \frac{3 - t_{j,k}^2}{2} + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \left(2 - \max(t_{i,k}, t_{j,k})\right).$$

As before, for an arbitrary sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ let

$$\overline{\mathrm{disc}}_{2,\gamma}(n,d) = \inf_{t_1,t_2,\dots,t_n\in[0,1)^d} \mathrm{disc}_{2,\gamma}(\{t_j\},\{n^{-1}\}), \tag{9.20}$$

$$\mathrm{disc}_{2,\gamma}(n,d) = \inf_{t_1,t_2,\dots,t_n\in[0,1)^d\,,\,a_1,a_2,\dots,a_n\in\mathbb{R}} \mathrm{disc}_{2,\gamma}(\{t_j\},\{a_j\}) \tag{9.21}$$

be the minimal weighted L_2 discrepancies. For $n = 0$, we obtain

$$\overline{\mathrm{disc}}^2_{2,\gamma}(0,d) = \mathrm{disc}^2_{2,\gamma}(0,d) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|}.$$

Observe that for the unweighted case, $\gamma_{d,\mathfrak{u}} = 1$, we have

$$\mathrm{disc}_{2,\{1\}}(0,d) = \left(\tfrac{4}{3}\right)^{d/2},$$

which is exponentially large in d.

For *product weights*, i.e., $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for some $\gamma_{d,j} \in [0,1]$, we have

$$\mathrm{disc}_{2,\gamma}(0,d) = \prod_{j=1}^{d} (1 + \tfrac{1}{3}\gamma_{d,j})^{1/2}.$$

It is easy to check that the initial discrepancy is uniformly bounded in d iff

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

For *product weights independent of* d, i.e., for $\gamma_{d,j} = \gamma_j$, the last condition simply means that $\sum_{j=1}^{\infty} \gamma_j < \infty$.

Hence, the initial L_2 discrepancy is exponentially small in d whereas the unweighted L_2 discrepancy is exponentially *large* in d. Both cases seem to be ill-normalized. We believe that the choice of the weight sequence γ should be such that the initial weighted L_2 discrepancy is of order d^q for some $q \ge 0$.

How small is the minimal weighted discrepancy? To answer this question we can average the square of the weighted L_2 discrepancy for the sample points t_j and coefficients $a_j = n^{-1}$, assuming that t_j are uniformly and independently distributed over $[0,1]^d$. We obtain

$$\begin{aligned}
&\int_{[0,1]^{nd}} \mathrm{disc}^2_{2,\gamma}\left(\{t_j\},\{n^{-1}\}\right) \mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_n \\
&\quad = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left(\frac{1}{3^{|\mathfrak{u}|}} - \frac{1}{2^{|\mathfrak{u}|-1}} \left(\frac{2}{3}\right)^{|\mathfrak{u}|} + \frac{1}{n^2} \left(n \left(\frac{1}{2}\right)^{|\mathfrak{u}|} + (n^2 - n) \left(\frac{1}{3}\right)^{|\mathfrak{u}|} \right) \right) \\
&\quad = \frac{1}{n} \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left(\left(\frac{1}{2}\right)^{|\mathfrak{u}|} - \left(\frac{1}{3}\right)^{|\mathfrak{u}|} \right).
\end{aligned}$$

By the mean value theorem we conclude that

$$\operatorname{disc}_{2,\gamma}(n,d) \le \overline{\operatorname{disc}}_{2,\gamma}(n,d) \le \frac{1}{n^{1/2}} \Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \big(2^{-|\mathfrak{u}|} - 3^{-|\mathfrak{u}|}\big)\Big)^{1/2}. \tag{9.22}$$

Applying Chebyshev's inequality with $C > 1$, we also conclude that the set of sample points

$$\Big\{(t_1,\dots,t_n) \,\Big|\, \operatorname{disc}_{2,\gamma}(\{t_j\},\{n^{-1}\}) \le C\, n^{-1/2}\big(\textstyle\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\big(2^{-|\mathfrak{u}|} - 3^{-|\mathfrak{u}|}\big)\big)^{1/2}\Big\}$$

has Lebesgue measure at least $1 - C^{-2}$. For the unweighted case, $\gamma_{d,\mathfrak{u}} \equiv 1$, we have

$$\operatorname{disc}_{2,\{1\}}(n,d) \le \overline{\operatorname{disc}}_{2,\{1\}}(n,d) \le n^{-1/2} \left((3/2)^d - (4/3)^d\right)^{1/2} \le n^{-1/2}\, 1.5^{d/2}.$$

We now assume product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$. Then dropping the negative terms $-3^{|\mathfrak{u}|}$ in (9.22) we obtain

$$\begin{aligned}
\operatorname{disc}_{2,\gamma}(n,d) \le \overline{\operatorname{disc}}_{2,\gamma}(n,d) &\le \frac{1}{n^{1/2}} \prod_{j=1}^{d} \left(1 + \tfrac{1}{2}\,\gamma_{d,j}\right) \\
&= n^{-1/2} \exp\Big(\sum_{j=1}^{d} \ln\left(1 + \tfrac{1}{2}\,\gamma_{d,j}\right)\Big) \le n^{-1/2} \exp\Big(\frac{1}{2}\sum_{j=1}^{d} \gamma_{d,j}\Big) \\
&= n^{-1/2} (d+1)^{\frac{1}{2}\sum_{j=1}^{d} \gamma_{d,j}/\ln(d+1)}.
\end{aligned}$$

It is easy to check that

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty \tag{9.23}$$

implies that the minimal discrepancies do not depend on d, i.e.,

$$\operatorname{disc}_{2,\gamma}(n,d) = \mathcal{O}(n^{-1/2}) \quad \text{and} \quad \overline{\operatorname{disc}}_{2,\gamma}(n,d) = \mathcal{O}(n^{-1/2})$$

with the factors in the $\mathcal{O}$ notation independent of d. Furthermore if

$$q^* := \limsup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty \tag{9.24}$$

then the minimal discrepancies depend polynomially on d. More precisely, for $q > q^*$ we have

$$\operatorname{disc}_{2,\gamma}(n,d) = \mathcal{O}(d^{\,q/2}\, n^{-1/2}) \quad \text{and} \quad \overline{\operatorname{disc}}_{2,\gamma}(n,d) = \mathcal{O}(d^{\,q/2}\, n^{-1/2}).$$

Furthermore, if

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0 \tag{9.25}$$

then the minimal discrepancies do *not* depend exponentially on d, i.e.,

$$\lim_{d\to\infty} \frac{\ln \mathrm{disc}_{2,\gamma}(n,d)}{d} = \lim_{d\to\infty} \frac{\ln \overline{\mathrm{disc}}_{2,\gamma}(n,d)}{d} = 0.$$

It turns out that (9.23), (9.24) and (9.25) are also necessary conditions to have the minimal discrepancies independent of d, polynomially dependent on d and not exponentially dependent on d, respectively. The first two properties are proved in [221], and the third one in [85]. We will also reprove these results in Chapter 11.

The consequences of the upper and lower bounds on the weighted L_2 discrepancy will be discussed later after we present relations between the weighted L_2 discrepancy and multivariate integration for some Sobolev spaces.

Remark 9.1. We end this subsection by a remark on the *limiting* discrepancy which is formally defined for $d = \infty$, see [277]. Here we assume product weights independent of d, i.e.,

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j \quad \text{for some } \gamma_j \in [0,1].$$

For points $t_i^{(\infty)} = [t_{i,1}, t_{i,2}, \dots] \in [0,1]^\infty$, let $t_i^{(d)} = [t_{i,1}, \dots, t_{i,d}] \in [0,1]^d$ denote their d-dimensional projections. Then the discrepancy $\mathrm{disc}_{2,\gamma}(\{t_i^{(d)}\}, \{n^{-1}\})$ is a non-decreasing function of d. The limiting discrepancy is then defined as

$$\mathrm{disc}_{2,\gamma}(\{t_i^{(\infty)}\}) = \lim_{d\to\infty} \mathrm{disc}_{2,\gamma}(\{t_i^{(d)}\}, \{n^{-1}\}).$$

We have

$$\mathrm{disc}_{2,\gamma}(\{t_i^{(\infty)}\}) < \infty \quad \text{iff} \quad \sum_{j=1}^{\infty} \gamma_j < \infty.$$

We stress that the last condition holds independently of the sample points $t_i^{(\infty)}$. We see once more that the condition $\sum_{j=1}^{\infty} \gamma_j < \infty$ is needed to have a finite limiting discrepancy. Error bounds for quasi-Monte Carlo algorithms in an infinite dimensional setting are studied by Hickernell and Wang [126].

9.3.1 Normalized Weighted L_2 Discrepancy

As for the normalized L_2 discrepancy, we define

$$\bar{n}_\gamma(\varepsilon, d) = \min\{\, n \mid \overline{\mathrm{disc}}_{2,\gamma}(n,d) \le \varepsilon\, \mathrm{disc}_{2,\gamma}(0,d)\,\}, \tag{9.26}$$

$$n_\gamma(\varepsilon, d) = \min\{\, n \mid \mathrm{disc}_{2,\gamma}(n,d) \le \varepsilon\, \mathrm{disc}_{2,\gamma}(0,d)\,\} \tag{9.27}$$

as the minimal number of points necessary to reduce the initial weighted discrepancy by a factor ε, either with the coefficients $a_j = n^{-1}$ or with optimally chosen a_j.

For the unweighted case, $\gamma = \{1\}$, it was proved in [221], see also Section 11.5.4 of Chapter 11, that for any positive $\varepsilon_0 < 1$ there exists a positive c such that

$$c\, 1.0202^d \le n_{\{1\}}(\varepsilon, d) \le \bar{n}_{\{1\}}(\varepsilon, d) \le 1.125^d\, \varepsilon^{-2} \quad \text{for all } d \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{9.28}$$

The upper bound on $n_{\{1\}}(\varepsilon, d)$ can be slightly improved using Plaskota, Wasilkowski and Zhao [248], see again Section 11.5.4 of Chapter 11, and we have

$$n_{\{1\}}(\varepsilon, d) \le \lceil (1.1143\dots)^d\, \varepsilon^{-2} \rceil.$$

Hence, for the normalized L_2 discrepancy, as well as for the normalized weighted L_2 discrepancy with $\gamma_{d,\mathfrak{u}} = 1$, we have an exponential dependence on d, and the corresponding $n(\varepsilon, d)$ and $n_{\{1\}}(\varepsilon, d)$ go exponentially fast with d to infinity. It is now natural to ask what are necessary and sufficient conditions on the weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ to not have an exponential dependence on d, and what we have to assume about γ to guarantee, say, polynomial dependence on d, or no dependence on d at all. We will study these questions later.

9.4 Multivariate Integration

We consider multivariate integration for real functions defined on the d-dimensional unit cube $[0,1]^d$ and belonging to a Hilbert space with a reproducing kernel $K_d\colon [0,1]^d \times [0,1]^d \to \mathbb{R}$. This space is denoted by $H(K_d)$ and its inner product by $\langle \cdot, \cdot \rangle_{H(K_d)}$. The basic information about such spaces can be found in Aronszajn [2] and in Berlinet and Thomas-Agnan [14]. Here, we only mention that $K_d(\cdot, x) \in H(K_d)$ for all $x \in [0,1]^d$, and that $(K_d(x_i, x_j))_{i,j=1,2,\dots,m}$ is a $m \times m$ symmetric positive semi-definite matrix for any choice of m and $x_j \in [0,1]^d$. Furthermore, and this is probably the most important property, for any function $f \in H(K_d)$ and any $x \in [0,1]^d$ we have

$$f(x) = \langle f, K_d(\cdot, x) \rangle_{H(K_d)}.$$

The space $H(K_d)$ is the completion of linear combinations of functions of the form

$$\sum_{j=1}^{m} a_j K_d(\cdot, x_j)$$

for any m, real a_j and $x_j \in [0,1]^d$.

We illustrate the reproducing kernel Hilbert spaces for two examples. For the first example and $d = 1$, we take a number $\beta \in [0,1]$ and define

$$K_1^{\beta}(x, y) = \tfrac{1}{2}\big[|x - \beta| + |y - \beta| - |x - y|\big] \quad \text{for } x, y \in [0,1]. \tag{9.29}$$

Note that for $\beta = 0$, we have

$$K_1^0(x, y) = \tfrac{1}{2}\big[x + y - |x - y|\big] = \min(x, y),$$

whereas for $\beta = 1$, we have

$$K_1^1(x,y) = \tfrac{1}{2}\big[1 - x + 1 - y - |x-y|\big] = 1 - \max(x,y).$$

For an arbitrary β, the kernel K_1^β vanishes for $x \le \beta \le y$ and $y \le \beta \le x$. This property is important for establishing lower bounds on the minimal errors for multivariate integration, see [221] and Chapter 11.

The space $H(K_1^\beta)$ consists of absolutely continuous functions vanishing at β and whose first derivatives are in $L_2([0,1])$. That is,

$$H(K_1^\beta) = \big\{\, f : [0,1] \to \mathbb{R} \;\big|\; f(\beta) = 0,\ f \text{ abs. cont. and } f' \in L_2([0,1]) \,\big\}$$

with the inner product

$$\langle f, g \rangle_{H(K_1^\beta)} = \int_0^1 f'(x)\, g'(x)\,\mathrm{d}x \quad \text{for all } f, g \in H(K_1^\beta).$$

For $d \ge 1$, we take a vector $\beta = [\beta_1, \beta_2, \dots, \beta_d] \in [0,1]^d$, and define $H(K_d^\beta)$ as the d-fold tensor product,

$$H(K_d^\beta) = H(K_1^{\beta_1}) \otimes H(K_1^{\beta_2}) \otimes \cdots \otimes H(K_1^{\beta_d}),$$

with the reproducing kernel given by

$$K_d^\beta(x,y) = \prod_{j=1}^d K_1^{\beta_j}(x_j, y_j) \quad \text{for } x, y \in [0,1]^d.$$

The space $H(K_d^\beta)$ consists of functions such that $f(x) = 0$ if there exists an index $j \in [d]$ such that $x_j = \beta_j$, and which are differentiable with respect to all variables, with first partial derivatives being in $L_2([0,1]^d)$. The inner product is given by

$$\langle f, g \rangle_{H(K_d^\beta)} = \int_{[0,1]^d} \frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} f(x)\, \frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} g(x)\,\mathrm{d}x$$

for $f, g \in H(K_d^\beta)$. The space $H(K_d^\beta)$ is called the *Sobolev space with mixed derivatives of order one anchored at* β, or shortly the *Sobolev space anchored at* β.

As the second example of a reproducing kernel Hilbert space, take an arbitrary weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} \ge 0$. Define the reproducing kernel as

$$K_{d,\gamma}^\beta(x,y) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}\, K_{\mathfrak{u}}^\beta(x,y)$$

with

$$K_{\mathfrak{u}}^\beta(x,y) = \prod_{j \in \mathfrak{u}} K_1^\beta(x_j, y_j) = \prod_{j \in \mathfrak{u}} \frac{|x_j - \beta_j| + |y_j - \beta_j| - |x_j - y_j|}{2}$$

for $x, y \in [0,1]^d$.

For the unweighted case, $\gamma_{d,\mathfrak{u}} = 1$, we have

$$K^{\beta}_{d,\{1\}}(x,y) = \prod_{j=1}^{d} \left(1 + K^{\beta}_{1}(x_j, y_j)\right)$$
$$= \prod_{j=1}^{d} \left[1 + \tfrac{1}{2}\left(|x_j - \beta_j| + |y_j - \beta_j| - |x_j - y_j|\right)\right].$$

The Hilbert space $H(K^{\beta}_{d,\gamma})$ is the sum of tensor product Hilbert spaces $H(K^{\beta}_{\mathfrak{u}})$ for all $\mathfrak{u}$ for which $\gamma_{d,\mathfrak{u}}$ is positive. If all $\gamma_{d,\mathfrak{u}}$ are positive, the inner product is given by

$$\langle f, g\rangle_{H(K^{\beta}_{d,\gamma})} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, \beta)\, \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x_{\mathfrak{u}}, \beta)\, \mathrm{d}x_{\mathfrak{u}}$$

for $f, g \in H(K_{d,\gamma})$ with the notation $\partial x_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \partial x_j$ and $\mathrm{d}x_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \mathrm{d}x_j$. Here, $(x_{\mathfrak{u}}, \beta)$ denotes the d-component vector whose jth component is x_j for $j \in \mathfrak{u}$ and β_j for $j \notin \mathfrak{u}$. In particular, for $\mathfrak{u} = \emptyset$ we have $(x_{\emptyset}, \beta) = \beta$, whereas for $\mathfrak{u} = [d]$ we have $(x_{[d]}, \beta) = x$.

For $\mathfrak{u} = \emptyset$, we have $K^{\beta}_{\emptyset} = 1$ and $H(K^{\beta}_{\emptyset}) = \mathrm{span}(1)$. The term in the inner product corresponding to $\mathfrak{u} = \emptyset$ is equal to $\gamma_{d,\emptyset}^{-1} f(\beta)\, g(\beta)$.

We have a unique decomposition of functions f from $H(K^{\beta}_{d,\gamma})$ given by

$$f = \sum_{\mathfrak{u}\subseteq[d]} f_{\mathfrak{u}} \quad \text{with } f_{\mathfrak{u}} \in H(K^{\beta}_{\mathfrak{u}}),$$

where the $f_{\mathfrak{u}}$'s are mutually orthogonal and

$$\|f\|^2_{H(K_{d,\gamma})} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \, \|f_{\mathfrak{u}}\|^2_{H(K^{\beta}_{\mathfrak{u}})}.$$

If one of the weights is zero, say $\gamma_{d,\mathfrak{u}} = 0$, then the corresponding term $f_{\mathfrak{u}} = 0$ and we interpret $0/0$ as 0. Hence, the inner product is the sum of terms for positive $\gamma_{d,\mathfrak{u}}$ with all $f_{\mathfrak{u}} = 0$ if $\gamma_{d,\mathfrak{u}} = 0$.

Observe that $f_{\mathfrak{u}}$ depends only on variables in $\mathfrak{u}$. In particular $f_{\emptyset} = f(\beta)$, and $f_{\{j\}}(x) = f(\beta_1, \beta_2, \dots, \beta_{j-1}, x_j, \beta_{j+1}, \dots, \beta_d) - f(\beta)$. It is shown in [155] that for any $\mathfrak{u} \subseteq [d]$ we have

$$f_{\mathfrak{u}}(x) = \sum_{\mathfrak{v}\subseteq\mathfrak{u}} (-1)^{|\mathfrak{u}|-|\mathfrak{v}|} f(x_{\mathfrak{v}}, \beta).$$

In general, the Hilbert space $H(K^{\beta}_{d,\gamma})$ is not a tensor product space. This holds if some $\gamma_{d,\mathfrak{u}} = 0$. However, for product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for some $\gamma_{d,j} \in$

[0, 1], we have

$$K_{d,\gamma}^{\beta}(x,y) = \prod_{j=1}^{d} \Big(1 + \gamma_{d,j} K_1^{\beta}(x_j, y_j)\Big)$$
$$= \prod_{j=1}^{d} \left[1 + \tfrac{1}{2}\gamma_{d,j} \big(|x_j - \beta_j| + |y_j - \beta_j| - |x_j - y_j|\big)\right].$$

Hence, $K_{d,\gamma}^{\beta}$ is of the product form, which implies that $H(K_{d,\gamma}^{\beta})$ is a tensor product space.

The space $H(K_{d,\gamma}^{\beta})$ is called the *weighted Sobolev space with mixed derivatives of order one anchored at* β, or shortly, the *weighted Sobolev space anchored at* β.

We are ready to define multivariate integration for functions from a general reproducing kernel Hilbert space $H(K_d)$. First of all, we need to assume that the space $H(K_d)$ consists of integrable functions. To guarantee that multivariate integration is a bounded linear functional, we need to assume that the function

$$h_d(x) = \int_{[0,1]^d} K_d(y,x)\,\mathrm{d}y \quad \text{for } x \in [0,1]^d$$

belongs to $H(K_d)$.

For $f \in H(K_d)$, we define the multivariate integration problem as approximation of

$$I_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x.$$

Since $f(x) = \langle f, K_d(\cdot,x)\rangle_{H(K_d)}$, we can rewrite $I_d(f)$ as

$$I_d(f) = \left\langle f, \int_{[0,1]^d} K_d(\cdot,x)\,\mathrm{d}x \right\rangle_{H(K_d)} = \langle f, h_d\rangle_{H(K_d)}\,.$$

Hence, multivariate integration is equivalent to approximating the inner product with the generator h_d. Clearly,

$$\|I_d\| := \sup_{\|f\|_{H(K_d)}\le 1} |I_d(f)| = \|h_d\|_{H(K_d)}.$$

This means that the norm of the multivariate integration functional I_d is the same as the norm of the function h_d. It is easy to check that

$$\|h_d\|_{H(K_d)}^2 = \int_{[0,1]^{2d}} K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

We approximate $I_d(f)$ by computing function values $f(t_j)$ at some sample points t_j. In general, these points can be chosen *adaptively*, i.e., the choice of t_j may depend on the previously computed function values $f(t_i)$ for $i = 1, 2, \dots, j-1$. Furthermore, knowing $f(t_j)$ for, say, $j = 1, 2, \dots, n$, we may take

$$\varphi_n(f(t_1), f(t_2), \dots, f(t_n))$$

as an approximation of $I_d(f)$ for some, in general, *nonlinear* function $\varphi_n : \mathbb{R}^n \to \mathbb{R}$. It turns out that neither adaption nor nonlinear choices of φ_n help, as proved by Bakhvalov (adaption) and by Smolyak (nonlinear φ_n), see the original paper of Bakhvalov [5] that presents both results. These results can be also found in Chapter 4 of Volume I. Hence, without loss of generality we may restrict ourselves to *linear and non-adaptive* approximations of the form

$$Q_{n,d}(f) = \sum_{j=1}^{n} a_j f(t_j) \tag{9.30}$$

for some real a_j and a priori (non-adaptively) given t_j from $[0,1]^d$. Usually, $Q_{n,d}$ is called a *linear* algorithm. If we let $a_j = n^{-1}$ then

$$Q_{n,d}(f) = \frac{1}{n}\sum_{j=1}^{n} f(t_j)$$

is called a QMC (Quasi-Monte Carlo) algorithm; these formulas are often used in numerical computational practice as approximations of multivariate integrals. This is especially the case when d is large.

The worst case error of $Q_{n,d}$ is defined as the largest error between $I_d(f)$ and $Q_{n,d}(f)$ over the unit ball of $H(K_d)$, so that

$$e^{\text{wor}}(Q_{n,d}; H(K_d)) = \sup_{\substack{f \in H(K_d) \\ \|f\|_{H(K_d)} \le 1}} \left| I_d(f) - Q_{n,d}(f) \right|.$$

Since $I_d - Q_{n,d}$ is linear, the worst case error is obviously the same as the norm $\|I_d - Q_{n,d}\|$. Furthermore, for any $f \in H(K_d)$ of arbitrary norm we have

$$\left| I_d(f) - Q_{n,d}(f) \right| \le e^{\text{wor}}(Q_{n,d}; H(K_d)) \, \|f\|_{H(K_d)}.$$

At first glance, it may seem surprising but there is an explicit formula for the worst case error $e^{\text{wor}}(Q_{n,d}; H(K_d))$. Indeed, we have

$$Q_{n,d}(f) = \Big\langle f, \sum_{j=1}^{n} a_j K_d(\cdot, t_j) \Big\rangle_{H(K_d)},$$

which yields

$$I_d(f) - Q_{n,d}(f) = \langle f, h_{d,n} \rangle_{H(K_d)} \quad \text{with } h_{d,n} = h_d - \sum_{j=1}^{n} a_j K_d(\cdot, t_j).$$

From this we easily conclude that

$$e^{\mathrm{wor}}(Q_{n,d}; H(K_d)) = \|I_d - Q_{n,d}\| = \|h_{d,n}\|_{H(K_d)}.$$

Using properties of the reproducing kernel K_d, we have

$$\|h_{d,n}\|_{H(K_d)}^2 = \|h_d\|_{H(K_d)}^2 - 2\sum_{j=1}^{n} a_j h_d(t_j) + \sum_{i,j=1}^{n} a_i a_j K_d(t_i, t_j). \tag{9.31}$$

We want to choose coefficients a_j and sample points t_j such that the worst case error of $Q_{n,d}$ is minimized. Let

$$\overline{e^{\mathrm{wor}}}(n, H(K_d)) = \inf\left\{e^{\mathrm{wor}}(Q_{n,d}) \;\middle|\; Q_{n,d} \text{ given by (9.30) with } a_j = n^{-1}\right\},$$
$$e^{\mathrm{wor}}(n, H(K_d)) = \inf\left\{e^{\mathrm{wor}}(Q_{n,d}) \;\middle|\; Q_{n,d} \text{ given by (9.30) with arbitrary } a_j\right\}.$$

Here, we use the abbreviation $e^{\mathrm{wor}}(Q_{n,d}) = e^{\mathrm{wor}}(Q_{n,d}; H(K_d))$.

In numerous articles the behavior of the minimal errors $\overline{e^{\mathrm{wor}}}(n, H(K_d))$ and $e^{\mathrm{wor}}(n, H(K_d))$ is studied for various spaces $H(K_d)$. The special emphasis is on finding sharp estimates of these quantities in terms of n and d. That is, we would like to know how fast these minimal errors go to zero as n approaches infinity, along with the dependence on d. In particular, we want to know if they depend polynomially or at least non-exponentially on d. We report many such estimates in this volume.

9.5 Relations Between Multivariate Integration and Various Notions of L_2 Discrepancy

In this section we show that multivariate integration defined over the Sobolev space anchored at β is related to the L_2 discrepancy. As we shall see, the test sets appearing in the definition of the L_2 discrepancy depend on the anchor β.

9.5.1 Discrepancy Anchored at the Origin

First, we want to show relations between multivariate integration and the most common discrepancy, which is the L_2 discrepancy at the origin.

Consider the Sobolev space $H(K_d^1)$ anchored at $\beta = 1 = [1, 1, \dots, 1]$. Then $K_1^1(x, y) = 1 - \max(x, y)$, and the reproducer $h_d = h_d^1$ of multivariate integration takes now the form

$$h_d^1(x) = 2^{-d} \prod_{j=1}^{d} \left(1 - x_j^2\right),$$

with $\|h_d^1\|_{H(K_d^1)} = 3^{-d/2}$. Note that the norm of h_d^1 is the same as the initial L_2 discrepancy anchored at the origin.

Consider a linear algorithm $Q_{n,d} f = \sum_{j=1}^{n} a_j f(t_j)$ and compute its worst case error given by (9.31). We have

$$\begin{aligned}\|h_{d,n}\|^2_{H(K_d^1)} \\ = 3^{-d} - 2^{-(d-1)} \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \left(1 - t_{j,k}^2\right) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \left(1 - \max(t_{i,k}, t_{j,k})\right),\end{aligned}$$

which is exactly the formula for the L_2 discrepancy (9.4). Hence,

$$e^{\text{wor}}(Q_{n,d}; H(K_d^1)) = \text{disc}_2(\{t_j\}, \{a_j\}),$$

proving that the L_2 discrepancy anchored at 0 is related to multivariate integration for the Sobolev space anchored at 1.

Do we really have to use different anchors for the L_2 discrepancy and the Sobolev space? Let us check what happens if we take the Sobolev space $H(K_d^0)$ anchored at 0. Then $K_1^0(x, y) = \min(x, y)$ and the reproducer $h_d = h_d^0$ is now

$$h_d^0(x) = \prod_{j=1}^{d} \left(x_j - \tfrac{1}{2} x_j^2\right)$$

with $\|h_d^0\|_{H(K_d^0)} = 3^{-d/2}$, as before. The worst case error of $Q_{n,d}$ is now equal to

$$\|h_{d,n}\|^2_{H(K_d^0)} = 3^{-d} - 2 \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \left(t_{j,k} - \tfrac{1}{2} t_{j,k}^2\right) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \min(t_{i,k}, t_{j,k}),$$

which, at first glance, does *not* seem to be related to the L_2 discrepancy anchored at 0. However, let us substitute

$$t_{j,k} = 1 - \tau_{j,k} \quad \text{for } k \in [d].$$

Since $\min(x, y) = 1 - \max(1 - x, 1 - y)$ we obtain

$$\begin{aligned}\|h_{d,n}\|^2_{H(K_d^0)} \\ = 3^{-d} - 2^{-(d-1)} \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \left(1 - \tau_{j,k}^2\right) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \left(1 - \max(\tau_{i,k}, \tau_{j,k})\right),\end{aligned}$$

which is the formula for the L_2 discrepancy anchored at 0 for the sample points τ_j. Hence,

$$e^{\text{wor}}(Q_{n,d}; H(K_d^0)) = \text{disc}_2(\{1 - t_j\}, \{a_j\}).$$

This means that L_2 discrepancy anchored at 0 is related to multivariate integration for both Sobolev spaces, but we either need to change the anchor of the Sobolev space from 0 to 1, or change the sample points from t_j to $1 - t_j$.

Similar relations hold also between the weighted L_2 discrepancy anchored at 0 and the weighted Sobolev spaces. For the weighted Sobolev space anchored at 1, we have the generator $h_d = h^1_{d,\gamma}$ given by

$$h^1_{d,\gamma}(x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, 2^{-|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \left(1 - x_j^2\right),$$

with

$$\|h^1_{d,\gamma}\|_{H(K_{d,\gamma})} = \Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|} \Big)^{1/2}.$$

From this we conclude that

$$e^{\mathrm{wor}}(Q_{n,d}; H(K^1_{d,\gamma})) = \mathrm{disc}_{2,\gamma}(\{t_j\}, \{a_j\}), \tag{9.32}$$

$$e^{\mathrm{wor}}(Q_{n,d}; H(K^0_{d,\gamma})) = \mathrm{disc}_{2,\gamma}(\{1 - t_j\}, \{a_j\}). \tag{9.33}$$

From these relations, it also follows that the minimal errors of multivariate integration and the minimal L_2 discrepancy anchored at 0 are the same. So we have

$$\begin{aligned}
\overline{e^{\mathrm{wor}}}(n, H(K^0_d)) &= \overline{e^{\mathrm{wor}}}(n, H(K^1_d)) = \overline{\mathrm{disc}}_2(n, d),\\
\overline{e^{\mathrm{wor}}}(n, H(K^0_{d,\gamma})) &= \overline{e^{\mathrm{wor}}}(n, H(K^1_{d,\gamma})) = \overline{\mathrm{disc}}_{2,\gamma}(n, d),\\
e^{\mathrm{wor}}(n, H(K^0_d)) &= e^{\mathrm{wor}}(n, H(K^1_d)) = \ \mathrm{disc}_2(n, d),\\
e^{\mathrm{wor}}(n, H(K^0_{d,\gamma})) &= e^{\mathrm{wor}}(n, H(K^1_{d,\gamma})) = \ \mathrm{disc}_{2,\gamma}(n, d).
\end{aligned}$$

Hence, the study of linear algorithms with the minimal worst case errors for multivariate integration over the (weighted) Sobolev spaces anchored at 0 or 1 is equivalent to the study of the minimal (weighted) L_2 discrepancy. We summarize the results of this section in the following corollary.

Corollary 9.2. *The* (*weighted*) L_2 *discrepancy anchored at* 0 *corresponds to multivariate integration in the worst case setting for the Sobolev spaces with mixed derivatives of order one. If the space is anchored at* 1 *then we use the same sample points, and if the space is anchored at* 0 *then we change the sample points from* t_j *to* $1 - t_j$*, see* (9.32) *and* (9.33)*.*

9.5.2 Discrepancy and Wiener Sheet Measure

So far, we have considered the worst case setting. In this subsection we consider multivariate integration in the average case setting and show its relations to L_2 discrepancy, see [346]. Namely, let $C = C([0,1]^d)$ be the space of continuous functions with the norm

$$\|f\| = \max_{x\in[0,1]^d} |f(x)|.$$

We equip the space C with the Wiener sheet measure w_d, which is a zero-mean Gaussian measure with the covariance function

$$K_d^*(x,y) := \int_C f(x)\, f(y)\, w_d(\mathrm{d}f) = \prod_{j=1}^{d} \min(x_j, y_j) \quad \text{for } x,y \in [0,1]^d.$$

Note that $\int_C f^2(x)\, w_d(\mathrm{d}f) = 0$ if at least one component of x is zero. Hence $f(x) = 0$ with probability one if $x_j = 0$ for some j.

As before, we want to approximate the integral $I_d(f)$ by a linear algorithm $Q_{n,d}(f)$. It is easy to see that the worst case error of any linear algorithm $Q_{n,d}$ (for the unit ball of the space C) must be at least one since there exist continuous functions that vanish at all t_j used by $Q_{n,d}$ and whose integrals and the norms are arbitrarily close to 1. This means that the space C is simply too large, and so the multivariate integration problem for this space cannot be solved in the worst case setting. That is why for the space C we switch to the average case setting, in which the average case error of $Q_{n,d}$ is defined by

$$e^{\mathrm{avg}}(Q_{n,d}) = \left(\int_C \left(I_d(f) - Q_{n,d}(f) \right)^2 \, w_d(\mathrm{d}f) \right)^{1/2}.$$

Note that also in the average case setting we have an explicit formula for the error. Indeed, we have

$$\begin{aligned}
&\int_C [I_d(f)]^2 \, w_d(\mathrm{d}f) \\
&= \int_{[0,1]^{2d}} \left[\int_C f(x)\, f(y)\, w_d(\mathrm{d}f) \right] \mathrm{d}x\, \mathrm{d}y = \int_{[0,1]^{2d}} \prod_{k=1}^{d} \min(x_k, y_k)\, \mathrm{d}x\, \mathrm{d}y = 3^{-d}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_C f(t)\, I_d(f)\, w_d(\mathrm{d}f) &= \int_{[0,1]^d} \left[\int_C f(t)\, f(x)\, w_d(\mathrm{d}f) \right] \mathrm{d}x \\
&= \int_{[0,1]^d} \prod_{k=1}^{d} \min(x_k, t_k)\, \mathrm{d}x = \prod_{k=1}^{d} \left(t_k - t_k^2/2 \right),
\end{aligned}$$

and

$$\int_C f(t_j)\, f(t_i)\, w_d(\mathrm{d}f) = \prod_{k=1}^{d} \min\left(t_{i,k}, t_{j,k} \right).$$

Combining these formulas we conclude that

$$e^{\mathrm{avg}}(Q_{n,d})^2 = 3^{-d} - 2 \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \left(t_{j,k} - t_{j,k}^2/2 \right) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \min\left(t_{i,k}, t_{j,k} \right).$$

Let $x_j = 1 - t_j$. Since $t - \frac{1}{2}t^2 = \frac{1}{2}(1 - x^2)$ for $x = 1 - t$ and again using $\min(t, y) = 1 - \max(1 - t, 1 - y)$, we may rewrite the last formula as

$$e^{\text{avg}}(Q_{n,d})^2$$
$$= 3^{-d} - 2^{-(d-1)} \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \left(1 - x_{j,k}^2\right) + \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \left(1 - \max(x_{i,k}, x_{j,k})\right).$$

But the last formula is the square of the L_2 discrepancy for the points x_j, see (9.4). Hence, as in [346], we have

$$e^{\text{avg}}(Q_{n,d}) = \text{disc}_2\left(\{1 - t_j\}, \{a_j\}\right).$$

Obviously, we can also obtain relations with the weighted L_2 discrepancy if we equip the space of continuous functions with the *weighted* Wiener sheet measure, which is a zero-mean Gaussian measure $w_{d,\gamma}$ whose covariance function is given by

$$K_{d,\gamma}(x, t) := \int_C f(x)\, f(t)\, w_{d,\gamma}(\mathrm{d}f) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \min(x_j, t_j).$$

Here, as always, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ is a weight sequence. Then

$$e^{\text{avg}}(Q_{n,d}) = \text{disc}_{2,\gamma}\left(\{1 - t_j\}, \{a_j\}\right). \tag{9.34}$$

Hence, the minimal L_2 discrepancy and weighted L_2 discrepancy also yield the minimal average case errors for multivariate integration for the space of continuous functions equipped with the standard Wiener sheet and weighted Wiener sheet measure, respectively.

We summarize the results of this section in the following corollary.

Corollary 9.3. *The* (*weighted*) L_2 *discrepancy anchored at* 0 *corresponds to multivariate integration in the average case setting for the space of continuous functions equipped with the (weighted) Wiener sheet measure with the change of the sample points from* t_j *to* $1 - t_j$, *see* (9.34).

9.5.3 Discrepancy Anchored at α

The L_2 discrepancy and weighted L_2 discrepancy anchored at the origin are defined by test sets $[0, x)$ for $x \in [0, 1]^d$. The L_2 discrepancy anchored at the point $\alpha = [\alpha_1, \dots, \alpha_d] \in [0, 1]^d$ is defined if we replace the boxes $[0, x)$ by the sets

$$J(x) = [\min(\alpha_1, x_1), \max(\alpha_1, x_1)) \times \cdots \times [\min(\alpha_d, x_d), \max(\alpha_d, x_d)).$$

That is, for $\alpha_j \le x_j$ for all $j \in [d]$, we have $J(x) = [\alpha, x) = [\alpha_1, x_1) \times \cdots \times [\alpha_d, x_d)$, whereas for $\alpha_j \ge x_j$ for all $j \in [d]$, we have $J(x) = [x, \alpha)$.

For an arbitrary α, the volume of $J(x)$ is given by

$$\mathrm{vol}(J(x)) = \prod_{j=1}^{d} \big[\max(\alpha_j, x_j) - \min(\alpha_j, x_j) \big] = \prod_{j=1}^{d} |x_j - \alpha_j|.$$

It is easy to compute the L_2 norm of $\mathrm{vol}(J(x))$, which is

$$\int_{[0,1]^d} [\mathrm{vol}(J(x))]^2 \, \mathrm{d}x = \prod_{j=1}^{d} \big(\tfrac{1}{3} - \alpha_j(1-\alpha_j) \big) \in \big[(12)^{-d}, 3^{-d} \big].$$

Note that $\|\mathrm{vol}(J)\|_{L_2}$ is always exponentially small in d.

We approximate the volume of the box $J(x)$ by a weighted sum of points t_j that are in $J(x)$, i.e.,

$$\mathrm{disc}^{\alpha}(x) = \mathrm{vol}(J(x)) - \sum_{j=1}^{n} a_j \, 1_{J(x)}(t_j).$$

The L_2 *discrepancy anchored at the point* α for points t_j and coefficients a_j is the L_2 norm of the function disc^{α}, i.e.,

$$\mathrm{disc}_2^{\alpha}(\{t_j\}, \{a_j\}) = \left(\int_{[0,1]^d} [\mathrm{disc}^{\alpha}(x)]^2 \, \mathrm{d}x \right)^{1/2}.$$

Obviously, for $\alpha = 0$ this notion coincides with the L_2 discrepancy studied before.

By direct integration we obtain an explicit formula,

$$\begin{aligned}
\mathrm{disc}_2^{\alpha}(\{t_j\}, \{a_j\})^2 &= \prod_{k=1}^{d} \big(\tfrac{1}{3} - \alpha_k(1-\alpha_k) \big) \\
&- 2 \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \frac{t_{j,k}(2\alpha_k - t_{j,k}) \, 1_{[0,\alpha_k)}(t_{j,k}) + (1-t_{j,k})(1+t_{j,k}-2\alpha_k) \, 1_{[\alpha_k,1]}(t_{j,k})}{2} \\
&+ \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \chi_{i,j,k},
\end{aligned}$$

where $\chi_{i,j,k}$ is given by

$$\big(\min(t_{i,k}, t_{j,k}) 1_{[0,\alpha_k)^2}((t_{i,k}, t_{j,k})) + \big(1 - \max(t_{i,k}, t_{j,k})\big) \; 1_{[\alpha_k,1]^2}((t_{i,k}, t_{j,k})) \big).$$

We now relate the L_2 discrepancy anchored at α to multivariate integration for the Sobolev space anchored at some β. Consider a linear algorithm $Q_{n,d}(f) = \sum_{j=1}^{n} a_j f(\tau_j)$ for some sample points $\tau_j \in [0,1]^d$. The worst case error of $Q_{n,d}$ is given by (9.31). We have

$$
\begin{aligned}
e^{\text{wor}}(Q_{n,d}; H(K_d^{\beta}))^2 &= \prod_{k=1}^{d} \left(\tfrac{1}{3} - \beta_k(1-\beta_k)\right) \\
&- 2\sum_{j=1}^{n} a_j \prod_{k=1}^{d} \frac{(\beta_k^2 - \tau_{j,k}^2)\, 1_{[0,\beta_k)}(\tau_{j,k}) + (\tau_{j,k}-\beta_k)(2-\tau_{j,k}-\beta_k)\, 1_{[\beta_k,1]}(\tau_{j,k})}{2} \\
&+ \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \frac{|\tau_{i,k}-\beta_k| + |\tau_{j,k}-\beta_k| - |\tau_{i,k}-\tau_{j,k}|}{2}.
\end{aligned}
$$

We want to check whether the L_2 discrepancy anchored at α for the points t_j and coefficients a_j can be the same as the worst case error of $Q_{n,d}$ with the sample points τ_j and coefficients a_j in the Sobolev space anchored at β. That is, for a given α and t_j we are looking for β and τ_j such that

$$e^{\text{wor}}(Q_{n,d}; H(K_d^{\beta})) = \text{disc}_2^{\alpha}(\{t_j\}, \{a_j\}).$$

Comparing the formulas for $e^{\text{wor}}(Q_{n,d})$ and $\text{disc}_2^{\alpha}(\{t_j\}, \{a_j\})$, we see that the equality holds if

$$t_j(2a - t_j)1_{[0,a)}(t_j) + (1-t_j)(1+t_j-2a)1_{[a,1]}(t_j)$$

is the same as

$$(b-\tau_j)(b+\tau_j)1_{[0,b)}(\tau_j) + (\tau_j - b)(2-\tau_j-b)1_{[b,1]}(\tau_j),$$

and

$$\min(t_i,t_j)1_{[0,a)^2}((t_i,t_j)) + (1-\max(t_i,t_j))1_{[a,1]^2}((t_i,t_j))$$

is the same as

$$2^{-1}\left(|\tau_i - b| + |\tau_j - b| - |\tau_i - \tau_j|\right).$$

Here, we suppress the dependence on k by taking $[t_j, a, \tau_j, b] = [t_{j,k}, \alpha_k, \tau_{j,k}, \beta_k]$. That is, for given t_j and a we want to find τ_j and b such that the corresponding equalities hold.

Assume first that $\alpha = 0$, which corresponds to the L_2 discrepancy anchored at the origin. Then we need to guarantee that

$$
\begin{aligned}
1 - t_j^2 &= (b-\tau_j)(b+\tau_j)1_{[0,b)}(\tau_j) + (\tau_j - b)(2-\tau_j-b)1_{[b,1]}(\tau_j), \\
1 - \max(t_i,t_j) &= 2^{-1}\left(|\tau_i - b| + |\tau_j - b| - |\tau_i - \tau_j|\right).
\end{aligned}
$$

Note that this indeed holds if we take $\tau_j = t_j$ and $b = 1$, or $\tau_j = 1 - t_j$ and $b = 0$. These two solutions correspond to the relations that we already discussed in Corollary 9.2.

Assume now that $\alpha = 1$. By symmetry, we again have two solutions. The first one is for $\tau_j = t_j$ and $b = 0$, and the second for $\tau_j = 1 - t_j$ and $b = 1$.

We now show that for any $a \in [0,1]$ the solution is given by

$$\tau_j = (a - t_j) \bmod 1 \quad \text{and} \quad b = a.$$

Indeed, let $t_j \in [0,a)$. Then $\tau_j = a - t_j$. The first formula for the discrepancy is now $t_j(2a - t_j)$, whereas for integration the first formula is

$$(b - \tau_j)(b + \tau_j) = t_j(2a - t_j),$$

so they agree. Similarly, for $t \in [a,1]$ we have $\tau_j = 1 + a - t_j$, and the first formula for the discrepancy is $(1 - t_j)(1 + t_j - 2a)$, whereas for integration it is

$$(\tau_j - b)(2 - \tau_j - b) = (1 - t_j)(1 + t_j + 2a),$$

so they again agree.

To check equality of the second formulas, note that both of them are zero if $t_i \le a \le t_j$ or $t_j \le a \le t_i$. For $t_i, t_j \in [0,a)$, we have $\min(t_i, t_j)$ for the discrepancy, and

$$2^{-1}\left(|\tau_i - b| + |\tau_j - b| - |\tau_i - \tau_j|\right) = 2^{-1}\left(t_i + t_j - |t_i - t_j|\right) = \min(t_i, t_j)$$

for integration.

For $t_i, t_j \in [a,1]$, we have $1 - \max(t_i, t_j)$ for the discrepancy, and

$$\begin{aligned} 2^{-1}\left(|\tau_i - b| + |\tau_j - b| - |\tau_i - \tau_j|\right) &= 2^{-1}\left(1 - t_i + 1 - t_j - |t_i - t_j|\right) \\ &= 1 - \max(t_i, t_j), \end{aligned}$$

as needed.

It is worth to notice that the choice $\tau_j = t_j$ cannot be made for $a \in (0,1)$. Indeed, assume that b is positive and take $t \in [0, \min(a,b))$. Then the function $t(2a - t)$ from the discrepancy cannot be equal to the function $b^2 - t^2$ from the integration. For $b = 0$, we take $t \in [a,1]$. Then the discrepancy function is $(1-t)(1+t-2a)$ and the integration function is $t(2-t)$. Again they are different.

Hence, for any $\alpha \in [0,1]^d$, the L_2 discrepancy for the points t_j and coefficients a_j is the same as the worst case error over the unit ball of the Sobolev space anchored at α of the linear algorithm $Q_{n,d}$ using the sample points τ_j such that $\tau_j = \left(\alpha - t_j\right) \bmod 1$, or equivalently

$$t_j = \left(\alpha - \tau_j\right) \bmod 1\,.$$

The last formula is understood component-wise, i.e., $t_{j,k} = (a_k - \tau_{j,k}) \bmod 1$ for $k \in [d]$. Then

$$e^{\mathrm{wor}}(Q_{n,d}; H(K_d^\alpha)) = \mathrm{disc}_2^\alpha(\{(\alpha - t_j) \bmod 1\}, \{a_j\}).$$

Proceeding as before, it is also possible to find relations between the weighted L_2 discrepancy anchored at α and multivariate integration for the weighted Sobolev space anchored at α. The weighted L_2 discrepancy anchored at α is defined as

$$\mathrm{disc}_2^\alpha(\{t_j\}, \{a\}) = \left(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} [\mathrm{disc}^{\alpha_\mathfrak{u}}(x_\mathfrak{u})]^2 \,\,\mathrm{d}x_\mathfrak{u}\right)^{1/2}.$$

Here, $\mathrm{disc}^{\alpha_{\mathfrak{u}}}$ is defined as disc^{α} for the $|\mathfrak{u}|$-dimensional case for the points $(t_j)_{\mathfrak{u}}$.

Then the worst case error of the previous $Q_{n,d}$ over the unit ball of $H(K^{\alpha}_{d,\gamma})$ is the same as the weighted L_2 discrepancy anchored at α for the points $(\alpha - t_j) \bmod 1$, i.e.,

$$e^{\mathrm{wor}}(Q_{n,d}; H(K^{\alpha}_{d,\gamma})) = \mathrm{disc}^{\alpha}_{2,\gamma}(\{(\alpha - t_j) \bmod 1\}, \{a_j\}). \tag{9.35}$$

We summarize the results of this section in the following corollary.

Corollary 9.4. *The* (*weighted*) L_2 *discrepancy anchored at* α *corresponds to multivariate integration in the worst case setting for the* (*weighted*) *Sobolev space with mixed derivatives of order one anchored at* α *with the change of the sample points from* t_j *to* $(\alpha - t_j)$ *mod* 1*, see* (9.35).

9.5.4 Quadrant Discrepancy at α

We showed that the L_2 discrepancy centered at α is related to multivariate integration for the Sobolev space anchored at α, but we needed to change the sample points from t_j to $(\alpha - t_j) \bmod 1$. We now show that the discrepancy studied by Hickernell, Sloan and Wasilkowski [123], see also Hickernell [118] for a special case, allows us to use the same points for the discrepancy and multivariate integration for the Sobolev space with the same anchor.

For a given $\alpha \in [0,1]^d$, we now consider test sets $Q(x)$ for $x \in [0,1]^d$ with

$$Q(x) = [w_1(x), z_1(x)) \times \cdots \times [w_d(x), z_d(x)),$$

where $[w_j(x), z_j(x)) = [0, x_j)$ if $x_j < \alpha_j$, and $[w_j(x), z_j(x)) = [x_j, 1)$ if $x_j \ge \alpha_j$. That is, the set of points $x \in [0,1]^d$ is partitioned into 2^d quadrants according to whether $x_j < \alpha_j$ or $x_j \ge \alpha_j$. The set $Q(x)$ denotes the box with one corner at x and the opposite corner defined by the unique vertex of $[0,1]^d$ that lies in the same quadrant as x. Note that for $\alpha = 1$, we have $Q(x) = [0, x)$ for $x \in [0,1)^d$, as with the L_2 discrepancy anchored at 0. For $\alpha = 0$, we have $Q(x) = [x, 1)$.

Let

$$\mathrm{disc}^{\alpha,\mathrm{quad}}(x) = \mathrm{vol}(Q(x)) - \sum_{j=1}^{n} a_j 1_{Q(x)}(t_j)$$

be the error of approximating the volume of $Q(x)$ by a weighted sum of points t_j that are in $Q(x)$.

The L_2 *same-quadrant discrepancy with anchor at* α, or shortly, the L_2 *quadrant discrepancy at* α, of points t_j and coefficients a_j is the L_2 norm of the function $\mathrm{disc}^{\alpha,\mathrm{quad}}$, i.e.,

$$\mathrm{disc}^{\alpha,\mathrm{quad}}_2(\{t_j\}, \{a_j\}) = \left(\int_{[0,1]^d} \left[\mathrm{disc}^{\alpha,\mathrm{quad}}(x)\right]^2 \,\mathrm{d}x \right)^{1/2}.$$

For $\alpha = [\frac{1}{2}, \dots, \frac{1}{2}]$, this type of discrepancy was studied by Hickernell [118], who called it the *centered discrepancy*. For general α, the quadrant discrepancy was studied in the L_∞ norm in Hickernell, Sloan and Wasilkowski [123], and we present its L_2 analog above.

The volume of $Q(x)$ is

$$\operatorname{vol}(Q(x)) = \prod_{j=1}^{d} \left[x_j 1_{[0,\alpha_j)}(x_j) + (1-x_j) 1_{[\alpha_j,1]}(x_j) \right],$$

and

$$\int_{[0,1]^d} [\operatorname{vol}(Q(x))]^2 \, \mathrm{d}x = \prod_{j=1}^{k} \left[\tfrac{1}{3} - \alpha_j(1-\alpha_j) \right] \in \left[(12)^{-d}, 3^{-d} \right].$$

To obtain an explicit formula for the L_2 quadrant discrepancy at α, note that

$$\begin{aligned} &\int_{[0,1]^d} \operatorname{vol}(Q(x)) 1_{Q(x)}(t_j) \, \mathrm{d}x \\ &= \prod_{k=1}^{d} \frac{(\alpha_k^2 - t_{j,k}^2) 1_{[0,\alpha_k)}(t_{j,k}) + (t_{j,k} - \alpha_k)(2 - t_{j,k} - a_k) 1_{[\alpha_k,1]}(t_{j,k})}{2}, \end{aligned}$$

whereas

$$\int_{[0,1]^d} 1_{Q(x)}(t_i) \, 1_{Q(x)}(t_j) \, \mathrm{d}x = \prod_{k=1}^{d} A_k = \prod_{k=1}^{d} \frac{|t_{i,k} - \alpha_k| + |t_{j,k} - \alpha_k| - |t_{i,j} - t_{j,k}|}{2}$$

with

$$\begin{aligned} A_k = &\left(\alpha_k - \max(t_{i,k}, t_{j,k})\right) 1_{[0,\alpha_k)^2}(t_{i,k}, t_{j,k}) \\ &+ \left(\min(t_{i,k}, t_{j,k}) - \alpha_k\right) 1_{[\alpha_k,1]^2}(t_{i,k}, t_{j,k}). \end{aligned}$$

Hence,

$$\begin{aligned} &\operatorname{disc}_2^{\alpha,\,\mathrm{quad}}(\{t_j\}, \{a_j\})^2 = \prod_{j=1}^{d} \left[\tfrac{1}{3} - \alpha_j(1-\alpha_j) \right] \\ &- 2 \sum_{j=1}^{n} a_j \prod_{k=1}^{d} \frac{(\alpha_k^2 - t_{j,k}^2) 1_{[0,\alpha_k)}(t_{j,k}) + (t_{j,k} - \alpha_k)(2 - t_{j,k} - a_k) 1_{[\alpha_k,1]}(t_{j,k})}{2} \\ &+ \sum_{i,j=1}^{n} a_i a_j \prod_{k=1}^{d} \frac{|t_{i,k} - \alpha_k| + |t_{j,k} - \alpha_k| - |t_{i,j} - t_{j,k}|}{2}. \end{aligned}$$

If we compare this formula to the formula for the worst case error of $Q_{n,d}$ for the Sobolev space anchored at α, we see that they are the same if we set

$$\beta = \alpha \quad \text{and} \quad \tau_j = t_j.$$

The same is also true for the weighted case. Hence, we have

$$e^{\rm wor}(Q_{n,d}; H(K_d^{\alpha})) = {\rm disc}_2^{\alpha,\,{\rm quad}}(\{t_j\},\{a_j\}). \tag{9.36}$$

We summarize the results of this section in the following corollary.

Corollary 9.5. *The* (*weighted*) L_2 *quadrant discrepancy at* α *corresponds to multivariate integration in the worst case setting for the* (*weighted*) *Sobolev space with mixed derivatives of order one anchored at* α *with the same sample points, see* (9.36).

9.5.5 Extreme or Unanchored Discrepancy

So far, we have used the boxes $[0,x)$, $J(x)$ or $Q(x)$ as our test sets. A different notion of L_2 discrepancy can be obtained if we use all boxes $[x,y)$ with $x \le y$ (componentwise) as test sets. That is, instead of (9.2) we now approximate the volume of $[x,y)$, which is obviously $\prod_{j=1}^d (y_j - x_j)$, by the weighted sum of points t_j belonging to the box $[x,y)$, so that

$$\mathrm{disc}^{\rm ex}(x,y) = (y_1-x_1)(y_2-x_2)\cdots(y_d-x_d) - \sum_{j=1}^d a_j 1_{[x,y)}(t_j), \tag{9.37}$$

where $1_{[x,y)}$ is the indicator function of the box $[x,y)$.

The L_2 *extreme* or *unanchored* discrepancy of points t_j and coefficients a_j is just the L_2 norm of the error function (9.37), i.e.,

$$\mathrm{disc}_2^{\rm ex}(\{t_j\},\{a_j\}) = \left(\int_{[0,1]^{2d},\ x\le y} [\mathrm{disc}^{\rm ex}(x,y)]^2 \, \mathrm{d}x\,\mathrm{d}y \right)^{1/2}. \tag{9.38}$$

Direct integration yields an explicit formula for the L_2 unanchored discrepancy,

$$\begin{aligned} \mathrm{disc}_2^{\rm ex}(\{t_j\},\{a_j\})^2 &= \frac{1}{12^d} - \frac{1}{2^{d-1}} \sum_{j=1}^n a_j \prod_{k=1}^d t_{j,k}(1-t_{j,k}) \\ &\quad + \sum_{i,j=1}^n a_i a_j \prod_{k=1}^d \left(\min(t_{i,k},t_{j,k}) - t_{i,k}t_{j,k} \right). \end{aligned} \tag{9.39}$$

For $n = 0$, we obtain the initial L_2 unanchored discrepancy, which is

$$\mathrm{disc}_2^{\rm ex}(0,d) = 12^{-d/2}.$$

Hence, as with the L_2 discrepancy anchored at α and the L_2 quadrant discrepancy anchored at α, the L_2 extreme discrepancy is exponentially small in d.

This type of L_2 discrepancy was introduced by Morokoff and Caflisch [191]. These authors preferred to use the unanchored discrepancy since it is "symmetric" and does not prefer a particular vertex, like the L_2 discrepancy anchored at the origin or at α. We will see that the extreme discrepancy is the error of multivariate integration for a class of periodic functions with a boundary condition.

We begin, however, with periodic functions without a boundary condition and summarize the analysis of Hickernell [118], [119] for this case. For $d = 1$ and $p = q = 2$, we consider the Sobolev space of periodic functions

$$F_{1,2} = \{f \in W_2^1([0,1]) \mid f(0) = f(1)\}$$

with the norm

$$\|f\|^2 = f(1)^2 + \|f'\|_{L_2}^2.$$

This is a rank-1 modification of the space $W_2^1([0,1])$. We obtain the kernel

$$K_1(x,y) = 1 + (\min(x,y) - xy)\,.$$

The analysis can be extended, by Hölder's inequality, to arbitrary q. The norm in $F_{1,q}$ is given by $\|f\|^q = |f(1)|^q + \|f'\|_{L_q}^q$. For arbitrary $d > 1$, we define $F_{d,q}$ by tensor products of factors $F_{1,q}$ with tensor product norms.

We now discuss the respective error (or discrepancy) of a QMC algorithm $Q_{n,d}$ for the space $F_{d,2}$. The error of $Q_{n,d}$ is given by

$$e(Q_{n,d}) = \left(\sum_{u \subseteq [d]} \int_{[0,1)^{2|u|},\, x_u \le y_u} \operatorname{disc}^2((x_u,0),(y_u,1))\, \mathrm{d}x_u \mathrm{d}y_u \right)^{1/2}.$$

For an arbitrary linear $Q_{n,d}$ we obtain

$$e(Q_{n,d})^2 = \frac{13^d}{12^d} - \sum_{i=1}^n 2a_i \prod_{k=1}^d \left(1 + \frac{t_{i,k}(1-t_{i,k})}{2}\right) + \sum_{i=1}^n \sum_{j=1}^n a_i a_j \prod_{k=1}^d (1 + \min(t_{i,k}, t_{j,k}) - t_{i,k} t_{j,k}).$$

Again we can modify this to cover spaces with a boundary condition. We start with $d = 1$ and $p = q = 2$, and we take the space

$$F_{1,2} = \{f \in W_2^1([0,1]) \mid f(0) = f(1) = 0\}.$$

The kernel for this subspace is given by

$$K_1(x,y) = \min(x,y) - xy.$$

For $d > 1$, we use tensor product kernels and norms. For $p = 2$, the error $e(Q_{n,d})$ of any linear $Q_{n,d}$ is

$$e(Q_{n,d}) = \left(\int_{[0,1)^{2d},\, x \le y} \operatorname{disc}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \right)^{1/2} = \operatorname{disc}_2^{\mathrm{ex}}(\{t_i\},\{a_i\}). \tag{9.40}$$

Hence, we see that the L_2 extreme discrepancy is also an error bound, this time for a class of periodic functions with a boundary condition. To prove the error bound (9.40) for the kernel

$$K_d(x,y) = \prod_{j=1}^{d} \left(\min(x_j, y_j) - x_j y_j\right),$$

we can simply use the general result (9.31), together with formula (9.39).

It is also possible to remove the boundary conditions $f(x) = 0$ if $x_j \in \{0,1\}$ for some $j \in [d]$, and consider the *weighted* L_2 unanchored discrepancy. In this case, we take

$$K^{\mathrm{ex}}_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \left(\min(x_j, y_j) - x_j y_j\right).$$

The Hilbert space $H(K^{\mathrm{ex}}_{d,\gamma})$ is the space of periodic functions with period one in each variable, and is the sum of the tensor products of $|\mathfrak{u}|$ copies of the space $H(K^{\mathrm{ex}}_1)$ with the inner product

$$\langle f, g\rangle_{H(K^{\mathrm{ex}}_{d,\gamma})} = \sum_{\mathfrak{u}\subseteq[d]} \gamma^{-1}_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x_{\mathfrak{u}}, 1)\, \mathrm{d}x_{\mathfrak{u}}$$

for $f, g \in H(K^{\mathrm{ex}}_{d,\gamma})$.

The weighted L_2 unanchored discrepancy is defined as

$$\mathrm{disc}^{\mathrm{ex}}_{2,\gamma}(\{t_j\},\{a_j\}) = \left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \int_{\substack{[0,1]^{2|\mathfrak{u}|}\\ x_{\mathfrak{u}}\le y_{\mathfrak{u}}}} \left[\mathrm{disc}^{\mathrm{ex}}\left((x_{\mathfrak{u}},1),(y_{\mathfrak{u}},1)\right)\right]^2 \mathrm{d}x_{\mathfrak{u}}\, \mathrm{d}y_{\mathfrak{u}}\right)^{1/2}.$$

Then

$$e^{\mathrm{wor}}(Q_{n,d}; H(K^{\mathrm{ex}}_{d,\gamma})) = \mathrm{disc}^{\mathrm{ex}}_{2,\gamma}(\{t_j\},\{a_j\}). \tag{9.41}$$

We summarize the results of this section in the following corollary.

Corollary 9.6. *The* (*weighted*) L_2 *unanchored discrepancy corresponds to multivariate integration in the worst case setting for the* (*weighted*) *space* $H(K^{\mathrm{ex}}_d)$ *with the same sample points, see* (9.41).

We end this subsection with a note on the classical *extreme discrepancy* for the space $F_{d,1}$, which corresponds to $p = \infty$. For $d = 1$, it can be checked that the norm of $F_{1,1}$ is given by $\|f\| = \frac{1}{2}\|f'\|_1$. We have

$$\mathrm{disc}^{\mathrm{ex}}_{\infty}(\{t_i\},\{a_i\}) = \sup_{x\le y} |\mathrm{disc}(x,y)|. \tag{9.42}$$

The extreme discrepancy (9.42) is polynomially tractable, see [115]. It is enough to use equal weights n^{-1}, for which we have an upper bound of the form

$$\inf_{t_1,\dots,t_n} \mathrm{disc}^{\mathrm{ex}}_{\infty}(\{t_i\},\{1/n\}) \le C \cdot d^{1/2} \cdot n^{-1/2}, \tag{9.43}$$

where the positive C does not depend on n or d. This bound is the same as for the star discrepancy, which corresponds to L_∞ discrepancy anchored at 0, except that we might have a different constant C. The star discrepancy will be discussed in Section 9.9.

9.6 Are They Always Related?

The examples of the previous section may suggest that for a specific kind of L_2 discrepancy, there always exists a reproducing kernel Hilbert space $H(K_d)$ for which the L_2 discrepancy and multivariate integration are related. We now show that, modulo natural assumptions, this is indeed true.

We first define what we mean by a general L_2 discrepancy. Let $D \subseteq \mathbb{R}^{\tau(d)}$ be a measurable set and let $\varrho\colon D \to \mathbb{R}$ be a non-negative measurable function such that $\int_D \varrho(x)\,\mathrm{d}x = 1$. Here, $\tau\colon \mathbb{N} \to \mathbb{N}$ is a given function, so that $\tau(d)$ is a given positive integer.

The choice of ϱ as a density function is quite natural. This allows us to properly normalize general L_2 discrepancies as well to use unbounded domains D.

We assume that for any $t \in D$ we have a measurable set $B(t) \subseteq \mathbb{R}^d$, and let $\mathrm{vol}(B(t))$ denote its Lebesgue measure (volume). We also assume that $\mathrm{vol}(B(\cdot))$ is a measurable function, and that

$$\int_D [\mathrm{vol}(B(t))]^2\ \varrho(t)\,\mathrm{d}t < \infty. \tag{9.44}$$

Let $g(t,x) = 1_{B(t)}(x)$ for $t \in D$ and $x \in \mathbb{R}^d$ be the indicator function. We also assume that g is measurable with respect to both arguments.

For example, take $D = [0,1]^d$ with $\tau(d) = d$, and $\varrho(t) = 1$. Then

- $B(t) = [0,t)$ will correspond to the discrepancy anchored at 0, whereas
- $B(t) = J(t)$ will correspond to the discrepancy anchored at α, and
- $B(t) = Q(t)$ will correspond to the quadrant discrepancy at α.

To obtain the unanchored discrepancy, we define

$$\tau(d) = 2d, \quad D = \{(x,y) \in [0,1]^{2d} \mid x \le y\}, \quad \text{and} \quad \varrho(t) = 2^d.$$

Then for $t = (x,y)$ and $B(t) = [x,y)$ we have the unanchored discrepancy modulo a normalizing factor. New examples of $B(t)$ will be presented later.

For given points $t_1, t_2, \dots, t_n \in \mathbb{R}^d$ and coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$, we approximate the volume of $B(t)$ by a weighted sum of the sample points t_j that belong to $B(t)$, so that

$$\mathrm{disc}(t) := \mathrm{vol}(B(t)) - \sum_{j=1}^n a_j\, 1_{B(t)}(t_j).$$

The L_2 *B-discrepancy* of points t_j and coefficients a_j, or shortly *B-discrepancy*, is the weighted L_2 norm of the last function, i.e.,

$$\mathrm{disc}_2^{\mathrm{B}}(\{t_j\},\{a_j\}) = \left(\int_D \Big[\mathrm{vol}(B(t)) - \sum_{j=1}^n a_j\, 1_{B(t)}(t_j)\Big]^2 \varrho(t)\,\mathrm{d}t\right)^{1/2}.$$

By direct integration we then have

$$\mathrm{disc}_2^{\mathrm{B}}(\{t_j\},\{a_j\})^2 = \int_D \mathrm{vol}(B(t))^2\,\varrho(t)\,\mathrm{d}t - 2\sum_{j=1}^n a_j \int_D \mathrm{vol}(B(t))1_{B(t)}(t_j)\,\varrho(t)\,\mathrm{d}t$$
$$+ \sum_{i,j=1}^n a_i a_j \int_D 1_{B(t)}(t_i)\,1_{B(t)}(t_j)\,\varrho(t)\,\mathrm{d}t.$$

A popular choice of coefficients is $a_j = n^{-1}$, which corresponds to *quasi-Monte Carlo* (QMC) algorithms for multivariate integration.

We now ask if it is possible to define a reproducing kernel Hilbert space $H(K_d)$ for which the B-discrepancy is related to multivariate integration. Here,

$$D_d = \overline{\bigcup_{t\in D} B(t)} \subseteq \mathbb{R}^d,$$

and $K_d\colon D_d \times D_d \to \mathbb{R}$ is a reproducing kernel. We assume that D_d is measurable.

Although we have so far only considered the reproducing kernel Hilbert spaces of functions and multivariate integration defined on $[0,1]^d$, it is easy to generalize everything to the domain D_d. We have

$$I_d(f) = \int_{D_d} f(x)\,\mathrm{d}x,$$

which can be written as $I_d(f) = \langle f, h_d\rangle_{H(K_d)}$ with

$$h_d(x) = \int_{D_d} K_d(y,x)\,\mathrm{d}y \quad \text{for } x \in D_d.$$

As before, we need to assume that

$$h_d \in H(K_d). \tag{9.45}$$

Then the worst case error of a linear algorithm $Q_{n,d}(f) = \sum_{j=1}^n a_j f(t_j)$ is

$$e^{\mathrm{wor}}(Q_{n,d}) = \sup_{f\in H(K_d),\,\|f\|_{H(K_d)}\le 1} \left|I_d(f) - Q_{n,d}(f)\right| = \|h_{d,n}\|_{H(K_d)},$$

with $h_{d,n} = h_d - \sum_{j=1}^n a_j K_d(\cdot, t_j)$ and

$$\|h_{d,n}\|^2_{H(K_d)} = \int_{D_d^2} K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y - 2\sum_{j=1}^n a_j \int_{D_d} K_d(x,t_j)\,\mathrm{d}x + \sum_{i,j=1}^n a_i a_j K_d(t_i,t_j). \tag{9.46}$$

If we compare this formula with the formula for the B-discrepancy we see that the candidate for the reproducing kernel is

$$K_d(x,y) = \int_D 1_{B(t)}(x)\,1_{B(t)}(y)\,\varrho(t)\,\mathrm{d}t \quad \text{for } x,y \in D_d. \tag{9.47}$$

Observe that $K_d(x,y)$ is well defined and $K_d(x,y) \in [0,1]$. It is easy to check that K_d is a reproducing kernel. Indeed, $K_d(x,y) = K_d(y,x)$, and so K_d is symmetric. Consider the $m \times m$ matrix $M = (K_d(x_i, x_j))_{i,j=1,\dots,m}$ for arbitrary points $x_j \in \mathbb{R}^d$. Then M is symmetric and it is also positive semi-definite since

$$(Ma,a) = \sum_{i,j=1}^{m} K_d(x_i,x_j)a_i a_j = \int_D \Big(\sum_{j=1}^{m} a_j 1_{B(t)}(t_j)\Big)^2 \varrho(t)\,\mathrm{d}t \ge 0.$$

So K_d is a reproducing kernel, as claimed.

To make sure that multivariate integration for $H(K_d)$ is well defined, we need to assume that the function

$$h_d(x) = \int_{D_d} K_d(x,y)\,\mathrm{d}y = \int_D \mathrm{vol}(B(t))\, 1_{B(t)}(x)\,\varrho(t)\,\mathrm{d}t \tag{9.48}$$

belongs to $H(K_d)$.

The choice of K_d by (9.47) will make the third terms in (9.46) and in the formula for the B-discrepancy equal. We obviously need to check that the first and second terms coincide. For the first term of (9.46), we have

$$\begin{aligned}\int_{D_d^2} K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_D \Big[\int_{D_d} 1_{B(t)}(x)\,\mathrm{d}x\Big]\Big[\int_{D_d} 1_{B(t)}(y)\,\mathrm{d}y\Big]\varrho(t)\,\mathrm{d}t\\ &= \int_D \mathrm{vol}(B(t))^2\,\varrho(t)\,\mathrm{d}t,\end{aligned}$$

which agrees with the first term for the B-discrepancy. Finally, for the second term of (9.46), we have

$$\begin{aligned}\int_{D_d} K_d(x,t_j)\,\mathrm{d}x &= \int_D \Big[\int_{D_d} 1_{B(t)}(x)\,\mathrm{d}x\Big] 1_{B(t)}(t_j)\,\varrho(t)\,\mathrm{d}t\\ &= \int_D \mathrm{vol}(B(t))\, 1_{B(t)}(t_j)\,\varrho(t)\,\mathrm{d}t\end{aligned}$$

which agrees with the second term for B-discrepancy.

As before, we can define the minimal B-discrepancy, $\mathrm{disc}_2^{\mathrm{B}}(n,d)$, and the minimal worst case error, $e^{\mathrm{wor}}(n, H(K_d))$, of multivariate integration by taking optimal sample points t_j and optimal coefficients a_j. Obviously they are the same, i.e.,

$$\mathrm{disc}_2^{\mathrm{B}}(n,d) = e^{\mathrm{wor}}(n, H(K_d)).$$

It is easy to show that the minimal B-discrepancy, or equivalently the minimal multivariate integration error, is at most of order $n^{-1/2}$ if we assume[2] that $\mathrm{vol}(D_d) < \infty$. Indeed, take an algorithm

$$Q_{n,d}(f) = \frac{\mathrm{vol}(D_d)}{n}\sum_{j=1}^{n} f(t_j)$$

[2] In Chapter 10 we present a more relaxed assumption based on Plaskota, Wasilkowski and Zhao [248].

for some sample points $t_j \in D_d$. To stress the dependence on t_j, we replace $Q_{n,d}$ by $Q_{n,d,\{t_j\}}$. The square of the worst case error $e(t_1, t_2, \dots, t_n)$ of $Q_{n,d,\{t_j\}}$ is given by (9.46) and takes the form

$$\int_{D_d^2} K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y - \frac{2\mathrm{vol}(D_d)}{n}\sum_{j=1}^{n}\int_{D_d} K_d(x,t_j)\,\mathrm{d}x + \frac{\mathrm{vol}^2(D_d)}{n^2}\sum_{i,j=1}^{n} K_d(t_i,t_j).$$

We now compute the average value of e assuming that t_j are independent and uniformly distributed over D_d. Using the standard proof technique that is also used for the study of Monte Carlo algorithms, we obtain

$$\begin{aligned}
&\frac{\int_{D_d^n} e(t_1,\dots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n}{\mathrm{vol}(D_d)^n}\\
&\quad= \frac{1}{n}\left(\mathrm{vol}(D_d)\int_{D_d} K_d(x,x)\,\mathrm{d}x - \int_{D_d^2} K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\qquad= \frac{1}{n}\left(\mathrm{vol}(D_d)\int_{D} \mathrm{vol}(B(t))\,\varrho(t)\,\mathrm{d}t - \int_{D} \mathrm{vol}(B(t))^2\,\varrho(t)\,\mathrm{d}t\right)\\
&\qquad\le \frac{\mathrm{vol}(D_d)^2}{n}.
\end{aligned}$$

By the mean value theorem, we conclude that there exists at least one choice of the sample points t_j for which the worst case error of $Q_{n,d}$ is at most the square root of the last value. Hence,

$$e^{\mathrm{wor}}(n, H(K_d)) \le \frac{1}{\sqrt{n}}\left(\mathrm{vol}(D_d)\int_{D_d} K_d(x,x)\,\mathrm{d}x\right)^{1/2} \le \frac{\mathrm{vol}(D_d)}{\sqrt{n}}.$$

We summarize the analysis of this section in the following theorem.

Theorem 9.7.

- *The worst case error of $Q_{n,d}$ in the space $H(K_d)$ with the reproducing kernel K_d given by (9.47) is equal to the B-discrepancy with the same sample points t_j and coefficients a_j.*
- *We also have*
$$\mathrm{disc}_2^{\mathrm{B}}(n,d) = e^{\mathrm{wor}}(n, H(K_d)).$$
- *If* $\mathrm{vol}(D_d) < \infty$ *then*
$$\mathrm{disc}_2^{\mathrm{B}}(n,d) \le n^{-1/2}\,\mathrm{vol}(D_d).$$

It seems interesting to check what kinds of reproducing kernels we obtain for various kinds of L_2 discrepancy. For the L_2 discrepancy anchored at 0 we have $D = [0,1]^d$,

$\varrho(t) = 1$ and $B(t) = [0, t)$. Therefore $D_d = [0, 1]^d$ and

$$K_d(x, y) = \prod_{j=1}^{d} \big(1 - \max(x_j, y_j)\big) \quad \text{for all } x, y \in D_d$$

which corresponds to the Sobolev space anchored at 1. This agrees with our previous results.

Consider now the L_2 discrepancy anchored at α. We now have $D = [0, 1]^d$, $\varrho(t) = 1$ and $B(t) = J(t)$, as given in Subsection 9.5.3. For $d = 1$, we have $B(t) = [t, \alpha)$ for $t \le \alpha$, and $B(t) = [\alpha, t)$ for $t > \alpha$. Therefore $D_1 = [0, 1]$ and $D_d = [0, 1]^d$. Furthermore,

$$\begin{aligned} K_1(x, y) &= \int_0^\alpha 1_{[t,\alpha)}(x) 1_{[t,\alpha)}(y)\, dt + \int_\alpha^1 1_{[\alpha,t)}(x) 1_{[\alpha,t)}(y)\, dt \\ &= \min(x, y)\, 1_{[0,\alpha)^2}((x, y)) + (1 - \max(x, y))\, 1_{[\alpha,1)^2}((x, y)). \end{aligned}$$

For $\alpha = 0$ we obtain the previous case, whereas for $\alpha = 1$, we have

$$K_1(x, y) = \min(x, y) \quad \text{for all } x, y \in [0, 1).$$

This corresponds to the Sobolev space anchored at 0.

Consider now $\alpha \in (0, 1)$. Then $H(K_1)$ is the space of functions f defined over $[0, 1]$ such that f vanishes at 0 and 1. Furthermore, f restricted to $[0, \alpha)$ is absolutely continuous with $f' \in L_2([0, \alpha))$, and f restricted to $[\alpha, 1]$ is absolutely continuous with $f' \in L_2([\alpha, 1])$. However, the function f may be *discontinuous* at α. The inner product for $f, g \in H(K_1)$ is

$$\langle f, g \rangle_{H(K_1)} = \int_0^\alpha f'(x) g'(x)\, dx + \int_\alpha^1 f'(x) g'(x)\, dx = \int_0^1 f'(x) g'(x)\, dx.$$

Here, the derivatives are meant point-wise almost everywhere.

Despite many similarities to a subspace of the Sobolev space anchored at α, the property that f may be discontinuous at α makes this space different than the Sobolev space. We stress that the difference between $H(K_1)$ and the Sobolev space is also necessary from a different point of view. Namely, in Subsection 9.5.3, we showed that we need to change the sample points from t_j to $(\alpha - t_j) \bmod 1$ to get a relation to the Sobolev space. Here, for the space $H(K_1)$ we used the same sample points for both discrepancy and integration. The last property requires the space $H(K_1)$ to be different than the Sobolev space.

For $d \ge 1$, we use the tensor product property and obtain

$$K_d(x, y) = \prod_{j=1}^{d} \big[\min(x_j, y_j)\, 1_{[0,\alpha_j)^2}((x_j, y_j)) + (1 - \max(x_j, y_j))\, 1_{[\alpha_j,1)^2}((x_j, y_j))\big].$$

We now turn to the L_2 quadrant discrepancy anchored at α. Again we have $D = [0,1]^d$, $\varrho(t) = 1$ and $B(t) = Q(t)$, as given in Subsection 9.5.4. For $d = 1$, we have $B(t) = [0,t)$ for $t < \alpha$, and $B(t) = [t,1)$ for $t \geq \alpha$. Therefore $D_1 = [0,1]$ and

$$\begin{aligned} K_1(x,y) &= \int_0^{\alpha} 1_{[0,t)}(x) 1_{[0,t)}(y)\,\mathrm{d}t + \int_{\alpha}^{1} 1_{[t,1)}(x) 1_{[t,1)}(y)\,\mathrm{d}t \\ &= (\alpha - \max(x,y))_+ + (\min(x,y) - \alpha)_+ \\ &= \tfrac{1}{2}\left(|x-\alpha| + |y-\alpha| - |x-y|\right). \end{aligned}$$

This and the tensor product property of $Q(t)$ yield that

$$K_d(x,y) = \prod_{j=1}^{d} \tfrac{1}{2}\left[|x_j - \alpha_j| + |y_j - \alpha_j| - |x_j - y_j|\right] \quad \text{for all } x,y \in D_d = [0,1]^d.$$

Hence, $H(K_d)$ is the Sobolev space anchored at α, which agrees with the results of Subsection 9.5.4.

We switch to unanchored discrepancy. We now have $\tau(d) = 2d$, $D = \{(x,y) \in [0,1]^{2d} \mid x \leq y\}$, $\varrho(t) = 2^d$ and $B(t) = [t_1,t_2)$ for $t = (t_1,t_2)$ with $t_i \in [0,1]^d$ and $t_1 \leq t_2$. Then $D_d = [0,1]^d$. For $d = 1$ we have

$$\begin{aligned} K_1(x,y) &= 2\int_0^1 \left(\int_{t_1}^{1} 1_{[t_1,t_2)}(x) 1_{[t_1,t_2)}(y)\,\mathrm{d}t_2\right)\mathrm{d}t_1 \\ &= 2\int_0^{\min(x,y)} \left(\int_{\max(x,y)}^{1} \mathrm{d}t_2\right)\mathrm{d}t_1 \\ &= 2\min(x,y)(1-\max(x,y)) = 2\left(\min(x,y) - xy\right). \end{aligned}$$

For $d \geq 1$, we use the tensor product property to obtain

$$K_d(x,y) = \prod_{j=1}^{d} 2\left(\min(x_j,y_j) - x_j y_j\right).$$

Hence, $K_d = 2^d K_d^{\mathrm{ex}}$, and $H(K_d)$ is the same as the space $H(K_d^{\mathrm{ex}})$ for the unanchored discrepancy, modulo the normalizing factor 2^d, so that

$$\langle f,g\rangle_{H(K_d)} = 2^{-d}\,\langle f,g\rangle_{H(K_d^{\mathrm{ex}})} \quad \text{for all } f,g \in H(K_d).$$

This agrees with the results of Subsection 9.5.5.

We can also have new examples of $B(t)$. For instance, let $\tau(d) = d+1$, and

$$D = \{\,(c,r) \mid c \in \mathbb{R}^d \text{ and } r \geq 0\,\}.$$

The weight function ϱ may be defined as $\varrho(c,r) = 1$ for $[c,r] \in [0,1]^{d+1}$ and zero otherwise. We may also consider the case for which ϱ is the density function of the

Gaussian measure on $\mathbb{R}^d \times \mathbb{R}_+$,

$$\varrho(c,r) = 2(2\pi)^{-(d+1)/2} \exp\Big(-r^2/2 - \sum_{j=1}^{d} c_j^2/2\Big).$$

Then for $t = (c,r)$, define

$$B(t) = \{x \in \mathbb{R}^d \mid \|x-c\|_p \le r\}$$

as the ball at center c and radius r in the usual l_p norm given by

$$\|x-c\|_p = \Big(\sum_{j=1}^{d} |x_j - c_j|^p\Big)^{1/p}$$

for $p \in [1,\infty)$ and $\|x-c\|_\infty = \max_{j\in[d]} |x_j - c_j|$. This corresponds to the *ball discrepancy in the l_p norm.*

We may also follow Chen and Travaglini [29] and define a *periodic ball discrepancy in the l_p case* as follows. For $x, y \in [0,1]^d$, let

$$\|x-y\|_p^* = \Big(\sum_{k=1}^{d} |x_k - y_k|_*^p\Big)^{1/p},$$

where $|x_k - y_k|_* = \min\{|x_k - y_k|, 1 - |x_k - y_k|\}$. Observe that

$$\|x-y\|_p^* \le \frac{d^{1/p}}{2}, \quad x, y \in [0,1]^d.$$

Now let $\tau(d) = d + 1$ and define

$$D = \Big\{(c,r) \mid c \in [0,1]^d,\ 0 \le r \le \frac{d^{1/p}}{2}\Big\}.$$

For $t = (c,r)$, we consider the set

$$B(t) = \{x \in [0,1]^d \mid \|x-c\|_p^* \le r\}.$$

To obtain an invariant kernel of the form $K_{d,p}(x,y) = k_{d,p}(x-y)$ we consider weights independent of c, i.e., of the form $\varrho(c,r) = \tilde{\varrho}(r)$. Formulas for the periodic ball discrepancy $\mathrm{disc}_2^{\mathrm{pball}}(n,d)$ in the case $p = 2$ can be found in Chen and Travaglini [29] in terms of the Fourier coefficients of the characteristic function of the ball of radius r. A lower bound of Beck [8] and Montgomery [190] states that for arbitrary d, there is a constant c_d such that

$$\mathrm{disc}_2^{\mathrm{pball}}(n,d) \ge c_d\, n^{-1/2-1/(2d)}$$

for all n. This bound is essentially sharp, as shown by Beck and Chen [10]. Note that for large d, the exponent of n^{-1} is close to $\frac{1}{2}$ which is the worst possible exponent for all B-discrepancies for which the sets $B(t)$ are subsets of D_d with $\mathrm{vol}(D_d) < \infty$, see Theorem 9.7.

We turn to the case $p = \infty$. Then we obtain $x \in B(t)$ iff

$$x_j - r \le c_j \le x_j + r \quad \text{or} \quad c_j \le x_j + r - 1 \quad \text{or} \quad c_j \ge 1 - r + x_j \quad \text{for all } j. \tag{9.49}$$

For given $x_j, y_j \in [0,1]$ and $r \in [0,1/2]$, let

$$\ell(x_j, y_j, r) = \int_0^1 1_{|x_j - c_j|_* \le r}(x_j)\, 1_{|y_j - c_j|_* \le r}(y_j)\, \mathrm{d}c_j .$$

Then (9.49) yields that $\ell(x_j, y_j, r)$ depends only on $\alpha = |x_j - y_j|_*$ and r, i.e., $\ell(x_j, y_j, r) = \ell(\alpha, r)$, and $\ell(\alpha, r) = 0$ if $r \le \alpha/2$, $\ell(\alpha, r) = 2r - \alpha$ if $\alpha/2 \le r \le 1/2 - \alpha/2$, and $\ell(\alpha, r) = -1 + 4r$ if $1/2 - \alpha/2 \le r \le 1/2$. Combining these results, we obtain the kernel

$$K_d(x, y) = \int_0^{1/2} \prod_{j=1}^{d} \ell(|x_j - y_j|_*, r)\, \tilde{\varrho}(r)\, \mathrm{d}r.$$

Observe that $K_d(x, y)$ only depends on the $|x_j - y_j|_*$. In particular, K_d is of the form $K_d(x, y) = k_d(x - y)$. In the case $\tilde{\varrho} = 2 \cdot 1_{[0,1/2]}$ and $d = 1$, we obtain the kernel $K_1(x, y) = \frac{1}{2} - |x - y|_* + |x - y|_*^2$.

We finally turn to the ball discrepancy in the l_∞ case. We now have $D = \mathbb{R}^d \times \mathbb{R}_+ = \{[c, r] \mid c \in \mathbb{R}^d, r \ge 0\}$ and we take $\varrho(c, r) = 1$ for $t = [c, r] \in [0, 1]^{d+1}$ and zero otherwise. The sets $B(t)$ are taken as the balls

$$B(t) = \{x \in \mathbb{R}^d \mid \max_{j \in [d]} |x_j - c_j| \le r\}.$$

Observe that $x \in B(t)$ means that $x_j - r \le c_j \le x_j + r$ for all $j \in [d]$. Hence, $x, y \in B(t)$ and $c \in [0, 1]^d$ yield that

$$c_j \in \left[\max(0, x_j - r, y_j - r), \min(1, x_j + r, y_j + r)\right].$$

This easily implies that

$$K_d(x, y) = \int_0^1 \prod_{j=1}^{d} \left[\min(1, x_j + r, y_j + r) - \max(0, x_j - r, y_j - r)\right]_+ \mathrm{d}r.$$

From this formula we conclude that $K_d(x, y) = 0$ if there exists j such that $x_j \ge 2$ or $y_j \ge 2$. Similarly, $K_d(x, y) = 0$ if there exists j such that $x_j \le -1$ or $y_j \le -1$. This means that the space $H(K_d)$ consists of functions that vanish outside $(-1, 2)^d$.

So far, we have defined B-discrepancy for the unweighted case. It is also possible to define *weighted* B-discrepancy by following the approach we used in defining the weighted L_2 discrepancy. We leave this to the reader as our next open problem.

Open Problem 35.

- Define weighted B-discrepancy analogously to the weighted L_2 discrepancy and find relations to multivariate integration defined over a reproducing kernel Hilbert space. In particular, generalize the formula (9.47) for the reproducing kernel in the weighted case.

This open problem was formulated in June 2009. Soon after that we sent this chapter for comments to many colleagues. Michael Gnewuch became interested in this problem and solved it in [80].

We showed that the B-discrepancy corresponds to multivariate integration for the reproducing kernel Hilbert space $H(K_d)$ with the reproducing kernel K_d given by (9.47). We now may ask the opposite question: for any reproducing kernel for which multivariate integration is well defined in the space $H(K_d)$, does there exist a family of sets $B(t)$ for which (9.47) holds? We can easily see that the answer is *no*. If K_d is given by (9.47) then its values are in [0, 1] which, in general, does not hold for reproducing kernels. For example, we may take the Korobov space of smooth periodic functions for $d = 1$, see e.g., Sloan and Joe [273]. For a specific smoothness, the kernel is

$$K_1(x,y) = 1 + 2\pi^2\,\mathrm{B}_2\,((x-y) \bmod 1) \quad \text{for all } x,y \in [0,1],$$

see Appendix A of Volume I with $\beta_1 = \beta_2 = r = 1$. Here, $\mathrm{B}_2(t) = t^2 - t + \frac{1}{6}$ is the Bernoulli polynomial of degree 2. We then have $K_1(x,x) = 1 + \pi^2/3 > 1$ and $K_1(\frac{1}{2}, 0) = 1 - \pi^2/6 < 0$.

9.7 Tractability

We now recall what is meant by *tractability*, see Volume I for background, history and motivation. To stress the role of QMC algorithms we also adapt the notions of tractability for this important class of algorithms and call it *QMC-tractability*.

Recall that $\overline{e^{\mathrm{wor}}}(n, H(K_d))$ and $e^{\mathrm{wor}}(n, H(K_d))$, defined in Section 9.4, denote the minimal worst case errors for multivariate integration in the reproducing kernel Hilbert space $H(K_d)$ for optimally chosen sample points and coefficients $a_j = n^{-1}$ or for optimally chosen coefficients a_j, respectively. For simplicity we denote both numbers $\overline{e^{\mathrm{wor}}}(n, H(K_d))$ and $e^{\mathrm{wor}}(n, H(K_d))$ by $e(n,d)$. These are the same for $n = 0$ with

$$e(0,d) = \overline{e^{\mathrm{wor}}}(0, H(K_d)) = e^{\mathrm{wor}}(0, H(K_d)) = \|I_d\|.$$

Here, $e(0,d)$ denotes the initial error.

For the absolute error criterion, we want to find the smallest n for which $e(n,d)$ is at most ε. For the normalized error criterion, we want to find the smallest n for which $e(n,d)$ is at most $\varepsilon\, e(0,d)$, that is, we want to reduce the initial error by a factor ε.

Let $\mathrm{CRI}_d = 1$ if we consider the absolute error, and $\mathrm{CRI}_d = e(0, d)$ if we consider the normalized error. Let

$$n(\varepsilon, d) = \min\{\, n \mid e(n, d) \le \varepsilon\, \mathrm{CRI}_d \,\}$$

denote the minimal number of sample points necessary to solve the problem to within ε. Note that both $e(n, d)$ and CRI_d may take two different values, and so we have four different cases of $n(\varepsilon, d)$.

Tractability means that $n(\varepsilon, d)$ does *not* depend exponentially on ε and d. There are obviously many different ways to measure the lack of exponential behavior, but we restrict ourselves to only three cases in this section.

By the multivariate problem $\mathrm{INT} = \{I_d\}_{d\in\mathbb{N}}$ we mean multivariate integration I_d defined on the reproducing kernel Hilbert space $H(K_d)$ for varying $d \in \mathbb{N}$.

We say that INT is *weakly tractable* iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0.$$

This means that $n(\varepsilon, d)$ is much smaller than $a^{\varepsilon^{-1}+d}$ for large $\varepsilon^{-1} + d$, and this holds for any $a > 1$. Hence, weak tractability implies that $n(\varepsilon, d)$ may go to infinity but slower than exponentially in $\varepsilon^{-1} + d$. If weak tractability does not hold then we say that INT is *intractable*. If $n(\varepsilon, d)$ is an exponential function of d for some ε then we say that INT suffers from the *curse of dimensionality*.

We say that INT is *polynomially tractable* iff there are three non-negative numbers C, p and q such that

$$n(\varepsilon, d) \le C\, \varepsilon^{-p}\, d^{q} \quad \text{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N}.$$

Hence, polynomial tractability means that $n(\varepsilon, d)$ may grow no faster than polynomially in ε^{-1} and d. If $q = 0$ in the bound above, that is,

$$n(\varepsilon, d) \le C\, \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N},$$

then we say that INT is *strongly polynomially tractable*, and the infimum of p satisfying the last bound is called the *exponent of strong polynomial tractability*.

Non-exponential behavior of $n(\varepsilon, d)$ can occur in many ways. As in [83], let $T : [1, \infty) \times [1, \infty) \to [1, \infty)$ be a non-decreasing function of the two arguments such that

$$\lim_{x+y\to\infty} \frac{\ln T(x, y)}{x + y} = 0.$$

We say that INT is *T-tractable* iff there are two non-negative numbers C and t such that

$$n(\varepsilon, d) \le C\, T(\varepsilon^{-1}\, d)^{t} \quad \text{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N}.$$

We say that INT is *strongly T-tractable* iff there are two non-negative number C and t such that

$$n(\varepsilon, d) \le C\, T(\varepsilon^{-1}\, 1)^{t} \quad \text{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N}.$$

The infimum of t satisfying one of the last two estimates is called the *exponent of T-tractability* and the *exponent of strong T-tractability*.

Note that for $T(x, y) = xy$, polynomial tractability and T-tractability are the same. Other interesting choices of T include

$$T(x, y) = \exp\left((1 + \ln x)\,(1 + \ln y)\right) \quad \text{or} \quad T(x, y) = \exp\left((x + y)^a\right) \qquad \text{for } a \in (0, 1).$$

For these last two examples, T tends to infinity faster than a polynomial of any degree.

Obviously, polynomial tractability or T-tractability implies weak tractability, and the lack of weak tractability implies the lack of polynomial and T-tractability.

To distinguish the case when $e(n, d) = \overline{e^{\text{wor}}}(n, H(K_d))$, i.e., when we use QMC algorithms for approximating the multivariate integrands, we will talk about *QMC-tractability* instead of tractability. Obviously, QMC-tractability implies tractability. To review our (elaborated) notation, note that we have weak, polynomial, strong polynomial, and T and strong T-tractability when we use arbitrary coefficients a_j, and we have all these concepts for QMC-tractability if we use $a_j = n^{-1}$. Furthermore, all these concepts are defined for the absolute or normalized error criterion.

The major problem studied in this volume is to verify for which spaces and for which linear or non-linear functionals we have weak tractability, polynomial tractability, strong polynomial tractability, and T-tractability. In particular, we will be interested in finding necessary and sufficient conditions on weights γ for which these notions of tractability hold.

In this section we wish to only illustrate a few tractability results for multivariate integration based on its relations with discrepancy. These tractability results will follow from discrepancy error bounds reported above as well as results obtained in a number of papers cited here. We mainly limit ourselves to multivariate integration for the (weighted) Sobolev space anchored at 0 or 1, which as we now know, is related to the (weighted) L_2 discrepancy anchored at 0. Even in this standard case, there are still open questions, which we will present as open problems. Much more will be presented in further chapters for different spaces and different linear as well as few non-linear functionals along with complete proofs.

We first consider the absolute error criterion. Note that we can now use Theorem 9.7 with $D_d = [0, 1]^d$ and conclude that INT is *strongly polynomially QMC-tractable* with exponent at most 2. From (9.12) we conclude that INT is *strongly polynomially tractable* with exponent at most 1.41274. This corresponds to the unweighted L_2 discrepancy.

We stress that the exponents of strong tractability for equal, positive and general coefficients a_j are not known; it is even not known if they are different from each other. By Matoušek's result [183], we know that the exponent of strong polynomial QMC-tractability must be at least 1.0669. This leads us to the following open problems.

Open Problem 36.

- Consider multivariate integration for the Sobolev space anchored at 0 or 1 in the worst case setting for the absolute error criterion.

– Find the exponent p^{str} of strong polynomial tractability allowing linear algorithms with arbitrary sample points and arbitrary coefficients. Today, we only know that
$$1 \le p^{\text{str}} \le 1.41274.$$

– Find the exponent p^{str} of strong polynomial tractability allowing linear algorithms with arbitrary sample points and arbitrary positive coefficients. Today, we only know that
$$1 \le p^{\text{str}} \le 2.$$

– Find the exponent p^{str} of strong polynomial tractability allowing linear algorithms with arbitrary sample points and coefficients $a_j = n^{-1}$. Today, we only know that
$$1.0669 \le p^{\text{str}} \le 2.$$

Open Problem 37.

- Consider multivariate integration for the Sobolev space anchored at 0 or 1 in the worst case setting for the absolute error criterion. Construct sample points for arbitrary, positive or equal coefficients achieving the exponent of strong polynomial tractability. As always, by construction we mean a polynomial time construction in ε^{-1} and d.

Note that the open problems 31 and 36 are related. Namely, the infimum of p^* in Open Problem 31 is also the solution of Open Problem 36 for the corresponding class of linear algorithms. Also, Open Problems 32 and 37 are related. The solution of Open Problem 32 with the minimal (or the infimum of) p is also the solution of Open Problem 37. Hence, Open Problems 36 and 37 are more difficult than Open Problems 31 and 32. Obviously, this may convince the reader to attack first Open Problems 31 and 32.

We now consider the normalized error criterion for multivariate integration for the Sobolev space anchored at 0 or 1. Since the initial error is now exponentially small in d, as we know it is $3^{-d/2}$, the tractability results are quite different. The lower bound in (9.17) proved in [221] means that INT is now *intractable*, even if we allow arbitrary coefficients. In this case, we have the curse of dimensionality. We will show this in Chapter 11. For positive weights, the proof of (9.16) is much easier, see [352] or Section 10.5, since the reproducing kernel is positive.

We now switch to multivariate integration for the weighted Sobolev space that corresponds to the L_2 weighted discrepancy defined in Section 9.3. More precisely, we consider a non-zero weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} \ge 0$. For the absolute error criterion, with $\text{CRI}_d = 1$, let
$$f_\gamma(d) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \big(2^{-|\mathfrak{u}|} - 3^{-|\mathfrak{u}|}\big),$$

whereas for the normalized error criterion, with $\mathrm{CRI}_d^2 = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, 3^{-|\mathfrak{u}|}$, let

$$f_\gamma(d) = \frac{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, \left(2^{-|\mathfrak{u}|} - 3^{-|\mathfrak{u}|}\right)}{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, 3^{-|\mathfrak{u}|}}.$$

For the unweighted case $\gamma = \{1\}$, the initial error is $(4/3)^{d/2} \ge 1$, which implies that multivariate integration for the absolute error is now much more difficult than for the normalized error. This along with the lower bound in (9.28) proved in [221], yield *intractability* and the curse of dimensionality of INT for both the absolute and normalized error criteria.

Hence for both these error criteria, we must consider decaying weights to obtain tractability. From the definition of f_γ, which depends on the error criteria, we can consider simultaneously both the absolute and normalized error criteria. From (9.22) we conclude that

$$n(\varepsilon, d) \le \left\lceil \frac{f_\gamma(d)}{\varepsilon^2} \right\rceil.$$

This yields that

$$\lim_{d\to\infty} \frac{\ln f_\gamma(d)}{d} = 0$$

implies *weak QMC-tractability* of INT, whereas

$$\limsup_{d\to\infty} \frac{\ln f_\gamma(d)}{\ln d} < \infty$$

implies *polynomial QMC-tractability* of INT, as well as *T-QMC-tractability* if we choose $T(x,y) = \exp((1+\ln x)\,(1+\ln y))$. For $T(x,y) = \exp((x+y)^a)$ with $a \in (0,1)$ we obtain *T-QMC-tractability* if

$$\limsup_{d\to\infty} \frac{\ln f_\gamma(d)}{d^a} < \infty.$$

Observe also that

$$\sup_d f_\gamma(d) < \infty$$

implies *strong polynomial QMC-tractability* of INT, with the exponent of strong tractability at most 2.

We now consider special classes of weights. The weights are called *finite-order* weights if

$$\gamma_{d,\mathfrak{u}} = 0 \quad \text{for all } |\mathfrak{u}| > \omega$$

for some integer ω independent of d. This concept was defined in [54], see Volume I for more information. Then we may have $\mathcal{O}(d^\omega)$ non-zero weights, which implies that for bounded finite-order weights, i.e., $\sup_{d\in\mathbb{N}} \gamma_{d,\mathfrak{u}} < \infty$, we have $f_\gamma(d) = \mathcal{O}(d^\omega)$ for the absolute error criterion with the factor in the $\mathcal{O}$ notation independent of d and γ. For the normalized error we have

$$f_\gamma(d) \le \frac{\sum_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} 2^{-|\mathfrak{u}|}}{\sum_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|}} = \frac{\sum_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le\omega} (3/2)^{|\mathfrak{u}|} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|}}{\sum_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|}} \le \left(\frac{3}{2}\right)^{\omega}.$$

This means that we have *polynomial QMC-tractability* for the absolute error criterion, and *strong polynomial QMC-tractability* for the normalized error criterion.

The weights are called *finite-diameter* weights if

$$\gamma_{d,\mathfrak{u}} = 0 \quad \text{for all } \operatorname{diam}(\mathfrak{u}) \ge \omega,$$

where $\operatorname{diam}(\mathfrak{u}) = \max_{i,j\in\mathfrak{u}} |i-j|$. This concept is due to Creutzig [36], see again Volume I for more information. Finite-diameter weights are a special case of finite-order weights but now we can have only $\mathcal{O}(d)$ non-zero weights. Therefore, $f_\gamma(d) = \mathcal{O}(d)$ for the absolute error and $f_\gamma(d) = \mathcal{O}(1)$ for the normalized error. Again, we have *polynomial QMC-tractability* for the absolute error criterion, and *strong polynomial QMC-tractability* for the normalized error criterion.

For finite-order weights we know bounds on the worst case errors of the QMC algorithms using Niederreiter, Halton or Sobol sample points. From [275] we know that

$$n(\varepsilon,d) \le d^{\tau}\,\frac{(C\,d\ln d)^{\omega}}{\varepsilon}\,\left(\ln\varepsilon^{-1} + \ln(C\,d\ln d)\right)^{\omega},$$

where $\tau = \omega$ for the absolute error and $\tau = 0$ for the normalized error, and C is a number greater than one independent of ε^{-1} and d.

Note that modulo logarithms we have the best dependence on ε^{-1}; indeed ε^{-1} is a lower bound since even for $d = 1$ we have $n(\varepsilon,1) = \Theta(\varepsilon^{-1})$. The last bound is especially interesting since the construction of the sample points does *not* depend on the finite-order weights. Still we have only polynomial dependence on d.

We may also use a shifted lattice rule

$$Q_{n,d}(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\left(\tfrac{k}{n}\,z + \Delta\right) \bmod 1\right)$$

with the generator vector $z \in \{1,2,\dots,n-1\}^d$ and $\Delta \in [0,1)^d$. Nuyens and Cools [226], [227], [228] proved that the generator z can be computed by the CBC (component-by-component) algorithm with cost $\mathcal{O}(d\,n\ln n)$, see Chapter 16. Then there exists a vector Δ such that for

$$n \le C_a \varepsilon^{-2/a} d^{\,\omega(1-1/a)},$$

the worst case error of $Q_{n,d}$ is at most ε for the normalized error criterion. Here $a \in [1,2)$ and C_a is a positive number depending only on a, see again [275]. This implies that for the normalized error criterion, we have

$$n(\varepsilon,d) \le C_a\,\varepsilon^{-2/a}\,d^{\,\omega(1-1/a)} \quad \text{for } a \in [1,2).$$

Note that for $a = 1$ we have *strong polynomial QMC-tractability* whereas for a close to 2 we have the best possible dependence on ε^{-1} and polynomial dependence on d. However, in this case, the choice of z and Δ depends on the finite-order weights.

We now consider *product weights*, which were the first type of weights studied for multivariate integration and other multivariate problems, see [277] where this concept was defined, and Volume I for more information.

Product weights take the form

$$\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_{d,j}$$

for

$$0 \le \gamma_{d,d} \le \gamma_{d,d-1} \le \cdots \le \gamma_{d,1} \le 1.$$

The main idea behind product weights is that $\gamma_{d,j}$ moderates the importance of the jth variable and that groups of $\mathfrak{u}$ variables are moderated by the product of weights of variables from $\mathfrak{u}$. The successive variables are ordered according to their importance, with the first variable being the most important one and so on.

For product weights, we have

$$f_\gamma(d) = \prod_{j=1}^{d} \left(1 + \tfrac{1}{2}\gamma_{d,j}\right) - \prod_{j=1}^{d} \left(1 + \tfrac{1}{3}\gamma_{d,j}\right)$$

for the absolute error, and

$$f_\gamma(d) = \prod_{j=1}^{d} \frac{1 + \frac{1}{2}\gamma_{d,j}}{1 + \frac{1}{3}\gamma_{d,j}} - 1 \;\in\; \Big[\prod_{j=1}^{d} \left(1 + \tfrac{1}{8}\gamma_{d,j}\right) - 1, \prod_{j=1}^{d} \left(1 + \tfrac{1}{6}\gamma_{d,j}\right) - 1\Big]$$

for the normalized error, since

$$1 + \tfrac{1}{8}x \le \frac{1 + \frac{1}{2}x}{1 + \frac{1}{3}x} \le 1 + \tfrac{1}{6}x$$

for all $x \in [0, 1]$. Obviously, the absolute error criterion is harder than the normalized error criterion.

For both the absolute and normalized error criterion, we obtain *strong polynomial QMC-tractability* if

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty,$$

and *polynomial QMC-tractability* if

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty,$$

see [277]. These conditions are also necessary for both the absolute and normalized error criteria for strong polynomial QMC-tractability and polynomial QMC-tractability, see again [277] for $\gamma_{d,j}$ independent of d. The same conditions are also necessary for strong tractability and polynomial tractability as proved in [221] for $\gamma_{d,j} \equiv \gamma_j$ independent of d, and in [85] for general $\gamma_{d,j}$.

From [85], for both the absolute and normalized error criteria, we have the following results:

- Weak tractability holds iff
$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$
- T-tractability holds iff
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(1+T(1,d))} < \infty,$$
and
$$\limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty.$$
- Strong T-tractability holds iff
$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty,$$
and
$$\limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty.$$

So far, we have discussed the L_2 discrepancy anchored at 0. Similar results hold for the L_2 discrepancy anchored at α, for the L_2 quadrant discrepancy and for the unanchored discrepancy. The main technical tool for lower bounds is to use the property that the corresponding reproducing kernels are decomposable or have finite rank decomposable parts, which allows us to use the results from [221]. This will be done in Chapter 11.

We now briefly address tractability for multivariate integration defined for $H(K_d)$ with the reproducing kernel given by (9.1),
$$K_d(x,y) = \int_D 1_{B(t)}(x) 1_{B(t)}(y)\, \varrho(t)\, \mathrm{d}t \quad \text{for all } x,y \in \mathbb{R}^d.$$
In this case multivariate integration is related to the B-discrepancy. For simplicity we consider only the absolute error criterion. As before, let $B(t) \subseteq D_d$ for all $t \in D$,

where $D_d \subseteq \mathbb{R}^d$ and $\mathrm{vol}(D_d) < \infty$. From Theorem 9.7, it is obvious that we have strong polynomial QMC-tractability with exponent at most 2 if $\mathrm{vol}(D_d)$ is uniformly bounded in d, and polynomial QMC-tractability if $\mathrm{vol}(D_d)$ is polynomially bounded in d. We leave the rest of the tractability problems related to the B-discrepancy as open problems.

Open Problem 38.

- Consider multivariate integration defined for the space $H(K_d)$, with K_d given by (9.1), in the worst case setting for the normalized error criterion. Provide necessary and sufficient conditions on weak, polynomial, strong polynomial and T-tractability.

The next problem is related to Open Problem 35, where the reader was asked to derive weighted B-discrepancy. Having done this, it is then natural to study tractability.

Open Problem 39.

- Consider multivariate integration defined for the Hilbert space related to weighted B-discrepancy. Consider the worst case setting for the absolute and normalized error criteria. Provide necessary and sufficient conditions on weights to get weak, polynomial, strong polynomial and T-tractability.

As mentioned before, in Chapter 10 and 11 we will be studying multivariate integration for spaces that are not necessarily related to discrepancy. In particular, we will do this for Hilbert spaces with general reproducing kernels.

9.8 L_p Discrepancy

In this section, we indicate how various notions of discrepancy can be also studied for the L_p norm with $p \in [1, \infty]$. We restrict ourselves to the L_p discrepancy anchored at 0 and to the centered L_p discrepancy. The case $p = \infty$ corresponds to the star discrepancy, which we will study in the next section. In this book we only study L_p norms of the discrepancy function, for other norms see the recent book of Triebel [312].

9.8.1 L_p Discrepancy Anchored at the Origin

The L_p discrepancy is defined analogously to the L_2 discrepancy anchored at 0. That is, we take the same sets $[0, x)$ for $x \in [0, 1]^d$ as in Section 9.2, and consider the *discrepancy function*

$$\mathrm{disc}(x) = x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i),$$

as in (9.2) with $a_j = \frac{1}{n}$.

The L_p discrepancy of the points $t_1, \ldots, t_n \in [0,1]^d$ is defined by the L_p norm of the discrepancy function disc, i.e., for $p \in [1,\infty)$ we let

$$\mathrm{disc}_p^*(t_1, t_2, \ldots, t_n) = \left(\int_{[0,1]^d} \left| x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i) \right|^p \mathrm{d}x \right)^{1/p}, \quad (9.50)$$

and for $p = \infty$ we let

$$\mathrm{disc}_\infty^*(t_1, t_2, \ldots, t_n) = \sup_{x \in [0,1]^d} \left| x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i) \right|. \quad (9.51)$$

It is customary to call the L_∞ discrepancy the *star* discrepancy. The main problem for the L_p discrepancy is to find points that minimize disc_p^*, and to study how this minimum $\mathrm{disc}_p^*(n,d)$ depends on d and n.

We now show that the L_p discrepancy is related to multivariate integration for all $p \in [1,\infty]$. Let

$$W_q^{\mathbf{1}} := W_q^{(1,1,\ldots,1)}([0,1]^d)$$

be the Sobolev space of functions defined on $[0,1]^d$ that are differentiable in each variable (in the distributional sense) and whose first derivatives have finite L_q-norm, where $1/p + 1/q = 1$. More precisely, for $d = 1$, W_q^1 is the space of absolutely continuous functions whose first derivatives are in $L_q([0,1])$, and for $d > 1$, $W_q^{\mathbf{1}}$ is a tensor product of factors from the univariate case. We consider first the subspace of functions that satisfy the boundary conditions $f(x) = 0$ if at least one component of x is 1 and under the norm

$$\|f\|_{d,q}^* = \left(\int_{[0,1]^d} \left| \frac{\partial^d}{\partial x} f(x) \right|^q \mathrm{d}x \right)^{1/q}$$

for $q \in [1,\infty)$ and

$$\|f\|_{d,\infty}^* = \sup_{x \in [0,1]^d} \left| \frac{\partial^d}{\partial x} f(x) \right|$$

for $q = \infty$. Here, $\partial x = \partial x_1 \partial x_2 \cdots \partial x_d$. That is, we consider the space

$$F_{d,q}^* = \left\{ f \in W_q^{\mathbf{1}} \mid f(x) = 0 \text{ if } x_j = 1 \text{ for some } j \in [1,d], \text{ and } \|f\|_{d,q}^* < \infty \right\}.$$

Note that for $q = 2$ we have $F_{d,2}^* = H(K_d^\beta)$ with $\beta = 0$ for the space $H(K_d^\beta)$ defined in Section 9.4.

Consider the multivariate integration problem

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \quad \text{for } f \in F_{d,q}^*.$$

We approximate $\mathrm{INT}_d(f)$ by *quasi-Monte Carlo* (QMC) algorithms, which are of the form

$$Q_{d,n}(f) = \frac{1}{n} \sum_{j=1}^{n} f(t_j)$$

for some points $t_j \in [0,1]^d$. We now recall Hlawka and Zaremba's identity, see Hlawka [135] and Zaremba [362], which states that for $f \in W_q^{\mathbf{1}}$ we have

$$\mathrm{INT}_d(f) - Q_{d,n}(f) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} (-1)^{|\mathfrak{u}|} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}(x_{\mathfrak{u}}, 1) \, \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1) \, \mathrm{d}x_{\mathfrak{u}}.$$

For $d = 1$ this identity has the form

$$\mathrm{INT}_1(f) - Q_{1,n}(f) = -\int_0^1 \mathrm{disc}(x) \, f'(x) \, \mathrm{d}x,$$

and can be simply proved using integration by parts.

Note that

$$\mathrm{disc}(x_{\mathfrak{u}}, 1) = \prod_{k \in \mathfrak{u}} x_k - \frac{1}{n} \sum_{j=1}^{n} 1_{[0, x_{\mathfrak{u}})} \left((t_j)_{\mathfrak{u}} \right).$$

For $f \in F_{d,q}^*$, due to the boundary conditions, all terms in Hlawka and Zaremba's identity vanish except the term for $\mathfrak{u} = [d] = \{1, 2, \ldots, d\}$. Hence, for $f \in F_{d,q}^*$ we have

$$\mathrm{INT}_d f - Q_{d,n} f = (-1)^d \int_{[0,1]^d} \mathrm{disc}(x) \, \frac{\partial^d}{\partial x} f(x) \, \mathrm{d}x.$$

Applying the Hölder inequality, we obtain that the worst case error of $Q_{n,d}$ is

$$e_q^{\mathrm{wor}}(Q_{d,n}) = \sup_{f \in F_{d,q}^*, \, \|f\|_{d,q}^* \le 1} \left| \mathrm{INT}_d f - Q_{d,n} f \right| = \mathrm{disc}_p^*(t_1, t_2, \ldots, t_n),$$

where p is chosen such that $1/p + 1/q = 1$. For $q = 1$ we have $p = \infty$ and for $q = \infty$ we have $p = 1$.

Hence, the worst case error of $Q_{n,d}$ for the unit ball of $F_{d,q}^*$ is the L_p discrepancy for the points $t_1, t_2, \ldots, t_d$ that are used by the QMC algorithm $Q_{d,n}$.

Now take $n = 0$ and define $Q_{d,0} = 0$. In this case we do not sample the function f. The error of this zero algorithm is the initial worst case error, which is the norm of the linear functional INT_d. It is easy to check that

$$e_q^{\mathrm{wor}}(0) = e_q^{\mathrm{wor}}(Q_{d,0}) = \|\mathrm{INT}_d\| = \left(\frac{1}{p+1} \right)^{d/p},$$

which is 1 for $p = \infty$.

Assume for now that $p < \infty$, or (equivalently) that we consider the multivariate integration problem for the class $F_{d,q}^*$ with $q > 1$. Then the initial error goes to zero exponentially fast with d. This means that the multivariate integration problem for the class $F_{d,q}$ is poorly scaled.

One might claim that this is because we introduced boundary conditions; perhaps it might be hard to find practical applications for which functions satisfy these boundary conditions. Let us agree with this criticism, and remove the boundary conditions. So we now consider the class

$$F_{d,q} = \left\{ f \in W_q^{\mathbf{1}} \mid \|f\|_{d,q} < \infty \right\},$$

where the norm is given by

$$\|f\|_{d,q} = \left(\sum_{\mathfrak{u} \subseteq [d]} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1) \right|^q \mathrm{d}x_{\mathfrak{u}} \right)^{1/q}.$$

The term for $\mathfrak{u} = \emptyset$ corresponds to $|f(1)|^q$.

We return to Hlawka and Zaremba's identity and again apply the Hölder inequality, this time for integrals and sums, and conclude that the worst case error of $Q_{n,d}$ for the unit ball of $F_{d,q}$ is

$$e^{\mathrm{wor}}(Q_{d,n}) = \sup_{f \in F_{d,q}\, \|f\|_{d,q} \le 1} \left| \mathrm{INT}_d f - Q_{d,n} f \right| = \mathrm{disc}_p(t_1, t_2, \dots, t_n),$$

with $1/p + 1/q = 1$, where the *combined* L_p discrepancy disc_p for $p \in [1, \infty]$ is given by

$$\mathrm{disc}_p(t_1, t_2, \dots, t_n) = \Big(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \left[\mathrm{disc}_p^* \left((t_1)_{\mathfrak{u}}, (t_2)_{\mathfrak{u}}, \dots, (t_n)_{\mathfrak{u}} \right) \right]^p \Big)^{1/p},$$

with the usual change to the maximum for $p = \infty$.

What is now the initial error? As before, it is the worst case error of the zero algorithm, which is again the norm of INT_d. However, this time the norm is given in the space $W_q^{\mathbf{1}}$ without boundary conditions, and

$$\begin{aligned} e^{\mathrm{wor}}(0) &:= e^{\mathrm{wor}}(Q_{d,0}) = \|\mathrm{INT}_d\| \\ &= \Big(\sum_{\mathfrak{u} \subseteq [d]} (p+1)^{-|\mathfrak{u}|} \Big)^{1/p} = \Big(\sum_{j=0}^{d} \binom{d}{j} (p+1)^{-j} \Big)^{1/p} \\ &= \left(1 + \frac{1}{p+1} \right)^{d/p}. \end{aligned}$$

So the initial error is now exponentially *large* in d for all $p < \infty$ or all $q > 1$. For $q = 1$ we have $p = \infty$ and the initial error is 1.

We now consider the normalized error criterion, i.e., we want to reduce the initial error by a factor ε and to solve the problem to within $\varepsilon\, e^{\mathrm{wor}}(0)$, under the natural assumption that $\varepsilon \in (0, 1)$. As usual, we define $n_q(\varepsilon, d)$ as the minimal number of function values needed to solve the problem to within $\varepsilon\, e^{\mathrm{wor}}(0)$ and ask again whether the integration problem is tractable.

The usual bounds on the L_p discrepancy are for a fixed dimension d and large n. It is well known that the asymptotic behavior of $\mathrm{disc}_p^*(n, d)$ with respect to n is of order at most $n^{-1}(\ln n)^{d-1}$, see once more Drmota and Tichy [61] and Niederreiter [201]. As before, points for which the L_p discrepancy has a bound proportional to $n^{-1}(\ln n)^d$ are called *low discrepancy points*. There is a deep and still evolving theory dealing with how to construct such low discrepancy points. This theory is mostly due to Niederreiter and his collaborators. The reader is referred to a recent monograph of Dick and Pillichshammer [53] for the state of the art of this subject.

We discuss the L_p discrepancy for uniformly distributed points. We consider only even p and define the average L_p discrepancy as

$$\mathrm{avg}_p(n,d) = \left(\int_{[0,1]^{nd}} \mathrm{disc}_p^*(t_1,t_2,\dots,t_n)^p \, \mathrm{d}t \right)^{1/p}, \qquad t = (t_1,t_2,\dots,t_n).$$

As shown in [115], the average L_p discrepancy depends on the Stirling numbers $s(k,i)$ of the first kind, and $S(k,i)$ of the second kind, see Riordan [250]. We have

$$\mathrm{avg}_p(n,d)^p = \sum_{r=p/2}^{p-1} C(r,p,d)\, n^{-r}, \tag{9.52}$$

where

$$C(r,p,d) = (-1)^r \sum_{i=0}^{p-r} \binom{p}{r+i} (-1)^i \sum_{k=i}^{i+r} (p+1-r+k-i)^{-d} s(k,i) S(i+r,k). \tag{9.53}$$

Furthermore,

$$|C(r,p,d)| \le \frac{(r+1)(4p)^p}{(p+1-r)^d},$$

and

$$\mathrm{avg}_p(n,d) \le \frac{4\sqrt{2}\; p\,(1+p/2)^{-d/p}}{n^{1/2}} \left(\sum_{i=0}^{p/2-1} n^{-i} \left(\frac{1+p/2}{1+p/2-i} \right)^d \right)^{1/p}.$$

Gnewuch [77] proved also the upper bound

$$\mathrm{avg}_p(n,d) \le 3^{2/3} 2^{5/2+d/p}\, p(p+2)^{-d/p} n^{-1/2}.$$

Hinrichs suggested to use symmetrization, see again Gnewuch [77] and [210], yielding

$$\mathrm{avg}_p(n,d) \le \begin{cases} 2^{3/2-d/p} n^{-1/2} & \text{if } p < 2d, \\ 2^{1/2+d/p} p^{1/2} (p+2)^{-d/p} n^{-1/2} & \text{if } p \ge 2d. \end{cases}$$

Formula (9.52) was generalized by Leobacher and Pillichshammer [171] to the weighted discrepancy. These authors then deduced conditions for tractability of the respective integration problems. Similar bounds for the average L_p extreme discrepancy were proved by Gnewuch [77].

From the mean value theorem, we know that there are points $t_1, t_2, \dots, t_n$ such that $\mathrm{disc}_p^*(t_1,t_2,\dots,t_n) \le \mathrm{avg}_p(n,d)$. However, it is *not* known how to construct them. This is our next open problem.

Open Problem 40.

- Construct in time polynomial in n and d, points $t_1, t_2, \dots, t_n$ for which

$$\mathrm{disc}_p^*(t_1, t_2, \dots, t_n) \le \mathrm{avg}_p(n, d).$$

Since it might be difficult to construct such points, we also propose a more modest problem. Observe that $\mathrm{avg}_p(n, d)$ is defined in terms of $n\, d$ random numbers. Our next question is whether we can obtain similar upper bounds on $\mathrm{avg}_p(n, d)$ when we use less random numbers, see [210]. More precisely, consider generalized lattices with shift defined by

$$M_n^{z,\Delta} = \{t_j = jz + \Delta \;(\mathrm{mod}\ 1) \mid j = 0, 1, \dots, n-1\}$$

with $z, \Delta \in [0,1]^d$. We will treat z and Δ as uniformly distributed and independent random vectors. Then two points $t_i, t_j \in M_n^{z,\Delta}$ are uniformly distributed in $[0,1]^d$ and are independent for $i \neq j$. Observe that $M_n^{z,\Delta}$ is given by $2d$ random numbers instead of $n\, d$ random numbers. Our next open problem is as follows.

Open Problem 41.

- Is it true that the inequality

$$\left(\int_{[0,1]^{2d}} \mathrm{disc}_p^*(M_n^{z,\Delta})^p \,\mathrm{d}z\,\mathrm{d}\Delta\right)^{1/p} \le \mathrm{avg}_p(n, d)$$

holds for all even p? It can be checked that this holds for $p = 2$.

A positive answer to this problem would mean that not much randomness is needed. This would be a first step towards derandomization of good sample points for L_p discrepancy.

9.8.2 Centered L_p Discrepancy

The centered discrepancy may be also considered in the L_p norm. This discrepancy is related to multivariate integration for the Sobolev space with first derivatives in the L_q norm, where as always $1/p + 1/q = 1$. We now present this relation, and for a change, we start first with multivariate integration.

For $d = 1$, we take the space $F_{1,q}$ as the Sobolev space of absolutely continuous functions whose first derivatives are in $L_q([0,1])$ and that vanish at $\frac{1}{2}$. The norm in $F_{1,q}$ is given by

$$\|f\|_{F_{1,q}} = \begin{cases} \left(\int_0^1 |f'(t)|^q \,\mathrm{d}t\right)^{1/q} & \text{if } q < \infty, \\ \mathrm{ess\,sup}_{t\in[0,1]} |f'(t)| & \text{if } q = \infty. \end{cases}$$

For $d > 1$, the space $F_{d,q}$ is taken as a tensor product of factors $F_{1,q}$. Then functions from $F_{d,q}$ vanish at x whenever at least one component of x is $\frac{1}{2}$. The norm in $F_{d,q}$ is given by

$$\|f\|_{F_{d,q}} = \|D^{\vec{1}} f\|_{L_q([0,1]^d} = \left(\int_{[0,1]^d} |D^{\vec{1}} f(x)|^q \, \mathrm{d}x \right)^{1/q},$$

where $\vec{1} = [1, 1, \ldots, 1]$ and $D^{\vec{1}} = \partial^d / \partial x_1 \cdots \partial x_d$.

We have

$$I_d(f) - Q_{n,d}(f) = \int_{[0,1]^d} D^{\vec{1}} f(t) \, D^{\vec{1}} \Big(h_d - \sum_{i=1}^{n} a_i K_d(\cdot, z_i) \Big)(t) \, \mathrm{d}t,$$

with

$$h_d(x) = 2^{-d} \prod_{j=1}^{d} \left(|x_j - \tfrac{1}{2}| - |x_j - \tfrac{1}{2}|^2 \right),$$

$$K_d(x,t) = 2^{-d} \prod_{j=1}^{d} \left(|x_j - \tfrac{1}{2}| + |t_j - \tfrac{1}{2}| - |x_j - t_j| \right).$$

From this we conclude that

$$e_q(Q_{n,d}) := \sup_{f \in F_{d,q}, \|f\|_{F_{d,q}} \le 1} \big| I_d(f) - Q_{n,d}(f) \big| = \tilde{d}_p^c(Q_{n,d}),$$

where $\tilde{d}_p^c(Q_{n,d})$ is the *centered L_p discrepancy* given by

$$\tilde{d}_p^c(Q_{n,d}) = \left(\int_{[0,1]^d} \Big| \prod_{j=1}^{d} \min(x_j, 1 - x_j) - \sum_{i=1}^{n} a_i \cdot 1_{J(b(x),x)}(z_i) \Big|^p \, \mathrm{d}x \right)^{1/p}.$$

If $q = 1$ then $p = \infty$ and, as usual, the integral is replaced by the essential supremum in the formula above.

Let $e(n, F_{d,q}) = \tilde{d}_p(n,d)$ denote the minimal error, or equivalently the minimal centered L_p discrepancy, that can be achieved by using n function values. The initial error, or the initial centered L_p discrepancy, is now given by

$$e(0, F_{d,q}) = \tilde{d}_p^c(0,d) = \begin{cases} 2^{-d} \, (p+1)^{-d/p} & \text{if } q > 1, \\ 2^{-d} & \text{if } q = 1. \end{cases}$$

Hence, for all values of q, the initial centered discrepancy is at most 2^{-d}.

The following result is from [221]. For $n < 2^d$ and $p < \infty$, we have

$$\tilde{d}_p^c(n,d) \ge \left(1 - n \, 2^{-d} \right)^{1/p} \, \tilde{d}_p^c(0,d).$$

Hence integration is intractable in $F_{d,q}$ for the normalized error criterion, since

$$n(\varepsilon, F_{d,q}) \geq (1-\varepsilon^p)2^d \quad \text{and} \quad \lim_{d\to\infty} \frac{\tilde{d}_p^c(\lceil C^d\, d\rceil, d)}{\tilde{d}_p^c(0,d)} = 1 \quad \text{for all } C \in (1,2).$$

Observe that for $q = 1$ we have $p = \infty$. Since $\tilde{d}_p^c(n,d)$ is a non-decreasing function of p, we have

$$\tilde{d}_\infty^c(n,d) = \tilde{d}_\infty^c(0,d) = 2^{-d} \quad \text{for all } n < 2^d.$$

Hence, unlike the star discrepancy and the extreme discrepancy, see (9.43), we find that integration and the centered discrepancy are intractable for the normalized error criterion when $q = 1$ and $p = \infty$.

It is known that $\tilde{d}_\infty^c(n,d)$ goes to zero at least like $n^{-1}(\ln n)^{d-1}$. However, in view of the previous property we must wait exponentially long in d to see this rate of convergence.

9.8.3 Spaces Without Boundary Values

We now remove the condition that $f(x) = 0$ if at least one component of x is $\frac{1}{2}$, which was imposed when we discussed multivariate integration related to the centered L_p discrepancy. That is, for $d = 1$ we take $D_1 = [0,1]$, and let $F_{1,q,\gamma}$ denote the Sobolev space $W_q^1([0,1])$ with the norm

$$\|f\|_{F_{1,q,\gamma}} = \left(\left|f(\tfrac{1}{2})\right|^q + \gamma^{-q/2} \int_0^1 |f'(x)|^q \,\mathrm{d}x \right)^{1/q},$$

where $\gamma > 0$. Observe that for $q = \infty$ we have

$$\|f\|_{F_{1,\infty,\gamma}} = \max\left(|f(\tfrac{1}{2})|, \gamma^{-1/2} \sup_{t\in[0,1]} |f'(t)| \right).$$

For $q = 2$, we have the Hilbert space with the kernel

$$K_{1,\gamma}(x,t) = 1 + \gamma\, 1_M(x,t) \min(|x - \tfrac{1}{2}|, |t - \tfrac{1}{2}|).$$

For $d > 1$ and $\gamma = \{\gamma_{d,\mathfrak{u}}\}$, we take $F_{d,q,\gamma} = W_q^{(1,1,\dots,1)}([0,1]^d)$ as the tensor product of $W_q^1([0,1])$. The norm in $F_{d,q,\gamma}$ is given by

$$\|f\|_{F_{d,q,\gamma}} = \left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-q/2} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1/2) \right|^q \mathrm{d}x_{\mathfrak{u}} \right)^{1/q}. \tag{9.54}$$

The formula for the error of the algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i f(z_i)$ takes the form,

$$I_d(f) - Q_{n,d}(f) \\ = \sum_{\mathfrak{u}\subseteq[d]} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, \tfrac{1}{2}) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} \Big(h_d - \sum_{i=1}^n a_i K_d(\cdot, z_i)\Big)(x_{\mathfrak{u}}, \tfrac{1}{2})\, \mathrm{d}x_{\mathfrak{u}},$$

see Hickernell [118], where h_d and the kernel K_d are given as before. Applying the Hölder inequality for integrals and sums to $I_d(f) - Q_{n,d}(f)$ we conclude that

$$e_q(Q_{n,d}) := \sup_{f\in F_{d,q,\gamma},\, \|f\|_{F_{d,q,\gamma}}\le 1} \big|I_d(f) - Q_{n,d}(f)\big| = d^c_{p,\gamma}(Q_{n,d}),$$

where, as always, $1/p+1/q=1$, and the weighted centered L_p discrepancy $d^c_{p,\gamma}(Q_{n,d})$ is given by

$$d^c_{p,\gamma}(Q_{n,d}) = \Bigg(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{p/2} \int_{[0,1]^{|\mathfrak{u}|}} |\mathrm{disc}^c(n,d)(x_{\mathfrak{u}}, 1/2)|^p \, \mathrm{d}x_{\mathfrak{u}}\Bigg)^{1/p}, \tag{9.55}$$

with

$$\mathrm{disc}^c(n,d)(x_{\mathfrak{u}}, 1/2) = \prod_{\ell\in\mathfrak{u}} \min(x_\ell, 1-x_\ell) - \sum_{i=1}^n a_i \cdot 1_{J(a(x_{\mathfrak{u}}),x_{\mathfrak{u}})}(t_i)_{\mathfrak{u}}.$$

Let $e(n, F_{d,q,\gamma}) = d_{p,\gamma}(n,d)$ denote the minimal error, or equivalently the minimal weighted centered L_p discrepancy, that can be achieved by using n function values. The initial error, or the initial centered weighted L_p discrepancy, is now given by

$$e(0, F_{d,q}) = d^c_{p,\gamma}(0,d) = \Bigg(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{p/2} \left(\frac{2^{-p}}{p+1}\right)^{|\mathfrak{u}|}\Bigg)^{1/p}.$$

For product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$, we obtain

$$e(0, F_{d,q}) = d^c_{p,\gamma}(0,d) = \prod_{j=1}^d \left(1 + \frac{2^{-p}}{p+1}\gamma_j^{p/2}\right)^{1/p}.$$

For $q=1$, we have $p=\infty$ and

$$e(0, F_{d,1}) = d^c_{\infty,\gamma}(0,d) = \max_{k=0,1,\dots,d} \left(2^{-k}(\gamma_1\gamma_2\cdots\gamma_k)^{1/2}\right).$$

It was proved in [221] that

$$d^c_{p,\gamma}(n,d) \ge \Bigg(\sum_{k=0}^d C_{d,p,k} \left(\frac{2^{-p}}{p+1}\right)^k \cdot (1 - n\,2^{-k})_+\Bigg)^{1/p},$$

where

$$C_{d,p,k} = \sum_{\mathfrak{u}\subseteq[d],|\mathfrak{u}|=k} \gamma_{\mathfrak{u}}^{p/2}.$$

For $n \le 2^m < 2^d$, this can be rewritten for $p < \infty$ as

$$d^c_{p,\gamma}(n,d) \ge 2^{-1/p} \left(1 - \frac{\sum_{k=0}^m C_{d,p,k}\,[2^{-p}/(p+1)]^k}{\sum_{k=0}^d C_{d,p,k}\,[2^{-p}/(p+1)]^k}\right)^{1/p} d^c_{p,\gamma}(0,d),$$

and for $p = \infty$ as

$$d^c_{\infty,\gamma} \ge 2^{-(m+1)}\,(\gamma_1\gamma_2\cdots\gamma_{m+1})^{1/2}.$$

For $q > 1$, i.e., for $p < \infty$, we can determine when tractability of integration does not hold, see [221]. More precisely, consider INT $= \{I_d\}$ with I_d defined over $F_{d,q,\gamma}$. Then $\sum_{j=1}^{\infty} \gamma_j^{p/2} = \infty$ implies

$$\lim_{d\to\infty} \frac{d^c_{p,\gamma}(n,d)}{d^c_{p,\gamma}(0,d)} = 1 \quad \text{for all } n,$$

and integration INT is *not* strongly polynomially tractable. If

$$\lim{}^*_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j^{p/2}}{\ln d} = \infty$$

then

$$\lim{}^*_{d\to\infty} \frac{d^c_{p,\gamma}(d^k,d)}{d^c_{p,\gamma}(0,d)} = 1 \quad \text{for all } k \in \mathbb{N},$$

and integration INT is *not* polynomially tractable, where $\lim^*$ is lim or lim sup. Furthermore, proceeding as in [85], we find that

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j^{p/2}}{d} \ne 0$$

implies that integration INT is intractable.

9.9 Star Discrepancy

We now consider multivariate integration for the spaces $F^*_{d,1}$ and $F_{d,1}$ with $q = 1$. As we know, these two problems correspond to the discrepancy for $p = \infty$, i.e., to the *star discrepancy*. Note that in this case, we have

$$\mathrm{disc}^*_\infty(t_1,t_2,\dots,t_n) = \mathrm{disc}_\infty(t_1,t_2,\dots,t_n),$$

and the multivariate problem is properly scaled, since the initial error is 1 for both spaces $F_{d,1}^{*}$ and $F_{d,1}$. Then

$$n(\varepsilon, d) = \min\{\, n \mid \mathrm{disc}_{\infty}^{*}(t_1, t_2, \dots, t_n) \le \varepsilon \text{ for some } t_1, t_2, \dots, t_n \in [0,1]^d \}$$

is the same for both spaces; it is just the inverse of the star discrepancy.

Hence tractability of integration $\mathrm{INT} = \{I_d\}$, with I_d defined over $F_{d,1}^{*}$ or $F_{d,1}$, depends on how the inverse of the star discrepancy behaves as a function of ε and d. Based on many negative results for classical spaces and on the fact that all variables play the same role for the star discrepancy, it would be natural to expect an exponential dependence on d, i.e., the curse of dimensionality and intractability of integration INT. Therefore it was quite a surprise when a positive result was proved in [115]. More precisely, let

$$\mathrm{disc}_{\infty}^{*}(n, d) = \inf_{t_1, t_2, \dots, t_n \in [0,1]^d} \mathrm{disc}_{\infty}^{*}(t_1, t_2, \dots, t_n)$$

denote the minimal star discrepancy that can be achieved with n points in the d dimensional case for coefficients $a_j = n^{-1}$. The main result of [115] is the following.

Theorem 9.8. *There exists a positive number C such that*

$$\mathrm{disc}_{\infty}^{*}(n, d) \le C\, d^{\,1/2}\, n^{-1/2} \quad \text{for all } n, d \in \mathbb{N}. \tag{9.56}$$

The proof of this bound follows directly from deep results in the theory of empirical processes. In particular, we use a result of Talagrand [292] combined with a result of Haussler [96], as well as a result of Dudley [62] on the VC (Vapnik–Chervonenkis) dimension of the family of rational cubes $[0, x)$. The proof is unfortunately non-constructive, and we do not know points for which this bound holds.

The slightly worse upper bound

$$\mathrm{disc}_{\infty}^{*}(n, d) \le 2\sqrt{2}\, n^{-1/2} \left(d \ln \left(\left\lceil \frac{d n^{1/2}}{2(\ln 2)^{1/2}} \right\rceil + 1 \right) + \ln 2 \right)^{1/2} \tag{9.57}$$

follows from Hoeffding's inequality and is quite elementary, see also Doerr, Gnewuch and Srivastav [58], and Gnewuch [78]. This proof is also non-constructive. However, using a probabilistic argument, it is easy to show that many points $t_1, t_2, \dots, t_n$ satisfy both bounds modulo a multiplicative factor greater than one, see [115] for details.

One can also use the results on the average behavior of the L_p discrepancy for an even integer p to obtain upper bounds for the star discrepancy, see again [115] and Gnewuch [77]. For concrete values of d and n, these upper bounds seem to be better than those presented above.

The upper bounds on $\mathrm{disc}_{\infty}^{*}(n, d)$ can be easily translated into upper bounds on $n(\varepsilon, d)$. In particular, we have

$$n(\varepsilon, d) \le \left\lceil C^2\, d \left(\frac{1}{\varepsilon} \right)^2 \right\rceil, \quad \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}. \tag{9.58}$$

This means that we have *polynomial tractability*. Furthermore it was also shown in [115] that there exists a positive number c such that

$$n(\varepsilon, d) \geq c\, d \ln \varepsilon^{-1} \quad \text{for all } \varepsilon \in (0, 1/64] \text{ and } d \in \mathbb{N}.$$

In fact, this lower bound holds not only for quasi-Monte Carlo algorithms but in full generality for all algorithms. The last bound was improved by Hinrichs [130] who showed that there exist positive numbers c and ε_0 such that

$$n(\varepsilon, d) \geq c\, d\, \varepsilon^{-1} \quad \text{for all } \varepsilon \in (0, \varepsilon_0] \text{ and } d \in \mathbb{N}.$$

The essence of the lower bounds is that we do *not* have strong polynomial tractability, and the factor d in the bounds on $n(\varepsilon, d)$ cannot be removed.

How about the dependence on ε^{-1}? This is open and seems to be a difficult problem. We know that for a fixed d, the minimal star discrepancy $\mathrm{disc}^*_\infty(n, d)$ behaves much better asymptotically in n. More precisely, we know that for arbitrary d, we have

$$\Omega\big(n^{-1}(\ln n)^{(d-1)/2}\big) = \mathrm{disc}^*_\infty(d, n) = \mathcal{O}\big(n^{-1}(\ln n)^{d-1}\big) \quad \text{as } n \to \infty.$$

The lower bound follows from the lower bound on the minimal L_2 discrepancy due to Roth [254], whereas the upper bound is due to Halton [94], see also Hammersley [93]. Another major open problem for the star discrepancy is to find the proper power of the logarithm of n in the asymptotic formula for $\mathrm{disc}^*_\infty(n, d)$.

Hence, modulo powers of logarithms, the star discrepancy behaves like n^{-1}, which is optimal since such behavior is already present for the univariate case $d = 1$. This means that $n(\varepsilon, d)$ grows at least as ε^{-1}. Furthermore, for any d, we have

$$\lim_{\varepsilon \to 0} \frac{n(\varepsilon, d)}{\varepsilon^{-(1+\delta)}} = 0 \quad \text{for any } \delta > 0.$$

This may suggest that the exponent 2 of ε^{-1} in the upper bound on $n(\varepsilon, d)$ in (9.58) can be lowered. However, we think that as long as we consider upper bounds of the form $n(\varepsilon, d) \leq C d^k \varepsilon^{-\alpha}$, the exponent $\alpha \geq 2$ and 2 cannot be improved. This is Open Problem 7 presented in Volume I.

It is not too difficult to modify the proof of (9.57) to obtain similar bounds that are achieved by "constructive" algorithms. However, the running time of such algorithms is extremely high since it is super-exponential in d. It seems to be very difficult to obtain constructive algorithms that have a reasonable running time that is polynomial in d and have error bounds comparable to (9.57).

Using the concept of *greedy approximations*, Temlyakov [299], [300] proves constructive upper bounds for points with a small L_p discrepancy. For the star discrepancy the constructive upper bound is

$$\mathrm{disc}^*_\infty(n, d) \leq C\, d^{3/2} \max(\ln d, \ln n)^{1/2}\, n^{-1/2}, \quad d, n \geq 2$$

with a positive number C independent of n and d. This bound is only slightly worse than (9.56), but there is no detailed analysis of the computing time, to obtain such points with small star discrepancy. Again we suspect that the computing time is at least exponential in d.

Another constructive approach is given in Doerr, Gnewuch and Srivastav [58]. For a given d and ε the authors can ensure a running time for the construction algorithm of order $C^d d^d (\ln d)^d \varepsilon^{-2(d+2)}$ which is too expensive for practical applications for large d. Improved bounds, but still exponential in d, are presented in the papers by Doerr and Gnewuch [56], Doerr, Gnewuch, Kritzer and Pillichshammer [57], Gnewuch [78] and Doerr, Gnewuch and Wahlström [59], [60]. The problem of constructing good n sample points for the star discrepancy in polynomial time in n and d is presented as Open Problem 6 in Volume I.

The upper bounds (9.56) and (9.57) were extended by other authors. We discuss some of these results. Dick [42] proved that there is a single sequence $t_1, t_2, \ldots \in [0,1]^\infty$ such that all projections $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n \in [0,1]^d$ to the first d coordinates have a small discrepancy. For example, a bound $C\sqrt{d \ln(n+1)/n}$ is possible for all $n, d \in \mathbb{N}$, with C independent of n and d. Further improvements can be found in Doerr, Gnewuch, Kritzer and Pillichshammer [57].

We now define *weighted star discrepancy* and present tractability results obtained by Hinrichs, Pillichshammer and Schmid [132]. It is natural to ask for which weights we can achieve strong polynomial tractability or when the exponent of d is smaller than 1. They proved that strong polynomial tractability indeed *does* hold for summable product weights, and does *not* hold if all weights corresponding to two variables are lower bounded by a positive number. They also provide a condition on the weights for which the star discrepancy depends *only* logarithmically on d.

As before for $\mathfrak{u} \subseteq [d]$, $\mathfrak{u} \neq \emptyset$, let $\gamma_{d,\mathfrak{u}}$ be a nonnegative real weight, $|\mathfrak{u}|$ the cardinality of $\mathfrak{u}$, and for a vector $x \in [0,1]^d$, let $x_{\mathfrak{u}}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of x whose indices are in $\mathfrak{u}$. By $(x_{\mathfrak{u}}, 1)$ we mean the vector x from $[0,1]^d$ with all components whose indices are not in $\mathfrak{u}$ replaced by 1. The *discrepancy function* is given, as before, by

$$\operatorname{disc}(x) = x_1 x_2 \cdots x_d - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(t_i).$$

The *weighted star discrepancy* $\operatorname{disc}_{\infty,\gamma}$ of sample points $t_1, t_2, \ldots, t_n \in [0,1]^d$ and given weights $\gamma = \{\gamma_{d,\mathfrak{u}} \mid \mathfrak{u} \subseteq [d], \mathfrak{u} \neq \emptyset\}$ is given by

$$\operatorname{disc}_{\infty,\gamma}(t_1, t_2, \ldots, t_n) = \sup_{x \in [0,1]^d} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} |\operatorname{disc}((x_{\mathfrak{u}}, 1))|. \tag{9.59}$$

It comes as no surprise that the weighted star discrepancy is also related to multivariate integration. Indeed, in this case, we consider the weighted space $F_{d,1,\gamma}$ defined as the space $F_{d,1}$ except that the norm is now given by

$$\|f\|_{F_{d,1,\gamma}} = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1) \right| \mathrm{d}x_{\mathfrak{u}}.$$

Then for any QMC algorithm $Q_{n,d}(f) = n^{-1}\sum_{j=1}^{n} f(t_j)$, Hlawka and Zaremba's identity yields

$$e(Q_{n,d}) = \sup_{f\in F_{d,1,\gamma},\|f\|_{F_{d,1,\gamma}}\le 1} |I_d(f) - Q_{n,d}(f)| = \mathrm{disc}_{\infty,\gamma}(t_1, t_2, \ldots, t_n).$$

Hence, weighted integration $\mathrm{INT}_\gamma = \{I_d\}$, with I_d defined over $F_{d,1,\gamma}$, is ultimately related to the weighted star discrepancy.

The following result is from Hinrichs, Pillichshammer and Schmid [132].

Theorem 9.9. *There is a positive number C such that for any n and d there exist points $t_1, t_2, \ldots, t_n$ from $[0, 1]^d$ for which*

$$\mathrm{disc}_{\infty,\gamma}(t_1, t_2, \ldots, t_n) \le C\,\frac{1 + \ln^{1/2} d}{\sqrt{n}} \max_{\emptyset\ne\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}|\mathfrak{u}|^{1/2}. \tag{9.60}$$

Proof. For given n and d, it was shown in Theorem 3 of [115] that the probability that an i.i.d. randomly chosen point set $t_1, t_2, \ldots, t_n$ from $[0, 1]^d$ has star discrepancy at most $\lambda\sqrt{d/n}$ is at least

$$1 - \left(K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^d,$$

for some number K independent of n and d and for all $\lambda \ge \max(1, K, \lambda_0)$, where λ_0 is such that $K\lambda^2 \le \mathrm{e}^{2\lambda^2}$ for all $\lambda \ge \lambda_0$.

Consider the set

$$A_{n,d} := \{\mathcal{P}_{n,d} \subseteq [0, 1]^d \mid \mathrm{disc}_\infty(\mathcal{P}_{n,d}(\mathfrak{u})) \le \lambda(|\mathfrak{u}|/n)^{1/2} \text{ for all } \mathfrak{u} \subseteq [d],\ \mathfrak{u} \ne \emptyset\},$$

where $\mathcal{P}_{n,d} = \{t_1, t_2, \ldots, t_n\}$ and $\mathcal{P}_{n,d}(\mathfrak{u}) := \{(t_1)_\mathfrak{u}, (t_2)_\mathfrak{u}, \ldots, (t_n)_\mathfrak{u}\}$. Furthermore, for $\mathfrak{u} \subseteq [d]$, $\mathfrak{u} \ne \emptyset$, we define

$$A_{d,\mathfrak{u}} := \left\{\mathcal{P}_{n,d} \subseteq [0, 1]^d \;\middle|\; \mathrm{disc}_\infty(\mathcal{P}_{n,d}(\mathfrak{u})) \le \lambda\sqrt{\tfrac{|\mathfrak{u}|}{n}}\right\}.$$

Then we have

$$A_{n,d} = \bigcap_{\emptyset\ne\mathfrak{u}\subseteq[d]} A_{d,\mathfrak{u}}.$$

Let $\mathbb{P}$ denote the Lebesgue measure in $[0, 1]^{nd}$, and let $A_{d,\mathfrak{u}}^c = [0, 1]^{nd} \setminus A_{d,\mathfrak{u}}$. Then

$$\begin{aligned}\mathbb{P}(A_{n,d}) &= \mathbb{P}\Big(\bigcap_{\emptyset\ne\mathfrak{u}\subseteq[d]} A_{d,\mathfrak{u}}\Big) = 1 - \mathbb{P}\Big(\bigcup_{\emptyset\ne\mathfrak{u}\subseteq[d]} A_{d,\mathfrak{u}}^c\Big) \ge 1 - \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \mathbb{P}\big(A_{d,\mathfrak{u}}^c\big)\\ &\ge 1 - \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \big(K\lambda^2\mathrm{e}^{-2\lambda^2}\big)^{|\mathfrak{u}|} = 1 - \sum_{u=1}^{d}\binom{d}{u}\big(K\lambda^2\mathrm{e}^{-2\lambda^2}\big)^u\\ &= 2 - \big(1 + K\lambda^2\mathrm{e}^{-2\lambda^2}\big)^d.\end{aligned}$$

Now we choose

$$\lambda := \alpha \max\left(1, \sqrt{(\ln d)/(\ln 2)}\right)$$

with $\alpha := \max(2, K, \lambda_0)$. Then for $d = 1$ we obtain

$$\mathbb{P}(A_{n,1}) > 1 - K\alpha^2 \mathrm{e}^{-2\alpha^2} \geq 0,$$

since $c \geq \lambda_0$. For $d \geq 2$ and $x := \alpha^2/\ln 2 > 5$ we have $x^2 \leq 2^x \leq d^x$ and $\ln d \leq d^{x-1}$. Therefore it follows that $x^2 \ln d \leq d^{2x-1}$ and hence

$$\frac{\alpha^3 \ln d}{(\ln 2) d^{2\alpha^2/(\ln 2)}} \leq \frac{\ln 2}{\alpha\, d}.$$

Let $d \geq 2$. From this inequality, we obtain

$$\begin{aligned}\mathbb{P}(A_d) &> 2 - \left(1 + K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^d \geq 2 - \left(1 + \frac{\alpha^3 \ln d}{(\ln 2) d^{2\alpha^2/(\ln 2)}}\right)^d \\ &\geq 2 - \left(1 + \frac{\ln 2}{\alpha\, d}\right)^d > 2 - \mathrm{e}^{(\ln 2)/\alpha} = 2 - 2^{1/\alpha} > 0.\end{aligned}$$

Hence for all $d \in \mathbb{N}$, we have $\mathbb{P}(A_d) > 0$. Thus, there exists a point set $\mathscr{P}_{n,d} \subseteq [0,1]^d$ such that for each $\emptyset \neq \mathfrak{u} \subseteq [d]$ we have

$$\mathrm{disc}_\infty(\mathscr{P}_{n,d}(\mathfrak{u})) \leq \alpha \max\left(1, \sqrt{\frac{\ln d}{\ln 2}}\right)\sqrt{\frac{|\mathfrak{u}|}{n}} \leq C\left(1 + \sqrt{\ln d}\right)\sqrt{\frac{|\mathfrak{u}|}{n}}.$$

For the weighted star discrepancy of this point set, we obtain

$$\mathrm{disc}_{\infty,\gamma}(\mathscr{P}_{n,d}) \leq C\,\frac{1 + \sqrt{\ln d}}{\sqrt{n}} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \sqrt{|\mathfrak{u}|},$$

which is the desired result. □

From Theorem 9.9 we obtain the following conclusion.

Corollary 9.10. *If*

$$C_\gamma := \sup_{d=1,2,\ldots} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \sqrt{|\mathfrak{u}|} < \infty, \tag{9.61}$$

then for the weighted star discrepancy of the point set from Theorem 9.9 we have

$$\mathrm{disc}_{\infty,\gamma}(\mathscr{P}_{n,d}) \leq C \cdot C_\gamma\, \frac{1 + \sqrt{\ln d}}{\sqrt{n}}, \tag{9.62}$$

where the unknown positive number C from Theorem 9.9 is independent of n and d. Hence

$$n(\varepsilon, d) \leq \left\lceil \frac{C^2 \cdot C_\gamma^2 \left(1 + \sqrt{\ln d}\right)^2}{\varepsilon^2} \right\rceil. \tag{9.63}$$

If condition (9.61) holds then (9.63) implies that weighted integration INT_γ, as well as the weighted star discrepancy, is polynomially tractable with d exponent zero and with ε^{-1} exponent at most 2. We stress that we do *not* obtain strong polynomial tractability in this case since we still have the logarithmic dependence on the dimension d. Condition (9.61) is always fulfilled for bounded finite order weights.

We now turn to the case of product weights (independent of the dimension), i.e., $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$ for some γ_j, and give a condition under which the weighted star discrepancy is strongly polynomially tractable. The following result of Hinrichs, Pillichshammer and Schmid [132] is an extension of Corollary 8 of Dick, Niederreiter and Pillichshammer [49].

Theorem 9.11. *Let $n, d \in \mathbb{N}$. For product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$, if*

$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

then there exist $t_1, t_2, \dots, t_n$ from $[0,1]^d$ such that for any $\delta > 0$ we have

$$\mathrm{disc}_{\infty,\gamma}(t_1, t_2, \dots, t_n) \le \frac{C_{\delta,\gamma}}{n^{1-\delta}},$$

where $C_{\delta,\gamma}$ is independent of d and n. Hence the weighted integration INT_γ *as well as the weighted star discrepancy is* strongly polynomially *tractable with the minimal exponent of strong polynomial tractability* 1.

We add that the points $t_1, t_2, \dots, t_n$ considered in Theorem 9.11 are obtained by a superposition of digital nets over $\mathbb{Z}_2$. However, the proof of Theorem 9.11 in [132] is still not constructive as it involves an averaging over all digital nets, see [49]. Constructive results, requiring stronger conditions on the weights, were proved by Wang [320], [321].

The following result of Hinrichs, Pillichshammer and Schmid [132] shows that the logarithmic factor in the dimension d in the tractability results is really needed.

Theorem 9.12. *If the weights $\gamma = \{\gamma_{d,\mathfrak{u}} \mid \mathfrak{u} \subseteq [d], \mathfrak{u} \ne \emptyset\}$ are such that*

$$c := \inf_{\{(d,\mathfrak{u})\,:\,\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le 2\}} \gamma_{d,\mathfrak{u}} > 0,$$

then for any $t_1, t_2, \dots, t_n \in [0,1]^d$ with $2^{n+1} \le d$ we have

$$\mathrm{disc}_{\infty,\gamma}(\mathcal{P}_{n,d}) \ge \frac{c}{12}.$$

In particular, for such weights the weighted integration INT_γ *as well as the weighted star discrepancy is* not *strongly polynomially tractable.*

To stress once more the difficulty of *constructing* points with small star discrepancy, we pose the following open problem:

Open Problem 42.

Find explicitly points $t_1, t_2, \ldots, t_n \in [0,1]^d$ with star discrepancy bounded by 1/4 and

- $n \le 1528$ for $d = 15$,
- $n \le 3187$ for $d = 30$, and
- $n \le 5517$ for $d = 50$.

The fact that such points exist follows from the bound (14) in Doerr, Gnewuch and Srivastav [58]. Actually, taking into account known results for $d < 10$, see Doerr, Gnewuch and Wahlström [60], one could conjecture that $n = 10d$ points are enough to obtain a star-discrepancy of at most 1/4.

9.10 Notes and Remarks

NR 9:1. This chapter is based on our papers [220] and [223]. In [220] we surveyed results obtained up to roughly the year 2000 for L_p discrepancy and multivariate integration including especially the case of star discrepancy. In [223] we introduce B-discrepancy and surveyed results on the L_2 discrepancy obtained up to the year 2007.

NR 9:2. More lower and upper bounds on discrepancy can be found in Chen and Travaglini [28], Dick, Leobacher and Pillichshammer [48], Dick and Pillichshammer [51], Hickernell, Sloan and Wasilkowski [124], Pirsic, Dick and Pillichshammer [243] and Temlyakov [298].

NR 9.2.2:1. The bounds on the exponent of the L_2 discrepancy obtained in [331] and [336] are based on relations between L_2 discrepancy, multivariate integration and approximation in the average case setting with a zero-mean Gaussian measure on the space of continuous function. The covariance kernel of the Gaussian measure is the same as the reproducing kernel of the Hilbert space of multivariate integration. The proofs use the results from Wasilkowski [327] and from [127], [329] and are *non-constructive*. Still the existing proof technique does not allow us to find the exact value of the exponent of the L_2 discrepancy.

NR 9.2.3:1. The main difficulty in the open problems of this subsection is determining the exact exponential dependence on d. Note that even today's lower and upper bounds are pretty close, and the numbers C_2^d appearing in the upper bounds are for C_2 not much larger than 1. This means that these upper bounds can be acceptable for relatively small d. However, as is always the case with exponential functions, the curse of dimensionality will eventually kick in. For example, take the upper bound with $C_2 = 4/3$. Then for $(4/3)^d = 10^{x_d}$ with $x_d = d \cdot 0.1249\ldots$. We have $x_{20} \approx 2.5$, $x_{50} \approx 6.25$ but $x_{360} \approx 45$.

NR 9.3:1. Weighted L_2 discrepancy appeared for the first time in [277] for product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$, followed by [332], which covered product weights of the more general form $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$. The first use of general weights $\gamma_{d,\mathfrak{u}}$ is probably in [128], see Example 1 on page 38 of [128], and the idea of using general weights or, equivalently, a general reproducing kernel of the Hilbert space for the corresponding multivariate integration, is attributed to A. Owen.

NR 9.3.1:1. For the unweighted case $\gamma_{d,\mathfrak{u}} = 1$, the upper bounds on $n_{\{1\}}(\varepsilon, d)$ are of the form $C_2^d \varepsilon^{-2}$ for C_2 even closer to 1 than before for the normalized L_2 discrepancy. In particular, we may take $C_2 = 1.1143\ldots$ and then $C_2^d = 10^{x_d}$ with $x_d = d \cdot 0.047\ldots$. We now have $x_{20} \approx 1.0$, $x_{50} \approx 2.35$ but $x_{360} \approx 17$ is still too large.

NR 9.4:1. The standard relations between multivariate integration and L_2 discrepancy are for QMC algorithms, that is when we use linear algorithms with coefficients $a_j = n^{-1}$. As we shall see later, this choice of coefficients is not good for some spaces, such as Korobov spaces. For Korobov spaces, although multivariate integration is properly normalized and the initial error is 1, the worst case error of an arbitrary QMC algorithm using n points in the d-dimensional case goes exponentially fast to infinity with d for a fixed n. We show in Chapter 10 that a small change of the coefficients n^{-1} will eliminate this bad property, and that the worst case error will then always be at most equal to the initial error.

NR 9.5:1. This section is entirely based on [223].

NR 9.5.2:1. In this subsection we presented relations between (weighted) L_2 discrepancy and multivariate integration in the average case setting. Since (weighted) L_2 discrepancy is also related to multivariate integration in the worst case setting, this implies that multivariate integration in the worst and average case settings are also related. Such relations have been known already for some time; they even hold for general linear functionals. In the worst case setting, we consider a linear functional defined on a reproducing kernel Hilbert space, whereas in the average case setting, we consider the same linear functional defined over a linear space equipped with a zero-mean Gaussian measure whose covariance function is equal to the reproducing kernel of the Hilbert space from the worst case setting. We will use such relations extensively. In this way, we will be able to translate tractability results for general linear functionals in the worst case setting to tractability results for the same problems in the average case setting.

NR 9.6:1. We demonstrated in this section that B-discrepancy is related to multivariate integration for a Hilbert space whose reproducing kernel is given by (9.47). We find this formula quite intriguing. In particular, this kernel takes only values from $[0, 1]$. This is a useful property since there are a number of results valid only for Hilbert spaces whose reproducing kernel is non-negative. In fact, this property was used in the first paper [277] for weighted spaces to establish a lower bound on the worst case error of QMC algorithms.

NR 9.7:1. The purpose of the tractability section is twofold. Firstly, to remind the reader the basic notions of tractability without necessarily reading Volume I. Secondly, to present a number of tractability results as a warm-up before the rest of Volume II. As always, we also presented a number of open problems, this time related to tractability.

NR 9.8:1. Much more is known about L_p discrepancy. Probably the case $p = \infty$ corresponding to the star discrepancy has been the most studied. Again we refer the reader to the books we mentioned earlier, and in particular, to the most recent monograph of Dick and Pillichshammer [53].

NR 9.9:1. In this short section we discussed the star discrepancy. This is one of a few natural examples for which we even have polynomial tractability for the unweighted case. The other known example is for multivariate integration defined over a Hilbert space whose kernel is related to an isotropic Wiener measure, see [128] as well Example 7 in Volume I, Chapter 3.

NR 9.9:2. We mention here a few more papers on star discrepancy. Further upper bounds for the star discrepancy (as well as on the extreme discrepancy) can be found in Doerr, Gnewuch and Srivastav [58], Gnewuch [77], [78] and Mhaskar [186]. We stress again that the computation and/or approximation of the star discrepancy of given points is very difficult, see Gnewuch [78], Gnewuch [79], Gnewuch, Srivastav and Winzen [82] and Thiémard [302], [303].

Further upper bounds for the weighted star discrepancy can be found in Dick, Niederreiter and Pillichshammer [49], Hinrichs, Pillichshammer and Schmid [132], Joe [139], Larcher, Pillichshammer and Scheicher [166], Sinescu [266], Sinescu and L'Ecuyer [267], Sinescu and Joe [268], [269], and Wang [320], [321].

Chapter 10

Worst Case: General Linear Functionals

10.1 Introduction

We now begin our study of the tractability of general (continuous) linear functionals in the worst case setting for the absolute and normalized error criteria. We assume that these functionals are defined over reproducing kernel Hilbert spaces. We study general kernels and their specific form is assumed only in the examples. In particular, the study of weighted kernels is deferred to the next chapters.

Obviously, linear functionals are trivial for the class Λ^{all} of all continuous linear functionals. This means that lower bounds established for the class Λ^{all} in Volume I are useless for such problems. We study the class Λ^{std} of function values and we consider algorithms that use finitely many function values. One of our goals is to establish sharp lower and upper error bounds. We mainly concentrate on upper bounds, whereas lower bounds will be studied in the next chapters.

As we shall see, the results are very rich in possibilities. For example, there are reproducing kernel Hilbert spaces, even of infinite dimension, for which all linear functionals can be computed with an arbitrarily small error by computing just *one* function value. It is possible that such spaces consist of continuous functions; their construction is related to Peano curves. We report these results in Section 10.3.

The information complexity for a linear functional defined over a reproducing kernel Hilbert space is always finite. This means that the nth minimal worst case error goes to zero as n tends to infinity. However, the speed of convergence very much depends on the space and on the linear functional. We show that there are spaces for which the nth minimal error for the d-dimensional case can go to zero as n goes to infinity arbitrarily slowly or quickly. That is, we can have arbitrarily bad or good convergence. Even for $d = 1$, some linear functionals may require exponentially many function values to compute an ε approximation. This peculiar result holds for many standard spaces. We illustrate this point for the Sobolev space anchored at zero, which (as we know from Chapter 9) is related to the L_2 discrepancy. In particular, we show that the sets of linear functionals with arbitrarily good and bad convergence are both dense. This is the subject of Section 10.4.

Not surprisingly, it is quite difficult to establish sharp lower bounds on the nth minimal errors. This problem becomes easier if we assume that the reproducing kernel is point-wise non-negative and we restrict ourselves to algorithms with non-negative coefficients. For positive linear functionals, such as multivariate integration, one might hope that the last assumption on algorithms is not restrictive. Then it is easy to derive a lower bound from which we can conclude necessary conditions for various kinds of tractability under the normalized error criterion. We will show this in Section 10.5.

We then return to the question of whether the use of algorithms with non-negative coefficients is restrictive. Surprisingly enough, sometimes this condition is very restrictive. To see this, we present a space of d-variate functions for which all linear functionals are easy; they can be approximated with an arbitrarily small error if we use at most $d + 1$ function values. This means that we always have polynomial tractability and the exponent of ε^{-1} is zero, whereas the exponent of d is at most 1. On the other hand, if we use algorithms with non-negative coefficients, then the number of function values needed to compute an ε-approximation is exponential in d. For example, for multivariate integration the number of function values must be at least proportional to 5^d. We admit that the space with these surprising results is quite esoteric. It is an interesting open problem to characterize such spaces and to check what happens for more standard spaces. This is the subject of Section 10.6.

We then study the problem of determining for which spaces and for which linear functionals the nth minimal error goes to zero at least as $n^{-1/2}$. This means that the order of convergence is independent of d and is the same as for the standard Monte Carlo algorithm for multivariate integration. However, we stress here that we are still in the worst case setting and our linear functionals are not necessarily multivariate integration. We present a number of error bounds with an explicit dependence on d and that are proportional to $n^{-1/2}$. From these error bounds we conclude sufficient conditions on tractability and present a number of multivariate problems for which these sufficient conditions hold. Typically these bounds hold only for the absolute error criterion when the initial error is less than one.

In Section 10.7 we study multivariate integration. In fact, we slightly generalize what is usually meant by multivariate integration; but this is only a small technical point, which is needed for our further study. We first study QMC (quasi-Monte-Carlo) algorithms, which are often used with much success for high-dimensional integration. We present a well known estimate on the worst case error of QMC algorithms in Subsection 10.7.1 and obtain sufficient conditions on tractability.

It is also known that for some spaces the coefficients of QMC algorithms are too large, see [280]. This is the case for the Korobov space, for which we know a priori that all integrals are in $[-1, 1]$ although the worst case error of any QMC algorithm using n function values tends exponentially fast to infinity with d. The last example motivates the need of a proper normalization for QMC algorithms, and this is the subject of Subsection 10.7.6. It is easy to find the best normalization coefficient. This leads to better tractability conditions, but only for the absolute error criterion. For tensor product spaces, we show that we obtain strong polynomial tractability of multivariate integration for *all* reproducing kernel Hilbert spaces, provided that the initial error for $d = 1$ is less than 1.

The previous subsections are based on the assumption that the reproducing kernel for equal arguments is integrable. For some spaces, this assumption does not hold, and the error bounds presented so far are not applicable. A recent paper of Plaskota, Wasilkowski and Zhao [248] relaxes this assumption by assuming only that its square root is integrable. We report their results in Subsection 10.7.9 for a slightly more general case. This leads to better error bounds, as well as relaxed conditions on tractability.

In Section 10.8 we show that some linear functionals are related to multivariate integration. For such functionals, we can apply the error bounds and tractability conditions obtained for multivariate integration. We illustrate this relation for a number of standard spaces.

We return to general linear functionals in Section 10.9. Based on the approach for multivariate integration, we present conditions under which the nth minimal errors go zero as fast as $n^{-1/2}$. It turns out that this is the case for a dense set of linear functionals. In general, this holds for linear functionals whose representers have a finite norm, the norm being defined in this section and we illustrate the construction of this new norm by a number of examples.

We continue to present open problems related to the subjects covered in this chapter. In this chapter we provide six open problems numbered from 43 to 48.

10.2 Linear Functionals

For fixed $d \geq 1$, let F_d be a reproducing kernel Hilbert space of real functions $f: D_d \to \mathbb{R}$ with $D_d \subseteq \mathbb{R}^d$. The reproducing kernel of F_d is denoted by

$$K_d : D_d \times D_d \to \mathbb{R},$$

and the inner product and the norm of F_d are denoted by $\langle \cdot, \cdot \rangle_{F_d}$ and $\| \cdot \|_{F_d}$. The basic information about reproducing kernel Hilbert spaces can be found in Aronszajn [2] and in the books of Berlinet and Thomas-Agnan [14], and Wahba [319]. In particular, we will often make use of the following facts:

- $f(t) = \langle f, K_d(\cdot, t) \rangle_{F_d}$ for all $f \in F_d$, $t \in D_d$.
- $K_d(x,t) = \langle K_d(\cdot, x), K_d(\cdot, t) \rangle_{F_d}$ for all $x, t \in D_d$.
- $\sqrt{K_d(t,t)} = \|K_d(\cdot, t)\|_{F_d}$ for all $t \in D_d$.
- $|K_d(x,t)| \leq \sqrt{K_d(x,x)}\,\sqrt{K_d(t,t)}$ for all $x, t \in D_d$.

We consider a (continuous) linear functional $I_d : F_d \to \mathbb{R}$. By Riesz's theorem, I_d takes the form

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d$$

for some $h_d \in F_d$. Clearly, the initial error is given by

$$e(0,d) = \|I_d\| = \|h_d\|_{F_d}.$$

Hence, $e(0,d) = 0$ only for trivial problems when $h_d = 0$ and $I_d(f) \equiv 0$.

As we know from Chapter 4, we can restrict our attention to non-adaptive information and linear algorithms. Let $A_{n,d}$ be a linear algorithm that uses at most n function values, so that

$$A_{n,d}(f) = \sum_{j=1}^{n} a_j f(t_j) \quad \text{for all } f \in F_d$$

for some real a_j and some $t_j \in D_d$. Note that

$$I_d(f) - A_{n,d}(f) = \Big\langle f, h_d - \sum_{j=1}^{n} a_j K_d(\cdot, t_j) \Big\rangle_{F_d}$$

and

$$\big| I_d(f) - A_{n,d}(f) \big| \le \|f\|_{F_d} \Big\| h_d - \sum_{j=1}^{n} a_j K_d(\cdot, t_j) \Big\|_{F_d}.$$

Furthermore the last estimate is sharp. This implies that the worst case error of $A_{n,d}$ is given by a well known formula

$$\begin{aligned} e^{\mathrm{wor}}(A_{n,d}) &:= \sup_{f \in F_d, \|f\|_{F_d} \le 1} \big| I_d(f) - A_{n,d}(f) \big| = \Big\| h_d - \sum_{j=1}^{n} a_j K_d(\cdot, t_j) \Big\|_{F_d} \\ &= \Big[\|h_d\|_{F_d}^2 - 2 \sum_{j=1}^{n} a_j h_d(t_j) + \sum_{i,j=1}^{n} a_i a_j K_d(t_i, t_j) \Big]^{1/2}. \end{aligned}$$

For fixed sample points t_j, the coefficients a_j that minimize the worst case error of $A_{n,d}$ are the solution of the $n \times n$ system of linear equations

$$Ma = b$$

with the symmetric and semi-positive definite matrix $M = (K_d(t_i, t_j))_{i,j=1}^n$, and the vectors $b = (h_d(t_j))_{j=1}^n$, $a = (a_j)_{j=1}^n$. The problem of choosing the sample points t_j that minimize the worst case error of $A_{n,d}$ is non-linear, in general, and therefore hard.

As in Volume I, we let $e(n,d) = e^{\mathrm{wor}}(n, I_d)$ denote the nth minimal worst case error, which in our case is equal to

$$\begin{aligned} e(n,d) &= \inf_{a_j \in \mathbb{R}, t_j \in D_d} \ \sup_{f \in F_d, \|f\|_{F_d} \le 1} \Big| I_d(f) - \sum_{j=1}^{n} a_j f(t_j) \Big| \\ &= \inf_{a_j \in \mathbb{R}, t_j \in D_d} \Big\| h_d - \sum_{j=1}^{n} a_j K_d(\cdot, t_j) \Big\|_{F_d}. \end{aligned} \tag{10.1}$$

Hence, the nth minimal worst case error is equal to the approximation error of the function h_d in the (at most) n dimensional subspace spanned by

$$K_d(\cdot, t_1),\ K_d(\cdot, t_2),\ \dots,\ K_d(\cdot, t_n)$$

for the best chosen sample points t_j from D_d.

By $n(\varepsilon, d) = n^{\mathrm{wor}}(\varepsilon, I_d)$ we mean the information complexity for the absolute error criterion, with $\mathrm{CRI}_d = 1$, or for the normalized error criterion, with $\mathrm{CRI}_d = \|I_d\|$, given by

$$n(\varepsilon, d) = \min\big\{ n \mid e(n,d) \le \varepsilon\, \mathrm{CRI}_d \big\}.$$

We stress that for the linear functionals studied in this chapter, the information complexity multiplied by the cost of one function value is practically the same as the total complexity, see Chapter 4 for details.

10.3 One Function Value

We begin analyzing tractability of linear functionals by taking $n = 1$. That is, we now consider the minimal error $e(1,d)$ when we use only one function value. It is obvious that for I_d of the form $I_d(f) = af(t)$ for some $a \in \mathbb{R}$ and $t \in D_d$, we have $h_d = aK_d(\cdot,t)$ and $e(1,d) = 0$. Surprisingly enough, there are spaces F_d of arbitrary dimension such that $e(1,d) = 0$ for *all* I_d. To show such an example of F_d we first derive the formula for $e(1,d)$, which will also be needed for further estimates. We are ready to present the theorem that was originally proved in [218].

Theorem 10.1.

- *For a reproducing kernel Hilbert space F_d, take $I_d(f) = \langle f, h_d\rangle_{F_d}$ for all $f \in F_d$ and for some $h_d \in F_d$. We have*

$$e^{\text{wor}}(1, I_d) = e(1,d) = \sqrt{e^2(0,d) - \sup_{t\in D_d} \frac{h_d^2(t)}{K_d(t,t)}} \tag{10.2}$$

  with the convention that $0/0 = 0$. Moreover,

$$e(0,d) > 0 \quad \textit{implies} \quad e(1,d) < e(0,d).$$

- *For any positive integer k or for $k = +\infty$, there exists a reproducing kernel Hilbert space F_d of dimension k such that*

$$e^{\text{wor}}(1, I_d) = 0$$

  for all linear functionals I_d.

Proof. We first prove the formula for $e(1,d)$. For arbitrary $t \in D_d$ and $a \in \mathbb{R}$, we have

$$\|h_d - a\,K_d(\cdot,t)\|_{F_d}^2 = e^2(0,d) - 2a\,h_d(t) + a^2 K_d(t,t).$$

Minimizing with respect to a we get $a = h_d(t)/K_d(t,t)$, and so

$$\inf_a \sup_{f\in F_d,\, \|f\|_{F_d}\le 1} \left|I_d(f) - a\,f(t)\right|^2 = e^2(0,d) - \frac{h_d^2(t)}{K_d(t,t)}.$$

Here, we use the convention that $0/0 = 0$. Indeed, observe that $K_d(t,t) = 0$ implies $K_d(t,t) = \|K_d(\cdot,t)\|_{F_d}^2 = 0$, so in turn $K_d(\cdot,t) = 0$, and $f(t) = 0$ for all $f \in F_d$. Hence, $K_d(t,t) = 0$ yields that $h_d(t) = 0$ and that the error is $e(0,d)$. This is consistent with our convention that $0/0 = 0$.

Minimizing with respect to t, we get

$$e^2(1,d) = e^2(0,d) - \sup_{t\in D_d} \frac{h_d^2(t)}{K_d(t,t)},$$

which yields (10.2).

Observe that $e(1,d)$ cannot be equal to $e(0,d)$ for positive $e(0,d)$. Indeed, $e(1,d) = e(0,d)$ implies that $h_d(t) = 0$ for all $t \in D_d$. Hence, $h_d = 0$ which contradicts our assumption that $e(0,d) = \|h_d\|_{F_d} > 0$.

We now turn to the second point. The dimension of F_d is to be k, hence we are looking for $F_d = \text{span}(e_1, e_2, \ldots, e_k)$ for some linearly independent functions $e_j : D_d \to \mathbb{R}$. We set $D_d = [-1,1]^d$. Let

$$\vec{e}\,(t) = [e_1(t), e_2(t), \ldots, e_k(t)] \quad \text{for all } t \in [-1,1]^d.$$

We choose the functions e_j such that $\vec{e}\,([-1,1]^d)$ is dense in $[-1,1]^k$. If $k = +\infty$ we use the l_2 norm, and we additionally assume that

$$\sum_{j=1}^{\infty} e_j^2(t) < +\infty \quad \text{for all } t \in [-1,+1]^d.$$

Clearly such functions exist since we do not impose any regularity assumptions on e_j. We may define the function $\vec{e}$ as follows. Let r_i be an ordered sequence of all rationals from $[-1,1]^d$, and let $\vec{p}_{i,k}$ be an ordered sequence of all rational vectors from $[-1,+1]^k$. For $k = +\infty$, we use the diagonal ordering of successive components such that each $\vec{p}_{i,\infty}$ has finitely many nonzero components. Define $\vec{e}(r_i) = \vec{p}_{i,k}$ and $\vec{e}(t) = 0$, otherwise. For $k = +\infty$, we see that $\sum_{j=1}^{\infty} e_j^2(t)$ equals zero for irrational t, and equals $\|\vec{p}_{\,i,\infty}\|_2 < +\infty$ for a rational $t = r_i$.

It is easy to check that these functions e_j are linearly independent. So we define $F_1 = \text{span}(e_1, e_2, \ldots, e_k)$, with the inner product chosen such that the functions e_j are orthonormal. The reproducing kernel K_d is then given by

$$K_d(x,t) = \sum_{j=1}^{k} e_j(x)\, e_j(t) \quad \text{for all } x, t \in [-1,+1]^d.$$

Indeed, $K_d(\cdot,t)$ belongs to F_1 since $\sum_{j=1}^{k} e_j^2(t) < \infty$, and $\langle f, K_d(\cdot,t)\rangle_{F_d} = f(t)$.

We now show that $e(1,d) = 0$ for an arbitrary linear functional

$$I_d(f) = \langle f, h_d\rangle_{F_d} \quad \text{with } h_d = \sum_{j=1}^{k} \alpha_j e_j \in F_d.$$

We have

$$e(0,d) = \|h_d\|_{F_d} = \Big[\sum_{j=1}^{k} \alpha_j^2\Big]^{1/2} < \infty.$$

If $e(0,d) = 0$ then $e(1,d) = 0$. So assume that $e(0,d) > 0$. Let $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_k]$. Then we have

$$\frac{1}{e(0,d)}\, \vec{\alpha} \in [-1,+1]^k \setminus \{\vec{0}\}.$$

Since $\vec{e}\,([-1,+1]^d)$ is dense in $[-1,+1]^k$, for any positive $\eta < 1$ there exists $t \in [-1,+1]^d$ such that

$$\left\| \vec{e}\,(t) - \vec{\alpha}/e(0,d) \right\|_2 \le \eta.$$

This means that for small η we get $\|\vec{e}\,(t)\|_2 > 0$, and the vectors $\vec{e}\,(t)$ and $\vec{\alpha}$ are almost parallel. We have

$$\frac{h_d^2(t)}{K_d(t,t)} = \frac{\left(\sum_{j=1}^k \alpha_j e_j(t)\right)^2}{\sum_{j=1}^k e_j^2(t)}.$$

Observe that

$$\sum_{j=1}^k \alpha_j e_j(t) = e(0,d)\Big(\sum_{j=1}^k e_j(t)^2 + \sum_{j=1}^k \left(\alpha_j/e(0,d) - e_j(t)\right)\, e_j(t)\Big).$$

Therefore

$$\Big|\sum_{j=1}^k \alpha_j e_j(t)\Big| \ge e(0,d)\, \sum_{j=1}^k e_j(t)^2 \left(1 - \|\vec{e}\,(t) - \vec{\alpha}/e(0,d)\|_2/\|\vec{e}\,(t)\,\|_2\right),$$

and

$$\frac{h_d^2(t)}{K_d(t,t)} \ge e^2(0,d)\, \|\vec{e}\,(t)\|_2^2\, \left(1 - \eta/\|\vec{e}\,(t)\|_2\right)^2.$$

Letting η go to zero, we get $\|\vec{e}\,(t)\|_2 \to 1$ and $\sup_{t\in D_d} h_d^2(t)/K_d(t,t) = e^2(0,d)$. Hence, (10.2) implies that $e(1,d) = 0$. This completes the proof. □

The space F_d in the proof of Theorem 10.1 consists of very irregular functions. We now show that F_d can be chosen as a subclass of the class $C([0,1]^d)$ of continuous functions. The construction of such F_d is as follows, see [204] and [218].

The interval $[0,1]^d$ is a Peano set, i.e., there exists a surjective continuous mapping

$$g = [g_1, g_2, \dots]\colon [0,1]^d \to [-1,1]^{\mathbb{N}},$$

see, e.g., Semadeni [260]. Such a mapping g is called a *Peano map* or a *Peano curve*. Here, g_j is the jth component of g and is a continuous function.

For a given integer k or $k = +\infty$, define

$$F_d = \left\{ f : [0,1]^d \to \mathbb{R} \mid f = \sum_{j=1}^k f_j g_j \text{ for which } \sum_{j=1}^k j^2 f_j^2 < +\infty \right\}$$

with the inner product

$$\langle f, h\rangle_{F_d} = \sum_{j=1}^k j^2 f_j h_j$$

for $h = \sum_{j=1}^k h_j g_j \in F_d$.

Observe that $f(t) = \sum_{j=1}^k f_j g_j(t)$ is well defined since $|g_j(t)| \le 1$ and

$$|f(t)| \le \sum_{j=1}^k |f_j| \le \Big(\sum_{j=1}^k j^2 f_j^2\Big)^{1/2} \Big(\sum_{j=1}^k j^{-2}\Big)^{1/2} \le \frac{\pi}{\sqrt{6}} \, \|f\|_{F_d}.$$

This also implies that f is a continuous function; hence $F_d \subseteq C([0,1]^d)$. It is easy to check that F_d is complete so F_d is a Hilbert space.

We now show that for any continuous linear functional I_d and any positive ε, there exist a nonnegative number β and $x \in [0,1]$ such that

$$|I_d(f) - \beta\, f(x)| \le \varepsilon, \quad \text{for all } f \in F_d,\ \|f\|_{F_d} \le 1. \tag{10.3}$$

That is, I_d can be recovered with arbitrarily small error by using at most one function value, so that $e(1,d) = 0$. Indeed, let us represent

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for some } h_d = \sum_{j=1}^k h_{d,j} g_j \in F_d.$$

The series $\sum_{j=1}^k j^2 h_{d,j}^2$ is convergent, and so there exists $m = m(\varepsilon)$ such that

$$\sum_{j=m+1}^k j^2 h_{d,j}^2 \le \varepsilon^2.$$

Let

$$\beta = \max_{j=1,2,\dots,m} |I_d(g_j)|.$$

If $\beta = 0$ then $\|I_d\| = \|h_d\|_{F_d} \le \varepsilon$ and (10.3) holds. Assume then that $\beta > 0$. Observe that $I_d(g_j) = j^2 h_{d,j}$ and since $|j h_{d,j}| \le \|h_d\|_{F_d}$ then $\beta \le m\|h_d\|_{F_d}$. Hence

$$u = \beta^{-1}\,[I_d(g_1), I_d(g_2), \dots, I_d(g_m)] \in [-1,1]^m.$$

Since g is surjective, there exists $x \in [0,1]$ such that $g(x) = [u, 0, 0, \dots]$. That is, $g_j(x) = \beta^{-1} I_d(g_j)$ for $j = 1, 2, \dots, m$, and $g_j(x) = 0$ for $j > m$. For $\|f\|_{F_d} \le 1$ we thus have

$$\begin{aligned} I_d(f) &= \sum_{j=1}^m f_j I_d(g_j) + \sum_{j=m+1}^k f_j I_d(g_j) = \beta \sum_{j=1}^m f_j g_j(x) + \sum_{j=m+1}^k j^2 f_j h_{d,j} \\ &= \beta \sum_{j=1}^k f_j g_j(x) + \sum_{j=m+1}^k j^2 f_j h_{d,j} = \beta\, f(x) + \sum_{j=m+1}^k j^2 f_j h_{d,j}. \end{aligned}$$

Hence,

$$|I_d(f) - \beta f(x)| \le \sum_{j=m+1}^k j^2 |f_j h_{d,j}| \le \|f\|_{F_d} \Big(\sum_{j=m+1}^k j^2 h_{d,j}^2\Big)^{1/2} \le \varepsilon,$$

as claimed in (10.3). Obviously, we can set $\varepsilon = 0$ in (10.3) if k is finite.

Theorem 10.1 states that for some reproducing kernel Hilbert spaces, all linear functionals I_d can be solved with arbitrarily small error by using just one function value. For such spaces the power of the class Λ^{all} and the class Λ^{std} is equal since

$$n^{\text{wor}}(\varepsilon, I_d, \Lambda^{\text{all}}) = n^{\text{wor}}(\varepsilon, I_d, \Lambda^{\text{std}}) = 1 \quad \text{for all } \varepsilon < \max\left(1, \|h_d\|_{F_d}\right),$$

whether we use the absolute or normalized error criterion. Obviously for such spaces we have strong polynomial tractability with exponent 0.

The reader may rightly think that this may happen only for esoteric spaces. Indeed, the construction of such spaces indicates this. As we shall see in the next section almost anything can happen, depending on the space F_d.

10.4 Bad or Good Convergence

Although there exist spaces for which $e(1,d) = 0$ for all linear functionals, for typical spaces and typical linear functionals we have that $e(n,d) > 0$ for all n.

First of all we notice that for all I_d we have

$$\lim_{n\to\infty} e(n,d) = 0.$$

Indeed, we have $I_d(f) = \langle f, h_d\rangle_{F_d}$ for some $h_d \in F_d = H(K_d)$. It is known that $H(K_d)$ is the completion of the union of finite dimensional subspaces

$$\text{span}(K_d(\cdot, t_1), K_d(\cdot, t_2), \dots, K_d(\cdot, t_m))$$

for an integer m and $t_1, t_2, \dots, t_m$ from D_d. This means that for any positive ε we may find a finite $m = m(\varepsilon)$, $a_j = a_j(m) \in \mathbb{R}$ and $t_j = t_j(m) \in D_d$ for $j = 1, 2, \dots, m$ such that

$$\Bigl\| h_d - \sum_{j=1}^{m} a_j K_d(\cdot, t_j) \Bigr\| \le \varepsilon.$$

From (10.1), this implies that $e(m,d) \le \varepsilon$. Since ε can be arbitrarily small and the sequence $\{e(n,d)\}$ is monotonically non-increasing in n, the limit of $e(n,d)$ is zero, as claimed.

So we always have convergence which implies that the information complexity $n(\varepsilon, d)$ is finite for all positive ε and all $d \in \mathbb{N}$. Furthermore, the argument above shows that the sets,

$$A_d = \left\{ h_d \mid e^{\text{wor}}(m, I_d) = 0 \text{ for some finite } m = m(I_d) \right\},$$
$$B_{d,p} = \left\{ h_d \mid \lim_{n\to\infty} e^{\text{wor}}(n, I_d)\, n^p = 0 \right\} \quad \text{with } p > 0,$$

are dense in F_1^*. Since we can identify linear functionals I_d with their representers h_d, we can also say that the properties above hold for dense sets of linear functionals.

This density result holds for *any* positive p. That is, for arbitrary large p, the set of linear functionals I_d for which

$$e^{\mathrm{wor}}(n, I_d) = \mathcal{O}(n^{-p})$$

is dense in F_d. Here, the factor in the $\mathcal{O}$ notation is independent of n but may depend on d through I_d.

This looks like a very good result stating that a dense set of linear functionals can be recovered exactly after computing finitely many function values, and that we can have an arbitrarily good rate of convergence. However, these properties are quite misleading. First of all, $m = m(I_d)$ can be arbitrarily large. Secondly, we may have to wait arbitrarily long for the good rate of convergence to happen. In Volume I, we saw many examples of multivariate problems with an excellent rate of convergence that suffer from the curse of dimensionality. Furthermore, these good properties are for a dense set of linear functionals; they tell us nothing about what can happen for the complement of this set. As we shall see, anything *good or bad* can happen. We already saw spaces F_d for which $e^{\mathrm{wor}}(1, I_d) = 0$ for all I_d. We now exhibit examples of spaces F_d for which the minimal errors $e(n, d) = e^{\mathrm{wor}}(n, I_d)$ have both bad or good properties, depending on I_d.

Indeed, the convergence of $e(n, d)$ to zero can be *arbitrarily slow* or *arbitrarily fast*, and therefore $n(\varepsilon, d)$ can go to infinity *arbitrarily fast* or *arbitrarily slow*; this can occur for arbitrary d. So, anything can really happen; it all depends on the space F_d and the linear functional I_d.

More precisely, let $g \colon [0, \infty) \to [0, 1]$ be an arbitrary convex decreasing function such that $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 0$. As in [218], we show that for any d there exist a reproducing kernel Hilbert space F_d and a continuous linear functional I_d defined on F_d for which

$$e(n, d) = \sqrt{g(n)} \quad \text{for all } n.$$

The function g can go to zero arbitrarily slowly. Indeed, take an integer k and define the function $g(x) = 1/\ln(k, x)$, where

$$\ln(k, x) = \ln \ln \cdots \ln(x + c_k)$$

with ln occurring k times and $c_k = \exp(\exp(\cdots \exp(1) \cdots))$ with exp also occurring k times. Then $g(0) = 1$, and it is decreasing to zero. It can be checked that g is convex since g'' is positive. In this case, we have

$$n(\varepsilon, d) = \left\lceil \exp(\exp(\cdots \exp(\varepsilon^{-2}) \cdots)) - c_k \right\rceil.$$

Hence, the information complexity is in this case a k-level exponential function, where k can be arbitrarily large. Obviously this means that the problem is intractable even for $d = 1$.

On the other hand, g can go to zero arbitrarily fast. In this case, the problem I_d is very easy. For example, if $g(x) = \exp(k, 0)/\exp(k, x)$, where

$$\exp(k, x) = \exp(\exp(\cdots \exp(x) \cdots))$$

with exp occurring k times, then g again is decreasing and convex and the information complexity

$$n(\varepsilon, d) = \left\lceil \ln(\ln(\cdots \ln\left(\exp(k,0)\,\varepsilon^{-2}\right)\,\cdots))\right\rceil$$

goes to infinity extremely slowly especially if k is large. These two cases of g indeed show that anything can happen.

We now provide two examples of spaces and linear functionals from [218] for which $e(n,d) = \sqrt{g(n)}$. These examples will also play an additional role of illustrating further estimates.

10.4.1 Example: Kernel for Non-Separable Space

Consider F_d as the space of functions defined on, say, $D_d = [0,1]^d$ with the reproducing kernel

$$K_d(t,t) = 1 \quad \text{and} \quad K_d(x,t) = 0 \quad \text{for } x \neq t.$$

Hence, F_d is the Hilbert space of functions $f\colon D_d \to \mathbb{R}$ such that

$$f = \sum_{j=1}^{\infty} a_j\, K_d(\cdot, t_j)$$

for some distinct t_j from $[0,1]^d$, with

$$\|f\|_{F_d}^2 = \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Hence we have $f(t_j) = a_j$ and $f(t) = 0$ for t distinct from all t_j, so that each function f from F_d vanishes almost everywhere.

For $g = \sum_{j=1}^{\infty} b_j K_d(\cdot, s_j)$, the inner product in F_d is

$$\langle f, g\rangle_{F_d} = \sum_{i,j=1}^{\infty} a_i b_j\, K_d(t_i, s_j).$$

Note that $K_d(\cdot, x)$ and $K_d(\cdot, t)$ are orthonormal for $x \neq t$. Hence, F_d has an uncountable orthonormal system, and therefore is *not* separable.

Consider now an arbitrary linear functional

$$I_d(f) = \langle f, h_d\rangle_{F_d} \qquad \text{for all } f \in F_d$$

with

$$h_d = \sum_{j=1}^{\infty} \alpha_j\, K_d(\cdot, t_j^*),$$

where t_j^* are distinct with

$$\sum_{j=1}^{\infty} \alpha_j^2 = 1 \quad \text{and} \quad |\alpha_1| \ge |\alpha_2| \ge \cdots .$$

Then $\|h_d\|_{F_d} = 1$ and the absolute and normalized error criteria coincide.

Take a linear algorithm $A_{n,d}(f) = \sum_{j=1}^n a_j f(t_j)$. Its worst case error is

$$e^{\mathrm{wor}}(A_{n,d}) = \Big\| h_d - \sum_{j=1}^n a_j K_d(\cdot, t_j) \Big\| = \Big[1 - 2\sum_{j=1}^n a_j h_d(t_j) + \sum_{j=1}^n a_j^2 \Big]^{1/2}.$$

We minimize the worst case error by taking $a_j = h_d(t_j)$, so that

$$e^{\mathrm{wor}}(A_{n,d}) = \Big[1 - \sum_{j=1}^n h_d^2(t_j) \Big]^{1/2}.$$

Since $h_d(t) = 0$ for t not equal to t_j^*, it is obvious that the sample points t_j that minimize the worst case error should be equal to some t_j^*. Since α_j^2 are non-increasing, the best choice is to take $t_j = t_j^*$. This means that the algorithm

$$A_{n,d}^*(f) = \sum_{j=1}^n h_d(t_j^*)\, f(t_j^*)$$

minimizes the worst case error among all algorithms that use n function values. Since $h_d(t_j^*) = \alpha_j$ we obtain

$$e(n,d) = \Big[1 - \sum_{j=1}^n \alpha_j^2 \Big]^{1/2} = \Big[\sum_{j=n+1}^{\infty} \alpha_j^2 \Big]^{1/2}.$$

For a given function $g \colon [0,\infty) \to \mathbb{R}$, which is convex and decreasing with $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 0$, define

$$\alpha_j = [g(j-1) - g(j)]^{1/2} \quad \text{for } j = 1, 2, \dots .$$

Then monotonicity of g yields that the α_j are well defined and positive, and convexity of g yields that $\alpha_j \ge \alpha_{j+1}$. Indeed, since

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y) \quad \text{for any } t \in [0,1] \text{ and } x, y \in [0,\infty),$$

it is enough to take $t = \frac{1}{2}$ and $x = j-1$, $y = j+1$. Then

$$g(j) = g\left(\tfrac{1}{2}(j-1) + \tfrac{1}{2}(j+1)\right) \le \tfrac{1}{2}\, (g(j-1) + g(j+1))$$

which is equivalent to $g(j-1)-g(j) \geq g(j)-g(j+1)$ or $\alpha_j \geq \alpha_{j+1}$, as needed. Finally, we have

$$e^2(n,d) = \sum_{j=n+1}^{\infty} [g(j-1)-g(j)] = g(n),$$

as claimed.

Hence, we can have arbitrarily slow/fast convergence, or equivalently, arbitrarily bad/good information complexity. □

10.4.2 Example: Unbounded Kernel

We present a separable Hilbert space F_d with an unbounded reproducing kernel K_d, and a continuous linear functional I_d for which $e(n,d) = \sqrt{g(n)}$. This is done by a simple modification of the space from the previous example. For simplicity, we restrict ourselves to $d = 1$.

Define F_1 as the space of functions $f : [0,1] \to \mathbb{R}$ which are constant over the intervals $(1/(j+1), 1/j]$ for $j = 1, 2, \ldots$. That is,

$$f(x) = \sum_{j=1}^{\infty} f(1/j)\, 1_{(1/(j+1),1/j]}(x),$$

where $1_{(a,b]}$ is the characteristic (indicator) function of the set $(a,b]$, i.e.,

$$1_{(a,b]}(t) = 1 \text{ if } t \in (a,b] \quad \text{and} \quad 1_{(a,b]}(t) = 0 \text{ if } t \notin (a,b].$$

We assume that $\sum_{j=1}^{\infty} f^2(1/j) < +\infty$, and define the inner product of F_1 as

$$\langle f, h \rangle_{F_1} = \sum_{j=1}^{\infty} f(1/j)\, h(1/j)\, j^{-1}(j+1)^{-1}.$$

Observe that

$$\int_0^1 f(x)h(x)\,\mathrm{d}x = \sum_{j=1}^{\infty} \int_{1/(j+1)}^{1/j} f(x)h(x)\,\mathrm{d}x = \sum_{j=1}^{\infty} f(1/j)\, h(1/j)\, j^{-1}(j+1)^{-1}.$$

Thus, $\langle f, h \rangle_{F_1} = \langle f, h \rangle_{L_2}$. This shows that $F_1 \subseteq L_2([0,1])$ and $\|f\|_{F_1} = \|f\|_{L_2}$.

We now show that F_1 is a reproducing kernel Hilbert space and find the reproducing kernel K_1. For any $t \in (1/(j+1), 1/j]$ we should have

$$f(t) = f(1/j) = \langle f, K_1(\cdot, 1/j) \rangle_{F_1} = \sum_{k=1}^{\infty} f(1/k) K_1(1/k, 1/j) k^{-1}(k+1)^{-1}.$$

This is satisfied for all f if

$$K_1(1/j, 1/k) = k(k+1)\,\delta_{j,k}.$$

Since $K_1(\cdot, t)$ should be piecewise constant we finally have

$$K_1(x,t) = j(j+1) \quad \text{if } T(x) = T(t) = j \text{ for some } j,$$

and $K_1(x,t) = 0$ otherwise. Here, $T(x) = k$ iff $x \in (1/(k+1), 1/k]$. Since

$$K_1(1/k, 1/k) = k(k+1) \quad \text{for all } k,$$

the kernel K_1 is unbounded.

Let $I_1(f) = \langle f, h_1 \rangle_{F_1}$ with $h_1 = \sum_{j=1}^{\infty} \alpha_j \, 1_{(1/(j+1),1/j]}$, and let

$$\|h\|_{F_1}^2 = \sum_{j=1}^{\infty} \alpha_j^2 j^{-1}(j+1)^{-1} = 1.$$

Consider the algorithm $A_1(f) = \sum_{j=1}^{n} a_j f(t_j)$. Since f is piecewise constant we may assume that $t_j = 1/k_j$ for some integers k_j. Since $K(\cdot, 1/i)$ and $K(\cdot, 1/j)$ are orthogonal for distinct i and j, it is easy to check that $a_j = \alpha_{k_j} k_j^{-1}(k_j+1)^{-1}$ minimizes the error. Then the worst case error of A_1 is

$$e^{\text{wor}}(A_1) = \Big[1 - \sum_{j=1}^{n} \alpha_{k_j}^2 k_j^{-1}(k_j+1)^{-1}\Big]^{1/2}.$$

The n best sample points correspond to the n largest numbers of the sequence $\alpha_j^2 j^{-1}(j+1)^{-1}$. Assume then that

$$\frac{\alpha_{j_1}^2}{j_1(j_1+1)} \ge \frac{\alpha_{j_2}^2}{j_2(j_2+1)} \ge \cdots \ge 0.$$

Then

$$e(n,1) = \Big[\sum_{i=n+1}^{\infty} \alpha_{j_i}^2 j_i^{-1}(j_i+1)^{-1}\Big]^{1/2}.$$

Similarly to the previous example, we define the coefficients α_j by

$$\alpha_j = [j(j+1)\,(g(j-1) - g(j))]^{1/2}$$

for any convex decreasing function g, with $g(0) = 0$ and $\lim_{x\to\infty} g(x) = 0$. Then

$$e(n,1) = \sqrt{g(n)},$$

as claimed. Hence, we can have arbitrarily slow/fast convergence or, equivalently, arbitrarily bad/good information complexity. □

The reader may think that the spaces of the previous two examples are a little contrived and hope that bad convergence will not occur for more standard spaces. Unfortunately, this is not true as illustrated in the next example for the Sobolev space anchored at zero which, as we know, is related to the L_2 discrepancy.

10.4.3 Example: Sobolev Space Anchored at 0

We now show that even for $d = 1$ there are linear functionals defined over the Sobolev space $F_1 = W_0^1([0,1])$ anchored at zero that are arbitrarily hard. That is, convergence of approximating them can be arbitrarily slow.

As we know, the reproducing kernel of the Sobolev space anchored at zero is

$$K_1(x,t) = \min(x,t) \quad \text{for all } x,t \in [0,1],$$

and the inner product is given by

$$\langle f, g\rangle_{F_1} = \int_0^1 f'(t)g'(t)\,\mathrm{d}t$$

for $f, g \in F_1$, i.e., for functions f and g that are absolutely continuous whose first derivatives are $L_1([0,1])$ and $f(0) = g(0) = 1$.

For $j = 1, 2, \dots,$ define

$$g_j(x) = \begin{cases} 0 & \text{for } x \in [0, 1/(j+1)] \cup [1/j, 1], \\ \sqrt{j(j+1)}\,[-x + 1/(j+1)] & \text{for } x \in [1/(j+1), \frac{1}{2}(1/(j+1) + 1/j)], \\ \sqrt{j(j+1)}\,[x - 1/j] & \text{for } x \in [\frac{1}{2}(1/(j+1) + 1/j), 1/j]. \end{cases}$$

The functions g_j are piecewise linear, the support of g_j is $[1/(j+1), 1/j]$, and these functions have disjoint supports. They are also normalized, i.e.,$\|g_j\|_{F_1} = 1$, and are orthonormal. Define

$$h_1(x) = \sum_{j=1}^{\infty} \alpha_j g_j(x) \quad \text{for all } x \in [0,1],$$

with

$$\|h_1\|_{F_1}^2 = \sum_{j=1}^{\infty} \alpha_j^2 \in (0,\infty) \quad \text{and} \quad \alpha_1 \ge \alpha_2 \ge \dots > 0.$$

Let $I_1(f) = \langle f, h_1\rangle_{F_1}$. We now prove that

$$e^{\mathrm{wor}}(n, I_1) \ge \Big[\sum_{j=n+1}^{\infty} \alpha_j^2 \Big]^{1/2}. \tag{10.4}$$

We know from Chapter 4 of Volume I that

$$e^{\mathrm{wor}}(n, I_1) = \inf_{t_1,t_2,\dots,t_n \in [0,1]} \ \sup_{\|f\|_{F_1} \le 1, f(t_j)=0, j=1,2,\dots,n} \langle f, h_1\rangle_{F_1}.$$

Take arbitrary $t_1, t_2, \dots, t_n$ from $[0,1]$. Without loss of generality we can assume that $t_j > 0$, since we know that all functions in F_1 are zero at 0. Let $J_k = (1/(k+1), 1/k]$

for $k = 1, 2, \ldots$. Then each $t_j \in J_{k_j}$ for some k_j. Let $A_n = \{k_1, k_2, \ldots, k_n\}$. Define the function

$$g(x) = \sum_{j \in [1,\infty):\ j \notin A_n} \alpha_j g_j(x) \quad \text{for all } x \in [0,1].$$

Then $g \in F_1$ and $g(t_j) = 0$ for all $j = 1, 2, \ldots, n$. Clearly, g is not zero. Finally define

$$f = \frac{1}{\|g\|_{F_1}}\, g.$$

Then $\|f\|_{F_1} = 1$, $f(t_j) = 0$ for $j = 1, 2, \ldots, n$ and

$$\langle f, h_1 \rangle_{F_1} = \Big[\sum_{j \in [1,\infty):\ j \notin A_n} \alpha_j^2 \Big]^{1/2} \ge \Big[\sum_{j=n+1}^{\infty} \alpha_j^2 \Big]^{1/2},$$

since A_n has at most n elements and the α_j's are ordered.

Obviously, we can now define the coefficients α_j as in the previous two examples to obtain arbitrarily bad convergence of approximating the linear functional with the representer h_1.

We may hope that this bad convergence happens only for some linear functionals. As we shall see now, this can happen for many linear functionals.

More precisely, for any $k \in \mathbb{N}$, we use the function $1/\ln(k, x)$, defined before, which goes to zero extremely slowly as the reciprocal of the k-iterated logarithm. Define

$$B_k = \big\{ h_1 \in F_1 \mid e^{\text{wor}}(n, I_1) \le \|h_1\|_{F_1} / \ln^{1/2}(k, n) \text{ for all } n \in \mathbb{N} \big\}$$

as the set of linear functionals $I_1(f) = \langle f, h_1 \rangle_{F_1}$ for $f \in F_1$ for which the squares of the nth minimal worst case errors tend to zero at least as fast as the reciprocal of the k-iterated logarithm. One might hope that at least for large k, the set B_k is the whole space or at least a good chunk of F_1. Unfortunately, B_k does not contain any ball in F_1 no matter how large k. That is, for any ball $B(h_1^*, r)$ with center h_1^* and positive radius r (no matter how small), there exists $h_1 \in B(h_1^*, r)$ that does not belong to B_k, i.e., for which convergence is slower than the reciprocal of the k-iterated logarithm.

To prove this bad property, take the ball $B(h_1^*, r)$. If $h_1^* \notin B_k$ we are done. Assume then that $h_1^* \in B_k$. If $h_1^* \ne 0$ we take $r \in (0, \|h_1^*\|_{F_1})$. We define

$$h_1 = h_1^* + r h_{1,k+1}(x),$$

where $h_{1,k+1}(x) = \sum_{j=1}^{\infty} \alpha_j g_j(x)$ is given as before with

$$\alpha_j = \left[\frac{1}{\ln(k+1, j-1)} - \frac{1}{\ln(k+1, j)} \right]^{1/2}.$$

Note that $\|h_{1,k+1}\|_{F_1} = 1$. Moreover $h_1 \ne 0$, since for $h_1^* \ne 0$ we have $\|h_1\|_{F_1} \ge \|h_1^*\|_{F_1} - r > 0$, and for $h_1^* = 0$ we have $\|h_1\|_{F_1} = r > 0$.

For $f \in F_1$, consider the three linear functionals

$$I_1(f) = \langle f, h_1 \rangle_{F_1}, \quad I_1^*(f) = \langle f, h_1^* \rangle_{F_1}, \quad I_{1,k+1}(f) = \left\langle f, h_{1,k+1} \right\rangle_{F_1}.$$

Then

$$e^{\mathrm{wor}}(n, I_1) \geq r\, e^{\mathrm{wor}}(n, I_{1,k+1}) - e^{\mathrm{wor}}(n, I_1^*).$$

From the previous estimate and the construction of the α_j's we know that

$$e^{\mathrm{wor}}(n, I_{1,k+1}) \geq \frac{1}{\ln^{1/2}(k+1, n)}.$$

Since $h_1^* \in B_k$ we conclude that

$$e^{\mathrm{wor}}(n, I_1) \geq \frac{r}{\ln^{1/2}(k+1, n)} - \frac{\|h_1^*\|_{F_1}}{\ln^{1/2}(k, n)}.$$

This shows that

$$e^{\mathrm{wor}}(n, I_1) \leq \frac{\|h_1\|_{F_1}}{\ln^{1/2}(k, n)}$$

cannot hold for large n. Hence, $h_1 \notin B_k$, as claimed.

Obviously, arbitrarily bad convergence means that the information complexity is arbitrarily large. Hence even for $d = 1$, in each ball we can find an example of a linear functional defined over the Sobolev space anchored at zero that has arbitrarily large information complexity, and therefore is intractable.

If we take into account the previous result that the set of linear functionals with arbitrarily good convergence is also dense in F_1, we can conclude that indeed anything can happen, since the sets of linear functionals with arbitrarily bad and good convergence are both dense in F_1.

Although we show this peculiar dependence only for the Sobolev space $W_0^1([0,1])$, it is clear that similar results are also true for many standard Sobolev or Korobov spaces, with more or less the same proof. In particular, increasing the smoothness in the function space will not change this property.

10.5 Non-negative Kernels and Algorithms

Sometimes it is relatively easy to prove lower bounds on the worst case errors for algorithms with non-negative coefficients, whereas it is usually much harder to prove similar lower bounds for linear algorithms with arbitrary and maybe negative coefficients. Lower bounds for algorithms with non-negative coefficients were studied in, e.g., [207], [208], [277], [352]. Here we present a general lower bound for Hilbert spaces whose reproducing kernel is point-wise non-negative.

We assume that the reproducing kernel $K_d : D_d \times D_d \to \mathbb{R}$ of the space F_d is such that

$$K_d(x, t) \geq 0 \quad \text{for all } x, t \in D_d. \tag{10.5}$$

The lower bound will be expressed in terms of K_d and h_d by the quantity

$$\kappa_d := \frac{1}{\|h_d\|_{F_d}} \sup_{x \in D_d} \frac{|h_d(x)|}{\sqrt{K_d(x,x)}}. \tag{10.6}$$

Since

$$|h_d(x)| = \left|\langle h_d, K_d(\cdot,x)\rangle_{F_d}\right| \le \|h_d\|_{F_d} \, \|K_d(\cdot,x)\|_{F_d} = \|h_d\|_{F_d} \sqrt{K_d(x,x)}$$

we have $\kappa_d \le 1$. If $\kappa_d = 1$ then I_d is trivial, since Theorem 10.1 then implies that $e(1,d) = 0$. This means that $I_d(f)$ can be approximated with arbitrarily small error using only one function value. So the only interesting case is when $\kappa_d < 1$. As we shall see later, the quantity κ_d is exponentially small in d for some spaces.

We will now consider linear algorithms $A_{n,d}$ with non-negative coefficients a_j. Obviously, the use of such coefficients can be justified only for some linear functionals. For example, assume that for $f \ge 0$ we have $I_d(f) \le 0$. Since $A_{n,d}(f) \ge 0$ then $|I_d(f) - A_{n,d}(f)| \ge |I_d(f)|$ and all algorithms $A_{n,d}$ are at most as good as the zero algorithm. However, we can hope that algorithms $A_{n,d}$ with non-negative coefficients can be good for positive linear functionals I_d, i.e., when $f \ge 0$ implies that $I_d(f) \ge 0$. In particular, the last property holds for multivariate integration that will be studied in the next subsection.

We present a lower bound on the worst case error that does not require I_d to be of a special form. In this section we will illustrate this bound for linear functionals that are not related to multivariate integration and for which the assumption that a_j's are non-negative is not restrictive, and in the next section for multivariate integration.

We are ready to prove the following theorem basically using the same argument that was presented for multivariate integration and for algorithms with $a_j \ge 0$ in [352], and for QMC algorithms (with $a_j = n^{-1}$) in [277].

Theorem 10.2. *Consider a linear functional I_d defined over the space F_d whose reproducing kernel is point-wise non-negative, see (10.5).*

Let $A_{n,d}(f) = \sum_{j=1}^n a_j f(t_j)$ be a linear algorithm with arbitrary $a_j \ge 0$ and arbitrary sample points t_j from D_d. Then

$$e^{\text{wor}}(A_{n,d}) \ge (1 - n\,\kappa_d^2)_+^{1/2} \|h_d\|_{F_d}.$$

Therefore, if $e^{\text{wor}}(A_{n,d}) \le \varepsilon \, \|h_d\|_{F_d}$ then

$$n \ge (1-\varepsilon^2)\,\kappa_d^{-2}.$$

Proof. The square of the worst case error of $A_{n,d}$ is

$$\left[e^{\text{wor}}(A_{n,d})\right]^2 = \|h_d\|_{F_d}^2 - 2\sum_{j=1}^n a_j\, h_d(t_j) + \sum_{i,j=1}^n a_i a_j\, K_d(t_i,t_j).$$

From the definition of κ_d we have

$$|h_d(x)| \le \kappa_d \, \|h_d\|_{F_d} \sqrt{K_d(x,x)} \quad \text{for all } x \in D_d.$$

Replacing $h_d(t_j)$ by its upper bound we obtain

$$\sum_{j=1}^{n} a_j\, h_d(t_j) \le \kappa_d\, \|h_d\|_{F_d} \sum_{j=1}^{n} |a_j| \sqrt{K_d(t_j,t_j)}$$
$$\le \kappa_d\, \|h_d\|_{F_d}\, \sqrt{n} \Big[\sum_{j=1}^{n} a_j^2\, K_d(t_j,t_j)\Big]^{1/2}.$$

Since $K_d(t_i,t_j)$ and a_j are non-negative, we have

$$\sum_{i,j=1}^{n} a_i a_j\, K_d(t_i,t_j) \ge \beta^2 := \sum_{j=1}^{n} a_j^2\, K_d(t_j,t_j).$$

We thus have

$$\big[e^{\mathrm{wor}}(A_{n,d})\big]^2 \ge \|h_d\|_{F_d}^2 - 2\kappa_d\, \|h_d\|_{F_d}\, \sqrt{n}\,\beta + \beta^2.$$

Minimizing with respect to β we obtain $\beta = \kappa_d \|h_d\|_{F_d} \sqrt{n}$, which gives the estimates of the theorem. □

The essence of Theorem 10.2 is the dependence on d for the normalized error criterion. If κ_d goes to zero as d tends to infinity then n also goes to infinity. This means that we cannot achieve strong polynomial tractability by using algorithms with non-negative coefficients. And what is more important, if κ_d goes exponentially fast to zero as d tends to infinity then n must go exponentially fast to infinity. In this case, we cannot achieve weak tractability by using algorithms with non-negative coefficients.

Therefore, necessary conditions to achieve various kinds of tractability for the normalized error criterion by using algorithms with non-negative coefficients are:

- strong polynomial tractability: $\liminf_{d\to\infty} \kappa_d > 0$,
- polynomial tractability: $\liminf_{d\to\infty} d^q \kappa_d > 0$,
- weak tractability: $\lim_{d\to\infty} \dfrac{\ln \kappa_d^{-1}}{d} = 0$.

For polynomial tractability, q is an arbitrary positive number.

Probably the best illustration of Theorem 10.2 is for tensor product spaces and tensor product linear functionals. We will thoroughly study these problems later. Here we only highlight their properties in the context of the lower bound of Theorem 10.2.

Hence, assume that $D_d = D_1^d$ and

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j,t_j) \quad \text{for all } x,t \in D_d,$$

where K_1 is the reproducing kernel of F_1 with $K_1 \geq 0$. Then $F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1$ is the tensor product of d copies of the space F_1. Let

$$h_d(x) = \prod_{j=1}^{d} h_1(x_j) \quad \text{for all } x \in D_d$$

for some non-zero function h_1 from F_1. Then $I_d = I_1 \otimes I_1 \otimes \cdots \otimes I_1$ is the tensor product of d copies of the linear functional $I_1(f) = \langle f, h_1 \rangle_{F_1}$.

It now follows that

$$\kappa_d = \kappa_1^d \quad \text{with } \kappa_1 = \frac{1}{\|h_1\|_{F_1}} \sup_{x \in D_1} \frac{|h_1(x)|}{\sqrt{K_1(x,x)}}.$$

Finally we assume that I_1 is not trivial, i.e., $\kappa_1 < 1$. If we use algorithms with non-negative coefficients and we want to reduce the initial error by ε, then $n \geq (1-\varepsilon^2)\, \kappa_1^{-2d}$. Hence, n is exponentially large in d, and the curse of dimensionality is present for such problems and algorithms. We summarize this in the corollary.

Corollary 10.3. *Consider a tensor product linear functional*

$$I_d = I_1 \otimes I_1 \otimes \cdots \otimes I_1, \quad d \textit{ times},$$

defined for a tensor product space

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1, \quad d \textit{ times},$$

where the reproducing kernel of F_1 is point-wise non-negative. Assume that the linear functional I_1 for the univariate case is not trivial, i.e., $\kappa_1 < 1$.

If we want to reduce the initial error by a factor ε by linear algorithms with non-negative coefficients, then the number n of function values must be exponential in d, i.e.,

$$n \geq (1 - \varepsilon^2)\, \kappa_1^{-2d}.$$

Hence such problems suffer from the curse of dimensionality in the class of linear algorithms with non-negative coefficients.

We now illustrate the last corollary by continuing our two examples and presenting new examples.

10.5.1 Example: Kernel for Non-Separable Space (Continued)

We apply the last estimate to the space F_d with the kernel $K_d(x,t) = \delta_{x,t}$ for $x, t \in [0,1]^d$, which can be viewed as a tensor product space with $K_1(x,t) = \delta_{x,t}$ for $x, t \in [0,1]$. Obviously such a kernel is non-negative.

We now take h_1 as

$$h_1(x) = \sum_{j=1}^{\infty} \alpha_j \, K_1(\cdot, t_j^*) \quad \text{for all } x \in [0,1],$$

with non-negative non-increasing α_j such that $\sum_{j=1}^{\infty} \alpha_j^2 = 1$. We define

$$h_d(x) = \prod_{j=1}^{d} h_1(x_j) \quad \text{for all } x \in [0,1]^d.$$

Then $h_d(x) \geq 0$, $\|h_d\|_{F_d} = 1$, and

$$\kappa_1 = \alpha_1 \quad \text{and} \quad \kappa_d = \alpha_1^d.$$

Note that $\kappa_1 < 1$ iff $\alpha_2 > 0$. Obviously, if $\alpha_2 = 0$ then the problem is trivial since $\alpha_1 = 1$ and $h_d = K_d(\cdot, [t_1^*, \dots, t_1^*])$. In this case, $I_d(f) = f(t_1^*, \dots, t_1^*)$ can be solved exactly using one function value, and so $n(\varepsilon, d) = 1$. Therefore it is natural to assume that $\alpha_2 > 0$, and then $\alpha_1 < 1$.

As we know, the best choice of a_j's that minimize the worst case error of

$$A_{n,d}(f) = \sum_{j=1}^{n} a_j f(t_j)$$

is $a_j = h_d(t_j) \geq 0$ for any sample points t_j from $[0,1]^d$. So in this case, the assumption that a_j are non-negative is not restrictive, and the bound presented in Theorem 10.2 also holds for the minimal worst case error $e(n, d)$. Hence, we have

$$n(\varepsilon, d) \geq (1 - \varepsilon^2)\, \alpha_1^{-2d}.$$

From this we conclude that $I = \{I_d\}$ is intractable and suffers from the curse of dimensionality. □

10.5.2 Example: Unbounded Kernel (Continued)

Basically, the situation is the same as for the previous example. We now take F_d with the reproducing kernel $K_d(x,t) = \prod_{j=1}^{d} K_1(x_j, t_j)$ for $x_j, t_j \in [0,1]$ and with $K_1(x,t) = \delta_{T(x),T(t)} T(t)(T(t)+1)$ as before. Let

$$h_1 = \sum_{j=1}^{\infty} \alpha_j \, 1_{(1/(j+1), 1/j]}$$

with non-negative and ordered α_j such that

$$\frac{\alpha_1^2}{2} \geq \frac{\alpha_2^2}{6} \geq \cdots \geq \frac{\alpha_j^2}{j(j+1)} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\alpha_j^2}{j(j+1)} = 1.$$

As in the previous example, for $h_d(x) = \prod_{j=1}^d h_1(x_j)$ we have

$$\kappa_1 = \alpha_1 \quad \text{and} \quad \kappa_d = \alpha_1^d.$$

The assumption that a_j's are non-negative is again not restrictive, since the optimal coefficients are positive with $a_j = \alpha_{k_j} k_j^{-1}(k_j+1)^{-1}$ for $t_j = k_j$. Therefore

$$n(\varepsilon, d) \ge (1-\varepsilon^2)\, \alpha_1^{-2d}.$$

Hence, $I = \{I_d\}$ is intractable and suffers from the curse of dimensionality iff we choose h_1 such that $\alpha_1 < 1$. □

10.5.3 Example: Kernels Related to Discrepancies

We now consider multivariate integration for a space F_d that is one of the Sobolev spaces considered in Chapter 9. We choose the parameters of the space F_d such that it is a tensor product Hilbert space with the reproducing kernel

$$K_d(x, y) = \prod_{j=1}^d K_1(x_j, y_j) \quad \text{for all } x, y \in D_d = [0,1]^d.$$

Here, K_1 is a reproducing kernel for the univariate case, and we consider a few examples of K_1, all related to various notions of L_2 discrepancy. First we take

$$K_1(x, y) = \min(x, y)\, 1_{[0,\alpha)^2}((x, y)) + (1 - \max(x, y))\, 1_{[\alpha,1)^2}((x, y)$$

for $x, y \in [0,1]$ and for some $\alpha \in [0,1]$. As we know from Chapter 9, this corresponds to the L_2 discrepancy anchored at α.

Clearly K_1 as well as K_d are point-wise non-negative so that we can apply Theorem 10.2. In this case we have

$$h_1(x) = \begin{cases} \alpha\, x - \frac{1}{2}x^2 & \text{for } x \in [0,\alpha), \\ \frac{1}{2}(1-x^2) - \alpha(1-x) & \text{for } x \in [\alpha, 1]. \end{cases}$$

Note that $h_1(0) = h_1(1) = 0$, and h_1 is discontinuous at α if $\alpha \neq \frac{1}{2}$, which agrees with Section 9.6 of Chapter 9. We have

$$\|h_1\|_{F_1}^2 = \tfrac{1}{3} - \alpha(1-\alpha).$$

These formulas allow us to compute

$$\kappa_1^2 = \kappa_1^2(\alpha) = \left(\frac{2}{3}\right)^3 \frac{\max^3(\alpha, 1-\alpha)}{\frac{1}{3} - \alpha(1-\alpha)}. \tag{10.7}$$

Note that

$$\kappa_1^2(\alpha) \in \left[\tfrac{4}{9}, \tfrac{8}{9}\right]$$

and the lower bound $\kappa_1^2(\alpha) = \frac{4}{9}$ is attained for $\alpha = \frac{1}{2}$, whereas the upper bound $\kappa_1^2(\alpha) = \frac{8}{9}$ is attained for $\alpha = 0$ and $\alpha = 1$. Obviously, $\kappa_1(\alpha) = \kappa_1(1-\alpha)$.

Hence, when we use linear algorithms with non-negative coefficients and want to reduce the initial error by a factor ε, we must use n function values with

$$n \geq (1-\varepsilon^2)\kappa_1(\alpha)^{-2d} \geq (1-\varepsilon^2)\left(\frac{9}{8}\right)^d.$$

Since the second lower bound is valid for all $\alpha \in [0,1]$, we can also consider F_d as the Sobolev space anchored at $\vec{\alpha}$ with arbitrary components α_j of $\vec{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_d]$ for $\alpha_j \in [0,1]$. Then the second lower bound on n holds for all such $\vec{\alpha}$ as long as we use linear algorithms with non-negative coefficients.

We now turn to the Sobolev space F_d related to the L_2 quadrant discrepancy anchored at $\vec{\alpha}$, see Section 9.6 of Chapter 9. We first assume that all components of $\vec{\alpha}$ are the same and equal to $\alpha \in [0,1]$, i.e., $\vec{\alpha} = [\alpha, \alpha, \dots, \alpha]$. Then F_d is the tensor product space with the reproducing kernel

$$K_1(x,t) = \tfrac{1}{2}\left[|x-\alpha| + |t-\alpha| - |x-t|\right] \quad \text{for all } x,t \in [0,1).$$

Again K_1 and K_d are point-wise non-negative. We now have

$$h_1(x) = \tfrac{1}{2}\left[|x-\alpha| + (\alpha - x)(\alpha + x - 1)\right] \quad \text{for all } x \in [0,1].$$

Note that $h_1(\alpha) = 0$ and

$$\|h_1\|_{F_1}^2 = \tfrac{1}{3} - \alpha(1-\alpha).$$

It is easy to check that in this case $\kappa_1 = \kappa_1(\alpha)$ is given by (10.7), just as for the L_2 discrepancy anchored at α. Hence, we can also consider $\vec{\alpha}$ with arbitrary components. For all $\vec{\alpha} \in [0,1]^d$, we have

$$n \geq (1-\varepsilon^2)\left(\frac{9}{8}\right)^d$$

for the class of linear algorithms with non-negative coefficients that use n function values with the worst case error at most equal to $\varepsilon\|h_d\|$.

Our next Sobolev space F_d is related to the unanchored discrepancy, see again Section 9.6 of Chapter 9. Then the reproducing kernel for the univariate case is

$$K_1(x,y) = \min(x,y) - xy \quad \text{for all } x,y \in [0,1].$$

This kernel is point-wise non-negative and we now have

$$h_1(x) = x - \tfrac{1}{2}(x - x^2).$$

Then $h_1(0) = h_1(1) = 0$ and

$$\|h_1\|_{F_1}^2 = \tfrac{1}{12}.$$

Note that this is the same as for the L_2 discrepancy anchored at $\frac{1}{2}$, and therefore in this case

$$\kappa_1^2 = \tfrac{4}{9}.$$

So in this case

$$n \ge (1-\varepsilon^2)\left(\frac{9}{4}\right)^d.$$

As our last space, we take

$$K_1(x,t) = 1 + \min(x,t),$$

which corresponds to the unweighted L_2 discrepancy anchored at 1. Then

$$h_1(x) = 1 + x - \tfrac{1}{2}x^2 \quad \text{and} \quad \|h_1\|_{F_1} = (4/3)^{1/2},$$

and

$$\kappa_1^2 = \tfrac{3}{4} \max_{x\in[0,1]} \frac{\left(1+x-\frac{1}{2}x^2\right)^2}{1+x} = \frac{29+4\sqrt{7}}{9(2+\sqrt{7})} = \frac{1}{1.0563058\ldots}.$$

In this case

$$n \ge (1-\varepsilon^2)(1.0563)^d.$$

In all cases considered in this example, we do not know whether linear algorithms with some negative coefficients are significantly better than linear algorithms with non-negative coefficients. This leads us to the next open problem.

Open Problem 43.

- Consider multivariate integration for the Sobolev spaces considered in this example in the worst case setting. Verify whether the minimal worst case error of linear algorithms with *arbitrary* coefficients is significantly smaller than the minimal worst case error of linear algorithm with *non-negative* coefficients.

That is why, unlike in the last two examples, we cannot yet claim intractability of multivariate integration considered in this example for the normalized error criterion. In fact, we do have intractability and the curse of dimensionality for the normalized error criterion and strong polynomial tractability for the absolute error criterion as we shall prove later. □

10.5.4 Example: Polynomials of Degree Two

Here we present an example from [208] that shows intractability of positive quadrature formulas for a very small class of polynomials. Let F_d be the linear space of $f\colon [0,1]^d \to \mathbb{R}$, where f is a polynomial of degree at most two in each variable. The space F_d is equipped with the norm

$$\|f\|_{F_d}^2 = \sum_{\alpha\in\{0,1,2\}^d} \|D^\alpha f\|_{L_2}^2.$$

For $d = 1$, define the polynomials

$$e_1 = 1, \quad e_2(x) = x - \tfrac{1}{2} \quad \text{and} \quad e_3(x) = (x-1/2)^2 - \tfrac{1}{12}.$$

These polynomials are orthogonal in F_1 with

$$\|e_1\|_{F_1} = 1, \quad \|e_2\|_{F_1} = \left(1 + \tfrac{1}{12}\right)^{1/2} \quad \text{and} \quad \|e_3\|_{F_1} = \left(4 + \tfrac{1}{3} + \tfrac{1}{180}\right)^{1/2}.$$

From this we obtain the positive reproducing kernel

$$\begin{aligned} K_1(x,y) &= \sum_{j=1}^{3} \frac{e_j(x)e_j(y)}{\|e_j\|_{F_1}^2} \\ &= 1 + \frac{12(x-1/2)(y-1/2)}{13} + \frac{((x-1/2)^2 - 1/12)((y-1/2)^2 - 1/12)}{4 + 1/3 + 1/180}. \end{aligned}$$

Univariate integration is clearly of the form

$$I_1(f) = \int_0^1 f(x)\,\mathrm{d}x = \langle f, 1 \rangle_{F_1}.$$

For $d > 1$, multivariate integration is a tensor product problem. We can apply Theorem 10.2 and Corollary 10.3. The number κ_1 is now

$$\kappa_1 = \sup_{x \in [0,1]} \frac{1}{\sqrt{K_1(x,x)}} = \frac{1}{\sqrt{K_1(\frac{1}{2}, \frac{1}{2})}} \approx 0.9992.$$

We conclude that integration is intractable for positive quadrature formulas.

A better lower bound on $e^{\mathrm{wor}}(A_{n,d})$ for positive quadrature formulas $A_{n,d}$ can be obtained in this case since $k_{\min} = \inf_{x,y} K_1(x,y) > 0$. One can easily prove that

$$e^{\mathrm{wor}}(A_{n,d})^2 \geq 1 - \frac{n\,\kappa_1^{2d}}{1 + (n-1)[\kappa_1^2 k_{\min}]^d},$$

see [208] for the details.

A similar example, with trigonometric polynomials instead of algebraic polynomials, has already been presented as Example 2 of Chapter 3 in Volume I. Based on this example we presented Open Problem 3. We add in passing that this problem was put into a much wider context (but not solved) by Hinrichs and Vybíral [133]. These authors found equivalent formulations of Open Problem 3, as well as more general open problems that are related to different fields of mathematics.

We again obtain an open problem when we allow arbitrary quadrature formulas.

Open Problem 44.

- Prove or disprove tractability of this integration problem. In particular, are general quadrature formulas better than positive quadrature formulas?

- Of course, it is also interesting to study the integration problem for the class of all $C^\infty([0,1]^d)$-functions with

$$\|f\|_{F_d}^2 = \sum_{\alpha \in \mathbb{N}_0^d} \|D^\alpha f\|_{L_2[0,1]^d}^2 < \infty.$$

  Again we have intractability for positive quadrature formulas. It is open what happens for general quadrature formulas. Answer the same questions as above for this problem.

We remind the reader that we have formulated a similar open problem in Volume I. This is Open Problem 2 for the infinity norm. In this case, strong polynomial tractability does not hold due to the result of J. Wojtaszczyk [345] and polynomial or weak tractability is still open.

10.6 Power of Negative Coefficients

In the previous section, we considered spaces with non-negative reproducing kernels and algorithms with non-negative coefficients. Obviously, for some linear functionals the choice of algorithms with non-negative coefficients is quite bad. However, it is interesting to ask if the assumption on non-negative coefficients is restrictive for positive linear functionals. By a *positive* linear functional I_d we mean that

$$f \geq 0 \quad \text{implies} \quad I_d(f) \geq 0.$$

So multivariate integration is a positive linear functional. In particular, if we consider multivariate integration for a space with non-negative kernel can we assume that coefficients of an optimal linear algorithm are positive?

One might be inclined to believe that the answer is yes. This guess is supported by the many successes of QMC algorithms (for which the coefficients are not only positive but all equal to n^{-1}) as well as by the discrepancy results reported in the previous chapter. Furthermore, the choice of positive coefficients is often recommended to guarantee numerical stability, and many practitioners simply refuse to use algorithms with some negative coefficients.

On the other hand, there is the very powerful Smolyak (or sparse grid) algorithm for general multivariate tensor product problems that uses some negative weights; in Chapter 15 this algorithm is thoroughly analyzed. In some cases the Smolyak algorithm for multivariate integration is better than all *explicitly known* algorithms with non-negative coefficients. Of course this does not prove that general linear algorithms are better than positive ones. Proving optimality of positive algorithms for natural function spaces is a challenging problem.

We now show that for multivariate integration and for some other positive functionals, linear algorithms with some negative coefficients may be exponentially better than algorithms with non-negative coefficients. We admit that our example is artificial; we still do not know whether optimal algorithms for classes such as classical Sobolev spaces use negative coefficients. In any case, this example shows that it is not *always* true that we can ignore algorithms allowing negative coefficients.

We consider a Hilbert space F_d of functions defined on $[0,1]^d$ that is a tensor product, $F_d = F_1 \otimes \cdots \otimes F_1$. The space F_1 is two-dimensional and generated by the orthonormal functions e_1 and e_2. To be more concrete, let us take

$$e_1(x) = 4\left|x - \tfrac{1}{2}\right| \quad \text{and} \quad e_2(x) = \left(x - \tfrac{1}{2}\right) \cdot g(x),$$

where g is Lebesgue measurable with $|g(x)| \le 1$, and g is symmetric about $\frac{1}{2}$, i.e., $g(\frac{1}{2} - x) = g(\frac{1}{2} + x)$, and takes infinitely many values. For instance, $g(x) = 1/2 + 2\,(x - 1/2)^2$ is an example of such a function. Note that

$$K_1(x,t) = e_1(x)e_1(t) + e_2(x)e_2(t)$$

is point-wise non-negative.

Let I_1 be a linear functional satisfying $e(1,1) > 0$. Note that $I_1(f) = \int_0^1 f(t)\,\mathrm{d}t$ is such a functional. Indeed, we now have

$$h_1(x) = \int_0^1 K_1(x,t)\,\mathrm{d}t = e_1(x) \quad \text{for all } x \in [0,1].$$

Furthermore, $\|h_1\|_{F_1} = 1$ and

$$\kappa_1 = \sup_{x\in[0,1]} \frac{e_1(x)}{\sqrt{K_1(x,x)}} = \max_{x\in[0,1]} \frac{1}{\sqrt{4(1+g^2(x))}} = \frac{\sqrt{5}}{5}.$$

For any such I_1, let $I_d = I_1 \otimes \cdots \otimes I_1$ be the d-fold tensor product functional. As long as we use algorithms with non-negative coefficients, we can achieve an error ε only if

$$n \ge (1-\varepsilon^2)\,\kappa_1^{-2d},$$

where $\kappa_1 < 1$. For multivariate integration, we have

$$n \ge (1-\varepsilon^2)\,5^d.$$

So the curse of dimensionality is present for this class of algorithms.

What can we achieve if we use general linear algorithms possibly with negative coefficients? The surprising answer is that

$$e(d+1,d) = 0$$

for *all* linear tensor product functionals.

That is, all linear functionals I_d, including multivariate integration, can be approximated with an arbitrarily small error using at most $d+1$ function values. This result has been proved in [218] and will be presented in Section 11.3 of Chapter 11.

This example shows that, at least for some spaces, allowing algorithms with negative coefficients breaks intractability of algorithms with non-negative coefficients. It would be of great practical importance to characterize for which spaces F_d this phenomenon does *not* occur. This leads us to the next open problem.

Open Problem 45.

- Characterize reproducing kernel Hilbert spaces for which positive linear functionals have optimal linear algorithms with non-negative coefficients.
- Characterize reproducing kernel Hilbert spaces for which tractability conditions for positive linear functionals are the same for the classes of linear algorithms with non-negative and arbitrary coefficients.

10.7 Multivariate Integration

This is the first section in which we present conditions on spaces F_d and linear functionals I_d under which $e(n,d)$ goes to zero at least as fast as $n^{-1/2}$.

We begin with multivariate integration, which is defined as follows. First of all, we assume that F_d is separable, consists of Lebesgue measurable functions and has a reproducing kernel K_d satisfying the condition

$$\int_{D_d^2} K_d(x,t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}y \;<\; \infty. \tag{10.8}$$

Here, ϱ_d is a non-negative function such that $\int_{D_d} \varrho_d(t)\, \mathrm{d}t = 1$, and g_d is a given real non-zero Lebesgue measurable function.

Then by *multivariate integration* we mean the functional $I_d : F_d \to \mathbb{R}$ defined as

$$I_d(f) = \int_{D_d} f(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t \quad \text{for all } f \in F_d.$$

Many papers assume that $g_d \equiv 1$ but for our purposes it is better to consider arbitrary functions g_d.

We now prove that I_d is a well-defined continuous linear functional. Indeed, for any orthonormal basis $\{\eta_j\}$ of F_d we have

$$K_d(x,t) = \sum_{j=1}^{\infty} \eta_j(x)\eta_j(t) \quad \text{for all } x,t \in D_d.$$

Since $f = \sum_{j=1}^{\infty} \langle f, \eta_j \rangle_{F_d} \eta_j$ with $\sum_{j=1}^{\infty} \langle f, \eta_j \rangle_{F_d}^2 < \infty$, we have

$$I_d(f) = \sum_{j=1}^{\infty} \langle f, \eta_j \rangle_{F_d} \int_{D_d} \eta_j(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t.$$

Hence, $I_d(f)$ is well defined iff

$$a := \sum_{j=1}^{\infty} \left(\int_{D_d} \eta_j(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t \right)^2 < \infty.$$

Note that (10.8) implies that

$$\begin{aligned} a &= \sum_{j=1}^{\infty} \int_{D_d^2} \eta_j(x)\, \eta_j(t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_{D_d^2} K_d(x,t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t < \infty. \end{aligned}$$

This also shows that

$$h_d(x) = \int_{D_d} K_d(x,t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t = \sum_{j=1}^{\infty} \left[\int_{D_d} \eta_j(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t \right] \eta_j(x)$$

is well defined and belongs to F_d. Furthermore,

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d,$$

and I_d is indeed a continuous linear functional with

$$\begin{aligned} \|I_d\| = \|h_d\|_{F_d} &= \left[\int_{D_d^2} K_d(x,t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t \right]^{1/2} \\ &= \left[\int_{D_d} h_d(t)\, g_d(dt)\, \varrho_d(t)\, \mathrm{d}t \right]^{1/2}. \end{aligned}$$

10.7.1 QMC Algorithms

We first consider QMC (quasi-Monte Carlo) algorithms. For $g \equiv 1$, these algorithms are defined by taking $a_j = n^{-1}$. For a general g_d, we take $a_j = n^{-1} g_d(t_j)$ and QMC algorithms take the form

$$A_{n,d}(f) = \frac{1}{n} \sum_{j=1}^{n} f(t_j)\, g_d(t_j) \quad \text{for all } f \in F_d$$

for some t_j from D_d. To stress the role of the sample points t_j, we sometimes write $\vec{t} = [t_1, t_2, \dots, t_n] \in D_d^n$ and $A_{n,d} = A_{n,d,\vec{t}}$.

Observe that QMC algorithms are similar to the standard MC (Monte Carlo) algorithms for approximating integrals. The important difference between QMC and MC is that the sample points t_j for QMC algorithms are *deterministic*, whereas for MC algorithms the sample points t_j are randomly selected. We return to MC algorithms in Chapter 17, where we study tractability in the randomized setting. In this chapter we study the worst case setting; the points t_j are always deterministic.

As in Volume I, we again stress that QMC algorithms have been used, mostly for $g_d \equiv 1$, with much success for many financial applications. The study of tractability was initiated in the 1990s to explain why QMC algorithms are so efficient for high dimensional integrals. The form of QMC algorithms may suggest that they can be used only for multivariate integration. As we shall see, this is not really the case, although multivariate integration is certainly the most important application for QMC algorithms.

We are ready to present an estimate, which is well known for $g_d \equiv 1$ on the worst case error of QMC algorithms. This estimate sometimes enables us to establish tractability of multivariate integration.

Theorem 10.4. *Consider the multivariate integration problem I_d defined for a separable reproducing Hilbert space F_d. Assume that*

$$C(K_d, g_d) := \int_{D_d} K_d(t,t)\, g_d^2(t)\, \varrho_d(t)\, \mathrm{d}t < \infty. \tag{10.9}$$

Then there exists $\vec{t} = [t_1, t_2, \dots, t_n] \in D_d^n$ for which the worst case error of the QMC algorithm $A_{n,d,\vec{t}}$ satisfies

$$e^{\mathrm{wor}}(A_{n,d,\vec{t}}) \le \frac{\sqrt{C(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}}.$$

Furthermore, for any $C > 1$, the Lebesgue measure $\lambda(Z)$ of the set

$$Z = \left\{ \vec{t} \in D_d^n \mid e^{\mathrm{wor}}(A_{n,d,\vec{t}}) \le C\, \frac{\sqrt{C(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right\}$$

satisfies

$$\lambda(Z) \ge 1 - C^{-2}.$$

Proof. First of all note that (10.9) implies that (10.8) holds and therefore multivariate integration is well defined. Indeed, since $K_d(x,t) \le [K_d(x,x)\, K_d(t,t)]^{1/2}$, we find

$$\begin{aligned}
&\int_{D_d^2} K_d(x,t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t \\
&\le \int_{D_d^2} \sqrt{K_d(x,x)}\, |g_d(x)|\, \sqrt{\varrho_d(x)\, \varrho_d(t)}\, \sqrt{K_d(t,t)}\, |g_d(t)|\, \sqrt{\varrho_d(x)\, \varrho_d(t)}\, \mathrm{d}x\, \mathrm{d}t \\
&\le \left[\int_{D_d^2} K_d(x,x) g_d^2(x) \varrho_d(x) \varrho_d(t) \mathrm{d}x \mathrm{d}t \int_{D_d^2} K_d(t,t) g_d^2(t) \varrho_d(x) \varrho_d(t) \mathrm{d}x \mathrm{d}t \right]^{1/2} \\
&= \int_{D_d} K_d(x,x)\, g_d^2(x)\, \varrho_d(x)\, \mathrm{d}x = C(K_d, g_d) < \infty.
\end{aligned}$$

We start with the worst case error of $A_{n,d,\vec{t}}$, which satisfies

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 = \|h_d\|_{F_d}^2 - \frac{2}{n}\sum_{j=1}^{n} h_d(t_j)\, g_d(t_j) + \frac{1}{n^2}\sum_{i,j=1}^{n} K_d(t_i,t_j)\, g_d(t_i)\, g_d(t_j).$$

We now treat the sample points t_j as independent identically distributed points over the set D_d with the density ϱ_d. We compute the expected value of the square of the worst case error of $A_{n,d,\vec{t}}$ with respect to such $\vec{t}$. Let

$$\mathbb{E} := \int_{D_d^n} \left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 \varrho_d(t_1)\ \ldots\ \varrho_d(t_n)\ \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

Since

$$\|h_d\|_{F_d}^2 = \int_{D_d} h_d(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t,$$
$$\|h_d\|_{F_d}^2 = \int_{D_d^2} K_d(x,t)\, g_d(x)\, g_d(t)\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t,$$

the integration of each $h_d(t_j)$, as well as of each $K_d(t_i,t_j)$ for $i \neq j$, yields $\|h_d\|_{F_d}^2$, whereas the integration of each $K_d(t_j,t_j)$ yields $C(K_d,g_d)$. So we obtain

$$\begin{aligned}\mathbb{E} &= \|h_d\|_{F_d}^2 - 2\|h_d\|_{F_d}^2 + \left(1-\frac{1}{n}\right)\|h_d\|_{F_d}^2 + \frac{1}{n}\, C(K_d,g_d)\\ &= \frac{C(K_d,g_d) - \|h_d\|_{F_d}^2}{n}.\end{aligned}$$

By the mean value theorem, there is a vector $\vec{t} = [t_1,t_2,\ldots,t_n] \in D_d^n$ for which $[e^{\mathrm{wor}}(A_{n,d,\vec{t}})]^2$ is at most equal to its average value. Hence,

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 \le \frac{C(K_d,g_d) - \|h_d\|_{F_d}^2}{n},$$

as claimed.

By Chebyshev's inequality we know that for any integrable function h with finite expectation $\mathbb{E} = \int_{D_d^n} h^2(\vec{t})\, \varrho_d(\vec{t})\, \mathrm{d}\vec{t}$, we have

$$\lambda(\{\vec{t} : h^2(\vec{t}) \le C^2\, \mathbb{E}\}) \ge 1 - C^{-2}.$$

Taking $h(\vec{t}) = \left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2$ and $\varrho_d(\vec{t}) = \prod_{j=1}^{n} \varrho_d(t_j)$ we get $\mathbb{E} = (C(K_d,g_d) - \|h_d\|_{F_d}^2)/n$, which completes the proof. □

In particular, Theorem 10.4 states that if $C(K_d,g_d) = \|h_d\|_{F_d}^2$ then multivariate integration can be solved exactly, even for $n = 1$. Unfortunately, the equality $C(K_d,g_d) = \|h_d\|_{F_d}^2$ does not often happen. For example, if $g_d \equiv 1$, then

$C(K_d, g_d) = \|h_d\|_{F_d}^2$ holds iff

$$\int_{D_d^2} K_d(x,t)\,\varrho_d(x)\,\varrho_d(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{D_d} K_d(t,t)\,\varrho_d(t)\,\mathrm{d}t.$$

This can happen iff $K(x,t) = \text{const}$ as an element of the space $L_{2,\varrho_d}(D_d)$; this easily follows from the fact that $K_d(x,t) \le [K_d(x,x)\,K_d(t,t)]^{1/2}$. Hence in this case, F_d is the space of constant functions, which is why multivariate integration is trivial. For all interesting spaces and functions g we have

$$C(K_d, g_d) > \|h_d\|_{F_d}^2;$$

as we shall see, for some spaces there is even an exponential difference between $C(K_d, g_d)$ and $\|h_d\|_{F_d}^2$.

From Theorem 10.4 we obtain sufficient conditions on tractability of multivariate integration. Namely, the bound in Theorem 10.4 yields an upper bound

$$n(\varepsilon,d) \le \left\lceil \frac{C(K_d,g_d) - \|h_d\|_{F_d}^2}{\mathrm{CRI}_d^2}\,\frac{1}{\varepsilon^2} \right\rceil \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N},$$

on the information complexity.

As earlier, $\mathrm{CRI}_d = 1$ for the absolute error criterion, and $\mathrm{CRI}_d = \|I_d\| = \|h_d\|_{F_d}$ for the normalized error criterion. From this we easily conclude the following corollary.

Corollary 10.5. *Consider the multivariate integration problem* INT $= \{I_d\}$ *in the worst case setting, where* I_d *is defined for a separable reproducing kernel Hilbert space* F_d*. Let*

$$A_d^{\mathrm{abs}} = \int_{D_d} K_d(t,t)\,g_d^2(t)\,\varrho_d(t)\,\mathrm{d}t - \int_{D_d^2} K_d(x,t)\,g_d(x)\,g_d(t)\,\varrho_d(x)\,\varrho_d(t)\,\mathrm{d}x\,\mathrm{d}t,$$

$$A_d^{\mathrm{nor}} = \frac{\int_{D_d} K_d(t,t)\,g_d^2(t)\,\varrho_d(t)\,\mathrm{d}t}{\int_{D_d^2} K_d(x,t)\,g_d(x)\,g_d(t)\,\varrho_d(x)\,\varrho_d(t)\,\mathrm{d}x\,\mathrm{d}t} - 1.$$

Let $x \in \{\text{abs}, \text{nor}\}$.

- *If there exists a number* $q \ge 0$ *such that*

$$C := \sup_{d\in\mathbb{N}} \frac{A_d^x}{d^q} < \infty$$

then INT *is polynomially tractable for the* x *error criterion, and*

$$n(\varepsilon,d) \le \lceil C\,d^q\,\varepsilon^{-2} \rceil \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

If $q = 0$ *then* INT *is strongly polynomially tractable for the* x *error criterion.*

- *If*
$$\lim_{d\to\infty} \frac{\ln A_d^x}{d} = 0$$
then INT *is weakly tractable for the* x *error criterion.*

- *If*
$$A := \limsup_{d\to\infty} \frac{\ln A_d^x}{\ln(1+T(1,d))} < \infty,$$
$$B := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty$$
then INT *is* T*-tractable for the* x *error criterion with the exponent of* T*-tractability at most* $A+2B$.

- *If* $\sup_d A_d^x < \infty$ *and* $B < \infty$ *then* INT *is strongly* T*-tractable for the* x *error criterion with the exponent of strong* T*-tractability at most* $2B$.

The bounds on the information complexity can be obtained by QMC algorithms.

Proof. The conditions on strong polynomial, polynomial and weak tractability are straightforward. For T-tractability we need to prove that there are some non-negative C_1 and t such that
$$\lceil A_d^x \varepsilon^{-2} \rceil \le C_1 T(\varepsilon^{-1},d)^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$
Since $T(\varepsilon^{-1},d) \ge 1$ we can replace $T(\varepsilon^{-1},d)$ by $1+T(\varepsilon^{-1},d)$ in the bound above with a modified C_1. Then $\ln(1+T(\varepsilon^{-1},d))$ is always positive. Taking the logarithms we conclude that the bound above holds if
$$\limsup_{\varepsilon^{-1}+d\to\infty} \left[\frac{\ln A_d^x}{\ln(1+T(\varepsilon^{-1},d))} + \frac{2\ln\varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},d))} \right] < \infty.$$
Note that
$$\max\big(T(1,d), T(\varepsilon^{-1},1)\big) \le T(\varepsilon^{-1},d).$$
Therefore the last limit superior is bounded by $A+2B$ and is finite. It is easy to see that the infimum of t is at most $A+2B$, as claimed. Strong T-tractability follows from the same argument. □

We illustrate Corollary 10.5 by a number of specific examples.

10.7.2 Example: Tensor Product Problems

We assume that F_d is a tensor product Hilbert space with the reproducing kernel
$$K_d(x,y) = \prod_{j=1}^{d} K_1(x_j,y_j) \quad \text{for all } x,y \in D_d = D_1 \times D_1 \times \cdots \times D_1$$

for some univariate reproducing kernel $K_1\colon D_1 \times D_1 \to \mathbb{R}$ with $D_1 \subseteq \mathbb{R}$. Assume also that

$$g_d(t) = \prod_{j=1}^{d} g_1(t_j) \quad \text{and} \quad \varrho_d(t) = \prod_{j=1}^{d} \varrho_1(t_j)$$

for some univariate functions $g_1, \varrho_1\colon D_1 \to \mathbb{R}$.

Consider multivariate integration for the absolute error criterion in the worst case setting. For tensor product problems, we have

$$A_d^{\mathrm{abs}} = \left[\int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t\right]^d - \left[\int_{D_1^2} K_1(x,t)\varrho_1(x) g_1(x) g_1(t) \varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t\right]^d.$$

For non-trivial spaces, we have

$$\int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t > \int_{D_1^2} K_1(x,t)\varrho_1(x) g_1(x) g_1(t) \varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t.$$

If

$$A_1 := \int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t \le 1 \tag{10.10}$$

then A_d^{abs} is uniformly bounded and we have strong polynomial tractability. Otherwise if $A_1 > 1$, then A_d^{abs} depends exponentially on d. Hence even the condition on weak tractability is not satisfied.

We stress that (10.10) holds for Sobolev spaces related to L_2 discrepancy. Indeed, let $g_1 = \varrho_1 = 1$. Then for the L_2 discrepancy anchored at α we have

$$A_1 = A_1(\alpha) = \tfrac{1}{2} - \alpha + \alpha^2 \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right],$$

for the L_2 quadrant discrepancy anchored at α we have the same $A_1 = A_1(\alpha)$ as above, and for the unanchored discrepancy $A_1 = \frac{1}{6}$. This proves the following corollary.

Corollary 10.6.

- *Multivariate integration defined on a separable tensor product space $H(K_d)$ with*

  $$\int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t \le 1$$

  is strongly polynomially tractable for the absolute error criterion in the worst case setting with exponent at most 2.

- *Multivariate integration with $g_1 = \varrho_1 = 1$ defined for Sobolev spaces related to the L_2 discrepancy anchored at α, quadrant discrepancy anchored at α and unanchored discrepancy is strongly polynomially tractable for the absolute error criterion in the worst case setting with exponent at most* 2. □

10.7.3 Example: Modified Sobolev Space

We begin with the Sobolev space anchored at 0 with the reproducing kernel

$$K_d(x,t) = \prod_{j=1}^{d} \min(x_j, t_j) \quad \text{for all } x,t \in [0,1]^d.$$

We take $D_d = [0,1]^d$ and $g_d(t) = \varrho_d(t) = 1$. For $f \in F_d$ we know that $f(x) = 0$ if at least one component of x is zero, and

$$\|f\|_{F_d}^2 = \int_{[0,1]^d} \left(\frac{\partial^d}{\partial x_1 \cdots \partial x_d} f(t) \right)^2 \mathrm{d}t.$$

As we know from Chapter 9, the worst case error of QMC algorithms is related to the L_2-discrepancy anchored at 1. Observe that we have $I_d(f) = \langle f, h_d \rangle_{F_d}$ with

$$h_d(x) = \int_{[0,1]^d} K_d(x,t)\,\mathrm{d}t = \prod_{j=1}^{d} \left(x_j - \tfrac{1}{2} x_j^2 \right).$$

Therefore

$$\|h_d\|_{F_d}^2 = \int_{[0,1]^{2d}} K_d(x,t)\,\mathrm{d}x\,\mathrm{d}t = 3^{-d},$$

$$C(K_d,1) = \int_{[0,1]^d} K_d(t,t)\,\mathrm{d}t = 2^{-d}.$$

Hence, multivariate integration is indeed strongly polynomially tractable for the absolute error criterion since $A_d^{\text{abs}} = 2^{-d} - 3^{-d}$ and $\sup_d A_d^{\text{abs}} = \frac{1}{6}$. For the normalized error criterion, note that

$$A_d^{\text{nor}} = 1.5^d - 1$$

is exponentially large in d. So even the condition for weak tractability does *not* hold. We shall see later that multivariate integration is intractable for the normalized error criterion. For the normalized error criterion, we have

$$n(\varepsilon,d) \le \left\lceil \left(\left(\frac{3}{2} \right)^d - 1 \right) \frac{1}{\varepsilon^2} \right\rceil \quad \text{for all } \varepsilon \in (0,1). \tag{10.11}$$

This estimate may be acceptable for relatively small d. For instance, for $d = 10, 20, 50$ we have

$$n(\varepsilon,10) \le 10^{1.76}\varepsilon^{-2} + 1, \;\; n(\varepsilon,20) \le 10^{3.54}\varepsilon^{-2} + 1, \;\; n(\varepsilon,50) \le 10^{8.805}\varepsilon^{-2} + 1.$$

We now partially remove the boundary conditions by taking the reproducing kernel $K_d = K_{d,a_d,b_d}$ as

$$K_{d,a_d,b_d}(x,t) = a_d + b_d \prod_{j=1}^{d} \min(x_j, t_j) \quad \text{for all } x,t \in [0,1]^d,$$

for some non-negative numbers a_d and b_d. Obviously we always assume that $a_d^2+b_d^2 > 0$ since otherwise $K_d \equiv 0$ and the space is trivial. For $a_d = 0$ and $b_d = 1$ we obtain the previous kernel.

For general non-negative a_d and b_d with $a_d^2 + b_d^2 > 0$, the space $F_d = F_{d,a_d,b_d}$ consists of functions $f(x) = c + h(x)$, where h belongs to the space $F_{d,0,1}$, and $c = f(0)$. For $f \in F_{d,a_d,b_d}$ we have

$$\|f\|^2_{F_{d,a_d,b_d}} = \frac{1}{a_d}\, f(0)^2 + \frac{1}{b_d} \int_{[0,1]^d} \left(\frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, f(t) \right)^2 \mathrm{d}t,$$

with the convention that $0/0 = 0$.

Multivariate integration $I_d(f) = \langle f, h_{d,a_d,b_d} \rangle_{F_{d,a_d,b_d}}$ is given by

$$h_{d,a_d,b_d}(x) = a_d + b_d \prod_{j=1}^{d} \left(x_j - \tfrac{1}{2}\, x_j^2 \right).$$

We have

$$C(K_{d,a_d,b_d}, 1) = a_d + b_d\, 2^{-d} \quad \text{and} \quad \|h_{d,a_d,b_d}\|^2_{F_{d,a_d,b_d}} = a_d + b_d\, 3^{-d}.$$

For the absolute error criterion, we have

$$A_d^{\mathrm{abs}} = b_d\, 2^{-d} \left(1 - \left(\frac{2}{3} \right)^d \right).$$

Therefore multivariate integration is

- strongly polynomially tractable if $b_d = \mathcal{O}(2^d)$,
- polynomially tractable if $b_d = \mathcal{O}(d^q 2^d)$ for some positive q,
- weakly tractable if $b_d = 2^d e^{o(d)}$, as d goes to infinity,
- T-tractable if

$$\limsup_{d \to \infty} \frac{\ln(b_d 2^{-d})}{\ln(1 + T(1,d))} < \infty \quad \text{and} \quad \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln(1 + T(\varepsilon^{-1}, 1))} < \infty.$$

Note that these conditions are independent of a_d.

For the normalized error criterion, we have

$$A_d^{\mathrm{nor}} = \frac{a_d + b_d 2^{-d}}{a_d + b_d 3^{-d}} - 1.$$

Then tractability conditions are the same as before, but with b_d replaced by b_d/a_d. In particular, multivariate integration is

- strongly polynomially tractable if $b_d/a_d = \mathcal{O}(2^d)$.
- polynomially tractable if $b_d/a_d = \mathcal{O}(d^q 2^d)$ for some positive q,
- weakly tractable if $b_d/a_d = 2^d e^{o(d)}$, as d goes to infinity.

These conditions now depend on both a_d and b_d.

Clearly, for various choices of a_d and b_d the tractability conditions may hold for one error criterion and not for the other. Indeed, for $a_d = 3^{-d}$ and $b_d = 1$, the condition of strong polynomial tractability holds for the absolute error criterion but even the condition for weak tractability does not hold for the normalized error criterion. On the other hand, for $a_d = b_d = 3^d$, the condition for weak tractability does not hold for the absolute error criterion, but the condition for strong polynomial tractability holds for the normalized error criterion. □

10.7.4 Example: Korobov Space with Varying Smoothness

As in [236], we consider the Korobov space F_d of functions that are r_j times differentiable with respect to the jth variable. Here $r = \{r_j\}$ is a given sequence of integers such that

$$1 \le r_1 \le r_2 \le \cdots \le r_j \le \dots .$$

That is, we may have increasing smoothness with respect to the successive variables. For which sequences r is multivariate integration strongly polynomially, polynomially or weakly tractable? This problem was studied in [236] for multivariate approximation for the class Λ^{all} in the worst case setting and it was proved that a logarithmic growth of r_j is needed for strong polynomial tractability. We want to verify if a similar result holds for multivariate integration for the class Λ^{std} in the worst case setting.

More precisely, let

$$F_d = H_{1,r_1} \otimes H_{1,r_2} \otimes \cdots \otimes H_{1,r_d},$$

where H_{1,r_j} is the Korobov space of univariate 1-periodic complex valued functions f defined on $[0,1]$ such that $f^{(r_j-1)}$ is absolutely continuous and $f^{(r_j)}$ belongs to $L_2([0,1])$. The space H_{1,r_j} is equipped with the norm

$$\|f\|^2_{H_{1,r_j}} = \left| \int_0^1 f(x)\,\mathrm{d}x \right|^2 + \int_0^1 |f^{(r_j)}(x)|^2\,\mathrm{d}x.$$

The space F_d is a reproducing kernel Hilbert space whose kernel is

$$K_d(x,t) = \prod_{j=1}^d \left(1 + \frac{2}{(2\pi)^{2r_j}} \sum_{h=1}^\infty \frac{\cos(2\pi\,h\,(x_j - t_j))}{h^{2r_j}} \right) \quad \text{for all } x, y \in [0,1]^d.$$

More about Korobov spaces for constant $r_j = r$ can be found in Appendix A of Volume I as well as in Chapter 16.

Multivariate integration is defined by taking $g_d(t) = \varrho_d(t) = 1$ for all $t \in [0,1]^d$, so that

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \text{for all } f \in F_d.$$

Note that $h_d = 1$ and $\|I_d\| = \|h_d\|_{F_d} = 1$, and so there is no difference between the absolute and normalized error criteria.

We now have

$$A_d := A_d^{\text{abs}} = A_d^{\text{nor}} = \prod_{j=1}^{d} \left(1 + \frac{2\,\zeta(2r_j)}{(2\pi)^{2r_j}}\right) - 1,$$

where, as always, ζ is the Riemann zeta function. As usual, we estimate

$$\prod_{j=1}^{d} \left(1 + \frac{2\,\zeta(2r_j)}{(2\pi)^{2r_j}}\right) \le \exp\left[2 \sum_{j=1}^{d} \zeta(2r_j)\,(2\pi)^{-2r_j}\right].$$

Since r_j's are non-decreasing, we have

$$1 \le \zeta(2r_j) \le \zeta(2r_1) \le \zeta(2) = \tfrac{1}{6}\pi^2.$$

Hence, tractability of multivariate integration depends on the behavior of

$$\sum_{j=2}^{d} (2\pi)^{-2r_j} = \sum_{j=2}^{d} j^{-2\ln(2\pi)\,r_j/\ln j}.$$

It is easy to check that

$$L^{\text{sup}} := \limsup_{j\to\infty} \frac{\ln j}{r_j} < 2\ln(2\pi) = 3.6757\ldots$$

implies that multivariate integration is strongly polynomially tractable with exponent at most 2. Indeed, for $p \in (L^{\text{sup}}, 2\ln(2\pi))$ and large j, we have $r_j \ge \ln(j)/p$ and therefore

$$\frac{2\ln(2\pi)\,r_j}{\ln j} \ge \frac{2\ln(2\pi)}{p} > 1.$$

This proves that

$$\sum_{j=2}^{\infty} j^{-2\ln(2\pi)\,r_j/\ln j} < \infty,$$

and $\sup_d A_d < \infty$. This yields strong tractability with exponent at most 2.

We now show that

$$L^{\text{sup}} = \limsup_{j\to\infty} \frac{\ln j}{r_j} < \infty$$

implies that multivariate integration is weakly tractable. Indeed, for large j and $p > \max(L^{\text{sup}}, 2\ln(2\pi))$, we have $r_j \ge \ln(j)/p$ and therefore

$$\sum_{j=2}^{d} j^{-2\ln(2\pi)\, r_j/\ln j} = \mathcal{O}(1) + \sum_{j=2}^{d} j^{-2\ln(2\pi)/p} = \mathcal{O}\big(d^{\,1-2\ln(2\pi)/p}\big).$$

Hence,

$$\lim_{d\to\infty} \frac{\ln A_d}{d} = \mathcal{O}\big(\lim_{d\to\infty} d^{-2\ln(2\pi)/p}\big) = 0,$$

and we have weak tractability.

We now comment on the polynomial tractability of multivariate integration. Let

$$L_{\text{inf}} := \liminf_{j\to\infty} \frac{\ln j}{r_j} \le 2\ln(2\pi).$$

We now show that in this case our upper bound is too weak to claim polynomial tractability. Indeed, if we take

$$r_j = \left\lceil \frac{\ln(j)}{2\ln(2\pi)} \right\rceil$$

for large j, then we have $L^{\text{sup}} = L_{\text{inf}} = 2\ln(2\pi)$. Furthermore

$$\sum_{j=2}^{d} j^{-2\ln(2\pi)\, r_j/\ln j} = \mathcal{O}(1) + \sum_{j=1}^{d} j^{-1} = \mathcal{O}(1) + \ln d,$$

and

$$A_d = \mathcal{O}\big(d^{2\zeta(2r_1)}\big).$$

This means polynomial tractability with a d exponent at most $2\zeta(2r_1)$.

On the other hand, if we take

$$r_j = \left\lceil \frac{\ln(j)}{2\ln(2\pi)} \left(1 - \frac{1}{\ln\ln j}\right) \right\rceil$$

for large j, then again $L^{\text{sup}} = L_{\text{inf}} = 2\ln(2\pi)$. However, the condition on polynomial tractability no longer holds. Indeed, we need to estimate

$$\sum_{j=j_0+1}^{d} j^{-1}\, j^{1/\ln\ln j}$$

for some j_0. Since $j^{1/\ln\ln j}$ goes to infinity with j, the last series cannot be bounded by a multiple of $\ln d$, and therefore $\ln A_d$ goes faster to infinity than any multiple of $\ln d$, which contradicts the condition on polynomial tractability. We now show that if

$L^{\sup} = L_{\inf} = 2\ln(2\pi)$, then strong polynomial tractability may even hold. Take the sequence

$$r_j = \left\lceil \frac{\ln(j)}{2\ln(2\pi)} \, (1 + f(j)) \right\rceil,$$

where

$$f(j) = \frac{\ln(\ln^2 j)}{\ln j} \quad \text{for large } j.$$

Then we need to estimate

$$\sum_{j=j_0+1}^{d} j^{-1-f(j)} = \sum_{j=j_0+1}^{d} \frac{1}{j \ln^2 j},$$

which is uniformly bounded in d and yields strong polynomial tractability.

We need also to consider the case when $L_{\inf} \in (2\ln(2\pi), \infty)$. Then for $p \in (2\ln(2\pi), L_{\inf})$ and for large j, we have

$$r_j \le \ln(j)/p \quad \text{and} \quad j^{-2\ln(2\pi)r_j/\ln j} \ge j^{-2\ln(2\pi)/p},$$

with the exponent $2\ln(2\pi)/p < 1$. Therefore

$$\sum_{j=2}^{d} j^{-2\ln(2\pi)\, r_j/\ln j} = \Omega\big(d^{1-2\ln(2\pi)/p}\big),$$

which contradicts the condition on polynomial tractability.

Let us summarize the tractability conditions. For $L^{\sup} < 2\ln(2\pi)$ we have strong tractability, for $L^{\sup} < \infty$ we have weak tractability, for $L_{\inf} > 2\ln(2\pi)$ we cannot claim polynomial tractability, and finally for $L_{\inf} = 2\ln(2\pi)$ we may or may not claim polynomial tractability.

In any case, if r_j grows faster than $2\ln(2\pi)\ln j$ then we have strong polynomial tractability, whereas when r_j goes as fast as $\ln j$ we have weak tractability. So the logarithmic growth of r_j ensures at least weak tractability. This can be compared with multivariate approximation for the class Λ^{all}, where the same condition on r_j yields strong tractability, see [236].

10.7.5 Example: Korobov Space with Fixed Smoothness

Consider the Korobov space F_d studied in the previous example with all $r_j = r \ge 1$. That is, we now assume the same smoothness for all variables. Multivariate integration is still properly normalized, since

$$\|I_d\| = \|h_d\|_{F_d} = 1,$$

but

$$C(K_d,1) = \int_{[0,1]^d} K_d(t,t)\,\mathrm{d}t = (1+A_r)^d \quad \text{with } A_r := \frac{2\,\zeta(2r)}{(2\pi)^{2r}},$$

is exponentially large in d. In particular, for $r=1$ we have $C(K_d,1) = (13/12)^d$.

The absolute and normalized error criteria coincide, and the condition for weak tractability does *not* hold. In fact, multivariate integration is *not* weakly tractable, as we will see later.

For this space, as in [280] for $r=1$, we can prove that the worst case error of *any* QMC algorithm with fixed n goes to infinity with d. Indeed, just now $h_d \equiv 1$ and therefore for any $A_{n,d,\vec{t}}$ we have

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 = -1 + \frac{1}{n^2}\sum_{i,j=1}^{n} K_d(t_i,t_j).$$

Observe that $\zeta(2r) \le \zeta(2) = \pi^2/6 \le (2\pi)^{2r}/2$ implies that

$$K_d(x,t) \ge 0 \quad \text{for all } x,t \in [0,1]^d$$

and $K_d(t,t) = (1+A_r)^d$. Therefore we can drop all terms for $i \ne j$ in the last formula and obtain

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 \ge -1 + \frac{(1+A_r)^d}{n}.$$

Hence, for fixed n, the worst case of $A_{n,d,\vec{t}}$ goes exponentially fast to infinity with d no matter how the sample points t_j are chosen. This means that any QMC algorithm is intractable. This is a very bad property, since the worst case error of the zero algorithm (which is arguably the most trivial) is just 1. So despite our a priori knowledge that $|I_d(f)| \le 1$ for all f from the unit ball of the space F_d, all QMC algorithms fail badly for large d relative to n. □

10.7.6 Properly Normalized QMC Algorithms

The last example shows that for some spaces QMC algorithms may significantly lose, even with the zero algorithm. The natural question is to determine what is wrong with QMC algorithms. The answer is that the coefficients $a_j = n^{-1}$ are sometimes quite inappropriate and for spaces such as the one in the last example, they are much too large.

Let us then ask why we want to use $a_j = n^{-1}$. There are two arguments for such a choice. The first one is that they make the implementation of linear algorithms easier, and allow the exact integration of constant functions if $g_d \equiv 1$. The second reason is that they are positive and guarantee numerical stability if we assume that $g_d \ge 0$.

So let us try to find positive and equal coefficients, $a_j = a$, to preserve the ease of implementation and numerical stability, with the new task of eliminating exponential dependence in d of the worst case error of QMC algorithms. It seems natural to repeat our reasoning from the proof of Theorem 10.4 with $a_j = a$ instead of $a_j = n^{-1}$, and to find the values of a such that the expected value $\mathbb{E}$ is minimized. More precisely, we now consider *normalized QMC algorithms*, that is, algorithms of the form

$$A_{n,d,\vec{t}}(f) = a \sum_{j=1}^{n} f(t_j)\, g_d(t_j) \quad \text{for all } f \in F_d$$

for some $a \in \mathbb{R}$ and t_j from D_d. Then

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 = \|h_d\|_{F_d}^2 - 2a \sum_{j=1}^{n} h_d(t_j)\, g_d(t_j) + a^2 \sum_{i,j=1}^{n} K_d(t_i,t_j)\, g_d(t_i)\, g_d(t_j).$$

Integrating over t_j, as before, we obtain

$$\mathbb{E} = \mathbb{E}(a) = \|h_d\|_{F_d}^2\,(1-a\,n)^2 + a^2\, n\left(C(K_d,g_d) - \|h_d\|_{F_d}^2\right).$$

Observe that the previous choice $a = n^{-1}$ makes the first term equal to zero, and $\mathbb{E}(n^{-1})$ is exactly the same as for QMC algorithms. However, if we want to minimize $\mathbb{E}(a)$ then we should take

$$a = \frac{\|h_d\|_{F_d}^2}{C(K_d,g_d) - \|h_d\|_{F_d}^2 + n\,\|h_d\|_{F_d}^2}. \tag{10.12}$$

This yields

$$\mathbb{E}^{\min} := \min_{a\in\mathbb{R}} \mathbb{E}(a) = \frac{\left(C(K_d,g_d) - \|h_d\|_{F_d}^2\right)\|h_d\|_{F_d}^2}{C(K_d,g_d) - \|h_d\|_{F_d}^2 + n\|h_d\|_{F_d}^2}. \tag{10.13}$$

Note that

$$\mathbb{E}^{\min} = b\,\min\left\{\|h_d\|_{F_d}^2,\ \frac{C(K_d,g_d) - \|h_d\|_{F_d}^2}{n}\right\} \quad \text{with } b \in \left[\tfrac{1}{2}, 1\right].$$

Furthermore, for fixed d and n tending to infinity we have

$$\mathbb{E}^{\min} = \frac{C(K_d,g_d) - \|h_d\|_{F_d}^2}{n}\,(1+o(1)).$$

Assume now that n is fixed and d tends to infinity. Assume also that $C(K_d,g_d) - \|h_d\|_{F_d}^2$ tends to infinity. (This holds for the space of the last example). Then

$$\mathbb{E}^{\min} = \|h_d\|_{F_d}^2\,(1+o(1)).$$

This choice of a yields an improved version of Theorem 10.4.

Theorem 10.7. *Consider the multivariate integration problem I_d defined for a separable reproducing kernel Hilbert space F_d. Assume that*

$$C(K_d, g_d) := \int_{D_d} K_d(t,t)\, g_d^2(t)\, \varrho_d(t)\,\mathrm{d}t < \infty.$$

Then there exists $\vec{t} = [t_1, t_2, \dots, t_n] \in D_d^n$ for which the worst case error of the properly normalized QMC algorithm

$$A_{n,d,\vec{t}}(f) = \frac{\|h_d\|_{F_d}^2}{C(K_d, g_d) - \|h_d\|_{F_d}^2 + n\,\|h_d\|_{F_d}^2} \sum_{j=1}^n f(t_j)\, g_d(t_j)$$

satisfies

$$e^{\mathrm{wor}}(A_{n,d,\vec{t}}) \le \min\left(\|h_d\|_{F_d}, \frac{\sqrt{C(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right).$$

Furthermore, for any $C > 1$, the Lebesgue measure $\lambda(Z)$ of the set

$$Z = \left\{ \vec{t} \in D_d^n \;:\; e^{\mathrm{wor}}(A_{n,d,\vec{t}}) \le C\, \min\left(\|h_d\|_{F_d}, \frac{\sqrt{C(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right) \right\}$$

satisfies

$$\lambda(Z) \ge 1 - C^{-2}.$$

Knowing Theorem 10.7 one can hope to improve Corollary 10.5. As we shall see in a moment, it will be indeed possible, but only for the absolute error criterion. Using (10.13) we should choose n such that

$$\frac{\big(C(K_d, g_d) - \|h_d\|_{F_d}^2\big)\|h_d\|_{F_d}^2}{C(K_d, g_d) - \|h_d\|_{F_d}^2 + n\|h_d\|_{F_d}^2} \le \varepsilon^2\, \mathrm{CRI}_d.$$

This yields

$$n(\varepsilon, d) \le n = \left\lceil \left(\frac{C(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1 \right) \frac{1}{\varepsilon^2} \left(\frac{\|h_d\|_{F_d}^2}{\mathrm{CRI}_d} - \varepsilon^2 \right)_+ \right\rceil.$$

For the normalized error criterion, $\|h_d\|_{F_d}^2/\mathrm{CRI}_d = 1$, so that the last factor is just $1 - \varepsilon^2$ for $\varepsilon \in (0,1)$. This cannot really help for tractability if $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 < 1$. Hence, in this case we have the same tractability conditions as before.

However, for the absolute error criterion the situation is quite different. For $\mathrm{CRI}_d = 1$, the last factor is $(\|h_d\|_{F_d}^2 - \varepsilon^2)_+$. We can thus restrict ε to be in $(0, \|h_d\|_{F_d})$. Let us substitute $\varepsilon^2 = t\|h_d\|_{F_d}^2$ for $t \in (0,1)$. Then we obtain (strong) polynomial tractability if there are non-negative C, q and p such that

$$n(\varepsilon, d) \le \left\lceil \left(\frac{C(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1 \right) \frac{1-t}{t} \right\rceil \le C\, d^q \varepsilon^{-p} = \frac{C\, d^q}{t^{p/2}\, \|h_d\|_{F_d}^p}$$

for all $t \in (0,1)$ and $d \in \mathbb{N}$. This is equivalent to

$$\sup_{t\in(0,1),\, d\in\mathbb{N}} t^{p/2}\, \|h_d\|_{F_d}^p \left[\left(\frac{C(K_d,g_d)}{\|h_d\|_{F_d}^2} - 1 \right) \frac{1-t}{t} \right] d^{-q} < \infty.$$

For $C(K_d,g_d) > \|h_d\|_{F_d}^2$ and fixed d, if we take t going to zero then the last inequality can hold only if $p \ge 2$. Similarly we obtain a condition on weak tractability. This analysis yields the improved version of Corollary 10.5.

Corollary 10.8. *Consider the multivariate integration problem* INT $= \{I_d\}$ *in the worst case setting and for the absolute error criterion. Here, I_d is defined for a separable reproducing Hilbert space F_d.*

- *If there exist a number $q \ge 0$ and a number $p \ge 2$ such that*

  $$C := \sup_{t\in(0,1),\, d\in\mathbb{N}} t^{p/2}\, \|h_d\|_{F_d}^p \left[\left(\frac{C(K_d,g_d)}{\|h_d\|_{F_d}^2} - 1 \right) \frac{1-t}{t} \right] d^{-q} < \infty$$

  then INT *is polynomially tractable, and*

  $$n(\varepsilon,d) \le C\, d^q\, \varepsilon^{-p} \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

  If $q = 0$ then INT *is strongly polynomially tractable.*

- *If*

  $$\lim_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{C(K_d,g_d)}{\|h_d\|_{F_d}^2} - 1 \right)}{d + t^{-1/2}\|h_d\|_{F_d}^{-1}} = 0$$

  then INT *is weakly tractable.*

- *If*

  $$A := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{C(K_d,g_d)}{\|h_d\|_{F_d}^2} - 1 \right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, d)} < \infty$$

  then INT *is T-tractable with the exponent of T-tractability at most A.*

- *If*

  $$B := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{C(K_d,g_d)}{\|h_d\|_{F_d}^2} - 1 \right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, 1)} < \infty$$

  then INT *is strongly T-tractable with the exponent of strong T-tractability at most B.*

The limits are for $d \in \mathbb{N}$ and $t \in (0,1)$. The bounds on the information complexity can be obtained by properly normalized QMC algorithms.

We illustrate Corollary 10.8 by continuing the previous examples.

10.7.7 Example: Tensor Product Problems (Continued)

Since K_d, h_d and g_d are now given as products of univariate functions, we can simplify the condition on C in Corollary 10.8. We have

$$C = \sup_{t\in(0,1),\, d\in\mathbb{N}} t^{p/2}\|h_1\|_{F_1}^{pd}\left\lceil\left[\left(\frac{C(K_1,g_1)}{\|h_1\|_{F_1}^2}\right)^d - 1\right]\frac{1-t}{t}\right\rceil d^{-q}.$$

We claim that for $\|h_1\|_{F_1} < 1$ we have strong polynomial tractability with the exponent at most

$$p = \begin{cases} 2 & \text{if } C(K_1,g_1) \le 1, \\ 2 + \frac{\ln C(K_1,g_1)}{\ln 1/\|h_1\|_{F_1}} & \text{if } C(K_1,g_1) > 1. \end{cases} \tag{10.14}$$

Indeed, we can bound the expression in the supremum of C by dropping -1 and taking $q = 0$, and then obtain

$$t^{p/2-1}(1-t)\left[\|h_1\|_{F_1}^{p-2}C(K_1,g_1)\right]^d + t^{p/2}\|h_1\|_{F_1}^{pd} \le 2.$$

We summarize this in the following corollary.

Corollary 10.9. *Multivariate integration defined for a separable tensor product space $H(K_d)$ with $\|h_1\|_{F_1} < 1$ and $C(K_1,g_1) < \infty$, is strongly polynomially tractable for the absolute error criterion in the worst case setting with the exponent at most p given by* (10.14).

We stress that Corollary 10.6 states that multivariate integration is strongly polynomially tractable if $C(K_1,g_1) \le 1$, whereas Corollary 10.8 states the same fact under a relaxed condition if $\|h_1\|_{F_1} < 1$. The exponent of strong tractability in both cases is at most 2. If $\|h_1\|_{F_1} < 1 < C(K_1,g_1)$ then Corollary 10.6 does *not* apply whereas Corollary 10.8 does at the expense of an exponent of strong tractability which is larger than 2. Note that the bound on the exponent of strong tractability can be large when $\|h_1\|_{F_1}$ is close to 1 or when $C(K_1,g_1)$ is large.

We return to the problem of the exponent of strong polynomial tractability in Chapter 15, where we discuss Smolyak's algorithm. As we shall see, sometimes we obtain better bounds than here. However, the problem of finding the exact value of the exponent is still open, and we present it as our next open problem.

Open Problem 46.

- Find the exact value of the exponent of strong tractability for multivariate integration defined on a separable tensor product space $H(K_d)$ in terms of h_1, g_1 and K_1 with $\|h_1\|_{F_1} < 1$ and $C(K_1,g_1) < \infty$ for the absolute error criterion in the worst case setting.

10.7.8 Example: Modified Sobolev Space (Continued)

As before, we take the kernel

$$K_{d,a_d,b_d}(x,t) = a_d + b_d \prod_{j=1}^{d} \min(x_j,t_j) \quad \text{for all } x,t \in [0,1]^d,$$

and consider only the absolute error criterion. For simplicity we restrict ourselves to polynomial and weak tractability.

We already know that we can achieve polynomial tractability by QMC algorithms if $b_d = \mathcal{O}(d^q 2^d)$; if $q = 0$ then strong polynomial tractability holds. Furthermore, weak tractability holds if $b_d = 2^d e^{o(d)}$. We want to check how much we can relax these conditions for properly normalized QMC algorithms.

To simplify the calculation, assume that $a_d = 0$ and $b_d = b^d$ for some positive b. Then the last conditions for QMC algorithms are equivalent and they hold iff $b \le 2$.

It is easy to see that the conditions on strong polynomial tractability, polynomial tractability and weak tractability for properly normalized QMC algorithms are equivalent and they hold iff $b < 3$. For $b < 3$ we have strong polynomial tractability with exponent bounded by

$$p = 2\,\max\left(1, \frac{\ln 3/2}{\ln 3/b}\right),$$

and then

$$n(\varepsilon,d) = \mathcal{O}(\varepsilon^{-p}) \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}.$$

Note that for $b \le 2$ we have $p = 2$, and for b tending to 3 the value of p tends to infinity.

Hence, for $b \in (2,3)$ we have an improvement at the expense of an increased p, and to achieve strong polynomial tractability we need to use properly normalized QMC algorithms. Note that $b < 3$ implies that $\|h_d\|_{F_d}^2 = (b/3)^d$, so that the initial error is exponentially small.

We now remove the boundary conditions by taking the kernel

$$K_d(x,t) = \prod_{j=1}^{d} \big(b + \min(x_j,t_j)\big) \quad \text{for all } x,t \in [0,1]^d,$$

for some positive b. The norm of f is now given by

$$\|f\|_{F_d}^2 = \frac{f^2(0)}{b^d} + \sum_{\mathfrak{u}\neq\emptyset,\mathfrak{u}\subseteq[d]} \frac{1}{b^{d-|\mathfrak{u}|}} \int_{[0,1]^{|\mathfrak{u}|}} \left(\frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(t_u,0)\right)^2 \mathrm{d}t.$$

In this case,

$$C(K_d,1) = \big(b+\tfrac12\big)^d \quad \text{and} \quad \|h_d\|_{F_d}^2 = \big(b+\tfrac13\big)^d.$$

Then the conditions on strong polynomial tractability, polynomial tractability and weak tractability for QMC are equivalent and hold if $b \le \frac{1}{2}$, whereas the conditions on strong polynomial tractability, polynomial tractability and weak tractability for properly normalized QMC algorithms are also equivalent but now hold if $b < \frac{2}{3}$. For $b < \frac{2}{3}$, the exponent of strong polynomial tractability is bounded by

$$p = 2 \max\left(1, \frac{\ln\left(b+\frac{1}{2}\right)/\left(b+\frac{1}{3}\right)}{\ln 1/\left(b+\frac{1}{3}\right)}\right),$$

and then

$$n(\varepsilon, d) = \mathcal{O}(\varepsilon^{-p}) \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}. \qquad \square$$

10.7.9 Algorithms with Arbitrary Coefficients

In Theorems 10.4 and 10.7 we assume that $C(K_d, g_d)$ is finite. As we shall see, for some multivariate problems it may happen that $C(K_d, g_d) = \infty$, even though the problem is well defined and (10.8) holds. In this case, Theorems 10.4 and 10.7 are not applicable.

We now report a recent result of Plaskota, Wasilkowski and Zhao [248] who presented a better bound on the worst case errors without assuming that $C(K_d, g_d)$ is finite, provided that we agree to use algorithms with arbitrary coefficients. They presented the bound for $g_d \equiv 1$ and for not necessarily properly normalized algorithms, although their proof technique can be applied for general g_d and properly normalized algorithms. That is, we now consider algorithms of the form

$$A_{n,d}(f) = \sum_{j=1}^{n} a_j f(t_j)$$

for some real a_j and t_j from D_d. As we shall see, the coefficients a_j will be non-negative for non-negative g_d.

Theorem 10.10. *Consider the multivariate integration I_d defined on a separable reproducing Hilbert space F_d. Assume that*

$$C^{\text{new}}(K_d, g_d) := \left[\int_{D_d} \sqrt{K_d(t,t)}\, |g_d(t)|\, \varrho_d(t)\, \mathrm{d}t\right]^2 < \infty. \qquad (10.15)$$

Let

$$a(t) = \left[\frac{C^{\text{new}}(K_d, g_d)}{K_d(t,t)}\right]^{1/2} \operatorname{sign}(g_d(t)) \quad \textit{for all } t \in D_d.$$

Then there exists $\vec{t} = [t_1, t_2, \dots, t_n] \in D_d^n$ for which the worst case error of the algorithm

$$A_{n,d,\vec{t}}(f) = \frac{\|h_d\|_{F_d}^2}{C^{\text{new}}(K_d, g_d) - \|h_d\|_{F_d}^2 + n\, \|h_d\|_{F_d}^2} \sum_{j=1}^{n} a(t_j) f(t_j),$$

with the convention $\infty \cdot 0 = 0$*, satisfies*

$$e^{\text{wor}}(A_{n,d,\vec{t}}) \leq \min\left(\|h_d\|_{F_d}, \frac{\sqrt{C^{\text{new}}(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right).$$

Furthermore, for any number $C > 1$*, the weighted Lebesgue measure* $\lambda_{\omega_d}(Z)$ *of the set*

$$Z = \left\{ \vec{t} \in D_d^n \ : \ e^{\text{wor}}(A_{n,d,\vec{t}}) \leq C \ \min\left(\|h_d\|_{F_d}, \frac{\sqrt{C^{\text{new}}(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right) \right\}$$

with

$$\omega_d(t) = \left[\frac{K_d(t,t)}{C^{\text{new}}(K_d, g_d)} \right]^{1/2} |g_d(t)| \, \varrho_d(t) \quad \textit{for all } t \in D_d,$$

satisfies

$$\lambda_{\omega_d}(Z) := \int_Z \omega_d(t_1) \cdots \omega_d(t_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \ \geq \ 1 - C^{-2}.$$

Proof. First of all note that (10.15) implies (10.8) and multivariate integration is well defined. This easily follows from the fact that

$$\begin{aligned}
&\int_{D_d^2} K_d(x,t) \, g_d(x) \, g_d(t) \, \varrho_d(x) \, \varrho_d(t) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq \int_{D_d^2} \sqrt{K_d(x,x)} \, |g_d(x)| \sqrt{K_d(t,t)} \, |g_d(t)| \; \varrho_d(x) \, \varrho_d(t) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \left[\int_{D_d} \sqrt{K_d(t,t)} \, |g_d(t)| \, \varrho_d(t) \, \mathrm{d}t \right]^2 = C^{\text{new}}(K_d, g_d) \\
&\quad \leq \int_{D_d} K_d(t,t) \, g_d^2(t) \, \varrho_d(t) \, \mathrm{d}t = C(K_d, g_d).
\end{aligned}$$

This also shows that $C^{\text{new}}(K_d, g_d) \leq C(K_d, g_d)$, and hence the error bounds of Theorem 10.10 are better than the error bounds of Theorem 10.7.

Consider algorithms of the form

$$B_{n,d,\vec{t}}(f) = a \sum_{j=1}^{n} a(t_j) \, f(t_j)$$

for some $a \in \mathbb{R}$ and $\vec{t} = [t_1, t_2, \ldots, t_d] \in D_d^n$. Observe that for $K_d(t,t) = 0$ we formally have $a(t) = \infty$. However $f(t) = 0$ for all $f \in F_d$ in this case, since $|f(t)| \leq \|f\|_{F_d} \sqrt{K_d(t,t)}$. Hence if $a(t_j) = \infty$, then we have in the sum above $a(t_j) f(t_j) = \infty \cdot 0$, which we interpret as zero. Therefore $B_{n,d,\vec{t}}$, as well as the algorithm $A_{n,d,\vec{t}}$ in Theorem 10.10, is well defined for any t_j.

The worst case error of $B_{n,d,\vec{t}}$ is now given by

$$\left[e^{\mathrm{wor}}(B_{n,d,\vec{t}})\right]^2 = \|h_d\|_{F_d}^2 - 2\,a \sum_{j=1}^{n} a(t_j)\,h_d(t_j) + a^2 \sum_{i,j=1}^{n} a(t_i)\,a(t_j)\,K_d(t_i,t_j).$$

We now treat the sample points t_j as independent identically distributed points over D_d, as in the proof of Theorem 10.4, but changing the density from ϱ_d to ω_d. Let

$$\mathbb{E}^{\mathrm{new}} := \int_{D_d^n} \left[e^{\mathrm{wor}}(B_{n,d,\vec{t}})\right]^2 \omega_d(t_1)\cdots\omega_d(t_d)\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

Since $a(t)\,\omega_d(t) = g_d(t)\,\varrho_d(t)$ we have

$$\int_{D_d} a(t_j)\,h_d(t_j)\,\omega_d(t_j)\,\mathrm{d}t_j = \int_{D_d} h_d(t)\,g_d(t)\,\varrho_d(t)\,\mathrm{d}t = \|h_d\|_{F_d}^2.$$

Similarly, for $i \neq j$ we have

$$\int_{D_d^2} a(t_i)\,a(t_j)\,K_d(t_i,t_j)\,\omega_d(t_i)\,\omega_d(t_j)\,\mathrm{d}t_i\,\mathrm{d}t_j$$
$$= \int_{D_d^2} K_d(x,t)\,g_d(x)\,g_d(t)\,\varrho_d(x)\,\varrho_d(t)\,\mathrm{d}x\,\mathrm{d}t = \|h_d\|_{F_d}^2,$$

whereas for $i = j$, we use

$$a^2(t)\,K_d(t,t)\omega_d(t) = \sqrt{C^{\mathrm{new}}(K_d,g_d)\,K_d(t,t)}\,|g_d(t)|\,\varrho_d(t),$$

and obtain

$$\int_{D_d} a^2(t_j)\;K_d(t_j,t_j)\,\omega_d(t_j)\,\mathrm{d}t_j$$
$$= \sqrt{C^{\mathrm{new}}(K_d,g_d)}\int_{D_d}\sqrt{K_d(t,t)}\,|g_d(t)|\,\varrho_d(t)\,\mathrm{d}t_j = C^{\mathrm{new}}(K_d,g_d).$$

Hence,

$$\mathbb{E}^{\mathrm{new}} = \mathbb{E}^{\mathrm{new}}(a) = \|h_d\|_{F_d}^2\,(1-a\,n)^2 + a^2\,n\left(C^{\mathrm{new}}(K_d,g_d) - \|h_d\|_{F_d}^2\right).$$

The rest of the argument is the same as before. That is, we minimize with respect to a and then use the mean value theorem and Chebyshev's inequality. □

In particular, Theorem 10.10 states that if $C^{\mathrm{new}}(K_d,g_d) = \|h_d\|_{F_d}^2$ then the worst case error of $A_{n,d,\vec{t}}$ is zero even for $n = 1$. As before, this can only happen in rare cases. For $g_d \equiv 1$, this happens only when F_d is the space of constant functions.

If we do not care about the dependence on d, then the worst case error of $A_{n,d,\vec{t}}$ for a well chosen $\vec{t}$ satisfies

$$e^{\mathrm{wor}}(A_{n,d,\vec{t}}) = \mathcal{O}(n^{-1/2}).$$

So multivariate integration enjoys an order of convergence, although not great, independent of d and equal to at least $\frac{1}{2}$. This is the same order of convergence as for Monte Carlo. However, Monte Carlo has this rate in the randomized setting, whereas here we consider the worst case setting.

As Theorem 10.7, Theorem 10.10 can be also used to obtain sufficient conditions on tractability for multivariate integration. The only difference between them is that $C(K_d, g_d)$ is used in Theorem 10.7 and $C^{\text{new}}(K_d, g_d)$ is used in Theorem 10.10. From this we easily conclude the obvious modifications of Corollary 10.5 and Corollary 10.8.

Corollary 10.11. *Consider the multivariate integration problem* INT $= \{I_d\}$ *in the worst case setting. Here,* I_d *is defined on a separable reproducing kernel Hilbert space* F_d.

- *Consider the absolute error criterion.*
  - *If there exist a number* $q \ge 0$ *and a number* $p \ge 2$ *such that*

$$C := \sup_{t\in(0,1),\, d\in\mathbb{N}} t^{p/2}\, \|h_d\|_{F_d}^{p} \left\lceil \left(\frac{C^{\text{new}}(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1 \right) \frac{1-t}{t} \right\rceil d^{-q} < \infty$$

  then INT *is polynomially tractable, and*

$$n(\varepsilon, d) \le C\, d^{q}\, \varepsilon^{-p} \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

  If $q = 0$ *then* INT *is strongly polynomially tractable.*
  - *If*

$$\lim_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln\frac{1-t}{t} + \ln\left(\frac{C^{\text{new}}(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1\right)}{d + t^{-1/2}\|h_d\|_{F_d}^{-1}} = 0$$

  then INT *is weakly tractable.*
  - *If*

$$A := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln\frac{1-t}{t} + \ln\left(\frac{C^{\text{new}}(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1\right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, d)} < \infty$$

  then INT *is* T*-tractable with exponent of* T*-tractability at most* A.
  - *If*

$$B := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln\frac{1-t}{t} + \ln\left(\frac{C^{\text{new}}(K_d, g_d)}{\|h_d\|_{F_d}^2} - 1\right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, 1)} < \infty$$

  then INT *is strongly* T*-tractable with exponent of strong* T*-tractability at most* B.

The limits are for $d \in \mathbb{N}$ *and* $t \in (0,1)$.

- *Consider the normalized error criterion. Let*

$$A_d^{\text{nor-new}} = \frac{\left[\int_{D_d} \sqrt{K_d(t,t)}\, |g_d(t)|\, \varrho_d(t)\, \mathrm{d}t\right]^2}{\int_{D_d^2} K_d(x,t)\, |g_d(x)|\, |g_d(t)|\, \varrho_d(x)\, \varrho_d(t)\, \mathrm{d}x\, \mathrm{d}t} - 1.$$

  - *If there exists a number $q \ge 0$ such that*

    $$C := \sup_{d \in \mathbb{N}} A_d^{\text{nor-new}}\, d^{\,-q} < \infty$$

    then INT *is polynomially tractable, and*

    $$n(\varepsilon, d) \le \left\lceil C\, d^q\, \varepsilon^{-2} \right\rceil \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

    If $q = 0$ then INT *is strongly polynomially tractable.*
  - *If*

    $$\lim_{d \to \infty} \frac{\ln A_d^{\text{nor-new}}}{d} = 0$$

    then INT *is weakly tractable.*
  - *If*

    $$A^{\text{new}} := \limsup_{d \to \infty} \frac{\ln A_d^{\text{nor-new}}}{\ln(1 + T(1,d))} < \infty,$$
    $$B^{\text{new}} := \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln(1 + T(\varepsilon^{-1}, 1))} < \infty$$

    then INT *is T-tractable with exponent of T-tractability at most $A^{\text{new}} + 2B^{\text{new}}$.*
  - *If $\sup_d A_d^{\text{nor-new}} < \infty$ and $B^{\text{new}} < \infty$ then* INT *is strongly T-tractable with exponent of strong T-tractability at most $2B^{\text{new}}$.*

The bounds on the information complexity can be obtained by algorithms with arbitrary coefficients presented in Theorem 10.10.

We first illustrate Corollary 10.11 by continuing the example for tensor product problems.

10.7.10 Example: Tensor Product Problems (Continued)

Proceeding exactly as before we check that $\|h_1\|_{F_1} < 1$ implies strong polynomial tractability for the absolute error criterion with the exponent bounded by

$$p = \begin{cases} 2 & \text{if } C^{\text{new}}(K_1, g_1) \le 1, \\ 2 + \frac{\ln C^{\text{new}}(K_1, g_1)}{\ln 1/\|h_1\|_{F_1}} & \text{if } C^{\text{new}}(K_1, g_1) > 1. \end{cases} \tag{10.16}$$

That is, if $\|h_1\|_{F_1} < 1$ then we achieve strong polynomial tractability for both properly normalized QMC algorithms and algorithms with arbitrary coefficients. However, the bound on the exponent of strong polynomial tractability can be smaller for algorithms with arbitrary coefficients. Indeed, take

$$K_1(x,t) = \tfrac{5}{9} + \min(x,t) \quad \text{for all } x,t \in [0,1],$$

as in the Sobolev space example with $b = \frac{5}{9}$. Then $\|h_1\|_{F_1} = (8/9)^{1/2} < 1$ and we have strong polynomial tractability. Since $C(K_1,1) = 19/18 = 1.055\ldots$, the exponent for properly normalized QMC algorithms is bounded by

$$p = 2 + 2\frac{\ln 19/18}{\ln 9/8} = 2.9180\ldots.$$

For algorithms with arbitrary coefficients,

$$C^{\text{new}}(K_1, g_1) = \frac{4}{9}\left[\left(\frac{14}{9}\right)^{3/2} - \left(\frac{5}{9}\right)^{3/2}\right]^2 = 1.0350\ldots.$$

and

$$p = 2 + 2\frac{\ln 1.0350\ldots}{\ln 9/8} = 2.5843\ldots.$$

But we still do not know the exact value of the exponent. What would be most interesting is to consider tensor products for which $C(K_1, g_1) = \infty$ and $C^{\text{new}}(K_1, g_1) < \infty$ and find the exact value of the exponent. This is our next open problem.

Open Problem 47.

- Find the exact value of the exponent of strong tractability for multivariate integration defined on a separable tensor product space $H(K_d)$ in terms of h_1, g_1 and K_1 with

  $$\|h_1\|_{F_1} < 1 \quad \text{and} \quad C^{\text{new}}(K_1, g_1) < C(K_1, g_1) = \infty$$

  for the absolute error criterion in the worst case setting.

We now illustrate Corollary 10.11 by an example for which

$$C^{\text{new}}(K_d, g_d) < C(K_d, g_d) = \infty,$$

so that Theorems 10.4 and 10.7 as well as Corollary 10.5 are not applicable. More such examples can be found in Plaskota, Wasilkowski and Zhao [248].

10.7.11 Example: Another Modified Sobolev Space (Continued)

We again consider the Sobolev space with the reproducing kernel

$$K_d(x,t) = \prod_{j=1}^{d} \min(x_j, t_j) \quad \text{for all } x,t \in [0,1]^d,$$

with $g_d = \varrho_d \equiv 1$. We still cannot achieve tractability for the normalized error criterion. However, since

$$C^{\text{new}}(K_d, 1) = \left(\int_0^1 \sqrt{t}\,\mathrm{d}t\right)^{2d} = \left(\frac{4}{9}\right)^d,$$

we can improve the bound (10.11) on $n(\varepsilon, d)$. More precisely, for the normalized error criterion we now have

$$n(\varepsilon, d) \le \left\lceil \left(\left(\frac{4}{3}\right)^d - 1\right) \frac{1}{\varepsilon^2} \right\rceil \quad \text{for all } \varepsilon \in (0,1). \tag{10.17}$$

We now extend the domain of functions and of the reproducing kernel K_d to $D_d = [0,\infty)^d$. We still want to have $g_d \equiv 1$. Note that in this case, we cannot let $\varrho_d \equiv 1$ since (10.8) would be violated. Instead we take

$$\varrho_d(t) = \prod_{j=1}^d \frac{2\sqrt{b}}{\pi} \frac{1}{1 + b\,t_j^2} \quad \text{for all } t \in D_d,$$

for some positive number b. It is easy to check that ϱ_d is indeed a density function, i.e., $\int_{D_d} \varrho_d(t)\,\mathrm{d}t = 1$.

The representer h_d of multivariate integration is now

$$h_d(x) = \prod_{j=1}^d \int_0^\infty \frac{2\sqrt{b}}{\pi} \frac{\min(x_j, t)}{1 + b\,t^2}\,\mathrm{d}t.$$

We have

$$\begin{aligned}\|h_d\|_{F_d}^2 &= \left[\frac{4b}{\pi^2} \int_{[0,\infty)^2} \frac{\min(x,t)}{(1+b\,x^2)(1+b\,t^2)}\,\mathrm{d}x\,\mathrm{d}t\right]^d \\ &= \left[\frac{4}{\pi^2\sqrt{b}} \int_{[0,\infty)^2} \frac{\min(x,t)}{(1+x^2)(1+t^2)}\,\mathrm{d}x\,\mathrm{d}t\right]^d.\end{aligned}$$

Since the last double integral is $2.17759\ldots$, computed by Mathematica, we finally obtain

$$\|h_d\|_{F_d}^2 = \left[\frac{0.88254\ldots}{\sqrt{b}}\right]^d.$$

Note that $C(K_d, 1) = \infty$. Indeed, we now have

$$K_d(t,t)\,\varrho_d(t) = \prod_{j=1}^d \frac{2\sqrt{b}}{\pi} \frac{t_j}{1 + b\,t_j^2}.$$

For large t_j, the last ratio is approximately proportional to $1/t_j$, which is *not* integrable. However $C^{\text{new}}(K_d, 1)$ *is* finite. In fact, we have

$$[C^{\text{new}}(K_d, 1)]^{1/2} = \prod_{j=1}^{d} \frac{2\sqrt{b}}{\pi} \int_0^\infty \frac{\sqrt{t}}{1 + b\,t^2}\,\mathrm{d}t.$$

Since $\int_0^\infty \sqrt{t}/(1 + b\,t^2)\,\mathrm{d}t = \pi/(b^{3/4}\sqrt{2})$, we obtain

$$C^{\text{new}}(K_d, 1) = \left[\frac{2}{\sqrt{b}}\right]^d.$$

Observe that

$$C^{\text{new}}(K_d, 1) \le 1 \quad \text{iff} \quad b \ge 4.$$

We know that multivariate integration is strongly polynomially tractable for the absolute error criterion if $\|h_1\|_{F_1} < 1$, which holds if

$$b > (0.88254\ldots)^2 = 0.7788\ldots.$$

For the normalized error criterion and for the absolute error criterion with $b \le 0.7788\ldots$, the condition of Corollary 10.11 for weak tractability does *not* hold. We return to these cases later when we establish lower bounds on the worst case errors. □

In the last example, we see that $C^{\text{new}}(K_d, g_d)/\|h_d\|_{F_d}^2$ is exponentially large in d and does not depend on b. In fact, for general spaces even a worse situation can happen: we may have $C^{\text{new}}(K_d, g_d) = \infty$ and $\|h_d\|_{F_d} = 1$. Such examples are presented in Plaskota, Wasilkowski and Zhao [248]. We now provide a similar example for the unbounded kernel that we studied earlier.

10.7.12 Example: Unbounded Kernel (Continued)

As before, we restrict our attention only to $d = 1$ and take $D_1 = [0, 1]$ and $g_1 \equiv \varrho_1 \equiv 1$. Recall that

$$K_1(x, t) = k(k+1) \quad \text{for all } x, t \in (1/(k+1), 1/k],$$

and $K_1(x, t) = 0$ otherwise. This implies that

$$\int_0^1 \sqrt{K_1(t, t)}\,\mathrm{d}t = \sum_{k=1}^{\infty} \sqrt{k(k+1)} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}} = \infty.$$

Hence, $C^{\text{new}}(K_d, 1) = \infty$. On the other hand,

$$I_1(f) = \int_0^1 f(t)\,\mathrm{d}t \quad \text{for all } f \in F_1$$

is well defined since its representer $h_1 \equiv 1$ and

$$\|h_d\|_{F_1}^2 = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} 1 \,\mathrm{d}t = \int_0^1 1 \,\mathrm{d}t = 1.$$

In this case, we know that the best sample points are $t_j = 1/j$ for $j = 1, 2, \dots, n$, the best algorithm that uses n function values is

$$A_{n,1}(f) = \sum_{j=1}^{n} \frac{1}{j(j+1)} f(1/j),$$

and the minimal worst case error $e(n,1)$ is given by

$$e(n,1) = \left[\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \right]^{1/2} = \left[\int_0^{1/(n+1)} 1 \,\mathrm{d}t \right]^{1/2} = \frac{1}{(n+1)^{1/2}}.$$

Hence, even though $C^{\text{new}}(K_d, 1) = \infty$, we still have convergence of order $n^{-1/2}$.

We add in passing that QMC algorithms for this space are quite bad. We now show that the minimal worst case error of QMC algorithms is 1. Indeed, the square of the worst case error of

$$A_{n,1}(f) = \frac{1}{n} \sum_{j=1}^{n} f(t_j)$$

is

$$e^2(A_{n,1}) = -1 + \frac{1}{n^2} \sum_{i,j=1}^{n} K_1(t_i, t_j).$$

Let $p_k = |\{j \in [1,n] \mid t_j \in (1/(k+1), 1/k]\}$ be the number of sample points in the subinterval $(1/(k+1), 1/k]$. Obviously, $\sum_{k=1}^{\infty} p_k = n$. Then

$$e^2(A_{n,1}) = -1 + \frac{1}{n^2} \sum_{k=1}^{\infty} p_k^2\, k(k+1) \ge -1 + 2 = 1,$$

as claimed. The best sample points for QMC algorithms are arbitrary points in the subinterval $(1/2, 1]$. Observe that for $t_j = 1/j$ we have

$$e^2(A_{n,1}) = -1 + \frac{1}{n} \sum_{j=1}^{n} j(j+1) = -1 + \frac{(n+1)(n+2)}{3} = \tfrac{1}{3} n^2 (1 + o(1))$$

and this increases quadratically with n. This shows that the change of the optimal coefficients $1/(j(j+1))$ to the QMC coefficients $1/j$ is is quite bad for this space. We finally observe that the properly normalized QMC algorithm is now zero since $C(K_d, 1) = \infty$ and its worst case error is just 1. □

The last example shows that for some d and some spaces F_d, it is possible to get convergence of order $n^{-1/2}$ even though $C^{\text{new}}(K_d, g_d) = \infty$. On the other hand, Plaskota, Wasilkowski and Zhao [248] showed examples of spaces or, equivalently reproducing kernels, for which $C^{\text{new}}(K_d, g_d) = \infty$ and the order of convergence for multivariate integration is arbitrarily bad. Hence, we can say that finite $C^{\text{new}}(K_d, g_d)$ implies that the order of convergence for multivariate integration is at least $n^{-1/2}$, whereas for infinite $C^{\text{new}}(K_d, g_d)$, the order may be the same or much worse depending on the space F_d.

Theorems 10.4, 10.7 and 10.10 are non-constructive, since we only know the existence of sample points with the specific error bounds. However, since we also know that the Lebesgue measure of such points is at least $1-C^{-1}$, which is sufficiently large for large C, this offers a *semi-construction* of good sample points.

More precisely, we can proceed as follows. Choose $C > 1$. Then pick up randomly and independently sample points $t_1, t_2, \dots, t_n$ from D_d with the density ϱ_d or ω_d. Compute the worst case error of the algorithm $A_{n,d,\vec{t}}$ from Theorem 10.4, 10.7 or 10.10. If $e^{\text{wor}}(A_{n,d,\vec{t}})$ does not exceed the error bound of the corresponding theorem multiplied by C, we are done. That is, for the case of Theorem 10.10 we check whether

$$e^{\text{wor}}(A_{n,d,\vec{t}}) \le C \min\left\{ \|h_d\|_{F_d}, \frac{\sqrt{C^{\text{new}}(K_d, g_d) - \|h_d\|_{F_d}^2}}{\sqrt{n}} \right\}.$$

If not, we repeat random selection of the sample points t_j. Since the probability of failure of k such random selections is at most C^{-2k}, we need only a few runs to be successful. If we are ready to tolerate a failure of measure $\delta > 0$, then we set

$$k = \left\lceil \frac{1}{2} \frac{\ln \delta^{-1}}{\ln C} \right\rceil.$$

Hence with probability of failure at most δ, we can compute sample points for which the worst case error exceeds C times the error bound presented in Theorem 10.4, 10.7 or 10.10. The cost of such construction will be of order of $k\, n^2\, d$ arithmetic operations needed to compute k times the worst case error and the cost of randomly selecting of $k\, n\, d$ numbers.

10.8 The Operator W_d

To continue the study of general linear functionals related to multivariate approximation, we need to define an operator W_d that will play a major role in this and the next volume. To do this, let us assume that F_d is separable and that its reproducing kernel satisfies

$$C(K_d) = \int_{D_d} K_d(t,t)\, \varrho_d(t)\, \mathrm{d}t \; < \; \infty.$$

Here, we simplify the notation by taking $C(K_d) = C(K_d, 1)$.

The last assumption implies that the space F_d is continuously embedded into the space $L_{2,\varrho_d} = L_{2,\varrho_d}(D_d)$. Indeed, for $f \in F_d$ we have $f(t) = \langle f, K_d(\cdot,t)\rangle_{F_d}$, which yields that

$$f^2(t) \le \|f\|_{F_d}^2 \|K_d(\cdot,t)\|_{F_d}^2 = \|f\|_{F_d}^2 K_d(t,t).$$

Then

$$\int_{D_d} f^2(t)\,\varrho_d(t)\,\mathrm{d}t \le \int_{D_d} \|f\|_{F_d}^2\, K_d(t,t)\,\varrho_d(t)\,\mathrm{d}t = \|f\|_{F_d}^2\, C(K_d).$$

Hence,

$$\|f\|_{L_{2,\varrho_d}} \le \sqrt{C(K_d)}\,\|f\|_{F_d} \quad \text{for all } f \in F_d,$$

as claimed.

We are ready to define the operator W_d. For $f \in F_d$, let

$$(W_d f)(x) = \int_{D_d} K_d(x,t)\, f(t)\,\varrho_d(t)\,\mathrm{d}t \quad \text{for all } x \in D_d. \tag{10.18}$$

We now establish a number of properties of W_d assuming[1] without loss of generality that $\dim(F_d) = \infty$.

- W_d is well defined.

  Indeed, for $x \in D_d$ we have

$$\begin{aligned}
|(W_d f)(x)| &\le \sqrt{K_d(x,x)} \int_{D_d} \sqrt{K_d(t,t)}\,|f(t)|\,\varrho_d(t)\,\mathrm{d}t \\
&\le \sqrt{K_d(x,x)} \left(\int_{D_d} K_d(t,t)\,\varrho_d(t)\,\mathrm{d}t \int_{D_d} f^2(t)\,\varrho_d(t)\,\mathrm{d}t\right)^{1/2} \\
&= \sqrt{K_d(x,x)}\,\sqrt{C(K_d)}\,\|f\|_{L_{2,\varrho_d}} \\
&\le C(K_d)\,\sqrt{K_d(x,x)}\,\|f\|_{F_d} < \infty.
\end{aligned}$$

- W_d maps F_d into F_d, and is a continuous linear operator.

  Since F_d is separable, there exists an orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$ of F_d, i.e., $\eta_j \in F_d$ and $\langle \eta_j, \eta_k\rangle_{F_d} = \delta_{j,k}$. Then

$$K_d(x,t) = \sum_{j=1}^{\infty} \eta_j(x)\,\eta_j(t) \quad \text{for all } x,t \in D_d.$$

  Observe that

$$\int_{D_d} K_d(t,t)\,\varrho_d(t)\,\mathrm{d}t = \sum_{j=1}^{\infty} \int_{D_d} \eta_j^2(t)\,\varrho_d(t)\,\mathrm{d}t = \sum_{j=1}^{\infty} \|\eta_j\|_{L_{2,\varrho_d}}^2.$$

[1] If $\dim(F_d) < \infty$ then we should vary the index j in all sums from 1 to $\dim(F_d)$ instead of from 1 to ∞.

Hence,

$$\sum_{j=1}^{\infty} \|\eta_j\|^2_{L_{2,\varrho_d}} = C(K_d). \tag{10.19}$$

We can rewrite $W_d f$ as

$$W_d f = \sum_{j=1}^{\infty} \left(\int_{D_d} \eta_j(t)\, f(t)\, \varrho_d(t)\, \mathrm{d}t \right) \eta_j .$$

Then $W_d f \in F_d$ iff

$$a := \sum_{j=1}^{\infty} \left(\int_{D_d} \eta_j(t)\, f(t)\, \varrho_d(t)\, \mathrm{d}t \right)^2 < \infty.$$

Since

$$\begin{aligned}\left(\int_{D_d} \eta_j(t)\, f(t)\, \varrho_d(t)\, \mathrm{d}t \right)^2 &\le \int_{D_d} \eta_j^2(t)\, \varrho_d(t)\, \mathrm{d}t \int_{D_d} f^2(t)\, \varrho_d(t)\, \mathrm{d}t \\ &= \|\eta_j\|^2_{L_{2,\varrho_d}} \|f\|^2_{L_{2,\varrho_d}},\end{aligned}$$

we have

$$a \le \|f\|^2_{L_{2,\varrho_d}} \sum_{j=1}^{\infty} \|\eta_j\|^2_{L_{2,\varrho_d}} = C^2(K_d)\, \|f\|^2_{F_d} < \infty.$$

Hence $W_d f \in F_d$. Clearly, W_d is linear and

$$\|W_d f\|_{F_d} \le C(K_d) \|f\|_{F_d},$$

so it is also continuous, as claimed.

- W_d is self-adjoint, positive semi-definite, and compact.

  Clearly, for $f, g \in F_d$ we have

$$\langle f, W_d g\rangle_{F_d} = \int_{D_d} f(t)\, g(t)\, \varrho_d(t)\, \mathrm{d}t = \langle W_d f, g\rangle_{F_d} .$$

Hence, W_d is self-adjoint. It is also positive semi-definite since

$$\langle f, W_d f\rangle_{F_d} = \int_{D_d} f^2(t)\, \varrho_d(t)\, \mathrm{d}t \ge 0 \quad \text{for all } f \in F_d .$$

To show that W_d is compact, for $m \in \mathbb{N}$ and $f \in F_d$ define

$$T_{m,d} f = \sum_{j=1}^{m} \left(\int_{D_d} \eta_j(t)\, f(t)\, \varrho_d(t)\, \mathrm{d}t \right) \eta_j .$$

Clearly, $T_{m,d}\colon F_d \to F_d$ and $T_{m,d}$ is a continuous linear and m-dimensional operator, i.e., $\dim(T_{m,d}(F_d)) = m$. Note that

$$W_d f - T_{m,d} f = \sum_{j=m+1}^{\infty} \left(\int_{D_d} \eta_j(t)\, f(t)\, \varrho_d(t)\, \mathrm{d}t \right) \eta_j,$$

and using the previous estimates we conclude that

$$\| W_d f - T_{m,d} f \|_{F_d} \le \left(\sum_{j=m+1}^{\infty} \|\eta_j\|^2_{L_{2,\varrho_d}} \right)^{1/2} \sqrt{C(K_d)}\, \|f\|_{F_d}.$$

Hence,

$$\| W_d - T_{m,d} \| \le \sqrt{C(K_d)} \left(\sum_{j=m+1}^{\infty} \|\eta_j\|^2_{L_{2,\varrho_d}} \right)^{1/2}.$$

Due to (10.19), the trace $\sum_{j=m+1}^{\infty} \|\eta_j\|^2_{L_{2,\varrho_d}}$ goes to zero as m goes to infinity. Hence, W_d can be approximated with arbitrarily small error by finite dimensional operators, and therefore is compact.

- W_d is a finite trace operator.

  As a self-adjoint, semi-positive and compact operator, W_d possesses eigenpairs $\{\eta_{d,j}, \lambda_{d,j}\}_{j\in\mathbb{N}}$,

$$W_d \eta_{d,j} = \lambda_{d,j}\, \eta_{d,j}$$

  such that the eigenvalues are ordered and convergent to zero,

$$\lambda_{d,1} \ge \lambda_{d,2} \ge \cdots \quad \text{and} \quad \lim_{j\to\infty} \lambda_{d,j} = 0.$$

  For simplicity we assume that all eigenvalues $\lambda_{d,j}$ are positive[2]. The eigenfunctions $\{\eta_{d,j}\}$ form an orthonormal basis of F_d, $\langle \eta_{d,j}, \eta_{d,k} \rangle_{F_d} = \delta_{j,k}$. They satisfy the equations

$$\int_{D_d} K_d(x,t)\, \eta_{d,j}(t)\, \varrho_d(t)\, \mathrm{d}t = \lambda_{d,j}\, \eta_{d,j}(x) \quad \text{for all } x \in D_d.$$

  This yields that for all $j,k \in \mathbb{N}$ we have

$$\begin{aligned} \lambda_{d,k}\delta_{j,k} = \langle \eta_{d,j}, \lambda_{d,k}\, \eta_{d,k} \rangle_{F_d} &= \int_{D_d} \eta_{d,j}(t)\, \eta_{d,k}(t)\, \varrho_d(t)\, \mathrm{d}t \\ &= \langle \eta_{d,j}, \eta_{d,k} \rangle_{L_{2,\varrho_d}}. \end{aligned}$$

  This proves that the $\eta_{d,j}$ are also orthogonal in L_{2,ϱ_d}. Furthermore,

$$\lambda_{d,j} = \|\eta_{d,j}\|^2_{L_{2,\varrho_d}} \quad \text{for all } j \in \mathbb{N}.$$

[2]Otherwise, we should replace F_d by its subspace spanned by all $\eta_{d,j}$ corresponding to positive eigenvalues.

Define

$$\eta^*_{d,j} = \frac{1}{\sqrt{\lambda_{d,j}}}\,\eta_{d,j}$$

and then the sequence $\{\eta^*_{d,j}\}$ is orthonormal in L_{2,ϱ_d}. In general, however, this set need not be a basis of L_{2,ϱ_d} since F_d can be a proper subspace of L_{2,ϱ_d}.

Finally observe that

$$\int_{D_d} K_d(t,t)\,\varrho_d(t)\,\mathrm{d}t = \sum_{j=1}^{\infty} \|\eta_{d,j}\|^2_{L_{2,\varrho_d}} = \sum_{j=1}^{\infty} \lambda_{d,j} = C(K_d) < \infty.$$

Hence, the sum of the eigenvalues of W_d is finite, thus W_d is a finite trace operator, as claimed.

For the eigenpairs $\{\eta_{d,j},\lambda_{d,j}\}$ of W_d we have

$$\langle f,\eta_{d,j}\rangle_{F_d} = \frac{1}{\lambda_{d,j}}\,\langle f,\eta_{d,j}\rangle_{L_{2,\varrho_d}} \qquad \text{for all } f \in F_d, \tag{10.20}$$

since

$$\begin{aligned}\langle f,\eta_{d,j}\rangle_{F_d} &= \frac{1}{\lambda_{d,j}}\left\langle f, \int_{D_d} K_d(\cdot,t)\eta_{d,j}(t)\,\varrho_d(t)\,\mathrm{d}t\right\rangle_{F_d}\\ &= \frac{1}{\lambda_{d,j}}\int_{D_d} f(t)\,\eta_{d,j}(t)\,\varrho_d(t)\,\mathrm{d}t = \frac{1}{\lambda_{d,j}}\,\langle f,\eta_{d,j}\rangle_{L_{2,\varrho_d}}.\end{aligned}$$

10.9 Relations to Multivariate Integration

The results on multivariate integration established in Section 10.7 may be also generalized to some other linear functionals defined over F_d for which $C(K_d) < \infty$. Consider a general continuous linear functional

$$I_d(f) = \langle f, h_d\rangle_{F_d} \qquad \text{for all } f \in F_d,$$

with h_d from F_d. We now assume that

$$\sum_{j=1}^{\infty} \frac{\langle h_d,\eta_{d,j}\rangle^2_{F_d}}{\lambda_{d,j}} < \infty. \tag{10.21}$$

Then

$$\tilde{h}_d := \sum_{j=1}^{\infty} \frac{\langle h_d,\eta_{d,j}\rangle_{F_d}}{\sqrt{\lambda_{d,j}}}\,\eta^*_{d,j} = \sum_{j=1}^{\infty} \frac{\langle h_d,\eta^*_{d,j}\rangle_{L_{2,\varrho_d}}}{\lambda_{d,j}}\,\eta^*_{d,j} \tag{10.22}$$

belongs to L_{2,ϱ_d} and its norm is

$$\|\tilde{h}_d\|^2_{L_{2,\varrho_d}} = \sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j}\rangle^2_{F_d}}{\lambda_{d,j}} = \sum_{j=1}^{\infty} \frac{\langle h_d, \eta^*_{d,j}\rangle^2_{L_{2,\varrho_d}}}{\lambda^2_{d,j}} < \infty.$$

We now claim that (10.21) implies that

$$I_d(f) = \int_{D_d} f(t)\, \tilde{h}_d(t)\, \varrho_d(t)\, \mathrm{d}t, \tag{10.23}$$

so that I_d is the same as multivariate integration with $g_d = \tilde{h}_d$. Indeed,

$$\begin{aligned} I_d(f) = \langle f, h_d\rangle_{F_d} &= \sum_{j=1}^{\infty} \langle f, \eta_{d,j}\rangle_{F_d} \, \langle h_d, \eta_{d,j}\rangle_{F_d} \\ &= \sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j}\rangle_{F_d}}{\lambda_{d,j}} \, \langle f, \eta_{d,j}\rangle_{L_{2,\varrho_d}} \\ &= \sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j}\rangle_{F_d}}{\sqrt{\lambda_{d,j}}} \, \langle f, \eta^*_{d,j}\rangle_{L_{2,\varrho_d}} \\ &= \langle f, \tilde{h}_d\rangle_{L_{2,\varrho_d}} = \int_{D_d} f(t)\, \tilde{h}_d(t)\, \varrho_d(t)\, \mathrm{d}t, \end{aligned}$$

as claimed.

In fact, the following opposite statement is also true. Namely, if there exists $g_d \in L_{2,\varrho_d}$ such that

$$I_d(f) = \langle f, h_d\rangle_{F_d} = \int_{D_d} f(t)\, g_d(t)\, \varrho_d(t)\, \mathrm{d}t \quad \text{for all } f \in F_d$$

then, in general, g_d is not unique but it can be chosen as $g_d = \tilde{h}_d$ and (10.21) holds. Indeed, since F_d is spanned by $\{\eta_{d,j}\}$, we can represent the function g_d as $g_d = g_{d,1} + g_{d,2}$ with $g_{d,2}$ orthogonal to all $\eta_{d,j}$ and $g_{d,1}$ being a linear combination of the $\eta_{d,j}$. Note that g_d is indistinguishable from $g_{d,1}$ in the formula above since f is a linear combination of all the $\eta_{d,j}$. So we can take $g_{d,2} = 0$ and then $g_d = g_{d,1}$. For $f = \eta_{d,j}$ we have

$$\langle \eta_{d,j}, h_d\rangle_{F_d} = \frac{1}{\lambda_{d,j}} \langle \eta_{d,j}, h_d\rangle_{L_{2,\varrho_d}} = \langle \eta_{d,j}, g_d\rangle_{L_{2,\varrho_d}}.$$

Hence

$$\begin{aligned} g_d &= \sum_{j=1}^{\infty} \langle g_d, \eta^*_{d,j}\rangle_{L_{2,\varrho_d}} \, \eta^*_{d,j} = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_{d,j}}} \, \langle g_d, \eta_{d,j}\rangle_{L_{2,\varrho_d}} \, \eta^*_{d,j} \\ &= \sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j}\rangle_{F_d}}{\sqrt{\lambda_{d,j}}} \, \eta^*_{d,j} = \tilde{h}_d, \end{aligned}$$

and obviously

$$\sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j} \rangle_{F_d}^2}{\lambda_{d,j}} = \|g_d\|_{L_{2,\varrho_d}}^2 < \infty,$$

as claimed.

To apply Theorem 10.10 and Corollary 10.11 we must check whether (10.15) holds for $g = \tilde{h}_d$, i.e., whether $\int_{D_d} \sqrt{K_d(t,t)}\,|\tilde{h}_d(t)|\,\varrho_d(t)\,\mathrm{d}t < \infty$. This is indeed the case since

$$\begin{aligned} \left[C^{\text{new}}(K_d, \tilde{h}_d)\right]^{1/2} &= \int_{D_d} \sqrt{K_d(t,t)}\,|\tilde{h}_d(t)|\,\varrho_d(t)\,\mathrm{d}t \\ &\le \left(\int_{D_d} K_d(t,t)\varrho_d(t)\,\mathrm{d}t\right)^{1/2} \|\tilde{h}_d\|_{L_{2,\varrho_d}} \\ &= C(K_d)^{1/2}\|\tilde{h}_d\|_{L_{2,\varrho_d}} < \infty. \end{aligned} \tag{10.24}$$

We summarize this analysis in the following corollary.

Corollary 10.12. *Consider a general continuous linear functional $I_d(f) = \langle f, h_d \rangle_{F_d}$ for all $f \in F_d$ with h_d satisfying (10.21), and with $\tilde{h}_d$ given by (10.22). Here we assume that the reproducing kernel K_d of the separable Hilbert space F_d is such that $C(K_d) < \infty$. Then I_d is the same as multivariate integration with $g_d = \tilde{h}_d$ and the results of Section 10.7 apply to I_d.*

We illustrate this relation to multivariate integration by an example.

10.9.1 Example: Linear Functionals in Korobov Space

We take the same Korobov space F_d as before with $r = 1$ and with $\varrho_d \equiv 1$ and $D_d = [0,1]^d$. Its reproducing kernel is of the form

$$K_d(x,t) = \prod_{j=1}^{d} \left(1 + \tfrac{1}{2} B_2(\{x_j - t_j\})\right) \quad \text{for all } x, t \in [0,1]^d,$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial of degree two, and $\{x\}$ denotes the fractional part of x. (This corresponds to $\alpha = r = 1$, $\beta_1 = 1$ and $\beta_2 = (2\pi)^{-2}$ in Appendix A of Volume I.)

The eigenpairs of W_d are known for this case, see e.g., Appendix A of Volume I. For $d = 1$ and $k = 1, 2, \dots,$ we have

$$\lambda_1 = 1, \quad \lambda_{2k} = \lambda_{2k+1} = \frac{1}{4\pi^2 k^2},$$

and

$$\eta_1 = 1, \quad \eta_{2k}(x) = \frac{1}{\sqrt{2}\,\pi\, k} \sin(2\pi\, kx), \quad \eta_{2k+1}(x) = \frac{1}{\sqrt{2}\,\pi\, k} \cos(2\pi\, kx).$$

Obviously, we also have

$$\eta_1^* = 1, \quad \eta_{2k}^*(x) = \sqrt{2}\,\sin(2\pi\,kx), \quad \eta_{2k+1}^*(x) = \sqrt{2}\,\cos(2\pi\,kx),$$

and the sequence $\{\eta_j^*\}$ is now an orthonormal basis of $L_2([0,1])$.

For $d \geq 1$, the tensor product structure of F_d implies that W_d has the eigenpairs $\{\lambda_{d,k}, \eta_{d,k}\}_{k=[k_1,k_2,\dots,k_d]\in\mathbb{N}^d}$ given by

$$\lambda_{d,k} = \prod_{j=1}^{d} \lambda_{k_j} \quad \text{and} \quad \eta_{d,k}(x) = \prod_{j=1}^{d} \eta_{k_i}(x_j).$$

Consider a linear tensor product functional

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{with } h_d(x) = \prod_{j=1}^{d} h_1(x_j)$$

for some $h_1 \in F_1$. Then (10.21) takes the form

$$\sum_{j=1}^{\infty} \frac{\langle h_d, \eta_{d,j}\rangle_{F_d}^2}{\lambda_{d,j}} = \Big[\sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j\rangle_{F_1}^2}{\lambda_j}\Big]^d < \infty.$$

Hence, we need to choose h_1 such that

$$\sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j\rangle_{F_1}^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j^*\rangle_{L_2([0,1])}^2}{\lambda_j^2} < \infty.$$

Then $\tilde{h}_d(x) = \prod_{j=1}^{d} \tilde{h}_1(x_j)$ with

$$\tilde{h}_1 = \sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j\rangle_{F_1}}{\sqrt{\lambda_j}}\,\eta_j^* \quad \text{and} \quad \|\tilde{h}_1\|_{L_2([0,1])}^2 = \sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j^*\rangle_{L_2([0,1])}^2}{\lambda_j^2} < \infty.$$

Since $h_1 \in F_1$ we know that

$$\sum_{j=1}^{\infty} \langle h_1, \eta_j\rangle_{F_1}^2 = \sum_{j=1}^{\infty} \frac{\langle h_1, \eta_j^*\rangle_{L_2([0,1])}^2}{\lambda_j} < \infty.$$

This means that I_d is related to multivariate integration if the Fourier coefficients of its representer h_1 tend to zero faster than is required in the space F_1. Equivalently, since $\lambda_j = \Theta(j^{-2})$, we can say that we know that

$$\sum_{j=1}^{\infty} j^2 \langle h_1, \eta_j^*\rangle_{L_2([0,1])}^2 < \infty$$

but to get a relation to multivariate integration we need to assume that

$$\sum_{j=1}^{\infty} j^4 \langle h_1, \eta_j^* \rangle_{L_2([0,1])}^2 < \infty.$$

This holds iff $h_1'' \in L_2([0,1])$. Hence, for $h_1 \in F_1$ we know that $h_1' \in L_2([0,1])$, and the assumption (10.21) means that we need to assume that $h_1'' \in L_2([0,1])$.

Note that the assumption (10.21) holds for a dense set of elements h_1 from F_1 since it holds for all h_1 with finitely many non-zero Fourier coefficients or for all polynomials h_1 which are also dense in F_1. □

In the last example we already noticed that the assumption (10.21) holds for a dense set of representers of linear functionals. In fact, this property is true in full generality. More precisely, let

$$A_d := \{ h_d \in F_d \mid h_d \text{ satisfies } (10.21) \}$$

be the set of linear functionals I_d (or, equivalently, the set of representers h_d) for which I_d is the same as multivariate integration for $g_d = \tilde{h}_d$. It is easy to see that the set A_d is dense in F_d since all h_d with finitely many non-zero $\langle h_d, \eta_{d,j} \rangle_{F_d}$ satisfy (10.21). Indeed, for any $h_d \in F_d$ take

$$h_{d,m} = \sum_{j=1}^{m} \langle h_d, \eta_{d,j} \rangle_{F_d} \eta_{d,j}$$

for some integer m. Then $h_{d,m} \in A$ and

$$\|h_d - h_{d,m}\|_{F_d}^2 = \sum_{j=m+1}^{\infty} \langle h_d, \eta_{d,j} \rangle_{F_d}^2$$

can be made arbitrarily small for large m since $\|h_d\|_{F_d}^2 = \sum_{j=1}^{\infty} \langle h_d, \eta_{d,j} \rangle_{F_d}^2 < \infty$, as claimed.

However, note that for the linear functional $I_{d,m}(f) = \langle f, h_{d,m} \rangle_{F_d}$ we have $e^{\mathrm{wor}}(m, I_{d,m}) = 0$ and for $n < m$, Theorem 10.10 and (10.24) yield

$$e^{\mathrm{wor}}(n, I_{d,m}) \le \frac{\sqrt{C(K_d)}\, \|\tilde{h}_{d,m}\|_{L_{2,\varrho_d}}}{\sqrt{n}}.$$

This proves the following corollary.

Corollary 10.13. *Consider a separable Hilbert space F_d whose reproducing kernel K_d satisfies $C(K_d) < \infty$. Then*

- *the set of continuous linear functionals that are the same as multivariate integration for some g_d is dense in F_d,*

- *for continuous linear functionals from this dense set, the worst case error of properly chosen linear algorithms is bounded by*

$$\frac{\sqrt{C(K_d)}\,\|\tilde{h}_d\|_{L_{2,\varrho_d}}}{\sqrt{n}}.$$

10.10 General Case

We now consider linear continuous functionals more general than multivariate integration and show conditions under which their nth minimal errors go to zero as fast as $n^{-1/2}$. Obviously, the general form of a linear functional is $I_d(f) = \langle f, h_d \rangle_{F_d}$ for an arbitrary h_d from F_d.

We know that h_d can be approximated with an arbitrarily small error by a linear combination of $K_d(\cdot, z_1), K_d(\cdot, z_2), \dots, K_d(\cdot, z_m)$ for some integer m and z_j from D_d. So as our first step, let us assume that

$$h_d = \sum_{j=1}^{m} \alpha_j\, K_d(\cdot, z_j). \tag{10.25}$$

In this case, the linear functional I_d takes the form

$$I_d(f) = \sum_{j=1}^{m} \alpha_j\, f(z_j) \quad \text{for all } f \in F_d.$$

Obviously,

$$\|h_d\|_{F_d}^2 = \sum_{i,j=1}^{m} \alpha_i \alpha_j\, K_d(z_i, z_j) = \sum_{j=1}^{m} \alpha_j\, h_d(z_j).$$

To eliminate the trivial problem we assume that $\|h_d\|_{F_d} > 0$. As we shall see, the quantity

$$\|h_d\|_{F_d}^* := \sum_{j=1}^{m} |\alpha_j|\, \sqrt{K_d(z_j, z_j)} \tag{10.26}$$

will be important for our analysis. Clearly, $\|h_d\|_{F_d} \le \|h_d\|_{F_d}^*$.

Let us consider linear algorithms

$$A_{n,d,\vec{t}}(f) = \sum_{j=1}^{n} a(t_j)\, f(t_j)$$

for some not necessarily non-negative real numbers $a(t_j)$ and some sample points t_j from D_d. As we know, the square of the worst case error of $A_{n,d,\vec{t}}$ is

$$\left[e^{\mathrm{wor}}(A_{n,d,\vec{t}})\right]^2 = \|h_d\|_{F_d}^2 - 2\sum_{j=1}^{n} a(t_j)\, h_d(t_j) + \sum_{i,j=1}^{m} a(t_i) a(t_j)\, K_d(t_i, t_j).$$

In analogy with the analysis of the previous sections, we now take independent and identically distributed random sample points t_j from the finite set $\{z_1, z_2, \ldots, z_m\}$, where the point z_k is taken with probability

$$\omega(z_j) = \frac{\sqrt{K_d(z_j, z_j)}}{\|h_d\|^*_{F_d}} \, |\alpha_j| \quad \text{for } j = 1, 2, \ldots, m.$$

Let $Z_m := \{z_1, z_2, \ldots, z_m\}$ and let

$$\mathbb{E} := \sum_{t_1 \in Z_m} \cdots \sum_{t_n \in Z_m} \omega(t_1) \cdots \omega(t_n) \left[e^{\text{wor}}(A_{n,d,\vec{t}})\right]^2$$

denote the expectation of the square of the worst average error of $A_{n,d,\vec{t}}$. Then

$$\begin{aligned}\mathbb{E} = \|h_d\|^2 - 2n \sum_{j=1}^{m} a(z_j) h_d(z_j)\, \omega(z_j) \\ + (n^2 - n) \sum_{i,j=1}^{m} a(z_i) a(z_j)\, K_d(z_i, z_j)\, \omega(z_i)\, \omega(z_j) \\ + n \sum_{j=1}^{n} a^2(z_j)\, K_d(z_j, z_j)\, \omega(z_j).\end{aligned}$$

We define $a(z_j)$ as

$$a(z_j) = a\, \frac{\alpha_j}{\omega(z_j)} = a\, \frac{\|h_d\|^*_{F_d}}{\sqrt{K_d(z_j, z_j)}} \, \text{sign}(\alpha_j) \quad \text{for } j = 1, 2, \ldots, m,$$

with the convention that $\infty \cdot 0 = 0$, and with a not yet specified positive number a. We stress that some $a(t_j)$ may be now negative. Then

$$\sum_{j=1}^{m} a(z_j) h_d(z_j)\, \omega(z_j) = a \sum_{j=1}^{m} \alpha_j\, h_d(z_j) = a\, \|h_d\|^2_{F_d},$$

$$\sum_{i,j=1}^{m} a(z_i) a(z_j)\, K_d(z_i, z_j)\, \omega(z_i)\, \omega(z_j) = \sum_{i,j=1}^{m} \alpha_i \alpha_j\, K_d(z_i, z_j) = a^2\, \|h_d\|^2_{F_d},$$

$$\sum_{j=1}^{n} a^2(z_j)\, K_d(z_j, z_j)\, \omega(z_j) = a^2 \left[\|h_d\|^*_{F_d}\right]^2.$$

This leads to

$$\mathbb{E} = \|h_d\|^2_{F_d} (1 - na)^2 + a^2 n^2 \left(\left[\|h_d\|^*_{F_d}\right]^2 - \|h_d\|^2_{F_d}\right).$$

Choosing a to minimize the last formula, we obtain

$$a = \frac{\|h_d\|^2_{F_d}}{\left[\|h_d\|^*_{F_d}\right]^2 - \|h_d\|^2_{F_d} + n\, \|h_d\|^2_{F_d}}.$$

As before, applying the mean value theorem we conclude that there exists $\vec{t}$ for which

$$e^{\text{wor}}(A_{n,d,\vec{t}}) \le \min\left(\|h_d\|_{F_d}, \frac{\sqrt{\left[\|h_d\|^*_{F_d}\right]^2 - \|h_d\|^2_{F_d}}}{\sqrt{n}} \right). \tag{10.27}$$

Consider the set

$$Z = \left\{ \vec{t} \in Z_m^n \;:\; e^{\text{wor}}(A_{n,d,\vec{t}}) \le C \; \min\left(\|h_d\|_{F_d}, \frac{\sqrt{\left[\|h_d\|^*_{F_d}\right]^2 - \|h_d\|^2_{F_d}}}{\sqrt{n}} \right) \right\}.$$

Applying Chebyshev's inequality we conclude that for any $C > 1$, a weighted measure $\lambda_\omega(Z)$ satisfies

$$\lambda_\omega(Z) := \sum_{\vec{t} \in Z} \omega(t_1) \cdots \omega(t_n) \;\ge\; 1 - C^{-2}. \tag{10.28}$$

We are ready to consider the next step where h_d is not necessary given by (10.25). For a general h_d from F_d, we generalize (10.26) to

$$\|h_d\|^*_{F_d} = \limsup_{\varepsilon \to 0} \; \inf\Big\{ \sum_{j=1}^m |\alpha_j| \sqrt{K_d(z_j, z_j)} \;\Big|\; \Big\| h_d - \sum_{j=1}^m \alpha_j K_d(\cdot, z_j) \Big\|_{F_d} \le \varepsilon \Big\}. \tag{10.29}$$

In the infimum above we vary m, α_j and z_j satisfying the last inequality. As we know, the set of such m, α_j and z_j is non-empty and we should choose these parameters to minimize the expression $\sum_{j=1}^m |\alpha_j| \sqrt{K_d(z_j, z_j)}$.

We stress that $\|h_d\|^*_{F_d}$ may be infinite. Indeed, take the non-separable space F_d as in the example we considered before with $K_d(x,t) = \delta(x,t)$ for $x, t \in [0,1]^d$. Then every h_d is of the form $h_d = \sum_{j=1}^\infty \alpha_j K_d(\cdot, t_j^*)$ for some t_j from $[0,1]^d$, and $\|h_d\|^2_{F_d} = \sum_{j=1}^\infty \alpha_j^2 < \infty$. In this case we have

$$\|h_d\|^*_{F_d} = \sum_{j=1}^\infty |\alpha_j|$$

and the last series does not have to be finite. This holds, for instance, for $\alpha_j = j^{-1}$.

For a general space F_d and h_d, we always have $\|h_d\|_{F_d} \le \|h_d\|^*_{F_d}$. In fact, it is easy to see that $\|\cdot\|^*_{F_d}$ is a norm on the linear subspace

$$X_d = \{ h_d \in F_d \mid \|h_d\|^*_{F_d} < \infty \}$$

of the space F_d. Note that all h_d satisfying (10.25) belong to X_d, and therefore X_d is dense in F_d. For h_d of the form (10.25), the definitions (10.26) and (10.29) coincide.

For multivariate integration satisfying (10.8), it is easy to check that

$$[\|h_d\|^*_{F_d}]^2 = C^{\text{new}}(K_d, 1);$$

as we know, this may be finite or infinite, depending on the space F_d.

Based on the analysis we did for h_d satisfying (10.25), we obtain the following theorem.

Theorem 10.14. *Consider the linear functional problem*

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d \text{ and some non-zero } h_d \in F_d.$$

Assume that

$$\|h_d\|^*_{F_d} < \infty. \tag{10.30}$$

Then

$$e(n,d) \le \min\left(\|h_d\|_{F_d}, \frac{\sqrt{\left[\|h_d\|^*_{F_d}\right]^2 - \|h_d\|^2_{F_d}}}{\sqrt{n}} \right).$$

For any $\delta \in (0,1)$, *take* $h_{d,\delta} = \sum_{j=1}^m \alpha_j K_d(\cdot, z_j)$ *with* $K_d(z_j, z_j) > 0$ *such that*

$$\|h_d - h_{d,\delta}\|_{F_d} \le \delta \, \|h_d\|_{F_d} \quad \text{and} \quad \|h_{d,\delta}\|^*_{F_d} \le (1+\delta)\, \|h_d\|^*_{F_d},$$

where $\|h_{d,\delta}\|^*_{F_d} = \sum_{j=1}^m |\alpha_j| \sqrt{K_d(z_j, z_j)}$.

Then there exists $\vec{t} = [t_1, t_2, \dots, t_n]$ *with* $t_j \in Z_m := \{z_1, z_2, \dots, z_m\}$ *for which the worst case error of the algorithm*

$$A_{n,d,\vec{t}}(f) = \frac{\|h_{d,\delta}\|^2_{F_d}}{\left[\|h_{d,\delta}\|^*_{F_d}\right]^2 - \|h_{d,\delta}\|^2_{F_d} + n\, \|h_{d,\delta}\|^2_{F_d}} \sum_{j=1}^n a_j f(t_j),$$

with

$$a_j = \frac{\|h_{d,\delta}\|^*_{F_d}}{\sqrt{K_d(z_j, z_j)}} \operatorname{sign}(\alpha_j),$$

satisfies

$$e^{\text{wor}}(A_{n,d,\vec{t}}) \le \delta \, \|h_d\|_{F_d} + E_{n,\delta},$$

where

$$E_{n,\delta} = (1+\delta) \min\left(\|h_d\|_{F_d}, \frac{\sqrt{\left[\|h_d\|^*_{F_d}\right]^2 - [(1-\delta)/(1+\delta)]^2 \|h_d\|^2_{F_d}}}{\sqrt{n}} \right).$$

Furthermore, for any $C > 1$, *the weighted measure*

$$\lambda_\omega(Z) = \sum_{\vec{t} \in Z} \omega(t_1) \cdots \omega(t_n),$$

where

$$\omega(z_k) = \frac{\sqrt{K_d(z_k, z_k)}}{\|h_{d,\delta}\|^*_{F_d}} \, |\alpha_k| \quad \text{for } k = 1, 2, \dots, m,$$

of the set

$$Z = \left\{ \vec{t} \in Z_m^n \mid e^{\text{wor}}(A_{n,d,\vec{t}}) \le \delta \, \|h_d\|_{F_d} + C\, E_{n,d} \right\}$$

satisfies

$$\lambda_\omega(Z) \ge 1 - C^{-2}.$$

Proof. First of all, note that we can always find $h_{d,\delta}$ satisfying the conditions presented in the theorem. Since $\|h_d\|^*_{F_d}$ is finite and positive, from the definition of $\|h_d\|^*_{F_d}$ we know that for any $\delta \in (0,1)$ there exists ε_δ such that

$$\|h_{d,\delta}\|^*_{F_d} \le \|h_d\|^*_{F_d} + \delta \|h_d\|^*_{F_d}$$

for all $h_{d,\delta}$ for which

$$\|h_d - h_{d,\delta}\|_{F_d} \le \min\left(\varepsilon_\delta, \delta \|h_d\|_{F_d}\right) \le \delta \|h_d\|_{F_d}.$$

Hence, $h_{d,\delta}$ exists. We then take the algorithm $A_{n,d,\vec{t}}$ for approximating $\langle f, h_{d,\delta}\rangle_{F_d}$. As we did before formulating the theorem, we know that for some $\vec{t}$, the worst case error of $A_{n,d,\vec{t}}$ satisfies (10.27) as well as (10.28) with h_d replaced by $h_{d,\delta}$. Note that $\|h_{d,\delta}\|_{F_d}/\|h_d\|_{F_d} \in [1-\delta, 1+\delta]$ and therefore

$$\sqrt{\left[\|h_{d,\delta}\|^*_{F_d}\right]^2 - \|h_{d,\delta}\|^2_{F_d}} \le (1+\delta)\sqrt{\left[\|h_d\|^*_{F_d}\right]^2 - [(1-\delta)/(1+\delta)]^2 \|h_d\|^2_{F_d}}.$$

This yields that $A_{n,d,\vec{t}}$ approximates $\langle f, h_{d,\delta}\rangle_{F_d}$ with worst case error at most $E_{n,\delta}$. Finally, we observe that

$$\left| \langle f, h_d\rangle_{F_d} - A_{n,d,\vec{t}} \right| \le \left| \langle f, h_d - h_{d,\delta}\rangle_{F_d} \right| + \left| \langle f, h_{d,\delta}\rangle_{F_d} - A_{n,d,\vec{t}} \right|$$

and the bound $\|h_d - h_{d,\delta}\|_{F_d} \le \delta \|h_d\|_{F_d}$ yields the error bound of $A_{n,d,\vec{t}}$. Letting δ go to zero, we obtain the estimate on $e(n,d)$. This completes the proof. □

Note that Theorem 10.14 is exactly of the same form as Theorem 10.10, but with $C^{\text{new}}(K_d, g_d)$ replaced by $[\|h_d\|^*_{F_d}]^2$. As we already mentioned for multivariate integration $C^{\text{new}}(K_d, 1) = [\|h_d\|^*_{F_d}]^2$, and therefore Theorem 10.14 generalizes Theorem 10.10, and is applicable to linear functionals more general than multivariate integration as long as $\|h_d\|^*_{F_d} < \infty$. Indeed, take the Hilbert space F_d with the reproducing kernel $K_d(x,t) = \delta(x,t)$. Then multivariate integration and Theorem 10.10 are not applicable since F_d is not separable, but Theorem 10.14 is applicable.

Therefore, we can generalize Corollary 10.11 and present tractability conditions in terms of $\|h_d\|^*_{F_d}$.

Corollary 10.15. *Consider the linear functional problem* $\mathrm{I} = \{I_d\}$ *in the worst case setting. Here,* $I_d(f) = \langle f, h_d\rangle_{F_d}$ *for* f *from a reproducing kernel Hilbert space* F_d *with* $\|h_d\|^*_{F_d} < \infty$ *for all* d.

- *Consider the absolute error criterion.*
  - *If there exist a number* $q \ge 0$ *and a number* $p \ge 2$ *such that*

$$C := \sup_{t\in(0,1),\, d\in\mathbb{N}} t^{p/2} \|h_d\|^p_{F_d} \left\lceil \left(\frac{[\|h_d\|^*_{F_d}]^2}{\|h_d\|^2_{F_d}} - 1 \right) \frac{1-t}{t} \right\rceil d^{-q} < \infty$$

then I *is polynomially tractable, and*

$$n(\varepsilon, d) \le C\, d^q\, \varepsilon^{-p} \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

If $q = 0$ *then* I *is strongly polynomially tractable.*

– *If*

$$\lim_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{[\|h_d\|_{F_d}^*]^2}{\|h_d\|_{F_d}^2} - 1\right)}{d + t^{-1/2}\|h_d\|_{F_d}^{-1}} = 0$$

then I *is weakly tractable.*

– *If*

$$A := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{[\|h_d\|_{F_d}^*]^2}{\|h_d\|_{F_d}^2} - 1\right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, d)} < \infty$$

then I *is* T*-tractable with the exponent of* T*-tractability at most* A.

– *If*

$$B := \limsup_{d+t^{-1/2}\|h_d\|_{F_d}^{-1}\to\infty} \frac{\ln \frac{1-t}{t} + \ln\left(\frac{[\|h_d\|_{F_d}^*]^2}{\|h_d\|_{F_d}^2} - 1\right)}{\ln T(t^{-1/2}\|h_d\|_{F_d}^{-1}, 1)} < \infty$$

then I *is strongly* T*-tractable with the exponent of strong* T*-tractability at most* B.

The limits are for $d \in \mathbb{N}$ *and* $t \in (0,1)$.

- *Consider the normalized error criterion. Let*

$$A_d^{\text{nor-gen}} = \left(\frac{\|h_d\|_{F_d}^*}{\|h_d\|_{F_d}}\right)^2 - 1.$$

– *If there exists a number* $q \ge 0$ *such that*

$$C := \sup_{d\in\mathbb{N}} A_d^{\text{nor-gen}}\, d^{-q} < \infty$$

then I *is polynomially tractable, and*

$$n(\varepsilon, d) \le \left\lceil C\, d^q\, \varepsilon^{-2} \right\rceil \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

If $q = 0$ *then* I *is strongly polynomially tractable.*

– *If*

$$\lim_{d\to\infty} \frac{\ln A_d^{\text{nor-gen}}}{d} = 0$$

then I *is weakly tractable.*

– *If*

$$A^{\text{gen}} := \limsup_{d\to\infty} \frac{\ln A_d^{\text{nor-gen}}}{\ln(1+T(1,d))} < \infty,$$

$$B^{\text{gen}} := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty$$

then I *is* T*-tractable with the exponent of* T*-tractability at most* $A^{\text{gen}} + 2B^{\text{gen}}$.

– *If* $\sup_d A_d^{\text{nor-gen}} < \infty$ *and* $B^{\text{gen}} < \infty$ *then* I *is strongly* T*-tractable with the exponent of strong* T*-tractability at most* $2B^{\text{gen}}$.

The bounds on the information complexity can be obtained by algorithms with arbitrary coefficients presented in Theorem 10.14.

We illustrate Corollary 10.15 by an example.

10.10.1 Example: Tensor Product Problems (Continued)

As before, we take K_d and h_d as the d-fold tensor product of K_1 and h_1, respectively. It is easy to check that we have strong polynomial tractability if $\|h_1\|_{F_1} < 1$ and $\|h_1\|_{F_1}^* < \infty$. Dropping -1 and taking $q = 0$, we estimate C by

$$t^{p/2-1}(1-t)\big[\|h_1\|_{F_1}^{(p-2)/2}\|h_1\|_{F_1}^*\big]^{2d} + t^{p/2}\|h_1\|_{F_1}^{pd} \le 2,$$

as long as we take

$$p = \begin{cases} 2 & \text{if } \|h_1\|_{F_1}^* \le 1, \\ 2 + 2\frac{\ln \|h_1\|_{F_1}^*}{\ln 1/\|h_1\|_{F_1}} & \text{if } \|h_1\|_{F_1}^* > 1. \end{cases} \tag{10.31}$$

We summarize this in the following corollary.

Corollary 10.16. *The tensor product linear functional problem* $I = \{I_d\}$ *with a tensor product* I_d *defined for a tensor product space* $H(K_d)$ *with*

$$\|h_1\|_{F_1} < 1 \quad \textit{and} \quad \|h_1\|_{F_1}^* < \infty$$

is strongly polynomially tractable for the absolute error criterion in the worst case setting with exponent bounded by (10.31).

We now take the non-separable space F_d with $K_d(x,t) = \delta(x,t)$ and

$$h_1 = a\sum_{j=1}^{\infty} q^{j-1} K_1(\cdot, t_j)$$

for a positive $a, q \in (0,1)$ and for distinct sample points t_j. Then

$$\|h_1\|_{F_1} = \frac{a}{\sqrt{1-q^2}} \quad \text{and} \quad \|h_1\|_{F_1}^* = \frac{a}{1-q}.$$

Hence for $a^2 + q^2 < 1$ we have $\|h_1\|_{F_1} < 1$. For $a + q > 1$ the exponent of strong polynomial tractability is bounded by

$$p = 2 + 2\,\frac{-\ln(1-q) + \ln a}{\frac{1}{2}\ln(1-q^2) - \ln a}.$$

For $a = q = 1/\sqrt{3}$ we obtain $\|h_1\|_{F_1} = \sqrt{2}/2$ and $\|h_1\|_{F_1}^* = (\sqrt{3}+1)/2$. This yields

$$p = 3.7999\ldots.$$

Note that this example cannot be analyzed by Corollary 10.11. □

Again, we do not know the exact value of the exponents. This is summarized in our last open problem of this chapter.

Open Problem 48.

- Find the exact value of the exponent of strong polynomial tractability for tensor product linear functionals in terms of h_1 with

  $$\|h_1\|_{F_1} < 1 \quad \text{and} \quad \|h_1\|_{F_1}^* < \infty$$

  for the absolute error criterion in the worst case setting.

Theorem 10.14 requires that $\|h_d\|_{F_d}^* < \infty$, that is, we must assume that $h_d \in X_d$. We know that the set X_d is dense in F_d. If we do not care about the dependence on d, we then have $e(n,d) = e^{\text{wor}}(n, I_d) = \mathcal{O}(n^{-1/2})$. This leads us to the next corollary.

Corollary 10.17. *Consider a reproducing kernel Hilbert space F_d. Then the set of continuous linear functionals I_d for which $\|h_d\|_{F_d}^* < \infty$ and*

$$e^{\text{wor}}(n, I_d) \le \frac{\|h_d\|_{F_d}^*}{\sqrt{n}}$$

is dense in F_d.

10.11 Notes and Remarks

NR 10.1:1. This chapter is based on a number of papers, as indicated in the Notes and Remarks of the successive sections.

NR 10.3:1. This section is based on [218]. Theorem 10.1 was needed and proved in [218] for $d = 1$. We present its version for arbitrary d with the same proof.

NR 10.4:1. This section is based on [218]. However, the example of the Sobolev space anchored at zero is new. We stress that arbitrarily bad convergence can happen even for $d = 1$. This means that there are intractable linear functionals even if we restrict d to take only one value $d = 1$. On the other hand, there are also linear functionals that are very easy to approximate. So indeed, anything can happen.

NR 10.4:2. Consider again the kernel $K_d(x,t) = \delta(x,t)$. Note that the matrix $(K_d(t_i,t_j))_{i,j=1,2,\dots,n}$ is just the identity matrix, independently of how the sample points t_j are chosen. From this point of view, this kernel seems to be the easiest. On the other hand, the Hilbert space with this reproducing kernel is non-separable. We think that this is probably the most natural example of a non-separable reproducing kernel Hilbert space. As we know, linear functionals defined over this space can be arbitrarily hard to approximate. This tells us how really hard this space is.

NR 10.5:1. This section is based on [277], [350]. Surprisingly, many reproducing kernels occurring in computational practice are point-wise non-negative. That is why the assumption (10.5) is not very restrictive. As we already discussed, many practitioners like to use algorithms with positive coefficients, and that is why the assumptions of Theorem 10.2 are not very restrictive. However, the message of Theorem 10.2 can be quite negative. This is especially the case for tensor product problems studied in Corollary 10.3. Obviously, this negative result holds for the unweighted tensor product problems and can be viewed as motivation to consider weighted tensor product as a way to vanquish the curse of dimensionality. This will be indeed done later.

NR 10.6:1. This section is based on [220]. We must admit that studying the power of negative coefficients is hard, which is probably why we do not know much about it. Nevertheless, the examples of linear functionals presented in this section once again show that anything can happen, even for positive linear functionals. Such results are quite counter-intuitive and, of course, they indicate that the general case is very hard to analyze.

NR 10.7:1. This section is formally new, although many of these facts are well known for $g_d \equiv 1$. The analysis for arbitrary g_d is exactly the same as for $g_d \equiv 1$. The need for general g_d is motivated in the next sections. Although, as we know by now, anything can happen with approximation of linear functionals, we believe that for "typical" linear functionals we have order of convergence at least $n^{-1/2}$ and this problem will be studied in many subsequent sections.

NR 10.7.6:1. In Corollary 10.6 we report the good news that as long as we consider multivariate integration for Sobolev spaces related to various notions of L_2 discrepancy, then we have strong polynomial tractability for the absolute error criterion. We stress that this result sounds much better than it really is. The reason is that for large d, both the L_2 discrepancy and multivariate integration are badly scaled. Take the standard example of the L_2 discrepancy anchored at 0. Then, as we already stressed, the initial error is $3^{-d/2}$. Assume now that $d = 360$. Then $3^{-180} = 10^{-85.88\dots}$. We cannot imagine that somebody would be interested in computing an approximation to such integrals with $\varepsilon < 10^{-85.88\dots}$. But if $\varepsilon > 10^{-85.88\dots}$ then the zero algorithm will do the job with cost 0. That is why it is much more practical to consider spaces (and L_2 discrepancy) for which the initial error is of order 1 or polynomially dependent on d. This will be done later.

NR 10.7.4:1. The example for the Korobov space with varying cardinality is based on [236], where multivariate approximation and diagonal linear multivariate problems were studied in the worst case setting for the class Λ^{all}. Using basically the same proof technique, in this section we found sufficient conditions for strong polynomial and weak tractability of multivariate integration. For multivariate approximation, strong polynomial and polynomial tractability are equivalent and hold if r_j tends to infinity at least as fast as $\ln j$. In our case, for multivariate integration, we have a different situation, at least in terms of sufficient tractability conditions. Strong tractability is not equivalent to polynomial tractability and the limit of $\ln(j)/r_j$ must be smaller than $2\ln(2\pi)$ to claim strong polynomial tractability.

NR 10.7.5:1. The example of bad behavior of any QMC algorithm for the Korobov space with $r = 1$ is taken from [280], and easily generalized here for arbitrary $r \geq 1$. This example shows that the choice of coefficients $a_j = n^{-1}$, although quite natural, can indeed be dangerous. Later, in the example for unbounded kernel, we show that an even worse property can happen for QMC algorithms for any d. Namely, even for $d = 1$, the error of any QMC algorithms is at least as large as the initial error. Furthermore, if we change the coefficients in the optimal algorithm to the QMC coefficients then the error goes to infinity with n.

NR 10.7.6:1. This subsection is new, although quite straightforward. We stress that it is generally necessary to use properly normalized QMC algorithms, although we must admit that this is rarely done in computational practice. Instead of n^{-1}, we should use

$$a = \frac{\|h_d\|_{F_d}^2}{C(K_d, g_d) - \|h_d\|_{F_d}^2 + n\|h_d\|_{F_d}^2}.$$

Usually it is not hard to compute the coefficient a, although it requires the knowledge of the space in which we want to approximate multivariate integrals. Note also that we have $a \approx n^{-1}$ asymptotically in n. Hence, the difference between QMC and properly normalized QMC disappears asymptotically. However, for initial n, it may be very important to use a instead of n^{-1}. We also stress that only for properly normalized QMC algorithms we can achieve strong polynomial tractability for tensor product problems with $\|h_1\|_{F_1} < 1$.

NR 10.7.9:1. This subsection is based on Plaskota, Wasilkowski and Zhao [248] who, however, assumed that $g_d \equiv 1$ and did not use normalized algorithms. Here, we consider algorithms that use arbitrary coefficients $a_j = a(t_j)$ given in Theorem 10.10. Note that the use of such coefficients is easy in general, and it does not require much computation. Furthermore for $g_d = 1$ or for any $g_d \geq 0$, all a_j are non-negative. As with properly normalized algorithms, we must, however, know the space in which we approximate multivariate integrals.

NR 10.8:1. This section is new. This is also the first section where the operator W_d is defined. This operator already played a major role in Volume I, where we studied algorithms using continuous linear functionals. The operator W_d is also very important for the class of standard information, in which we only use function values. We will use this operator extensively also in Volume III, especially for multivariate approximation.

NR 10.9:1. This section is new. We believe that relations with multivariate integration for a dense set of continuous linear functionals makes the study of multivariate integration even more appealing.

NR 10.10:1. This section is new. We find it interesting that separability of F_d is not needed in this section.

Chapter 11

Worst Case: Tensor Products and Decomposable Kernels

11.1 Introduction

We continue the study of linear functionals in the worst case setting. In this chapter we study linear tensor product functionals defined on linear tensor product reproducing kernel Hilbert spaces. The precise definition of these problems is given in Section 11.2. As we know from the previous chapter, for some reproducing kernel Hilbert spaces it may happen that all linear functionals can be solved with arbitrary small error by using just one function value. To eliminate such spaces, we assume that $e(1,1) > 0$, i.e., the minimal worst case error for the univariate case $d = 1$ is positive if we use one function value, $n = 1$. Then for the normalized error criterion in the worst case setting, it turns out that strong polynomial tractability is impossible to obtain, and polynomial tractability may hold only with a d exponent at least equal to 1. Furthermore, in full generality, these last two properties are sharp. That is, there exists a reproducing kernel Hilbert space for which the $(d+1)$st minimal worst case errors are zero for all linear tensor product functionals. This is done in Section 11.3 based on [218].

In Section 11.4, we introduce the notion of a *decomposable* kernel from [221]. This notion allows us to find lower bounds that are sharp in many cases. In this chapter we study decomposable kernels for the unweighted case, whereas in the next chapter we study them for the weighted case. The reproducing kernel for a linear tensor product space depends only on the reproducing kernel for the univariate case. Decomposability is defined by a property of the reproducing kernel for $d = 1$. This property means that each function in the space may be characterized by two orthogonal functions (components) over two domains that are either disjoint or have at most one point in common. If the domains have one point in common then functions vanish at this point. For $d \ge 1$, decomposability of the reproducing kernel means that a linear tensor product functional can be decomposed into 2^d independent linear functionals whose representers have disjoint supports. This property allows us to prove the curse of dimensionality for the normalized error criterion in the worst case setting. The curse holds for all linear tensor product functionals for which the representer for $d = 1$ has two non-zero components, see Theorem 11.8. For the absolute error criterion, the situation is more complicated. If the norm of the representer for $d = 1$ is at least one, then obviously the absolute error criterion is harder than the normalized one, and therefore we also have the curse of dimensionality. If, however, it is less than one, then anything can happen. We may even have strong polynomial tractability as already shown by many examples in the previous chapter, or we may have intractability even for $d = 1$. We illustrate Theorem 11.8 by several examples showing that the lower estimates in this theorem are sharp in general. In particular, we discuss Gaussian integration for the Sobolev space, which for $d = 1$ consists of r times differentiable functions. It turns out that Gaussian integration is intractable and suffers from the curse of dimensionality for the normalized error criterion and is strongly tractable for the absolute error criterion. We stress that the curse of dimensionality for the normalized error criterion holds no matter how large r we have. We also consider uniform integration related to the centered discrepancy for the L_q norm with $q \in [1, \infty]$.

Many reproducing kernels are *not* decomposable. However, many kernels can be written as the sum of two kernels with one of them being decomposable. This is done in Section 11.5. This allows us to generalize Theorem 11.8 and again prove the curse of dimensionality for the normalized error criterion in the worst case setting under the same assumption that the part of the representer for $d = 1$ corresponding to the decomposable part has two non-zero components, see Theorems 11.12 and 11.14. However, the proofs are more complicated. As described before, the situation is different for the absolute error criterion and anything can happen.

Hence, we obtain lower bounds also for reproducing kernels that have a decomposable part for $d = 1$. This significantly enlarges the applicability of these results. We illustrate this for several examples. The choice of spaces is now more natural than before since decomposable kernels usually require spaces with sometimes unnatural boundary conditions. We again study uniform integration related to the centered discrepancy for a variety of norms.

In Section 11.6, we want to characterize tractability of all linear tensor product functionals defined on tensor product spaces. Based on the error estimates for reproducing kernels with decomposable parts, we present Theorem 11.15, which states that either a linear tensor product is trivial and can be solved exactly using at most one function value or it suffers the curse of dimensionality and is intractable. This holds for the normalized error criterion and for the absolute error criterion if the representer for $d = 1$ has norm at least one. In this theorem, we assume that we can identify a decomposable part by a rank one modification of the original reproducing kernel and this holds for all points from the domain of functions. We then check that this assumption holds for a number of standard Sobolev spaces of non-periodic functions with the smoothness parameter $r = 1$. We also identify Sobolev spaces of periodic functions with smoothness parameter $r \geq 2$, for which Theorem 11.15 is not applicable. As always we present a number of open problems related to the subjects covered in this chapter. We have eleven open problems numbered from 49 to 59.

11.2 Linear Tensor Product Functionals

We now define linear functionals studied in this chapter. For $d = 1$, we assume that F_1 is a class of univariate functions defined over $D_1 \subseteq \mathbb{R}$, and F_1 is a reproducing kernel Hilbert space whose reproducing kernel is $K_1 : D_1 \times D_1 \to \mathbb{R}$. The inner product of

F_1 is denoted by $\langle \cdot, \cdot \rangle_{F_1}$. The (continuous) linear functional I_1 now takes the form

$$I_1(f) = \langle f, h_1 \rangle_{F_1} \quad \text{for all } f \in F_1$$

for some function h_1 from F_1.

For $d > 1$, we take

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1$$

as the d-fold tensor product of F_1. Then F_d is a class of multivariate functions defined over $D_d = D_1 \times D_1 \times \cdots \times D_1$ (d times). The class F_d is a Hilbert space whose reproducing kernel $K_d : D_d \times D_d \to \mathbb{R}$ is given by

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j, t_j) \quad \text{for all } x, t \in D_d.$$

The inner product of F_d is denoted by $\langle \cdot, \cdot \rangle_{F_d}$. Finally, the (continuous) linear functional $I_d = I_1 \otimes \cdots \otimes I_1$ is the d-fold tensor product of I_1. This means that

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d, \quad \text{with } h_d(t) = h_1(t_1)h_1(t_2)\cdots h_1(t_d).$$

Clearly, the initial error is given by

$$e(0,d) = \|I_d\| = \|h_d\|_{F_d} = \|h_1\|_{F_1}^d.$$

Hence, $e(0,d) = 0$ only for trivial problems in which $h_1 \equiv 0$, so that $I_d \equiv 0$. As we shall see, many results will depend on whether $\|h_1\|_{F_1}$ is less than 1 or at least equal to 1. Note, however, that if $\|h_1\|_{F_1} < 1$ then $\|I_d\| = \|h_1\|_{F_1}^d$ is exponentially *small* in d, whereas $\|h_1\|_{F_1} > 1$ implies that $\|I_d\| = \|h_1\|_{F_1}^d$ is exponentially *large* in d. Hence, the only case for which the problem is well-normalized is when $\|h_1\|_{F_1} = 1$.

As in the previous chapter, $e(n,d) = e^{\text{wor}}(n, I_d)$ denotes the nth minimal worst case error, and $n(\varepsilon, d) = n^{\text{wor}}(\varepsilon, I_d)$ denotes the information complexity for the absolute error criterion, $\text{CRI}_d = 1$, or for the normalized error criterion, $\text{CRI}_d = \|I_d\|$.

The only range for ε that really matters is $\varepsilon \in \big(0, \|h_d\|_{F_d}\big)$ for the absolute error criterion, and $\varepsilon \in (0,1)$ for the normalized error criterion. Otherwise, the zero algorithm solves the problem and $n(\varepsilon, d) = 0$ for $\varepsilon \geq \|h_d\|_{F_d}$ under the absolute error criterion, and for $\varepsilon \geq 1$ under the normalized error criterion.

11.3 Preliminary Error Estimates

Theorem 10.1 states the formula for the minimal error when we use one function value,

$$e^2(1,d) = e^2(0,d) - \sup_{t \in D_d} \frac{h_d^2(t)}{K_d(t,t)}$$

with the convention that $0/0 = 0$. Since F_d and I_d have tensor product structure, this formula now takes the form given in the following lemma.

Lemma 11.1. *We have*

$$\begin{aligned} e(1,d) &= e(0,d)\sqrt{1-\left(1-\frac{e^2(1,1)}{e^2(0,1)}\right)^d} \\ &= \sqrt{e^{2d}(0,1)-(e^2(0,1)-e^2(1,1))^d} \\ &= \sqrt{\|h_1\|_{F_1}^{2d}-\left(\sup_{t\in D_1}\frac{h_1^2(t)}{K_1(t,t)}\right)^d}, \end{aligned} \tag{11.1}$$

again with the convention that $0/0=0$.

Hence, $e(1,1)=0$ *implies* $e(1,d)=0$ *for all* $d\in\mathbb{N}$, *and* $n(\varepsilon,d)\le 1$ *for all* $\varepsilon>0$, $d\in\mathbb{N}$. *In this case, the problem is strongly polynomial tractable with exponent zero.*

Proof. Observe that $h_d(t)/K_d(t,t)=\prod_{i=1}^d h_1(t_i)/K_1(t_i,t_i)$, and therefore

$$\sup_{t\in D_d}\frac{h_d^2(t)}{K_d(t,t)}=\left(\sup_{t\in D_1}\frac{h_1^2(t)}{K_1(t,t)}\right)^d=\left(e^2(0,1)-e^2(1,1)\right)^d,$$

by the formula for $e(1,1)$ in Theorem 10.1. Again using the same formula for $e(1,d)$ we obtain the formulas presented in the lemma. From this it is obvious that $e(1,1)=0$ implies that $e(1,d)=0$, which completes the proof. □

Hence if we can solve the problem for $d=1$ with arbitrary small error by using one function value, then the same is true for all d. As we know from the previous chapter, there are reproducing kernel Hilbert spaces even of infinite dimension for which this indeed holds, see Section 10.3 of Chapter 10.

To omit this trivial case, from now on we assume that $e(1,1)>0$. Observe that Theorem 10.1 now states that $e(1,d)<e(0,d)$ for all d. We present from [218] several estimates for the sequence $e(n,d)$ in terms of $e(0,1)$, $e(1,1)$ and $e(0,d)=e^d(0,1)$.

Theorem 11.2. *Assume that* $e(1,1)>0$. *Let*

$$\tau=1-\frac{e^2(1,1)}{e^2(0,1)}\in(0,1).$$

Then

$$e(d,d)\ge e^d(1,1)>0, \tag{11.2}$$

$$\frac{e(n,n\,d)}{e(0,n\,d)}\ge\left(1-\tau^d\right)^{n/2}, \tag{11.3}$$

$$\lim_{d\to\infty}\frac{e(n,d)}{e(0,d)}=1 \quad \textit{for all } n, \tag{11.4}$$

$$\lim_{d\to\infty}\frac{e(\lceil d^p\rceil,d)}{e(0,d)}=1 \quad \textit{for all } p\in[0,1). \tag{11.5}$$

Proof. We recall from Chapter 4 of Volume I that

$$e(n,d) = \inf_{\substack{t_j \in D_d \\ j=1,2,\dots,n}} \sup_{\substack{f \in F_d,\, \|f\|_{F_d} \le 1 \\ f(t_j)=0,\, j=1,2,\dots,n}} \langle f, h_d \rangle_{F_d}. \tag{11.6}$$

In particular,

$$e(1,1) = \inf_{t \in D_1} \sup_{\substack{f \in F_1,\, \|f\|_{F_1} \le 1 \\ f(t)=0}} \langle f, h_1 \rangle_{F_1}.$$

Let $\eta \in (0, e(1,1))$. Then for every $t \in D_1$ there exists $f_t \in F_1$ with $\|f_t\|_{F_1} = 1$, such that $f_t(t) = 0$ and $\langle f_t, h_1 \rangle_{F_1} \ge e(1,1) - \eta$.

To prove (11.2), take $n = d$ and arbitrary points $t_1, t_2, \dots, t_d \in D_d$. Let $t_{j,j} \in D_1$ denote the jth component of the point t_j. Define the function

$$f(x) = f_{t_{1,1}}(x_1) f_{t_{2,2}}(x_2) \cdots f_{t_{d,d}}(x_d) \quad \text{for all } x = [x_1, x_2, \dots, x_d] \in D_d.$$

Then $f \in F_d$, $\|f\|_{F_d} = 1$, and $f(t_j) = 0$ for $j = 1, 2, \dots, d$. Furthermore,

$$\langle f, h_d \rangle_{F_d} = \prod_{j=1}^{d} \langle f_{t_{j,j}}, h_1 \rangle_{F_1} \ge (e(1,1) - \eta)^d.$$

Since this holds for arbitrary t_j, we have $e(d,d) \ge (e(1,1) - \eta)^d$ from (11.6). Letting η go to zero we obtain (11.2).

To prove (11.3) we proceed similarly. This time let $\eta \in (0, e(1,d))$. From (11.6) with $n = 1$, for any $t \in D_d$ there exists $f_t \in F_d$, $\|f\|_{F_d} = 1$ such that $f_t(t) = 0$ and

$$\langle f_t, h_d \rangle_{F_d} \ge e(1,d) - \eta = e(0,d)\,(1 - \tau^d)^{1/2} - \eta,$$

where we used the first formula from Lemma 11.1.

Take arbitrary points $t_1, t_2, \dots, t_n \in D_{nd}$. Let $t_{j,d} \in D_d$ denote the components from $(j-1)d + 1$ to $j\,d$ of the point t_j. For $x = [x_1, x_2, \dots, x_n] \in D_{nd}$ with $x_j \in D_d$ for $j = 1, 2, \dots, n$, define the function

$$f(x) = f_{t_{1,d}}(x_1)\, f_{t_{2,d}}(x_2) \cdots f_{t_{n,d}}(x_n) \quad \text{for all } x \in D_{nd}.$$

Then $f \in F_{nd}$, $\|f\|_{F_{nd}} = 1$, $f(t_j) = 0$ for $j = 1, 2, \dots, n$, and

$$\begin{aligned}
\langle f, h_{nd} \rangle_{F_{nd}} &= \prod_{j=1}^{n} \langle f_{t_{j,d}}, h_d \rangle_{F_d} \ge \left(e(0,d)\,(1-\tau^d)^{1/2} - \eta \right)^n \\
&= e^n(0,d) \left((1-\tau^d)^{1/2} - \frac{\eta}{e(0,d)} \right)^n \\
&= e(0,nd) \left((1-\tau^d)^{1/2} - \frac{\eta}{e(0,d)} \right)^n.
\end{aligned}$$

Letting η go to zero we obtain (11.3).

The last estimates (11.4) and (11.5) follow easily from (11.3). Indeed, $e(n,d) \le e(0,d)$ and by letting d go to infinity in (11.3) we get (11.4). Finally, using $d = \Theta(\lceil d^p \rceil\, d^{1-p})$ and (11.3) for $n = \lceil d^p \rceil$ we have for large d,

$$\frac{e(\lceil d^p \rceil, d)}{e(0,d)} \ge \left(1 - \tau^{c\, d^{1-p}}\right)^{\lceil d^p \rceil/2}$$

for some positive c. Since $1-p$ is positive and $\tau^c < 1$, the logarithm of the right hand side of the last inequality is of order $d^p (\tau^c)^{d^{1-p}}$ and goes to zero as d approaches infinity. Thus, the right hand side goes to one. This completes the proof. □

Observe that from the last two estimates of Theorem 11.2 we can conclude the following corollary.

Corollary 11.3. *If $e(1,1) > 0$ then the linear tensor product functional problem $I = \{I_d\}$ is not strongly polynomially tractable for the normalized error criterion. If the problem is polynomially tractable then its exponent with respect to d is at least one.*

In particular, Theorem 11.2 says that $e(d,d)$ is positive. We now prove that this estimate cannot be improved in general, i.e., it can fail if d is replaced by $d+1$. Furthermore, we also show that the last estimate of Theorem 11.2 is somewhat sharp, in the sense that it can fail for arbitrary $p > 1$.

Theorem 11.4. *There exists a reproducing kernel Hilbert space F_1 for which the following hold:*

- *For all linear tensor product functionals we have*

$$e(d+1,d) = 0 \quad \text{for all } d \in \mathbb{N}.$$

- *There exist linear functionals I_1 defined over F_1 with $e(1,1) > 0$ for which*

$$e(d,d) > 0 \quad \text{and} \quad e(d+1,d) = 0 \quad \text{for all } d \in \mathbb{N}.$$

Therefore for both the absolute and normalized error criterion we have

$$n(\varepsilon,d) \le d+1 \quad \text{for all } \varepsilon \ge 0 \text{ and } d \in \mathbb{N}.$$

This means polynomial tractability with the exponent with respect to ε^{-1} equal to zero, and with the exponent with respect to d equal to at most one.

Proof. We construct a two-dimensional space $F_1 = \mathrm{span}(e_1, e_2)$, where e_1 and e_2 are two linearly independent functions defined on $D_1 = [0,1]$. We choose an inner product in such a way that e_j are orthonormal.

Take an arbitrary linear functional

$$I_1 f = \langle f, h_1 \rangle_{F_1} \quad \text{with } h_1 = \alpha_1 e_1 + \alpha_2 e_2.$$

Without loss of generality[1], assume that $\|h_1\|_{F_1}^2 = \alpha_1^2 + \alpha_2^2 = 1$. We first check for which α_j's we have $e(1,1) = 0$. The reproducing kernel of F_1 is given by

$$K_1(t,x) = e_1(t)e_1(x) + e_2(t)e_2(x).$$

Let

$$g(t) = \frac{\alpha_1\, e_1(t) + \alpha_2\, e_2(t)}{\sqrt{e_1^2(t) + e_2^2(t)}} \quad \text{for all } t \in [0,1] \text{ (as always } 0/0 = 0).$$

We know that

$$e(1,1) = \big[1 - \sup_{t\in[0,1]} g^2(t)\big]^{1/2},$$

and $e(1,1) = 0$ iff $\sup_{t\in[0,1]} g^2(t) = 1$. By application of the Cauchy–Schwartz inequality for l_2, this holds if there exists $t \in [0,1]$ such that

$$\alpha_1 e_2(t) = \alpha_2 e_1(t). \tag{11.7}$$

Let $Z_2 = \{t \in [0,1] \mid e_2(t) = 0\}$ denote the zero set of e_2. Consider the function

$$r := e_1/e_2 \colon [0,1] \setminus Z_2 \;\to \mathbb{R}.$$

Then (11.7) holds for some $t \in [0,1]$ for arbitrary α_1 and α_2 iff $r([0,1] \setminus Z_2) = \mathbb{R}$. In this case, we have $e(1,1) = 0$ for all linear functionals of F_1.

From now on, we assume that the functions e_1 and e_2 are chosen in such a way that $\overline{r([0,1] \setminus Z_2)}$ is a proper subset of $\mathbb{R}$. This implies that there exist linear functionals for which $e(1,1) > 0$. They are characterized by the condition

$$\alpha_1/\alpha_2 \notin \overline{r([0,1] \setminus Z_2)}.$$

For such functionals we know from (11.2) that $e(d,d) > 0$.

We now prove that no matter how h_1 is chosen we always have $e(d+1,d) = 0$. To do this, we need to assume one more condition on the choice of functions e_1 and e_2. Namely, that

$$r\,([0,1] \setminus Z_2) \;\; \text{has infinitely many elements.} \tag{11.8}$$

Obviously there exist functions e_1 and e_2 satisfying all these assumptions. For instance, one can take $e_1(t) = t$ and $e_2(t) = t^2 + 1$.

For $d \ge 2$, we have $I_d(f) = \langle f, h_d\rangle_{F_d}$ with

$$h_d(x) = h_d(x_1, x_2, \dots, x_d) = \prod_{j=1}^{d} \big(\alpha_1 e_1(x_j) + \alpha_2 e_2(x_j)\big).$$

[1] For $\|h_1\|_{F_1} = 1$ we construct a linear algorithm $A_{d+1,d}$ that uses $d+1$ function values and that recovers I_d exactly. For general h_1 it is enough to multiply $A_{d+1,d}$ by $\|h_1\|_{F_1}^d$.

We approximate $I_d(f)$ by a linear algorithm

$$A_{d+1,d}(f) = \sum_{j=1}^{d+1} a_j\, f(t_j, t_j, \dots, t_j) \quad \text{for all } f \in F_d, \tag{11.9}$$

for some $a_j \in \mathbb{R}$ and $t_j \in [0,1]$. We stress that $A_{d+1,d}$ uses $d+1$ function values at points whose components are all equal. The worst case error of $A_{d+1,d}$ is $e(A_{d+1,d}) = \|g_d\|_{F_d}$ with

$$g_d = h_d - \sum_{j=1}^{d+1} a_j\, K_d(\cdot, [t_j, t_j, \dots, t_j]).$$

That is, we have

$$g_d = \prod_{j=1}^{d} \big(\alpha_1 e_{1,j} + \alpha_2 e_{2,j}\big) - \sum_{i=1}^{d+1} a_i \prod_{j=1}^{d} \big(e_1(t_i)\, e_{1,j} + e_2(t_i)\, e_{2,j}\big),$$

where $e_{i,j}(x) = e_i(x_j)$.

Define the set

$$J_k = \big\{\, \vec{j} = [j_1, j_2, \dots, j_d] \mid j_i \in \{1,2\}, \text{ and the number of } i \text{ with } j_i = 1 \text{ is } k \,\big\},$$

for $k = 0, 1, \dots, d$. The cardinality of the set J_k is obviously $\binom{d}{k}$. We now decompose the first term in g_d as

$$\prod_{j=1}^{d} \big(\alpha_1 e_{1,j} + \alpha_2 e_{2,j}\big) = \sum_{k=0}^{d} \alpha_1^k\, \alpha_2^{d-k}\, e_k^*,$$

where

$$e_k^* = \sum_{\vec{j} \in J_k} e_{j_1,1} e_{j_2,2} \cdots e_{j_d,d}.$$

Similarly we have

$$\prod_{j=1}^{d} \big(e_1(t_i)\, e_{1,j} + e_2(t_i)\, e_{2,j}\big) = \sum_{k=0}^{d} e_1^k(t_i)\, e_2^{d-k}(t_i)\, e_k^*.$$

Substituting these expressions into the expression above for g_d, we obtain

$$g_d = \sum_{k=0}^{d} \Big(\alpha_1^k \alpha_2^{d-k} - \sum_{i=1}^{d+1} a_i\, e_1^k(t_i) e_2^{d-k}(t_i)\Big)\, e_k^*.$$

Hence $e(A_{d+1,d}) = 0$ iff $\|g_d\|_{F_d} = 0$, which in turn holds, because the $e_0^*, e_1^*, \dots, e_d^*$ are linearly independent, iff

$$\sum_{i=1}^{d+1} a_i\, e_1^k(t_i) e_2^{d-k}(t_i) = \alpha_1^k \alpha_2^{d-k} \quad \text{for } k = 0, 1, \dots, d.$$

We have a system of $d+1$ linear equations and $d+1$ unknown coefficients a_i. We can find a_i's for arbitrary α_i's iff the matrix

$$M = \big(e_1^k(t_i)e_2^{d-k}(t_i)\big) = (m_{k,i}) \quad \text{for } k = 0,1,\dots,d,\ i = 1,2,\dots,d+1,$$

is nonsingular.

Take points t_i for which $e_2(t_i)$ are nonzero and $q_i = r(t_i)$ are distinct for all $i = 1,2,\dots,d+1$. Such points exist due to (11.8).

We claim that for these points t_i the matrix M is nonsingular. Indeed, let

$$W = \operatorname{diag}\big(e_2^{-d}(t_1), e_2^{-d}(t_2), \dots, e_2^{-d}(t_{d+1})\big)$$

be a diagonal matrix. By our assumptions it is nonsingular. Moreover, $M\,W = (a_{k,i})$ is a Vandermonde matrix with $a_{k,i} = q_i^k$. Since the q_i are distinct, the matrix $M\,W$ is nonsingular, and therefore so is M. This completes the proof. □

We stress that the points t_i in the proof of Theorem 11.4 do not depend on the functionals I_d. More precisely, in the space F_d used in the proof of Theorem 11.4, let

$$N_j(f) = [f(t_1,\dots,t_1), f(t_2,\dots,t_2),\dots, f(t_j,\dots,t_j)]$$

be the information, with numbers t_i for which $e_2(t_i)$ are all nonzero and $e_1(t_i)/e_2(t_i)$ are distinct for all i. Then for any linear tensor product functional I_d, we have $r^{\text{wor}}(N_{d+1}) = 0$, where $r^{\text{wor}}(N_{d+1})$ stands for the radius of information in the worst case setting, see Chapter 4 of Volume I.

In fact, for an arbitrary linear tensor product functional I_d and any choice of $t_1, t_2, \dots, t_{d+1}$ as above, we showed that there exist numbers $a_j = a_j(I_d)$ for $j = 1,2,\dots,d+1$, such that

$$I_d(f) = A_{d+1,d}(f) = \sum_{j=1}^{d+1} a_j f(t_j,\dots,t_j) \quad \text{for all } f \in F_d.$$

The proof of Theorem 11.4 presents a two-dimensional univariate space F_1 for which all linear tensor product functionals are tractable. It is possible to generalize the proof of Theorem 11.4 for spaces F_1 of dimension $p \ge 2$. Namely, assume that $F_1 = \operatorname{span}(e_1, e_2, \dots, e_p)$ for orthonormal e_i defined on D_1. For given points $t_i \in D$ consider the $n \times n$ matrix

$$M = \big(e_1^{k_1}(t_i)\, e_2^{k_2}(t_i)\, \cdots\, e_p^{k_p}(t_i)\big)$$

for nonnegative k_j such that

$$k_1 + k_2 + \cdots + k_p = d \quad \text{and} \quad i = 1,2,\dots,n = \binom{d+p-1}{p-1}.$$

We prove that if there exist points $t_1, t_2, \dots, t_n$ such that M is nonsingular then

$$e(n,d) = 0 \tag{11.10}$$

for all linear tensor product functionals. In this case, the problem is polynomially tractable and the exponent with respect to ε^{-1} is zero whereas the exponent with respect to d is at most $p-1$.

Indeed, we now have

$$h_d = \prod_{j=1}^{d} \left(\alpha_1 e_{1,j} + \alpha_2 e_{2,j} + \cdots + \alpha_p e_{p,j}\right).$$

To decompose the last expression, let

$$A_{p,d} = \left\{\vec{k} = [k_1, k_2, \ldots, k_p], \text{ for integers } k_i \ge 0 \text{ with } \textstyle\sum_{i=1}^{p} k_i = d\right\}.$$

The cardinality of the set $A_{p,d}$ is $n = \binom{d+p-1}{p-1}$. For each $\vec{k} \in A_{p,d}$ define the set

$$J_{\vec{k}} = \left\{\vec{j} = [j_1, j_2, \ldots, j_d] : \text{ the number of } j_i = m \text{ is } k_m\right\}.$$

Here $j_i \in \{1, 2, \ldots, p\}$ and $m \in [1, p]$. Then

$$h_d = \sum_{\vec{k} \in A_{p,d}} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_p^{k_p} \, e^*_{\vec{k}},$$

where

$$e^*_{\vec{k}} = \sum_{\vec{j} \in J_{\vec{k}}} e_{j_1,1} e_{j_2,2} \cdots e_{j_d,d}.$$

Consider $A_{n,d}$ given by (11.9) with $n = \binom{d+p-1}{p-1}$ function values. As before we can show that the error $e(A_{n,d}) = \|g_d\|_{F_d}$ with

$$g_d = \sum_{\vec{k} \in A_{p,d}} \left(\alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_p^{k_p} - \sum_{i=1}^{n} a_i \, e_1^{k_1}(t_i) e_2^{k_2}(t_i) \cdots e_p^{k_p}(t_i)\right) e^*_{\vec{k}}.$$

To guarantee that $\|g_d\|_{F_d} = 0$ we require that the a_i's satisfy the system of linear equations

$$\sum_{i=1}^{n} a_i \, e_1^{k_1}(t_i) e_2^{k_2}(t_i) \cdots e_p^{k_p}(t_i) = \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_p^{k_p} \quad \text{for all } \vec{k} \in A_{p,d}.$$

If the matrix M of this system is nonsingular, we can find a_i for arbitrary α_i. This completes the proof of (11.10).

It is natural to ask for which points t_i the matrix M is nonsingular. An example is provided for $D = [0, +\infty)$ and $e_i(t) = t^{\sqrt{q_i}}$, where q_i is the ith prime number, with $q_1 = 1$. Then

$$e_1^{k_1}(t_i)\, e_2^{k_2}(t_i) \cdots e_p^{k_p}(t_i) = t_i^{k_1 + k_2\sqrt{q_2} + \cdots + k_p\sqrt{q_p}}.$$

Clearly, the exponents $u_{\vec{k}} = k_1 + k_2\sqrt{q_2} + \cdots + k_p\sqrt{q_p}$ are different for different vectors $\vec{k} = [k_1, k_2, \ldots, k_p]$.

We use induction on n to check non-singularity of $M = \big(t_i^{u_{\vec{k}}}\big)$. The induction hypothesis is that for $1 \le m < n$ the $m \times m$ sub-matrices of M that involve only $t_1, \ldots, t_m$ can all be made nonsingular by appropriate choice of $t_1, \ldots, t_m$. If the result holds for sub-matrices of size $m = \nu - 1$ then for each sub-matrix M_ν of size ν we expand the determinant along the appropriate row to find that

$$\det(M_\nu) = a\, t_\nu^\beta + o(t_\nu^\beta), \quad \text{as } t_\nu \to +\infty$$

for some nonzero a and β. Hence, we can take a large t_ν for which each $\det(M_\nu)$ is nonzero. From this it follows that choices of points always exist for which M is non-singular.

The preceding theorem says that in some spaces all linear tensor product functionals are polynomially tractable. We now show that the opposite can also happen.

Theorem 11.5. *There exists a reproducing kernel Hilbert space for which all linear tensor product functionals with* $e(1,1) > 0$ *are intractable and suffer from the curse of dimensionality under the normalized error criterion, and under the absolute error criterion if* $e(0,1) \ge 1$.

Proof. Take the non-separable Hilbert space F_1 that was used in the previous chapter. That is, F_1 is the space of functions defined on $[0,1]$ with the reproducing kernel $K_1(t,t) = 1$ and $K_1(t,x) = 0$ for $x \ne t$.

Consider an arbitrary linear functional $I_1(f) = \langle f, h_1 \rangle_{F_1}$ for some $h_1 \in F_1$. Then

$$h_1 = \sum_{j=1}^{\infty} \alpha_j\, K_1(\cdot, t_j)$$

for some $\alpha_j \in \mathbb{R}$ and distinct $t_j \in [0,1]$, with

$$e^2(0,1) = \|h_1\|_{F_1}^2 = \sum_{j=1}^{\infty} \alpha_j^2 < +\infty.$$

Assume for a notational convenience that the α_j are ordered, i.e., $\alpha_1^2 \ge \alpha_2^2 \ge \cdots \ge \alpha_n^2$. It is easy to show that $e(1,1)$ is now given by

$$e^2(1,1) = e^2(0,1) - \alpha_1^2 = \sum_{j=2}^{\infty} \alpha_j^2.$$

Hence, $e(1,1) > 0$ iff $\alpha_2 > 0$, i.e., at least two α_j's are nonzero.

For $d \ge 2$, we have

$$h_d = \sum_{j \in \mathbb{N}^d} \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_d}\, K_d(\cdot, [t_{j_1}, t_{j_2}, \ldots, t_{j_d}]).$$

From the form of h_d it is clear that the best we can do when we use n function values is to eliminate the n largest coefficients of the sequence $\{\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_d}\}_{j_i\in\mathbb{N}}$. The largest element in this sequence is α_1^d and we can decrease the square of the initial error $e^2(0,d) = \|h_1\|_{F_1}^{2d}$ by at most $n\,\alpha_1^{2d}$. Hence,

$$e^2(n,d) \ge e^2(0,d) - n\,\alpha_1^{2d}.$$

To guarantee that $e(n,d) \le \varepsilon\,\mathrm{CRI}_d$ we must take n satisfying[2] the inequality

$$n \ge \left(\frac{\sum_{j=1}^{\infty}\alpha_j^2}{\alpha_1^2}\right)^d \left(1 - \frac{\mathrm{CRI}_d^2\,\varepsilon^2}{e^2(0,d)}\right). \tag{11.11}$$

For the normalized error $\mathrm{CRI}_d = e(0,d)$, we have

$$n(\varepsilon,d) \ge \left(1 + \frac{\alpha_2^2}{\alpha_1^2} + \frac{\sum_{j=3}^{\infty}\alpha_j^2}{\alpha_1^2}\right)^d (1-\varepsilon^2),$$

which is exponential in d for all $\varepsilon < 1$. For the absolute error criterion $\mathrm{CRI}_d = 1$, we assumed that $e(0,1) \ge 1$, so that $e^2(0,d) = e^{2d}(0,1) \ge 1$ and again we have an exponential dependence on d for all $\varepsilon < 1$. So the problem suffers from the curse of dimensionality and is intractable. This completes the proof. □

From the results we presented so far, we can say that tractability of linear functionals very much depends on the Hilbert space. For some spaces, all nontrivial linear functionals are intractable, whereas for other spaces, all linear functionals are polynomially tractable with ε^{-1} exponent zero and d exponent at most one. Such results hold for very special spaces. We believe that in "typical" reproducing kernel Hilbert spaces, some linear functionals are tractable and some others are not. The next theorem presents conditions under which we can find tractable and intractable linear functionals in a given space, see [218] where this theorem was originally presented.

Theorem 11.6. *Let F_1 be a reproducing Hilbert space of real-valued functions on a domain $D_1 \subseteq \mathbb{R}$.*

(i) *For two distinct t_1 and t_2 from D_1, let e_1 and e_2 be orthonormal elements from*

$$\mathit{span}\,(K_1(\cdot,t_1), K_1(\cdot,t_2)).$$

If the function e_1/e_2 takes infinitely many values then all linear functionals

$$I_1(f) = \langle f, h_1\rangle_{F_1} \quad \text{with } h_1 \in \mathit{span}(e_1,e_2)$$

are polynomially tractable with exponents zero and at most one, since

$$e(d+1,d) = 0 \quad \text{and} \quad n(\varepsilon,d) \le d+1.$$

[2] For $\|h_1\|_{F_1} = 1$, the bound (11.11) reads $n \ge (1-\varepsilon^2)\alpha_1^{-2d}$, which was already presented in the Example: Kernel for Non-separable Space in Chapter 10. Hence, (11.11) generalizes the bound from Chapter 10 for arbitrary non-zero h_1.

(ii) *If there exist two orthonormal elements e_1 and e_2 from the space F_1 which have disjoint supports then for all linear tensor product functionals with*

$$h_1 = \alpha\, e_1 + \sqrt{1-\alpha^2}\, e_2 \quad \textit{for } \alpha \in (0,1)$$

we have

$$n(\varepsilon, d) \ge (1-\varepsilon^2) \left[\frac{1}{\max(\alpha^2, 1-\alpha^2)} \right]^d .$$

Hence, these linear functionals are intractable and suffer from the curse of dimensionality.

Proof. The first part can be proved by using exactly the same proof as in Theorem 11.4, with $\mathrm{span}(e_1, e_2)$ now playing the role of F_1. The second part was proven in [329] p. 53, and we now repeat this short instructive proof to see for the first time how disjoint supports allow us to obtain lower bounds on the worst case errors.

Note that $\|h_1\|_{F_1} = 1$ which, of course, implies that $\|h_d\|_{F_d} = 1$ for all d, so that there is no difference between the absolute and normalized error criteria.

For $j = [j_1, j_2, \dots, j_d]$ with $j_i \in \{1, 2\}$, let $1(j)$ denote the number of indices j_i equal to 1, and let $2(j)$ denote the number of indices j_i equal to 2. Clearly, $1(j) + 2(j) = d$ for all 2^d such integer vectors j. For $x = [x_1, x_2, \dots, x_d] \in D_d$, let

$$e_{j,1}(x) = \prod_{i:\, j_i = 1} e_1(x_i) \quad \text{and} \quad e_{j,2}(x) = \prod_{i:\, j_i = 2} e_2(x_i).$$

Then

$$\begin{aligned} h_d(x) &= \prod_{j=1}^{d} \left(\alpha e_1(x_j) + \sqrt{1-\alpha^2} e_2(x_j) \right) \\ &= \sum_{j=[j_1, j_2, \dots, j_d]:\ j_i \in \{1,2\}} \alpha^{1(j)} (1-\alpha^2)^{2(j)/2}\, e_{j,1}(x) e_{j,2}(x). \end{aligned}$$

Let A_k be the support of e_k for $k = 1, 2$. We assumed that $A_1 \cap A_2 = \emptyset$. Then the support of the function $e_{j,1} e_{j,2}$ is

$$A_j := A_{j_1} \times A_{j_2} \times \dots \times A_{j_d}.$$

Therefore the functions $e_{j,1} e_{j,2}$ have disjoint supports for distinct j. Equivalently the 2^d sets $A_{d,j}$ are disjoint and are subsets of D_d. Note also that the functions $e_{j,1} e_{j,2}$ are orthonormal.

Take an arbitrary algorithm that uses function values $f(t_j)$ for arbitrary sample points t_j from D_d for $j = 1, 2, \dots, n$. We assume that $n < 2^d$. The points t_j belong to at most n sets A_j, and let $J_{d,n}$ be the set of indices j for which

$$\{t_1, t_2, \dots, t_n\} \cap A_j = \emptyset.$$

Obviously, the cardinality $|J_{d,n}|$ of $J_{d,n}$ is at least $2^d - n \ge 1$. Define the function

$$f = a_{d,n} \sum_{j \in J_{d,n}} \alpha^{1(j)} (1-\alpha^2)^{2(j)/2} e_{j,1} e_{j,2},$$

where

$$a_{d,n} = \Big(\sum_{j \in J_{d,n}} \alpha^{2 \cdot 1(j)} (1-\alpha^2)^{2(j)} \Big)^{-1/2}.$$

Then $f \in F_d$, $\|f\|_{F_d} = 1$, and most importantly, $f(t_j) = 0$ for all $j = 1, 2, \dots, n$. Furthermore,

$$I_d(f) = a_{d,n} \sum_{j \in J_{d,n}} \alpha^{2 \cdot 1(j)} (1-\alpha^2)^{2(j)} = \Big(\sum_{j \in J_{d,n}} \alpha^{2 \cdot 1(j)} (1-\alpha^2)^{2(j)} \Big)^{1/2}.$$

Note that

$$\sum_{j \in J_{d,n}} \alpha^{2 \cdot 1(j)} (1-\alpha^2)^{2(j)} = 1 - \sum_{j \notin J_{d,n}} \alpha^{2 \cdot 1(j)} (1-\alpha^2)^{2(j)} \ge 1 - n \left[\max(\alpha^2, 1-\alpha^2) \right]^d.$$

Let $\beta = \max(\alpha^2, 1-\alpha^2)$. Then $\alpha \in (0,1)$ implies that $\beta \in [1/2, 1)$, and

$$I_d(f) \ge (1 - n\beta^d)_+^{1/2}.$$

Again using (11.6), we conclude that

$$e(n,d) \ge (1 - n\beta^d)_+^{1/2}$$

and that

$$n(\varepsilon, d) \ge (1-\varepsilon^2)\, \beta^{-d} \quad \text{for all } \varepsilon \in (0,1),$$

which shows the curse of dimensionality. This completes the proof. □

We believe that for "typical" spaces F_1 the assumptions of (i) and (ii) are satisfied. For example, this holds for Sobolev spaces $F_1 = W^r([0,1])$. Hence, the classes of tractable linear functionals and intractable linear functionals are both non-empty in general. The trouble is that for the fixed problem $I_1 f = \langle f, h_1 \rangle_{F_1}$ (like integration or weighted integration) we do not know whether the problem is tractable or intractable. We will return to this problem later when more powerful lower bounds will be established.

We now consider a sequence of linear tensor product functionals $I = \{I_d\}$ with $\|I_1\| > 1$. Obviously, then $\|I_d\| = \|I_1\|^d$ is exponentially large. Can we claim intractability of such problems for the absolute error criterion? Theorem 11.4 says that for some spaces F_1 the initial norm $\|I_1\|$ is irrelevant, and we have polynomial tractability for all I, including problems with $\|I_1\| > 1$. However, the spaces F_1 constructed in the proof of this theorem (and after the proof) are finite dimensional, and then $e(n,1) = 0$ for $n \ge \dim(F_1)$. It is natural to ask what happens if $\dim(F_1) = \infty$

and $e(n,1) > 0$ for all $n \in \mathbb{N}$. We strengthen the last assumption by assuming that $e(n,1) = \Omega(n^{-\alpha})$ for some positive (maybe arbitrarily large) α. This case covers many applications of smooth functions but not C^∞ or analytic functions. As we shall now prove, then we have intractability.

Theorem 11.7. *Consider a sequence of linear tensor product functionals $I = \{I_d\}$ with $\|I_1\| > 1$. Assume that for the univariate problem we have*

$$e(n,1) = \Omega(n^{-\alpha}) \quad \textit{for some } \alpha > 0.$$

Then I is intractable for the absolute error criterion.

Proof. We know that

$$e(n,d) = \inf_{x_1,x_2,\dots,x_n \in D_d} \sup_{f \in F_d,\, \|f\|_{F_d}=1,\, f(x_j)=0,\, j=1,2,\dots,n} |I_d(f)|.$$

For arbitrary points x_j, we can take

$$f(x) = f_n(x_1)\,\frac{h_1(x_2)}{\|h_1\|_{F_1}} \cdots \frac{h_1(x_d)}{\|h_1\|_{F_1}},$$

where f_n is chosen such that $f_n \in F_1$ with $\|f_n\|_{F_1} = 1$ and $f_n(x_{j,1}) = 0$ for $j = 1,2,\dots,n$ with $|I_1(f_n)| \ge \frac{1}{2}e(n,1)$. Here, $x_{j,1}$ is the first component of x_j. Clearly, $\|f\|_{F_d} = 1$ and

$$|I_d(f)| = |I_1(f_n)|\,\|I_1\|^{d-1} \ge \tfrac{1}{2}\,e(n,1)\,\|I_1\|^{d-1}.$$

Since $e(n,1) \ge 2c\,n^{-\alpha}$ for some positive c, we have

$$c\,n^{-\alpha}\|I_1\|^{d-1} \le e(n,d).$$

Hence, $e(n,d) \le \varepsilon$ implies

$$n(\varepsilon,d) \ge c^{1/\alpha}\,\varepsilon^{-1/\alpha}\,\|I_1\|^{(d-1)/\alpha}.$$

Therefore $n(\varepsilon,d)$ depends exponentially on d and we have intractability, as claimed. □

It is not clear what can happen if we relax the assumption $e(n,1) = \Omega(n^{-\alpha})$ and permit sequences $e(n,1)$ that go to zero faster than polynomially in n^{-1}. For instance, if $e(n,1) = q^n$ for $q \in (0,1)$ or even $e(n,1) = q_1^{q_2^n}$ for $q_1 \in (0,1)$ and $q_2 > 1$. This is the subject of the next open problem.

Open Problem 49.

Consider a sequence of linear tensor product functionals $I = \{I_d\}$ with $\|I_1\| > 1$ and $e(n,1) > 0$ for all $n \in \mathbb{N}$.

- Is it possible that such a problem is polynomially or weakly tractable for the absolute error criterion?
- If so, characterize such problems for which we have polynomial or weak tractability.

We now discuss the case $I = \{I_d\}$ with $\|I_1\| \le 1$ and $e(n,1) = \mathcal{O}(n^{-\alpha})$ for some positive α. We shall see in Chapter 15, that such problems with $\|I_1\| < 1$ are even strongly polynomially tractable and this can be achieved by the Smolyak or sparse grid algorithm. Assume then that $\|I_1\| = 1$ and $e(n,1) = \Omega(n^{-\alpha})$. We know from Corollary 11.3 that such problems are *not* strongly polynomially tractable. Note also that Theorem 11.4 is *not* applicable since $e(2,1) > 0$. For all known cases of such problems we have intractability. But it is not clear that this holds in general. This is our next open problem.

Open Problem 50.

Consider a sequence of linear tensor product functionals $I = \{I_d\}$ with $\|I_1\| = 1$ such that $e(n,1) = \Omega(n^{-\alpha})$ for some $\alpha > 0$.

- Is it possible that such a problem is polynomially or weakly tractable?
- If so, characterize such problems that are polynomially or weakly tractable.

11.4 Decomposable Kernels

In this section we introduce the notion of decomposable reproducing kernels. As we shall see, this notion will allow us to obtain lower bounds on the worst case minimal errors and to conclude intractability for certain tensor product linear functionals. This section is based on [221].

For $a^* \in \mathbb{R}$, define

$$D_{(0)} = \{x \in D_1 : x \le a^*\} \quad \text{and} \quad D_{(1)} = \{x \in D_1 : x \ge a^*\}.$$

Obviously $D_1 = D_{(0)} \cup D_{(1)}$ and $D_{(0)} \cap D_{(1)}$ is either empty or consists of one point a^* if $a^* \in D_1$. We also have

$$D_1^2 = D_{(0)} \times D_{(0)} \,\cup\, D_{(0)} \times D_{(1)} \,\cup\, D_{(1)} \times D_{(0)} \,\cup\, D_{(1)} \times D_{(1)}.$$

We say that the reproducing kernel K_1 is *decomposable* iff there exists a number $a^* \in \mathbb{R}$ such that $D_{(0)}$ and $D_{(1)}$ are nonempty and

$$K_1(x,t) = 0 \quad \text{for } (x,t) \in D_{(0)} \times D_{(1)} \,\cup\, D_{(1)} \times D_{(0)}, \tag{11.12}$$

or equivalently iff

$$K_1(x,t) = 0 \quad \text{for all } (x,t) \in D_1^2 \text{ such that } (x-a^*)(t-a^*) \le 0.$$

To stress the role of the number a^* for which the last equality holds, we sometimes say that K_1 is *decomposable at* a^*.

The essence of this property is that the function K_1 may take non-zero values only if x and t belong to the same quadrant $D_{(0)}^2 = D_{(0)} \times D_{(0)}$ or $D_{(1)}^2 = D_{(1)} \times D_{(1)}$.

Observe that if K_1 is decomposable at $a^* \in D_1$ then $K_1(\cdot, a^*) = 0$. This implies that all functions in F_1 vanish at a^*, since $f(a^*) = \langle f, K_1(\cdot, a^*)\rangle_{F_1} = 0$. For arbitrary d, since $K_d(x,t) = \prod_{j=1}^d K_1(x_j, t_j)$ for all $x, t \in D_d$ we also have

$$K_d(\cdot, x) = 0 \quad \text{if } \ x_j = a^* \text{ for some } j.$$

Then all functions from F_d vanish at $x \in D_d$ if $x_j = a^*$ for some j.

Furthermore, if K_1 is decomposable then $K_d(x,t) = 0$ if x and t belong to different quadrants. More precisely, for a Boolean vector $b \in \{0,1\}^d$, define

$$D_{(b)} = D_{(b_1)} \times D_{(b_2)} \times \cdots \times D_{(b_d)}.$$

Then $D_d = \bigcup_{b\in\{0,1\}^d} D_{(b)}$. If $x \in D_{(b_x)}$ and $t \in D_{(b_t)}$ for different Boolean vectors b_x and b_t then $K_d(x,t) = 0$. If $a^* \notin D_1$ then $D_{(b)}$'s are disjoint.

Again take $d = 1$. If K_1 is decomposable, then the space F_1 can be decomposed as the direct sum of Hilbert spaces $F_{(0)}$ and $F_{(1)}$ of univariate functions defined over $D_{(0)}$ and $D_{(1)}$ with reproducing kernels $K_1\big|_{D_{(0)}^2}$ and $K_1\big|_{D_{(1)}^2}$, respectively. Indeed, functions of the form $f = \sum_{j=1}^m \beta_j K_1(\cdot, t_j)$, for some m, real β_j and $t_j \in D_1$, are dense in F_1. Then for all $t \in D_1$ we have

$$f(t) = \sum_{(t,t_j)\in D_{(0)}^2} \beta_j K_1(t, t_j) + \sum_{(t,t_j)\in D_{(1)}^2} \beta_j K_1(t, t_j) = f_{(0)}(t) + f_{(1)}(t),$$

where $f_{(0)} := \sum_{j:\, t_j\in D_{(0)}} \beta_j K_1(\cdot, t_j) \in F_{(0)}$ and $f_{(1)} \sum_{j:\, t_j\in D_{(1)}} \beta_j K_1(\cdot, t_j) \in F_{(1)}$. It is easy to see that

$$f_{(0)} = f\big|_{D_{(0)}} \quad \text{and} \quad f_{(1)} = f\big|_{D_{(1)}}.$$

The spaces $F_{(0)}$ and $F_{(1)}$ can be treated as subspaces of the space F_1. Indeed, take for instance $f = \sum_{j=1}^k \beta_j K_1(\cdot, t_j)$ from $F_{(0)}$. Then $t_j \in D_{(0)}$ and $f(t)$ is well defined for *all* $t \in D_1$. Furthermore $f \in F_1$ and $f(t) = 0$ for $t \in D_{(1)}$. This also shows that

$$\|f\|_{F_{(0)}} = \|f\|_{F_1} \quad \text{for all } f \in F_{(0)}.$$

Obviously, $\|f\|_{F_{(1)}} = \|f\|_{F_1}$ for all $f \in F_{(1)}$. The subspaces $F_{(0)}$ and $F_{(1)}$ are orthogonal, since $\langle f, g\rangle_{F_1} = 0$ for all $f \in F_{(0)}$ and $g \in F_{(1)}$. Hence,

$$F_1 = F_{(0)} \oplus F_{(1)} \quad \text{and} \quad \|f\|_{F_1}^2 = \|f_{(0)}\|_{F_1}^2 + \|f_{(1)}\|_{F_1}^2 \text{ for all } f \in F_1.$$

For $d > 1$, we can similarly decompose the space F_d for decomposable K_1. For any Boolean vector b, we define $F_{(b)}$ as the Hilbert space of multivariate functions defined over D_d with the reproducing kernel

$$K_{d,(b)}(x,t) = \begin{cases} K_d(x,t) & \text{if } (x,t) \in D^2_{(b)}, \\ 0 & \text{if } (x,t) \notin D^2_{(b)}. \end{cases}$$

Clearly, for $f \in F_{(b)}$ we have $f(t) = 0$ for all $t \notin D_{(b)}$.

Then $F_{(b)}$ is a subspace of F_d and

$$\|f\|_{F_{(b)}} = \|f\|_{F_d} \quad \text{for all } f \in F_{(b)}.$$

Furthermore the spaces $F_{(b)}$ are mutually orthogonal and

$$F_d = \bigoplus_{b \in \{0,1\}^d} F_{(b)}.$$

If $f_{(b)} = f\big|_{D_{(b)}}$ denotes the restriction of the function f from F_d to the domain $D_{(b)}$, then $f_{(b)} \in F_{(b)}$. Moreover $f = \sum_{b \in \{0,1\}^d} f_{(b)}$ and

$$\|f\|^2_{F_d} = \sum_{b \in \{0,1\}^d} \|f_{(b)}\|^2_{F_d} \quad \text{for all } f \in F_d. \tag{11.13}$$

We now apply the last formula to the function h_d that defines the linear functional I_d, i.e., for $I_d(f) = \langle f, h_d \rangle_{F_d}$ for all $f \in F_d$. For $d = 1$, we have

$$h_1 = h_{1,(0)} + h_{1,(1)} \quad \text{with } \|h_1\|^2_{F_1} = \|h_{1,(0)}\|^2_{F_1} + \|h_{1,(1)}\|^2_{F_1}.$$

For $d > 1$, we use the fact that the function h_d is the tensor product of the function h_1 and obtain

$$h_{d,(b)}(x) = \prod_{j \in [1,d]:\, b_j = 0} h_{1,(0)}(x_j) \prod_{j \in [1,d]:\, b_j = 1} h_{1,(1)}(x_j).$$

From this we get

$$\|h_{d,(b)}\|_{F_d} = \|h_{1,(0)}\|^{0(b)}_{F_1} \, \|h_{1,(1)}\|^{1(b)}_{F_1}, \tag{11.14}$$

where

$$0(b) = |\{j \mid b_j = 0\}| \quad \text{and} \quad 1(b) = |\{j \mid b_j = 1\}|$$

are defined as the number of zeros and ones in the Boolean vector b. Obviously,

$$0(b) + 1(b) = d \quad \text{for any } b \in \{0,1\}^d.$$

The essence of (11.13) is that the problem of approximating

$$I_d(f) = \langle f, h_d \rangle_{F_d} = \sum_{b \in \{0,1\}^d} I_{d,(b)}(f)$$

can be decomposed into 2^d independent subproblems of approximating

$$I_{d,(b)}(f) = \langle f, h_{d,(b)} \rangle_{F_d} = \langle f_{(b)}, h_{d,(b)} \rangle_{F_d} \quad \text{for all } f \in F_d$$

whose initial errors are given by (11.14).

Assume for a moment that $a^* \notin D_1$. If we use n function values at some points $t_j \in D_d$ with $n < 2^d$ then the sample points t_j can belong to at most n different sets $D_{(b)}$. Hence, at least $2^d - n$ of the subproblems $I_{d,(b)}$ have to be approximated without knowing function values of $f_{(b)}$. The best we can then do is to approximate $I_{d,(b)}$ by zero. If all $h_{d,(b)}$ are non-zero this will lead to intractability for the normalized error criterion as well as for the absolute error criterion if $\|h_1\|_{F_1} \ge 1$. The proof of this fact is presented below also for the case when $a^* \in D_1$. As always,

$$e(n,d) = e^{\text{wor}}(n, I_d)$$

denotes the nth minimal worst case error of approximating I_d by n function values. For $n = 0$ we obtain the initial error. By

$$n(\varepsilon, d) = n^{\text{wor}}(\varepsilon, I_d),$$

we mean the information complexity for the absolute or normalized error criterion; it will be clear from the context which error criterion we are using.

Theorem 11.8. *If K_1 is a decomposable kernel then*

$$e(n,d) \ge \left(1 - n\,\alpha^d\right)_+^{1/2} e(0,d),$$

where

$$\alpha = \frac{\max\left(\|h_{1,(0)}\|_{F_1}^2, \|h_{1,(1)}\|_{F_1}^2\right)}{\|h_{1,(0)}\|_{F_1}^2 + \|h_{1,(1)}\|_{F_1}^2}, \quad \textit{with the convention } 0/0 = 1.$$

Assume that both $h_{1,(0)}$ and $h_{1,(1)}$ are non-zero, so that $\alpha \in [\frac{1}{2}, 1)$. Then

$$\lim_{d\to\infty} \frac{e\left(\lfloor C^d \rfloor, d\right)}{e(0,d)} = 1 \quad \textit{for all } C \in (1, 1/\alpha).$$

- *Consider the absolute error criterion, and assume that $\|h_1\|_{F_1} \ge 1$.*

  Then $I = \{I_d\}$ suffers from the curse of dimensionality and is intractable, since

$$n(\varepsilon, d) \ge \left(1 - \frac{\varepsilon^2}{\|h_1\|_{F_1}^{2d}}\right)\left(\frac{1}{\alpha}\right)^d \quad \textit{for all } \varepsilon \in \left(0, \|h_1\|_{F_1}^d\right).$$

- *Consider the absolute error criterion, and assume that* $\|h_1\|_{F_1} < 1$. *Then*

$$n(\varepsilon, d) = 0 \quad \textit{for all } \varepsilon > \|h_1\|_{F_1}^d,$$
$$n(\varepsilon, d) \ge \left(1 - \frac{\varepsilon^2}{\|h_1\|_{F_1}^{2d}}\right)\left(\frac{1}{\alpha}\right)^d \quad \textit{for all } \varepsilon \in \big(0, \|h_1\|_{F_1}^d\big).$$

Let $x \in (0,1)$ *and* $\varepsilon_d = x\|h_1\|_{F_1}^d$. *Then*

$$n(\varepsilon_d, d) \ge (1 - x^2)\, x^p\, \varepsilon_d^{-p}$$

with

$$p = \frac{\ln \alpha^{-1}}{\ln \|h_1\|_{F_1}^{-1}}.$$

If $I = \{I_d\}$ *is strongly polynomially tractable then its exponent satisfies*

$$p^{\text{str-wor}}(I) \ge p.$$

- *Consider the normalized error criterion.*

  Then $I = \{I_d\}$ *suffers from the curse of dimensionality and is intractable since*

$$n(\varepsilon, d) \ge (1 - \varepsilon^2)\left(\frac{1}{\alpha}\right)^d \quad \textit{for all } \varepsilon \in (0,1).$$

Proof. First of all note that the lower bound on $e(n,d)$ is trivial if $n\alpha^d \ge 1$. Hence, it is enough to prove it for n such that $n\alpha^d < 1$. Since $\alpha \ge \frac{1}{2}$ this means that $n < 2^d$.

We know that nonlinear algorithms and adaption do not help for approximating linear problems defined over Hilbert spaces in the worst case setting, see Chapter 4 of Volume I. So without loss of generality, we can consider linear algorithms. Take an arbitrary linear algorithm $Q_{n,d}(f) = \sum_{j=1}^n a_j f(z_j)$. We know that the square of the worst case error of $Q_{n,d}$ is given by

$$e^2(Q_{n,d}) = \|h_d\|_{F_d}^2 - 2\sum_{j=1}^n a_j h_d(z_j) + \sum_{i,j=1}^n a_i a_j K_d(z_i, z_j). \tag{11.15}$$

Note that although the sets $D_{(b)}$ do not have to be disjoint, the only elements that belong to their intersections are vectors with at least one component equal to a^*. For such vectors the value of h_d, as well as of K_d, is zero. Therefore without loss of generality we may assume that no component of any sample point z_j equals a^*. We then have

$$\sum_{j=1}^n a_j h_d(z_j) = \sum_{b \in \{0,1\}^d} \; \sum_{j \in [1,n]:\, z_j \in D_{(b)}} a_j h_d(z_j).$$

This and decomposability of the kernel K_1 yield

$$\sum_{i,j=1}^{n} a_i a_j K_d(z_i, z_j) = \sum_{b\in\{0,1\}^d} \sum_{i,j\in[1,n]^2:\, z_i,z_j\in D_{(b)}^2} a_i a_j K_d(z_i, z_j).$$

From (11.13) we can rewrite $e^2(Q_{n,d})$ as

$$e^2(Q_{n,d}) = \sum_{b\in\{0,1\}^d} e^2(b),$$

where

$$e^2(b) = \|h_{d,(b)}\|_{F_d}^2 - 2\sum_{j\in[1,n]:\, z_j\in D_{(b)}} a_j h_d(z_j) + \sum_{i,j\in[1,n]^2:\, z_i,z_j\in D_{(b)}^2} a_i a_j K_d(z_i, z_j).$$

Since $h_d(z_j) = h_{d,(b)}(z_j)$ for $z_j \in D_{(b)}$, we see that $e^2(b)$ is just the square of the worst case error of approximating the continuous linear functional

$$I_{d,(b)}(f) = \left\langle f_{(b)}, h_{d,(b)} \right\rangle_{F_{(b)}} = \left\langle f, h_{d,(b)} \right\rangle_{F_d}$$

by using sample points $z_j \in D_{(b)}$. Let

$$n_{(b)} = |\{ j \in [1,n] \mid z_j \in D_{(b)} \}|$$

be the cardinality of the sample points z_j in the set $D_{(b)}$. Obviously,

$$n = \sum_{b\in\{0,1\}^d} n_{(b)}.$$

If we define $e(n_{(b)}, F_{(b)})$ as the $n_{(b)}$th minimal error of approximating the functional $I_{d,(b)}$ then clearly $e^2(b) \ge e^2(n_{(b)}, F_{(b)})$. Hence,

$$e^2(Q_{n,d}) \ge \sum_{b\in\{0,1\}^d} e^2(n_{(b)}, F_{(b)}).$$

The last sum has 2^d terms and $n < 2^d$. Therefore at least $2^d - n$ numbers $n_{(b)}$ must be equal to zero. Observe also that $e^2(0, F_{(b)}) = \|h_{d,(b)}\|_{F_d}^2$ which, due to (11.14), is equal to $a_0^{0(b)} a_1^{1(b)}$ with $a_0 = \|h_{1,(0)}\|_{F_1}^2$ and $a_1 = \|h_{1,(1)}\|_{F_1}^2$. From this we conclude

$$\begin{aligned} e^2(Q_{n,d}) &\ge \sum_{b\in\{0,1\}^d:\, n_{(b)}=0} e^2(0, F_{(b)}) \\ &\ge \sum_{b\in\{0,1\}^d} e^2(0, F_{(b)}) - n \max_{b\in\{0,1\}^d} e^2(0, F_{(b)}). \end{aligned}$$

The first sum is just $\|h_d\|_{F_d}^2 = e^2(0, F_d) = (a_0 + a_1)^d$, whereas

$$\max_{b\in\{0,1\}^d} e^2(0, F_{(b)}) = \max_{b\in\{0,1\}^d} a_0^{0(b)} a_1^{1(b)} = [\max(a_0, a_1)]^d .$$

This proves that

$$e^2(Q_{n,d}) \ge \left[1 - n\left(\frac{\max(a_0, a_1)}{a_0 + a_1}\right)^d\right] e^2(0, F_d) = \left(1 - n\,\alpha^d\right) e^2(0, F_d).$$

Since this holds for any linear algorithm that uses n function values, we conclude that the same lower bound holds for $e^2(n, d)$, as claimed.

If both a_0 and a_1 are positive, then α is strictly less than one; moreover the initial error $e(0, d) = \|h_1\|_{F_1}^d$ is positive. If $n = n_d = \lfloor C^d \rfloor$ with $C > 1$ and $C < 1/\alpha$, then $n_d \alpha^d \le (C\alpha)^d$ goes to zero since $C\alpha < 1$. Since $e(n, d) \le e(0, d)$ for all n, the ratio $e(n_d, d)/e(n, 0)$ goes to 1 as d tends to infinity, as claimed.

Consider now the absolute error criterion. For $\|h_1\|_{F_1} \ge 1$, the bound on $n(\varepsilon, d)$ is straightforward. For $\|h_1\|_{F_1} < 1$, the initial error, which is also the error of the zero algorithm, is $\|h_1\|_{F_1}^d$ that is exponentially small in d. Thus $n(\varepsilon, d) = 0$ for all $\varepsilon > \|h_1\|_{F_1}^d$. For $\varepsilon \in (0, \|h_1\|_{F_1}^d)$, the bound on $n(\varepsilon, d)$ is straightforward. For $\varepsilon = \varepsilon_d$ we have $d = \ln(x/\varepsilon_d)/\ln(1/\|h_1\|_{F_1})$ and $\alpha^{-d} = (x/\varepsilon_d)^p$. Therefore $n(\varepsilon_d, d) = (1 - x^2)x^p \varepsilon_d^{-p}$, as claimed.

Assume now that the problem I is strongly polynomially tractable. Then

$$n(\varepsilon_d, d) \le C_\tau\, \varepsilon_d^{-\tau} \quad \text{for all } \varepsilon \in (0, 1) \text{ and } d \in \mathbb{N},$$

where $\tau > p^{\text{str-wor}}$. For a fixed x and d tending to infinity, we conclude that $\tau \ge p$. For τ tending to $p^{\text{str-wor}}$ we conclude the bound $p^{\text{str-wor}} \ge p$, as claimed.

Consider the normalized error criterion. We now want to reduce the initial error by a factor ε. Therefore $n(\varepsilon, d) \ge (1-\varepsilon^2)\alpha^{-d}$, as claimed. This completes the proof. □

The essence of Theorem 11.8 is intractability for the normalized error criterion and for the absolute error criterion with $\|h_1\|_{F_1} \ge 1$. Observe that for the absolute error criterion with $\|h_1\|_{F_1} > 1$ we can even take an exponentially large ε, say, $\varepsilon = \beta\|h_1\|_{F_1}^d$ for some $\beta \in (0, 1)$; we will still need to compute at least exponentially many function values to solve the problem since,

$$n\big(\beta\|h_1\|_{F_1}^d, d\big) \ge (1 - \beta^2)\alpha^{-d}.$$

From Chapter 10, we know that if $\|h_1\|_{F_1} < 1$ then the problem I may be strongly polynomially tractable depending on h_1 as well as on the space F_1. In particular, this holds if $\|h_1\|_{F_1}^* < \infty$, see Corollary 10.16. Clearly, this cannot be true for all h_1 since the assumption $\|h_1\|_{F_1} < 1$ is not enough to guarantee strong polynomial tractability; we know that even the univariate problem, $d = 1$, may cause intractability. But if we assume that the minimal worst case errors for $d = 1$ behave like $\mathcal{O}(n^{-k})$ for some positive k, then strong polynomial tractability indeed holds and can be achieved by the Smolyak or sparse grid algorithm, see [329], and this will be reported in Chapter 15.

The lower bound on the exponent of strong polynomial tractability is quite loose since it does not depend on, in particular, how hard the problem is for $d = 1$. Nevertheless, this bound shows that the exponent may be arbitrarily large. For example, this is the case if $\alpha = \frac{1}{2}$ and $\|h_1\|_{F_1}$ is close to one.

As already remarked in the proof of Theorem 11.8, the number α belongs to $[\frac{1}{2}, 1]$. In fact, α can take any value from this interval as we shall see later in examples. For $\alpha = 1$, Theorem 11.8 is trivial, since the lower bound of $e(n, F_d)$ is zero for $n \geq 1$. This should come as no surprise, since there exist trivial problems with $\alpha = 1$. For instance, take $I_1(f) = f(a)$ for an arbitrary $a \in D_1$. Then $I_d(f) = f(a, a, \dots, a)$ can be solved exactly using one function value. That means that $e(n, d) = 0$ for all $n \geq 1$. For this problem we have $h_1 = K_1(\cdot, a)$ and $h_{1,(0)} = 0$ if $a \leq a^*$, and $h_{1,(1)} = 0$ otherwise. In either case, $\alpha = 1$.

If α is smaller than one but is close to one then the exponential dependence on d becomes less drastic, since d must be sufficiently large to be hurt by α^{-d}. The largest lower bound on $e(n, d)$ is when $\alpha = \frac{1}{2}$. We now show that this holds for decomposable K_1 when the set D_1, the kernel K_1 and the function h_1 are *symmetric* with respect to a^*. That is, when

$$
\begin{gathered}
a^* - x \in D_1 \;\Rightarrow\; a^* + x \in D_1, \\
K_1(a^* - x, a^* - t) = K_1(a^* + x, a^* + t), \\
h_1(a^* - x) = h_1(a^* + x).
\end{gathered}
$$

Indeed, symmetry of K_1 implies that the spaces $F_{(0)}$ and $F_{(1)}$ are essentially the same. More precisely, define the mapping $P(x) = 2a^* - x$ for $x \in D_1$. Then $P^2(x) = x$ and $P(D_{(0)}) = D_{(1)}$, $P(D_{(1)}) = D_{(0)}$. Then $f \in F_{(0)}$ iff $fP \in F_{(1)}$ and $\|f\|_{F_{(0)}} = \|fP\|_{F_{(1)}}$. For symmetric functions $f \in F_1$ we have

$$
\|f\big|_{D_{(0)}}\|_{F_{(0)}} = \|f\big|_{D_{(1)}}\|_{F_{(1)}}.
$$

Since h_1 is symmetric we have

$$
\|h_{1,(0)}\|_{F_1} = \|h_{1,(1)}\|_{F_1} = \|h_1\|_{F_1}/\sqrt{2},
$$

and for non-zero h_1 we have $\alpha = \frac{1}{2}$, as claimed. We summarize this in a corollary.

Corollary 11.9. *Let decomposable K_1, D_1 and h_1 be symmetric with respect to $a^* \in \mathbb{R}$. Then*

$$
e(n, d) \geq \left(1 - n\, 2^{-d}\right)_+^{1/2} e(0, d).
$$

The problem $I = \{I_d\}$ is intractable for the absolute error criterion if $e(0, d) \geq 1$, and for the normalized error criterion if $e(0, d) > 0$. For both error criteria we have

$$
n(\varepsilon, F_d) \geq (1 - \varepsilon^2)\, 2^d \quad \textit{for all } \varepsilon \in (0, 1).
$$

We now illustrate Theorem 11.8 and Corollary 11.9 by several examples. In particular, we show that the estimates of Theorem 11.8 are sharp in general.

11.4.1 Example: Weighted Integration

Let $r \in \mathbb{N}$. We take $F_1 = W_0^r(\mathbb{R})$ as the Sobolev space of functions defined over $\mathbb{R}$ whose $(r-1)$st derivatives are absolutely continuous, with the rth derivatives belonging to $L_2(\mathbb{R})$ and their derivatives up to the $(r-1)$st at zero being zero. That is, we now have $D_1 = \mathbb{R}$ and

$$F_1 = \{f : \mathbb{R} \to \mathbb{R} \mid f^{(j)}(0) = 0,\ j \in [0, r-1],\ f^{(r-1)} \text{ abs. cont. and } f^{(r)} \in L_2(\mathbb{R})\}.$$

The inner product of F_1 is given as

$$\langle f, g \rangle_{F_1} = \int_{\mathbb{R}} f^{(r)}(t) g^{(r)}(t)\, \mathrm{d}t.$$

It is known, and not hard to check, that this Hilbert space has the reproducing kernel

$$K_1(x,t) = 1_M(x,t) \int_0^\infty \frac{(|t| - u)_+^{r-1} (|x| - u)_+^{r-1}}{[(r-1)!]^2}\, \mathrm{d}u,$$

where 1_M is the characteristic (indicator) function of the set $M = \{(x,t) \mid xt \geq 0\}$. For $r = 1$, we have

$$K_1(x,t) = 1_M(x,t)\ \min(|t|, |x|).$$

For $r \geq 1$, observe that this kernel is *decomposable* at $a^* = 0$. Indeed,

$$D_{(0)} = \{x \in \mathbb{R} \mid x \leq 0\} \quad \text{and} \quad D_{(1)} = \{x \in \mathbb{R} \mid x \geq 0\},$$

and $M = D_{(0)}^2 \cup D_{(1)}^2$. The kernel K_1 is also symmetric since $K_1(x,t) = K_1(-x,-t)$.

For $d > 1$, we obtain

$$F_d = W_0^{r,r,\ldots,r}(\mathbb{R}^d) = W_0^r(\mathbb{R}) \otimes \cdots \otimes W_0^r(\mathbb{R})$$

as the d-fold tensor product of $W_0^r(\mathbb{R})$. Hence, F_d is the Sobolev space of smooth functions defined over $D_d = \mathbb{R}^d$ such that $D^\alpha f(x) = 0$ if at least one component of x is zero for any multi-index $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_d]$ with integers $\alpha_j \in [0, r-1]$. As always, D^α is the partial differential operator, $D^\alpha f = \partial^{|\alpha|} f / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d$. The inner product of F_d is given by

$$\langle f, g \rangle_{F_d} = \int_{\mathbb{R}^d} D^{[r,r,\ldots,r]} f(x)\, D^{[r,r,\ldots,r]} g(x)\, \mathrm{d}x.$$

Consider the *weighted integration* problem. That is,

$$I_1(f) = \int_{\mathbb{R}} \varrho(t)\, f(t)\, \mathrm{d}t$$

for some measurable non-zero weight function $\varrho : \mathbb{R} \to \mathbb{R}_+$. It is easy to check that I_1 is a continuous linear functional iff the function

$$h_1(x) = \int_{\mathbb{R}} \varrho(t) K_1(x,t)\, \mathrm{d}t$$

belongs to F_1, which holds iff

$$\int_{\mathbb{R}^2} \varrho(t)\varrho(x) K_1(x,t)\,\mathrm{d}t\,\mathrm{d}x < \infty. \tag{11.16}$$

It is easy to check that $K_1(x,t) = \mathcal{O}(|t\,x|^{r-1/2})$, and (11.16) holds if

$$\int_{\mathbb{R}} \varrho(t)\,|t|^{r-1/2}\,\mathrm{d}t < \infty.$$

The last condition imposes a restriction on the behavior of the weight ϱ at infinity. If (11.16) holds, then

$$I_1(f) = \langle f, h_1\rangle_{F_1} \quad \text{and} \quad \|I_1\| = \|h_1\|_{F_1} = \left(\int_{\mathbb{R}^2} \varrho(t)\varrho(x) K_1(x,t)\,\mathrm{d}t\,\mathrm{d}x\right)^{1/2} < \infty.$$

We also have

$$h_{1,(0)}(x) = \int_{-\infty}^{0} \varrho(t) K_1(x,t)\,\mathrm{d}t \quad \text{and} \quad h_{1,(1)}(x) = \int_{0}^{\infty} \varrho(t) K_1(x,t)\,\mathrm{d}t.$$

For $d > 1$, we have

$$I_d(f) = \int_{\mathbb{R}^d} \varrho_d(t)\, f(t)\,\mathrm{d}t \quad \text{with } \varrho_d(t) = \varrho(t_1)\varrho(t_2)\cdots\varrho(t_d).$$

We are ready to apply Theorem 11.8 for weighted integration. Observe that $D_{(0)} = \mathbb{R}_- \cup \{0\}$ and $D_{(1)} = \mathbb{R}_+ \cup \{0\}$. If the weight ϱ does not vanish (in the L_2 sense) over $\mathbb{R}_-$ and $\mathbb{R}_+$ then the norms of $h_{1,(0)}$ and $h_{1,(1)}$ are positive. Hence, weighted integration is intractable for the normalized error criterion and for the absolute error criterion if $\|h_1\|_{F_1} \ge 1$. In particular, if we take a nonzero symmetric ϱ, i.e., $\varrho(t) = \varrho(-t)$, then Corollary 11.9 holds and $\alpha = \frac{1}{2}$.

This is the case for *Gaussian integration*, since

$$\varrho(t) = (2\pi)^{-1/2} \exp(-t^2/2)$$

is symmetric. Hence, Gaussian integration is *intractable* for the normalized error criterion since

$$n(\varepsilon, F_d) \ge (1-\varepsilon^2)\,2^d \quad \text{for all } \varepsilon \in (0,1).$$

We stress that this intractability result holds independently of the assumed smoothness of functions, i.e., this holds even if r is arbitrarily large.

We now consider Gaussian integration for the absolute error criterion. As we know,

$K_1(x,t) \le \sqrt{K_1(x,x)}\sqrt{K_1(t,t)}$, and therefore we have

$$\begin{aligned}
\|h_1\|_{F_1} &= \int_{\mathbb{R}^2} \varrho(t)\varrho(x)K_1(x,t)\,\mathrm{d}t\,\mathrm{d}x \le \int_{\mathbb{R}^2} \varrho(t)\varrho(x)\sqrt{K_1(x,x)}\sqrt{K_1(t,t)}\,\mathrm{d}t\,\mathrm{d}x \\
&= \left(\int_{\mathbb{R}} \varrho(t)\sqrt{K_1(t,t)}\,\mathrm{d}t\right)^2 \le \int_{\mathbb{R}} \varrho(t)K_1(t,t)\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\,\frac{1}{((r-1)!)^2}\int_0^\infty e^{-t^2/2}\int_0^t (t-u)^{2(r-1)}\,\mathrm{d}u\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\,\frac{1}{((r-1)!)^2(2r-1)}\int_0^\infty e^{-t^2/2}t^{2r-1}\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\,\frac{2^{r-1}}{(2r-1)(r-1)!} < 1.
\end{aligned}$$

Hence, $\|h_1\|_{F_1} < 1$ for all $r \ge 1$, which opens the possibility that Gaussian integration might be strongly polynomially tractable. In fact, it *is* strongly polynomially tractable since we can apply Corollary 10.9 with

$$C(K_1,\varrho_1) = \int_{\mathbb{R}} \varrho(t)K_1(t,t)\,\mathrm{d}t < 1,$$

as shown above. Then (10.14) states that the exponent of strong polynomial tractability is at most 2. More precisely, we have

$$e(n,d) \le n^{-1/2}\left(\int_{\mathbb{R}} \varrho(t)K_1(t,t)\,\mathrm{d}t\right)^{d/2} \le n^{-1/2} \tag{11.17}$$

which leads to $n(\varepsilon,d) \le \lceil \varepsilon^{-2} \rceil$.

We summarize these properties in the following corollary.

Corollary 11.10. *Consider Gaussian integration for the Sobolev space*

$$F_d = W_0^{r,r,\dots,r}(\mathbb{R}^d),$$

with $r \ge 1$, in the worst case setting.

- *For the normalized error criterion, Gaussian integration suffers from the curse of dimensionality and is intractable no matter how large r we have, and*

$$n(\varepsilon, F_d) \ge (1-\varepsilon^2)2^d \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

- *For the absolute error criterion, Gaussian integration is strongly polynomially tractable with exponent at most 2, and*

$$n(\varepsilon, F_d) \le \lceil \varepsilon^{-2} \rceil \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } d \in \mathbb{N}.$$

Let us check the lower bound on the exponent given in Theorem 11.8. For $r = 1$, it is not so hard to check that

$$\|h_1\|_{F_1}^2 = \frac{1}{\pi} \int_{\mathbb{R}_+^2} e^{-(t^2+x^2)/2} \min(t,x)\, \mathrm{d}t\, \mathrm{d}x = \frac{\sqrt{2}-1}{\sqrt{\pi}}.$$

Theorem 11.8 states that

$$p^{\text{str-wor}} \ge \frac{\ln 2}{\ln 1/\|h_1\|_{F_1}} = \frac{2\ln 2}{\ln \sqrt{\pi}/(\sqrt{2}-1)} = 0.953606\ldots.$$

The last bound is poor, since we know, see e.g., [333], that for $d = 1$ we have $e(n,1) = \Theta(n^{-1})$ and therefore $p^{\text{str-wor}} \ge 1$. The actual value of $p^{\text{str-wor}}$ is not known, which leads us to the next open problem.

Open Problem 51.

- Find the exponent of strong polynomial tractability of Gaussian integration for the absolute error criterion in the worst case setting.

We now take the constant weight ϱ over the interval $[a,b]$, where $a \le 0 < b$. More specifically, we assume that $\varrho(t) = 1$ for $t \in [a,b]$ and $\varrho(t) = 0$ otherwise. For simplicity, we only consider $r = 1$. Then

$$h_{1,(0)}(x) = \int_a^0 K_1(x,t)\,\mathrm{d}t = \left[a \max(x,a) - \tfrac{1}{2} \max(x,a)^2\right]_+,$$
$$h_{1,(1)}(x) = \int_0^b K_1(x,t)\,\mathrm{d}t = \left[b \min(x,b) - \tfrac{1}{2} \min(x,b)^2\right]_+.$$

From this we conclude that

$$\|h_{1,(0)}\|_{F_1}^2 = |a|^3/3 \quad \text{and} \quad \|h_{1,(1)}\|_{F_1}^2 = b^3/3.$$

Hence,

$$\alpha = \frac{\max(|a|^3, b^3)}{|a|^3 + b^3}.$$

Taking $b = 1$ and $|a| = t \in [0,1]$ we obtain $\alpha = \alpha(t) = 1/(1+t^3)$, which varies continuously with t and takes all values from $[\frac{1}{2}, 1]$. This shows that α in Theorem 11.8 can be an arbitrary number from $[\frac{1}{2}, 1]$. Obviously, for $a = 0$ Theorem 11.8 is trivial. This means that we cannot yet claim anything about tractability of the problem

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \text{for all } f \in F_d.$$

We now take $a = -b$ so that $\alpha = \frac{1}{2}$, and as before $r = 1$. Then the problem $I = \{I_d\}$ is always intractable for the normalized error criterion.

Consider then the absolute error criterion. Note that the initial error is now

$$e(0,d) = \left(\frac{2b^3}{3}\right)^{d/2}.$$

Hence for $b^3 \geq \frac{3}{2}$ the initial error is at least one and therefore $I = \{I_d\}$ is intractable.

Assume then that $b^3 < \frac{3}{2}$. Theorem 11.8 states that we can have even strong polynomial tractability. We now use an upper bound on $e(n,d)$ from Theorem 10.10 based on Plaskota, Wasilkowski and Zhao [248], which states that

$$e(n,d) \leq \frac{1}{\sqrt{n}} \int_{D_d} \varrho_d(t) \sqrt{K_d(t,t)}\, \mathrm{d}t. \tag{11.18}$$

In our case, the last inequality becomes

$$e(n,d) \leq \frac{1}{\sqrt{n}} \left(\int_{-b}^{b} \sqrt{K_1(t,t)}\, \mathrm{d}t\right)^d = \frac{1}{\sqrt{n}} \left(\frac{16\, b^3}{9}\right)^{d/2}.$$

Hence for $b^3 \leq \frac{9}{16}$ we have

$$n(\varepsilon,d) \leq \lceil \varepsilon^{-2} \rceil \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N},$$

which means strong polynomial tractability with $p^{\text{str-wor}} \leq 2$. From Theorem 11.8 we obtain for $b^3 = \frac{9}{16}$ that

$$p^{\text{str-wor}} \geq \frac{2 \ln 2}{\ln 8/3} = 1.41339\ldots.$$

The exact value of $p^{\text{str-wor}}$ is not known.

The case $b^3 \in (\frac{9}{16}, \frac{3}{2})$ has not been studied and it is not clear if $I = \{I_d\}$ is strongly polynomially or polynomially tractable for the absolute error criterion. This leads us to the next open problem.

Open Problem 52.

- Consider weighted integration for $r = 1$ and $\varrho = 1_{[-b,b]}$ for the absolute error criterion in the worst case setting. Find all b for which strong polynomial tractability holds and for such b determine its exponent.

11.4.2 Example: Uniform Integration

We now define $D_1 = [0,1]$ and take $F_1 = W_a^1([0,1])$ as the Sobolev space of absolutely continuous functions whose first derivatives are in $L_2([0,1])$ and whose function values are zero at the point a of the interval $[0,1]$. That is,

$$F_1 = \{f : [0,1] \to \mathbb{R} \mid f(a) = 0,\ f \text{ abs. cont. and } f' \in L_2([0,1])\}$$

with the inner product $\langle f, g\rangle_{F_1} = \int_0^1 f'(t)g'(t)\,\mathrm{d}t$. As we know from Chapter 9, this Hilbert space is related to the L_2 discrepancy anchored at a, and has the reproducing kernel

$$K_1(x,t) = \tfrac{1}{2}(|x-a| + |t-a| - |x-t|),$$

which can be rewritten as

$$K_1(x,t) = 1_M(x,t)\cdot \min(|x-a|, |t-a|),$$

where $M = [0,a]\times[0,a] \cup [a,1]\times[a,1]$.

Hence, the kernel K_1 is *decomposable* at $a^* = a$. It is symmetric only if $a = \frac{1}{2}$. We have $D_{(0)} = [0,a]$ and $D_{(1)} = [a,1]$.

For $d > 1$, we obtain

$$F_d = W_a^{1,1,\dots,1}([0,1]^d) = W_a^1([0,1]) \otimes \cdots \otimes W_a^1([0,1]), \quad d \text{ times},$$

as the Sobolev space of smooth functions f defined over $D_d = [0,1]^d$ such that $f(x) = 0$ if at least one component of x is a. The inner product of F_d is given by

$$\langle f, g\rangle_{F_d} = \int_{[0,1]^d} \frac{\partial^d}{\partial x_1\cdots\partial x_d} f(x) \frac{\partial^d}{\partial x_1\cdots\partial x_d} g(x)\,\mathrm{d}x.$$

Consider the *uniform integration* problem. That is,

$$I_1(f) = \int_0^1 f(t)\,\mathrm{d}t.$$

It is easy to compute

$$h_{1,(0)}(x) = \int_0^a \min(a-x, a-t)\,\mathrm{d}t = \tfrac{1}{2}\,(a-x)(a+x) \qquad \text{for all } x \in [0,a],$$

$$h_{1,(1)}(x) = \int_a^1 \min(x-a, t-a)\,\mathrm{d}t = \tfrac{1}{2}\,(x-a)(2-a-x) \quad \text{for all } x \in [a,1].$$

Furthermore,

$$\|h_{1,(0)}\|_{F_1}^2 = \tfrac{1}{3}\,a^3 \quad \|h_{1,(1)}\|_{F_1}^2 = \tfrac{1}{3}\,(1-a)^3.$$

Hence, we have

$$\alpha = \frac{\max(a^3, (1-a)^3)}{a^3 + (1-a)^3}.$$

From Theorem 11.8 we conclude that the problem $I = \{I_d\}$ is intractable for the normalized error criterion if $a \in (0,1)$. Observe also that if a is close to zero then $\alpha^{-1} = 1 + a^3 + O(a^4)$ is barely larger than 1. This means that although α^{-d} goes exponentially fast to infinity with d we must take really large d to get large α^{-d}.

For the absolute error criterion, we have

$$\|h_1\|_{F_1}^2 = \frac{a^3 + (1-a)^3}{3} \le \frac{1}{3}.$$

From (11.17) or (11.18) we easily conclude that $n(\varepsilon, d) \le \lceil \varepsilon^{-2} \rceil$, so $I = \{I_d\}$ is strongly polynomially tractable for all $a \in [0,1]$ with the exponent $p^{\text{str-wor}} \le 2$. Again, the lower bound on $p^{\text{str-wor}}$ is poor. For $d = 1$, we know that $e(n,1) = \Theta(n^{-1})$, and therefore $p^{\text{str-wor}} \ge 1$. The exact value of $p^{\text{str-wor}}$ is not known. For $a = 0$ or $a = 1$, the problem of finding the exponent is presented in Chapter 9 as Open Problem 36. We now generalize this open problem to arbitrary a.

Open Problem 53.

- Find the exponent of strong polynomial tractability of uniform integration for the absolute error criterion in the worst case setting for arbitrary $a \in [0,1]$.

11.4.3 Example: Centered Discrepancy

We now specialize the previous example by taking $a = \frac{1}{2}$, which corresponds to centered discrepancy, see Chapter 9. Then the kernel K_1 and the function h_1 are symmetric. Corollary 11.9 now applies with $\alpha = \frac{1}{2}$ and $\|h\|_{F_1} = 12^{-1/2}$. In this case, the worst case error of linear algorithms

$$Q_{n,d}(f) = \sum_{j=1}^{n} a_j f(z_j)$$

is given by

$$e^2(Q_{n,d}) = \int_{[0,1]^d} \Big| \prod_{j=1}^{d} \min(x_j, 1-x_j) - \sum_{j=1}^{n} a_j \cdot 1_{J(b(x),x)}(z_j) \Big|^2 \, dx, \tag{11.19}$$

where $J(b(x),x)$ is the cube generated by x and the vertex $b(x)$ of $[0,1]^d$ that is closest to x in the sup-norm. That is, $x \in D_{(b)}$ iff $b(x) = b$ for almost all $x \in [0,1]^d$. Essentially the same formulas were presented by Hickernell [118], who considered spaces similar to F_d without assuming the condition $f(\frac{1}{2}) = 0$ and considered algorithms with $a_j = n^{-1}$.

For the space F_d with the condition $f(\frac{1}{2}) = 0$ we denote the centered discrepancy by $\tilde{d}_2^c(n,d)$. In the next section we remove this condition and consider the centered discrepancy $d_2^c(Q_{n,d})$ (without the tilde), as originally studied by Hickernell [118]. The relation between uniform integration for $a = \frac{1}{2}$ and the centered discrepancy means that

$$\tilde{d}_2^c(n,d) = e(Q_{n,d}).$$

The minimal centered discrepancy is defined as

$$\tilde{d}_2^c(n,d)^2 = \inf_{\substack{a_j, z_j \\ j=1,2,\dots,n}} \int_{[0,1]^d} \Big| \prod_{j=1}^{d} \min(x_j, 1-x_j) - \sum_{j=1}^{n} a_j \cdot 1_{J(b(x),x)}(z_j) \Big|^2 \, dx. \tag{11.20}$$

Corollary 11.9 states that

$$\tilde{d}_2^c(n,d) \geq \left(1 - n\,2^{-d}\right)_+^{1/2} \tilde{d}_2^c(0,d) \quad \text{with } \tilde{d}_2^c(0,d) = 12^{-d/2}. \tag{11.21}$$

The centered discrepancy and uniform integration for $a = \frac{1}{2}$ may be also considered for the space $L_q([0,1]^d)$ with $q \in [1,\infty]$. More precisely, for $d = 1$ we take the space $F_{1,q}$ as the Sobolev space of absolutely continuous functions whose first derivatives are in $L_q([0,1])$ and that vanish at $\frac{1}{2}$. The norm in $F_{1,q}$ is given by

$$\|f\|_{F_{1,q}} = \left(\int_0^1 |f'(t)|^q \,\mathrm{d}t\right)^{1/q} \quad \text{for } q < \infty,$$

and

$$\|f\|_{F_{1,\infty}} = \operatorname*{ess\,sup}_{t\in[0,1]} |f'(t)| \quad \text{for } q = \infty.$$

For $d > 1$, the space $F_{d,q}$ is taken as a d-fold tensor product of $F_{1,q}$. Then functions from $F_{d,q}$ vanish at x whenever at least one component of x is $\frac{1}{2}$ and their norm is

$$\|f\|_{F_{d,q}} = \|D^{\vec{1}} f\|_{L_q([0,1]^d} = \left(\int_{[0,1]^d} |D^{\vec{1}} f(x)|^q \,\mathrm{d}x\right)^{1/q},$$

where $\vec{1} = [1,1,\dots,1]$. From

$$I_d(f) - Q_{n,d}(f) = \int_{[0,1]^d} D^{\vec{1}} f(t)\, D^{\vec{1}}\Big(h_d - \sum_{i=1}^n a_i K_d(\cdot, z_i)\Big)(t)\,\mathrm{d}t$$

we conclude that

$$e(Q_{n,d}) := \sup_{f\in F_{d,q}:\,\|f\|_{F_{d,q}}\leq 1} \big|I_d(f) - Q_{n,d}(f)\big| = \tilde{d}_p^c(Q_{n,d}),$$

where $1/p + 1/q = 1$ and $\tilde{d}_p^c(Q_{n,d})$ is the centered p-discrepancy given by

$$\tilde{d}_p^c(Q_{n,d}) = \left(\int_{[0,1]^d} \Big|\prod_{j=1}^d \min(x_j, 1-x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b(x),x)}(z_i)\Big|^p \,\mathrm{d}x\right)^{1/p}.$$

If $q = 1$ then $p = \infty$ and, as usual, the integral is replaced by the essential supremum in the formula above.

Let

$$e(n,d,q) = \tilde{d}_p(n,d)$$

denote the minimal error or, equivalently, the minimal centered p-discrepancy that can be achieved by using n function values. The initial error, or the initial centered discrepancy, is now given by

$$e(0,d,q) = \tilde{d}_p^c(0,d) = \frac{2^{-d}}{(p+1)^{d/p}}$$

for $q > 1$, and $e(0, d, 1) = 2^{-d}$. Similarly, let $n(\varepsilon, d, q)$ denote the information complexity for the absolute or normalized error criterion, i.e., the smallest n for which $e(n, d, q) \le \varepsilon\,\mathrm{CRI}_d$ with $\mathrm{CRI}_d = 1$ for the absolute error criterion, and $\mathrm{CRI}_d = e(0, d, q)$ for the normalized error criterion.

Hence, for all values of q, the initial centered discrepancy is at most 2^{-d}. We are ready to prove the following corollary, which easily follows from the proof of Theorem 11.8.

Corollary 11.11. *For $n < 2^d$ and $p < \infty$, we have*

$$\tilde{d}_p^c(n, d) \ge \left(1 - n\,2^{-d}\right)^{1/p} \tilde{d}_p^c(0, d).$$

Hence, uniform integration is intractable for the normalized error criterion since

$$n(\varepsilon, d, q) \ge (1 - \varepsilon^p) 2^d \quad \textit{and} \quad \lim_{d\to\infty} \frac{\tilde{d}_p^c(\lfloor C^d \rceil, d)}{\tilde{d}_p^c(0, d)} = 1 \quad \textit{for all } C \in (1, 2).$$

Proof. As in the proof of Theorem 11.8, take a Boolean vector $b = [b_1, \dots, b_d] \in \{0, 1\}^d$ and an algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i f(z_i)$. The worst case error of $Q_{n,d}$ is the centered discrepancy $\tilde{d}_p^c(Q_{n,d})$. The formula for $\tilde{d}_p^c(Q_{n,d})$ can be written as

$$\tilde{d}_p^c(Q_{n,d})^p = \sum_{b\in\{0,1\}^d} \int_{D_{(b)}} \left| \prod_{j=1}^d \min(x_j, 1 - x_j) - \sum_{i=1}^n a_i \cdot 1_{J(b,x)}(z_i)) \right|^p \mathrm{d}x.$$

Observe that for each sample point z_j and for each fixed x, there is at most one b such that $z_j \in J(b, x)$. Hence, at least $2^d - n$ terms in the last sum with respect to b have zero contribution from the algorithm $Q_{n,d}$. Since

$$\int_{D_{(b)}} \prod_{j=1}^d \min(x_j, 1 - x_j)^p \mathrm{d}x = \left(\frac{2^{-p-1}}{p+1}\right)^d$$

does not depend on b, we obtain

$$\tilde{d}_p^c(Q_{n,d})^p \ge (2^d - n)\, 2^{-d} \left(\frac{2^{-p}}{p+1}\right)^d = \left(1 - n\,2^{-d}\right) \tilde{d}_p^c(0, d)^p,$$

as claimed. The formula for $n(\varepsilon, d)$ and the value for the limit easily follow from the lower bound on $\tilde{d}_p^c(n, d)$. This completes the proof. □

Observe that for $q = 1$ we have $p = \infty$, and since $\tilde{d}_p^c(n, d)$ is a nondecreasing function of p we have

$$\tilde{d}_\infty^c(n, d) = \tilde{d}_\infty^c(0, d) = 2^{-d} \quad \text{for all } n < 2^d.$$

Again this means that uniform integration is intractable for the normalized error criterion since

$$n(\varepsilon, d, 1) \geq 2^d \quad \text{for all } \varepsilon \in (0, 1).$$

It is known that $\tilde{d}_\infty^c(n, d)$ goes to zero at least like $n^{-1}(\log n)^{d-1}$. However, in view of the previous property, we must wait exponentially long in d to enjoy this rate of convergence.

The absolute error criterion has not been yet thoroughly studied for $q \neq 2$. However for $q \geq 2$, we can use the results for $q = 2$, which state that we have strong polynomial tractability and this can be achieved by the Smolyak or sparse grid algorithm, see Chapter 15. Hence, uniform integration is also strongly polynomially tractable for $q \geq 2$.

For $q = 1$, we obtain a classical discrepancy problem, which is not harder than the extreme discrepancy where we take all cubes. But we know that even the extreme discrepancy is polynomially tractable, and its information complexity is at most linear in d. For $q \in (1, 2)$, we can use bounds for $q = 1$ and conclude polynomial tractability. However, it is not clear if we can have strong polynomial tractability in this case, which leads us to the next open problem.

Open Problem 54.

- Verify strong polynomial tractability of uniform integration (or the centered discrepancy) for the space $F_{d,q}$ with $q \in (1, 2)$ for the absolute error criterion in the worst case setting.

11.4.4 Example: Two Function Values

We now show that the estimate of Theorem 11.8 is, in general, sharp. This will be done for a seemingly simple problem I_1 defined by only two function values. More precisely, consider the space F_1 with symmetric D_1 around $a^* = 0$ such that $\{-1, 1\} \subseteq D_1$. Let the reproducing kernel K_1 of F_1 be symmetric and decomposable at $a^* = 0$. We assume that $K_1(1, 1) > 0$. Define

$$I_1(f) = f(-1) + f(1).$$

Then $h_1(x) = K_1(x, -1) + K_1(x, 1)$, and

$$h_{1,(0)}(x) = K_1(x, -1) \quad \text{with } \|h_{1,(0)}\|_{F_1}^2 = K_1(-1, -1) = K_1(1, 1),$$
$$h_{1,(1)}(x) = K_1(x, 1) \quad \text{with } \|h_{1,(1)}\|_{F_1}^2 = K_1(1, 1).$$

We show that

$$e(n, d) = \left(1 - n\, 2^{-d}\right)_+^{1/2} e(0, d).$$

By Theorem 11.8, it is enough to find a matching upper bound, that is, an algorithm such that $e(Q_{n,d}) \leq \left(1 - n\, 2^{-d}\right)_+^{1/2} e(0, d)$. Observe that

$$I_d(f) = \sum_{b \in \{-1,1\}^d} f(b)$$

and $h_d(x) = \sum_{b\in\{-1,1\}^d} K_d(x,b)$. From this, we have

$$\|h_d\|_{F_d}^2 = \sum_{b\in\{-1,1\}^d} K_d(b,b) = 2^d K_d(\vec{1},\vec{1}) = (2K_1(1,1))^d .$$

Define the algorithm

$$Q_{n,d}(f) = \sum_{b\in A_n} f(b),$$

where A_n is a subset of $\{-1,1\}^d$ and has $\min(n,2^d)$ elements. Then

$$I_d(f) - Q_{n,d}(f) = \sum_{b\in\{-1,1\}^d\setminus A_n} f(b)$$

and

$$\begin{aligned} e^2(Q_{n,d}) &= \sum_{b\in\{-1,1\}^d - A_n} K_d(b,b) \\ &= \left(2^d - \min(n,2^d)\right) K_1^d(1,1) = \left(1 - n\,2^{-d}\right)_+ e^2(0,d), \end{aligned}$$

as claimed. So in this case, the first estimate of Theorem 11.8 holds with equality. □

11.5 Non-decomposable Kernels

In the previous section, we presented lower bounds on the minimal worst case errors for decomposable kernels. In this section, we show similar lower bounds for certain non-decomposable kernels.

For $d = 1$, we consider two reproducing kernels R_1 and R_2 each defined over the set D_1^2. Let $H(R_i)$ denote the Hilbert space with the reproducing kernel R_i. We assume that

$$H(R_1) \cap H(R_2) = \{0\}.$$

Define the reproducing kernel K_1 as the sum of the two kernels, i.e.,

$$K_1 = R_1 + R_2. \tag{11.22}$$

The Hilbert space F_1 with the kernel K_1 is the space of functions of the form $f_1 + f_2$ for $f_i \in H(R_i)$ with the inner product

$$\langle f,g\rangle_{F_1} = \langle f_1,g_1\rangle_{H(R_1)} + \langle f_2,g_2\rangle_{H(R_2)},$$

where $f = f_1 + f_2$ and $g = g_1 + g_2$, see Aronszajn [2], p. 353.

As in the previous section, we define $I_1(f) = \langle f,h_1\rangle_{F_1}$ for a given function h_1 from F_1. The function h_1 has the unique decomposition

$$h_1 = h_{1,1} + h_{1,2}, \quad \text{with } h_{1,j} \in H(R_j),\ j = 1,2.$$

We will assume that the kernel R_2 is decomposable. Hence, the kernel K_1 has one term R_1, which may be non-decomposable, and one decomposable term R_2. As in the previous section, by $h_{1,2,(0)}$ and $h_{1,2,(1)}$ we denote the restriction of the function $h_{1,2}$ to the sets $D_{(0)}$ and $D_{(1)}$, respectively.

For $d > 1$, we take as before

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1$$

as the d-fold tensor product of F_1. The reproducing kernel of F_d is

$$K_d(x,t) = \prod_{j=1}^{d} \left[R_1(x_j,t_j) + R_2(x_j,t_j) \right].$$

The continuous linear functional I_d is defined (without any change) as the d-fold tensor products of I_1. That is,

$$I_d(f) = \langle f, h_d \rangle_{F_d},$$

where

$$h_d(x) = \prod_{j=1}^{d} h_1(x_j) = \prod_{j=1}^{d} \left[h_{1,1}(x_j) + h_{1,2}(x_j) \right].$$

The initial error is given by

$$e^2(0,d) = \|h_d\|_{F_d}^2 = \left(\|h_{1,1}\|_{H(R_1)}^2 + \|h_{1,2}\|_{H(R_2)}^2 \right)^d.$$

We are ready to present a lower bound on the nth minimal worst case error $e(n,d)$ of approximating I_d over the unit ball of the space F_d.

Theorem 11.12. *Assume that*

$$H(R_1) \cap H(R_2) = \{0\} \quad \textit{and} \quad R_2 \textit{ is decomposable.}$$

Then

$$e^2(n,d) \ge \sum_{k=0}^{d} \binom{d}{k} \left(1 - n\,\alpha^k\right)_+ \alpha_1^{d-k} \alpha_2^k, \quad \textit{with } 0^0 = 1,$$

where

$$\alpha_1 = \|h_{1,1}\|_{H(R_1)}^2, \quad \alpha_2 = \|h_{1,2}\|_{H(R_2)}^2,$$

and

$$\alpha = \frac{\max\left(\|h_{1,2,(0)}\|_{H(R_2)}^2, \|h_{1,2,(1)}\|_{H(R_2)}^2 \right)}{\|h_{1,2,(0)}\|_{H(R_2)}^2 + \|h_{1,2,(1)}\|_{H(R_2)}^2} \in \left[\tfrac{1}{2}, 1\right].$$

Proof. For $x \in D_1^d$ and

$$\mathfrak{u} \subseteq [d] := \{1, 2, \ldots, d\},$$

we let $x_{\mathfrak{u}}$ denote the vector from $D_1^{|\mathfrak{u}|}$ whose components corresponding to indices in $\mathfrak{u}$ are the same as the components of the vector x, i.e., $x_{\mathfrak{u},j} = x_j$ for all $j \in \mathfrak{u}$. We let $x_{\bar{\mathfrak{u}}}$ denote the vector $x_{[d]\setminus\mathfrak{u}}$.

Define the functions $h_{\mathfrak{u},1} \colon D_1^{d-|\mathfrak{u}|} \to \mathbb{R}$ and $h_{\mathfrak{u},2} \colon D_1^{|\mathfrak{u}|} \to \mathbb{R}$ by

$$h_{\mathfrak{u},1}(x_{\bar{\mathfrak{u}}}) = \prod_{j \notin \mathfrak{u}} h_{1,1}(x_j), \quad \text{and} \quad h_{\mathfrak{u},2}(x_{\mathfrak{u}}) = \prod_{j \in \mathfrak{u}} h_{1,2}(x_j).$$

For $\mathfrak{u} = \emptyset$ or $\mathfrak{u} = [d]$, we take $h_{\emptyset,2} = 1$ and $h_{[d],1} = 1$. Let

$$R_{\mathfrak{u},1}(x_{\bar{\mathfrak{u}}}, t_{\bar{\mathfrak{u}}}) = \prod_{j \notin \mathfrak{u}} R_1(x_j, t_j) \quad \text{and} \quad R_{\mathfrak{u},2}(x_{\mathfrak{u}}, t_{\mathfrak{u}}) = \prod_{j \in \mathfrak{u}} R_2(x_j, t_j)$$

be the reproducing kernels of the Hilbert spaces $H(R_{\mathfrak{u},1})$ and $H(R_{\mathfrak{u},2})$. Then $h_{\mathfrak{u},i} \in H(R_{\mathfrak{u},i})$, and

$$\|h_{\mathfrak{u},1}\|_{H(R_{\mathfrak{u},1})} = \|h_{1,1}\|_{H(R_1)}^{d-|\mathfrak{u}|} \quad \text{and} \quad \|h_{\mathfrak{u},2}\|_{H(R_{\mathfrak{u},2})} = \|h_{1,2}\|_{H(R_2)}^{|\mathfrak{u}|}.$$

For the function h_d we have

$$h_d(x) = \prod_{j=1}^{d} (h_{1,1}(x_j) + h_{1,2}(x_j)) = \sum_{\mathfrak{u} \subseteq [d]} h_{\mathfrak{u},1}(x_{\bar{\mathfrak{u}}})\, h_{\mathfrak{u},2}(x_{\mathfrak{u}}). \tag{11.23}$$

Furthermore,

$$\|h_d\|_{F_d}^2 = \prod_{j=1}^{d} \left(\|h_{1,1}\|_{H(R_1)}^2 + \|h_{1,2}\|_{H(R_2)}^2\right) = \sum_{\mathfrak{u} \subseteq [d]} \|h_{\mathfrak{u},1}\|_{H(R_{\mathfrak{u},1})}^2 \|h_{\mathfrak{u},2}\|_{H(R_{\mathfrak{u},2})}^2. \tag{11.24}$$

Consider now the reproducing kernel K_d of the space F_d. We have

$$K_d(x,t) = \prod_{j=1}^{d} \left(R_1(x_j,t_j) + R_2(x_j,t_j)\right) = \sum_{\mathfrak{u} \subseteq [d]} R_{\mathfrak{u},1}(x_{\bar{\mathfrak{u}}}, t_{\bar{\mathfrak{u}}})\, R_{\mathfrak{u},2}(x_{\mathfrak{u}}, t_{\mathfrak{u}}). \tag{11.25}$$

We are ready to give a lower bound on the worst case error of an arbitrary linear algorithm $Q_{n,d}(f) = \sum_{j=1}^{n} a_i f(z_j)$. As always, we have

$$e^2(Q_{n,d}) = \|h_d\|_{F_d}^2 - 2\sum_{j=1}^{n} a_j h_d(z_j) + \sum_{i,j=1}^{n} a_i a_j K_d(z_i, z_j).$$

Using (11.23), (11.24) and (11.25) we get

$$e^2(Q_{n,d}) = \sum_{\mathfrak{u} \subseteq [d]} e_{\mathfrak{u}}^2,$$

where

$$e_{\mathfrak{u}}^2 = \|h_{\mathfrak{u},1}\|^2_{H(R_{\mathfrak{u},1})}\|h_{\mathfrak{u},2}\|^2_{H(R_{\mathfrak{u},2})} - 2\sum_{j=1}^n a_j h_{\mathfrak{u},1}((z_j)_{\bar{\mathfrak{u}}})h_{\mathfrak{u},2}((z_j)_{\mathfrak{u}})$$
$$+ \sum_{i,j=1}^n a_i a_j R_{\mathfrak{u},1}((z_i)_{\bar{\mathfrak{u}}},(z_j)_{\bar{\mathfrak{u}}})R_{\mathfrak{u},2}((z_i)_{\mathfrak{u}},(z_j)_{\mathfrak{u}}).$$

There exists an orthonormal basis $\{r_k\} = \{r_{k,\mathfrak{u}}\}$ of the space $H(R_{\mathfrak{u},1})$, where $k \in \mathcal{I}$ and $\mathcal{I}$ is an index set that is at most countable if the space is separable. It is known that the reproducing kernel $R_{\mathfrak{u},1}$ may be written as

$$R_{\mathfrak{u},1}(x_{\bar{\mathfrak{u}}},t_{\bar{\mathfrak{u}}}) = \sum_{k\in\mathcal{I}} r_k(x_{\bar{\mathfrak{u}}})\, r_k(t_{\bar{\mathfrak{u}}}).$$

Even for an uncountable set $\mathcal{I}$, the last series has always at most a countable number of positive terms. Furthermore,

$$h_{\mathfrak{u},1} = \sum_{k\in\mathcal{I}} \langle h_{\mathfrak{u},1}, r_k\rangle_{H(R_{\mathfrak{u},1})}\, r_k \quad \text{and} \quad \|h_{\mathfrak{u},1}\|^2_{H(R_{\mathfrak{u},1})} = \sum_{k\in\mathcal{I}} \langle h_{\mathfrak{u},1}, r_k\rangle^2_{H(R_{\mathfrak{u},1})}.$$

We can thus rewrite $e_{\mathfrak{u}}^2$ as $e_{\mathfrak{u}}^2 = \sum_{k\in\mathcal{I}} e_{\mathfrak{u},k}^2$ with

$$e_{\mathfrak{u},k}^2 = \langle h_{\mathfrak{u},1}, r_k\rangle^2_{H(R_{\mathfrak{u},1})}\, \|h_{\mathfrak{u},2}\|^2_{H(R_{\mathfrak{u},2})}$$
$$- 2\sum_{j=1}^n a_j\, \langle h_{\mathfrak{u},1}, r_k\rangle_{H(R_{\mathfrak{u},1})}\, r_k((z_j)_{\bar{\mathfrak{u}}})h_{\mathfrak{u},2}((z_j)_{\mathfrak{u}})$$
$$+ \sum_{i,j=1}^n a_i a_j r_k((z_i)_{\bar{\mathfrak{u}}}) r_k((z_j)_{\bar{\mathfrak{u}}}) R_{\mathfrak{u},2}((z_i)_{\mathfrak{u}},(z_j)_{\mathfrak{u}}).$$

Assume for a moment that $\langle h_{\mathfrak{u},1}, r_k\rangle_{H(R_{\mathfrak{u},1})} \neq 0$, and define

$$a'_j = \frac{a_j r_k((z_j)_{\bar{\mathfrak{u}}})}{\langle h_{\mathfrak{u},1}, r_k\rangle_{H(R_{\mathfrak{u},1})}}.$$

Then $e_{\mathfrak{u},k}^2 = \langle h_{\mathfrak{u},1}, r_k\rangle^2_{H(R_{\mathfrak{u},1})}\, [e'_{\mathfrak{u},k}]^2$ with

$$\left[e'_{\mathfrak{u},k}\right]^2 = \|h_{\mathfrak{u},2}\|^2_{H(R_{\mathfrak{u},2})} - 2\sum_{j=1}^n a'_j h_{\mathfrak{u},2}((z_j)_{\mathfrak{u}}) + \sum_{i,j=1}^n a'_i a'_j R_{\mathfrak{u},2}((z_i)_{\mathfrak{u}},(z_j)_{\mathfrak{u}}).$$

Observe that $e'_{\mathfrak{u},k}$ is the worst case error of approximating $I_{\mathfrak{u}}(f) = \langle f, h_{\mathfrak{u},2}\rangle_{H(R_{\mathfrak{u},2})}$ over the unit ball of the space $H(R_{\mathfrak{u},2})$. The reproducing kernel $R_{\mathfrak{u},2}$ is a tensor product of R_2 which is decomposable. Therefore we may apply Theorem 11.8 to obtain

$$e_{\mathfrak{u},k}^2 \geq \langle h_{\mathfrak{u},1}, r_k\rangle^2_{H(R_{\mathfrak{u},1})}\, \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \|h_{\mathfrak{u},2}\|^2_{H(R_{\mathfrak{u},2})}.$$

Observe that the last inequality is also trivially true if $\langle h_{\mathfrak{u},1}, r_k\rangle_{H(R_{\mathfrak{u},1})} = 0$.

Summing with respect to k, we obtain

$$e_{\mathfrak{u}}^2 \ge \|h_{\mathfrak{u},1}\|_{H(R_{\mathfrak{u},1})}^2 \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \|h_{\mathfrak{u},2}\|_{H(R_{\mathfrak{u},2})}^2 = \alpha_1^{d-|\mathfrak{u}|}\left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_2^{|\mathfrak{u}|}.$$

The lower bound on $e_{\mathfrak{u}}^2$ is the same for all $\mathfrak{u}$ of the same cardinality. Hence if $|\mathfrak{u}| = k$ then we have $\binom{d}{k}$ such $\mathfrak{u}$'s. Summing with respect to $\mathfrak{u}$, we obtain

$$e^2(Q_{n,d}) \ \ge \ \sum_{k=0}^{d} \binom{d}{k} \left(1 - n\,\alpha^{-k}\right)_+ \alpha_1^{d-k}\alpha_2^k.$$

Since this holds for any algorithm $Q_{n,d}$, the same lower bound holds for the nth minimal error $e^2(n,d)$. This completes the proof. □

Observe that Theorem 11.12 generalizes Theorem 11.8. Indeed, it is enough to take $h_{1,1} = 0$ (which always holds for $R_1 = 0$). Then $\alpha_1 = 0$ and the sum in Theorem 11.12 consists of only one term for $k = d$. Thus

$$e(n,d) \ge \left(1 - n\,\alpha^d\right)_+^{1/2} e(0,d), \tag{11.26}$$

as in Theorem 11.8.

Theorem 11.12 presents a lower bound on the nth minimal error. It is easy to relate this lower bound to the initial error. We have

$$e^2(0,d) = (\alpha_1 + \alpha_2)^d = \sum_{\mathfrak{u}\subseteq[d]} \alpha_1^{d-|\mathfrak{u}|}\alpha_2^{|\mathfrak{u}|} = \sum_{k=0}^{d} \binom{d}{k} \alpha_1^{d-k}\,\alpha_2^k.$$

If we assume that

$$n \le a\,\alpha^{-m} \quad \text{for some } a \in (0,1) \text{ with } m < d,$$

we can estimate $(1 - n\alpha^k)_+$ from below by zero for $k < m$ and by $1 - a$ for $k \ge m$. This yields the following corollary.

Corollary 11.13. *Let*

$$n \le a\alpha^{-m} \quad \textit{for some } a \in (0,1) \textit{ with } m < d.$$

Under the assumptions of Theorem 11.12 we have

$$e(n,d) \ge (1-a)^{1/2} \left[1 - \frac{\sum_{k=0}^{m-1}\binom{d}{k}\alpha_1^{d-k}\alpha_2^k}{\sum_{k=0}^{d}\binom{d}{k}\alpha_1^{d-k}\alpha_2^k}\right]^{1/2} e(0,d).$$

We are ready to discuss tractability of $I = \{I_d\}$.

Theorem 11.14. *Assume that*

$$H(R_1) \cap H(R_2) = \{0\} \quad \textit{and} \quad R_2 \textit{ is decomposable.}$$

Assume also that $h_{1,1}$ *as well as both* $h_{1,2,(0)}$ *and* $h_{1,2,(1)}$ *are non-zero. Then*

$$\lim_{d\to\infty} \frac{e(\lfloor C^d \rfloor, d)}{e(0,d)} = 1 \quad \textit{for all } C \in (1, \alpha^{-\alpha_3/(1+\alpha_3)}),$$

where, as before,

$$\alpha = \frac{\max(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)})}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}} \in \left[\tfrac{1}{2}, 1\right),$$

and

$$\alpha_3 = \frac{\|h_{1,2}\|^2_{H(R_2)}}{\|h_{1,1}\|^2_{H(R_1)}} = \frac{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}}{\|h_{1,1}\|^2_{H(R_1)}} > 0.$$

- *Consider the absolute error criterion, and assume that* $\|h_1\|_{F_1} \geq 1$.

  Then $I = \{I_d\}$ *suffers from the curse of dimensionality and is intractable. For an arbitrary* $\beta \in (0,1)$, $\varepsilon \in \left(0, \beta \|h_1\|^d_{F_1}\right)$ *and* C *as above we have*

  $$n(\varepsilon, d) \geq C^d\,(1 + o(1)) \quad \textit{as } d \to \infty.$$

- *Consider the absolute error criterion, and assume that* $\|h_1\|_{F_1} < 1$.

  Let $x \in (0,1)$ *and* $\varepsilon_d = x\|h_1\|^d_{F_1}$. *Then for* C *as above we have*

  $$n(\varepsilon_d, d) \geq x^p\ \varepsilon_d^{-p}\,(1 + o(1)) \quad \textit{as } d \to \infty$$

  with

  $$p = \frac{\ln C}{\ln \|h_1\|^{-1}_{F_1}}.$$

  If $I = \{I_d\}$ *is strongly polynomially tractable then its exponent satisfies*

  $$p^{\text{str-wor}}(I) \geq \frac{\alpha_3 \ln \alpha^{-1}}{(1+\alpha_3) \ln \|h_1\|^{-1}_{F_1}}.$$

- *Consider the normalized error criterion.*

  Then $I = \{I_d\}$ *suffers from the curse of dimensionality and is intractable. For* $\varepsilon \in (0,1)$ *and* C *as above we have*

  $$n(\varepsilon, d) \geq C^d\,(1 + o(1)) \quad \textit{as } d \to \infty.$$

Proof. Define $C_{d,k} = \alpha_3^k \binom{d}{k}$ and remember that $\alpha_3 = \alpha_2/\alpha_1$. Then Theorem 11.12, the form of the initial error, and dividing $e^2(n,d)$ and $e^2(0,d)$ by α_1^d, yield

$$1 \geq \frac{e^2(n,d)}{e^2(0,d)} \geq \frac{\sum_{k=0}^{d} C_{d,k}(1-n\alpha^k)_+}{\sum_{k=0}^{d} C_{d,k}}.$$

We take an arbitrary $a \in (0,1)$ and $n = \lfloor C^d \rfloor$ with $C \in (1, \alpha^{-\alpha_3/(1+\alpha_3)})$. This means that $C\alpha^{\alpha_3/(1+\alpha_3)} < 1$. Then there exists a positive c such that

$$c < \frac{\alpha_3}{1+\alpha_3} \quad \text{and} \quad C\alpha^c < 1.$$

Take $k(d) = \lfloor c\, d \rfloor$. Then for sufficiently large d, we have

$$n\,\alpha^k \leq C^d \alpha^{cd-1} = \alpha^{-1}(C\alpha^c)^d \leq a \quad \text{for all } k \in (k(d), d].$$

This implies that

$$\frac{e^2(n,d)}{e^2(0,d)} \geq (1-a)\frac{\sum_{k=k(d)+1}^{d} C_{d,k}}{\sum_{k=0}^{d} C_{d,k}} = (1-a)[1-\alpha(d)],$$

where

$$\alpha(d) = \frac{\sum_{k=0}^{k(d)} C_{d,k}}{\sum_{k=0}^{d} C_{d,k}} = \frac{\sum_{k=0}^{k(d)} \alpha_3^k \binom{d}{k}}{(1+\alpha_3)^d}.$$

Let $f(k) = \alpha_3^k \binom{d}{k}$. Then $f(0) = 1$ and $f(1) = \alpha_3 d$. For large d, we have $f(1) \geq f(0)$ and $k(d) \geq 1$. Furthermore, for $k \geq 1$ we have

$$\frac{f(k)}{f(k-1)} = \alpha_3 \frac{d-k+1}{k} \geq 1 \quad \text{iff} \quad k \leq \frac{\alpha_3}{1+\alpha_3}(d+1).$$

Hence, f as a function of k increases for $k \leq k(d)$. Therefore we have

$$\alpha(d) \leq x(d) := \frac{(\lfloor cd \rfloor + 1)\alpha_3^{\lfloor cd \rfloor}\binom{d}{\lfloor cd \rfloor}}{(1+\alpha_3)^d}.$$

In the proof of Theorem 5.5 in Chapter 5, we used Stirling's formula to show that

$$\ln \binom{d}{\lfloor cd \rfloor} = d\left[c \ln \frac{1}{c} + (1-c)\ln \frac{1}{1-c}\right] - \tfrac{1}{2}\ln d + \mathcal{O}(1).$$

This implies that

$$\ln x(d) = d\, g(c) + \mathcal{O}(\ln d),$$

where

$$g(c) = c \ln \frac{1}{c} + (1-c)\ln \frac{1}{1-c} + c \ln \alpha_3 - \ln(1+\alpha_3).$$

We now show that $g(c) < 0$. Indeed, note that $g(0) = -\ln(1+\alpha_3) < 0$ and $g(\alpha_3/(1+\alpha_3)) = 0$. Furthermore

$$g'(x) = \ln \frac{(1-x)\,\alpha_3}{x} > 0 \quad \text{iff} \quad x < \frac{\alpha_3}{1+\alpha_3}.$$

Hence, g is increasing in $[0, \alpha_3/(1+\alpha_3)]$. Since we took $c < \alpha_3/(1+\alpha_3)$ then we have $g(c) < g(\alpha_3/(1+\alpha_3)) = 0$, as claimed.

This means that $\ln x(d)$ goes to $-\infty$ as d goes to infinity, so that both $x(d)$ and $\alpha(d)$ go to zero. So for any positive δ, we can find a (large) integer $d(\delta)$ such that for all $d \ge d(\delta)$, we have

$$1 \ge \frac{e^2(\lfloor C^d \rfloor, d)}{e^2(0,d)} \ge (1-a)(1-\delta).$$

Since a and δ can be arbitrarily small, the limit of $e(\lfloor C^d \rfloor, d)/e(0,d)$ must be one, as claimed.

The rest is easy. Consider the absolute error criterion with $\|h_1\|_{F_1} \ge 1$. For large d and $\varepsilon \le \beta \|h_1\|_{F_1}^d$, with $\beta \in (0,1)$, the worst case error $e(\lfloor C^d \rfloor, d)$ is arbitrarily close to $e(0,d) = \|h_1\|_{F_1}^d$, and therefore larger than ε. Hence $n(\varepsilon, d) \ge C^d(1+o(1))$. For the normalized error criterion, we take $\varepsilon \in (0,1)$ and for large d we have $\varepsilon \|h_1\|_{F_1}^d < e(\lfloor C^d \rfloor, d) = \|h_1\|_{F_1}^d (1+o(1))$ and therefore $n(\varepsilon, d) \ge C^d(1+o(1))$. Since $C > 1$, in both cases this obviously means the curse of dimensionality.

For the absolute error criterion with $\|h_1\|_{F_1} < 1$ take $\varepsilon = \varepsilon_d = x\|h_1\|_{F_1}^d$. Since x is fixed and less than one, we know that $\lfloor C^d \rfloor$ function values are not enough for large d. Note that now $d = \ln(x/\varepsilon_d)/\ln(\|h_1\|_{F_1}^{-1})$ and $C^d = (x/\varepsilon_d)^p$, which yields the bound on $n(\varepsilon_d, d)$.

Finally, if $I = \{I_d\}$ is strongly polynomially tractable, then $n(\varepsilon_d, d) \le C_\tau \varepsilon_d^{-\tau}$ for all $d \in \mathbb{N}$ with τ arbitrarily close to the exponent $p^{\text{str-wor}}$. This implies that $p^{\text{str-wor}} \ge p$, and we can maximize p by taking C tending to $\alpha^{-\alpha_3/(1+\alpha_3)}$. This completes the proof. □

The essence of Theorem 11.14 is intractability for the normalized error criterion and for the absolute error criterion with $\|h_1\|_{F_1} \ge 1$. We stress that Theorem 11.14 is more general than Theorem 11.8, since the reproducing univariate kernel K_1 is now assumed to have a decomposable part and can be itself non-decomposable. As before, the case $\|h_1\|_{F_1} < 1$ may indeed lead to strong polynomial tractability, and the comment after Theorem 11.8 applies also for non-decomposable reproducing kernels.

We now illustrate Theorems 11.12 and 11.14 by continuing the examples of the previous section.

11.5.1 Example: Weighted Integration (Continued)

We now remove the boundary conditions $f^{(j)}(0)$ from the definition of the Sobolev space $W_0^r(\mathbb{R})$ and consider $F_1 = W^r(\mathbb{R})$ as the Sobolev space equipped with the inner product

$$\langle f, g\rangle_{F_1} = \sum_{j=0}^{r-1} f^{(j)}(0)\, g^{(j)}(0) \;+\; \int_{\mathbb{R}} f^{(r)}(t) g^{(r)}(t)\,\mathrm{d}t.$$

The reproducing kernel K_1 takes now the form

$$K_1(x,t) = \sum_{j=0}^{r-1} \frac{x^j}{j!}\,\frac{t^j}{j!} + 1_M(x,t) \int_{\mathbb{R}_+} \frac{(|t|-u)_+^{r-1}\,(|x|-u)_+^{r-1}}{((r-1)!)^2}\,\mathrm{d}u,$$

where, as before, 1_M is the characteristic (indicator) function of the set

$$M = \{(x,t) :\; xt \ge 0\}.$$

We can now take

$$R_1(x,t) = \sum_{j=0}^{r-1} \frac{x^j}{j!}\,\frac{t^j}{j!},$$

$$R_2(x,t) = 1_M(x,t) \int_{\mathbb{R}_+} \frac{(|t|-u)_+^{r-1}\,(|x|-u)_+^{r-1}}{[(r-1)!]^2}\,\mathrm{d}u.$$

The space $H(R_1)$ is the space of polynomials of degree at most $r-1$, which has dimension r. The space $H(R_2) = W_0^r(\mathbb{R})$ is the Sobolev space with the conditions $f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, r-1$. Clearly,

$$H(R_1) \cap H(R_2) = \{0\}.$$

The kernel R_2 is decomposable at $a^* = 0$ (and is also symmetric), so all the assumptions of Theorem 11.12 hold.

For $d > 1$, F_d is the d-fold tensor product Sobolev space $W^r(\mathbb{R}) \otimes \cdots \otimes W^r(\mathbb{R})$ whose inner product $\langle f, g\rangle_{F_d}$ is a sum of $(r+1)^d$ terms. In the case $r = 1$ and $d = 2$ we have

$$\begin{aligned}\langle f, g\rangle_{F_2} = {} & f(0)g(0) \\ & + \int_{\mathbb{R}} \frac{\partial f}{\partial x_1}(x_1,0)\frac{\partial g}{\partial x_1}(x_1,0)\,\mathrm{d}x_1 + \int_{\mathbb{R}} \frac{\partial f}{\partial x_1}(0,x_2)\frac{\partial g}{\partial x_1}(0,x_2)\,\mathrm{d}x_2 \\ & + \int_{\mathbb{R}^2} \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2)\frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

Consider the *weighted integration* problem given by

$$I_1(f) = \int_{\mathbb{R}} \varrho(t)\, f(t)\,\mathrm{d}t \quad \text{with} \quad \int_{\mathbb{R}} \varrho(t)\,|t|^{r-1/2}\,\mathrm{d}t < \infty.$$

We now have

$$h_{1,1}(x) = \sum_{j=0}^{r-1} \frac{x^j}{j!} \int_{\mathbb{R}} \frac{t^j}{j!} \varrho(t)\, \mathrm{d}t,$$

and $h_{1,2} = h_{1,2,(0)} + h_{1,2,(1)}$ with

$$h_{1,2,(0)}(x) = \int_{-\infty}^{0} \varrho(t) R_2(x,t)\, \mathrm{d}t \quad \text{and} \quad h_{1,2,(1)}(x) = \int_{0}^{\infty} \varrho(t) R_2(x,t)\, \mathrm{d}t.$$

Hence, $h_{1,1}$ is a polynomial of degree at most $r-1$. Note that $h_{1,1}$ is not zero since the coefficient of $h_{1,1}$ for $j = 0$ is $\int_{\mathbb{R}} \varrho(t)\, \mathrm{d}t$, which is positive. We assume that ϱ is non-zero over $\mathbb{R}_-$ and $\mathbb{R}_+$, which yields that both of the $h_{1,2,(i)}$ are also non-zero and $\alpha < 1$. For symmetric ϱ, we have $\alpha = \frac{1}{2}$.

Consider first the normalized error criterion. Theorem 11.14 states that weighted integration is intractable no matter how large r we have.

Consider then the absolute error criterion. It seems natural to assume that ϱ is a density of a probability measure. If so, then $\int_{\mathbb{R}} \varrho(t)\, \mathrm{d}t = 1$. This implies that $\|h_{1,1}\|_{H(R_1)} \geq 1$, and therefore the initial error is at least one. Theorem 11.14 again implies that weighted integration is intractable. Hence, the only case for which we can break intractability is for a not-so-natural weight ϱ when $\int_{\mathbb{R}} \varrho(t)\, \mathrm{d}t < 1$. □

11.5.2 Example: Uniform Integration

We now remove the condition $f(a) = 0$ from the example of uniform integration studied in the previous section. We now have $D_1 = [0,1]$ and F_1 is the Sobolev space $W^1([0,1])$ with the inner product

$$\langle f, g \rangle_{F_{1,\gamma}} = f(a)g(a) + \int_0^1 f'(x)g'(x)\, \mathrm{d}x.$$

The reproducing kernel is

$$K_1(x,t) = 1 + 1_M(x,t) \min(|x-a|, |t-a|)$$

with $M = [0,a] \times [0,a] \cup [a,1] \times [a,1]$.

For $d > 1$, we take $F_d = W^{(1,1,\dots,1)}([0,1]^d)$ as the d-fold tensor product of $W^1([0,1])$. The inner product of F_d is now given by

$$\langle f, g \rangle_{F_d} = \sum_{\mathfrak{u} \subseteq [d]} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, a) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x_{\mathfrak{u}}, a)\, \mathrm{d}x_{\mathfrak{u}}.$$

Here $(x_{\mathfrak{u}}, a)$ is the vector $x \in [0,1]^d$ with all components whose indices not in $\mathfrak{u}$ are replaced by a.

We consider uniform integration

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t.$$

We now have $R_1 = 1$ and

$$R_2(x,t) = 1_M(x,t)\min(|x-a|, |t-a|).$$

This implies that $h_{1,1} = 1$, and $\|h_{1,1}\|_{H(R_1)} = 1$. The functions $h_{1,2,(0)} = h_{1,(0)}$ and $h_{1,2,(1)} = h_{1,(1)}$ are as in the previous sections. Therefore all the assumptions of Theorem 11.14 are satisfied as long as $a \in (0,1)$. It is easy to check that the initial error is now

$$e(0,d) = \left(1 + \tfrac{1}{3}\left(a^3 + (1-a)^3\right)\right)^{d/2} \geq \left(\frac{13}{12}\right)^{d/2},$$

which is exponentially large in d.

Hence, uniform integration is intractable for both the absolute and normalized error criteria as along as the anchor $a \in (0,1)$. We return to the case $a = 0$ and $a = 1$ later. □

11.5.3 Example: Centered Discrepancy (Continued)

We now remove the condition $f(\frac{1}{2}) = 0$. As before, for this special case we study the more general case of the L_q norm. We restrict ourselves to $q > 1$. That is, $D_1 = [0,1]$ and $F_{1,q}$ is the Sobolev space $W_q^1([0,1])$ with the norm

$$\|f\|_{F_{1,q}} = \left(\left|f(\tfrac{1}{2})\right|^q + \int_0^1 |f'(x)|^q\,\mathrm{d}x\right)^{1/q}.$$

Observe that for $q = \infty$ we have

$$\|f\|_{F_{1,\infty}} = \max\left(|f(\tfrac{1}{2})|, \operatorname*{ess\,sup}_{t\in[0,1]} |f'(t)|\right).$$

For $q = 2$, we have the Hilbert space with the kernel

$$K_1(x,t) = 1 + 1_M(x,t)\min\left(|x-\tfrac{1}{2}|, |t-\tfrac{1}{2}|\right).$$

For $d > 1$, we take $F_{d,q} = W_q^{(1,1,\dots,1)}([0,1]^d)$ as the tensor product of $W_q^1([0,1])$. The norm in $F_{d,q}$ is given by

$$\|f\|_{F_{d,q}} = \left(\sum_{\mathfrak{u}\subseteq[d]} \int_{[0,1]^{|\mathfrak{u}|}} \left|\frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, 1/2)\right|^q \mathrm{d}x_{\mathfrak{u}}\right)^{1/q}. \tag{11.27}$$

We consider uniform integration

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t.$$

The formula for the error of the algorithm $Q_{n,d}(f) = \sum_{i=1}^n a_i\, f(z_i)$ takes the form,

$$\begin{aligned} &I_d(f) - Q_{n,d}(f) \\ &\quad = \sum_{\mathfrak{u}\subseteq[d]} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, \tfrac{1}{2}) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} \Big(h_d - \sum_{i=1}^n a_i\, K_d(\cdot, z_i)\Big)(x_{\mathfrak{u}}, \tfrac{1}{2})\,\mathrm{d}x_{\mathfrak{u}}, \end{aligned}$$

where h_d and the kernel K_d are given as before by

$$\begin{aligned} h_d(x) &= 2^{-d} \prod_{j=1}^d \left(|x_j - \tfrac{1}{2}| - |x_j - \tfrac{1}{2}|^2\right), \\ K_d(x,t) &= 2^{-d} \prod_{j=1}^d \left(|x_j - \tfrac{1}{2}| + |t_j - \tfrac{1}{2}| - |x_j - t_j|\right), \end{aligned}$$

see Hickernell [118]. Applying the Hölder inequality for integrals and sums to $I_d(f) - Q_{n,d}(f)$ we conclude that

$$e(Q_{n,d}) := \sup_{f\in F_{d,q}:\, \|f\|_{F_{d,q}}\le 1} \left|I_d(f) - Q_{n,d}(f)\right| = d_p^c(Q_{n,d}),$$

where $1/p + 1/q = 1$ and the centered p-discrepancy $d_p^c(Q_{n,d})$ is given by

$$d_p^c(Q_{n,d}) = \left(\sum_{\mathfrak{u}\subseteq[d]} \int_{[0,1]^{|\mathfrak{u}|}} |\mathrm{disc}(n,d)^c(x_{\mathfrak{u}}, \tfrac{1}{2})|^p\,\mathrm{d}x_{\mathfrak{u}} \right)^{1/p}, \tag{11.28}$$

with

$$\mathrm{disc}(n,d)^c(x_{\mathfrak{u}}, \tfrac{1}{2}) = \prod_{\ell\in\mathfrak{u}} \min(x_\ell, 1 - x_\ell) - \sum_{i=1}^n a_i \cdot 1_{J(a(x_{\mathfrak{u}}), x_{\mathfrak{u}})}(t_i)_{\mathfrak{u}}.$$

Since we assume that $q > 1$, we now have $p < \infty$.

Let $e(n,d,q) = d_p(n,d)$ denote the nth minimal error or, equivalently, the minimal nth centered p-discrepancy, that can be achieved by using n function values. The initial error, or the initial centered p-discrepancy, is given by

$$e(0,d,q) = d_p^c(0,d) = \left(\sum_{\mathfrak{u}\subseteq[d]} \left(\frac{2^{-p}}{p+1}\right)^{|\mathfrak{u}|} \right)^{1/p} = \left(1 + \frac{2^{-p}}{p+1}\right)^{d/p}.$$

From Corollary 11.11 we conclude that

$$d_p^c(n,d) \geq \left(\sum_{k=0}^{d} \binom{d}{k} \left(\frac{2^{-p}}{p+1} \right)^k \cdot (1 - n\,2^{-k})_+ \right)^{1/p}.$$

For $n \leq 2^m < 2^d$, this can be rewritten as

$$d_p^c(n,d) \geq 2^{-1/p} \left(1 - \frac{\sum_{k=0}^{m} \binom{d}{k} (2^{-p}/(p+1))^k}{\sum_{k=0}^{d} \binom{d}{k} (2^{-p}/(p+1))^k} \right)^{1/p} d_p^c(0,d).$$

Hence, we can check intractability of $I = \{I_d\}$ for the normalized as well as the absolute error criterion by using the proof of Theorem 11.14.

The case $q = 1$ can be analyzed similarly as the case of the star discrepancy. We leave details to the reader.

11.5.4 Example: Sobolev Space Anchored at 0

As we have already explained, the L_2 discrepancy is related to uniform integration defined on the Sobolev space $F_d = W_0^{1,1,\dots,1}([0,1]^d)$ anchored at 0. We now show how to apply the results of this section to obtain interesting lower bounds on multivariate integration, as well as on the L_2 discrepancy.

We begin with the boundary case. For $d = 1$, the reproducing kernel of F_1 is now

$$K_1(x,t) = \min(x,t).$$

This kernel is formally decomposable at $a^* = 0$. But then $D_{1,(0)} = \{0\}$ and $h_{1,(0)} = 0$ since all functions in F_1 vanish at zero. Therefore $\alpha = 1$, and so Theorem 11.8 is trivial in this case.

However, Theorem 11.12 offers an alternative approach. If we can present the kernel K_1 as the sum of the reproducing kernels

$$K_1 = R_{1,a^*} + R_{2,a^*}$$

with a decomposable R_{2,a^*} for $a^* \in (0,1)$ we may have a chance of concluding that uniform integration is intractable for the Sobolev space $W_0^{1,1,\dots,1}([0,1]^d)$, and obtain interesting bounds for the L_2 discrepancy.

We now show that this approach works. For an arbitrary $a^* \in (0,1)$, consider the subspace F_{a^*} of F_1 by taking functions for which $f(a^*) = 0$. That is,

$$F_{a^*} = \{\, f \in W_0^1([0,1]) \mid f(a^*) = 0 \,\}.$$

Note that the projection $f - f(a^*)K_1(\cdot,a^*)/K_1(a^*,a^*)$ belongs to F_{a^*} for any $f \in F_1$. This implies that

$$F_{a^*} = H(R_{2,a^*})$$

is the Hilbert space with the reproducing kernel

$$R_{2,a^*}(x,t) = K_1(x,t) - \frac{K_1(x,a^*)K_1(t,a^*)}{K_1(a^*,a^*)}.$$

What is important for us is that

$$R_{2,a^*}(x,t) = 0 \quad \text{for all } (x,t) \in [0,a^*] \times [a^*,1] \cup [a^*,1] \times [0,a^*].$$

Hence, R_{2,a^*} is decomposable at a^*. We also have

$$R_{1,a^*}(x,t) = \frac{K_1(t,a^*)\,K_1(x,a^*)}{K_1(a^*,a^*)},$$

and so the one-dimensional Hilbert space $H(R_{1,a^*})$ consists of functions of the form

$$c\ \min(\cdot,a^*) \quad \text{with real } c.$$

Obviously $H(R_{1,a^*}) \cap H(R_{2,a^*}) = \{0\}$, since all functions in $H(R_{2,a^*})$ vanish at a^* and the only function from $H(R_{1,a^*})$ that vanishes at a^* is the zero function. Clearly,

$$F_1 = H(R_{1,a^*}) \oplus H(R_{2,a^*}).$$

For uniform integration, h_1 is given by $h_1(x) = x - \frac{1}{2}x^2$. We thus have

$$\begin{aligned} h_{1,1}(x) &= h_1(a^*)\frac{K_1(x,a^*)}{K_1(a^*,a^*)} = \left(1 - \tfrac{1}{2}a^*\right)\min(x,a^*), \\ h_{1,2,(0)}(x) &= 1_{[0,a^*]}(x)\ \left(\tfrac{1}{2}a^* x - \tfrac{1}{2}x^2\right), \\ h_{1,2,(1)}(x) &= 1_{[a^*,1]}(x)\ \left(x - \tfrac{1}{2}x^2 - a^*\left(1 - \tfrac{1}{2}a^*\right)\right). \end{aligned}$$

From this we compute

$$\begin{aligned} \|h_{1,1}\|^2_{H(R_{1,a^*})} &= \left(1 - \tfrac{1}{2}a^*\right)^2 a^*, \\ \|h_{1,2,(0)}\|^2_{H(R_{2,a^*})} &= \frac{(a^*)^3}{12}, \\ \|h_{1,2,(1)}\|^2_{H(R_{2,a^*})} &= \frac{(1-a^*)^3}{3}. \end{aligned}$$

All the assumptions of Theorem 11.12 now hold. In particular, we may choose a^* such that

$$[a^*]^3 = 4(1-a^*)^3,$$

which corresponds to $a^* = 4^{1/3}/(1+4^{1/3}) = 0.613511790\ldots$. Then the parameters of Theorem 11.12 are

$$\alpha = \tfrac{1}{2}, \quad \alpha_1 = (1-a^*/2)^2 a^*, \quad \alpha_2 = [a^*]^3/6, \quad \text{and} \quad \alpha_3 = [a^*]^2/[6(1-a^*/2)^2],$$

and we obtain

$$\alpha_1 = 0.294846029\ldots, \quad \alpha_2 = 0.0384873039\ldots \quad \alpha_3 = 0.130533567\ldots.$$

Then

$$\alpha^{-\alpha_3/(1+\alpha_3)} = 1.083321840\ldots.$$

For the normalized error criterion, uniform integration is intractable and for all $\varepsilon \in (0,1)$ we have

$$1.0833^d\,(1+o(1)) \le n(\varepsilon,d) \le \left\lceil (4/3)^d\,\varepsilon^{-2}\right\rceil \quad \text{as } d \to \infty.$$

The lower bound follows from Theorem 11.14, whereas the upper bound from Plaskota, Wasilkowski and Zhao [248], see (11.18), since in our case we have

$$e(n,d) \le \frac{1}{\sqrt{n}}\left(\frac{4}{9}\right)^{d/2},$$

which leads to the upper bound on $n(\varepsilon,d)$.

As we know, for the absolute error criterion, uniform integration is strongly polynomially tractable.

We turn to the case without boundary conditions. That is, we now have the space $F_d = W^{1,1,\ldots,1}([0,1]^d)$. For $d = 1$, the reproducing kernel is

$$K_1(x,t) = 1 + \min(x,t),$$

which is *not* decomposable. The initial error is now $(\frac{4}{3})^{d/2}$, which is exponentially large in d.

We can modify the representation of K_1 as

$$K_1 = (1 + R_{1,a^*}) + R_{2,a^*}.$$

From this form we conclude that K_1 satisfies the assumptions of Theorems 11.12 and 11.14 with $R_1 = 1 + R_{1,a^*}$ and $R_2 = R_{2,a^*}$.

For uniform integration we now have

$$h_1(x) = 1 + x - \tfrac{1}{2}x^2 \quad \text{and} \quad \|h_1\|^2_{H(R_1)} = 1 + (1 - \tfrac{1}{2}a^*)a^*.$$

The functions $h_{1,2}$, $h_{1,2,(0)}$ and $h_{1,2(1)}$ are unchanged. For $a^* = 4^{1/3}/(1+4^{1/3})$ we obtain

$$\alpha = \tfrac{1}{2},\ \ \alpha_1 = 1.294846029\ldots,\ \ \alpha_2 = 0.0384873039\ldots,\ \ \alpha_3 = 0.0297234598\ldots.$$

Then

$$\alpha^{-\alpha_3/(1+\alpha_3)} = 1.020209526\ldots.$$

Hence for both the absolute and normalized error criterion, uniform integration is intractable. More precisely, for both criteria we have

$$n(\varepsilon,d) \ge 1.0202^d\,(1+o(1)),$$

whereas

$$n(\varepsilon, d) \le \left\lceil \left[\frac{4}{9} (9 - 4\sqrt{2}) \right]^d \varepsilon^{-2} \right\rceil = \lceil (1.4858\ldots)^d \varepsilon^{-2} \rceil \quad \text{as } d \to \infty$$

for the absolute error criterion, and

$$n(\varepsilon, d) \le \left\lceil \left[\frac{9 - 4\sqrt{2}}{3} \right]^d \varepsilon^{-2} \right\rceil = \lceil (1.1143\ldots)^d \varepsilon^{-2} \rceil \quad \text{as } d \to \infty$$

for the normalized error criterion.

Again, the lower bounds follows from Theorem 11.14, whereas the upper bounds from Plaskota, Wasilkowski and Zhao [248], see (11.18), and the fact that

$$\int_{[0,1]^d} \sqrt{K(t,t)}\, \mathrm{d}t = \left[\frac{4}{9} \left(9 - 4\sqrt{2}\right) \right]^{d/2}.$$

We add in passing that we can obtain slightly better lower bounds if we restrict ourselves to algorithms $Q_{n,d} f = \sum_{j=1}^n a_j f(t_j)$ with non-negative a_j. This class contains the class of quasi Monte Carlo algorithms, for which $a_j = n^{-1}$, and which are widely used for many applications of high-dimensional integration. Then for the class $W_0^{1,1,\ldots,1}([0,1]^d)$ with the boundary conditions the lower bound takes the form

$$n(\varepsilon, d) \ge \left(\tfrac{9}{8}\right)^d (1 - \varepsilon^2) \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N},$$

see Section 10.5 of Chapter 10 as well as [352] where this result is proved for arbitrary non-negative a_j and [277] for $a_j = n^{-1}$.

Hence, instead of the bound 1.0833^d we now have a slightly better bound $(\frac{9}{8})^d = 1.125^d$, at the expense of restricting the class of algorithms.

Without boundary conditions, we have the class $W^{1,1,\ldots,1}([0,1]^d)$. Then for algorithms with non-negative coefficients we have

$$n(\varepsilon, d) \ge (1.0563)^d (1 - \varepsilon^2) \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N},$$

as computed in Section 10.5 of Chapter 10.

Hence, instead of the lower bound 1.0202^d we now have 1.0563^d with the upper bound for the same class of algorithms with 1.1143^d. □

We end this section by the following remark that leads to an open problem. Decomposability of a reproducing kernel means that the space F_1 contains two orthogonal subspaces $F_{(0)}$ and $F_{(1)}$ such that for all $f \in F_1$ we have $f = f_{(0)} + f_{(1)}$ with $f_{(j)} \in F_{(j)}$ and

$$f_{(0)}(x) = 0 \quad \text{for all } x \notin D_{(1)},$$
$$f_{(1)}(x) = 0 \quad \text{for all } x \notin D_{(0)}.$$

This property does not hold for spaces of analytic functions, and therefore not every reproducing kernel can be expressed with a decomposable part. For example, take

$$K_1(x,y) = \sum_{j=0}^{\infty} \frac{x^j}{j!}\,\frac{y^j}{j!} \quad \text{for all } x,y \in \mathbb{R}. \tag{11.29}$$

This reproducing kernel corresponds to a Hilbert space F_1 of entire functions for which $\sum_{j=0}^{\infty} \left[f^{(j)}(0)\right]^2 < \infty$ with the inner product

$$\langle f, g\rangle_{F_1} = \sum_{j=0}^{\infty} f^{(j)}(0)\, g^{(j)}(0) \quad \text{for all } f,g \in F_1.$$

The results of this section do not apply for this space. Therefore we have the following open problem, which is similar to Open Problem 44 in Section 10.5.4, as well as to Open Problem 3 in Volume I for trigonometric polynomials.

Open Problem 55.

- Characterize intractability of linear tensor product functionals defined over the tensor product of F_1 with the reproducing kernel K_1 given by (11.29) for the normalized error criterion in the worst case setting.

11.6 Which Linear Tensor Product Functionals Are Tractable?

We return to the question of characterizing linear tensor product functionals that are weakly, polynomially, or strongly polynomially tractable or that are intractable. As we know, the answer to this question depends very much on the space F_d. As we shall now see, we may answer this question for some spaces based on the previous results on decomposable kernels.

As before, let $F_d = H(K_d)$ be a tensor product reproducing kernel Hilbert space of real functions defined on $D_d = D_1 \times D_1 \times \cdots \times D_1$ with the kernel

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j,t_j) \quad \text{for all } x,t \in D_d.$$

We consider a linear tensor product functional $I_d(f) = \langle f, h_d\rangle_{F_d}$ with the representer

$$h_d(x) = \prod_{j=1}^{d} h_1(x_j) \quad \text{for all } x \in D_d.$$

For simplicity, we assume that the function $h_1^2(t)/K_1(t,t)$ attains its maximum, i.e., there exists a point $t^* \in D_1$ such that

$$\sup_{t\in D_1} \frac{h_1^2(t)}{K_1(t,t)} = \frac{h_1^2(t^*)}{K_1(t^*,t^*)}. \tag{11.30}$$

As before, we use the convention that $0/0 = 0$.

For any $a^* \in D_1$, let

$$F_{1,a^*} = \{f \in F_1 \mid f(a^*) = 0\}.$$

As we already observed, the subspace F_{1,a^*} has the reproducing kernel

$$R_{2,a^*}(x,t) = K_1(x,t) - R_{1,a^*}(x,t) \quad \text{with } R_{1,a^*}(x,t) = \frac{K_1(x,a^*)K_1(a^*,t)}{K_1(a^*,a^*)} \tag{11.31}$$

for all $x, t \in D_1$.

Theorem 11.15. *Consider a linear tensor product functional $I_d(f) = \langle f, h_d\rangle_{F_d}$ for f from the tensor product space F_d defined as above. Assume that (11.30) holds, and that for any a^* from D_1 the reproducing kernel R_{2,a^*} given by (11.31) is decomposable at a^*. For the absolute error criterion we additionally assume that $\|h_1\|_{F_1} \geq 1$. Let $I = \{I_d\}$. Then*

I is intractable for the absolute and normalized error criterion iff $e(1,1) > 0$.

More precisely, for $\varepsilon \in (0,1)$,

$$\text{either } \ n(\varepsilon,d) \leq 1 \quad \text{or} \quad n(\varepsilon,d) \ \text{ is exponentially large in } d.$$

Furthermore, $n(\varepsilon,d) \leq 1$ holds iff

$$I_d(f) = a^d f(t,t,\dots,t) \quad \text{for some real } a \text{ and } t \in [0,1].$$

Proof. We consider several cases for h_1.

Case 1. Assume that $h_1 = 0$. Then $I_d \equiv 0$, and obviously $n(\varepsilon,d) = 0$.

Case 2. Assume that $h_1 \neq 0$ and $e(1,1) = 0$. From the formula for $e(1,1)$ and the assumption (11.30), this may only happen if

$$\frac{h_1^2(t^*)}{K_1(t^*,t^*)} = \|h_1\|_{F_1}^2 > 0.$$

Hence

$$h_1^2(t^*) = \langle h_1, K_1(\cdot,t^*)\rangle_{F_1}^2 = \|h_1\|_{F_1}^2 K_1(t^*,t^*).$$

Since $\|K_1(\cdot,t^*)\|_{F_1} = \sqrt{K_1(t^*,t^*)}$, this means that h_1 is parallel to $K_1(\cdot,t^*)$. Since we are in a Hilbert space, two elements are parallel iff they are multiple of each other. Hence, $h_1 = \alpha K_1(\cdot,t^*)$ with $\alpha = h_1(t^*)/K_1(t^*,t^*)$. Therefore

$$h_1(x) = \frac{h_1(t^*)}{K_1(t^*,t^*)} K_1(x,t^*) \quad \text{for all } x \in D_1.$$

Then $I_1(f) = a\, f(t)$ for all $f \in F_1$, with $a = h_1(t^*)/K_1(t^*,t^*)$ and $t = t^*$. Due to the tensor product structure, $I_d(f) = a^d f(t,t,\dots,t)$ for all $f \in F_d$, and therefore $n(\varepsilon,d) \leq 1$.

Case 3. Assume finally that h_1 is such that $e(1,1) > 0$. For any $a^* \in D_1$, decompose

$$h_1 = h_{a^*,1,1} + h_{a^*,1,2} \quad \text{with } h_{a^*,1,1} \in H(R_{1,a^*}) \text{ and } h_{a^*,1,2} \in H(R_{2,a^*}).$$

We have

$$h_{a^*,1,1}(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)}\, K_1(x,a^*) \quad \text{for all } x \in D_1,$$

and

$$h_{a^*,1,2}(x) = h_1(x) - h_{a^*,1,1}(x) \quad \text{for all } x \in D_1.$$

Hence, $e(1,1) > 0$ implies that no matter how we choose a^*, the function $h_{a^*,1,2}$ is non-zero.

Since R_{2,a^*} is decomposable at a^*, Theorem 11.14 for $h_{a^*,1,1} \neq 0$ and (11.26) for $h_{a^*,1,1} = 0$ imply that if $h_{a^*,1,2,(0)}$ and $h_{a^*,1,2,(1)}$ are both non-zero then indeed I is intractable. This holds for the normalized error criteria as well as for the absolute error criterion since we assume in this case that $e(0,1) = \|h_1\|_{F_1} \geq 1$.

Let

$$D_{a^*,(0)} = \{x \in D_1 \mid x \leq a^*\} \quad \text{and} \quad D_{a^*,(1)} = \{x \in D_1 \mid x \geq a^*\}.$$

Since $h_{a^*,1,2,(0)}$ is equal to $h_{a^*,1,2}$ over $D_{a^*,(0)}$, and $h_{a^*,1,2,(1)}$ is equal to $h_{a^*,1,2}$ over $D_{a^*,(1)}$, with $h_{a^*,1,2}(a^*) = 0$, it is enough to prove that there exists $a^* \in D_1$ such that $h_{a^*,1,2}$ is not zero over both $D_{a^*,(0)}$ and $D_{a^*,(1)}$. Assume by contradiction that for every $a^* \in D_1$ we have

$$h_{a^*,1,2}\big|_{D_{a^*,(0)}} = 0 \quad \text{or} \quad h_{a^*,1,2}\big|_{D_{a^*,(1)}} = 0.$$

This means that for every $a^* \in D_1$ we have

$$h_1(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)} K_1(x,a^*) \quad \text{for all } x \in D_{a^*,(0)} \tag{11.32}$$

or

$$h_1(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)} K_1(x,a^*) \quad \text{for all } x \in D_{a^*,(1)}. \tag{11.33}$$

Define

$$\begin{aligned} a_L^* &= \sup\{a^* \in D_1 \mid \text{for which (11.32) holds}\},\\ a_R^* &= \inf\{a^* \in D_1 \mid \text{for which (11.33) holds}\}. \end{aligned}$$

Note that we cannot have $a_L^* < a_R^*$ since this contradicts that (11.32) or (11.33) holds for all $a^* \in D_1$. Hence, $a_L^* \geq a_R^*$. If $a_L^* > a_R^*$ then there is a^* for which both (11.32) and (11.33) hold. Then

$$h_1(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)} K_1(x,a^*) \quad \text{for all } x \in D_1,$$

and $e(1,1) = 0$, which is a contradiction.

Assume finally that $a_L^* = a_R^*$. Then

$$h_1(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)} K_1(x,a^*) \quad \text{for all } x \in D_1 \setminus \{a^*\}.$$

Since $a^* \in D_1$ then $h_{a^*,1,2}(a^*) = 0$ and $h_1(a^*) = h_{a^*,1,1}(a^*)$, so that

$$h_1(x) = \frac{h_1(a^*)}{K_1(a^*,a^*)} K_1(x,a^*) \quad \text{for all } x \in D_1,$$

and again $e(1,1) = 0$. This completes the proof. □

The essence of Theorem 11.15 is that for the normalized error criterion and for the absolute error criterion with $\|h_1\|_{F_1} \ge 1$, we have only two options, and these two options are not good. Either the problem is trivial and can be solved by the use of at most one function value, or the problem suffers from the curse of dimensionality. Nothing in between can happen.

For the absolute error criterion with $\|h_1\|_{F_1} < 1$ the situation is different. As we know, the problem may still be intractable even for $d = 1$. On the other hand, we also saw many problems that are strongly polynomially tractable in this case.

We now show that the assumptions of Theorem 11.15 hold for a number of standard spaces.

11.6.1 Example: Sobolev Spaces with $r = 1$ over $[0,1]^d$

We begin with the standard tensor product Sobolev space F_d, which has the reproducing kernel

$$K_d(x,t) = \prod_{j=1}^{d} \big(1 + \min(x_j,t_j)\big) \quad \text{for all } x,t \in [0,1]^d.$$

As we know, the inner product is now given by

$$\langle f,g \rangle_{F_d} = \sum_{\mathfrak{u} \subseteq \{1,2,\dots,d\}} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}},0)\, \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x_{\mathfrak{u}},0)\, \mathrm{d}x_u,$$

where we differentiate once with respect to variables in $\mathfrak{u}$ and the rest of variables are replaced by 0.

Note that now the function

$$g(t) = \frac{h_1^2(t)}{K_1(t,t)} = \frac{h_1^2(t)}{1+t} \quad \text{for all } t \in D_1 := [0,1],$$

is a continuous function defined on a compact set and therefore it attains its maximal and minimal values. Hence, (11.30) holds.

Furthermore, for $a^* \in D_1$ we have

$$R_{2,a^*}(x,t) = 1+\min(x,t)-\frac{(1+\min(x,a^*))(1+\min(t,a^*))}{1+a^*} \quad \text{for all } x,t \in [0,1].$$

For $x \le a^* \le t$ we have

$$R_{2,a^*}(x,t) = 1 + x - \frac{(1+x)(1+a^*)}{1+a^*} = 0,$$

and indeed R_{2,a^*} is decomposable at a^*. Hence, Theorem 11.15 is applicable for this space.

We add in passing that for this space and for the absolute error criterion with $\|h_1\|_{F_1} < 1$, we may have intractability and exponential dependence on ε^{-1} even for $d = 1$. This was shown in Example 10.4.3 of Section 10.4 of Chapter 10 for the Sobolev space anchored at 0. In fact, intractability holds for a dense set of linear functionals in this space. Note that the Sobolev space anchored at 0 is a subspace of the Sobolev space considered here.

We now turn to the Sobolev space whose reproducing kernel for $d = 1$ is

$$K_{1,\alpha}(x,t) = 1 + \tfrac{1}{2}\left[|x-\alpha| + |t-\alpha| - |x-t|\right] \quad \text{for all } x,t \in [0,1]$$

for some $\alpha \in [0,1]$. Obviously, for $\alpha = 0$ we have the previous case. This Sobolev space was called the second weighted Sobolev space in Appendix A of Volume I. Here, we have the unweighted case corresponding to $\gamma = 1$; the weighted case will be studied later in Chapter 12.

As we know, this space corresponds to the unweighted L_2 discrepancy anchored at α. We leave to the reader to check that the assumptions of Theorem 11.15 now hold for all α.

We now switch to the Sobolev space whose reproducing kernel for $d = 1$ is given by

$$K_1(x,t) = \min(x,t) - xt \quad \text{for all } x,t \in [0,1].$$

As we know, this space is related to the L_2 unanchored discrepancy.

Let us check the assumption (11.30). We now have

$$g(t) = \frac{h_1^2(t)}{t(1-t)} \quad \text{for all } t \in [0,1].$$

Keeping in mind that $h_1 \in F_1$ implies that $h_1(0) = h_1(1) = 0$, we see that $g(0) = g(1) = 0$, and g is continuous. That is why (11.30) holds. The reader may check that R_{2,a^*} is decomposable for all $a^* \in [0,1]$ and again Theorem 11.15 is applicable.

We now take the Sobolev space of absolutely continuous functions whose first derivatives are in $L_2([0,1])$ with the inner product

$$\langle f, g\rangle_{F_1} = \int_0^1 f(t)g(t)\,\mathrm{d}t + \int_0^1 f'(t)g'(t)\,\mathrm{d}t.$$

This space has the intriguing reproducing kernel,

$$K_1(x,t) = \beta\, f(1-\max(x,t))\, f(\min(x,t)) \quad \text{for all } x,t \in [0,1],$$

see Thomas-Agnan [304], where $\beta = 1/\sinh(1) = 2/(e-1/e)$, and $f(y) = \cosh(y)$.

This Sobolev space was called the first weighted Sobolev space in Appendix A of Volume I. As before, here we study the unweighted case, and defer the study of the weighted case to Chapter 12.

Note that the kernel values $K_1(x,t)$ are always positive, and for $h \in F_1$, the function $h_1^2(t)/K_1(t,t)$ as a continuous function over $[0,1]$ attains its maximum. So (11.30) holds.

For this kernel and for $a^*, x, t \in [0,1]$, we have

$$\begin{aligned} R_{2,a^*}(x,t) &\\ &= \beta\, f(1-\max(x,t))\, f(\min(x,t)) \\ &\quad - \beta\, \frac{f(1-\max(x,a^*))\, f(\min(x,a^*))\, f(1-\max(t,a^*))\, f(\min(t,a^*))}{f(1-a^*)\, f(a^*)}. \end{aligned}$$

Take $x \le a \le t$. Then

$$R_{2,a^*}(x,t) = \beta \left[f(1-t)\, f(x) - \frac{f(1-a^*)\, f(x)\, f(1-t)\, f(a^*)}{f(1-a^*)\, f(a^*)} \right] = 0,$$

so that R_{2,a^*} is decomposable at a^* and this holds for all $a^* \in [0,1]$.

Hence, Theorem 11.15 is applicable and only trivial linear tensor product functionals are tractable. In particular, multivariate integration is intractable and suffers from the curse of dimensionality. Note that for multivariate integration we now have $h_1 \equiv 1$ and $\|h_1\|_{F_1} = 1$, so the absolute and normalized error criteria coincide. The intractability of multivariate integration was known and proved originally in [280].

We stress that the specific form of f was not really used. It is only important that f is chosen such that K_1 is a reproducing kernel and that (11.30) holds. Decomposability at any a^* holds with no extra conditions on f.

It is then natural to ask for which functions $f : [0,1] \to \mathbb{R}$, the function

$$K_1(x,t) = f(1-\max(x,t))\, f(\min(x,t)) \quad \text{for all } x,t \in [0,1],$$

is a reproducing kernel. We now show that this holds iff

- $f(1-t)\, f(t) \ge 0$ for all $t \in [0,1]$,
- $|f(1-t)\, f(t)| \le |f(1-x)\, f(x)|$ for all $x \le t$ from $[0,1]$.

Indeed, we must have $K_1(t,t) = f(1-t)f(t) \ge 0$ for all $t \in [0,1]$. Furthermore, we must have $|K_1(x,t)| \le \sqrt{K_1(x,x)}\sqrt{K_1(t,t)}$, which for $x \le t$ implies

$$|f(1-t)\, f(t)| \le \sqrt{f(1-x)\, f(x)}\, \sqrt{f(1-t)\, f(t)}.$$

This condition can be simplified to

$$|f(1-t)\,f(t)| \le |f(1-x)\,f(x)| \quad \text{for all } x \le t \text{ from } [0,1].$$

This shows that the two conditions above are necessary. We now show that they are also sufficient for K_1 to be a reproducing kernel. Clearly, K_1 is symmetric so it is enough to prove that

$$\sum_{i,j=1}^{m} a_i a_j K_1(x_i, x_j) \ge 0$$

for all choices of integer m, real a_j and $x_j \in [0,1]$. Without loss of generality we can order x_j such that

$$0 \le x_1 \le x_2 \le \cdots \le x_m \le 1,$$

and then we need to show that

$$\sum_{i,j=1}^{m} a_i a_j\, f(1 - x_{\max(i,j)})\, f(x_{\min(i.j)})$$
$$= 2\sum_{i=1}^{m} a_i\, f(x_i) \sum_{j=i+1}^{m} a_j\, f(1-x_j) + \sum_{j=1}^{m} a_j^2\, f(x_j)\, f(1-x_j) \ge 0.$$

We can also assume that $f(1-x_j)$ are nonzero for all $j = 1, 2, \ldots, m$, since else we can eliminate the zero terms and reduce m. Then the second condition $|f(1-t)\,f(x)| \le |f(1-x)\,f(t)|$ is equivalent to the condition

$$0 \le \frac{f(x_i)}{f(1-x_i)} \le \frac{f(x_j)}{f(1-x_j)} \quad \text{for all } i \le j.$$

We now substitute

$$a_j = \frac{b_j}{f(1-x_j)} \quad \text{and} \quad z_j = \frac{f(x_j)}{f(1-x_j)} \quad \text{for } j = 1, 2, \ldots, m.$$

Note that $z_1 \le z_2 \le \cdots \le z_m$. Therefore

$$\sum_{i,j=1}^{m} a_i a_j\, f(1 - x_{\max(i,j)})\, f(x_{\min(i.j)}) = \sum_{i,j=1}^{m} b_i b_j \min(z_i, z_j) \ge 0$$

since $\min(z_i, z_j)$ corresponds to the function $\min(x,t)$ for $x, t \in [0, \infty)$ which is known to be the reproducing kernel. This completes the proof.

Examples of f satisfying the two conditions above include

- $f(x) = x$, and then

$$K_1(x,t) = (1 - \max(x,t))\,\min(x,t) = \min(x,t) - xt$$

  corresponds, as we know, to the L_2 unanchored discrepancy, and

- $f(x) = \sqrt{\beta}\,\cosh(x)$ that was our original motivating example.

Observe that the second condition always holds for $x = 1$, and it holds for $x = 0$ if $f(0) = 0$. For f positive over $(0, 1)$, this condition is equivalent to the condition

$$\frac{f(x)}{f(1-x)} \quad \text{is nondecreasing as a function of } x \in (0, 1).$$

For example, this holds for $f(x) = x^\alpha$ for $\alpha > 0$.

Hence, for all functions f satisfying the two conditions above and for which (11.30) holds, Theorem 11.15 applies for the univariate space F_1 with the kernel

$$K_1(x,t) = f(1 - \max(x,t))\, f(\min(x,t)) \quad \text{for all } x, t \in [0, 1].$$

For this space, only trivial linear functionals are tractable. □

11.6.2 Example: Sobolev Space with $r \geq 1$ over $\mathbb{R}^d$

We first consider the case of $r = 1$. For $d = 1$, consider the Sobolev space $F_1 = W^1(\mathbb{R})$ of real univariate functions defined over $D_1 := \mathbb{R}$ that are absolutely continuous and whose first derivative are in $L_2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle_{F_1} = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t + \int_{\mathbb{R}} f'(t)g'(t)\,\mathrm{d}t.$$

The reproducing kernel of this space was found in Thomas-Agnan [304] and is equal to

$$K_1(x,t) = \tfrac{1}{2}\exp(-|x-t|) \quad \text{for all } x, t \in \mathbb{R}.$$

For $d \geq 1$, define $F_d = W^{1,1,\dots,1}(\mathbb{R}^d)$ as the d-fold tensor product of F_1. Then the inner product of F_d is given by

$$\langle f, g\rangle_{F_d} = \sum_{\mathfrak{u}\subseteq\{1,2,\dots,d\}} \int_{\mathbb{R}^d} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x)\,\mathrm{d}x,$$

where we differentiate once with respect to variables present in the subset $\mathfrak{u}$.

Note that now the function

$$g(t) = \frac{h_1^2(t)}{K_1(t,t)} = 2\,h_1^2(t) \quad \text{for all } t \in \mathbb{R}.$$

Although the function g is defined on a non-compact set, it attains the supremum since h_1 is continuous and $\int_0^1 h_1^2(t)\,\mathrm{d}t < \infty$, and therefore h_1 vanishes at infinity. Hence, (11.30) holds.

Furthermore, for $a^* \in D_1$ we have

$$R_{1,a^*}(x,t) = \frac{K_1(x,a^*)K_1(t,a^*)}{K_1(a^*,a^*)} = \tfrac{1}{2}\exp\left(-|x-a^*| - |t-a^*|\right)$$

for all $x, t \in \mathbb{R}$. Then

$$R_{2,a^*}(x,t) = K_1(x,t) - R_{1,a^*}(x,t)$$
$$= \tfrac{1}{2}\left[\exp(-|x-t|) - \exp(-|x-a^*| - |t-a^*|)\right]$$

for all $x, t \in \mathbb{R}$. For $x \le a^* \le t$ we have

$$R_{2,a^*}(x,t) = \tfrac{1}{2}\left[\exp(x-t) - \exp(x - a^* + a^* - t)\right] = 0,$$

and R_{2,a^*} is decomposable at a^*. Hence, Theorem 11.15 is applicable also for this space. □

We now show two natural examples for which Theorem 11.15 is *not* applicable. The first example is for a tensor product Sobolev space of periodic functions defined on $[0,1]^d$ whose reproducing kernel for $d = 1$ is given by

$$K_1(x,t) = 1 + \min(x,t) - xt \quad \text{for all } x, t \in [0,1]. \tag{11.34}$$

As in Section 9.5.5 of Chapter 9, the space F_1 is then

$$F_1 = \{ f \in W_2^1([0,1]) \mid f(0) = f(1) \},$$

with the inner product

$$\langle f, g \rangle_{F_1} = f(1)g(1) + \int_0^1 f'(t)g'(t)\,\mathrm{d}t.$$

We may convince ourselves that R_{2,a^*} is *not* decomposable for all $a^* \in (0,1)$ even without formally checking that indeed decomposability does not hold. Decomposability at a^* means that for any function f from F_1, the behavior of f over $[0,a^*)$ is independent of its behavior over $(a^*,1)$. But in our case f is periodic and has the same value at 0 and 1, and this property contradicts decomposability. So, for this space Theorem 11.15 cannot be applied.

The second example is for the Sobolev space that was called the third Sobolev space in Appendix A of Volume I. This is the space F_1 of absolutely continuous functions whose first derivatives are in $L_2([0,1])$ with the inner product

$$\langle f, g \rangle_{F_1} = \int_0^1 f(t)\,\mathrm{d}t \int_0^1 g(t)\,\mathrm{d}t + \int_0^1 f'(t)g'(t)\,\mathrm{d}t.$$

The reproducing kernel of this space is

$$K_1(x,t) = 1 + \tfrac{1}{2}B_2(|x-t|) + (x-\tfrac{1}{2})(t-\tfrac{1}{2}) \quad \text{for all } x, t \in [0,1], \tag{11.35}$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial of degree 2.

Again we may check that R_{2,a^*} is not decomposable for all $a^* \in (0,1)$. Indeed, suppose that R_{2,a^*} is decomposable at some a^*. Then for any $f \in H(R_{2,a^*})$ we have

$f = f_{(0)} + f_{(1)}$ with $f_{(0)}$ vanishing over $[a^*, 1]$ and $f_{(1)}$ vanishing over $[0, a^*]$, and $\|f\|_{F_1}^2 = \|f_{(0)}\|_{F_1}^2 + \|f_{(1)}\|_{F_1}^2$. Hence

$$\|f\|_{F_1}^2 = \left[\int_0^{a^*} f_{(0)}(t)\,\mathrm{d}t\right]^2 + \left[\int_{a^*}^1 f_{(1)}(t)\,\mathrm{d}t\right]^2 + \int_0^{a^*} \left[f'_{(0)}(t)\right]^2 \mathrm{d}t + \int_{a^*}^1 \left[f'_{(1)}(t)\right]^2 \mathrm{d}t.$$

On the other hand,

$$\|f\|_{F_1}^2 = \left[\int_0^1 f(t)\,\mathrm{d}t\right]^2 + \int_0^1 \left[f'(t)\right]^2\,\mathrm{d}t.$$

This implies that

$$\int_0^{a^*} f_{(0)}(t)\,\mathrm{d}t \int_{a^*}^1 f_{(1)}(t)\,\mathrm{d}t = 0 \quad \text{for all } f \in H(R_{2,a^*}),$$

which can happen only if $a^* = 0$ or $a^* = 1$. So for this space, Theorem 11.15 cannot be applied.

These two examples lead us to the next open problem.

Open Problem 56.

- Characterize tractability of linear tensor product functionals for the tensor product Sobolev space F_d with the univariate kernel given for $d = 1$ by (11.34).
- Characterize tractability of linear tensor product functionals for the tensor product Sobolev space F_d with the univariate kernel given for $d = 1$ by (11.35).

We now briefly consider the case $r \geq 2$. Theorem 11.15 uses a rank one modification of the original reproducing kernel, which usually is not enough for $r \geq 2$. Indeed, take the Sobolev space considered in the weighted integration example with $r \geq 2$. For $d = 1$, the reproducing kernel is given by

$$K_1(x,t) = \sum_{j=1}^{r-1} \frac{x^j}{j!}\frac{t^j}{j!} + 1_{M(x,t)} \int_{\mathbb{R}_+} \frac{(|t|-u)_+^{r-1}(|x|-u)_+^{r-1}}{((r-1)!)^2}\,\mathrm{d}u \tag{11.36}$$

for all $x, t \in \mathbb{R}$ and $M = \{(x,t) \mid xt \geq 0\}$. Then we needed to modify K_1 by the rank r construction of

$$R_1(x,t) = \sum_{j=1}^{r-1} \frac{x^j}{j!}\frac{t^j}{j!}$$

to obtain a decomposable kernel $R_2 = K_1 - R_1$. Hence, for this space Theorem 11.15 is also not applicable. We present this as our next open problem.

Open Problem 57.

- Characterize tractability of linear tensor product functionals for the tensor product Sobolev space F_d with the univariate kernel given by (11.36) for $r \geq 2$.

A natural modification of the Sobolev space considered in this example is a tensor product Sobolev space of smooth functions for which the norm for $d = 1$ is given by

$$\|f\| = \Big(\sum_{i=1}^{r} \|f^{(i)}\|^2_{L_2(D_1)}\Big)^{1/2} \tag{11.37}$$

for $r \geq 2$, where $D_1 = [0, 1]$ or $D_1 = \mathbb{R}$.

The reproducing kernel in this case is not explicitly known. Obviously, this makes the tractability analysis even harder. This leads us to the next open problem.

Open Problem 58.

- Characterize tractability of linear tensor product functionals for the tensor product Sobolev space F_d with the univariate norm given by (11.37) for $r \geq 2$.

We end this chapter with the following remark about decomposable kernels. The notion of decomposability is based on the assumption that D_1 is a subset of $\mathbb{R}$ and tensor products for the d-variate case are defined as the d-fold tensor products of the univariate case. We already mentioned in Volume I that some linear tensor product functionals or (more generally) operators may be defined by assuming that the univariate case is replaced by the m-variate case as naturally happens for some applications with $m > 1$, see Kuo and Sloan [154], Kwas [161], Kwas and Li [162], and Li [173] and NR 5.2.1 of Volume I. For such problems the notion of a decomposable kernel should be defined for D_1 being a subset of $\mathbb{R}^m$ with $m > 1$. This can be probably done by assuming that $D_1 = D_{(0)} \cup D_{(1)}$ for some disjoint sets $D_{(0)}$ and $D_{(1)}$ or some sets whose intersections has measure zero, and by assuming that the reproducing kernel $K_m \colon D_1 \times D_1 \to \mathbb{R}$ has the property that $K_m(x, t) = 0$ if $x \in D_{(0)}$ and $t \in D_{(1)}$. Then the point a^* formally disappears and it is not clear how the results based on the point a^* may be recovered. For instance, the role of a^* was important in Theorem 11.15 of this section where we assume the decomposable part of the kernel for all a^* from $D_1 \subseteq \mathbb{R}$. Also the case of complex numbers has not yet been covered by the notion of decomposable kernels. This leads us to the next open problem.

Open Problem 59.

- Generalize the results of this chapter for tensor product linear functionals defined as here but with D_1 being a subset of the m dimensional real or complex Euclidean space with $m > 1$.

11.7 Notes and Remarks

NR 11.1:1. This chapter is mostly based on the results originally obtained in the two papers [218] and [221], as well as a few new results that will be indicated in the successive sections.

NR 11.3:1. This section is based on [218], whereas the second part of Theorem 11.6 is taken from [329]. In this section, we presented reproducing kernel Hilbert spaces for which some linear tensor product functionals are polynomially tractable with ε^{-1} exponent equal to 0 and d exponent equal to 1, whereas other linear tensor product functionals suffer from the curse of dimensionality.

NR 11.4:1. This section is based entirely on [221]. The notion of a decomposable kernel was introduced in [221]. Theorem 11.8 is a slightly modified and strengthened version of Theorem 1 in [221] in which only the normalized error criterion was considered.

In the proof of Theorem 11.8, we approximated the corresponding part of the linear functional with zero error for a sample point t_j belonging to $D_{(b)}$. In general, this estimate is sharp, as illustrated for the example of two function values. However, for many practically important functionals the zero estimate is not sharp since one function value is not enough to approximate the corresponding part of the linear functionals with zero error. Therefore, one can strengthen Theorem 11.8 by assuming that each part of the linear functional can be approximated with a lower bound, say, $\|h_{d,(b)}\|_{F_d}(n_{(b)}+1)^{-p(d)}$, where $n_{(b)}$ is the number of function values with sample points from $D_{(b)}$. This may improve the total lower bound. We opted for simplicity and did not present this line of thought in the text. However, we encourage the reader to pursue this point.

NR 11.5:1. This section is also based on [221]. The main idea for obtaining a decomposable part of the reproducing kernel is to use a finite rank modification of the kernel. Equivalently, this means that we decompose the space into a finite dimensional space plus a space with a decomposable kernel. For the weighted integration example, a rank r modification was used, whereas for the uniform integration example a rank one modification was enough. We stress that a finite rank modification does not always work, as mentioned at the end of this section.

NR 11.5:2. Theorem 11.14 is a slightly improved version of Theorem 3 from [221]. In the proof of Theorem 11.14 we used a more accurate estimate of $\ln \binom{d}{\lfloor c\,d \rfloor}$ and obtain a better estimate on C.

NR 11.5.4:1. For uniform integration defined for the Sobolev space anchored at 0, we proved that

$$n(\varepsilon, d) \geq 1.0833^d (1 + o(1)).$$

This is slightly better than the result in [221], where the base of the exponent is 1.0628.

For uniform integration defined for the Sobolev space without boundary conditions, we proved that

$$n(\varepsilon, d) \geq 1.0202^d (1 + o(1)).$$

In Section 5 of [221] we performed similar computations but forgot that the number c in point 4 of Theorem 3 in [221] must satisfy the two inequalities and obtained wrongly $C = 1.0463\ldots$. The correct value of C is $1.0202\ldots$, as reported in this section.

NR 11.6:1. This section is new. It seems interesting to generalize Theorem 11.15 for more general tensor product spaces. One possibility may include the use of higher rank modifications of the original reproducing kernel to get decomposable parts. This would probably require us to assume a little more about the univariate case, i.e., that not only $e(1, 1)$ is positive but that at least a couple of initial errors $e(n, 1)$ are positive; such an assumption is usually not restrictive.

Chapter 12
Worst Case: Linear Functionals on Weighted Spaces

12.1 Introduction

In the previous chapters we studied linear functionals in the worst case setting for unweighted spaces. We obtained many intractability results, especially for the normalized error criterion. We showed that the curse of dimensionality is indeed present for the approximation of many linear functionals.

We now begin to extend the worst case analysis for approximation of linear functionals for weighted spaces. Our main goal is to find sharp conditions on the weights that yield various tractability results, and break the curse of dimensionality.

We define weighted linear functionals in Section 12.2, just as we did in Volume I for weighted linear operators. We consider general weights, which are specialized in the subsequent sections.

Section 12.3 is the first section in which we present lower bounds on the nth minimal worst case errors for weighted tensor product functionals defined over weighted reproducing kernel Hilbert spaces. The key assumption is that the reproducing kernel K_1 for $d = 1$ has a decomposable part. More precisely, we assume that

$$K_1 = R_1 + R_2$$

for two reproducing kernels for which the Hilbert spaces $H(R_j)$ have only the zero element in common and R_2 is decomposable. We generalize Theorems 11.12 and 11.14 from Chapter 11 by showing how weights affect lower bounds. This allows us to present conditions on the weights that disallow certain kinds of tractability. These conditions tell us that if the weights do not decay sufficiently fast then tractability cannot hold. The tractability conditions are mainly for the normalized error criterion, as well as for the absolute error criterion whenever the initial error is at least one. The last assumption is quite natural, since we have already seen that some linear functionals are tractable for the absolute error criterion with the initial error less than one, even for the unweighted case. As always, we illustrate our bounds by a number of examples.

The next Section 12.4 is on product weights. For such weights, we obtain more explicit tractability conditions. Roughly speaking, strong polynomial or strong T-tractability does not hold if the sum of product weights is not bounded as a function of d, whereas polynomial tractability does not hold if the sum of product weights divided by $\ln d$ is unbounded as a function of d. We also present a condition that rules out T-tractability. This condition states that T-tractability does not hold if the sum of product weights divided by $\ln T(\varepsilon^{-1}, d)$ is unbounded as a function of d for ε close to 1. We also present lower bounds on the d exponent whenever polynomial or T-tractability holds.

Our estimates are derived for bounded product weights. We show that the assumption on boundedness is indeed needed, and for unbounded product weights the tractability results are quite different.

Section 12.5 relaxes the assumption that the reproducing kernel R_2 should be decomposable. Instead, we assume that R_2 can be decomposed as $R_2 = R_{2,1} + R_{2,2}$ with the Hilbert spaces $H(R_{2,j})$ having only the zero element in common and with a decomposable $R_{2,2}$. It turns out that under this relaxed assumption we obtain practically the same tractability conditions as before. We illustrate the results of this section for weighted integration of smooth functions. We also show that tractability may depend on details of how the weighted spaces are defined. For product weights, it turns out that our necessary and sufficient conditions on tractability are the same only for $R_1 = 1$, whereas for more general R_1 they are different; in fact, tractability is open in this case. We are inclined to believe that $R_1 = 1$ is indeed needed, and that otherwise weighted integration is intractable and suffers from the curse of dimensionality. This is presented as Open Problem 62.

Section 12.6 deals with upper bounds on the nth minimal errors. Our approach is parallel to the approach in Chapter 10, where we derived upper bounds first for multivariate integration and then for linear functionals whose representers have a finite star norm $\| \cdot \|^*$, see Section 10.10 of Chapter 10. For multivariate integration in the weighted case, we obtain sufficient tractability conditions that resemble necessary tractability conditions. For the absolute error criterion these conditions are less restrictive than for the normalized error criterion, in particular, they hold if the representers have a small norm. For the normalized error criterion we need to assume a quite restrictive assumption that tells us that $H(R_1)$ must be a one-dimensional space and that the part of the representer must be related to R_1, see (12.11). For $R_1 \equiv 1$, these assumptions are satisfied only for standard multivariate integration. Nevertheless, in this case we obtain the same necessary and sufficient conditions on product weights for a number of multivariate integration problems defined over standard spaces. For arbitrary finite-order or finite-diameter weights, we obtain strong polynomial tractability for the normalized error criterion. As usual, we only have bounds on the exponent of strong tractability. For some spaces we know that the minimal value of the exponent can be achieved if we additionally assume more severe conditions on weights. Whether these extra conditions on weights are indeed needed is an open question, which is presented as Open Problem 63.

Section 12.7 deals with upper bounds for more general linear functionals. This generality is achieved at the expense of more severe conditions on weights. The reason for these extra conditions is that the star norm $\| \cdot \|^*_{F_d}$ is not a Hilbert norm, and so we can no longer say that parts of a representer are orthogonal. The tractability conditions are expressed in terms of the square root of weights. In this case, our tractability results are not so strong as for multivariate integration. In particular, for finite-order weights of order ω and the normalized error criterion, we obtain polynomial tractability with a d exponent at most 2ω instead of ω as before. For product weights and still for the normalized error criterion, the difference between necessary and sufficient conditions is even more significant. For example, if strong polynomial tractability holds, then the sum of product weights must be uniformly bounded in d; however, we are able to show strong polynomial tractability only if the sum of the squares of product weights is uniformly bounded in d. It is not clear if the last condition is really needed; we present this as Open Problem 64.

There are five open problems presented in this chapter, numbered from 60 to 64.

12.2 Weighted Linear Functionals

In the previous chapter we studied linear (unweighted) tensor product functionals. They were defined on tensor product spaces

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1, \quad d \text{ times},$$

where $F_1 = H(K_1)$ was a reproducing kernel Hilbert space with a reproducing kernel $K_1\colon D_1 \times D_1 \to \mathbb{R}$ and $D_1 \subseteq \mathbb{R}$. We considered

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d,$$

with

$$h_d(x) = \prod_{j=1}^{d} h_1(x_j) \quad \text{for } x \in D_d = D_1 \times D_1 \times \cdots D_1,\ d \text{ times},$$

and for $h_1 \in F_1$. Then we assumed that

$$K_1 = R_1 + R_2$$

for two reproducing kernels R_1 and R_2 whose reproducing kernel Hilbert spaces $H(R_j)$ satisfied

$$H(R_1) \cap H(R_2) = \{0\}.$$

Note that in this case, the reproducing kernel K_d of the space F_d was given by

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j,t_j) = \sum_{\mathfrak{u}\subseteq[d]} \prod_{j\notin\mathfrak{u}} R_1(x_j,t_j) \prod_{j\in\mathfrak{u}} R_2(x_j,t_j) \quad \text{for all } x,t \in D_d.$$

As always, $[d] = \{1,2,\dots,d\}$. For $\mathfrak{u} = \emptyset$ and $\mathfrak{u} = [d]$ the value of the product with the empty range is taken as 1.

This allowed us to decompose $f \in F_d$ as

$$f(x) = \sum_{\mathfrak{u}\subseteq[d]} f_{\bar{\mathfrak{u}},1}(x_{\bar{\mathfrak{u}}})\, f_{\mathfrak{u},2}(x_{\mathfrak{u}}),$$

where

$$f_{\bar{\mathfrak{u}},1} \in H(R_{\bar{\mathfrak{u}},1}) \quad \text{with } R_{\bar{\mathfrak{u}},1}(x_{\bar{\mathfrak{u}}}, t_{\bar{\mathfrak{u}}}) = \prod_{j\notin\mathfrak{u}} R_1(x_j,t_j),$$

and

$$f_{\mathfrak{u},2} \in H(R_{\mathfrak{u},2}) \quad \text{with } R_{\mathfrak{u},2}(x_{\mathfrak{u}}, t_{\mathfrak{u}}) = \prod_{j\in\mathfrak{u}} R_2(x_j, t_j).$$

Again, for $\mathfrak{u} = \emptyset$ and $\mathfrak{u} = [d]$ we take $f_{\emptyset,2} = 1$ and $f_{\overline{[d]},1} = 1$.

The inner product of f, g from F_d can be written as

$$\langle f, g\rangle_{F_d} = \sum_{\mathfrak{u}\subseteq[d]} \langle f_{\bar{\mathfrak{u}},1}, g_{\bar{\mathfrak{u}},1}\rangle_{H(R_{\bar{\mathfrak{u}},1})} \langle f_{\mathfrak{u},2}, g_{\mathfrak{u},2}\rangle_{H(R_{\mathfrak{u},2})}$$

with $\langle f_{\overline{[d]},1}, g_{\overline{[d]},1}\rangle_{H(R_{\overline{[d]},1})} = 1$ and $\langle f_{\emptyset,2}, g_{\emptyset,2}\rangle_{H(R_{\emptyset,2})} = 1$.

We are ready to generalize these notions for a weighted case. Assume that a sequence

$$\gamma = \{\gamma_{d,\mathfrak{u}}\}_{d\in\mathbb{N},\mathfrak{u}\subseteq[d]}$$

of *weights* is given with $\gamma_{d,\mathfrak{u}} \ge 0$. To omit the trivial problem, we always assume that there is a non-zero $\gamma_{d,\mathfrak{u}} > 0$ for every d.

Then we define the *weighted* space $F_{d,\gamma}$ as the Hilbert space with the reproducing kernel

$$K_{d,\gamma}(x,t) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\notin\mathfrak{u}} R_1(x_j, t_j) \prod_{j\in\mathfrak{u}} R_2(x_j, t_j) \quad \text{for all } x, t \in D_d.$$

The inner product in $F_{d,\gamma}$ takes the form

$$\langle f, g\rangle_{F_{d,\gamma}} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \langle f_{\bar{\mathfrak{u}},1}, g_{\bar{\mathfrak{u}},1}\rangle_{H(R_{\bar{\mathfrak{u}},1})} \langle f_{\mathfrak{u},2}, g_{\mathfrak{u},2}\rangle_{H(R_{\mathfrak{u},2})}.$$

As always, we adopt the convention that if $\gamma_{d,\mathfrak{u}} = 0$ then $f_{\mathfrak{u},2} = 0$ for all $f_{\mathfrak{u},2} \in H(R_{\mathfrak{u},2})$, and interpret $0/0 = 0$.

Obviously, if all $\gamma_{d,\mathfrak{u}} > 0$ then $F_{d,\gamma} = F_d$ and

$$\frac{1}{\max_{\mathfrak{u}} \gamma_{d,\mathfrak{u}}} \|f\|_{F_d} \le \|f\|_{F_{d,\gamma}} \le \frac{1}{\min_{\mathfrak{u}} \gamma_{d,\mathfrak{u}}} \|f\|_{F_d} \quad \text{for all } f \in F_d.$$

However, if some $\gamma_{d,\mathfrak{u}} = 0$, then $F_{d,\gamma}$ is a proper subspace of F_d.

For general weights, $F_{d,\gamma}$ is not a tensor product space, even though we started from the tensor product space F_d. However, if we consider *product* weights, i.e.,

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \text{for all } \mathfrak{u} \subseteq [d],$$

for some non-negative $\gamma_{d,j}$ and with $\gamma_{d,\emptyset} = 1$, then

$$K_d(x,t) = \prod_{j=1}^{d} \left[R_1(x_j, t_j) + \gamma_{d,j} R_2(x_j, t_j)\right] \quad \text{for all } x, t \in D_d,$$

and

$$F_{d,\gamma} = H(R_1 + \gamma_{d,1} R_2) \otimes H(R_1 + \gamma_{d,2} R_2) \otimes \cdots \otimes H(R_1 + \gamma_{d,d} R_2)$$

becomes a tensor product space.

We want to study the linear tensor product functional $I_d(f) = \langle f, h_d \rangle_{F_d}$ over the space $F_{d,\gamma}$. Note that $h_1 \in F_1$ means that h_1 has an orthogonal decomposition $h_1 = h_{1,1} + h_{1,2}$ with $h_{1,j} \in H(R_j)$ for $j = 1, 2$, and

$$\|h_1\|_{F_1}^2 = \|h_{1,1}\|_{H(R_1)}^2 + \|h_{1,2}\|_{H(R_2)}^2.$$

Define

$$h_{d,\gamma}(x) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \notin \mathfrak{u}} h_{1,1}(x_j) \prod_{j \in \mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d,$$

with the value of products for empty sets taken as 1. Clearly, $h_{d,\gamma} \in F_{d,\gamma}$. It is easy to see that

$$\langle f, h_d \rangle_{F_d} = \left\langle f, h_{d,\gamma} \right\rangle_{F_{d,\gamma}} \quad \text{for all } f \in F_{d,\gamma}.$$

Indeed, it is enough to check the last equality for f of the form

$$f = \sum_{i=1}^{m} a_i K_{d,\gamma}(\cdot, x_i)$$

for arbitrary integer m, reals a_i and points x_i from D_d. Then

$$\begin{aligned} \langle f, h_d \rangle_{F_d} &= \sum_{i=1}^{m} a_i \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \notin \mathfrak{u}} h_{1,1}((x_i)_j) \prod_{j \in \mathfrak{u}} h_{1,2}((x_i)_j) \\ &= \sum_{i=1}^{m} a_i h_{d,\gamma}(x_i) = \left\langle f, h_{d,\gamma} \right\rangle_{F_{d,\gamma}}, \end{aligned}$$

as claimed.

Hence, we will study in this chapter *weighted* linear functionals

$$I_{d,\gamma}(f) = \left\langle f, h_{d,\gamma} \right\rangle_{F_{d,\gamma}} \quad \text{for all } f \in F_{d,\gamma}.$$

Obviously, $I_{d,\gamma}(f) = I_d(f)$ for all $f \in F_{d,\gamma}$. This means that $I_{d,\gamma}$ is the restriction of the previous linear functional I_d to the weighted space $F_{d,\gamma}$. We change the notation from I_d to $I_{d,\gamma}$ to stress the change of the domain to the space $F_{d,\gamma}$ and the role of the weights. Observe that for $\gamma_{d,\mathfrak{u}} \equiv 1$ we have the unweighted case studied in the previous chapter. For general weights γ, we have

$$\|I_{d,\gamma}\|^2 = \|h_{d,\gamma}\|_{F_{d,\gamma}}^2 = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)} \|h_{1,2}\|_{H(R_2)}^{2|\mathfrak{u}|}, \quad \text{with } 0^0 = 1.$$

We now want to determine which weights γ make the problem $I_\gamma = \{I_{d,\gamma}\}$ tractable. In the next section we study lower bounds on the minimal worst case errors of $I_{d,\gamma}$ and necessary conditions on tractability of I_γ, and later sections contain upper bounds and sufficient conditions on tractability of I_γ.

We will address T-tractability, whose definition we recall from Section 9.7 of Chapter 9. We repeat that $T\colon [1,\infty) \times [1,\infty) \to [1,\infty)$ is a non-decreasing function of the two variables and it is non-exponential, that is,

$$\lim_{x+y\to\infty} \frac{T(x,y)}{x+y} = 0.$$

Without loss of generality we restrict ourselves to functions T for which

$$\lim_{y\to\infty} T(x,y) = \infty \quad \text{for all } x > 1.$$

Let $n(\varepsilon,d) = n^{\mathrm{wor}}(\varepsilon, I_{d,\gamma})$ denote the information complexity, that is, the minimal number of function values needed to compute an approximation with the worst case error $\varepsilon\, \mathrm{CRI}_d$, where $\mathrm{CRI}_d = 1$ for the absolute error criterion and $\mathrm{CRI}_d = \|I_{d,\gamma}\|$ for the normalized error criterion.

We say that I_γ is *T-tractable* if the information complexity $n(\varepsilon,d)$ satisfies

$$n(\varepsilon,d) \le C\, T(\varepsilon^{-1},d)^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}$$

for some non-negative numbers C and t. We say that I_γ is *strongly T-tractable* if

$$n(\varepsilon,d) \le C\, T(\varepsilon^{-1},1)^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}$$

for some non-negative numbers C and t. The infimum of t satisfying the last inequality is called the exponent of strong T-tractability.

For $T(x,y) = xy$, we recover polynomial tractability. In this case, we may distinguish between the exponents of ε^{-1} and d. That is, if

$$n(\varepsilon,d) \le C\, \varepsilon^{-p}\, d^q \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}$$

then p is called the ε^{-1} exponent and q the d exponent. We stress that they are not uniquely defined.

Finally, we say that I_γ is *weakly tractable* if[1]

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} = 0.$$

If weak tractability does not hold then we say that I_γ is *intractable*. If $n(\varepsilon,d)$ is exponential in d for some ε then we say that I_γ suffers from the curse of dimensionality. Obviously, the curse of dimensionality implies intractability.

[1]We remind the reader that we adopt the convention $\ln \mathbf{0} = \mathbf{0}$. Hence, for the trivial problem in which $n(\varepsilon,d) \equiv \mathbf{0}$ the condition on weak tractability is automatically satisfied.

12.3 Lower Bounds

In this section we generalize Theorems 11.12 and 11.14 from Chapter 11 by showing the corresponding lower bounds for the weighted case. As before, we first assume that the reproducing kernel R_2 is *decomposable*, i.e., there exists a real number a^* such that

$$R_2(x,t) = 0 \quad \text{for all } x,t \in D_1 \text{ such that } (x-a^*)(t-a^*) \le 0.$$

Since $h_1 = h_{1,1} + h_{1,2}$ with $h_{1,j} \in H(R_j)$, we can decompose

$$h_{1,2} = h_{1,2,(0)} + h_{1,2,(1)},$$

as in Chapter 11, with $h_{1,2,(0)} = h_{1,2}$ over $D_1 \cap \{x : x \le a^*\}$, and $h_{1,2,(1)} = h_{1,2}$ over $D_1 \cap \{x : x \ge a^*\}$. Then $h_{1,2,(j)} \in H(R_2)$ and $\langle h_{1,2,(0)}, h_{1,2,(1)} \rangle_{H(R_2)} = 0$.

We let $e(n,d) = e^{\text{wor}}(n, I_{d,\gamma})$ denote the nth minimal worst error of approximating $I_{d,\gamma}$ over the unit ball of $F_{d,\gamma}$.

Theorem 12.1. *Assume that*

$$H(R_1) \cap H(R_2) = \{0\} \quad \textit{and} \quad R_2 \textit{ is decomposable}.$$

Then

$$e^2(n,d) \ge \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_1^{d-|\mathfrak{u}|} \alpha_2^{|\mathfrak{u}|}, \quad \textit{with } 0^0 = 1,$$

where

$$\alpha_1 = \|h_{1,1}\|^2_{H(R_1)}, \quad \alpha_2 = \|h_{1,2}\|^2_{H(R_2)},$$

and

$$\alpha = \frac{\max\left(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)}\right)}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}} \in \left[\tfrac{1}{2}, 1\right].$$

Proof. We basically proceed as in the proof of Theorem 11.12. We decompose the square of the worst case error of an arbitrary linear algorithm $Q_{n,d}$ as

$$e^2(Q_{n,d}) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}\, e^2_{\mathfrak{u}},$$

where

$$e^2_{\mathfrak{u}} \ge \|h_{\bar{\mathfrak{u}},1}\|^2_{H(R_{\bar{\mathfrak{u}},1})} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \|h_{\mathfrak{u},2}\|^2_{H(R_{\mathfrak{u},2})} = \alpha_1^{d-|\mathfrak{u}|}\left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_2^{|\mathfrak{u}|},$$

as before. Since this holds for any algorithm $Q_{n,d}$, the same lower bound holds for the nth minimal error $e^2(n,d)$, which completes the proof. □

For $\gamma_{d,\mathfrak{u}} \equiv 1$, the sum over $\mathfrak{u}$ can be rewritten as the sum over the cardinality of $\mathfrak{u}$, and so Theorem 12.1 reduces to Theorem 11.12. In fact, for the *order-dependent* weights, when

$$\gamma_{d,\gamma} = \Gamma_{d,|\mathfrak{u}|}$$

for some non-negative sequence $\{\Gamma_{d,j}\}_{j\in[d],d\in\mathbb{N}}$, we have

$$e^2(n,d) \ge \sum_{k=0}^{d} \Gamma_{d,k} \binom{d}{k} \left(1 - n\,\alpha^k\right)_+ \alpha_1^{d-k}\alpha_2^k.$$

For general weights, we can rewrite the estimate on $e^2(n,d)$ as

$$e^2(n,d) \ge \sum_{k=0}^{d} \Big(\sum_{\mathfrak{u}:\,|\mathfrak{u}|=k} \gamma_{d,k} \Big) \left(1 - n\,\alpha^k\right)_+ \alpha_1^{d-k}\alpha_2^k.$$

Assume for a moment that $h_{1,1} = 0$ (which always holds for $R_1 = 0$). Then $\alpha_1 = 0$, and so Theorem 12.1 yields

$$e^2(n,d) \ge \gamma_{d,[d]}(1 - n\alpha^d)_+\,\alpha_2^d = (1 - n\alpha^d)_+\, e^2(0,d).$$

For $\alpha < 1$ and $e(0,d) > 0$, we then have intractability and the curse of dimensionality of I_γ for the normalized error criterion, since

$$n(\varepsilon,d) = n^{\mathrm{wor}}(\varepsilon, I_{d,\gamma}) \ge \alpha^{-d}\,(1-\varepsilon^2) \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}.$$

For the absolute error criterion, we have

$$n(\varepsilon,d) \ge \alpha^{-d}\Big(1 - \frac{\varepsilon^2}{\gamma_{d,[d]}\alpha_2^d}\Big).$$

Hence, we again have intractability and the curse of dimensionality if $\alpha < 1$ and $e(0,d) = \left(\gamma_{d,[d]}\alpha_2^d\right)^{1/2} \ge 1$.

From now on, we assume that $h_{1,1} \ne 0$, and denote

$$\alpha_3 = \frac{\alpha_2}{\alpha_1} = \frac{\|h_{1,2}\|^2_{H(R_2)}}{\|h_{1,1}\|^2_{H(R_1)}}. \tag{12.1}$$

Then the lower estimate of Theorem 12.1 can be rewritten as

$$\frac{e^2(n,d)}{e^2(0,d)} \ge \frac{\sum_{k=0}^{d} \left(\alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}\right)(1-n\alpha^k)_+}{\sum_{k=0}^{d} \left(\alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}\right)}.$$

For $d \in \mathbb{N}$ and $m \in \{0,1,\dots,d\}$, define

$$f(m,d) = \frac{\sum_{k=0}^{m} \alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\sum_{k=0}^{d} \alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}.$$

This function will be important for studying the tractability of I_γ. Clearly $f(m,d) \in [0,1]$. We are ready to formulate conditions on the lack of various notions of tractability of I_γ in terms of the function f.

Theorem 12.2. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in Section 12.2 in the worst case setting for the normalized error criterion or for the absolute error criterion with $e(0,d) \ge 1$. Assume that*

$$H(R_1) \cap H(R_2) = \{0\} \quad \text{and} \quad R_2 \text{ is decomposable}.$$

Assume also that $h_{1,1}$ as well as $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero.

- *If*
$$\limsup_{m\to\infty} \liminf_{d\to\infty} f(m,d) < 1$$
then I_γ is not *strongly T-tractable for any tractability function T.*

- *If*
$$\limsup_{m\to\infty} \liminf_{d\to\infty} f(\lceil m \ln d \rceil, d) < 1$$
then I_γ is not *polynomially tractable.*

- *For $\varepsilon \in (0,1)$, let*
$$a_\varepsilon := \limsup_{m\to\infty} \liminf_{d\to\infty} f(\lceil m \ln T(\varepsilon^{-1}, d) \rceil, d).$$
If[2]
$$\lim_{\varepsilon\to 1^-} a_\varepsilon < 1$$
then I_γ is not *T-tractable.*

- *If*
$$\liminf_{d\to\infty} f(\lceil c\, d \rceil, d) < 1 \quad \text{for some positive } c < 1$$
then I_γ is intractable and suffers from the curse of dimensionality.

Proof. It is enough to consider the normalized error criterion, since $e(0,d) \ge 1$ implies that the absolute error criterion is harder. Since $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero, we know that

$$\alpha = \frac{\max\left(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)}\right)}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}} \in \left[\tfrac{1}{2}, 1\right).$$

Take $a \in (0,1)$. For any integer n, define $s(a,n)$ as the smallest integer k for which

$$n\, \alpha^k \le a.$$

Obviously,

$$s(a,n) = \left\lceil \frac{\ln n/a}{\ln 1/\alpha} \right\rceil$$

[2] Clearly, a_ε is a non-increasing function of ε and therefore the limit of a_ε exists as ε goes to 1 from below.

and goes to infinity if a goes to zero or if n goes to infinity.

Using the proof of Theorem 11.14, we know that for $d \ge m \ge s(a,n)$ we have

$$\frac{e^2(n,d)}{e^2(0,d)} \ge (1-a)(1-f(m,d)).$$

Assume that

$$l^* := \limsup_{m\to\infty} \liminf_{d\to\infty} f(m,d) < 1.$$

Take $\delta \in (0, 1-l^*)$. Then there exists an integer m_δ such that

$$\liminf_{d\to\infty} f(m,d) < l^* + \delta \quad \text{for all } m \ge m_\delta.$$

This implies that

$$\liminf_{d\to\infty} \frac{e^2(n,d)}{e^2(0,d)} \ge (1-a)(1-l^*-\delta).$$

Since this holds for arbitrarily small positive a and δ, we also have

$$\liminf_{d\to\infty} \frac{e^2(n,d)}{e^2(0,d)} \ge 1-l^*.$$

We stress that this holds for arbitrarily large n. This proves that

$$\limsup_{d\to\infty} n(\varepsilon,d) = \infty \quad \text{for all } \varepsilon^2 \in (0,1-l^*),$$

and contradicts strong T-tractability, i.e., that $n(\varepsilon,d) \le C\, T(\varepsilon,1)^t$ for all d.

Assume now that

$$l^* := \limsup_{m\to\infty} \liminf_{d\to\infty} f(\lceil m \ln d \rceil, d) < 1.$$

Take an arbitrary positive integer q and let $n = d^q$. Choose $a > 0$. Then no matter how small a and how large q may be, for large d and large $m \in [d]$ we have

$$\lceil m \ln d \rceil \ge s(a, d^q) = \left\lceil \frac{q \ln d + \ln 1/a}{\ln 1/\alpha} \right\rceil.$$

Similarly as before, this proves that

$$\liminf_{d\to\infty} \frac{e^2(d^q,d)}{e^2(0,d)} \ge 1-l^* \quad \text{for all } q \in \mathbb{N},$$

and contradicts polynomial tractability, i.e., that $n(\varepsilon,d) \le C\, \varepsilon^{-p^*} d^{\,q^*}$ for some p^* and q^* and all $\varepsilon^2 \in (0,1-l^*)$ and $d \in \mathbb{N}$.

Assume that

$$l^* := \lim_{\varepsilon\to 1^-} a_\varepsilon < 1.$$

Then for any $\delta \in (0, 1 - l^*)$ there exists a positive ε_δ such that

$$a_\varepsilon = \limsup_{m\to\infty} \liminf_{d\to\infty} f(\lceil m \ln T(\varepsilon^{-1}, d))\rceil, d) \le l^* + \delta \quad \text{for all } \varepsilon \in (0, \varepsilon_\delta).$$

Take an arbitrary positive t and C, and let

$$n = \lceil C\, T(\varepsilon^{-1}, d)^t \rceil = C_{1,d} T(\varepsilon^{-1}, d)^t$$

with $C_{1,d} \ge C$. Note that since $T(\varepsilon^{-1}, d)$ goes to infinity with d, we have

$$C_{1,d} = C + o(d).$$

Then no matter how small a and large t and C may be, for large d and large $m \in [d]$ we have

$$\lceil m \ln T(\varepsilon^{-1}, d)\rceil \ge s(a, \lceil C\, T(\varepsilon^{-1}, d)^t\rceil) = \left\lceil \frac{t \ln T(\varepsilon^{-1}, d) + \ln C_{1,d}/a}{\ln 1/\alpha} \right\rceil.$$

Then we conclude as before that

$$\liminf_{d\to\infty} \frac{e^2(\lceil C\, T(\varepsilon^{-1}, d)^t\rceil), d)}{e(0, d)} \ge 1 - l^* - \delta \quad \text{for all real positive } t \text{ and } C.$$

This holds for all $\varepsilon \le \varepsilon_\delta$. Therefore this contradicts T-tractability, i.e., that $n(\varepsilon, d) \le C^*\, T(\varepsilon^{-1}\, d)^{t^*}$ for some C^* and t^* and all $\varepsilon^2 \in (0, 1 - l^*)$ and $d \in \mathbb{N}$.

Assume that

$$l^* := \liminf_{d\to\infty} f(\lceil c\, d\rceil, d) < 1 \quad \text{for some positive } c < 1.$$

For $a \in (0, 1)$, let

$$n = \lfloor a\alpha^{-c\,d} \rfloor.$$

Then $s(a, \lfloor a\alpha^{-c\,d}\rfloor) \le \lceil c\, d\rceil$, and therefore

$$\liminf_{d\to\infty} \frac{e(\lfloor a\alpha^{-c\,d}\rfloor, d)}{e(0, d)} \ge (1 - a)^{1/2}(1 - l^*)^{1/2}.$$

Hence, for $\varepsilon^* = \frac{1}{2}(1 - a)^{1/2}(1 - l^*)^{1/2}$ we have

$$n(\varepsilon^*, d) \ge a\, \alpha^{-c\,d}(1 + o(1)) \quad \text{as } d \to \infty$$

which shows that $n(\varepsilon^*, d)$ is exponential in d and I_γ suffers from the curse of dimensionality. This completes the proof. □

We now illustrate Theorem 12.2 by several examples.

12.3.1 Example: Constant and Almost Constant Weights

Assume first that $\gamma_{d,\mathfrak{u}} = \beta_d > 0$ for all $\mathfrak{u} \subseteq [d]$. Then $e(0,d) = \beta_d^{1/2}\,\|h_1\|_{F_1}^d$, and $e(0,d) \ge 1$ for $\beta_d \ge \|h_1\|_{F_1}^{-2d}$.

For the normalized error criterion, a positive β_d does not matter since the function f is invariant under a scaling of the weights, and we now have

$$f(m,d) = \frac{\sum_{k=0}^m \alpha_3^k \binom{d}{k}}{\sum_{k=0}^d \alpha_3^k \binom{d}{k}} = \frac{\sum_{k=0}^m \alpha_3^k \binom{d}{k}}{(1+\alpha_3)^d}.$$

We have the same situation as in Theorem 11.14, and therefore the last condition of Theorem 12.2 holds for any $c < \alpha_3/(1+\alpha_3)$. Hence, we have the curse of dimensionality.

Assume now that $\gamma_{d,\emptyset} = 1$ and $\gamma_{d,\mathfrak{u}} = \beta_d > 0$ for all non-empty $\mathfrak{u} \subseteq [d]$. Then

$$e^2(0,d) = \|h_{1,1}\|_{H(R_1)}^{2d}(1-\beta_d) + \beta_d\|h_1\|_{F_1}^{2d}.$$

Clearly, $e(0,d) > 1$ if $\|h_{1,1}\|_{H(R_1)} \ge 1$. For the normalized error criterion, we now have

$$f(m,d) = \frac{1+\beta_d \sum_{k=1}^m \alpha_3^k \binom{d}{k}}{1+\beta_d[(1+\alpha_3)^d - 1]}.$$

Take now $\beta_d = \beta^d$. Then for $\beta(1+\alpha_3) \ge 1$ we claim that I_γ is intractable. Indeed, as we did before in the proof of Theorem 11.14, it is easy to check that for small positive c and $m = \lceil c\,d \rceil$, the terms $\alpha_3^k \binom{d}{k}$ are increasing for all $k \le \lceil c\,d \rceil$, and therefore

$$\sum_{k=1}^m \alpha_3^k \binom{d}{k} \le \lceil c\,d \rceil \alpha_3^{\lceil c\,d \rceil} \binom{d}{\lceil c\,d \rceil} = \exp\left(c\,(\ln 1/c)\,d(1+o(c))\right).$$

Hence

$$\beta_d \sum_{k=1}^m \alpha_3^k \binom{d}{k} \le \exp\left([\ln(\beta) + c\ln 1/c]\ d(1+o(c))\right).$$

For $\beta(1+\alpha_3) > 1$ and small positive c, we have

$$\lim_{d\to\infty} f(\lceil c\,d \rceil) \le \lim_{d\to\infty} \exp\left([-\ln(1+\alpha_3) + c\ln 1/c]\ d(1+o(c))\right) = 0,$$

whereas for $\beta(1+\alpha_3) = 1$, we have

$$\liminf_{d\to\infty} f(\lceil c\,d \rceil) \le \frac{1 + \lim_{d\to\infty} \exp([-\ln(1+\alpha_3) + c\ln(1/c)]d(+o(c)))}{2} = \frac{1}{2}.$$

In both cases, the fourth case of Theorem 12.2 implies intractability.

For $\beta(1+\alpha_3) < 1$, observe that even for m independent of d, the values $f(m,d)$ go to 1 as d approaches infinity, and the condition on the lack of strong tractability is not satisfied. As we shall see in the next example, the condition $\beta(1+\alpha_3) < 1$ may even yield strong polynomial tractability. □

12.3.2 Example: Three Function Values

For $d = 1$, we take $D = [-1, 1]$, $R_1 \equiv 1$, and R_2 as a reproducing kernel such that $1 \notin H(R_2)$, R_2 is symmetric and decomposable at 0, and

$$R_2(-1, -1) = R_2(1, 1) = 1.$$

Then $K_1 = R_1 + R_2$ and $F_1 = H(K_1)$.

An example of such an R_2 is

$$R_2(x, t) = \begin{cases} \min(|x|, |t|) & \text{for } xt \geq 0, \\ 0 & \text{for } xt < 0. \end{cases}$$

In this case, the inner product of F_1 is

$$\langle f, g \rangle_{F_1} = f(0)g(0) + \int_{-1}^{1} f'(t)\, g'(t)\, \mathrm{d}t \quad \text{for all } f, g \in F_1.$$

Let

$$I_1(f) = f(0) + f(-1) + f(1) \quad \text{for all } f \in F_1$$

be the *sum of three function values* functional. Clearly,

$$I_1(f) = \langle f, h_1 \rangle_{F_1} \quad \text{with } h_1(x) = 1 + R_2(x, -1) + R_2(x, 1) \text{ for all } x \in D.$$

We now have $h_{1,1} = 1$, $h_{1,2} = R_2(\cdot, -1) + R_2(\cdot, 1)$ with $h_{1,2,(0)} = R_2(\cdot, -1)$ and $h_{1,2,(1)} = R_2(\cdot, 1)$, and therefore

$$\alpha_1 = \|h_{1,1}\|^2_{H(R_1)} = 1, \quad \alpha_2 = \|h_{1,2}\|^2_{H(R_2)} = 2, \quad \alpha_3 = 2, \quad \alpha = \tfrac{1}{2}.$$

For $d \geq 1$, we then have

$$I_d(f) = \sum_{b \in \{-1,0,1\}^d} f(b) \quad \text{for all } f \in F_d = F_1 \otimes F_1 \cdots \otimes F_1, \ d \text{ times.}$$

Hence, I_d consists of 3^d function values, and $I_d(f) = \langle f, h_d \rangle_{F_1}$ with

$$h_d(x) = \prod_{j=1}^{d} \left[1 + R_2(x_j, -1) + R_2(x_j, 1)\right] \quad \text{for all } x \in D_d = [-1, 1]^d.$$

We also have

$$\|I_d\| = \|h_d\|_{F_d} = 3^{d/2}.$$

For the unweighted case, Theorem 11.14 states that I_γ is intractable for both the absolute and normalized error criteria.

We now turn to the weighted case, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\emptyset} = 1$. We have $I_{d,\gamma}(f) = \langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}}$ with

$$h_{d,\gamma}(x) = 1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in [-1,1]^d.$$

The initial error is now

$$\|I_{d,\gamma}\| = \|h_{d,\gamma}\|_{F_{d,\gamma}} = \Big[1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} 2^{|\mathfrak{u}|}\Big]^{1/2}.$$

We need to have an upper bound on the information complexity $n(\varepsilon, d)$. Consider the following algorithm. For $n \le 3^d$, choose a subset A_n of $\{-1,0,1\}^d$ of cardinality n. Then

$$A_{n,d}(f) = \sum_{b \in A_n} f(b).$$

Clearly,

$$I_d(f) - A_{n,d}(f) = \sum_{b \in \{-1,0,1\}^d \setminus A_n} f(b) = \Big\langle f, h_{d,\gamma} - \sum_{b \in A_n} K_{d,\gamma}(\cdot, b) \Big\rangle_{F_{d,\gamma}}.$$

The square of the worst case error of $A_{n,d}$ is then

$$e^2(A_{n,d}) = \Big\| h_{d,\gamma} - \sum_{b \in A_n} K_{d,\gamma}(\cdot, b) \Big\|^2_{F_{d,\gamma}} = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} 2^{|\mathfrak{u}|} - \sum_{b \in A_n} \gamma_{d,\mathfrak{u}_b} 2^{|\mathfrak{u}_b|},$$

where $\mathfrak{u}_b = \{ j \in [d] \mid b_j \neq 0 \}$.

Based on this error formula, it is clear how to select the set A_n. First we order the sequence $\{\gamma_{d,\mathfrak{u}} 2^{|\mathfrak{u}|}\}$ such that

$$\gamma_{d,\mathfrak{u}_1} 2^{|\mathfrak{u}_1|} \ge \gamma_{d,\mathfrak{u}_2} 2^{|\mathfrak{u}_2|} \ge \cdots \ge \gamma_{d,\mathfrak{u}_{2^d}} 2^{|\mathfrak{u}_{2^d}|}.$$

Then we take

$$A_n = \big\{ \gamma_{d,\mathfrak{u}_1} 2^{|\mathfrak{u}_1|}, \gamma_{d,\mathfrak{u}_2} 2^{|\mathfrak{u}_2|}, \dots, \gamma_{d,\mathfrak{u}_n} 2^{|\mathfrak{u}_n|} \big\}$$

as the n largest normalized weights. For such a choice of A_n, we have

$$e^2(A_{n,d}) = e^2(0,d) - \sum_{j=1}^{n} \gamma_{d,\mathfrak{u}_j} 2^{|\mathfrak{u}_j|}.$$

Consider now the (almost constant) weights $\gamma_{d,\mathfrak{u}} = \beta^d > 0$ for all non-empty $\mathfrak{u} \subseteq [d]$. In the previous example, we already checked that the problem is intractable whenever $\beta(1+\alpha_3) = 3\beta \ge 1$. We want to show that for $3\beta < 1$, the problem is strongly polynomially tractable for both the absolute and normalized error criteria. Note that

$$e^2(0,d) = 1 + \beta^d (3^d - 1) = 1 + (3\beta)^d \, (1 - 3^{-d}) \to 1 \quad \text{as } d \to \infty,$$

so that there is no essential difference between the two error criteria for large d.

For the algorithm $A_{n,d}$ the largest normalized weight is $\gamma_{d,\mathfrak{u}_1} 2^{|\mathfrak{u}_1|} = 1$, and so

$$e^2(A_{n,d}) = 1 + \beta^d (3^d - 1) - 1 - \beta^d \sum_{k=1}^{n-1} 2^{|\mathfrak{u}_j|} \le (3\beta)^d - \beta^d \, (n-1).$$

We want to choose n such that $e(A_{n,d}) \le \varepsilon$. For $\varepsilon^2 \ge (3\beta)^d$ we can take $n = 1$. For $\varepsilon^2 = x(3\beta)^d$ with $x \in (0,1)$ we take

$$n = 1 + \lceil 3^d (1-x) \rceil = 1 + \left\lceil \varepsilon^{-2p} \, x^p (1-x) \, [\,3(3\beta)^p\,]^d \, \right\rceil.$$

If we take p such that $3(3\beta)^p = 1$, i.e.,

$$p = \frac{\ln 3}{\ln 1/(3\beta)},$$

then we have

$$n \le 1 + \left\lceil \varepsilon^{-2p} \right\rceil.$$

This proves strong polynomial tractability. However, note that p goes to infinity as β approaches $\frac{1}{3}$, and approaches zero as β goes to zero. Of course, for β going to zero, the problem becomes trivial, which is reflected in the bound on the exponent of strong polynomial tractability. □

12.3.3 Example: Order-Dependent Weights

We now assume that

$$\gamma_{d,\mathfrak{u}} = \Gamma_{d,|\mathfrak{u}|} \quad \text{for all } \mathfrak{u} \subseteq [d],$$

where $\Gamma_{d,j}$ are non-negative for $j \in [d]$ and $d \in \mathbb{N}$. Take first

$$\Gamma_{d,j} = \Gamma^j \quad \text{for all } j \in [d],\ d \in \mathbb{N},$$

for some positive Γ. We now have

$$e(0,d) = \left(\|h_{1,1}\|^2_{H(R_1)} + \Gamma \, \|h_{1,2}\|^2_{H(R_2)} \right)^{d/2}.$$

Hence, $e(0,d) \ge 1$ if $\|h_{1,1}\|_{H(R_1)} \ge 1$, independently of Γ.

Furthermore,

$$f(m,d) = \frac{\sum_{k=0}^{m} (\alpha_3 \Gamma)^k \binom{d}{k}}{(1+\alpha_3\Gamma)^d}.$$

This is exactly the same situation as in Theorem 11.14, but with α_3 replaced by $\alpha_3\Gamma$. Therefore

$$\lim_{d\to\infty} \frac{e(\lfloor C^d \rfloor, d)}{e(0,d)} = 1 \quad \text{for all } C \in \left(1, \alpha^{-\alpha_3\Gamma/(1+\alpha_3\Gamma)}\right).$$

Hence in this case, we have intractability and the curse of dimensionality no matter how large or small Γ may be. However, for small Γ the curse holds roughly for $C = \alpha^{-\alpha_3\Gamma/(1+\alpha_3\Gamma)}$ which goes to 1 as Γ goes to zero. So the curse of dimensionality is delayed for small Γ.

In any case, we will need smaller order-dependent weights if we want to vanquish the curse. Assume then that

$$\Gamma_{d,j} = \frac{\Gamma^j}{\binom{d}{j}\, d^{\alpha j}} \quad \text{for all } j \in [d],\ d \in \mathbb{N},$$

for some $\alpha > 0$. For such order-dependent weights we have

$$f(m,d) = \frac{1 - (\alpha_3\Gamma/d^\alpha)^{m+1}}{1 - (\alpha_3\Gamma/d^\alpha)^{d+1}}.$$

Hence,

$$\limsup_{d\to\infty} \liminf_{d\to\infty} f(m,d) = 1$$

and the condition guaranteeing the lack of T-tractability does *not* hold. Indeed, we may now even have strong polynomial tractability. To see this, let us return to the three function values example. We now have $\alpha_3 = 2$ and

$$e^2(0,d) = \frac{1 - (2\Gamma/d^\alpha)^{d+1}}{1 - 2\Gamma/d^\alpha} = 1 + o(1) \quad \text{as } d \to \infty,$$

so that there is no essential difference between the absolute and normalized error criteria for large d.

We again consider the algorithm $A_{d,n}$ with

$$n = \sum_{j=0}^{k} \binom{d}{k} = \frac{d^k}{k!}(1 + o(1)) \quad \text{as } d \to \infty,$$

for some integer $k \ge 0$ independent of d. Then the set $A_n = \{\mathfrak{u} \mid |\mathfrak{u}| \le k\}$, and so the square of the worst case error of $A_{n,d}$ is

$$e^2(A_{n,d}) = (2\Gamma/\, d^\alpha)^{k+1} \frac{1 - (2\Gamma/d^\alpha)^{d-k}}{1 - 2\Gamma/d^\alpha} = (2\Gamma/\, d^\alpha)^{k+1}(1 + o(1)) \quad \text{as } d \to \infty.$$

Without loss of generality we assume that d is so large that $2\Gamma/d^\alpha < 1$. For every $\varepsilon \in (0,1)$ there exists a unique non-negative integer $k^* = k^*(\varepsilon)$ such that

$$\varepsilon^2 \in \big[(2\Gamma/d^\alpha)^{k^*}, (2\Gamma/d^\alpha)^{k^*-1}\big).$$

Hence, $e(A_{n,d}) \le \varepsilon\, e(0,d)$ if we take $k = k^* - 1$. Note that for large d, we have

$$n = \frac{d^{k^*-1}(1 + o(1))}{(k^* - 1)!} \le C \left(\frac{d^\alpha}{2\Gamma}\right)^{p(k^*-1)/2} < C\, \varepsilon^{-p}$$

if we take large C and $p > 2/\alpha$. This indeed proves strong polynomial tractability with exponent at most $2/\alpha$.

In summary, we see that for some order-dependent weights we have the curse of dimensionality and for some other order-dependent weights we have strong polynomial tractability. It would be of interest to have a full characterization of order-dependent weights for which various notions of tractability hold. This leads us to the next open problem.

Open Problem 60.

- Characterize order-dependent weights for which I_γ is weakly tractable, polynomially tractable, T-tractable, strongly polynomially tractable or strongly T-tractable.
- Do tractability conditions for order-dependent weights depend on a given problem I_γ?

12.4 Product Weights

In this section we consider product weights. That is, we now assume that $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\emptyset} = 1$ and

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \text{for all non-empty } \mathfrak{u} \subseteq [d],$$

where $\gamma_{d,j}$ are non-negative for $j \in [d]$ and $d \in \mathbb{N}$. For product weights, we can extend Theorem 12.2 and find more explicit conditions on weights for which tractability does not hold. This is the subject of the next theorem.

Theorem 12.3. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in Section 12.2 in the worst case setting for the normalized error criterion or for the absolute error criterion with $e(0,d) \ge 1$. Here, γ is a sequence of bounded product weights, i.e.,*

$$\Gamma := \sup_{d\in\mathbb{N},\, j=1,2,\dots,d} \gamma_{d,j} < \infty.$$

Assume that

$$H(R_1) \cap H(R_2) = \{0\}, \quad \textit{with decomposable } R_2,$$

and that $h_{1,1}$ as well as $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero. Let $\lim^* \in \{\lim, \limsup\}$.

- *If*

$$\lim_{d\to\infty}{}^{*} \sum_{j=1}^{d} \gamma_{d,j} = \infty$$

then

$$\lim_{d\to\infty}{}^{*}\ \frac{e(n,d)}{e(0,d)} = 1 \quad \textit{for all } n \in \mathbb{N}$$

and I_γ is not *strongly T-tractable for any tractability function T.*

- *Let*

  $$a^* := \lim_{d\to\infty}{}^{*}\ \frac{\sum_{j=1}^{d}\gamma_{d,j}}{\ln d}.$$

  If $a^* = \infty$ then I_γ is not *polynomially tractable.*

  If $a^* \in (0,\infty)$ then define

  $$\beta := \sup\left\{ x \in \left[\frac{1}{1+\alpha_3\Gamma}, 1\right] \;\Big|\; \lim_{d\to\infty}{}^{*}\ \frac{\prod_{j=1}^{d}(1+\alpha_3\gamma_{d,j})}{\exp\left(x\alpha_3\sum_{j=1}^{d}\gamma_{d,j}\right)} \ge 1 \right\},$$

  where α_3 is given by (12.1), and $\beta^* = 1$ if $\beta > 1/e$, and $\beta^* \ln 1/\beta^* = \beta$ if $\beta \le 1/e$. Then

  $$\lim_{d\to\infty}{}^{*}\ \frac{e(\lceil d^q\rceil, d)}{e(0,d)} = 1 \quad \textit{for all } q \in \left(0, a^*\beta^*\alpha_3 \ln 1/\alpha\right),$$

  where α is as in Theorem 12.1.

  If polynomial tractability holds then the d exponent is at least

  $$a^*\beta^*\alpha_3 \ln 1/\alpha.$$

- *For $\varepsilon \in (0,1)$, let*

  $$a_\varepsilon^* := \lim_{d\to\infty}{}^{*}\ \frac{\sum_{j=1}^{d}\gamma_{d,j}}{\ln T(\varepsilon^{-1},d)}.$$

  If[3]

  $$\lim_{\varepsilon\to 1^-} a_\varepsilon^* = \infty$$

  then I_γ is not *T-tractable. If $\lim_{\varepsilon\to 1^-} a_\varepsilon^* \in (0,\infty)$ then*

  $$\lim_{d\to\infty}{}^{*}\ \frac{e(\lceil T(\varepsilon^{-1},d)^t\rceil, d)}{e(0,d)} = 1 \quad \textit{for all } t \in \left(0, a_\varepsilon^*\beta^*\alpha_3 \ln 1/\alpha\right)$$

  with β^* defined as before. Hence, if T-tractability holds then the exponent of T-tractability is at least

  $$\left[\lim_{\varepsilon\to 1^-} a_\varepsilon^*\right]\beta^*\alpha_3 \ln 1/\alpha.$$

[3] Now a_ε^* is a non-decreasing function of ε and therefore the limit of a_ε exists as ε goes to 1 from below.

- *If*

$$\gamma_{d,j} \geq \gamma^* \quad \textit{for all } j \in [d],\ d \in \mathbb{N}$$

then I_γ *is intractable and suffers from the curse of dimensionality since*

$$\lim_{d\to\infty} \frac{e(\lfloor C^d \rfloor, d)}{e(0,d)} = 1 \quad \textit{for all } C \in \left(1, \alpha^{-\gamma^*\alpha_3/(1+\gamma^*\alpha_3)}\right).$$

- *If*

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} > 0$$

then I_γ *is intractable and suffers from the curse of dimensionality.*

Proof. For $k = 0, 1, \dots, d$, define

$$C_{d,k} = \sum_{\mathfrak{u}\subseteq[d]:\ |\mathfrak{u}|=k} \alpha_3^k \prod_{j\in\mathfrak{u}} \gamma_{d,j}.$$

Note that $C_{d,0} = 1$. Then we can rewrite the lower bound on $e(n,d)/e(0,d)$ from the proof of Theorem 12.1 as

$$1 \geq \frac{e^2(n,d)}{e^2(0,d)} \geq \frac{\sum_{k=0}^d C_{d,k}(1-n\alpha^k)_+}{\sum_{k=0}^d C_{d,k}}.$$

Let

$$s_d = \alpha_3 \sum_{j=1}^d \gamma_{d,j}.$$

We now claim that

$$C_{d,k} \leq \frac{s_d^k}{k!} \quad \text{for } k = 0, 1, \dots, d.$$

Indeed, this clearly holds for $k = 0$. For $k \geq 1$, we have

$$s_d^k = \sum_{i_1,i_2,\dots,i_k\in[d]} \prod_{j=1}^k \alpha_3 \gamma_{d,i_j}.$$

Note that each term in $C_{d,k}$ is indexed by $\mathfrak{u} = \{u_1, u_1, \dots, u_k\}$ for distinct integers u_i and $u_i \in [d]$. Such terms appear in s_d^k. In fact, each term appears $k!$ times for all permutations of the u_i's. Therefore $k!C_{d,k} \leq s_d^k$, as claimed. Furthermore, the last inequality is not sharp for $k \geq 2$ since there are terms in s_d^k with no distinct i_j's that are not present in $C_{d,k}$.

For given positive integer n and $a \in (0,1)$, we have

$$n\,\alpha^k \leq a \quad \text{for } k \geq k(n,a) := \left\lceil \frac{\ln n/a}{\ln 1/\alpha} \right\rceil.$$

As before, we bound $e(n,d)/e(0,d)$ from below for $d > k(n,a)$ by

$$\frac{e^2(n,d)}{e^2(0,d)} \ge (1-a)\frac{\sum_{k=k(n,a)+1}^{d} C_{d,k}}{\sum_{k=0}^{d} C_{d,k}} = (1-a)\bigl(1-\alpha_{d,n,a}\bigr),$$

where

$$\alpha_{d,n,a} = \frac{\sum_{k=0}^{k(n,a)} C_{d,k}}{\sum_{k=0}^{d} C_{d,k}}.$$

Note that

$$\sum_{k=0}^{d} C_{d,k} = \prod_{j=1}^{d}\bigl(1+\alpha_3\gamma_{d,j}\bigr) = \exp\Bigl[\sum_{j=1}^{d}\ln(1+\alpha_3\gamma_{d,j})\Bigr].$$

For $x \in [0,c]$ with an arbitrary $c > 0$, we have

$$\frac{1}{1+c}\,x \le \ln(1+x) \le x.$$

In our case, $\gamma_{d,j} \in [0,\Gamma]$, so we can take $c = \alpha_3\Gamma$ and estimate

$$\exp(s_d/(1+\alpha_3\Gamma)) \le \prod_{j=1}^{d}\bigl(1+\alpha_3\gamma_{d,j}\bigr) \le \exp(s_d).$$

These two estimates show that β in Theorem 12.3 is well defined and at least equal to $1/(1+\alpha_3\Gamma)$. As we shall see later, depending on $\gamma_{d,j}$ it may take larger values.

To prove the first case of Theorem 12.3 it is enough to show that

$$\lim_{d\to\infty}\alpha_{d,n,a} = 0$$

for all $a \in (0,1)$ and $n \in \mathbb{N}$. Indeed, this implies that for large d, the ratio $e(n,d)/e(0,d)$ is greater than roughly $(1-a)^{1/2}$ and since a can be arbitrarily small, the limit is one, as claimed.

So we need to prove that $\alpha_{d,n,a}$ goes to zero as d approaches infinity. Since $C_{d,k} \le s_d^k/k!$ we have

$$\alpha_{d,n,a} \le \exp\Bigl[-\frac{s_d}{1+\alpha_3\Gamma}\Bigr]\sum_{k=0}^{k(n,a)}\frac{s_d^k}{k!}.$$

We now know that s_d goes to infinity for a subsequence of d tending to infinity in the case of $\lim^* = \limsup$ or for d tending to infinity in the case of $\lim^* = \lim$. The first factor of the bound above goes to zero exponentially fast with s_d, whereas the second factor is a polynomial in s_d of the fixed degree $k(n,a)$. Therefore the product of these two factors goes to zero. Hence, $\lim^*_{d\to\infty} s_d = \infty$ implies that

$\lim^*_{d\to\infty} e(n,d)/e(0,d) = 1$ for all n. This clearly yields that $\lim^*_{d\to\infty} n(\varepsilon,d) = \infty$ for all $\varepsilon \in (0,1)$, and contradicts strong T-tractability.

We now proceed to the second case of Theorem 12.3. Take now $n = n_d = \lceil d^q \rceil$ for $q \in (0, a^*\beta^*\alpha_3 \ln 1/\alpha)$. For $a^* = \infty$, this means that q can be arbitrarily large. For any finite or infinite a^*, we have

$$k(n,a) = k(n_d,a) = \left\lceil \frac{\ln\lceil d^q\rceil + \ln 1/a}{\ln 1/\alpha} \right\rceil = \frac{q}{\ln 1/\alpha}\,(1+o(1))\,\ln d \quad \text{as } d \to \infty.$$

Observe that $\beta^* \le 1$ and therefore $q < a^*\alpha_3 \ln 1/\alpha$.

We need to show that $\lim^*_{d\to\infty} \alpha_{d,n_d,a} = 0$. We now know that

$$\lim_{d\to\infty}{}^* \frac{s_d}{\ln d} = a^*\alpha_3.$$

This implies that for large d or for a subsequence of large d, we have

$$s_d > k(n_d,a).$$

Note that the function $s_d^k/k!$ of k is increasing over the interval $[0, k(n_d,a)]$. Therefore we can estimate

$$\sum_{k=0}^{k(n_d,a)} \frac{s_d^k}{k!} \le 1 + k(n_d,a)\,\frac{s_d^{k(n_d,a)}}{k(n_d,a)!} = 1 + \exp\big[k(n_d,a)\ln s_d - \ln(k(n_d,a)-1)!\big].$$

Using Stirling's formula for factorials we conclude that

$$\sum_{k=0}^{k(n_d,a)} \frac{s_d^k}{k!} \le 1 + \exp\left[\frac{q(1+o(1))}{\ln 1/\alpha}\,\ln(d)\,\ln\frac{s_d}{q\ln(d)/\ln 1/\alpha}\right].$$

Let $\delta \in (0,\beta)$. For large d or for a subsequence of large d, we also have

$$\prod_{j=1}^{d} \big(1+\alpha_3\gamma_{d,j}\big)^{-1} \le \exp(-(\beta-\delta)\,s_d).$$

This yields

$$\alpha_{d,n_d,a} \le \exp\left(-\ln d\left[\frac{(\beta-\delta)\,s_d}{\ln d} - \frac{q(1+o(1))}{\ln 1/\alpha}\ln\frac{s_d}{q\ln(d)/\ln 1/\alpha}\right]\right).$$

Assume that $a^* = \infty$. This means that $s_d/\ln d$ goes to infinity for a subsequence of d or for d tending to infinity depending on the meaning of $\lim^*$. In this case, the expression in the square bracket goes to infinity in the same sense, and $\alpha_{d,n_d,a}$ goes to zero, as needed.

Assume then that $a^* \in (0, \infty)$. Since δ can be arbitrarily small, the expression in the square bracket goes to

$$\beta\, a^* \alpha_3 - \frac{q}{\ln 1/\alpha} \ln \frac{a^* \alpha_3 \ln 1/\alpha}{q} = a^* \alpha_3 \Big(\beta - x \ln \frac{1}{x} \Big),$$

where $x := q/(a^* \alpha_3 \ln 1/\alpha) \in [0, 1]$. The maximum of the function $f(x) := x \ln 1/x$ is $1/e$. This implies that for $\beta > 1/e$, the right-hand side of the last formula is positive for all $q < \alpha^* \alpha_3 \ln 1/\alpha$. If $\beta \le 1/e$ then $\beta^* \le \beta$ and the function f is increasing over $[0, \beta^*]$ and its maximum is $\beta^* \ln 1/\beta^* = \beta$. Therefore, again the right-hand side of the last formula is positive for all $q < \alpha^* \beta^* \alpha_3 \ln 1/\alpha$.

Hence, in the same sense $\alpha_{d,n_d,a}$ goes to zero and proves the second case of Theorem 12.3. Therefore

$$\lim_{d\to\infty}{}^* \frac{e(\lceil d^q \rceil, d)}{e(0, d)} = 1$$

for all $q \in (0, a^* \beta^* \alpha_3 \ln 1/\alpha)$. The rest of this case is easy.

The third case of Theorem 12.3 is analogous to the previous case. We now take $n = n_{\varepsilon,d} = \lceil T(\varepsilon^{-1}, d)^t \rceil$, replace q by t, and $\ln d$ by $\ln T(\varepsilon^{-1}, d)$ and apply the previous argument.

We turn to the fourth case of Theorem 12.3. We now show that $\alpha_{d,n,a}$ is a non-increasing function of $\gamma_{d,j}$. Indeed, for $k = 0$ we have $C_{d,0} = 1$ and for $k \ge 1$, we have

$$\begin{aligned} C_{d,k} &= \alpha_3 \gamma_{d,d} \sum_{\substack{\mathfrak{u} \subseteq [d-1] \\ |\mathfrak{u}| = k-1}} \prod_{j \in \mathfrak{u}} \alpha_3 \gamma_{d,j} + \sum_{\substack{\mathfrak{u} \subseteq [d-1] \\ |\mathfrak{u}| = k}} \prod_{j \in \mathfrak{u}} \alpha_3 \gamma_{d,j} \\ &= \alpha_3 \gamma_{d,d} C_{d-1,k-1} + C_{d-1,k} \quad \text{with } C_{d-1,d} = 0. \end{aligned}$$

Therefore

$$\alpha_{d,n,a} = \frac{1 + \alpha_3 \gamma_{d,d} \sum_{k=1}^{k(n,a)} C_{d-1,k-1} + \sum_{k=1}^{k(n,a)} C_{d-1,k}}{1 + \alpha_3 \gamma_{d,d} \sum_{k=1}^{d} C_{d-1,k-1} + \sum_{k=1}^{d} C_{d-1,k}}.$$

From this formula it easily follows that $\alpha_{d,n,a}$ is a non-increasing function of $\gamma_{d,d}$. Since all $\gamma_{d,j}$'s play the same role in $\alpha_{d,n,a}$, we conclude that $\alpha_{d,n,a}$ is maximized for the smallest values of $\gamma_{d,j}$. Since now we assume that $\gamma_{d,j} \ge \gamma^*$, we have

$$\alpha_{d,n,a} \le \frac{\sum_{k=0}^{k(n,a)} (\alpha_3 \gamma^*)^k \binom{d}{k}}{\sum_{k=0}^{d} (\alpha_3 \gamma^*)^k \binom{d}{k}} = \frac{\sum_{k=0}^{k(n,a)} (\alpha_3 \gamma^*)^k \binom{d}{k}}{(1 + \alpha_3 \gamma^*)^d}.$$

The upper bound is exactly what we studied in the proof of Theorem 11.14 in Chapter 11 with α_3 replaced by $\alpha_3 \gamma^*$. The rest of the proof is the same as before.

Finally, the last fifth case of Theorem 12.3 assumes that

$$\gamma^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} > 0.$$

This means that for a subsequence of d we have

$$\gamma^* d(1+o(1)) = \sum_{j=1}^{d} \gamma_{d,j}.$$

Without loss of generality, we can order the weights so that $\gamma_{d,k} \ge \gamma_{d,k+1}$. Remembering that we consider bounded weights, so that $\gamma_{d,j} \le \Gamma$, then for every $k \in (1,d)$ we have

$$\gamma^* d(1+o(1)) \le (k-1)\Gamma + (d-k+1)\gamma_{d,k}.$$

Hence, for all $j \in [1,k]$ we can estimate

$$\gamma_{d,j} \ge \gamma_{d,k} \ge \gamma^*(1+o(1))\,\frac{d}{d-k+1} - \frac{(k-1)\Gamma}{d-k+1}.$$

Take, say, $k = \lceil d/s \rceil$ for $s = (2\Gamma/\gamma^*(1+o(1))+1)^{1/2}$. Then for large d,

$$\gamma_{d,j} \ge \tfrac{1}{2}\gamma^* \quad \text{for all } j \in [1, \lceil d/s \rceil].$$

We now choose $n = \lfloor C^d \rfloor$ with $C > 1$. For C close to 1 and large d, we have $k(n,a) \le \lceil d/s \rceil$. Then due to the property that $\alpha_{n,d,a}$ is a non-increasing function of $\gamma_{d,j}$, we obtain

$$\alpha_{d,n,a} \le \frac{\sum_{k=0}^{k(n,a)} [\alpha_3\gamma^*/2]^k \binom{\lceil d/s \rceil}{k}}{\sum_{k=0}^{\lceil d/s \rceil} [\alpha_3\gamma^*/2]^k \binom{\lceil d/s \rceil}{k}} = \frac{\sum_{k=0}^{k(n,a)} [\alpha_3\gamma^*/2]^k \binom{\lceil d/s \rceil}{k}}{(1+\alpha_3\gamma^*(1+o(1)))^{\lceil d/s \rceil}}.$$

This means that we have exactly the same situation as in the proof of Theorem 11.14 in Chapter 11, but with α_3 replaced by $\alpha_3\gamma^*/2$ and d replaced by $\lceil d/s \rceil$. We still have an exponential dependence on d, which means that I_γ is intractable and suffers from the curse of dimensionality. This completes the proof. □

Theorem 12.3 presents necessary conditions on various kinds of tractability of I_γ for bounded product weights. In particular, for

$$s_d = \alpha_3 \sum_{j=1}^{d} \gamma_{d,j}$$

we have

- strong tractability of I_γ may hold only if s_d is uniformly bounded,
- polynomial tractability of I_γ may hold only if s_d is bounded by a multiple of $\ln d$, and
- weak tractability of I_γ may hold only if s_d is essentially less then d.

This means that the bounded product weights must decay sufficiently fast if we want to obtain some notions of tractability of I_γ. As we shall see soon, for some functionals I_γ these conditions are also sufficient.

Theorem 12.3 also says that s_d must be a reasonably small multiple of $\ln d$ if we want to have polynomial tractability with a reasonable exponent of d. Indeed, if a^* is large then $\lceil d^q \rceil$ function values are not enough whenever q is proportional to a^*. The exponent q also depends on β^*. In general, β^* can be arbitrarily small if Γ is large enough. For example, for a positive number a and $d > 2$, take

$$\gamma_{d,j} = \begin{cases} a & \text{if } j = 1, 2, \dots, \lceil \ln d \rceil, \\ 0 & \text{if} j = \lceil \ln d \rceil + 1, \dots, d. \end{cases}$$

Then it is easy to check that $a^* = \Gamma = a$ and $\beta = 1/(a\alpha_3) \ln(1 + a\alpha_3)$. Hence, β and β^* are arbitrarily small for large a.

On the other hand, for $\gamma_{d,j} = \gamma_j$ independent of d with $\gamma_j \ge \gamma_{j+1}$ for all j, and $a^* = \lim_d{}^* \sum_{j=1}^d \gamma_j / \ln d > 0$, we always have $\beta = \beta^* = 1$. That is, β attains its maximal value. Indeed, we now have $\lim_j \gamma_j = 0$. Then for any positive δ there is an integer $d(\delta)$ such that

$$\ln(1 + \alpha_3 \gamma_j) \ge (1 - \delta)\alpha_3 \gamma_j \quad \text{for all } j \ge d(\delta).$$

Take now $x < 1 - \delta$. Then

$$\frac{\prod_{j=1}^d (1 + \alpha_3 \gamma_j)}{\exp\left(x\alpha_3 \sum_{j=1}^d \gamma_j\right)} \ge \frac{\prod_{j=1}^{d(\delta)} (1 + \alpha_3 \gamma_j)}{\exp\left(x\alpha_3 \sum_{j=1}^{d(\delta)} \gamma_j\right)} \exp\left[(1 - \delta - x)\alpha_3 \sum_{j=d(\delta)+1}^{d} \gamma_j\right].$$

Since $\sum_{j=1}^d \gamma_j$ and $\sum_{j=d(\delta)+1}^d \gamma_j$ both behave like $a^* \ln d$ and go to infinity with d or a subsequence of d, the right-hand side of the last formula goes to infinity as well. This proves that $\beta \ge x$ and since x can be arbitrarily close to 1, we have $\beta = 1$, as claimed. Hence, for product weights independent of d, the exponent q can be arbitrarily close to $a^* \alpha_3 \ln 1/\alpha$.

12.4.1 Example: Unbounded Weights

So far, we have discussed bounded product weights, $\gamma_{d,j} \le \Gamma < \infty$. We now show that this assumption is significant, and that Theorem 12.3 does not hold for unbounded weights.

For bounded weights, the conditions on s_d can be translated into conditions on the individual weights $\gamma_{d,j}$, as we did in the proof of Theorem 12.3. For example, we used

$$\exp(s_d/(1 + \alpha_3 \Gamma)) \le \prod_{j=1}^d \left(1 + \alpha_3 \gamma_{d,j}\right) \le \exp(s_d),$$

showing that $\prod_{j=1}^{d}(1+\alpha_3\gamma_{d,j})$ depends exponentially on s_d.

For unbounded weights, this is no longer true. Consider the weights $\gamma_{d,1} = d$ and $\gamma_{d,j} = 0$ for $j = 2, 3, \dots$. For these weights, we have the univariate case and polynomial tractability holds for many linear functionals, such as integration. However, $s_d = \alpha_3 d$ satisfies all the assumptions of Theorem 12.3 and should imply the curse of dimensionality. In this case, we have

$$\prod_{j=1}^{d}(1+\alpha_3\gamma_{d,j}) = 1 + \alpha_3 d = 1 + s_d.$$

So it is not exponential in s_d, and the proof of Theorem 12.3 breaks down.

12.4.2 Example: Weighted Integration (Continued)

We return to weighted integration, which was studied in Chapter 11. We now partially remove the boundary conditions by taking

$$R_1 \equiv 1 \quad \text{and} \quad R_2(x,t) = 1_M(x,t)\int_0^\infty \frac{(|t|-u)_+^{r-1}(|x|-u)_+^{r-1}}{[(r-1)!]^2}\,\mathrm{d}u$$

with $M = \{(x,t) : xt \ge 0\}$ and $r \in \mathbb{N}$. This corresponds to the space $F_1 = H(R_1 + R_2)$ of real functions f defined on $\mathbb{R}$ whose $(r-1)$st derivatives are absolutely continuous, and $f^{(r)} \in L_2(\mathbb{R})$ as well as f satisfies the boundary conditions, $f^{(j)}(0) = 0$ for $j = 1, 2, \dots, r-1$. The inner product in F_1 is given by

$$\langle f, g\rangle_{F_1} = f(0)g(0) + \int_{\mathbb{R}} f^{(r)}(t) g^{(r)}(t)\,\mathrm{d}t \quad \text{for all } f, g \in F_1.$$

The case when all boundary conditions are removed will be considered later.

As we know, R_2 is decomposable at 0. Since $r \ge 1$, the space $H(R_2)$ does not contain constant functions. Moreover, $H(R_1) = \mathrm{span}(1)$, and therefore $H(R_1) \cap H(R_2) = \{0\}$, as needed.

The weighted integration problem is given by

$$I_1(f) = \int_{\mathbb{R}} \varrho(t)\, f(t)\,\mathrm{d}t \quad \text{for all } f \in F_1,$$

where the non-negative weight function ϱ satisfies the conditions $\int_{\mathbb{R}} \varrho(t)\,\mathrm{d}t = 1$ and $\int_{\mathbb{R}} \varrho(t)\,|t|^{r-1/2}\,\mathrm{d}t < \infty$. This implies that

$$\int_{\mathbb{R}} \varrho(t)\, R_2(t,t)\,\mathrm{d}t < \infty.$$

We also assume that ϱ is a symmetric function, i.e., $\varrho(t) = \varrho(-t)$ for all $t \in \mathbb{R}$. It is easy to check that

$$h_{1,1} \equiv 1 \quad \text{and} \quad h_{1,2}(x) = \int_{\mathbb{R}} \varrho(t)\, R_2(x,t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}.$$

Furthermore, $\|h_{1,2,(0)}\|_{H(R_2)} = \|h_{1,2,(1)}\|_{H(R_2)} > 0$ and therefore $\alpha = \frac{1}{2}$. Obviously, $\alpha_1 = 1$ and

$$\alpha_3 = \alpha_2 = \|h_{1,2}\|^2_{H(R_2)} = \int_{\mathbb{R}^2} \varrho(t)\, \varrho(x)\, R_2(x,t)\, \mathrm{d}t\, \mathrm{d}x,$$

which depends on ϱ and r. Hence, all the assumptions of Theorem 12.3 are satisfied.

We consider product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for bounded $\gamma_{d,j} \in [0,\Gamma]$. Note that the initial error is now given by

$$e^2(0,d) = \prod_{j=1}^{d} \Big(1 + \gamma_{d,j} \int_{\mathbb{R}^2} \varrho(t)\, \varrho(x)\, R_2(x,t)\, \mathrm{d}t\, \mathrm{d}x\Big).$$

Hence, the initial error is always at least one, so that the absolute error is harder than the normalized error.

We now check that the tractability conditions presented in Theorem 12.3 are also sufficient for tractability of weighted integration for both the absolute and normalized error criteria. We need to get matching upper bounds. It is enough to get upper bounds for the absolute error criterion, since they will be also valid for the normalized error criterion. We may use one of the upper bounds presented in Chapter 10. It will be enough to use the most simple bound given in Theorem 10.4. From this theorem (with $g_d \equiv 1$) we know that there exists a QMC algorithm $A_{n,d}$ such that

$$e(n,d) \le e^{\mathrm{wor}}(A_{n,d}) \le \prod_{j=1}^{d} \Big(1 + \gamma_{d,j} \int_{\mathbb{R}} \varrho(t)\, R_2(t,t)\, \mathrm{d}t\Big)^{1/2} n^{-1/2}. \tag{12.2}$$

This estimate is enough for our purpose. Indeed, let us first consider strong polynomial tractability of I_γ. From Theorem 12.3 we know that $\limsup_d \sum_{j=1}^d \gamma_{d,j} < \infty$ is a necessary condition. If this holds then

$$A := \sup_{d\in\mathbb{N}} \prod_{j=1}^{d} \Big(1 + \gamma_{d,j} \int_{\mathbb{R}} \varrho(t)\, R_2(t,t)\, \mathrm{d}t\Big) < \infty,$$

and (12.2) yields

$$n(\varepsilon,d) \le \left\lceil \frac{A}{\varepsilon^2} \right\rceil$$

for the absolute error criterion. Hence, strong polynomial tractability holds with an exponent at most 2.

Similarly, strong T-tractability holds if we assume that

$$B := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty$$

and then the exponent is at most $2B$. It is easy to see that $B < \infty$ is also a necessary condition, not only for strong T-tractability but also for T-tractability whenever we assume that

$$\varrho(t) \ge c > 0 \quad \text{for } t \in [a,b] \text{ for some } a < b.$$

Indeed, the assumption on ϱ implies that the nth minimal error of the weighted integration problem is $\Omega(n^{-r})$, and therefore we have $n(\varepsilon, 1) = \Omega(\varepsilon^{-r})$. The lower bound on $n(\varepsilon, 1)$ can be bounded by $C\, T(\varepsilon^{-1}, 1)^t$ for some C and t only if $B < \infty$.

We now consider polynomial tractability. Theorem 12.3 states that we now must have

$$a^* = \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln d} < \infty.$$

Let $C := \int_{\mathbb{R}} \varrho(t)\, R_2(t,t)\, \mathrm{d}t$. Then for any positive δ we have

$$\prod_{j=1}^d \bigl(1+\gamma_{d,j} C\bigr) \le \exp\Bigl(C \sum_{j=1}^d \gamma_{d,j}\Bigr) = d^{\,C\sum_{j=1}^d \gamma_{d,j}/\ln d} \le C_\delta\bigl(d^{\,C(a^*+\delta)}\bigr),$$

where C_δ does not depend on d. This and (12.2) imply that

$$n(\varepsilon, d) \le C_\delta d^{\,C(a^*+\delta)} \varepsilon^{-2},$$

and so we have polynomial tractability. We now discuss the exponent of d. For simplicity, we take product weights independent of d for which $\beta^* = 1$, as already shown. Theorem 12.3 tells us that the exponent of d must be at least $a^* \alpha_3 \ln 2$, whereas the last upper bound has an exponent of d arbitrarily close to $C\, a^*$. Observe that $\alpha_3 \le C$, and therefore the lower and upper bounds on the exponent of d differ by a factor $\alpha_3 \ln(2)/C = \alpha_3 0.693147\ldots/C < 1$.

For T-tractability, we know that

$$l^* := \lim_{\varepsilon\to 1^-} \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)} < \infty.$$

Proceeding similarly as before, we conclude that

$$n(\varepsilon, d) \le C_\delta T(\varepsilon^{-1}, d)^{C(l^*+\delta)} \varepsilon^{-2}.$$

For small ε, we can estimate ε^{-1} by $T(\varepsilon^{-1}, 1)^{B+\delta} \le T(\varepsilon^{-1}, d)^{B+\delta}$. Therefore for any positive δ there exists a number M_δ such that

$$n(\varepsilon, d) \le M_\delta T(\varepsilon^{-1}, d)^{C\, l^* + 2B + \mathcal{O}(\delta)} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Hence, the problem is T-tractable.

Finally, we address weak tractability. From Theorem 12.3 we now must have $\limsup_d \sum_{j=1}^d \gamma_{d,j}/d = 0$. Then (12.2) implies that

$$n(\varepsilon, d) \le \Bigl\lceil \varepsilon^{-2} \prod_{j=1}^d (1+\gamma_{d,j} C) \Bigr\rceil \le 2\varepsilon^{-2} \prod_{j=1}^d (1+\gamma_{d,j} C)$$

and

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln \max(1, n(\varepsilon, d))}{\varepsilon^{-1}+d} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln 2 + 2\ln \varepsilon^{-1} + C\ \sum_{j=1}^{d} \gamma_{d,j}}{\varepsilon^{-1}+d} = 0.$$

Hence we have weak tractability. We summarize these properties in the following corollary.

Corollary 12.4. *Consider weighted integration I_γ for bounded product weights defined as in this example. For T-tractability we additionally assume that*

$$\varrho(t) \ge c > 0 \quad \textit{for } t \in [a,b] \textit{ for some } a, b \textit{ and } c \textit{ with } a < b.$$

Then tractability conditions for the absolute and normalized error criteria are the same. More precisely,

- *I_γ is strongly polynomially tractable iff* $\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$.
- *I_γ is strongly T-tractable iff*

$$\limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty \quad \textit{and} \quad \limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

- *I_γ is polynomially tractable iff*

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty.$$

- *I_γ is T-tractable iff*

$$\limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty \quad \textit{and} \quad \lim_{\varepsilon\to 1^-} \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)} < \infty.$$

- *I_γ is weakly tractable iff*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$

We also indicated some upper bounds on the exponents of various kinds of tractability. Lower bounds on these exponents are much harder to obtain. In particular, the exponents may depend on the weight function ϱ. Similar problems were considered in [333], but only for the univariate case in terms of the order of convergence. It turns out that the order of convergence may indeed vary depending on ϱ. The tractability exponents in terms of ϱ have not been yet analyzed. This leads us to the next open question.

Open Problem 61.

- Find the exponents of strong polynomial tractability and strong T-tractability for weighted integration considered in this example at least for some weight functions ϱ.
- Study the dependence of these exponents in terms of the weight function ϱ.

12.5 Further Lower Bounds

In the previous section we assumed that the reproducing kernel R_2 was decomposable. For many linear functionals, the kernel R_2 is *not* decomposable, but has a decomposable part. That is, we have

$$R_2 = R_{2,1} + R_{2,2}$$

where $R_1, R_2 \colon D_1 \times D_1 \to \mathbb{R}$ are reproducing kernels such that

$$H(R_{2,1}) \cap H(R_{2,2}) = \{0\} \quad \text{and} \quad R_{2,2} \text{ is decomposable.}$$

The subject of this section is to extend lower bounds obtained in Section 12.3 to these more general reproducing kernels.

The weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$, the space $F_{d,\gamma}$ and the linear functionals $I_{d,\gamma}$ are defined as in Section 12.2. In particular, we have $I_{d,\gamma}(f) = \langle f, h_{d,\gamma}\rangle_{F_{d,\gamma}}$ with

$$h_{d,\gamma}(x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\notin\mathfrak{u}} h_{1,1}(x_j) \prod_{j\in\mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d.$$

Here, $h_1 = h_{1,1} + h_{1,2}$ with $h_{1,j} \in H(R_j)$ for $j = 1, 2$. We now decompose

$$h_{1,2} = h_{1,2,1} + h_{1,2,2} \quad \text{with } h_{1,2,j} \in H(R_{2,j}).$$

Furthermore, since $R_{2,2}$ is decomposable, say at a^*, we have

$$h_{1,2,2} = h_{1,2,2,(0)} + h_{1,2,2,(1)}$$

with

$$h_{1,2,2,(0)}(x) = \begin{cases} h_{1,2,2}(x) & \text{for } x \in D_1 \cap \{x : x \le a^*\} \\ 0 & \text{for } x \in D_1 \cap \{x : x \ge a^*\}, \end{cases}$$

and

$$h_{1,2,2,(1)}(x) = \begin{cases} h_{1,2,2}(x) & \text{for } x \in D_1 \cap \{x : x \ge a^*\} \\ 0 & \text{for } x \in D_1 \cap \{x : x \le a^*\}, \end{cases}$$

and $h_{1,2,2,(j)} \in H(R_{2,2})$. We introduce the following notation

$$\begin{aligned} \alpha_1 &= \|h_{1,1}\|^2_{H(R_1)}, & \alpha_2 &= \|h_{1,2}\|^2_{H(R_2)}, \\ \alpha_{2,1} &= \|h_{1,2,1}\|^2_{H(R_{2,1})}, & \alpha_{2,2} &= \|h_{1,2,2}\|^2_{H(R_{2,2})}. \end{aligned}$$

Obviously, $\alpha_2 = \alpha_{2,1} + \alpha_{2,2}$ and

$$\alpha_{2,2} = \|h_{1,2,2,(0)}\|^2_{H(R_{2,2})} + \|h_{1,2,2,(0)}\|^2_{H(R_{2,2})}.$$

We now have

$$\begin{aligned} e^2(0,d) &= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \alpha_1^{d-|\mathfrak{u}|} \alpha_2^{|\mathfrak{u}|} \\ &= \sum_{k=0}^{d} \Big(\sum_{\substack{\mathfrak{u}\subseteq[d] \\ |\mathfrak{u}|=k}} \gamma_{d,\mathfrak{u}} \Big) \alpha_1^{d-k} \sum_{j=0}^{k} \binom{k}{j} \alpha_{2,1}^{k-j} \alpha_{2,2}^{j}. \end{aligned}$$

We are ready to generalize Theorem 11.14.

Theorem 12.5. *Assume that*

$$H(R_1) \cap H(R_2) = H(R_{2,1}) \cap H(R_{2,2}) = \{0\} \quad \textit{and} \quad R_{2,2} \textit{ is decomposable}.$$

Then

$$e^2(n,d) \ge \sum_{k=0}^{d} \Big(\sum_{\substack{\mathfrak{u}\subseteq[d] \\ |\mathfrak{u}|=k}} \gamma_{d,\mathfrak{u}} \Big) \alpha_1^{d-k} \sum_{j=0}^{k} \binom{k}{j} \left(1 - n\,\alpha^j\right)_+ \alpha_{2,1}^{k-j} \alpha_{2,2}^{j} \quad \textit{with } 0^0 = 1,$$

where

$$\alpha = \frac{\max\left(\|h_{1,2,2,(0)}\|^2_{H(R_{2,2})}, \|h_{1,2,2,(1)}\|^2_{H(R_{2,2})} \right)}{\|h_{1,2,2,(0)}\|^2_{H(R_{2,2})} + \|h_{1,2,2,(1)}\|^2_{H(R_{2,2})}} \in \left[\tfrac{1}{2}, 1\right].$$

Proof. As in the proof of Theorem 11.12, we decompose the square of the worst case error of a linear algorithm $Q_{n,d}$ as

$$e^2(Q_{n,d}) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} e_{\mathfrak{u}}^2$$

where $e_{\mathfrak{u}}^2 = \sum_{k\in\mathcal{I}} e_{\mathfrak{u},k}^2$ with $e_{\mathfrak{u},k}^2 = \langle h_{\mathfrak{u},1}, r_k\rangle^2_{H(R_{\mathfrak{u},1})} [e'_{\mathfrak{u},k}]^2$ and $e'_{\mathfrak{u},k}$ is the worst case error of approximating the linear functional $\langle f, h_{\mathfrak{u}}\rangle_{H(R_{\mathfrak{u},2})}$, where

$$h_{\mathfrak{u}}(x) = \prod_{j\in\mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d.$$

Since $R_2 = R_{2,1} + R_{2,2}$ with $H(R_{2,1}) \cap H(R_{2,,2}) = \{0\}$ and a decomposable $R_{2,2}$, we can apply Theorem 11.12 to estimate the error $e'_{\mathfrak{u},k}$ with the obvious changes. We now have $R_{2,1}$ instead of R_1, $R_{2,2}$ instead R_2, $|\mathfrak{u}|$ instead of d, $\alpha_{2,1}$ instead of α_1, $\alpha_{2,2}$ instead of α_2; moreover, we use the definition of α as in the current theorem. Therefore we have

$$e_{\mathfrak{u},k}^2 \ge \sum_{j=0}^{|\mathfrak{u}|} \binom{|\mathfrak{u}|}{j} (1 - n\,\alpha^j)_+ \alpha_{2,1}^{|\mathfrak{u}|-j} \alpha_{2,2}^{j}.$$

Summing over $k \in \mathcal{I}$, we obtain

$$e_{\mathfrak{u}}^2 \geq \alpha_1^{d-|\mathfrak{u}|} \sum_{j=0}^{|\mathfrak{u}|} \binom{|\mathfrak{u}|}{j} (1 - n\,\alpha^j)_+ \alpha_{2,1}^{|\mathfrak{u}|-j} \alpha_{2,2}^j.$$

Finally, summing now over $\mathfrak{u}$, we obtain the desired estimate on $e^2(n,d)$. This completes the proof. □

Observe that for a decomposable reproducing kernel R_2 we can take $R_{2,1} = 0$; then $h_{1,2,1} = 0$ yields $\alpha_{2,1} = 0$ and $\alpha_{2,2} = \alpha_2$. In this case the sum over j in Theorem 12.5 drops to the one term $(1 - n\alpha^k)_+ \alpha_2^k$ and Theorem 12.5 reduces to Theorem 12.1. The same situation holds if $h_{1,2,1} = 0$, independently of whether $R_{2,1}$ is zero or not.

Theorem 12.5 is trivial if $\alpha = 1$. That is why we need to assume that both $h_{1,2,2,(0)}$ and $h_{1,2,2,(1)}$ are non-zero to guarantee that $\alpha < 1$. Obviously, the best case is when $\alpha = \frac{1}{2}$.

Assume for a moment that $h_{1,1} = 0$. Then $\alpha_1 = 0$ and the sum over k in Theorem 12.5 drops to one term and we have

$$e^2(n,d) \geq \gamma_{d,[d]} \sum_{j=0}^{d} \binom{d}{j} (1 - n\,\alpha^j)_+ \alpha_{2,1}^{k-j} \alpha_{2,2}^j.$$

Hence modulo $\gamma_{d,[d]}$ we have the same situation as in Theorem 11.12. The square of the initial error is now

$$e^2(0,d) = \gamma_{d,[d]} \alpha_2^d$$

and we may apply Theorem 11.14 with the obvious changes. In particular, for non-zero $\gamma_{d,[d]}$, $h_{1,2,2,(0)}$ and $h_{1,2,2,(1)}$ we have the curse of dimensionality for the normalized error criterion.

Assuming that $h_{1,1} \neq 0$ and $h_{1,2} \neq 0$, we denote, as before,

$$\alpha_3 = \frac{\alpha_2}{\alpha_1} = \frac{\|h_{1,2}\|^2_{H(R_2)}}{\|h_{1,1}\|^2_{H(R_1)}},$$

and

$$\beta_1 = \frac{\alpha_{2,1}}{\alpha_2} = \frac{\|h_{1,2,1}\|^2_{H(R_{2,1})}}{\|h_{1,2}\|^2_{H(R_1)}} \quad \text{and} \quad \beta_2 = \frac{\alpha_{2,2}}{\alpha_2} = \frac{\|h_{1,2,2}\|^2_{H(R_{2,1})}}{\|h_{1,2}\|^2_{H(R_1)}}.$$

Obviously, $\beta_1 + \beta_2 = 1$. Then we can rewrite the estimate of Theorem 12.5 as

$$\frac{e^2(n,d)}{e^2(0,d)} \geq \frac{\sum_{k=0}^{d} \left(\alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}\right) \sum_{j=0}^{k} \binom{k}{j} (1 - n\alpha^j)_+ \beta_1^{k-j} \beta_2^j}{\sum_{k=0}^{d} \left(\alpha_3^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}\right)}.$$

We now show that the case studied in this section can be reduced to the case studied in the previous section if we properly change the weights. This will allow us to use all the results of Section 12.3 for reproducing kernels R_2 that are not decomposable but have a decomposable part.

Corollary 12.6. *Let $\alpha_1 > 0$. For an arbitrary positive c_d, define*

$$\gamma'_{d,\mathfrak{u}} = c_d \sum_{\substack{\mathfrak{v}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \gamma_{d,\mathfrak{v}} \left(\frac{\alpha_{2,1}}{\alpha_1}\right)^{|\mathfrak{v}|-|\mathfrak{u}|} \quad \textit{for all } \mathfrak{u}\subseteq[d], \textit{ with } 0^0 = 1.$$

Under the assumptions of Theorem 12.5 we have

$$e^2(n,d) \ge \frac{1}{c_d} \sum_{\mathfrak{u}\subseteq[d]} \gamma'_{d,\mathfrak{u}} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_1^{d-|\mathfrak{u}|}\,\alpha_{2,2}^{|\mathfrak{u}|},$$

with equality for $n = 0$. Therefore

$$\frac{e^2(n,d)}{e^2(0,d)} \ge \frac{\sum_{\mathfrak{u}\subseteq[d]} \gamma'_{d,\mathfrak{u}} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_1^{d-|\mathfrak{u}|}\,\alpha_{2,2}^{|\mathfrak{u}|}}{\sum_{\mathfrak{u}\subseteq[d]} \gamma'_{d,\mathfrak{u}} \alpha_1^{d-|\mathfrak{u}|}\,\alpha_{2,2}^{|\mathfrak{u}|}}.$$

Proof. Note that

$$\begin{aligned}
&\frac{1}{c_d} \sum_{\mathfrak{u}\subseteq[d]} \gamma'_{d,\mathfrak{u}} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_1^{d-|\mathfrak{u}|}\,\alpha_{2,2}^{|\mathfrak{u}|} \\
&\quad = \sum_{\mathfrak{u}\subseteq[d]} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_1^{d-|\mathfrak{u}|}\,\alpha_{2,2}^{|\mathfrak{u}|} \sum_{\substack{\mathfrak{v}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \gamma_{d,\mathfrak{v}} (\alpha_{2,1}/\alpha_1)^{|\mathfrak{v}|-|\mathfrak{u}|} \\
&\quad = \sum_{\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \alpha_1^{d-|\mathfrak{v}|} \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \alpha_{2,2}^{|\mathfrak{u}|} \left(1 - n\,\alpha^{|\mathfrak{u}|}\right)_+ \alpha_{2,1}^{|\mathfrak{v}|-|\mathfrak{u}|} \\
&\quad = \sum_{\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \alpha_1^{d-|\mathfrak{v}|} \sum_{j=0}^{|\mathfrak{v}|} \sum_{\substack{\mathfrak{u}:\,\mathfrak{u}\subseteq\mathfrak{v}\\ |\mathfrak{u}|=j}} \alpha_{2,2}^{j} \left(1 - n\,\alpha^{j}\right)_+ \alpha_{2,1}^{|\mathfrak{v}|-j} \\
&\quad = \sum_{\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \alpha_1^{d-|\mathfrak{v}|} \sum_{j=0}^{|\mathfrak{v}|} \binom{|\mathfrak{v}|}{j} \alpha_{2,2}^{j} \left(1 - n\,\alpha^{j}\right)_+ \alpha_{2,1}^{|\mathfrak{v}|-j} \\
&\quad = \sum_{k=0}^{d} \Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ |\mathfrak{u}|=k}} \gamma_{d,\mathfrak{u}} \Bigg) \alpha^{d-k} \sum_{j=0}^{k} \binom{k}{j} \left(1 - n\,\alpha^{j}\right)_+ \alpha_{2,1}^{k-j}\alpha_{2,2}^{j}.
\end{aligned}$$

Hence, we have obtained the right-hand side of the estimate on $e^2(n,d)$ in Theorem 12.5. For $n = 0$, the factor $1 - n\alpha^j = 1$, and we end up with the square of the initial error. This completes the proof □

Observe that the estimate in Corollary 12.6 has the same form as the estimate in Theorem 12.1, but with $\gamma_{d,\mathfrak{u}}$ replaced by $\gamma'_{d,\mathfrak{u}}$ and α_2 replaced by $\alpha_{2,2}$. Also, the parameter α has different meanings in Theorem 12.1 and in Corollary 12.6. But the analysis we perform after Theorem 12.1 also goes through for the case of this section.

We now consider several special cases and indicate how to define c_d to switch from the weights $\gamma_{d,j}$ to the weights $\gamma'_{d,\mathfrak{u}}$. We also show that some properties of the weights $\gamma_{d,\mathfrak{u}}$ are also preserved for the weights $\gamma'_{d,j}$.

- Assume that $h_{1,2,1} = 0$. For example, this holds when $R_{2,1} = 0$.

  Then $\alpha_{2,1} = 0$ and
$$\gamma'_{d,\mathfrak{u}} = c_d\, \gamma_{d,\mathfrak{u}}.$$
  In this case, we set $c_d = 1$ and the weights are not changed. This is correct since this case is the same as the case of Section 12.3.

- Assume product weights, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for all $\mathfrak{u} \subseteq [d]$. Let $c^* = \alpha_{2,1}/\alpha_1$. Then
$$\begin{aligned}\sum_{\substack{\mathfrak{v}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \gamma_{d,\mathfrak{v}}[c^*]^{|\mathfrak{v}|-|\mathfrak{u}|} &= \prod_{j\in\mathfrak{u}} \gamma_{d,j} \sum_{\mathfrak{v}\subseteq[d]:\,\mathfrak{u}\subseteq\mathfrak{v}} [c^*]^{|\mathfrak{v}|-|\mathfrak{u}|} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} \gamma_{d,j}\\ &= \prod_{j\in\mathfrak{u}} \gamma_{d,j} \sum_{\mathfrak{v}\subseteq[d]\setminus\mathfrak{u}} [c^*]^{|\mathfrak{v}|} \prod_{j\in\mathfrak{v}} \gamma_{d,j}\\ &= \prod_{j\in\mathfrak{u}} \gamma_{d,j} \prod_{j\in[d]\setminus\mathfrak{u}} \left(1 + c^*\gamma_{d,j}\right).\end{aligned}$$
  Hence
$$\sum_{\substack{\mathfrak{v}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \gamma_{d,\mathfrak{v}}[c^*]^{|\mathfrak{v}|-|\mathfrak{u}|} = \prod_{j\in\mathfrak{u}} \frac{\gamma_{d,j}}{1 + c^*\gamma_{d,j}} \prod_{j=1}^{d} \left(1 + c^*\gamma_{d,j}\right).$$
  If we take $c_d = \prod_{j=1}^{d}(1 + c^*\gamma_{d,j})^{-1}$ and
$$\gamma'_{d,j} = \frac{\gamma_{d,j}}{1 + c^*\gamma_{d,j}}$$
  then $\gamma' = \{\gamma'_{d,\mathfrak{u}}\}$ with $\gamma'_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma'_{d,j}$ is also a sequence of product weights. Furthermore, if $c^* > 0$, then these new weights are always bounded by $1/c^*$, independently of whether the product weights γ are or are not bounded. For $c^* = 0$, i.e., when $h_{1,2,1} = 0$, we have $c_d = 1$ and $\gamma'_{d,j} = \gamma_{d,j}$ as in the previous point. If the product weights γ are bounded by Γ, then the product weights γ' are bounded by $\Gamma/(1 + c^*\Gamma) \le \Gamma$.

- Assume order-dependent weights, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} = \Gamma_{d,|\mathfrak{u}|}$ for all $\mathfrak{u} \subseteq [d]$. Then
$$\gamma'_{d,\mathfrak{u}} = c_d \sum_{\substack{\mathfrak{v}\subseteq[d]\\ \mathfrak{u}\subseteq\mathfrak{v}}} \Gamma_{d,|\mathfrak{u}|}[c^*]^{|\mathfrak{v}|-|\mathfrak{u}|} = c_d \sum_{j=|\mathfrak{u}|}^{d} \binom{d}{j-|\mathfrak{u}|} \Gamma_{d,j}[c^*]^{j-|\mathfrak{u}|}.$$

Hence, $\gamma_{d,\mathfrak{u}}$ depends only on $|\mathfrak{u}|$, and no matter how we define c_d we also have order-dependent weights $\gamma' = \{\gamma'_{d,\mathfrak{u}}\}$.

- Assume finite-order weights, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$.

  Then $\gamma'_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$ no matter how c_d is defined. Hence, again $\gamma' = \{\gamma'_{d,\mathfrak{u}}\}$ is a sequence of finite-order weights with the same order ω.

- Assume finite-diameter weights, $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} = 0$ for all $\operatorname{diam}(\mathfrak{u}) \geq q$.

  Then $\gamma'_{d,\mathfrak{u}} = 0$ for all $\operatorname{diam}(\mathfrak{u}) \geq q$ no matter how c_d is defined, and $\gamma' = \{\gamma'_{d,\mathfrak{u}}\}$ is a sequence of finite-diameter weights with the same order q.

The essence of Corollary 12.6 and the discussion after it is to convince the reader that we may now apply all the results of Section 12.3 for the weights γ'. In particular, consider product weights. We now have $\gamma'_{d,j} = \gamma_{d,j}/(1 + c^*\gamma_{d,j})$. Formally, we can apply Theorem 12.3 and express the lack of tractability in terms of $\sum_{j=1}^d \gamma'_{d,j}$. However, it is easy to check that the situation is even simpler and we can still use the original $\sum_{j=1}^d \gamma_{d,j}$. The reason is that the conditions involving the sums $\sum_{j=1}^d \gamma'_{d,j}$ and $\sum_{j=1}^d \gamma_{d,j}$ are the same. For example,

$$\limsup_{d\to\infty}{}^* \sum_{j=1}^d \gamma_{d,j} = \limsup_{d\to\infty}{}^* \sum_{j=1}^d \gamma'_{d,j}.$$

Indeed, if the first sum is infinite then also the second sum is infinite. This simply follows from the following argument. If $\gamma_{d,j}$ does not converge to zero then $\gamma_{d,j} \geq c > 0$ for some subsequence of d. But then $\gamma'_{d,j} \geq c/(1 + c^*c)$, which implies that the second sum is infinite. On the other hand, if the first sum is still infinite but $\gamma_{d,j}$ goes to zero, then $\gamma'_{d,j} \geq (1-\delta)\gamma_{d,j}$ for small δ and large d, and the second sum is again infinite. Similarly, one can show that a^* and a^*_ε are the same for both $\gamma'_{d,j}$ and $\gamma_{d,j}$. Also the conditions $\limsup_d \sum_{j=1}^d \gamma'_{d,j}/d > 0$ and $\limsup_d \sum_{j=1}^d \gamma_{d,j}/d > 0$ are equivalent.

Obviously there are some differences between the cases with $\gamma'_{d,j}$ and $\gamma_{d,j}$. The most important is the difference between α's. For $\gamma_{d,j}$, we have

$$\alpha = \frac{\max\left(\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)}\right)}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}},$$

whereas for $\gamma'_{d,j}$ we have

$$\alpha' = \frac{\max\left(\|h_{1,2,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,2,(1)}\|^2_{H(R_2)}\right)}{\|h_{1,2,2(0)}\|^2_{H(R_2)} + \|h_{1,2,2,(1)}\|^2_{H(R_2)}}.$$

In general, α and α' are not related.

Also α_3 and β may be different for the two cases of $\gamma'_{d,j}$ and $\gamma_{d,j}$. We have

$$\alpha'_3 = \frac{\|h_{1,2,2}\|^2_{H(R_{2,2})}}{\|h_{1,1}\|^2_{H(R_1)}} \quad \text{and} \quad \alpha_3 = \frac{\|h_{1,2}\|^2_{H(R_2)}}{\|h_{1,1}\|^2_{H(R_1)}}.$$

Clearly, the different α'_3 and α_3 may result in different β' and β. This means that the exponents presented in Theorem 12.3 may be different for $\gamma'_{d,j}$ and $\gamma_{d,j}$.

For the convenience of the reader we now summarize this discussion in the following corollary.

Corollary 12.7. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in Section 12.2 in the worst case setting for the normalized error criterion or for the absolute error criterion with $e(0,d) \ge 1$. Here, γ is a sequence of bounded product weights, i.e.,*

$$\Gamma := \sup_{d \in \mathbb{N},\, j=1,2,\dots,d} \gamma_{d,j} < \infty.$$

Assume that

$$H(R_1) \cap H(R_2) = H(R_{2,1}) \cap H(R_{2,2}) = \{0\} \quad \textit{and} \quad R_{2,2} \textit{ is decomposable.}$$

and that $h_{1,1}$ as well as $h_{1,2,2,(0)}$ and $h_{1,2,2,(1)}$ are non-zero. Let

$$\alpha' = \frac{\max\left(\|h_{1,2,2,(0)}\|^2_{H(R_{2,2})}, \|h_{1,2,2,(1)}\|^2_{H(R_{2,2})}\right)}{\|h_{1,2,2,(0)}\|^2_{H(R_{2,2})} + \|h_{1,2,2,(1)}\|^2_{H(R_{2,2})}} \in \left[\tfrac{1}{2}, 1\right),$$

$$\alpha'_3 = \frac{\|h_{1,2,2}\|^2_{H(R_{2,2})}}{\|h_{1,1}\|^2_{H(R_1)}}, \quad c^* = \frac{\|h_{1,2,1}\|^2_{H(R_{2,1})}}{\|h_{1,1}\|^2_{H(R_1)}}, \quad \gamma'_{d,j} = \frac{\gamma_{d,j}}{1 + c^*\gamma_{d,j}},$$

and let $\lim^* \in \{\lim, \limsup\}$.

- *If*

  $$\lim_{d\to\infty}{}^* \sum_{j=1}^{d} \gamma_{d,j} = \infty$$

  then

  $$\lim_{d\to\infty}{}^* \frac{e(n,d)}{e(0,d)} = 1 \quad \textit{for all } n \in \mathbb{N}$$

  and I_γ is not *strongly T-tractable for any tractability function T.*

- *Let*

  $$a^* := \lim_{d\to\infty}{}^* \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d}.$$

  If $a^* = \infty$ then I_γ is not *polynomially tractable.*

If $a^* \in (0, \infty)$ *then define*

$$\beta' := \sup\left\{ x \in \left[\frac{1}{1+\alpha_3'\Gamma}, 1\right] \;\middle|\; \lim_{d\to\infty}{}^* \frac{\prod_{j=1}^d (1+\alpha_3'\gamma_{d,j}')}{\exp\left(x\alpha_3' \sum_{j=1}^d \gamma_{d,j}'\right)} \ge 1 \right\},$$

and $\beta^* = 1$ *if* $\beta' > 1/e$, *and* $\beta^* \ln 1/\beta^* = \beta'$ *if* $\beta' \le 1/e$. *Then*

$$\lim_{d\to\infty}{}^* \frac{e(\lceil d^q \rceil, d)}{e(0,d)} = 1 \quad \textit{for all } q \in \left(0, a^*\beta^*\alpha_3' \ln 1/\alpha'\right).$$

If polynomial tractability holds then the d *exponent is at least*

$$a^*\beta^*\alpha_3' \ln 1/\alpha'.$$

- *For* $\varepsilon \in (0,1)$, *let*

$$a_\varepsilon^* := \lim_{d\to\infty}{}^* \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)}.$$

If

$$\lim_{\varepsilon\to 1^-} a_\varepsilon^* = \infty$$

then I_γ *is* not T*-tractable. If* $\lim_{\varepsilon\to 1^{-1}} a_\varepsilon^* \in (0,\infty)$ *then*

$$\lim_{d\to\infty}{}^* \frac{e(\lceil T(\varepsilon^{-1}, d)^t \rceil, d)}{e(0,d)} = 1 \quad \textit{for all } t \in \left(0, a_\varepsilon^*\beta^*\alpha_3' \ln 1/\alpha'\right)$$

with β^* *defined as before. If* T*-tractability holds then the exponent of* T*-tractability is at least*

$$\left[\lim_{\varepsilon\to 1^-} a_\varepsilon^*\right] \beta^*\alpha_3' \ln 1/\alpha'.$$

- *If*

$$\gamma_{d,j} \ge \gamma^* \quad \textit{for all } j \in [d],\ d \in \mathbb{N}$$

then I_γ *is intractable and suffers from the curse of dimensionality since*

$$\lim_{d\to\infty} \frac{e(\lfloor C^d \rfloor, d)}{e(0,d)} = 1 \quad \textit{for all } C \in \left(1, [\alpha']^{-\gamma^*\alpha_3/(1+\gamma^*(c^*+\alpha_3))}\right).$$

- *If*

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} > 0$$

then I_γ *is intractable and suffers from the curse of dimensionality.*

We illustrate Corollary 12.7 for several examples.

12.5.1 Example: Weighted Integration (Continued)

We now remove all boundary conditions in the space F_1 by defining the inner product as

$$\langle f, g\rangle_{F_1} = \sum_{j=0}^{r-1} f^{(j)}(0)g^{(j)}(0) + \int_{\mathbb{R}} f^{(r)}(t)g^{(r)}(t)\,\mathrm{d}t \quad \text{for all } f, g \in F_1.$$

The reproducing kernel of this space is

$$K_1(x,t) = \sum_{j=0}^{r-1} \frac{x^j}{j!}\frac{t^j}{j!} + 1_M(x,t)\int_0^\infty \frac{(|t|-u)_+^{r-1}(|x|-u)_+^{r-1}}{[(r-1)!]^2}\,\mathrm{d}u \quad \text{for all } x, t \in \mathbb{R},$$

with $M = \{(x,t) \mid xt \geq 0\}$ and $r \in \mathbb{N}$, as before.

If $r \geq 2$ then the kernel K_1 can be decomposed as

$$K_1 = R_1 + R_{2,1} + R_{2,2},$$

and we consider several such decompositions.

For the first decomposition we take

$$R_1 \equiv 1 \quad \text{and} \quad R_2 = R_{2,1} + R_{2,2},$$

where

$$R_{2,1}(x,t) = \sum_{j=1}^{r-1} \frac{x^j}{j!}\frac{t^j}{j!}$$

and

$$R_{2,2}(x,t) = 1_M(x,t)\int_0^\infty \frac{(|t|-u)_+^{r-1}(|x|-u)_+^{r-1}}{[(r-1)!]^2}\,\mathrm{d}u.$$

Since $r \geq 2$ the reproducing kernel $R_{2,1}$ is not zero and

$$H(R_{2,1}) = \mathrm{span}(t, t^2, \dots, t^{r-1})$$

is a space of polynomials of dimension $r-1$.

Clearly, $H(R_1) \cap H(R_2) = H(R_{2,1}) \cap H(R_{2,2}) = \{0\}$ and $R_{2,2}$ is decomposable at 0. We now have

$$h_{1,1} \equiv 1 \quad \text{and} \quad h_{1,2,1}(x) = \sum_{j=1}^{r-1} \frac{x^j}{j!} \int_{\mathbb{R}} \varrho(t)\,\frac{t^j}{j!}\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R},$$

whereas

$$h_{1,2,2}(x) = \int_{\mathbb{R}} \varrho(t)\, R_2(x,t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}.$$

Note that $h_{1,1}$ is a polynomial of degree at most $r-1$, which is not zero since the coefficient for $j = 0$ is 1. The rest of the terms in the sum over j may be zero, depending on ϱ. For a non-zero ϱ over $\{x \in \mathbb{R} : x \le 0\}$ and $\{x \in \mathbb{R} : x \ge 0\}$, both $h_{1,2,2,(0)}$ and $h_{1,2,2,(1)}$ are non-zero and $\alpha' < 1$. Additionally, if ϱ is symmetric then $\alpha' = \frac{1}{2}$. Hence, all the assumptions of Corollary 12.7 are satisfied.

The square of the initial error of weighted integration with product weights for this decomposition is

$$e^2(0,d) = \prod_{j=1}^{d} \left[1 + \gamma_{d,j} \left(\sum_{j=1}^{r-1} \left[\int_{\mathbb{R}} \varrho(t) \frac{t^j}{j!} \, \mathrm{d}t \right]^2 + \int_{\mathbb{R}^2} \varrho(t)\, \varrho(x)\, R_{2,2}(x,t) \, \mathrm{d}t \, \mathrm{d}x \right) \right].$$

Hence, the initial error is always at least 1 for all weights $\gamma_{d,j}$.

As before, we check that the tractability conditions in Corollary 12.7 are also sufficient for tractability of weighted integration for both the absolute and normalized error criteria. For simplicity we now assume a little more about the weight ϱ, namely that

$$\int_{\mathbb{R}} \varrho(t)\, t^{2(r-1)} \, \mathrm{d}t < \infty.$$

Then for the QMC algorithm $A_{n,d}$ considered in Theorem 10.4 of Chapter 10, we have

$$e^{\mathrm{wor}}(A_{n,d}) \le \prod_{j=1}^{d} \left[1 + \gamma_{d,j} \left(\sum_{j=1}^{r-1} \int_{\mathbb{R}} \varrho(t) \frac{t^{2j}}{[j!]^2} \, \mathrm{d}t + \int_{\mathbb{R}} \varrho(t)\, R_2(t,t) \, \mathrm{d}t \right) \right] n^{-1/2}.$$

Then we can apply the same reasoning as before, with

$$C = \sum_{j=1}^{r-1} \int_{\mathbb{R}} \varrho(t) \frac{t^{2j}}{[j!]^2} \, \mathrm{d}t + \int_{\mathbb{R}} \varrho(t)\, R_2(t,t) \, \mathrm{d}t < \infty.$$

Hence we obtain the following corollary, which is analogous to Corollary 12.4.

Corollary 12.8. *Consider weighted integration I_γ for bounded product weights and R_1, $R_{2,1}$ and $R_{2,2}$ defined as above. For T-tractability we additionally assume that*

$$\varrho(t) \ge c > 0 \quad \textit{for } t \in [a,b] \textit{ for some } a, b \textit{ and } c \textit{ with } a < b.$$

Then tractability conditions for the absolute and normalized error criteria are the same. More precisely,

- *I_γ is strongly polynomially tractable iff* $\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$.
- *I_γ is strongly T-tractable iff*

$$\limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty \quad \textit{and} \quad \limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

- *I_γ is polynomially tractable iff*

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln d} < \infty.$$

- *I_γ is T-tractable iff*

$$\limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty \quad \textit{and} \quad \lim_{\varepsilon\to 1-} \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln T(\varepsilon^{-1},d)} < \infty.$$

- *I_γ is weakly tractable iff*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} = 0.$$

We now turn to different decomposition of K_1 for $r \ge 2$. Namely, take an integer $k \in \{0, 1, \dots, r-1\}$ and define

$$R_1(x,t) = \sum_{j=0}^{k} \frac{t^j}{j!}\frac{x^j}{j!} \quad \text{and} \quad R_{2,1}(x,t) = \sum_{j=k+1}^{r-1} \frac{t^j}{j!}\frac{x^j}{j!}$$

with the same $R_{2,2}$ as above. So we have $r-1$ decompositions parameterized by k. For $k = 0$, this is the decomposition we already analyzed.

We stress that the weighted integration problem depends on k, which is why we may have $r-1$ different weighted integration problems. Indeed, we now have

$$\begin{aligned} h_{1,1}(x) &= 1 + \sum_{j=1}^{k} \frac{x^j}{j!} \int_{\mathbb{R}} \varrho(t)\, \frac{t^j}{j!}\, \mathrm{d}t, \\ h_{1,2,1}(x) &= \sum_{j=k+1}^{r-1} \frac{x^j}{j!} \int_{\mathbb{R}} \varrho(t)\, \frac{t^j}{j!}\, \mathrm{d}t, \\ h_{1,2,2}(x) &= \int_{\mathbb{R}} \varrho(t)\, R_2(x,t)\, \mathrm{d}t. \end{aligned}$$

Let

$$\begin{aligned} A_k &:= \sum_{j=1}^{k} \left[\int_{\mathbb{R}} \varrho(t) \frac{t^j}{j!}\, \mathrm{d}t \right]^2, \\ B_k &:= \sum_{j=k+1}^{r-1} \left[\int_{\mathbb{R}} \varrho(t) \frac{t^j}{j!}\, \mathrm{d}t \right]^2 + \int_{\mathbb{R}^2} \varrho(t)\, \varrho(x)\, R_{2,2}(x,t)\, \mathrm{d}t\, \mathrm{d}x. \end{aligned}$$

Note that $A_0 = 0$. We see that $A_1 = 0$ whenever the integral $\int_{\mathbb{R}} \varrho(t)\, t\, \mathrm{d}t = 0$. Of course, this depends on the choice of the function ϱ; for instance, this condition holds for symmetric ϱ. However, for $k \ge 2$, we have positive A_k since the integral $\int_{\mathbb{R}} \varrho(t)\, t^2\, \mathrm{d}t > 0$.

The square of the initial error is now

$$e^2(0,d) = \prod_{j=1}^{d} \left[1 + A_k + \gamma_{d,j}\, B_k\right].$$

If ϱ is chosen such that all integrals $\int_{\mathbb{R}} \varrho(t)\, t^j\, \mathrm{d}t$ for $j = 1, 2, \dots, r-1$ are non-zero then for non-zero weights $\gamma_{d,j}$, the initial errors are different for different k.

Observe that all the assumptions of Corollary 12.7 are satisfied independently of k, which is why we have the same necessary conditions on product weights for various kinds of tractability.

However, the upper bounds are different and significantly depend on k. Consider once again the QMC algorithm $A_{n,d}$ and let

$$A'_k := \sum_{j=1}^{k} \int_{\mathbb{R}} \varrho(t)\, \frac{t^{2j}}{[j!]^2}\, \mathrm{d}t,$$

$$B'_k := \sum_{j=k+1}^{r-1} \int_{\mathbb{R}} \varrho(t) \frac{t^{2j}}{[j!]^2}\, \mathrm{d}t + \int_{\mathbb{R}} \varrho(t)\, R_{2,2}(t,t)\, \mathrm{d}t.$$

Theorem 10.4 of Chapter 10 states that we now have

$$e^{\mathrm{wor}}(A_{n,d}) \le \prod_{j=1}^{d} \left(1 + A'_k + \gamma_{d,j}\, B'_k\right)^{1/2}\, n^{-1/2}.$$

Formally we can apply the same reasoning as before and obtain

$$n(\varepsilon, d) \le \left\lceil \varepsilon^{-2}\, C_d \right\rceil,$$

where

$$C_d := \prod_{j=1}^{d} \left(1 + A'_k + \gamma_{d,j}\, B'_k\right).$$

Observe that for $k \ge 1$, we have positive A'_k and therefore C_d is *exponentially* large in d even if all $\gamma_{d,j} = 0$. Hence, we cannot claim tractability of weighted integration. One might hope that we can improve this unfortunate situation by using more sophisticated upper bounds from Chapter 10. Unfortunately, all of them suffer from the same bad property and we always have an exponential dependence on d. Does it really mean that we have intractability of weighted integration? Not necessarily, since for weights $\gamma_{d,j}$ that decay sufficiently fast, we may satisfy the necessary tractability conditions. However, since our upper bounds may be not good enough we cannot be sure whether tractability holds. We are inclined to believe that indeed we do *not* have tractability and this is presented as our next open problem.

Open Problem 62.

- Consider weighted integration as defined here with $r \ge 2$ and $k \ge 1$. Verify whether weighted integration suffers from the curse of dimensionality.
- Verify if the first point holds for all ϱ for which weighted integration is well defined or whether for some ϱ we have intractability and for some other ϱ we have at least weak tractability.

12.6 Upper Bounds for Multivariate Integration

We presented several lower bounds for approximation of linear functionals in the previous sections. We now derive upper bounds and then compare them with lower bounds. We use the results of Chapter 10, where we established a number of upper bounds for approximating $I_{d,\gamma}(f) = \langle f, h_{d,\gamma}\rangle_{F_{d,\gamma}}$ for all $f \in F_{d,\gamma}$. As we know, we have

$$h_{d,\gamma}(x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\notin\mathfrak{u}} h_{1,1}(x_j) \prod_{j\in\mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d. \tag{12.3}$$

Here $h_1 = h_{1,1} + h_{1,2}$ with $h_{1,j} \in H(R_j)$, where $F_1 = H(K_1) = H(R_1) \cup H(R_2)$ with

$$K_1 = R_1 + R_2 \quad \text{and} \quad H(R_1) \cap H(R_2) = \{0\}.$$

We stress that we are *not* assuming that R_2 is decomposable. Without loss of generality we may assume that $h_1 \neq 0$ since otherwise $h_{d,\gamma} = 0$, and so $I_{d,\gamma}(f) = 0$ for all $f \in F_{d,\gamma}$, i.e., $I_{d,\gamma}$ is trivial.

We now specify h_1 to obtain the multivariate integration problem that was studied in Section 10.7. For $d = 1$ we take two real Lebesgue integrable functions ϱ_1 and g_1, where $\varrho_1 \ge 0$ with $\int_{D_1} \varrho_1(t)\,\mathrm{d}t = 1$ and g_1 is non-zero. We assume that F_1 is separable and consists of Lebesgue measurable functions, and that its reproducing kernel K_1 satisfies the condition

$$\int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t < \infty. \tag{12.4}$$

Then

$$I_1(f) = \int_{D_1} f(t) g_1(t) \varrho_1(t)\,\mathrm{d}t = \langle f, h_1\rangle_{F_1} \quad \text{for all } f \in F_1,$$

where

$$h_1(x) = \int_{D_1} K_1(x,t) g_1(t) \varrho_1(t)\,\mathrm{d}t \quad \text{for all } x \in D_1.$$

We have

$$h_{1,1}(x) = \int_{D_1} R_1(x,t) g_1(t) \varrho_1(t)\,\mathrm{d}t \quad \text{for all } x \in D_1,$$

$$h_{1,2}(x) = \int_{D_1} R_2(x,t) g_1(t) \varrho_1(t)\,\mathrm{d}t \quad \text{for all } x \in D_1.$$

For $d \geq 1$, we now take

$$g_d(x) = \prod_{j=1}^{d} g_1(x_j) \quad \text{for all } x \in D_d$$

$$\varrho_d(x) = \prod_{j=1}^{d} \varrho_1(x_j) \quad \text{for all } x \in D_d$$

and obtain the *multivariate integration* problem defined by

$$I_{d,\gamma}(f) = \int_{D_d} f(t) g_d(t) \varrho_d(t) \, \mathrm{d}t = \langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}} \quad \text{for all } f \in F_d, \tag{12.5}$$

where $h_{d,\gamma}$ has the form (12.3) with $h_{1,1}$ and $h_{1,2}$ as above. Indeed, (12.5) follows from the following calculation. We have

$$\langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}} = \sum_{\mathfrak{u} \subseteq [d]} \frac{1}{\gamma_{d,\mathfrak{u}}} \gamma_{d,\mathfrak{u}} a_{\mathfrak{u}} \, b_{\mathfrak{u}},$$

where

$$a_{\mathfrak{u}} = \Big\langle f_{\bar{\mathfrak{u}},1}, \int_{D_1^{d-|\mathfrak{u}|}} \prod_{j \notin \mathfrak{u}} g_1(t_j) \varrho_1(t_j) \prod_{j \notin \mathfrak{u}} R_1(\cdot, t_j) \, \mathrm{d}t_{\bar{\mathfrak{u}}} \Big\rangle_{H_{R_{\bar{\mathfrak{u}},1}}},$$

$$b_{\mathfrak{u}} = \Big\langle f_{\mathfrak{u},2}, \int_{D_1^{|\mathfrak{u}|}} \prod_{j \in \mathfrak{u}} g_1(t_j) \varrho_1(t_j) \prod_{j \in \mathfrak{u}} R_2(\cdot, t_j) \, \mathrm{d}t_{\mathfrak{u}} \Big\rangle_{H_{R_{\mathfrak{u},2}}}.$$

Clearly,

$$a_{\mathfrak{u}} = \int_{D_1^{d-|\mathfrak{u}|}} \prod_{j \notin \mathfrak{u}} g_1(t_j) \varrho_1(t_j) \; f_{\bar{\mathfrak{u}},1}(t_{\bar{\mathfrak{u}}}) \, \mathrm{d}t_{\bar{\mathfrak{u}}},$$

$$b_{\mathfrak{u}} = \int_{D_1^{|\mathfrak{u}|}} \prod_{j \in \mathfrak{u}} g_1(t_j) \varrho_1(t_j) \; f_{\mathfrak{u},2}(t_{\mathfrak{u}}) \, \mathrm{d}t_{\mathfrak{u}}.$$

Then

$$a_{\mathfrak{u}} \, b_{\mathfrak{u}} = \int_{D_1^d} g_d(t) \varrho_d(t) \; f_{\bar{\mathfrak{u}},1}(t_{\bar{\mathfrak{u}}}) f_{\mathfrak{u},2}(t_{\mathfrak{u}}) \, \mathrm{d}t,$$

and so

$$\begin{aligned} \langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}} &= \sum_{\mathfrak{u} \subseteq [d]} \int_{D_1^d} g_d(t) \varrho_d(t) \; f_{\bar{\mathfrak{u}},1}(t_{\bar{\mathfrak{u}}}) \; f_{\mathfrak{u},2}(t_{\mathfrak{u}}) \, \mathrm{d}t \\ &= \int_{D_1^d} g_d(t) \varrho_d(t) \sum_{\mathfrak{u} \subseteq [d]} f_{\bar{\mathfrak{u}},1}(t_{\bar{\mathfrak{u}}}) \; f_{\mathfrak{u},2}(t_{\mathfrak{u}}) \, \mathrm{d}t \\ &= \int_{D_d} g_d(t) \varrho_d(t) \; f(t) \, \mathrm{d}t, \end{aligned}$$

as claimed.

We stress that multivariate integration (12.5) can be also obtained as $I_d(f) = \langle f, h_d \rangle_{F_d}$, where h_1 satisfies (10.21) for $d = 1$. More precisely, we need to assume that

$$C(K_1) = \int_{D_1} K_1(t,t)\varrho_1(t)\,\mathrm{d}t < \infty, \tag{12.6}$$

and to find eigenpairs $(\lambda_{1,j}, \eta_{1,j})$ of the operator

$$(W_1 f)(x) = \int_{D_1} K_1(x,t) f(t)\varrho_1(t)\,\mathrm{d}t \quad \text{for all } x \in D_1.$$

We additionally assume that

$$\sum_{j=1}^{\infty} \frac{\langle h_1, \eta_{1,j}\rangle_{F_1}^2}{\lambda_{1,j}} < \infty. \tag{12.7}$$

Then

$$\tilde{h}_d = \sum_{j=1}^{\infty} \frac{\langle h_1, \eta_{1,j}\rangle_{F_1}}{\lambda_{1,j}}\,\eta_{1,d} \in L_{2,\varrho_1},$$

and we can take $g_1 = \tilde{h}_1$. Hence, (12.6) and (12.7) imply that I_d is equivalent to multivariate integration.

We want to estimate the nth minimal error $e(n,d)$ for approximating $I_{d,\gamma}$, where $I_{d,\gamma}(f) = \langle f, h_{d,\gamma}\rangle_{F_{d,\gamma}}$ for all $f \in F_{d,\gamma}$. To apply Theorem 10.7, we need to consider

$$C(K_{d,\gamma}, g_d) = \int_{D_d} K_{d,\gamma}(t,t) g_d^2(t)\varrho_d(t)\,\mathrm{d}t.$$

Let

$$\alpha_1^* = \int_{D_1} R_1(t,t) g_1^2(t)\varrho_1(t)\,\mathrm{d}t$$

and

$$\alpha_2^* = \int_{D_1} R_2(t,t) g_1^2(t)\varrho_1(t)\,\mathrm{d}t.$$

Note that the α_j^* are finite by (12.4). They should be compared with the α_j that we have often used before. The α_j take now the form

$$\alpha_1 = \|h_{1,1}\|_{H(R_1)}^2 = \int_{D_1^2} R_1(x,t) g_1(x) g_1(t)\varrho_1(x)\varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\alpha_2 = \|h_{1,2}\|_{H(R_2)}^2 = \int_{D_1^2} R_2(x,t) g_1(x) g_1(t)\varrho(x)\varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Clearly, $\alpha_j \le \alpha_j^*$ for $j = 1, 2$. We have

$$\begin{aligned} C(K_d, g_d) &= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \int_{D_d} g_d^2(t)\varrho_d(t) \prod_{j\notin\mathfrak{u}} R_1(t_j,t_j) \prod_{j\in\mathfrak{u}} R_2(t_j,t_j)\,\mathrm{d}t \\ &= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\alpha_1^*\right]^{d-|\mathfrak{u}|} \left[\alpha_2^*\right]^{|\mathfrak{u}|} \\ &= \sum_{k=0}^{d} \left[\alpha_1^*\right]^{d-k} \left[\alpha_2^*\right]^{k} \sum_{\mathfrak{u}\subseteq[d];\ |\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}. \end{aligned}$$

This should be again compared with the square of the initial error,

$$e^2(0,d) = \|h_{d,\gamma}\|_{F_{d,\gamma}}^2 = \sum_{k=0}^{d} [\alpha_1]^{d-k}\ [\alpha_2]^k \sum_{\mathfrak{u}\subseteq[d];\ |\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}.$$

As always, $0^0 = 1$.

From Theorem 10.7, we know that

$$e(n,d) \le \min\left(\|h_{d,\gamma}\|_{F_{d,\gamma}},\ \frac{\sqrt{C(K_{d,\gamma}, g_d) - \|h_{d,\gamma}\|_{F_{d,\gamma}}^2}}{\sqrt{n}} \right). \tag{12.8}$$

We stress that the error bound above can be achieved by a properly normalized QMC algorithm.

Let

$$f^*(d) = \frac{\sum_{k=0}^{d} \left[\alpha_1^*\right]^{d-k} \left[\alpha_2^*\right]^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\mathrm{CRI}_d^2}. \tag{12.9}$$

As always, $\mathrm{CRI}_d = 1$ for the absolute error criterion, and $\mathrm{CRI}_d = e(0,d)$ for the normalized error criterion.

For the absolute error criterion, (12.8) implies that for $\varepsilon \ge \|h_{d,\gamma}\|_{F_{d,\gamma}}$ we have $n(\varepsilon,d) = 0$, whereas for $\varepsilon < \|h_{d,\gamma}\|_{F_{d,\gamma}}$ we have

$$n(\varepsilon,d) \le \left\lceil \frac{f^*(d) - \|h_{d,\gamma}\|_{F_{d,\gamma}}^2}{\varepsilon^2} \right\rceil \le \frac{f^*(d)}{\varepsilon^2}.$$

For the normalized error criterion, (12.8) can be rewritten as

$$\frac{e(n,d)}{e(0,d)} \le \min\left(1,\ \frac{\sqrt{f^*(d) - 1}}{\sqrt{n}}\right).$$

Therefore, for $\varepsilon \ge 1$ we have $n(\varepsilon,d) = 0$, whereas for $\varepsilon < 1$ we have

$$n(\varepsilon,d) \le \left\lceil \frac{f^*(d) - 1}{\varepsilon^2} \right\rceil \le \frac{f^*(d)}{\varepsilon^2}.$$

The two cases of the absolute and normalized error criteria can be combined and we obtain

$$n(\varepsilon, d) \le \frac{f^*(d)}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}. \tag{12.10}$$

From this estimate we easily obtain conditions on tractability of I_γ. We stress that the function f^* depends on the error criterion. That is why the conditions on f^* also depend on the error criterion. The theorem below can be applied for the absolute and normalized error criteria by taking the function f^* corresponding to the error criterion we wish to consider.

Theorem 12.9. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ with a non-zero h_1 such that (12.4) holds. We consider the worst case setting for the absolute and normalized error criteria.*

- *If*

$$\sup_{d\in\mathbb{N}} f^*(d) < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$p^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2p^*$.

- *If there exists a non-negative q such that*

$$\limsup_{d\to\infty} f^*(d)\, d^{-q} < \infty$$

then I_γ is polynomially tractable with ε^{-1} exponent at most 2 and d exponent at most q.

- *If*

$$t^* := \limsup_{\substack{\varepsilon < \min(1, \|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d) \\ \varepsilon^{-1}+d\to\infty}} \frac{\ln f^*(d) + \ln \varepsilon^{-2}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ is T-tractable with exponent at most t^*.

- *If*

$$\lim_{d\to\infty} \frac{\ln \max(1, f^*(d))}{d} = 0$$

then I_γ is weakly tractable.

Proof. Clearly, if f^* is uniformly bounded then (12.10) yields strong polynomial tractability with exponent at most 2. Additionally, if $p^* < \infty$ then for an arbitrarily small positive δ there exists a positive ε_δ such that $\varepsilon^{-2} \le T(\varepsilon^{-1},1)^{2(p^*+\delta)}$ for all $\varepsilon \le \varepsilon_\delta$. Since $T(\varepsilon^{-1},1) \ge 1$ for all $\varepsilon \in (0,1)$, we have

$$\varepsilon^{-2} \le \varepsilon_\delta^{-2}\, T(\varepsilon^{-1},1)^{2(p^*+\delta)} \quad \text{for all } \varepsilon \in (0,1).$$

This means strong T-tractability with exponent at most $2p^*$.

The next point on polynomial tractability is also clear. If

$$C := \limsup_d f^*(d) d^{-q} < \infty$$

then for any positive δ there is a positive integer d_δ such that $f^*(d) \le (C+\delta)\, d^q$ for all $d \ge d_\delta$. Therefore

$$f^*(d) \le \max\Big(\max_{k=1,2,\dots,d_\delta-1} f^*(k), C+\delta\Big)\, d^q \quad \text{for all } d \in \mathbb{N}.$$

This implies polynomial tractability with d exponent at most q.

Assume now that $t^* < \infty$. Note that

$$\|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d = \begin{cases} \|h_{d,\gamma}\|_{F_{d,\gamma}} & \text{for the absolute error criterion,} \\ 1 & \text{for the normalized error criterion.} \end{cases}$$

Hence, for $\varepsilon \ge \|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d$ we have $n(\varepsilon,d) = 0$.

For any positive δ there is a positive integer M_δ such that

$$f^*(d)\, \varepsilon^{-2} \le T(\varepsilon^{-1},d)^{t^*+\delta}$$

for all $\varepsilon < \min(1, \|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d)$ and $\varepsilon^{-1}+d \ge M_\delta$.

For the remaining case of $(\varepsilon^{-1},d) \in (1,\infty) \times \mathbb{N}$, we have

$$\max_{\varepsilon^{-1}+d \le M_\delta} f^*(d)\, \varepsilon^{-2} \le \max_{\varepsilon^{-1} \le M_\delta} \varepsilon^{-2} \max_{d \le M_\delta} f^*(d) \le C := M_\delta^2 \max_{d=1,2,\dots,M_\delta} f^*(d).$$

Therefore for all cases, we have

$$n(\varepsilon,d) \le f^*(d)\, \varepsilon^{-2} \le \max(1,C)\, T(\varepsilon^{-1},d)^{t^*+\delta} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Hence, the problem is T-tractable with exponent at most t^*.

Finally, weak tractability follows from

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln \max(1, f^*(d)) + \ln \varepsilon^{-2}}{\varepsilon^{-1}+d} = 0.$$

This completes the proof. □

We now check the conditions of Theorem 12.9 for $h_{1,1} = 0$. Then $\alpha_1^* = \alpha_1 = 0$ and since we assume that $h_1 \ne 0$ we have $h_{1,2} \ne 0$ and $\alpha_2 > 0$. In this case,

$$f^*(d) = \begin{cases} \left[\alpha_2^*\right]^d \gamma_{d,[d]} & \text{for the absolute error criterion,} \\ \left(\alpha_2^*/\alpha_2\right)^d & \text{for the normalized error criterion.} \end{cases}$$

For the absolute error criterion we have

$$\sup_{d\in\mathbb{N}} f^*(d) < \infty \quad \text{iff} \quad \gamma_{d,[d]} = \mathcal{O}\big(\big[\alpha_2^*\big]^{-d}\big),$$

$$\limsup_{d\to\infty} f^*(d)d^{-q} < \infty \quad \text{iff} \quad \gamma_{d,[d]} = \mathcal{O}\big(d^q\,\big[\alpha_2^*\big]^{-d}\big),$$

$$\lim_{d\to\infty} \frac{\ln f^*(d)}{d} = 0 \quad \text{iff} \quad \gamma_{d,[d]} = \exp\big(o(d) - d\,\ln\alpha_2^*\big).$$

For the normalized error criterion, the situation is even simpler. Since $\alpha_2^* \ge \alpha_2$, the only case for which the conditions of Theorem 12.9 hold is when $\alpha_2^* = \alpha_2$. In what follows we assume that $h_{1,1} \neq 0$.

Theorem 12.9 is quite straightforward but has interesting applications for finite-order, finite-diameter and product weights. We remind the reader that finite-diameter weights are also finite-order weights; the main difference between them is that we can have $\mathcal{O}(d^{\,\omega})$ non-zero finite-order weights of order ω, whereas we can have $2^{q-1}d + \mathcal{O}(1)$ non-zero finite-diameter weights of order q, as d goes to infinity. This is the subject of our next theorem.

Theorem 12.10. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ with non-zero $h_{1,1}$ and $h_{1,2}$ such that (12.4) holds. We consider the worst case setting for the absolute and normalized error criteria.*

- *Consider finite-order weights*

$$\gamma_{d,\mathfrak{u}} = 0 \quad \textit{for all } \mathfrak{u} \textit{ with } |\mathfrak{u}| > \omega.$$

For the absolute error criterion and bounded finite-order weights, i.e.,

$$\sup_{d\in\mathbb{N}} \gamma_{d,\mathfrak{u}} < \infty,$$

we have the following:

  – *If*

$$\alpha_1^* := \int_{D_1} R_1(t,t) g_1^2(t) \varrho_1(t)\,\mathrm{d}t < 1$$

  then I_γ is strongly polynomially tractable with exponent at most 2.

  – *If*

$$\alpha_1^* < 1 \quad \textit{and} \quad t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty$$

  then I_γ is strongly T-tractable with exponent at most $2t^*$.

  – *If*

$$\alpha_1^* = 1$$

  then I_γ is polynomially tractable with d exponent at most ω and ε^{-1} exponent at most 2; for bounded finite-diameter weights the d exponent is at most 1.

– *If*

$$\alpha_1^* = 1 \quad \textit{and} \quad t^* := \limsup_{\substack{\varepsilon < \|h_{d,\gamma}\|_{F_{d,\gamma}} \\ \varepsilon^{-1}+d\to\infty}} \frac{k \ln d + 2\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ is T-tractable with exponent at most t^*. Here $k = \omega$ for finite-order weights, and $k = 1$ for finite-diameter weights.

For the normalized error criterion and finite-order weights we have the following:

– *If*

$$\alpha_1^* = \alpha_1 := \|h_{1,1}\|_{H(R_1)}$$

then I_γ is strongly polynomially tractable with exponent at most 2.

– *If*

$$\alpha_1^* = \alpha_1 \quad \textit{and} \quad t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

- *Consider product weights*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \textit{for all } \mathfrak{u} \subseteq [d].$$

For the absolute error criterion and product weights we have the following:

– *If*

$$\alpha_1^* + \alpha_2^* \cdot \sup_{d\in\mathbb{N}} \max_{j=1,2,\dots,d} \gamma_{d,j} \le 1$$

or

$$\alpha_1^* < 1 \quad \textit{and} \quad \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln d} < \infty$$

or

$$\alpha_1^* = 1 \quad \textit{and} \quad \limsup_{d\to\infty} \sum_{j=1}^d \gamma_{d,j} < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

– *If*

$$\alpha_1^* = 1 \quad \textit{and} \quad q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln d} < \infty$$

then I_γ is polynomially tractable with d exponent at most arbitrarily close to $\alpha_2^* q^*$ and ε^{-1} exponent at most 2.

– *If* $\alpha_1^* \le 1$ *and*

$$t^* := \limsup_{\substack{\varepsilon^2 < \prod_{j=1}^d (\alpha_1^* + \alpha_2^* \gamma_{d,j}) \\ \varepsilon^{-1} + d \to \infty}} \frac{\ln \varepsilon^{-2} + d \ln \alpha_1^* + (\alpha_2^*/\alpha_1^*) \sum_{j=1}^d \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ *is* T*-tractable with exponent at most* t^*.

– *If* $\alpha_1^* \le 1$ *and*

$$\lim_{d \to \infty} \frac{\left[d \ln \alpha_1^* + (\alpha_2^*)/(\alpha_1^*) \sum_{j=1}^d \gamma_{d,j}\right]_+}{\ln d} = 0$$

then I_γ *is weakly tractable.*

For the normalized error criterion and product weights assume that

$$\alpha_1^* = \alpha_1.$$

Then we have the following:

– *If*

$$\limsup_{d \to \infty} \sum_{j=1}^d \gamma_{d,j} < \infty$$

then I_γ *is strongly polynomially tractable with exponent at most* 2. *Additionally, if*

$$t^* := \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ *is strongly* T*-tractable with exponent at most* $2t^*$.

– *If*

$$q^* := \limsup_{d \to \infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln d} < \infty$$

then I_γ *is polynomially tractable with* d *exponent at most arbitrarily close to* $(\alpha_2^* - \alpha_2)\, q^*/\alpha_1$ *and* ε^{-1} *exponent at most* 2.

– *If*

$$t^* := \limsup_{\varepsilon < 1:\, \varepsilon^{-1} + d \to \infty} \frac{\ln \varepsilon^{-2} + (\alpha_2^* - \alpha_2)/\alpha_1 \sum_{j=1}^d \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)}$$

then I_γ *is* T*-tractable with exponent at most* t^*.

– *If*

$$\lim_{d \to \infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} = 0$$

then I_γ *is weakly tractable.*

Proof. For the absolute error criterion and bounded finite-order weights, where $\gamma_{d,\mathfrak{u}} \le \Gamma$, we have

$$f^*(d) = \sum_{k=0}^{d} \left[\alpha_1^*\right]^{d-k} \left[\alpha_2^*\right]^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}$$
$$\le \left[\alpha_1^*\right]^{d} \sum_{k=0}^{\omega} \left(\alpha_2^*/\alpha_1^*\right)^{k} \binom{d}{k} \Gamma$$
$$= \left[\alpha_1^*\right]^{d} \, p_\omega(d),$$

where p_ω is a polynomial of degree at most ω.

Thus for $\alpha_1^* < 1$ the function f^* is uniformly bounded; for $\alpha_1^* = 1$, the values $f^*(d)$ are bounded by a multiple of $d^{\,\omega}$.

For finite-diameter weights, the values $f^*(d)$ are bounded by a multiple of $[\alpha_1^*]^d \, d$. Therefore for $\alpha_1^* = 1$, they are bounded by a multiple of d. This and Theorem 12.9 yield the first part of Theorem 12.10.

For the normalized error criterion and finite-order weights, we assumed that $\alpha_1^* = \alpha_1$. Since $\alpha_2 > 0$ and at least one $\gamma_{d,\mathfrak{u}}$ is positive we have

$$f^*(d) = \frac{\left[\alpha_1^*\right]^{d} \sum_{k=0}^{\omega} \left(\alpha_2^*/\alpha_1^*\right)^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\left[\alpha_1\right]^{d} \sum_{k=0}^{\omega} \left(\alpha_2/\alpha_1\right)^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}$$
$$= \frac{\sum_{k=0}^{\omega} \left(\alpha_2/\alpha_1\right)^{k} \left(\alpha_2^*/\alpha_2\right)^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\sum_{k=0}^{\omega} \left(\alpha_2/\alpha_1\right)^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}$$
$$\le \left(\frac{\alpha_2^*}{\alpha_2}\right)^{\omega}.$$

Hence, f^* is now uniformly bounded. Theorem 12.9 now yields the second part of Theorem 12.10.

For the absolute error criterion and product weights we have

$$f^*(d) = \prod_{j=1}^{d} \left(\alpha_1^* + \alpha_2^* \gamma_{d,j}\right).$$

Clearly,

$$\alpha_1^* + \alpha_2^* \gamma_{d,j} \le \alpha_1^* + \alpha_2^* \cdot \sup_{d\in\mathbb{N}} \max_{j=1,2,\dots,d} \gamma_{d,j}.$$

Hence if the last bound is at most 1, the function f^* is uniformly bounded and we have strong tractability.

Let $\alpha_1^* \le 1$. Since $\alpha_1^* \ge \alpha_1 > 0$, we rewrite f^* as

$$f^*(d) = \left[\alpha_1^*\right]^{d} \prod_{j=1}^{d} \left[1 + \frac{\alpha_2^*}{\alpha_1^*} \gamma_{d,j}\right] \le \left[\alpha_1^*\right]^{d} \exp\left(\frac{\alpha_2^*}{\alpha_1^*} \sum_{j=1}^{d} \gamma_{d,j}\right).$$

Note that $q^* := \limsup_d \sum_{j=1}^d \gamma_{d,j} / \ln d < \infty$ implies that for any positive δ there is a positive C_δ such that

$$\exp\left(\frac{\alpha_2^*}{\alpha_1^*} \sum_{j=1}^d \gamma_{d,j}\right) \le C_\delta \, d^{\,\alpha_2^*(q+\delta)/\alpha_1^*} \quad \text{for all } d \in \mathbb{N}.$$

This yields that

$$f^*(d) \le C_\delta \left[\alpha_1^*\right]^d \, d^{\,\alpha_2^*(q+\delta)/\alpha_1^*} \quad \text{for all } d \in \mathbb{N}.$$

Hence f^* is uniformly bounded for $\alpha_1^* < 1$, and polynomially bounded for $\alpha_1^* = 1$. The exponent of d in the latter case is at most $\alpha_2^*(q+\delta)$ and δ can be arbitrarily small.

Obviously for $\alpha_1^* = 1$ and $\limsup_d \sum_{j=1}^d \gamma_{d,j} < \infty$, we have

$$\sup_{d \in \mathbb{N}} \prod_{j=1}^d \left(1 + \alpha_2^* \gamma_{d,j}\right) < \infty.$$

Hence f^* is uniformly bounded, and so strong polynomial tractability holds. This covers the proof of all cases except the last one of the third part of the theorem. For the last case, we indeed can restrict ε^2 to be less than $e^2(0,d) = \prod_{j=1}^d (\alpha_1^* + \alpha_2^* \gamma_{d,j})$ since otherwise $n(\varepsilon, d) = 0$.

For the normalized error and product weights, it is enough to observe that

$$f^*(d) = \prod_{j=1}^d \frac{1 + (\alpha_2^*/\alpha_1^*)\,\gamma_{d,j}}{1 + (\alpha_2/\alpha_1)\,\gamma_{d,j}} \le \prod_{j=1}^d \left(1 + (\alpha_2^* - \alpha_2)/\alpha_1 \, \gamma_{d,j}\right).$$

The rest is easy. □

We now comment on Theorem 12.10. For the absolute error criterion, we must assume that $\alpha_1^* \le 1$ to satisfy one of the tractability conditions. In particular, this holds if g_1 is sufficiently small. For the normalized error criterion we need a more restrictive assumption that $\alpha_1^* = \alpha_1$, i.e.,

$$\int_{D_1} R_1(t,t)\, g_1^2(t)\, \varrho_1(t)\, \mathrm{d}t = \int_{D_1^2} R_1(x,t)\, g_1(x)\, g_1(t)\, \varrho_1(x)\, \varrho_1(t)\, \mathrm{d}x\, \mathrm{d}t. \quad (12.11)$$

Obviously, this holds for $g_1 = 0$ or for $R_1 = 0$. For $g_1 = 0$ the problem $I_\gamma = 0$ is trivial. For $R_1 = 0$ we have $h_{1,1} = 0$ and this case has been already considered. Therefore we now restrict ourselves to non-zero g_1 and R_1.

We prove that (12.11) holds iff

$$R_1(x,t) = e_1(x) e_1(t) \quad \text{and} \quad e_1\, g_1 \equiv \text{constant} \ne 0 \text{ in the space } L_{2,\varrho_1}(D_1),$$

for some non-zero function e_1.

Note that the condition on R_1 means that $\dim(H(R_1)) = 1$ and $H(R_1) = \operatorname{span}(e_1)$. Equality in the space $L_{2,\varrho_1} = L_{2,\varrho_1}(D_1)$ means that the function $e_1 g_1$ is constant over the support of ϱ_1. If $1 \in H(R_1)$ then (12.11) holds iff

$$R_1 \equiv \text{constant} > 0 \quad \text{and} \quad g_1 \equiv \text{constant} \neq 0,$$

again with equality understood in the space L_{2,ϱ_1}.

Obviously, it is enough to show that (12.11) implies that $R_1(x,t) = e_1(x)e_1(t)$ and $e_1 g_1$ is constant. Let us take an orthonormal basis $\{e_k\}_{k\in\mathcal{I}}$ of $H(R_1)$, where $\mathcal{I}$ is an index set that is at most countable if the space $H(R_1)$ is separable. Then

$$R_1(x,t) = \sum_{k\in\mathcal{I}} e_k(x)e_k(t),$$

where the last series is convergent and consists of at most a countable number of positive terms. We rewrite (12.11) as

$$\sum_{k\in\mathcal{I}} \int_{D_1} e_k^2(t)g_1^2(t)\varrho_1(t)\,\mathrm{d}t = \sum_{k\in\mathcal{I}} \left(\int_{D_1} e_k(t)g_1(t)\varrho_1(t)\mathrm{d}t\right)^2.$$

Since the kth term of the left hand side is at least equal to the kth term of the right hand side, we conclude that for all $k \in \mathcal{I}$ we have

$$\int_{D_1} e_k^2(t)g_1^2(t)\varrho_1(t)\,\mathrm{d}t = \left(\int_{D_1} e_k(t)g_1(t)\varrho_1(t)\mathrm{d}t\right)^2.$$

Since L_{2,ϱ_1} is separable, we take $\{\eta_j\}_{j\in\mathbb{N}}$ as its orthonormal basis with $\eta_1 \equiv 1$. Then

$$e_k g_1 = \sum_{j=1}^{\infty} \langle e_k g_1, \eta_j\rangle_{L_{2,\varrho_1}}$$

yields

$$\sum_{j=1}^{\infty} \langle e_k g_1, \eta_j\rangle_{L_{2,\varrho_1}}^2 = \langle e_k g_1, \eta_1\rangle_{L_{2,\varrho_1}}^2 .$$

Therefore $\langle e_k g_1, \eta_j\rangle_{L_{2,\varrho_1}} = 0$ for all $j \geq 2$ and

$$e_k g_1 = \langle g_1, \eta_1\rangle_{L_{2,\varrho_1}}\, e_1 = \text{constant} \neq 0.$$

This holds for all $k \in \mathcal{I}$. If $|\mathcal{I}| > 2$ then $e_1 g_1 = c_1$ and $e_2\, g_2 = c_2$ for non-zero c_j and therefore $e_1 = (c_1/c_2)e_2$. Hence, e_1 and e_2 are linearly dependent which contradicts the fact that they form a part of the basis of $H(R_1)$. This proves that $|\mathcal{I}| = 1$ and $R_1(x,t) = e_1(x)e_1(t)$ as well as that $e_1 g_1 = \text{constant} \neq 0$, as claimed.

Although the assumption (12.11) for the normalized error criterion may seem to be restrictive, it is satisfied for a number of problems related to standard multivariate integration. We now illustrate this by several examples.

12.6.1 Example: Weighted Integration (Continued)

This example was recently analyzed in Subsection 12.5.1. We have $g_1 \equiv 1$ and

$$R_1(x,t) = \sum_{j=0}^{k} \frac{x^j}{j!}\,\frac{t^j}{j!} \quad \text{for all } x,t \in \mathbb{R},$$

for some $k \in \{0,1,\dots,r-1\}$. Note that for $r = 1$, the only option is for k to be zero and then $R_1 \equiv 1$. For $r \geq 2$, if we take $k = 0$ then again $R_1 \equiv 1$. In these two cases, (12.11) holds. It is easy to see that Corollary 12.4 is a special case of Theorem 12.10. For $r \geq 2$ and $k \geq 1$, Theorem 12.10 does not apply, and that is why we still have Open Problem 62. □

12.6.2 Example: Uniform Integration (Continued)

This example

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \text{for all } f \in F_d = W_a^{1,1,\dots,1}([0,1]^d)$$

was analyzed in Section 11.5.2. For this example, we have

$$R_1 = g_1 = \varrho_1 = 1,$$

so that (12.11) holds.

For the unweighted case $\gamma_{d,\mathfrak{u}} \equiv 1$, we already checked that multivariate integration is intractable for both the absolute and normalized error criteria, as long as the anchor $a \in (0,1)$.

We now consider the weighted case for $a \in [0,1]$. We have

$$R_2(x,t) = 1_M(x,t)\min(|x-a|,|t-a|) \quad \text{for all } x,t \in [0,1],$$

where $M = \{(x,t) \mid (x-a)(t-a) \geq 0,\ x,t \in [0,1]\}$. For $a \in (0,1)$, the kernel R_2 is decomposable at a and

$$h_{1,2,(0)}(x) = \begin{cases} \frac{1}{2}(a-x)(a+x) & \text{for } x \in [0,a], \\ 0 & \text{for } x \in [a,1], \end{cases}$$

$$h_{1,2,(1)}(x) = \begin{cases} 0 & \text{for } x \in [0,a], \\ \frac{1}{2}(x-a)(2-a-x) & \text{for } x \in [a,1]. \end{cases}$$

Furthermore

$$\|h_{1,2,(0)}\|_{F_1} = \tfrac{1}{3}a^3 \quad \text{and} \quad \|h_{1,2,(1)}\|_{F_1} = \tfrac{1}{3}(1-a)^3.$$

Then Theorems 12.2, 12.3, 12.9 and 12.10 can be applied.

For $a \in \{0, 1\}$ we proceed as follows. It is enough to consider $a = 0$ since the case $a = 1$ can be done analogously. We now have

$$R_2(x,t) = \min(x,t) \quad \text{for all } x,t \in [0,1],$$

and $H(R_2) = W_0^1([0,1])$ is the Sobolev space of absolutely continuous functions with first derivatives in $L_2([0,1])$ and vanishing at 0. Take $a^* \in (0,1)$ and consider the subspace

$$F_{a^*} = \{f \in H(R_2) \mid f(a^*) = 0\}.$$

Note that the projection

$$f - f(a^*)\frac{R_2(\cdot, a^*)}{R_2(a^*, a^*)}$$

belongs to F_{a^*} for all $f \in H(R_2)$. This implies that F_{a^*} is the reproducing kernel Hilbert space with the kernel

$$R_{2,2}(x,t) = R_2(x,t) - \frac{R_2(x,a^*)\,R_2(t,a^*)}{R_2(a^*,a^*)} = \min(x,t) - \frac{\min(x,a^*)\,\min(t,a^*)}{a^*}$$

for all $x,t \in [0,1]$. Note that $R_2(x,t) = 0$ for all $(x-a^*)(t-a^*) \le 0$ which means that R_2 is decomposable at a^*. We also have

$$R_{2,1}(x,t) = \frac{R_2(x,a^*)\,R_2(t,a^*)}{R_2(a^*,a^*)} = \frac{\min(x,a^*)\,\min(t,a^*)}{a^*} \quad \text{for all } x,t \in [0,1]$$

and $H(R_{2,1})$ is the one-dimensional reproducing kernel Hilbert space of functions

$$c\,\min(\cdot, a^*) \quad \text{for all } c \in \mathbb{R}.$$

This implies that $R_2 = R_{2,1} + R_{2,2}$ and

$$H(R_1) \cap H(R_2) = H(R_{2,1}) \cap H(R_{2,1}) = \{0\},$$

as needed in Theorem 12.5 and Corollary 12.7. Furthermore, for all $x \in [0,1]$ we have

$$\begin{aligned}
h_{1,1}(x) &= 1,\\
h_{1,2,1}(x) &= \left(1 - \tfrac{1}{2}a^*\right)\,\min(x,a^*),\\
h_{1,2,2,(0)}(x) &= 1_{[0,a^*]}(x)\,\left(\tfrac{1}{2}a^* x - \tfrac{1}{2}x^2\right),\\
h_{1,2,2,(1)}(x) &= 1_{[a^*,1]}(x)\,\left(x - \tfrac{1}{2}x^2 - a^*\left(1 - \tfrac{1}{2}a^*\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
\|h_{1,1}\|^2_{H(R_1)} &= 1,\\
\|h_{1,2,1}\|^2_{H(R_{2,1})} &= \left(1 - \tfrac{1}{2}a^*\right)^2 a^*,\\
\|h_{1,2,2,(0)}\|^2_{H(R_{2,2})} &= \frac{[a^*]^3}{12}\\
\|h_{1,2,2,(1)}\|^2_{H(R_{2,2})} &= \frac{(1-a^*)^3}{3}.
\end{aligned}$$

We also have

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_2 = \tfrac{1}{3}\left(a^3 + (1-a)^3\right) \in \left[\tfrac{1}{12}, \tfrac{1}{3}\right],$$
$$\alpha_1^* = 1 \quad \text{and} \quad \alpha_2^* = \tfrac{1}{2}\left(a^2 + (1-a)^2\right) \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right].$$

Hence, all the assumptions of Corollary 12.7 are satisfied.

From this and Theorem 12.10 we obtain the following corollary.

Corollary 12.11. *Consider uniform integration $I_\gamma = \{I_{d,\gamma}\}$ as defined in this example. To omit the trivial case, we assume that there is at least one non-zero weight $\gamma_{d,\mathfrak{u}}$ with $|\mathfrak{u}| > 0$ for every d. We consider the worst case setting for the absolute and normalized error criteria.*

- *Consider finite-order weights*

$$\gamma_{d,\mathfrak{u}} = 0 \quad \textit{for all } \mathfrak{u} \textit{ with } |\mathfrak{u}| > \omega.$$

For the absolute error criterion and finite-order weights we have the following:

  - *If*

$$\sup_{d\in\mathbb{N}} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} < \infty \tag{12.12}$$

  then I_γ is strongly polynomially tractable with exponent in $[1,2]$,

  - *If (12.12) holds then I_γ is strongly T-tractable iff*

$$t^* = \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty.$$

  If this holds, then the exponent belongs to $[t^*, 2t^*]$.

  - *If finite-order weights are polynomially bounded, i.e., there exists a positive k such that*

$$\sup_{d\in\mathbb{N}} \gamma_{d,\mathfrak{u}} d^{-k} < \infty,$$

  then I_γ is polynomially tractable with d exponent at most $\omega + k$ and ε^{-1} exponent at most 2; for finite-diameter weights the d exponent is at most $k+1$.

  - *If finite-order weights are polynomially bounded and*

$$t^* := \limsup_{\varepsilon<1,\ \varepsilon^{-1}+d\to\infty} \frac{k\ln d + 2\ln\varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

  then I_γ is T-tractable with exponent at most t^*. Here, $k = \omega$ for finite-order weights and $k = 1$ for finite-diameter weights.

For the normalized error criterion and finite-order weights we have the following:

– *I_γ is strongly polynomially tractable with exponent in* $[1,2]$*,*
– *I_γ is strongly T-tractable iff*

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty.$$

If this holds then the exponent belongs to $[t^*, 2t^*]$.

• *Consider product weights*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \textit{for all } \mathfrak{u} \subseteq [d].$$

For the absolute and normalized error criteria and product weights we have the following:

– *I_γ is strongly polynomially tractable iff*

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$$

If this holds then the exponent belongs to $[1,2]$.
– *I_γ is strongly T-tractable iff*

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty \quad \textit{and} \quad t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty.$$

If this holds then the exponent belongs to $[t^*, 2t^*]$.
– *I_γ is polynomially tractable iff*

$$q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty.$$

If this holds then the d exponent is at most arbitrarily close to

$$\left(\tfrac{1}{2} - a + a^2\right) q^* \in \left[\tfrac{1}{4} q^*, \tfrac{1}{2} q^*\right],$$

and the ε^{-1} exponent belongs to $[1,2]$*,*
– *I_γ is T-tractable iff*

$$\limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty \quad \textit{and} \quad \lim_{\varepsilon\to 1-} \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln T(\varepsilon^{-1},d)} < \infty.$$

If this holds then

$$t^* := \limsup_{\varepsilon\to\infty,\ \varepsilon^{-1}+d\to\infty} \frac{\ln \varepsilon^{-2} + (\frac{1}{2} - a + a^*)\ \sum_{j=1}^{d} \gamma_{d,j}}{\ln T(\varepsilon^{-1},d)} < \infty$$

and I_γ is T-tractable with exponent at most t^*.

– *I_γ is weakly tractable iff*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} = 0.$$

We now comment on Corollary 12.11. For the absolute error criterion and finite-order weights, we need to assume that the sum of the weights or the weights are bounded to get strong polynomial or polynomial tractability. Indeed, take $\gamma_{d,\{1\}} = w(d)$ and $\gamma_{d,\mathfrak{u}} = 0$ for all $\mathfrak{u} \neq \{1\}$. Suppose that $\lim_{d\to\infty} w(d) = \infty$. Clearly, this corresponds to the univariate case, for which it is known that

$$n(\varepsilon, 1) = \Omega(\sqrt{w(d)}\varepsilon^{-1}).$$

This contradicts strong polynomial tractability. Similarly, if the $\gamma_{d,\mathfrak{u}}$ are not polynomially bounded then we can take the function w such that $\lim_{d\to\infty} w(d)d^{-k} = \infty$ for all k and polynomial tractability does not hold.

The condition on strong T-tractability is also necessary, since for $d = 1$ we have $\gamma_{1,\{1\}} > 0$ and $n(\varepsilon, 1) = \Omega(\sqrt{\gamma_{1,\{1\}}}\,\varepsilon^{-1})$, which yields that t^* must be finite; said t^* is also a lower bound on the exponent of strong T-tractability. The last argument shows that the exponent of strong polynomial tractability and the ε^{-1} exponent of polynomial tractability are at least 1. The rest follows, more or less, directly from the results we mentioned before.

We stress that the exponent of strong polynomial tractability, as well as strong T-tractability, are not known exactly. As we shall see later, it will be possible to show that the exponent of strong polynomial tractability takes its smallest value 1 if we assume more about the product weights. Namely, the exponent is 1 if

$$\sup_d \sum_{j=1}^{d} \sqrt{\gamma_{d,j}} < \infty.$$

It is unknown whether this last condition on the product weights is really needed. This leads us to the next open problem.

Open Problem 63.

- Find necessary and sufficient conditions on finite-order, finite-diameter and product weights for which the exponent of strong polynomial tractability of multivariate integration considered in the example is 1.

- Do the conditions on weights depend on the anchor a?

12.6.3 Example: Bounds on Weighted L_2 Discrepancy

In Section 11.5.4 we showed bounds on uniform integration for the Sobolev space $F_d = W^{1,1,\dots,1}([0,1]^d)$ with no boundary conditions. For the unweighted case, we showed that uniform integration is intractable for both the absolute and normalized error criteria. As we know, this problem is equivalent to unweighted L_2 discrepancy.

So we now consider the weighted case $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ and show conditions on the weights to get various kinds of tractability for weighted L_2 discrepancy.

This corresponds to the previous example with $a = 0$, and then we obtain $\alpha_1 = \alpha_1^* = 1$, $\alpha_2 = \frac{1}{3}$ and $\alpha_2^* = \frac{1}{2}$. As before, let $\mathrm{disc}_{2,\gamma}(n,d)$ denote the minimal L_2 discrepancy anchored at 0 when we use n points in the d-dimensional case. We already mentioned in Section 9.3 of Chapter 9 that

$$\mathrm{disc}_{2,\gamma}^2(0,d) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, 3^{-|\mathfrak{u}|},$$
$$\mathrm{disc}_{2,\gamma}^2(n,d) \le n^{-1} \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left(2^{-|\mathfrak{u}|} - 3^{-|\mathfrak{u}|}\right).$$

For bounded finite-order weights, $\gamma_{d,\mathfrak{u}} \le \Gamma$ and $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| \ge \omega$, we have

$$\mathrm{disc}_{2,\gamma}^2(n,d) \le n^{-1}\,\Gamma \sum_{k=0}^{\omega} \binom{d}{k} 2^{-k} = \mathcal{O}\left(n^{-1}\,\Gamma\, d^{\,\omega}\right).$$

We stress that the bounds on $\mathrm{disc}_{2,\gamma}(n,d)$ can be attained by QMC algorithms.

For bounded finite-diameter weights of order q, i.e., $\gamma_{d,\mathfrak{u}} = 0$ for all $\mathrm{diam}(\mathfrak{u}) \ge q$, we have

$$\mathrm{disc}_{2,\gamma}^2(n,d) \le n^{-1}\,\Gamma \sum_{\mathfrak{u}\subseteq[d]\,:\,\mathrm{diam}(\mathfrak{u})\le q} \gamma_{d,\mathfrak{u}} 2^{-k} = \mathcal{O}\left(n^{-1}\,\Gamma\, d\right).$$

Furthermore, for finite-order weights (not necessarily bounded) of order ω we have

$$\frac{\mathrm{disc}_{2,\gamma}^2(n,d)}{\mathrm{disc}_{2,\gamma}^2(0,d)} \le n^{-1} \frac{\sum_{|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} 2^{-|\mathfrak{u}|}}{\sum_{|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}|}} \le n^{-1} \left(\frac{3}{2}\right)^{\omega},$$

whereas for finite-diameter weights (not necessarily bounded) of order q we have $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > q$ and therefore

$$\frac{\mathrm{disc}_{2,\gamma}^2(n,d)}{\mathrm{disc}_{2,\gamma}^2(0,d)} \le n^{-1} \frac{\sum_{\mathrm{diam}(\mathfrak{u})\ge q} \gamma_{d,\mathfrak{u}} 2^{-|\mathfrak{u}}}{\sum_{\mathrm{diam}(\mathfrak{u})\ge q} \gamma_{d,\mathfrak{u}} 3^{-|\mathfrak{u}}} \le n^{-1} \left(\frac{3}{2}\right)^{q}.$$

Hence, we have polynomial bounds for the absolute error criterion and strongly polynomial bounds for the normalized error criterion.

For product weights, we have

$$\mathrm{disc}_{2,\gamma}^2(0,d) = \prod_{j=1}^{d} \left(1 + \tfrac{1}{3}\gamma_{d,j}\right),$$

$$\mathrm{disc}_{2,\gamma}^2(n,d) \le n^{-1} \prod_{j=1}^{d} \left(1 + \tfrac{1}{2}\gamma_{d,j}\right).$$

From this and Corollary 12.11 we conclude the following:

- There exists a positive number C such that

$$\mathrm{disc}_{2,\gamma}^2(n,d) \le C\, n^{-1} \quad \text{for all} \quad n,d \in \mathbb{N} \quad \text{iff} \quad \sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

- There exist positive numbers C and q such that

$$\mathrm{disc}_{2,\gamma}^2(n,d) \le C\, d^{\,q}\, n^{-1} \quad \text{for all} \quad n,d \in \mathbb{N}$$

  iff

$$q^* := \limsup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty.$$

  If $q^* < \infty$ then for all $q > q^*$ we have

$$\mathrm{disc}_{2,\gamma}^2(n,d) = \mathcal{O}\big(d^{\,q} n^{-1}\big) \quad \text{for all } n,d \in \mathbb{N}.$$

- Furthermore

$$\lim_{d\to\infty} \frac{\ln \mathrm{disc}_{2,\gamma}(n,d)}{d} = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{iff} \quad \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$

This proves that the conditions (9.23), (9.24) and (9.25) presented in Chapter 9 are also necessary.

12.6.4 Upper Bounds Based on Theorem 10.10

The upper bounds for multivariate integration presented so far in this section are based on Theorem 10.7 that requires that

$$\int_{D_1} K_1(t,t) g_1^2(t) \varrho_1(t)\, \mathrm{d}t < \infty.$$

As we know, for some multivariate integration problems the last integral is infinite and then the upper bounds obtained so far are not applicable.

As in Chapter 10, we know that we can relax the last assumption by assuming that

$$\int_{D_1} \sqrt{K_1(t,t)}\,|g_1(t)|\varrho_1(t)\,\mathrm{d}t < \infty, \tag{12.13}$$

and then apply Theorem 10.10. More precisely, we must guarantee that

$$C^{\text{new}}(K_d, g_d) := \left[\int_{D_d} \sqrt{K_d(t,t)}\,|g_d(t)|\,\varrho_d(t)\,\mathrm{d}t\right]^2 < \infty.$$

We claim that (12.13) yields that $C^{\text{new}}(K_d, g_d)$ is finite. Indeed, $K_1 = R_1 + R_2$ implies that $R_1(t,t) + R_2(t,t) = K_1(t,t)$. Obviously $R_j(t,t) \ge 0$, so that

$$\beta_j^* := \int_{D_1} \sqrt{R_j(t,t)}\,|g_1(t)|\varrho_1(t)\,\mathrm{d}t \le \int_{D_1} \sqrt{K_1(t,t)}\,|g_1(t)|\varrho_1(t)\,\mathrm{d}t < \infty$$

for $j = 1,2$. Let $a = [C^{\text{new}}(K_d, g_d)]^{1/2}$. Since $\sqrt{\sum_j x_j} \le \sum_j x_j^{1/2}$ for any non-negative x_j, we have

$$\begin{aligned}
a &= \int_{D_d} \Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\notin\mathfrak{u}} R_1(t_j,t_j) g_1^2(t_j)\varrho_1^2(t_j) \prod_{j\in\mathfrak{u}} R_2(t,t) g_1^2(t_j)\varrho_1^2(t_j)\Big)^{1/2} \mathrm{d}t \\
&\le \int_{D_d} \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/2} \prod_{j\notin\mathfrak{u}} \sqrt{R_1(t_j,t_j)}|g_1(t_j)|\varrho_1(t_j) \prod_{j\in\mathfrak{u}} \sqrt{R_2(t,t)}|g_1(t_j)|\varrho_1(t_j)\,\mathrm{d}t \\
&= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/2} [\beta_1^*]^{d-|\mathfrak{u}|} [\beta_2^*]^{|\mathfrak{u}|} < \infty,
\end{aligned}$$

as claimed.

Note that $\beta_j^* \le [\alpha_j^*]^{1/2}$, where α_j^* are given as in Section 12.6, that is, $\alpha_j^* = \int_{D_1} R_j(t,t)\,g_1^2(t)\,\varrho_1(t)\,\mathrm{d}t$. Let

$$g^*(d) = \frac{\big(\sum_{k=0}^{d} [\beta_1^*]^{d-k} [\beta_2^*]^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}^{1/2}\big)^2}{\mathrm{CRI}_d}.$$

The function g^* should be compared with the function f^* defined by (12.9). We have

$$g^*(d) \le |\{\gamma_{d,\mathfrak{u}} \mid \gamma_{d,\mathfrak{u}} > 0\}|\; f^*(d) \le 2^d\, f^*(d).$$

For product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$, we have

$$\frac{g^*(d)}{f^*(d)} = \prod_{j=1}^{d} \frac{(\beta_1^* + \beta_2^* \gamma_{d,j}^{1/2})^2}{\alpha_1^* + \alpha_2^* \gamma_{d,j}}.$$

For general weights γ, we know from Theorem 10.10 that

$$e(n,d) \le \min\left(\|h_{d,\gamma}\|_{F_{d,\gamma}}, \frac{\sqrt{C^{\text{new}}(K_{d,\gamma}, g_d) - \|h_{d,\gamma}\|^2_{F_{d,\gamma}}}}{\sqrt{n}} \right),$$

and this error bound can be achieved by the algorithm $A_{n,d,\vec{t}}$ of Section 10.7.9.

Using this estimate we can repeat the argument of Section 12.6 and conclude that for the absolute and normalized error criteria we have

$$n(\varepsilon, d) \le \frac{g^*(d)}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Then we can use the proof of Theorem 12.9 for the function g^* instead of the function f^* to get the following theorem.

Theorem 12.12. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ with a non-zero h_1 such that (12.13) holds. We consider the worst case setting for the absolute and normalized error criteria.*

- *If*
$$\sup_{d \in \mathbb{N}} g^*(d) < \infty$$
then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if
$$p^* := \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$
then I_γ is strongly T-tractable with exponent at most $2p^*$.

- *If there exists a non-negative q such that*
$$\limsup_{d \to \infty} g^*(d)\, d^{-q} < \infty$$
then I_γ is polynomially tractable with ε^{-1} exponent at most 2 and d exponent at most q.

- *If*
$$t^* := \limsup_{\substack{\varepsilon < \min(1, \|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d) \\ \varepsilon^{-1} + d \to \infty}} \frac{\ln g^*(d) + \ln \varepsilon^{-2}}{\ln T(\varepsilon^{-1}, d)} < \infty$$
then I_γ is T-tractable with exponent at most t^*.

- *If*
$$\lim_{d \to \infty} \frac{\ln \max(1, g^*(d))}{d} = 0$$
then I_γ is weakly tractable.

Proceeding similarly as in Section 12.6, we now specialize Theorem 12.12 for finite-order and product weights.

Theorem 12.13. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ with non-zero $h_{1,1}$ and $h_{1,2}$ such that (12.13) holds. We consider the worst case setting for the absolute and normalized error criteria.*

- *Consider finite-order weights*

$$\gamma_{d,\mathfrak{u}} = 0 \quad \textit{for all } \mathfrak{u} \textit{ with } |\mathfrak{u}| > \omega.$$

For the absolute error criterion and bounded finite-order weights, i.e.,

$$\sup_{d \in \mathbb{N}} \gamma_{d,\mathfrak{u}} < \infty,$$

we have the following:

  - *If*

$$\beta_1^* := \int_{D_1} \sqrt{R_1(t,t)}\, |g_1(t)|\, \varrho_1(t)\, \mathrm{d}t < 1$$

  then I_γ is strongly polynomially tractable with exponent at most 2.
  - *If*

$$\beta_1^* < 1 \quad \textit{and} \quad t^* := \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

  then I_γ is strongly T-tractable with exponent at most $2t^*$.
  - *If*

$$\beta_1^* = 1$$

  then I_γ is polynomially tractable with d exponent at most 2ω and ε^{-1} exponent at most 2*; for bounded finite-diameter weights the d exponent is at most* 2.
  - *If*

$$\beta_1^* = 1 \quad \textit{and} \quad t^* := \limsup_{\substack{\varepsilon < \|h_{d,\gamma}\|_{F_{d,\gamma}} \\ \varepsilon^{-1} + d \to \infty}} \frac{k \ln d + 2 \ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

  then I_γ is T-tractable with exponent at most t^*. Here $k = 2\omega$ for finite-order weights, and $k = 2$ for finite-diameter weights.

For the normalized error criterion and finite-order weights we have the following:

  - *If*

$$[\beta_1^*]^2 = \alpha_1 := \|h_{1,1}\|_{H(R_1)}$$

  then I_γ is polynomially tractable with exponent at most ω for finite-order weights and at most 1 *for finite-diameter weights.*
  - *If*

$$[\beta_1^*]^2 = \alpha_1 \quad \textit{and} \quad t^* := \limsup_{\varepsilon^{-1} + d \to \infty} \frac{k \ln d + 2 \ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

  then I_γ is T-tractable with exponent at most t^*. Here, as before, $k = \omega$ for finite-order weights and $k = 1$ for finite-diameter weights.

- *Consider product weights*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \textit{for all } \mathfrak{u} \subseteq [d].$$

For the absolute error criterion and product weights we have the following:

  - *If*

$$\beta_1^* + \beta_2^* \cdot \sup_{d\in\mathbb{N}} \max_{j=1,2,\dots,d} \gamma_{d,j}^{1/2} \le 1$$

or

$$\beta_1^* < 1 \quad \textit{and} \quad \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}^{1/2}}{\ln d} < \infty$$

or

$$\beta_1^* = 1 \quad \textit{and} \quad \limsup_{d\to\infty} \sum_{j=1}^d \gamma_{d,j}^{1/2} < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

  - *If*

$$\beta_1^* = 1 \quad \textit{and} \quad q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}^{1/2}}{\ln d} < \infty$$

then I_γ is polynomially tractable with d exponent at most arbitrarily close to $2\beta_2^* q^*$ and ε^{-1} exponent at most 2.

  - *If $\beta_1^* \le 1$ and*

$$t^* := \limsup_{\substack{\varepsilon^2 < \prod_{j=1}^d (\beta_1^* + \beta_2^* \gamma_{d,j}^{1/2})^2 \\ \varepsilon^{-1} + d \to \infty}} \frac{\ln \varepsilon^{-2} + 2d \ln \beta_1^* + 2(\beta_2^*/\beta_1^*) \sum_{j=1}^d \gamma_{d,j}^{1/2}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ is T-tractable with exponent at most t^*.

  - *If $\beta_1^* \le 1$ and*

$$\lim_{d\to\infty} \frac{\left[2d \ln \beta_1^* + 2(\beta_2^*)/(\beta_1^*) \sum_{j=1}^d \gamma_{d,j}^{1/2}\right]_+}{\ln d} = 0$$

then I_γ is weakly tractable.

For the normalized error criterion and product weights assume that

$$[\beta_1^*]^2 = \alpha_1.$$

Then we have the following:

- *If*

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}^{1/2} < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2*. Additionally, if*

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

- *If*

$$q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{1/2}}{\ln d} < \infty$$

then I_γ is polynomially tractable with d exponent at most arbitrarily close to $2\,q^*\beta_2^*/\beta_1^*$ and ε^{-1} exponent at most 2*.*

- *If*

$$t^* := \limsup_{\varepsilon<1:\,\varepsilon^{-1}+d\to\infty} \frac{\ln \varepsilon^{-2} + 2\,q^*\,\beta_2^*/\beta_1^*\,\sum_{j=1}^{d} \gamma_{d,j}}{\ln T(\varepsilon^{-1}, d)}$$

then I_γ is T-tractable with exponent at most t^*.

- *If*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{1/2}}{d} = 0$$

then I_γ is weakly tractable.

Proof. Basically we need to repeat the reasoning of Theorem 12.10 with the obvious changes.

For the absolute error criterion and bounded finite-order weights, we have $g^*(d) = [\beta_1^*]^{2d}\,p_{2k}(d)$, where p_{2k} is a polynomial of degree $2k$ with $k = \omega$ for finite-order weights of order ω, and $k = 1$ for finite-diameter weights. This implies all the statements for this error criterion.

For the normalized error criterion and finite-order weights, we assumed that $[\beta_1^*]^2 = \alpha_1$. Since $\alpha_2 > 0$ and at least one $\gamma_{d,\mathfrak{u}}$ is positive we have

$$g^*(d) = \frac{[\beta_1^*]^{2d} \left(\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le k} \gamma_{d,\mathfrak{u}}^{1/2} (\beta_2^*/\beta_1^*)^{|\mathfrak{u}|}\right)^2}{[\alpha_1]^d \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le k} \gamma_{d,\mathfrak{u}} \, (\alpha_2/\alpha_1)^{|\mathfrak{u}|}}$$
$$= \frac{\left(\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le k} \gamma_{d,\mathfrak{u}}^{1/2} (\alpha_2/\alpha_1)^{|\mathfrak{u}|/2} \left((\alpha_1/\alpha_2)^{1/2} (\beta_2^*/\beta_1^*)\right)^{|\mathfrak{u}|}\right)^2}{\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le k} \gamma_{d,\mathfrak{u}} \, (\alpha_2/\alpha_1)^{|\mathfrak{u}|}}$$
$$\le \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le k} \left[\frac{\alpha_1^{1/2}}{\alpha_2^{1/2}} \frac{\beta_2^*}{\beta_1^*}\right]^{|\mathfrak{u}|} = \mathcal{O}(d^k).$$

Hence, f^* is now bounded by a multiple of d^k, again with $k = \omega$ for finite-order weights, and $k = 1$ for finite-diameter weights. This yields the next point of the theorem.

For product weights and the absolute error criterion, we have

$$g^*(d) = \prod_{j=1}^d \left(\beta_1^* + \beta_2^* \gamma_{d,j}^{1/2}\right)^2.$$

Then we proceed as in the proof of Theorem 12.10 with the obvious changes, and obtain the next point of the theorem.

For product weights and the normalized error, we have

$$g^*(d) = \prod_{j=1}^d \frac{(1 + (\beta_2^*/\beta_1^*)\, \gamma_{d,j}^{1/2})^2}{1 + (\alpha_2/\alpha_1)\, \gamma_{d,j}} \le \exp\left(2 \frac{\beta_2^*}{\beta_1^*} \sum_{j=1}^d \gamma_{d,j}^{1/2}\right).$$

The rest is easy. □

The main difference between Theorems 12.10 and 12.13 for finite-order weights is the difference between the d exponents. For the absolute error criterion, the d exponents are double if we switch from Theorem 12.10 to Theorem 12.13. For the normalized error criterion, we have strong polynomial tractability in the case of Theorem 12.10, and only polynomial tractability in the case of Theorem 12.13. We are not sure if this is caused by an overestimating $\sqrt{\sum_j x_j}$ by $\sum_j x_j^{1/2}$, or it is the consequence of replacing the assumption 12.4 by the weaker assumption 12.13. This leads us to the next open problem.

Open Problem 64.

Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ for finite-order and finite-diameter weights with non-zero $h_{1,2}$ and $h_{1,2}$ such that

$$\int_{D_1} K_1(t,t)\, g_1^2(t)\, \varrho_1(t)\, \mathrm{d}t = \infty \quad \text{and} \quad \int_{D_1} \sqrt{K_1(t,t)}\, |g_1(t)|\, \varrho_1(t)\, \mathrm{d}t < \infty.$$

- For the absolute error criterion, find the exact d exponent of polynomial tractability.
- For the normalized error criterion, verify strong polynomial tractability. If strong polynomial tractability does not hold, find a d exponent of polynomial tractability.
- Do the exponents depend on the kernel K_1 and the functions g_1 and ϱ_1? □

For product weights and normalized error criterion, the estimates in Theorem 12.13 are also not completely satisfactory since they are expressed in terms of $\gamma_{d,j}^{1/2}$, whereas lower bound estimates of Chapter 11 are given in terms of $\gamma_{d,j}$. We now show that at least sometimes we can obtain similar estimates as in Theorem 12.13 in terms of $\gamma_{d,j}^{s}$ for s that can be arbitrarily close to 1.

For simplicity let us assume that $R_1 = 1$. Then the reproducing kernel is

$$K_d(x,t) = \prod_{j=1}^{d} \big(1 + \gamma_{d,j} R_2(x_j,t_j)\big) \quad \text{for all } x,t \in D_1^d.$$

We need to find a better estimate of

$$C^{\text{new}}(K_d, g_d) = \bigg(\prod_{j=1}^{d} \int_{D_1} \big(1 + \gamma_{d,j} R_2(t,t)\big)^{1/2} \, |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t\bigg)^2.$$

It is easy to check that for any $s \in (\frac{1}{2}, 1)$ there exists a positive C_s such that

$$(1+x)^{1/2} \le 1 + C_s x^s \quad \text{for all } x \in [0,\infty).$$

Indeed, by squaring both sides, the last inequality is equivalent to

$$1 \le g(x) := 2C_s x^{s-1} + C_s^2 x^{2s-1}.$$

Clearly, $g(0) = g(\infty) = \infty$ since the first term goes to infinity as x goes to zero, whereas the second term goes to infinity as x goes to infinity. It is enough to check if $1 \le g(x)$ for x minimizing g. This x is obviously characterized by $g'(x) = 0$ and is given by

$$x = \left[\frac{2(1-s)}{C_s(2s-1)}\right]^{1/s}.$$

Then

$$\min_{x\in[0,\infty)} g(x) = C_s^{1/s}\left[2\left(\frac{2(1-s)}{2s-1}\right)^{1-1/s} + \left(\frac{2(1-s)}{2s-1}\right)^{2-1/s}\right] = 1$$

if we take

$$C_s = \left[2\left(\frac{2(1-s)}{2s-1}\right)^{1-1/s} + \left(\frac{2(1-s)}{2s-1}\right)^{2-1/s}\right]^{-s}.$$

Therefore,

$$C^{\text{new}}(K_d, g_d) \le \left(\prod_{j=1}^{d} \int_{D_1} \left(1 + C_s \gamma_{d,j}^s \, R_2^s(t,t)\right) |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t \right)^2.$$

Based on this estimate we are ready to prove the following theorem.

Theorem 12.14. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in this section for product weights and for the normalized error criterion. Assume that*

$$\int_{D_1} |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t = \left| \int_{D_1} g_1(t) \, \varrho_1(t) \, \mathrm{d}t \right| > 0,$$

and that there exists $s \in (\frac{1}{2}, 1)$ such that

$$\int_{D_1} R_2^s(t,t) \, |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t < \infty.$$

- *If*

  $$\sup_{d \in \mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^s < \infty$$

  then I_γ is strongly polynomially tractable with exponent at most 2.

- *If*

  $$q^* := \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^s}{\ln d} < \infty$$

  then I_γ is polynomially tractable with ε exponent at most 2, and d exponent at most arbitrarily close to

  $$2 q^* C_s \, \frac{\int_{D_1} R_2^s(t,t) \, |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t}{\int_{D_1} |g_1(t)| \, \varrho_1(t) \, \mathrm{d}t}.$$

- *If*

  $$\lim_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^s}{d} = 0$$

  then I_γ is weakly tractable.

Proof. We now have

$$n(\varepsilon, d) \le \frac{a}{\varepsilon^2} \quad \text{with } a := \frac{C^{\text{new}}(K_d, g_d)}{e^2(0,d)}.$$

We estimate a as follows

$$a \le \prod_{j=1}^{d} \frac{\left(\int_{D_1} |g_1(t)|\varrho_1(t)\,\mathrm{d}t + C_s\gamma_{d,j}^s \int_{D_1} R^s(t,t)|g_1(t)|\varrho_1(t)\,\mathrm{d}t\right)^2}{\left(\int_{D_1} g_1(t)\varrho_1(t)\,\mathrm{d}t\right)^2 + \gamma_{d,j}\int_{D_1^2} R_2(x,t)g_1(x)g_1(x)g_1(t)\varrho_1(x)\varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t}$$
$$= \prod_{j=1}^{d} \frac{(1+a_s\gamma_{d,j}^s)^2}{1+b\gamma_{d,j}} \le \prod_{j=1}^{d}(1+a_s\gamma_{d,j}^s)^2,$$

where

$$a_s = C_s\,\frac{\int_{D_1} R_2^s(t,t)\,|g_1(t)|\,\varrho_1(t)\,\mathrm{d}t}{\int_{D_1} |g_1(t)|\,\varrho_1(t)\,\mathrm{d}t},$$
$$b = \frac{\int_{D_1^2} R_2(x,t)g_1(x)g_1(x)g_1(t)\varrho_1(x)\varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t}{\left(\int_{D_1} g_1(t)\varrho_1(t)\,\mathrm{d}t\right)^2}.$$

From the last estimate we easily obtain all points of the theorem. □

We illustrate the analysis of this section by an example.

12.6.5 Example: Integration and Unbounded Kernel

We already discussed uniform integration for the Sobolev space with the reproducing kernel for $d = 1$ of the form $K_1(x,t) = 1 + \min(x,t)$ for $x,t \in [0,1]$. We now generalize this space by taking

$$K_1(x,t) = 1 + \min(x,t) \quad \text{for all } x,t \in [0,\infty).$$

This corresponds to the space $F_1 = H(K_1)$ of absolutely continuous functions with first derivatives in $L_2([0,\infty))$ and the inner product

$$\langle f, g\rangle_{F_1} = f(0)g(0) + \int_0^\infty f'(t)g'(t)\,\mathrm{d}t \quad \text{for all } f,g \in F_1.$$

We now have $R_1 = g_1 = 1$ and $R_2(x,t) = \min(x,t)$ for all $x,t \in D_1 := [0,\infty)$.

Let $\varrho_1(t) = (1+t)^{-2}$ for $t \in [0,\infty)$. It is easy to check that

$$\int_{[0,\infty)^2} K_1(x,t)\varrho_1(x)\varrho_1(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \frac{1+\ln(1+x)}{(1+x)^2}\,\mathrm{d}x = 2.$$

This means that the univariate integration problem

$$I_1(f) = \int_0^\infty f(t)\varrho_1(t)\,\mathrm{d}t \quad \text{for all } f \in F_1$$

is well defined, with $\|I_1\| = \sqrt{2}$. Furthermore,

$$I_1(f) = \langle f, h_1 \rangle_{F_1} \quad \text{with } h_1(x) = 1 + \ln(1 + x).$$

Clearly, $h_{1,1} = 1$ and $h_{1,2}(x) = \ln(1 + x)$ and $\|h_{1,1}\|_{F_1} = \|h_{1,2}\|_{F_1} = 1$.

For $d \geq 1$, we take tensor products and obtain

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1, \quad d \text{ times},$$

and

$$I_d(f) = \int_{[0,\infty)^d} f(t) \varrho_d(t) \, \mathrm{d}t$$

with $\varrho_d(t) = \prod_{j=1}^d \varrho_1(t_j)$ for $t = [t_1, t_2, \dots, t_d] \in [0, \infty)^d$.

Since $K_1(t,t)\varrho_1(t) = (1 + t)/(1 + t)^2 = 1/(1 + t)$, we have

$$\int_{[0,\infty)} K_1(t,t)\varrho_1(t) \, \mathrm{d}t = \infty,$$

and (12.4) with $g_1 = 1$ is indeed not satisfied and therefore Theorems 12.9 and 12.10 are not applicable.

On the other hand $\sqrt{K_1(t,t)}\varrho_1(t) = \sqrt{1 + t}/(1+t)^2 = (1+t)^{-3/2}$ and therefore

$$\int_0^\infty \sqrt{K_1(t,t)}\varrho_1(t) \, \mathrm{d}t = \int_0^\infty \frac{1}{(1 + t)^{3/2}} \, \mathrm{d}t = 2,$$

so that 12.13 holds, and we can apply the results presented in this section. We now have

$$\beta_1^* = \alpha_1 = \alpha_1^* = 1,$$

$$\beta_2^* = \int_0^\infty \frac{\sqrt{t}}{(1 + t)^2} \, \mathrm{d}t = \left(-\frac{\sqrt{t}}{1 + t} + \arctan(\sqrt{t}) \right)\Big|_0^\infty = \frac{\pi}{2}.$$

Note that

$$\int_0^\infty R_2(t,t)^s |g_1(t)| \varrho_1(t) \, \mathrm{d}t = \int_0^\infty \tfrac{t^s}{(1+t)^2} \, \mathrm{d}t < \infty$$

for all $s \in [\frac{1}{2}, 1)$ so that we can use Theorem 12.14 for product weights.

We summarize the results obtained in this section for this example assuming the normalized error criterion and finite-order, finite-diameter and product weights. We also mention a few lower bounds from Chapter 11.

- For finite-order weights of order ω, the problem is polynomially tractable with d exponent at most ω and ε^{-1} exponent at most 2.
- For finite-diameter weights of order q, the problem is polynomially tractable with d exponent at most 1 and ε^{-1} exponents at most 2.
- Consider product weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_{d,j}$.

- If
$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}^{s} < \infty$$
for some $s \in [\frac{1}{2}, 1)$ then the problem is strongly polynomially tractable.
- If the problem is strongly polynomially tractable then
$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$
- If
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{s}}{\ln d} < \infty$$
for some $s \in [\frac{1}{2}, 1)$ then the problem is polynomially tractable.
- If the problem is polynomially tractable then
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty.$$
- If
$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{s}}{d} = 0$$
for some $s \in [\frac{1}{2}, 1)$ then the problem is weakly tractable.
- If the problem is weakly tractable then
$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$

Clearly, there is a small gap between the necessary and sufficient conditions for product weights since we cannot take $s = 1$.

12.7 Upper Bounds for the General Case

In the previous section we studied multivariate integration. For the normalized error criterion we needed to assume (12.11) to determine conditions on tractability.

In this section we study more general linear functionals, not merely multivariate integration. This will be done at the expense of more restrictive assumptions on the weights needed for various kinds of tractability.

As we know, some linear functionals are arbitrarily hard to approximate even for $d = 1$. To eliminate such cases, we need to assume that we deal with linear functionals that are relatively easy to approximate for $d = 1$. More precisely, since we still want to use general upper bounds from Chapter 10, we restrict ourselves to linear functionals $I_{d,\gamma}$ for which a non-zero $h_1 \in F_1$ is chosen such that

$$\|h_{1,1}\|^*_{F_1} < \infty \quad \text{and} \quad \|h_{1,2}\|^*_{F_1} < \infty. \tag{12.14}$$

For the convenience of the reader we recall the definition of the norm $\|\cdot\|^*_{F_1}$ from Section 10.10 of Chapter 10. For $f \in F_1$, let

$$b_\varepsilon = \inf\left\{ \sum_{j=1}^m |\beta_j| \sqrt{K_1(z_j,z_j)} \mid \left\| f - \sum_{j=1}^m \beta_j K_1(\cdot,z_j)\right\|_{F_1} \le \varepsilon \right\}.$$

Then

$$\|f\|^*_{F_1} = \limsup_{\varepsilon\to 0} b_\varepsilon.$$

Obviously, $\|f\|_{F_1} \le \|f\|^*_{F_1}$. If $f = \beta_1 K_1(\cdot,z_1)$ for some real $\beta_1 \in \mathbb{R}$ and some point $z_1 \in D_1$ then we have

$$\|f\|_{F_1} = |\beta_1| \sqrt{K_1(z_1,z_1)} = \|f\|^*_{F_1}.$$

Note that if $\dim(F_1) = 1$ then we have $\|f\|^*_{F_1} = \|f\|_{F_1}$ for all $f \in F_1$, since each $f = \beta_1 K_1(\cdot,z_1)$ for some real β_1 and $z_1 \in D_1$.

Since $\|\cdot\|^*_{F_1}$ is a norm and $h_1 = h_{1,1} + h_{1,2}$, we have

$$\|h_1\|^*_{F_1} \le \|h_{1,1}\|^*_{F_1} + \|h_{1,2}\|^*_{F_2}.$$

As before, $\alpha_j = \|h_{1,j}\|^2_{H(R_j)} = \|h_{1,j}\|^2_{F_1}$. Analogously, we denote

$$\alpha_1^* = \left[\|h_{1,1}\|^*_{F_1}\right]^2 \quad \text{and} \quad \alpha_2^* = \left[\|h_{1,2}\|^*_{F_1}\right]^2.$$

For $\mathfrak{u} \subseteq [d]$, let

$$h_{\mathfrak{u}}(x) = \prod_{j\notin\mathfrak{u}} h_{1,1}(x_j) \prod_{j\in\mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d.$$

Then $h_{\mathfrak{u}}$ belongs to the tensor product space F_d, and we have

$$\|h_{\mathfrak{u}}\|^*_{F_{d,\gamma}} = \frac{1}{\sqrt{\gamma_{d,\mathfrak{u}}}} \|h_{\mathfrak{u}}\|^*_{F_d} \le \frac{1}{\sqrt{\gamma_{d,\mathfrak{u}}}} \left[\|h_{1,1}\|^*_{F_1}\right]^{d-|\mathfrak{u}|} \left[\|h_{1,2}\|^*_{F_1}\right]^{|\mathfrak{u}|}.$$

Therefore

$$\|h_{d,\gamma}\|^*_{F_{d,\gamma}} \le \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \|h_{\mathfrak{u}}\|^*_{F_{d,\gamma}} \le \sum_{\mathfrak{u}\subseteq[d]} \sqrt{\gamma_{d,\mathfrak{u}}} \left[\|h_{1,1}\|^*_{F_1}\right]^{d-|\mathfrak{u}|} \left[\|h_{1,2}\|^*_{F_1}\right]^{|\mathfrak{u}|}.$$

Hence

$$\|h_{d,\gamma}\|^*_{F_{d,\gamma}} \le \sum_{\mathfrak{u}\subseteq[d]} \sqrt{\gamma_{d,\mathfrak{u}}} \left[\alpha_1^*\right]^{(d-|\mathfrak{u}|)/2} \left[\alpha_2^*\right]^{|\mathfrak{u}|/2}.$$

This inequality should be compared with

$$\|h_{d,\gamma}\|_{F_{d,\gamma}} = \Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, [\alpha_1]^{d-|\mathfrak{u}|}\, [\alpha_2]^{|\mathfrak{u}|} \Big)^{1/2}.$$

By Jensen's inequality, we find that

$$\|h_{d,\gamma}\|_{F_{d,\gamma}} \le \sum_{\mathfrak{u}\subseteq[d]} \sqrt{\gamma_{d,\mathfrak{u}}}\, [\alpha_1]^{(d-|\mathfrak{u}|)/2}\, [\alpha_2]^{|\mathfrak{u}|/2},$$

and since $\alpha_j \le \alpha_j^*$ we have $\|h_{d,\gamma}\|_{F_{d,\gamma}} \le \|h_{d,\gamma}\|^*_{F_{d,\gamma}}$. As always, we interpret $0^0 = 1$.

From Theorem 10.14 we know that

$$e(n,d) \le \min\left(\|h_{d,\gamma}\|_{F_{d,\gamma}}, \frac{\sqrt{\big[\|h_{d,\gamma}\|^*_{F_{d,\gamma}}\big]^2 - \|h_{d,\gamma}\|^2_{F_{d,\gamma}}}}{\sqrt{n}} \right). \tag{12.15}$$

Let

$$f^*(d) = \frac{\Big(\sum_{k=0}^{d} \big[\alpha_1^*\big]^{(d-k)/2} \big[\alpha_2^*\big]^{k/2} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\Big)^2}{\mathrm{CRI}_d^2}, \tag{12.16}$$

where, as always, $\mathrm{CRI}_d = 1$ for the absolute error criterion, and

$$\mathrm{CRI}_d^2 = e^2(0,d) = \|h_{d,\gamma}\|^2_{F_{d,\gamma}} = \sum_{k=0}^{d} [\alpha_1]^{d-k}\, [\alpha_2]^{k} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}$$

for the normalized error criterion.

As in the previous section, we can check that we have

$$n(\varepsilon,d) \le \frac{f^*(d)}{\varepsilon^2} \quad \text{for all } \varepsilon\in(0,1),\ d\in\mathbb{N} \tag{12.17}$$

for the absolute and normalized error criteria.

We can now apply exactly the same reasoning as in the proof of Theorem 12.9 to obtain the analogous theorem.

Theorem 12.15. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in Section 12.2 with a non-zero h_1 such that $\|h_{1,1}\|^*_{F_1} < \infty$ and $\|h_{1,2}\|^*_{F_1} < \infty$. We consider the worst case setting for the absolute and normalized error criteria.*

- *If*

$$\sup_{d\in\mathbb{N}} f^*(d) < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$p^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2p^*$.

- *If there exists a non-negative q such that*

$$\limsup_{d\to\infty} f^*(d)\, d^{-q} < \infty$$

then I_γ is polynomially tractable with ε^{-1} exponent at most 2 and d exponent at most q.

- *If*

$$t^* := \limsup_{\varepsilon<\min(1,\|h_{d,\gamma}\|_{F_{d,\gamma}}/\mathrm{CRI}_d)\,:\,\varepsilon^{-1}+d\to\infty} \frac{\ln f^*(d) + \ln \varepsilon^{-2}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ is T-tractable with exponent at most t^*.

- *If*

$$\lim_{d\to\infty} \frac{\ln \max(1, f^*(d))}{d} = 0$$

then I_γ is weakly tractable.

The case $h_{1,1} = 0$ in Theorem 12.15 can be checked as in the previous section. For $h_{1,1} \neq 0$, we can apply Theorem 12.15 for finite-order and product weights. This is the subject of our next theorem.

Theorem 12.16. *Consider $I_\gamma = \{I_{d,\gamma}\}$ defined as in Section 12.2 in the worst case setting with a non-zero $h_{1,1}$ such that $\|h_{1,1}\|^*_{F_1} < \infty$ and $\|h_{1,2}\|^*_{F_1} < \infty$.*

- *Consider finite-order weights*

$$\gamma_{d,\mathfrak{u}} = 0 \quad \textit{for all } \mathfrak{u} \textit{ with } |\mathfrak{u}| > \omega.$$

For the absolute error criterion and bounded finite-order weights, i.e.,

$$\sup_{d\in\mathbb{N}} \gamma_{d,\mathfrak{u}} < \infty,$$

we have the following:

  - *If*

$$\|h_{1,1}\|^*_{H(R_1)} < 1$$

then I_γ is strongly polynomially tractable with exponent at most 2.

  - *If*

$$\|h_{1,1}\|^*_{H(R_1)} < 1 \quad \textit{and} \quad t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

– *If*

$$\|h_{1,1}\|^*_{H(R_1)} = 1$$

then I_γ *is polynomially tractable with* d *exponent at most* 2ω *and* ε^{-1} *exponent at most* 2*; for bounded finite-diameter weights the* d *exponent is at most* 2*.*

– *If*

$$\|h_{1,1}\|^*_{H(R_1)} = 1 \quad and \quad t^* := \limsup_{\substack{\varepsilon < \|h_{d,\gamma}\|_{F_{d,\gamma}} \\ \varepsilon^{-1}+d\to\infty}} \frac{k \ln d + 2\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ *is* T*-tractable with exponent at most* t^**. Here,* $k = 2\omega$ *for finite-order weights, and* $k = 2$ *for finite-diameter weights.*

For the normalized error criterion and finite-order weights we have the following:

– *If*

$$\|h_{1,1}\|^*_{H(R_1)} = \|h_{1,1}\|_{H(R_1)}$$

(which always holds for $\dim H(R_1) = 1$*) then* I_γ *is polynomially tractable with* ε^{-1} *exponent at most* 2 *and* d *exponent at most* ω*; for finite-diameter weights the* d *exponent is at most* 1*.*

– *If*

$$\|h_{1,1}\|^*_{H(R_1)} = \|h_{1,1}\|_{H(R_1)}$$

and

$$t^* := \limsup_{\varepsilon<1,\ \varepsilon^{-1}+d\to\infty} \frac{k \ln d + 2\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ *is* T*-tractable with exponent at most* t^**. Here,* $k = \omega$ *for finite-order weights, and* $k = 1$ *for finite-diameter weights.*

- *Consider product weights*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \text{for all } \mathfrak{u} \subseteq [d].$$

For the absolute error criterion and product weights we have the following:

– *If*

$$\|h_{1,1}\|^*_{H(R_1)} + \|h_{1,2}\|^*_{H(R_2)} \cdot \sup_{d\in\mathbb{N}} \max_{j=1,2,\dots,d} \sqrt{\gamma_{d,j}} \le 1$$

or

$$\|h_{1,1}\|^*_{H(R_1)} < 1 \quad and \quad \limsup_{d\to\infty} \frac{\sum_{j=1}^d \sqrt{\gamma_{d,j}}}{\ln d} < \infty$$

or

$$\|h_{1,1}\|^*_{H(R_1)} = 1 \quad \textit{and} \quad \limsup_{d\to\infty} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}} < \infty,$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

- *If*

$$\|h_{1,1}\|^*_{H(R_1)} = 1 \quad \textit{and} \quad q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \sqrt{\gamma_{d,j}}}{\ln d} < \infty,$$

then I_γ is polynomially tractable with d exponent at most arbitrarily close to $2\sqrt{\alpha_2^*}\, q^*$ and ε^{-1} exponent at most 2.

- *If $\|h_{1,1}\|^*_{H(R_1)} \le 1$ and*

$$t^* := \limsup_{\substack{\varepsilon^2 < \prod_{j=1}^{d}(\alpha_1+\alpha_2\gamma_{d,j}) \\ \varepsilon^{-1}+d\to\infty}} \frac{\ln \varepsilon^{-2} + d \ln \alpha_1^* + 2(\alpha_2^*/\alpha_1^*)^{1/2} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}}}{\ln T(\varepsilon^{-1}, d)} < \infty$$

then I_γ is T-tractable with exponent at most t^*.

- *If $\|h_{1,1}\|^*_{H(R_1)} \le 1$ and*

$$\lim_{d\to\infty} \frac{\left[d \ln \alpha_1^* + 2(\alpha_2^*/\alpha_1^*)^{1/2} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}}\right]_+}{\ln d} = 0$$

then I_γ is weakly tractable.

For the normalized error criterion and product weights, assume that

$$\|h_{1,1}\|^*_{H(R_1)} = \|h_{1,1}\|_{H(R_1)}.$$

We have the following:

- *If*

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}} < \infty$$

then I_γ is strongly polynomially tractable with exponent at most 2. Additionally, if

$$t^* := \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

then I_γ is strongly T-tractable with exponent at most $2t^*$.

– *If*

$$q^* := \limsup_{d\to\infty} \frac{\sum_{j=1}^d \sqrt{\gamma_{d,j}}}{\ln d}$$

then I_γ is polynomially tractable with d exponent at most arbitrarily close to $2\sqrt{\alpha_2^*}\, q^*/\alpha_1$ and ε^{-1} exponent at most 2.

– *If*

$$t^* := \limsup_{\varepsilon<1:\,\varepsilon^{-1}+d\to\infty} \frac{\ln \varepsilon^{-2} + 2(\alpha_2^*/\alpha_1)^{1/2} \sum_{j=1}^d \sqrt{\gamma_{d,j}}}{\ln T(\varepsilon^{-1}, d)}$$

then I_γ is T-tractable with exponent at most t^*.

– *If*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^d \sqrt{\gamma_{d,j}}}{d} = 0$$

then I_γ is weakly tractable.

Proof. For bounded finite-order weights, when $\gamma_{d,\mathfrak{u}} \le \Gamma$, and for the absolute error criterion, we have

$$\begin{aligned} f^*(d) &= \Big(\sum_{k=0}^{d} \big[\alpha_1^*\big]^{(d-k)/2} \big[\alpha_2^*\big]^{k/2} \sum_{\substack{\mathfrak{u}\subseteq[d]\\ |\mathfrak{u}|=k}} \sqrt{\gamma_{d,\mathfrak{u}}}\Big)^2 \\ &\le \big[\alpha_1^*\big]^d \Big(\sum_{k=0}^{\omega} (\alpha_2^*/\alpha_1^*)^{k/2} \binom{d}{k} \sqrt{\Gamma}\Big)^2 \\ &= \big[\alpha_1^*\big]^d \; p_{2\omega}(d), \end{aligned}$$

where $p_{2\omega}$ is a polynomial of degree at most 2ω.

For bounded finite-diameter weights of order q, i.e., $\gamma_{d,\mathfrak{u}} \le \Gamma$ and $\gamma_{d,\mathfrak{u}} = 0$ for all $\mathfrak{u}$ such that $\operatorname{diam}(\mathfrak{u}) \ge 1$, the condition $|\mathfrak{u}| > q$ implies $\operatorname{diam}(\mathfrak{u}) \ge q$ and therefore $\gamma_{d,\mathfrak{u}} = 0$. Hence, we can estimate f^* by

$$f^*(d) \le \big[\alpha_1^*\big]^d \max\big(1, (\alpha_2^*/\alpha_1)^q\big)\; \Gamma\Big(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}\ne 0}} 1\Big)^2 = \big[\alpha_1^*\big]^d \; p_2(d),$$

where p_2 is a polynomial of degree at most 2.

Hence for $\alpha_1^* = \|h_{1,1}\|_{H(R_1)}^* < 1$ the function f^* is uniformly bounded, and for $\alpha_1^* = 1$, the values $f^*(d)$ are bounded by a multiple of $d^{2\omega}$ for finite-order weights, and by a multiple of d^2 for finite-diameter weights. This and Theorem 12.9 yield the first part of Theorem 12.10.

For the normalized error criterion and finite-order weights we assumed that $\alpha_1^* = \alpha_1$, and therefore

$$
\begin{aligned}
f^*(d) &= \frac{[\alpha_1^*]^d \, \left(\sum_{k=0}^{\omega} (\alpha_2^*/\alpha_1^*)^{k/2} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\right)^2}{[\alpha_1]^d \, \sum_{k=0}^{\omega} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&= \frac{\left(\sum_{k=0}^{\omega} (\alpha_2^*/\alpha_2)^{k/2} \binom{d}{k}^{1/2} \left[(\alpha_2/\alpha_1)^{k/2}/\binom{d}{k}^{1/2}\right] \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\right)^2}{\sum_{k=0}^{\omega} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \frac{\sum_{k=0}^{\omega} (\alpha_2^*/\alpha_2)^k \binom{d}{k} \sum_{k=0}^{\omega} \left[(\alpha_2/\alpha_1)^k/\binom{d}{k}\right] \left(\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\right)^2}{\sum_{k=0}^{\omega} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \frac{\sum_{k=0}^{\omega} (\alpha_2^*/\alpha_2)^k \binom{d}{k} \sum_{k=0}^{\omega} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\sum_{k=0}^{\omega} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \sum_{k=0}^{\omega} \left(\frac{\alpha_2^*}{\alpha_2}\right)^k \binom{d}{k} = \mathcal{O}(d^{\omega}).
\end{aligned}
$$

Hence, f^* is bounded by a polynomial of degree at most ω.

For finite-diameter weights of order q, we proceed similarly, finding that

$$
\begin{aligned}
f^*(d) &= \frac{\left(\sum_{k=0}^{q} (\alpha_2^*/\alpha_2)^{k/2} (\alpha_2/\alpha_1)^{k/2} \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\right)^2}{\sum_{k=0}^{q} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \frac{\sum_{k=0}^{q} (\alpha_2^*/\alpha_2)^k \sum_{k=0}^{q} (\alpha_2/\alpha_1)^k \left(\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \sqrt{\gamma_{d,\mathfrak{u}}}\right)^2}{\sum_{k=0}^{q} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \frac{\sum_{k=0}^{q} (\alpha_2^*/\alpha_2)^k \, |\{\mathfrak{u} \mid \gamma_{d,\mathfrak{u}} > 0\}| \sum_{k=0}^{q} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}}{\sum_{k=0}^{q} (\alpha_2/\alpha_1)^k \sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|=k} \gamma_{d,\mathfrak{u}}} \\
&\le \sum_{k=0}^{q} \left(\frac{\alpha_2^*}{\alpha_2}\right)^k |\{\mathfrak{u} \mid \gamma_{d,\mathfrak{u}} > 0\}| = \mathcal{O}(d).
\end{aligned}
$$

Hence, f^* is bounded by a polynomial of degree at most 1. Theorem 12.9 yields the second part of Theorem 12.10.

For the absolute error criterion and product weights, we have

$$
f^*(d) = \prod_{j=1}^{d} \left(\sqrt{\alpha_1^*} + \sqrt{\alpha_2^*}\sqrt{\gamma_{d,j}}\right)^2.
$$

Clearly,

$$
\sqrt{\alpha_1^*} + \sqrt{\alpha_2^*}\sqrt{\gamma_{d,j}} \le \|h_{1,1}\|_{H(R_1)}^* + \|h_{1,2}\|_{H(R_2)}^* \cdot \sup_{d\in\mathbb{N}} \max_{j=1,2,\dots,d} \sqrt{\gamma_{d,j}}.
$$

Hence if the last bound is at most 1, the function f^* is uniformly bounded and I_γ is strongly polynomially tractable.

If $\alpha_1^* \le 1$ then $\alpha_1^* \ge \alpha_1 > 0$. We rewrite f^* as

$$f^*(d) = \left[\alpha_1^*\right]^d \prod_{j=1}^{d} \left(1 + \left(\frac{\alpha_2^*}{\alpha_1^*}\right)^{1/2} \sqrt{\gamma_{d,j}}\right)^2$$
$$\le \left[\alpha_1^*\right]^d \exp\left(2\left(\frac{\alpha_2^*}{\alpha_1^*}\right)^{1/2} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}}\right).$$

Note that $q^* := \limsup_d \sum_{j=1}^d \sqrt{\gamma_{d,j}} / \ln d < \infty$ implies that for any positive δ there is a positive C_δ such that

$$\exp\left(2\left(\frac{\alpha_2^*}{\alpha_1^*}\right)^{1/2} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}}\right) \le C_\delta\, d^{\,2(\alpha_2^*/\alpha_1^*)^{1/2}(q^*+\delta)} \quad \text{for all } d \in \mathbb{N}.$$

This yields that

$$f^*(d) \le C_\delta \left[\alpha_1^*\right]^d \, d^{\,2(\alpha_2^*/\alpha_1^*)^{1/2}(q^*+\delta)} \quad \text{for all } d \in \mathbb{N}.$$

Hence, f^* is uniformly bounded for $\alpha_1^* < 1$, and polynomially bounded for $\alpha_1^* = 1$. The exponent of d in the latter case is at most $2\sqrt{\alpha_2^*}(q^*+\delta)$, where δ can be arbitrarily small.

Obviously for $\alpha_1^* = 1$ and $\limsup_d \sum_{j=1}^d \sqrt{\gamma_{d,j}} < \infty$, we have

$$\sup_{d\in\mathbb{N}} \prod_{j=1}^{d} \left(1 + \alpha_2^* \sqrt{\gamma_{d,j}}\right) < \infty.$$

So f^* is uniformly bounded and strong polynomial tractability holds. Note that for T-tractability we can restrict ε^2 to be less than $e^2(0,d) = \prod_{j=1}^d (\alpha_1^* + \alpha_2^* \gamma_{d,j})$ since otherwise $n(\varepsilon, d) = 0$.

For the normalized error criterion and product weights with the assumption that $\alpha_1^* = \alpha_1$ we have

$$f^*(d) = \prod_{j=1}^{d} \frac{\left(1 + \sqrt{\alpha^*/\alpha_1}\,\sqrt{\gamma_{d,j}}\right)^2}{1 + (\alpha_2/\alpha_1)\,\gamma_{d,j}} \le \exp\left(2\sqrt{\alpha_2^*/\alpha_1} \sum_{j=1}^{d} \sqrt{\gamma_{d,j}}\right).$$

The rest is easy. This completes the proof □

We stress that Theorem 12.16 can be applied to problems that are not necessarily related to multivariate integration. However, the results are usually weaker than for multivariate integration. In particular, for the normalized error criterion and for product weights, we need to assume that

$$\|h_{1,1}\|^*_{H(R_1)} = \|h_{1,1}\|_{H(R_1)}.$$

As already discussed, the last assumption is quite restrictive and holds if $\dim H(R_1) = 1$ or if $h_{1,1} = \beta_1 R_1(\cdot, z_1)$ for any dimension of $H(R_1)$.

We believe that much more refined analysis is needed to get sharper tractability results, especially for linear functionals that are not necessarily related to multivariate integration.

12.8 Notes and Remarks

NR 12.1:1. This chapter is based on the results originally obtained in [221]. There are also a number of new results, as will be indicated in what follows.

NR 12.3:1. Theorem 12.2 generalizes Theorem 3 in [221], since Theorem 12.2 is for general weights whereas Theorem 3 in [221] was only for product weights independent of d. The examples are new.

NR 12.4:1. Theorem 12.3 generalizes Theorem 3 in [221], since product weights may now depend on d. The case of finite a^* that leads to lower bounds on the d exponent is also new. The bound on C in the fourth case is also slightly improved. The fifth case corresponds to Theorem 1 in [85]. The example of weighted integration is basically from [221].

NR 12.5:1. Theorem 12.5 corresponds to Theorem 2 in [221], which was proved only for product weights independent of d. Corollary 12.6 about the change of weights is new. Its application for product weights leads to the change of weights that had already been used in Theorem 4 in [221]. Corollary 12.7 is related to Theorem 4 in [221] for product weights independent of d.

NR 12.6:1. This section is new. We are somewhat disappointed that the results for multivariate integration for the normalized error criterion really apply only for standard integration.

NR 12.6:2. Similar results on finite-order weights can be found in a number of papers. For the normalized error criterion and for linear functionals, said results were obtained for standard multivariate integration, see the first paper [54] on finite-order weights, as well as [275]. For the same error criterion and for linear operators, see [334]. It is worthwhile to mention that the titles of the last two papers state that finite-order weights imply (polynomial) tractability. The absolute error criterion had not yet been formally studied for finite-order weights. However, for the multivariate integration problem studied in [54] and for the approximation problem studied in [343], the initial errors are 1; therefore there is no difference between the absolute and normalized error criteria. In this case, the assumption on bounded finite-order weights was needed. Finite-order weights were also studied for some nonlinear problems, see [341], [342]. Obviously, the case of finite-diameter weights introduced by Creutzig in 2007 is only present in the newest papers.

NR 12.6:3. Product weights had usually been analyzed for the normalized error criterion. In the first papers, starting from [277], product weights independent of d were considered, the dependence on d was then allowed in [332]. The strong polynomial and polynomial tractability conditions were as in Theorem 12.10.

NR 12.7:1. This section is new. We again stress that more general results for linear functionals with finite star norms require more severe conditions on the weights. This area is not yet well studied, much more refined analysis is probably needed to obtain sharp tractability conditions. In particular, it would be of great interest to answer the questions of Open Problem 64 for more general linear functionals than the integration problem considered in this example.

Chapter 13
Average Case Setting

13.1 Introduction

In Chapters 10, 11 and 12, we studied tractability of linear functionals in the worst case setting. In this chapter we turn to tractability of linear functionals in the average case setting. The reader may be afraid that the average case setting will need as many pages as the worst case setting, or even more pages since the average case setting seems more demanding than the worst case setting. But we have a nice surprise for our readers. It turns out that the study of the average case setting for linear functionals is practically the same as the study of the worst case setting. More precisely, the study of a linear functional I_d in the average case setting defined over a separable Banach space F_d equipped with a zero mean Gaussian measure μ_d is practically equivalent to the study of the restriction of I_d in the worst case setting defined over a corresponding reproducing kernel Hilbert space H_{μ_d}. We stress that the Hilbert space H_{μ_d} is constructed from the space F_d and the measure μ_d that appear in the average case setting. This is a vast simplification, since we can now use all the tractability results for linear functionals developed in the worst case setting for tractability of linear functionals in the average case setting. Since only a few pages are needed to show how the average case setting is related to the worst case setting, this chapter is very short. There is also one open problem 65.

13.2 Basics of the Average Case Setting

Let F_d be a separable Banach space of real functions $f\colon D_d \to \mathbb{R}$ with $D_d \subseteq \mathbb{R}^d$. We assume that function values are continuous linear functionals in the norm of the space F_d. That is, the linear functional $L_x(f) = f(x)$, defined for $x \in D_d$ and for all $f \in F_d$, is continuous, $|L_x(f)| \le \|L_x\|\,\|f\|_{F_d}$ for all $x \in D_d$.

The space F_d is equipped with a Gaussian measure μ_d with *mean* zero and *covariance operator* C_{μ_d}. That is,

$$\int_{F_d} L(f)\,\mu_d(\mathrm{d}f) = 0 \quad \text{for all } L \in F_d^*,$$

and $C_{\mu_d}\colon F_d^* \to F_d$ is given by

$$L_1(C_{\mu_d} L_2) = \int_{F_d} L_1(f) L_2(f)\,\mu_d(\mathrm{d}f) \quad \text{for all } L_1, L_2 \in F_d^*.$$

The covariance operator is linear and symmetric, that is, $L_1(C_{\mu_d} L_2) = L_2(C_{\mu_d} L_1)$ for all $L_1, L_2 \in F_d^*$, and positive, that is, $L(C_\mu L) \ge 0$ for all $L \in F_d^*$.

The reader is referred to the books of Kuo [160] and Vakhania, Tarieladze and Chobanyan [316] for basic information and properties of Gaussian measures. A short introduction to Gaussian measure may be also found in Appendix B of Volume I.

For linear functionals $L_x(f) = f(x)$ and $L_t(f) = f(t)$, let

$$K_{\mu_d}(x,t) = L_x(C_{\mu_d} L_t) = \int_{F_d} f(x)\, f(t)\, \mu_d(\mathrm{d}f) \quad \text{for all } x,t \in D_d,$$

denote the *covariance kernel* of the measure μ_d. Note that K_{μ_d} has all the properties of a reproducing kernel. Indeed, $K_{\mu_d}(x,t) = K_{\mu_d}(t,x)$ for all $x,t \in D_d$, and the symmetric matrix $M = (K_{\mu_d}(x_i,x_j))_{i,j=1,2,\dots,m}$ is semi-positive definite for all choices of m and points x_j from D_d. The last statement follows from the easy argument

$$\begin{aligned} 0 &\le \int_{F_d} \Big(\sum_{j=1}^m a_j f(x_j)\Big)^2 \mu_d(\mathrm{d}f) = \sum_{i,j=1}^m a_i a_j \int_{F_d} f(x_i) f(x_j)\, \mu_d(\mathrm{d}f) \\ &= \sum_{i,j=1}^m a_i a_j K_{\mu_d}(x_i,x_j) \quad \text{for any real numbers } a_j. \end{aligned}$$

For $L \in F_d^*$, define the function $h_L \colon D_d \to \mathbb{R}$ by

$$h_L(x) = (C_{\mu_d} L)(x) = L_x(C_{\mu_d} L) = \int_{F_d} f(x)\, L(f)\, \mu(\mathrm{d}f) \quad \text{for all } x \in D_d.$$

Clearly, $h_L \in C_{\mu_d}(F_d^*)$ for all $L \in F_d^*$.

From Proposition 1.6, p. 152, of Vakhania, Tarieladze and Chobanyan [316], we know that there exists a unique Hilbert space H_{μ_d} such that

- $H_{\mu_d} \subseteq F_d$,
- there is a positive C for which $\|f\|_{F_d} \le C\, \|f\|_{H_{\mu_d}}$ for all $f \in H_{\mu_d}$,
- $C_{\mu_d}(F_d^*)$ is dense in H_{μ_d}, and
- $\langle h_{L_1}, h_{L_2}\rangle_{H_{\mu_d}} = L_1(C_{\mu_d} L_2)$ for all $L_1, L_2 \in F_d^*$.

Note that the Hilbert space H_{μ_d} is a reproducing kernel Hilbert space and its reproducing kernel is K_{μ_d}. Indeed, we have $K_{\mu_d}(\cdot,t) = C_{\mu_d} L_t = h_{L_t}$ and

$$\langle h_L, K_{\mu_d}(\cdot,t)\rangle_{H_{\mu_d}} = \langle h_L, h_{L_t}\rangle_{H_{\mu_d}} = L_t(C_{\mu_d} L) = h_L(t) \quad \text{for all } L \in F_d^*,\ t \in D_d.$$

We illustrate the concepts of this section by the following example.

13.2.1 Example: Wiener Measure

Let $F_d = C([0,1]^d)$ be the space of real continuous functions defined on $[0,1]^d$ with the norm $\|f\| = \max_{x\in[0,1]^d} |f(x)|$. Let $\mu_d = w_d$ be the Wiener sheet measure that corresponds to the covariance kernel $K_{w_d}(x,t) = \prod_{j=1}^d \min(x_j, t_j)$. This means that

$$\int_{C([0,1]^d)} f(x) f(t)\, w_d(\mathrm{d}f) = \prod_{j=1}^{d} \min(x_j, t_j) \quad \text{for all } x, t \in [0,1]^d.$$

In particular, this means that $f(x) = 0$ with probability 1 if at least one component of x is zero, and the average value of $f^2(x)$ is $\prod_{j=1}^d x_j$.

The Hilbert space H_{w_d} is now the Sobolev space which is the d fold tensor product of univariate functions that vanish at zero, are absolutely continuous, and whose first derivatives are in $L_2([0,1])$. The reproducing kernel of H_{w_d} is K_{w_d} and the inner product of $f, g \in H_{w_d}$ is given by

$$\langle f, g\rangle_{H_{w_d}} = \int_{[0,1]^d} \frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} f(x) \frac{\partial^d}{\partial x_1 \partial x_2 \cdots \partial x_d} g(x)\, \mathrm{d}x. \qquad \square$$

As we shall see in the next section, the average case setting for a continuous linear functional I_d defined over the separable Banach space F_d will be closely related to the worst case setting for the restriction of I_d to the Hilbert space H_{μ_d}.

13.3 Linear Functionals

Our problem is to approximate a continuous linear functional $I_d : F_d \to \mathbb{R}$. The initial error in the average case setting is defined as

$$e^{\mathrm{avg}}(0; I_d) = \left(\int_{F_d} I_d^2(f)\, \mu_d(\mathrm{d}f) \right)^{1/2} = \left[I_d(C_{\mu_d} I_d) \right]^{1/2}.$$

Let $h_d = h_{I_d} = C_{\mu_d} I_d$. That is,

$$h_d(x) = L_x(C_{\mu_d} I_d) = \int_{F_d} f(x) I_d(f)\, \mu(\mathrm{d}f) \quad \text{for all } x \in D_d.$$

We have

$$\|h_d\|_{H_{\mu_d}} = \left[I_d(C_{\mu_d} I_d) \right]^{1/2} = e^{\mathrm{avg}}(0; I_d).$$

Let $\nu_d = \mu_d I_d^{-1}$. Then ν_d is a univariate Gaussian measure defined on Borel sets of $\mathbb{R}$ with zero mean and variance

$$\sigma_d = I_d(C_{\mu_d} I_d) = \left[e^{\mathrm{avg}}(0; I_d) \right]^2.$$

This means that for any Borel set $A \subseteq \mathbb{R}$ we have

$$\nu_d(A) = \mu_d(\{f \in F_d \mid I_d(f) \in A\}) = \frac{1}{\sqrt{2\pi\,\sigma_d}} \int_A \exp\big(-t^2/(2\sigma_d)\big)\,\mathrm{d}t.$$

The measure ν_d tells us about the distribution of elements $I_d(f)$.

We first consider linear algorithms for approximating I_d. That is, let

$$A_{n,d}(f) = \sum_{j=1}^{n} a_j f(x_j) \quad \text{for all } f \in F_d,$$

for some real a_j and $x_j \in D_d$. The average case error of $A_{n,d}$ is defined as

$$e^{\mathrm{avg}}(A_{n,d}; I_d) = \left(\int_{F_d} \big[I_d(f) - A_{n,d}(f)\big]^2 \,\mu(\mathrm{d}f)\right)^{1/2}.$$

It is easy to compute the square of $e^{\mathrm{avg}}(A_{n,d}; I_d)$. Indeed, we have

$$\begin{aligned}
\big[e^{\mathrm{avg}}(A_{n,d}; I_d)\big]^2 &= \int_{F_d} I_d^2(f)\,\mu(\mathrm{d}f) - 2\sum_{j=1}^{n} a_j \int_{F_d} I_d(f) f(x_j)\,\mu(\mathrm{d}f) \\
&\quad + \sum_{i,j=1}^{n} a_i a_j \int_{F_d} f(x_i) f(x_j)\,\mu(\mathrm{d}f) \\
&= I_d(C_{\mu_d} I_d) - 2\sum_{i,j=1}^{n} L_{x_j}(C_{\mu_d} I_d) + \sum_{i,j=1}^{n} a_i a_j L_{x_i}(C_{\mu_d} L_{x_j}).
\end{aligned}$$

Hence, for an arbitrary linear algorithm $A_{n,d}$ we have

$$e^{\mathrm{avg}}(A_{n,d}; I_d) = \left[\|h_d\|^2_{H_{\mu_d}} - 2\sum_{j=1}^{n} a_j h_d(x_j) + \sum_{i,j=1}^{n} a_i a_j K_{\mu_d}(x_i, x_j)\right]^{1/2}.$$

13.4 Relations to the Worst Case Setting

The last formula that we obtained in the previous section should be quite familiar to the reader. In particular, this formula can be found in Section 10.2 of Chapter 10, where we studied the *worst case error* of approximating[1]

$$I_d^{\mathrm{res}}(f) = \langle f, h_d\rangle_{H_{\mu_d}} \quad \text{for all } f \in H_{\mu_d}.$$

More precisely, if we define

$$e^{\mathrm{wor}}(A_{n,d}; I_d^{\mathrm{res}}) = \sup_{f \in H_{\mu_d},\, \|f\|_{H_{\mu_d}} \le 1} \big|I_d^{\mathrm{res}}(f) - A_{n,d}(f)\big|$$

then we obtain the following[2] corollary.

[1] In a moment, we explain our notation I_d^{res}.

Corollary 13.1. *For an arbitrary linear algorithm* $A_{n,d}$ *we have*

$$e^{\text{avg}}(A_{n,d}; I_d) = e^{\text{wor}}(A_{n,d}; I_d^{\text{res}}).$$

How are the continuous linear functionals I_d and I_d^{res} related? First of all, the domain F_d of I_d is usually a much larger space than its subspace H_{μ_d} that is the domain of I_d^{res}. However, if we restrict ourselves to H_{μ_d} then we have

$$I_d^{\text{res}}(f) = I_d(f) \quad \text{for all } f \in H_{\mu_d}. \tag{13.1}$$

This means that I_d^{res} is the restriction of I_d to the reproducing kernel Hilbert space H_{μ_d} with kernel K_{μ_d}, which explains why we denote this restriction by I_d^{res}.

To show (13.1) we proceed as follows. For every $f \in H_{\mu_d}$ and every $\varepsilon > 0$, there exists $f_\varepsilon = \sum_{j=1}^m a_j K_{\mu_d}(\cdot, t_j) \in H_{\mu_d}$ such that $\|f - f_\varepsilon\|_{H_{\mu_d}} \le \varepsilon$. Then

$$I_d^{\text{res}}(f_\varepsilon) = \langle f_\varepsilon, h_d \rangle_{H_{\mu_d}} = \sum_{j=1}^m a_j h_d(t_j).$$

On the other hand,

$$I_d(f_\varepsilon) = \sum_{j=1}^m a_j I_d\left(K_{\mu_d}(\cdot, t_j)\right).$$

Since $K_{\mu_d}(\cdot, t) = C_{\mu_d} L_t$ we have

$$I_d\left(K_{\mu_d}(\cdot, t_j)\right) = I_d(C_{\mu_d} L_{t_j}) = L_{t_j}(C_{\mu_d} I_d) = L_{t_j}(h_d) = h_d(t_j).$$

This implies that $I_d^{\text{res}}(f_\varepsilon) = I_d(f_\varepsilon)$. Finally,

$$\begin{aligned}
\left|I_d(f) - I_d^{\text{res}}(f)\right| &= \left|I_d(f - f_\varepsilon) - I_d^{\text{res}}(f - f_\varepsilon)\right| \\
&\le \|I_d\|_{F_d \to F_d} \|f - f_\varepsilon\|_{F_d} + \|I_d^{\text{res}}\|_{H_{\mu_d} \to H_{\mu_d}} \|f - f_\varepsilon\|_{H_{\mu_d}} \\
&\le \left[C \, \|I_d\|_{F_d \to F_d} + \|h_d\|_{H_{\mu_d}}\right] \|f - f_\varepsilon\|_{H_{\mu_d}} = \mathcal{O}(\varepsilon).
\end{aligned}$$

For ε tending to zero, we conclude that $I_d(f) = I_d^{\text{res}}(f)$ for all $f \in H_{\mu_d}$, as claimed.

Corollary 13.1 and (13.1) mean that as long as we consider linear algorithms then their average case errors for approximating I_d in the separable Banach space F_d equipped with a zero mean Gaussian measure μ_d are the same as their worst case errors for approximating the restriction of I_d in the reproducing Hilbert space H_{μ_d}. Hence for linear algorithms, the average case setting for (F_d, μ_d, I_d) is equivalent to the worst case setting for $(H_{\mu_d}, I_d^{\text{res}})$, where the Hilbert space H_{μ_d} depends crucially on the space F_d and the Gaussian measure μ_d. This means that all results obtained for linear algorithms in the worst case setting for reproducing kernel Hilbert spaces in the previous chapters are also applicable for linear algorithms in the average case

[2]Corollary 13.1 is valid not only for Gaussian measures μ. It holds if we assume that the first moment (mean) is zero and the second moment of μ is finite. The assumption that μ is Gaussian is needed later. For simplicity, we restrict ourselves in this chapter to Gaussian measures.

setting. In particular, we know from Section 10.4 of Chapter 10 that all continuous linear functionals can be approximated with an arbitrarily small worst case error by linear algorithms. Thus, the same is also true in the average case setting.

Can we thus claim the same tractability results for $I^{\text{avg}} = \{I_d\}$ in the average case setting and for $I^{\text{wor}} = \{I_d^{\text{res}}\}$ in the worst case setting? Not yet, since tractability depends on the behavior of optimal algorithms, and so far we only considered linear algorithms. However, we know that linear algorithms are optimal in the worst case setting for linear functionals as discussed in Chapter 4 of Volume I. In the average case setting, the situation is a little more complicated. Although linear algorithms are not necessarily optimal, we shall see that tractability depends only on the behavior of optimal linear algorithms.

To show this, we proceed as follows. First of all, note that the problem I_d in the average case setting and the problem I_d^{res} in the worst case setting have the same initial errors. Indeed,

$$e^{\text{wor}}(0; I_d^{\text{res}}) = \|h_d\|_{H_{\mu_d}} = \big[I_d(C_{\mu_d} I_d)\big]^{1/2} = e^{\text{avg}}(0; I_d),$$

as claimed. This will enable us to consider the absolute and normalized error criteria for both problems.

We need to discuss general algorithms in the average case setting, see Chapter 4 of Volume I. The general form of a (nonlinear) algorithm $A_{n,d}$ is,

$$A_{n,d}(f) = \varphi_{n(f)}(f(x_1), f(x_2), \dots, f(x_{n(f)})), \tag{13.2}$$

where x_j can be chosen adaptively, and the average value of $n(f)$ is at most n, i.e.,

$$\int_{F_d} n(f)\, \mu(\mathrm{d}f) \le n.$$

Here, $\varphi_n \colon \bigcup_{k=1}^{\infty} \mathbb{R}^k \to \mathbb{R}$. For simplicity, we may assume that φ_n is measurable. The average case error of $A_{n,d}$ is given by

$$e^{\text{avg}}(A_{n,d}; I_d) = \bigg[\int_{F_d} (I_d(f) - A_{n,d}(f))^2\, \mu(\mathrm{d}f)\bigg]^{1/2}.$$

As always, let

$$n^{\text{avg}}(\varepsilon, I_d) = \min\big\{n \mid \text{there exists } A_{n,d} \text{ with } e^{\text{avg}}(A_{n,d}; I_d) \le \varepsilon\, \mathrm{CRI}_d\big\}$$

denote the minimal number of function values needed to get an average case error at most $\varepsilon\, \mathrm{CRI}_d$. For the absolute error criterion we take $\mathrm{CRI}_d = 1$ and for the normalized error criterion we take $\mathrm{CRI}_d = e^{\text{avg}}(0; I_d)$.

Let

$$n^{\text{avg-lin}}(\varepsilon, I_d) = \min\big\{n \mid \text{there exists a linear } A_{n,d} \text{ with } e^{\text{avg}}(A_{n,d}; I_d) \le \varepsilon\, \mathrm{CRI}_d\big\}$$

denote the minimal number of function values to get an average case error at most $\varepsilon\,\mathrm{CRI}_d$ when we restrict ourselves to *linear* algorithms. Clearly,

$$n^{\mathrm{avg}}(\varepsilon, I_d) \le n^{\text{avg-lin}}(\varepsilon, I_d) \quad \text{for all } \varepsilon \in (0,1).$$

Surprisingly enough, $n^{\text{avg-lin}}(\varepsilon, I_d)$ and $n^{\mathrm{avg}}(\varepsilon, I_d)$ are closely related for Gaussian measures. Based on the results from Wasilkowski [325], the estimate

$$\sup_{x>1} \min\left(n^{\text{avg-lin}}(x\varepsilon, I_d), \frac{x^2-1}{x^2}\, n^{\text{avg-lin}}(\varepsilon, I_d)\right) \le n^{\mathrm{avg}}(\varepsilon, I_d) \quad \text{for all } \varepsilon \in (0,1)$$

is presented as Theorem 5.7.2 on page 249 of [305].

Since $n^{\text{avg-lin}}(x\varepsilon, I_d) \le n^{\text{avg-lin}}(\varepsilon, I_d)$ for all $x \ge 1$, the last estimate implies that

$$\frac{x^2-1}{x^2}\, n^{\text{avg-lin}}(x\varepsilon, I_d) \le n^{\mathrm{avg}}(\varepsilon, I_d) \quad \text{for all } x>1,\ \varepsilon \in (0,1).$$

This bound is enough for our purpose. Indeed, suppose that we have (strong) polynomial tractability for the absolute or normalized error criterion. Then there are non-negative numbers C, p and q such that

$$n^{\mathrm{avg}}(\varepsilon, I_d) \le C\, \varepsilon^{-p}\, d^{q} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

This implies that

$$n^{\text{avg-lin}}(\varepsilon, I_d) \le C\, \frac{x^{p+2}}{x^2-1}\, \varepsilon^{-p}\, d^{q} \quad \text{for all } x>1,\ \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

We can take x that minimizes $x^{p+2}/(x^2-1)$, i.e., $x = [(p+2)/p]^{1/2}$ and obtain

$$n^{\text{avg-lin}}(\varepsilon, I_d) \le C\, \frac{p}{2} \left(\frac{p+2}{p}\right)^{(p+2)/2} \varepsilon^{-p}\, d^{q} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Hence, modulo a different factor, we have the same (strong) polynomial tractability bounds with the same exponents p and q when we use linear algorithms. Similarly, weak tractability means that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{avg}}(\varepsilon, I_d)}{\varepsilon^{-1}+d} = 0,$$

and this easily implies that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{avg-lin}}(\varepsilon, I_d)}{\varepsilon^{-1}+d} = 0.$$

For T-tractability we need to assume that $T((x\varepsilon)^{-1}, d)$ and $T(\varepsilon^{-1}, d)$ behave similarly for some $x>1$. For instance, if we assume that there is a number $M \ge 1$ such that $T(x\,\varepsilon^{-1}, d) \le M\, T(\varepsilon^{-1}, d)$ for all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, then

$$n^{\mathrm{avg}}(\varepsilon, I_d) \le C\, \left[T(\varepsilon^{-1}, d)\right]^{t} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$

which implies that

$$n^{\text{avg-lin}}(\varepsilon, I_d) \le C\, \frac{M\, x^2}{x^2-1}\, \big[T(\varepsilon^{-1}, d)\big]^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Similarly, the same is true for strong T-tractability.

In summary, we see that as long as we are interested in tractability we can restrict ourselves to linear algorithms and study $n^{\text{avg-lin}}(\varepsilon, I_d)$. Due to Corollary 13.1, we know that $n^{\text{avg-lin}}(\varepsilon, I_d) = n^{\text{wor}}(\varepsilon, I_d^{\text{res}})$, that is, the minimal number of function values for approximating I_d in the average case setting is the same as the minimal number of function values for approximating its restriction I_d^{res} in the worst case setting. Hence,

$$\frac{x^2-1}{x^2}\, n^{\text{wor}}(x\varepsilon, I_d^{\text{res}}) \le n^{\text{avg}}(\varepsilon, I_d) \le n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) \quad \text{for all } x > 1.$$

These bounds can be written more concisely as

$$n^{\text{avg}}(\varepsilon, I_d) = (1-a_x)\, n^{\text{wor}}(b_x\varepsilon, I_d^{\text{res}})$$

with $a_x \in [-1/x^2, 0]$ and $b_x \in [1, x]$ for all $x > 1$.

Hence, tractabilities in the average and worst case settings are equivalent. We summarize this analysis in the following theorem.

Theorem 13.2. *Consider $I^{\text{avg}} = \{I_d\}$ in the average case setting and $I^{\text{wor}} = \{I_d^{\text{res}}\}$ in the worst case setting, defined as in this chapter for the absolute or normalized error criterion. Then*

$$n^{\text{avg-lin}}(\varepsilon, I_d) = n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) \quad \textit{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$

and

$$n^{\text{avg}}(\varepsilon, I_d) = (1-a_x)\, n^{\text{wor}}(b_x\varepsilon, I_d^{\text{res}})$$

with $a_x \in [-1/x^2, 0]$, $b_x \in [1, x]$ for all $x > 1$. Let

$$y \in \{\textit{strongly polynomially, polynomially, weakly}\}.$$

Then

- *I^{avg} is y tractable in the average case setting iff I^{wor} is y tractable in the worst case setting. Furthermore, the exponents of strong polynomial and polynomial tractability are the same for I^{avg} and I^{wor}.*
- *Let T be a tractability function. Suppose there are numbers $x > 1$ and $M \ge 1$ such that*

  $$T(x\,\varepsilon^{-1}, d) \le M\, T(\varepsilon^{-1}, d) \quad \textit{for all } \varepsilon \in (0,1],\ d \in \mathbb{N}.$$

  Then I^{avg} is (strongly) T-tractable in the average case setting iff I^{wor} is (strongly) T-tractable in the worst case setting. Again, this holds with the same exponents for I^{avg} and I^{wor}.

This theorem tells us that we can use all the tractability results of the worst case setting for approximating I^{wor} presented in Chapters 10, 11, and 12 to conclude tractability of I^{avg} in the average case setting.

For some tractability functions T, the assumption

$$T(x\varepsilon^{-1}, d) \le M\, T(\varepsilon^{-1}, d) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}, \tag{13.3}$$

in Theorem 13.2, is *not* satisfied. This is the case, for example, when

$$T(x, y) = \exp\left(x^p + y^q\right) \quad \text{with } p, q \in (0,1).$$

However, even if T is a general tractability function that does not satisfy (13.3), it is easy to see that (strong) T-tractability of I^{wor} in the worst case setting still implies (strong) T-tractability of I^{avg} in the average case setting, but we do not know if the converse statement is true. This is the subject of our next open problem.

Open Problem 65.

- Verify whether (strong) T-tractability of I^{avg} in the average case setting implies (strong) T-tractability of I^{wor} in the worst case setting for a tractability function T for which there do *not* exist $x > 1$ and $M \ge 1$ such that

$$T(x\varepsilon^{-1}, d) \le M\, T(\varepsilon^{-1}, d) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

13.5 Notes and Remarks

NR 13.1:1. The relations between the average case and worst case settings for approximating linear functionals by linear algorithms are not new. They can be found in a number of papers and books especially for multivariate integration, see e.g., [305] and Ritter [251].

NR 13.4:1. We used Theorem 5.7.2 of [305] for linear functionals. In fact, this theorem is more general and also holds for linear operators. It was already used in [347] to relate the powers of non-adaptive and adaptive information in the average case setting for linear operators and polynomial tractability.

Chapter 14
Probabilistic Setting

14.1 Introduction

In this chapter we analyze the probabilistic setting for approximating linear functionals. The error of an algorithm is now defined as in the worst case setting, but disregarding a set of measure $\delta \in (0, 1)$. Hence, in the probabilistic setting we have one extra parameter δ, and so we will now be studying tractability also with respect to this parameter. As we shall see, the role of the parameter δ greatly depends on the error criterion. For the absolute error criterion, the information complexity of approximating a linear functional only depends weakly on δ, through $\sqrt{2 \ln \delta^{-1}}\,(1 + o(1))$. This will enable us to obtain tractability even if we allow polynomial dependence on $\ln \delta^{-1}$. For the normalized error criterion, we have even a more surprising situation. Namely, the information complexity does *not* depend on δ at all, and the parameter δ does not play any role in this case. This property was already indicated in Example 3.2.5 of Chapter 3 in Volume I. The reason for this surprising behavior is that the initial error depends weakly on δ through $\sqrt{2 \ln \delta^{-1}}\,(1 + o(1))$ and goes slowly to infinity as δ goes to zero. This means that we have a trade-off for the normalized error criterion. The probabilistic error of an algorithm and the initial error increase with δ, and a priori it is not clear which of these two behaviors is more important. It turns out that for the information complexity they cancel and therefore there is no dependence on δ.

In this chapter we also consider the *relative* error criterion. As we have already mentioned in Volume I, the relative error criterion for linear functionals in the worst case and average case setting leads to negative results. More precisely, the information complexity is infinite for linear functionals that cannot be solved exactly by using a finite number of function values. This means that such linear functionals can *not* be solved for the relative error criterion in the worst case and average case settings. We only have positive results in the probabilistic setting. However, the parameter δ now plays a much more important role. Indeed, the information complexity now depends as much on δ as on ε, and is usually a polynomial in δ^{-1}. Therefore we obtain tractability only if we allow polynomial dependence on δ^{-1}.

The analysis of the probabilistic setting for linear functionals is quite straightforward because of its close relation to the average case setting, which in turn is related to the worst case setting, see Chapter 13. More precisely, the probabilistic setting for a linear functional I_d with the parameters (ε, δ) is equivalent to the average case setting for the same linear functional I_d with only one parameter ε_δ, where

$$\varepsilon_\delta = \begin{cases} \varepsilon & \text{for the normalized error criterion,} \\ \frac{\varepsilon}{\sqrt{2 \ln \delta^{-1}}}(1 + o(1)) & \text{for the absolute error criterion,} \\ \varepsilon\, \frac{\pi}{2}\, \delta\, (1 + o(1)) & \text{for the relative error criterion.} \end{cases}$$

Here, the $1 + o(1)$ factor is with respect to ε and δ tending to zero. This also explains the different roles of the parameter δ for the three error criteria.

Knowing the relations between the probabilistic setting and the average case setting for the linear functional I_d, we can then apply the relations between the average case setting for I_d and the worst case setting for its restriction I_d^{res} as shown in Chapter 13. In this way we relate the information complexity for I_d in the probabilistic setting to the information complexity of I_d^{res} in the worst case setting. This allows us to apply tractability results obtained in the worst case setting to the probabilistic setting. The only difference is that we need to replace I_d and (ε, δ) in the probabilistic setting by I_d^{res} and ε_δ in the worst case setting. This relation explains why also this chapter is relatively short. There is also one open problem 66.

14.2 Tractability in the Probabilistic Setting

As in Chapter 13, we consider a separable Banach space F_d of real valued functions $f : D_d \to \mathbb{R}$ with $D_d \subseteq \mathbb{R}^d$. We assume that linear functionals $L_x(f) = f(x)$ for $f \in F_d$ are continuous for all $x \in D_d$. The space F_d is equipped with a zero mean Gaussian measure μ_d whose covariance operator is C_{μ_d} and whose covariance kernel is K_{μ_d}.

Let $I_d : F_d \to \mathbb{R}$ be a continuous linear functional. Let $A_{n,d}$ be an algorithm of the form (13.2) for approximating I_d. The cost of $A_{n,d}$ is defined as in the worst case setting, that is, its cost is

$$n = \sup_{f \in F_d} n(f).$$

This means that varying the cardinality of information does not help in the probabilistic setting, and without loss of generality we can consider only algorithms for which $n(f) \equiv n$ for all $f \in F_d$.

Let $\delta \in (0, 1)$. The error of $A_{n,d}$ in the probabilistic setting is defined as

$$e^{\text{prob}}(A_{n,d}, \delta, I_d) = \inf_{B \subseteq F_d,\, \mu_d(B) \le \delta} \ \sup_{f \in F_d \setminus B} |I_d(f) - A_{n,d}(f)|.$$

Hence, we take the worst case error modulo a set B of measure at most δ. Clearly, the error is a non-increasing function of δ.

As in all the previous settings, the information complexity in the probabilistic setting is defined as the minimal number of function values that is needed to guarantee that the error is at most $\varepsilon\,\text{CRI}_d$, i.e.,

$$n^{\text{prob}}(\varepsilon, \delta, I_d) = \min\left\{ n \mid \text{ there exists } A_{n,d} \text{ with } e^{\text{prob}}(A_{n,d}, \delta, I_d) \le \varepsilon\,\text{CRI}_d \right\},$$

where, as always, $\text{CRI}_d = 1$ for the absolute error criterion and

$$\text{CRI}_d = e^{\text{prob}}(0, \delta, I_d)$$

for the normalized error criterion. We stress that for the normalized error criterion, the initial error depends on δ, and as we shall see in a moment, it usually goes slowly to infinity as δ goes to zero. The relative error criterion will be studied later in Section 14.4.

We now address tractability in the probabilistic setting. Since we now have three parameters ε, δ and d, we need to study how information complexity depends on all of them. It is easy to modify the concept of polynomial tractability.

We say that $I = \{I_d\}$ is *polynomially tractable* in the probabilistic setting for the absolute/normalized error criterion iff there exist non-negative numbers C, p, q and s such that

$$n^{\text{prob}}(\varepsilon, \delta, I_d) \le C\, \varepsilon^{-p}\, d^q\, \delta^{-s} \quad \text{for all } \varepsilon, \delta \in (0,1) \text{ and } d \in \mathbb{N}.$$

If $q = 0$ then we say that I is *strongly* polynomially tractable with respect to d, and if $s = 0$ then we say that I is *strongly* polynomially tractable with respect to δ.

Analogously, weak tractability is defined when the information complexity is *not* exponential in ε^{-1}, δ^{-1} and d. We measure the lack of exponential dependence as before, and this leads us to the following definition.

We say that $I = \{I_d\}$ is *weakly tractable* in the probabilistic setting for the absolute/normalized error criterion iff

$$\lim_{\varepsilon^{-1}+\delta^{-1}+d\to\infty} \frac{\ln n^{\text{prob}}(\varepsilon, \delta, I_d)}{\varepsilon^{-1} + \delta^{-1} + d} = 0.$$

As we shall see soon, sometimes the dependence on the parameter δ^{-1} is weaker than polynomial. Indeed, sometimes the information complexity depends logarithmically on δ^{-1}. This explains the following modification of the definitions of polynomial and weak tractability.

We say that $I = \{I_d\}$ is *poly-log tractable* in the probabilistic setting for the absolute/normalized error criterion if there exist non-negative numbers C, p, q and s such that

$$n^{\text{prob}}(\varepsilon, \delta, I_d) \le C\, \varepsilon^{-p}\, d^q\, \left[1 + \ln \delta^{-1}\right]^s \quad \text{for all } \varepsilon, \delta \in (0,1) \text{ and } d \in \mathbb{N}.$$

If $q = 0$ then we say that I is *strongly* poly-log tractable with respect to d, and if $s = 0$ then we say that I is *strongly* poly-log tractable with respect to δ.

We say that $I = \{I_d\}$ is *weakly-log tractable* in the probabilistic setting for the absolute/normalized error criterion iff

$$\lim_{\varepsilon^{-1}+\ln\delta^{-1}+d\to\infty} \frac{\ln n^{\text{prob}}(\varepsilon, \delta, I_d)}{\varepsilon^{-1} + \ln\delta^{-1} + d} = 0.$$

Clearly, (strong) poly-log tractability implies (strong) polynomial tractability, and weak-log tractability implies weak tractability, but the converse is not true in general.

Similarly, we can generalize T-tractability. For simplicity, we restrict ourselves to the unrestricted case, in which $\Omega = [1,\infty) \times [1,\infty) \times \mathbb{N}$. We now assume that

$$T\colon [1,\infty) \times [1,\infty) \times [1,\infty) \to [1,\infty)$$

depends on three variables, and is non-decreasing with respect to all of them. As before, we assume that T grows slower than exponentially, that is,

$$\lim_{x,y,z\geq 1,\; x+y+z\to\infty} \frac{\ln T(x,y,z)}{x+y+z} = 0.$$

Then we say that $I = \{I_d\}$ is *T-tractable* if there exist non-negative numbers C and t such that

$$n^{\text{prob}}(\varepsilon,\delta,I_d) \leq C\, T(\varepsilon^{-1},d,\delta^{-1})^t \quad \text{for all } \varepsilon,\delta\in(0,1) \text{ and } d\in\mathbb{N}.$$

We say that $I = \{I_d\}$ is *strongly T-tractable* if there exist non-negative numbers C and t such that

$$n^{\text{prob}}(\varepsilon,\delta,I_d) \leq C\, T(\varepsilon^{-1},1,\delta^{-1})^t \quad \text{for all } \varepsilon,\delta\in(0,1) \text{ and } d\in\mathbb{N}.$$

The infimum of the t satisfying those bounds is called the *exponent* of T-tractability, and the *exponent* of strong T-tractability, respectively.

Finally, we say that $I = \{I_d\}$ is *T-log-tractable* if there exist non-negative numbers C and t such that

$$n^{\text{prob}}(\varepsilon,\delta,I_d) \leq C\, T(\varepsilon^{-1},d,1+\ln\delta^{-1})^t \quad \text{for all } \varepsilon,\delta\in(0,1) \text{ and } d\in\mathbb{N}.$$

Similarly, we say that $I = \{I_d\}$ is *strongly T-log-tractable* if there exist non-negative numbers C and t such that

$$n^{\text{prob}}(\varepsilon,\delta,I_d) \leq C\, T(\varepsilon^{-1},1,1+\ln\delta^{-1})^t \quad \text{for all } \varepsilon,\delta\in(0,1) \text{ and } d\in\mathbb{N}.$$

The infimum of the exponents t is called the *exponent* of T-log-tractability and strong T-log-tractability, respectively.

Clearly, (strong) T-log-tractability implies (strong) T-tractability, but the converse is not true in general.

We comment on T-tractability in different settings. In the worst case and average case settings, the tractability function T depends on two variables $x = \varepsilon^{-1}$ and $y = d$, whereas in the probabilistic setting it depends on an additional variable $z = \delta^{-1}$. To distinguish the tractability functions in different settings, we sometimes indicate the setting by writing $T = T^{\text{wor}}$, $T = T^{\text{avg}}$ and $T = T^{\text{prob}}$. As we shall see later, tractability conditions are related or even equivalent in the worst case and probabilistic settings.

14.3 Relations to the Worst Case Setting

In this section we show how the probabilistic setting is related to the average and worst case settings. We are especially interested in relations to the worst case setting, since we can then apply tractability results in the worst case setting developed in Chapters 10, 11 and 12 to the probabilistic setting.

We begin with the initial error $e^{\text{prob}}(0, \delta, I_d)$, see also Chapter 8 of [305]. Let

$$\psi(z) = \sqrt{\frac{2}{\pi}} \int_0^z \exp(-\tfrac{1}{2}t^2)\, \mathrm{d}t \quad \text{for all } z \in [0, \infty)$$

be the probability integral. Clearly, $\psi(z) \in [0, 1)$ and $\lim_{z\to\infty} \psi(z) = 1$. We have

$$\psi(z) = 1 - (2/\pi)^{1/2} z^{-1} \exp(-z^2/2)(1 + o(1)) \quad \text{as } z \to \infty,$$

and therefore

$$\psi^{-1}(1-\delta) = \sqrt{2 \ln \delta^{-1}}\, (1 + o(1)) \quad \text{as } \delta \to 0.$$

Obviously, both ψ and ψ^{-1} are increasing functions.

Let $\nu_d = \mu_d I_d^{-1}$ be defined as in Chapter 13. That is, ν_d is a univariate Gaussian measure defined on Borel sets of $\mathbb{R}$, with mean zero and variance

$$\sigma_d = I_d(C_{\mu_d} I_d) = \left[e^{\text{avg}}(0, I_d)\right]^2.$$

We change variables by $t = I_d(f)$ and obtain

$$e^{\text{prob}}(0, \delta, I_d) = \inf_{B \subseteq F_d,\, \mu_d(B) \le \delta} \ \sup_{f \in F_d \setminus B} |I_d(f)| = \inf_{A \subseteq \mathbb{R},\, \nu_d(A) \le \delta} \ \sup_{t \in \mathbb{R}\setminus A} |t|.$$

Clearly, we should take A such that $A = (-\infty, a_\delta) \cup (a_\delta, \infty)$, so that $\mathbb{R}\setminus A = [-a_\delta, a_\delta]$, where a_δ is chosen such that

$$\begin{aligned}
\delta = \nu_d(A) &= \sqrt{\frac{2}{\pi\, \sigma_d}} \int_{a_d}^{\infty} \exp\left(-t^2/(2\sigma_d)\right)\, \mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}} \int_{a_\delta/\sqrt{\sigma_d}}^{\infty} \exp\left(-\tfrac{1}{2}t^2\right)\, \mathrm{d}t = 1 - \psi\left(\frac{a_\delta}{\sqrt{\sigma_d}}\right).
\end{aligned}$$

Hence, $a_\delta = \psi^{-1}(1-\delta)\, \sqrt{\sigma_d}$, and

$$e^{\text{prob}}(0, \delta, I_d) = a_\delta = \psi^{-1}(1-\delta)\, e^{\text{avg}}(0, I_d).$$

This shows that, modulo the factor $\psi^{-1}(1-\delta)$, the initial errors in the probabilistic and average case settings are the same.

It turns out that a similar relation holds for optimal linear algorithms, see Corollary 5.3.2 of Chapter 8 in [305]. More precisely, let $A_{n,d} = \sum_{j=1}^n a_j f(t_j)$ be a linear algorithm. Define the minimal errors

$$e^{\text{prob-lin}}(n, \delta, I_d) = \inf_{\text{linear } A_{n,d}} e^{\text{prob}}(A_{n,d}, \delta, I_d),$$

and

$$e^{\text{avg-lin}}(n, I_d) = \inf_{\text{linear } A_{n,d}} e^{\text{avg}}(A_{n,d}, I_d)$$

for linear algorithms in the probabilistic and average case settings. Then

$$e^{\text{prob-lin}}(n,\delta,I_d) = \psi^{-1}(1-\delta)\, e^{\text{avg-lin}}(n,I_d) \quad \text{for all } n,d \in \mathbb{N}. \tag{14.1}$$

Due to Corollary 13.1 we can rewrite (14.1) as

$$e^{\text{prob-lin}}(n,\delta,I_d) = \psi^{-1}(1-\delta)\, e^{\text{wor}}(n,I_d^{\text{res}}) \quad \text{for all } n,d \in \mathbb{N}.$$

For small δ we have

$$e^{\text{prob-lin}}(n,\delta,I_d) = \sqrt{2\ln\delta^{-1}}\,(1+o(1))\, e^{\text{avg-lin}}(n,I_d),$$

which goes slowly to infinity as δ goes to zero if $e^{\text{avg}}(n,I_d)$ is positive.

This again means that, modulo the factor $\psi^{-1}(1-\delta)$, the minimal errors for linear algorithms are the same in the probabilistic and average case settings. It is also known, see Chapter 6 and 8 of [305], that the same sample points t_j and the same coefficients a_j minimize the probabilistic and average case errors. Hence, the same linear algorithms enjoy optimality properties in both settings.

Similarly to the average case setting, define $n^{\text{prob}}(\varepsilon,\delta,I_d)$ as the minimal number of function values needed to guarantee that the error in the probabilistic setting is at most $\varepsilon\,\mathrm{CRI}_d$ when we use general algorithms, i.e.,

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = \min\left\{ n \mid \text{there exists } A_{n,d} \text{ with } e^{\text{prob}}(A_{n,d},I_d) \le \varepsilon\,\mathrm{CRI}_d \right\}.$$

As proved in [138], see also Corollary 5.3.1 of Chapter 8 in [305], adaption does not help in the probabilistic setting. Since it is also known that linear algorithms are optimal in the probabilistic setting, we have

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = \min\left\{ n \mid e^{\text{prob-lin}}(n,\delta,I_d) \le \varepsilon\,\mathrm{CRI}_d \right\},$$

as long as the corresponding infimum for $e^{\text{prob-lin}}(n,\delta,I_d)$ is attained.

Similarly to Chapter 13, we can relate the information complexity in the probabilistic setting to the information complexity in the average and worst case settings.

We first consider the absolute error criterion. Let

$$\varepsilon_\delta := \frac{\varepsilon}{\psi^{-1}(1-\delta)} = \frac{\varepsilon}{\sqrt{2\ln\delta^{-1}}}\,(1+o(1)).$$

From (14.1), we have the relation

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = n^{\text{avg-lin}}(\varepsilon_\delta,I_d)$$

between the probabilistic and average case settings.

From Chapter 13 we also conclude that for all $x>1$, we have

$$n^{\text{avg}}(\varepsilon_\delta,I_d) \le n^{\text{prob}}(\varepsilon,\delta,I_d) \le \frac{x^2}{x^2-1}\, n^{\text{avg}}(\varepsilon_\delta/x,I_d).$$

This can be concisely written as

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = (1+a_x)\, n^{\text{avg}}(b_x\, \varepsilon_\delta, I_d)$$

with $a_x \in [0, 1/(x^2-1)]$ and $b_x \in [1/x, 1]$ for all $x > 1$.

From Chapter 13, we have the relation

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = n^{\text{wor}}\left(\varepsilon_\delta, I_d^{\text{res}}\right)$$

between the probabilistic and worst case settings.

We turn to the normalized error criterion. Note that

$$e^{\text{prob-lin}}(n,\delta,I_d) \le \varepsilon\, e^{\text{prob-lin}}(0,\delta,I_d) \quad \text{iff} \quad e^{\text{avg-lin}}(n,I_d) \le \varepsilon\, e^{\text{avg-lin}}(0,I_d).$$

Hence, the parameter δ disappears and, proceeding as before, we have the bounds

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = n^{\text{avg-lin}}(\varepsilon,I_d) = n^{\text{wor}}(\varepsilon, I_d^{\text{res}}),$$

and

$$n^{\text{avg}}(\varepsilon,I_d) \le n^{\text{prob}}(\varepsilon,\delta,I_d) \le \frac{x^2}{x^2-1}\, n^{\text{avg}}(\varepsilon/x, I_d)$$

on the information complexity for all $x > 1$. These last bounds can be written as

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = (1+a_x)\, n^{\text{avg}}(b_x\varepsilon, I_d)$$

with $a_x \in [0, 1/(x^2-1)]$ and $b_x \in [1/x, 1]$ for all $x > 1$.

We stress that the relations between the probabilistic and worst case settings are especially pleasing. These relations will be heavily used to obtain tractability results in the probabilistic setting.

The bounds presented above can be used to verify under which conditions we have equivalence of tractability in the probabilistic and average case settings for the linear functional I_d and for the absolute/normalized error criterion, as well as equivalence of tractability in the probabilistic setting for I_d and tractability in the worst case setting for I_d^{res} also for the absolute/normalized error criterion. As already mentioned, we are especially interested in relations to the worst case setting.

We now elaborate on the equivalence of tractability in the probabilistic and the worst case settings. To simplify the presentation, by tractability of

$$I^{\text{wor}} = \{I_d^{\text{res}}\}$$

we mean tractability of I^{wor} in the worst case setting, and by tractability of

$$I^{\text{prob}} = \{I_d\}$$

we mean tractability of I^{prob} in the probabilistic setting.

14.3.1 Absolute Error Criterion

We consider several cases of tractability for the absolute error criterion.

• Suppose that we have polynomial tractability of $I^{\rm prob}$. That is,

$$n^{\rm prob}(\varepsilon,\delta,I_d) \le C\,\varepsilon^{-p}\,d^{q}\,\delta^{-s} \quad \text{for all } \varepsilon,\delta\in(0,1),\ d\in\mathbb{N}.$$

We claim that $p=0$ or $s=0$ can occur only if $I_d^{\rm res}$ can be approximated using a finite number of function values with an arbitrary small error. Indeed, recall that

$$e^{\rm prob}(n,\delta,I_d) = \psi^{-1}(1-\delta)\,e^{\rm wor}(n,I_d^{\rm res}).$$

Strong tractability with respect to δ, i.e., $s=0$, means that for a fixed d and ε there exists n such that

$$e^{\rm prob}(n,\delta,I_d) = \psi^{-1}(1-\delta)\,e^{\rm wor}(n,I_d^{\rm res}) \le \varepsilon \quad \text{for } \delta\to 0.$$

Since $\psi^{-1}(1-\delta)$ goes to infinity, this can happen only if $e^{\rm wor}(n,I_d^{\rm res})=0$. This means that we can approximate $I_d^{\rm res}$ with an arbitrarily small worst case error by a linear algorithm that uses at most n function values. Similarly, if $p=0$ then we can take ε tending to zero, and again it can only happen if $e^{\rm wor}(n,I_d^{\rm res})=0$.

Hence, for all linear functionals for which $e^{\rm wor}(n,I_d^{\rm res})$ is positive for all n and d, we have $p>0$ and $s>0$, i.e., strong polynomial tractability with respect to δ cannot happen.

For $p>0$ and $s>0$, we apply the bound on $n^{\rm prob}(\varepsilon,\delta,I_d)=n^{\rm wor}(\varepsilon_\delta,I_d^{\rm res})$, and conclude that

$$n^{\rm wor}(\varepsilon,I_d^{\rm res}) \le C\,\varepsilon_1^{-p}\,d^{q}\,\left[\psi^{-1}(1-\delta_1)\right]^{-p}\,\delta_1^{-s},$$

for all $\varepsilon_1\in(0,1)$ and $\delta_1\in(0,1)$ for which $\varepsilon=\varepsilon_1/\psi^{-1}(1-\delta)\in(0,1)$. Substituting $\varepsilon_1=\varepsilon\,\psi^{-1}(1-\delta)$ we have

$$n^{\rm wor}(\varepsilon,I_d^{\rm res}) \le C\,\varepsilon^{-p}\,d^{q}\,\min_{\delta_1\in(0,1)} f(\delta_1),$$

where

$$f(\delta_1) = \frac{1}{[\psi^{-1}(1-\delta_1)]^{2p}\ \delta_1^{s}}.$$

Note that positive p and s imply that $f(\delta_1)$ tends to infinity as δ_1 tends to 0 or to 1. Therefore the minimum of f is positive, and we obtain polynomial tractability of $I^{\rm wor}$ with the same exponents p and q as in the probabilistic setting. This also means that strong polynomial tractability of $I^{\rm prob}$ with respect to d, i.e., $q=0$, implies strong polynomial tractability of $I^{\rm wor}$.

• Suppose that we have poly-log tractability of $I^{\rm prob}$. That is,

$$n^{\rm prob}(\varepsilon,\delta,I_d) \le C\,\varepsilon^{-p}\,d^{q}\,(1+\ln\delta^{-1})^{s} \quad \text{for all } \varepsilon,\delta\in(0,1),\ d\in\mathbb{N}.$$

Similarly as before, we then have

$$n^{\rm wor}(\varepsilon, I_d^{\rm res}) \le C\,\varepsilon_1^{-p}\,\frac{(1+\ln\delta_1^{-1})^s}{[\psi^{-1}(1-\delta_1)]^p},$$

where $\varepsilon = \varepsilon_1/\psi^{-1}(1-\delta_1)$. Taking, say, $\delta = \frac{1}{2}$ and $\varepsilon_1 = \varepsilon\,\psi^{-1}(\frac{1}{2})$ we conclude that $I^{\rm wor}$ is polynomially tractable with exponents p and q. Again, $q = 0$ implies that $I^{\rm wor}$ is strongly polynomially tractable.

• Suppose that we have weak tractability of $I^{\rm prob}$. Then we take $\delta = \frac{1}{2}$ and

$$\frac{\ln n^{\rm wor}(\varepsilon, I_d^{\rm res})}{\varepsilon^{-1}+d} = \frac{\ln n^{\rm prob}(\psi^{-1}\big(\frac{1}{2}\big)\varepsilon, \frac{1}{2}, I_d)}{\big[\psi^{-1}\big(\frac{1}{2}\big)\varepsilon\big]^{-1}+2+d}\,\frac{\big[\psi^{-1}\big(\frac{1}{2}\big)\varepsilon\big]^{-1}+2+d}{\varepsilon^{-1}+d}.$$

Since the first factor goes to zero as $\varepsilon^{-1}+d$ goes to infinity, and the second factor is uniformly bounded, the limit of the left-hand side is zero as $\varepsilon^{-1}+d$ approaches infinity, and so $I^{\rm wor}$ is weakly tractable.

• Suppose that we have weak-log tractability of $I^{\rm prob}$. Then we take $\delta = \frac{1}{2}$ and

$$\frac{\ln n^{\rm wor}(\varepsilon, I_d^{\rm res})}{\varepsilon^{-1}+d} = \frac{\ln n^{\rm prob}(\psi^{-1}\big(\frac{1}{2}\big)\varepsilon, \frac{1}{2}, I_d)}{\big[\psi^{-1}\big(\frac{1}{2}\big)\varepsilon\big]^{-1}+\ln 2+d}\,\frac{\big[\psi^{-1}\big(\frac{1}{2}\big)\varepsilon\big]^{-1}+\ln 2+d}{\varepsilon^{-1}+d}.$$

As before, the first factor goes to zero as $\varepsilon^{-1}+d$ goes to infinity, and the second factor is uniformly bounded. Thus $I^{\rm wor}$ is weakly tractable.

• Suppose that we have (strong) T-tractability of $I^{\rm prob}$. As already explained, we denote $T = T^{\rm prob}$ to stress that the function $T^{\rm prob}$ is used in the probabilistic setting and depends on three variables. Hence, we have

$$n^{\rm prob}(\varepsilon,\delta,I_d) \le C\,T^{\rm prob}(\varepsilon^{-1}, d_s, \delta^{-1})^t \quad \text{for all } \varepsilon,\delta\in(0,1),\ d\in\mathbb{N},$$

where $d_s = 1$ for strong $T^{\rm prob}$-tractability, and $d_s = d$ for $T^{\rm prob}$-tractability.

We want to translate the $T^{\rm prob}$-tractability of $I^{\rm prob}$ into $T^{\rm wor}$-tractability of $I^{\rm wor}$. In particular, we need to define $T^{\rm wor}$ in the worst case setting that depends only on two variables. We take $\delta = 1-\psi(1)$, so that $\psi^{-1}(1-\delta) = 1$, and let

$$T^{\rm wor}(x,y) = T^{\rm prob}\big(x, y, [1-\psi(1)]^{-1}\big) \quad \text{for all } x,y\in[1,\infty).$$

Note that $T^{\rm wor}$ is a tractability function, since it is non-decreasing in both the variables and grows sub-exponentially. Then

$$\begin{aligned} n^{\rm wor}(\varepsilon, I_d^{\rm res}) &= n^{\rm prob}\big(\psi^{-1}(\psi(1))\,\varepsilon, 1-\psi(1), I_d\big) = n^{\rm prob}(\varepsilon, 1-\psi(1), I_d) \\ &\le C\,T^{\rm prob}\big(\varepsilon^{-1}, d_s, [1-\psi(1)]^{-1}\big)^t = C\,T^{\rm wor}(\varepsilon^{-1}, d_s)^t. \end{aligned}$$

This means (strong) $T^{\rm wor}$-tractability of $I^{\rm wor}$ with the same exponent t.

• Suppose that we have (strong) T^{prob}-log-tractability of I^{prob}. We now take

$$T^{\text{wor}}(x,y) = T^{\text{prob}}\left(x,y,1+\ln[1-\psi(1)]^{-1}\right) \quad \text{for all } x,y \in [1,\infty),$$

which again is a tractability function, and then

$$\begin{aligned} n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) &= n^{\text{prob}}(\psi^{-1}(\psi(1))\,\varepsilon, 1-\psi(1), I_d) \\ &\le C\, T^{\text{prob}}\left(\varepsilon^{-1}, d_s, 1+\ln[1-\psi(1)]^{-1}\right)^t = C\, T^{\text{wor}}(\varepsilon^{-1}, d_s)^t. \end{aligned}$$

This means (strong) T^{wor}-tractability of I^{wor} with the same exponent t.

• Suppose that we have polynomial tractability of I^{wor}. That is,

$$n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) \le C\,\varepsilon^{-p}\,d^q \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Then

$$n^{\text{prob}}(\varepsilon,\delta,I_d) \le C\,\varepsilon^{-p}\,d^q\,\left[\psi^{-1}(1-\delta)\right]^p \quad \text{for all } \varepsilon,\delta \in (0,1),\ d \in \mathbb{N}.$$

Since $\psi^{-1}(1-\delta) = \mathcal{O}((1+\ln\delta^{-1})^{1/2})$, we obtain polynomial tractability of I^{prob} with the same exponents p and q. Moreover, the exponent s is positive, but it can be arbitrarily small. We also obtain poly-log tractability of I^{prob} with the exponents p, q and $s = p/2$.

• Suppose that we have weak tractability of I^{wor}. Then

$$\frac{\ln n^{\text{prob}}(\varepsilon,\delta,I_d)}{\varepsilon^{-1}+\delta^{-1}+d} = \frac{\ln n^{\text{wor}}(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}})}{\psi^{-1}(1-\delta)\,\varepsilon^{-1}+d}\;\frac{\psi^{-1}(1-\delta)\,\varepsilon^{-1}+d}{\varepsilon^{-1}+\delta^{-1}+d}.$$

Note that the first factor goes to zero; however, the second factor is *not* uniformly bounded in general. Indeed, for a fixed d and $\delta = \varepsilon$, the second factor for small ε is of order $\sqrt{\ln\varepsilon^{-1}}$ and goes to infinity. In fact, weak tractability of I^{wor} might *not* imply weak tractability of I^{prob}. Indeed, assume that I^{wor} is "barely" weakly tractable, with

$$n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) = \Theta\left(d^q\,\exp\left(\varepsilon^{-1}/\sqrt{\ln\varepsilon^{-1}}\right)\right) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Then for $\delta = \varepsilon$ and small ε, we have

$$n^{\text{prob}}(\varepsilon,\varepsilon,I_d) = \Theta\Big(d^q\,\exp\left(\left[\sqrt{2}\,\varepsilon\right]^{-1}(1+o(1))\right)\Big),$$

and I^{prob} is indeed not weakly tractable.

To guarantee weak tractability of I^{prob}, we must assume a little more about the information complexity of I^{wor}. Namely, let us assume that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{wor}}(\varepsilon/\sqrt{\ln\varepsilon^{-1}}, I_d^{\text{res}})}{\varepsilon^{-1}+d} = 0. \tag{14.2}$$

Let $x = \max(\varepsilon^{-1}, \delta^{-1})$. Since $\varepsilon \geq 1/x$ and $1/\sqrt{\ln \delta^{-1}} \geq 1/\sqrt{\ln x}$, we have

$$\frac{\varepsilon}{\psi^{-1}(1-\delta)} = \frac{\varepsilon(1+o(1))}{\sqrt{2\ln \delta^{-1}}} \geq \frac{1+o(1)}{x\sqrt{2\ln x}}$$

for small δ. This implies that

$$n^{\text{wor}}\left(\frac{\varepsilon}{\psi^{-1}(1-\delta)}, I_d^{\text{res}}\right) \leq n^{\text{wor}}\left(\frac{1+o(1)}{x\sqrt{2\ln x}}, I_d^{\text{res}}\right),$$

and

$$\begin{aligned}\frac{\ln n^{\text{prob}}(\varepsilon,\delta,I_d)}{\varepsilon^{-1}+\delta^{-1}+d} &= \frac{\ln n^{\text{wor}}\big(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}}\big)}{\varepsilon^{-1}+\delta^{-1}+d} \\ &\leq \frac{\ln n^{\text{wor}}\big((1+o(1))/(x\sqrt{2\ln x}), I_d^{\text{res}}\big)}{x+d}.\end{aligned}$$

Due to (14.2), the right-hand side goes to zero as $x+d$ goes to infinity. This yields weak tractability of I^{prob}.

When can we claim weak-log tractability of I^{prob}? For this to hold, we need to prove that

$$\frac{\ln n^{\text{prob}}(\varepsilon,\delta,I_d)}{\varepsilon^{-1}+\ln\delta^{-1}+d} = \frac{\ln n^{\text{wor}}\big(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}}\big)}{\varepsilon^{-1}+\ln\delta^{-1}+d}$$

goes to zero as $\varepsilon^{-1}+\ln\delta^{-1}+d$ goes to infinity. It is easy to see that (14.2) is too weak in this case. Indeed, take $\delta = \exp(-1/\varepsilon)$ so that $\ln\delta^{-1} = \varepsilon^{-1}$. Then for small ε, we have

$$\frac{\ln n^{\text{wor}}(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}})}{\varepsilon^{-1}+\ln\delta^{-1}+d} = \frac{\ln n^{\text{wor}}\big(\varepsilon^{3/2}/\sqrt{2}\,(1+o(1)), I_d^{\text{res}}\big)}{2\varepsilon^{-1}+d}.$$

Hence, we must strengthen (14.2) by assuming that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{wor}}\big(\varepsilon^{3/2}, I_d^{\text{res}}\big)}{\varepsilon^{-1}+d} = 0. \tag{14.3}$$

Indeed, to show weak-log tractability of I^{prob}, take $x = \max(\varepsilon^{-1}, \ln\delta^{-1})$. Then

$$\frac{\varepsilon}{\psi^{-1}(1-\delta)} = \frac{\varepsilon(1+o(1))}{\sqrt{2\ln\delta^{-1}}} \geq \frac{1+o(1)}{x^{3/2}\sqrt{2}},$$

and

$$\frac{\ln n^{\text{wor}}\big(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}}\big)}{\varepsilon^{-1}+\ln\delta^{-1}+d} \leq \frac{\ln n^{\text{wor}}\big((1+o(1))/(x^{3/2}\sqrt{2}), I_d^{\text{res}}\big)}{x+d}.$$

Due to (14.3), the right-hand side goes to zero as $x+d$ goes to infinity. This implies weak-log tractability of I^{prob}, as claimed.

• Suppose that we have (strong) T^{wor}-tractability of I^{wor}. That is,

$$n^{\text{wor}}\big(\varepsilon, I_d^{\text{res}}\big) \le C\, T^{\text{wor}}(\varepsilon^{-1}, d_s)^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$

where $d_s = 1$ for strong T^{wor}-tractability, and $d_s = d$ for T^{wor}-tractability, as before.

We now want to translate the T^{wor}-tractability of I^{wor} into T^{prob}-tractability of I^{prob}. In particular, we need to define a function T^{prob} that now depends on three variables. We take

$$T^{\text{prob}}(x,y,z) = T^{\text{wor}}\big(\max(1, \psi^{-1}(1-z^{-1})\,x), y\big) \quad \text{for all } x,y,z \in [1,\infty). \tag{14.4}$$

We must check that T^{prob} is a tractability function. First of all, T^{prob} is well defined since $\max(1, \psi^{-1}(1-z^{-1})\,x) \ge 1$ belongs to the domain of T^{wor}. Clearly, T^{prob} is non-decreasing in all variables. We only need to check that T^{prob} is non-exponential, i.e., that

$$\lim_{x,y,z\ge 1,\ x+y+z\to\infty} \frac{\ln T^{\text{prob}}(x,y,z)}{x+y+z} = 0.$$

For a general tractability function T^{wor} this does not hold. So we need to assume a little more on T^{wor}, namely that

$$\lim_{x,y\ge 1,\ x+y\to\infty} \frac{\ln T^{\text{wor}}(x\sqrt{\ln x}, y)}{x+y} = 0. \tag{14.5}$$

Indeed, similarly as we did for weak tractability, let $w = \max(x,z)$. Then for large z we have

$$\psi^{-1}(1-z^{-1})\,x = \sqrt{2\ln z}\,(1+o(1))\,x \le w\sqrt{2\ln w}\,(1+o(1)),$$

and

$$T^{\text{wor}}(\psi^{-1}(1-z^{-1})\,x, y) \le T^{\text{wor}}(w\sqrt{2\ln w}\,(1+o(1)), y).$$

Therefore

$$\frac{\ln T^{\text{prob}}(x,y,z)}{x+y+z} = \frac{\ln T^{\text{wor}}(\psi^{-1}(1-z^{-1})\,x, y)}{x+y+z} \le \frac{\ln T^{\text{wor}}(w\sqrt{2\ln w}\,(1+o(1)), y)}{w+y}.$$

Due to (14.5), the right-hand side goes to zero as $w+y$ goes to infinity. Hence, T^{prob} is indeed a tractability function.

The rest is easy since

$$\begin{aligned} n^{\text{prob}}(\varepsilon,\delta,I_d) &= n^{\text{wor}}\big(\varepsilon/\psi^{-1}(1-\delta), I_d^{\text{res}}\big) \le n^{\text{wor}}\big(\min(1, \varepsilon/\psi^{-1}(1-\delta)), I_d^{\text{res}}\big) \\ &\le C\, T^{\text{wor}}\big(\max(1, \psi^{-1}(1-\delta)\,\varepsilon^{-1}), d_s\big)^t = C\, T^{\text{prob}}(\varepsilon^{-1}, d_s, \delta^{-1})^t. \end{aligned}$$

This means that we have (strong) T^{prob}-tractability of I^{prob} with the same exponent t.

We turn to (strong) T^{prob}-log tractability. We now take

$$T^{\text{prob}}(x,y,z) = T^{\text{wor}}\big(\max(1, \psi^{-1}(1-\exp(1-z))\,x), y\big) \quad \text{for all } x,y,z \in [1,\infty). \tag{14.6}$$

Then T^{prob} is a tractability function if we assume that

$$\lim_{x,y\ge 1,\; x+y\to\infty} \frac{\ln T^{\text{wor}}(x^{3/2}, y)}{x+y} = 0. \tag{14.7}$$

Indeed, for the same $w = \max(x, z)$ and large z, we have

$$\psi^{-1}(1-\exp(1-z)) = \sqrt{2\ln\exp(z-1)}(1+o(1) = \sqrt{2\,z/e}(1+o(1)),$$

and

$$\begin{aligned}\frac{\ln T^{\text{prob}}(x,y,z)}{x+y+z} &= \frac{T^{\text{wor}}\big(\sqrt{2\,z/e}(1+o(1))\,x, y\big)}{x+y+z}\\ &\le \frac{T^{\text{wor}}\big(w^{3/2}\sqrt{2/e}(1+o(1)), y\big)}{w+y}.\end{aligned}$$

Due to (14.7), the right-hand side goes to zero as $w+y$ goes to infinity. Hence, T^{prob} is a tractability function, as claimed. Finally,

$$\begin{aligned}n^{\text{prob}}(\varepsilon,\delta,I_d) &= n^{\text{wor}}\left(\varepsilon/\psi^{-1}(1-\delta), d\right) \le n^{\text{wor}}\left(\min(1,\varepsilon/\psi^{-1}(1-\delta)), d\right)\\ &\le C\,T^{\text{wor}}(\max(1,\psi^{-1}(1-\delta)\,\varepsilon^{-1}), d_s)^t\\ &= C\,T^{\text{prob}}(\varepsilon^{-1}, d_s, 1+\ln\delta^{-1})^t.\end{aligned}$$

Hence, we have (strong) T^{prob}-log-tractability of I^{prob} with the same exponent t.

14.3.2 Normalized Error Criterion

For the normalized error criterion, we have

$$n^{\text{prob}}(\varepsilon,\delta,I_d) = n^{\text{wor}}(\varepsilon, I_d^{\text{res}}) \quad \text{for all } \varepsilon,\delta\in(0,1),\; d\in\mathbb{N},$$

and δ does not play any role. That is why the following statements are true.

- Polynomial tractability of I^{prob} is equivalent to polynomial tractability of I^{wor}. Furthermore, the exponent $s = 0$, and the exponents p and q are the same in the tractability bounds for I^{prob} and I^{wor}. In particular, strong polynomial tractability with respect to d is equivalent for I^{prob} and I^{wor}.
- Pol-log tractability of I^{prob} is equivalent to polynomial tractability of I^{prob} and polynomial tractability of I^{wor}.
- Weak tractability of I^{prob} is equivalent to weak tractability of I^{wor}.
- Weak-log tractability of I^{prob} is equivalent to weak tractability of I^{prob} and weak tractability of I^{wor}.

- (Strong) T^{prob}-tractability of I^{prob} implies *(*strong*)* T^{wor}-tractability of I^{wor} if
$$T^{\text{wor}}(x,y) = T^{\text{prob}}(x,y,1) \quad \text{for all } x,y \in [1,\infty).$$
- (Strong) T^{wor}-tractability of I^{wor} implies (strong) T^{prob}-tractability of I^{prob}-tractability of I^{prob} if
$$T^{\text{prob}}(x,y,z) = T^{\text{wor}}(x,y) \quad \text{for all } x,y,z \in [1,\infty). \tag{14.8}$$
- (Strong) T^{prob}-log-tractability of I^{prob} is equivalent to (strong) T^{prob}-tractability of I^{prob}.

14.3.3 Summary

We summarize the analysis of the previous two subsections as follows.

Theorem 14.1. *Consider $I^{\text{prob}} = \{I_d\}$ in the probabilistic setting and $I^{\text{wor}} = \{I_d^{\text{res}}\}$ in the worst case setting for the absolute or normalized error criterion, defined as in this chapter. Then*
$$n^{\text{prob}}(\varepsilon, I_d) = n^{\text{wor}}(\varepsilon_d, I_d^{\text{res}}) \quad \textit{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$
where $\varepsilon_d = \varepsilon/\psi^{-1}(1-\delta)$ for the absolute error criterion, and $\varepsilon_d = \varepsilon$ for the normalized error criterion.
Consider the absolute error criterion.

- *Polynomial tractability of I^{wor} implies polynomial tractability of I^{prob} with the same exponents p and q, and with an arbitrarily small positive exponent s, as well as poly-log tractability of I^{prob} with the same exponents p and q, and with $s = p/2$. Hence, strong polynomial tractability of I^{wor} implies strong polynomial tractability of I^{prob} with respect to d.*
- *Weak tractability of I^{wor} and* (14.2) *imply weak tractability of I^{prob}.*
- *Weak tractability of I^{wor} and* (14.3) *imply weak-log tractability of I^{prob}.*
- (*Strong*) *T^{wor}-tractability of I^{wor} with the function T^{wor} satisfying* (14.5) *implies* (*strong*) *T^{prob}-tractability of I^{prob} for T^{prob} defined by* (14.4).
- (*Strong*) *T^{wor}-log-tractability of I^{wor} with the function T^{wor} satisfying* (14.7) *implies* (*strong*) *T^{prob}-tractability of I^{prob} for T^{prob} defined by* (14.6).

Consider the normalized error criterion.

- *Polynomial tractability of I^{wor} implies polynomial tractability, as well as poly-log tractability of I^{prob} with the same exponents p and q, and $s = 0$.*
- *Weak tractability of I^{wor} implies weak as well as weak-log tractability of I^{prob}.*
- (*Strong*) *T^{wor}-tractability of I^{wor} implies* (*strong*) *T^{prob}-tractability as well as* (*strong*) *T^{prob}-log tractability of I^{prob} with T^{prob} defined by* (14.8).

14.4 Relative Error

This section deals with the relative error criterion. As already mentioned a few times, the relative error in the worst and average case settings leads to negative results for approximating linear functionals[1]. We can obtain positive results and tractability only in the probabilistic setting.

Let $A_{n,d}$ be an algorithm given by (13.2) with $n(f) \equiv n$. The relative error of $A_{n,d}$ in the probabilistic setting is given by

$$e^{\text{prob-rel}}(A_{n,d}, \delta, I_d) = \inf_{B \subseteq F_d,\, \mu_d(B) \le \delta} \; \sup_{f \in F_d \setminus B} \frac{|I_d(f) - A_{n,d}(f)|}{|I_d(f)|},$$

with the convention that $0/0 = 0$. Note that for $I_d \neq 0$, the initial error is

$$e^{\text{prob-rel}}(0, \delta, I_d) = 1,$$

so that the problem is always well normalized. As always, the information complexity is defined as the minimal number of function values needed to find an algorithm with error at most ε, i.e.,

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = \min \left\{ n \mid \text{there exists } A_{n,d} \text{ with } e^{\text{prob-rel}}(A_{n,d}, \delta, I_d) \le \varepsilon \right\}.$$

It was proved in [138], see also Section 6.1 of Chapter 6 in [305], that adaption does not help and that

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \le n^{\text{avg-lin-nor}} \left(\frac{\varepsilon \, \tan(\delta\, \pi/2)}{\sqrt{1 + \varepsilon^2 \tan^2(\delta\, \pi/2)}}, I_d \right),$$

[1] The relative error of an algorithm $A_{n,d}$ in the worst case setting is defined as

$$e^{\text{wor-rel}}(A_{n,d}, I_d) = \sup_{f \in F_d} \frac{|I_d(f) - A_{n,d}(f)|}{|I_d(f)|},$$

with the convention that $0/0 = 0$. Then as long as I_d cannot be approximated exactly using finitely many function values, $e^{\text{wor-avg}}(A_{n,d}, I_d) \ge 1$ and the information complexity $n^{\text{wor}}(\varepsilon, I_d) = \infty$ for all $\varepsilon \in (0, 1)$. In fact, this result holds not only for linear functionals but also for linear operators that cannot be approximated exactly by finitely many function values, see Section 6.1 of Chapter 6 in [305].

In the average case setting with a zero mean Gaussian measure μ_d, the relative error of an algorithm $A_{n,d}$ is defined as

$$e^{\text{avg-rel}}(A_{n,d}, I_d) = \left(\int_{F_d} \frac{|I_d(f) - A_{n,d}(f)|^2}{|I_d(f)|^2} \, \mu(\mathrm{d}f) \right)^{1/2}.$$

Again, $e^{\text{avg-rel}}(A_{n,d}, I_d) \ge 1$ and the information complexity $n^{\text{avg}}(\varepsilon, I_d) = \infty$ for all $\varepsilon \in (0, 1)$ as long as I_d cannot be approximated exactly by using finitely many function values. This result holds for linear operators I_d for which $\dim(I_d(F_d)) \le 2$, again assuming that I_d cannot be approximated exactly by finitely many function values, see Section 6.1 of Chapter 6 in [305].

The negative results in the worst and average case setting can be overcome by a modified relative error. This is beyond the scope of this book. The reader interested in this subject is referred to Sections 6 of Chapter 4 and 6 in [305].

where $n^{\text{avg-lin-nor}}(\varepsilon, I_d)$ denotes the information complexity of approximating I_d in the average case setting for the normalized error criterion when we use only linear algorithms. Furthermore, the last bound is sharp for small ε since

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \ge n^{\text{avg-lin-nor}}\left(\frac{\varepsilon\ \tan(\delta_1\,\pi/2)}{\sqrt{1+\varepsilon^2\tan^2(\delta_1\,\pi/2)}}, I_d\right)$$

with

$$\delta_1 := \min\left(1, \delta\left[1-\sqrt{\frac{\pi\,\varepsilon}{4}\,\ln\frac{1+\varepsilon}{1-\varepsilon}}\,\right]_+^{-1}\right) = \delta\,(1+\mathcal{O}(\varepsilon)) \quad \text{as } \varepsilon \to 0. \tag{14.9}$$

For small ε, we thus have

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = n^{\text{avg-lin-nor}}\left(\varepsilon\ \tan(\delta\,\pi/2\,(1+o(1)))\,(1+o(1)), I_d\right).$$

Due to Theorem 13.2 in Chapter 13, we know that

$$n^{\text{avg-lin-nor}}(\varepsilon, I_d) = n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}),$$

where $n^{\text{wor-nor}}$ stands for the information complexity in the worst case setting for the normalized error criterion. Let

$$g(\varepsilon, d) = \frac{\varepsilon\ \tan(\delta\,\pi/2)}{\sqrt{1+\varepsilon^2\tan^2(\delta\,\pi/2)}} \quad \text{for all } \varepsilon, \delta \in (0,1).$$

Note that for small ε and δ we have

$$g(\varepsilon, \delta) = \frac{1}{2}\,\pi\,\varepsilon\,\delta\,(1+o(1)).$$

Therefore there exist numbers ε^* and δ^* from $(0, 1/\pi)$ such that

$$\tfrac{1}{4}\,\pi\,\varepsilon\,\delta \le g(\varepsilon, \delta) \le \pi\,\varepsilon\,\delta \quad \text{for all } \varepsilon \in (0, \varepsilon^*] \text{ and } \delta \in (0, \delta^*].$$

Additionally, we choose ε^* and δ^* such that

$$\delta_1 \le 2\,\delta \quad \text{for all } \delta \in (0, \delta^*],$$

where δ_1 is given by (14.9). We have

$$n^{\text{wor-nor}}(g(\varepsilon, \delta_1), I_d^{\text{res}}) \le n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \le n^{\text{wor-nor}}(g(\varepsilon, \delta), I_d^{\text{res}}) \tag{14.10}$$

for all $\varepsilon, \delta \in (0,1)$. For small ε, we have

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = n^{\text{wor-nor}}\big(\varepsilon\ \tan\left(\tfrac{1}{2}\delta\,\pi\,(1+o(1))\right), I_d^{\text{res}}\big), \tag{14.11}$$

and for small ε and δ, we have

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = n^{\text{wor-nor}}\big(\tfrac{1}{2}\,\pi\,\varepsilon\,\delta\,(1+o(1)), I_d^{\text{res}}\big). \tag{14.12}$$

This shows that the probabilistic setting for the relative error criterion is also closely related to the average and worst case settings for the normalized error criterion. However, note that the role of the parameter δ is now much more important than in the previous cases. For small ε and δ, the parameter δ is just as important as the parameter ε.

Tractability in the probabilistic setting for the relative error criterion is defined exactly as for the absolute or normalized error criterion. The only difference is that we now bound $n^{\text{prob-rel}}(\varepsilon, \delta, I_d)$ instead of $n^{\text{prob}}(\varepsilon, \delta, I_d)$.

Tractability of $I^{\text{prob-rel}} = \{I_d\}$ means tractability of $\{I_d\}$ in the probabilistic setting for the relative error criterion. By $T^{\text{prob-rel}}$ we denote a tractability function that is used in the probabilistic setting for the relative error. Again, tractability in the probabilistic setting for the relative error criterion is closely related to tractability in the worst case setting for the normalized error criterion. Proceeding similarly as before, we easily check that the following statements are true:

- Suppose that we have polynomial tractability of $I^{\text{prob-rel}}$. That is,

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \le C\, \varepsilon^{-p}\, d^q\, \delta^{-s} \quad \text{for all } \varepsilon, \delta \in (0,1),\ d \in \mathbb{N}.$$

Then $\varepsilon \ge \min(\varepsilon, \varepsilon^*) \ge g(\min(\varepsilon, \varepsilon^*), \delta^*)$ since $g(\min(\varepsilon, \varepsilon^*), \delta^*) \le \pi\, \min(\varepsilon, \varepsilon^*)\, \delta^*$ and $\pi \delta^* \le 1$. This and the left-hand side of (14.10) yield that

$$\begin{aligned} n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) &\le n^{\text{wor-nor}}\big(\min(\varepsilon, \varepsilon^*), I_d^{\text{res}}\big) \\ &\le n^{\text{wor-nor}}\big(g(\min(\varepsilon, \varepsilon^*), \delta^*), I_d^{\text{res}}\big) \\ &\le n^{\text{wor-nor}}\big(g(\min(\varepsilon, \varepsilon^*), \tfrac{1}{2}\, \delta_1^*), I_d^{\text{res}}\big) \\ &\le n^{\text{prob-rel}}(\min(\varepsilon, \varepsilon^*), \tfrac{1}{2}\, \delta^*, I_d) \le C\, \min(\varepsilon, \varepsilon^*)^{-p}\, d^q\, \big[\tfrac{1}{2}\, \delta^*\big]^{-s} \\ &\le C\, \big[\tfrac{1}{2}\, \delta^*\big]^{-s} \max\big(\varepsilon^{-p}, [\varepsilon^*]^{-p}\big)\, d^q \le C\, \big[\tfrac{1}{2}\, \delta^*\big]^{-s}\, [\varepsilon^*]^{-p}\, \varepsilon^{-p}\, d^q. \end{aligned}$$

This means that we have polynomial tractability of I^{wor} with the same exponents p and q as in the probabilistic setting for the relative error. This also means that strong polynomial tractability of $I^{\text{prob-rel}}$ with respect to d, i.e., $q = 0$, implies strong polynomial tractability of I^{wor}.

- Suppose that we have poly-log tractability of $I^{\text{prob-rel}}$. That is,

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \le C\, \varepsilon^{-p}\, d^q\, (1 + \ln \delta^{-1})^s \quad \text{for all } \varepsilon, \delta \in (0,1),\ d \in \mathbb{N}.$$

We now have

$$\begin{aligned} n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) &\le n^{\text{prob-rel}}(\min(\varepsilon, \varepsilon^*), \delta^*, I_d) \\ &\le C\, (1 + \ln[\delta^*]^{-1})^s\, [\varepsilon^*]^{-p}\, \varepsilon^{-p}\, d^q. \end{aligned}$$

This means polynomial tractability of I^{wor} with the exponents p and q. Again, $q = 0$ implies strong polynomial tractability of I^{wor}.

- Suppose that we have weak tractability of $I^{\text{prob-rel}}$. Then

$$\frac{\ln n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}})}{\varepsilon^{-1} + d} \le \frac{\ln n^{\text{prob-rel}}\,(\min(\varepsilon, \varepsilon^*), \delta^*, I_d)}{[\min(\varepsilon, \varepsilon^*)]^{-1} + [\delta^*]^{-1} + d}\, \frac{[\min(\varepsilon, \varepsilon^*]^{-1} + [\delta^*]^{-1} + d}{\varepsilon^{-1} + d}.$$

Since the first factor goes to zero as $\varepsilon^{-1} + d$ goes to infinity and the second factor is uniformly bounded, the limit is zero and we obtain weak tractability of I^{wor}.

• Suppose that we have weak-log tractability of $I^{\text{prob-rel}}$. Then

$$\frac{\ln n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}})}{\varepsilon^{-1}+d} \le \frac{\ln n^{\text{prob-rel}}(\min(\varepsilon,\varepsilon^*),\delta^*, I_d)}{[\min(\varepsilon,\varepsilon^*)]^{-1} + \ln[\delta^*]^{-1} + d}\;\frac{[\min(\varepsilon,\varepsilon^*]^{-1} + \ln[\delta^*]^{-1} + d}{\varepsilon^{-1}+d}.$$

As before, the first factor goes to zero as $\varepsilon^{-1}+d$ goes to infinity, and the second factor is uniformly bounded. Thus we obtain weak tractability of I^{wor}.

• Suppose that we have (strong) $T^{\text{prob-rel}}$-tractability of $I^{\text{prob-rel}}$. That is,

$$n^{\text{prob-rel}}(\varepsilon,\delta,I_d) \le C\,T^{\text{prob}}(\varepsilon^{-1}, d_s, \delta^{-1})^t \quad \text{for all } \varepsilon,\delta \in (0,1),\ d \in \mathbb{N},$$

where $d_s = 1$ for strong $T^{\text{prob-rel}}$-tractability, and $d_s = d$ for $T^{\text{prob-rel}}$-tractability. Define

$$T^{\text{wor-nor}}(x,y) = T^{\text{prob-rel}}\big(\max(x,[\varepsilon^*]^{-1}), y, [\delta^*]^{-1}\big) \quad \text{for all } x,y \in [1,\infty).$$

Note that T^{wor} is a tractability function since it is non-decreasing in both variables and grows sub-exponentially. Then

$$\begin{aligned} n^{\text{wor-rel}}(\varepsilon, I_d^{\text{res}}) &\le n^{\text{prob-rel}}\big(\min(\varepsilon,\varepsilon^*),\delta^*, I_d\big) \\ &\le C\,T^{\text{prob-rel}}\big(\max(\varepsilon^{-1},[\varepsilon^*]^{-1}), d_s, [\delta^*]^{-1}\big)^t \\ &= C\,T^{\text{wor-nor}}(\varepsilon^{-1}, d_s)^t. \end{aligned}$$

This means (strong) T^{wor}-tractability of I^{wor} with the same exponent t.

• Suppose that we have (strong) $T^{\text{prob-rel}}$-log-tractability of I^{prob}. We now take

$$T^{\text{wor-nor}}(x,y) = T^{\text{prob-rel}}\big(\max(x,[\varepsilon^*]^{-1}), y, 1+\ln[\delta^*]^{-1}\big) \quad \text{for all } x,y \in [1,\infty),$$

which is also a tractability function, and then

$$\begin{aligned} n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) &\le n^{\text{prob-rel}}\big(\min(\varepsilon,\varepsilon^*),\delta^*, I_d\big) \\ &\le C\,T^{\text{prob-rel}}\big(\max(\varepsilon^{-1},[\varepsilon^*]^{-1}), d_s, 1+\ln[\delta^*]^{-1}\big)^t \\ &= C\,T^{\text{wor}}(\varepsilon^{-1}, d_s)^t. \end{aligned}$$

This means (strong) T^{wor}-tractability of I^{wor} with the same exponent t.

• Suppose that we have polynomial tractability of $I^{\text{wor-nor}}$. That is,

$$n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) \le C\,\varepsilon^{-p}\,d^q \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Then the right-hand side of (14.10) yields

$$
\begin{aligned}
n^{\text{prob-rel}}(\varepsilon,\delta,I_d) &\le n^{\text{prob-rel}}\big(\min(\varepsilon,\varepsilon^*),\min(\delta,\delta^*),I_d\big)\\
&\le n^{\text{wor-nor}}\big(g(\min(\varepsilon,\varepsilon^*),\min(\delta,\delta^*)),I_d^{\text{res}}\big)\\
&\le n^{\text{wor-nor}}\big(\tfrac14\,\pi\,\min(\varepsilon,\varepsilon^*)\,\min(\delta,\delta^*),I_d^{\text{res}}\big)\\
&\le C\,(4/\pi)^p\,\max(\varepsilon^{-p},[\varepsilon^*]^{-p})\,\max(\delta^{-p},[\delta^*]^{-p})\,d^q\\
&\le C\,(4/\pi)^p\,[\varepsilon^*]^{-p}\,[\delta^*]^{-p}\,\varepsilon^{-p}\,d^q\,\delta^{-p}
\end{aligned}
$$

for all $\varepsilon,\delta\in(0,1)$, $d\in\mathbb{N}$.

Hence, we obtain polynomial tractability of $I^{\text{prob-rel}}$ with the same exponents p and q, whereas the exponent $s=p$.

However, note that we do *not* obtain poly-log tractability of $I^{\text{prob-rel}}$ unless $p=0$ which happens only for trivial problems. Indeed, $p=0$ means that for all d we can approximate I_d^{res} with an arbitrarily small error by using at most $C\,d^q$ function values.

• Suppose that we have weak tractability of $I^{\text{wor-nor}}$. In general, we cannot claim weak tractability of $I^{\text{prob-rel}}$ since for $\delta=\varepsilon$ and small ε, from (14.12) we have

$$
n^{\text{prob-rel}}(\varepsilon,\varepsilon,I_d)=n^{\text{wor-nor}}\big(\tfrac12\,\pi\,\varepsilon^2(1+o(1)),I_d^{\text{res}}\big)
$$

and $n^{\text{wor-nor}}(\varepsilon^2,I_d^{\text{res}})$ may be an exponential function of ε^{-1}. To guarantee weak tractability of $I^{\text{prob-rel}}$ we must assume that

$$
\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\text{wor-nor}}(\varepsilon^2,I_d^{\text{res}})}{\varepsilon^{-1}+d}=0. \tag{14.13}
$$

Indeed, let $x=2^{-1/2}\min(\varepsilon,\delta,\varepsilon^*,\delta^*)$. Then

$$
\tfrac14\,\pi\,\min(\varepsilon,\varepsilon^*)\,\min(\delta,\delta^*)\ge\tfrac14\,\pi\,(\sqrt2\,x)^2=\tfrac12\,\pi\,x^2\ge x^2.
$$

Furthermore, it is easy to check that

$$
x^{-1}\le C\,(\varepsilon^{-1}+\delta^{-1})\quad\text{with } C:=\sqrt2\,\max(1/\varepsilon^*,1/\delta^*)\ge1.
$$

This in turn implies that

$$
\frac{1}{\varepsilon^{-1}+\delta^{-1}+d}\le\frac{C}{x^{-1}+d}\quad\text{for all }\varepsilon,\delta\in(0,1),\ d\in\mathbb{N}.
$$

We already showed that

$$
n^{\text{prob-rel}}(\varepsilon,\delta,I_d)\le n^{\text{wor-nor}}\big(\tfrac14\,\pi\,\min(\varepsilon,\varepsilon^*)\,\min(\delta,\delta^*),I_d^{\text{wor}}\big).
$$

Therefore

$$
\begin{aligned}
\frac{\ln n^{\text{prob-rel}}(\varepsilon,\delta,I_d)}{\varepsilon^{-1}+\delta^{-1}+d} &\le \frac{\ln n^{\text{wor-nor}}\big(\tfrac14\,\pi\,\min(\varepsilon,\varepsilon^*)\,\min(\delta,\delta^*),I_d^{\text{res}}\big)}{\varepsilon^{-1}+\delta^{-1}+d}\\
&\le \frac{\ln n^{\text{wor-nor}}\big(x^2,I_d^{\text{res}}\big)}{\varepsilon^{-1}+\delta^{-1}+d}\\
&\le C\,\frac{\ln n^{\text{wor-nor}}\big(x^2,I_d^{\text{res}}\big)}{x^{-1}+d}.
\end{aligned}
$$

Due to (14.13), the last right-hand side goes to zero as $x^{-1} + d$ goes to infinity. This implies weak tractability of I^{prob}, as claimed.

Obviously, (14.13) holds when $I^{\text{wor-nor}}$ is polynomially tractable since the dependence of $n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}})$ is then polynomial in ε^{-1}, so that $\ln n^{\text{wor-nor}}(\varepsilon^2, I_d^{\text{res}})$ depends logarithmically on ε^{-1}. Even if

$$n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) = \mathcal{O}\big(\exp(d^q)\ \exp(\varepsilon^{-p})\big)$$

then (14.13) holds for $q < 1$ and $p < \frac{1}{2}$. On the other hand, the conditions $q < 1$ and $p < 1$ are needed to guarantee weak tractability of $I^{\text{wor-nor}}$.

When can we claim that $I^{\text{prob-rel}}$ is weakly-log tractable? To claim this, we need to prove that

$$\lim_{\varepsilon^{-1}+\ln\delta^{-1}+d\to\infty} \frac{\ln n^{\text{prob-rel}}(\varepsilon, \delta, I_d)}{\varepsilon^{-1} + \ln\delta^{-1} + d} = 0.$$

In this case (14.13) is too weak in general. Taking $\delta = \exp(-1/\varepsilon)$, so that $\ln\delta^{-1} = \varepsilon^{-1}$, we see that we need to assume that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{wor}}(\varepsilon\ \exp(-1/\varepsilon), I_d^{\text{res}})}{\varepsilon^{-1} + d} = 0. \tag{14.14}$$

Indeed, (14.14) yields weak-log tractability of I^{prob}. To show this, take

$$x = 2^{-1/2}\ \min\big(\varepsilon, \varepsilon^*, -1/\ln\min(\delta, \delta^*)\big) \quad \text{and} \quad t = \sqrt{2}\,x.$$

Then $\min(\varepsilon, \varepsilon^*) \ge \sqrt{2}\,x = t$, $\min(\delta, \delta^*) \ge \exp(-1/(\sqrt{2}\,x)) = \exp(-1/t)$ and

$$\frac{1}{\varepsilon^{-1} + \ln\delta^{-1} + d} \le C\,\frac{1}{1/(\sqrt{2}x) + d} = C\,\frac{1}{t^{-1} + d}$$

with $C := \max(1/\varepsilon^*, \ln 1/\delta^*) \ge 1$. Since

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) \le n^{\text{wor-nor}}\left(\tfrac{1}{4}\,\pi\,\min(\varepsilon, \varepsilon^*)\ \min(\delta\,\delta^*), I_d^{\text{res}}\right),$$

then

$$\frac{\ln n^{\text{prob-rel}}(\varepsilon, \delta, I_d)}{\varepsilon^{-1} + \ln\delta^{-1} + d} \le C\,\frac{n^{\text{wor-nor}}\left(t\ \exp(-1/t), I_d^{\text{res}}\right)}{t^{-1} + d}.$$

Due to (14.14), the last right-hand side goes to zero as $t^{-1} + d$ goes to infinity. This implies weak-log tractability of I^{prob}, as claimed.

We stress that (14.14) is quite demanding and does *not* hold for many problems. Even for $d = 1$, the typical behavior of $n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}})$ as a function of ε is polynomial in ε^{-1}, and so (14.14) does not hold. If

$$n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) = \Theta\big(d^q\ [\ln\varepsilon^{-1}]^p\big) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}, \tag{14.15}$$

for some non-negative p and q, then (14.14) holds iff $p < 1$. The logarithmic dependence on ε^{-1} usually holds if the problem $I^{\text{wor-nor}}$ is defined on analytic functions or C^∞ functions. The full characterization of problems for which (14.14) or (14.15) holds is not known and is the subject of the next open problem.

Open Problem 66.

- Characterize problems $I^{\text{prob-rel}}$ in the probabilistic setting for the relative error criterion that are weakly-log tractable.
- Equivalently, characterize problems $I^{\text{wor-nor}}$ in the worst case setting for the normalized error criterion for which (14.14) holds. In particular, characterize problems for which (14.15) holds. □

- Suppose that we have (strong) $T^{\text{wor-nor}}$-tractability of $I^{\text{wor-nor}}$. That is,

$$n^{\text{wor-nor}}(\varepsilon, I_d^{\text{res}}) \le C\, T^{\text{wor-nor}}(\varepsilon^{-1}, d_s)^t \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$

with $d_s = 1$ for strong $T^{\text{wor-nor}}$-tractability, and $d_s = d$ for $T^{\text{wor-nor}}$-tractability, as before.

We now define

$$T^{\text{prob-rel}}(x,y,z) = T^{\text{wor-nor}}\left(\tfrac{4}{\pi}\, \max(x, 1/\varepsilon^*)\, \max(z, 1/\delta^*), y\right) \tag{14.16}$$

for all $x, y, z \in [1, \infty)$. This is a non-decreasing function of three variables. It is also non-exponential if we additionally assume that

$$\lim_{x,y \ge 1,\ x+y \to \infty} \frac{\ln T^{\text{wor-rel}}(x^2, y)}{x+y} = 0. \tag{14.17}$$

Indeed, for $w = 2\,\pi^{-1/2} \max(x, z, 1/\varepsilon^*, 1/\delta^*)$ we have

$$\frac{1}{x+y+z} \le \frac{C}{w+y} \quad \text{with } C := 2\,\pi^{-1/2} \max(1/\varepsilon^*, 1/\delta^*) \ge 1.$$

Since $w \ge 2\,\pi^{-1/2} \max(x, 1/\varepsilon^*)$ and $w \ge 2\,\pi^{-1/2} \max(z, 1/\delta^*)$ then

$$w^2 \ge \frac{4}{\pi} \max(x, 1/\varepsilon^*)\, \max(z, 1/\delta^*),$$

and therefore

$$\frac{\ln T^{\text{prob-rel}}(x,y,z)}{x+y+z} \le C\, \frac{\ln T^{\text{wor-nor}}(w^2, y)}{w+y}.$$

Due to (14.17), the right-hand side goes to zero as $w + d$ goes to infinity. This means that $T^{\text{prob-rel}}$ is non-exponential, and hence it is a tractability function.

Furthermore, we have

$$\begin{aligned} n^{\text{prob-rel}}(\varepsilon, \delta, I_d) &\le n^{\text{wor-rel}}\left(\tfrac{1}{4}\pi\, \min(\varepsilon, \varepsilon^*)\, \min(\delta, \delta^*), I_d^{\text{res}}\right) \\ &\le C\, T^{\text{wor-nor}}\left(\tfrac{4}{\pi}\, \max(\varepsilon^{-1}, 1/\varepsilon^*)\, \max(\delta^{-1}, 1/\delta^*), d_s\right)^t \\ &= C\, T^{\text{prob-rel}}(\varepsilon^{-1}, d_s, \delta^{-1})^t. \end{aligned}$$

Hence, we have (strong) $T^{\text{prob-rel}}$-tractability of $I^{\text{prob-rel}}$ with the same exponent t.

We turn to (strong) $T^{\text{prob-rel}}$-log-tractability of $I^{\text{prob-rel}}$. We now redefine the function $T^{\text{prob-rel}}$ as

$$T^{\text{prob-rel}}(x,y,z) = T^{\text{wor-nor}}\big(\tfrac{4}{\pi}\ \max(x,1/\varepsilon^*)\ \max(\exp(z-1),1/\delta^*),y\big) \tag{14.18}$$

for all $x,y,z \in [1,\infty)$.

We need to assume much more about the tractability function $T^{\text{wor-nor}}$ to prove that $T^{\text{prob-rel}}$ is also a tractability function. Namely, let

$$\lim_{x,y\ge 1,\,x+y\to\infty} \frac{\ln T^{\text{wor-nor}}(x\ \exp(x),y)}{x+y} = 0. \tag{14.19}$$

Then for $w = 4\pi^{-1}\ \max(x, z-1, 1/\varepsilon^*, \ln 1/\delta^*)$, we have

$$\frac{1}{x+y+z} \le \frac{C}{w+y} \quad \text{with } C := 4\,\pi^{-1}\ \max(1/\varepsilon^*, \ln 1/\delta^*) \ge 1,$$

and $w \ge 4\,\pi^{-1}\ \max(x, 1/\varepsilon^*)$ and

$$w \ge 4\,\pi^{-1}\max(z-1,\ln 1/\delta^*) \ge \max(z-1,\ln 1/\delta^*),$$

so that

$$\exp(w) \ge \max\left(\exp(z-1), 1/\delta^*\right).$$

Then

$$\lim_{\substack{x,y,z\ge 1\\ x+y+z\to\infty}} \frac{\ln T^{\text{prob-rel}}(x,y,z)}{x+y+z} \le C \lim_{\substack{w,y\ge 1\\ w+y\to\infty}} \frac{\ln T^{\text{wor-nor}}(w\ \exp(w),y)}{w+y}.$$

Due to (14.19), the right-hand side goes to zero, as $w+y$ goes to infinity. Hence, $T^{\text{prob-rel}}$ is a tractability function. Furthermore,

$$\begin{aligned}
n^{\text{prob-rel}}(\varepsilon,\delta,I_d) &\le n^{\text{wor-nor}}\left(\tfrac{1}{4}\,\pi\ \min(\varepsilon,\varepsilon^*)\ \min(\delta,\delta^*), I_d^{\text{res}}\right)\\
&\le C\ T^{\text{wor-nor}}\left(\tfrac{4}{\pi}\ \max(\varepsilon^{-1},1/\varepsilon^*)\ \max(\delta^{-1},1/\delta^*),d_s\right)^t\\
&= C\ T^{\text{prob-rel}}(\varepsilon^{-1},d_s,1+\ln\delta^{-1})^t.
\end{aligned}$$

Hence, we have (strong) $T^{\text{prob-rel}}$-log-tractability of $I^{\text{prob-rel}}$ with the same exponent t.

We summarize this analysis in the following theorem.

Theorem 14.2. *Consider $I^{\text{prob-rel}} = \{I_d\}$ in the probabilistic setting for the relative error criterion, and $I^{\text{wor}} = \{I_d^{\text{res}}\}$ in the worst case setting for the normalized error criterion, defined as in this chapter. Then*

- *The probabilistic setting with the relative error for $I^{\text{prob-rel}}$ is closely related to the worst case setting with the normalized error criterion for $I^{\text{wor-nor}}$.*

  For small ε, we have

$$n^{\text{prob-rel}}(\varepsilon,\delta,I_d) = n^{\text{wor-nor}}\big(\varepsilon\ \tan\left(\tfrac{1}{2}\delta\,\pi\,(1+o(1))\right), I_d^{\text{res}}\big).$$

For small ε and δ, we have

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = n^{\text{wor-nor}}\big(\tfrac{1}{2}\,\pi\,\varepsilon\,\delta\,(1+o(1)), I_d^{\text{res}}\big),$$

showing that the parameter δ is as important as the parameter ε.

- *Polynomial tractability of $I^{\text{wor-nor}}$ implies polynomial tractability of $I^{\text{prob-rel}}$ with the same exponents of p and q, and with the exponent $s = p$. Hence, strong polynomial tractability of $I^{\text{wor-nor}}$ implies strong polynomial tractability of $I^{\text{prob-rel}}$ with respect to d. However, if $p > 0$ then poly-log tractability of $T^{\text{prob-rel}}$ does* not *hold.*
- *Weak tractability of $I^{\text{wor-nor}}$ and* (14.13) *imply weak tractability of $I^{\text{prob-rel}}$.*
- *Weak tractability of $I^{\text{wor-nor}}$ and* (14.14) *imply weak-log tractability of $I^{\text{wor-nor}}$.*
- (*Strong*) *$T^{\text{wor-nor}}$-tractability of $I^{\text{wor-nor}}$ with the function $T^{\text{wor-nor}}$ satisfying* (14.17) *implies* (*strong*) *$T^{\text{prob-nor}}$-tractability of $I^{\text{prob-rel}}$ for T^{prob} defined by* (14.16).
- (*Strong*) *$T^{\text{wor-nor}}$-log-tractability of $I^{\text{wor-nor}}$ with the function $T^{\text{prob-rel}}$ satisfying* (14.19) *implies* (*strong*) *$T^{\text{prob-rel}}$-tractability of $I^{\text{prob-rel}}$ for $T^{\text{prob-rel}}$ defined by* (14.18).

14.5 Notes and Remarks

NR 14.1:1. The relations between the probabilistic and average case setting hold not only for linear functionals but also for linear operators, see Chapter 8 of [305] which is based on Lee and Wasilkowski [169], Wasilkowski [325], [326] and [351]. The relations for linear functionals and the relative error criterion are from [138].

NR 14.2:1. The definition of poly-log tractability in the probabilistic setting for the absolute error criterion was given in [348]. It is also shown in [348] that for linear operators and for classes Λ^{all} and Λ^{std}, tractability in the probabilistic setting is equivalent to tractability in the average case setting with a zero mean Gaussian measure in both settings. These relations for linear operators will be used in Volume III.

As already mentioned in Volume I, there are two interesting papers by Lifshits and Tulyakova [174] and Lifshits and Zani [175] with negative results for multivariate approximation. For Gaussian integration, Examples 3.2.5 and 3.2.6 of Chapter 3 in Volume I are probably the first examples of linear functionals with tractability results in the probabilistic setting for the normalized error criterion.

The concepts of polynomial tractability, weak tractability and T-log tractability are formally new. However, they are a direct analog of the corresponding concepts in the previous settings.

Chapter 15
Smolyak/Sparse Grid Algorithms

15.1 Introduction

In 1963, Smolyak published a four page long paper [283] (in Russian), where he outlined an algorithm for approximating linear tensor product problems. Today this algorithm is called the *Smolyak algorithm*. Antecedents of his idea may be found earlier in the work of Babenko [3]. There are literally hundreds of papers with important modifications of this algorithm and they go under different names such as *blending* algorithms, *Boolean* algorithms, *hyperbolic cross point* algorithms, or *sparse grid* algorithms. The latter name is probably most popular and well explains the essence of the basic algorithm. That is why we decided to name this chapter as "Smolyak/Sparse Grid Algorithms". A partial list of papers dealing with Smolayk/sparse grid algorithms is given in NR 15.1.

The Smolyak/sparse grid algorithms are used for linear tensor product problems. Their essence is that it is enough to know how to solve such problems for the univariate case[1] $d = 1$. Then the algorithms for arbitrary d are fully determined in terms of tensor products of the algorithms for $d = 1$. In general, the univariate algorithms may use function values or arbitrary linear functionals. Furthermore, the univariate algorithms do not have to be optimal, although a poor choice of these algorithms makes the error of the algorithm larger for arbitrary d.

Since this volume is primarily devoted to approximation of linear functionals, we restrict our presentation of the Smolyak algorithm only to linear tensor products of univariate linear functionals defined over reproducing kernel Hilbert spaces. In view of the results for the average case and probabilistic settings in Chapters 13 and 14, it is enough to only consider the worst case setting. In Volume III, we will revisit the Smolyak algorithm for general linear tensor product operators in different settings.

In our case, univariate algorithms use only function values. Then the Smolyak and sparse grid algorithms also use function values in the d-variate case chosen at the so called *hyperbolic cross* sample points. The geometry of such points resembles a sparse grid and this property was used as the motivation of the name: "*sparse grid* algorithms". Such information has been initiated by Babenko [3], who studied approximation of periodic functions by polynomials that use Fourier coefficients whose indices are from a hyperbolic cross. There are many papers studying the power of hyperbolic cross information for a number of problems in different settings, see the papers cited in NR 15.1.

[1] It is also possible to start the construction of the Smolyak or sparse grid algorithms by using algorithms for m-variate problems and obtain algorithms for dm-variate problems. Here m can be an arbitrary positive integer. For simplicity, we restrict ourselves in this chapter to $m = 1$, although it is quite straightforward how to generalize the construction for arbitrary m.

We will begin with the unweighted case, as was originally done by Smolyak [283]. Our emphasis will be on the explicit dependence on d as in [329]. This will allow us to analyze tractability of unweighted linear tensor product functionals. Suppose we consider a non-trivial linear functional for $d = 1$, i.e., a linear functional that cannot be solved exactly by using one function value. Then strong polynomial tractability holds for the absolute error criterion if the norm of the univariate linear functional is less than one and the nth univariate error goes polynomially fast (in n^{-1}) to zero. This result sounds better than it really is. The reason is simple. If the norm for the univariate case is $\beta \in (0, 1)$, then the norm for the d-variate case is β^d, and is exponentially *small* in d. This means that we are approximating a linear functional with an exponentially small norm. For large d, it seems to us that for most ε of practical importance we will have $\varepsilon \geq \beta^d$, and then the (trivial) zero algorithm will do the job. On the other hand, if $\beta > 1$, then the norm for the d-variate case is exponentially *large* in d. From Theorem 11.7 of Chapter 11 we know that the problem is then intractable (still for the absolute error criterion) and suffers from the curse of dimensionality if the nth minimal worst case error behaves polynomially. Hence, it seems that only the case $\beta = 1$ is of interest. As we already know from the previous chapters, tractability for $\beta = 1$ depends on the univariate linear functional and the underlying space and anything can happen.

Then we study the weighted case for arbitrary weights. Similarly to [332], we analyze a class of algorithms that are called *weighted Smolyak* or *weighted tensor product* algorithms, or shortly, WTP algorithms. They are a generalization of the Smolyak and sparse grid algorithms for weighted linear tensor product problems. In this chapter, we analyze WTP algorithms for linear functionals, leaving the case of weighted linear tensor product operators to Volume III. The class of WTP algorithms depends on several parameters characterizing weights, spaces and linear functionals. The values of these parameters determine the efficiency of WTP algorithms. In Section 15.3.1 we show that the WTP algorithm is a weighted sum of the Smolyak algorithm, and we obtain an explicit form as well as explicit error and cost bounds of the WTP algorithm.

We then address when the WTP algorithms yield tractability bounds for both the absolute and normalized error criterion. We restrict ourselves to finite-order and product weights, leaving the case of general weights as an open problem to the reader.

For finite-order weights we propose two classes of WTP algorithms. The first class depends on all the finite-order weights, whereas the second class is much less dependent on the finite-order weights. In particular, for the normalized error criterion, it is enough to know only the order ω of finite-order weights to define the second class of WTP algorithms. The main result for finite-order weights is that we have polynomial tractability for the normalized error criterion if we only assume that the univariate case can be solved with polynomial cost. Under some conditions on the finite-order weights, even strong polynomial tractability can be achieved, although only for the first class of WTP algorithms. This means that the price of a more lenient dependence on finite-order weights is the lack of strong polynomial tractability bounds for the second class of WTP algorithms. We illustrate finite-order weights for perturbed Coulomb potentials, and finish this subsection by several open problems related to finite-order weights.

We then turn our attention to product weights, which were studied in [332]. We present conditions on product weights for which the WTP algorithms yield tractability bounds. In particular, we present bounds on the exponent of strong polynomial tractability. Usually these bounds are not sharp. It would be desirable to improve the choice of parameters defining the WTP algorithms such that the bounds on the exponent are smaller. As always, we illustrate the results for several examples including uniform integration. We also discuss how to choose product weights for Cobb Douglas functions that occur in economics.

For both finite-order and product weights we briefly discuss the robustness of WTP algorithms. In particular, we study what happens if we use a function that does not belong to a space equipped with finite-order weights or if we use wrong product weights. As originally proved in [335], we show a remarkable property of the WTP algorithm for finite-order weights. Namely, the WTP algorithm approximates the part of a function outside the space equipped with finite-order weights by zero, and approximates the part of a function for which it has been designed. Hence, if the part of a function outside the space is small or negligible, the WTP algorithms still provide good approximations.

As in all chapters, we propose several open problems. In this chapter we have six open problems numbered 67 through 72.

15.2 Unweighted Case: Algorithms

As in Section 11.2 of Chapter 11, we consider linear tensor product functionals. We briefly recall their definition. For $d = 1$, let F_1 be a reproducing kernel Hilbert space of real functions defined on $D_1 \subseteq \mathbb{R}$. The reproducing kernel of F_1 is denoted by K_1. The (continuous) linear functional I_1 is of the form

$$I_1(f) = \langle f, h_1 \rangle_{F_1} \quad \text{for all } f \in F_1,$$

and for some non-zero function h_1 from F_1. For $d > 1$, we take

$$F_d = F_1 \otimes F_1 \otimes \cdots \otimes F_1 \quad \text{and} \quad I_d = I_1 \otimes I_1 \otimes \cdots \otimes I_1$$

as the d-fold tensor product of F_1 and I_1. The class F_d is a reproducing kernel Hilbert space of real functions defined on $D_d = D_1^d$ whose kernel is

$$K_d(x,t) = \prod_{j=1}^{d} K_1(x_j, t_j) \quad \text{for all } x, t \in D_d.$$

The (continuous) linear functional I_d is of the form

$$I_d(f) = \langle f, h_d \rangle_{F_d} \quad \text{for all } f \in F_d,$$

where

$$h_d(t) = \prod_{j=1}^{d} h_1(t_j) \quad \text{for all } t \in D_d.$$

We consider approximation of I_d in the worst case setting for the absolute and normalized error criteria. As we know, the initial error is

$$e(0,d) = \|I_d\| = \|h_d\|_{F_d} = \|h_1\|_{F_1}^d.$$

We are ready to define the Smolyak/sparse grid algorithms. As already mentioned, these algorithms for $d > 1$ are given by certain combinations of tensor products of univariate algorithms. That is why we need first to discuss the univariate case $d = 1$.

We assume therefore that we know linear algorithms U_i, $i \geq 1$, that approximate the linear functional I_1 such that $\lim_{i\to\infty} \|I_1 - U_i\| = 0$. We denote

$$\Delta_0 = U_0 = 0, \quad \Delta_i = U_i - U_{i-1} \quad \text{for all } i \in \mathbb{N}.$$

Clearly,

$$I_1(f) = \sum_{i=1}^{\infty} \Delta_i(f) \quad \text{for all } f \in F_1.$$

For $d > 1$, we take a non-negative integer q and approximate the linear tensor product functionals I_d by the algorithm $A(q,d)\colon F_d \to \mathbb{R}$ given by

$$A(q,d) = \sum_{0 \leq i_1 + i_2 + \cdots + i_d \leq q} \Delta_{i_1} \otimes \cdots \otimes \Delta_{i_d}. \tag{15.1}$$

To get familiar with the formula (15.1), take $d = 1$. Then

$$A(q,1) = \sum_{i_1=0}^{q} \Delta_{i_1} = U_q.$$

For $d \geq 1$, take f of the form

$$f(t) = f_1(t_1)\, f_2(t_2) \cdots f_d(t_d) \quad \text{for some } f_i \in F_1 \text{ and all } t \in D_d.$$

Then we have

$$(A(q,d)\, f)(t) = \sum_{0 \leq i_1 + i_2 + \cdots + i_d \leq q} \big(\Delta_{i_1} f_1\big)(t_1) \big(\Delta_{i_2} f_2\big)(t_2) \cdots \big(\Delta_{i_d} f_d\big)(t_d).$$

Observe that for $q < d$, all terms in (15.1) are zero. Indeed, $0 \leq i_1 + i_2 + \cdots + i_d \leq q$ implies that one of the indices is zero, say $i_j = 0$, and $\Delta_{i_j} = 0$. Hence, $q < d$ implies that $A(q,d) = 0$. Therefore the only interesting case is when $q \geq d$ and only then we can take all $i_j \geq 1$.

The algorithm $A(q,d)$ given by (15.1) is the celebrated *Smolyak algorithm*[2] already presented by Smolyak [283]. It is indeed defined entirely by tensor products of the differences $U_{i_j} - U_{i_j-1}$ of univariate algorithms.

[2] If I_1 is a linear operator, and I_d the d-fold tensor product of I_1, the form of $A(q,d)$ is the same, with U_i now approximating a linear operator I_1.

In his original paper, Smolyak proved the following error estimate. Assume that there exists a positive α such that

$$\|I - U_j\| = \mathcal{O}\big(2^{-j\,\alpha}\big) \quad \text{for all } j \in \mathbb{N}.$$

Then for all d we have

$$\|I_d - A(q,d)\| = \mathcal{O}\Big(\,[\ln 2^{\alpha\, q}]^{d-1}\ \ 2^{-q\,\alpha}\Big) \quad \text{for all } q \in \mathbb{N}.$$

Here, the factor in the big $\mathcal{O}$ notation depends on all parameters of the problem, including d, but it is independent of q.

The essence of this error bound for $d > 1$ is that modulo a power of the logarithm of $2^{\alpha\, q}$ we have the same rate of convergence as for $d = 1$. For a relatively small d, this is an excellent result and that is why this algorithm is so popular for approximating linear tensor product problems.

However, for large d and tractability analysis, we cannot ignore the power of the logarithm as well as we must know how the factor in the big $\mathcal{O}$ notation in the error bounds depends on d. This indicates the need to track down the dependence on d. This painful job was done in [329].

It is also not yet clear what is the cost of the algorithm $A(q,d)$. Obviously, we must assume that the univariate algorithms U_i use function values of f_1. Let n_i be the (minimal) number of function values used by U_i for which $\|I_1 - U_i\| = \mathcal{O}(2^{-i\,\alpha})$. Then we need to find out how many function values of f_d are used by $A(q,d)$, and relate the cost of $A(q,d)$ to its error bound.

In his original paper, Smolyak showed for a number of standard spaces and linear operators that the error for the d-variate case is of order $n^{-r}[\ln n]^{(d-1)\beta}$ if we use n function values. Here, n^{-r} measures the univariate error when we use n function values, and β is some positive number independent of n and d. Hence, modulo powers of logarithm, we have the same rate of convergence for all d. We stress again that this property made the Smolyak algorithm famous.

Back to tractability of $I = \{I_d\}$. We will be able to verify whether we obtain tractability by the use of the algorithm $A(q,d)$ if we know:

- an explicit form of $A(q,d)$,
- an explicit error bound of $A(q,d)$,
- an explicit cost bound of $A(q,d)$.

These points will be addressed in the subsequent subsections based on [329].

15.2.1 Explicit Form

We want to find an explicit form of the algorithm $A(q,d)$ in terms of algorithms U_i. We will use the following notation.

For $\vec{i} = [i_1, i_2, \dots, i_d]$ with non-negative integers i_j, let $|\vec{i}| = \sum_{k=1}^{d} i_k$. We write $\vec{i} \ge \vec{j}$ if $i_k \ge j_k$ for all k. Furthermore, by $Q(q,d)$ we mean the set

$$Q(q,d) = \big\{\vec{i} = [i_1, i_2, \dots, i_d] \mid \vec{1} \le \vec{i},\ |\vec{i}\,| \le q\big\}, \quad \text{with } \vec{1} = [1, 1, \dots, 1].$$

The cardinality of the set $Q(q,d)$ is equal to $\binom{q}{d}$. We have

$$\begin{aligned} A(q,d) &= \sum_{\vec{i}\in Q(q,d)} \bigotimes_{k=1}^{d} \Delta_{i_k} = \sum_{\vec{i}\in Q(q-1,d-1)} \Big(\bigotimes_{k=1}^{d-1} \Delta_{i_k}\Big) \otimes \sum_{i_d=1}^{q-|\vec{i}\,|} \Delta_{i_d} \\ &= \sum_{\vec{i}\in Q(q-1,d-1)} \Big(\bigotimes_{k=1}^{d-1} \Delta_{i_k}\Big) \otimes U_{q-|\vec{i}\,|}, \end{aligned} \tag{15.2}$$

since $\sum_{i=1}^{m} \Delta_i = U_m$ for any $m \geq 1$. Observe that

$$\bigotimes_{k=1}^{d} \big(U_{i_k} - U_{i_k-1}\big) = \sum_{\vec{\alpha}\in\{0,1\}^d} (-1)^{|\vec{\alpha}\,|} \bigotimes_{k=1}^{d} U_{i_k-\alpha_k}.$$

Note that $\bigotimes_{k=1}^{d} U_{j_k}$ appears in $A(q,d)$ for all indices $\vec{i}$ for which $i_k = j_k + \alpha_k$ with $\vec{\alpha} \in \{0,1\}^d$ and $|\vec{\alpha}\,| \leq q - |\vec{j}\,|$. Furthermore, the sign of $\bigotimes_{k=1}^{d} U_{j_k}$ in this case is $(-1)^{|\vec{\alpha}\,|}$. Let

$$b(i,d) = \sum_{\substack{\vec{\alpha}\in\{0,1\}^d \\ |\vec{\alpha}\,|\leq i}} (-1)^{|\vec{\alpha}\,|}.$$

This and (15.2) yield

$$A(q,d) = \sum_{\vec{j}\in Q(q,d)} b(q-|\vec{j}\,|,d) \bigotimes_{k=1}^{d} U_{j_k}.$$

We now compute $b(i,d)$. Clearly, we can sum up with respect to $|\vec{\alpha}\,| = 0,1,\ldots,d$. Since $|\vec{\alpha}\,| = j$ corresponds to $\binom{d}{j}$ terms, we have

$$b(i,d) = \sum_{j=0}^{\min\{i,d\}} \binom{d}{j} (-1)^j = (-1)^i \binom{d-1}{i}.$$

In particular, $b(i,d) = 0$ for $i \geq d$. This yields an explicit form of $A(q,d)$ which is summarized in the following lemma.

Lemma 15.1.

$$A(q,d) = \sum_{\vec{i}\in P(q,d)} (-1)^{q-|\vec{i}\,|} \binom{d-1}{q-|\vec{i}\,|} \bigotimes_{k=1}^{d} U_{i_k},$$

where

$$P(q,d) = \big\{\vec{i} \mid \vec{1} \leq \vec{i},\ q-d+1 \leq |\vec{i}\,| \leq q\big\}.$$

In particular, for

$$U_i(f) = \sum_{j=1}^{m_i} a_{i,j}\, f(x_{i,j}) \quad \textit{for all } f \in F_1 \tag{15.3}$$

with $a_{i,j} \in \mathbb{R}$ *and* $x_{i,j} \in D_1$, *we have*

$$A(q,d)(f) = \sum_{\vec{i} \in P(q,d)} (-1)^{q-|\vec{i}\,|} \binom{d-1}{q-|\vec{i}\,|} \sum_{\vec{1} \le \vec{j} \le \vec{m}_{\vec{i}}} a_{\vec{i},\vec{j}}\, f(x_{i_1,j_1}, \dots, x_{i_d,j_d}), \tag{15.4}$$

where $a_{\vec{i},\vec{j}} = \prod_{k=1}^{d} a_{i_k,j_k}$ *and* $\vec{m}_{\vec{i}} = [m_{i_1}, \dots, m_{i_d}]$. □

We now comment on the information

$$N_{q,d}(f) = \{ f(x_{i_1,j_1}, \dots, x_{i_d,j_d}) \mid \vec{i} \in P(q,d),\ \vec{j} \le \vec{m}_{\vec{i}} \}$$

used by the algorithm $A(q,d)$.

Assume that $m_i \le M^i$ for some M. Then $\vec{j} \le \vec{m}_{\vec{i}}$ yields

$$j_1 j_2 \cdots j_d \le m_{i_1} m_{i_2} \cdots m_{i_d} \le M^{|\vec{i}|} \le M^q. \tag{15.5}$$

Hence, the index $\vec{j}$ satisfies the so called *hyperbolic* inequality (15.5). However, not every index $\vec{j}$ satisfying (15.5) must correspond to some $(x_{i_1,j_1}, \dots, x_{i_d,j_d})$. For example, take $d = M = 2$ and $m_i = M^i$ with $q = 5$. Then $\vec{j} = [5,5]$ is a counter-example.

Thus, the indices of the functionals of the information $N_{q,d}$ are a subset of the solution of the hyperbolic inequality (15.5). The solution set of (15.5) forms part of a hyperbolic cross. This explains why the information $N_{q,d}$ is called *hyperbolic cross* information.

We stress that coefficients in (15.4) are of different signs even if we use algorithms U_i with, say, positive coefficients $a_{i,j}$. The use of coefficients with different signs may result in a lack of numerical stability. We return to this point in Subsection 15.2.9, where we discuss a couple of implementation issues of the Smolyak algorithm. Following [214] we show that for a relatively small d, the sum of the absolute values of all coefficients is not large, and in this case numerical stability is not in jeopardy.

15.2.2 Explicit Error Bound

We analyze the worst case error of the algorithm $A(q,d)$ in terms of its error for $d = 1$. The following assumptions will be used for $d = 1$:

$$\|I_1\| \le B, \tag{15.6}$$

$$\|I_1 - U_i\| \le C\, D^i \quad \text{for all } i \ge 0, \tag{15.7}$$

$$\|\Delta_i\| = \|U_i - U_{i-1}\| \le E\, D^i \quad \text{for all } i \ge 1. \tag{15.8}$$

Here, the number B bounds the norm of the operator I_1, and therefore we can take $B = \|I_1\| = \|h_1\|_{F_1}$. Since $I_1 \neq 0$, we have $B > 0$. The numbers C, D, and E describe how well U_i approximates I_1. Of course, only $D < 1$ is of interest.

For $i = 0$ in (15.7), we get $\|I_1\| \le C$. Therefore we can assume that $B \le C$. Similarly, letting $i = 1$ in (15.8), we get

$$\|I_1\| \le \|I_1 - U_1\| + \|U_1\| \le D\,(C + E\,).$$

Hence, we can assume that $B \le D(C + E)$. By the same argument,

$$C(D^{-1} - 1) \le E \le C(D^{-1} + 1).$$

Since $B > 0$, then C, E, and D have to be positive.

We stress that, in general, we do not assume any optimality properties of linear algorithms U_i for $d = 1$. We also do not assume any relation between the information used by successive U_i. In Subsection 15.2.2.1, we derive an upper bound on the error of $A(q,d)$ in this general case. In Subsection 15.2.2.2, we assume "nested" information and optimality of U_i. Under these assumptions we improve the error bounds of $A(q,d)$ and also conclude optimality of $A(q,d)$.

15.2.2.1 Non-nested information. Recall that $e(A(q,d)) = \|I_d - A(q,d)\|$ is the worst case error of the algorithm $A(q,d)$ in the d dimensional case. Similarly, $e(U_i) = \|I_1 - U_i\|$ is the worst case error of U_i for the univariate case.

For $q < d$, we have $A(q,d) = 0$ and $e(A(q,d)) = \|I_d\| \le B^d$. For $q \ge d$, we present the following estimates.

Lemma 15.2. *If* (15.6), (15.7), *and* (15.8) *hold then for* $q \ge d$ *we have*

$$\begin{aligned} e(A(q,d)) &\le C\,B^{d-1}\,D^{q-d+1}\sum_{j=0}^{d-1}\left(\frac{E\,D}{B}\right)^j\binom{q-d+j}{j} \\ &\le CH^{d-1}D^q\binom{q}{d-1}, \end{aligned} \tag{15.9}$$

with $H = \max(B/D, E)$.

Proof. For $d = 1$, Lemma 15.2 coincides with (15.7). Assume by induction that Lemma 15.2 holds for d. Due to (15.2), we have

$$\begin{aligned} &I_{d+1} - A(q+1, d+1) \\ &\quad = I_{d+1} - \sum_{\vec{i}\in Q(q,d)}\left(\bigotimes_{k=1}^{d}\Delta_{i_k}\right)\otimes U_{q+1-|\vec{i}|} \end{aligned}$$

$$= I_{d+1} + \sum_{\vec{i} \in Q(q,d)} \Big(\bigotimes_{k=1}^{d} \Delta_{i_k} \Big) \otimes \big(I_1 - U_{q+1-|\vec{i}|} \big) - A(q,d) \otimes I_1$$

$$= \sum_{\vec{i} \in Q(q,d)} \Big(\bigotimes_{k=1}^{d} \Delta_{i_k} \Big) \otimes \big(I_1 - U_{q+1-|\vec{i}|} \big) + \big(I_d - A(q,d) \big) \otimes I_1.$$

Hence, due to (15.6), (15.7), and (15.8), we have

$$e(A(q+1,d+1)) \le \sum_{\vec{i} \in Q(q,d)} E^d D^{|\vec{i}|} C D^{q+1-|\vec{i}|} + B e(A(q,d))$$
$$= C E^d D^{q+1} \binom{q}{d} + B e(A(q,d)).$$

This and the inductive assumption complete the proof of the first inequality. Estimating $(ED/B)^j$ by $\max^{d-1}(1, ED/B)$ and using the fact that

$$\sum_{j=0}^{d-1} \binom{q-d+j}{j} = \binom{q}{d-1},$$

we obtain the second inequality. □

15.2.2.2 Nested information and optimal algorithms. In this subsection we assume that the algorithms

$$U_i(f) = \sum_{j=1}^{m_i} a_{i,j} f(x_{i,j})$$

use *nested* information $N_i(f) = \big[f(x_{i,1}), f(x_{i,2}), \dots, f(x_{i,m_i}) \big]$ for $f \in F_1$. That is,

$$\{x_{i,1}, x_{i,2}, \dots, x_{i,m_i}\} \subseteq \{x_{i+1,1}, x_{i+1,2}, \dots, x_{i+1,m_{i+1}}\} \quad \text{for all } i \in \mathbb{N}.$$

Nested information means that there exists a sequence $\{x_i\}$, with $x_i \in D_1$, such that

$$N_i(f) = [f(x_1), f(x_2), \dots, f(x_{m_i})] \quad \text{for all } f \in F_1,\ i \in \mathbb{N}. \tag{15.10}$$

We also assume that the algorithms U_i are optimal, i.e., that they minimize the worst case error among all algorithms that use the information N_i. It is well known, see also Chapter 4 of Volume I, that U_i is optimal if

$$U_i = I_1 \mathcal{P}_i, \tag{15.11}$$

where $\mathcal{P}_i$ is the orthogonal projection on the linear subspace

$$\operatorname{span}\{K_1(\cdot, x_j) \mid j = 1, 2, \dots, m_i\} = (\ker N_i)^{\perp}.$$

We show that (15.11) implies optimality of the algorithm $A(q,d)$ for any d.

Lemma 15.3. *For nested information N_i of the form* (15.10) *and optimal U_i of the form* (15.11)*, we have*

$$A(q,d) = I_d \mathcal{P}(q,d),$$

where $\mathcal{P}(q,d)$ is the orthogonal projection on the linear subspace $(\ker(N_{q,d}))^{\perp}$*. Thus, in particular, $A(q,d)$ minimizes the error among all algorithms that use the same information $N_{q,d}$.*

Proof. From (15.1) and (15.11),

$$A(q,d) = I_d R(q,d) \quad \text{for } R(q,d) = \sum_{\vec{i} \in Q(q,d)} \bigotimes_{k=1}^{d} R_{i_k} \text{ with } R_{i_k} = \mathcal{P}_{i_k} - \mathcal{P}_{i_k - 1}.$$

Let $h_i \in F_1$ be such that $h_i(x_j) = \delta_{j,i}$. Then $R_\ell h_{j_k} = \delta_{\ell, j_k} h_{j_k}$. Hence, on the subspace spanned by the functions

$$h(t_1, t_2, \dots, t_d) = \prod_{k=1}^{d} h_{j_k}(t_k) \quad \text{with } \vec{j} \le \vec{m}_i \text{ and } \vec{i} \in Q(q,d),$$

the operator $R(q,d)$ is the identity. Moreover, $R(q,d) = 0$ for the orthogonal complement of this subspace. This proves that $R(q,d) = \mathcal{P}(q,d)$. As for $d = 1$, it is known that this form of $A(q,d)$ yields the minimal worst case error. □

We add in passing that the projection form of the algorithm $A(q,d)$ implies additional error properties. In particular, $A(q,d)$ minimizes all local errors, see, e.g., [187], [305], [307]. Such algorithms are sometimes referred to as being *central* or *strongly optimal*, see also Chapter 4 of Volume I.

We now estimate the error of $A(q,d)$ using the property that the N_i are nested and U_i optimal. Consider first $d = 1$. Then

$$U_i f = I_1 \mathcal{P}_i f = \langle \mathcal{P}_i f, h_1 \rangle_{F_1} = \langle f, \mathcal{P}_i h_1 \rangle_{F_1} \quad \text{for all } f \in F_1.$$

This implies that $I_1 f - U_i f = \langle f, (I - \mathcal{P}_i) h_1 \rangle_{F_1}$ with the identity $If = f$, and

$$e(U_i) = \|I_1 - U_i\| = \|(I - \mathcal{P}_i) h_1\|_{F_1} \quad \text{for all } i \in \mathbb{N}.$$

Furthermore

$$(U_i - U_{i-1}) f = \langle f, (\mathcal{P}_i - \mathcal{P}_{i-1}) h_1 \rangle_{F_1} \quad \text{for all } f \in F_1.$$

Therefore

$$\begin{aligned}
\|U_i - U_{i-1}\|^2 &= \|\mathcal{P}_i h_1 - \mathcal{P}_{i-1} h_1\|_{F_1}^2 = \|(I - \mathcal{P}_{i-1}) h_1 - (I - \mathcal{P}_i) h_1\|_{F_1}^2 \\
&= \|(I - \mathcal{P}_{i-1}) h_1\|_{F_1}^2 - 2 \langle (I - \mathcal{P}_i) h_1, (I - \mathcal{P}_{i-1}) h_1 \rangle_{F_1} \\
&\qquad + \|(I - \mathcal{P} i) h_1\|_{F_1}^2 = \|(I - \mathcal{P}_{i-1}) h_1\|_{F_1}^2 - \|(I - \mathcal{P}_i) h_1\|_{F_1}^2 \\
&= e^2(U_{i-1}) - e^2(U_i).
\end{aligned}$$

For $d \geq 2$, we have $I_d(f) - A(q,d)(f) = \langle f, h_d - \mathcal{P}(q,d)h_d \rangle_{F_d}$, and

$$\begin{aligned} e^2(A(q,d)) &= \|h_d - \mathcal{P}(q,d)h_d\|_{F_d}^2 \\ &= \|h_d\|_{F_d}^2 - 2\,\langle h_d, \mathcal{P}(q,d)h_d\rangle_{F_d} + \|\mathcal{P}(q,d)h_d\|_{F_d}^2 \\ &= \|h_d\|_{F_d}^2 - \|\mathcal{P}(q,d)h_d\|^2 F_d, \end{aligned}$$

where

$$\mathcal{P}(q,d)h_d = \sum_{\vec{i}\in Q(q,d)} \bigotimes_{k=1}^{d} (\mathcal{P}_{i_k} - \mathcal{P}_{i_k-1})h_1.$$

Note that $\left\langle (\mathcal{P}_{i_k} - \mathcal{P}_{i_k-1})h_1, (\mathcal{P}_{j_k} - \mathcal{P}_{j_k-1})h_1 \right\rangle_{F_1} = 0$ for $i_k \neq j_k$. Therefore the elements $\bigotimes_{k=1}^{d} (\mathcal{P}_{i_k} - \mathcal{P}_{i_k-1})h_1$ are orthogonal for different $\vec{i}$'s. Therefore

$$\begin{aligned} \|\mathcal{P}(q,d)h_d\|_{F_d}^2 &= \sum_{\vec{i}\in Q(q,d)} \prod_{k=1}^{d} \|(\mathcal{P}_{i_k} - \mathcal{P}_{i_k-1})h_1\|_{F_1}^2 \\ &= \sum_{\vec{i}\in Q(q,d)} \prod_{k=1}^{d} \left[e^2(U_{i_k-1}) - e^2(U_{i_k})\right]. \end{aligned}$$

This yields

$$e^2(A(q,d)) = \|h_1\|_{F_1}^{2d} - \sum_{\vec{i}\in Q(q,d)} \prod_{k=1}^{d} \left[e^2(U_{i_k-1}) - e^2(U_{i_k})\right]. \tag{15.12}$$

Note that

$$\begin{aligned} &\sum_{\vec{i}\in Q(q,d)} \prod_{k=1}^{d} \left[e^2(U_{i_k-1}) - e^2(U_{i_k})\right] \\ &= \sum_{\vec{i}\in Q(q-1,d-1)} \left[e^2(U_0) - e^2(U_{q-|\vec{i}|})\right] \prod_{k=1}^{d-1} \left[e^2(U_{i_k-1}) - e^2(U_{i_k})\right] \\ &= \|h_1\|_{H_1}^2 \left(\|h_1\|_{F_1}^{2(d-1)} - e^2(A(q-1,d-1)\right) \\ &\qquad - \sum_{\vec{i}\in Q(q-1,d-1)} e^2(U_{q-|\vec{i}|}) \prod_{k=1}^{d-1} \left[e^2(U_{i_k-1}) - e^2(U_{i_k})\right]. \end{aligned}$$

This gives the formula

$$\begin{aligned} e^2(A(q,d)) &= \|h_1\|_{F_1}^2\, e^2(A(q-1,d-1)) \\ &\qquad + \sum_{\vec{i}\in Q(q-1,d-1)} e^2(U_{q-|\vec{i}|}) \prod_{k=1}^{d-1} \left(e^2(U_{i_k-1}) - e^2(U_{i_k})\right) \end{aligned} \tag{15.13}$$

for the error of $A(q,d)$.

Observe that $e^2(A(q,d))$ is a non-decreasing function of $e(U_i)$. Indeed, for $j = 1, 2, \ldots, d$ define vectors $x_j = (x_{0,j}, x_{1,j}, \ldots, x_{q-d+1,j}) \in [0,1)^{q-d+2}$, and $x = (x_1, x_2, \ldots, x_d)$, and consider the function

$$f_d(x) = \prod_{k=1}^{d} x_{0,k} - \sum_{\vec{i} \in Q(q,d)} \prod_{k=1}^{d} (x_{i_k-1,k} - x_{i_k,k}).$$

Obviously, for $x_{j,k} = e^2(U_j)$ we have from (15.12) that $f_d(x)$ is equal to $e^2(A(q,d))$. We now show that f_d is non-decreasing in each variable as long as $0 \le x_{j,k} \le x_{j-1,k}$ for all j, k holds. Take the pth component of the jth vector x_j, i.e. $x_{p,j}$. Then we can rewrite f_d as

$$f_d(x) = x_{0,j} \prod_{k \ne j} x_{0,k} + \sum_{\vec{i} \in Q(d-1,q-1)} \left[-x_{0,j} + x_{q-|\vec{i}|,j} \right] \prod_{k \ne j} (x_{i_k-1,k} - x_{i_k,k}).$$

Hence, the terms of the last sum depend on $x_{p,j}$ only when $|\vec{i}| = q - p$. This means that $g(x_{p,j}) := f_d(x)$, with all fixed values of variables x except $x_{p,j}$, has the form

$$\begin{aligned} g(x_{p,j}) = x_{0,j} \prod_{k \ne j} x_{0,k} + \left(x_{p,j} - x_{0,j}\right) \sum_{\substack{\vec{i} \in Q(d-1,q-1) \\ |\vec{i}|=q-p}} \prod_{k \ne j} (x_{i_k-1,k} - x_{i_k,k}) \\ + \sum_{\substack{\vec{i} \in Q(d-1,q-1) \\ |\vec{i}| \ne q-p}} \left[-x_{0,j} + x_{q-|\vec{i}|,j} \right] \prod_{k \ne j} (x_{i_k-1,k} - x_{i_k,k}). \end{aligned}$$

Hence, g is a linear function. Note that $x_{p,j}$ is multiplied by a non-negative number and therefore g non-decreasing. This yields the desired property.

Due to the monotonicity property we estimate the error of $A(q,d)$ given by (15.13) using (15.6) and (15.7) and the inequality $B \le C$,

$$\begin{aligned} e^2(A(q,d)) &\le C^2 e^2(A(q-1,d-1)) \\ &\quad + C^{2d}(1-D^2)^{d-1} D^{2(q-d+1)} \sum_{\vec{i} \in Q(q-1,d-1)} 1 \\ &= C^2 e^2(A(q-1,d-1)) + C^{2d}(1-D^2)^{d-1} D^{2(q-d+1)} \binom{q-1}{d-1}. \end{aligned}$$

Therefore,

$$e^2(A(q,d)) \le C^{2d} D^{2(q-d+1)} s(x), \tag{15.14}$$

where

$$s(x) := \sum_{i=0}^{d-1} (1-x)^{d-1-i} \binom{q-i-1}{d-i-1} \quad \text{with } x = D^2.$$

It is easy to check that for $q > (d - D^2)/(1 - D^2)$, the largest term in the last sum is for $i = 0$. Therefore, in this case, one can estimate the last sum by d times the term for $i = 0$.

We find a closed form for the function s. We have

$$s(x) = \sum_{i=0}^{d-1} \binom{q-i-1}{q-d} (1-x)^{d-1-i} = \sum_{j=0}^{d-1} \binom{q+j-d}{q-d} (1-x)^j.$$

Using the binomial formula for $(1-x)^j$ and changing the order of summation, we get

$$s(x) = \sum_{k=0}^{d-1} (-x)^k \sum_{j=k}^{d-1} \binom{j}{k} \binom{q+j-d}{q-d}.$$

Since the product of the two binomial coefficients in the second sum is equal to

$$\binom{q+k-d}{k} \binom{q-d+j}{q-d+k},$$

and since the sum with respect to j of the latter coefficient is $\binom{q}{d-1-k}$, we get

$$s(x) = \sum_{k=0}^{d-1} (-x)^k \binom{q+k-d}{k} \binom{q}{d-1-k}.$$

The product of the last two binomial coefficients is

$$\frac{q\binom{q-1}{d-1}\binom{d-1}{k}}{q+k-d+1}.$$

Hence,

$$s(x) = q\binom{q-1}{d-1} x^{-q+d-1} g(x), \text{ where } g(x) := \sum_{k=0}^{d-1} (-1)^k \frac{x^{k+q-d+1}}{k+q-d+1} \binom{d-1}{k}.$$

Obviously, $x^{k+q-d+1}/(k+q-d+1) = \int_0^x t^{k+q-d}\, \mathrm{d}t$. Therefore,

$$g(x) = \int_0^x t^{q-d} (1-t)^{d-1}\, \mathrm{d}t. \tag{15.15}$$

Clearly, $1-t \le 1$ and $g(x) \le x^{q-d+1}$. For small x the last estimate is sharp.

We summarize this analysis in the following lemma.

Lemma 15.4. *Let* (15.6) *and* (15.7) *hold. For nested information N_i of the form* (15.10) *and optimal U_i of the form* (15.11) *we have:*

- *For $q \ge d$,*

$$\begin{aligned} e(A(q,d)) &\le C^d \sqrt{q\binom{q-1}{d-1} \int_0^{D^2} t^{q-d}(1-t)^{d-1}\, \mathrm{d}t} \\ &\le C^d D^{q-d+1} \sqrt{\binom{q}{d-1}}. \end{aligned} \tag{15.16}$$

Moreover, if $\|S_1 - U_i\| = CD^i$ *for all* $i \geq 0$*, then the first inequality in* (15.16) *becomes an equality.*

- *For* $q \geq (d - D^2)/(1 - D^2)$*,*

$$e(A(q,d)) \leq \sqrt{d}\, C^d\, D^{q-d+1}\, (1 - D^2)^{(d-1)/2} \sqrt{\binom{q-1}{d-1}}. \tag{15.17}$$

15.2.3 Explicit Cost Bound

Just as we did for the error bounds, we estimate the cost of the algorithm $A(q,d)$ for arbitrary d by the cost of the algorithms U_i for $d = 1$. Let $m(q,d)$ be the number of function values used by the algorithm $U(q,d)$. Since $A(q,d)$ is a linear algorithm, its cost is roughly equal to the cost of computing $m(q,d)$ function values.

We now discuss the number m_i of function values used by the algorithms U_i for $d = 1$, see (15.3). In Section 15.2.2 we assume that the error of U_i is of order D^i with $D \in (0,1)$. Hence, we want to define m_i such that this error estimate holds. For many problems, the error depends on some power of the reciprocal of m_i,

$$\|I_1 - U_i\| = \mathcal{O}\big(m_i^{-p}\big)$$

for some positive p. Hence, to satisfy (15.7) we have to take

$$m_i = \mathcal{O}\big(D^{-i/p}\big).$$

Hence, m_i depends exponentially on i. More specifically, we assume that

$$m_i \leq M_0\big(M^i - 1\big) \tag{15.18}$$

for some numbers $M > 1$ and $M_0 > 0$. The minus 1 in the formula above is taken to simplify further estimates. Moreover, it makes the bound sharp for $i = 0$, since $U_0 = 0$ and $m_0 = 0$.

Of course, $m(q,d)$ depends on M_0 via M_0^d. Thus, to simplify notation assume for a moment that $M_0 = 1$. For non-nested information, (15.4) in Lemma 15.1 implies

$$m(q,d) \leq \sum_{\vec{i} \in P(q,d)} \prod_{k=1}^{d} M^{i_k} = \sum_{\vec{i} \in P(q,d)} M^{|\vec{i}\,|} = \sum_{i=\max\{d,q-d+1\}}^{q} M^i \binom{i-1}{d-1}.$$

To estimate the last sum, observe that $\binom{i-1}{d-1}$ is a non-decreasing function of i and so we can replace $\binom{i-1}{d-1}$ by $\binom{q-1}{d-1}$. From this we have

$$m(q,d) \leq \frac{M^{q+1} - M^{\max\{d,q-d+1\}}}{M-1} \binom{q-1}{d-1} \leq \frac{M}{M-1}\, M^q \binom{q-1}{d-1}.$$

We now analyze the case of nested information. For any $\vec{s} = [s_1, \ldots, s_d]$, let $U_{\vec{s}} = \bigotimes_{i=1}^{d} U_{s_i}$. Since the algorithm $A(q,d)$ is a combination of the $U_{\vec{s}}$'s, see

Lemma 15.1, and for $|\vec{s}| < q$, $U_{\vec{s}}$ uses information contained in another $U_{\vec{r}}$ with $\vec{s} \le \vec{r}$ and $|\vec{r}| = q$, we only need to consider the case $|\vec{s}| = q$.

For $|\vec{s}| = q$, let $\vec{s}\,' = [s_1, \dots, s_{d-1}]$. If $|\vec{s}\,'| = d-1$ then $s_d = q-d+1$ and hence $U_{\vec{s}}$ requires at most $(M-1)^{d-1}(M^{q-d+1}-1)$ functionals. For $|\vec{s}\,'| = p \ge d$, there are at most

$$(M^{q-p}-1)\prod_{i=1}^{d-1}(M^{s_i}-M^{s_i-1}) = (M^{q-p}-1)M^{p-d+1}(M-1)^{d-1}$$

function values used by $U_{\vec{s}}$ that are not used by any other $U_{\vec{v}}$ with $|\vec{v}| = q$. For a fixed p, there are $\binom{p-1}{d-2}$ of different $\vec{s}\,'$ with $|\vec{s}\,'| = p$. Since $p \le q-1$, the cardinality $m(q,d)$ is bounded by

$$\bar{m}(q,d) = (M-1)^{d-1}\sum_{p=d-1}^{q-1}\left(M^{q-d+1}-M^{p-d+1}\right)\binom{p-1}{d-2}.$$

Hence

$$\begin{aligned}\bar{m}(q,d)-\bar{m}(q-1,d) &= M^{q-d}(M-1)^d\sum_{p=d-1}^{q-1}\binom{p-1}{d-2}\\ &= M^{q-d}(M-1)^d\binom{q-1}{d-1},\end{aligned}$$

where the latter equality follows from 0.151.1 of Gradshtein and Ryzhik [88]. Hence

$$\bar{m}(q,d) = (M-1)^d\sum_{j=0}^{q-d}M^j\binom{j+d-1}{d-1} \le (M-1)^{d-1}M^{q-d+1}\binom{q-1}{d-1}.$$

We summarize the cost estimate in the following lemma.

Lemma 15.5. *Let (15.18) hold and $q \ge d$. The number $m(q,d)$ of function values used by $A(q,d)$ is bounded as follows.*

- *For non-nested information*

$$m(q,d) \le M_0^d\,\frac{M}{M-1}\,M^q\binom{q-1}{d-1}. \tag{15.19}$$

- *For nested information*

$$\begin{aligned}m(q,d) &\le M_0^d(M-1)^d\sum_{j=0}^{q-d}M^j\binom{j+d-1}{d-1}\\ &\le \left(\frac{M-1}{M}\right)^{d-1}M_0^d\,M^q\binom{q-1}{d-1}.\end{aligned} \tag{15.20}$$

For $m_i = M_0(M^i-1)$ the first inequality in (15.20) becomes an equality.

We stress that the bounds for $m(q,d)$ for non-nested and nested information differ by an exponentially small factor $(1-1/M)^{d-1}$. This will make the further steps of the cost analysis easier.

15.2.4 ε-Cost Bound

We want to determine a possibly minimal q for which the error of $A(q,d)$ is at most ε, and estimate the number $\mathrm{cost}(\varepsilon,d)$ of function values used by such $A(q,d)$. That is, we now consider the absolute error criterion. Obviously, the bounds for the absolute error criterion can be also used for the normalized error criterion if we replace ε by $\varepsilon\, e(0,d) = \varepsilon\,\|I_1\|^d$.

First of all, observe that $e(U_0) \le B^d$ implies that

$$\mathrm{cost}(\varepsilon,d) = 0 \quad \text{for } \varepsilon \ge B^d.$$

We also mention the easy case $d = 1$. For $d = 1$ we have $e(U_q) \le C\,D^q$ and $\mathrm{cost}(A(q,1)) \le M_0 M^q$. Hence, $e(A(q,1)) \le \varepsilon$ for $q = \lceil(\ln C/\varepsilon)/(\ln D^{-1})\rceil$ and

$$\mathrm{cost}(A(q,1)) \le M_0\, M \left(\frac{C}{\varepsilon}\right)^{\ln M/\ln D^{-1}}.$$

In what follows, we consider the remaining case when $q \ge d \ge 2$.

We begin with non-nested information. Let $q = q(d,\varepsilon)$ be the minimal integer for which the error bound (15.9) does not exceed ε. Hence

$$\binom{q}{d-1} \le \frac{\varepsilon}{C H^{d-1} D^q} \tag{15.21}$$

and so (15.19) implies

$$m(q,d) \le \frac{M_0^d M}{M-1}\,\frac{q-d+1}{q}\,\frac{\varepsilon}{C H^{d-1}}\left(\frac{M}{D}\right)^q. \tag{15.22}$$

To estimate q we proceed as follows. Let $q = x(d-1)$ with $x \ge d/(d-1)$. It is relatively easy to check that

$$\begin{aligned}\binom{q}{d-1} &\le \left(x\left(1+\frac{1}{x-1}\right)^{x-1}\right)^{d-1}\sqrt{\frac{x}{2\pi(d-1)(x-1)}}\\ &\le (xe)^{d-1}\,\frac{1}{\sqrt{(2\pi(d-1))}}\,\sqrt{\frac{x}{x-1}} \le (xe)^{d-1}\sqrt{\frac{d}{2\pi(d-1)}}.\end{aligned}$$

Let $x = t/\ln D^{-1}$. Using the last right-hand side instead of the left-hand side of (15.21), we get

$$t \ge \ln t + \ln h \quad \text{with } h = h(\varepsilon,d) = \frac{eH}{\ln D^{-1}}\left(\frac{C}{\varepsilon}\sqrt{\frac{d}{2\pi(d-1)}}\right)^{1/(d-1)}. \tag{15.23}$$

Observe that $h > e\, D^{-1} / \ln D^{-1} \ge e^2$ since

$$\varepsilon \le B^d \le CB^{d-1} \le C(HD)^{d-1} < C(eHD)^{d-1}\sqrt{d/(2\pi(d-1))}.$$

Consider the following sequence of $t_k = t_k(h)$:

$$t_0 = \frac{e}{e-1}\ln h \quad \text{and} \quad t_{k+1} = \ln(ht_k). \tag{15.24}$$

It is easy to verify that t_0 satisfies (15.23), and then one can show by a simple induction that all the t_k's also satisfy this inequality. Hence

$$t^* \le t_k \quad \text{for all } k,$$

where t^* is the unique solution of the equation $t^* = \ln t^* + \ln h$. Clearly, $t^* > \ln D^{-1}$ since $x > 1$.

The sequence of t_k's converges monotonically to t^*. Indeed, consider $y_k = \ln(hy_{k-1})$ with $y_0 = \ln h$. Then $y_k \le t^*$. It can be easily checked that $t_k - y_k$ converges to zero. Moreover the convergence is quite fast, and thus this can be used in an algorithmic implementation when computing t^*.

Hence, $q(\varepsilon, d) \le \lceil t_{k+1}(d-1)/\ln D^{-1}\rceil = x(d-1)$ with $x \ge d/(d-1)$, and

$$q(\varepsilon, d) \le 1 + t_{k+1}\frac{d-1}{\ln D^{-1}} \quad \text{for all } k \ge 0.$$

Using this in (15.22), we get

$$m(q,d) \le \frac{M_0^d M^2}{M-1}\,\frac{1}{DH^{d-1}}\left(\sqrt{\frac{d}{2\pi(d-1)}}\right)^{\alpha+1}\left(\frac{eH}{\ln D^{-1}}t_k\right)^{(\alpha+1)(d-1)}\left(\frac{C}{\varepsilon}\right)^{\alpha},$$

where

$$\alpha = \frac{\ln M}{\ln D^{-1}}.$$

For simplicity we use the last inequality only for t_0. Then

$$m(q,d) \le \beta\left(\frac{C}{\varepsilon}\right)^{\alpha}\left(\ln\left(\frac{eH}{\ln D^{-1}}\right)+\frac{\ln(\sqrt{d/(2\pi(d-1))})+\ln(C/\varepsilon)}{d-1}\right)^{(\alpha+1)(d-1)},$$

where

$$\beta = \frac{M_0 M^2}{(M-1)D}\left(\frac{d}{2\pi(d-1)}\right)^{(\alpha+1)/2}\left(\frac{e^2 M_0^{1/(\alpha+1)}H^{\alpha/(\alpha+1)}}{(e-1)\ln D^{-1}}\right)^{(\alpha+1)(d-1)}.$$

For nested information, the cardinality $m(q,d)$ has the same bounds as above, multiplied by $((M-1)/M)^d$. For nested information and optimal U_i we can use (15.16) of Lemma 15.4 to estimate the error of $A(q,d)$. To guarantee that $e(A(q,d)) \le \varepsilon$, we can take $q = q(\varepsilon, d)$ such that

$$C^{2d} D^{2(q-d+1)} \binom{q}{d-1} = C^2 \left(\frac{C^2}{D^2}\right)^{d-1} D^{2q} \binom{q}{d-1} \le \varepsilon^2.$$

This leads to the same analysis as for non-nested information with ε, C, D and H replaced by ε^2, C^2 and C^2/D^2, respectively. In particular, $h(\varepsilon, d)$ from (15.23) and (15.24) is now

$$\frac{eC^2}{2D^2 \ln D^{-1}} \left(C^2 \varepsilon^{-2} \sqrt{d/(2\pi(d-1))}\right)^{1/(d-1)}.$$

The analysis above and the relation between the cardinality $m(q,d)$ and the cost lead to the following theorem. In this theorem, we use the following abbreviations: *non-nested* stands for non-nested information, and *nested* for nested information and optimal U_i.

Theorem 15.6. *Assume that* (15.6), (15.7), *and* (15.8) *hold if non-nested information is considered. Assume that* (15.6) *and* (15.7) *hold if nested information and optimal* U_i *are considered. Assume also that* (15.18) *holds.*

Let $d \ge 2$ *and* $\varepsilon \le B^d$. *Define*

$$t_{k+1} = \ln(h t_k) \quad \text{with } t_0 = \frac{e}{e-1} \ln h$$

with $h = h(\varepsilon, d)$ *given by* (15.23). *For*

$$q = \left\lceil t_{k+1} \frac{d-1}{\ln D^{-1}} \right\rceil$$

with an arbitrary $k \ge 0$, *the algorithm* $A_\varepsilon(d) = A(q,d)$ *has error at most* ε *and its cost is bounded by*

$$\operatorname{cost}(A_\varepsilon(d)) \le \alpha_0(d) \left[\alpha_1 + \alpha_2 \frac{\ln(\sqrt{d/(2\pi(d-1))}) + \ln(C/\varepsilon)}{d-1}\right]^{\alpha_3(d-1)} \left(\frac{C}{\varepsilon}\right)^{\alpha}.$$

Here

$$\alpha = \frac{\ln M}{\ln D^{-1}},$$

and depending on the specific case, the values of h, α_1, α_2, *and* α_3 *are given by*

$$h = h(\varepsilon, d) = \begin{cases} \frac{eH}{\ln D^{-1}} \left(\frac{C}{\varepsilon} \sqrt{\frac{d}{2\pi(d-1)}}\right)^{1/(d-1)}, & \textit{non-nested}, \\ \frac{eC^2}{2D^2 \ln D^{-1}} \left(\frac{C^2}{\varepsilon^2} \sqrt{\frac{d}{2\pi(d-1)}}\right)^{1/(d-1)}, & \textit{nested}, \end{cases}$$

$$\alpha_0(d) = \begin{cases} \left(\frac{d}{2\pi(d-1)}\right)^{(\alpha+1)/2} \frac{M_0 M^2}{(M-1)D}, & \textit{non-nested,} \\ \left(\frac{d}{2\pi(d-1)}\right)^{(\alpha+2)/4} \frac{M_0 M}{D^2}, & \textit{nested,} \end{cases}$$

$$\alpha_1 = \begin{cases} \alpha_2 \ln\left(\frac{eH}{\ln D^{-1}}\right), & \textit{non-nested,} \\ \alpha_2 \ln\left(\frac{e\,C^2}{2D^2 \ln D^{-1}}\right)/2, & \textit{nested,} \end{cases}$$

$$\alpha_2 = \begin{cases} \frac{e^2 M_0^{1/(\alpha+1)} H^{\alpha/(\alpha+1)}}{(e-1)\ln D^{-1}}, & \textit{non-nested,} \\ \left(\frac{M_0(M-1)}{M}\right)^{2/(\alpha+2)} \frac{e^2}{(e-1)\ln D^{-1}} \left(\frac{C}{D}\right)^{2\alpha/(\alpha+2)}, & \textit{nested,} \end{cases}$$

$$\alpha_3 = \begin{cases} \alpha + 1, & \textit{non-nested,} \\ \alpha/2 + 1, & \textit{nested,} \end{cases}$$

where $H = \max\{B/D, E\}$.

We now comment on Theorem 15.6. The essence of the estimates of this theorem is that for arbitrary d the cost of computing an ε-approximation is fully determined by the parameters from the univariate case, $d = 1$. To focus on the dependence on d, we slightly simplify the estimate on $\mathrm{cost}(A_\varepsilon(d))$. Since $\alpha_0(d)$ is decreasing in d we have

$$\mathrm{cost}(A_\varepsilon(d)) \le \beta_1 \left(\beta_2 + \beta_3 \frac{\ln 1/\varepsilon}{d-1}\right)^{\beta_4(d-1)} \left(\frac{1}{\varepsilon}\right)^{\beta_5}, \tag{15.25}$$

where $\beta_1 = \alpha_0(2)C^\alpha$, $\beta_2 = \alpha_1 + \alpha_2\left(\ln C/\sqrt{2\pi}\right)$, $\beta_3 = \alpha_2$, $\beta_4 = \alpha_3$ and $\beta_5 = \alpha$.

Observe that the leading factor $(1/\varepsilon)^{\beta_5}$ of the cost has the same exponent for all d. The value of β_5 depends on the quality of the information N_i and on the quality of the algorithms U_i for $d = 1$. Sometimes we can choose them in such a way that β_5 is minimized. The next leading factor of the cost is of the form

$$\mathrm{cost}_2(\varepsilon, d) = \left(\beta_2 + \beta_3 \frac{\ln 1/\varepsilon}{d-1}\right)^{\beta_4(d-1)}.$$

For fixed d and ε tending to zero we have

$$\mathrm{cost}_2(\varepsilon, d) = \left(\frac{\beta_3}{d-1}\right)^{\beta_4(d-1)} \left(\ln \frac{1}{\varepsilon}\right)^{\beta_4(d-1)} (1 + o(1)),$$

where the factor in the o notation may depend on d. Observe that the asymptotic constant goes to zero super-exponentially with d. We stress that the exponent β_4 is sometimes too large. That is, for some problems, there exist algorithms for which the cost of computing an ε-approximation is

$$C_d \left(\ln \frac{1}{\varepsilon}\right)^{\beta_6(d-1)} \left(\frac{1}{\varepsilon}\right)^{\beta_5} \quad \text{with } \beta_6 \in (0, \beta_4).$$

This indicates that in general the algorithm does not minimize the cost of computing an ε-approximation, although the loss is usually only in a power of $\ln 1/\varepsilon$. Moreover, for those problems, the dependence of C_d on d is unknown and it could happen that C_d might even grow super-exponentially with d.

We now fix ε and vary d. Observe the very interesting dependence of $\mathrm{cost}_2(\varepsilon, d)$ on d. With increasing d, the power grows but $\beta_3 \ln 1/\varepsilon$ is divided by increasing numbers. For $\beta_2 > 0$, due to $(1 + x/(d-1))^{d-1} \le e^x$ we have

$$\mathrm{cost}_2(\varepsilon, d) \le \beta_2^{\beta_4(d-1)}\, \varepsilon^{-\beta_5-\beta_3\beta_4/\beta_2}.$$

Hence, for $\beta_2 = 1$ the dependence on d disappears, for $\beta_2 < 1$ it goes exponentially fast to zero, and for $\beta_2 > 1$ it goes exponentially fast to infinity.

This indicates that the quality of the algorithm $A(q, d)$ may depend on β_2. However, it is not clear whether the condition $\beta_2 > 1$ is an overestimate of the cost of the algorithm $A(q, d)$, or that the algorithm $A(q, d)$ is not good, or that the tensor product problem is hard. This is discussed in the next section.

15.2.5 Tractability

We now check when the error and cost bounds of the algorithm $A(q, d)$ yield tractability of $I = \{I_d\}$. For $d = 1$, the assumptions (15.7) and (15.18) mean that we have a polynomial rate of convergence, since

$$\|I_1 - U_i\| \le C\, D^i = C\, M^{-ir} \le M_0^r\, C\, m_i^{-r} \quad \text{with } r = \alpha^{-1} = \frac{\ln D^{-1}}{\ln M}.$$

It is then natural to ask when we can claim polynomial or even strong polynomial tractability of I. In Chapter 11 we have seen examples of (unweighted) linear tensor product functionals for which we have polynomial error bounds for the univariate case $d = 1$, and which are intractable and suffer from the curse of dimensionality. This was for the normalized error criterion and for the absolute error criterion, when the initial error was at least one. Hence, the only hope to get polynomial tractability of $I = \{I_d\}$ for a general I_1 and for the absolute error criterion is to assume that $\|I_1\| < 1$. Indeed, then we have not only polynomial but also strong polynomial tractability. This is the subject of the next theorem.

Theorem 15.7. *Consider $I = \{I_d\}$ defined as in this section in the worst case setting for the absolute error criterion. Assume that (15.6), (15.7), (15.8) as well as (15.18) hold. We also assume that*

$$\|I_1\| < 1.$$

- *If we take $B \in (\|I_1\|, 1)$ in (15.6) then the algorithm $A_\varepsilon(d) = A(q, d)$ with q defined as in Theorem 15.6 has error at most ε and its cost is bounded by*

$$\mathrm{cost}(A_\varepsilon(d)) \le K\, \varepsilon^{-p} \quad \textit{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N},$$

where $K = \max\{M\, C^{\alpha}, K_1\}$ *with*

$$K_1 = \alpha_0(2) \max\left\{1, \left(\frac{C}{\sqrt{2\pi}\, B}\right)^{\alpha_2\alpha_3/(\alpha_1+\alpha_2 \ln B^{-1})}\right\} \left(\frac{C}{B}\right)^{\alpha} B^{p}$$

and

$$p = \alpha + \alpha_2\alpha_3\, q^*,$$

where q^* *is given by*

$$q^* = \begin{cases} \frac{\ln \gamma_1}{\ln \gamma_2} & \text{if } \gamma_1 \ln \gamma_1 - \ln \gamma_2 \geq 0, \\ q^{**} & \text{if } \gamma_1 \ln \gamma_1 - \ln \gamma_2 < 0, \end{cases}$$

with $\gamma_1 = \alpha_1 + \alpha_2 \ln B^{-1}$ *and* $\gamma_2 = B^{-\alpha_2}$ *(we always have* $\gamma_1 \geq 0$ *and* $\gamma_2 > 1$*), and* q^{**} *is the unique solution from* $(0, 1)$ *of the equation*

$$q\, \alpha_1 = 1 + \ln q.$$

The other parameters are as in Theorem 15.6.

- *This means that* I *is strongly polynomially tractable and the exponent of strong polynomial tractability is at most* p.

Proof. For $d = 1$, the bound on $\mathrm{cost}(A_\varepsilon(1))$ is trivial. For $d \geq 2$, Theorem 15.6 yields

$$\mathrm{cost}(A_\varepsilon(d)) \leq K_1 \left(\alpha_1 + \alpha_2 \frac{\ln B/\varepsilon}{d-1}\right)^{\alpha_3(d-1)} \left(\frac{B}{\varepsilon}\right)^{\alpha} \frac{1}{B^p}.$$

Letting $B/\varepsilon = (x/B^{\alpha_2})^{(d-1)/\alpha_2}$, it is enough to verify whether

$$\gamma_1 + \ln x \leq (\gamma_2\, x)^{q^*} \quad \text{for all } x \geq 1. \tag{15.26}$$

Observe that $\gamma_1 \geq 0$, due to the definitions of α_1 and α_2, and the relations between B, C, D, H. Clearly, $B < 1$ and $\alpha_2 > 0$ imply $\gamma_2 > 1$. Thus, (15.26) has a solution. We now show that q^* is the smallest solution of (15.26).

Substituting $y = \ln x + \gamma_1$ and $a = \gamma_1 - \ln \gamma_2$, we need to show that

$$q^* = \sup_{y \geq \gamma_1} g(y) > 0 \quad \text{with } g(y) := \frac{\ln y}{y - a}.$$

Note that $\lim_{\infty} g(y) = 0$ and $g'(y) = h(y)/(y(y-a)^2)$ with $h(y) = y - a - y \ln y$. Moreover, $h'(y) = -\ln y$.

First, suppose that $\ln \gamma_2 - \gamma_1 \ln \gamma_1 \leq 0$ which implies $\gamma_1 \geq 1$. Then $y \geq \gamma_1 \geq 1$, and so the function g attains its maximum at $y^* = \gamma_1$ since $h'(y)$ is always negative and $h(\gamma_1) \leq 0$. Hence $q^* = g(\gamma_1) = (\ln \gamma_1)/(\ln \gamma_2)$, as claimed.

Now suppose that $\ln \gamma_2 - \gamma_1 \ln \gamma_1 > 0$. Since $g'(\gamma_1) > 0$, the maximum of g is attained at a critical point y^* that is a root of h. Such a root is unique. Indeed, for $\gamma_1 \geq 1$, $h'(y) \leq 0$; and for $\gamma_1 < 1$, h does not have a root in $[\gamma_1, 1]$ since h' is positive and $h(\gamma_1) > 0$. Therefore $q^* = g(y^*)$, where y^* is the unique solution of $y^* - a = y^* \ln y^*$; or equivalently of $(\ln y^*)/(y^* - a) = 1/y^*$. From the definition of g, we get $g(y^*) = (\ln y^*)/(y^* - a) = 1/y^*$. Substituting y^* by $1/g(y^*)$ in the definition of $g(y^*)$ we finally conclude that $g(y^*) = q^*$ is the unique solution of $1 - aq = \ln q$, as claimed. □

As before, the number K and the exponent p in Theorem 15.7 are fully determined by the parameters for $d = 1$. However, the exponent p is usually too large, since we are using some overestimates on the error and on the cardinality of the algorithm $A_\varepsilon(d)$. We now show that sometimes the exponent p can be lowered by a different approach.

We consider nested information N_i and optimal U_i satisfying (15.6), (15.7) and (15.18). For $x \geq 1/(1 - D^2)$ and $d \geq 2$, let

$$f(x) = M_0(M-1)M^{x-1}\frac{x^x}{(x-1)^{x-1}}, \quad g(x) = CD^{x-1}\sqrt{\frac{x^x(1-D^2)}{(x-1)^{x-1}}},$$

and

$$a(d) = C\,\max\left\{\left(\frac{D^2 d^2}{2\pi(d-1)}\right)^{1/4}, \left(\frac{B}{C}\right)^d\right\}, \quad b(d) = \frac{M_0 M^2}{D^3\sqrt{2\pi(d-1)}}.$$

Let

$$p^* = \max_{x \geq 1/(1-D^2)} \frac{\ln f(x)}{\ln(1/g(x))}. \tag{15.27}$$

Since for $h(x) = \ln f(x)/\ln(1/g(x))$ we have $h(x) > \ln M/\ln D^{-1} = \alpha$ for large x and $\lim_{x\to\infty} h(x) = \alpha$, the number p^* exists and $p^* \in (\alpha, +\infty)$.

Lemma 15.8. *Consider $I = \{I_d\}$ in the worst case setting with nested information N_i and optimal U_i for which (15.6), (15.7), and (15.18) hold with $B < 1$. For $\varepsilon < B^d$ with $d \geq 2$, define the algorithm $A_\varepsilon(d) = A(q, d)$ with $q = \lceil x(d-1) + 1 \rceil$, where x is the unique solution of*

$$g^{d-1}(x)\,a(d) = \varepsilon.$$

Then the error of $A_\varepsilon(d)$ is at most ε, and for any positive η the cost of $A_\varepsilon(d)$ is bounded by

$$\mathrm{cost}(A_\varepsilon(d)) \leq K_d \left(\frac{1}{\varepsilon}\right)^{p^*+\eta} \leq C_\eta \left(\frac{1}{\varepsilon}\right)^{p^*+\eta}, \tag{15.28}$$

where

$$K_d = b(d)\,a^{p^*}(d)\,B^{d\eta} \quad \textit{and} \quad C_\eta = \max_{d\geq 2} K_d < +\infty. \tag{15.29}$$

Moreover, if

$$\frac{M_0(M-1)}{1-D^2}\left(\frac{M}{D^2}\right)^{D^2/(1-D^2)} > C^2 \tag{15.30}$$

then

$$p^* = \frac{\ln M + \ln \frac{x^*}{x^*-1}}{\ln 1/D - \frac{1}{2}\ln \frac{x^*}{x^*-1}}, \tag{15.31}$$

where $x^* > 1/(1-D^2)$ *is the unique solution of*

$$\frac{\ln f(x)}{\ln g(x)} = \frac{\ln(M\,x/(x-1))}{\ln(D(x/(x-1))^{1/2})}. \tag{15.32}$$

Proof. Note that the function g is decreasing and $g(1/(1-D^2)) = C$. Hence, $g^{d-1}(1/(1-D^2))a(d) \geq B^d > \varepsilon$ and the equation $g^{d-1}(x)a(d) = \varepsilon$ has a unique solution $x > 1/(1-D^2)$. From the definition of q we obtain that $q = (x+y)(d-1)+1$ with $x \geq 1$ and $y \in [0, 1/(d-1)]$. We now show that

$$e(A(q,d)) \leq g^{d-1}(x)a(d) \quad \text{and} \quad m(q,d) \leq f^{d-1}(x)b(d). \tag{15.33}$$

Indeed, for $x \geq 1/(1-D^2)$ we have $q \geq (d-D^2)/(1-D^2)$, and (15.17) of Lemma 15.4 yields

$$\begin{aligned} e(A(q,d)) &\leq C^d D^{(x-1)(d-1)+1}(1-D^2)^{(d-1)/2}\sqrt{d\binom{x(d-1)}{d-1}} \\ &\leq g^{d-1}(x)CD\left(\frac{d^2x}{2\pi(x-1)(d-1)}\right)^{1/4} \leq g^{d-1}(x)C\left(\frac{D^2d^2}{2\pi(d-1)}\right)^{1/4}, \end{aligned}$$

as claimed in the first inequality of (15.33).

To show the second inequality, observe that for $h(x) = x^x/(x-1)^{x-1}$ we have

$$h(x+y) \leq h(x)\left(\frac{x}{x-1}\right)^y.$$

Using this and Lemma 15.5, we obtain

$$\begin{aligned} m(q,d) &\leq M_0(M-1)^{d-1}F^{(x+y-1)(d-1)+1}\binom{(x+y)(d-1)}{d-1} \\ &\leq f^{d-1}(x)M_0M^2\left(\frac{x}{x-1}\right)^{y(d-1)}\sqrt{\frac{x+y}{2\pi(d-1)(x+y-1)}} \\ &\leq f^{d-1}(x)\frac{M_0M^2}{D^2}\sqrt{\frac{1}{2\pi D^2(d-1)}}. \end{aligned}$$

This proves (15.33).

We now prove (15.28). It follows from (15.33) that the error of the algorithm $A_\varepsilon(d)$ is at most ε. To estimate the cardinality $m(q,d)$ of $A_\varepsilon(d)$ observe that the definition of p^* yields $f(x) \leq (1/g(x))^{p^*}$. Hence, (15.33) yields

$$\begin{aligned} m(q,d) &\leq b(d)\,g^{-p^*(d-1)}(x) = b(d)\,a(d)^{p^*}\,\varepsilon^{-p^*} \\ &\leq b(d)\,a(d)^{p^*}\,\varepsilon^{\eta}\,\varepsilon^{-p^*-\eta} \leq b(d)\,a(d)^{p^*}\,B^{d\,\eta}\,\varepsilon^{-p^*-\eta} \leq C_\eta\,\varepsilon^{-p^*-\eta}. \end{aligned}$$

Clearly, C_η is finite since $B < 1$. By direct calculation we can bound C_η as in (15.29).

To show the second part of the Lemma, denote $p(x) = -\ln f(x)/\ln g(x)$. Of course,

$$p^* = \max_{x \ge 1/(1-D^2)} p(x).$$

Observe that $p'(x) = -c(x)/\ln^2 g(x)$ with

$$c(x) = \ln\left(M\frac{x}{x-1}\right)\ln(g(x)) - \ln\left(D\sqrt{\frac{x}{x-1}}\right)\ln(f(x)).$$

Obviously, $c(1/(1-D^2)) = \ln(M/D^2)\ln(C)$ is negative. On the other hand,

$$\begin{aligned} c(x) &= \ln(x)\ln(M^{1/2}/D) + \ln(M\,x/(x-1))\ln(C(1-D^2)^{1/2}) \\ &\quad - \ln(D(x/(x-1))^{1/2})\ln(M_0(M-1)), \end{aligned}$$

which implies $\lim_{x\to\infty} c(x) = +\infty$. This means that the equation $p'(x) = 0$ has a solution x^*. Of course, this equation is equivalent to (15.32). To show the uniqueness of x^*, note that $c'(x) = -(x(x-1))^{-1}\ln\left(g(x)/\sqrt{f(x)}\right)$ is always positive since $g(x)/\sqrt{f(x)} < 1$ due to (15.30). This completes the proof. □

Remark 15.9. Observe that for $p^* \le 2$ we can set $\eta = 0$ in (15.28). Indeed, this easily follows from (15.29), since $C_0 \le \max_d\{b(d)a^{p^*}(d)\}$ and $b(d)a^{p^*}(d) = O(d^{(p^*-2)/4})$. For $p^* > 2$, we need $\eta > 0$. However, the maximum in (15.29) is then attained for $d \le (p^*-2)/(4\eta\ln B^{-1})$. □

Remark 15.10. Theorem 15.6 and Lemma 15.8 describe two different definitions of the parameter q for which the algorithm $A_\varepsilon(d) = A(q,d)$ allows to attain strong polynomial tractability. As already mentioned, the exponent $p^* + \eta$ given in Lemma 15.8 is usually smaller. For a fixed d, we can estimate the cost of the algorithm $A_\varepsilon(d)$ in Theorem 15.6 by

$$\mathrm{cost}(A_\varepsilon(d)) \le C_d\,\varepsilon^{-p^*-\eta},$$

where

$$C_d = \max_{\varepsilon \le B^d} \alpha_0(d)\,C^\alpha\left(\alpha_1 + \alpha_2\frac{\ln(\sqrt{d/(2\pi(d-1))}) + \ln(C/\varepsilon)}{d-1}\right)^{\alpha_3(d-1)} \varepsilon^{p^*+\eta-\alpha}.$$

Clearly, C_d is finite since $p^* > \alpha$. On the other hand, C_d goes to infinity with d. Still, for some small d, the inequality $C_d < K_d$ might hold.

Hence, if we define $A_\varepsilon(d) = A(q,d)$ with q as in Theorem 15.6 if $C_d < K_d$ and with q as in Lemma 15.8 if $C_d \ge K_d$, then

$$\mathrm{cost}(A_\varepsilon(d)) \le \min(C_d, K_d)\,\varepsilon^{-p^*-\eta}.$$

As we shall see later, this estimate will be used to lower the estimates of the cost. □

Theorem 15.7 states strong polynomial tractability for the absolute error criterion when we assume, in particular, that $\|I_1\| < 1$. From Corollary 11.3 we know that the assumption $\|I_1\| < 1$ is needed for all non-trivial problems. Indeed, if the problem I_d cannot be solved by using just one function value, i.e., the minimal worst case error $e(1,1) > 0$, then $I = \{I_d\}$ is not strongly polynomially tractable for the normalized error criterion. For $\|I_1\| \geq 1$ the absolute error criterion is harder than the normalized error criterion, and therefore strong polynomial tractability also does not hold for the absolute error criterion.

Can we have polynomial tractability if $\|I_1\| \geq 1$? Yes, we can. As we know from Theorem 11.4 of Chapter 11, there is a reproducing kernel Hilbert space for which $e(d+1,d) = 0$ for all linear tensor product functionals. That is, we always have polynomial tractability with the exponent of at most 1. On the other hand, Theorem 11.6 states that for *all* reproducing kernel Hilbert spaces having two orthonormal elements with disjoint supports, some linear tensor product functionals are intractable.

The Smolyak algorithm has been studied so far only for the case when the minimal worst case errors for $d = 1$ decay polynomially, that is, when $e(n,1) = \mathcal{O}(n^{-\alpha})$ for some $\alpha > 0$. This obviously covers the most frequent cases occurring in computational practice. However, the more difficult case when $e(n,1)$ decays slower than polynomially is also of interest. Obviously we must assume that $e(n,1) = o(\ln^{-1} n)$ since otherwise weak tractability cannot hold. It is not clear if the Smolyak algorithm leads to weak tractability bounds for such $e(n,1)$, even if we assume that $\|I_1\| < 1$. We leave this as an open problem.

Open Problem 67.

Consider $I = \{I_d\}$ defined as in this section in the worst case setting for the absolute error criterion with $\|I_1\| < 1$. Assume that $e(n,1)$ decays slower than polynomially.

- What are necessary and sufficient conditions on the decay of $e(n,1)$ so that the Smolyak algorithm yields weak tractability bounds?
- More specifically, let $e(n,1) = \mathcal{O}(\ln^{-p} n)$ with $p > 1$. For which values of p, does the Smolyak algorithm yield weak tractability bounds?
- For a given tractability function T, what do we need to assume about $e(n,1)$ so that the Smolyak algorithm yields T-tractability bounds?

We now illustrate the results of this section for three examples of linear tensor products. Tensor products are defined by the univariate case for scalar functions. In general, one may consider functions $f\colon [a,b] \to \mathbb{R}$ for an arbitrary interval $[a,b]$. Clearly, with an obvious change of variables we can assume that $a = 0$, and so the new interval becomes $[0,\beta] = [0, b-a]$. That's why we choose to work with functions defined over $[0,\beta]$ with $\beta > 0$. Then for $d \geq 2$, the domain of the functions is $[0,\beta]^d$. We are interested in both large and modest d. For large d, we would like to have strong polynomial tractability. As we shall see, strong polynomial tractability will depend on β. For modest d, tractability is irrelevant and the parameter β does not matter.

15.2.6 Example: Integration of Smooth Periodic Functions

In this subsection we consider an integration problem. As we know, the worst and average case settings for linear functionals are closely related. To again illustrate this relation, we present this integration problem in its two equivalent formulations in the average and worst case settings.

For $d = 1$, let $F_1 = \widetilde{C}^r([0, \beta])$ be the Banach space of periodic r-times continuously differentiable functions with period β and equipped with the norm

- for $r = 0$,

$$\|f\|_{F_1} = \max_{t\in[0,\beta]} |f(t)|,$$

- for $r \geq 1$,

$$\|f\|_{F_1} = |f(0)| + \max_{t\in[0,\beta]} |f^{(r)}(t)|.$$

We now explain how a Gaussian measure μ_1 on the space F_1 is chosen. Let

$$B_1 = \{f \mid f \in C^r([0, \beta]) \text{ and } f(0) = f'(0) = \cdots = f^{(r)}(0) = 0\}$$

be quipped with the same norm as F_1. First we take w_r as the classical Wiener measure w placed on rth derivatives, so that

$$w_r(B) = w(\{f^{(r)} \mid f \in B\}) \quad \text{for all Borel sets } B \subseteq B_1.$$

Recall that w is a Gaussian measure with mean zero and covariance function $R_w(x, t) = \min(x, t)$. Since we deal with periodic functions we also must have $f^{(j)}(\beta) = 0$ for $j = 0, 1, \ldots, r$ with probability 1. To satisfy these boundary conditions, we take the measure μ_1 as the conditional measure $w_r\{\,\cdot\,|f^{(j)}(\beta) = 0,\ j = 0, 1, \ldots, r\,\}$.

Let $G_1 = \mathbb{R}$ and

$$I_1(f) = \int_0^\beta f(t)\,\mathrm{d}t \quad \text{for all } f \in F_1.$$

This integration problem in the average case setting is equivalent to the integration problem in the worst case setting for the reproducing kernel space $H(R_{\mu_1})$, see Chapter 13. The space $H(R_{\mu_1})$ is the Sobolev space of periodic functions f vanishing at 0 and β, i.e., $f^{(j)}(0) = f^{(j)}(\beta) = 0$ for $j = 0, 1, \ldots, r$, and whose rth derivative is absolutely continuous and whose $(r + 1)$st derivative is in $L_2([0, \beta])$. The norm in $H(R_{\mu_1})$ is

$$\|f\|_{H(R_{\mu_1})} = \left[\int_0^\beta \left[f^{(r+1)}(t)\right]^2 \mathrm{d}t\right]^{1/2}.$$

For $d \geq 2$, we take the tensor products $\{F_d, G_d, I_d\}$. That is, F_d is now the Banach space of periodic (in each variable) functions f with continuous mixed derivatives $f^{(j_1,j_2,\ldots,j_d)}$ for $j_i \leq r$. The measure μ_d is Gaussian with mean zero and covariance function $R_{\mu_d}(x,t) = \prod_{j=1}^{d} R_{\mu_1}(x_j,t_j)$, where R_{μ_1} is the covariance function of the measure μ_1. With probability 1, the following boundary conditions hold: $f^{(j_1,j_2,\dots,j_d)}(t) = 0$ for all t with at least one component equal to zero or β, for all $j_i \le r$. For such functions, we have

$$\|f\|_{F_d} = \max_{t\in[0,\beta]^d} |f^{(r,r,\cdots,r)}(t)|.$$

Clearly, $G_d = \mathbb{R}$ and

$$I_d(f) = \int_{[0,\beta]^d} f(t)\,\mathrm{d}t.$$

For the worst case setting and for $d \ge 2$, this problem corresponds to the integration problem for the d-fold tensor product H_d of $H(R_{\mu_1})$,

$$H_d = H(R_{\mu_1}) \otimes H(R_{\mu_1}) \otimes \cdots \otimes H(R_{\mu_1}).$$

We now turn to the algorithm $A(q,d)$, which is formally defined only for the space H_d. But since $A(q,d)$ is linear, we can extend its domain to F_d. For $d = 1$, we need to define the information N_i and the algorithms U_i. We take

$$N_i(f) = \left[f\left(\frac{\beta}{m_i+1}\right), f\left(\frac{2\beta}{m_i+1}\right), \dots, f\left(\frac{m_i\beta}{m_i+1}\right)\right]$$

and U_i as the trapezoidal algorithm,

$$U_i(f) = \frac{\beta}{m_i}\sum_{j=1}^{m_i} f\left(\frac{j\beta}{m_i+1}\right).$$

From Sections 2.1 of Chapters 5 and 7 of [305], it follows that the algorithm U_i is optimal and its average/worst case error is given by

$$e(U_i) = \frac{C_r\beta^{(2r+3)/2}}{(m_i+1)^{r+1}}, \quad i \ge 0,$$

where $C_r = \sqrt{|B_{2r+2}|/(2r+2)!}$ with the Bernoulli constant B_{2r+2}. (Recall that $U_0 = 0$.) Observe that for

$$m_i = 2^i - 1$$

the information N_i is nested, and the assumptions (15.6), (15.7) and (15.18) hold as equalities. Indeed, we have

$$B = C = C_r\beta^{(2r+3)/2}, \quad D = 2^{-(r+1)}, \quad M = 2, \quad \text{and} \quad M_0 = 1.$$

Hence, the algorithm $A_\varepsilon(d)$ yields strongly polynomial bounds if

$$\beta < C_r^{-2/(2r+3)}.$$

For instance, for $r = 0$, this holds if

$$\beta < 12^{1/3} = 2.2894\dots\,.$$

We now estimate the cost of the algorithm $A_\varepsilon(d)$. We first compute the parameters which appear in Theorem 15.6. We have

$$\begin{aligned}
\alpha &= \frac{1}{r+1},\\
\alpha_0(d) &= \left(\frac{d}{2\pi(d-1)}\right)^{\frac{2r+3}{4(r+1)}} 2^{2r+3} \le \left(\frac{1}{\pi}\right)^{\frac{2r+3}{4(r+1)}} 2^{2r+3},\\
\alpha_1 &= \frac{\alpha_2}{2}\ln\left(\frac{e\,2^{2r+1}C_r^2\beta^{2r+3}}{(r+1)\ln 2}\right),\\
\alpha_2 &= \frac{e^2\beta}{(e-1)(r+1)\ln 2}\,C_r^{\frac{2}{2r+3}},\\
\alpha_3 &= \frac{2r+3}{2(r+1)}.
\end{aligned}$$

Then we can use the estimates of Theorems 15.6 and 15.7 with these values.

We specialize these estimates for $r = 0$ and assume for simplicity that $\beta = 1$. This corresponds to a Brownian bridge with

$$R_{\mu_1}(x,t) = \min(x,t) - xt \quad \text{for all } x,t \in [0,1].$$

We now have

$$B = C = \frac{1}{2\sqrt{3}} = 0.288675\dots, \quad D^{-1} = F = 2, \quad M_0 = 1,$$

and

$$\begin{aligned}
\alpha &= 1,\\
\alpha_0(d) &= 8\left(\frac{d}{2\pi(d-1)}\right)^{3/4} \le 8\left(\frac{1}{\pi}\right)^{3/4} = 3.39021\dots,\\
\alpha_1 &= \left(\frac{1}{12}\right)^{1/3}\frac{e^2}{2(e-1)\ln 2}\ln\frac{e}{6\ln 2} = -0.57617289\dots,\\
\alpha_2 &= \left(\frac{1}{12}\right)^{1/3}\frac{e^2}{(e-1)\ln 2} = 2.7098298\dots,\\
\alpha_3 &= 1.5.
\end{aligned}$$

As explained before, we need only to consider $\varepsilon < B^d = 12^{-d/2}$. This inequality corresponds to modest d or, if d is large, to an unusually high precision. For $d \ge 2$, Theorem 15.6 yields

$$\operatorname{cost}(A_\varepsilon(d)) \le 0.9787\left(-0.576 + 2.71\,\frac{-1.8148 + \ln 1/\varepsilon}{d-1}\right)^{1.5(d-1)}\frac{1}{\varepsilon}.$$

It is known[3] that the average case complexity of computing an ε-approximation for this integration problem is $\Theta(\varepsilon^{-1}(\ln \varepsilon^{-1})^{(d-1)/2})$ where the factors in the Θ notation depend on d. Hence, for fixed d and ε tending to zero the cost of the algorithm $A_\varepsilon(d)$ agrees with the leading term of the complexity, however, the exponent of $\ln 1/\varepsilon$ is too large.

Computing K and p from Theorem 15.7 we obtain

$$\operatorname{cost}(A_\varepsilon(d)) \le 0.2064\, \varepsilon^{-2.253} \quad \text{for } \varepsilon \le 12^{-d/2}. \tag{15.34}$$

We now comment on the last estimate. Since the covariance function $R_{\mu_d}(t,t)$ is uniformly bounded in d, we know that the complexity of computing an ε-approximation for integration in the average/worst case setting is $\mathcal{O}(\varepsilon^{-2})$ with the factor in the $\mathcal{O}$ notation independent of d. This means that (15.34) is not satisfactory. Indeed, it is possible to improve this bound. It can be verified that we can now use (15.17) of Lemma 15.4. This formally corresponds to replacing C by $C(1-D^2)^{1/2} = 1/4$. Computing p for this new value of C we get

$$\operatorname{cost}(A_\varepsilon(d)) = \mathcal{O}\big(\varepsilon^{-2.0569}\big)$$

which is better but still not satisfactory. (Of course, the factor in the $\mathcal{O}$ notation does not depend on d.) This can be improved by using Lemma 15.8. Indeed, using Newton's iteration it was found out in [329] that $p^* \le 1.850698$. Hence, as explained in Remark 15.9, we can take $\eta = 0$. Computing C_0 it yields

$$\operatorname{cost}(A_\varepsilon(d)) \le 1.28068\, \varepsilon^{-1.850698}.$$

The exact value of the exponent is unknown and it yields us to the next open problem.

Open Problem 68.

Consider the integration problem $I = \{I_d\}$ of smooth periodic functions defined as in this example for the average/worst case setting and for the absolute error criterion.

- For which β is the problem strongly polynomially tractable?
- Find the exponent of strong polynomial tractability as a function of β for all $\beta \in (0, C_r^{-2/(2r+3)})$.

15.2.7 Example: Integration of Non-Periodic Functions

In this subsection we also consider an integration problem, but this time for non-smooth and non-periodic functions. As before, we consider both the average and worst case settings.

[3] This follows from the fact that periodicity does not change the dependence on ε, and without periodicity the bound on the average case complexity is derived in [346].

For $d = 1$, we define $\{F_1, G_1, S_1\}$ as in Subsection 15.2.6 with $r = 0$ but without assuming periodicity of functions. That is, $F_1 = C([0, \beta])$ is now the Banach space of continuous functions equipped with the norm $\|f\|_{F_1} = \max_{t\in[0,\beta]} |f(t)|$. As the measure μ_1 we take the Wiener measure $w_0 = w$, which is a zero Gaussian measure with the covariance function

$$R_{\mu_1}(x, t) = \min(x, t) \quad \text{for all } x, t \in [0, 1].$$

This corresponds to the worst case setting for the Sobolev space $H(R_{\mu_1})$ of absolutely continuous functions vanishing at zero and with square integrable first derivatives and the norm

$$\|f\|^2_{H_{\mu_1}} = \int_0^\beta \left[f'(t)\right]^2 \, dt.$$

For $d \geq 2$, the Banach space F_d is the class of continuous functions with the sup norm, the measure μ_d is the classical Wiener sheet measure, which is Gaussian with mean zero and covariance function

$$R_{\mu_d}(x, t) = \prod_{j=1}^{d} \min(x_j, t_j) \quad \text{for all } x, t \in [0, 1]^d.$$

This corresponds to the worst case setting for the space $H_d(R_{\mu_d})$ of functions for which $f(x) = 0$ if at least one component of x is zero, and with the norm

$$\|f\|^2_{H(R_{\mu_d})} = \int_{[0,1]^d} \left[\frac{\partial^d f}{\partial t_1 \, \partial t_2 \cdots \partial t_d}(t)\right]^2 dt.$$

To define the algorithm $A(q, d)$, we take for $d = 1$ the information

$$N_i(f) = \left[f\left(\frac{2\beta}{2m_i + 1}\right), f\left(\frac{4\beta}{2m_i + 1}\right), \ldots, f\left(\frac{2m_i\beta}{2m_i + 1}\right)\right] \tag{15.35}$$

and the algorithms

$$U_i(f) = \frac{2\beta}{2m_i + 1} \sum_{j=1}^{m_i} f\left(\frac{2j\beta}{2m_i + 1}\right). \tag{15.36}$$

It is known, see Lee [168], that the algorithm U_i is optimal and its average/worst case error is

$$e(U_i) = \|S_1 - U_i\|_{\mu_1} = \frac{\beta^{3/2}}{\sqrt{3}(2m_i + 1)}, \quad i \geq 0.$$

Observe that for

$$m_i = \tfrac{1}{2}\left(3^i - 1\right)$$

the information N_i is nested, and the assumptions (15.6) and (15.7) hold as equalities. Indeed, we have

$$B = C = \frac{\beta^{3/2}}{\sqrt{3}}, \quad D = 1/3, \quad M = 3, \quad \text{and} \quad M_0 = 1/2.$$

Hence, the algorithm $A_\varepsilon(d)$ yields strongly polynomial bounds if

$$\beta < 3^{1/3} = 1.4422\ldots.$$

Assume for simplicity that $\beta = 1$. To estimate the cost of the algorithm, we first compute the parameters of Theorem 15.6. We have

$$\begin{aligned}
\alpha &= 1,\\
\alpha_0(d) &= 13.5\left(\frac{d}{2\pi(d-1)}\right)^{\frac{3}{4}} \le 13.5\left(\frac{1}{\pi}\right)^{\frac{3}{4}} = 5.72099...,\\
\alpha_1 &= \frac{e^2}{2\cdot 3^{1/3}(e-1)\ln 3}\ \ln\frac{3\,e}{2\ln 3} = 1.77958...,\\
\alpha_2 &= \frac{e^2}{3^{1/3}(e-1)\ln 3} = 2.71399...,\\
\alpha_3 &= 1.5.
\end{aligned}$$

Similarly to Subsection 15.2.6, for $\varepsilon \ge 3^{-d/2}$ the problem is trivial. For $\varepsilon < 3^{-d/2}$ and $d \ge 2$, Theorem 15.6 yields

$$\mathrm{cost}(A_\varepsilon(d)) \le 3.304\left(1.77959 + 2.714\,\frac{-1.12167 + \ln 1/\varepsilon}{d-1}\right)^{1.5(d-1)}\frac{1}{\varepsilon}.$$

We compare this bound with the average case complexity

$$\Theta(\varepsilon^{-1}(\ln\varepsilon^{-1})^{(d-1)/2}),$$

where the factors in the Θ notation depend on d, see [346]. As in Subsection 15.2.6, the exponent of $1/\varepsilon$ in the cost estimate of the algorithm $A(q,d))$ agrees with the power of $1/\varepsilon$ in the average case complexity, however, the power of $\ln 1/\varepsilon$ is too large.

Computing K and p from Theorem 15.7 we obtain

$$\mathrm{cost}(A_\varepsilon(d)) \le 0.558477\,\varepsilon^{-4.23568} \quad \text{for } \varepsilon \le 3^{-d/2}.$$

As in Subsection 15.2.6 the last estimate is not satisfactory since the average case complexity of computing an ε-approximation is $\mathcal{O}(\varepsilon^{-2})$, with the factor in the $\mathcal{O}$ notation independent of d. As before, it is possible to improve this bound. For instance, by using (15.17) of Lemma 15.4 we can replace C by $C(1-D^2)^{1/2} = 2\sqrt{2}/(3\sqrt{3})$ to get

$$\mathrm{cost}(A_\varepsilon(d)) = \mathcal{O}\big(\varepsilon^{-4}\big).$$

The exponent p can be further lowered by using a modified Lemma 15.8. The corresponding p^* was computed in [329] and it turns out that $p^* \le 2.452616$. Similarly, C_η and K_d were computed for $\eta = 10^{-3}$ following Remark 6, and

$$\mathrm{cost}(A_\varepsilon(d)) \le K_d\,\varepsilon^{-2.454},$$

where $K_1 = 0.39$ and $\{K_d\}$, for $d \geq 2$, is monotonically decreasing to zero with $C_\eta = K_2 = 12.59$, $K_3 = 8.9$, $K_4 = 7.26$ and $K_5 = 6.29$.

Following Remark 7, these estimates can be improved for $d = 2$ and $d = 3$. Indeed, $C_2 = 1.76$ and $C_3 = 4.76$. Hence, if we define $A_\varepsilon(d) = A(q,d)$ with q from Theorem 15.6 for $d = 2$ and $d = 3$, and with q from Lemma 15.8 for $d \geq 4$ then

$$\operatorname{cost}(A_\varepsilon(d)) \leq 7.26\, \varepsilon^{-2.454} \quad \text{for all } d,\ \varepsilon \leq 1.$$

Still the exponent is larger than 2.

A number of different choices of the parameters m_i, as well as different information N_i and algorithms U_i, were used in [329]. All these choices yield exponents larger than two. The exact value of the exponent is unknown and yields us to the next open problem.

Open Problem 69.

Consider the integration problem $I = \{I_d\}$ of smooth non-periodic functions defined as in this example for the average/worst case setting and for the absolute error criterion.

- For which β is the problem strongly polynomially tractable?
- Find the exponent of strong polynomial tractability as a function of β for all $\beta \in (0, 3^{1/2})$.

15.2.8 Example: Discrepancy

As we know from Chapter 9, discrepancy in the L_2 norm is related to multivariate integration in the average case setting with the Wiener sheet measure and in the worst case setting for the Sobolev space $H(R_{\mu_d})$ studied in the previous subsection. That is why the bounds presented for multivariate integrations can be also used to bound discrepancy in the L_2 norm.

More precisely, take N_i and U_i as in (15.35) and (15.36) with $\beta = 1$ and $m_i = (3^i - 1)/2$. Then (15.4) yields

$$A(q,d)(f) = \sum_{\vec{i} \in P(q,d)} c_{\vec{i}} \sum_{\vec{j} \leq \vec{m}_{\vec{i}}} f\left(x_{\vec{i},\vec{j}}\right),$$

where $P(q,d) = \left\{ \vec{i} \mid \vec{1} \leq \vec{i},\ q - d + 1 \leq |\vec{i}\,| \leq q \right\}$ and

$$c_{\vec{i}} = \frac{(-1)^{q-|\vec{i}\,|}\, 2^d}{3^{|\vec{i}\,|}} \binom{d-1}{q-|\vec{i}\,|} \quad \text{and} \quad x_{\vec{i},\vec{j}} = \left[\frac{2\, j_1}{3^{i_1}}, \frac{2\, j_2}{3^{i_2}}, \dots, \frac{2\, j_d}{3^{j_d}} \right]$$

with $\vec{m}_{\vec{i}} = \left[(3^{i_1} - 1)/2, (3^{i_2} - 1)/2, \dots, (3^{i_d} - 1)/2 \right]$.

Let $n = m(q,d)$ be the number of function values used by the algorithm $A(q,d)$; we have due to (15.20), so that

$$n = \sum_{j=0}^{q-d} 3^j \binom{j+d-1}{d-1}.$$

Define n points $z_{\vec{i},\vec{j}} = \vec{1} - x_{\vec{i},\vec{j}}$ and let

$$\operatorname{disc}(t) = \sum_{\vec{i} \in P(q,d)} c_{\vec{i}} \sum_{\vec{j} \le \vec{m}_{\vec{i}}} \chi_{[0,t)}(z_{\vec{i},\vec{j}}) \; - \; t_1 t_2 \cdots t_d,$$

where $\chi_{[0,t)}$ is the characteristic (indicator) function of

$$[0,t) = [0,t_1) \times [0,t_2) \times \cdots \times [0,t_d).$$

The discrepancy of the n points $z_{\vec{i},\vec{j}}$ is given by

$$\|\operatorname{disc}_2(z_{\vec{i},\vec{j}})\| = \left(\int_{[0,1]^d} \operatorname{disc}^2(t)\, \mathrm{d}t \right)^{1/2}.$$

We know that

$$\|\operatorname{disc}_2(z_{\vec{i},\vec{j}})\| = e(A(q,d)).$$

For $n = 0$, we have $\|\operatorname{disc}_2(0,d)\| = 3^{-d/2}$. Hence, the equality above also holds for $q < d$. From Lemma 15.4 we have

$$\|\operatorname{disc}_2(z_{\vec{i},\vec{j}})\| \le \left(\frac{1}{\sqrt{3}}\right)^d \left(\frac{1}{3}\right)^{q-d+1} \sqrt{\binom{q}{d-1}}.$$

If we choose q to guarantee that $e(A(q,d)) \le \varepsilon \le 3^{-d/2}$ as in Theorem 15.7 then

$$\|\operatorname{disc}_2(z_{\vec{i},\vec{j}})\| \le \varepsilon$$

for $n = n(\varepsilon, d)$ such that

$$n(\varepsilon,d) \le 3.304 \left(1.77959 + 2.714 \, \frac{-1.12167 + \ln 1/\varepsilon}{d-1} \right)^{1.5(d-1)} \frac{1}{\varepsilon}.$$

From Subsection 15.2.7 we also have

$$n(\varepsilon,d) \le \gamma_d \, \varepsilon^{-2.454} \le 7.26\, \varepsilon^{-2.454} \quad \text{for all } d, \ \varepsilon \le 1,$$

with $\gamma_1 = 0.39$, $\gamma_2 = 1.76$, $\gamma_3 = 4.76$, $\gamma_4 = 7.26$ and $\{\gamma_d\}$ monotonically decreasing to zero starting with $d = 4$. This is exactly what we reported in Section 9.2.1 of Chapter 9.

As we know, for every ε, there exist $n(\varepsilon) = \mathcal{O}(\varepsilon^{-2})$ points with discrepancy at most ε for all d. Construction of these points is open and presented as Open Problems 31 and 32.

15.2.9 Implementation Issues

In this short section we discuss implementation of the Smolyak algorithm for approximating (unweighted) linear tensor functionals. We will restrict ourselves only to the absolute error criterion with $\|I_1\| < 1$ since this is the only case for which we know that the Smolyak algorithm yields strong polynomial tractability. We will address three issues:

- initial error and polynomial tractability,
- gaps in the cardinality,
- numerical stability.

We shall argue that as long as d is relatively small all is fine with the Smolyak algorithm, whereas for large d the situation is quite different and the Smolyak algorithm fails quite badly. This means that the use of the Smolyak algorithms for the unweighted case should be recommended only for relatively small d. We now discuss these three issues in turn.

- Initial error and polynomial tractability.

The initial error for the d variate case is $\|I_d\| = \|I_1\|^d$. As long as d is relatively small, the initial error is fine independently of what is the value of $\|I_1\|$. For large d and the absolute error criterion, the initial error $\|I_1\|^d$ becomes exponentially small. For instance assume that $\|I_1\| = \frac{1}{2}$ and $d = 100$. Then the initial error is 2^{-100}. As already discussed, it is really hard to believe that there is a practically important computational problem for which we would be interested in computing an ε-approximation for the absolute error criterion with $\varepsilon < 2^{-100}$. On the other hand, if $\varepsilon > 2^{-100}$ then our problem is trivial since the zero algorithm will do the job. Hence, for large d it seems to us that the domain of practical ε will preclude us from using the Smolyak algorithm for the absolute error criterion. Obviously, the situation is quite different for the normalized error criterion since for all $\varepsilon < 1$, the zero algorithm does not solve the problem and we have to find out a non-trivial algorithm that would reduce the initial error by a factor of ε algorithm. As we know, sometimes (and probably in most practical cases) such problems are intractable, but even if they are tractable we do not know whether the Smolyak algorithm will do the job.

- Gaps in the cardinality.

The Smolyak algorithm $A(q,d)$ is defined for $q, d \in \mathbb{N}$. The cardinality of $A(q,d)$ is the number of function values used in $A(q,d)$ and, as before, is denoted by $m(q,d)$. If for $d = 1$ we use $m_i = M_0(M^i - 1)$ function values for the algorithm U_i then Lemma 15.19 states that for nested information we have

$$m(q,d) = M_0^d (M-1)^d \sum_{j=0}^{q-d} M^j \binom{j+d-1}{d-1}.$$

Suppose we change the value of q by one. Then the cardinality will change from $m(q,d)$ to $m(q+1,d)$, and

$$m(q+1,d) - m(q,d) = M_0^d\,(M-1)^d\, M^{q-d+1} \binom{q}{d-1}.$$

Again, as long as d is modest, there is nothing special in the change of cardinality. However, if d is large then the gap between $m(q+1,d)$ and $m(q,d)$ is huge. It is obvious that for $M_0 > 1$ or for $M > 2$ it is exponentially large in d. But even for $M_0 = 1$ and $M = 2$ we have

$$m(q+1,d) - m(q,d) = 2^{q-d+1} \binom{q}{d-1}.$$

The first q of interest is $q = d$ and we have

$$\begin{aligned} m(d,d) &= 1, \\ m(d+1,d) - m(d,d) &= 2d, \\ m(d+2,d) - m(d+1,d) &= 2d(d+1), \\ m(d+3,d) - m(d+2,d) &= \tfrac{4}{3}\,(d+2)(d+1)d. \end{aligned}$$

For $d = 360$, which is used quite often in finance, we have

$$\begin{aligned} m(d,d) &= 1, \\ m(d+1,d) - m(d,d) &= 360, \\ m(d+2,d) - m(d+1,d) &= 259\,200, \\ m(d+3,d) - m(d+2,d) &= 62\,727\,360 \end{aligned}$$

and it is already quite dubious if we can perform the fourth step of the Smolyak algorithm.

For large d, the gaps in the cardinality of the Smolyak algorithm become so large that we can perform only very few steps. This should be contrasted with QMC algorithms for which there are no gaps in the cardinality. This again precludes us from using the Smolyak algorithm for large d.

- Numerical stability

The Smolyak algorithm $A(q,d)$ is linear and therefore can be written as

$$A(q,d)(f) = \sum_{j=1}^{m(q,d)} a_j\, f(x_j)$$

for some real numbers a_j and sample points $x_j \in D_d$. It is well known that numerical stability of the linear algorithm $A(q,d)$ holds if

$$C(q,d) := \sum_{j=1}^{m(q,d)} |a_j|$$

is relatively small. That is why, it is a common knowledge that, if possible, we should use coefficients a_j of the same sign. Then the sum $C(q,d)$ of their absolute values is bounded by the sum $|I_d(1)|$ plus the error for $f = 1$. For large $m(q,d)$ this is usually almost the same as $|I_d(1)|$.

But the Smolyak algorithm uses coefficients a_j of different signs even when we use algorithms U_i with, say, non-negative coefficients $a_{i,j}$, see Lemma 15.1. So what can we say about the sum of the absolute values of a_j for the Smolyak algorithm?

For simplicity, assume that all U_i's are QMC algorithms so that $a_{i,j} = 1/m_i$. Then

$$C(q,d) = \sum_{k=q-d+1}^{q} \binom{d-1}{q-k}\binom{k-1}{d-1} = \sum_{j=0}^{d-1} \binom{d-1}{j}\binom{q-j-1}{d-1}.$$

For large d we are in trouble. Again, even if we take the fourth step of the Smolyak algorithm $q = d+3$ we have

$$C(d+3,d) = \frac{d(d+1)(d+2)}{6} + \frac{(d-1)d(d+1)}{2} + \frac{(d-2)(d-1)d}{2} + \frac{(d-3)(d-2)(d-1)}{6} = \tfrac{4}{3}\,d^3 + \mathcal{O}(d^2).$$

For $d = 360$ we have

$$C(d+3,d) = 61\,949\,759$$

which is already quite large.

For $q \ge 2d-1$ we have $\binom{q-j-1}{d-1} \ge 1$ and therefore

$$C(q,d) \ge \sum_{j=0}^{d-1} \binom{d-1}{j} = 2^{d-1}$$

is exponentially large in d.

We now estimate $C(q,d)$ for arbitrary d following [214]. Since the numbers $\binom{q-j-1}{d-1}$ decrease with j, we have obvious estimates

$$C(q,d) \in \left[\binom{q-1}{d-1}, 2^d \binom{q-1}{d-1}\right].$$

Since $\binom{q-1}{d-1} \le (q-1)^{d-1}/(d-1)!$ and $2^{d-1}/(d-1)! \le \exp(2)$ we obtain

$$C(q,d) \le 2\exp(2)\,(q-1)^{d-1}.$$

The last upper bound can be compared with the cardinality $m(q,d)$ of the Smolyak algorithm. Assuming $m_i = M_0(M^i - 1)$ and non-nested information, the cardinality $m(q,d)$ can be upper bounded by the estimate of the first point of Lemma 15.5, and it can be lower bounded by the last term for $j = q-d$ for nested information, so that

$$m(q,d) = c_d\,M^q \binom{q-1}{d-1} \quad \text{with } c_d \in \left[M_0^d\left(1-M^{-1}\right)^d, M_0^d\left(1-M^{-1}\right)\right].$$

Note that for modest d, the coefficient c_d is irrelevant. From this we get

$$q = \frac{\ln m(q,d)}{\ln M}\,(1+o(1)) \quad \text{as } q \to \infty.$$

Hence

$$C(q,d) = \mathcal{O}\big([\ln m(q,d)]^{d-1}\big),$$

where the factor in the big $\mathcal{O}$ notation depends on d, but it is irrelevant for a modest d.

Hence, if d is modest we can accept the logarithmic growth of $C(q,d)$ and, in this relaxed sense, numerical stability of the Smolyak algorithm is nearly achieved.

This shows that the Smolyak/sparse grid algorithms are not really efficient or applicable for approximating linear tensor product functionals for the *unweighted* case with large d. Therefore in the next section, we turn to linear *weighted* tensor product functionals and check for which weights the Smolyak/sparse grid algorithms are efficient and lead to tractability bounds.

15.3 Weighted Case: Algorithms

In this section we deal with linear weighted tensor product functionals. For such problems, we extend the definition of the Smolyak/sparse grid algorithms and obtain *weighted Smolyak algorithms*, which are also called *weighted tensor product* (WTP) algorithms, see [332], [335] where the WTP algorithms have been introduced for product and finite-order weights.

We define linear weighted tensor product functionals following the approach presented in Section 11.5 of Chapter 11 and Section 12.2 of Chapter 12.

As before for $d = 1$, we consider the reproducing kernel Hilbert space $F_1 = H(K_1)$. Without loss of generality we may assume that $K_1 \neq 0$, and choose a point $a \in D_1$ for which $K_1(a,a) > 0$. Let

$$\eta_1(x) = \frac{1}{\sqrt{K_1(a,a)}}\,K_1(x,a) \quad \text{for all } x \in D_1.$$

Then $\eta_1 \in F_1$ and $\|\eta_1\|_{F_1} = 1$. Let

$$R_1(x,t) = \eta_1(x)\,\eta_1(t) \quad \text{for all } x,t \in D_1.$$

Clearly, $H(R_1) = \operatorname{span}(\eta_1)$ is one dimensional and $f \in H(R_1)$ means that

$$f(x) = \frac{f(a)}{\sqrt{K_1(a,a)}}\,\eta_1(x) \quad \text{for all } x \in D_1$$

with $\|f\|_{H(R_1)} = |f(a)|/\sqrt{K_1(a,a)}$. Let

$$R_2(x,t) = K_1(x,t) - \frac{K_1(x,a)\,K_1(a,t)}{K_1(a,a)} \quad \text{for all } x,t \in D_1.$$

Then R_2 is a reproducing kernel with $R_2(a,t) = 0$ for all $t \in D_1$. This yields that

$$H(R_2) = \{\, f \in H(K_1) \mid f(a) = 0 \,\}$$

is a subspace of $H(K_1)$ of functions vanishing at a. Clearly,

$$K_1 = R_1 + R_2 \quad \text{with } H(R_1) \cap H(R_2) = \{0\},$$

so that the assumptions of Section 11.5 of Chapter 11 and Section 12.2 of Chapter 12 are satisfied.

We now define the weighted space $F_{d,\gamma}$ for $d \geq 1$ as in Section 12.2 of Chapter 12. That is, for a given sequence

$$\gamma = \{\gamma_{d,\mathfrak{u}}\}_{\mathfrak{u} \subseteq [d], d \in \mathbb{N}}$$

of weights, we have $F_{d,\gamma} = H(K_{d,\gamma})$ with the reproducing kernel

$$K_{d,\gamma}(x,t) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \notin \mathfrak{u}} R_1(x_j, t_j) \prod_{j \in \mathfrak{u}} R_2(x_j, t_j) \quad \text{for all } x, t \in D_d.$$

We turn to linear weighted tensor product functionals

$$I_\gamma = \{I_{d,\gamma}\}.$$

For $d = 1$, we have $I_1(f) = \langle f, h_1 \rangle_{F_1}$ for all $f \in F_1$ and a non-zero h_1. We decompose $h_1 = h_{1,1} + h_{1,2}$ with $h_{1,i} \in H(R_i)$ for $i = 1, 2$. We have

$$h_{1,1}(x) = \frac{h_1(a)}{\sqrt{K_1(a,a)}}\, \eta_1(x),$$
$$h_{1,2}(x) = h_1(x) - \frac{h_1(a)}{\sqrt{K_1(a,a)}}\, \eta_1(x)$$

for all $x \in D_1$. Furthermore,

$$\|h_{1,1}\|_{H(R_1)} = \frac{|h_1(a)|}{\sqrt{K_1(a,a)}} \quad \text{and} \quad \|h_{1,2}\|_{H(R_2)} = \left(\|h_1\|^2_{H(K_1)} - \frac{h_1^2(a)}{K_1(a,a)} \right)^{1/2}.$$

From Section 12.2 of Chapter 12 we know that $I_{d,\gamma} = I_d$ and

$$I_{d,\gamma}(f) = \langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}} \quad \text{for all } f \in F_{d,\gamma}$$

with

$$h_{d,\gamma}(x) = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \notin \mathfrak{u}} h_{1,1}(x_j) \prod_{j \in \mathfrak{u}} h_{1,2}(x_j) \quad \text{for all } x \in D_d.$$

The initial error is now given by

$$\|I_{d,\gamma}\| = \|h_{d,\gamma}\|_{F_{d,\gamma}} = \Big(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \|h_{1,1}\|^{2(d-|\mathfrak{u}|)}_{H(R_1)} \|h_{1,2}\|^{2|\mathfrak{u}|}_{H(R_2)} \Big)^{1/2}.$$

This concludes the definition of linear weighted tensor product functionals.

We turn our attention to algorithms for approximating $I_\gamma = \{I_{d,\gamma}\}$. As always we begin with $d = 1$. It will be convenient to let $P_1 : H(K_1) \to H(R_1)$ be the projection operator defined by

$$P_1 f = \frac{f(a)}{\sqrt{K_1(a,a)}}\, \eta_1 \quad \text{for all } f \in H(K_1).$$

Then

$$(P_1 f)(a) = f(a), \quad P_1\eta_1 = \eta_1, \quad h_{1,1} = P_1 h_1 \quad \text{and} \quad h_{1,2} = (I - P_1)h_1$$

with the identity mapping $I : F_1 \to F_1$, i.e., $I(f) = f$ for all $f \in F_1$.

We also have

$$I_1(f) = \langle f, h_1\rangle_{H(K_1)} = \langle f, h_{1,1}\rangle_{H(R_1)} + \langle f, h_{1,2}\rangle_{H(R_2)} \quad \text{for all } f \in H(K_1).$$

For $f \in H(R_1)$, we have

$$I_1(f) = \langle f, h_{1,1}\rangle_{H(R_1)} = \frac{h_1(a)}{K_1(a,a)}\, f(a).$$

This means that this subproblem can be solved exactly by using one function value at a. Observe also that

$$(I_1 P_1)(f) = \frac{h_1(a)}{K_1(a,a)}\, f(a) \quad \text{for all } f \in F_1 = H(K_1).$$

For $f \in H(R_2)$, we have

$$I_1(f) = \langle f, h_{1,2}\rangle_{H(R_2)}\,.$$

Note that for $h_{1,2} = 0$, the problem trivializes, since

$$I_d(f) = \left[\frac{h_1(a)}{K_1(a,a)}\right]^d f(a,a,\dots,a) \quad \text{for all } f \in F_d$$

and so I_d can be computed exactly using at most one function value. Therefore from now on we assume that $h_{1,2} \neq 0$.

For this subproblem and its (unweighted) tensor products we will be using the Smolyak algorithm from the previous section. That is, we assume that we have a sequence of linear algorithms $\{U_i\}_{i=0,1,\dots}$, $U_i : H(R_2) \to \mathbb{R}$, that approximate the linear functional $I_1\big|_{H(R_2)}$ such that $U_0 = 0$ and

$$\lim_{i\to\infty} \|I_1 - U_i\|_{H(R_2)\to\mathbb{R}} = 0.$$

As in Section 15.2, we denote

$$\Delta_0 = U_0 = 0, \quad \Delta_i = U_i - U_{i-1} \quad \text{for all } i \in \mathbb{N}.$$

Then

$$I_1(f) = \sum_{i=1}^{\infty} \Delta_i(f) \quad \text{for all } f \in H(R_2).$$

We now extend the definition of algorithms U_i to the space $F_1 = H(K_1)$ by defining linear algorithms $\{A_i\}_{i=0,1,\dots}$. We set $A_0 = 0$ and

$$A_i(f) = \frac{h_1(a)}{K_1(a,a)}\, f(a) + U_{i-1}\left(f - \frac{f(a)}{\sqrt{K_1(a,a)}}\,\eta_1\right) \quad \text{for all } f \in F_1 \text{ and } i \in \mathbb{N}.$$

Note that $f - (f(a)/\sqrt{K_1(a,a)})\,\eta_1 \in H(R_2)$ and therefore both U_{i-1} and A_i are well defined. The algorithm A_i can be also written as

$$A_i = I_1 P_1 + U_{i-1}(I - P_1) \quad \text{for all } i \in \mathbb{N}.$$

Clearly,

$$I_1 - A_i = I_1 P_1 + I_1(I - P_1) - I_1 P_1 - U_{i-1}(I - P_1) = (I_1 - U_{i-1})(I - P_1),$$

and therefore

$$\lim_{i\to\infty} \|I_1 - A_i\|_{H(K_1)\to\mathbb{R}} = 0.$$

We also have

$$I_1 = A_1 + \sum_{i=1}^{\infty} \Delta_i(I - P_1).$$

Indeed,

$$\begin{aligned} I_1 &= \sum_{i=1}^{\infty}(A_i - A_{i-1}) = A_1 + \sum_{i=2}^{\infty}(A_i - A_{i-1}) \\ &= A_1 + \sum_{i=2}^{\infty}(I_1 P_1 - U_{i-1}(I - P_1) - I_1 P_1 - U_{i-2}(I - P_1)) \\ &= A_1 + \sum_{i=1}^{\infty}(U_i - U_{i-1})(I - P_1) = A_1 + \sum_{i=1}^{\infty} \Delta_i(I - P_1), \end{aligned}$$

as claimed.

We briefly elaborate on the algorithms A_i. Since $U_0 = 0$, for $i = 1$ we have

$$A_1(f) = \frac{h_1(a)}{K_1(a,a)}\, f(a) \quad \text{for all } f \in F_1.$$

We know that $A_1 = I_1 P_1$ and therefore

$$A_1(f) = I_1(f) \quad \text{for all } f \in H(R_1),$$
$$A_1(f) = 0 \qquad \text{for all } f \in H(R_2).$$

We also have

$$\|A_1\|_{H(K_1)\to\mathbb{R}} = \|I_1\|_{H(R_1)\to\mathbb{R}} = \|h_{1,1}\|_{H(R_1)} = \frac{|h_1(a)|}{\sqrt{K_1(a,a)}}.$$

Furthermore,

$$A_i(f) = A_1(f) \qquad \text{for all } i \in \mathbb{N} \text{ and } f \in H(R_1),$$
$$A_i(f) = U_{i-1}(f) \quad \text{for all } i \in \mathbb{N} \text{ and } f \in H(R_2).$$

We are ready to consider $d \geq 1$. Let $\mathfrak{u} \subseteq [d]$. For $\mathfrak{u} = \emptyset$, define

$$\Delta_\emptyset = \bigotimes_{k=1}^{d} A_1, \quad \text{i.e.,} \quad \Delta_\emptyset(f) = \left[\frac{h_1(a)}{K_1(a,a)}\right]^d f(\vec{a})$$

with $\vec{a} = [a, a, \dots, a] \in D_d$.

For $\mathfrak{u} \neq \emptyset$, define the set

$$Q(d,\mathfrak{u}) = \left\{ i = [i_1, i_2, \dots, i_d] \in \mathbb{N}^d \mid \text{with } i_k = 1 \text{ for } k \notin \mathfrak{u} \text{ and } i_k \geq 2 \text{ for } k \in \mathfrak{u} \right\},$$

as the set of integer vectors whose components from $\bar{\mathfrak{u}}$ are one and from $\mathfrak{u}$ are at least 2.

For non-empty $\mathfrak{u}$ and $i = [i_1, i_2, \dots, i_d] \in Q(d,\mathfrak{u})$ define

$$\Delta_{\mathfrak{u},i} = \bigotimes_{k=1}^{d} (A_{i_k} - A_{i_k-1}) = \bigotimes_{k=1}^{d} \left[1_{\bar{\mathfrak{u}}}(k)\, A_1 + 1_{\mathfrak{u}}(k)\, (A_{i_k} - A_{i_k-1}) \right]. \tag{15.37}$$

That is, in the tensor product we take A_1 if $k \notin \mathfrak{u}$ and $A_{i_k} - A_{i_k-1}$ if $k \in \mathfrak{u}$.

For $\mathfrak{v} \subseteq [d]$, define the reproducing kernel

$$K_{d,\mathfrak{v}}(x,t) = \prod_{j=1}^{d} \left[1_{\bar{\mathfrak{v}}}(j)\, R_1(x_j,t_j) \;+\; 1_{\mathfrak{v}}(j)\, R_2(x_j,t_j) \right] \quad \text{for all } x,t \in D_d.$$

That is, we now take $R_1(x_j,t_j)$ if $j \notin \bar{\mathfrak{v}}$ and $R_2(x_j,t_j)$ if $j \in \mathfrak{v}$.

For $\mathfrak{v} \neq \emptyset$, we have

$$\Delta_\emptyset(H(K_{d,\mathfrak{v}})) = \{0\}.$$

Indeed, it is enough to check that $\Delta_\emptyset(K_{d,\mathfrak{v}}(\cdot,t)) = 0$ for all $t \in D_d$. We have

$$\Delta_\emptyset(K_{d,\mathfrak{v}}(\cdot,t)) = \prod_{j=1}^{d} \left[1_{\bar{\mathfrak{v}}}(j)\, A_1 R_1(\cdot,t_j) + 1_{\mathfrak{v}}(t_j)\, A_1 R_2(\cdot,t_j) \right].$$

Since $\mathfrak{v} \neq \emptyset$, at least one factor is $A_1 R_2(\cdot, t_j) = 0$, and therefore the whole product is zero, as claimed.

For $\mathfrak{v} = \emptyset$, we have

$$K_{d,\emptyset}(x,t) = \prod_{j=1}^{d} R_1(x_j, t_j) = \prod_{j=1}^{d} \eta_j(x_j)\,\eta_j(t_j) \quad \text{for all } x, t \in D_d.$$

The space $H(K_{d,\emptyset})$ is one dimensional and $H(K_{d,\emptyset}) = \text{span}(\eta_d)$ with $\eta_d(x) = \prod_{j=1}^{d} \eta_1(x_j)$ for $x \in D_d$. We have

$$\Delta_\emptyset(f) = I_d(f) \quad \text{for all } f \in H(K_{d,\emptyset}).$$

We now check that for all $\mathfrak{u} \neq \emptyset$ and $i \in Q(d, \mathfrak{u})$ we have

$$\Delta_{\mathfrak{u},i}(H(K_{d,\mathfrak{v}})) = \{0\} \quad \text{for all } \mathfrak{v} \neq \mathfrak{u}. \tag{15.38}$$

As before, it is enough to verify that $\Delta_{\mathfrak{u},i}(K_{d,\mathfrak{v}}(\cdot, t)) = 0$ for all $t \in D_d$. We have

$$\Delta_{\mathfrak{u},i}(K_{d,\mathfrak{v}}(\cdot,t)) = \prod_{k=1}^{d} \left[1_{\bar{\mathfrak{u}}}(k) A_1 \left(K_{d,\mathfrak{v}}(\cdot,t)\right)_k + 1_{\mathfrak{u}}(k)(A_{i_k} - A_{i_k-1}) \left(K_{d,\mathfrak{v}}(\cdot,t)\right)_k \right],$$

where

$$\left(K_{d,\mathfrak{v}}(\cdot,t)\right)_k = 1_{\bar{\mathfrak{v}}}(k) R_1(\cdot, t_k) + 1_{\mathfrak{v}}(k) R_2(\cdot, t_k).$$

For $\mathfrak{v} = \emptyset$, we have $\left(K_{d,\mathfrak{v}}(\cdot,t)\right)_k = R_1(\cdot, t_k)$ for all k. For $k \in \mathfrak{u}$ we then have

$$(A_{i_k} - A_{i_k-1}) R_1(\cdot, t_k) = A_1 R_1(\cdot, t_k) - A_1 R_1(\cdot, t_k) = 0.$$

For $\mathfrak{v} \neq \emptyset$, there is an index k such that $k \in \mathfrak{v}$ and $k \notin \mathfrak{u}$, or $k \notin \mathfrak{v}$ and $k \in \mathfrak{u}$. If $k \in \mathfrak{v}$ and $k \notin \mathfrak{u}$ then the kth factor of $\Delta_{\mathfrak{u},i}(K_{d,\mathfrak{v}}(\cdot,t))$ is $A_1 R_2(\cdot, t_k) = 0$. Otherwise, if $k \notin \mathfrak{v}$ and $k \in \mathfrak{u}$ then the kth factor is

$$(A_{i_k} - A_{i_k-1}) R_1(\cdot, t_k) = (U_{i_k-1} - U_{i_k-2})(I - P_1) R_1(\cdot, t_k) = 0.$$

Hence (15.38) holds, as claimed.

For $f \in F_{d,\gamma}$, we have $f = \sum_{\mathfrak{u} \subseteq [d]} f_{\mathfrak{u}}$ with $f_{\mathfrak{u}} \in H(K_{d,\mathfrak{u}})$ and

$$f(x) = \sum_{\mathfrak{u} \subseteq [d]} f_{\mathfrak{u}}(x_{\mathfrak{u}}, a) \quad \text{for all } x \in D_d,$$

where $y = (x_{\mathfrak{u}}, a) \in D_d$ with $y_j = a$ for $j \notin \mathfrak{u}$ and $y_j = x_j$ for $j \in \mathfrak{u}$. Then (15.38) yields

$$\Delta_{\mathfrak{u},i}(f) = \Delta_{\mathfrak{u},i}(f_{\mathfrak{u}}) = \beta^{d-|\mathfrak{u}|} \bigotimes_{k \in \mathfrak{u}} \left(U_{i_k} - U_{i_k-1}\right) \left(f_{\mathfrak{u}}(\cdot, a)\right) \quad \text{for all } f \in F_{d,\gamma}, \tag{15.39}$$

where

$$\beta = \frac{h_1(a)}{K_1(a,a)}.$$

We now show that

$$I_{d,\gamma} = I_d = \Delta_\emptyset + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \sum_{i \in Q(d,\mathfrak{u})} \Delta_{\mathfrak{u},i}. \tag{15.40}$$

Indeed, the first equality holds and the right-hand side of (15.40) is

$$\bigotimes_{k=1}^{d} A_1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \sum_{i \in Q(d,\mathfrak{u})} \bigotimes_{k=1}^{d} \left[1_{\bar{\mathfrak{u}}}(k)\, A_1 + 1_{\mathfrak{u}}(k)\, (A_{i_k} - A_{i_k-1}) \right]$$
$$= \bigotimes_{k=1}^{d} A_1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \sum_{i \in Q(d,\mathfrak{u})} \bigotimes_{k=1}^{d} \left[1_{\bar{\mathfrak{u}}}(k)\, A_1 + 1_{\mathfrak{u}}(k)\, (U_{i_k-1} - U_{i_k-2})(I - P_1) \right]$$
$$= \bigotimes_{k=1}^{d} A_1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \bigotimes_{k=1}^{d} [1_{\bar{\mathfrak{u}}}(k)\, A_1 + 1_{\mathfrak{u}}(k)\, I_1(I - P_1)]$$
$$= \bigotimes_{k=1}^{d} [A_1 + I_1(I - P_1)] = \bigotimes_{k=1}^{d} I_1 = I_d,$$

as claimed.

We are ready to define algorithms for approximating $I_{d,\gamma}(f)$ for $f \in F_{d,\gamma}$. These algorithms will depend on several parameters, but most of them will be suppressed, and we list only the dependence on d and $q = \{q(\mathfrak{u})\}_{\mathfrak{u} \subseteq [d]}$ for some non-negative integers $q(\mathfrak{u})$ that will be specified later.

The algorithms are based on the formula (15.40). Note that the set $Q(d,\mathfrak{u})$ in (15.40) is infinite. We obtain the algorithms $A_w(q,d)$ by truncating the set $Q(d,\mathfrak{u})$ to a finite set

$$Q(\mathfrak{u}) = \left\{ i \in Q(d,\mathfrak{u}) \mid \textstyle\sum_{k \in \mathfrak{u}} i_k \le q(\mathfrak{u}) + |\mathfrak{u}| \right\} = \left\{ i \in Q(d,\mathfrak{u}) \,\middle|\, |i| \le q(\mathfrak{u}) + d \right\}.$$

We later choose integers $q(\mathfrak{u})$ to guarantee that the worst case error of $A_w(q,d)$ is at most ε for the absolute or normalized error criterion. Obviously, the set $Q(\mathfrak{u}) = \emptyset$ iff $q(\mathfrak{u}) < |\mathfrak{u}|$.

The algorithms $A_w(q,d)$ are therefore of the form

$$A_w(q,d) = \Delta_\emptyset + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \sum_{i \in Q(\mathfrak{u})} \Delta_{\mathfrak{u},i}. \tag{15.41}$$

We call the algorithms $A_w(q,d)$ as the *weighted Smolyak* algorithms or the *weighted tensor product* (WTP) algorithms. In the next subsection we show how they are related to the Smolyak algorithms.

For $f \in F_{d,\gamma}$, we have

$$A_w(q,d)(f) = \beta^d \ f(\vec{a}) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \ \sum_{i \in Q(\mathfrak{u})} \bigotimes_{k=1}^{d} \left[1_{\bar{\mathfrak{u}}}(k) A_1 + 1_{\mathfrak{u}}(k)(A_{i_k} - A_{i_k-1}) \right] (f).$$

To get familiar with the last formula, assume for a moment that

$$f(x) = \prod_{j=1}^{d} f_j(x_j) \quad \text{for all } x \in D_d \text{ with } f_j \in F_1.$$

Then

$$A_w(d,q)(f) = \beta^d \ f(\vec{a}) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \ \sum_{i \in Q(\mathfrak{u})} \beta^{d-|\mathfrak{u}|} \prod_{k \notin \mathfrak{u}} f_k(a) \ \prod_{k \in \mathfrak{u}} (A_{i_k} - A_{i_k-1})(f_k).$$

15.3.1 Explicit Form

We first show a relation between the WTP and Smolyak algorithms. From (15.39) for $f \in F_{d,\gamma}$ we obtain

$$A_w(q,d)(f) = \beta^d f(\vec{a}) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} \sum_{i \in Q(\mathfrak{u})} \bigotimes_{k \in \mathfrak{u}} (U_{i_k-1} - U_{i_k-2}) \ (f_{\mathfrak{u}}(\cdot, a)).$$

Note that

$$\begin{aligned} \sum_{i \in Q(\mathfrak{u})} \bigotimes_{k \in \mathfrak{u}} (U_{i_k-1} - U_{i_k-2}) \ f_{\mathfrak{u}}(\cdot, a) &= \sum_{i_j \geq 2, \ i_1+i_2+\cdots+i_{|\mathfrak{u}|} \leq q(\mathfrak{u})+|\mathfrak{u}|} \ \bigotimes_{k \in \mathfrak{u}} \Delta_{i_k-1} \\ &= \sum_{i_j \geq 1, \ i_1+i_2+\cdots+i_{|\mathfrak{u}|} \leq q(\mathfrak{u})} \ \bigotimes_{k \in \mathfrak{u}} \Delta_{i_k}. \end{aligned}$$

The last sum is just the Smolyak algorithm (15.1) applied for approximating

$$I_{\mathfrak{u}} = \bigotimes_{k \in \mathfrak{u}} I_1 \big|_{H(R_2)}$$

with $d = |\mathfrak{u}|$ and $q = q(\mathfrak{u})$. We denote this Smolyak algorithm by $A_{\mathfrak{u}}(q(\mathfrak{u}), |\mathfrak{u}|)$ to stress the dependence on active variables from the subset $\mathfrak{u}$. For $\mathfrak{u} = \emptyset$ we set $q(\emptyset) = 0$ and

$$A_{\emptyset}(0,0) \ (f_{\emptyset}(\cdot, a)) = \beta^d \ f(\vec{a}).$$

Then we can rewrite the WTP algorithm as a weighted sum of the Smolyak algorithms,

$$A_w(q,d)(f) = \sum_{\mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} A_{\mathfrak{u}}(q(\mathfrak{u}), |\mathfrak{u}|) \, (f_{\mathfrak{u}}(\cdot, a)) \quad \text{for all } f \in F_{d,\gamma}. \tag{15.42}$$

Since we know the explicit form of $A_{\mathfrak{u}}(q(\mathfrak{u}), |\mathfrak{u}|)$ from Lemma 15.1 we easily find the explicit form of the WTP algorithm $A_w(q,d)$ in terms of algorithms A_1 and U_i. We summarize this explicit form in the following lemma.

Lemma 15.11. *The WTP algorithm has the form*

$$A_w(q,d) = \bigotimes_{k=1}^{d} A_1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} \sum_{\vec{i} \in P_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)} (-1)^{q(\mathfrak{u})-|\vec{i}|} \bigotimes_{k \in \mathfrak{u}} U_{i_k},$$

where

$$P_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|) = \left\{ \vec{i} \in \mathbb{N}^{|\mathfrak{u}|} \;\middle|\; q(\mathfrak{u}) - |\mathfrak{u}| + 1 \leq |\vec{i}| \leq q(\mathfrak{u}) \right\}.$$

If

$$U_i(f) = \sum_{j=1}^{m_i} a_{i,j} f(x_{i,j}) \quad \textit{for all } f \in H(R_2)$$

with $a_{i,j} \in \mathbb{R}$ *and* $x_{i,j} \in D_1$ *then*

$$A_w(q,d)(f) = \beta^d \, f(\vec{a}) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} \beta_{\mathfrak{u}}$$

with

$$\beta_{\mathfrak{u}} = \sum_{\vec{i} \in P_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)} (-1)^{q(\mathfrak{u})-|\vec{i}|} \binom{|\mathfrak{u}|-1}{q(\mathfrak{u}) - |\vec{i}|} \sum_{\vec{1} \leq \vec{j} \leq \vec{m}_{\vec{i}}} a_{\vec{i},\vec{j}} f\left((x_{\vec{i},\vec{j}})_{\mathfrak{u}}, a\right),$$

where

$$a_{\vec{i},\vec{j}} = \prod_{k=1}^{d} a_{i_j,j_k} \quad \textit{and} \quad \vec{m}_{\vec{i}} = [m_{i_1}, m_{i_2}, \ldots, m_{i_d}],$$

and the vector $y = ((x_{\vec{i},\vec{j}})_{\mathfrak{u}}, a)$ *has components* $y_k = x_{i_k,j_k}$ *for* $k \in \mathfrak{u}$ *and* $y_k = a$ *for* $k \notin \mathfrak{u}$.

15.3.2 Explicit Error Bound

It is easy to obtain an explicit error bound of the WTP algorithms using (15.42) and the explicit error bounds of the Smolyak algorithms established in Section 15.2.2. More precisely, we have

$$I_{d,\gamma}(f) = \sum_{\mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} I_{\mathfrak{u}}\left(f_{\mathfrak{u}}(\cdot,a)\right) \quad \text{for all } f \in F_{d,\gamma}.$$

This and (15.42) yield

$$I_{d,\gamma}(f) - A_w(q,d)(f) = \sum_{\mathfrak{u} \subseteq [d]} \beta^{d-|\mathfrak{u}|} \left[I_{\mathfrak{u}}\left(f_{\mathfrak{u}}(\cdot,a)\right) - A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)\left(f_{\mathfrak{u}}(\cdot,a)\right) \right]$$

for all $f \in F_{d,\gamma}$.

Denote with $e(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))$ the worst case error of the Smolyak algorithm $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)$ for approximating $I_{\mathfrak{u}}$. The estimates on $e(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))$ are given in Section 15.2.2. For $q(\mathfrak{u}) < |\mathfrak{u}|$ we have $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|) = 0$. Therefore in this case $e(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)) = \|I_{\mathfrak{u}}\| = \|h_{1,2}\|_{H(R_2)}^{|\mathfrak{u}|}$.

For all $f \in F_{d,\gamma}$ we obtain

$$\begin{aligned}
&\left|I_{d,\gamma}(f) - A_w(q,d)(f)\right| \\
&\quad\le \sum_{\mathfrak{u}\subseteq[d]} |\beta|^{d-|\mathfrak{u}|}\, |I_{\mathfrak{u}}\,(f_{\mathfrak{u}}(\cdot,a)) - A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)\,(f_{\mathfrak{u}}(\cdot,a))| \\
&\quad\le \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} |\beta|^{d-|\mathfrak{u}|}\, e(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))\, \|f_{\mathfrak{u}}(\cdot,a)\|_{H(K_{d,\mathfrak{u}})} \\
&\quad= \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} |\beta|^{d-|\mathfrak{u}|}\, e(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))\, \sqrt{\gamma_{d,\mathfrak{u}}}\, \|f_{\mathfrak{u}}\|_{F_{d,\gamma}} \\
&\quad\le \Bigg[\sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, |\beta|^{2(d-|\mathfrak{u}|)}\, e^2(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))\Bigg]^{1/2} \|f\|_{F_{d,\gamma}}.
\end{aligned}$$

By $e(A_w(q,d))$ we mean the worst case error of the WTP algorithm $A_n(q,d)$. Since the last estimates are sharp we obtain the following lemma.

Lemma 15.12. *The square of the worst case error of the WTP algorithm is given by*

$$e^2(A_w(q,d)) = \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}\, |\beta|^{2(d-|\mathfrak{u}|)}\, e^2(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)).$$

We are ready to apply the estimates on $e^2(A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|))$ from Section 15.2.2. To do this we need to assume the error estimates for the algorithms U_i. We proceed as in Section 15.2.2 with the space F_1 replaced now by the space $H(R_2)$. Hence, we assume

$$\|I_1\|_{H(R_2)\to\mathbb{R}} \le B, \tag{15.43}$$

$$\|I_1 - U_i\|_{H(R_2)\to\mathbb{R}} \le C\, D^i \quad \text{for all } i \ge 0, \tag{15.44}$$

$$\|\Delta_i\| = \|U_i - U_{i-1}\|_{H(R_2)\to\mathbb{R}} \le E\, D^i \quad \text{for all } i \ge 1. \tag{15.45}$$

Then we apply Lemma 15.2 if the algorithms U_i use non-nested information or we apply Lemma 15.3 if the algorithms U_i use nested information (15.10) and are optimal, i.e.,

$$U_i = I_1 P_i$$

with the orthogonal projection P_i on the linear subspace

$$\operatorname{span}\{R_2(\cdot,x_j) \mid j = 1,2,\dots,m_i\},$$

see (15.11), and obtain the following lemma.

Lemma 15.13.

- *Let* (15.43), (15.44) *and* (15.45) *hold. Then*

$$e^2(A_w(q,d)) \le \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} |\beta|^{2(d-|\mathfrak{u}|)} \beta_{\mathfrak{u}},$$

where

$$\beta_{\mathfrak{u}} = \begin{cases} \left[C\, H^{|\mathfrak{u}|-1}\, D^{\,q(\mathfrak{u})} \binom{q(\mathfrak{u})}{|\mathfrak{u}|-1}\right]^2 & \text{if } q(\mathfrak{u}) \ge |\mathfrak{u}|, \\ \|h_{1,2}\|_{H(R_2)}^{2\,|\mathfrak{u}|} & \text{if } q(\mathfrak{u}) < |\mathfrak{u}| \end{cases}$$

with $H = \max(B/D, E)$.

- *Assume that* (15.44) *holds. For nested information* N_i *of* (15.10) *and optimal* U_i *we have*

$$e^2(A_w(q,d)) \le \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} |\beta|^{2(d-|\mathfrak{u}|)} \beta_{\mathfrak{u}},$$

where

$$\beta_{\mathfrak{u}} = \begin{cases} C^{2\,|\mathfrak{u}|}\, D^{\,2(q(\mathfrak{u})-|\mathfrak{u}|+1)} \binom{q(\mathfrak{u})}{|\mathfrak{u}|-1} & \text{if } q(\mathfrak{u}) \ge |\mathfrak{u}|, \\ \|h_{1,2}\|_{H(R_2)}^{2\,|\mathfrak{u}|} & \text{if } q(\mathfrak{u}) < |\mathfrak{u}|. \end{cases}$$

15.3.3 Explicit Cost Bound

We now derive bounds on the number of function values used by the WTP algorithms. As before, we assume that the algorithms U_i use at most m_i function values such that (15.18) holds, i.e.,

$$m_i \le M_0\,(M^i - 1) \tag{15.46}$$

for some $M > 1$ and $M_0 > 0$.

Let $m_w(q,d)$ denote the number of function values used by the WTP algorithm $A_w(q,d)$. Then we apply Lemma 15.5 to conclude the following lemma.

Lemma 15.14. *Let* (15.18) *hold.*

- *For non-nested information*

$$m_w(q,d) \le 1 + \frac{M}{M-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} M_0^{|\mathfrak{u}|}\, M^{q(\mathfrak{u})} \binom{q(\mathfrak{u})-1}{|\mathfrak{u}|-1}.$$

- *For nested information*

$$m_w(q,d) \le 1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} M_0^{|\mathfrak{u}|}\, M^{q(\mathfrak{u})} \left(\frac{M-1}{M}\right)^{|\mathfrak{u}|-1} \binom{q(\mathfrak{u})-1}{|\mathfrak{u}|-1}.$$

We add in passing that for $q(\mathfrak{u}) < |\mathfrak{u}|$ the binomial coefficients above are zero, and this agrees with the fact that the cardinality of $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|) = 0$ is indeed zero. This explains why we do not have to consider the two cases depending on whether or not $q(\mathfrak{u}) \geq |\mathfrak{u}|$ holds as was needed for the error bounds.

15.3.4 ε-Cost Bound

We want to minimize the number of function values used by the WTP algorithm $A_w(q,d)$ such that the worst case error of $A_w(q,d)$ is at most ε for the absolute or normalized error criterion. More precisely, we want to determine $q = \{q(\mathfrak{u})\}$ such that $e(A_w(q,d)) \leq \varepsilon\,\mathrm{CRI}_d$ and $m_w(q,d)$ is as small as possible. Here, as always, $\mathrm{CRI}_d = 1$ for the absolute error criterion and $\mathrm{CRI}_d = \|I_{d,\gamma}\|$ for the normalized error criterion.

From now on we assume that we use nested information and optimal algorithms U_i. For all non-empty $\mathfrak{u} \subseteq [d]$, we want to approximate $I_{\mathfrak{u}}$ by $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)$ to within $\varepsilon_{\mathfrak{u}}$. The non-negative parameters $\varepsilon_{\mathfrak{u}}$ will be specified later in terms of ε and the error criterion.

To get the error $\varepsilon_{\mathfrak{u}}$ we need to define $q(\mathfrak{u})$. If $\varepsilon_{\mathfrak{u}} \geq \|I_{\mathfrak{u}}\| = \|h_{1,2}\|_{H(R_2)}^{|\mathfrak{u}|}$ then we set $q(\mathfrak{u}) = 0$ so that $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|) = 0$. If $\varepsilon_{\mathfrak{u}} < \|I_{\mathfrak{u}}\|$ then we define $q(\mathfrak{u})$ as in Theorem 15.6. Then the number of function values $m(q(\mathfrak{u}),\mathfrak{u})$ used by $A_{\mathfrak{u}}(q(\mathfrak{u}),|\mathfrak{u}|)$ is bounded in Theorem 15.6, see also (15.25), as

$$m(q(\mathfrak{u}),\mathfrak{u}) \leq \beta_1 \left(\beta_2 + \beta_3 \, \frac{\ln 1/\varepsilon_{\mathfrak{u}}}{|\mathfrak{u}|-1}\right)^{\beta_4(|\mathfrak{u}|-1)} \varepsilon_{\mathfrak{u}}^{-\beta_5}$$

for β_i fully determined by the parameters C, D, M_0 and M for the univariate problem $I_1\big|_{H(R_2)}$, see (15.44) and (15.46). In particular,

$$\beta_5 = \ln M / \ln D^{-1}.$$

For $|\mathfrak{u}| = 1$, we formally have ∞^0 above, and we formally set $\infty^0 = 1$.

From Lemma 15.12 we obtain

$$e^2(A_w(q,d)) \leq \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} |\beta|^{2(d-|\mathfrak{u}|)} \varepsilon_{\mathfrak{u}}^2, \tag{15.47}$$

again with $0\,\infty = 0$. Let

$$\kappa(\mathfrak{u},\varepsilon_{\mathfrak{u}}) = \begin{cases} 1 & \text{if } \varepsilon_{\mathfrak{u}} < \|h_{1,2}\|_{H(R_2)}^{|\mathfrak{u}|}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the number of function values $m_w(q,d)$ used by the algorithm $A_w(q,d)$ is bounded by

$$m_w(q,d) \leq 1 + \beta_1 \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \kappa(\mathfrak{u},\varepsilon_{\mathfrak{u}}) \left(\beta_2 + \beta_3 \, \frac{\ln 1/\varepsilon_{\mathfrak{u}}}{|\mathfrak{u}|-1}\right)^{\beta_4(|\mathfrak{u}|-1)} \varepsilon_{\mathfrak{u}}^{-\beta_5}. \tag{15.48}$$

We hope that it is clear what we should do next. We should select $\varepsilon_{\mathfrak{u}}$ to minimize the right-hand side of (15.48) subject to the condition that the right-hand side of (15.47) is at most $\varepsilon^2\,\mathrm{CRI}_d^2$. This is, however, a highly nonlinear problem and the system of nonlinear equations for such $\varepsilon_{\mathfrak{u}}$ is quite ugly. To simplify further calculations we bound the right-hand side of (15.48) to get rid of logarithms. We need the following lemma.

Lemma 15.15. *For any positive δ define*

$$C_\delta = \beta_4 \left(\ln \frac{\beta_3\,\beta_4}{\delta\,\beta_2} - 1 + \frac{\delta\,\beta_2}{\beta_3\,\beta_4}\right)_+ + \delta\,\beta_4 \ln \beta.$$

Then

$$\left(\beta_2 + \beta_3\,\frac{\ln 1/\varepsilon}{k}\right)^{\beta_4\,k} \le C_\delta^k\,\varepsilon^{-\delta} \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } k \in \mathbb{N}.$$

Proof. Taking the logarithms we need to show that

$$\beta_4\,k\left[\ln \beta_2 + \ln\left(1 + \frac{\beta_3}{\beta_2}\,\frac{\ln 1/\varepsilon}{k}\right)\right] \le k\,C_\delta + \delta \ln 1/\varepsilon,$$

or, equivalently, that

$$\ln\left(1 + \frac{\beta_3}{\beta_2}\,\frac{\ln 1/\varepsilon}{k}\right) \le \frac{C_\delta - \beta_4 \ln \beta_2}{\beta_4} + \frac{\delta}{\beta_4}\,\frac{\ln 1/\varepsilon}{k}.$$

Let $x = k^{-1} \ln 1/\varepsilon$ and $c = \beta_3/\beta_2$. We need to show that

$$f(x) := \ln(1 + c\,x) - \delta\,\beta_4^{-1}\,x \le \beta_4^{-1}\,(C_\delta - \delta\,\beta_4 \ln \beta_2) \quad \text{for all } x \in \mathbb{R}_+.$$

Since $\lim_{x\to\infty} f(x) = -\infty$ we conclude that $\sup_{x\in\mathbb{R}_+} f(x) < \infty$. In fact, by standard argument, we find out that

$$\sup_{x\in\mathbb{R}_+} f(x) = \left(\ln \frac{c\beta_4}{\delta} - 1 + \frac{\delta}{c\,\beta_4}\right)_+.$$

Therefore for

$$C_\delta = \beta_4 \sup_{x\in\mathbb{R}_+} f(x) + \delta\,\beta_4 \ln \beta_2,$$

the last inequality holds. This completes the proof. □

Applying Lemma 15.15 to the right-hand side of (15.48) we obtain

$$m_w(q,d) \le 1 + \beta_1 \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \kappa(\mathfrak{u},\varepsilon_{\mathfrak{u}})\; C_\delta^{|\mathfrak{u}|-1}\,\varepsilon_{\mathfrak{u}}^{-(\beta_5+\delta)}. \tag{15.49}$$

The next lemma has a standard proof and therefore its proof is omitted.

Lemma 15.16. *For non-negative a_i, b_i with $i = 1, 2, \dots, k$, and positive τ and ε, consider the following minimization problem*

$$\min := \min_{\varepsilon_i\,:\,\varepsilon_i \ge 0,\ \sum_{i=1}^k a_i\,\varepsilon_i^2 \le \varepsilon^2} \ \sum_{i=1}^k b_i\,\varepsilon_i^{-\tau}.$$

Then

$$\min = \Big(\sum_{i=1}^k a_i^{\tau/(\tau+2)}\, b_i^{2/(\tau+2)}\Big)^{(\tau+2)/2}\ \varepsilon^{-\tau}$$

and is achieved for

$$\varepsilon_i = \varepsilon\,\frac{(b_i/a_i)^{1/(\tau+2)}}{\big(\sum_{j=1}^k a_j^{\tau/(\tau+2)}\, b_j^{2/(\tau+2)}\big)^{1/2}} \quad \textit{for all } i = 1, 2, \dots, k.$$

If $a_i = 0$ then we formally take $\varepsilon_i = \infty$ and $0 \cdot \infty = 0$.

We are ready to apply Lemma 15.16 for our minimization problem,

$$\min_{\varepsilon_{\mathfrak{u}}:\ \ \varepsilon_{\mathfrak{u}} \ge 0,\ \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}\,|\beta|^{2(d-|\mathfrak{u}|)}\,\varepsilon_{\mathfrak{u}}^2 \le \varepsilon^2\,\mathrm{CRI}_d^2} \ \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} C_\delta^{|\mathfrak{u}|-1}\,\varepsilon_{\mathfrak{u}}^{-(\beta_5+\delta)}.$$

Here, δ can be an arbitrary positive number.

Using Lemma 15.16 with $k = 2^d - 1$, the coefficients a_i given by $\gamma_{d,\mathfrak{u}}|\beta|^{2(d-|\mathfrak{u}|)}$, where $|\beta| = \|h_{1,1}\|_{H(R_1)}$, the coefficients β_i given by $C_\delta^{|\mathfrak{u}|-1}$, $\tau = p > \beta_5$ and ε replaced by $\varepsilon\,\mathrm{CRI}_d$, we obtain the following theorem.

Theorem 15.17. *Assume that algorithms U_i are optimal, we use nested information and (15.44) and (15.46) are satisfied. Let*

$$p > \frac{\ln M}{\ln D^{-1}}.$$

Consider the WTP algorithms $A_w(q,d)$ with $\varepsilon_{\mathfrak{u}}$ defined for all non-empty $\mathfrak{u} \subseteq [d]$ by

$$\varepsilon_{\mathfrak{u}} = \varepsilon\,\mathrm{CRI}_d\,\frac{\Big[\dfrac{C_\delta^{|\mathfrak{u}|-1}}{\gamma_{d,\mathfrak{u}}\,\|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)}}\Big]^{1/(p+2))}}{\big(\sum_{\emptyset \ne \mathfrak{v} \subseteq [d]} \gamma_{d,\mathfrak{v}}^{p/(p+2)}\,\|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{v}|)\,p/(p+2)}\,C_\delta^{2(|\mathfrak{v}|-1)/(p+2)}\big)^{1/2}}.$$

Here, $\delta = p - \ln M/\ln D^{-1}$. Then

$$e(A_w(q,d)) \le \varepsilon\,\mathrm{CRI}_d$$

and

$$m_w(q,d) \le 1 + \beta_1\,C(d,\gamma)\,\varepsilon^{-p}, \tag{15.50}$$

where

$$C(d,\gamma) = \Big(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)p/(p+2)} \, C_\delta^{2(|\mathfrak{u}|-1)/(p+2)} \Big)^{(p+2)/2}$$

for the absolute error criterion, and

$$C(d,\gamma) = \frac{\big(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)p/(p+2)} \, C_\delta^{2(|\mathfrak{u}|-1)/(p+2)} \big)^{(p+2)/2}}{\big(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)} \, \|h_{1,2}\|_{H(R_2)}^{2|\mathfrak{u}|} \big)^{p/2}}$$

for the normalized error criterion.

We stress that $\varepsilon_{\mathfrak{u}}$ in the last theorem depends on the weight $\gamma_{d,\mathfrak{u}}$. Therefore, the use of the WTP algorithm defined as in Theorem 15.17 requires the knowledge of the weights $\gamma_{d,\mathfrak{u}}$ for all $\mathfrak{u} \subseteq [d]$. Later, we will discuss what happens if we use incorrect weights. That is, we use $\gamma_{d,\mathfrak{u}}$ although our functions belong to $H(K_{d,\eta})$ for some weights $\eta = \{\eta_{d,\mathfrak{u}}\}$ not necessarily equal to γ.

In the next subsections we use the error and cost estimates obtained so far to determine when the WTP algorithms yield tractability bounds for $I_\gamma = \{I_{d,\gamma}\}$. We limit our interest only to finite-order weights and product weights.

For general weights, various kinds of tractability hold under appropriate conditions on $C(d,\gamma)$. It would be of interest to obtain more explicit conditions on $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ that guarantee tractability. This leads us to the next open problem.

Open Problem 70.

Consider the WTP algorithms for general weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ for approximation of $I_\gamma = \{I_{d,\gamma}\}$, defined as in this subsection.

- Find the most lenient explicit conditions on the weight sequence γ for which the WTP algorithm yields strong polynomial, polynomial, T-tractable or weakly tractable bounds for I_γ.

15.3.5 Tractability for Finite-Order Weights

In this subsection we assume that $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ is a sequence of *finite-order weights* of order ω, i.e., that

$$\gamma_{d,\mathfrak{u}} = 0 \quad \text{for all } d \text{ and for all } \mathfrak{u} \text{ with } |\mathfrak{u}| > \omega.$$

To make the problem non-trivial we assume that $\omega \geq 1$.

We analyze two classes of WTP algorithms for finite-order weights. The first class is specified by the choice of $\varepsilon_{\mathfrak{u}}$ given by Theorem 15.17. Then this choice of $\varepsilon_{\mathfrak{u}}$ leads to the definition of $q(\mathfrak{u})$ as in Theorem 15.6. As we shall see, for the first class of WTP algorithms we obtain polynomial dependence on ε^{-1} with an exponent $p > p^*$, where

$p^* = \ln M / \ln D^{-1}$ is the exponent of the univariate case. We also obtain conditions on finite-order weights for which we have strong polynomial and polynomial tractability. We stress again that in this case, we have to know all the weights $\gamma_{d,\mathfrak{u}}$ for $|\mathfrak{u}| \le \omega$, as is clear from the definition of $\varepsilon_{\mathfrak{u}}$ in Theorem 15.17.

The second class of WTP algorithms is specified by a different choice of $q(\mathfrak{u})$, which depends much less on the finite-order weights. This will be especially clear for the normalized error criterion where the dependence is only through ω, the order of the finite-order weights. For the absolute error criterion, the dependence is only through ω and the logarithm of the norm of $\|I_{d,\gamma}\|$. The logarithmic dependence means that we only need a rough upper bound on the norm of $\|I_{d,\gamma}\|$. For the second class of WTP algorithms we obtain a polynomial dependence on ε^{-1} with an exponent $p = p^*$ modulo some powers of $\ln \varepsilon^{-1}$ independent of d. We also obtain conditions on strong polynomial and polynomial tractability.

It is interesting to compare the tractability results for the two classes of WTP algorithms. For the normalized error criterion we always have polynomial tractability for both classes, however, strong polynomial tractability holds only for the first class under a suitable condition on finite-order weights. Hence, the less demanding dependence on finite-order weights for the second class is at the expense of losing strong polynomial tractability.

Later, we briefly discuss what happens if we do not have correct information about finite-order weights or about their order. That is when we use the WTP algorithms for finite order weights $\gamma_{d,\mathfrak{u}}$ of order ω for functions that belong to a space equipped with finite-order weights $\eta = \{\eta_{d,\mathfrak{u}}\}$ with η not necessarily equal to γ and of order not necessarily equal to ω.

15.3.6 The First Class of WTP Algorithms

We consider the WTP algorithms $A_w(q,d)$ with optimal algorithms U_i that use nested information and for which the $\varepsilon_{\mathfrak{u}}$'s are defined in Theorem 15.17. Note that $\varepsilon_{\mathfrak{u}} = \infty$ for all $\mathfrak{u}$ with $|\mathfrak{u}| > \omega$. Then $q(\mathfrak{u}) = 0$ and the part of the WTP algorithm $A_{\mathfrak{u}}(q(\mathfrak{u}), |\mathfrak{u}|) = 0$. This means that there are no contributions for such terms, which is quite natural since $f_{\mathfrak{u}} = 0$ and there is no need to approximate $I_{d,\gamma}(f_{\mathfrak{u}}) = 0$. We now specify the estimates presented in Theorem 15.17 for finite-order weights of order ω.

We first address the absolute error criterion. The number $C(d,\gamma)$ is now

$$C(d,\gamma) = \frac{\|h_{1,1}\|_{H(R_1)}^{dp}}{C_\delta} \Bigg[\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}} \right)^{2|\mathfrak{u}|/(p+2)} \Bigg]^{(p+2)/2}.$$

Let

$$C_{\mathrm{abs},\delta} = \frac{1}{\|h_{1,1}\|_{H(R_1)}^{p}} \max\left(1, \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}}\right)^{\omega-1}\right).$$

We stress that $C_{\mathrm{abs},\delta}$ is independent of d. Then

$$C(d,\gamma) \le C_{\mathrm{abs},\delta}\, \|h_{1,1}\|_{H(R_1)}^{dp} \Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \Bigg)^{(p+2)/2}.$$

This estimate on $C(d,\gamma)$ will be used to find conditions on the lack of exponential dependence on d and tractability conditions on $I_\gamma = \{I_{d,\gamma}\}$ for the absolute error criterion.

We now turn to the normalized error criterion. The number $C(d,\gamma)$ from Theorem 15.17 is now of the form

$$C(d,\gamma) = \frac{\|h_{1,1}\|_{H(R_1)}^{dp} \left[\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}} \right)^{2|\mathfrak{u}|/(p+2)} \right]^{(p+2)/2}}{C_\delta\, \|h_{1,1}\|_{H(R_1)}^{dp} \left[\sum_{\mathfrak{u}\subseteq[d],\ |\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \left(\frac{\|h_{1,2}\|_{H(R_2)}}{\|h_{1,1}\|_{H(R_1)}} \right)^{2\,|\mathfrak{u}|} \right]^{p/2}}.$$

Let

$$C_{\mathrm{nor},\delta} = \frac{1}{\|h_{1,2}\|_{H(R_2)}^{p}}\ \frac{\max\left(1, \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}}\right)^{\omega-1}\right)}{\min\left(1, \left(\frac{\|h_{1,2}\|_{H(R_2)}}{\|h_{1,1}\|_{H(R_1)}}\right)^{p(\omega-1)}\right)}.$$

We also stress that $C_{\mathrm{nor},\delta}$ is independent of d. Then

$$C(d,\gamma) \le C_{\mathrm{nor},\delta}\, \frac{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \right)^{(p+2)/2}}{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \right)^{p/2}}.$$

This estimate on $C(d,\gamma)$ will be used to find conditions on the lack of exponential dependence on d and tractability conditions on $I_\gamma = \{I_{d,\gamma}\}$ for the normalized error criterion.

Theorem 15.18. *Consider the problem $I_\gamma = \{I_{d,\gamma}\}$ for finite-order weights of order $\omega \ge 1$. Let the WTP algorithm $A_w(q,d)$ be defined with optimal algorithms U_i that use nested information and satisfy (15.44) and (15.46), and with $\varepsilon_{\mathfrak{u}}$ defined as in Theorem 15.17. Let*

$$p > p^* := \frac{\ln M}{\ln D^{-1}},$$

where p^* is the exponent for approximation of the univariate problem I_1. Then

$$e(A_w(q,d)) \le \varepsilon\, \mathrm{CRI}_d$$

and the number $m_w(q,d)$ of function values used by $A_w(q,d)$ is bounded as follows.

- *For the absolute error criterion, we have*

$$m_w(q,d) = \mathcal{O}\left(\|h_{1,1}\|_{H(R_1)}^{dp} \Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \Bigg)^{(p+2)/2} \varepsilon^{-p} \right)$$

  for all $\varepsilon < \|I_{d,\gamma}\|$.

- *For the normalized error criterion, we have*

$$m_w(q,d) = \mathcal{O}\left(\frac{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)}\right)^{(p+2)/2}}{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}\right)^{p/2}}\ \varepsilon^{-p} \right)$$

  for all $\varepsilon < 1$.

The factors in the big $\mathcal{O}$ *notation are independent of* d *and* ε^{-1}. *This implies the following tractability conditions.*

- *For the absolute error criterion:*
  - *If for some* q *we have*

$$\|h_{1,1}\|_{H(R_1)}^{dp} \Big(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}^{p/(p+2)}\Big)^{(p+2)/2} = \mathcal{O}(d^q) \quad \textit{for all } d \in \mathbb{N}$$

    then I_γ *is polynomially tractable, with an* ε^{-1} *exponent at most* p *and a* d *exponent at most* q. *In particular, if* $q = 0$ *then* I_γ *is strongly polynomially tractable with an exponent at most* p.
  - *If*

$$\lim_{d\to\infty} \frac{\ln\ \|h_{1,1}\|_{H(R_1)}^{dp} \left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)}\right)^{(p+2)/2}}{d} = 0$$

    then I_γ *is weakly tractable.*

- *For the normalized error criterion:*
  - *The problem* I_γ *is polynomially tractable for arbitrary finite-order weights of order* ω. *The* ε^{-1} *exponent is at most* p *for any* $p > p^*$, *and the* d *exponent is at most* ω. *If there is a number* $k \in [0,\omega)$ *such that*

$$\left|\{\gamma_{d,\mathfrak{u}} \mid \gamma_{d,\mathfrak{u}} > 0\,\}\right| = \mathcal{O}(d^k) \quad \textit{for all } d \in \mathbb{N}$$

    then the d *exponent is at most* k.
  - *If for some* $q \in [0,\omega)$ *we have*

$$\frac{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)}\right)^{(p+2)/2}}{\left(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}\right)^{p/2}} = \mathcal{O}(d^q) \quad \textit{for all } d \in \mathbb{N}$$

    then I_γ *is polynomially tractable with an* ε^{-1} *exponent is at most* p *and a* d *exponent is at most* q. *In particular, if* $q = 0$ *then* I_γ *is strongly polynomially tractable with an exponent at most* p.

All tractability bounds are achieved by WTP algorithms.

Proof. All points are quite clear except the point for the normalized error criterion, which states that we always have polynomial tractability for arbitrary finite-order weights. In this case, observe that Hölder's inequality with $p' = (p+2)/p$ and $q' = (p+2)/2$ yields

$$\Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \Bigg)^{(p+2)/2} \le \Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ 1\le|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}} \Bigg)^{p/2} \sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} 1.$$

The last sum is the cardinality of the finite-order weights of order ω, and as we know from [335], see also Volume I, p. 196, it is at most $2d^{\,\omega}$. Therefore

$$\frac{\big(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{p/(p+2)} \big)^{(p+2)/2}}{\big(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \big)^{p/2}} \le 2\,d^{\,\omega}.$$

This argument holds for all $p > p^*$. Obviously, if we eliminate zero weights $\gamma_{d,\mathfrak{u}}$ then the same argument yields the bound in terms of the total number of non-zero weights and if this cardinality is of order d^k then the d exponent is at most k. This completes the proof. □

The tractability conditions of Theorem 15.18 can be illustrated for the absolute error criterion as follows.

- For $\|h_{1,1}\|_{H(R_1)} < 1$ and uniformly bounded finite-order weights, i.e.,

  $$\sup_{d\in\mathbb{N}} \gamma_{d,\mathfrak{u}} < \infty,$$

  the problem I_γ is strongly polynomially tractable with an exponent at most p^*.

- For $\|h_{1,1}\|_{H(R_1)} = 1$ and uniformly bounded finite-order weights, the problem I_γ is polynomially tractable with an ε^{-1} exponent at most p and with a d exponent at most $\omega(p+2)/2$ for any $p > p^*$.

- For $\|h_{1,1}\|_{H(R_1)} > 1$ and uniformly bounded finite-order weights, we cannot claim even weak tractability of the problem I_γ.

For the normalized error criterion, we stress that there may be a tradeoff between the tractability exponents p and q. Depending on the weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$, we may be forced to choose large p to get a (strong) polynomial dependence on d and this will result in a large exponent with respect to ε^{-1}. In particular we may ask for which γ we get strong polynomial tractability. For the normalized error criterion, it is easy to rewrite the condition in Theorem 15.18 as the condition that was already presented in [335]. Namely, let

$$r^* = \sup\left(r \ge 1 \;\Big|\; \sup_{d\in\mathbb{N}} \frac{\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{1/r}}{\big(\sum_{\mathfrak{u}\subseteq[d],\ 1\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \big)^{1/r}} < \infty \right).$$

Then $r^* > 1$ implies that I_γ is strongly polynomially tractable with an exponent at most

$$\max\left(p^*, \frac{2}{r^*-1}\right).$$

Hence, this exponent may be arbitrarily large if r^* is arbitrarily close to 1.

We illustrate Theorem 15.18 for the normalized error criterion and for the finite-order weights of the form

$$\gamma_{d,\mathfrak{u}} = \binom{d}{|\mathfrak{u}|}^{-a} \quad \text{for all } |\mathfrak{u}| \le \omega$$

for some $a \in \mathbb{R}$. Then it is easy to check that

- if $a > 1$ then I_γ is strongly polynomially tractable with an exponent at most
$$\max\left(p^*, \frac{2}{a-1}\right),$$
- for all real a the problem I_γ is polynomially tractable with an ε^{-1} exponent at most p for any $p > p^*$ and a d exponent at most ω.

15.3.7 The Second Class of WTP Algorithms

In the previous subsection we considered WTP algorithms that fully depend on the finite-order weights. We also used $p > p^*$ and the corresponding numbers $C_{\text{abs},\delta}$ and $C_{\text{nor},\delta}$ with $\delta = p - p^*$ tend to infinity as δ goes to zero. In fact, this has to be so. The reason is that for $d > 2$ there is, in general, also the logarithmic dependence on ε^{-1}. The total cost is of order $\varepsilon^{-p^*}[\ln \varepsilon^{-1}]^a$ for some positive a that may be a linear function of d. The main point of introducing $p > p^*$ was to eliminate the powers of $\ln \varepsilon^{-1}$.

For finite-order weights, the powers of $\ln \varepsilon^{-1}$ are independent of d and it is tempting to allow $p = p^*$ at the expense of some powers of $\ln \varepsilon^{-1}$. The subject of this subsection is to work with $p = p^*$ and to analyze WTP algorithms with a weaker dependence on the finite-order weights.

We again consider the WTP algorithms $A_w(q,d)$ with optimal algorithms U_i that use nested information but we define $q(\mathfrak{u})$ differently as before without introducing $\varepsilon_{\mathfrak{u}}$. As before, we obviously define

$$q(\mathfrak{u}) = 0 \quad \text{for all } \mathfrak{u} \text{ with } |\mathfrak{u}| \ge \omega.$$

For simplicity, we also assume that

$$q(\mathfrak{u}) \ge |\mathfrak{u}| \quad \text{for all } |\mathfrak{u}| \in [1,\omega].$$

From Lemma 15.12 we then have

$$e^2(A_w(q,d)) \le \sum_{\mathfrak{u}\subseteq[d],\ 0<|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}\, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)}\, C^{2|\mathfrak{u}|}\, D^{2(q(\mathfrak{u})-|\mathfrak{u}|+1)} \binom{q(\mathfrak{u})}{|\mathfrak{u}|-1}.$$

From Stirling's formula we know that $k! \ge (k/e)^k$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned}\binom{q(\mathfrak{u})}{|\mathfrak{u}|-1} &\le \frac{[q(\mathfrak{u})]^{|\mathfrak{u}|-1}}{(|\mathfrak{u}|-1)!} \le [q(\mathfrak{u})]^{|\mathfrak{u}|-1} \left(\frac{e}{|\mathfrak{u}|-1+\delta_{|\mathfrak{u}|,1}}\right)^{|\mathfrak{u}|-1}\\ &\le [q(\mathfrak{u})]^{|\mathfrak{u}|-1} \left(\frac{e}{|\mathfrak{u}|-1+\delta_{|\mathfrak{u}|,1}}\right)^{-1} \left(\frac{\sqrt{e}}{\sqrt{|\mathfrak{u}|-1+\delta_{|\mathfrak{u}|,1}}}\right)^{2|\mathfrak{u}|}\\ &\le \frac{\omega}{e}\, [q(\mathfrak{u})]^{|\mathfrak{u}|-1} \left(\frac{\sqrt{e}}{\sqrt{|\mathfrak{u}|-1+\delta_{|\mathfrak{u}|,1}}}\right)^{2|\mathfrak{u}|},\end{aligned}$$

where $\delta_{|\mathfrak{u}|,1}$ is the Kronecker delta.

Let

$$M = \max_{k\in\mathbb{N}} M_k \quad \text{for } M_k = \left(\frac{C\,\sqrt{e}}{D\,\|h_{1,2}\|_{H(R_2)}\,\sqrt{k-1+\delta_{k,1}}}\right)^{2k}.$$

Note that M is well defined and finite since M_k tends to zero as k goes to infinity.

Let

$$r_{q(\mathfrak{u})} = D^{2\,q(\mathfrak{u})}\,[q(\mathfrak{u})]^{|\mathfrak{u}|-1} \quad \text{for all } \mathfrak{u}\subseteq[d],\ 0<|\mathfrak{u}|\le\omega,$$

and let

$$r_q = \max_{\mathfrak{u}\subseteq[d],0<|\mathfrak{u}|\le\omega} r_{q(\mathfrak{u})}.$$

Then we can estimate $e^2(A_w(q,d))$ by

$$\begin{aligned}\frac{e^2(A_w(q,d))}{\omega\, D^2/e} &\le \sum_{\substack{\mathfrak{u}\subseteq[d]\\ 0<|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}\, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)}\, \|h_{1,2}\|_{H(R_2)}^{2|\mathfrak{u}|}\, M_k\, r_{q(\mathfrak{u})}\\ &\le M\, r_q \sum_{\substack{\mathfrak{u}\subseteq[d]\\ 0<|\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}}\, \|h_{1,1}\|_{H(R_1)}^{2(d-|\mathfrak{u}|)}\, \|h_{1,2}\|_{H(R_2)}^{2|\mathfrak{u}|}.\end{aligned}$$

Note that the last sum is just the square of the initial error. Therefore we have

$$e(A_w(q,d)) \le \sqrt{\omega\, D^2\, M/e}\left(\max_{\substack{\mathfrak{u}\subseteq[d]\\ 0<|\mathfrak{u}|\le\omega}} D^{2\,q(\mathfrak{u})}\,[q(\mathfrak{u})]^{|\mathfrak{u}|-1}\right)^{1/2} \|I_{d,\gamma}\|. \tag{15.51}$$

We want to define $q(\mathfrak{u})$ in such a way that the error of $A_1(q,d)$ is at most ε for the absolute or normalized error criterion. We need the following lemma.

Lemma 15.19. *Let $\tau \in (0,1)$ and let ω be a positive integer. There exists a positive number C_0, depending only on τ and ω, such that for all $\delta \in (0,\tau]$ we have*

$$C_0 - 1 \geq \frac{\omega-1}{\ln \tau^{-1}} \ln\left(1 + \frac{\omega-1}{\ln \delta^{-1}} \ln \frac{\ln \delta^{-1}}{\ln \tau^{-1}} + C_0 \frac{\ln \tau^{-1}}{\ln \delta^{-1}}\right). \tag{15.52}$$

For such a C_0, we have

$$\tau^k k^{\omega-1} \leq \delta$$

for

$$k = \begin{cases} 1 & \text{if } \delta \geq \tau, \\ \left\lfloor C_0 + \frac{\ln \delta^{-1}}{\ln \tau^{-1}} + \frac{\omega-1}{\ln \tau^{-1}} \ln \frac{\ln \delta^{-1}}{\ln \tau^{-1}} \right\rfloor & \text{if } \delta < \tau. \end{cases}$$

Proof. First of all, note that $x^{-1} \ln x \leq 1/e$ for all $x \geq 1$ implies that the right-hand side of (15.52) is bounded by

$$\frac{\omega-1}{\ln \tau^{-1}} \ln\left(1 + \frac{\omega-1}{e \ln \tau^{-1}} + C_0\right),$$

and so we can clearly choose sufficiently large C_0 such that the last expression is at most $C_0 - 1$. This shows that there exists C_0 satisfying (15.52).

Consider now the inequality $\tau^k k^{\omega-1} \leq \delta$. For $\delta \geq \tau$, this inequality holds for $k = 1$. For $\delta < \tau$, the inequality $\tau^k k^{\omega-1} \leq \delta$ is equivalent to

$$k - \frac{\omega-1}{\ln \tau^{-1}} \ln k \geq \frac{\ln \delta^{-1}}{\ln \tau^{-1}}.$$

We now take

$$\begin{aligned} k^* &= C_0 + \frac{\ln \delta^{-1}}{\ln \tau^{-1}} + \frac{\omega-1}{\ln \tau^{-1}} \ln \frac{\ln \delta^{-1}}{\ln \tau^{-1}} \\ &= \frac{\ln \delta^{-1}}{\ln \tau^{-1}} \left(1 + \frac{\omega-1}{\ln \delta^{-1}} \ln \frac{\ln \delta^{-1}}{\ln \tau^{-1}} + C_0 \frac{\ln \tau^{-1}}{\ln \delta^{-1}}\right), \end{aligned}$$

and $k = \lfloor k^* \rfloor = k^* + a_\delta$, where $a_\delta \in (-1, 0]$. Then

$$k - \frac{\omega-1}{\ln \tau^{-1}} \ln k \geq k^* - 1 - \frac{\omega-1}{\ln \tau^{-1}} \ln k^*.$$

The inequality

$$k^* - 1 - \frac{\omega-1}{\ln \tau^{-1}} \ln k^* \geq \frac{\ln \delta^{-1}}{\ln \tau^{-1}}$$

is equivalent to the inequality (15.52) and is satisfied by the choice of C_0. This completes the proof. □

We are ready to apply Lemma 15.19 for the algorithm $A_w(q,d)$. We define

$$q(\mathfrak{u}) = k \quad \text{for all } \mathfrak{u} \subseteq [d],\ 0 < |\mathfrak{u}| \leq \omega$$

for some not yet specified k. From (15.51) we have

$$e^2(A_w(q,d)) \le \omega/e\, D^2\, M\, D^{2k}\, k^{\omega-1}\, \|I_{d,\gamma}\|^2 \le \varepsilon^2\, \mathrm{CRI}_d^2,$$

if we take

$$\tau = D^2 \quad \text{and} \quad \delta = \frac{e\,\varepsilon^2\,\mathrm{CRI}_d^2}{\omega\, D^2\, M\, \|I_{d,\gamma}\|^2}.$$

Then take k as in Lemma 15.19. That is, we have

- for the absolute error criterion

$$q(\mathfrak{u}) = \frac{\ln \|I_{d,\gamma}\|\,\varepsilon^{-1}}{\ln D^{-1}} + \frac{\omega-1}{2\ln D^{-1}} \ln\ln \|I_{d,\gamma}\|\,\varepsilon^{-1} + \mathcal{O}(1),$$

- for the normalized error criterion

$$q(\mathfrak{u}) = \frac{\ln \varepsilon^{-1}}{\ln D^{-1}} + \frac{\omega-1}{2\ln D^{-1}} \ln\ln \varepsilon^{-1} + \mathcal{O}(1).$$

We stress that now the dependence on finite-order weights is much less than before for the first class of WTP algorithms. For the normalized error criterion we need to only know the order ω, and for the absolute error criterion we also need to know the logarithm of $\|I_{d,\gamma}\|$.

We now turn to the cost of the algorithm $A_w(q,d)$. Assuming (15.46), and keeping in mind that nested information is used, we have from Lemma 15.14

$$m_w(q,d) \le 1 + \sum_{\mathfrak{u}\subseteq[d],\ 0<|\mathfrak{u}|\le\omega} M_0^{|\mathfrak{u}|}\, M^{q(u)} \left(\frac{M-1}{M}\right)^{|\mathfrak{u}|-1} \binom{q(\mathfrak{u})-1}{|\mathfrak{u}|-1}.$$

Similarly as before, we estimate

$$\binom{q(\mathfrak{u})-1}{|\mathfrak{u}|-1} \le [q(\mathfrak{u})]^{\omega-1} \left(\frac{e}{|\mathfrak{u}|-1+\delta_{|\mathfrak{u}|,1}}\right)^{|\mathfrak{u}|-1}.$$

Since $q(\mathfrak{u}) = k$, we obtain

$$m_w(q,d) \le 1 + M_0\, M^k\, k^{\omega-1} \sum_{k=1}^{\omega} \binom{d}{k} \left(\frac{M_0(M-1)\,e}{M(k-1+\delta_{k,1})}\right)^{k-1}.$$

Note that the binomial coefficients $\binom{d}{k}$ are multiplied by numbers that are uniformly bounded in k. That is why the sum above is of order $d^{\,\omega}$.

For $p = \ln M/\ln D^{-1}$ we have

$$M^k = \left(\varepsilon\,\frac{\mathrm{CRI}_d}{\|I_{d,\gamma}\|}\right)^{-p} \left(\ln\left(\varepsilon\,\frac{\mathrm{CRI}_d}{\|I_{d,\gamma}\|}\right)^{-1}\right)^{(\omega-1)\,p/2} e^{\mathcal{O}(1)},$$

whereas

$$k^{\omega-1} = \mathcal{O}\left(\left[\ln\left(\varepsilon \frac{\mathrm{CRI}_d}{\|I_{d,\gamma}\|}\right)^{-1}\right]^{\omega-1}\right).$$

Therefore for all $\varepsilon < \mathrm{CRI}_d$ we have

$$m_w(q,d) = \mathcal{O}\left(d^{\,\omega}\left(\varepsilon \frac{\mathrm{CRI}_d}{\|I_{d,\gamma}\|}\right)^{-p}\left(\ln\left(\varepsilon \frac{\mathrm{CRI}_d}{\|I_{d,\gamma}\|}\right)^{-1}\right)^{(\omega-1)(1+p/2)}\right).$$

Here, the factor in the big $\mathcal{O}$ notation is independent of d and ε^{-1}.

From this we easily obtain the following theorem about tractability bounds achieved by the WTP algorithm.

Theorem 15.20. *Consider the problem $I_\gamma = \{I_{d,\gamma}\}$ for finite-order weights of order $\omega \ge 1$. Let the WTP algorithm $A_w(q,d)$ be defined as in this subsection with optimal algorithms U_i that use nested information and satisfy (15.44) and (15.46). Let*

$$p = \frac{\ln M}{\ln D^{-1}}$$

be the exponent for approximation of the univariate problem I_1. Then

$$e(A_w(q,d)) \le \varepsilon\, \mathrm{CRI}_d$$

and the number $m_w(q,d)$ of function values used by $A_w(q,d)$ is bounded as follows.

- *For the absolute error criterion:*

$$m_w(q,d) = \mathcal{O}\Big(d^{\,\omega}\big(\|I_{d,\gamma}\|^p\,\varepsilon^{-p}\,\big(\ln(\|I_{d,\gamma}\|\,\varepsilon^{-1})^{(\omega-1)(1+p/2)}\big)\big)\Big)$$

  for all $\varepsilon < \|I_{d,\gamma}\|$,

- *For the normalized error criterion:*

$$m_w(q,d) = \mathcal{O}\Big(d^{\,\omega}\,\varepsilon^{-p}\,\big(\ln \varepsilon^{-1}\big)^{(\omega-1)(1+p/2)}\Big)$$

  for all $\varepsilon < 1$.

The factors in the big $\mathcal{O}$ notation are independent of d and ε^{-1}. This implies the following tractability conditions.

- *For the absolute error criterion:*
  - *If for some q we have*

$$\|I_{d,\gamma}\| = \mathcal{O}(d^q) \quad \textit{for all } d \in \mathbb{N}$$

    then I_γ is polynomially tractable. Then the ε^{-1} exponent is at most p and the d exponent is at most $\omega + q$.

– *If*

$$\lim_{d\to\infty} \frac{\ln \max\big(1, \|I_{d,\gamma}\|\big)}{d} = 0$$

then I_γ *is weakly tractable.*

- *For the normalized error criterion:*

  – *The problem* I_γ *is polynomially tractable for arbitrary finite-order weights of order* ω*. The* ε^{-1} *exponent is at most* p *and the* d *exponent is at most* ω*. If there is a number* $k \in [0, \omega)$ *such that*

$$\big|\{\gamma_{d,\mathfrak{u}} \mid \gamma_{d,\mathfrak{u}} > 0\}\big| = \mathcal{O}(d^k) \quad \textit{for all } d \in \mathbb{N}$$

  then the d *exponent is at most* k*.*

The main point of Theorem 15.20 is that for finite-order weights we achieve the same ε^{-1} exponent for all d at the expense of some powers of $\ln \varepsilon^{-1}$ that are independent of d. As before, for the normalized error criterion we have polynomial tractability for arbitrary finite-order weights and we can achieve this by using the WTP algorithm. For the absolute error criterion we need to control the initial error. If the finite-order weights are chosen so that the initial error depends polynomially on d then we also have polynomial tractability, whereas if the finite-order weights are chosen so that the initial error is not exponentially dependent on d then we have weak tractability.

15.3.8 Example: Perturbed Coulomb Potential

Coulomb or perturbed Coulomb potentials are a natural example for which finite-order weights occur. In this case, functions are given as small perturbation of the sum of Coulomb pair potentials, see [76],

$$f_\alpha(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_\ell) = \sum_{1\le i<j\le \ell} \frac{1}{\sqrt{\|\vec{x}_i - \vec{x}_j\|^2 + \alpha}} \qquad \text{for } \vec{x}_i \in \Omega \subseteq \mathbb{R}^3.$$

Here $\alpha \ge 0$. For $\alpha = 0$ we obtain the Coulomb potential but then the function f_α is not well defined for $\vec{x}_i = \vec{x}_j$. To make the function f_α well defined for all $\vec{x}_i \in \Omega$, we assume that α is small but positive.

We stress that f_α only depends on groups of two variables, each being a 3-dimensional vector $\vec{x}_i \in \mathbb{R}^3$. From this it is clear that we should embed such functions into a space equipped with finite-order weights of order $\omega = 6$.

Letting $\mathbf{x} = [\vec{x}_1, \dots, \vec{x}_\ell]$, we can view f_α as a function of 3ℓ variables, i.e.,

$$d = 3\,\ell.$$

(Note that the perturbed Coulomb force $g_\alpha(\mathbf{x}) = \sum_{1\le i<j\le\ell}(\|\vec{x}_i - \vec{x}_j\|^2 + \alpha)^{-1}$ is even simpler than f_α, and the analysis of this section may also be applied to g_α.)

For simplicity we consider only bounded domains Ω. Since the function

$$g(t_1, t_2, \dots, t_6) := \Big(\alpha + \sum_{i=1}^{3} (t_i - t_{i+3})^2\Big)^{-1/2} \quad \text{for all } t_i \in \mathbb{R},$$

is infinitely differentiable, so is f_α. Therefore, the (perturbed) Coulomb potential function can be viewed as an element of many different reproducing kernel spaces of various smoothness.

In what follows, for the sake of brevity, we will illustrate the results of this subsection using a relatively low degree smoothness, although extensions to higher smoothness is straightforward. For simplicity, we also assume that Ω is a Cartesian product of three identical subsets of $\mathbb{R}$, e.g., Ω is a cube in $\mathbb{R}^3$.

We will be using the results of this subsection for an arbitrary sequence $I_\gamma = \{I_{d,\gamma}\}$ of linear functionals for which the assumptions of Theorems 15.18 and 15.20 hold. In particular, we assume that p^* is determined by the univariate case for approximating I_1, and to omit the powers of logarithms of ε^{-1} we express the cost bounds in terms of ε^{-p} for $p > p^*$.

Let

$$\Omega = D \times D \times D,$$

where, for simplicity of presentation, we take $D = [0,1]$. As for the kernel K_1 we choose

$$K_1(x,t) = 1 + \min(x,t) \quad \text{for all } x,t \in [0,1],$$

so that $H(K_1)$ is the Sobolev space $W_2^1[0,1]$ and

$$\langle f, g\rangle_{H(K_1)} = f(0)\, g(0) + \int_0^1 f'(t)\, g'(t)\, \mathrm{d}t \quad \text{for all } f, g \in H(K_1).$$

The (perturbed) Coulomb potential function f_α belongs to $H(K_{d,\gamma})$ for

$$K_{d,\gamma}(\mathbf{x},\mathbf{y}) = \sum_{u \in \mathcal{U}_d} K_{d,u}(\mathbf{x},\mathbf{y})$$

with $d = 3\,\ell$, and $K_{d,\emptyset} = 1$ and $K_{d,\mathfrak{u}}(\mathbf{x},\mathbf{y}) = \prod_{j\in\mathfrak{u}} \min(x_j, y_j)$ for non-empty $\mathfrak{u}$. The set $\mathcal{U}_d$ consists of subsets

$$\mathfrak{u} = \{3i-2, 3i-1, 3i, 3j-2, 3j-1, 3j\} \quad \text{for } i < j,$$

i.e., $\mathfrak{u}$ contains indices of coefficients corresponding to two different vectors $\vec{x}_i$ and $\vec{x}_j$. This means that we set

$$\gamma_{d,u} = 1 \ \text{ for } u \in \mathcal{U}_d \quad \text{and} \quad \gamma_{d,u} = 0 \ \text{ for } u \notin \mathcal{U}_d.$$

Of course, such weights are finite-order with order $\omega^* = 6$. Furthermore, the cardinality of non-zero weights is

$$|\mathcal{U}_d| = \frac{(\ell-1)\ell}{2} < \frac{d^2}{18}.$$

This implies that when we use Theorem 15.18 or Theorem 15.20 for approximating $I_\gamma = \{I_{d,\gamma}\}$ then the dependence on d is at most quadratic. More precisely, if the assumptions of Theorem 15.20 are satisfied then for the normalized error criterion we can find an ε-approximation by the WTP algorithm using $m_w(q,d)$ function values and

$$m_w(q,d) = \mathcal{O}\big(d^{\,2}\,\varepsilon^{-p}\big)$$

where the factor in the big $\mathcal{O}$ notation does not depend on d and ε^{-1}, whereas $p > p^*$ with p^* determined by the univariate case for I_1.

We now estimate the norm $\|f_\alpha\|_{H(K_{d,\gamma})}$. Consider

$$g(\vec{x},\vec{y}) = (\|\vec{x}-\vec{y}\|^2 + \alpha)^{-1/2} \quad \text{for all } \vec{x},\vec{y} \in \mathbb{R}^3.$$

Let $\vec{z} = [z_1,\dots,z_6]$ with $z_i = x_i$ for $i \le 3$ and $z_i = y_{i-3}$ for $i > 3$. It is easy to verify that for a non-empty subset $\mathfrak{u}$ of $\{1,2,\dots,6\}$, we have

$$\frac{\partial^{|\mathfrak{u}|} g}{\prod_{j\in\mathfrak{u}} \partial z_j}(\vec{z}) = (-1)^{|\mathfrak{u}|} \prod_{j=1}^{|\mathfrak{u}|-1} (j+\tfrac12)\, \frac{\prod_{j\in\mathfrak{u}:\, j\le 3}(x_j-y_j)\,\prod_{j\in\mathfrak{u}:\, j>3}(y_{j-3}-x_{j-3})}{(\|\vec{x}-\vec{y}\|^2+\alpha)^{|\mathfrak{u}|+1/2}},$$

with the convention that the product over the empty set is taken as one. Consider now $\mathfrak{u} = \{1,2,\dots,6\}$, and let

$$\begin{aligned} a &:= \int_{[0,1]^6} \left(\frac{\partial^6}{\prod_{j=1}^6 \partial z_j} g(\vec{z})\right)^2 \mathrm{d}\vec{z} \\ &= \Theta\left(\int_{[0,1]^6} \frac{(z_1-z_4)^4(z_2-z_5)^4(z_3-z_6)^4}{((z_1-z_4)^2+(z_2-z_5)^2+(z_3-z_6)^2+\alpha)^{13}}\right) \mathrm{d}\vec{z}. \end{aligned}$$

By changing variables $z_i = t_i\sqrt{\alpha}$ we immediately conclude that $a = \Theta(\alpha^{-4})$ as $\alpha \to 0$. This implies that

$$\|f_\alpha\|_{H(K_{d,\gamma})} = \Theta(d\,\alpha^{-2}) \quad \text{as } \alpha \to 0,$$

with the factor in the Θ-notation independent of d.

Hence, for $\hat{\varepsilon} = \varepsilon\,\|f_\alpha\|^{-1}_{H(K_{d,\gamma})}$ and for the normalized error criterion, we obtain

$$|I_{d,\gamma}(f_\alpha) - A_w(q,d)(f_\alpha)| \le \varepsilon\,\|I_{d,\gamma}\|$$

with

$$m_w(q,d) = \mathcal{O}\big(d^{\,2+p}\,\alpha^{-2p}\,\varepsilon^{-p}\big).$$

15.3.9 Open Problems for Finite-Order Weights

We briefly mention several open problems for finite-order weights. First of all, note that most of the lower bound results presented in the previous chapters do not apply for finite-order weights. This is not too surprising, since for the normalized error criterion we have polynomial tractability for all finite-order weights. However, for the absolute and/or normalized error criterion, it is not clear what conditions on γ are necessary for strong polynomial, polynomial, T-tractability or weak tractability of the problem $I_\gamma = \{I_{d,\gamma}\}$. It is also not clear what the minimal exponents of ε^{-1} and d are and whether there is the tradeoff between them, as we observed for the WTP algorithms. We summarize these questions in the next open problems

Open Problem 71.

- What are necessary and sufficient conditions on finite-order weights of order ω for which the problem $I_\gamma = \{I_{d,\gamma}\}$ is
  - strongly polynomially tractable,
  - polynomially tractable
  - T-tractable,
  - weakly tractable

  for the absolute and normalized error criteria?
- Is the exponent of strong polynomial tractability always the same as the exponent for the univariate case?
- What are the exponents of tractability?
- Is there the tradeoff between the ε^{-1} and d exponents?

15.3.10 Tractability for Product Weights

In this subsection we assume that $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ is a sequence of *product* weights, i.e.,

$$\gamma_{d,\emptyset} = 1, \quad \gamma_{d\,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \text{for all non-empty } \mathfrak{u} \subseteq [d].$$

Here, $\{\gamma_{d,j}\}_{d\in\mathbb{N},j\in[d]}$ is a given sequence of non-negative numbers.

We consider the WTP algorithms as defined in Theorem 15.17. As before, tractability holds when the numbers $C(d,\gamma)$ given in Theorem 15.17 are *not* exponential in d.

For the absolute error criterion, we have

$$\begin{aligned} &C(d,\gamma)\\ &= \frac{1}{C_\delta}\Big(\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \big(\|h_{1,1}\|_{H(R_1)}^{2p/(p+2)}\big)^{d-|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \gamma_{d,j}^{p/(p+2)}\, C_\delta^{2/(p+2)}\Big]^{(p+2)/2}\\ &= \frac{1}{C_\delta}\Big[\prod_{j=1}^{d} \big(\|h_{1,1}\|_{H(R_1)}^{2p/(p+2)} + \gamma_{d,j}^{p/(p+2)}\, C_\delta^{2/(p+2)}\big) - \|h_{1,1}\|_{H(R_1)}^{2pd/(p+2)}\Big]^{(p+2)/2} \end{aligned}$$

$$= \frac{\|h_{1,1}\|_{H(R_1)}^{pd}}{C_\delta} \left[\prod_{j=1}^{d} \left(1 + \gamma_{d,j}^{p/(p+2)} \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}} \right)^{2/(p+2)} \right) - 1 \right]^{(p+2)/2}.$$

Replacing $1 + x$ by $\exp(x)$ for non-negative x, we obtain

$$C(d,\gamma) \le \frac{\|h_{1,1}\|_{H(R_1)}^{pd}}{C_\delta} \exp\left[\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}} \left(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)} \right)^{(p+2)/2} \right]$$

$$= \|h_{1,1}\|_{H(R_1)}^{pd} \exp\left[\mathcal{O}\left(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)} \right)^{(p+2)/2} \right].$$

The last bound will be used for tractability analysis for the absolute error criterion.

For the normalized error criterion, we have

$$C(d,\gamma) = \frac{1}{C_\delta} \frac{\left[\prod_{j=1}^{d} \left(1 + \gamma_{d,j}^{p/(p+2)} \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}} \right)^{2/(p+2)} \right) - 1 \right]^{(p+2)/2}}{\left[\prod_{j=1}^{d} \left(1 + \gamma_{d,j} \frac{\|h_{1,2}\|_{H(R_2)}^2}{\|h_{1,1}\|_{H(R_1)}^2} \right) \right]^{p/2}}$$

$$\le \frac{1}{C_\delta} \prod_{j=1}^{d} \frac{\left(1 + \gamma_{d,j}^{p/(p+2)} \left(\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}} \right)^{2/(p+2)} \right)^{(p+2)/2}}{\left(1 + \gamma_{d,j} \frac{\|h_{1,2}\|_{H(R_2)}^2}{\|h_{1,1}\|_{H(R_1)}^2} \right)^{p/2}}.$$

Since the denominator is at least 1 we obtain

$$C(d,\gamma) \le \frac{1}{C_\delta} \exp\left[\frac{C_\delta}{\|h_{1,1}\|_{H(R_1)}^{p}} \left(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)} \right)^{(p+2)/2} \right]$$

$$= \exp\left[\mathcal{O}\left(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)} \right)^{(p+2)/2} \right].$$

This bound will be used for tractability analysis for the normalized error criterion.

From Theorem 15.18 and the bounds on $C(d,\gamma)$ we obtain the following theorem.

Theorem 15.21. *Consider the problem $I_\gamma = \{I_{d,\gamma}\}$ for product weights. Let the WTP algorithm $A_w(q,d)$ be defined with optimal algorithms U_i that use nested information and satisfy (15.44) and (15.46), and with $\varepsilon_{\mathfrak{u}}$ defined as in Theorem 15.17. Let*

$$p > p^* := \frac{\ln M}{\ln D^{-1}},$$

where p^* is the exponent for approximation of the univariate problem I_1. Then

$$e(A_w(q,d)) \le \varepsilon \, \mathrm{CRI}_d$$

and the number $m_w(q,d)$ of function values used by $A_w(q,d)$ is bounded as follows.

• *For the absolute error criterion, we have*

$$m_w(q,d) = \|h_{1,1}\|_{H(R_1)}^{pd} \, \exp\Big[\mathcal{O}\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2}\Big]\, \varepsilon^{-p}$$

for all $\varepsilon < \|I_{d,\gamma}\|$.

• *For the normalized error criterion, we have*

$$m_w(q,d) = \exp\Big[\mathcal{O}\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2}\Big]\, \varepsilon^{-p}$$

for all $\varepsilon < 1$.

The factors in the big $\mathcal{O}$ *notation are independent of* d *and* ε^{-1}.

This implies the following tractability conditions.

• *For the absolute error criterion:*

  – *If* $\|h_{1,1}\|_{H(R_1)} \le 1$ *and*

$$\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2} = \mathcal{O}\big(1 + d\, \ln \|h_{1,1}\|_{H(R_1)}^{-1}\big)$$

  as $d \to \infty$ *then* I_γ *is strongly polynomially tractable with an exponent at most* p.

  – *If* $\|h_{1,1}\|_{H(R_1)} \le 1$ *and*

$$\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2} = \mathcal{O}\big(1 + \ln d + d\, \ln \|h_{1,1}\|_{H(R_1)}^{-1}\big)$$

  as $d \to \infty$ *then* I_γ *is polynomially tractable. Hence, for* $\|h_{1,1}\|_{H(R_1)} < 1$ *the conditions on strong polynomial and polynomial tractability are the same, and we have strong polynomial tractability with an exponent at most* p.

  For $\|h_{1,1}\|_{H(R_1)} = 1$ *strong polynomial and polynomial tractability may differ and* I_γ *is polynomially tractable with an* ε^{-1} *exponent at most* p *and a* d *exponent at most* q*, with* q *given as for the normalized error criterion.*

  – *If* $\|h_{1,1}\|_{H(R_1)} \le 1$ *and*

$$\lim_{d\to\infty} \left(p \ln \|h_{1,1}\|_{H(R_1)} + \frac{C_\delta \big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\big)^{(p+2)/2}}{d\, \|h_{1,1}\|_{H(R_1)}^{p}} \right)_+ = 0$$

  then I_γ *is weakly tractable.*

- *For the normalized error criterion:*
  - *If*

$$\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2} = \mathcal{O}(1) \quad \textit{as } d \to \infty$$

then I_γ is strongly polynomially tractable with an exponent at most p.

  - *If*

$$a^* = \limsup_{d\to\infty} \frac{\big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\big)^{(p+2)/2}}{\ln d} < \infty$$

then I_γ is polynomially tractable with an ε^{-1} exponent at most p and a d exponent at most q, where $q > a^*\|h_{1,1}\|_{H(R_1)}/C_\delta$.

  - *If*

$$\lim_{d\to\infty} \frac{\big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\big)^{(p+2)/2}}{d} = 0$$

then I_γ is weakly tractable.

All tractability bounds are attained by WTP algorithms.

Proof. The bounds on $m_w(q,d)$ easily follow from the bounds on $C(d,\gamma)$. We also know that the factor in the big $\mathcal{O}$ notation in the exponential function is $M := C_\delta/\|h_{1,1}\|_{H(R_1)}^p$. For the absolute error criterion it is clear that $M_1 := \|h_{1,1}\|_{H(R_1)} > 1$ makes $m_w(q,d)$ exponentially dependent on d and so there is no chance to get tractability of I_γ. That is why we need to assume that $M_1 = \|h_{1,1}\|_{H(R_1)} \le 1$. To get tractability conditions we take logarithms, and for polynomial tractability we need to check when

$$p\, d \ln M_1 + M \Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2} \le M_2 + q \ln d$$

for some M_2 and q.

This means that

$$\Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2} \le M^{-1}\,\big(M_2 + q\ln d + p\, d \ln M_1^{-1}\big) = \mathcal{O}\big(1 + \ln d + d \ln M_1^{-1}\big),$$

as claimed.

For weak tractability we must guarantee that $d^{-1}\ln m_w(q,d)$ goes to zero. This is equivalent to the condition that

$$\bigg(p \ln M_1 + d^{-1}\, M \Big(\sum_{j=1}^{d} \gamma_{d,j}^{p/(p+2)}\Big)^{(p+2)/2}\bigg)_+$$

goes to zero, as claimed.

For the normalized error criterion, we proceed as before, taking $M_1 = 1$. This completes the proof. □

We now simplify tractability conditions in Theorem 15.21 for the normalized error criterion by recalling the notion of the sum-exponent p_γ for product weights, see [332] as well as Volume I, page 201. Namely,

$$p_\gamma = \inf \left\{ \tau \geq 0 \;\middle|\; \limsup_{d\to\infty} \textstyle\sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty \right\}$$

with the convention that $\inf \emptyset = \infty$.

Then I_γ is strongly polynomially tractable if $p_\gamma < 1$. Indeed, in this case we can define p such that $p/(p+2)$ is arbitrarily close to p_γ. That is, p is arbitrarily close to $2p_\gamma/(1-p_\gamma)$ and the exponent of strong polynomial tractability is at most

$$\max\left(p^*, \frac{2p_\gamma}{1-p_\gamma}\right).$$

Hence, if $p_\gamma \leq \frac{p^*}{p^*+2}$ we obtain the exponent of strong polynomial tractability p^* as for the univariate case. This agrees with the original results obtained in [332]. We illustrate this point by continuing the example of uniform integration.

15.3.11 Example: Uniform Integration (Continued)

We want to apply the WTP algorithms for multivariate integration for the weighted Sobolev space with the reproducing kernel

$$K_{d,\gamma}(x,t) = \prod_{j=1}^{d} \left(1 + \gamma_{d,j} \min(x_j, t_j)\right) \quad \text{for all } x,t \in [0,1]^d.$$

We now have $a = 0$ which implies that $\eta_1 = 1$, $R_1(x,t) = 1$ and $R_2(x,t) = \min(x,t)$ for $x,t \in [0,1]$. We also have $h_1(x) = 1 + x - x^2/2$ and $h_{1,1} = 1$ with $\|h_1\|_{H(R_1)} = 1$, and $h_{1,2}(x) = x - x^2/2$ with $\|h_{1,2}\|_{H(R_2)} = 3^{-1/2}$.

We now specify algorithms U_i. As always, $U_0 = 0$ and for $f \in H(R_2)$ we set

$$U_i(f) = \frac{1}{2^{i-1}}\left[\tfrac{1}{2}\, f(1) + \sum_{j=1}^{2^{i-1}-1} f(j2^{-(i-1)})\right] \quad \text{for all } i \in \mathbb{N}.$$

Hence U_i is the trapezoid rule, since $f(0) = 0$ for $f \in H(R_2)$. This means that $U_i(f)$ is equal to the integral of piecewise linear functions interpolating f at $0, 2^{-(i-1)}, \ldots, 1$. It is known that U_i is optimal and clearly uses nested information of cardinality 2^{i-1}. This implies that we can take $M_0 = 1$ and $M = 2$. It is also known that

$$\|I_1 - U_i\|_{H(R_2)\to\mathbb{R}} = \Theta(2^{-i})$$

and therefore we can take $D = 1/2$. Thus $p = 1$.

For this problem we know that the minimal $p^* = 1$. We have already showed that

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$$

is necessary for strong polynomial tractability of multivariate integration. Under this assumption on the product weights, we know that some QMC algorithms achieve strong tractability bounds proportional to ε^{-2}, although it is not clear whether the exponent 2 of ε^{-1} can be lowered.

Consider product weights with the sum exponent p_γ. If we have strong polynomial tractability then $p_\gamma \le 1$. However, we need to assume more if we want to guarantee that the WTP algorithms achieve strong tractability error bounds. Namely, we must assume that $p_\gamma < 1$. This looks like an unimportant extra assumption on the product weights. However, if $p_\gamma < 1$ then the exponent of ε^{-1} is

$$\max\left(1, \frac{2p_\gamma}{1-p_\gamma}\right),$$

which can be arbitrarily large if p_γ is arbitrarily close to 1. Only for $p_\gamma \le \frac{1}{3}$ we achieve that best possible exponent 1.

In the next chapter we analyze different algorithms that require more lenient conditions on p_γ to get the exponent of strong tractability equal to 1. As we shall see, it would be enough to assume that $p_\gamma \le \frac{1}{2}$. However, we still do not know what is the minimal condition on p_γ to get the exponent of strong tractability equal to 1.

There is a conjecture that the exponent of strong tractability does depend on the sum-exponent p_γ of the product weight, see [353]. We repeat this conjecture here.

Open Problem 72.

Consider multivariate integration for the weighted Sobolev space with the reproducing kernel for product weights as in this subsection. Assume that

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$$

so that strong polynomial tractability holds. Let p_γ be the sum exponent of product weights.

- Prove that the exponent p of strong polynomial tractability is

$$p = \max(1, 2p_\gamma).$$

- A weaker version is to prove that

$$p = 1 \quad \text{iff} \quad \limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}^{1/2} < \infty.$$

15.3.12 Example: Weights for Cobb Douglas Functions

We still consider the weighted Sobolev space as in the previous subsection but for a special sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ of product weights suggested by a problem from economics. There is a well known family of functions studied by economists, called the Cobb Douglas family, see e.g, [258], which consists of the functions of the form

$$f(x) = \prod_{k=1}^{d} (x_k + a_{d,k})^{\alpha_{d,k}} \quad \text{for all } x \in [0,1]^d,$$

where $a_{d,k}$'s are non-negative and $\alpha_{d,k} \geq 0$ with $\sum_{k=1}^{d} \alpha_{d,k} = 1$.

Obviously, multivariate integration for such functions is trivial since we can integrate these functions explicitly. However, we may need to approximate other linear functionals such as weighted multivariate integration. Perhaps we can get good results by using WTP algorithms. Hence, we want to check if such functions f belong to the weighted Sobolev space $H(K_{d,\gamma})$, and how to choose product weights to guarantee that the norms of such functions f are not too large.

We now estimate the norm f in the space $H(K_{d,\gamma})$ for arbitrary product weights. We have

$$f_{\mathfrak{u}}(x_{\mathfrak{u}}) = f(x_{\mathfrak{u}}, 0) = \prod_{k\in\mathfrak{u}} (x_k + a_{d,k})^{\alpha_{d,k}} \prod_{k\notin\mathfrak{u}} a_{d,k}^{\alpha_{d,k}}.$$

Therefore

$$\int_{[0,1]^{|\mathfrak{u}|}} \Big(\prod_{k\in\mathfrak{u}} \frac{\partial}{\partial x_k} f_{\mathfrak{u}}(x) \Big)^2 \prod_{k\in\mathfrak{u}} \frac{\mathrm{d}x_k}{\gamma_{d,k}} = \prod_{k\in\mathfrak{u}} \frac{\alpha_{d,k}^2}{\gamma_{d,k}} \int_0^1 (t + a_{d,k})^{2(\alpha_{d,k}-1)} \mathrm{d}t \prod_{k\notin\mathfrak{u}} a_{d,k}^{2\alpha_{d,k}}.$$

Since $2(\alpha_{d,k} - 1) \leq 0$ and $(t + a_{d,k})^{2(\alpha_{d,k}-1)} \leq a_{d,k}^{2(\alpha_{d,k}-1)}$ we have

$$\int_{[0,1]^{|\mathfrak{u}|}} \Big(\prod_{k\in\mathfrak{u}} \frac{\partial}{\partial x_k} f_{\mathfrak{u}}(x) \Big)^2 \prod_{k\in\mathfrak{u}} \frac{\mathrm{d}x_k}{\gamma_{d,k}} \leq \prod_{k\in\mathfrak{u}} \frac{\alpha_{d,k}^2}{a_{d,k}^2 \gamma_{d,k}} \prod_{k=1}^{d} a_{d,k}^{2\alpha_{d,k}}.$$

Hence,

$$\|f\|_{H(K_{d,\gamma})}^2 \leq \prod_{k=1}^{d} a_{d,k}^{2\alpha_{d,k}} \Big(1 + \sum_{\mathfrak{u}\neq\emptyset} \prod_{k\in\mathfrak{u}} \frac{\alpha_{d,k}^2}{a_{d,k}^2 \gamma_{d,k}} \Big).$$

For any non-negative numbers β_k we have

$$\sum_{\mathfrak{u}\neq\emptyset} \prod_{k\in\mathfrak{u}} \beta_k = \prod_{j=1}^{d} (1 + \beta_k) - 1 = \sum_{j=1}^{d} \beta_j \prod_{k=j+1}^{d} (1 + \beta_k),$$

where the last equality can be shown inductively on d, see formula (40) in [277]. Hence,

$$\sum_{\mathfrak{u}\neq\emptyset} \prod_{k\in\mathfrak{u}} \beta_k \leq \Big(\sum_{j=1}^{d} \beta_j \Big) \exp\Big(\sum_{j=1}^{d} \beta_j \Big).$$

Therefore,

$$\|f\|^2_{H(K_{d,\gamma})} \le \prod_{k=1}^{d} a_{d,k}^{2\alpha_{d,k}} \left(1 + \sum_{j=1}^{d} \frac{\alpha_{d,j}^2}{a_{d,j}^2 \gamma_{d,j}} \prod_{k=j+1}^{d} \left(1 + \frac{\alpha_k^2}{a_k^2 \gamma_{d,k}}\right)\right). \tag{15.53}$$

We now consider two choices of $\gamma_{d,k}$ for which $\|f\|_{H(K_{d,\gamma})}$ is not too large.

- Let $a_d = \min_{j\in[d]} a_{d,j}$ and $b_d = \max_{j\in[d]} a_{d,j}$. Since $\sum_{j=1}^d \alpha_{d,j} = 1$ we have
$$\|f\|^2_{H(K_{d,\gamma})} \le b_d^2 + \frac{b_d^2}{a_d^2}\left(\sum_{j=1}^{d} \frac{\alpha_{d,j}^2}{\gamma_{d,j}}\right) \exp\left(\frac{1}{a_d^2} \sum_{j=1}^{d} \frac{\alpha_{d,j}^2}{\gamma_{d,j}}\right).$$
Setting $\gamma_{d,j} = \alpha_{d,j}$ we conclude that
$$\|f\|^2_{H(K_{d,\gamma})} \le b_d^2 + \frac{b_d^2}{a_d^2} \exp\left(\frac{1}{a_d^2}\right).$$
Hence, $\|f\|_{H(K_{d,\gamma})}$ has a bound which is not too large if a_d is not too small and if b_d is not too large as functions of d.
- Assume for simplicity that $a_{d,j} = a_d$ for all $j \in [d]$ and all $d \in \mathbb{N}$. Setting now $\gamma_{d,j} = \alpha_{d,j}^2$ we conclude from (15.53) that
$$\|f\|^2_{H(K_{d,\gamma})} \le a_d^2 \left(1 + \frac{1}{a_d^2} \sum_{j=1}^{d} \left(1 + a_d^{-2}\right)^{d-j}\right) = a_d^2 \left(1 + a_d^{-2}\right)^d.$$
For $a_d = \sqrt{d}$, say, we have
$$\|f\|^2_{H(K_{d,\gamma})} \le d\left(1 + d^{-1}\right)^d \le d\, e.$$
Hence, $\|f\|_{H(K_{d,\gamma})} \le \sqrt{d\, e}$ depends only linearly on $\sqrt{d}$.

This example from economics suggests to consider product weights for which

$$\sum_{k=1}^{d} \gamma_{d,k}^q = 1 \quad \text{for all } d \in \mathbb{N}$$

for some positive q. Note that $q = 1$ and $q = \frac{1}{2}$ were used above. For instance, $\gamma_{d,k} = 1/d^{1/q}$ (for all k) or $\gamma_{d,k} = \delta_{d,k}$ are two extreme examples of such sequences. Depending on the sequence $\{\gamma_{d,k}\}$, the sum-exponent can be any number between 0 and q,

$$0 \le p_\gamma \le q.$$

Note that $p_\gamma = q$ when, e.g., $\gamma_{d,k} = 1/d^{1/q}$, and that $p_\gamma = 0$ when, e.g., $\gamma_{d,k} = \delta_{d,k}$.

For such product weights, we can apply the results of this subsection for approximating $I_\gamma = \{I_{d,\gamma}\}$. In particular, we know that I_γ is strongly polynomially tractable if $p_\gamma < 1$.

15.3.13 Example: Integration of Smooth Functions

We now discuss multivariate integration for classes of smooth functions, see Example 5.14 in Volume I. For $d = 1$, let $R_1 = 1$ and

$$R_2(x,t) = \int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!}\,\frac{(t-u)_+^{r-1}}{(r-1)!}\,\mathrm{d}u \quad \text{for all } x,t \in [0,1].$$

Then $H(R_1) = \operatorname{span}(1)$ and

$$H(R_2) = \{\, f\colon [0,1] \to \mathbb{R} \mid f^{(j)}(0) = 0, \text{ for all } j < r, \\ f^{(r-1)} \text{ is abs. cont.}, f^{(r)} \in L_2([0,1]) \,\}$$

with

$$\langle f, g\rangle_{H(R_2)} = \int_0^1 f^{(r)}(x)\, g^{(r)}(x)\,\mathrm{d}x \quad \text{for all } f,g \in H(R_2).$$

Here r is a positive integer that measures regularity of functions f, and $r = 1$ corresponds to the uniform integration example studied before. As always, $K_1 = R_1 + R_2$.

For $d \geq 2$, through the tensor product construction, we obtain $H(K_{d,\gamma})$ for product weights γ, and multivariate integration is defined as

$$I_{d,\gamma}(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \text{for all } f \in H(K_{d,\gamma}).$$

We now have $h_{1,1} = 1$ and $\|h_{1,1}\|_{H(R_1)} = 1$, whereas $h_{1,2}(x) = \int_0^1 R_2(x,t)\,\mathrm{d}t$ and

$$\|h_{1,2}\|_{H(R_2)} = \frac{1}{(2r+1)\,[r!]^2}.$$

We now specify algorithms U_i. For $i = 0$ we have $U_0 = 0$ and for $i \geq 1$, the algorithm U_i samples the function f at $j2^{-(i-1)}$ for $j = 1,2,\dots,2^{i-1}$, and

$$U_i(f) = \int_0^1 \sigma_i(x)\,\mathrm{d}x,$$

where $\sigma_i = \sigma_i(f)$ is a spline that minimizes $\|\sigma_i^{(r)}\|_2$ among all functions from $H(R_2)$ that interpolate f at the points $j2^{-(i-1)}$. The choice of the spline σ_i guarantees that U_i is optimal. (In fact, U_i is also central.) It is well known that

$$\|I_1 - U_i\| = \mathcal{O}(2^{-ir}),$$

and therefore $p^* = 1/r$, which is also optimal.

Let p_γ denote the sum-exponent of product weights. Then I_γ is strongly tractable if $p_\gamma < 1$ and then the exponent of strong polynomial tractability is at most

$$\max\left(\frac{1}{r}, \frac{2p_\gamma}{1-p_\gamma}\right),$$

and this can be achieved by the WTP algorithm defined in Theorem 15.21.

15.3.14 Robustness of WTP Algorithms

We briefly discuss the robustness of WTP algorithms. That is, we want to verify what happens if we use the WTP algorithm designed for the weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ for functions that belong to the space $H(K_{d,\eta})$ with $\eta = \{\eta_{d,\mathfrak{u}}\}$ not necessarily equal to γ. We hope that if η is not much different from γ then the WTP algorithm will still behave properly, even for functions for which it has not been designed. This property is usually called as *robustness*.

We need to relate the norms of functions from $H(K_{d,\gamma})$ and $H(K_{d,\eta})$. For $f \in H(K_{d,\gamma})$ or $f \in H(K_{d,\eta})$ we have $f = \sum_{\mathfrak{u}\subseteq[d]} f_u$ for $f_\mathfrak{u} \in H(K_{d,\mathfrak{u}})$, and

$$\|f\|^2_{H(K_{d,\gamma})} = \sum_{\mathfrak{u}\subseteq[d]} \frac{1}{\gamma_{d,\mathfrak{u}}} \|f_\mathfrak{u}\|^2_{H(K_{d,\mathfrak{u}})} = \sum_{\mathfrak{u}\subseteq[d]} \frac{\eta_{d,\mathfrak{u}}}{\gamma_{d,\mathfrak{u}}} \frac{1}{\eta_{d,\mathfrak{u}}} \|f_\mathfrak{u}\|^2_{H(K_{d,\mathfrak{u}})}.$$

Therefore for all $f \in H(K_{d,\gamma})$ we have

$$C_{d,\min} \|f\|_{H(K_{d,\eta})} \le \|f\|_{H(K_{d,\gamma})} \le C_{d,\max} \|f\|_{H(K_{d,\eta})},$$

where

$$C_{d,\min} = \min_{\mathfrak{u}\subseteq[d]} \frac{\eta^{1/2}_{d,\mathfrak{u}}}{\gamma^{1/2}_{d,\mathfrak{u}}} \quad \text{and} \quad C_{d,\max} = \max_{\mathfrak{u}\subseteq[d]} \frac{\eta^{1/2}_{d,\mathfrak{u}}}{\gamma^{1/2}_{d,\mathfrak{u}}}.$$

Here, by convention $0/0 = 0$. That is, if $\gamma_{d,\mathfrak{u}} = 0$ then we must have $\eta_{d,\mathfrak{u}} = 0$ to guarantee that $C_{d,\max}$ is finite.

Assume for a moment that both $C_{d,\min}$ and $C_{d,\max}$ are positive and finite. Then $H(K_{d,\gamma}) = H(K_{d,\eta})$ and their norms are equivalent. Furthermore, we know that $I_{d,\gamma}(f) = I_{d,\eta}(f) = I_d(f)$ for all $f \in H(K_{d,\gamma})$.

Consider a WTP algorithm $A_w(q,d)$ designed for the space $H(K_{d,\gamma})$ and apply this algorithm for functions from the space $H(K_{d,\eta})$. Then for a non-zero f from $H(K_{d,\eta})$ and $h_f = f/\|f\|_{H(K_{d,\eta})}$ we have

$$\begin{aligned}
&|I_{d,\eta}(f) - A_w(q,d)(f)| \\
&\quad \le \|f\|_{H(K_{d,\eta})} \left|I_{d,\eta}(h_f) - A_w(q,d)(h_f)\right| \\
&\quad \le \|f\|_{H(K_{d,\eta})} \sup_{\|h\|_{H(K_{d,\eta})}\le 1} |I_{d,\eta}(h) - A_w(q,d)(h)| \\
&\quad \le \|f\|_{H(K_{d,\eta})} \sup_{\|h\|_{H(K_{d,\gamma})}\le C_{d,\max}} |I_{d,\gamma}(h) - A_w(q,d)(h)| \\
&\quad = C_{d,\max} \|f\|_{H(K_{d,\eta})} \sup_{\|h\|_{H(K_{d,\eta})}\le 1} |I_{d,\gamma}(h) - A_w(q,d)(h)| \\
&\quad \le \frac{C_{d,\max}}{C_{d,\min}} \|f\|_{H(K_{d,\omega})} \sup_{\|h\|_{H(K_{d,\eta})}\le 1} |I_{d,\gamma}(h) - A_w(q,d)(h)|.
\end{aligned}$$

Note that the last estimate formally holds if $C_{d,\min} = 0$ or $C_{d,\max} = \infty$, although in this case $C_{d,\max}/C_{d,\min} = \infty$.

Let $e(A_w(q,d); H(K_{d,\psi}))$ denote the worst case error of the algorithm $A_w(q,d)$ over the unit ball of the space $H(K_{d,\psi})$ with the weight sequence $\psi = \{\psi_{d,\mathfrak{u}}\}$. The last estimate proves the following lemma[4].

Lemma 15.22. *We have*

$$e(A_w(q,d); H(K_{d,\eta})) \le \frac{C_{d,\max}}{C_{d,\min}}\, e(A_w(q,d); H(K_{d,\gamma}).$$

Hence, as long as $C_{d,\max}/C_{d,\min}$ are uniformly bounded or polynomially bounded in d then tractability results based on the WTP algorithms for the weight sequence γ also apply for the weight sequence η.

The main point of the last lemma is that we do not have to know the weights exactly to use the WTP algorithm. The price we pay for not knowing the exact weights is measured by the ratio $C_{d,\max}/C_{d,\min}$.

For finite-order weights γ of order ω, the last lemma is applicable only if $\eta_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$. Hence, it is not clear what happens if we do not use the correct value of the order and apply the WTP algorithms for functions belonging to the space with finite-order weights of order larger than ω, or for the space with not necessarily finite-order weights.

We now show that the WTP algorithms are still robust, although in a different sense than that described in the last lemma, see Remark 2 in [335] where this property was originally proved.

The WTP algorithms designed for finite-order weights of order ω neglect terms $f_{\mathfrak{u}}$ for $|\mathfrak{u}| > \omega$. These algorithms are of the form

$$A_w(q,d) = \Delta_{\emptyset} + \sum_{\substack{\mathfrak{u}\subseteq[d] \\ 1\le|\mathfrak{u}|\le\omega}} \sum_{i\in Q(\mathfrak{u})} \Delta_{\mathfrak{u},i},$$

see (15.41). Furthermore

$$\Delta_{\mathfrak{u},i}(f_{\mathfrak{v}}) = 0 \quad \text{for all } f_{\mathfrak{v}} \in H(K_{d,\mathfrak{v}}) \quad \text{with } \mathfrak{v} \ne \mathfrak{u},$$

see (15.38).

For $f \in H(K_{d,\gamma})$ we have $f = f_{\omega} + f_{-\omega}$, where

$$f_{\omega} = \sum_{\substack{\mathfrak{u}\subseteq[d] \\ 0\le|\mathfrak{u}|\le\omega}} f_{\mathfrak{u}} \quad \text{and} \quad f_{-\omega} = \sum_{\substack{\mathfrak{v}\subseteq[d] \\ \omega<|\mathfrak{v}|\le\omega}} f_{\mathfrak{v}}$$

with $f_{\mathfrak{u}} \in H(K_{d,\mathfrak{u}})$.

[4] In fact, Lemma 15.22 holds not only for the WTP algorithm but also for all linear algorithms.

For any $|\mathfrak{u}| \le \omega$, we have $\Delta_{\mathfrak{u},i}(f_{\mathfrak{v}}) = 0$ for all $|\mathfrak{v}| > \omega$. Therefore $\Delta_{\mathfrak{u},i}(f_{-\omega}) = 0$, and

$$A_w(q,d)(f_{-\omega}) = 0 \quad \text{for all } f_{-\omega}.$$

This property makes the WTP algorithms robust. More precisely, assume that we apply the WTP algorithms designed for finite-order weights of order ω for functions with non-zero components $f_{\mathfrak{v}}$ for $|\mathfrak{v}| > \omega$. That is, we apply the WTP algorithm $A_w(q,d)$ for incorrect functions. Formally, anything could happen. However, we have

$$I_{d,\gamma} - A_w(q,d)(f) = I_{d,\gamma}(f_\omega) - A_w(q,d)(f_\omega) + I_{d,\gamma}(f_{-\omega}).$$

Hence, the WTP algorithm $A_w(q,d)$ leaves the part of the problem $I_{d,\gamma}(f_{-\omega})$ untouched and does approximate the part of the problem $I_{d,\gamma}(f_\omega)$ for which it has been designed. Obviously,

$$\begin{aligned} |I_{d,\gamma} - A_w(q,d)(f)| &\le |I_{d,\gamma}(f_\omega) - A_w(q,d)(f_\omega)| + |I_{d,\gamma}(f_{-\omega})| \\ &\le e(A_w(q,d))\, \|f_\omega\|_{H(K_{d,\gamma})} + \|I_{d,\gamma}\big|_{H_{-\omega}}\|\, \|f_{-\omega}\|_{H(K_{d,\gamma})}, \end{aligned}$$

where $H_{-\omega} = \bigotimes_{\mathfrak{v}\subseteq[d],|\mathfrak{v}|>\omega} H(K_{d,\mathfrak{v}})$.

Hence, if $\|f_\omega\|_{H(K_{d,\gamma})}$ or $|I_{d,\gamma}(f_{-\omega})|$ is small then we can still obtain a pretty good approximation of $I_{d,\gamma}$ by using the WTP algorithm $A_w(q,d)$. More precisely, suppose that $|I_{d,\gamma}(f_{-\omega})| \le \varepsilon\, \mathrm{CRI}_d$ for some (small) ε. Then applying the WTP algorithm $A_w(q,d)$ as defined in Theorem 15.17 we obtain

$$|I_{d,\gamma}(f) - A_w(q,d)(f)| \le 2\varepsilon\, \mathrm{CRI}_d.$$

Hence, the WTP algorithms can be also used for functions with small components $f_{-\omega}$ outside of the space for which it has been designed.

We finally add a few words about the robustness of WTP (and linear) algorithms for product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$. Assume that η is also a product weight sequence, $\eta_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \eta_{d,j}$ with $\gamma_{d,\emptyset} = \eta_{d,\emptyset} = 1$. Then

$$\frac{C_{d,\max}}{C_{d,\min}} = \frac{\max_{\mathfrak{u}\subseteq[d]} \frac{\eta_{d,j}}{\gamma_{d,j}}}{\min_{\mathfrak{u}\subseteq[d]} \frac{\eta_{d,j}}{\gamma_{d,j}}}.$$

For example, if $\eta_{d,j} = c\,\gamma_{d,j}$ for some positive c then

$$\frac{C_{d,\max}}{C_{d,\min}} = \frac{\max(1,c^d)}{\min(1,c^d)} = \left[\max(c^{-1},c)\right]^d.$$

Only for $c = 1$, i.e., for $\eta = \gamma$, we do not have an exponential dependence on d.

On the other hand, if $\eta_{d,j} = (1 + c_{d,j})\gamma_{d,j}$ for some non-negative $c_{d,j}$ then

$$\frac{C_{d,\max}}{C_{d,\min}} = \prod_{j=1}^{d} \left(1 + c_{d,j}\right),$$

which is uniformly bounded in d iff $\sup_d \sum_{j=1}^d c_{d,j} < \infty$ and polynomially bounded iff $\sup_d d^{-q} \sum_{j=1}^d c_{d,j} < \infty$ for some non-negative positive q.

15.4 Notes and Remarks

NR 15:1. It is impossible to cite all the papers where Smolyak or sparse grid algorithms are used. These algorithms have been applied for specific problems usually defined on spaces of functions with bounded mixed derivatives in the worst and average case settings. We limit ourselves to Bungartz and Griebel [20], [21], Delvos [39], Delvos and Schempp [40], Frank and Heinrich [68], Frank, Heinrich and Pereverzev [69], Garcke and Hegland [73], Genz [74], Gerstner and Griebel [75], Gnewuch, Lindloh, Schneider and Srivastav [81], Griebel [89], Griebel and Hamaekers [90], [91], Griebel, Schneider and Zenger [92], Hang and Li [95], Heinrich [97], Klimke and Wohlmuth [150], Leentvaar and Oosterlee [167], Paskov [238], Pereverzev [239], Petras [240], Plaskota [244], Plaskota and Wasilkowski [245], [246], Sickel [261], Sickel and Ullrich [262], Sprengel [287], Steinbauer [289], Temirgaliev, Kudaibergenov and Shomanova [293], Temlyakov [294], [295], [296], [297], Trigub and Belinsky [313], Ullrich [315], Wahba [319], Werschulz [338], Yserentant [357], [358], as well as [6], [33], [212], [213], [214], [215], [216], [217], [247], [252], [329], [332], [347]. The reader is also referred to the recent survey of Griebel [89], where many more references can be found.

In the majority of these papers, only asymptotic error bounds are provided, usually for arbitrary fixed d. As we know, such error bounds are not enough to establish tractability or intractability of a problem. The first paper with an explicit dependence of error bounds on d and with tractability results was probably [329] for the unweighted case, and [332] for the weighted case, see also [334], [335] for finite-order weights.

NR 15.1:1. As already indicated, this chapter is mostly based on [329], [332], [334], [335]. The section for unweighted linear tensor product functionals is more or less the same as in [329]. However, the subsection on implementation issues is new.

The section for weighted linear tensor product functionals is based on [329], [332], [334], however none of these papers analyzed general weights. More precisely, [329] studies product weights, whereas [334], [335] study finite-order weights. The criterion of choosing some parameters of WTP algorithms are also different here, however, the results are basically the same. The section on the robustness of WTP algorithms is new.

NR 15.1:2. We wish to stress again that the Smolyak/sparse grid algorithms are not restricted to linear functionals. We presented here these algorithms only for linear functionals since that is the focus of this volume. In Volume III we will revisit the Smolyak/sparse grid algorithms for linear operators.

There are literally hundreds of papers devoted to the Smolyak/sparse grid algorithms for approximation of linear functionals and operators. This algorithm is indeed one of the major computational tools for approximate solutions of multivariate problems. As long as d is not very large, say at most 5 to 10, there is no need to study tractability, and the analysis done in most papers is enough although the dependence on d is not explicitly known. Only for large d do we need to study the dependence on d, to find conditions under which the cost bounds of the Smolyak/sparse grid algorithms do not suffer from the curse of dimensionality.

NR 15.2.5:1. One reason why the Smolyak algorithm does not lead to optimal tractability bounds is that it does not use enough information about the problem we want to approximate. Indeed, its efficiency depends on the choice of parameters and they, in turn, depend only on the error bounds for the univariate case. More precisely, suppose we have two multivariate problems for which the assumptions (15.6), (15.7), (15.8) are satisfied for the same B, C, D and E. Then the crucial parameter q in the Smolyak algorithm, as well as all error and cost bounds, will be the same. A possible remedy would be to use more information about the behavior of the problem. How to do this is presently unknown.

NR 15.3:1. We stress that the WTP algorithm is a weighted sum of the Smolyak algorithm applied for approximation of the unweighted problems $I_{\mathfrak{u}}$. This allows us to apply the results obtained for the unweighted problems and relate then the accuracy $\varepsilon_{\mathfrak{u}}$ to the total error ε. This is especially visible in Theorem 15.17 in the definition of $\varepsilon_{\mathfrak{u}}$, which is proportional to ε and inversely proportional to the weight $\gamma_{d,\mathfrak{u}}$. In particular for zero or small weights, $\varepsilon_{\mathfrak{u}}$ is infinity or very large. Of course, the problem of approximating $I_{\mathfrak{u}}$ with such a large error is easy.

NR 15.3.5:1. As already indicated in Volume I, we believe that finite-order weights properly model many multivariate problems that are computationally important. Furthermore, the order ω of finite-order weights is usually small. Hence, the polynomial dependence on $d^{\,\omega}$ should be quite acceptable in computational practice.

NR 15.3.10:1. We again stress that the bounds on the exponent of strong polynomial tractability are quite loose. This is especially the case when the sum-exponent of product weights is close to 1, and then the bound of the exponent is huge. We hope that a more refined analysis will eliminate this artifact.

NR 15.3.11:1. We add a comment on Open Problem 72. Suppose for a moment that the exponent of strong polynomial tractability is indeed $p = \max(1, 2p_{\gamma})$. Then it continuously varies from 1 to 2 and depends on the decay of the product weights. However, we do not even know the weaker property that the exponent varies with the product weights through its sum-exponent. It might be true, and very desirable from a computational point of view, that the exponent is always 1 as long as $\limsup_d \sum_{j=1}^{d} \gamma_{d,j} < \infty$. However, it was recently shown in [356] that the exponent of strong polynomial tractability of multivariate integration does depend on product weights for the weighted class whose unweighted analog is presented in Example 3 of Volume I. By analogy, we can thus expect that the same property is also true for the weighted Sobolev class considered here.

NR 15.3.14:1. The problem of selecting proper weights is important and hard, see [54]. Algorithms that use only partial information about weights are quite desirable, and algorithms that can also work for "wrong" weights are definitely preferable.

For finite-order weights, we find the property that the WTP algorithm leaves unchanged the part of the problem outside the space for which it has been designed quite remarkable. However, as pointed out in Sloan [282], we lose convergence to zero in this case. That is, if $m_q(q,d)$ goes to infinity then the WTP algorithm $A_w(q,d)$ goes to $I_{d,\gamma}(f\omega)$,

$$\lim_{m_w(q,d)\to\infty} \big[I_{d,\gamma}(f) - A_w(q,d)(f)\big] = I_{d,\gamma}(f_{-\omega}).$$

On the other hand, for product weights, the situation is different since we always have convergence to zero but at the expense of possible exponential dependence on d, see again Sloan [282] and Lemma 15.22.

In view of our interest of tractability, it is not surprising that we prefer a fast convergence to $I_{d,\gamma}(f) + I_{d,\gamma}(f_{-\omega})$ instead of convergence to $I_{d,\gamma}(f)$ with a running time that may depend exponentially on d.

Chapter 16
Multivariate Integration for Korobov and Related Spaces

16.1 Introduction

The reader may be surprised to see that this chapter is also devoted to multivariate integration. After all, multivariate integration has been already studied in many sections, subsections and examples throughout Volume II. However, this was mostly done to illustrate points studied in a given part of Volume II or to find relations between arbitrary linear functionals and multivariate integration.

As we shall see, multivariate integration is so rich that even adding a long chapter with about 90 pages does not entirely cover this subject. The reader should keep in mind that there are literally thousands of recent research papers on multivariate integration and there are streams of international conferences devoted to this subject. For example, MCQMC (Monte Carlo and Quasi-Monte Carlo) international conferences have been held every two years in different parts of the world. Each MCQMC conference has its own proceedings, and the last proceedings of the MCQMC in Montreal edited by L'Ecuyer and Owen, see [164], with about 670 pages is a vivid proof how much multivariate integration is still being extensively studied.

This chapter is mostly devoted to multivariate integration defined over weighted Korobov spaces. We consider periodic functions with arbitrary smoothness measured by the decay of their Fourier coefficients. As usual, we consider arbitrary weights that monitor the importance of successive variables and groups of variables. Multivariate integration is properly normalized for weighted Korobov spaces. The initial error is always 1 independently of the smoothness parameter and weights. In particular, this means that the absolute and normalized error criteria coincide.

The main focus in this chapter is on *lattice rules*. They are a special case of QMC algorithms with sample points given by a *generator*, which is a vector with integer components. Lattice rules are classical algorithms that have been thoroughly studied in the past. The reader may consult the monograph of Sloan and Joe [273] to see the history and theoretical foundations of lattice rules as of 1994. Our presentation on lattice rules follows the analysis done in [54].

Many results on lattice rules are *non-constructive*. That is, a typical result is that we know there is a generator for which the worst case error of the lattice rule is appropriately small. For instance, there is a generator for which the lattice rule achieves nearly optimal convergence rate as well as allows us to achieve various tractability bounds under appropriate conditions on weights. We also show that some of these conditions are necessary for tractability if we consider QMC algorithms.

For d-variate integration we study lattice rules that use n function values. For simplicity we always assume that n is a prime; however as indicated in the appropriate sections, there are papers where this assumption is relaxed. We believe that the assumption on primality is not very restrictive since, as we know, for every integer m there is a prime in the interval $[m, 2m]$. Hence, instead of using m function values we can at most double m and switch to n function values with a prime n.

The generators of lattice rules for the d-variate case with n points are in the set $\{1, 2, \dots, n-1\}^d$. So there are $(n-1)^d$ generators. For small d, we can formally perform the complete search of all generators, compute the worst case error for each of them, and choose the generator that leads to the minimal worst case error. By theoretical arguments we know that this minimal error is small, and all looks fine. But if d is large, which is the main point of our tractability study, the complete search *cannot* be done since its cost is exponential in d. In fact, it is highly exponential in d since n cannot be too small. For instance, take very modest values of n and d, namely $n = 101$ and $d = 10$. Then $(n-1)^d = 10^{20}$ is quite a formidable number.

So how can we find a good generator of the lattice rule if d is large? For many years it looked like a hopeless problem, and indeed the study of lattice rules has been considered purely theoretical for many years. The big and beautiful surprise came from the Australian school of Ian H. Sloan and it was named the *CBC* (*component-by component)* algorithm. The main idea is to search for the successive components of a good generator one component at a time. That is, we take without loss of generality the first component of the generator as 1, and assuming that the first k components are already known we look for the $(k+1)$th component by searching the set $\{1, 2, \dots, n-1\}$. We do this for $k = 1, 2, \dots, d-1$. In this way, we end up with a generator by searching through the set of $(n-1)(d-1)$ elements instead of $(n-1)^d$ elements.

The CBC algorithm looks very promising, but it is by far not clear whether such a generator is good and leads to a small worst case error. This was first deemed not likely to be a successful strategy, see Niederreiter [201] p. 987 but later it was shown feasible. We quote Ian Sloan from [270]:

> For me the idea for construction started in 1999, at the Hong Kong Workshop for the Complexity of Multivariate Problems. By the time of that Workshop it was already accepted that weighted spaces gave a good setting for the non-constructive proof of the existence of good QMC rules. Also, Henryk and I were about to submit the paper on the existence of good lattice rules that later appeared as [279]. At that time nothing was known about construction, but I remember that there was some vigorous discussion at the Workshop on the desirability of constructive proofs. During a coffee break I sat down together with Stephen Joe of the University of Waikato, and we said to each other something along these lines: "Now that we know that a good lattice exists, is it thinkable that we can construct such a thing one component at a time?" Fortunately, we had at that moment forgotten the conventional wisdom for classical lattice rules, that it is folly to attempt to construct a good lattice rule in d dimension from one in $d-1$ dimensions. I say fortunately, because in the context of weighted spaces

> it turns out perfectly possible to build up good lattices one coordinate at a time. The work that Stephen Joe and I did in that coffee break led in due course to what is now called the "component-by-component" or CBC construction of good lattice rules, see Sloan, Kuo and Joe [271], Sloan, Kuo and Joe [272], Sloan and Reztsov [274].

The first constructions of the CBC algorithm were done for the weighted Sobolev space with smoothness parameter 1 and yielded $\mathcal{O}(n^{-1/2})$ error bounds. The choice of this Sobolev space is not merely happenstance since it is related to the weighted Korobov space with $\alpha = 1$ via the shift-invariant kernel technique introduced by Hickernell [117], see also [128]. More will be said later in this introduction.

Kuo [152] was the first one to obtain better rates of convergence for lattice rules with generators computed by the CBC algorithm. She obtained up to the best order of $\mathcal{O}(n^{-1})$ for the weighted Sobolev space and up to the best order $\mathcal{O}(n^{-\alpha})$ for the weighted Korobov space with the smoothness parameter $\alpha > \frac{1}{2}$. First, mostly product weights were used for the CBC algorithm, and gradually the study of the CBC algorithm was done for general weights, see [54].

To make the CBC algorithm really practical, we must know that the cost of construction of a generator is relatively small. The first constructions for product weights required $\mathcal{O}(d\,n^2)$ arithmetic operations. Nuyens and Cools made another breakthrough by showing that the CBC algorithm can be done using $\mathcal{O}(d\,n\ln n)$ arithmetic operations, see [226], [227], [228]. More precisely, they prove that the CBC algorithm is equivalent to a matrix-vector multiplication and that the matrix can be permuted to become a circulant matrix for which the fast Fourier transform can be used. Note that for the lattice rule we need to compute n function values at sample points of d components. That is why the cost of the lattice rule is at least proportional to $n\,d$. Hence, the cost of the CBC algorithm is at most only slightly larger, and modulo $\ln n$ we match the lower bound. The fast implementation of the CBC algorithm permits an online algorithm that constructs the generator and at the same time computes an approximation to a multivariate integral.

The error bounds of the lattice rules with the generators computed by the CBC algorithm allow us to prove tractability under some conditions on the weights. We study various notions of tractability such as strong polynomial, polynomial, strong T-tractability, T-tractability and weak tractability. Some of these conditions on weights are sharp for QMC algorithms.

In the second part of this chapter we ask whether tractability conditions obtained through the use of lattice rules can be relaxed if we use arbitrary algorithms. From general IBC results presented in Chapter 4 of Volume I, we know that we can restrict ourselves to linear algorithms. In Chapter 12 we presented a number of lower error bounds for arbitrary linear algorithms under the assumption that the reproducing kernel of the space is decomposable or it has a decomposable part. Unfortunately, the assumptions on decomposability do not hold for the weighted Korobov spaces. To overcome this problem we show as in [129] that multivariate integration over the weighted Korobov spaces is not easier than multivariate integration over certain weighted Sobolev spaces, and for the latter the idea of decomposable kernels works. In this way, we prove that for product weights the tractability conditions are the same for the classes of arbitrary and QMC algorithms.

More precisely, the last results are obtained by applying shift-invariant kernels, see again Hickernell [117] and [128]. We know that multivariate integration for the space with the reproducing kernel K_d is not harder than multivariate integration for the space with the shift-invariant $K_{\mathrm{sh},d}$ kernel of K_d. Informally, we write this as

$$\mathrm{INT}(K_d) \le \mathrm{INT}(K_{\mathrm{sh},d}).$$

Of course, this relation can be used in two ways. Lower bounds on $\mathrm{INT}(K_d)$ can be applied to $\mathrm{INT}(K_{\mathrm{sh},d})$, and upper bounds on $\mathrm{INT}(K_{\mathrm{sh},d})$ can be applied to $\mathrm{INT}(K_d)$. Indeed, we use this relation in both ways.

First, we want to solve an inverse problem. That is, we look for K_d for which $K_{\mathrm{sh},d}$ is the reproducing kernel for the weighted Korobov space. It turns out that K_d is the kernel for a weighted Sobolev space with certain boundary conditions. Furthermore, K_d has a decomposable part and we can apply lower bounds on $\mathrm{INT}(K_d)$ as well as on $\mathrm{INT}(K_{\mathrm{sh},d})$.

Secondly, we take K_d as the kernel of a typical weighted Sobolev space and compute $K_{\mathrm{sh},d}$. For some K_d corresponding to the smoothness parameter $r = 1$, the kernel $K_{\mathrm{sh},d}$ corresponds to the weighted Korobov space with the smoothness parameter $\alpha = 1$. Then we apply upper bounds for $\mathrm{INT}(K_{\mathrm{sh},d})$ to $\mathrm{INT}(K_d)$. In this way we obtain *shifted lattice rules* for the weighted Sobolev space that enjoy the same error bounds as lattice rules for the weighted Korobov spaces.

Although lattice rules and shifted lattice rules have good theoretical properties, they do depend on the weights. That is, we must know a priori weights to construct the generator, and for different weights we may have different generators. From one point of view, this property is quite natural since the weights define the space, and a good algorithm must, in general, depend on the space for which it should work well. On the other hand, it is quite demanding to know exactly the weights and we would definitely prefer to have algorithms with some degree of *universality*, that is, algorithms that work well for some classes of weights. We briefly discussed this point already in Chapter 15. We return to this point in this chapter in the context of the weighted Sobolev spaces. In this case, we can not only use shifted lattice rules but also low discrepancy points or sequences. The latter are not dependent on weights, and as we know there is a huge, deep and beautiful theory of QMC algorithms that use such points. As in [275], we check in the final subsection of this chapter that low discrepancy points and sequences also work well for weighted spaces. We show error estimates for the Niederreiter sequence in the case of finite-order weights. Surprisingly enough, for the normalized error criterion these error bounds depend only on the order ω of finite-order weights. Hence, if we know ω or an upper bound on ω then the QMC algorithm that uses the Niederreiter sequence will work for *all* finite-order weights of order ω. We also mention that similar results hold for Halton and Sobol sequences.

As in all chapters, we include several open problems. There are five of them and they are numbered from 73 to 77.

16.2 Weighted Korobov Spaces

Weighted Korobov spaces are defined in Appendix A.1.1 of Volume I. We will recall a few facts about these spaces that will be needed in this chapter.

We use the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ for general non-negative weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$, where $d \in \mathbb{N}$ and $\mathfrak{u}$ is an arbitrary subset of

$$[d] := \{1, 2, \ldots, d\}.$$

We assume that $\gamma_{d,\emptyset} = 1$. The weighted Korobov spaces also depend on two positive parameters β_1 and β_2. We take $\beta_1 = 1$ and $\beta_2 = (2\pi)^{-2\alpha}$, since this choice allows to express the norm of functions from $H_{d,\alpha,\gamma}$ nicely in terms of their derivatives, as we shall see later.

For $h = [h_1, h_2, \ldots, h_d] \in \mathbb{Z}^d$, let $\mathfrak{u}_h = \{j \in [d] \mid h_j \neq 0\}$ and

$$\varrho_{d,\alpha,\gamma}(h) = \frac{1}{\gamma_{d,\mathfrak{u}_h}} \prod_{j \in \mathfrak{u}_h} |2\pi\, h_j|^{2\alpha}.$$

For $\gamma_{d,\mathfrak{u}_h} = 0$ we formally set $\varrho_{d,\alpha,\gamma}(h) = \infty$. For $h = 0$ we have $\mathfrak{u}_0 = \emptyset$ and

$$\varrho_{d,\alpha,\gamma}(0) = 1.$$

The weighted Korobov space $H_{d,\alpha,\gamma}$ consists of periodic complex-valued functions defined on $[0,1]^d$ for which

$$\|f\|_{H_{d,\alpha,\gamma}} := \Big(\sum_{h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}(h)\, |\hat{f}(h)|^2 \Big)^{1/2} < \infty$$

with $\hat{f}(h)$ denoting the Fourier coefficient of f, and given by

$$\hat{f}(h) = \int_{[0,1]^d} \exp(-2\pi\, \mathrm{i}\, h \cdot x)\, f(x)\, \mathrm{d}x,$$

where $\mathrm{i} = \sqrt{-1}$ and $h \cdot x = h_1 x_1 + h_2 x_2 + \cdots + h_d x_d$.

The inner product is defined for $f, g \in H_{d,\alpha,\gamma}$ as

$$\langle f, g \rangle_{H_{d,\alpha,\gamma}} := \sum_{h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}(h)\, \hat{f}(h)\, \overline{\hat{g}(h)}.$$

If $\varrho_{d,\alpha,\gamma}(h) = \infty$ then we assume that $\hat{f}(h) = 0$ for all $f \in H_{d,\alpha,\gamma}$, and interpret $\infty \cdot 0 = 0$, so that the corresponding term in the sums above disappears.

For this choice of β_1 and β_2, and for $d = 1$, $\alpha = r$ with a positive integer r, and $\gamma_{1,\{1\}} = 1$ we have the pleasing relation

$$\|f\|^2_{H_{d,\alpha,\gamma}} = \left| \int_0^1 f(x)\, \mathrm{d}x \right|^2 + \int_0^1 |f^{(r)}(x)|^2\, \mathrm{d}x.$$

For $d \ge 1$, similar relations are exhibited in Appendix A.1.1 of Volume I.

Note that $\varrho_{d,\alpha,\gamma}$ is a non-decreasing function of α. Therefore, the norm of f is also non-decreasing. That is, for $\alpha \le \beta$ we have

$$\|f\|_{H_{d,\alpha,\gamma}} \le \|f\|_{H_{d,\beta,\gamma}} \quad \text{for all } f \in H_{d,\beta,\gamma}.$$

This means that the unit ball of $H_{d,\beta,\gamma}$ is a subset of the unit ball of $H_{d,\alpha,\gamma}$.

Note also that $\varrho_{d,\alpha,\gamma}$ is a non-increasing function of γ. That is, if we have two weight sequences $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ and $\eta = \{\eta_{d,\mathfrak{u}}\}$ such that

$$\gamma_{d,\mathfrak{u}} \le \eta_{d,\mathfrak{u}} \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d],$$

then $\varrho_{d,\alpha,\gamma}(h) \ge \varrho_{d,\alpha,\eta}(h)$ for all $h \in \mathbb{Z}^d$ and

$$\|f\|_{H_{d,\alpha,\eta}} \le \|f\|_{H_{d,\alpha,\gamma}} \quad \text{for all } f \in H_{d,\alpha,\gamma}.$$

This means that the unit ball of $H_{d,\alpha,\gamma}$ is a subset of the unit ball of $H_{d,\alpha,\eta}$, and monotonically decreasing weights shrink the unit ball of the weighted Korobov spaces.

It is also of interest to ask when the unit balls B_d of the spaces $H_{d,\alpha,\gamma}$ are non-decreasing, so that

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_d \subseteq \cdots .$$

Here we assume that a function $f \in H_{d,\alpha,\gamma}$ of d variables is treated as a function of $d+k$ variables and is independent of the $d+1, d+2, \dots, d+k$ variables for all $k \in \mathbb{N}$. Assume for a moment that all weights $\gamma_{d,\mathfrak{u}}$ are positive. Note that for $f \in H_{d,\alpha,\gamma}$ we have

$$\begin{aligned} \|f\|^2_{H_{d+k,\alpha,\gamma}} &= \sum_{h \in \mathbb{Z}^{d+k}} \varrho_{d+k,\alpha,\gamma}(h)\, |\hat{f}(h)|^2 \\ &= \sum_{h=[h_1,h_2,\dots,h_d,0,0,\dots,0] \in \mathbb{Z}^{d+k}} \varrho_{d+k,\alpha,\gamma}(h)\, |\hat{f}(h)|^2. \end{aligned}$$

For $h \in \mathbb{Z}^d$ and $0 \in \mathbb{Z}^k$ we have

$$\varrho_{d+k,\alpha,\gamma}([h,0]) = \frac{1}{\gamma_{d+k,\mathfrak{u}_h}} \prod_{j \in \mathfrak{u}_h} |2\pi\, h_j|^{2\alpha} = \frac{\gamma_{d,\mathfrak{u}_h}}{\gamma_{d+k,\mathfrak{u}_h}}\, \varrho_{d,\alpha,\gamma}(h).$$

Therefore

$$\|f\|^2_{H_{d+k,\alpha,\gamma}} = \sum_{h \in \mathbb{Z}^d} \frac{\gamma_{d,\mathfrak{u}_h}}{\gamma_{d+k,\mathfrak{u}_h}}\, \varrho_{d,\alpha,\gamma}(h)\, |\hat{f}(h)|^2 \le \max_{\mathfrak{u} \subseteq [d]} \frac{\gamma_{d,\mathfrak{u}}}{\gamma_{d+k,\mathfrak{u}}}\, \|f\|^2_{H_{d,\alpha,\gamma}} < \infty.$$

Hence, $f \in H_{d,\alpha,\gamma}$ implies that $f \in H_{d+k,\alpha,\gamma}$ for all $k \in \mathbb{N}$. Furthermore, if

$$\gamma_{d,\mathfrak{u}} \le \gamma_{d+1,\mathfrak{u}} \quad \text{for all } d \in \mathbb{N} \text{ and } \mathfrak{u} \subseteq [d], \tag{16.1}$$

then

$$\|f\|_{H_{d+k,\alpha,\gamma}} \le \|f\|_{H_{d,\alpha,\gamma}} \quad \text{for all } f \in H_{d,\alpha,\gamma} \text{ and } k \in \mathbb{N}.$$

Additionally, if we have equality in (16.1) then

$$\|f\|_{H_{d+k,\alpha,\gamma}} = \|f\|_{H_{d,\alpha,\gamma}} \quad \text{for all } f \in H_{d,\alpha,\gamma} \text{ and } k \in \mathbb{N}.$$

As in [55], we say that the weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ is *nested* if (16.1) holds. In this case, we also say that the weights $\gamma_{d,\mathfrak{u}}$ are *nested*.

Obviously, if the weights $\gamma_{d,\mathfrak{u}}$ are independent of d, i.e., $\gamma_{d,\mathfrak{u}} = \gamma_{\mathfrak{u}}$, then they are nested. In particular, product weights independent of d have the form

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j \quad \text{for all } d \in \mathbb{N} \text{ and } \mathfrak{u} \subseteq [d],$$

and are nested. Similarly, order-dependent weights, i.e., $\gamma_{d,\mathfrak{u}} = \Gamma_{d,|\mathfrak{u}|}$, are nested if $\Gamma_{d,|\mathfrak{u}|} = \Gamma_{|\mathfrak{u}|}$ does not depend on d. Finally, finite-order weights are nested if they do not depend on d, i.e., $\gamma_{d,\mathfrak{u}} = \gamma_{\mathfrak{u}}$ and $\gamma_{\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$.

We stress that weights independent of d are not only nested, but also that (16.1) holds with equality. This means that in this case the norms of $f \in H_{d,\alpha,\gamma}$ are the same for all spaces $H_{d+k,\alpha,\gamma}$ with $k \in \mathbb{N}$.

For nested weights, we have $B_d \subseteq B_{d+1}$ and the unit balls are non-decreasing. This also means that the weighted Korobov spaces do not decrease with d, i.e.,

$$H_{1,\alpha,\gamma} \subseteq H_{2,\alpha,\gamma} \subseteq \cdots \subseteq H_{d,\alpha,\gamma} \subseteq \cdots .$$

The weighted Korobov space $H_{d,\alpha,\gamma}$ is a reproducing kernel Hilbert space with the kernel

$$\begin{aligned} K_{d,\alpha,\gamma}(x,y) &= \sum_{h\in\mathbb{Z}^d} \varrho^{-1}_{d,\alpha,\gamma}(h)\, \exp\left(2\pi\, \mathrm{i}\, h\cdot(x-y)\right) \\ &= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \frac{2}{(2\pi)^{2\alpha}} \sum_{h=1}^{\infty} \frac{\cos(2\pi\, h\,(x_j-y_j))}{h^{2\alpha}} \end{aligned}$$

for all $x, y \in [0,1]^d$. For $x = y$ we obtain

$$K_{d,\alpha,\gamma}(x,x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \quad \text{for all } x \in [0,1]^d,$$

where, as always, ζ is the Riemann zeta function. Note that $K_{d,\alpha,\gamma}(x,x)$ is well defined since we assume that $\alpha > \frac{1}{2}$ and then $\zeta(2\alpha) < \infty$.

16.3 Multivariate Integration

Multivariate integration is given as

$$I_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \text{for all } f \in H_{d\,\alpha,\gamma}.$$

Clearly,

$$I_d(f) = \hat{f}(0) = \langle f, 1\rangle_{H_{d,\alpha,\gamma}} \quad \text{for all } f \in H_{d\,\alpha,\gamma}.$$

Hence, the initial error is

$$e(0,d) = \|I_d\| = 1,$$

and there is no difference between the absolute and normalized error criteria.

Consider a QMC algorithm

$$Q_{n,d}(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_k) \quad \text{for all } f \in H_{d,\alpha,\gamma} \tag{16.2}$$

for some sample points $x_k \in [0,1]^d$. We know that the square of the worst case error of $Q_{n,d}$ in the space $H_{d,\alpha,\gamma}$ is

$$e^2(Q_{n,d}; H_{d,\alpha,\gamma}) = -1 + \frac{1}{n^2}\sum_{k,s=1}^{n} K_{d,\alpha,\gamma}(x_k, x_s).$$

Using the formula for $K_{d,\alpha,\gamma}(x,y)$ we see that the term for $\mathfrak{u} = \emptyset$ is 1 and therefore

$$\begin{aligned}
&e^2(Q_{n,d}; H_{d,\alpha,\gamma}) \\
&\quad = \frac{1}{n^2}\sum_{k,s=1}^{n}\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]}\gamma_{d,\mathfrak{u}}\prod_{j\in\mathfrak{u}}\frac{1}{(2\pi)^{2\alpha}}\sum_{0\neq h\in\mathbb{Z}}^{\infty}\frac{\exp(2\pi\,\mathrm{i}\,h(x_{k,j}-x_{s,j})}{|h|^{2\alpha}} \\
&\quad = \frac{1}{n^2}\sum_{k,s=1}^{n}\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]}\gamma_{d,\mathfrak{u}}\prod_{j\in\mathfrak{u}}\frac{2}{(2\pi)^{2\alpha}}\sum_{h=1}^{\infty}\frac{\cos(2\pi\,h\,(x_{k,j}-x_{s,j}))}{h^{2\alpha}},
\end{aligned}$$

where $x_{k,j}$ and $x_{s,j}$ denote the jth components of x_k and x_s.

The formula above is a non-increasing function of α. This simply follows from the fact that the unit ball of $H_{d,\beta,\gamma}$ is a subset of the unit ball of $H_{d,\alpha,\gamma}$ for $\alpha \le \beta$ and therefore

$$e(Q_{n,d}; H_{d,\beta,\gamma}) \le e(Q_{n,d}; H_{d,\alpha,\gamma}) \quad \text{for all } n, d \in \mathbb{N}.$$

This means that the smoothness, measured by α, helps to decrease the worst case error, and so multivariate integration in $H_{d,\beta,\gamma}$ is not harder than multivariate integration in $H_{d,\alpha,\gamma}$.

Similarly, for the two sequences of weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ and $\eta = \{\eta_{d,\mathfrak{u}}\}$ with $\gamma_{d,\mathfrak{u}} \le \eta_{d,\mathfrak{u}}$ for all $d \in \mathbb{N}$ and for all $\mathfrak{u} \subseteq [d]$, we have

$$e(Q_{n,d}; H_{d,\alpha,\gamma}) \le e(Q_{n,d}; H_{d,\alpha,\eta}) \quad \text{for all } n, d \in \mathbb{N}.$$

This means that multivariate integration in $H_{d,\alpha,\gamma}$ is not harder than multivariate integration in $H_{d,\alpha,\eta}$, or (equivalently) that the monotonically decreasing weights make multivariate integration easier. In the extreme case when we take $\gamma_{d,\mathfrak{u}} = 0$ for all non-empty $\mathfrak{u}$ then

$$e(Q_{n,d}; H_{d,\alpha,\gamma}) = 0 \quad \text{for all } n \in \mathbb{N},$$

and multivariate integration is trivial. This is why we always assume that at least one $\gamma_{d,\mathfrak{u}}$ is positive for a non-empty $\mathfrak{u}$.

For nested weights, it is clear that d-variate integration is not harder than $(d+1)$-variate integration since the unit ball B_d of $H_{d,\alpha,\gamma}$ is a subset of the unit ball B_{d+1} of $H_{d+1,\alpha,\gamma}$, i.e.,

$$e(Q_{n,d}; H_{d,\alpha,\gamma}) \le e(Q_{n,d+1}; H_{d+1,\alpha,\gamma}) \quad \text{for all } n, d \in \mathbb{N}.$$

The property that integrating functions with more variables is not easier than integrating functions with fewer variables is quite natural.

16.4 Lattice Rules

We now specify sample points x_k used by the QMC algorithm $Q_{n,d}$ to be given by the rank-1 *lattice* point set

$$\left\{\left\{\frac{k\mathbf{z}}{n}\right\} \mid k = 0, 1, \dots, n-1\right\}.$$

Here we assume that n is prime and $\mathbf{z} = [z_1, z_2, \dots, z_d]$ is an integer vector whose components are from the set

$$\mathbb{Z}_n := \{1, 2, \dots, n-1\}.$$

By $\{k\mathbf{z}/\mathbf{n}\}$ we denote the vector $[\{kz_1/n\}, \{kz_2/n\}, \dots, \{kz_d/n\}]$ with $\{kz_j/n\}$ being the fractional part of kz_j/n.

The assumption that n is prime can be relaxed but we do not pursue this point in this chapter. For non-prime number n similar results can be established by applying the approach used in Dick [41] and Kuo and Joe [153].

Hence, for lattice rules we have

$$x_{k+1} = \left\{\frac{k\,\mathbf{z}}{n}\right\} \quad \text{for all } k = 0, 1, \dots, n-1,$$

and the integer vector $\mathbf{z}$ is called a *generator* of the lattice rule. To stress the dependence on $\mathbf{z}$, we denote the worst case error of the lattice rule $Q_{n,d}$ with generator $\mathbf{z}$ as

$$e_{n,d}(\mathbf{z}) = e(Q_{n,d}; H_{d,\alpha,\gamma}).$$

We have

$$\begin{aligned} e_{n,d}^2(\mathbf{z}) &= -1 + \frac{1}{n^2} \sum_{k,s=1}^{n} K_{d,\alpha,\gamma}(x_k, x_s) \\ &= -1 + \frac{1}{n^2} \sum_{k,s=0}^{n-1} \sum_{h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}^{-1}(h) \exp\big(2\pi\, \mathrm{i}\, (k-s)\, h \cdot \mathbf{z}/n\big) = \end{aligned}$$

$$= -1 + \frac{1}{n^2} \sum_{s=0}^{n-1} \Big(\sum_{h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}^{-1}(h) \sum_{k=0}^{n-1} \exp\big(2\pi\,\mathrm{i}\,(k-s)\,h \cdot \mathbf{z}/n\big) \Big).$$

Note that

$$\sum_{k=0}^{n-1} \exp\big(2\pi\,\mathrm{i}\,(k-s)\,h\cdot\mathbf{z}/n\big) = \begin{cases} n & \text{if } h\cdot\mathbf{z} \equiv 0 \mod n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this means that the last sum does not depend on s. Therefore each term in the sum above over s has the same value, and we can simplify the expression for $e_{n,d}^2(\mathbf{z})$ as follows,

$$\begin{aligned} e_{n,d}^2(\mathbf{z}) &= -1 + \frac{1}{n} \sum_{k=0}^{n-1} \sum_{h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}^{-1} \prod_{j=1}^{d} \exp\big(2\pi\,\mathrm{i}\,k\,h_j z_j/n\big) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{0 \neq h \in \mathbb{Z}^d} \varrho_{d,\alpha,\gamma}^{-1} \prod_{j=1}^{d} \exp\big(2\pi\,\mathrm{i}\,k\,h_j z_j/n\big) \qquad (16.3) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \Bigg[\sum_{0 \neq h \in \mathbb{Z}} \frac{\exp\big(2\pi\,\mathrm{i}\,hkz_j/n\big)}{(2\pi\,|h|)^{2\alpha}} \Bigg]. \end{aligned}$$

16.4.1 The Existence of Good Lattice Rules

By a good lattice rule we mean a lattice rule, or equivalently a generator $\mathbf{z}$, with a small worst case error. We now show the existence of good lattice rules by an averaging argument over $\mathbf{z} \in \mathbb{Z}_n^d$, where $\mathbb{Z}_n^d$ denotes the d-fold copy of $\mathbb{Z}_n$. Define

$$M_{n,d}(\alpha) := \frac{1}{(n-1)^d} \sum_{\mathbf{z} \in \mathbb{Z}_n^d} e_{n,d}^2(\mathbf{z}), \qquad (16.4)$$

as the mean square worst case error taken over all possible $\mathbf{z} \in \mathbb{Z}_n^d$. The value of $M_{n,d}(\alpha)$ was found for product weights in [279], and for general weights in [55]. In our case, a few details are different from the study done in [55], where β_2 is taken as 1 and α corresponds to our 2α. In any case, following the proof from [55], it is easy to find $M_{n,d}(\alpha)$ for general weights in our case.

Theorem 16.1. *Let n be a prime number and $\alpha > \frac{1}{2}$.*

- *We have*

$$M_{n,d}(\alpha) = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\,\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|} + \frac{n-1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}\, T^{|\mathfrak{u}|}(\alpha),$$

where

$$T(\alpha) = -\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \, \frac{(1-n^{1-2\alpha})}{n-1}. \tag{16.5}$$

- *We have*

$$M_{n,d}(\alpha) \le \frac{1}{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\,\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|}.$$

- *There exists a generator* $\mathbf{z} \in \mathbb{Z}_n^d$ *such that*

$$e_{n,d}(\mathbf{z}) \le \frac{1}{\sqrt{n-1}} \left(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\,\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|} \right)^{1/2}.$$

- *Let* μ *be the uniform distribution on* $\mathbb{Z}_n^d$*, i.e.,* $\mu(\mathbf{z}) = (n-1)^{-d}$ *for all* $\mathbf{z} \in \mathbb{Z}_n^d$*. For* $c > 1$*, define the set*

$$\mathbb{Z}_c = \left\{ \mathbf{z} \in \mathbb{Z}_n^d \mid e_{n,d}(\mathbf{z}) \le \frac{c}{\sqrt{n-1}} \left(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\,\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|} \right)^{1/2} \right\}.$$

Then

$$\mu(\mathbb{Z}_c) \ge 1 - c^{-2}.$$

Proof. We use the formula (16.3) for $e_{n,d}^2(\mathbf{z})$ and average it over all the $(n-1)^d$ values of $\mathbf{z} \in \mathbb{Z}_n^d$. We obtain

$$\begin{aligned} M_{n,d}(\alpha) &= \frac{1}{(n-1)^d} \sum_{\mathbf{z} \in \mathbb{Z}_n^d} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left[\sum_{0 \ne h \in \mathbb{Z}} \frac{e^{2\pi\,\mathrm{i}\,hkz_j/n}}{(2\pi\,|h|)^{2\alpha}} \right] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \frac{\gamma_{d,\mathfrak{u}}}{(n-1)^{|\mathfrak{u}|}} \sum_{\mathbf{z}_\mathfrak{u} \in \mathbb{Z}_n^{|\mathfrak{u}|}} \prod_{j \in \mathfrak{u}} \left(\sum_{0 \ne h \in \mathbb{Z}} \frac{e^{2\pi\,\mathrm{i}\,hkz_j/n}}{(2\pi\,|h|)^{2\alpha}} \right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left(\frac{1}{n-1} \sum_{z_j \in \mathbb{Z}_n} \sum_{0 \ne h \in \mathbb{Z}} \frac{e^{2\pi i hkz_j/n}}{(2\pi\,|h|)^{2\alpha}} \right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} T_\alpha^{|\mathfrak{u}|}(k,n), \end{aligned} \tag{16.6}$$

where

$$T_\alpha(k,n) := \frac{1}{(n-1)} \sum_{z=1}^{n-1} \sum_{0 \ne h \in \mathbb{Z}} \frac{e^{2\pi i hkz/n}}{(2\pi\,|h|)^{2\alpha}}.$$

We now show that

$$T_\alpha(k,n) = \begin{cases} \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} & \text{if } k \text{ is a multiple of } n, \\ T(\alpha) & \text{otherwise,} \end{cases}$$

where $T(\alpha)$ is given by (16.5). Indeed, if k is a multiple of n we need to sum up

$$\sum_{0\neq h\in\mathbb{Z}} \frac{1}{(2\pi\,|h|)^{2\alpha}} = \frac{2}{(2\pi)^{2\alpha}} \sum_{j=1}^{\infty} \frac{1}{j^{2\alpha}} = \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}},$$

as claimed.

If k is not a multiple of n then we separate out the terms in the sum over h in which h is a multiple of n and obtain

$$\begin{aligned} a &:= \frac{1}{(2\pi)^{2\alpha}} \sum_{z=1}^{n-1} \left(\sum_{h=jn,\,0\neq j\in\mathbb{Z}} \frac{1}{|h|^{2\alpha}} + \sum_{h\not\equiv 0 \bmod n} \frac{e^{2\pi\,\mathrm{i}hkz/n}}{|h|^{2\alpha}} \right) \\ &= \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \frac{n-1}{n^{2\alpha}} + \frac{1}{(2\pi)^{2\alpha}} \sum_{h\not\equiv 0 \bmod n} \frac{1}{|h|^{2\alpha}} \sum_{z=1}^{n-1} e^{2\pi\,\mathrm{i}hkz/n}. \end{aligned}$$

For $q = e^{2\pi\,\mathrm{i}hk/n}$ we have $q \neq 1$ since n is prime, and $q^n = 1$. Therefore

$$\sum_{z=1}^{n-1} e^{2\pi\,\mathrm{i}hkz/n} = \sum_{z=1}^{n-1} q^z = \frac{1-q^n}{1-q} - 1 = -1.$$

Hence

$$\begin{aligned} \sum_{h\not\equiv 0 \bmod n} \frac{1}{|h|^{2\alpha}} \sum_{z=1}^{n-1} e^{2\pi\,\mathrm{i}hkz/n} &= -\sum_{h\not\equiv 0 \bmod n} \frac{1}{|h|^{2\alpha}} \\ &= -\left(\sum_{0\neq h\in\mathbb{Z}} \frac{1}{|h|^{2\alpha}} - \sum_{h=jn,\,0\neq j\in\mathbb{Z}} \frac{1}{|h|^{2\alpha}} \right) \\ &= -2\zeta(2\alpha)(1-n^{-2\alpha}), \end{aligned}$$

and

$$a = -\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \left[1 - n^{-2\alpha} - (n-1)n^{-2\alpha}\right] = -\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \left(1 - n^{1-2\alpha}\right).$$

This proves that $T_\alpha(k,n) = T(\alpha)$, as claimed.

We return to the formula (16.6) for $M_{n,d}(\alpha)$. Separating out the $k = 0$ term in the expression of $M_{n,d}(\alpha)$, we have

$$M_{n,d}(\alpha) = \frac{1}{n} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} + \frac{n-1}{n} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} T^{|\mathfrak{u}|}(\alpha).$$

This completes the first part of the theorem.

To prove the second point, write $M_{n,d}(\alpha)$ as

$$M_{n,d}(\alpha) = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|} (1 + R(n, \mathfrak{u}, \alpha)),$$

where

$$R(n, \mathfrak{u}, \alpha) = (-1)^{|\mathfrak{u}|}(n-1) \left(\frac{1 - n^{1-2\alpha}}{n-1} \right)^{|\mathfrak{u}|}.$$

If $|\mathfrak{u}|$ is odd then $R(n, \mathfrak{u}, \alpha) \in (-1, 0)$, while if $|\mathfrak{u}|$ is even then $|\mathfrak{u}| \geq 2$ and

$$R(n, \mathfrak{u}, \alpha) \in \left(0, (n-1) \left(\frac{1 - n^{1-2\alpha}}{n-1} \right)^2 \right] \subseteq \left(0, \frac{1}{n-1} \right).$$

Therefore for all $|\mathfrak{u}|$, we have

$$\begin{aligned} M_{n,d}(\alpha) &\leq \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|} \left(1 + \frac{1}{n-1} \right) \\ &= \frac{1}{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|}. \end{aligned}$$

This completes the proof of the estimate of $M_{n,d}(\alpha)$.

The remaining parts of Theorem 16.1 follow from an easy application of the mean value theorem and Chebyshev's inequality applied to (16.4). □

Theorem 16.1 presents the formula and an upper bound for the mean square worst case error in terms of the number of sample points n, the smoothness parameter α and the weights $\gamma_{d,\mathfrak{u}}$ of the Korobov space $H_{d,\alpha,\gamma}$.

We stress that Theorem 16.1 is *not* constructive since we only claim the existence of a generator $\mathbf{z}$ for which

$$e_{n,d}(\mathbf{z}) = \mathcal{O}(n^{-1/2}).$$

The convergence rate is not great. However, it is independent of d. The last part of this theorem states that for large c, say $c = 10$, we have a large probability, at least 0.99 for $c = 10$, that randomly selected generators from $\mathbb{Z}_n^d$ satisfy the bound on the mean worst case error modulo a factor c.

The next theorem establishes a faster rate of convergence than is apparent in Theorem 16.1. It shows that for a properly chosen generator $\mathbf{z}$ we may have the rate of convergence arbitrarily close to α, i.e., $e_{n,d}(\mathbf{z}) = \mathcal{O}(n^{-\tau})$ for $\tau < \alpha$. We stress that the rate α is best possible since even for the univariate case the minimal worst case errors tend to zero like $n^{-\alpha}$, see [273]. The main tool to obtain such a result is Jensen's inequality,

$$\sum_{k=1}^{\infty} a_k \leq \left(\sum_{k=1}^{\infty} a_k^{\lambda} \right)^{1/\lambda} \quad \text{for all } \lambda \in (0, 1],$$

where a_k are arbitrary non-negative numbers, and which has been already used many times in Volume I as well as in Volume II.

In order to apply Jensen's inequality we need to find a special form of the square of the worst case error of the lattice rule with a generator $\mathbf{z}$. This form should be expressed as a sum of infinitely many non-negative terms.

It will be convenient to use an alternative notation $e_{n,d}(\alpha,\gamma,\mathbf{z})$ for the worst case error $e_{n,d}(\mathbf{z})$, in order to stress the dependence on the parameter α and the weight sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$. Let $\mathbf{z}_\mathfrak{u}$ and $h_\mathfrak{u}$ denote the $|\mathfrak{u}|$-dimensional vectors containing the components of $\mathbf{z}$ and h with indices in $\mathfrak{u}$. Let

$$\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$$

denote the set of non-zero integers. Then we have the following lemma.

Lemma 16.2. *For a prime n and $\alpha > \frac{1}{2}$, we have*

$$e_{n,d}^2(\alpha,\gamma,\mathbf{z}) = \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \sum_{\substack{h_\mathfrak{u}\in\mathbb{Z}_0^{|\mathfrak{u}|}\\ h_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u}\equiv 0 \bmod n}} \frac{\gamma_{d,\mathfrak{u}}}{\prod_{j\in\mathfrak{u}}(2\pi\,|h_j|)^{2\alpha}}.$$

Proof. We know from (16.3) that

$$e_{n,d}^2(\alpha,\gamma,\mathbf{z}) = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \Big[\sum_{0\neq h\in\mathbb{Z}} \frac{\exp\big((2\pi\,\mathrm{i}\,hkz_j/n\big)}{(2\pi\,|h|)^{2\alpha}}\Big].$$

We exchange the order of the product and the last sum, and obtain

$$\begin{aligned} e_{n,d}^2(\alpha,\gamma,\mathbf{z}) &= \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \sum_{h_\mathfrak{u}\in\mathbb{Z}_0^{|\mathfrak{u}|}} \frac{\exp(2\pi ikh_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u}/n)}{\prod_{j\in\mathfrak{u}}(2\pi\,|h_j|)^{2\alpha}} \\ &= \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \sum_{h_\mathfrak{u}\in\mathbb{Z}_0^{|\mathfrak{u}|}} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\exp(2\pi ikh_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u}/n)}{\prod_{j\in\mathfrak{u}}(2\pi\,|h_j|)^{2\alpha}}. \end{aligned}$$

This allows us to sum up with respect to k. Since

$$\frac{1}{n}\sum_{k=0}^{n-1}\exp(2\pi ikh_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u}/n) = \begin{cases} 1 & \text{if } h_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u} \equiv 0 \bmod n\ , \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$e_{n,d}^2(\alpha,\gamma,\mathbf{z}) = \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \sum_{\substack{h_\mathfrak{u}\in\mathbb{Z}_0^{|\mathfrak{u}|}\\ h_\mathfrak{u}\cdot\mathbf{z}_\mathfrak{u}\equiv 0 \bmod n}} \frac{\gamma_{d,\mathfrak{u}}}{\prod_{j\in\mathfrak{u}}(2\pi\,|h_j|)^{2\alpha}},$$

as claimed. □

Lemma 16.2 expresses the square of the worst case of the lattice rules as an infinite sum of non-negative terms. Note also that the number n now affects the terms in the sum over $h_{\mathfrak{u}}$, and the factor n^{-1} present in the previous formula formally disappeared.

We are ready to apply Jensen's inequality and to get estimates with a better rate of convergence.

Theorem 16.3. *Let n be a prime number and $\alpha > \frac{1}{2}$.*

- *Then there exists a rank-1 lattice rule with a generator $\mathbf{z}^* \in \mathbb{Z}_n^d$ such that for all $\tau \in [\frac{1}{2}, \alpha)$ we have*

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \;\le\; C(d,\tau)\,(n-1)^{-\tau},$$

  where

$$C(d,\tau) = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \right]^{|\mathfrak{u}|} \right)^{\tau}. \tag{16.7}$$

- *Let μ be the same probability measure as in Theorem 16.1. For $\tau \in [\frac{1}{2}, \alpha)$ and $c > 1$, define*

$$\mathbb{Z}_c(\tau) = \left\{ \mathbf{z} \in \mathbb{Z}_n^d \;:\; e_{n,d}(\mathbf{z}) \le c\, C(d,\tau)\,(n-1)^{-\tau} \right\}.$$

  Then

$$\mu\left(\mathbb{Z}_c(\tau)\right) \ge 1 - c^{-1/\tau}.$$

Proof. Applying Jensen's inequality to the expression on $e_{n,d}^2(\alpha,\gamma,\mathbf{z})$ given by Lemma 16.2, we obtain for $\lambda \in (1/(2\alpha), 1]$ that

$$\begin{aligned} e_{n,d}^2(\alpha,\gamma,\mathbf{z}) &\le \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \sum_{\substack{h_{\mathfrak{u}} \in \mathbb{Z}_0^{|\mathfrak{u}|} \\ h_{\mathfrak{u}} \cdot \mathbf{z}_{\mathfrak{u}} \equiv 0 \bmod n}} \frac{\gamma_{d,\mathfrak{u}}^{\lambda}}{\prod_{j \in \mathfrak{u}} (2\pi\,|h_j|)^{2\alpha\lambda}} \right)^{1/\lambda} \\ &= \left[e_{n,d}^2(\alpha\lambda, \gamma^{\lambda}, \mathbf{z}) \right]^{1/\lambda}, \end{aligned}$$

where γ^{λ} denotes the weight sequence with values $\gamma_{d,\mathfrak{u}}^{\lambda}$ for each $\mathfrak{u} \subseteq [d]$.

This means that

$$e_{n,d}(\alpha,\gamma,\mathbf{z}) \le e_{n,d}^{1/\lambda}(\alpha\lambda, \gamma^{\lambda}, \mathbf{z}) \quad \text{for all } \lambda \in (1/(2\alpha), 1]. \tag{16.8}$$

For $\lambda < 1$ we have $1/\lambda > 1$, and we estimate the worst case error of the lattice rule for the original weighted Korobov space by a higher power of the worst case error of the same lattice rule for the weighted Korobov space with the modified smoothness parameter and the modified weight sequence.

We now apply Theorem 16.1 with α replaced by $\alpha\lambda$ and γ replaced by γ^λ. Then there exists a generator $\mathbf{z}_\lambda \in \mathbb{Z}_n^d$ such that

$$e_{n,d}(\alpha\lambda, \gamma^\lambda, \mathbf{z}_\lambda) \le \frac{1}{\sqrt{n-1}} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^\lambda \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right]^{|\mathfrak{u}|} \right)^{1/2}.$$

Now let $\mathbf{z}^* \in \mathbb{Z}_n^d$ be such that

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \le e_{n,d}(\alpha, \gamma, \mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathbb{Z}_n^d.$$

Then for all $\lambda \in (1/(2\alpha), 1]$ we have

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \le e_{n,d}(\alpha, \gamma, \mathbf{z}_\lambda) \le e_{n,d}^{1/\lambda}(\alpha\lambda, \gamma^\lambda, \mathbf{z}_\lambda),$$

and therefore

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \le (n-1)^{-1/(2\lambda)} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^\lambda \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right]^{|\mathfrak{u}|} \right)^{1/(2\lambda)}. \tag{16.9}$$

For $\tau \in [\frac{1}{2}, \alpha)$, take $\lambda = 1/(2\tau)$. Then $\lambda \in (1/(2\alpha), 1]$, and we rewrite (16.9) as

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \le (n-1)^{-\tau} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \right]^{|\mathfrak{u}|} \right)^{\tau}.$$

This completes the proof of the first part.

We now prove the second part. For $\lambda \in (1/(2\alpha), 1]$, we know from Theorem 16.1 that the set of $\mathbf{z} \in \mathbb{Z}_n^d$ for which

$$e_{n,d}(\alpha\lambda, \gamma^\lambda, \mathbf{z}) \le \frac{c^\lambda}{\sqrt{n-1}} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^\lambda \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right]^{|\mathfrak{u}|} \right)^{1/2}$$

has measure at least $1 - c^{-2\lambda}$. Since $e_{n,d}^\lambda(\alpha, \gamma, \mathbf{z}) \le e_{n,d}(\alpha\lambda, \gamma^\lambda, \mathbf{z})$, then the set of $\mathbf{z}$ for which

$$e(\alpha, \gamma, \mathbf{z}) \le \frac{c}{(n-1)^{1/(2\lambda)}} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^\lambda \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right]^{|\mathfrak{u}|} \right)^{1/(2\lambda)}$$

also has measure at least $1 - c^{-2\lambda}$. Substituting $\tau = 1/(2\lambda)$ we obtain the second part. This completes the proof. □

Theorem 16.3 tells us that for arbitrarily large d there is a rank-1 lattice rule whose error is of order $n^{-\tau}$. Since τ can be arbitrarily close to α we may achieve almost the same speed of convergence as for the univariate case, which is $n^{-\alpha}$, and it is known that this bound is sharp, see again [273]. Hence, as long as we control the factors $C(d, \tau)$, the difficulty of the d-variate integration is roughly the same as for the univariate case. Furthermore, if we choose c such that $c^{-1/\tau}$ is small, we have large probability that the vectors $\mathbf{z}$ from $\mathbb{Z}_n^d$ satisfy the error bound of order $n^{-\tau}$.

We now show that $C(d,\tau)$ is a non-decreasing function of $\tau \in [\frac{1}{2},\alpha)$. Indeed, again by Jensen's inequality, for $\tau_1, \tau_2 \in [\frac{1}{2},\alpha)$ with $\tau_1 \le \tau_2$, we take $\lambda = \tau_1/\tau_2 \le 1$ and

$$\begin{aligned} C(d,\tau_1) &= \Bigg(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \left[\frac{\gamma_{d,\mathfrak{u}}}{(2\pi)^{2\alpha|\mathfrak{u}|}} \right]^{1/(2\tau_1)} \sum_{j_1,j_2,\dots,j_{|\mathfrak{u}|} \in \mathbb{Z}_0} \prod_{k=1}^{|\mathfrak{u}|} j_k^{-\alpha/\tau_1} \Bigg)^{\tau_1} \\ &\le \Bigg(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \left[\frac{\gamma_{d,\mathfrak{u}}}{(2\pi)^{2\alpha|\mathfrak{u}|}} \right]^{\lambda/(2\tau_1)} \sum_{j_1,j_2,\dots,j_{|\mathfrak{u}|} \in \mathbb{Z}_0} \prod_{k=1}^{|\mathfrak{u}|} j_k^{-\lambda\alpha/\tau_1} \Bigg)^{\tau_1/\lambda} \\ &= \Bigg(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \left[\frac{\gamma_{d,\mathfrak{u}}}{(2\pi)^{2\alpha|\mathfrak{u}|}} \right]^{1/(2\tau_2)} \sum_{j_1,j_2,\dots,j_{|\mathfrak{u}|} \in \mathbb{Z}_0} \prod_{k=1}^{|\mathfrak{u}|} j_k^{-\alpha/\tau_2} \Bigg)^{\tau_2} \\ &= C(d,\tau_2), \end{aligned}$$

as claimed. This property implies that we have a tradeoff in the estimate

$$e_{n,d}(\mathbf{z}^*) \le C(d,\tau)\,(n-1)^{-\tau}.$$

Namely, the increase of τ improves the rate of convergence at the expense of the increase of the factor $C(d,\tau)$. Obviously, for a relatively small d, the larger value of τ is preferable since even an exponential dependence on d can be tolerated. Of course, this point cannot be taken to the limit, since for τ tending to α the factor $C(d,\tau)$ goes to infinity. This must be so since for $d \ge 2$ we know that even the nth minimal worst case error behaves asymptotically as $n^{-\alpha}(\ln n)^{a(d-1)}$ for some positive a. The presence of $\ln n$ makes it impossible to let $\tau = \alpha$; equivalently, for τ tending to α the factor $C(d,\tau)$ must blow up.

The factor $C(d,\tau)$ blows up to infinity as τ approaches α since the argument of the Riemann zeta function then tends to one, and it behaves like $\zeta(1+\delta) \approx \delta^{-1}$ for small positive δ. If we set $\tau = \alpha(1-\delta)$, then we have

$$C(d,\alpha(1-\delta)) \approx \Bigg(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\alpha)} \frac{1}{[\delta^{-1}\pi]^{|\mathfrak{u}|}} \Bigg)^{\alpha}.$$

It is natural to ask what is the largest τ for which we still have a meaningful estimate of $e_{n,d}(\mathbf{z}^*)$. Of course, this depends on how fast $C(d,\tau)$ grows. We would like to choose the largest τ for which $C(d,\tau)$ is *not* exponential in d. That is, we would like to maximize the rate of convergence and still have tractability. This depends on the weights. In the next sections we find conditions on the weights to get various kinds of tractability.

16.4.2 Tractability for General Weights

We are now ready to discuss tractability of multivariate integration in the weighted Korobov spaces $H_{d,\alpha,\gamma}$ for general weights. From Theorem 16.3 we easily conclude sufficient conditions on the weights for (strong) polynomial tractability, T-tractability and weak tractability. We also provide matching necessary conditions, assuming that the weights are chosen such that the reproducing kernel is point-wise non-negative and we use arbitrary QMC algorithms. For this case, we use the same analysis as in Section 6.2 of [277]. The case of arbitrary algorithms is much harder and will be considered later in Section 16.8.

We remind the reader that the initial error is one, and there is no difference between the absolute and normalized error criteria. Therefore for brevity, we do not mention the error criteria for tractability results. As always, by $n(\varepsilon, d)$ we denote the minimal number of function values that are needed to find a (linear) algorithm with the worst case error at most ε. By $n_{\mathrm{QMC}}(\varepsilon, d)$ we denote the minimal number of function values that are needed to find a QMC algorithm with the worst case error at most ε. Obviously, $n(\varepsilon, d) \le n_{\mathrm{QMC}}(\varepsilon, d)$.

Theorem 16.4. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined for the weighted Korobov spaces $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$.*

For $\tau \in [\frac{1}{2}, \alpha)$ and $q \ge 0$ define

$$B_{\tau,q} := \sup_{d \in \mathbb{N}} \left[\frac{1}{d^q} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \right]^{|\mathfrak{u}|} \right]. \tag{16.10}$$

- *If there exists a non-negative q such that $B_{1/2,q} < \infty$ then I_γ is polynomially tractable. If*

$$B_{\tau,q} < \infty \quad \text{for some } \tau \in \left[\tfrac{1}{2}, \alpha\right) \text{ and a non-negative } q$$

then I_γ is polynomially tractable with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most q,

$$n_{\mathrm{QMC}}(\varepsilon, d) = \mathcal{O}\left(\varepsilon^{-1/\tau} d^q\right)$$

with the factor in the big $\mathcal{O}$ notation independent of ε^{-1} and d.

- *In particular, if*

$$B_{1/2,0} < \infty$$

then I_γ is strongly polynomially tractable. If

$$B_{\tau,0} < \infty \quad \text{for some } \tau \in \left[\tfrac{1}{2}, \alpha\right)$$

then the ε^{-1} exponent of strong polynomial tractability is at most $1/\tau$. If

$$B_{\tau,0} < \infty \quad \text{for all } \tau \in \left[\tfrac{1}{2}, \alpha\right)$$

then the ε^{-1} exponent of strong polynomial tractability reaches the minimal value $1/\alpha$.

- *Assume that the weights* $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ *are chosen such that*

$$K_{d,\alpha,\gamma}(x,y) \ge 0 \quad \textit{for all } x,y \in [0,1]^d. \tag{16.11}$$

  Then I_γ *is strongly polynomially tractable for QMC algorithms iff*

$$B_{1/2,0} < \infty,$$

  and I_γ *is polynomially tractable for QMC algorithms iff*

$$B_{1/2,q} < \infty \quad \textit{for some } q.$$

- *Let* T *be a tractability function. If* $B_{1/2,0} < \infty$ *and*

$$t^* = \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1},1)} < \infty$$

  then I_γ *is strongly* T*-tractable.*

  If $B_{\tau,0} < \infty$ *for some* $\tau \in [\frac{1}{2},\alpha)$ *and* $t^* < \infty$ *then the exponent of strong* T*-tractability is at most* t^*/τ.

- *Let* T *be a tractability function. If*

$$t^* := \inf_{\tau \in [1/2,\alpha)} \limsup_{\varepsilon^{-1}+d \to \infty} \frac{\ln\Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \big[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\big]^{|\mathfrak{u}|}\Big) + \ln \varepsilon^{-1/\tau}}{1 + \ln T(\varepsilon^{-1},d)} < \infty$$

  then I_γ *is* T*-tractable with the exponent* $t > t^*$. *If* (16.11) *holds then the last condition is also necessary for* T*-tractability for QMC algorithms.*

- *If*

$$\lim_{d\to\infty} \frac{\ln\Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \big[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\big]^{|\mathfrak{u}|}\Big)}{d} = 0$$

  then I_γ *is weakly tractable. If* (16.11) *holds then the last condition is also necessary for weak tractability for QMC algorithms.*

*The corresponding tractability bounds can be achieved by rank-*1 *lattice rules.*

Proof. We prove the first part. Assume that $B = B_{\tau,q}$ is finite for some $\tau \in [\frac{1}{2},\alpha)$ and some non-negative q. Then we have

$$\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|} \le B\, d^q \quad \text{for all } d \in \mathbb{N}.$$

Hence, (16.7) yields

$$C(d,\tau) \le B^\tau\, d^{\,q\tau} \quad \text{for all } d \in \mathbb{N}.$$

Therefore, Theorem 16.3 implies that there exists a generator $\mathbf{z}^* \in \mathbb{Z}_n^d$ such that

$$e_{n,d}(\mathbf{z}^*) \le C(d,\tau)\,(n-1)^{-\tau} \le B^\tau\, d^{\,q\tau}\,(n-1)^{-\tau}.$$

This implies that

$$n_{\rm QMC}(\varepsilon, d) \le \left\lceil B\, d^q\, \varepsilon^{-1/\tau}\right\rceil + 1.$$

Thus, we have polynomial tractability with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most q.

To prove the second part, we see that if $B_{\tau,0} < \infty$ for some $\tau \in [\frac{1}{2}, \alpha)$ then we have strong polynomial tractability with an ε^{-1} exponent at most $1/\tau$. If $B_{\tau,0} < \infty$ for all $\tau \in [\frac{1}{2}, \alpha)$, then the ε^{-1} exponent of strong polynomial tractability is at most

$$\inf\left\{1/\tau \mid \tau \in \left[\tfrac{1}{2}, \alpha\right)\right\} = 1/\alpha.$$

As already mentioned, $n^{-\alpha}$ is the best possible rate of convergence for $d = 1$, and therefore the ε^{-1} exponent of strong polynomial tractability is minimal.

We now turn to the third part. Consider an arbitrary QMC algorithm $Q_{n,d}$, see (16.2). Then the square of its worst case error is

$$e^2\left(Q_{n,d}; H_{d,\alpha,\gamma}\right) = -1 + \frac{1}{n^2}\sum_{k,s=1}^{n} K_{d,\alpha,\gamma}(x_k, x_s).$$

We bound $e^2\left(Q_{n,d}; H_{d,\alpha,\gamma}\right)$ from below by dropping all terms with $k \ne s$ since $K_{d,\alpha,\gamma}(x_k, x_s) \ge 0$. Furthermore, for $k = s$ we have

$$K_{d,\alpha,\gamma}(x_k, x_k) = K_{d,\alpha,\gamma}(0,0) = 1 + \sum_{\emptyset \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}.$$

Therefore

$$\begin{aligned} e^2\left(Q_{n,d}; H_{d,\alpha,\gamma}\right) &\ge -1 + \frac{1}{n} K_{d,\alpha,\gamma}(0,0) \\ &\ge -1 + \frac{1}{n} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}. \end{aligned}$$

Assume that we have (strong) polynomial tractability for QMC algorithms. That is, there is a QMC algorithm $Q_{n,d}$ such that

$$e(Q_{n,d}; H_{d,\alpha,\gamma}) \le \varepsilon \quad \text{for } n = n(\varepsilon, H_{d,\alpha,\gamma}) \le C\,\varepsilon^{-p} d^q$$

for some non-negative C, p and q, where $q = 0$ when we consider strong polynomial tractability. Take, say, $\varepsilon = \frac{1}{2}$. Then the lower bound on the worst case error of $Q_{n,d}$ for $n = n(\frac{1}{2}, H_{d,\alpha,\gamma})$ yields

$$\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \le \left(1 + e^2(Q_{n,d}, H_{d,\alpha,\gamma})\right) n \le (1 + .25)\, C\, 2^p\, d^q.$$

Hence

$$\sup_{d\in\mathbb{N}} \frac{1}{d^q} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \le 5\,C\,2^{p-2} < \infty.$$

This means that $B_{1/2,q} < \infty$, and the third part is proven.

We now turn to strong T-tractability. The assumption $B_{1/2,0} < \infty$ means that $n_{\mathrm{QMC}}(\varepsilon, d) = \mathcal{O}(\varepsilon^{-2})$. This can be bounded by $C\,T^t(\varepsilon^{-1}, d)$ if we take $d = 1$ and use the second assumption that the corresponding limit superior is finite. Hence, we have strong T-tractability, as claimed. If $B_{\tau,0} < \infty$ for some $\tau \in [\frac{1}{2}, \alpha)$ then $n_{\mathrm{QMC}}(\varepsilon, d) = \mathcal{O}(\varepsilon^{-1/\tau})$. This can be bounded by $C\,T^t(\varepsilon^{-1}, 1)$ if we take $t > t^*/\tau$. This proves that the exponent of strong T-tractability is at most t^*/τ.

We now analyze T-tractability. Assume that $t^* < \infty$. This means that the corresponding limit superior is finite for some $\tau \in [\frac{1}{2}, \alpha)$. For this τ, we know that

$$n_{\mathrm{QMC}}(\varepsilon, d) \le \left\lceil C(d,\tau)^{1/\tau}\,\varepsilon^{-1/\tau}\right\rceil + 1 \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

By taking logarithms, it is easy to check that $n_{\mathrm{QMC}}(\varepsilon, d) \le C\,T^t(\varepsilon^{-1}, d)$ holds if

$$r = \limsup_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(1 + C(d,\tau)^{1/\tau}\right) + \ln \varepsilon^{-1/\tau}}{1 + \ln T(\varepsilon^{-1}, d)} < \infty.$$

Substituting the formula for $C(d,\tau)$, we see that r is indeed finite, since $t^* < \infty$. Hence, we have T-tractability with an exponent $t > r$. Since r can be arbitrarily close to t^*, this shows that $t > t^*$ and completes the proof of this part.

Assume now that (16.11) holds. We want to show that T-tractability for QMC algorithms implies that $t^* < \infty$. Let

$$A := \sup_{\varepsilon\in(0,1),\,d\in\mathbb{N}} \frac{\ln\left(1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}\right) + \ln \varepsilon^{-2}}{1 + \ln T(\varepsilon^{-1}, d)}.$$

We first show that $A < \infty$. For $d = 1$, we know that $n_{\mathrm{QMC}}(\varepsilon, 1) = \Theta(\varepsilon^{-1/\alpha})$. Then $n_{\mathrm{QMC}}(\varepsilon, 1) \le C\,T^t(\varepsilon^{-1}, 1) \le C\,T^t(\varepsilon^{-1}, d)$ implies that

$$\sup_{\varepsilon\in(0,1),\,d\in\mathbb{N}} \frac{\ln \varepsilon^{-2}}{1 + \ln T(\varepsilon^{-1}, d)} < \infty.$$

For $d \ge 1$, we know that

$$\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \le \left(1 + \varepsilon^2\right)\, n_{\mathrm{QMC}}(\varepsilon, d).$$

Then $n_{\mathrm{QMC}}(\varepsilon, d) \le C\,T^t(\varepsilon, d)$ implies that

$$\sup_{\varepsilon\in(0,1),\,d\in\mathbb{N}} \frac{\ln\left(1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}\right)}{1 + \ln T(\varepsilon^{-1}, d)} < \infty.$$

Adding up the last two suprema, we conclude that $A < \infty$. The limit superior in the definition of t^* for $\tau = \frac{1}{2}$ is at most A, and therefore t^* is finite, as needed.

Finally, for weak tractability it is enough to show that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n_{\mathrm{QMC}}(\varepsilon, d)}{\varepsilon^{-1}+d} = 0.$$

Since $n_{\mathrm{QMC}}(\varepsilon, d) \le C(d,\tau)^{1/\tau}\,\varepsilon^{-\tau} + 2$ for all $\tau \in [\frac{1}{2}, \alpha)$, the last condition holds if there is a number $\tau \in [\frac{1}{2}, \alpha)$ such that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(1 + C(d,\tau)^{1/\tau}\right) + \tau^{-1}\ln\varepsilon^{-1}}{\varepsilon^{-1}+d} = 0.$$

In turn, this holds if

$$\lim_{d\to\infty} \frac{\ln\left(1 + C(d,\tau)^{1/\tau}\right)}{d} = \lim_{d\to\infty} \frac{\ln\left(1 + C(d,\tau)\right)}{d} = 0.$$

Since $C(d,\tau)$ is a non-decreasing function of τ, its minimum is attained for $\tau = \frac{1}{2}$. We only need to guarantee that

$$\lim_{d\to\infty} \frac{\ln\left(1 + C(d,1/2)\right)}{d} = 0,$$

which is indeed satisfied due to our assumption.

Assume now that (16.11) holds and we have weak tractability for QMC algorithms. That is, there exists a QMC algorithm with worst case error at most ε that uses $n_{\mathrm{QMC}}(\varepsilon, d)$ function values, and that $\ln n_{\mathrm{QMC}}(\varepsilon, d) = o(\varepsilon^{-1} + d)$. Again we use

$$\sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \le \left(1+\varepsilon^2\right)\, n_{\mathrm{QMC}}(\varepsilon, d).$$

Then we take logarithms for a fixed ε and for d tending to infinity, and conclude that

$$\lim_{d\to\infty} \frac{\ln\left(1 + \sum_{\emptyset\ne\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}\right)}{d} = 0,$$

as claimed. This completes the proof. □

Theorem 16.4 states matching necessary and sufficient conditions for (strong) polynomial tractability of multivariate integration for QMC algorithms under the assumption (16.11). That is, when we assume that the reproducing kernel is point-wise non-negative, $K_{d,\alpha,\gamma}(x,y) \ge 0$. We now discuss this assumption for product weights.

For the product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in u} \gamma_{d,j}$, the reproducing kernel is

$$K_{d,\alpha,\gamma}(x,y) = \prod_{j=1}^{d}\left(1 + \frac{2}{(2\pi)^{2\alpha}}\,\gamma_{d,j}\, D_\alpha(x_j - y_j)\right),$$

where

$$D_\alpha(t) = \sum_{h=1}^{\infty} \frac{\cos(2\pi\, h\, t)}{h^{2\alpha}} \quad \text{for all } t \in \mathbb{R}. \tag{16.12}$$

Obviously, D_α is a periodic function, with period 1.

It is known that D_α is related to the Bernoulli polynomial $B_{2\alpha}$ if α is an integer. Then

$$B_{2\alpha}(t) = \frac{2\,(-1)^{\alpha+1}\,(2\alpha)!}{(2\pi)^{2\alpha}}\, D_\alpha(t).$$

Clearly,

$$D_\alpha(t) \le D_\alpha(0) = \zeta(2\alpha).$$

It has been proven in Brown, Chandler, Sloan and Wilson [19] that the minimum of the function D_α on the interval $[0,1]$ is attained at $\frac{1}{2}$. Since $D_\alpha(\frac{1}{2})$ is given as an alternating series we have

$$-1 < D_\alpha\big(\tfrac{1}{2}\big) < -1 + 2^{-2\alpha} \quad \text{for all } 2\alpha > 1. \tag{16.13}$$

For product weights, we then have

$$K_{d,\alpha,\gamma}(x,y) \ge 0 \quad \text{for all } x,y \in [0,1]^d$$

iff $1 + 2\gamma_{d,j}\, D_\alpha(\frac{1}{2})/(2\pi)^{2\alpha} \ge 0$, or equivalently iff

$$\gamma_{d,j} \le a_\alpha := \frac{(2\pi)^{2\alpha}}{2|D_\alpha(1/2)|} < \frac{(2\pi)^{2\alpha}}{2(1-2^{-2\alpha})} \quad \text{for } j \in [d]. \tag{16.14}$$

That is, the assumption (16.11) holds if we take all $\gamma_{d,j}$ no larger than a_α.

For general weights, it is easy to check that

$$K_{d,\alpha,\gamma}(x,y) \ge 1 - \sum_{\emptyset \ne \mathfrak{u} \in [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}.$$

Therefore

$$\sum_{\emptyset \ne \mathfrak{u} \in [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \le 1$$

implies that $K_{d,\alpha,\gamma}(x,y) \ge 0$ for all $x,y \in [0,1]^d$.

The reader may ask why we put $1 + \ln T(\varepsilon^{-1},d)$ instead of $\ln T(\varepsilon^{-1},d)$ in the denominator of the limit superior defining t^*. Although this is a small technical point, this extra 1 is needed at least for some tractability functions T. Indeed, take $T(x,y) = (x-1)y + 1$ for all $x,y \in [1,\infty)$. Clearly, T is a tractability function although $\ln T(1,d) = \ln 1 = 0$, and without the extra 1, the denominator of the limit superior would be zero. Take now the weights $\gamma_{d,\emptyset} = \gamma_{d,\{1\}} = 1$ and the rest of them $\gamma_{d,\mathfrak{u}} = 0$. Then the numerator of the limit superior is always positive, and without the extra 1, we would have $t^* = \infty$. However, the problem is T-tractable. Indeed, for these weights we really have the univariate case, and for $d \ge 1$ with $\tau = \frac{1}{2}$ and $c = \ln(1 + 2\zeta(2\alpha)/(2\pi)^{2\alpha}) > 0$, we have

$$\limsup_{\varepsilon^{-1}+d\to\infty} \frac{c + \ln \varepsilon^{-2}}{1 + \ln[(\varepsilon^{-1} - 1)d + 1]} = \max(2, c) < \infty,$$
$$\limsup_{\varepsilon^{-1}+d\to\infty} \frac{c + \ln \varepsilon^{-2}}{\ln[(\varepsilon^{-1} - 1)d + 1]} = \infty.$$

We stress that Theorem 16.4 is also non-constructive. We only know that the tractability bounds can be achieved by rank-1 lattice rules for some generators but we do not know yet how to efficiently compute such generators. We address this issue in the section on CBC algorithms. However, before doing that we discuss tractability bounds for specific weights.

16.4.3 Tractability for Product Weights

We now show how to obtain necessary and sufficient conditions on (strong) polynomial tractability for product weights and the space $H_{d,\alpha,\gamma}$ for arbitrary $\alpha > \frac{1}{2}$, without assuming (16.14). As before, by necessary conditions we mean conditions for QMC algorithms, leaving the much harder case of arbitrary algorithms for later, see Section 16.8.

For product weights, the case $\alpha = 1$ has been analyzed in [54], and for general $\alpha > \frac{1}{2}$ in [55]. Again, for our case a few details are different but basically we repeat the reasoning from [55]. We prove the following theorem.

Theorem 16.5. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined for the weighted Korobov spaces $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ and product weights $\gamma = \{\gamma_{d,j}\}$,*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \textit{for all } d \textit{ and for all } \mathfrak{u} \in [d],$$

with $\gamma_{d,1} \ge \gamma_{d,2} \ge \cdots$.

We only consider QMC algorithms and necessary tractability conditions refer only to the class of QMC algorithms.

Assume that

$$A := \sup_{d\in\mathbb{N}} \max_{j\in[d]} \gamma_{d,j} < \infty.$$

- *I_γ is strongly polynomially tractable iff*

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty. \tag{16.15}$$

If

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} < \infty \quad \textit{for some } \tau \in \left[\tfrac{1}{2}, \alpha\right)$$

then the ε^{-1} exponent of strong polynomial tractability is at most $1/\tau$. If

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} < \infty \quad \textit{for all } \tau \in \left[\tfrac{1}{2}, \alpha\right)$$

then the ε^{-1} exponent of strong polynomial tractability reaches the minimal value $1/\alpha$.

- *I_γ is polynomially tractable iff*

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(d+1)} < \infty. \tag{16.16}$$

If

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)}}{\ln(d+1)} < \infty \quad \textit{for some } \tau \in \left[\tfrac{1}{2}, \alpha\right)$$

then the ε^{-1} exponent of polynomial tractability is at most $1/\tau$, and the d exponent of polynomial tractability is at most q, where

$$q > \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)}}{\ln(d+1)}.$$

- *I_γ is strongly T-tractable iff*

$$\sup_{d\in\mathbb{N}} \gamma_{d,j} < \infty \quad \textit{and} \quad t^* = \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty.$$

If

$$\sup_{d\in\mathbb{N}} \gamma_{d,j}^{1/(2\tau)} < \infty \quad \textit{for some } \tau \in \left[\tfrac{1}{2}, \alpha\right) \textit{ and } t^* < \infty$$

then the exponent of strong T-tractability is at most t^*/τ.

If

$$\sup_{d\in\mathbb{N}} \gamma_{d,j}^{1/(2\tau)} < \infty \quad \textit{for all } \tau \in \left[\tfrac{1}{2}, \alpha\right) \textit{ and } t^* < \infty$$

then the exponent of strong T-tractability is t^*/α.

- *I_γ is T-tractable iff*

$$\sup_{\varepsilon\in(0,1),\ d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j} + \ln \varepsilon^{-1/2}}{1 + \ln T(\varepsilon^{-1}, d)} < \infty.$$

Then the exponent of T-tractability t satisfies

$$t > \inf_{\tau \in [1/2,\alpha)} \limsup_{\varepsilon^{-1}+d\to\infty} \frac{\left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right] \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} + \ln \varepsilon^{-1/\tau}}{1 + \ln T(\varepsilon^{-1}, d)}.$$

- I_γ *is weakly tractable iff*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$

Proof. For product weights, the third part of Theorem 16.1 states that there is a lattice rule $Q_{n,d}$ such that

$$e^2(Q_{n,d}; H_{d,\alpha,\gamma}) \le \frac{1}{n-1} \prod_{j=1}^{d} \left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \gamma_{d,j}\right).$$

Hence we obtain (strong) polynomial tractability if

$$\sup_{d\in\mathbb{N}} \frac{\prod_{j=1}^{d} \left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \gamma_{d,j}\right)}{(d+1)^q} < \infty$$

for some positive q, and we obtain strong polynomial tractability if $q = 0$ in the condition above. By taking logarithms, this condition is equivalent to

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \ln\left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \gamma_{d,j}\right)}{1 + \kappa_{q,0} \ln(d+1)} < \infty,$$

where $\kappa_{q,0} = 0$ for $q = 0$, and 1 otherwise.

It is easy to show that there exists a positive $b = b(\alpha)$ such that

$$b\,x \le \ln(1+x) \le x \quad \text{for all } x \in \left[0, \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} A\right].$$

Hence that last condition on the weights $\gamma_{d,j}$ is equivalent to

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{1 + \kappa_{q,0} \ln(d+1)} < \infty. \tag{16.17}$$

We then see that (16.17) is equivalent to (16.15) with $q = 0$, and to (16.16) with $q > 0$. This proves that (16.17) implies (strong) polynomial tractability.

We now need to show that strong polynomial tractability implies (16.15) and polynomial tractability implies (16.16) as long as we only use QMC algorithms. This can be done as follows. We can decrease the weights by switching from $\gamma_{d,j}$ to

$$\eta_{d,j} = c^* \gamma_{d,j} \quad \text{with } c^* = \min\left(1, \frac{(2\pi)^{2\alpha}}{2A}\right) > 0.$$

Clearly, decreasing the weights does not make multivariate integration any harder. Then

$$\eta_{d,j} \le \frac{(2\pi)^{2\alpha}}{2A}\,\gamma_{d,j} \le \frac{(2\pi)^{2\alpha}}{2} \le \frac{(2\pi)^{2\alpha}}{2|D_\alpha(1/2)|},$$

since $1 < 1/|D_\alpha(\frac{1}{2})|$ due to (16.13).

Hence, the kernel $K_{d,\alpha,\eta}$ for the product weights $\eta_{d,j}$ is point-wise non-negative and we can apply the last part of Theorem 16.4 for the weights $\eta_{s,j} = c^*\gamma_{s,j}$. For such weights, strong polynomial tractability for QMC algorithms implies that $B_{1/2,0} < \infty$. We now have

$$B_{1/2,0} = \sup_{d\in\mathbb{N}} \prod_{j=1}^{d} \left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\eta_{d,j}\right) - 1.$$

Hence, $B_{1/2,0} < \infty$ is equivalent to $\sup_d \sum_{j=1}^d \eta_{d,j} < \infty$, which in turn is equivalent to $\sup_d \sum_{j=1}^d \gamma_{d,j} < \infty$. Therefore, strong polynomial tractability implies (16.15).

Polynomial tractability for weights $\eta_{d,j}$ implies that $B_{1/2,q} < \infty$ for some positive q. We now have

$$B_{1/2,q} = \sup_{d\in\mathbb{N}} \frac{1}{d^q}\left(\prod_{j=1}^{d}\left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\eta_{d,j}\right) - 1\right).$$

Hence, $B_{1/2,q} < \infty$ is equivalent to

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(d+1)} < \infty,$$

as claimed.

We now discuss the exponents of (strong) polynomial tractability. If

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} < \infty$$

then

$$B_{\tau,0} = \sup_{d\in\mathbb{N}} \prod_{j=1}^{d}\left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\,\gamma_{d,j}\right) - 1 < \infty.$$

So we can apply Theorem 16.4 and conclude that the ε^{-1} exponent of strong polynomial tractability is at most $1/\tau$, as claimed.

On the other hand, if $\sup_d \sum_{j=1}^d \gamma_{d,j}^{1/(2\tau)}/\ln(d+1) < \infty$ then

$$B_{\tau,q} = \sup_{d\in\mathbb{N}} \frac{1}{d^q}\left(\prod_{j=1}^{d}\left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\,\gamma_{d,j}^{1/(2\tau)}\right) - 1\right)$$

is finite if we take q such that

$$q > \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)}}{\ln(d+1)}.$$

The last fact easily follows by taking logarithms and bounding $\ln(1+cx)$ by cx. Then again Theorem 16.4 yields polynomial tractability with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most q.

We now turn to strong T-tractability. Note that $\sup_d \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} < \infty$ implies that $B_{\tau,0} < \infty$, and so it is enough to use the corresponding part of Theorem 16.4. The exact value of the exponent of strong T-tractability follows from the fact that for the univariate case we know that $n_{\mathrm{QMC}}(\varepsilon, 1) = \Theta(\varepsilon^{-1/\alpha})$.

We now analyze T-tractability. We know that it is enough to use $n_{\mathrm{QMC}}(\varepsilon, d)$ function values to obtain an ε-approximation, where

$$n_{\mathrm{QMC}}(\varepsilon, d) \le \varepsilon^{-1/\tau} \prod_{j=1}^{d} \left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \gamma_{d,j}^{1/(2\tau)}\right) + 2$$

for some $\tau \in [\frac{1}{2}, \alpha)$.

We can estimate $n_{\mathrm{QMC}}(\varepsilon, d) \le C\, T^t(\varepsilon^{-1}, d)$ for some positive C and t if, by taking logarithms,

$$t \ge \frac{\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \sum_{j=1}^{d} \gamma_{d,j}^{1/(2\tau)} + (1 + o(1)) \ln \varepsilon^{-1/\tau}}{1 + \ln T(\varepsilon^{-1}, d)}$$

for all $\varepsilon \in (0,1)$, $d \in \mathbb{N}$. Such a finite number t exists since we assumed that the corresponding supremum is finite. This also proves the bound on t.

Assume now that we have T-tractability or weak tractability. Then we repeat the reasoning of Theorem 16.4, and replace $\ln(1+x)$ by $\Theta(x)$ for non-negative and bounded x. We leave details to the reader. This completes the proof. □

We comment on Theorem 16.5. Consider nested weights

$$\gamma_{d,j} = j^{-\beta} \quad \text{for all } j \in [d] \text{ and some non-negative } \beta.$$

Then we have the following.

- For $\beta > 1$, we have strong polynomial tractability with an ε^{-1} exponent at most $\max(1/\alpha, 2/\beta)$. Hence for $\beta \ge 2\alpha$ we achieve the minimal ε^{-1} exponent $1/\alpha$.

- For $\beta = 1$, we have polynomial tractability with an ε^{-1} exponent at most 2 and a d exponent at most $2\zeta(2\alpha)/(2\pi)^{2\alpha}$. Note that the d exponent tends to zero exponentially fast with α going to infinity.

- For $\beta \in (0,1)$, we have weak tractability. For

$$T(x,y) = x \exp(d^{\beta^*}) \quad \text{for all } x, y \in [1, \infty)$$

with $\beta^* \in [0,1)$, we have T-tractability if $\beta^* \geq 1-\beta$. For

$$T(x,y) = \exp(\ln(1+x)\ln(1+y)) \quad \text{for all } x,y \in [1,\infty),$$

we do not have T-tractability for QMC algorithms.

- For $\beta = 0$, we have the curse of dimensionality and I_γ is intractable as long as we use QMC algorithms.

In Theorem 16.5 we consider bounded weights. This assumption was needed in the proof when we replace $\ln(1+x)$ by $\Theta(x)$ for non-negative and bounded x. Consider now unbounded weights, i.e., $A = \infty$. Since

$$\prod_{j=1}^{d} \left(1 + \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\gamma_{d,j}\right) \leq (1+\varepsilon^2)\, n_{\mathrm{QMC}}(\varepsilon,d)$$

then strong polynomial tractability cannot hold. Indeed for a fixed ε and varying d, the left hand-side goes to infinity, showing that $n_{\mathrm{QMC}}(\varepsilon,d)$ cannot be uniformly bounded in d. This contradicts strong polynomial tractability. Hence, the condition (16.15) is necessary and sufficient for strong polynomial tractability for general product weights.

However, for polynomial tractability the situation is different. For example, take

$$\gamma_{d,1} = d^{\,2\beta} \quad \text{and} \quad \gamma_{d,j} = 0 \quad \text{for all } j = 2,3,\dots,$$

for some positive β. Then the integration problem is univariate and

$$n_{\mathrm{QMC}}(\varepsilon,d) = n_{\mathrm{QMC}}(\varepsilon,1) = \Theta\big(d^{\,\beta}\,\varepsilon^{-\alpha}\big).$$

This means that we have polynomial tractability with the minimal ε^{-1} exponent $1/\alpha$ and the d exponent β. However, in this case, the condition (16.16) does *not* hold. This means that this point of Theorem 16.5 is not valid for unbounded weights. Similarly, the point of Theorem 16.5 for weak tractability is not valid if $\beta \geq \frac{1}{2}$. Also the point for T-tractability is not valid at least for some tractability functions. We leave the problem of finding conditions on tractability for unbounded product weights to the reader.

Open Problem 73.

Consider multivariate integration I_γ for the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ and unbounded product weights.

- Find necessary and sufficient conditions on γ for which I_γ is polynomially tractable, T-tractable or weakly tractable for QMC algorithms.

- Do the same for arbitrary algorithms.

16.4.4 Tractability for Weights Independent of *d*

We now give another sufficient condition for (strong) polynomial tractability and weak tractability for weights independent of the dimension d, i.e.,

$$\gamma_{d,\mathfrak{u}} = \gamma_{\mathfrak{u}} \quad \text{for all } \mathfrak{u} \subseteq [d].$$

In this case, the weights are nested, see (16.1). For $\tau \in [\frac{1}{2}, \alpha)$, let

$$D(j,\tau) := \sum_{\mathfrak{u} \subseteq [j-1]} \gamma_{\mathfrak{u} \cup \{j\}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \right]^{|\mathfrak{u}|+1} \quad \text{for all } j \in \mathbb{N}.$$

The factor $C(d,\tau)$ that is present in the error bound in Theorem 16.3, see (16.7), can now be written as

$$C(d,\tau) = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [d-1]} \gamma_{\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}} \right]^{|\mathfrak{u}|} + D(d,\tau) \right)^{\tau} = \left(\sum_{j=1}^{d} D(j,\tau) \right)^{\tau}.$$

Assume that there exists a number $A(\tau)$, such that

$$D(j,\tau) \le A(\tau)\, j^{q-1} \quad \text{for all } j \in [d]. \tag{16.18}$$

Then we have the following:

- If $q > 0$ then there exists a number $A_1(\tau)$ such that

  $$C(d,\tau) \le A_1(\tau) d^{\,q\tau}.$$

  Arguing as in the proof of Theorem 16.4, we have polynomial tractability with an ε^{-1} exponent $1/\tau$ and a d exponent q.

- If $q < 0$ then there exists a number $A_2(\tau)$ such that

  $$C(d,\tau) \le A_2(\tau) \quad \text{for all } d \in \mathbb{N},$$

  and we have strong polynomial tractability with an ε^{-1} exponent $1/\tau$.

- If $q = 0$ then there exists a number $A_3(\tau)$ such that

  $$C(d,\tau) \le A_3(\tau)[\ln d]^{\tau},$$

  and we have polynomial tractability with an ε^{-1} exponent $1/\tau$ and an arbitrarily small d exponent, since

  $$n_{\mathrm{QMC}}(\varepsilon,d) = \mathcal{O}\big(\varepsilon^{-1/\tau} \ln d\big).$$

An example of weights that satisfy the condition (16.18) is given by

$$\gamma_{\mathfrak{u}\cup\{j\}} = \mathcal{O}\big(j^{\,2\tau(q-1-|\mathfrak{u}|)}\big) \quad \text{for all } \mathfrak{u} \subseteq [j-1] \text{ and } j \in \mathbb{N}.$$

Indeed, for such weights we have

$$\begin{aligned} D(j,\tau) &= \mathcal{O}\left(j^{\,q-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq[j-1]} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}\, j}\right]^{|\mathfrak{u}|}\right) \\ &= \mathcal{O}\left(j^{\,q-1}\left(1+\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}\, j}\right)^{j}\right) = \mathcal{O}\big(j^{\,q-1}\big) \\ &\le A_4(\tau)\, j^{\,q-1} \end{aligned}$$

for some number $A_4(\tau)$. Thus, the condition (16.18) is satisfied, as claimed.

We now discuss weak tractability. Assume that

$$\gamma_{\mathfrak{u}\cup\{j\}} = j^{\,-f(j)\,(|\mathfrak{u}|+1)} \quad \text{for all } \mathfrak{u} \subseteq [j-1] \text{ and } j \in \mathbb{N}.$$

Here, $f:\mathbb{N}\to(0,\infty)$ is a non-increasing function. For QMC algorithms, we show that I_γ is weakly tractable iff

$$\lim_{j\to\infty} f(j)\ln j = \infty.$$

We have

$$\begin{aligned} D(j,\tfrac12) &= \sum_{\mathfrak{u}\subseteq[j-1]} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}\, j^{\,f(j)}}\right]^{|\mathfrak{u}|+1} \\ &= \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}\, j^{\,f(j)}}\left(1+\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}\, j^{\,f(j)}}\right)^{j-1}. \end{aligned}$$

Assume that we have weak tractability. Then

$$\begin{aligned} \frac{\ln C(d,1/2)}{2d} &\ge \frac{\ln D(d,1/2)}{2d} \\ &\ge \frac{d-1}{2d}\ln\left(1+\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\frac{1}{d^{\,f(d)}}\right) + \frac{1}{2d}\ln\frac{1}{d^{\,f(d)}} + \Omega(1). \end{aligned}$$

Since now $\lim_d[\ln C(d,\frac12)]/(2d) = 0$, we conclude that $\lim_d d^{\,f(d)} = \infty$, which is equivalent to $\lim_d f(d)\ln d = \infty$, as claimed.

Assume now that $\lim_d f(d)\ln d = \infty$, or equivalently that $\lim_d 1/d^{\,f(d)} = 0$. Then

$$\begin{aligned} C(d,\tfrac12)^{1/2} &= \sum_{j=1}^{d} D(j,\tfrac12) \le \sum_{j=1}^{d}\left(1+\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\frac{1}{j^{\,f(j)}}\right)^{j} \\ &\le \sum_{j=1}^{d}\exp\left(\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\, j^{\,1-f(j)}\right). \end{aligned}$$

Note that if $f(j) \ge 1$ then $j^{1-f(j)} \le 1$. If $f(j) < 1$ then $j^{1-f(j)} \le d^{1-f(j)}$, and since f is non-increasing $d^{1-f(j)} \le d^{1-f(d)}$. Hence,

$$j^{1-f(j)} \le \max\left(1, d^{1-f(d)}\right).$$

From this we have

$$C(d, \tfrac{1}{2})^{1/2} \le d \exp\left(\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \max\left(1, d^{1-f(d)}\right)\right).$$

Then

$$\frac{\ln C(d,1/2)^{1/2}}{d} = \mathcal{O}\left(\max\left(\frac{1}{d}, \frac{1}{d^{f(d)}}\right)\right)$$

and the right hand-side goes to zero as d goes to infinity. This implies weak tractability, as claimed.

We leave the case of T-tractability as on open problem to the reader.

Open Problem 74.

Consider multivariate integration I_γ for the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ and weights $\gamma_{d,\mathfrak{u}} = \gamma_{\mathfrak{u}}$ independent of d.

- Find necessary and sufficient conditions on such weights for which I_γ is strongly T-tractable and T-tractable for QMC algorithms.
- Do the same for arbitrary algorithms.

16.4.5 Tractability for Finite-Order and Order-Dependent Weights

We now specialize tractability conditions presented in Theorem 16.4 to finite-order weights, finite-diameter weights and order-dependent weights.

We begin with finite-order and finite-diameter weights. As we know, finite-order weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ of order ω satisfy

$$\gamma_{d,\mathfrak{u}} = 0 \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d] \text{ with } |\mathfrak{u}| > \omega.$$

As before, we assume that $\gamma_{d,\emptyset} = 1$. Finite-diameter weights of order ω, for which

$$\gamma_{d,\mathfrak{u}} = 0 \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d] \text{ with } \operatorname{diam}(\mathfrak{u}) \ge \omega,$$

are a subclass of finite-order weights.

Here, $\operatorname{diam}(\mathfrak{u}) = \max_{k,s\in\mathfrak{u}} |k - s|$ for a non-empty $\mathfrak{u}$, and $\operatorname{diam}(\emptyset) = 0$. The reader may consult Section 5.3.2 of Volume I for more information on such weights. In particular, for finite-order weights of order ω we have at most

$$2\, d^{\,\omega} \quad \text{non-zero weights,}$$

and for finite-diameter weights of order ω we have at most

$$2^{\min(\omega,d)-1}\left[d+2-\min(\omega,d)\right]=\mathcal{O}(d)\ \text{ non-zero weights.}$$

Note that for finite-diameter weights of order ω, for $\gamma_{d,\mathfrak{u}}\neq 0$ we must have $|\mathfrak{u}|\le\omega$. This means that such weights are also finite-order weights of order ω.

Obviously, $\omega=0$ means that non-zero weights only occur for $\mathfrak{u}=\emptyset$, so that multivariate integration is trivial. Hence, we always consider $\omega\ge 1$.

Theorem 16.6. *Consider multivariate integration $I_\gamma=\{I_{d,\gamma}\}$ defined for the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha>\frac{1}{2}$ and with bounded finite-order or finite-diameter weights $\gamma=\{\gamma_{d,\mathfrak{u}}\}$ of order $\omega\ge 1$, so that*

$$A:=\sup_{d\in\mathbb{N}}\max_{\mathfrak{u}\subseteq[d]}\gamma_{d,\mathfrak{u}}<\infty.$$

Let

$$q=\begin{cases}\omega & \textit{if finite-order weights of order } \omega \textit{ are considered,}\\ 1 & \textit{if finite-diameter weights of order } \omega \textit{ are considered.}\end{cases}$$

- *There exists a rank-1 lattice rule with a generator $\mathbf{z}^*\in\mathbb{Z}_n^d$ such that*

$$e_{n,d}(\alpha,\gamma,\mathbf{z})=\mathcal{O}\big(d^{\,q\tau}\,\varepsilon^{-1/\tau}\big)\quad \textit{for all } \tau\in\big[\tfrac{1}{2},\alpha\big), \tag{16.19}$$

  with the factor in the big $\mathcal{O}$ notation independent of d and n and depending on τ. Hence, I_γ is polynomially tractable with an ε^{-1} exponent $1/\tau$, which can be arbitrarily close to the minimal value $1/\alpha$, and a d exponent q. The tractability bounds are achieved by the resulting lattice rule.

- *If*

$$\sup_{d\in\mathbb{N}}\sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}\neq 0}}\gamma_{d,\mathfrak{u}}<\infty$$

  then I_γ is strongly polynomially tractable. If

$$\sup_{d\in\mathbb{N}}\sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}\neq 0}}\gamma_{d,\mathfrak{u}}^{1/(2\tau)}<\infty\quad \textit{for some } \tau\in\big[\tfrac{1}{2},\alpha\big)$$

  then the ε^{-1} exponent of strong polynomial tractability is at most $1/\tau$. If

$$\sup_{d\in\mathbb{N}}\sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}\neq 0}}\gamma_{d,\mathfrak{u}}^{1/(2\tau)}<\infty\quad \textit{for all } \tau\in\big[\tfrac{1}{2},\alpha\big)$$

  then the ε^{-1} exponent of strong polynomial tractability achieves the minimal value $1/\alpha$.

Proof. We prove the first part. For any $\tau \in [\frac{1}{2}, \alpha)$ we have

$$\begin{aligned}\sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|} &\le A^{1/(2\tau)} \sum_{\mathfrak{u}\colon \gamma_{d,\mathfrak{u}} \neq 0} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|} \\ &\le A^{1/(2\tau)} \max\left(\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}, \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{\omega}\right) \sum_{\mathfrak{u}\colon \gamma_{d,\mathfrak{u}} \neq 0} 1 \\ &= \mathcal{O}\big(d^{q}\big)\end{aligned}$$

with the factor in the big $\mathcal{O}$ notation independent of d and depending on τ. Therefore, from (16.7) the corresponding factor $C(d,\tau)$ in the error bound satisfies

$$C(d,\tau) = \mathcal{O}\big(d^{\,q\tau}\big).$$

This implies that

$$n_{\mathrm{QMC}}(\varepsilon, d) = \mathcal{O}\big(d^{q}\,\varepsilon^{-1/\tau}\big),$$

and we have polynomial tractability of I_γ with exponents as indicated in the theorem.

We now turn to the second part. This part easily follows from Theorem 16.4. Indeed, to get strong polynomial tractability we need to guarantee that $B_{1/2,0} < \infty$ with $B_{\tau,q}$ given by (16.10). Since in our case we have

$$\left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|}$$

with $|\mathfrak{u}| \in [\omega]$, these factors cannot change whether the $B_{\tau,0}$'s are finite or not, and therefore they can be dropped. This completes the proof. □

In Theorem 16.20, we assume that the weights are bounded. As for product weights, the assumption on bounded weights is essential. The same example, with $\gamma_{d,\{1\}} = d^{2\beta}$, where $\beta > 0$, and $\gamma_{d,\mathfrak{u}} = 0$ for all $\mathfrak{u} \notin \{\emptyset, \{1\}\}$, shows that we have unbounded finite-order weights of order 1, and polynomial tractability with the d exponent β. This shows that the first point of Theorem 16.6 is now not valid.

We briefly turn to order-dependent weights,

$$\gamma_{d,\mathfrak{u}} = \Gamma_{d,|\mathfrak{u}|} \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d].$$

Here $\Gamma_{d,0} = 1$ and $\Gamma_{d,j} \ge 0$ for all $j \in [d]$.

In this case, for $\tau \in [\frac{1}{2}, \alpha)$ we have

$$A := \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|} = \sum_{k=1}^{d} \binom{d}{k} \Gamma_{d,k}^{1/(2\tau)} \left[\frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\right]^{|\mathfrak{u}|}.$$

If

$$\Gamma_{d,j} \in \big[A_{\mathrm{low}}^{j}, A_{\mathrm{upp}}^{j}\big] \quad \text{with } 0 < A_{\mathrm{low}} \le A_{\mathrm{upp}}$$

then

$$A \in \left[\left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\, A_{\text{low}}^{1/(2\tau)}\right)^d - 1, \left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\, A_{\text{upp}}^{1/(2\tau)}\right)^d - 1\right].$$

Hence, we have an exponential dependence on d, and we cannot even claim weak tractability. On the other hand, if

$$\Gamma_{d,j} = \gamma_{d,j}^{j} \quad \text{for all } d \in \mathbb{N} \text{ and for all } j \in [d]$$

then

$$A = \prod_{j=1}^{d} \left(1 + \frac{2\zeta(\alpha/\tau)}{(2\pi)^{\alpha/\tau}}\, \gamma_{d,j}\right) - 1.$$

So we have the same situation as for product weights, and Theorem 16.5 can now be applied.

16.5 Worst Case Error for α Approaching to 1/2

The simple proof technique needed to show the last part of Theorem 16.4 may also be applied to the case when the parameter α goes to $\frac{1}{2}$. As we have already seen, upper error bounds of lattice rules depend on the factor $\zeta(2\alpha)$, which goes to infinity as α goes to $\frac{1}{2}$. For α going to $\frac{1}{2}$, we are losing smoothness of functions in $H_{d,\alpha,\gamma}$, and its reproducing kernel degenerates. We now prove that the error of any QMC algorithm goes to infinity if α goes to $\frac{1}{2}$, independently of how many function values are used.

Theorem 16.7. *Consider multivariate integration for the weighted Korobov spaces $H_{d,\alpha,\gamma}$ for arbitrary weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ independent of α. We assume that at least one $\gamma_{d,\mathfrak{u}}$ for a non-empty $\mathfrak{u}$ is non-zero.*

For arbitrary n and arbitrary QMC algorithm $Q_{n,d}$ using sample points x_k that may depend on α, we have

$$\lim_{\alpha \to 1/2+} e(Q_{n,d}; H_{d,\alpha,\gamma}) = \infty.$$

Proof. We have

$$e^2(Q_{n,d}; H_{d,\alpha,\gamma}) = -1 + \frac{1}{n^2} \sum_{k,s=1}^{n} K_{d,\alpha,\gamma}(x_k, x_s),$$

where the points $x_k = x_k(\alpha)$ are arbitrary. This can be rewritten as

$$e^2(Q_{n,d}; H_{d,\alpha,\gamma}) = -(c+1) + \frac{1}{n^2} \sum_{k,s=0}^{n-1} \left[c + K_{d,\alpha,\gamma}(x_k, x_s)\right] \quad \text{for all } c \in \mathbb{R}. \tag{16.20}$$

The kernel $K_{d,\alpha,\gamma}$ can be written as

$$K_{d,\alpha,\gamma}(x,y) = 1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}\, K_{\mathfrak{u},\alpha,\gamma}(x_{\mathfrak{u}}, y_{\mathfrak{u}}) \quad \text{for all } x, y \in [0,1]^d,$$

where

$$K_{\mathfrak{u},\alpha,\gamma}(x_{\mathfrak{u}}, y_{\mathfrak{u}}) = \prod_{j \in \mathfrak{u}} \left(\sum_{0 \neq h \in \mathbb{Z}} \frac{e^{2\pi\, h(x_j - y_j)}}{(2\pi\, |h|)^{2\alpha}} \right) = \prod_{j \in \mathfrak{u}} \frac{2}{(2\pi)^{2\alpha}}\, D_\alpha(\{x_j - y_j\}).$$

We already indicated that

$$-1 < D_\alpha(t) \leq \zeta(2\alpha) \quad \text{for all } t \in [0,1].$$

This implies that $K_{\mathfrak{u},\alpha}(x_{\mathfrak{u}}, y_{\mathfrak{u}}) \geq -2(2\zeta(2\alpha)/((2\pi)^{2\alpha}))^{|\mathfrak{u}|-1}$ for all $x_{\mathfrak{u}}$ and $y_{\mathfrak{u}}$, and

$$K_{d,\alpha,\gamma}(x,y) \geq c^* := 1 - 2 \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|-1}.$$

Setting $c = -c^*$ in (16.20), we conclude that the kernel $c + K_{d,\alpha,\gamma}(x,y)$ is point-wise non-negative. Thus we can drop off-diagonal elements, $k \neq s$, in (16.20) and obtain

$$e^2(Q_{n,d}; H_{d,\alpha,\gamma}) \geq 2 \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|-1} \left(\frac{1}{n} \frac{\zeta(2\alpha)}{(2\pi)^{2\alpha}} - \left(1 - \frac{1}{n}\right) \right).$$

Since $\zeta(2\alpha)$ goes to infinity as α goes to $\frac{1}{2}$, and at least one $\gamma_{d,\mathfrak{u}}$ is positive and independent of α, the error goes to infinity, as claimed. □

We remind the reader that the initial error is 1, independent of the value of $\alpha > \frac{1}{2}$. This means that the trivial zero algorithm has worst case error 1 even when α goes to $\frac{1}{2}$. This should be contrasted with the worst case error of any QMC algorithm that goes to infinity as α approaches $\frac{1}{2}$. The reason for this bad behavior is that the coefficient n^{-1} for QMC algorithms is too large. We already discussed this point in Section 10.7.6 of Chapter 10 as a motivation to study properly normalized QMC algorithms.

16.6 CBC Algorithm

We showed in the previous section the existence of "good" lattice rules, see Theorems 16.1–16.4. These theorems are unfortunately *not constructive*, although the choice of generators of the lattice rules is from a finite set. A global optimal generator $\mathbf{z}^*$ from the finite set $\mathbb{Z}_n^d$ defined by

$$e_{n,d}(\alpha, \gamma, \mathbf{z}^*) \leq e_{n,d}(\alpha, \gamma, \mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathbb{Z}_n^d,$$

satisfies the optimal error bound of Theorem 16.3. However, the full search over all $(n-1)^d$ different $\mathbf{z} \in \mathbb{Z}_n^d$ is impossible for large d and even moderate n. In this section we will show that the generator obtained by carrying out the construction one component at a time still satisfies the optimal error bound of Theorem 16.3.

We now present the *CBC* (component-by component) algorithm of selecting generators of good lattice rules. The history of this powerful algorithm is given in the introduction of this chapter. In this section we present this algorithm for the weighted Korobov spaces with general weights. In Section 16.9.1 we will study this algorithm for some Sobolev spaces of non-periodic functions.

CBC Algorithm

Suppose n is a prime number, $\alpha > \frac{1}{2}$ and the weights $\{\gamma_{d,\mathfrak{u}}\}$ are given.
The generator

$$\bar{z} = [\bar{z}_1, \bar{z}_2, \dots, \bar{z}_d]$$

is found as follows:

- Set the first component $\bar{z}_1$ of the generator $\bar{z}$ to 1.
- For $s = 2, 3, \dots, d$ and known $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{s-1}$, find $\bar{z}_s \in \mathbb{Z}_n$ such that the worst case error

$$e_{n,s}^2(1, \bar{z}_2, \dots, \bar{z}_{s-1}, \bar{z}_s) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left[2 \sum_{h=1}^{\infty} \frac{\cos(2\pi\, hk\bar{z}_j/n)}{(2\pi\, |h|)^{2\alpha}} \right] \tag{16.21}$$

is minimized.

The cost of the CBC algorithm will be discussed later. We now show that the lattice rule with the generator constructed by the CBC algorithm has good theoretical properties.

Theorem 16.8. *For a prime n, $\alpha > \frac{1}{2}$ and given weights $\{\gamma_{d,\mathfrak{u}}\}$, let*

$$\bar{z} = [1, \bar{z}_2, \dots, \bar{z}_d]$$

be found by the CBC algorithm. Then for $s = 1, 2, \dots, d$ and for any $\tau \in [\frac{1}{2}, \alpha)$, we have

$$e_{n,s}(1, \bar{z}_2, \dots, \bar{z}_s) \le C(s, \tau)\, (n-1)^{-\tau}, \tag{16.22}$$

where $C(d, \tau)$ is given in (16.7).

Proof. Let $\lambda = 1/(2\tau) \in (1/(2\alpha), 1]$. We prove by induction on s that

$$e_{n,s}^2(1, \bar{z}_2, \dots, \bar{z}_s) \le (n-1)^{-1/\lambda} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{d,\mathfrak{u}}^{\lambda} \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right]^{|\mathfrak{u}|} \right)^{1/\lambda}. \tag{16.23}$$

Let $s = 1$. Since $\bar{z}_1 = 1$, we now use the sample points $x_k = \{(k-1)\bar{z}_1/n\} = (k-1)/n$. Using Lemma 16.2 and Jensen's inequality, we have

$$\begin{aligned} e_{n,1}^2(1) &= \sum_{0 \neq h \equiv 0 \bmod n} \frac{\gamma_{d,\{1\}}}{(2\pi\,|h|)^{2\alpha}} \le \left(\sum_{0 \neq h \equiv 0 \bmod n} \frac{\gamma_{d,\{1\}}^{\lambda}}{(2\pi\,|h|)^{2\alpha\lambda}} \right)^{1/\lambda} \\ &= \frac{1}{n^{2\alpha}}\,\gamma_{d,\{1\}}\,\frac{[2\zeta(2\alpha\lambda)]^{1/\lambda}}{(2\pi)^{2\alpha}} \le (n-1)^{-1/\lambda} \left(\gamma_{d,\{1\}}^{\lambda}\,\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}} \right)^{1/\lambda}, \end{aligned}$$

as needed.

Suppose now that $s \ge 2$, and that the generator $\bar{\mathbf{z}}_s = [1, \bar{z}_2, \ldots, \bar{z}_s]$ found by the CBC algorithm satisfies (16.23). We prove that the $(s+1)$-dimensional vector $[\bar{\mathbf{z}}_s, \bar{z}_{s+1}]$ with $\bar{z}_{s+1}$ again found by the CBC algorithm satisfies (16.23) with s replaced by $s+1$.

From (16.3) we have

$$\begin{aligned} e_{n,s+1}^2(\bar{\mathbf{z}}_s, \bar{z}_{s+1}) &= \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s+1]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left(\sum_{0 \neq h \in \mathbb{Z}} \frac{e^{2\pi i h k \bar{z}_j/n}}{(2\pi\,|h|)^{2\alpha}} \right) \\ &= e_{n,s}^2(\bar{\mathbf{z}}_s) + \vartheta(\alpha, \gamma, \bar{\mathbf{z}}_s, \bar{z}_{s+1}), \end{aligned} \tag{16.24}$$

where

$$\vartheta(\alpha, \gamma, \bar{\mathbf{z}}_s, z_{s+1}) = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\substack{\mathfrak{u} \subseteq [s+1] \\ s+1 \in \mathfrak{u}}} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left(\sum_{0 \neq h \in \mathbb{Z}} \frac{e^{2\pi i h k \bar{z}_j/n}}{(2\pi\,|h|)^{2\alpha}} \right). \tag{16.25}$$

We need the following lemma.

Lemma 16.9. *Under the assumptions of Theorem 16.8, there exists $z_{s+1}^* \in \mathbb{Z}_n$ such that*

$$\vartheta(\alpha, \gamma, \bar{\mathbf{z}}_s, z_{s+1}^*) \le \frac{1}{n-1} \sum_{\substack{\mathfrak{u} \subseteq [s+1] \\ s+1 \in \mathfrak{u}}} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}} \right]^{|\mathfrak{u}|}.$$

Proof. We average $\vartheta(\alpha, \gamma, \bar{\mathbf{z}}_s, z_{s+1})$ over all possible values of $z_{s+1} \in \mathbb{Z}_n$. For $s+1 \subseteq \mathfrak{u} \subseteq [s+1]$, let $\mathbf{z}_{\mathfrak{u}}$ and $h_{\mathfrak{u}}$ denote the $|\mathfrak{u}|$-dimensional vectors containing the components of $(\bar{\mathbf{z}}_s, z_{s+1})$ and (h, h_{s+1}) with indices in $\mathfrak{u}$. We obtain

$$\begin{aligned} \Phi(\alpha, \gamma, \bar{\mathbf{z}}_s) &:= \frac{1}{n-1} \sum_{z_{s+1}=1}^{n-1} \vartheta(\alpha, \gamma, \bar{\mathbf{z}}_s, z_{s+1}) \\ &= \frac{1}{n-1} \sum_{z_{s+1}=1}^{n-1} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\substack{\mathfrak{u} \subseteq [s+1] \\ s+1 \in \mathfrak{u}}} \gamma_{d,\mathfrak{u}} \sum_{h_{\mathfrak{u}} \in \mathbb{Z}_0^{|\mathfrak{u}|}} \frac{e^{2\pi i k h_{\mathfrak{u}} \cdot \mathbf{z}_{\mathfrak{u}}/n}}{\prod_{j \in \mathfrak{u}} (2\pi\,|h_j|)^{2\alpha}} \\ &= \frac{1}{n-1} \sum_{\substack{\mathfrak{u} \subseteq [s+1] \\ s+1 \in \mathfrak{u}}} \gamma_{d,\mathfrak{u}} \sum_{h_{\mathfrak{u} \setminus \{s+1\}} \in \mathbb{Z}_0^{|\mathfrak{u}|-1}} \prod_{j \in \mathfrak{u} \setminus \{s+1\}} \frac{S(h_{\mathfrak{u}}, \mathbf{z}_{\mathfrak{u}})}{(2\pi\,|h_j|)^{2\alpha}}, \end{aligned}$$

where

$$S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) = \sum_{z_{s+1}=1}^{n-1} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{h_{s+1}\in\mathbb{Z}_0} \frac{e^{2\pi i k h_{\mathfrak{u}}\cdot \mathbf{z}_{\mathfrak{u}}/n}}{(2\pi\,|h_{s+1}|)^{2\alpha}}.$$

For $\mathfrak{u} = \{s+1\}$ we use the convention that

$$\sum_{h_{\mathfrak{u}\setminus\{s+1\}}\in\mathbb{Z}_0^{|\mathfrak{u}|-1}} \prod_{j\in\mathfrak{u}\setminus\{s+1\}} (2\pi\,|h_j|)^{-2\alpha}\; S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) = S(h_{\{s+1\}},\mathbf{z}_{\{s+1\}}).$$

We show that for all $h_{\mathfrak{u}}$ and $\mathbf{z}_{\mathfrak{u}}$ we have

$$S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) \in \left[0,\, \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]. \tag{16.26}$$

Indeed, for $s+1 \subseteq \mathfrak{u} \subseteq [s+1]$, we can write $h_{\mathfrak{u}}\cdot\mathbf{z}_{\mathfrak{u}} = c + h_{s+1}z_{s+1}$ for some integer c, and

$$\begin{aligned} S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) &= \sum_{z=1}^{n-1} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{h\in\mathbb{Z}_0} \frac{e^{2\pi i k(c+hz)/n}}{(2\pi\,|h|)^{2\alpha}} \\ &= \sum_{z=1}^{n-1} \sum_{\substack{h\in\mathbb{Z}_0 \\ c+hz\equiv 0 \bmod n}} \frac{1}{(2\pi\,|h|)^{2\alpha}} \\ &= \frac{1}{(2\pi)^{2\alpha}} \sum_{z=1}^{n-1} \sum_{m\in\mathbb{Z},\, mn-cz^{-1}\neq 0} \frac{1}{|mn-cz^{-1}|^{2\alpha}} \end{aligned}$$

where $z^{-1}\in\mathbb{Z}_n$ is the inverse element of z satisfying $zz^{-1}\equiv 1 \bmod n$. In the last step we have used the fact that $c+hz\equiv 0 \bmod n$ is equivalent to $h\equiv -cz^{-1} \bmod n$, i.e., $h = mn - cz^{-1}$ for some integer m.

Now if $c\equiv 0 \bmod n$, then

$$(2\pi)^{2\alpha}\, S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) = \sum_{z=1}^{n-1} \sum_{m\in\mathbb{Z}_0} \frac{1}{|mn|^{2\alpha}} = \frac{(n-1)}{n^{2\alpha}} \sum_{m\in\mathbb{Z}_0} \frac{1}{|m|^{2\alpha}} < 2\zeta(2\alpha).$$

If $c\not\equiv 0 \bmod n$, then the set $\{cz^{-1} \bmod n \mid z\in[n-1]\} = [n-1]$. Note that the numbers $mn-b$ for $b\in[n-1]$ and $m\in\mathbb{Z}$ are distinct and the set

$$\{mn-b \mid b\in[n-1],\; m\in\mathbb{Z}\}$$

is a proper subset of $\mathbb{Z}_0$. Therefore

$$(2\pi)^{2\alpha}\, S(h_{\mathfrak{u}},\mathbf{z}_{\mathfrak{u}}) = \sum_{b=1}^{n-1} \sum_{m\in\mathbb{Z}} \frac{1}{|mn-b|^{2\alpha}} < \sum_{m\in\mathbb{Z}_0} \frac{1}{|m|^{2\alpha}} = 2\zeta(2\alpha),$$

proving (16.26).

Now return to the expression for $\Phi(\alpha, \gamma, \bar{z}_s)$. Using (16.26), we have

$$\Phi(\alpha,\gamma,\bar{z}_s) \le \frac{1}{n-1} \sum_{\mathfrak{u}\subseteq[s+1],\, s+1\in\mathfrak{u}} \gamma_{d,\mathfrak{u}} \Bigg(\prod_{j\in\mathfrak{u}\setminus\{s+1\}} \sum_{h\in\mathbb{Z}_0} \frac{1}{(2\pi\,|h|)^{-2\alpha}} \Bigg) \frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}$$
$$= \frac{1}{n-1} \sum_{\mathfrak{u}\subseteq[s+1],\, s+1\in\mathfrak{u}} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|}.$$

Since there exists a $z^*_{s+1} \in \mathbb{Z}_n$ such that $\vartheta(\alpha,\gamma,\bar{z}_s,z^*_{s+1})$ is no larger than the average $\Phi(\alpha,\gamma,\bar{z}_s)$, the proof of Lemma 16.9 is complete. □

We now continue the proof of Theorem 16.8. The quantity $\vartheta(\alpha,\gamma,\bar{z}_s,z_{s+1})$, see (16.25), satisfies

$$\vartheta(\alpha,\gamma,\bar{z}_s,z_{s+1}) \le \vartheta^{1/\lambda}(\alpha\lambda,\gamma^\lambda,\bar{z}_s,z_{s+1}) \quad \text{for all } \lambda \in (1/(2\alpha),1]. \tag{16.27}$$

The last estimate can be proved using the same Jensen's inequality argument as in the proof of Lemma 16.2. Indeed, comparing the expression for $\vartheta(\alpha,\gamma,\bar{z}_s,z_{s+1})$ in (16.25) and the expression for $e^2_{n,s}(\alpha,\gamma,\mathbf{z})$ in (16.3), we see that the only difference is that for the former the sum is over those $\mathfrak{u}$ for which $s+1 \subseteq \mathfrak{u} \subseteq [s+1]$, while for the latter, the sum is over those $\mathfrak{u}$ for which $\emptyset \ne \mathfrak{u} \subseteq [d]$. This changes the cardinality of the sums but Jensen's inequality is independent of the cardinality of the sum.

For the known $\bar{z}_s = [1,\bar{z}_2,\dots,\bar{z}_s]$ and with the parameters α and γ replaced by $\alpha\lambda$ and γ^λ, it follows from Lemma 16.9 that for every $\lambda \in (1/(2\alpha),1]$ there exists $z_{s+1,\lambda} \in \mathbb{Z}_n$ such that

$$\vartheta(\alpha\lambda,\gamma^\lambda,\bar{z}_s,z_{s+1,\lambda}) \le \frac{1}{n-1} \sum_{\mathfrak{u}\subseteq[s+1]\, s+1\in\mathfrak{u}} \gamma^\lambda_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}}\right]^{|\mathfrak{u}|}. \tag{16.28}$$

For this $z_{s+1,\lambda}$, from (16.24), (16.27) (16.28), and the induction assumption we have

$$\begin{aligned}
e^2_{n,s+1}(\bar{z}_s,z_{s+1,\lambda}) &= e^2_{n,s}(\bar{z}_s) + \vartheta(\alpha,\gamma,\bar{z}_s,z_{s+1,\lambda})\\
&\le e^2_{n,s}(\bar{z}_s) + \vartheta^{1/\lambda}(\alpha\lambda,\gamma^\lambda,\bar{z}_s,z_{s+1,\lambda})\\
&\le \left(\frac{1}{n-1}\sum_{\emptyset\ne\mathfrak{u}\subseteq[s+1]} \gamma^\lambda_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}}\right]^{|\mathfrak{u}|}\right)^{1/\lambda}\\
&\quad + \left(\frac{1}{n-1}\sum_{\mathfrak{u}\subseteq[s+1],\, s+1\in\mathfrak{u}} \gamma^\lambda_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}}\right]^{|\mathfrak{u}|}\right)^{1/\lambda}\\
&\le \left(\frac{1}{n-1}\sum_{\emptyset\ne\mathfrak{u}\subseteq[s+1]\}} \gamma^\lambda_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha\lambda)}{(2\pi)^{2\alpha\lambda}}\right]^{|\mathfrak{u}|}\right)^{1/\lambda},
\end{aligned}$$

where in the last step we again used Jensen's inequality. Since $\bar{z}_{s+1}$ is obtained by minimizing $e^2_{n,s+1}(\bar{z}_s,z_{s+1})$ with respect to z_{s+1}, we have

$$e^2_{n,s+1}(\bar{z}_s,\bar{z}_{s+1}) \le e^2_{n,s+1}(\bar{z},z_{s+1,\lambda})$$

for all $\lambda \in (1/(2\alpha), 1]$, and the result (16.23) follows. Substituting $\tau = 1/(2\alpha) \in [\frac{1}{2}, \alpha)$ we obtain (16.22). This completes the proof of Theorem 16.8. □

We stress that Theorems 16.4, 16.5 and 16.6 are based on error bounds from the non-constructive Theorem 16.3. Since exactly the same error bounds hold for the lattice rules with the generator obtained by the CBC algorithm, all tractability bounds presented in these Theorems are valid for such lattice rules. We summarize this in the following corollary.

Corollary 16.10. *Bounds for various notions of tractability presented in Theorems 16.4, 16.5 and 16.6 are achieved by the rank-1 lattice rules with the generators given by the CBC algorithm.*

16.6.1 Cost of the CBC Algorithm

We now discuss the computational cost of the CBC algorithm. In particular, for general α we must know how to compute

$$\prod_{j\in\mathfrak{u}} \left(2 \sum_{h=1}^{\infty} \frac{\cos(2\pi\, hk\bar{z}_j/n)}{(2\pi\, |h|)^{2\alpha}} \right) = \prod_{j\in\mathfrak{u}} \frac{2}{(2\pi)^{2\alpha}}\, D_\alpha\left(\frac{k\bar{z}_j}{n}\right),$$

with D_α given by (16.12), i.e.,

$$D_\alpha(t) = \sum_{h=1}^{\infty} \frac{\cos(2\pi\, h\, t)}{h^{2\alpha}} \quad \text{for all } t \in \mathbb{R}.$$

Hence, we need to know how to compute $D_\alpha(t)$. As we already mentioned, for an integer α the function D_α is related to the Bernoulli polynomial of degree 2α, and can be computed in cost proportional to α. For other values of α, we can compute only an approximation to D_α. The latter problem is obviously easier for large α, since the corresponding series defining D_α converges faster when α is large. More precisely, assume that we approximate $D_\alpha(t)$ for $t \in [0, 1]$ by the first k terms, i.e.,

$$D_{\alpha,k}(t) = \sum_{h=1}^{k} \frac{\cos(2\pi\, h\, t)}{h^{2\alpha}}.$$

Then

$$|D_\alpha(t) - D_{\alpha,k}(t)| \le \sum_{h=k+1}^{\infty} \frac{1}{h^{2\alpha}} \le \int_k^{\infty} \frac{1}{x^{2\alpha}}\, \mathrm{d}x = \frac{1}{2\alpha - 1}\, \frac{1}{k^{2\alpha-1}}.$$

For

$$k = \left\lceil \left(\frac{1}{(2\alpha - 1)\, \varepsilon} \right)^{1/(2\alpha-1)} \right\rceil$$

we obtain

$$|D_\alpha(t) - D_{\alpha,k}(t)| \le \varepsilon.$$

The cost of computing $D_{\alpha,k}(t)$ to within ε is proportional to

$$k \le \frac{1}{(2\alpha-1)^{1/(2\alpha-1)}}\,\varepsilon^{-1/(2\alpha-1)} + 1.$$

As long as α is not too close to $\frac{1}{2}$, the cost is a moderate function of ε^{-1}. For instance, for $\alpha \ge 1$ the cost is at most $\varepsilon^{-1} + 1$. Furthermore, the cost decreases as α increases. Hence, even if α is an integer and we can compute $D_\alpha(t)$ exactly based on the Bernoulli polynomial of degree 2α with cost proportional to α, it is easier to compute an ε-approximation if α is large and ε not too small.

Even when we know how to compute D_α exactly or approximately, we face another and more difficult computational problem when arbitrary weights are considered. Indeed, we have to sum up $2^d - 1$ terms as part of Step 2 of the CBC algorithm and each term is independent since it is proportional to presumably different $\gamma_{d,\mathfrak{u}}$. The total cost of the CBC algorithm would require then $\mathcal{O}(n^2 2^d\, d)$ operations, which is unfortunately exponential in d, making the algorithm impossible to use for large d. So the cost of the CBC algorithm can only be reasonable for special families of weights.

16.6.2 Cost for Product Weights

We analyze the cost of the CBC algorithm for product weights,

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d].$$

For simplicity, we assume that we can compute $D_\alpha(t)$ at a time dependent on α and concentrate on how the cost of the CBC algorithm compares to the cost of the lattice rule. Since we need to compute n points

$$x_{k+1} = \left\{\frac{k}{n}\,\mathbf{z}\right\} \quad \text{for all } k = 0, 1, \dots, n-1,$$

and each point x_k has d components, the cost of the lattice rule must be at least of order $n\,d$.

We follow the analysis of Nuyens and Cools who showed that the cost of the CBC algorithm for product weights is $\mathcal{O}(n\,d\ln n)$, see [226], [227], [228]. This is a surprising result since we need to compute $e^2_{n,s}(\bar{z}_s)$ for n points and $s \in [d]$. The formula for $e^2_{n,s}(\bar{z}_s)$ suggests that the cost of its computation should be proportional to at least to n, making the total cost at least of order $n^2\,d$. The result of Nuyens and Cools matches the cost of the lattice rule modulo at most a factor of order $\ln n$, which is not really large for most n occurring in computational practice. That is why this result allows us to use the CBC algorithm for really large n and d. Numerical experiments with the CBC algorithm can be found in many papers.

The first step of Nuyens and Cools is to formulate the CBC algorithm as a matrix-vector product. Then they show how to use the fast Fourier transform for this matrix-vector product.

For product weights, the square of the error $e^2_{n,s}(\bar{z}_s)$ given by (16.21) can be rewritten as

$$\begin{aligned} e^2_{n,s}(\bar{z}_s) &= -1 + \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{s} \left[1 + \gamma'_{d,j}\, D_\alpha\left(k\bar{z}_j/n\right)\right] \\ &= -1 + \frac{1}{n} \sum_{k=0}^{n-1} p_{s-1}(k)\big[1 + \gamma'_{d,s}\, D_\alpha\,(k\bar{z}_s/n)\big], \end{aligned}$$

where

$$\gamma'_{d,j} = \frac{2}{(2\pi)^{2\alpha}}\, \gamma_{d,j},$$

and

$$\begin{aligned} p_0(k) &= 1 \\ p_s(k) &= p_{s-1}(k)\big[1 + \gamma'_{d,s}\, D_\alpha\,(k\bar{z}_s/n)\big]. \end{aligned}$$

We need to compute $e^2(\bar{z}_{s-1}, \bar{z}_s)$ for fixed $\bar{z}_{s-1}$ and all $\bar{z} \in \mathbb{Z}_n$. We are ready to present the matrix-vector product representation of the CBC algorithm. Denote

$$\begin{aligned} \vec{p}_s &= [p_s(0), p_s(1), \dots, p_s(n-1)]^T \in \mathbb{R}^n, \\ \vec{e}_s^{\,2} &= [e^2_{n,s}(\bar{z}_{s-1}, 1), e^2_{n,s}(\bar{z}_{s-1}, 2), \dots, e^2_{n,s}(\bar{z}_{s-1}, n-1)]^T \in \mathbb{R}^{n-1}, \\ \Omega_n &= \big(D_\alpha(k\, z/n)\big)_{z\in[n-1],\, k\in\{0,1,\dots,n-1\}} \in \mathbb{R}^{n-1,n}. \end{aligned}$$

Hence, Ω_n is the $(n-1) \times n$ matrix with elements $D_\alpha(k\, z/n)$.

Let $1_{k,s}$ denote the $k \times s$ matrix having all coefficients equal to 1, and let further $\mathrm{diag}_n(a_k)_{k=0,1,\dots,n-1}$ be the $n \times n$ diagonal matrix having the elements a_k on the main diagonal. Then the CBC algorithm can be written as follows.

CBC Algorithm

- $\vec{p}_0 = \vec{1}_{n,1}$,
- for $s = 1, 2, \dots, d$ do

$$\begin{aligned} \vec{e}_s^{\,2} &= -1_{n-1,1} + \frac{1}{n}\big(1_{n-1,n} + \gamma'_{d,s}\, \Omega_n\big)\, \vec{p}_{s-1} \\ \bar{z}_s &= \mathrm{argmin}\, \vec{e}_s^{\,2} \\ \vec{p}_s &= \mathrm{diag}_n\big(1 + \gamma'_{d,s}\, D_\alpha(k\bar{z}_s/n)\big)_{k=0,1,\dots,n-1}\, \vec{p}_{s-1}. \end{aligned}$$

The cost of this form of the CBC algorithm is clearly $\mathcal{O}(n)$ memory, and $\mathcal{O}(n\,d)$ operations plus the cost of d matrix-vector multiplications

$$\Omega_n\,\vec{p}_{s-1}.$$

Obviously, each matrix-vector multiplication can be done with $\mathcal{O}(n^2)$ operations without using the special form of the matrix Ω_n. Note that Ω_n has at most n, not $(n-1)n$, different elements, which allows us to hope that $\Omega_n\,\vec{p}_{s-1}$ can be computed in $\mathcal{O}(n \ln n)$ operations.

Indeed, this is the case as shown by Nuyens and Cools [226]. The precise way how this can be achieved is beyond the scope of this volume. We only mention that it is enough to drop the first column of the matrix Ω and switch to the square $(n-1)\times(n-1)$ matrix

$$\Omega'_{n-1} = \big(D_\alpha(kz/n)\big)_{z,k\in[n-1]}.$$

It turns out that there exists a permutation of rows and columns of Ω'_{n-1} such that the resulting matrix C_{n-1} is a circulant matrix. It is known that a circulant matrix has a similarity transform $C_{n-1} = F_{n-1}^{-1}\Lambda_{n-1}F_{n-1}$, where F_{n-1} is the Fourier transform and $\Lambda = \mathrm{diag}_{n-1}(F_{n-1}c)$ with c being the first column of C_{n-1}. This allows us to apply FFT and therefore the cost is indeed $\mathcal{O}(n \ln n)$.

16.6.3 Cost for Finite-Order and Finite-Diameter Weights

We now analyze the cost of the CBC algorithm for finite-order and finite-diameter weights of order ω. We rewrite (16.3) as

$$e_{n,d}^2(\bar{z}) = \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]\,:\,\gamma_{d,\mathfrak{u}}\neq 0} \gamma'_{d,\mathfrak{u}}\, e_{n,\mathfrak{u}}^2(\bar{z}_{\mathfrak{u}}),$$

where

$$\gamma'_{d,\mathfrak{u}} = \left[\frac{2}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|},$$

and

$$e_{n,\mathfrak{u}}^2(\bar{z}_{\mathfrak{u}}) = \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}} D_\alpha(k\bar{z}_j/n).$$

Note that $e_{n,\mathfrak{u}}^2(\bar{z}_{\mathfrak{u}})$ is basically the same as $e_{n,s}(\bar{z}_s)$ with $s = |\mathfrak{u}|$ and $\gamma'_{d,j} = 1$, with the missing term -1 and the missing term $+1$ in the product. These changes are not important and we can still find the matrix-vector representation of the CBC algorithm with a fast matrix-vector multiplication. This allows us to compute and store all values of $e_n^2(\bar{z}_{\mathfrak{u}})$ using $\mathcal{O}(|\mathfrak{u}|\,n \ln n)$ operations and $\mathcal{O}(n)$ memory. In general, we need to repeat this calculation for each non-negative weight. Since we now have $\mathcal{O}(d^q)$ non-negative weights, where $q = \omega$ for finite-order weights and $q = 1$ for finite-diameter weights, the total cost of this part of the algorithm is $\mathcal{O}(\omega\,d^q\,n \ln n)$ operations and

$\mathcal{O}(d^q\, n)$ memory. Then we compute each $e_n^2(\bar{z}_s)$ by summing up $\gamma'_{d,\mathfrak{u}} e^2_{n,\mathfrak{u}}(\bar{z}_{\mathfrak{u}})$ using $\mathcal{O}(d^q)$ operations and finally compute $\bar{z}_s$ that minimizes $e^2_{n,s}(\mathbf{z}_s)$ with the total cost of $\mathcal{O}(\omega\, d^q\, n \ln n)$ operations and with $\mathcal{O}(d^q n)$ memory.

We stress that the computational cost is exponential in ω for finite-order weights. In general, this must be so since we have of order $d^{\,\omega}$ non-negative weights and each of them must be used. On the other hand, the order ω is usually small so that the exponential dependence on ω is not so dangerous.

16.6.4 Cost for Order-Dependent Weights

As in [54], we now consider order-dependent weights, i.e.,

$$\gamma_{d,\mathfrak{u}} = \Gamma_{d,|\mathfrak{u}|} \quad \text{for all } d \in \mathbb{N} \text{ and for all } \mathfrak{u} \subseteq [d]$$

with $\Gamma_{d,0} = 1$ and $\Gamma_{d,j} \ge 0$ for all $j \in \mathbb{N}$. Then

$$\begin{aligned} e^2_{n,d}(\mathbf{z}) &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma'_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} D_\alpha\left(k\bar{z}_j/n\right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=1}^{d} \Gamma'_{d,m} \sum_{\mathfrak{u} \subseteq [d],\, |\mathfrak{u}|=m} \prod_{j \in \mathfrak{u}} D_\alpha\left(k\bar{z}_j/n\right) = \sum_{m=1}^{d} \Gamma'_{d,m} D_{(m)}, \end{aligned}$$

where

$$\begin{aligned} \gamma'_{d,\mathfrak{u}} &= \left[\frac{2}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \gamma_{d,\mathfrak{u}}, \\ \Gamma'_{d,m} &= \left[\frac{2}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} \Gamma_{d,m} \\ D_{(m)} &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\mathfrak{u} \subseteq [d],\, |\mathfrak{u}|=m} \prod_{j \in \mathfrak{u}} D_\alpha\left(k\bar{z}_j/n\right). \end{aligned}$$

Note that $D_{(m)}$ can be considered as an overall measure of the quality of the m-dimensional projections of the lattice rule with generator $\mathbf{z}$. The quantity $D_{(1)}$ has the same value for all rank-1 lattice rules if n is prime, since every one-dimensional projection of such a lattice rule is just $\{k/n : k = 0, 1, \dots, n-1\}$.

The formula for $D_{(\ell)}$ involves quantities of the form

$$\sum_{\mathfrak{u} \subseteq [d],\, |\mathfrak{u}|=m} \prod_{j \in \mathfrak{u}} D_\alpha\left(k\bar{z}_j, n\right).$$

We give a recursive formula to compute such quantities. Define

$$T_k(s,m) = \sum_{\mathfrak{u} \subseteq [s],\, |\mathfrak{u}|=m} \prod_{j \in \mathfrak{u}} D_\alpha\left(k\bar{z}_j, n\right) \quad \text{for all } s \in [d] \text{ and } m \in [s]. \tag{16.29}$$

We can view $T_k = (T_k(s,m))$ as a $d \times d$ lower triangular matrix. Obviously,

$$T_k(s,1) = \sum_{j=1}^{s} D_\alpha\left(k\bar{z}_j, n\right) \quad \text{and} \quad T_k(s,s) = \prod_{j=1}^{m} D_\alpha\left(k\bar{z}_j, n\right)$$

for $s = 1, 2, \ldots, d$. The elements of the dth row of T_k are used to compute

$$e_{n,d}^2(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} \Big(\sum_{m=1}^{d} \Gamma_{d,m}\, T_k(d,m) \Big).$$

From (16.29) we get

$$T_k(s,m) = T_k(s-1,m) + D_\alpha\,(k\bar{z}_m, n)\; T_k(s-1,m-1) \quad \text{for all } s \ge 2,\ m \ge 2.$$

This allows us to compute $T_k(s,m)$ by the following algorithm:

$$T_k(1,1) = D_\alpha\,(k\bar{z}_1, n)\,, \quad \text{for } s = 2, 3, \ldots, d;$$

$$T_k(s,1) = \sum_{j=1}^{s} D_\alpha\left(k\bar{z}_j, n\right) \quad \text{and} \quad T_k(s,s) = \prod_{j=1}^{s} D_\alpha\left(k\bar{z}_j, n\right),$$

for $m = 2, 3, \ldots, s-1$;

$$T_k(s,m) = T_k(s-1,m) + D_\alpha\,(k\bar{z}_m, n)\; T_k(s-1,m-1).$$

This algorithm is especially convenient in the case when we successively increase $d = 2, 3, \ldots, s$, and therefore is especially well suited to the CBC algorithm. If the $T_k(d-1,m)$'s have been computed, then the next step to compute all $T_k(d,m)$ as well as $\sum_{m=1}^{d} \Gamma_{d,m} T_k(d,m)$ requires only $\mathcal{O}(d)$ operations. The computation of $e_{n,d+1}^2(\mathbf{z})$ requires $\mathcal{O}(n\,d)$ operations, and therefore the total cost of the CBC algorithm is $O(n^2\,d^{\,2})$. Finally, if the order-dependent weights are also of finite-order of order ω, then the total cost of the algorithms is reduced to $O(n^2\,\omega\,d)$.

We do not know how to reduce the cost of the CBC algorithm in terms of the dependence of n. So far we have an n^2 dependence and it is not clear how to achieve $\mathcal{O}(n \ln n)$ by fast matrix-vector multiplication. We leave this as an open problem for the reader.

Open Problem 75.

Consider the CBC algorithm for order-dependent weights.

- Is it possible to find an implementation of the CBC algorithm such that the generator $\mathbf{z}$ can be computed using $\mathcal{O}(d^{\,2}\, n \ln n)$ operations?

16.7 Weighted Korobov and Sobolev Spaces

So far we have considered multivariate integration for periodic functions from the weighted Korobov spaces, and we have only analyzed QMC algorithms. In particular, we presented a number of lower bounds on the worst case error but only for the class of QMC algorithms.

It is natural to ask whether *arbitrary* linear algorithms can be much better than QMC algorithms and to seek possibly sharp lower bounds on the worst case errors for arbitrary algorithms. In Chapter 11 we presented lower bounds on the worst case error for arbitrary algorithms, assuming that the reproducing kernel is decomposable or it has a decomposable part without boundary conditions. It is easy to see that for spaces of periodic functions, the reproducing kernels cannot be decomposable or they cannot have a decomposable part. The reason is simple: a decomposable part of the reproducing kernel means that for the univariate case, functions behave independently on the subintervals. In the case of periodic functions we know that they behave exactly the same at the end points of the domain interval.

That is why for periodic spaces of functions, such as the weighted Korobov spaces, we need a different proof technique to get lower bounds on the worst case error for arbitrary algorithms.

This is the subject of the next sections. As in [129], we show that multivariate integration for weighted Korobov spaces is closely related to multivariate integration for non-periodic functions from certain Sobolev spaces. This will allow us to apply the proof technique of decomposable kernels to these Sobolev spaces, and to also determine lower bounds for weighted Korobov spaces. In particular, we show that some tractability conditions for QMC algorithms are the same as for arbitrary algorithms. In this way we obtain matching necessary and sufficient conditions on various kinds of tractability for multivariate integration for weighted Korobov spaces.

16.7.1 Kernels and Shift-Invariant Kernels

In this section we present some basic properties of reproducing kernels and shift-invariant reproducing kernels. These properties will be needed to relate weighted Korobov spaces to certain Sobolev spaces

Let $H_d = H_d(K_d)$ be an arbitrary Hilbert space of functions defined over $[0,1]^d$ with real-valued reproducing kernel K_d. If this kernel is absolutely integrable, then we may compute its Fourier series coefficients, which are given by

$$\widehat{K_d}(h,\tau) = \int_{[0,1]^{2d}} K_d(x,t) \exp\left[-2\pi i (h\cdot x + \tau\cdot t)\right] \,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } h,\tau \in \mathbb{Z}^d. \tag{16.30}$$

If these Fourier coefficients are absolutely summable, then we may write the kernel as

$$K_d(x,t) = \sum_{h,\tau\in\mathbb{Z}^d} \widehat{K_d}(h,\tau) \exp\left(2\pi\,\mathrm{i}\,(h\cdot x + \tau\cdot t)\right).$$

For weighted Korobov spaces and Sobolev spaces with certain boundary conditions the reproducing kernel may be written in the form above. However, reproducing kernels that are not periodic cannot be written as an absolutely summable series, although the Fourier coefficients may be well defined.

A particularly interesting case occurs when

$$\widehat{K_d}(h,\tau) = 0 \quad \text{for all } \tau \neq -h.$$

Let $\widehat{K_d}(h) = \widehat{K_d}(h,-h)$. Then

$$K_d(x,t) = \sum_{h\in\mathbb{Z}^d} \widehat{K_d}(h) \exp\left(2\pi\, \mathrm{i}\, h\cdot(x-t)\right) \quad \text{with} \quad \sum_{h\in\mathbb{Z}^d} |\widehat{K_d}(h)| < \infty. \tag{16.31}$$

Since our reproducing kernel is real-valued and symmetric in its arguments, one may show that

$$\widehat{K_d}(h) = \overline{\widehat{K_d}(h)} = \widehat{K_d}(-h).$$

Note that K_d of the form (16.31) depends only on the fractional parts of the successive components of the vector $x-y$, i.e., on

$$\{x-y\} = [\{x_1-y_1\}, \{x_2-y_2\}, \dots, \{x_d-y_d\}] \in [0,1]^d.$$

It is easy to check that the inner product of $H(K_d)$ for reproducing kernels of the form (16.31) is

$$\langle f,g\rangle_{H(K_d)} = \sum_{h\in\mathbb{Z}^d} \widehat{K_d}^{-1}(h)\,\hat f(h)\,\overline{\hat g(h)} \quad \text{for all } f,g\in H(K_d), \tag{16.32}$$

with the convention that for $\widehat{K_d}(h) = 0$ we have $\hat f(h) = 0$ for all $f \in H(K_d)$ and interpret $0/0 = 0$.

The kernel $K_{d,\alpha,\gamma}$ of the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ is of the form (16.31) with

$$\widehat{K_d}(h) = \varrho_{d,\alpha,\gamma}^{-1}(h) \quad \text{and then} \quad \sum_{h\in\mathbb{Z}^d} |\widehat{K_d}(h)| = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \left[\frac{2\zeta(2\alpha)}{(2\pi)^{2\alpha}}\right]^{|\mathfrak{u}|} < \infty. \tag{16.33}$$

We need the following lemma.

Lemma 16.11. *A function K_d of the form (16.31) is a real-valued reproducing kernel iff $\widehat{K_d}(h) \ge 0$ for all $h \in \mathbb{Z}^d$.*

Proof. This lemma easily follows from the facts that $f(x) = K_d(x,0)$ is periodic and positive definite[1], and that $f \in L_1([0,1]^d)$ is positive definite iff its Fourier coefficients $\widehat{K_d}(h)$ are non-negative, see Edwards [64], pp. 149–150. □

[1]The function f is *positive definite* iff

$$\int_{[0,1]^{2d}} f(x-y)\overline{u(x)}u(y)\,\mathrm{d}x\,\mathrm{d}y \ge 0$$

for all continuous complex-valued functions u.

We need to recall a relation between arbitrary reproducing kernels and shift-invariant reproducing kernels. As in Hickernell [117] and [128], we say that K_d is a *shift-invariant* kernel iff

$$K_d(x,t) = K_d(\{x+\Delta\},\{t+\Delta\}) \quad \text{for all } x,t,\Delta \in [0,1]^d. \tag{16.34}$$

It is easy to check that K_d is shift-invariant iff

$$K_d(x,t) = K_d(\{x-t\},0) \quad \text{for all } x,t \in [0,1]^d.$$

Thus, reproducing kernels of the form (16.31) are shift-invariant. This includes the kernel $K_{d,\alpha,\gamma}$ of the weighted Korobov space $H_{d,\alpha,\gamma}$.

As in Hickernell [117] and [128], for an arbitrary kernel K_d we define the *shift-invariant kernel* $K_{\mathrm{sh},d}$ *associated to* K_d by

$$K_{\mathrm{sh},d}(x,t) = \int_{[0,1]^d} K_d(\{x+\Delta\},\{t+\Delta\})\, d\Delta \quad \text{for all } x,t \in [0,1]^d. \tag{16.35}$$

Indeed, $K_{\mathrm{sh},d}$ is shift-invariant, and if K_d is shift-invariant then $K_{\mathrm{sh},d} = K_d$. The definition above can be used to show that the Fourier series coefficients for the associated shift-invariant kernel are related to the Fourier series coefficients of the original kernel by

$$\widehat{K_{\mathrm{sh},d}}(h) = \widehat{K_d}(h,-h) \quad \text{for all } h \in \mathbb{Z}^d.$$

As in [128], see Theorem 2, we now consider multivariate integration for the spaces $H(K_d)$ and $H(K_{\mathrm{sh},d})$. We first compute the initial errors, i.e., the worst case errors of the zero algorithm in these spaces. We have

$$\begin{aligned} e^2(0; H(K_{\mathrm{sh},d})) &= \int_{[0,1]^{2d}} K_{\mathrm{sh},d}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{[0,1]^{3d}} K_d\left(\{x+\Delta\},\{t+\Delta\}\right)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\Delta. \end{aligned}$$

Since the function $f(x,t) = K_d(\{x+\Delta\},\{t+\Delta\})$ is periodic, we can drop Δ in the inside integral of the last formula and obtain

$$e^2(0; H(K_{\mathrm{sh},d}) = \int_{[0,1]^{2d}} K_d(x,t)\,\mathrm{d}x\,\mathrm{d}t = e^2(0; H(K_d)).$$

This means that the initial errors are equal.

We now show that multivariate integration over the space $H(K_d)$ is not harder than multivariate integration over the space $H(K_{\mathrm{sh},d})$. As always, let $e(n; H(K_d))$ and $e(n, H(K_{\mathrm{sh},d}))$ denote the minimal worst case errors for multivariate integration in the spaces $H(K_d)$ and $H(K_{\mathrm{sh},d})$, respectively. We show that

$$e(n; H(K_d)) \le e(n; H(K_{\mathrm{sh},d})) \quad \text{for all } n \in \mathbb{N}.$$

We have already shown equality when $n = 0$. So, let $n \ge 1$. We know that it is enough to consider linear algorithms for both spaces. Take then an arbitrary linear algorithm

$$A_{n,d}(f) = \sum_{j=1}^{n} a_j f(x_j)$$

for the space $H(K_{\mathrm{sh},d})$. Formally, the a_j's could be complex numbers. However, since K_d and $K_{\mathrm{sh},d}$ are real-valued, it is easy to show that the worst case error of $A_{n,d}$ is minimized if we take real a_j. Hence, without loss of generality we assume that the coefficients a_j are real. For $\Delta \in [0,1]^d$, define the algorithm

$$A_{n,d,\Delta}(f) = \sum_{j=1}^{n} a_j f\big(\{x_j + \Delta\}\big) \quad \text{for all } f \in H(K_d).$$

That is, for a linear algorithm $A_{n,d}$ for the space $H(K_{\mathrm{sh},d})$ we switch to the algorithm $A_{n,d,\Delta}$ for the space $H(K_d)$ by using the same coefficients a_j and changing the sample points x_j to $\{x_j + \Delta\}$ for some not yet specified Δ. We know the formulas for the worst case errors of these algorithms. Namely,

$$\begin{aligned} e^2(A_{n,d}; H(K_{\mathrm{sh},d})) = e^2(0; H(K_{\mathrm{sh},d})) - 2\sum_{j=1}^{n} a_j \int_{[0,1]^d} K_{\mathrm{sh},d}(x, x_j)\,\mathrm{d}x \\ + \sum_{j,k=1}^{n} a_j a_k K_{\mathrm{sh},d}(x_j, x_k). \end{aligned}$$

Note that the last integral does not depend on x_j. Indeed, since $K_{\mathrm{sh},d}$ is shift-invariant and periodic with period 1 in each variable, then

$$K_{\mathrm{sh},d}(x, x_j) = K_{\mathrm{sh},d}(\{x - x_j\}, 0)$$

and when we integrate over x we can drop x_j, getting

$$\begin{aligned} \int_{[0,1]^d} K_{\mathrm{sh},d}(x, x_j)\,\mathrm{d}x = \int_{[0,1]^d} K_{\mathrm{sh},d}\big(\{x - x_j\}, 0\big)\,\mathrm{d}x = \int_{[0,1]^d} K_{\mathrm{sh},d}(x, 0)\,\mathrm{d}x \\ = e^2(0; H(K_{\mathrm{sh},d})) = e^2(0; H(K_d)). \end{aligned}$$

Therefore

$$e^2(A_{n,d}; H(K_{\mathrm{sh},d})) = e^2(0; H(K_{\mathrm{sh},d}))\Big(1 - 2\sum_{j=1}^{n} a_j\Big) + \sum_{j,k=1}^{n} a_j a_k K_{\mathrm{sh},d}(x_j, x_k).$$

Consider now $e^2(A_{n,d,\Delta}; H(K_d))$ and compute its average value

$$a := \int_{[0,1]^d} e^2(A_{n,d,\Delta}; H(K_d))\,\mathrm{d}\Delta$$

with respect to uniformly distributed Δ. We have

$$
\begin{aligned}
a &= e^2(0, H(K_d)) - 2\sum_{j=1}^{n} a_j \int_{[0,1]^{2d}} K_d(\{x_j + \Delta\}, x)\, \mathrm{d}\Delta\, \mathrm{d}x \\
&\quad + \sum_{j,k=1}^{n} a_j a_k \int_{[0,1]^d} K_d\left(\{x_j + \Delta\}, \{x_k + \Delta\}\right) \mathrm{d}\Delta \\
&= e^2(0, H(K_d)) - 2\sum_{j=1}^{n} a_j \int_{[0,1]^{2d}} K_d(\Delta, x)\, \mathrm{d}\Delta\, \mathrm{d}x \\
&\quad + \sum_{j,k=1}^{n} a_j a_k K_{\mathrm{sh},d}(x_j, x_k) \\
&= e^2(0; H(K_d))\Big(1 - 2\sum_{j=1}^{n} a_j\Big) + \sum_{j,k=1}^{n} a_j a_k K_{\mathrm{sh},d}(x_j, x_k) \\
&= e^2(0; H(K_{\mathrm{sh},d}))\Big(1 - 2\sum_{j=1}^{n} a_j\Big) + \sum_{j,k=1}^{n} a_j a_k K_{\mathrm{sh},d}(x_j, x_k) \\
&= e^2(A_{n,d}; H(K_{\mathrm{sh},d})).
\end{aligned}
$$

This means that the average value of the square of the worst case error of $A_{n,d,\Delta}$ for the space $H(K_d)$ is equal to the square of the worst case error of $A_{n,d}$ for the space $H(K_{\mathrm{sh},d})$. From the mean value theorem, there exists an $\Delta \in [0,1]^d$ such that

$$
e(A_{n,d,\Delta}; H(K_d)) \le e(A_{n,d}, H(K_{\mathrm{sh},d})). \tag{16.36}
$$

Since this holds for any linear algorithm $A_{n,d}$, we have

$$
e(n; H(K_d)) \le e(n; H(K_{\mathrm{sh},d})),
$$

as claimed.

Let $n(\varepsilon, H(K_d)$ and $n(\varepsilon, H(K_{\mathrm{sh},d}))$ denote the information complexity for the normalized error criterion for the corresponding spaces, that is, the minimal number of function values needed to reduce the initial error by a factor of ε. The last result implies that

$$
n(\varepsilon, H(K_d)) \le n(\varepsilon, H(K_{\mathrm{sh},d})) \quad \text{for all } \varepsilon \in (0,1). \tag{16.37}
$$

In fact, a little more can be shown, which is the subject of the following lemma.

Lemma 16.12. *Let K_d be a reproducing kernel defined on $[0,1]^{2d}$ with Fourier coefficients $\widehat{K_d}(h,\tau)$, as defined in (16.30). Let $\widetilde{K}_d$ be a shift-invariant kernel defined on $[0,1]^{2d}$ with Fourier coefficients $\widehat{\widetilde{K}_d}(h)$, as defined in (16.31). If these Fourier coefficients satisfy the inequality*

$$
\frac{\widehat{K_d}(h,-h)}{\widehat{K_d}(0,0)} \le \frac{\widehat{\widetilde{K}_d}(h)}{\widehat{\widetilde{K}_d}(0)} \quad \textit{for all } h \in \mathbb{Z}^d, \tag{16.38}
$$

then multivariate integration for the normalized error criterion is not harder for the space $H(K_d)$ than it is for the space $H(\widetilde{K}_d)$, i.e.,

$$n(\varepsilon, H(K_d)) \le n(\varepsilon, H(\widetilde{K}_d)) \quad \textit{for all } \varepsilon \in (0,1).$$

Proof. By (16.37) it follows that $n(\varepsilon, H(K_d)) \le n(\varepsilon, H(K_{\mathrm{sh},d}))$. Now consider the spaces $H(K_{\mathrm{sh},d})$ and $H(\widetilde{K}_d)$. Assumption (16.38) implies that

$$\widehat{K_{\mathrm{sh},d}}(h)/\widehat{K_{\mathrm{sh},d}}(0) \le \widehat{\widetilde{K}_d}(h)/\widehat{\widetilde{K}_d}(0) \quad \text{for all } h \in \mathbb{Z}^d.$$

Multivariate integration in the space $H(K_{\mathrm{sh},d})$ is $I_d(f) = \langle f, \eta_d \rangle_{H(K_{\mathrm{sh},d})}$ with

$$\eta_d(y) = \int_{[0,1]^d} K_{\mathrm{sh},d}(y,t)\,\mathrm{d}t \quad \text{for all } y \in [0,1]^d.$$

For any linear algorithm $Q_{n,d}(f) = \sum_{j=1}^n a_j f(x_j)$ with real a_j, the representer of the quadrature error for the space $H(K_{\mathrm{sh},d})$ is

$$\begin{aligned}
\xi_{n,d}(y) &:= \eta_d(y) - \sum_{j=1}^n a_j K_{\mathrm{sh},d}(y, x_j) \\
&= \widehat{K_{\mathrm{sh},d}}(0)\Big(1 - \sum_{j=1}^n a_j\Big) - \sum_{0 \ne h \in \mathbb{Z}^d} \widehat{K_{\mathrm{sh},d}}(h) \sum_{j=1}^n a_j \exp\left[2\pi i h \cdot (y - x_j)\right],
\end{aligned}$$

and by (16.32) the square worst case error is

$$\begin{aligned}
&e^2(Q_{n,d}; H(K_{\mathrm{sh},d})) \\
&\quad = \|\xi_{n,d}\|^2_{H(K_{\mathrm{sh},d})} \\
&\quad = \widehat{K_{\mathrm{sh},d}}(0)\Big|1 - \sum_{j=1}^n a_j\Big|^2 + \sum_{0 \ne h \in \mathbb{Z}^d} \widehat{K_{\mathrm{sh},d}}(h)\Big|\sum_{j=1}^n a_j \exp\left(2\pi i h \cdot x_j\right)\Big|^2.
\end{aligned}$$

Due to Lemma 16.11, all Fourier coefficients $\widehat{K_{\mathrm{sh},d}}$ in the sum above are non-negative. Also, the square of the initial error is

$$e^2(0; H(K_{\mathrm{sh},d})) = \widehat{K_{\mathrm{sh},d}}(0).$$

The worst case error for the space $H(\widetilde{K}_d))$ has similar expressions. The inequality above relating the Fourier coefficients for the two kernels then implies that

$$\frac{e(Q_{n,d}; H(K_{\mathrm{sh},d}))}{e(0; H(K_{\mathrm{sh},d}))} \le \frac{e(Q_{n,d}; H(\widetilde{K}_d))}{e(0; H(\widetilde{K}_d))}.$$

Since this holds for any $Q_{n,d}$, the desired result then follows. □

16.7.2 Relations to Weighted Sobolev Space

We will use Lemma 16.12 by identifying $H(\widetilde{K}_d)$ as the weighted Korobov space, and constructing a weighted Sobolev space that plays the role of $H(K_d)$. To satisfy the assumptions of Lemma 16.12, the Sobolev space must have certain boundary conditions.

16.7.2.1 Unweighted univariate case. We first begin with the unweighted univariate case, $d = 1$. Let $W_2^r([0,1])$ be the classical Sobolev space of functions defined on $[0,1]$ with $(r-1)$ absolutely continuous derivatives and with rth derivatives belonging to the space $L_2([0,1])$. We study the Sobolev space H_r with boundary conditions, defined by

$$H_r = \{\, f \in W_2^r([0,1]) \mid f^{(k)}(0) = f^{(k)}(1) = 0 \quad \text{for } k = 0, 1, \dots, r-1 \,\}$$

and with the inner product

$$\langle f, g \rangle_{H_r} = \int_0^1 f^{(r)}(t) g^{(r)}(t)\,\mathrm{d}t \quad \text{for all } f, g \in H_r.$$

We want to find the reproducing kernel K_r of H_r. For $r = 1$, it is known that

$$K_1(x,t) = \min(x,t) - xt = \tfrac{1}{2}\big(B_2(\{x-t\}) - B_2(x) - B_2(t) + \tfrac{1}{6}\big). \tag{16.39}$$

For $r \ge 2$, let r_e denote the smallest even integer $\ge r+1$, and r_o denote the smallest odd integer $\ge r+1$. Define the vectors and matrices:

$$\gamma_{r,e}(x) = \left[\frac{B_{2r}(x)}{(2r)!}, \frac{B_{2r-2}(x)}{(2r-2)!}, \dots, \frac{B_{r_e}(x)}{r_e!}, 1\right]^T, \tag{16.40}$$

$$\gamma_{r,o}(x) = \left[\frac{B_{2r-1}(x)}{(2r-1)!}, \frac{B_{2r-3}(x)}{(2r-3)!}, \dots, \frac{B_{r_o}(x)}{r_o!}\right]^T, \tag{16.41}$$

$$\Gamma_{r,e} = \begin{pmatrix} \gamma_{r,e}^{(1)T}(0) \\ \gamma_{r,e}^{(2)T}(0) \\ \vdots \\ \gamma_{r,e}^{(r_0-3)T}(0) \\ (1,0,\dots,0) \end{pmatrix}^{-1}, \qquad \Gamma_{r,o} = \begin{pmatrix} \gamma_{r,o}^{(1)T}(0) \\ \gamma_{r,o}^{(3)T}(0) \\ \vdots \\ \gamma_{r,o}^{(r_e-3)T}(0) \end{pmatrix}^{-1}. \tag{16.42}$$

Here the numerical superscript denotes the order of the derivative.

Of course, the $B_j(x)$ are Bernoulli polynomials of degree j. The matrices $A_{r,e}$ and $A_{r,o}$ whose inverses define $\Gamma_{r,e}$ and $\Gamma_{r,o}$ are nonsingular, as shown in the proof of the lemma below. Note that the matrices $\Gamma_{r,e}$ and $\Gamma_{r,o}$ are symmetric.

As in [129], we claim that the reproducing kernel for H_r can be written in terms of a finite rank modification of

$$K_{1,r,1}(x,t) = 1 + (-1)^{r+1} \frac{B_{2r}(\{x-t\})}{(2r)!}, \tag{16.43}$$

the kernel for the unweighted univariate Korobov space with smoothness parameter $\alpha = r$.

Lemma 16.13. *The reproducing kernel for H_r with $r \geq 2$ is*

$$K_r(x,t) = \frac{(-1)^{r+1}}{(2r)!} B_{2r}(\{x-t\}) + (-1)^r \gamma_{r,e}^T(x)\Gamma_{r,e}\gamma_{r,e}(t) + (-1)^r \gamma_{r,o}^T(x)\Gamma_{r,o}\gamma_{r,o}(t). \tag{16.44}$$

Proof. Assume for a moment that $\Gamma_{r,e}$ and $\Gamma_{r,o}$ are well defined. Then K_r given by (16.44) is also well defined. To prove that this is indeed the reproducing kernel, we first show that $K_r(\cdot,t) \in H_r$ for every fixed t. For $j = 0,1,\dots,r-1$, we have $[B_k(x)/(k)!]^{(j)} = B_{k-j}(x)/(k-j)!$ for $k \geq j$, and the jth derivative of $K_r(\cdot,t)$ is

$$K_r^{(j,0)}(x,t) = \frac{(-1)^{r+1}}{(2r-j)!} B_{2r-j}(\{x-t\}) + (-1)^r \gamma_{r,e}^{(j)T}(x)\Gamma_{r,e}\gamma_{r,e}(t) + (-1)^r \gamma_{r,o}^{(j)T}(x)\Gamma_{r,o}\gamma_{r,o}(t).$$

Hence, the $(r-1)$st derivative $K_r^{(r-1,0)}(\cdot,t)$ is obviously absolutely continuous, and the rth derivative $K_r^{(r,0)}(\cdot,t)$ is square integrable because $K_{1,r,1}^{(r,0)}(\cdot,t)$, given by (16.43), is square integrable and polynomials are, of course, square integrable over finite intervals. Next, we check the boundary conditions.

For even derivatives, $j = 0,2,\dots,r_o-3$, the vector $\gamma_{r,o}^{(j)T}(x)$ consists of the derivatives of the Bernoulli polynomials of odd degrees that are at least $r_o - j \geq 3$. Hence, for $x = 0$ and $x = 1$ it follows from the properties of Bernoulli polynomials that $\gamma_{r,o}^{(j)T}(0) = \gamma_{r,o}^{(j)T}(1) = 0$. So by the definition of $\gamma_{r,e}(x)$ and $\Gamma_{r,e}$, we have

$$\begin{aligned} K_r^{(j,0)}(1,t) = K_r^{(j,0)}(0,t) &= \frac{(-1)^{r+1}}{(2r-j)!} B_{2r-j}(\{0-t\}) \\ &\quad + (-1)^r \gamma_{r,e}^{(j)T}(0)\Gamma_{r,e}\gamma_{r,e}(t) \\ &= \frac{(-1)^{r+1}}{(2r-j)!} B_{2r-j}(t) + \frac{(-1)^r}{(2r-j)!} B_{2r-j}(t) = 0. \end{aligned}$$

A similar argument shows that $K_r^{(j,0)}(1,t) = K_r^{(j,0)}(0,t) = 0$ for $j = 1,3,\dots,r_e-3$. Since $\max(r_e,r_o) - 3 = r-1$, we have $K_r^{(j,0)}(1,t) = K_r^{(j,0)}(0,t) = 0$ for $j = 0,1,\dots,r-1$.

Finally, we must show that $K_r(\cdot,t)$ has the reproducing property. Any function f in the Sobolev space $H(K_r)$ must also be in the weighted Korobov space $H_{1,r,1}$, so

$$f(t) = \langle f, K_{1,r,1}(\cdot,t)\rangle_{H_{1,r,1}} = \int_0^1 f(x)\,\mathrm{d}x + \int_0^1 K_{1,r,1}^{(r,0)}(x,t) f^{(r)}(x)\,\mathrm{d}x.$$

Since $B_n'(x) = nB_{n-1}(x)$ we have $B_{2r}^{(r)}(x)/[(2r)!] = B_r(x)/[r!]$ and therefore,

$$
\begin{aligned}
\langle f, K_r(\cdot,t)\rangle_{H_r} &= \int_0^1 K_r^{(r,0)}(x,t) f^{(r)}(x)\,\mathrm{d}x \\
&= \int_0^1 K_{1,r,1}^{(r,0)}(x,t) f^{(r)}(x)\,\mathrm{d}x \\
&\quad + (-1)^r \left[\int_0^1 \gamma_{r,e}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x\right] \Gamma_{r,e}\gamma_{r,e}(t) \\
&\quad + (-1)^r \left[\int_0^1 \gamma_{r,o}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x\right] \Gamma_{r,o}\gamma_{r,o}(t) \\
&= f(t) - \int_0^1 f(x)\,\mathrm{d}x \\
&\quad + (-1)^r \left[\int_0^1 \gamma_{r,e}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x\right] \Gamma_{r,e}\gamma_{r,e}(t) \\
&\quad + (-1)^r \left[\int_0^1 \gamma_{r,o}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x\right] \Gamma_{r,o}\gamma_{r,o}(t).
\end{aligned}
$$

Since f and its first $r-1$ derivatives vanish at 0 and 1, one can integrate by parts to show that

$$
\begin{aligned}
\int_0^1 \frac{B_j(x)}{j!} f^{(r)}(x)\,\mathrm{d}x &= \int_0^1 \frac{(-1)^r B_j^{(r)}(x)}{j!} f(x)\,\mathrm{d}x \\
&= \begin{cases} 0 & j < r, \\ (-1)^r \int_0^1 f(x)\,\mathrm{d}x & j = r. \end{cases}
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
&\int_0^1 \gamma_{r,e}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x = (-1)^r \left[\int_0^1 f(x)\,\mathrm{d}x, 0, \dots, 0\right]^T, \\
&\int_0^1 \gamma_{r,o}^{(r)T}(x) f^{(r)}(x)\,\mathrm{d}x = 0.
\end{aligned}
$$

We also have

$$
[1,0,\dots,0]^T \Gamma_{r,e} = [0,\dots,0,1]^T \quad \text{and} \quad [1,0,\dots,0]^T \Gamma_{r,e}\gamma_{r,e}(t) = 1,
$$

which holds for all t. Substituting these expressions into the formula above completes the proof that $K_r(\cdot,t)$ has the reproducing property.

We now prove that $\Gamma_{r,e}$ and $\Gamma_{r,o}$ are well defined. We need to show that matrices $A_{r,e}$ and $A_{r,o}$ defining $\Gamma_{r,e}$ and $\Gamma_{r,o}$ are nonsingular. Let c_e and c_o be any vectors satisfying

$$
A_{r,e}c_e = 0 \quad \text{and} \quad A_{r,o}c_o = 0. \tag{16.45}
$$

Define the polynomial $p(x) = c_e^T \gamma_{r,e}(x) + c_o^T \gamma_{r,o}(x)$. By the definitions of $\gamma_{r,e}(x)$ and $\gamma_{r,o}(x)$, this polynomial has degree at most $2r$. However, the bottom row of the first condition in (16.45) is $(1, 0 \ldots, 0)c_e = 0$. This implies that the first element of c_e vanishes, so p has degree of at most $2r - 1$. The other conditions in (16.45) imply that $p(0) = \cdots = p^{(r-1)}(0) = p(1) = \cdots = p^{(r-1)}(1) = 0$ by an argument similar to that above. Thus, the polynomial p has two zeros, each with multiplicity r. Since the degree of p is at most $2r - 1$, p must be the zero polynomial, i.e., $p(x) = 0$ for all x. Since the polynomials comprising $\gamma_{r,e}(x)$ and $\gamma_{r,e}$ are linearly independent, it follows that $c_e = 0$ and $c_o = 0$. Thus, $A_{r,e}$ and $A_{r,o}$ must be nonsingular. □

Note that another formula for the reproducing kernel K_r appears in Ritter [251]. However, the formula in Lemma 16.13 in terms of Bernoulli polynomials makes it easier to derive bounds on the Fourier coefficients of K_r, which are needed to eventually apply Lemma 16.12.

Lemma 16.14. *For $r \geq 1$, the Fourier coefficients of the reproducing kernel K_r have the bounds*

$$\widehat{K_r}(0,0) > 0 \quad \textit{and} \quad 0 \leq \widehat{K_r}(h,-h) \leq G_r h^{-2r} \quad \textit{for all } h \neq 0$$

for some positive G_r.

Proof. The Fourier coefficient $\widehat{K_r}(0,0)$ is simply the square of the norm of multivariate integration. We know that this is nonzero because there are integrands in H_r with nonzero integrals, e.g., $x^r(1-x)^r$. Lemma 16.11 implies that $\widehat{K_r}(h,-h) \geq 0$. Thus, we only need to prove upper bounds on $\widehat{K_r}(h,-h)$.

Assume first that $r \geq 2$. For $h \neq 0$, the Fourier coefficient $\widehat{B_j}(h)$ of the jth degree Bernoulli polynomial is known to be $-j!(2\pi i h)^{-j}$. Due to (16.44), the Fourier coefficient $\widehat{K_r}(h,-h)$ satisfies

$$\begin{aligned}\widehat{K_r}(h,-h) = \frac{1}{(2\pi h)^{2r}} &+ (-1)^r \widehat{\gamma_{r,e}}^T(h)\Gamma_{r,e}\widehat{\gamma_{r,e}}(-h) \\ &+ (-1)^r \widehat{\gamma_{r,o}}^T(h)\Gamma_{r,o}\widehat{\gamma_{r,o}}(-h),\end{aligned} \tag{16.46}$$

where $\widehat{\gamma_{r,e}}(h)$ and $\widehat{\gamma_{r,o}}(h)$ are the Fourier coefficients of $\gamma_{r,e}(x)$ and $\gamma_{r,o}(x)$, respectively. This implies that for large h we have

$$(-1)^r \widehat{\gamma_{r,e}}^T(h)\Gamma_{r,e}\widehat{\gamma_{r,e}}(-h) = \mathcal{O}(h^{-2r_e}),$$
$$(-1)^r \widehat{\gamma_{r,o}}^T(h)\Gamma_{r,o}\widehat{\gamma_{r,o}}(-h) = \mathcal{O}(h^{-2r_o}).$$

Since both r_e and r_o are larger than r, the dominant term in (16.46) is the first one.

For $r = 1$, note that (16.39) is the same as (16.44) if we omit the last term with $\gamma_{r,o}$. Therefore the proof for $r = 1$ is the same as that for $r \geq 2$, with an obvious change. □

16.7.2.2 Multivariate weighted case. We now turn to multivariate weighted Sobolev spaces that are built from the univariate space H_r with the reproducing kernel K_r given in Lemma 16.13. The multivariate weighted Sobolev space $H^{\mathrm{sob}}_{d,r,\eta}$ is defined as a reproducing kernel Hilbert space whose kernel is

$$K^{\mathrm{sob}}_{d,r,\eta}(x,t) = 1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} K_r(x_j, t_j) \quad \text{for all } x, t \in [0,1]^d. \tag{16.47}$$

Here, r is a positive integer, and $\eta = \{\eta_{d,\mathfrak{u}}\}$ is a weight sequence with $\eta_{d,\emptyset} = 1$ and with non-negative $\eta_{d,\mathfrak{u}}$, for all non-empty $\mathfrak{u} \subseteq [d]$.

For product weights η, i.e., $\eta_{d,j} = \prod_{j \in \mathfrak{u}} \eta_{d,j}$ for some non-negative and non-increasing $\eta_{d,j}$, we have

$$K^{\mathrm{sob}}_{d,r,\eta}(x,t) = \prod_{j=1}^{d} \left[1 + \eta_{d,j}\, K_r(x_j, t_j)\right] \quad \text{for all } x, t \in [0,1]^d.$$

In this case, the space $H^{\mathrm{sob}}_{d,r,\eta}$ is the tensor product of the spaces $H^{\mathrm{sob}}_{1,r,\eta_{d,j}}$ of univariate functions defined on $[0,1]$ with the reproducing kernel

$$K^{\mathrm{sob}}_{1,r,\eta_{d,j}}(x,t) = 1 + \eta_{d,j}\, K_r(x,t) \quad \text{for all } x, t \in [0,1].$$

The space $H^{\mathrm{sob}}_{1,r,\eta_{d,j}}$ consists of functions whose $(r-1)$st derivatives are absolutely continuous, rth derivatives are in $L_2([0,1])$, and satisfying the following boundary conditions

$$f(0) = f(1) \quad \text{and} \quad f^{(j)}(0) = f^{(j)}(1) = 0 \quad \text{for } j = 1, 2, \dots, r-1.$$

The inner product of $H^{\mathrm{sob}}_{1,r,\eta_{d,j}}$ is

$$\langle f, g \rangle_{H^{\mathrm{sob}}_{1,r,\eta_{d_j}}} = f(0)g(0) + \frac{1}{\eta_{d,j}} \int_0^1 f^{(r)}(t) g^{(r)}(t)\, \mathrm{d}t.$$

If $\eta_{d,j} = 0$ then we assume that $f^r \equiv 0$ and then the space $H^{\mathrm{sob}}_{1,r,\eta_{d_j}} = \mathrm{span}(1)$ consists of constant functions.

For arbitrary weights $\eta_{d,\mathfrak{u}}$, the inner product of the multivariate Sobolev space $H^{\mathrm{sob}}_{d,r,\eta}$ is given by

$$\langle f, g \rangle_{H_{d,r,\hat{\gamma}}} = f(0)g(0) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \prod_{j \in \mathfrak{u}} \eta_{d,j}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} f^{(r_{\mathfrak{u}})}(x_{\mathfrak{u}}, 0) g^{(r_{\mathfrak{u}})}(x_{\mathfrak{u}}, 0)\, \mathrm{d}x_{\mathfrak{u}},$$

where $r_{\mathfrak{u}} = [(r_{\mathfrak{u}})_1, \dots, (r_{\mathfrak{u}})_d]$ is a vector with $(r_{\mathfrak{u}})_j = 1$ if $j \in \mathfrak{u}$ and $(r_{\mathfrak{u}})_j = 0$ otherwise. The vector $x_{\mathfrak{u}}$ is a $|\mathfrak{u}|$-dimensional vector with components x_j for $j \in \mathfrak{u}$, and $(x_{\mathfrak{u}}, 0)$ is a d-dimensional vector with components x_j if $j \in \mathfrak{u}$ and $x_j = 0$ otherwise. As always, $\eta_{d,\mathfrak{u}} = 0$ implies that $f^{(r_{\mathfrak{u}})} \equiv 1$, and we interpret $0/0 = 0$.

The Fourier coefficients of the reproducing kernel $K_{d,r,\eta}^{\text{sob}}$ are

$$\widehat{K_{d,r,\eta}^{\text{sob}}}(h,\tau) = \widehat{1}(h,\tau) + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \widehat{K_r}(h_j,\tau_j) \prod_{j \notin \mathfrak{u}} \widehat{1}(h_j,\tau_j) \quad \text{for all } h,\tau \in \mathbb{Z}^d.$$

Recall that the set of non-zero h_j is denoted by $\mathfrak{u}_h = \{j \in [d] \mid h_j \neq 0\}$. If $\mathfrak{u}_h$ is not a subset of $\mathfrak{u}$ then there is an index j such that $j \in \mathfrak{u}_h$ and $j \notin \mathfrak{u}$. Since $h_j \neq 0$ then $\widehat{1}(h_j,\tau_j) = 0$ and the $\mathfrak{u}$ term in the sum above is zero. Therefore

$$\frac{\widehat{K_{d,r,\eta}^{\text{sob}}}(h,-h)}{\widehat{K_{d,r,\eta}^{\text{sob}}}(0,0)} = \frac{\sum_{\mathfrak{u}_h \subseteq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \widehat{K_r}(h_j,h_j)}{1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|}} \quad \text{for all } h \in \mathbb{Z}^d.$$

Note that for $h = 0$ the last ratio is one. For non-zero $h \in \mathbb{Z}^d$, we apply Lemma 16.14 and obtain

$$\begin{aligned} A_h &:= \frac{\widehat{K_{d,r,\eta}^{\text{sob}}}(h,-h)}{\widehat{K_{d,r,\eta}^{\text{sob}}}(0,0)} \\ &= \frac{\sum_{\mathfrak{u}_h \subseteq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u} \setminus \mathfrak{u}_h} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|-|\mathfrak{u}_h|} \prod_{j \in \mathfrak{u}_h} \widehat{K_r}(h_j,h_j)}{1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|}} \\ &\le \frac{\sum_{\mathfrak{u}_h \subseteq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u} \setminus \mathfrak{u}_h} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|-|\mathfrak{u}_h|} G_r^{|\mathfrak{u}_h|} \prod_{j \in \mathfrak{u}_h} h_j^{-2r}}{1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \eta_{d,\mathfrak{u}} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|}}. \end{aligned}$$

For the weighted Korobov space $H_{d,\alpha,\gamma}$ we have $\widehat{K_{d,\alpha,\gamma}}(0,0) = 1$, and for non-zero $h \in \mathbb{Z}^d$, we have

$$\widehat{K_{d,\alpha,\gamma}}(h,-h) = \gamma_{d,\mathfrak{u}_h} \prod_{j \in \mathfrak{u}_h} (2\pi |h_j|)^{-2\alpha}.$$

To satisfy the conditions of Lemma 16.12 we need to guarantee that the inequality $A_h \le \widehat{K_{d,\alpha,\gamma}}(h,-h)$ holds for all $h \in \mathbb{Z}^d$. This holds for $h = 0$, so we need to consider only non-zero $h \in \mathbb{Z}^d$.

For given α and $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ we take

$$r = \lceil \alpha \rceil,$$

and denote

$$\gamma'_{d,\mathfrak{u}} := \gamma_{d,\mathfrak{u}} \left[\frac{\widehat{K_r}(0,0)}{(2\pi)^{2\alpha} G_r}\right]^{|\mathfrak{u}|} \quad \text{and} \quad \eta'_{d,\mathfrak{u}} := \eta_{d,\mathfrak{u}} \left[\widehat{K_r}(0,0)\right]^{|\mathfrak{u}|}.$$

We assume that $\eta = \{\eta_{d\mathfrak{u}}\}$ is chosen such that

$$\frac{\sum_{\mathfrak{v}\,:\,\mathfrak{u} \subseteq \mathfrak{v}} \eta'_{d,\mathfrak{v}}}{1 + \sum_{\emptyset \neq \mathfrak{v} \subseteq [d]} \eta'_{d,\mathfrak{v}}} \le \gamma'_{d,\mathfrak{u}} \quad \text{for all } \mathfrak{u} \subseteq [d]. \tag{16.48}$$

It is obvious that (16.48) is equivalent to $A_h \le \widehat{K_{d,\alpha,\gamma}}(h,-h)$ for all $h \in \mathbb{Z}^d$.

We now show how to choose the sequence η so that (16.48) holds for a number of specific sequences γ.

- Consider product weights $\gamma_{d,j} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$. Take η also as a sequence of product weights, $\eta_{d,j} = \prod_{j\in\mathfrak{u}} \eta_{d,j}$ with

$$\eta_{d,j} = \frac{\gamma_{d,j}}{(2\pi)^{2\alpha}\, G_r} \quad \text{for all } j \in [d].$$

Then η' is also a sequence of product weights, $\eta'_{d,\mathfrak{v}} = \prod_{j\in\mathfrak{v}} \eta'_{d,j}$ with $\eta'_{d,j} = \eta_{d,j}\, \widehat{K_r}(0,0)$ and

$$\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}} \eta'_{d,\mathfrak{v}} = \prod_{j\in\mathfrak{u}} \eta'_{d,j} \sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} \eta'_{d,j} = \prod_{j\in\mathfrak{u}} \eta'_{d,j} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} (1+\eta'_{d,j}).$$

Therefore

$$\frac{\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}} \eta'_{d,\mathfrak{v}}}{1+\sum_{\emptyset\neq\mathfrak{v}\subseteq[d]} \eta'_{d,\mathfrak{v}}} = \frac{\prod_{j\in\mathfrak{u}} \eta'_{d,j} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}}(1+\eta'_{d,j})}{\prod_{j=1}^{d}(1+\eta'_{d,j})} \le \prod_{j\in\mathfrak{u}} \eta'_{d,j}$$
$$= \prod_{j\in\mathfrak{u}} \gamma_{d,j} \frac{\widehat{K_r}(0,0)}{(2\pi)^{2\alpha}\, G_r} = \gamma'_{d,\mathfrak{u}},$$

as claimed

- Consider finite-order weights of order ω such that

$$\gamma_{d,\mathfrak{u}} = \begin{cases} \left[\frac{(2\pi)^{2\alpha}\, G_r}{\widehat{K_r}(0,0)}\right]^{|\mathfrak{u}|} & \text{if } |\mathfrak{u}| \le \omega, \\ 0 & \text{if } |\mathfrak{u}| > \omega. \end{cases}$$

Take

$$\eta_{d,\mathfrak{u}} = \frac{\gamma_{d,\mathfrak{u}}}{[(2\pi)^{2\alpha}\, G_r]^{|\mathfrak{u}|}} \quad \text{for all } \mathfrak{u} \subseteq [d].$$

Then $\eta'_{d,\mathfrak{u}} = \gamma'_{d,\mathfrak{u}} = 1$, and

$$\frac{\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}} \eta'_{d,\mathfrak{v}}}{1+\sum_{\emptyset\neq\mathfrak{v}\subseteq[d]} \eta'_{d,\mathfrak{v}}} \le 1 = \gamma'_{d,\mathfrak{u}}$$

trivially holds.

We summarize the analysis performed above in the following theorem.

Theorem 16.15. *For the weighted Korobov space $H_{d,\alpha,\gamma}$ construct the weighted Sobolev space $H^{\text{sob}}_{d,r,\eta}$ with $r = \lceil\alpha\rceil$ and η chosen such that (16.48) is satisfied.*

Then multivariate integration over this weighted Sobolev space is not harder than multivariate integration over this weighted Korobov space, i.e.,

$$n(\varepsilon, H^{\text{sob}}_{d,r,\eta}) \le n(\varepsilon, H_{d,\alpha,\gamma}) \quad \textit{for all } \varepsilon \in (0,1).$$

The theorem above says that lower bounds on multivariate integration for the weighted Sobolev space $H^{\text{sob}}_{d,r,\eta}$ are also lower bounds on multivariate integration for the weighted Korobov space $H_{d,\alpha,\gamma}$ under the appropriate assumptions on the choice of r and η. For the weighted Sobolev space $H^{\text{sob}}_{d,r,\eta}$ we obtain lower bounds based on the proof technique of decomposable kernels, and in this way we also obtain lower bounds for the weighted Korobov space $H_{d,\alpha,\gamma}$.

16.8 Tractability for Weighted Korobov Spaces

In this section we find necessary and sufficient conditions on tractability of multivariate integration defined over weighted Korobov spaces $H_{d,\alpha,\gamma}$. For simplicity we consider only bounded product weights, leaving the case of more general weights to the reader. In Sections 16.4.2, 16.4.3 and 16.4.4 we already discussed tractability, but only for the class of QMC algorithms. In this section we consider general algorithms. Necessary conditions on tractability of multivariate integration will be achieved by switching to weighted Sobolev spaces $H^{\text{sob}}_{d,r,\eta}$ and applying the results developed in Chapter 12 based on the notion of decomposable reproducing kernels. Sufficient conditions will be achieved by lattice rules and estimates presented in this chapter. Surprisingly enough, these necessary and sufficient conditions match.

Theorem 16.16. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined over the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ and bounded product weights*

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \quad \textit{for all } \mathfrak{u} \subseteq [d]$$

with $\sup_{d\in\mathbb{N},\, j=1,2,\dots,d} \gamma_{d,j} < \infty$.

- *I_γ is strongly polynomially tractable iff*

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$

- *I_γ is polynomially tractable iff*

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(d+1)} < \infty.$$

- *I_γ is strongly T-tractable iff*

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty \quad \textit{and} \quad \limsup_{\varepsilon\to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty.$$

- *I_γ is T-tractable iff*

$$\limsup_{\varepsilon^{-1}+d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j} + \ln \varepsilon^{-1}}{1+\ln T(\varepsilon^{-1},d)} < \infty.$$

- *I_γ is weakly tractable iff*

$$\lim_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} = 0.$$

Before we present the proof of Theorem 16.16 we need to elaborate on multivariate integration defined over the weighted Sobolev spaces $H^{\mathrm{sob}}_{d,r,\eta}$. We also consider product weights η, given by

$$\eta_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \eta_{d,j},$$

where $\eta_{d,1} \ge \eta_{d,2} \ge \cdots \ge 0$.

The reproducing kernel of $H^{\mathrm{sob}}_{d,r,\eta}$ is now of the form

$$K^{\mathrm{sob}}_{d,r,\eta}(x,t) = \prod_{j=1}^d \big(1+\eta_{d,j} K_r(x_j,t_j)\big) \quad \text{for all } x,t \in [0,1]^d.$$

As we know from Chapter 12, necessary tractability conditions are given in terms of the univariate case, $d=1$. We need to represent the univariate reproducing kernel

$$K_{1,r,\eta_{d,j}} = 1 + \eta_{d,j} K_r$$

as

$$K_{1,r,\eta_{d,j}} = R_1 + \eta_{d,j}(R_2+R_3),$$

where the R_j are reproducing kernels of the Hilbert space $H(R_j)$ of univariate functions such that

$$(H(R_j)\otimes H(R_k)) \cap H(R_m) = \{0\}$$

for pairwise different j,k and m with $j,k,m = 1,2,3$. We must also guarantee that the reproducing kernel R_2 is *decomposable* at a, that is,

$$R_2(x,t) = 0 \quad \text{for all } 0 \le x \le a \le t \le 1.$$

As in [129], from the form of $K_{1,r,\eta}$ it is natural to take

$$R_1 = 1 \quad \text{and} \quad H(R_1) = \mathrm{span}(1).$$

We need to decompose K_r as

$$K_r = R_2 + R_3 \tag{16.49}$$

with a decomposable R_2 and with $(H(R_j) \otimes H(R_k)) \cap H(R_m) = \{0\}$ for pairwise different j, k and m with $j, k, m = 1, 2, 3$.

We take R_3 as the reproducing kernel of the Hilbert space

$$H_{1,3} = \operatorname{span}\big(K_r^{(0,0)}(\cdot, \tfrac{1}{2}), K_r^{(0,1)}(\cdot, \tfrac{1}{2}), \dots, K_r^{(0,r-1)}(\cdot, \tfrac{1}{2})\big).$$

The space $H_{1,3}$ is equipped with the inner product of the Hilbert space $H_r = H(K_r)$. Let g_j^* be an orthonormal basis of $H_{1,3}$. Since $\dim(H_{1,3}) = r$ we have

$$R_3(x,t) = \sum_{j=0}^{r-1} g_j^*(x)\, g_j^*(t).$$

We now decompose the space H_r as

$$H_r = H_{1,3} \oplus H_{1,2}$$

with the Hilbert space

$$H_{1,2} = \{ f \in W_2^r([0,1]) \;:\; f^{(j)}(0) = f^{(j)}(\tfrac{1}{2}) = f^{(j)}(1) = 0,\ j = 0, 1, \dots, r-1 \}.$$

Then the reproducing kernel R_2 of $H_{1,2}$ is

$$R_2 = K_r - R_3.$$

Hence, (16.49) holds. Moreover, we have $(H(R_j) \otimes H(R_k)) \cap H(R_m) = \{0\}$ for pairwise different j, k and m with $j, k, m = 1, 2, 3$, as needed.

We claim that R_2 is decomposable at $\frac{1}{2}$. We need the following lemma.

Lemma 16.17. *Assume that $K\colon [0,1]^2 \to \mathbb{R}$ is an arbitrary reproducing kernel that is decomposable at $a \in (0,1)$. Let*

$$A = \{ f \in H(K) \mid f(b_1) = f(b_2) = \dots = f(b_k) = 0 \},$$

where $a \le b_j \le 1$ for all $j = 1, 2, \dots, k$ or $0 \le b_j \le a$ for all $j = 1, 2, \dots, k$.

Let K_A be the reproducing kernel of A. Then K_A is also decomposable at a.

Proof. We have

$$f(b_j) = \langle f, K(\cdot, b_j) \rangle_{H(K)}.$$

Let g_j be an orthonormal basis of $\operatorname{span}(K(\cdot, b_1), \dots, K(\cdot, b_k))$. Then

$$K_A(x,t) = K(x,t) - \sum_{j=1}^{k} g_j(x) g_j(t).$$

If $a \le b_j$ for all j, then $g_j(x) = 0$ for all $x \le a$ since $K(x, b_j) = 0$. This yields $K_A(x,t) = 0$ for all $x \le a \le t$. Similarly, for $b_j \le a$ we have $g_j(t) = 0$ for all $a \le t$ and $K_A(x,t) = 0$ for all $x \le a \le t$. In either case, K_A is decomposable at a, as claimed. $\square$

We now take the Sobolev space

$$H(K) = \big\{ f \in W_2^r([0,1]) \mid f(0) = \cdots = f^{(r-1)}(0) = 0 \\ \text{and } f\big(\tfrac12\big) = \cdots = f^{(r-1)}\big(\tfrac12\big) = 0 \big\}$$

As in [221], we now show the following lemma.

Lemma 16.18. *The reproducing kernel K of the space defined above is decomposable at $a = \frac12$.*

Proof. Without the conditions $f^{(j)}(\frac12) = 0$, but with the boundary conditions $f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, r-1$, the space $W_2^r([0,1])$ has the reproducing kernel, say, R of the form

$$R(x,t) = \int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!} \, \frac{(t-u)_+^{r-1}}{(r-1)!} \, \mathrm{d}u.$$

We show an explicit form of R. It is enough to consider $x \le t$. We use $t - u = t - x + x - u$ and $(t-u)^{r-1} = \sum_{j=0}^{r-1} \binom{r-1}{j} (t-x)^{r-1-j} (x-u)^j$ to conclude by simple integration that

$$R(x,t) = \frac{x^r}{r!} \sum_{j=0}^{r-1} \frac{r}{r+j} \, \frac{x^j}{j!} \, \frac{(t-x)^{r-1-j}}{(r-1-j)!}.$$

This means that for a fixed x, the function $R(x,\cdot)$ is a polynomial in $t \in [x,1]$ of degree at most $r-1$.

Note that $H(K)$ is a subspace of $H(R)$ consisting of functions $f \in H(R)$ for which $f^{(i)}(\frac12) = 0$ for $i = 0, 1, \ldots, r-1$. Observe that

$$f^{(i)}\big(\tfrac12\big) = \big\langle f, R^{(i,0)}\big(\tfrac12, \cdot\big)\big\rangle_{H(R)}.$$

Hence, $f \in H(K)$ iff $f \in H(R)$ and is orthogonal to

$$A_{r-1} = \operatorname{span}\big(R^{(0,0)}\big(\tfrac12,\cdot\big), R^{(1,0)}\big(\tfrac12,\cdot\big), \ldots, R^{(r-1,0)}\big(\tfrac12,\cdot\big)\big).$$

Let $g_i \in A_{r-1}$ be an orthonormal basis of A_{r-1} for $i = 0, 1, \ldots, r-1$. Since each $R^{(i,0)}(\frac12, \cdot)$ is a polynomial of degree at most $r-1$, the same holds for g_i in the interval $[\frac12, 1]$.

The reproducing kernel K of $H(K)$ is of the form

$$K(x,t) = R(x,t) - \sum_{j=0}^{r-1} g_j(x) g_j(t).$$

For $x \le \frac12 \le t$, we need to show that $R_2(x,t) = 0$. Indeed, by Taylor's theorem we have

$$K(x,t) = \sum_{j=0}^{r-1} \frac{1}{j!} K^{(0,j)}\big(x, \tfrac12\big) \big(t - \tfrac12\big)^j + \int_{1/2}^t K^{(0,r)}(x,u) \frac{(t-u)^{r-1}}{(r-1)!} \, \mathrm{d}u.$$

Observe that the terms of the sum vanish since $K^{(0,j)}(\cdot,\frac{1}{2}) \equiv 0$, due to the fact that

$$0 = f^{(j)}\big(\tfrac{1}{2}\big) = \big\langle f, K^{(0,j)}\big(\cdot,\tfrac{1}{2}\big)\big\rangle_{H(K)} \quad \text{for all } f \in H(K).$$

To show that the last term is also zero, it is enough to prove that $K^{(0,r)}(x,u) = 0$ for all $u \ge \frac{1}{2} \ge x$. This follows from the fact that all g_j are polynomials of degree at most $r-1$ in the interval $[\frac{1}{2},1]$, and therefore

$$K^{(0,r)}(x,u) = R^{(0,r)}(x,u) = \frac{\partial^r}{\partial u^r}\int_0^x \frac{(x-v)^{r-1}}{(r-1)!}\frac{(u-v)^{r-1}}{(r-1)!}\,\mathrm{d}v = 0,$$

as claimed. Thus, we have $R = R_{1,1/2}+R_{2,1/2}$ with $R_{1,1/2}(x,t) = \sum_{j=1}^{r-1} g_j(x)g_j(t)$ and $R_{2,1/2} = K$ is decomposable at $\frac{1}{2}$, as claimed. □

Observe that for the space $H_{1,2}$ we have

$$H_{1,2} = \{f \in H(K) \ :\ f(1) = \cdots = f^{(r-1)}(1) = 0\}.$$

Due to Lemma 16.17, the reproducing kernel R_2 of $H_{1,2}$ is therefore also decomposable at $\frac{1}{2}$ with the limiting case of b_j tending to $\frac{1}{2}$ and with $k = r$.

Consider univariate integration

$$I_1(f) = \int_0^1 f(x)\,\mathrm{d}x = \langle f, g_1\rangle_{H(K_{1,r,\eta_{d,j}})}\,.$$

We can decompose g_1 as

$$g_1 = g_{1,1} + \eta_{d,j}(g_{1,2}+g_{1,3}),$$

where $g_{1,k} \in H(R_k)$. We also denote

$$g_{1,2} = g_{1,2,(0)} + g_{1,2,(1)}$$

with $g_{1,2,(0)}(x) = g_{1,2}(x)$ for $x \in [0,\frac{1}{2}]$, and $g_{1,2,(0)}(x) = 0$ for $x \in [\frac{1}{2},1]$. Similarly, $g_{1,2,(1)}(x) = 0$ for $x \in [0,\frac{1}{2}]$, and $g_{1,2,(1)}(x) = g_{1,2}(x)$ for $x \in [\frac{1}{2},1]$.

Multivariate integration for the space $H^{\mathrm{sob}}_{d,r,\eta}$ with the reproducing kernel $K^{\mathrm{sob}}_{d,r,\eta}$ is defined as

$$I_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x = \langle f, g_d\rangle_{H^{\mathrm{sob}}_{d,r,\eta}}$$

with

$$g_d(x) = \prod_{j=1}^d \big(g_{1,1}(x_j) + \eta_{d,j}[g_{1,2}(x_j)+g_{1,3}(x_j)]\big).$$

We now check that $g_{1,2,(0)}$ and $g_{1,2,(1)}$ are both non-zero. We have

$$I_1(f) = \int_0^1 f(x)\,\mathrm{d}x = \big\langle f, g_{1,2,(j)}\big\rangle_{H_{1,2}}$$

for $f \in H_{1,2}$ and $f(x) = 0$ for all $x \in [0, \frac{1}{2}]$ when $j = 0$, and $f(x) = 0$ for all $x \in [\frac{1}{2}, 1]$ when $j = 1$. Observe that for

$$f_1(x) = \begin{cases} x^r \left(x - \frac{1}{2}\right)^r & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ (x-1)^r \left(x - \frac{1}{2}\right)^r & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

we have $f_j \in H_{1,2}$, f_1 vanishes over $[\frac{1}{2}, 1]$, and f_2 vanishes over $[0, \frac{1}{2}]$. Since their integrals are positive $g_{1,2,(j)}$ must be non-zero for $j = 0$ and $j = 1$.

We are ready to prove Theorem 16.16.

Proof of Theorem 16.16. Sufficient tractability conditions follow from Theorem 16.5 and are achieved by lattice rules.

We turn to necessary conditions. We take $r = \lceil \alpha \rceil$ and $\eta_{d,j} = \gamma_{d,j}/((2\pi)^{2\alpha} G_r)$ and apply Theorem 16.15. We know that it is enough to apply necessary conditions on tractability for the weighted Sobolev space $H^{\text{sob}}_{d,r,\eta}$ and for the normalized error criterion. We now apply the results of Section 12.5 with $R_{2,2} = R_2$ and $R_{2,1} = R_3$. The reproducing kernel $R_{2,2}$ is decomposable at $\frac{1}{2}$. Furthermore, $h_{1,2,2,(0)} = g_{1,2,(0)}$ and $h_{1,2,2,(1)} = g_{1,2,(1)}$ are non-zero. Hence, Corollary 12.7 applies and this completes the proof. □

We comment on Theorems 16.5 and 16.16. These theorems present necessary and sufficient conditions on various kinds of tractability for QMC algorithms and for arbitrary algorithms. We stress that these conditions are the same as long as we use bounded product weights. This is indeed good news for QMC algorithms: although they are much simpler and much easier to use than general (linear) algorithms, they enjoy the same tractability conditions as needed for a much larger class of arbitrary (linear) algorithms. Furthermore, we may use lattice rules of rank-1 whose generators can be computed by the CBC algorithms with almost linear time as QMC algorithms.

That is why we think we may say that for multivariate integration defined over weighted Korobov spaces equipped with bounded product weights, the main tractability problems are solved.

For weights other than bounded product weights, the situation is different. Obviously, for bounded finite-order or finite-diameter weights we always have polynomial or even strong polynomial tractability. However, it is not yet clear what are necessary and sufficient conditions on general weights for which various notions of tractability hold. This leads us to the next open problem.

Open Problem 76.

Consider multivariate integration defined over the weighted Korobov space $H_{d,\alpha,\gamma}$ with $\alpha > \frac{1}{2}$ and general weights γ.

- Find necessary and sufficient conditions on the weights γ to obtain various notions of tractability.
- Are these conditions different for QMC algorithms and arbitrary (linear) algorithms?

16.9 Weighted Sobolev Spaces

We know that multivariate integration over certain weighted Sobolev spaces is not harder than multivariate integration over weighted Korobov spaces. We have used this relation so far to find lower bounds on multivariate integration over weighted Korobov spaces. Obviously, we can also use this relation the other way around to find good upper bounds on multivariate integration over some weighted Sobolev spaces in terms of bounds established for weighted Korobov spaces. The reader may remember that the weighted Sobolev spaces, which are related to weighted Korobov spaces, are defined by some boundary conditions that are not natural per se. However, if we restrict ourselves to the relatively small smoothness $r = 1$, this problem disappears. So in this section, we analyze the anchored and unanchored weighted Sobolev spaces with the smoothness parameter $r = 1$ and we follow the analysis done in [275].

More specifically, we consider the weighted Sobolev spaces that are reproducing kernel Hilbert spaces $H(K_{d,\gamma})$ with the reproducing kernel

$$K_{d,\gamma}(x, y) = 1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \eta_j(x_j, y_j), \tag{16.50}$$

where

$$\eta_j(x, y) = \tfrac{1}{2} B_2(\{x - y\}) + \left(x - \tfrac{1}{2}\right)\left(y - \tfrac{1}{2}\right) + \mu_j(x) + \mu_j(y) + m_j, \tag{16.51}$$

and $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ is a weight sequence of non-negative numbers $\gamma_{d,\mathfrak{u}}$ with $\gamma_{d,\emptyset} = 1$, and

$$B_2(x) := x^2 - x + \tfrac{1}{6} = \frac{1}{\pi^2} \sum_{h=1}^{\infty} \frac{\cos(2\pi x)}{h^2}$$

is the Bernoulli polynomial of degree 2. Further, μ_j is a function with bounded derivative in [0, 1] such that $\int_0^1 \mu_j(x)\,\mathrm{d}x = 0$, and the number m_j is given by

$$m_j := \int_0^1 (\mu_j'(x))^2 \mathrm{d}x.$$

We will study the following two choices for the function μ_j in (16.51):

(A) $\mu_j(x) = \max(x, a_j) - \frac{1}{2}x^2 - \frac{1}{2}a_j^2 - \frac{1}{3}$ with arbitrary $a_j \in [0, 1]$ for $j \in [d]$,

(B) $\mu_j(x) = 0$ for all $j \in [d]$.

These two choices lead to two different kinds of Sobolev spaces:

- The choice (**A**) leads to an *anchored Sobolev kernel*, denoted by $K_{d,\gamma,A}$, and is given by (16.50) with

$$\eta_j(x,y) = \begin{cases} \min(|x-a_j|, |y-a_j|) & \text{for } (x-a_j)(y-a_j) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{16.52}$$

  This reproducing kernel Hilbert space is called the *anchored Sobolev space*, and is denoted by $H(K_{d,\gamma,A})$.

  Note that $\eta_j(a_j, y) = \eta_j(x, a_j) = 0$. The point $\boldsymbol{a} = [a_1, \dots, a_d]$ is called the *anchor*. In this case, $m_j = a_j^2 - a_j + \frac{1}{3}$. Clearly, $m_j \in [\frac{1}{12}, \frac{1}{3}]$.

- The choice (**B**) leads to an *unanchored Sobolev kernel*, denoted by $K_{d,\gamma,B}$, and is given by (16.50) with

$$\eta_j(x,y) = \tfrac{1}{2} B_2(\{x-y\}) + \left(x - \tfrac{1}{2}\right)\left(y - \tfrac{1}{2}\right).$$

  Note that $\int_0^1 \eta_j(x,y)\,\mathrm{d}x = 0$ for all $y \in [0,1]$.

  This reproducing kernel Hilbert space is called the *unanchored Sobolev space*, and is denoted by $H(K_{d,\gamma,B})$.

For general weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$, it can be checked that the inner product in the space $H(K_{d,\gamma,A})$ is

$$\langle f, g \rangle_{H(K_{d,\gamma,A})} = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f(x_{\mathfrak{u}}, \boldsymbol{a}_{-\mathfrak{u}})}{\partial x_{\mathfrak{u}}} \, \frac{\partial^{|\mathfrak{u}|} g(x_{\mathfrak{u}}, \boldsymbol{a}_{-\mathfrak{u}})}{\partial x_{\mathfrak{u}}} \,\mathrm{d}x_{\mathfrak{u}},$$

where x_u denotes the $|\mathfrak{u}|$-dimensional vector of components x_j with $j \in \mathfrak{u}$, and $x_{-\mathfrak{u}}$ denotes the vector $x_{[d]\setminus\mathfrak{u}}$; moreover $(x_{\mathfrak{u}}, \boldsymbol{a}_{-\mathfrak{u}})$ denotes a d-dimensional vector whose jth component is x_j if $j \in \mathfrak{u}$ and a_j if $j \notin \mathfrak{u}$. For $\mathfrak{u} = \emptyset$, we use the convention that $\int_{[0,1]^\emptyset} f(x_\emptyset, \boldsymbol{a}_{-\emptyset})\,\mathrm{d}x_\emptyset = f(\boldsymbol{a})$.

For the space $H(K_{d,\gamma,B})$ with general weights, it can be checked that the inner product is

$$\langle f, g \rangle_{H(K_{d,\gamma,B})} = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f(x)}{\partial x_{\mathfrak{u}}} \,\mathrm{d}x_{-\mathfrak{u}} \right) \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g(x)}{\partial x_{\mathfrak{u}}} \,\mathrm{d}x_{-\mathfrak{u}} \right) \mathrm{d}x_{\mathfrak{u}},$$

with the term corresponding to $\mathfrak{u} = \emptyset$ interpreted as $\int_{[0,1]^d} f(x)\,\mathrm{d}x \int_{[0,1]^d} g(x)\,\mathrm{d}x$.

The difference between the inner products is for terms indexed by $\mathfrak{u}$. The components of x not in $\mathfrak{u}$ are anchored at $\boldsymbol{a}$ for the space $H(K_{d,\gamma,A})$, while the same components are integrated over $[0,1]$ for the space $H(K_{d,\gamma,B})$.

Obviously, multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ with

$$I_{d,\gamma}(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \quad \text{for all } f \in H(K_{d,\gamma})$$

is well defined, where here and later $K_{d,\gamma}$ represents either the anchored Sobolev kernel $K_{d,\gamma,A}$ or the unanchored Sobolev kernel $K_{d,\gamma,B}$. The square of the initial error $e^2(0; H(K_{d,\gamma}))$ is given by

$$e^2(0; H(K_{d,\gamma})) = \int_{[0,1]^{2d}} K_{d,\gamma}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j. \tag{16.53}$$

In particular, for the unanchored Sobolev space $H(K_{d,\gamma,B})$ the initial error is always $\gamma_{d,\emptyset} = 1$, independent of the weights since the m_j's are zero. Hence, the absolute and normalized error are identical for $H(K_{d,\gamma,B})$, and are different for $H(K_{d,\gamma,A})$.

As the first step we recall what happens if we take QMC algorithms

$$A_{n,d}(f) = \frac{1}{n}\sum_{j=1}^{n} f(x_j) \quad \text{for all } f \in H(K_{d,\gamma})$$

for some sample points $x_j \in [0,1]^d$. When we compute the square of the worst case error of $A_{n,d}$ and then take the average with respect to uniform distribution of x_j then we can apply Theorem 10.4 from Chapter 10. This theorem states that there exists a QMC algorithm $A_{n,d}$ such that

$$e(A_{n,d}; H(K_{d,\gamma})) \le \frac{C_{d,\gamma}}{\sqrt{n}}, \tag{16.54}$$

where

$$\begin{aligned} C_{d,\gamma}^2 &= \int_{[0,1]^d} K_{d,\gamma}(x,x)\,\mathrm{d}x - \int_{[0,1]^{2d}} K_{d,\gamma}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \left(m_j + \tfrac{1}{6}\right) - \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j. \end{aligned}$$

This yields the following corollary.

Corollary 16.19. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined over the weighted Sobolev space $H(K_{d,\gamma})$ with arbitrary weights.*

- **(A)** *There exists a QMC algorithm $A_{n,d}$ such that*

$$e(A_{n,d}; H(K_{d,\gamma,A})) \le \frac{1}{\sqrt{n}}\Big(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \left(m_j + \tfrac{1}{6}\right) - \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in u} m_j\Big)^{1/2}.$$

Therefore, if

$$\sup_{d\in\mathbb{N}} \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \left(m_j + \tfrac{1}{6}\right) < \infty \quad \textit{for the absolute error,}$$

$$\sup_{d\in\mathbb{N}} \left(\frac{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} (m_j + \frac{1}{6})}{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j} \right) < \infty \quad \textit{for the normalized error,}$$

then I_γ*, defined for spaces* $H(K_{d,\gamma,A})$*, is strongly polynomially tractable for the absolute or normalized error criterion, respectively, with an* ε^{-1} *exponent at most* 2*.*

- **(B)** *There exists a QMC algorithm* $A_{n,d}$ *such that*

$$e(A_{n,d}; H(K_{d,\gamma,B})) \le \frac{1}{\sqrt{n}} \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} 6^{-|\mathfrak{u}|} \right)^{1/2}.$$

Therefore, if

$$\sup_{d\in\mathbb{N}} \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} 6^{-|\mathfrak{u}|} \right) < \infty,$$

then I_γ*, defined for spaces* $H(K_{d,\gamma,B})$*, is strongly polynomially tractable for both the absolute and normalized error criteria, with an* ε^{-1} *exponent at most* 2*.*

16.9.1 Shifted Lattice Rules

Corollary 16.19 indicates the existence of a QMC algorithm whose convergence order is $n^{-1/2}$, which is obviously not optimal. To improve the convergence order we will present a *shifted lattice rule*, which will allow us to obtain an optimal order of convergence. This will be done by switching to weighted Korobov spaces and using Theorem 16.15.

A shifted lattice rule has the form

$$A_{n,d,\Delta}(f) = \frac{1}{n} \sum_{j=1}^{n} f\left(\left\{ \frac{j}{n} \mathbf{z} + \Delta \right\} \right) \quad \text{for all } f \in H(K_{d,\gamma})$$

for some generator $\mathbf{z} \in [n-1]^d$ and $\Delta \in [0,1)^d$ with a prime n. Note that for $\Delta = 0$ we obtain lattice rules, which were studied before. For $\Delta \neq 0$, we shift the sample points that are used by the lattice rule, which motivates the name.

First, we need to find the associated shift-invariant kernel $K_{\text{sh},d,\gamma}$ of the original kernel $K_{d,\gamma}$. Consider first the unanchored Sobolev space $H(K_{d,\gamma,B})$. Its associated shift-invariant kernel can be easily found,

$$
\begin{aligned}
K_{\mathrm{sh},d,\gamma,B}(x,y) &:= \int_{[0,1]^d} K_{d,\gamma,B}(\{x+\Delta\},\{y+\Delta\})\,\mathrm{d}\Delta \\
&= 1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} B_2(\{x_j-y_j\}) \\
&= 1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma^{\mathrm{B}}_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \frac{2}{(2\pi)^2} \sum_{h=1}^{\infty} \frac{\cos(2\pi\, h(x_j-y_j))}{h^2},
\end{aligned}
$$

where

$$
\gamma^{\mathrm{B}}_{d,\mathfrak{u}} = 2^{|\mathfrak{u}|}\,\gamma_{d,\mathfrak{u}}. \tag{16.55}
$$

This means that the shift-invariant kernel $K_{\mathrm{sh},d,\gamma,B}$ of the unanchored Sobolev space is just the reproducing kernel of the weighted Korobov space with the weights $\gamma^{\mathrm{B}} = \{\gamma^{\mathrm{B}}_{d,\mathfrak{u}}\}$ and with $\alpha = 1$, see Section 16.2, i.e.,

$$
K_{\mathrm{sh},d,\gamma,B} = K_{d,1,\gamma^{\mathrm{B}}}. \tag{16.56}
$$

For the anchored Sobolev kernel $K_{d,\gamma,A}(x,y)$, its associated shift-invariant kernel can also be found after some computations,

$$
\begin{aligned}
K_{\mathrm{sh},d,\gamma,A}(x,y) &= 1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} [B_2(\{x_j-y_j\}) + m_j] \\
&= \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \sum_{\mathfrak{v}:\,\mathfrak{v}\subseteq\mathfrak{u}} \prod_{j\in\mathfrak{v}} B_2(\{x_j-y_j\}) \prod_{j\in\mathfrak{u}\setminus\mathfrak{v}} m_j \\
&= \sum_{\mathfrak{v}\subseteq[d]} \Big[\sum_{\mathfrak{u}:\,\mathfrak{v}\subseteq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}\setminus\mathfrak{v}} m_j \Big] \prod_{j\in\mathfrak{v}} B_2(\{x_j-y_j\}) \\
&= \sum_{\mathfrak{u}\subseteq[d]} \Big[\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j \Big] \prod_{j\in\mathfrak{u}} B_2(\{x_j-y_j\}) \\
&= \sum_{\mathfrak{u}\subseteq[d]} 2^{|\mathfrak{u}|} \Big[\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j \Big] \prod_{j\in\mathfrak{u}} \frac{2}{(2\pi)^2} \sum_{h=1}^{\infty} \frac{\cos(2\pi\,(x_j-y_j))}{h^2} \\
&= e^2(0;H(K_{d,\gamma,A})) \sum_{\mathfrak{u}\subseteq[d]} \gamma^{\mathrm{A}}_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \frac{2}{(2\pi)^2} \sum_{h=1}^{\infty} \frac{\cos(2\pi\,(x_j-y_j))}{h^2},
\end{aligned}
$$

where

$$
\gamma^{\mathrm{A}}_{d,\mathfrak{u}} = \frac{2^{|\mathfrak{u}|}}{e^2(0;H(K_{d,\gamma,A}))} \Big[\sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j \Big]. \tag{16.57}
$$

The initial error $e(0;H(K_{s,A}))$ is given by (16.53). Note that $\gamma^{\mathrm{A}}_{d,\emptyset} = 1$. Thus apart from the factor $e^2(0;H(K_{d,\gamma,A}))$, the shift-invariant kernel $K_{\mathrm{sh},d,\gamma,A}$ is just the Korobov reproducing kernel for the weights $\gamma^{\mathrm{A}} = \{\gamma^{\mathrm{A}}_{d,u}\}$ and the parameter $\alpha = 1$, i.e.,

$$
K_{\mathrm{sh},d,\gamma,A} = e^2(0;H(K_{d,\gamma,A}))\,K_{d,1,\gamma^{\mathrm{A}}}. \tag{16.58}
$$

The worst case errors of any QMC algorithm $A_{n,d}$ in the spaces $H(K_{\mathrm{sh},d,\gamma,A})$ and $H(K_{d,1,\gamma^{\mathrm{A}}})$ are therefore related by

$$\frac{e(A_{n,d}; H(K_{\mathrm{sh},d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} = e(A_{n,d}; H(K_{d,1,\gamma^{\mathrm{A}}})). \tag{16.59}$$

We summarize the analysis of this subsection in the following lemma.

Lemma 16.20. *The shift-invariant kernel of the anchored Sobolev kernel $K_{d,\gamma,A}$ or of the unanchored Sobolev kernel $K_{d,\gamma,B}$ is related to the weighted Korobov kernel $K_{d,1,\beta}$ by (16.58) or (16.56), respectively, with the weights $\beta = \{\beta_{d,\mathfrak{u}}\}$ given by*

$$\beta_{d,\mathfrak{u}} := \begin{cases} \gamma_{d,\mathfrak{u}}^{\mathrm{A}} & \text{if } K_{d,\gamma} = K_{d,\gamma,A}, \\ \gamma_{d,\mathfrak{u}}^{\mathrm{B}} & \text{if } K_{d,\gamma} = K_{d,\gamma,B}. \end{cases}$$

We now combine the constructive results for lattice rules with generators computed by the CBC algorithm for multivariate integration defined over weighted Korobov spaces and apply them to multivariate integration defined over the anchored and unanchored Sobolev spaces. We obtain the following theorem.

Theorem 16.21. *Consider multivariate integration defined over the weighted Sobolev space $H(K_{d,\gamma})$. For a prime number n, let $\mathbf{z}$ be the generator constructed by the CBC algorithm with the parameter $\alpha = 1$ and the weights $\gamma_{d,\mathfrak{u}}^{A}$ if $K_{d,\gamma} = K_{d,\gamma,A}$ and with the weights $\gamma_{d,\mathfrak{u}}^{B}$ if $K_{d,\gamma} = K_{d,\gamma,B}$.*

For $\Delta \in [0,1)^d$, consider the shifted rank-1 lattice rule

$$A_{n,d,\Delta}(f) = \frac{1}{n} \sum_{j=1}^{n} f\left(\left\{\frac{j}{n}\mathbf{z} + \Delta\right\}\right).$$

- **(A)** *There exists a shift $\Delta \in [0,1)^d$ such that for any $\tau \in [\frac{1}{2}, 1)$, we have*

$$e(A_{n,d,\Delta}; H(K_{d,\gamma,A})) \le \frac{\left(\sum_{\mathfrak{u} \subseteq [d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \prod_{j \in \mathfrak{u}} \left(\frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}} + m_j^{1/(2\tau)}\right)\right)^{\tau}}{(n-1)^{\tau}}.$$

- **(B)** *There exists a shift $\Delta \in [0,1)^d$ such that for any $\tau \in [\frac{1}{2}, 1)$ we have*

$$e(A_{n,d,\Delta}; H(K_{d,\gamma,B})) \le \frac{\left(\sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \left[\frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}}\right]^{|\mathfrak{u}|}\right)^{\tau}}{(n-1)^{\tau}}.$$

Proof. We first prove Part (A). From the mean value theorem, see (16.36), and from Lemma 16.20 we know that there exists $\Delta \in [0,1)^d$ such that

$$\frac{e(A_{n,d,\Delta}; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} \le e_{n,d}(1, \gamma^{\mathrm{A}}, \mathbf{z}),$$

where $e_{n,d}(1,\gamma^{\mathrm{A}},\mathbf{z})$ is the worst case error of the lattice rule with generator $\mathbf{z}$ for the weighted Korobov space $H_{d,1,\gamma^{\mathrm{A}}}$. From Theorem 16.3, we have

$$e_{n,d}(1,\gamma^{\mathrm{A}},\mathbf{z}) \le C_A\,(n-1)^{-\tau},$$

with

$$C_A := \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}}^{A}\right]^{1/(2\tau)} \left[\frac{2\zeta(1/\tau)}{(2\pi)^{1/\tau}}\right]^{|\mathfrak{u}|}\right)^{\tau}.$$

Let $W_\tau = 2\zeta(1/\tau)/(2\pi)^{1/\tau}$. Inserting the expression (16.57) for $\gamma_{d,u}^{\mathrm{A}}$ into the expression for $C_A(d,\tau)$ and then using Jensen's inequality with $\lambda = 1/(2\tau) \le 1$, we have

$$\begin{aligned}
C_A &= \frac{1}{e(0;H(K_{d,\gamma,A}))}\left(\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} W_\tau^{|\mathfrak{u}|}\Big(2^{|\mathfrak{u}|} \sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j\Big)^{1/(2\tau)}\right)^{\tau}\\
&\le \frac{1}{e(0;H(K_{d,\gamma,A}))}\left(\sum_{\mathfrak{u}\subseteq[d]} 2^{|\mathfrak{u}|/(2\tau)}\, W_\tau^{|\mathfrak{u}|} \sum_{\mathfrak{v}:\,\mathfrak{u}\subseteq\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}}^{1/(2\tau)} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j^{1/(2\tau)}\right)^{\tau}\\
&= \frac{1}{e(0;H(K_{d,\gamma,A}))}\left(\sum_{\mathfrak{v}\subseteq[d]} \gamma_{d,\mathfrak{v}}^{1/(2\tau)} \sum_{\mathfrak{u}:\,\mathfrak{u}\subseteq\mathfrak{v}} \big(W_\tau 2^{1/(2\tau)}\big)^{|\mathfrak{u}|} \prod_{j\in\mathfrak{v}\setminus\mathfrak{u}} m_j^{1/(2\tau)}\right)^{\tau}\\
&= \frac{1}{e(0;H(K_{d,\gamma,A}))}\left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \prod_{j\in\mathfrak{u}} \left(\frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}} + m_j^{1/(2\tau)}\right)\right)^{\tau}.
\end{aligned}$$

This proves Part (A). Part (B) follows by the same argument, but with $\gamma_{d,\mathfrak{u}}^{\mathrm{B}} = 2^{|\mathfrak{u}|}\gamma_{d,\mathfrak{u}}$. This completes the proof. □

We now address tractability. We have three cases since for the space $H(K_{d,\gamma,A})$ we have the absolute and normalized error criteria, and for the space $H(K_{d,\gamma,B})$ the absolute and normalized error criteria are the same. We introduce $x \in \{1,2,3\}$ to distinguish between these three cases.

- $x = 1$ means the space $H(K_{d,\gamma,A})$ and the absolute error criterion,
- $x = 2$ means the space $H(K_{d,\gamma,A})$ and the normalized error criterion,
- $x = 3$ means the space $H(K_{d,\gamma,B})$.

For $\tau \in [\frac{1}{2},1)$ and $q \ge 0$ define

$$B_{x,\tau,q,d} := \left[\frac{1}{d^q}\, \frac{\sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{1/(2\tau)} \prod_{j\in\mathfrak{u}} \left(\frac{2\zeta(1/\tau)}{(\sqrt{2}\pi^2)^{1/\tau}} + \delta_{x,\{1,2\}}\, m_j^{1/(2\tau)}\right)}{\delta_{x,\{1,3\}} + \delta_{x,\{2\}} \left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j\right)^{1/(2\tau)}} \right], \tag{16.60}$$

and

$$B_{x,\tau,q} := \sup_{d\in\mathbb{N}} B_{x,\tau,q,d}. \tag{16.61}$$

Here $\delta_{x,M} = 1$ if $x \in M$ and $\delta_{x,M} = 0$ if $x \notin M$.

This means that for $x \in \{1,2\}$ we add m_j in the numerator because then we consider the spaces $H(K_{d,\gamma,A})$, and for $x = 3$ the term m_j is missing, since we then deal with the spaces $H(K_{d,\gamma,B})$. Similarly for $x \in \{1,3\}$ we deal with the absolute error criterion and therefore the denominator is 1. For $x = 2$ we deal with the absolute error criterion for the spaces $H(K_{d,\gamma,A})$ and therefore we divide by a proper power of the initial error.

By tractability of I_γ, we will mean tractability for a fixed x that will be clear from the context. Hence, for $x = 1$ we mean tractability for the anchored Sobolev spaces and for the absolute error criterion, for $x = 2$ we mean tractability for still the anchored Sobolev spaces but now for the normalized error criterion, and finally for $x = 3$ we mean tractability for the unanchored Sobolev spaces in either absolute or normalized error criterion since they coincide.

Using this notation, we easily obtain the following theorem from the last theorem and Theorem 16.4.

Theorem 16.22. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ for the anchored Sobolev space $H(K_{d,\gamma,A})$ with an arbitrary anchor $\boldsymbol{a}$ or the unanchored Sobolev space $H(K_{d,\gamma,B})$. For $x \in \{1,2,3\}$, $\tau \in [\frac{1}{2},1)$ and $q \ge 0$, let $B_{x,\tau,q,d}$ and $B_{x,\tau,q}$ be defined by* (16.60) *and* (16.61).

- *If there exists a non-negative q such that $B_{x,1/2,q} < \infty$ then I_γ is polynomially tractable. If*

$$B_{x,\tau,q} < \infty \quad \text{for some } \tau \in \left[\tfrac{1}{2}, 1\right) \text{ and a non-negative } q$$

  then I_γ is polynomially tractable with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most q.

- *In particular, if $B_{x,1/2,0} < \infty$ then I_γ is strongly polynomially tractable. If*

$$B_{x,\tau,0} < \infty \quad \text{for some } \tau \in \left[\tfrac{1}{2}, 1\right)$$

  then the ε^{-1} exponent of strong polynomial tractability is at most $1/\tau$. If

$$B_{x,\tau,0} < \infty \quad \text{for all } \tau \in \left[\tfrac{1}{2}, 1\right)$$

  then the ε^{-1} exponent of strong polynomial tractability reaches the minimal value 1.

- *If $B_{x,1/2,0} < \infty$ and*

$$t^* = \limsup_{\varepsilon \to 0} \frac{\ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty$$

  then I_γ is strongly T-tractable.

  If $B_{x,\tau,0} < \infty$ for some $\tau \in [\frac{1}{2},1)$ and $t^* < \infty$ then the exponent of strong T-tractability is at most t^*/τ.

• *If*

$$t^* := \inf_{\tau \in [1/2,\alpha)} \limsup_{\varepsilon^{-1}+d\to\infty} \frac{\ln B_{x,\tau,0,d} + \ln \varepsilon^{-1/\tau}}{1+\ln T(\varepsilon^{-1},d)} < \infty$$

then I_γ is T-tractable with the exponent $t > t^*$.

• *If*

$$\lim_{d\to\infty} \frac{\ln B_{x,1/2,0,d}}{d} = 0$$

then I_γ is weakly tractable.

Theorems 16.21 and 16.22 state that for arbitrarily large d, shifted lattice rules with the generator constructed by the CBC algorithm and a suitable shift converge as $n^{-\tau}$. If τ can be arbitrarily close to 1, we may achieve almost the same convergence as for the univariate case, which is n^{-1}, and the difficulty of the d-dimensional integration is roughly the same as for the univariate one.

We stress that the CBC algorithm described above is *not* fully constructive, since we only know that there exists a shift for which the generator computed by the CBC algorithm leads to desired error bounds. The simultaneous construction of both a generator and a shift with a polynomial cost is given in Sloan, Kuo and Joe [271] for the anchored Sobolev space with $\alpha = 1$ and for product weights. However, the proven convergence rate for this construction is only $n^{-1/2}$. The construction of a shift vector preserving better rates of convergence is open, and left for future research. This is the subject of our next open problem

Open Problem 77.

Consider multivariate integration defined over the anchored or unanchored Sobolev spaces for general weights.

- Construct a shift and a generator of the shifted lattice rule for which the convergence rate is $n^{-\tau}$ for $\tau \in (\frac{1}{2}, 1)$.
- For which weights is it possible to do this in time $\mathcal{O}(d\, n\, \ln n)$?

16.9.2 Tractability for Finite-Order Weights

The theorems of the previous section are for general weights. In particular, we may apply them to finite-order or finite-diameter weights. As we shall see, the tractability conditions greatly simplify for such weights.

We now show that for the anchored Sobolev spaces and for the normalized error criterion, strong polynomial tractability holds for *arbitrary* finite-order weights. For the anchored Sobolev space and the absolute error criterion as well as for the unanchored Sobolev space we get polynomial tractability, not strong polynomial tractability. This holds under the additional (reasonable) assumption that the finite-order weights are bounded.

Theorem 16.23. **(A)** *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined over the anchored Sobolev space $H(K_{d,\gamma,A})$. For arbitrary finite-order weights of order ω, there exists a QMC algorithm $A_{n,d}$ such that the following holds:*

- *For the absolute error criterion, we have*

$$e(A_{n,d}; H(K_{d,\gamma,A})) \le \frac{1}{\sqrt{n}} \left[\max_{\mathfrak{u}\subseteq[d]:\, |\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \sum_{\mathfrak{u}:\, \gamma_{d,\mathfrak{u}}\neq 0} 2^{-|\mathfrak{u}|} \right]^{1/2}.$$

- *For the normalized error criterion, we have*

$$\begin{aligned} \frac{e(A_{n,d}; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} &\le \frac{1}{\sqrt{n}} \left[\max_{\mathfrak{u}:\, |\mathfrak{u}|\le\omega} \prod_{j\in\mathfrak{u}} \left(1 + \frac{1}{6m_j}\right) - 1 \right]^{1/2} \\ &\le \frac{1}{\sqrt{n}} \left(3^\omega - 1\right)^{1/2}. \end{aligned}$$

Hence, the minimal number $n_{\mathrm{QMC}}(\varepsilon, H(K_{s,A}))$ of function values needed to compute an ε-approximation by a QMC algorithm is bounded as follows:

- *For the absolute error criterion, we have*

$$n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A})) \le \left\lceil \varepsilon^{-2} \max_{\mathfrak{u}\subseteq[d]:\, |\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \sum_{\mathfrak{u}:\, \gamma_{d,\mathfrak{u}}\neq 0} 2^{-|\mathfrak{u}|} \right\rceil.$$

- *For the normalized error criterion, we have*

$$n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A})) \le \left\lceil \varepsilon^{-2} \left(3^\omega - 1\right) \right\rceil.$$

This implies the following results:

- *For the absolute error criterion and bounded finite-order weights, I_γ is polynomially tractable with an ε^{-1} exponent at most 2 and a d exponent at most ω and at most 1 if finite-order weights are also finite-diameter weights.*
- *For the normalized error criterion and arbitrary finite-order weights, I_γ is strongly polynomially tractable with an ε^{-1} exponent at most 2.*

(B) *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined over the unanchored Sobolev space $H(K_{d,\gamma,B})$. For arbitrary finite-order weights of order ω, there exists a QMC algorithm $A_{n,d}$ such that*

$$e(A_{n,d}; H(K_{d,\gamma,B})) \le \frac{1}{\sqrt{n}}\, G(d),$$

where

$$G(d) = \Big[\max_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \sum_{\mathfrak{u}\,:\,\gamma_{d,\mathfrak{u}}\neq 0} 6^{-|\mathfrak{u}|} \Big]^{1/2}$$
$$\le \max_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{1/2} \Big[\sum_{k=0}^{\omega} \binom{d}{k} 6^{-k} \Big]^{1/2}$$
$$= \max_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{1/2} \frac{d^{\,\omega/2}}{[\omega!\, 6^{\omega}]^{1/2}} \big(1 + \mathcal{O}(d^{-1})\big).$$

Hence,

$$n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,B})) \le \big\lceil \varepsilon^{-2} G^2(d) \big\rceil .$$

Thus, for bounded finite-order weights, I_γ is polynomially tractability with an ε^{-1} exponent at most 2 *and a d exponent at most ω and at most* 1 *if finite-order weights are also finite-diameter weights.*

Proof. For the weighted Sobolev space $H(K_{d,\gamma})$, we have from (16.54) that there exists a QMC algorithm $A_{n,d}$ such that

$$e\left(A_{n,d}; H(K_{d,\gamma})\right) \le \frac{1}{\sqrt{n}}\, C_{d,\gamma},$$

where

$$C_{d,\gamma}^2 \le \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \left(m_j + \tfrac{1}{6}\right).$$

Here $m_j \in [\frac{1}{12}, \frac{1}{3}]$ for the anchored case, and $m_j = 0$ for the unanchored case. Clearly, for the anchored case $m_j + \frac{1}{6} \le \frac{1}{2}$ and

$$C_{d,\gamma}^2 \le \max_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \sum_{\mathfrak{u}\,:\,\gamma_{d,\mathfrak{u}}\neq 0} 2^{-|\mathfrak{u}|}.$$

For the unanchored case, $m_j = 0$ and we have the same bound on $C_{d,\gamma}$ as above, with $2^{-|\mathfrak{u}|}$ replaced by $6^{-|\mathfrak{u}|}$. This proves the bounds for the anchored case and the absolute error criterion as well as for the unanchored case.

For the anchored case and the normalized error criterion, we have

$$e^2\left(A_{n,d}; H(K_{d,\gamma,A})\right) \le \frac{\varrho_d - 1}{n}\, e^2(0, H(K_{d,\gamma,A})),$$

where

$$\varrho_d = \frac{\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} (m_j + 1/6)}{\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j}$$
$$= \frac{\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \big(\prod_{j\in\mathfrak{u}} m_j\big)\big(\prod_{j\in\mathfrak{u}} (1 + \frac{1}{6m_j})\big)}{\sum_{\mathfrak{u}\subseteq[d]:\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j}$$
$$\le \max_{\mathfrak{u}:\,|\mathfrak{u}|\le\omega} \prod_{j\in\mathfrak{u}} \left(1 + \frac{1}{6m_j}\right).$$

Since $m_j \in [\frac{1}{12}, \frac{1}{3}]$, we have $\varrho_d \le 3^\omega$ independently of d and independently of the finite-order weights $\gamma_{d,\mathfrak{u}}$. Therefore

$$\frac{e^2(A_{n,d}; H(K_{d,\gamma,A}))}{e^2(0; H(K_{d,\gamma,A}))} \le \frac{1}{n}\left[\max_{\mathfrak{u}:\,|\mathfrak{u}|\le\omega} \prod_{j\in\mathfrak{u}}\left(1+\frac{1}{6m_j}\right)-1\right] \le \frac{1}{n}\,(3^\omega-1)\,.$$

All the rest is easy. This completes the proof. □

For the anchored Sobolev space and for the normalized error criterion we have strong tractability for arbitrary finite-order weights. However, both the error bounds and the minimal number $n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A}))$ of function values depend exponentially on ω. Hence, if the order ω is large, the corresponding minimal number may be huge. We now show that the exponential growth is indeed present for both the absolute and normalized error criteria and for some finite-order weights of order ω, and this holds for *any* QMC algorithm.

As in [275], we provide a lower bound on the worst case error of any QMC algorithm in the space $H(K_{d,\gamma,A})$, and conclude that for both the absolute and normalized error criteria, the minimal number $n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A}))$ of function values must depend exponentially on ω. The proof technique used in the next theorem is based on the reproducing kernel being point-wise non-negative as in [277], see also Section 10.5 of Chapter 10.

This assumption is obviously true for the anchored Sobolev space since we have $K_{d,\gamma,A}(x,y) \ge 1$ for all $x, y \in [0,1]^d$. For the unanchored Sobolev space, the kernel also takes negative values, and therefore we are unable to provide a corresponding lower bound in this case.

Theorem 16.24. *Consider multivariate integration defined over the anchored Sobolev spaces $H(K_{d,\gamma,A})$ with an arbitrary anchor $\boldsymbol{a} = [a_1,\dots,a_d]$. There are finite-order weights $\{\gamma_{d,\mathfrak{u}}\}$ of arbitrary order ω such that for any QMC algorithm $A_{n,d}$ with $d \ge \omega$ we have*

$$\frac{e^2(A_{n,d}; H(K_{s,A}))}{e^2(0; H(K_{d,\gamma,A}))} \ge 1 - 2c_\omega\, n \ge 1 - 2\left(\frac{8}{9}\right)^\omega n,$$

where

$$c_\omega = \min_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|=\omega} \prod_{j\in\mathfrak{u}} \frac{8\max(a_j^3, (1-a_j)^3)}{27(a_j^2-a_j)+9} \in \left[\left(\frac{4}{9}\right)^\omega, \left(\frac{8}{9}\right)^\omega\right].$$

Hence, for both the absolute and normalized error criterion, we have

$$n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A}) \ge \frac{1-\varepsilon^2}{2c_\omega} \ge \frac{1-\varepsilon^2}{2}\left(\frac{9}{8}\right)^\omega,$$

which depends exponentially on ω.

Proof. Since the kernel $K_{d,\gamma,A}(x,y)$ is always positive, see (16.52), we may use Theorem 10.2 of Chapter 10 or Lemma 4 of [277]. This theorem states that

$$\frac{e^2(A_{n,d};H(K_{d,\gamma,A}))}{e^2(0;H(K_{d,\gamma,A}))} \ge 1 - n\,\kappa_d^2, \tag{16.62}$$

where

$$\kappa_d^2 = \max_{x\in[0,1]^d} \frac{h_d^2(x)}{e^2(0;H(K_{d,\gamma,A}))\;K_{d,\gamma,A}(x,x)}, \tag{16.63}$$

with

$$h_d(x) = \int_{[0,1]^d} K_{d,\gamma,A}(x,y)\,\mathrm{d}y \quad \text{and} \quad K_{d,\gamma,A}(x,x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} |x_j - a_j|.$$

By direct computation we have

$$h_d(x) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} |x_j - a_j|\, w_j(x_j), \tag{16.64}$$

where

$$w_j(x) := \begin{cases} 1 - \frac12 x - \frac12 a_j, & \text{if } x > a_j,\\ \frac12 x + \frac12 a_j, & \text{if } x \le a_j. \end{cases}$$

For $\mathfrak{u}\subseteq[d]$, define

$$a_{d,\mathfrak{u}} = \gamma_{d,\mathfrak{u}}^{1/2} \prod_{j\in\mathfrak{u}} |x_j - a_j|^{1/2} \quad \text{and} \quad b_{d,\mathfrak{u}} = \gamma_{d,\mathfrak{u}}^{1/2} \prod_{j\in\mathfrak{u}} |x_j - a_j|^{1/2} w_j(x_j).$$

From (16.64), using Cauchy's inequality we have

$$h_d^2(x) = \Big(\sum_{\mathfrak{u}\subseteq[d]} a_{d,\mathfrak{u}}\, b_{d,\mathfrak{u}}\Big)^2 \le \sum_{\mathfrak{u}\subseteq[d]} a_{d,\mathfrak{u}}^2 \sum_{\mathfrak{u}\subseteq[d]} b_{d,\mathfrak{u}}^2 = K_{d,\gamma,A}(x,x) \sum_{\mathfrak{u}\subseteq[d]} b_{d,\mathfrak{u}}^2.$$

Based on this and (16.63) we find that

$$\begin{aligned} \kappa_d^2 &\le \max_{x\in[0,1]^d} \frac{\sum_{\mathfrak{u}\subseteq[d]} b_{d,\mathfrak{u}}^2}{e^2(0;H(K_{d,\gamma,A}))} \\ &= \max_{x\in[0,1]^d} \frac{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} |x_j - a_j|\, w_j^2(x_j)}{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j} \\ &\le \frac{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} W_j}{\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j}, \end{aligned} \tag{16.65}$$

where

$$W_j := \max_{x\in[0,1]} |x - a_j|\, w_j^2(x) = \max\left(\frac{8a_j^3}{27},\ \frac{8(1-a_j)^3}{27}\right).$$

For the anchored space $H(K_{d,\gamma,A})$ we have $m_j = a_j^2 - a_j + \frac{1}{3}$. As functions of a_j, both W_j and m_j are symmetric with respect to $a_j = \frac{1}{2}$; moreover, for $a_j \in [\frac{1}{2}, 1]$, we have

$$\frac{W_j}{m_j} = \frac{8a_j^3}{27\,(a_j^2 - a_j + \frac{1}{3})} \in \left[\tfrac{4}{9}, \tfrac{8}{9}\right] \quad \text{for all } j \in [d].$$

The minimal value of this ratio is obtained for $a_j = \frac{1}{2}$, and the maximal value for $a_j = 1$.

We are ready to define the finite-order weights for which Theorem 16.24 holds. As always, $\gamma_{d,\emptyset} = 1$. Let $\mathfrak{u}^*$ be a subset for which c_ω is attained, i.e.,

$$c_\omega = \prod_{j\in\mathfrak{u}^*} \frac{8\max(a_j^3, (1-a_j)^3)}{27(a_j^2 - a_j) + 9}.$$

Then we take the weights

$$\gamma_{d,\mathfrak{u}^*} = \beta \text{ for some } \beta > 0 \quad \text{and} \quad \gamma_{d,\mathfrak{u}} = 0 \text{ for all other } \mathfrak{u}.$$

From (16.65) we have

$$\kappa_d^2 \le \frac{1 + \beta\prod_{j\in\mathfrak{u}^*} W_j}{1 + \beta\prod_{j\in\mathfrak{u}^*} m_j}.$$

Take β so large[2] that

$$\frac{1 + \beta\prod_{j\in\mathfrak{u}^*} W_j}{1 + \beta\prod_{j\in\mathfrak{u}^*} m_j} \le 2\prod_{j\in u^*} \frac{W_j}{m_j} = 2c_\omega. \tag{16.66}$$

Then

$$\kappa_d^2 \le 2c_\omega \in \left[2\left(\tfrac{4}{9}\right)^\omega, 2\left(\tfrac{8}{9}\right)^\omega\right].$$

From (16.62) we have that

$$\frac{e^2(A_{n,d}; H(K_{d,\gamma,A}))}{e^2(0; H(K_{d,\gamma,A}))} \ge 1 - n\,\kappa_s^2 \ge 1 - 2n\,c_\omega \ge 1 - 2\left(\tfrac{8}{9}\right)^\omega n,$$

as claimed. Finally, note that $e(0, H(K_{d,\gamma,A})) \ge 1$ and the normalized error criterion is not harder than the absolute error criterion. Therefore the bound on $n_{\text{QMC}}(\varepsilon, H(K_{d,\gamma,A}))$ is obvious. This completes the proof. □

Combining this theorem with Theorem 16.23, we see that for the normalized error criterion and for the anchored Sobolev spaces and some finite-order weights of order ω, the minimal number $n_{\text{QMC}}(\varepsilon, H(K_{d,\gamma,A}))$ of function values is bounded for any anchor α and for $d \ge \omega$ by

$$\frac{1-\varepsilon^2}{2}\left(\frac{9}{8}\right)^\omega \le n_{\text{QMC}}(\varepsilon, H(K_{d,\gamma,A})) \le \frac{3^\omega - 1}{\varepsilon^2}.$$

[2] Clearly, the number 2 in (16.66) can be replaced by any number greater than 1.

Moreover, for any anchor α for which the first ω components have the value $\frac{1}{2}$, the minimal number is bounded by

$$\frac{1-\varepsilon^2}{2}\left(\frac{9}{4}\right)^{\omega} \le n_{\mathrm{QMC}}(\varepsilon, H(K_{d,\gamma,A})) \le \frac{3^{\omega}-1}{\varepsilon^2}.$$

These bounds depend exponentially on ω. Theoretically, if ω is large the minimal number of function values is huge. For example, for $d \ge \omega = 300$, $\varepsilon = \frac{1}{2}$ and the anchor $\left[\frac{1}{2}, \dots, \frac{1}{2}\right]$, we have

$$n_{\mathrm{QMC}}\left(\tfrac{1}{2}, H(K_{d,\gamma,A})\right) \ge \tfrac{3}{8}\left(\tfrac{9}{4}\right)^{\omega} > 1.5 \times 10^{105}.$$

However, for many practical problems ω is small, say, $\omega \le 3$ or 5. In such cases, we may be able to tolerate exponential dependence on ω.

16.9.3 Shifted Lattice Rules for Finite-Order Weights

The next theorem, which is a corollary of Theorem 16.21, shows that lattice rules constructed by the CBC algorithm for finite-order weights achieve polynomial tractability or strong polynomial tractability error bounds with high order of convergence under similar conditions on the weights as in Theorem 16.23.

Theorem 16.25. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined over the Sobolev space $H(K_{d,\gamma})$ with arbitrary finite-order weights $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ of order ω. Let $q = 1$ if finite-order weights are also finite-diameter weights, otherwise let $q = \omega$. Define*

$$\Gamma_d := \max_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}$$

$$N_d := \left|\{\mathfrak{u} \subseteq [d] \ : \ \gamma_{d,\mathfrak{u}} \ne 0\}\right| = \mathcal{O}(d^q).$$

For a prime number n, let $\mathbf{z}$ be the generator found by the CBC algorithm with the parameter $\alpha = 1$ and the weights $\gamma_{d,\mathfrak{u}}^{\mathrm{A}}$ if $K_{d,\gamma} = K_{d,\gamma,A}$, and with the weights $\gamma_{d,\mathfrak{u}}^{\mathrm{B}}$ if $K_{d,\gamma} = K_{d,\gamma,B}$, see (16.57) and (16.55).

For $\Delta \in [0,1)^d$, consider the shifted rank-1 lattice rule

$$A_{n,d,\Delta}(f) = \frac{1}{n}\sum_{j=1}^{n} f\left(\left\{\frac{j}{n}\mathbf{z} + \Delta\right\}\right).$$

- **(A)** *For the absolute error criterion, there exists a shift $\Delta \in [0,1)^d$ such that for any $\tau \in [\frac{1}{2}, 1)$ we have*

$$\begin{aligned} e(A_{n,d,\Delta}; H(K_{d,\gamma,A})) &\le \frac{\Gamma_d^{1/2} \max\left(1, \frac{2\zeta(1/\tau)}{(\sqrt{2}\pi)^{1/\tau}} + 3^{-1/(2\tau)}\right)^{\omega\tau} N_d^{\tau}}{(n-1)^{\tau}} \\ &= \mathcal{O}\left(\Gamma_d^{1/2}\, d^{\,q\,\tau}\, n^{-\tau}\right) \end{aligned}$$

with the factor in the big $\mathcal{O}$ notation independent of d and n but dependent on τ.

Hence, for polynomially bounded finite order weights, $\Gamma_d = \mathcal{O}(d^{\,s})$, the integration problem I_γ is polynomially tractable with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most $q + s/(2\tau)$.

- **(A)** *For the normalized error criterion, there exists a shift $\Delta \in [0,1)^d$ such that for any $\tau \in [\frac{1}{2}, 1)$ we have*

$$\frac{e(A_{n,d,\Delta}; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} \le \frac{\big(1 + 2\zeta(1/\tau)\,(\sqrt{6}/\pi)^{1/\tau}\big)^{\omega\,\tau}\, N_d^{\tau-1/2}}{(n-1)^{\tau}} = \mathcal{O}\big(d^{\,q(\tau-1/2)}\, n^{-\tau}\big)$$

with the factor in the big $\mathcal{O}$ notation independent of d and n but dependent on τ.

Hence, for arbitrary finite-order weights of order ω, the integration problem I_γ is strongly polynomially with an ε^{-1} exponent at most 2, as well as it is polynomially tractability with an ε^{-1} exponent $1/\tau$ and a d exponent $q(1 - 1/(2\tau))$.

(B) *For the absolute (and normalized) error criteria, there exists a shift $\Delta \in [0,1)^s$ such that for any $\tau \in [\frac{1}{2}, 1)$ we have*

$$e(A_{n,d}, \Delta; H(K_{d,\gamma,B})) \le \frac{\Gamma_d^{1/2}\, \max\big(1, \frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}}\big)^{\omega\,\tau}\, N_d^{\tau}}{(n-1)^{\tau}} = \mathcal{O}\big(\Gamma_d^{1/2}\, d^{\,q\,\tau}\, n^{-\tau}\big)$$

with the factor in the big $\mathcal{O}$ notation independent of d and n but dependent on τ.

Hence, for polynomially bounded finite order weights, $\Gamma_d = \mathcal{O}(d^{\,s})$, the integration problem I_γ is polynomially tractable with an ε^{-1} exponent at most $1/\tau$ and a d exponent at most $q + s/(2\tau)$.

Proof. Consider the case (A) for the absolute error criterion. From Theorem 16.21 we know that there exists a shift Δ such that

$$e(A_{n,d,\Delta}; H(K_{d,\gamma,A})) \le \frac{C_A}{(n-1)^{\tau}},$$

where

$$C_A := \Bigg(\sum_{\mathfrak{u}\subseteq[d]} \big[\gamma_{d,\mathfrak{u}}\big]^{1/(2\tau)} \prod_{j\in\mathfrak{u}} \bigg(\frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}} + m_j^{1/(2\tau)} \bigg) \Bigg)^{\tau}.$$

Since $m_j \le \frac{1}{3}$ and $|\mathfrak{u}| \le \omega$ for non-zero weights, we estimate

$$C_A \le \Gamma_d^{1/2} \max\bigg(1, \frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}} + 3^{-1/(2\tau)}\bigg)^{\omega\,\tau} N_d^{\tau},$$

and the rest is easy.

Consider the case (A) for the normalized error criterion. We need to estimate

$$C_N := \frac{\left(\sum_{\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \prod_{j\in\mathfrak{u}} \left(\frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}} + m_j^{1/(2\tau)}\right)\right)^{\tau}}{\left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j\right)^{1/2}}.$$

We have

$$C_N = \frac{\left(\sum_{\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \prod_{j\in\mathfrak{u}} m_j^{1/(2\tau)} \prod_{j\in\mathfrak{u}} \left(1 + \frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}\, m_j^{1/(2\tau)}}\right)\right)^{\tau}}{\left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j\right)^{1/2}}.$$

Since $m_j \ge \frac{1}{12}$ and $|\mathfrak{u}| \le \omega$, we have

$$\prod_{j\in\mathfrak{u}} \left(1 + \frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}\, m_j^{1/(2\tau)}}\right) \le \left(1 + \frac{2\zeta(1/\tau)\, 6^{1/(2\tau)}}{\pi^{1/\tau}}\right)^{\omega}.$$

This yields

$$C_N \le \left(1 + \frac{2\zeta(1/\tau)\, 6^{1/(2\tau)}}{\pi^{1/\tau}}\right)^{\omega\tau} \frac{\left(\sum_{\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \prod_{j\in\mathfrak{u}} m_j^{1/(2\tau)}\right)^{\tau}}{\left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j\right)^{1/2}}.$$

Using Hölder's inequality with $2\tau \ge 1$, we obtain

$$\frac{\left(\sum_{\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}}\right]^{1/(2\tau)} \prod_{j\in\mathfrak{u}} m_j^{1/(2\tau)}\right)^{\tau}}{\left(\sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j\right)^{1/2}} \le \left(\sum_{\mathfrak{u}\,:\,\gamma_{d,\mathfrak{u}}\neq 0} 1\right)^{\tau(1-1/(2\tau))} = N_d^{\tau-1/2},$$

as claimed. The rest is easy. The case (B) is done just as the case (A) for the absolute error criterion. This completes the proof. □

For $\tau = \frac{1}{2}$, the convergence rate of shifted lattice rules is $n^{-1/2}$. For the normalized error criterion, the error bounds are independent of the dimension d for the anchored Sobolev spaces, whereas for the absolute error criterion, they are polynomially dependent for the anchored and unanchored Sobolev space. For $\tau > \frac{1}{2}$, the rate of convergence of shifted lattice rules is improved to $n^{-\tau}$ but the error bounds depend polynomially on the dimension d for both the anchored and unanchored Sobolev spaces.

Hence, we have a tradeoff for the normalized error criterion and the anchored Sobolev space. Strong polynomial tractability is possible with a smaller rate of convergence, whereas polynomial tractability allows a better rate of convergence. Obviously, for a specific d and ε we can choose which is better and apply τ that minimizes the error bound.

We stress that Theorem 16.25 holds for *arbitrary* finite-order weights in the case of the anchored Sobolev spaces and the normalized error criterion, and for *arbitrary polynomially bounded* finite-order weights otherwise. Better results than those presented in Theorem 16.25 are possible to obtain if we assume stronger conditions on the finite-order weights as in Theorem 16.22.

16.9.4 Tractability Using Low Discrepancy Sequences

Lattice rules constructed by the CBC algorithm have good theoretical properties. However, these lattice rules depend on n as well as on the weights, since we minimize the worst case error that depends on both n and the weights, see also Sloan, Kuo and Joe [271]. In general, when the weights change, the lattice rules also change. These properties may make those rules inconvenient for applications, since for different problems even for fixed n we may need different weights and therefore different lattice rules. It may be a challenging problem to construct a "universal" lattice rule which is "good" for all, or at least for many, choices of weights.

An alternative approach is to fix a sequence of sample points $\{x_k\} \in [0,1]^d$ for $k = 0, 1, \dots,$ apply the QMC algorithms $A_{n,d}$ that use the first n points x_k, i.e., the points from the set $P_n := \{x_0, x_1, \dots, x_{n-1}\}$, and then investigate the worst case error bounds for anchored or unanchored Sobolev spaces with different weights. To stress the point set P_n we denote the worst case error of $A_{n,d}$ as

$$e(P_n; H(K_{d,\gamma})) = e(A_{n,d}; H(K_{d,\gamma})).$$

It is natural to study the point sets given by well known low discrepancy sequences such as Halton, Sobol or Niederreiter sequences, see e.g. Niederreiter [201] for their precise definition. This approach has been already proposed by Hickernell and Wang in [126] and by Wang in [321], see also Yue and Hickernell [359]. We use this approach for both general and finite-order weights, choosing to study explicitly the Niederreiter sequence. We make use of a lemma proved in Wang [321], involving the L_∞-star discrepancy of projections of $P_n = \{x_0, x_1, \dots, x_{n-1}\}$. We recall that the L_∞-star discrepancy of P_n is defined by

$$D^*(P_n) = \sup_{t \in [0,1)^d} |\mathrm{disc}(t; P_n)|,$$

where $\mathrm{disc}(t; P_n)$ is the local discrepancy given by

$$\mathrm{disc}(t; P_n) := \frac{\big|\{j : x_j \in [0,t)\}\big|}{n} - \prod_{j=1}^{d} t_j \quad \text{for all } t = [t_1, t_2, \dots, t_d] \in [0,1]^d.$$

Lemma 16.26. *Let b be a prime number, and let P_n be the first n points of the d-dimensional Niederreiter sequence in base b, which is based on the first irreducible polynomial over the finite field F_b. Let $P_n^{\mathfrak{u}}$ be the projection of P_n on the lower dimensional space $[0,1]^{|\mathfrak{u}|}$. Then for any non-empty subset $\mathfrak{u} \subseteq [d]$, the L_∞-star discrepancy of $P_n^{\mathfrak{u}}$ satisfies*

$$D^*(P_n^{\mathfrak{u}}) \le \frac{1}{n} \prod_{j \in \mathfrak{u}} [C_0\, j \log_2(j+b) \log_2(b\,n)],$$

where C_0 is a positive number independent of n, $\mathfrak{u}$ and d.

Using this lemma, we will prove the following theorem.

Theorem 16.27. *Let $H(K_{d,\gamma})$ be the anchored Sobolev space $H(K_{d,\gamma,A})$ with an arbitrary anchor $\boldsymbol{a}$, or the unanchored Sobolev space $H(K_{d,\gamma,B})$. Let P_n be the point set consisting of the first n points of the d-dimensional Niederreiter sequence in base b. Then*

$$e^2(P_n; H(K_{d,\gamma})) \le \frac{1}{n^2} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \left[C_1 j \log_2(j+b) \log_2(b\,n)\right]^2, \qquad (16.67)$$

where $C_1 = 2C_0$ is a positive number independent of n and d.

Proof. For simplicity, we first consider the anchored Sobolev space with the anchor $\boldsymbol{a} = [1, 1, \dots, 1]$. The corresponding kernel is, see (16.50) and (16.52),

$$K_{s,A}(x, y) = 1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \min(1 - x_j, 1 - y_j).$$

The square of the worst case error is in this case equal to the square of the weighted L_2-discrepancy, see Chapter 9, and is equal to

$$e^2(P_n; H(K_{s,A})) = \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}^2((x_{\mathfrak{u}}, 1); P_n)\, \mathrm{d}x_{\mathfrak{u}}. \qquad (16.68)$$

Obviously,

$$\int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}^2((x_{\mathfrak{u}}, 1); P_n)\, \mathrm{d}x_{\mathfrak{u}} \le [D^*(P_n^{\mathfrak{u}})]^2,$$

where $P_n^{\mathfrak{u}}$ is the projection of P_n on $[0,1]^{|\mathfrak{u}|}$. From Lemma 16.26 we have

$$\int_{[0,1]^{|\mathfrak{u}|}} \mathrm{disc}^2((x_{\mathfrak{u}}, 1); P_n)\, \mathrm{d}x_{\mathfrak{u}} \le \frac{1}{n^2} \prod_{j \in \mathfrak{u}} [C_0 j \log_2(j+b) \log_2(b\,n)]^2.$$

Thus from (16.68) we have

$$e^2(P_n; H(K_{d,\gamma,A})) \le \frac{1}{n^2} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [C_0 j \log_2(j+b) \log_2(b\,n)]^2,$$

which proves the result for the case $\boldsymbol{a} = [1, 1, \dots, 1]$.

For an arbitrary anchor $\boldsymbol{a} = [a_1, \dots, a_d]$, the proof is similar. It is useful to introduce some notation from Hickernell, Sloan and Wasilkowski [125]. The unit cube $[0,1]^d$ is partitioned into 2^d quadrants (some of them possibly degenerate) by the planes $x_j = a_j$ for $j = 1, 2, \dots, d$. Given x in the interior of one of these quadrants, let $B(x; \boldsymbol{a})$ denote the box with one corner at x and the opposite corner given by the unique vertex of $[0,1]^d$ that lies in the same quadrant as x. Let $B_{\mathfrak{u}}(x_{\mathfrak{u}}; \boldsymbol{a}_{\mathfrak{u}})$ be the projection of $B(x;\boldsymbol{a})$ on $[0,1]^{|\mathfrak{u}|}$. Instead of (16.68), we now have, see also Hickernell [118] and Hickernell, Sloan and Wasilkowski [125],

$$e^2(P_n; H(K_{d,\gamma,A})) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} R^2_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}})\, \mathrm{d}x_{\mathfrak{u}}, \tag{16.69}$$

with

$$R_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}}) = \frac{1}{n}\left|P_n^{\mathfrak{u}} \cap B_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}})\right| - \mathrm{Vol}\,(B_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}})) = (A_{n,d} - I_d)\chi_{B_{\mathfrak{u}}(x_{\mathfrak{u}};\alpha_{\mathfrak{u}})},$$

where χ_S denotes the indicator function for the set S. Clearly,

$$\begin{aligned}
\int_{[0,1]^{|\mathfrak{u}|}} R^2_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}})\, \mathrm{d}x_{\mathfrak{u}} &\le \sup_{x_{\mathfrak{u}} \in [0,1)^{\mathfrak{u}}} R^2_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}}) \\
&\le \sup_{x_{\mathfrak{u}} < y_{\mathfrak{u}}} \left(\frac{1}{n}\left|P_n^{\mathfrak{u}} \cap [x_{\mathfrak{u}}, y_{\mathfrak{u}})\right| - \mathrm{Vol}\,([x_{\mathfrak{u}}, y_{\mathfrak{u}}))\right)^2 \\
&\le 4^{|\mathfrak{u}|}\,(D^*(P_n^{\mathfrak{u}})^2.
\end{aligned}$$

The last step follows from the relation of the extreme discrepancy to the L_∞-star discrepancy, see Niederreiter [201]. It then follows from (16.69) and Lemma 16.26 that

$$e^2(P_n; H(K_{d,\gamma,A})) \le \frac{1}{n^2} \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [2C_0 j \log_2(j+b) \log_2(b\, n)]^2.$$

We now consider the unanchored Sobolev space $H(K_{d,\gamma,B})$. It is known, see e.g., Hickernell [118] and [277], that the worst case error $e(P_n; H(K_{s,B}))$ is the norm of the worst case integrand

$$\xi(x) := I_d(K_{d,\gamma,B}(x,\cdot)) - A_{n,d}(K_{d,\gamma,B}(x,\cdot)).$$

By computing its norm, we find that (16.69) is now replaced by

$$e^2(P_n; H(K_{d,\gamma,B})) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{|\mathfrak{u}|}} \widetilde{R}_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}})\, \mathrm{d}\boldsymbol{a}_{\mathfrak{u}}\right)^2 \mathrm{d}x_{\mathfrak{u}}, \tag{16.70}$$

where

$$\widetilde{R}_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}}) := \left(\prod_{j \in \mathfrak{u}} \tau_j(x_j, a_j)\right) R_{\mathfrak{u}}(x_{\mathfrak{u}};\boldsymbol{a}_{\mathfrak{u}}),$$

and

$$\tau_j(x_j, a_j) := \begin{cases} 1 & \text{if } x_j < a_j, \\ 0 & \text{if } x_j = a_j, \\ -1 & \text{if } x_j > a_j. \end{cases}$$

In verifying (16.70) it may help to observe that for fixed $x \in [0,1)^d$ the quantity $\widetilde{R}_{\mathfrak{u}}(x_{\mathfrak{u}}; \boldsymbol{a}_{\mathfrak{u}})$ is a piecewise-constant function of $\boldsymbol{a}_{\mathfrak{u}}$.

Similarly to the above argument, we now use

$$\begin{aligned}\int_{[0,1]^{|\mathfrak{u}|}} &\left(\int_{[0,1]^{|\mathfrak{u}|}} \widetilde{R}_{\mathfrak{u}}(x_{\mathfrak{u}}; \boldsymbol{a}_{\mathfrak{u}})\, \mathrm{d}\boldsymbol{a}_{\mathfrak{u}}\right)^2 \mathrm{d}x_{\mathfrak{u}} \\ &\le \sup_{x_{\mathfrak{u}} \in [0,1)^{|\mathfrak{u}|}} \sup_{\boldsymbol{a}_{\mathfrak{u}} \in [0,1)^{|\mathfrak{u}|}} R^2_{\mathfrak{u}}(x_{\mathfrak{u}}; \boldsymbol{a}_{\mathfrak{u}}) \\ &\le \sup_{x_{\mathfrak{u}} < y_{\mathfrak{u}}} \left(\frac{1}{n} |P_n^{\mathfrak{u}} \cap [x_{\mathfrak{u}}, y_{\mathfrak{u}})| - \mathrm{Vol}\,([x_{\mathfrak{u}}, y_{\mathfrak{u}}))\right)^2 \\ &\le 4^{|\mathfrak{u}|}\,(D^*(P_n^{\mathfrak{u}}))^2.\end{aligned}$$

It follows from (16.70) and Lemma 16.26 that

$$e^2(P_n; H(K_{d,\gamma,B})) \le \frac{1}{n^2} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [2C_0\, j \log_2(j+b) \log_2(b\,n)]^2.$$

This completes the proof. □

We are ready to prove that the QMC algorithm using the Niederreiter sequence achieves a tractability or strong tractability error bound for finite-order weights.

Theorem 16.28. *Let P_n be the point set of the first n points of the d-dimensional Niederreiter sequence in base b.*

(A) *Consider the anchored Sobolev space $H(K_{d,\gamma,A})$ with an arbitrary anchor $\boldsymbol{a}$.*

- *For arbitrary finite-order weights $\{\gamma_{d,\mathfrak{u}}\}$ of order ω, we have*

$$\frac{e(P; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} \le \frac{C_2\, d^{\,\omega} \log_2^{\omega}(d+b) \log_2^{\omega}(b\,n)}{n},$$

  where C_2 is a positive number independent of d and n.

  Hence, we have optimal convergence order, and polynomial tractability for the normalized error criterion with an ε^{-1} exponent arbitrarily close to 1, and a d exponent arbitrarily close to ω.

- *If the finite-order weights $\{\gamma_{d,\mathfrak{u}}\}$ of order ω satisfy*

$$\mathcal{M} := \sup_{d \in \mathbb{N}} \left(\frac{\sum_{\mathfrak{u} \subseteq [d],\, |\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [j \log_2(j+b)]^2}{\sum_{\mathfrak{u} \subseteq [d],\, |\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} m_j}\right) < \infty \tag{16.71}$$

  then for arbitrary $\delta > 0$ there exists a positive number C_δ independent of d and n such that

$$\frac{e(P_n; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} \le C_\delta\, n^{-1+\delta}.$$

  Hence, for the normalized error criterion we have strong polynomial tractability with an ε^{-1} exponent of strong tractability 1.

(B) *Consider the unanchored Sobolev space* $H(K_{d,\gamma,B})$.

- *For arbitrary bounded finite-order weights* $\{\gamma_{d,\mathfrak{u}}\}$ *of order* ω *we have*

$$e(P; H(K_{d,\gamma,B})) \le C_3\, d^{\,\omega}\, \log_2^{\omega}(d+b)\, \log_2^{\omega}(b\,n)\, n^{-1},$$

  where C_3 *is a positive number independent of* d *and* n.

  Hence, we have optimal order of convergence, and tractability with an ε^{-1} *exponent arbitrarily close to* 1*, and a* d *exponent arbitrarily close to* ω.

- *If the finite-order weights* $\{\gamma_{d,\mathfrak{u}}\}$ *of order* ω *satisfy*

$$\sup_{d\in\mathbb{N}} \Big(\sum_{\mathfrak{u}\subseteq[d],\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} [j\,\log_2(j+b)]^2 \Big) < \infty,$$

  then for arbitrary $\delta > 0$ *there exists a positive number* C'_δ *independent of* d *and* n *such that*

$$e(P_n; H(K_{d,\gamma,B})) \le C'_\delta\, n^{-1+\delta}.$$

  Hence, we have strong polynomial tractability with an ε^{-1} *exponent* 1.

Proof. Consider the anchored Sobolev space $H(K_{d,\gamma,A})$. As we know, the square of the initial error is

$$e^2(0; H(K_{d,\gamma,A})) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j. \tag{16.72}$$

For arbitrary finite-order weights $\{\gamma_{d,\mathfrak{u}}\}$ of order ω, from Theorem 16.27 we have

$$\begin{aligned}
\frac{e^2(P_n; H(K_{d,\gamma,A}))}{e^2(0; H(K_{d,\gamma,A}))} &\le \frac{1}{n^2}\, \frac{\sum_{0<|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} [C_1\, j\,\log_2(j+b)\log_2(b\,n)]^2}{\sum_{0\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} m_j} \\
&\le \frac{12^{\omega}}{n^2}\, \frac{\sum_{0<|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} [C_1\, j\,\log_2(j+b)\log_2(b\,n)]^2}{\sum_{0\le|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}} \\
&\le \frac{12^{\omega}}{n^2} \max_{\mathfrak{u}:\,|\mathfrak{u}|\le\omega} \prod_{j\in\mathfrak{u}} [C_1\, j\,\log_2(j+b)\log_2(b\,n)]^2 \\
&\le \frac{12^{\omega} C_1^{2\omega}}{n^2} (d\,\log_2(d+b))^{2\omega}\, (\log_2(b\,n))^{2\omega}.
\end{aligned}$$

Therefore,

$$\frac{e(P_n; H(K_{d,\gamma,A}))}{e(0; H(K_{d,\gamma,A}))} \le 2^{\omega}\sqrt{3^{\omega}}\, C_1^{\omega} d^{\,\omega}\, \log_2^{\omega}(d+b)\, \log_2^{\omega}(b\,n)\, n^{-1},$$

as claimed.

Now consider finite-order weights of order ω satisfying (16.71). Clearly, the bound (16.67) in Theorem 16.27 can be rewritten as

$$\begin{aligned} e^2(P_n; H(K_{d,\gamma,A})) &\le \frac{1}{n^2} \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [C_1 \, j \, \log_2(j+b) \log_2(b\,n)]^2 \\ &= \frac{1}{n^2} \sum_{\ell=1}^{\omega} \Big([C_1 \log_2(b\,n)]^{2\ell} \sum_{|\mathfrak{u}|=\ell} \Big[\gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} [j \log_2(j+b)]^2 \Big] \Big). \end{aligned} \tag{16.73}$$

For an arbitrary $\delta > 0$ define

$$B_\delta = \max_{\ell=1,2,\dots,\omega} \left[\left(\frac{C_1 \log_2 e}{2\delta} \right)^{2\ell} (2\ell)! \right].$$

It now follows from (16.72), (16.73) and (16.71) that for the anchored case we have

$$\begin{aligned} C_A &:= \frac{e^2(P_n; H(K_{d,\gamma,A}))}{e^2(0; H(K_{d,\gamma,A}))} \\ &\le \frac{1}{n^2} \sum_{\ell=1}^{\omega} \left([C_1 \log_2(b\,n)]^{2\ell} \frac{\sum_{|\mathfrak{u}|=\ell} \{\gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} [j \log_2(j+b)]^2\}}{\sum_{0 \le |\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} m_j} \right) \\ &\le \frac{\mathcal{M}}{n^2} \sum_{\ell=1}^{\omega} [C_1 \log_2(b\,n)]^{2\ell} \le \frac{B_\delta \mathcal{M}}{n^2} \sum_{\ell=1}^{\omega} \frac{[2\delta \log_e(b\,n)]^{2\ell}}{(2\ell)!} \\ &\le \frac{B_\delta \mathcal{M}}{n^2} \exp\,[2\delta \log_e(b\,n)] = C_\delta^2 \, n^{-2+2\delta}, \end{aligned}$$

where $C_\delta = \sqrt{B_\delta \mathcal{M}}\, b^\delta$. The case of the unanchored Sobolev space can be proven similarly. □

Results similar to those in Theorem 16.28 can be established for the Halton and Sobol sequences. Indeed, let P_n be the first n points of the d-dimensional Halton sequence based on the first d prime numbers, see Halton [94]. Then it is proved in Hickernell and Wang [126] that

$$D^*(P_n^{\mathfrak{u}}) \le \frac{1}{n} \prod_{j \in \mathfrak{u}} [C_{\text{Hal}} \, j \log_2(j+1) \log_2(e\,n)],$$

for any non-empty subset $\mathfrak{u} \subseteq [d]$, with C_{Hal} being independent of $\mathfrak{u}$ and d. For the Sobol sequence based on the first primitive polynomial, see Sobol [284], a similar bound is proved in Wang [321], namely

$$D^*(P_n^{\mathfrak{u}}) \le \frac{1}{n} \prod_{j \in \mathfrak{u}} [C_{\text{Sob}} \, j \log_2(j+1) \log_2 \log_2(j+3) \log_2(2n)]$$

with C_{Sob} independent of $\mathfrak{u}$ and d. These bounds are similar to the bound for the Niederreiter sequence. Therefore similar polynomial tractability and strong polynomial tractability results to those in Theorem 16.28 hold for the Halton and Sobol sequences.

16.10 Notes and Remarks

NR 16:1. In this chapter we study multivariate integration on Hilbert spaces. Banach spaces for multivariate integration have been also studied, see the papers by Sobol [285], Yue and Hickernell [360] and [276], [278].

There is a very important stream of research for spaces based on Walsh functions, see Dick and Pillichshammer [50]. Walsh spaces are Hilbert spaces with many interesting relations to Korobov spaces.

There is also the recent paper [47] for multivariate integration defined over weighted Korobov spaces with exponentially decaying Fourier coefficients. This allows us to obtain tractability results with polylog dependence on ε^{-1} and polynomial dependence on d, i.e., there are non-negative p and q such that

$$n(\varepsilon, d) = \mathcal{O}\big(\big(1 + \ln \varepsilon^{-1}\big)^{p}\, d^{q}\big) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

NR 16.1:1. The first part of this chapter (till Section 16.7) is based on [54]. Of course, the subsection on the cost of the CBC algorithm for product weights is based on Nuyens and Cools [226], [227], [228]. Sections 16.7 and 16.8 are based on [128] and [129], although only product weights are considered in these papers. Finally, Section 16.9 is based on [275].

NR 16.4 and 16.6:1. We have already mentioned in the text a number of papers where lattice rules are studied for large d. The list is by far not complete. More information on lattice rules can be found in Dick and Kuo [43], [44], [45], Dick, Kuo, Pillichshammer and Sloan [46], Dick and Pillichshammer [52], Hickernell and Niederreiter [121], Hickernell, Sloan and Wasilkowski [122], Kritzer and Pillichshammer [151], Kuo, Wasilkowski and Waterhouse [159], Lyness [180], Wang, Sloan and Dick [322], Waterhouse, Kuo and Sloan [337] as well as in [157], [158].

NR 16.4.2:1. T-tractability conditions presented in this section are new but quite straightforward. Formally, conditions on weak-tractability are also new but the approach is similar to the approach in [85].

NR 16.9:1. Corollary 16.19 is from [275]. For the anchor $\boldsymbol{a} = [1, 1, \dots, 1]$, Corollary 16.19 (A) reduces to a result in [129]. For product weights, Corollary 16.19 reduces to the results in [54]; furthermore, if the product weights $\{\gamma_{d,j}\}$ are independent of the dimension d, i.e., $\gamma_{d,j} = \gamma_j$, then Corollary 16.19 reduces to the results in [277] for the anchored space with anchor $\boldsymbol{a} = [1, 1, \dots, 1]$, and reduces to the results in [280] for the unanchored space.

Chapter 17
Randomized Setting

In this chapter we switch to the randomized setting for linear functionals defined on Hilbert spaces. We already know from Chapter 7 of Volume I that there is a close relationship between the worst case and randomized settings for the class Λ^{all}, and randomization does not really help. Here we study the class Λ^{std} of information consisting of function values and obtain very different results.

A significant part of this chapter is devoted to *multivariate integration* which is a continuous linear functional. This problem is probably one of the most important continuous problems with applications in many diverse areas including financial mathematics, physics, chemistry, statistics and numerical analysis.

In Section 17.1 we restrict ourselves to the (standard) Monte Carlo algorithm for multivariate integration. This is a classical algorithm due to Metropolis and Ulam [185], which is probably the first randomized algorithm for continuous problems. Today Monte Carlo is widely used in many areas of applied science. It is well known that the speed of convergence of Monte Carlo is independent of the number of variables. However, it is often overlooked that the randomized error of Monte Carlo may depend on d through the variance of a function. We stress that the variance can be, in fact, an arbitrary function of d. To claim good dependence on d we must know how the variances of functions behave. We will study Monte Carlo for a number of typical spaces with the emphasis on the dependence on d. Based on [281], we provide necessary and sufficient conditions on the randomized error of Monte Carlo to be independent of d, polynomially dependent d, or at least non-exponentially dependent on d. As we shall see, all results can happen depending on the class of functions and on the choice of the absolute or normalized error criterion. In particular, tractability conditions of Monte Carlo for the Sobolev spaces studied in this chapter are more lenient than tractability conditions in the worst case setting. However, for general reproducing kernel Hilbert spaces, the opposite may happen. In fact, there are spaces for which Monte Carlo depends exponentially on d, although multivariate integration is trivial even in the worst case setting.

Obviously, we are interested in the best possible algorithms in the randomized setting, and it is clear that Monte Carlo cannot be always a good choice especially if one considers classes of very smooth functions. Unfortunately, not much is known about tractability of multivariate integration in the randomized setting. We hope that this will be a major research subject in the future. This problem is quite technically demanding and we definitely need new proof techniques to attack this problem successfully.

At the end of Section 17.1, we present lower bounds on the minimal randomized errors for a space that is basically the L_2 space of functions equipped with a weighted norm. For this space Monte Carlo is nearly optimal. Hence, Monte Carlo tractability conditions are exactly the same as the tractability conditions for multivariate integration.

In Section 17.2, we present improvements of Monte Carlo based on *importance sampling*. To illustrate this approach we start with a simple example and then present results from Wasilkowski [328] and Plaskota, Wasilkowski and Zhou [248] with new, and more lenient, sufficient conditions to obtain tractability in the randomized setting. A space for which Monte Carlo can be improved is a periodic variant of the weighted Sobolev space for which Wasilkowski [328] proved that tractability conditions of Monte Carlo can be relaxed by designing a different randomized algorithm.

We also report a recent result of Hinrichs [131] on the power of importance sampling. From his result we know that multivariate integration is strongly polynomially tractable for the normalized error criterion when it is defined over a reproducing kernel Hilbert space with a point-wise non-negative kernel. We stress that this result holds even for unweighted spaces[1].

In Section 17.3, we present results of Muller [193] and Motoo [192] on the local solution of the Dirichlet problem for the Laplace equation. If we assume that the solution of this problem is Lipschitz then the complexity of the problem is at most of the order $d^2\varepsilon^{-2}\ln\varepsilon^{-1}$. Hence the problem is polynomially tractable in the randomized setting. We find it interesting that results on the spherical process that are over 50 years old yield polynomial tractability for the randomized setting. We stress that this problem is not polynomially tractable in the worst case setting. There are three open problems numbered from 78 to 80.

17.1 Monte Carlo for Multivariate Integration

This is the first section where we study the randomized setting for the class Λ^{std}. It seems appropriate to begin with the widely used and the most famous randomized algorithm for continuous problems. This is of course the celebrated *Monte Carlo* algorithm for approximation of multivariate integrals introduced in the 1940s by Metropolis and Ulam, see [185]. We already discussed the Monte Carlo algorithm in Example 11 of Chapter 3 in Volume I for multivariate integration over the d-dimensional unit cube, and in this section we consider more general domains.

We now consider a general multivariate integration following the approach of [281]. Let D_d be a (Lebesgue) measurable subset of $\mathbb{R}^d$, and let $\varrho_d : D_d \to \mathbb{R}_+$ be a weight function, $\int_{D_d} \varrho_d(x)\,\mathrm{d}x = 1$. By $L_{2,\varrho_d}(D_d)$ we mean the Hilbert space of (Lebesgue) measurable functions $f : D_d \to \mathbb{R}$ such that

$$\|f\|_{2,\varrho_d} = \left(\int_{D_d} \varrho_d(x)\, f^2(x)\,\mathrm{d}x\right)^{1/2} < \infty.$$

Consider *multivariate integration*

$$\mathrm{INT}_d(f) = \int_{D_d} \varrho_d(x)\, f(x)\,\mathrm{d}x \quad \text{for all } f \in L_{2,\varrho_d}(D_d).$$

[1] Added in proof: It turns out that for some spaces the result of Hinrichs is optimal as recently proved in [225].

Note that $\mathrm{INT}_d(f)$ is well defined over the space $L_{2,\varrho_d}(D_d)$.

For example, we can take $D_d = [0,1]^d$ and $\varrho_d(x) = 1$. In this case we obtain *uniform integration*. On the other hand, if we take $D_d = \mathbb{R}^d$ and

$$\varrho_d(x) = \prod_{j=1}^{d} \frac{\exp(-x_j^2/(2\sigma_{d,j}))}{(2\pi\sigma_{d,j})^{1/2}},$$

as a density of a Gaussian measure, then we obtain *Gaussian integration*. Here, the non-negative $\sigma_{d,j}$ is the variance of the jth variable.

The Monte Carlo algorithm is of the form

$$\mathrm{MC}_{x_1,x_2,\dots,x_n}(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_k),$$

where the sample points $x_1, x_2, \dots, x_n$ are independent random variables that are distributed over D_d with density ϱ_d. Taking the expectation $\mathbb{E}$ with respect to the sample points x_k we obtain the well known and famous formula

$$\left[\,\mathbb{E}\left(\,|\,\mathrm{INT}_d(f) - \mathrm{MC}_{x_1,\dots,x_n}(f)|^2\,\right)\right]^{1/2} = \frac{\sqrt{\mathrm{var}_d(f)}}{n^{1/2}}, \tag{17.1}$$

where

$$\mathrm{var}_d(f) = \mathrm{INT}_d(f^2) - \mathrm{INT}_d^2(f) = \mathrm{INT}_d\left((f - \mathrm{INT}_d(f))^2\right)$$

is the variance of the function f. The derivation of this formula is easy and can be done exactly as it was done in Example 11 of Chapter 3 of Volume I.

Hence, the randomized error of Monte Carlo for a function f goes to zero as $n^{-1/2}$ independently of d which is the most important property of this algorithm. The randomized error also depends on d through the variance of a function. To address the dependence on d, let us assume that H_d is a normed linear space which is a subset of $L_{2,\varrho_d}(D_d)$, so that $\mathrm{INT}_d(f)$ is still well defined for all functions $f \in H_d$. The randomized error of Monte Carlo is

$$e^{\mathrm{mc}}(n, H_d) = \left[\sup_{f\in H_d,\, \|f\|_{H_d}\le 1} \mathbb{E}\left(\left|I_d(f) - \mathrm{MC}_{x_1,\dots,x_n}(f)\right|^2\right)\right]^{1/2}.$$

From (17.1) we have

$$e^{\mathrm{mc}}(n, H_d) = \left(\frac{\mathrm{var}(H_d)}{n}\right)^{1/2},$$

where

$$\mathrm{var}(H_d) = \sup_{f\in H_d, \|f\|_{H_d}\le 1} \mathrm{var}_d(f)$$

is the largest variance of a function from the unit ball of H_d. Hence, the randomized error of Monte Carlo is the square root of the largest variance divided by the square root of the number n of randomized samples. To guarantee that $\mathrm{var}(H_d)$ is finite we assume that H_d is continuously embedded in $L_{2,\varrho_d}(D_d)$, i.e., there exists a number C_d such that $\|f\|_{2,\varrho_d} \le C_d \|f\|_{H_d}$ for all $f \in H_d$. Then

$$\mathrm{var}(H_d) \le \sup_{f \in H_d, \|f\|_{H_d} \le 1} \|f\|_{2,\varrho_d} \le C_d.$$

Under the assumption that H_d is continuously embedded in $L_{2,\varrho_d}(D_d)$, the linear functional INT_d becomes continuous over H_d, and $\|\mathrm{INT}_d\| \le C_d$.

For $n = 0$, we formally set $\mathrm{MC}(f) = 0$ and then $e^{\mathrm{mc}}(0, H_d) = \|I_d\|$. This means that $\|I_d\|$ is the initial error that can be achieved without sampling the function. If $\|I_d\| = 0$, which can happen for some spaces H_d, then multivariate integration is trivial, since then $I_d(f) = 0$ for all $f \in H_d$. Therefore from now on we assume that $\|I_d\| > 0$.

As always, we consider the absolute error criterion, for which $\mathrm{CRI}_d = 1$, and the normalized error criterion, for which $\mathrm{CRI}_d = \|\mathrm{INT}_d\|$. Let $n^{\mathrm{mc}}(\varepsilon, H_d)$ denote the minimal number of function values used by the Monte Carlo algorithm which is needed to guarantee that its randomized error is at most $\varepsilon\,\mathrm{CRI}_d$ for $\varepsilon \in (0,1)$. That is, the minimal n for which

$$e^{\mathrm{mc}}(n, H_d) \le \varepsilon\,\mathrm{CRI}_d.$$

We say that Monte Carlo is a *polynomially tractable* (*PT*) algorithm if $n^{\mathrm{mc}}(\varepsilon, H_d)$ is bounded by a polynomial in d and ε^{-1} for all d and all $\varepsilon \in (0,1)$. That is, if there exist non-negative numbers C, q and p such that

$$n^{\mathrm{mc}}(\varepsilon, H_d) \le C\, d^q\, \varepsilon^{-p} \quad \text{for all } d = 1, 2, \dots \text{ and for all } \varepsilon \in (0,1).$$

If $q = 0$ in the inequality above then we say that Monte Carlo is a *strongly polynomially tractable* (*SPT*) algorithm. In this case, the bound on the minimal number of function values is *independent* of d and polynomially dependent on ε^{-1}.

We say that Monte Carlo is a *weakly tractable* (*WT*) algorithm if

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{mc}}(\varepsilon, H_d)}{\varepsilon^{-1} + d} = 0,$$

meaning that $n^{\mathrm{mc}}(\varepsilon, H_d)$ does not depend exponentially on ε^{-1} and d.

From the form of the randomized error of Monte Carlo given above, we conclude that the Monte Carlo algorithm is strongly polynomially tractable iff

$$C := \sup_{d=1,2,\dots} \frac{\mathrm{var}(H_d)}{\mathrm{CRI}_d^2} < \infty,$$

is polynomially tractable iff there exist non-negative C and q that such

$$\frac{\mathrm{var}(H_d)}{\mathrm{CRI}_d^2} \le C\, d^q \quad \text{for all } d = 1, 2, \dots,$$

and is weakly tractable iff

$$\lim_{d\to\infty} \frac{\ln \mathrm{var}(H_d)/\mathrm{CRI}_d^2}{d} = 0.$$

If so, then we have respectively

$$\begin{aligned} n^{\mathrm{mc}}(\varepsilon, H_d) &\le \lceil C\,\varepsilon^{-2}\rceil && \text{and Monte Carlo is SPT,}\\ n^{\mathrm{mc}}(\varepsilon, H_d) &\le \lceil C\,d^q\,\varepsilon^{-2}\rceil && \text{and Monte Carlo is PT,}\\ n^{\mathrm{mc}}(\varepsilon, H_d) &\le \exp\left(o(d)\right)\,\varepsilon^{-2} && \text{and Monte Carlo is WT.}\end{aligned}$$

Hence, $\mathrm{var}(H_d)/\mathrm{CRI}_d^2$ determines whether Monte Carlo is a strongly polynomially, polynomially or weakly tractable algorithm. In general, the dependence of $n^{\mathrm{mc}}(\varepsilon, H_d)$ is proportional to ε^{-2}. This is the case when $\mathrm{var}(H_d) > 0$.

As in [281], we now show that $\mathrm{var}(H_d)$ can be expressed as the largest eigenvalue of a certain symmetric semi-positive operator. This will be done by assuming that H_d is a Hilbert space which is not necessarily a reproducing kernel Hilbert space but continuously embedded in $L_{2,\varrho_d}(D_d)$. Its inner product will be denoted by $\langle\cdot,\cdot\rangle_{H_d}$. Since INT_d is a continuous linear functional, by Riesz's theorem there exists a $h_d \in H_d$ such that

$$\mathrm{INT}_d(f) = \langle f, h_d\rangle_{H_d} \quad \text{for all } f \in H_d.$$

Consider the embedding operator $\mathrm{Im}\colon H_d \to L_{2,\varrho_d}(D_d)$ given by $\mathrm{Im}\, f = f$. The operator Im is again continuous and $\|\mathrm{Im}\, f\|_{2,\varrho_d} \le C_d \|f\|_{H_d}$. This is the same operator which is studied for multivariate approximation, as will be done in Volume III.

Let $\mathrm{Im}^*\colon L_{2,\varrho_d}(D_d) \to H_d$ denote the adjoint of Im, i.e.,

$$\langle \mathrm{Im}^* f, g\rangle_{H_d} = \langle f, \mathrm{Im}\, g\rangle_{2,\varrho_d} \quad \text{for all } f \in L_{2,\varrho_d}(D_d) \text{ and } g \in H_d.$$

Then

$$W_d = \mathrm{Im}^*\mathrm{Im}\colon H_d \to H_d$$

is a symmetric and semi-positive operator.

If H_d is a reproducing kernel Hilbert space, $H_d = H(K_d)$, we know that it is enough to assume

$$\int_{D_d} \varrho_d(x)\, K_d(x,x)\,\mathrm{d}x < \infty$$

which guarantees that $H(K_d)$ is continuously embedded in $L_{2,\varrho_d}(D_d)$. Then

$$h_d(t) = \int_{D_d} \varrho_d(x)\, K_d(t,x)\,\mathrm{d}x$$

and the operator W_d takes the form of an integral operator,

$$W_d f(t) = \int_{D_d} \varrho_d(x)\, K_d(t,x)\, f(x)\,\mathrm{d}x.$$

For a general Hilbert space H_d, we now consider a rank-one modification of the operator W_d,

$$V_d f = W_d f - \mathrm{INT}_d(f)\, h_d.$$

The operator $V_d : H_d \to H_d$ is symmetric and semi-positive since

$$\langle V_d f, g\rangle_{H_d} = \mathrm{INT}_d(fg) - \mathrm{INT}_d(f)\,\mathrm{INT}_d(g)$$

and $\langle V_d f, f\rangle_{H_d} = \mathrm{INT}_d(f^2) - \mathrm{INT}_d^2(f) = \mathrm{var}_d(f) \ge 0$. This also proves that

$$\mathrm{var}(H_d) = \sup_{\substack{f\in H_d\\ \|f\|_{H_d}\le 1}} \mathrm{var}_d(f) = \sup_{\substack{f\in H_d\\ \|f\|_{H_d}\le 1}} \langle V_d f, f\rangle_{H_d} = \lambda_1(V_d),$$

where $\lambda_j(M)$ denotes the jth largest eigenvalue of a linear operator M.

The largest eigenvalue $\lambda_1(V_d)$ is related to the two largest eigenvalues of the operator W_d. Indeed, since V_d differs from W_d by a rank one operator, Weyl's monotonicity theorem says

$$\lambda_2(W_d) \le \lambda_1(V_d) \le \lambda_1(W_d). \tag{17.2}$$

This means that we can analyze the randomized error of Monte Carlo by studying the two largest eigenvalues of W_d.

If $H_d = H(K_d)$ is a reproducing kernel Hilbert space then

$$\begin{aligned}\langle W_d f, f\rangle_{H_d} = \mathrm{INT}_d(f^2) &= \int_{D_d} \varrho_d(x)\, \langle K_d(\cdot,x), f\rangle_d^2\,\mathrm{d}x\\ &\le \|f\|_{H_d}^2 \int_{D_d} \varrho_d(x)\, K_d(x,x)\,\mathrm{d}x,\end{aligned}$$

which implies that

$$\lambda_1(W_d) \le \int_{D_d} \varrho_d(x)\, K_d(x,x)\,\mathrm{d}x.$$

Hence, it is enough to have a polynomial bound in d on

$$\int_{D_d} \varrho_d(x)\, K_d(x,x)\,\mathrm{d}x / \mathrm{CRI}_d^2$$

to guarantee that Monte Carlo is polynomially tractable.

For a general Hilbert space H_d, an important special case is when h_d is an eigenfunction of W_d, that is, when $W_d h_d = \lambda^*\, h_d$ for some λ^*. Then

$$\mathrm{INT}_d(h_d^2) = \langle W_d h_d, h_d\rangle_{H_d} = \lambda^*\, \langle h_d, h_d\rangle_{H_d} = \lambda^*\,\mathrm{INT}_d(h_d).$$

Hence, $\lambda^* = \mathrm{INT}_d(h_d^2)/\mathrm{INT}_d(h_d) \ge \mathrm{INT}_d(h_d)$. The function h_d is also an eigenfunction of V_d, with the eigenvalue $\lambda^* - \mathrm{INT}_d(h_d)$. Observe that the rest of the eigenpairs of W_d are also eigenpairs of V_d. Indeed, if f is an eigenfunction of W_d different from h_d then, by the self-adjointness of W_d, f is orthogonal to h_d, that is

$\langle f, h_d \rangle_{H_d} = 0$, implying $\mathrm{INT}_d(f) = 0$, and $V_d f = W_d f$. Hence, if h_d is an eigenfunction of W_d then we have

$$\lambda_1(V_d) = \begin{cases} \lambda_1(W_d) & \text{if } \lambda_1(W_d) \neq \lambda^*, \\ \max(\lambda_1(W_d) - \mathrm{INT}_d(h_d), \lambda_2(W_d)) & \text{if } \lambda_1(W_d) = \lambda^*. \end{cases} \tag{17.3}$$

We summarize this analysis in the following theorem.

Theorem 17.1.

- *The largest variance in the unit ball of a Hilbert space H_d, $\mathrm{var}(H_d)$, is equal to the largest eigenvalue $\lambda_1(V_d)$ of the operator V_d. Monte Carlo is strongly polynomially, polynomially or weakly tractable iff the ratio*

$$\frac{\mathrm{var}(H_d)}{\mathrm{CRI}_d^2} = \frac{\lambda_1(V_d)}{\mathrm{CRI}_d^2}$$

is bounded uniformly in d or polynomially in d or is of order $\exp(o(d))$.

- *Let $H_d = H(K_d)$ be a reproducing kernel Hilbert space.*

Consider the absolute error criterion.

Monte Carlo is strongly polynomially, polynomially or weakly tractable if

$$\int_{D_d} \varrho_d(x)\, K_d(x,x)\, \mathrm{d}x$$

is uniformly bounded in d, polynomial in d or is of order $\exp(o(d))$.

Consider the normalized error criterion.

Monte Carlo is strongly polynomially, polynomially or weakly tractable if

$$\frac{\int_{D_d} \varrho_d(x)\, K_d(x,x)\, \mathrm{d}x}{\int_{D_d^2} \varrho_d(x)\varrho_d(t)\, K_d(x,t)\, \mathrm{d}t\, \mathrm{d}x}$$

is uniformly bounded in d, polynomial in d or is of order $\exp(o(d))$.

We add in passing that the conditions in the second part of Theorem 17.1 also guarantee that some quasi Monte Carlo algorithms are strongly polynomially, polynomially or weakly tractable in the worst case setting, as we have seen in Section 10.7.1.

We now consider the case when the space H_d is given as a tensor product of the d copies of a Hilbert space H_1 of univariate functions defined on $D_1 \subseteq \mathbb{R}$. In this case, we have $D_d = D_1^d$. To preserve the tensor product structure of multivariate integration we also assume that $\varrho_d(x) = \prod_{j=1}^d \varrho_1(x_j)$, where ϱ_1 is a weight function for univariate integration for the space H_1. Clearly, $\|\mathrm{INT}_d\| = \|\mathrm{INT}_1\|^d$, where INT_1 is univariate integration for the space H_1. That is why $\mathrm{CRI}_d = \mathrm{CRI}_1^d$.

Let $\lambda_j = \lambda_j(W_1)$ for $j = 1, 2$ denote the two largest eigenvalues of W_1. Assume that $\lambda_2 > 0$, i.e., the operator W_1 has rank at least two. Clearly, the largest eigenvalue is $\lambda_1(W_d) = \lambda_1^d$, and the second largest eigenvalue $\lambda_2(W_d) = \lambda_1^{d-1}\lambda_2$. From (17.2) we therefore obtain

$$\frac{\lambda_2}{\lambda_1}\left(\frac{\lambda_1}{\mathrm{CRI}_1^2}\right)^d \le \frac{\mathrm{var}_d(H_d)}{\mathrm{CRI}_d^2} = \frac{\lambda_1(V_d)}{\mathrm{CRI}_d^2} \le \left(\frac{\lambda_1}{\mathrm{CRI}_1^2}\right)^d. \tag{17.4}$$

From these inequalities we immediately obtain the following theorem.

Theorem 17.2. *Consider multivariate integration for tensor product Hilbert spaces H_d continuously embedded in $L_{2,\varrho_d}(D_d)$ and with the tensor product weight ϱ_d. Assume that the second largest eigenvalue of the operator W_1 is positive.*

Then the notions of Monte Carlo being strongly polynomially, polynomially or weakly tractable are equivalent, and Monte Carlo is strongly polynomially tractable iff

$$\lambda_1(W_1) \le \mathrm{CRI}_d^2.$$

The essence of Theorem 17.2 is that in the tensor product case it is enough to analyze the univariate case to conclude whether Monte Carlo is strongly polynomially tractable. As we shall see, the inequality $\lambda_1(W_1) \le \mathrm{CRI}_d^2$ guaranteeing that Monte Carlo is strongly polynomially tractable may or may not hold, depending on the space H_1.

Indeed, take for an example, $H_d = L_{2,\varrho_d}(D_d)$. Then $\lambda_1(W_1) = \|\mathrm{INT}_1\| = 1$. Hence, the absolute and normalized error criteria coincide and Monte Carlo is strongly polynomially tractable. In the next sections, we provide many examples of Hilbert spaces for which $\lambda_1(W_1) > \|\mathrm{INT}_1\| = 1$, and Monte Carlo is not even weakly tractable.

17.1.1 Uniform Integration

In this section we consider uniform integration. That is, we take $D_d = [0,1]^d$ and $\varrho_d(x) = 1$. We analyze a number of weighted Sobolev spaces H_d that often occur in computational practice, and check whether Monte Carlo is strongly polynomially, polynomially or weakly tractable by using Theorem 17.1 or Theorem 17.2. We also compare tractability conditions for Monte Carlo with tractability conditions in the worst case setting. The weighted Sobolev spaces considered in this section were discussed in the context of quasi Monte Carlo algorithms in [280] and are presented in Appendix A of Volume I.

17.1.1.1 The first Sobolev space. Consider $H_d = H_{d,\gamma}$ as the first weighted Sobolev space given in Section A.2.1 of Volume I. This is the Hilbert space with the inner product

$$\langle f, g\rangle_{H_{d,\gamma}} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^d} \frac{\partial^{|\mathfrak{u}|} f}{\partial x_{\mathfrak{u}}}(x)\frac{\partial^{|\mathfrak{u}|} g}{\partial x_{\mathfrak{u}}}(x)\,\mathrm{d}x \tag{17.5}$$

for an arbitrary sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ of weights with $\gamma_{d,\emptyset} = 1$.

For multivariate uniform integration we have

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x = \langle f, 1\rangle_{H_{d,\gamma}}.$$

Since $\|1\|_{H_{d,\gamma}} = 1$ we have $\|\mathrm{INT}_d\| = 1$ independently of the weights $\gamma_{d,\mathfrak{u}}$. Hence, the absolute and normalized error criteria coincide.

Let us now consider Monte Carlo for this weighted Sobolev space $H_{d,\gamma}$. Clearly, $\mathrm{INT}_d(f^2) \le \|f\|^2_{H_{d,\gamma}}$ for any $\gamma_{d,\mathfrak{u}}$. Thus, the variance of f from the unit ball of $H_{d,\gamma}$ is at most 1. Since $\|\mathrm{INT}_d\|$ is also 1, this means that Monte Carlo is strongly polynomially tractable for any weights γ with $\gamma_{d,\emptyset} = 1$.

The same result holds if we consider the periodic variant of this weighted Sobolev space $H_{d,\gamma}$. That is, if we assume that $f \in H_{d,\gamma}$ is periodic with respect to all variables with period 1. Uniform integration is again given by $\mathrm{INT}_d(f) = \langle f, 1\rangle_{H_{d,\gamma}}$ since the function 1 is periodic. Therefore the norm of INT_d is 1, and the variances of periodic functions from the unit ball of $H_{d,\gamma}$ are bounded by 1. Hence, Monte Carlo is again strongly polynomially tractable independently of γ with $\gamma_{d,\emptyset} = 1$.

We summarize the results of this subsection in the following corollary.

Corollary 17.3. *Let $H_{d,\gamma}$ be the Sobolev space of non-periodic functions or periodic functions with the inner product (17.5) and with arbitrary weights $\gamma_{d,\mathfrak{u}}$. Then Monte Carlo is strongly polynomially tractable independently of the weights $\gamma_{d,\mathfrak{u}}$, and*

$$n^{\mathrm{mc}}(\varepsilon, H_d) \le \lceil \varepsilon^{-2} \rceil \quad \textit{for all } d = 1, 2, \dots, \textit{ and } \varepsilon \in (0,1).$$

We add in passing that in the worst case setting for the product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$, we must assume that

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$$

to have strong polynomial tractability, and

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(d+1)} < \infty$$

to have polynomial tractability, and

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0,$$

to have weak tractability, see [54], [355], [84], respectively, and Section 12.4 of Chapter 12. In particular, for the standard unweighted Sobolev space, $\gamma_{d,j} = 1$, we have exponential dependence on d in the worst case setting whereas Monte Carlo is strongly polynomially tractable. Hence, the switch to the randomized setting *breaks* the curse of dimensionality of multivariate integration for this space.

17.1.1.2 The second Sobolev space. Consider $H_d = H_{d,\gamma}$ as the second weighted Sobolev space anchored at a given in Section A.2.2 of Volume I. This is the Hilbert space of non-periodic functions with the reproducing kernel

$$K_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \big[|x_j - a_j| + |y_j - a_j| - |x_j - y_j|\big] \tag{17.6}$$

for an arbitrary sequence $\gamma = \{\gamma_{d,\mathfrak{u}}\}$ of weights. For the periodic case, the space $H_{d,\gamma}$ has the reproducing kernel given by

$$\tilde{K}_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \big[|x_j - a_j| + |y_j - a_j| - |x_j - y_j| - 2(x_j - a_j)(y_j - a_j)\big]. \tag{17.7}$$

Multivariate uniform integration for the non-periodic or periodic cases takes the form

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x = \langle f, h_d\rangle_{H_{d,\gamma}}$$

with

$$h_d(x) = \sum_{\mathfrak{u}\in[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \big[|x_j - a_j| + a_j^2 - a_j - x_j^2 + x_j - 1_{\mathrm{per}}(x_j - a_j)(1 - 2a_j)\big],$$

where $1_{\mathrm{per}} = 0$ for the non-periodic case, and $1_{\mathrm{per}} = 1$ for the periodic case. The norm of INT_d is

$$\|\mathrm{INT}_d\|^2 = \int_{[0,1]^d} h_d(x)\,\mathrm{d}x = \sum_{\mathfrak{u}\in[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \big[1 - 2a_j + 2a_j^2 - \tfrac{1}{3} - \tfrac{1}{2}\, 1_{\mathrm{per}}\, (1 - 2a_j)^2\big].$$

Consider first the absolute error criterion. Define

$$f_\gamma^{\text{abs, non-per}}(d) := \int_{[0,1]^d} K_{d,\gamma}(x,x)\,\mathrm{d}x = \sum_{\mathfrak{u}\in[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \big[1 - 2a_j + 2a_j^2\big],$$

$$f_\gamma^{\text{abs, per}}(d) := \int_{[0,1]^d} \tilde{K}_{d,\gamma}(x,x)\,\mathrm{d}x = \sum_{\mathfrak{u}\in[d]} \frac{\gamma_{d,\mathfrak{u}}}{6^{|\mathfrak{u}|}}.$$

Then we need to guarantee that $f_\gamma^{\text{abs}, y}(d)$ is polynomial or not exponential in d, where $y \in \{\text{non-per}, \text{per}\}$.

Consider now the normalized error criterion. Define

$$f_\gamma^{\text{nor, non-per}}(d) := \frac{f_\gamma^{\text{abs, non-per}}(d)}{\|\mathrm{INT}_d\|^2} = \frac{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}}[1 - 2a_j + 2a_j^2]}{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}}[1 - 2a_j + 2a_j^2 - \frac{1}{3}]},$$

$$f_\gamma^{\text{nor, per}}(d) := \frac{f_\gamma^{\text{abs, per}}(d)}{\|\mathrm{INT}_d\|^2} = \frac{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{6^{|\mathfrak{u}|}}}{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{(12)^{|\mathfrak{u}|}}}.$$

Now we need to guarantee that $f_\gamma^{\text{nor},\,y}(d)$ is polynomial or not exponential in d, where $y \in \{\text{non-per},\ \text{per}\}$.

From this and the second point of Theorem 17.1 it is easy to obtain the following corollary.

Corollary 17.4. *Let $H_{d,\gamma}$ be the weighted Sobolev space anchored at a of non-periodic or periodic functions with the reproducing kernel kernel (17.6) or (17.7), and with arbitrary weights $\gamma_{d,\mathfrak{u}}$. Consider the absolute or normalized error criterion. Let $x \in \{abs,\ nor\}$, and $y \in \{non\text{-}per,\ per\}$. Then*

- *Monte Carlo is strongly polynomially tractable if*

$$C = \sup_d f_\gamma^{x,y}(d) < \infty,$$

and then

$$n^{\text{mc}}(\varepsilon, H_d) \le \lceil C\, \varepsilon^{-2} \rceil \quad \textit{for all } d = 1, 2, \dots, \textit{ and } \varepsilon \in (0,1).$$

- *Monte Carlo is polynomially tractable if there is a non-negative q such that*

$$C = \sup_d d^{-q} f_\gamma^{x,y}(d) < \infty,$$

and then

$$n^{\text{mc}}(\varepsilon, H_d) \le \lceil C\, d^q\, \varepsilon^{-2} \rceil \quad \textit{for all } d = 1, 2, \dots, \textit{ and } \varepsilon \in (0,1).$$

- *Monte Carlo is weakly tractable if*

$$\lim_{d\to\infty} \frac{\ln f_\gamma^{x,y}(d)}{d} = 0,$$

and then

$$n^{\text{mc}}(\varepsilon, H_d) \le \exp(o(d))\, \varepsilon^{-2} \quad \textit{for all } d = 1, 2, \dots, \textit{ and } \varepsilon \in (0,1).$$

Note that Corollary 17.4 covers four cases indexed by x and y. For example, if we consider the periodic variant of $H_{d,\gamma}$ for the normalized error criterion then we need to study the function $f_\gamma^{\text{per}\,,\,\text{nor}}$.

Assume now finite-order weights, $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$. For the absolute error criterion, we need to assume that the weights $\gamma_{d,\mathfrak{u}}$ are uniformly bounded, say, $\gamma_{d,\mathfrak{u}} \le M$ for all d and all $\mathfrak{u}$. Then for $y \in \{\text{non-per, per}\}$ we have

$$f_\gamma^{\text{abs},\,y}(d) \le M \sum_{k=0}^{\omega} \binom{d}{k} \frac{1}{(2 + 4 \cdot 1_{\text{per}})^k} = \mathcal{O}(M\, d^{\omega}).$$

For the normalized error criterion we do not need to assume anything more about finite-order weights. For the periodic case, we obtain

$$f_\gamma^{\text{nor, per}}(d) = \frac{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{12^{|\mathfrak{u}|}}\, 2^{|\mathfrak{u}|}}{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{(12)^{|\mathfrak{u}|}}} \le 2^{\omega}.$$

For the non-periodic case, we obtain

$$f_\gamma^{\text{nor, non-per}}(d) = \frac{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}}[1-2a_j+2a_j^2-\frac{1}{3}] \prod_{j\in\mathfrak{u}} \frac{1-2a_j+2a_j^2}{1-2a_j+2a_j^2-\frac{1}{3}}}{\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}}[1-2a_j+2a_j^2-\frac{1}{3}]}.$$

Note that for any $a \in [0,1]$ we have

$$\frac{1-2a+2a^2}{1-2a+2a^2-\frac{1}{3}} \le 3,$$

and the last estimate is sharp for $a = \frac{1}{2}$. Therefore

$$f_\gamma^{\text{nor ,non}-\text{per}}(d) \le 3^{\omega}.$$

This proves the following corollary.

Corollary 17.5. *Let $H_{d,\gamma}$ be the weighted Sobolev space anchored at a of non-periodic or periodic functions with the reproducing kernel kernel (17.6) or (17.7), and with finite-order weights $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$. Consider the absolute or normalized error criterion, and for the absolute error criterion assume additionally that $\gamma_{d,\mathfrak{u}} \le M$ for all d and $\mathfrak{u} \subseteq [d]$.*

- *For the absolute error criterion, Monte Carlo is polynomially tractable and*

  $$n^{\text{mc}}(\varepsilon, H_d) = \mathcal{O}\big(M\, d^{\,\omega}\, \varepsilon^{-2}\big) \quad \text{for all } d = 1,2,\dots, \text{ and } \varepsilon \in (0,1),$$

  with the factor in the $\mathcal{O}$ notation independent of ε^{-1}, d and M.

- *For the normalized error criterion, Monte Carlo is strongly polynomially tractable and*

  $$n^{\text{mc}}(\varepsilon, H_d) \le \big\lceil (3-1_{\text{per}})^{\omega}\, \varepsilon^{-2} \big\rceil \quad \text{for all } d = 1,2,\dots, \text{ and } \varepsilon \in (0,1).$$

Corollaries 17.4 and 17.5 supply sufficient conditions on weights to obtain tractability for Monte Carlo. In particular, these conditions do *not* hold for the unweighted case, $\gamma_{d,\mathfrak{u}} = 1$. Therefore, it is not clear if the unweighted case leads to intractability of Monte Carlo.

For simplicity, we now take $a = 0$ and consider product weights. Product weights for the normalized error criterion were studied in [280], and necessary as well as sufficient conditions for Monte Carlo to be strongly polynomially or polynomially tractable were found. We extend the analysis of [280] for the absolute error criterion and weak tractability. We prove the following theorem.

Theorem 17.6. *Let $H_{d,\gamma}$ be the weighted Sobolev space anchored at* 0 *of non-periodic or periodic functions with the reproducing kernel kernel* (17.6) *or* (17.7)*, and with product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for which $\gamma_{d,j+1} \le \gamma_{d,j}$.*

Then tractability conditions for Monte Carlo are the same for the non-periodic and periodic cases.

- *Consider the absolute error criterion.*
  - *Monte Carlo is strongly polynomially tractable iff*
  $$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$
  - *Monte Carlo is polynomially tractable iff*
  $$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty.$$
  - *Monte Carlo is weakly tractable iff*
  $$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0.$$
- *Consider the normalized error criterion.*
  - *Monte Carlo is strongly polynomially tractable iff*
  $$\limsup_{d\to\infty} \sum_{j=1}^{d} \min(\gamma_{d,j}^2, 1) < \infty$$
  - *Monte Carlo is polynomially tractable iff*
  $$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \min(\gamma_{d,j}^2, 1)}{\ln d} < \infty.$$
  - *Monte Carlo is weakly tractable iff*
  $$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \min(\gamma_{d,j}^2, 1)}{d} = 0.$$

Proof. We want to apply Theorem 17.1 and (17.4). That is why we need to derive estimates on the two largest eigenvalues of the operator W_d for the space $H_{d,\gamma}$. The space $H_{d,\gamma} = H_{1,\gamma_{d,1}} \otimes H_{1,\gamma_{d,2}} \otimes \cdots \otimes H_{1,\gamma_{d,d}}$ is now a tensor product space. For the non-periodic case, we now have

$$h_d(x) = \prod_{j=1}^{d} \left(1 + \gamma_{d,j}(x_j - \tfrac{1}{2}x_j^2)\right), \quad \|I_d\| = \|h_d\|_{H_{d,\gamma}} = \prod_{j=1}^{d} \left(1 + \tfrac{1}{3}\gamma_{d,j}\right)^{1/2}.$$

For the periodic case, we have

$$h_d(x) = \prod_{j=1}^{d} \left(1 + \tfrac{1}{2}\gamma_{d,j}(x_j - x_j^2)\right), \quad \|I_d\| = \|h_d\|_{H_{d,\gamma}} = \prod_{j=1}^{d} (1 + \tfrac{1}{12}\gamma_{d,j})^{1/2}.$$

Note that $\|I_d\|$ is uniformly bounded in d for both cases iff $\sup_d \sum_{j=1}^{d} \gamma_{d,j} < \infty$.

As usual for tensor product spaces, the kernel of W_d has a factorized form. Thus the eigenvalues of W_d are the products of the eigenvalues of the univariate operators

$$W_{\gamma_j} f(t) = \int_0^1 K_{\gamma_j}(t,x)\, f(x)\, \mathrm{d}x,$$

with the reproducing kernel $K_\gamma(t,x) = 1 + \gamma \min(t,x)$ for the non-periodic case, and $K_\gamma(t,x) = 1 + \gamma\,(\min(t,x) - tx)$ for the periodic case.

We first consider the non-periodic case. Then $W_\gamma f = \lambda f$ simplifies to

$$\int_0^1 f(x)\, \mathrm{d}x + \gamma \int_0^1 \min(t,x) f(x)\, \mathrm{d}x = \lambda\, f(t) \quad \text{for all } t \in [0,1].$$

The eigenpairs of W_γ were found in [332], and are easily seen as the solution of the eigenvalue problem of the differential equation

$$-\gamma\, f(x) = \lambda\, f''(x) \quad \text{for all } x \in (0,1),$$

with the boundary conditions

$$\int_0^1 f(x)\, \mathrm{d}x = \lambda\, f(0) \quad \text{and} \quad f'(1) = 0. \tag{17.8}$$

The eigenpairs are $\lambda_i = \gamma/\alpha_i^2$ and $f_i(x) = \cos(\alpha_i(1-x))$, where α_i is the unique solution of

$$x \tan x = \gamma, \quad x \in ((i-1)\pi, i\pi),\ i = 1, 2, \dots .$$

In what follows, by $\|f\|_1$ we mean $\|f\|_{H_{1,\gamma}}$ for $f \in H_{1,\gamma}$. The largest eigenvalue $\lambda_1 = \lambda_{1,\gamma}$ satisfies

$$1 + \tfrac{1}{3}\gamma \le \lambda_1 \le 1 + \tfrac{1}{2}\gamma. \tag{17.9}$$

Indeed, since $h_1(x) = 1 + \gamma(x - \frac{1}{2}x^2)$ we have $\|h_1\|_1^2 = I_1(h_1) = 1 + \frac{1}{3}\gamma$, and

$$\langle W_\gamma h_1, h_1 \rangle_1 = \int_0^1 h_1^2(x)\,\mathrm{d}x = 1 + \tfrac{2}{3}\,\gamma + \tfrac{2}{15}\,\gamma^2.$$

Hence,

$$\lambda_1 \geq \frac{\langle W_\gamma h_1, h_1 \rangle_1}{\|h_1\|_1^2} = \frac{1 + \frac{2}{3}\gamma + \frac{2}{15}\gamma^2}{1 + \frac{1}{3}\gamma} > 1 + \tfrac{1}{3}\,\gamma.$$

On the other hand, $\int_0^1 K_\gamma(x,x)\,\mathrm{d}x = 1 + \frac{1}{2}\gamma$ implies that $\lambda_1 \leq 1 + \frac{1}{2}\gamma$, as claimed.

We note in passing that if all $\gamma_{d,j} = \gamma$ then because $\lambda_1 > 1 + \frac{1}{3}\gamma = \|I_1\|^2$ and λ_2 is positive, Theorem 17.2 implies that Monte Carlo is *not* weakly tractable.

We now check what happens if we have non-constant weights $\gamma_{d,j}$. The largest eigenvalue $\lambda_{1,\gamma}$ of W_γ is a smooth function of γ. Define

$$u(\gamma) = \frac{\lambda_{1,\gamma}}{1 + \frac{1}{3}\gamma}.$$

Then u is continuous, and from (17.9) we have $u(\gamma) \in [1, 1.5]$ and $u(\gamma) > 1$ for positive γ. Moreover, by direct calculation we easily find that for all $\gamma \geq 0$,

$$u(\gamma) \geq \frac{1 + \frac{2}{3}\gamma + \frac{2}{15}\gamma^2}{(1 + \frac{1}{3}\gamma)^2} \geq 1 + \tfrac{1}{80}\min(\gamma^2, 1). \tag{17.10}$$

For γ tending to infinity, it is easy to check that $\lambda_{1,\gamma} = \frac{4}{\pi^2}\gamma(1+o(1))$, and therefore

$$u(\gamma) = \tfrac{12}{\pi^2}\big(1 + o(1)\big) \quad \text{as } \gamma \to \infty.$$

This proves that for any positive γ^* we have

$$\inf_{\gamma \in [\gamma^*, \infty)} u(\gamma) > 1.$$

On the other hand, for γ tending to zero we can find the asymptotic expansion of $u(\gamma)$ by showing that $x \tan x = \gamma$ for $x \in [0, \pi]$ implies

$$x^2 = \gamma - \tfrac{1}{3}\gamma^2 + \tfrac{4}{45}\gamma^3 + \mathcal{O}(\gamma^4).$$

This yields

$$\lambda_{1,\gamma} = 1 + \tfrac{1}{3}\gamma + \tfrac{1}{45}\gamma^2 + \mathcal{O}(\gamma^3), \tag{17.11}$$

and hence

$$u(\gamma) = 1 + \tfrac{1}{45}\gamma^2\big(1 + \mathcal{O}(\gamma)\big). \tag{17.12}$$

This analysis shows that there exists a positive number C such that for all $\gamma \geq 0$ we can write

$$u(\gamma) = 1 + \tfrac{1}{45}\min(\gamma^2, 1)\big(1 + C_\gamma \min(\gamma, 1)\big) \quad \text{with } |C_\gamma| \leq C. \tag{17.13}$$

We will also need to know how the normalized eigenfunction $\eta_{1,\gamma} = f_1/\|f_1\|_1$ behaves for small γ. We have, see also [332] p. 410,

$$\|f_1\|_1^2 = \cos^2(\alpha_1) + \frac{\alpha_1}{2\gamma}\left(\alpha_1 - \tfrac{1}{2}\sin(2\alpha_1)\right).$$

Since $\alpha_1^2 = \gamma/\lambda_1$ we can easily check that for γ tending to zero

$$\|f_1\|_1 = 1 - \tfrac{1}{3}\gamma + \tfrac{2}{15}\gamma^2 + \mathcal{O}(\gamma^3).$$

Due to the first boundary condition (17.8) we have

$$\int_0^1 \eta_{1,\gamma}(x)\,\mathrm{d}x = \lambda_1 \frac{f_1(0)}{\|f_1\|_1} = 1 + \tfrac{1}{6}\gamma - \tfrac{1}{72}\gamma^2 + \mathcal{O}(\gamma^3)$$

which yields

$$\left(\int_0^1 \eta_{1,\gamma}(x)\,\mathrm{d}x\right)^2 = 1 + \tfrac{1}{3}\gamma + \mathcal{O}(\gamma^3). \tag{17.14}$$

We now consider the periodic case. Then $W_\gamma f = \lambda f$ simplifies to

$$\int_0^1 f(x)\,\mathrm{d}x + \gamma \int_0^1 (\min(t,x) - tx)\, f(x)\,\mathrm{d}x = \lambda f(t) \quad \text{for all } t \in [0,1].$$

This is equivalent to the same differential equation $-\gamma f = \lambda f''$ but with the different boundary conditions

$$\lambda\, f(0) = \lambda\, f(1) = \int_0^1 f(x)\,\mathrm{d}x.$$

It is easy to check that the eigenpairs are now $\gamma_i/(4\alpha_i^2)$, $f_i(x) = \cos(2\alpha_i x - \alpha_i)$, where α_i is the unique solution of

$$x \tan x = \tfrac{1}{4}\gamma.$$

The square of the norm of univariate integration is now $1 + \frac{1}{12}\gamma$, and thus we need to consider the function

$$u(\gamma) = \frac{\lambda_{1,\gamma}}{1 + \frac{1}{12}\gamma/12}.$$

Hence, the previous analysis applies when we change γ to $\frac{1}{4}\gamma$.

Note that the conditions presented in Theorem 17.6 do not depend on scaling of $\gamma_{d,j}$. It is obvious for the absolute case, whereas for the normalized case it follows from

$$\tfrac{1}{16}\min(\gamma_{d,j}^2, 1) \le \min\left(\tfrac{1}{16}\gamma_{d,j}^2, 1\right) \le \min(\gamma_{d,j}^2, 1) \quad \text{for all } \gamma_{d,j}.$$

That is why it is enough to prove the theorem only for the non-periodic case.

We consider the two largest eigenvalues of W_d. We have

$$\lambda_1(W_d) = \prod_{j=1}^{d} \lambda_{1,\gamma_j},$$

$$\lambda_2(W_d) = \left(\max_{1\le j\le d} \frac{\lambda_2(W_{\gamma_j})}{\lambda_1(W_{\gamma_j})} \right) \prod_{j=1}^{d} \lambda_{1,\gamma_j}.$$

Consider now the absolute error criterion. We have

$$\operatorname{var}(H_{d,\gamma}) \le \lambda_1(W_d) \le \prod_{j=1}^{d} \left(1 + \tfrac{1}{2}\gamma_{d,j}\right) = \exp\left(\sum_{j=1}^{d} \ln\left(1 + \tfrac{1}{2}\gamma_{d,j}\right)\right)$$

$$\le \exp\left(\frac{1}{2} \sum_{j=1}^{d} \gamma_{d,j} \right) = d^{\,\frac{1}{2}\sum_{j=1}^{d}\gamma_{d,j}/\ln d}.$$

Hence, if $\gamma_{d,j}$'s satisfy the conditions of Theorem 17.6, Monte Carlo is strongly polynomially, polynomially or weakly tractable.

To prove that these conditions are also necessary, note that $\alpha_2 \in (\pi, 2\pi)$, and we have $\gamma/(4\pi^2) < \lambda_2(W_\gamma) < \gamma/\pi^2$. This implies that for $c_d = \gamma_{d,1}/(4\pi^2(1+\frac{1}{2}\gamma_{d,1}))$ we obtain, using (17.9),

$$\max_{1\le j\le d} \frac{\lambda_2(W_{\gamma_j})}{\lambda_1(W_{\gamma_j})} \ge \frac{\lambda_2(W_{\gamma_1})}{\lambda_1(W_{\gamma_1})} \ge c_d.$$

Suppose first that for some positive γ^* we have $\gamma_{d,1} \ge \gamma^*$ for all d. Then $c_d \ge c^* := \gamma^*/(4\pi^2(1+\frac{1}{2}\gamma^*))$, and we have

$$\operatorname{var}(H_{d,\gamma}) \ge \lambda_2(W_d) \ge c^* \prod_{j=1}^{d} \left(1 + \tfrac{1}{3}\gamma_{d,j}\right).$$

Hence, if Monte Carlo is strongly polynomially tractable then $\sup_d \operatorname{var}(H_{d,\gamma}) < \infty$ implies that

$$\limsup_d \prod_{j=1}^{d} \left(1 + \tfrac{1}{3}\gamma_{d,j}\right) < \infty,$$

and this yields that $\limsup_d \sum_{j=1}^{d} \gamma_{d,j}$ is finite.

Similarly, if Monte Carlo is polynomially tractable then $\operatorname{var}(H_{d,\gamma}) = \mathcal{O}(d^q)$ for some q, and this implies that $C := \limsup_d \sum_{j=1}^{d} \gamma_{d,j} / \ln d$ is finite. In fact, then we may take q sufficiently close to $\frac{1}{3}C$ for the non-periodic case, and $\frac{1}{12}C$ for the periodic case.

Weak tractability of Monte Carlo implies $\operatorname{var}(H_{d,\gamma}) = \exp(o(d))$, and this yields that $\sum_{j=1}^{d} \gamma_{d,j} = o(d)$, as needed.

We need to consider the remaining case for which for an arbitrary positive γ^* there exists an index d such that $\gamma_{d,1} < \gamma^*$. This case may happen, for example, if $\gamma_{d,j} = d^{-\alpha}$ for all $j \le d$ and for some positive α. Let $\eta_{1,d}(x) = \prod_{j=1}^{d} \eta_{1,\gamma_{d,j}}(x_j)$ be the normalized eigenfunction of W_d. We have

$$\langle V_d \eta_{1,d}, \eta_{1,d} \rangle = \langle W_d \eta_{1,d}, \eta_{1,d} \rangle - I_d^2(\eta_{1,d}) = \lambda_1(W_d) - \prod_{j=1}^{d} \left(\int_0^1 \eta_{1,\gamma_{d,j}}(x)\, \mathrm{d}x \right)^2.$$

Since all $\gamma_{d,j} \le \gamma_{d,1} < \gamma^*$ and γ^* can be arbitrarily small, we apply (17.11) and (17.14) to conclude that

$$\langle V_d \eta_{1,d}, \eta_{1,d} \rangle = \prod_{j=1}^{d} \left[1 + \tfrac{1}{3}\gamma_{d,j} + \tfrac{1}{45}\gamma_{d,j}^2 (1 + o(1)) \right] - \prod_{j=1}^{d} \left[1 + \tfrac{1}{3}\gamma_{d,j} + \mathcal{O}(\gamma_{d,j}^3) \right],$$

which can be rewritten as

$$\langle V_d \eta_{1,d}, \eta_{1,d} \rangle = \prod_{j=1}^{d} \left[1 + \tfrac{1}{3}\gamma_{d,j} + \tfrac{1}{45}\gamma_{d,j}^2 (1 + o(1)) \right] \Big(1 - \prod_{j=1}^{d} \left[1 - \tfrac{1}{45}\gamma_{d,j}^2 (1 + o(1)) \right] \Big).$$

Since $\ln(1+x) = x(1+o(1))$ for small $|x|$ we conclude that

$$\langle V_d \eta_{1,d}, \eta_{1,d} \rangle \ge \exp\left(\frac{1}{6} \sum_{j=1}^{d} \gamma_{d,j} \right) \left(1 - \exp\left(\frac{1}{6} \gamma^* \sum_{j=1}^{d} \gamma_{d,j} \right) \right).$$

From this formula, the same conditions on $\gamma_{d,j}$ easily follow. This completes the proof for the absolute error criterion.

We now consider the normalized error. Assume first that $\gamma_{d,1} \ge \gamma^* > 0$ for all d. Then $c_d \ge c^*$ and

$$c^* \prod_{j=1}^{d} u(\gamma_j) \le c_d \prod_{j=1}^{d} u(\gamma_j) \le \frac{\lambda_2(W_d)}{\|I_d\|^2} \le \frac{\lambda_1(V_d)}{\|I_d\|^2} \le \frac{\lambda_1(W_d)}{\|I_d\|^2} = \prod_{j=1}^{d} u(\gamma_j).$$

The error of Monte Carlo depends on the behavior of $\prod_{j=1}^{d} u(\gamma_{d,j})$. From (17.13) we have

$$\prod_{j=1}^{d} u(\gamma_{d,j}) = \prod_{j=1}^{d} \left[1 + \tfrac{1}{45} \min(\gamma_{d,j}^2, 1) \big(1 + C_{\gamma_{d,j}} \min(\gamma_{d,j}, 1) \big) \right], \quad \left| C_{\gamma_{d,j}} \right| \le C.$$

Monte Carlo is strongly polynomially tractable iff the last product is uniformly bounded in d. This holds iff $\limsup_d \sum_{j=1}^{d} \min(\gamma_{d,j}^2, 1) < \infty$, as claimed. We also have

$$\prod_{j=1}^{d} u(\gamma_{d,j}) = d^{\sum_{j=1}^{d} \ln u(\gamma_{d,j}) / \ln d}.$$

From (17.13) and (17.10) we know that

$$1 + \tfrac{1}{80}\,\min(\gamma_{d,j}^2, 1) \le u(\gamma_{d,j}) \le 1 + \tfrac{1}{45}\,(1 + C)\,\min(\gamma_{d,j}^2, 1).$$

Since $\ln(1 + x) \le x$ for all $x \ge 0$, and $x/(1 + \beta) \le \ln(1 + x)$ for all $x \in [0, \beta]$ and all positive β, we take $\beta = \frac{1}{80}$ and conclude that

$$\tfrac{1}{81}\,\min(\gamma_{d,j}^2, 1) \le \ln u(\gamma_{d,j}) \le \tfrac{1+C}{45}\,\min(\gamma_{d,j}^2, 1).$$

This proves that Monte Carlo is polynomially tractable iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \min(\gamma_{d,j}^2, 1)}{\ln d} < \infty.$$

Similarly, Monte Carlo is weakly tractable iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \min(\gamma_{d,j}^2, 1)}{d} = 0.$$

For the remaining case, $\gamma_{d,1} \le \gamma^*$ for sufficiently small γ^*, we apply (17.12) and (17.14) to conclude that

$$\frac{\langle V_d\eta_{1,d}, \eta_{1,d}\rangle}{\|I_d\|^2} = \prod_{j=1}^d \left[1 + \tfrac{1}{45}\gamma_{d,j}^2(1 + o(1))\right] - \prod_{j=1}^d \left[1 + \mathcal{O}(\gamma_{d,j}^3)\right].$$

From this formula, the same conditions on $\gamma_{d,j}$ easily follow. This completes the proof. □

We stress the difference in the tractability conditions of Monte Carlo for the absolute and normalized error criteria. For the absolute error, we must assume that the sum of the first powers of $\gamma_{d,j}$ behaves properly, whereas for the normalized error, we must assume the same conditions for the sum of $\min(\gamma_{d,j}^2, 1)$. Hence, a few arbitrarily large weights do not matter for the normalized error and Monte Carlo can be still strongly polynomially tractable. For example, let $\gamma_{d,j} = 2^d$ for $j = 1, 2, \dots, p$, with $p \ge 0$, and $\gamma_{d,j} = j^{-s}$ for $j \ge p + 1$ and $s > 0$. Then for the absolute error criterion, Monte Carlo is *not* even weakly tractable for $p \ge 1$, whereas it is weakly tractable if $p = 0$, and is polynomially tractable for $p = 0$ and $s = 1$, and strongly polynomially tractable if $p = 0$ and $s > 1$. For the normalized error criterion, Monte Carlo is weakly tractable for *all* p and s, whereas it is polynomially tractable for all p and $s = \frac{1}{2}$, and strongly polynomially tractable for all p and $s > \frac{1}{2}$.

It is known, see [54], [279], [355], [84], respectively, and Chapter 12, that strong polynomial tractability holds in the worst case setting for the space $H_{d,\gamma}$ and for the normalized error criterion in both the non-periodic and periodic cases iff

$$\limsup_{d} \sum_{j=1}^d \min(\gamma_{d,j}, 1) < \infty,$$

polynomial tractability holds iff

$$\limsup_d \sum_{j=1}^d \min(\gamma_{d,j}, 1)/\ln d < \infty,$$

and weak tractability holds iff

$$\lim_d \sum_{j=1}^d \min(\gamma_{d,j}, 1)/d = 0.$$

As we see, the conditions for polynomial behavior of Monte Carlo are more lenient. For example, for $\gamma_{d,j} = j^{-2/3}$ there are no polynomially tractable algorithms in the worst case setting whereas Monte Carlo is strongly polynomially tractable.

We finally discuss sharp estimates on the exponent q if Monte Carlo is polynomially tractable, i.e., when $n^{\mathrm{mc}}(\varepsilon, H_d) \le C\, d^q \varepsilon^{-2}$. For simplicity assume that there exists a positive γ^* such that $\gamma_{d,1} \ge \gamma^*$, and consider only the normalized error criterion. From the proof of Theorem 17.6 we know that q must be chosen such that $\prod_{j=1}^d u(\gamma_{d,j}/\beta)/d^q$ is uniformly bounded in d, where $\beta = 1$ for the non-periodic case, and $\beta = 4$ for the periodic case. Let

$$A = \limsup_{d\to\infty} \sum_{j=1}^d \ln u(\gamma_{d,j}/\beta) \,/\, \ln d\, .$$

Then Monte Carlo is polynomial iff $A < \infty$ and q in the bound of $n^{\mathrm{mc}}(\varepsilon, H_d)$ cannot be smaller than A. The number A can be computed more easily if we consider weights independent of d, i.e., when $\gamma_{d,j} = \gamma_j$ for all d and $j = 1, 2, \dots, d$. Without loss of generality we assume that $\gamma_1 \ge \gamma_2 \ge \cdots$. The case $\gamma_1 = 0$ is trivial since then $H_d = \mathrm{span}(1)$ and $\mathrm{var}_d(f) = 0$ for all $f \in H_d$. Assume thus that $\gamma_1 > 0$. Then $A < \infty$ implies that $\lim_d \gamma_j = 0$, and using the asymptotic expansion (17.12) of u, we conclude that Monte Carlo is polynomially tractable iff

$$a := \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j^2}{\ln d} < \infty.$$

Then $A = \frac{1}{45} a/\beta^2$, and for any $q > \frac{1}{45}a$ for the non-periodic case, and for $q > \frac{1}{720}a$ for the periodic case, there exists a positive C such that

$$n^{\mathrm{mc}}(\varepsilon, H_d) \le C d^q \varepsilon^{-2} \quad \text{for all } d = 1, 2, \dots,\ \varepsilon \in (0, 1).$$

Furthermore, the exponent q cannot be smaller than $\frac{1}{45}a$ for the non-periodic case and $\frac{1}{720}a$ for the periodic case.

17.1.1.3 The third Sobolev space. The third Sobolev space $H_{d,\gamma}$ is algebraically the same as the spaces in Sections 17.1.1.1 and 17.1.1.2 but has a different inner product and norm, see Section A.2.3 of Appendix A, Volume I. Its reproducing kernel is

$$K_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} K(x_j, y_j), \tag{17.15}$$

where

$$K(x,y) = \big(B_2(|x-y|) + 2\big(x-\tfrac{1}{2}\big)\big(y-\tfrac{1}{2}\big)\big) \quad \text{for all } x,y\in[0,1] \tag{17.16}$$

with B_2 the Bernoulli polynomial of degree 2, $B_2(x) = x^2 - x + \frac{1}{6}$.

The Sobolev space $H_{d,\gamma}$ has the inner product

$$\langle f,g\rangle_{H_{d,\gamma}} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial x_{\mathfrak{u}}}(x)\, \mathrm{d}x_{-\mathfrak{u}} \right) \times \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} g}{\partial x_{\mathfrak{u}}}(x)\, \mathrm{d}x_{-\mathfrak{u}} \right) \mathrm{d}x_{\mathfrak{u}}.$$

Here, $x_{-\mathfrak{u}}$ denotes the vector $x_{[d]\setminus\mathfrak{u}}$. For $\mathfrak{u} = \emptyset$, the integral $\int_{[0,1]^0} f(x)\,\mathrm{d}x_{\mathfrak{u}}$ should be replaced by $f(x)$.

The periodic variant of the space $H_{d,\gamma}$ is obtained as before by assuming that for $d = 1$ we impose the periodicity condition $f(0) = f(1)$. Then the reproducing kernel is

$$\tilde{K}_d(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{d,\mathfrak{u}}}{2^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} B_2(|x_j - y_j|) \quad \text{for all } x,y\in[0,1]^d. \tag{17.17}$$

For multivariate integration we have $\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x = \gamma_{d,\emptyset}\,\langle f,1\rangle_{H_{d,\gamma}}$. For the non-periodic and periodic cases, the representer of multivariate integration is $h_d \equiv \gamma_{d,\emptyset}$, with $\|\mathrm{INT}_d\| = \|h_d\|_{H_{d,\gamma}} = \gamma_{d,\emptyset}^{1/2}$, and we need to assume that $\gamma_{d,\emptyset} > 0$.

It is shown in [54] that this space of functions is related to the *anova* decomposition of functions which is a popular tool to analyze the error of integration, see e.g., Efron and Stein [65], and Sobol [286].

We analyze Monte Carlo for this space. We need to find the largest eigenvalue of the operator V_d. First we find the largest eigenvalues for $d = 1$ of the operator

$$(Uf)(t) = \int_0^1 K(t,x)\, f(x)\,\mathrm{d}x.$$

Note that for $f \equiv 1$, we have $Uf = 0$ which means that zero is one of the eigenvalues. We now consider $Uf = \lambda f$ for a non-zero f orthogonal 1, $\int_0^1 f(x)\,\mathrm{d}x = 0$. Since the kernel is $B_2(|t-x|)) + 2\beta(t-\frac{1}{2})(x-\frac{1}{2})$, where $\beta = 1$ for the non-periodic case and $\beta = 0$ for the periodic case, the equation $Uf = \lambda f$ simplifies to

$$\int_0^1 (x^2 - \beta x) f(x)\,\mathrm{d}x + 2(\beta-1)\,t \int_0^1 x f(x)\,\mathrm{d}x - \int_0^1 |t-x|\, f(x)\,\mathrm{d}x = \lambda f(t)$$

for $t \in [0,1]$. On differentiating and setting $t = 0$, this yields $f'(0) = 0$ for $\beta = 1$. By double differentiating we conclude that

$$-2\, f(x) = \lambda f''(x)$$

and the boundary condition $\int_0^1 f(x)\,\mathrm{d}x = 0$, and $f'(0) = 0$ for the non-periodic case ($\beta = 1$), and $f(0) = f(1)$ for the periodic case ($\beta = 0$). This has the solution $f(x) = \cos(k\pi x)$ with the eigenvalue $\lambda = 2/(\pi^2 k^2)$ for the non-periodic case, and $f(x) = \sin(2k\pi x)$ with the eigenvalue $\lambda = 2/(4\pi^2 k^2)$ for the periodic case; here $k = 1, 2, \ldots$. Hence, the largest eigenvalue of U is obtained for $k = 1$ and is equal to $2/\alpha$, where $\alpha = \pi^2$ for the non-periodic case, and $\alpha = 4\pi^2$ for the periodic case, compare also with [279].

We now find the eigenvalues of W_d for $d \ge 1$. Note that $W_d h_d = \lambda^* h_d$ with $\lambda^* = \gamma_{d,\emptyset}$. For an arbitrary non-empty $\mathfrak{u} \subseteq [d]$, let

$$f_{\mathfrak{u}}(x) = \prod_{j \in \mathfrak{u}} \big(\beta\, \cos(\pi x_j) + (1-\beta)\sin(2\pi x_j)\big)\,.$$

Then $W_d f = \alpha^{-|\mathfrak{u}|}\, \gamma_{d,\mathfrak{u}}\, f$. Hence, the largest eigenvalue of W_d is

$$\lambda_1(W_d) = \max_{\mathfrak{u} \subseteq [d]} \alpha^{-|\mathfrak{u}|}\, \gamma_{d,\mathfrak{u}}.$$

Note that for $\mathfrak{u} = \emptyset$ we have $\lambda_1(W_d) \ge \gamma_{d,\emptyset}$.

We are ready to find the largest eigenvalue of V_d. Note that $\lambda^* - \mathrm{INT}_d(h_d) = 0$ is an eigenvalue of V_d. If $\lambda_1(W_d) = \lambda^*$ then (17.3) yields

$$\lambda_1(V_d) = \lambda_2(W_d) = \max_{\emptyset \ne \mathfrak{u} \subseteq [d]} \alpha^{-|\mathfrak{u}|}\, \gamma_{d,\mathfrak{u}}.$$

If $\lambda_1(W_d) > \lambda^*$ then (17.3) yields $\lambda_1(V_d) = \lambda_1(W_d)$. This implies that

$$\lambda_1(V_d) = \max_{\emptyset \ne \mathfrak{u} \subseteq [d]} \alpha^{-|\mathfrak{u}|}\, \gamma_{d,\mathfrak{u}}.$$

From this we easily conclude the following theorem.

Theorem 17.7. *Let $H_{d,\gamma}$ be the Sobolev space of non-periodic or periodic functions with the reproducing kernel (17.15) or (17.17), respectively. Let $\alpha = \pi^2$ for the non-periodic case, and $\alpha = 4\pi^2$ for the periodic case, and $CRI_d = 1$ for the absolute error criterion, and $CRI_d = \gamma_{d,\emptyset}^{1/2} > 0$ for the normalized error criterion. Then*

- *Monte Carlo is strongly polynomially tractable iff there is a non-negative number C such that*

  $$\gamma_{d,\mathfrak{u}} \le C\; CRI_d\; \alpha^{|\mathfrak{u}|} \quad \text{for all } d \text{ and } \mathfrak{u} \subseteq [d].$$

  When this holds then

  $$n^{\mathrm{mc}}(\varepsilon, H_d) \le \lceil C\, \varepsilon^{-2} \rceil \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}.$$

- *Monte Carlo is polynomially tractable iff there are non-negative numbers C and q such that*
$$\gamma_{d,\mathfrak{u}} \le C\ CRI_d\ d^q\ \alpha^{|\mathfrak{u}|} \quad \text{for all } d \text{ and } \mathfrak{u} \subseteq [d].$$
If this holds then
$$n^{\rm mc}(\varepsilon, H_d) \le \lceil C\ d^q\ \varepsilon^{-2} \rceil \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}.$$
- *Monte Carlo is weakly tractable iff*
$$\gamma_{d,\mathfrak{u}} \le CRI_d\ \exp\left(o(d)\right)\ \alpha^{|\mathfrak{u}|} \quad \text{for all } d \text{ and } \mathfrak{u} \subseteq [d].$$

Note that for $\gamma_{d,\mathfrak{u}} = a^{|\mathfrak{u}|}$, Monte Carlo is strongly tractable if $a \le \alpha$ and not weakly tractable if $a > \alpha$. Consider now product weights which have been studied in [279], $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$. Then $\gamma_{d,\emptyset} = 1$, and the absolute and normalized error criteria coincide. Then
$$\gamma_{d,\mathfrak{u}}/\alpha^{|\mathfrak{u}|} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}/\alpha$$
is uniformly bounded, polynomially bounded or non-exponential iff
$$\prod_{j=1}^{d} \max\left(1, \gamma_{d,j}/\alpha\right) = \exp\Big(\sum_{j=1}^{d} \left(\ln \gamma_{d,j}/\alpha\right)_+\Big)$$
is uniformly bounded, polynomially bounded or non-exponential. Hence, Monte Carlo is strongly polynomially tractable iff
$$\limsup_{d\to\infty} \sum_{j=1}^{d} \ln\left(\gamma_{d,j}/\alpha\right)_+ < \infty,$$
is polynomially tractable iff
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \ln\left(\gamma_{d,j}/\alpha\right)_+}{\ln d} < \infty,$$
or weakly tractable iff
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \ln\left(\gamma_{d,j}/\alpha\right)_+}{d} = 0.$$
The first two conditions coincide with the conditions found in [279].

Note that these conditions hold if the weights have the property that $\gamma_{d,j} \le \alpha$ for almost all j. For the worst case setting, multivariate integration for the space $H_{d,\gamma}$ in both the non-periodic and periodic cases is strongly polynomially tractable iff
$$\limsup_{d} \sum_{j=1}^{d} \gamma_{d,j} < \infty,$$

and is polynomially tractable iff

$$\limsup_{d} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln d} < \infty,$$

and is weakly tractable iff

$$\lim_{d} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} = 0,$$

see [54], [279], [355] and Chapter 12.

Again, the conditions on polynomial behavior on Monte Carlo are much more lenient. For example, for $\gamma_{d,j} = \alpha$, multivariate integration is intractable in the worst case setting, whereas Monte Carlo is strongly polynomially tractable.

Take now $\gamma_{d,j} = 4\pi^2$. Then Monte Carlo is not weakly tractable in the non-periodic case, and it is strongly polynomially tractable in the periodic case. For such weights, periodicity is a very powerful property and makes Monte Carlo strongly polynomially tractable. As we shall see in Section 17.1.3, even a more extreme example is possible. Namely, for some spaces Monte Carlo is not weakly tractable in the non-periodic case for *all* weights, whereas it becomes strongly polynomially tractable in the periodic case for some weights. However, the opposite case may also happen. That is, as we shall see in Section 17.1.4, for some spaces Monte Carlo is strongly polynomially tractable in the non-periodic case for some weights whereas it becomes not weakly tractable in the periodic case for *all* weights. This means that the tractability behavior of Monte Carlo essentially depends on the space and its periodic version, and in general, there are no relations between them.

17.1.2 Gaussian Integration

In this section we consider Gaussian integration. That is, we take $D_d = \mathbb{R}^d$ and

$$\varrho_d(x) = \prod_{j=1}^{d} \exp(-x_j^2/(2\sigma_j))/(2\pi\sigma_j)^{1/2} \quad \text{for all } x \in \mathbb{R}^d$$

as the density of a Gaussian measure, with the variance $\sigma_j = \sigma_{d,j} > 0$ of the jth variable. We consider two spaces and check tractability conditions of Monte Carlo.

17.1.2.1 Sobolev space over $\mathbb{R}^d$. For simplicity we consider now only the normalized error criterion and product weights. As in [128], we take $H_{d,\gamma}$ as the weighted Sobolev space of m times differentiable functions with respect to each variable, with the reproducing kernel

$$K_{d,\gamma}(x,y) = \prod_{j=1}^{d} \gamma_{d,j} R_m\left(\gamma_{d,j}|x_j - y_j|\right), \tag{17.18}$$

where $\gamma_{d,j} > 0$ and

$$R_m(x) = \frac{1}{2m} \sum_{k=0}^{m-1} (-1)^k \exp\big(- |x| e^{\mathrm{i}\pi(2k+1-m)/(2m)} + \mathrm{i}\pi(2k+1-m)/(2m) \big),$$

with $\mathrm{i} = \sqrt{-1}$. This space has the inner product, see Thomas-Agnan [304],

$$\langle f, g \rangle_{H_{d,\gamma}} = \sum_{u \subseteq [d]} \prod_{j=1}^{d} \gamma_{d,j}^{-2m} \int_{\mathbb{R}^d} \frac{\partial^{m|u|} f}{\partial x_u^m}(x) \frac{\partial^{m|u|} g}{\partial x_u^m}(x)\, \mathrm{d}x.$$

For $m = 1$ and $d = 1$, we have $R_1(x) = \frac{1}{2} \exp(-|x|)$, and

$$\|f\|_{H_{1,\gamma}}^2 = \|f\|_{L_2(\mathbb{R})}^2 + \gamma_{1,1}^{-2} \|f'\|_{L_2(\mathbb{R})}^2.$$

We have the following theorem.

Theorem 17.8. *Let $H_{d,\gamma}$ be the weighted Sobolev space with kernel (17.18). Consider the normalized error criterion.*

- *Monte Carlo is strongly polynomially tractable if*

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \sigma_{d,j} \gamma_{d,j} < \infty.$$

- *Monte Carlo is polynomially tractable if*

$$\limsup_{d} \frac{\sum_{j=1}^{d} \sigma_{d,j} \gamma_{d,j}}{\ln d} < \infty.$$

- *Monte Carlo is weakly tractable if*

$$\limsup_{d} \frac{\sum_{j=1}^{d} \sigma_{d,j} \gamma_{d,j}}{d} = 0.$$

Proof. It is known, see Corollary 4 of [128], that

$$\limsup_{d} \sum_{j=1}^{d} \sigma_{d,j} \gamma_{d,j} < \infty$$

implies that Monte Carlo is strongly polynomially tractable. Using the same proof it is easy to show that

$$\limsup_{d} \sum_{j=1}^{d} \sigma_{d,j} \gamma_{d,j} / \ln d < \infty$$

implies that Monte Carlo is polynomially tractable, and that

$$\limsup_{d} \sum_{j=1}^{d} \sigma_{d,j}\gamma_{d,j}/d = 0$$

implies that Monte Carlo is weakly tractable. □

We stress that the same conditions also guarantee that Gaussian integration in the worst case setting is strongly polynomially tractable, polynomially tractable and weakly tractable. It is not known if these conditions are also necessary for tractability of Monte Carlo. This yields us to the next open problem.

Open Problem 78.

- Find necessary and sufficient conditions for Monte Carlo to be strongly polynomially, polynomially, or weakly tractable for Gaussian integration over the space $H_{d,\gamma}$ considered in this subsection.

17.1.2.2 Isotropic Sobolev space. We now consider an isotropic space H_d for which all variables play the same role and still Monte Carlo is strongly polynomially tractable for arbitrary variances $\sigma_{d,j}$ and for the normalized error criterion. This is the Sobolev space H_d with the kernel

$$K_d(x,y) = \frac{\|A_d\, x\|_2 + \|A_d\, y\|_2 + \|A_d(x-y)\|_2}{2} \quad \text{for all } x,y \in \mathbb{R}^d, \tag{17.19}$$

where A_d is any $d \times d$ nonsingular matrix, and $\|\cdot\|_2$ denotes the Euclidean norm of vectors. This kernel is related to the isotropic Wiener measure (usually with A_d being the identity), and is sometimes called Brownian motion in the Lévy sense.

The inner product of this space was characterized by Molchan [189] for odd d, and later by Ciesielski [32] for arbitrary d, see also the book of Stein [288], and is given by

$$\langle f,g\rangle_d = a_d \left\langle (-\Delta)^{(d+1)/4} f, (-\Delta)^{(d+1)/4} g \right\rangle_{L_2(\mathbb{R}^d)},$$

for f and g which have finite support, vanish at zero and are infinitely many times differentiable. The constant a_d is known, Δ is the Laplace operator, and for $d+1$ not divisible by 4, the operator $(-\Delta)^{(d+1)/4}$ is understood in the generalized sense.

It is known, see Corollary 5 of [128], that the hypothesis in part (ii) of Theorem 17.1 holds (with $q = 0$ and $C = 2 + \sqrt{2}$ independently of $\sigma_{d,j}$), see also Example 7 of Chapter 3 of Volume I. Hence, Monte Carlo as well as some deterministic algorithms are strongly polynomially tractable. We summarize the results of this subsection in the following theorem.

Theorem 17.9. *Let H_d be the weighted Sobolev space with kernel* (17.19)*. Then Monte Carlo is strongly polynomially tractable for any variances of Gaussian integration.*

17.1.3 Periodicity May Help

We now provide an example of a space $H_{d,\gamma}$ defined on $[0,1]^d$ for which Monte Carlo is not weakly tractable for uniform integration in the non-periodic case for all product weights, whereas it becomes strongly polynomially tractable in the periodic case for some product weights. For such weights periodicity of functions makes Monte Carlo strongly polynomially tractable.

We define the reproducing kernel space $H_{d,\gamma}$ by its kernel. For $d = 1$, we take

$$K_\gamma(x,y) = K_1(x,y) + \gamma\, K_2(x,y) \quad \text{for all } x,y \in [0,1],$$

where

$$K_1(x,y) = g_1(x)g_1(y) + g_2(x)g_2(y), \quad K_2(x,y) = B_2(|x-y|),$$

with $g_1(x) = a(x - 1/2)$ and $g_2(x) = 1$ for $x \in [0,1]$, with $a > 2\sqrt{3}$. As before, $B_2(x) = x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial of degree 2.

Observe that K_i are reproducing kernels, and they generate Hilbert spaces $H(K_i)$ such that

$$H(K_1) = \text{span}(g_1, g_2), \quad H(K_2) = \{ f \in W_1 : \text{INT}_1(f) = 0 \}.$$

The space $H(K_1)$ is two dimensional, and it can easily be checked that g_1 and g_2 are orthonormal. Hence, for $f = c_1 g_1 + c_2 g_2$ we have $\|f\|^2_{H(K_1)} = c_1^2 + c_2^2$. We also have $\text{INT}_1(f) = c_2$, and $\text{INT}_1(f^2) = \frac{1}{12}c_1^2 a^2 + c_2^2$. Hence, for the function $f = g_1$ we have $\text{var}_1(g_1) = \frac{1}{12}a^2 > 1$.

The space $H(K_2)$ is a subspace of the periodic space which was considered in Section 17.1.1.3 with $\gamma_1 = 2$. Therefore the inner product in $H(K_2)$ is

$$\langle f, g \rangle_{H(K_2)} = \frac{1}{2} \int_0^1 f'(x) g'(x) \, \mathrm{d}x.$$

Consider now univariate integration. That is, $\text{INT}_1(f) = \langle f, h_1 \rangle_{H_{1,\gamma}}$ with $h_1(x) = \int_0^1 K_\gamma(x,t)\, \mathrm{d}t = 1 = g_2(x)$ of norm 1. We stress that h_1 has a zero component in $H(K_2)$.

For arbitrary d, we take the tensor product.

$$H_{d,\gamma} = H(K_{\gamma_{d,1}}) \otimes \cdots \otimes H(K_{\gamma_{d,d}}),$$

which has the reproducing kernel

$$K_d(x,y) = \prod_{j=1}^d K_{\gamma_{d,j}}(x_j, y_j).$$

For multivariate integration, we have $\text{INT}_d(f) = \langle f, h_d \rangle_{H_{d,\gamma}}$ with $h_d(x) = 1$ and $\|\text{INT}_d\| = 1$.

Take now the function $f(x) = g_1(x_1)g_1(x_2)\cdots g_1(x_d)$. Then $\|f\|_{H_{d,\gamma}} = 1$ and $\mathrm{INT}_d(f) = 0$. Thus

$$\mathrm{var}_d(f) = \mathrm{INT}_d(f^2) = \mathrm{INT}_1(g_1^2)^d = \left(\tfrac{1}{12}a^2\right)^d$$

which is exponentially large in d. This proves that Monte Carlo is not weakly tractable for any choice of the product weights $\gamma_{d,j}$.

We now turn to the periodic case. That is, for $d = 1$ we take

$$H(\tilde{K}_\gamma) = \{\, f \in H(K_\gamma) \mid f(0) = f(1)\,\}.$$

It is easy to check that

$$\tilde{K}_\gamma(x,y) = 1 + \gamma\, B_2(|x-y|),$$

which is the same as in Section 17.1.1.3 with the weight replaced by 2γ.

For $d \ge 1$, we take

$$\tilde{H}_{d,\gamma} = H(\tilde{K}_{1,\gamma_1}) \otimes \cdots \otimes H(\tilde{K}_{1,\gamma_d}).$$

The representer of multivariate integration is still 1, with norm one. Since, the multivariate integration problem over $\tilde{H}_d$ is the same as in Section 17.1.1.3 for the space weights $2\gamma_{d,j}$, from Theorem 17.7 we know, in particular, that Monte Carlo is strongly polynomially tractable iff $\limsup_d \sum_{j=1}^d \left(\ln(\gamma_{d,j}/(2\pi^2))\right)_+ < \infty$. For such product weights $\gamma_{d,j}$, the periodic case makes Monte Carlo strongly polynomially tractable.

17.1.4 Periodicity May Hurt

We now present a weighted space and multivariate integration for which for the normalized error criterion Monte Carlo is strongly polynomially tractable for the non-periodic case for some product weights, and it is not polynomially tractable for the periodic case for all non-zero product weights.

Let $D_d = [0,1]^d$, and $\varrho_d(x) = \prod_{j=1}^d \varrho_1(x_j)$ with $\varrho_1(x_j) = 2$ for $x_j \in [0, \frac{1}{2}]$ and $\varrho_1(x_j) = 0$ for $x_j \in (\frac{1}{2}, 1]$.

For $d = 1$, we take the kernel K_γ of the space $H(K_\gamma)$ to be

$$K_\gamma(t,x) = h(t)h(x) + \gamma\, g(t)g(x),$$

where $h(x) = \frac{1}{2}\varrho_1(x)$ and $g(x) = 6x(1-x)$. Then $h(0) = 1, h(1) = g(0) = g(1) = 0$ and $I_1(h) = 1$ as well as $I_1(g) = 1$ and $I_1(g^2) = \frac{6}{5}$. Observe that $K_\gamma(t,1) = 0$ for all $t \in [0,1]$ and this implies that $f(1) = 0$ for all $f \in H(K_\gamma)$. Obviously, $H(K_\gamma)$ is a two dimensional space.

For $d \ge 1$, we take the tensor product

$$H_{d,\gamma} = H(K_{\gamma_1}) \otimes \cdots \otimes H(K_{\gamma_d})$$

with the reproducing kernel $K_{d,\gamma}(x,y) = \prod_{j=1}^{d} K_{\gamma_j}(x_j,y_j)$. The space $H_{d,\gamma}$ has dimension 2^d.

For multivariate integration we have $\mathrm{INT}_d(f) = \langle f, h_d\rangle_{H_{d,\gamma}}$ with

$$h_d(t) = \int_{[0,1]^d} \varrho_d(x) K_d(t,x)\,\mathrm{d}x = \prod_{j=1}^{d} \big(h(t_j) + \gamma_j\, g(t_j)\big),$$

and

$$\|\mathrm{INT}_d\|^2 = \mathrm{INT}_d(h_d) = \prod_{j=1}^{d} \big(1 + \gamma_{d,j}\big).$$

On the other hand,

$$\int_{D_d} \varrho_d(x) K_d(x,x)\,\mathrm{d}x = \prod_{j=1}^{d} \big(1 + \tfrac{6}{5}\,\gamma_{d,j}\big).$$

Hence, for $\sup_d \sum_{j=1}^{d} \gamma_{d,j} < \infty$, (ii) of Theorem 17.1 implies that Monte Carlo is strongly polynomial for the non-periodic case and for the absolute and normalized error criteria.

We now turn to the periodic case. For $d = 1$, we already have $f(1) = 0$, and therefore we need only to assume that $f(0) = 0$. That is, we switch to the subspace $\tilde{H}_1 = \{f \in H(K_\gamma) : f(0) = 0\}$ which is of dimension one and has the kernel

$$\tilde{K}_\gamma(x,t) = K_\gamma(x,t) - \frac{K_\gamma(x,0)K_\gamma(t,0)}{K_\gamma(0,0)} = K_\gamma(x,t) - h(x)h(t) = \gamma\, g(t)g(x).$$

For $d \ge 1$ we have $\tilde{H}_d = H(\tilde{K}_{\gamma_{d,1}}) \otimes \cdots \otimes H(\tilde{K}_{\gamma_{d,d}})$ with the reproducing kernel $\tilde{K}_d(x,y) = \gamma_{d,1}\cdots\gamma_{d,d} \prod_{j=1}^{d} g(t_j)g(x_j)$ and multivariate integration INT_d has the norm

$$\|\mathrm{INT}_d\| = \big(\gamma_{d,1}\cdots\gamma_{d,d}\big)^{1/2}.$$

We assume that all $\gamma_{d,j}$ are positive and therefore $\|\mathrm{INT}_d\| > 0$.

Take now the function

$$f(x) = K_d\big(\tfrac12, x\big)/K_d\big(\tfrac12,\tfrac12\big)^{1/2} = \prod_{j=1}^{d} \gamma_{d,j}^{1/2} g_j(x_j).$$

Then $\|f\|_{H_{d,\gamma}} = 1$ and $\mathrm{var}_d(f)/\|\mathrm{INT}_d\| = \mathrm{INT}_1(g^2)^d - 1 = (6/5)^d - 1$ is exponentially large in d. Therefore Monte Carlo is not even weakly tractable for the normalized error criterion.

Obviously, for the absolute error criterion everything depends on

$$A_d = \left(\left(\frac{6}{5}\right)^d - 1\right)^{1/2} \prod_{j=1}^{d} \gamma_{d,j}^{1/2}.$$

If A_d is uniformly bounded in d then MC is strongly polynomially tractable, if A_d is polynomially bounded in d then MC is polynomially tractable, and if $A_d = \exp(o(d))$ then MC is weakly tractable.

We also add that this multivariate integration problem is trivial for deterministic algorithms. Indeed, for any $f \in \tilde{H}_d$ we have $f = \alpha g_d$ with $g_d(x) = \prod_{j=1}^d g(x_j)$ and $\alpha = f(x^*)(\frac{4}{3})^d$ for $x^* = [\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}]$. Furthermore $\mathrm{INT}_d(f) = f(x^*)(\frac{4}{3})^d$ and it can be computed exactly using one function value. As we shall see in the next section, this can happen even for infinite dimensional spaces.

17.1.5 Monte Carlo May Lose

We have seen a number of spaces for which conditions on Monte Carlo being polynomially tractable are more lenient than for the worst case setting. In this section, we show that the opposite may happen as well. That is, for some infinite dimensional spaces Monte Carlo is *not* weakly tractable, yet uniform integration is trivial in the worst case setting.

For $d = 1$, let $g_j : [0,1] \to \mathbb{R}$ for $j = 1, 2, \dots,$ be linearly independent real-valued functions. We consider the reproducing kernel K_γ of the form

$$K_\gamma(x,t) = g_1(x)g_1(t) + \gamma \sum_{j=2}^{\infty} g_j(x)g_j(t), \quad x,t \in [0,1].$$

Then $H(K_\gamma) = \mathrm{span}(g_1, g_2, \dots)$ and the g_j's are orthonormal. We assume that the functions g_j are chosen such that there is a number $x^* \in [0,1]$ and

$$\begin{aligned} \int_0^1 g_1(t)\,\mathrm{d}t &= g_1(x^*) > 0, \\ g_j(x^*) &= 0, \qquad & j &= 2, 3, \dots, \\ \int_0^1 g_j(t)\,\mathrm{d}t &= 0, \qquad & j &= 2, 3, \dots. \end{aligned}$$

For instance, we can take $g_1(t) = 2t$ and $g_j(x) = \sin(2\pi j x)/j$ for $j \geq 2$. Then the last assumptions hold with $x^* = \frac{1}{2}$.

For $f \in H(K_\gamma)$ we have $f(x) = \sum_{j=1}^\infty c_j g_j(x)$ with $\sum_{j=1}^\infty c_j^2 < \infty$. Taking $x = x^*$ we get $c_1 = f(x^*)/g_1(x^*)$. Consider now univariate integration INT_1 over $H(K_\gamma)$,

$$\mathrm{INT}_1(f) = \int_0^1 f(t)\,\mathrm{d}t = c_1 \int_0^1 g_1(t)\,\mathrm{d}t = c_1 g_1(x^*) = f(x^*) \quad \text{for all } f \in H(K_\gamma).$$

Hence, $\mathrm{INT}_1(f) = \langle f, h_1 \rangle_{H_{1,\gamma}}$ with $h_1(x) = K_\gamma(x,x^*) = g_1(x)g_1(x^*)$ is a continuous linear functional which can be exactly computed using just one function value at x^*. We also have $\|\mathrm{INT}_1\| = g_1(x^*)$.

For $d \ge 1$, we take $H_{d,\gamma} = H(K_{\gamma_{d,1}}) \otimes \cdots \otimes H(K_{\gamma_{d,d}})$ with the reproducing kernel

$$K_d(x, y) = \prod_{j=1}^{d} K_{\gamma_{d,j}}(x_j, y_j).$$

Multivariate integration INT_d is now of the form

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(t)\,\mathrm{d}t = \langle f, h_d \rangle_{H_{d,\gamma}} = f(x^*, x^*, \dots, x^*) \quad \text{for all } f \in H_d.$$

This follows from the fact that $h_d(x) = \prod_{j=1}^{d} h_1(x_j) = \prod_{j=1}^{d} K_{\gamma_{d,j}}(x_j, x^*)$, and $\|\mathrm{INT}_d\| = g_1^d(x^*)$. Hence, multivariate integration is trivial since it can be computed exactly using one function value at $(x^*, x^*, \dots, x^*)$.

On the other hand, Monte Carlo may be not weakly tractable for arbitrary product weights $\gamma_{d,j}$. Indeed, if we take the functions g_j as before, i.e.,

$$g_1(t) = 2t \quad \text{and} \quad g_j(x) = \sin(2\pi j x)/j \text{ for } j \ge 2$$

with $x^* = \frac{1}{2}$, then $g_1(x^*) = 1$ and $\|\mathrm{INT}_d\| = 1$. For $f(x) = \prod_{j=1}^{d} g_1(x_j)$ we have $\|f\|_{H_{d,\gamma}} = 1$, yet the variance of f, namely

$$\mathrm{INT}_d(f^2) - \mathrm{INT}_d^2(f) = \left(\int_0^1 g_1^2(t)\,\mathrm{d}t\right)^d - \left(\int_0^1 g_1(t)\,\mathrm{d}t\right)^{2d} = \left(\frac{4}{3}\right)^d - 1,$$

is exponentially large in d. This proves that Monte Carlo is not weakly tractable.

17.1.6 Can Monte Carlo Be Improved?

We studied Monte Carlo for multivariate integration and presented necessary and sufficient conditions to guarantee that Monte Carlo is strongly polynomially, polynomially or weakly tractable.

It is natural to ask if tractability conditions of Monte Carlo can be weakened for some other randomized algorithms, and what are tractability conditions for optimal randomized algorithms for multivariate integration. In fact, not much is known about these problems.

We present partial answers for some specific classes of functions. Section 17.2 deals with *importance sampling* and we report the result of Wasilkowski [328] who improved tractability conditions of Monte Carlo for the periodic variant of the second Sobolev space and for the normalized error criterion. We also present results from the paper of Plaskota, Wasilkowski and Zhao [248], and from the paper of Hinrichs [131].

There are classes of functions for which Monte Carlo is nearly optimal. The simplest such class is the space $L_2([0,1]^d)$. Monte Carlo has the error $n^{-1/2}$, independent of d, and is almost optimal. This follows from Lemma 17.10 that is presented below, see also, e.g., Mathé [181].

To prove optimality of Monte Carlo, we need lower bounds on randomized errors for arbitrary algorithms. So far, all lower bounds use the technique of Bakhvalov that was already described in Section 4.3.3 of Volume I for general solution operators S. This technique uses the fact that the randomized error of any algorithm is lower bounded by an average case error with respect to any probability measure. For specially constructed probability measures, we sometimes obtain sharp bounds on the minimal randomized errors.

More precisely, as in Section 4.3.3 of Volume I, let A_n be a randomized algorithm that uses at most n function values on the average for functions belonging to F_d that is a subset of a normed space. By $e^{\text{ran}}(A_n) = e^{\text{ran}}(A_n, F_d)$ we mean the randomized error of A_n. By

$$e^{\text{ran}}(n, F_d) = \inf_{A_n} e^{\text{ran}}(A_n, F_d)$$

we denote the nth minimal randomized error. Lemma 4.37 of Volume I states that

$$e^{\text{ran}}(n, F_d) \ge \frac{\sqrt{2}}{2}\, e^{\text{avg}}(2n, F_d, \varrho), \tag{17.20}$$

where $e^{\text{avg}}(2n, F_d, \varrho)$ denotes the $2n$th minimal average case error for an atomic measure ϱ on F_d.

We prove two more lower bounds. The first bound will be slightly better than the estimate (17.20), and the second bound will relate the d-variate case to the univariate case. Then we present a class where Monte Carlo is nearly optimal. Hence, depending on the space of functions, Monte Carlo may enjoy optimality or be far away from being optimal.

17.1.6.1 Lower bounds. We slightly modify Lemma 1 from [205], p. 63.

Lemma 17.10. *Let F_d be a set of integrable real functions defined on D_d and let $f_1, f_2, \dots, f_N$ be a family of functions with the following properties:*

- *$f_i \in F_d$ and $-f_i \in F_d$ for all $i = 1, 2, \dots, N$,*
- *the functions f_i have disjoint supports and satisfy $\text{INT}_d(f_i) = \eta > 0$.*

Then for $n < N$ we have

$$e^{\text{ran}}(n, F_d) \ge \left(1 - \frac{n}{N}\right)^{1/2} \eta.$$

Proof. We apply the idea of Bakhvalov [4] and switch to multivariate integration in the average case setting on the set $M = \{\pm f_i \mid i = 1, 2, \dots, N\}$ with the uniform distribution ϱ. That is, the average case error of an algorithm A is now

$$e^{\text{avg}}(A) = \left[\frac{1}{2N}\sum_{i=1}^{N}\left[(\text{INT}_d(f_i) - A(f_i))^2 + (\text{INT}_d(-f_i) - A(-f_i)^2\right]\right]^{1/2}$$

$$= \left[\frac{1}{2N} \sum_{i=1}^{N} \left[(\eta - A(f_i))^2 + (-\eta - A(-f_i)^2 \right] \right]^{1/2}.$$

Suppose first that A uses k function values, $k < N$. Then at least $N - k$ supports of f_i's are missed and for these functions $A(f_i) = A(-f_i)$. Then

$$(\eta - A(f_i))^2 + (-\eta - A(-f_i)^2 \geq 2\eta^2,$$

and therefore

$$\left[e^{\text{avg}}(A) \right]^2 \geq \frac{1}{2N}(N-k)2\eta^2 = \left(1 - \frac{k}{N} \right) \eta^2.$$

Next, let A use k function values with probability p_k such that $\sum_{k=1}^{\infty} p_k = 1$ and $\sum_{k=1}^{\infty} k p_k \leq n$. Then

$$\left[e^{\text{avg}}(A) \right]^2 \geq \sum_{k=1}^{\infty} p_k \left(1 - \frac{k}{N} \right) \eta^2 = \left(1 - \frac{\sum_{k=1}^{\infty} k p_k}{N} \right) \eta^2 \geq \left(1 - \frac{n}{N} \right) \eta^2.$$

Since this holds for any algorithm using n function values on the average, we conclude that $e^{\text{avg}}(n, F_d, \varrho) \geq (1 - n/N)_+^{1/2} \eta$.

Take now an arbitrary randomized algorithm A_n that uses n function values on the average. The square of its randomized error is

$$\begin{aligned}
e^2(A_n) &= \sup_{f \in F_d} \int_{\Omega}^{*} (\text{INT}_d(f) - A_{n,\omega}(f))^2 \, \mu(\mathrm{d}\,\omega) \\
&\geq \int_{\Omega}^{*} \left(\frac{1}{2N} \sum_{i=1}^{N} \left[(\eta - A_{n,\omega}(f_i))^2 + (-\eta - A_{n,\omega})^2 \right] \right) \mu(\mathrm{d}\,\omega) \\
&\geq \int_{\Omega}^{*} \left[e^{\text{avg}}(n, F_d, \varrho) \right]^2 \mu(\mathrm{d}\,\omega) = \left[e^{\text{avg}}(n, F_d, \varrho) \right]^2 \\
&\geq \left(1 - \frac{n}{N} \right)_+^2 \eta^2.
\end{aligned}$$

This completes the proof. □

We now assume that F_1 is a normed space of univariate functions and univariate integration $\text{INT}_1 \colon F_1 \to \mathbb{R}$ is a linear functional that we approximate by (deterministic or randomized) algorithms that involve n function values (on the average for the randomized setting).

For $d > 1$, we assume that F_d has the tensor product property, i.e., if $f_i \in F_1$ then $f_1 \cdot f_2 \cdots f_d \in F_d$ and the underlying norm of F_d has the property

$$\| f_1 \cdot f_2 \cdots f_d \|_{F_d} = \prod_{i=1}^{d} \| f_i \|_{F_1}. \tag{17.21}$$

By $f_1 \cdot f_2 \cdots f_d$ we mean the tensor product function, defined by

$$(f_1 \cdot f_2 \cdots f_d)(x) = \prod_{i=1}^{d} f_i(x_i).$$

Clearly,

$$\mathrm{INT}_d(f_1 \cdot f_2 \cdots f_d) = \prod_{i=1}^{d} \mathrm{INT}_1(f_i). \tag{17.22}$$

Observe that the multivariate problem is not necessarily completely defined by (17.22) and (17.21). We only know that tensor products (with factors in F_1) belong to F_d and we know how INT_d, as well as the norm on F_d, are defined for tensor products. This is enough, however, for the proof of certain *lower* bounds on the minimal errors.

We consider deterministic and randomized algorithms A_n, and we denote by $e^{\mathrm{wor}}(n, F_d)$ and $e^{\mathrm{ran}}(n, F_d)$ the nth minimal error of algorithms A_n in the worst case and randomized settings for multivariate integration.

Theorem 17.11. *Let F_d satisfy* (17.21). *Then*

$$e^{\mathrm{wor}}(n, F_d) \ge e^{\mathrm{wor}}(n, F_1) \cdot \|\mathrm{INT}_1\|^{d-1},$$
$$e^{\mathrm{ran}}(n, F_d) \ge e^{\mathrm{ran}}(n, F_1) \cdot \|\mathrm{INT}_1\|^{d-1}.$$

Proof. Assume that A_n is a (randomized or deterministic) algorithm for INT_d on F_d using at most n function values (on the average in the randomized setting). Then, in particular, the error of $e(A_n)$ is lower bounded if we consider all functions of the form

$$f(x) = \tilde{f}(x_1) f_\delta(x_2) f_\delta(x_3) \dots f_\delta(x_d), \tag{17.23}$$

where $\tilde{f} \in F_1$ with $\|\tilde{f}\|_{F_1} = 1$, and $f_\delta \in F_1$ with $\|f_\delta\|_{F_1} = 1$ and $\mathrm{INT}_1(f_\delta) \ge \|\mathrm{INT}_1\| - \delta$ for $\delta \in (0, \|\mathrm{INT}_1\|)$. Then $f \in F_d$ and $\|f\|_{F_d} = 1$. Observe that

$$\mathrm{INT}_d(f) = \mathrm{INT}_1(\tilde{f}) \cdot \mathrm{INT}_1(f_\delta)^{d-1} \ge \mathrm{INT}_1(\tilde{f}) \cdot (\|\mathrm{INT}_1\| - \delta)^{d-1}.$$

Denote all functions f of the form (17.23) by $\widetilde{F}_d$. Clearly, $\widetilde{F}_d \subseteq F_d$.

We define the algorithm $\widetilde{A}_n$ for functions from F_1 by

$$\widetilde{A}_n(\tilde{f}) = A_n(f) \cdot \mathrm{INT}_1(f_\delta)^{1-d}.$$

We now show that

$$e(\widetilde{A}_n) \le e(A_n) \cdot (\|\mathrm{INT}_1\| - \delta)^{1-d}.$$

Indeed, consider the randomized setting. Then

$$\begin{aligned} e^2(\widetilde{A}_n) &= \sup_{\tilde{f} \in F_1, \|\tilde{f}\|_{F_1} \le 1} \int_\Omega^* \left(\mathrm{INT}_1(\tilde{f}) - A_{n,\omega}(\tilde{f})\right)^2 \mu(\mathrm{d}\,\omega) \\ &= \mathrm{INT}_1(f_\delta)^{2(1-d)} \sup_{f \in \widetilde{F}_d} \int_\Omega^* (\mathrm{INT}_d(f) - A_{n,\omega}(f))^2 \mu(\mathrm{d}\,\omega) \\ &\le \mathrm{INT}_1(f_\delta)^{2(1-d)} e^2(A_n) \le (\|I_1\| - \delta)^{2(1-d)} e^2(A_n), \end{aligned}$$

as claimed. For the worst case setting we proceed similarly.

Since $e(\widetilde{A}_n) \ge e^{\text{sett}}(n, F_d)$ for sett $\in \{\text{wor}, \text{ran}\}$, respectively, we have

$$e(A_n) \ge e^{\text{sett}}(n, F_d)\ (\|I_d\| - \delta)^{d-1}.$$

This holds for any algorithm A_n and for all small δ. Therefore

$$e^{\text{sett}}(n, F_d) \ge e^{\text{sett}}(n, F_d)\, \|I_d\|^{d-1},$$

as claimed. □

A few comments are in order. We have to work with f_δ since the existence of $f_0 \in F_1$ with $\|f_0\| = 1$ and $\text{INT}_1(f_0) = \|\text{INT}_1\|$ cannot be guaranteed. An example would be the star-discrepancy with $\|\text{INT}_1\| = 1$.

For the worst case setting, Theorem 17.11 is similar to Theorem 11.7 in Chapter 11. However, Theorem 17.11 is more general since it applies to more general spaces F_d. Indeed, Theorem 11.7 assumes that F_d is a reproducing kernel Hilbert space, where Theorem 17.11 can be applied for normed spaces equipped with (weighted) L_p norms for $p \in [1, \infty]$ for $D_d = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_d, b_d]$ for some finite or infinite a_j and b_j.

From Theorems 11.7 and 17.11, we cannot conclude much about tractability if $\|\text{INT}_1\| \le 1$. Even if $\|\text{INT}_1\| > 1$ then all depends on the sequence $e^{\text{sett}}(n, F_1)$. If, for example, $e^{\text{sett}}(n, F_1) = \gamma^n$ with $\gamma < 1$ then it would be possible only to conclude that

$$n^{\text{sett}}(\varepsilon, d) \ge \left\lceil \frac{(d-1)\ln \|\text{INT}_1\| + \ln \varepsilon^{-1}}{\ln \gamma^{-1}} \right\rceil$$

for the absolute error criterion. Hence, the problem may be still tractable. However, if the univariate problem is not "too easy" then we get intractability. The following result holds for the randomized and for the deterministic case.

Corollary 17.12. *Let F_d satisfy* (17.21) *and* sett $\in \{\text{wor}, \text{ran}\}$. *Assume that*

$$e^{\text{sett}}(n, F_1) \ge C\, n^{-\alpha} \quad \textit{with } C, \alpha > 0 \quad \textit{and} \quad \|\text{INT}_1\| > 1.$$

Then for the absolute error criterion we have

$$n^{\text{sett}}(\varepsilon, d) \ge C^{1/\alpha}\, \|\text{INT}_1\|^{(d-1)/\alpha}\, \varepsilon^{-1/\alpha},$$

i.e., the problem is intractable and suffers from the curse of dimensionality.

Proof. This follows directly from Theorem 17.11. □

Remark 17.13. Theorem 17.11 and Corollary 17.12 can be slightly generalized for the weighted case, and this will be needed later in Chapter 20. Assume that

$$F_{d,\gamma} = F_{1,\gamma_{d,1}} \otimes F_{1,\gamma_{d,2}} \otimes \cdots \otimes F_{d,\gamma_{d,d}}$$

be the tensor product of the Hilbert spaces $F_{1,\gamma_{d,j}}$ of univariate functions with the norms depending on non-negative weights $\gamma_{d,j}$. Then for $f_j \in H_{1,\gamma_{d,j}}$ we have

$$\|f_1 f_2 \cdot f_d\|_{F_{d,\gamma}} = \prod_{i=1}^{d} \|f_i\|_{F_{1,\gamma_{d,i}}},$$

and Theorem 17.11 generalizes to

$$e^{\mathrm{wor}}(n, F_{d,\gamma}) \ge e^{\mathrm{wor}}(n, F_{1,\gamma_{d,1}}) \cdot \prod_{i=2}^{d} \|\mathrm{INT}_{1,\gamma_{d,i}}\|,$$
$$e^{\mathrm{ran}}(n, F_{d,\gamma}) \ge e^{\mathrm{ran}}(n, F_{1,\gamma_{d,1}}) \cdot \prod_{i=2}^{d} \|\mathrm{INT}_{1,\gamma_{d,i}}\|.$$

Similarly, Corollary 17.12 generalizes as follows. For sett $\in \{\mathrm{wor}, \mathrm{ran}\}$, assume that

$$e^{\mathrm{sett}}(n, F_{1,\gamma_{d,1}}) \ge C_{\gamma_{d,1}}\, n^{-\alpha} \quad \text{with } C_{\gamma_{d,1}}, \alpha > 0.$$

Then for the absolute error criterion we have

$$n^{\mathrm{sett}}(\varepsilon, d) \ge C_{\gamma_{d,1}}^{1/\alpha} \left(\prod_{i=2}^{d-1} \|\mathrm{INT}_{1,\gamma_{d,i}}\| \right)^{1/\alpha} \varepsilon^{-1/\alpha}.$$

If

$$C_{\gamma_{d,1}}^{1/\alpha} \left(\prod_{i=2}^{d-1} \|\mathrm{INT}_{1,\gamma_{d,i}}\| \right)^{1/\alpha}$$

goes exponentially fast to infinity with d, then the problem is intractable and suffers from the curse of dimensionality.

17.1.6.2 Monte Carlo cannot be improved. We consider the space F_{d,γ_d} of integrable real functions defined on $[0,1]^d$ which is algebraically the same as the space $L_2([0,1]^d)$ with the norm

$$\|f\|_d^2 = \alpha_d^{-1} \left(\int_{[0,1]^d} f(x)\,\mathrm{d}x \right)^2 + \beta_d^{-1} \int_{[0,1]^d} f^2(x)\,\mathrm{d}x,$$

where $\gamma_d = [\alpha_d, \beta_d]$ with positive weights α_d and β_d. Consider multivariate integration

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \quad \text{for all } f \in F_{d,\gamma}.$$

Let us start with the initial error which, as always, is $e^{\mathrm{ran}}(0, F_{d,\gamma_d}) = \|\mathrm{INT}_d\|$. Since $|\mathrm{INT}_d(f)| \le \|f\|_{L_2}$ with equality for constant functions, it is enough to consider a constant function $f_c \equiv c$ and compute c such that

$$\|f_c\|_d^2 = \alpha_d^{-1}\, c^2 + \beta_d^{-1}\, c^2 = 1.$$

From this we conclude that

$$e^{\text{ran}}(0, F_{d,\gamma_d}) = \left(\alpha_d^{-1} + \beta_d^{-1}\right)^{-1/2}.$$

We will prove error bounds for Monte Carlo and lower bounds for all randomized algorithms that use n function evaluations on the average. Let $e^{\text{MC}}(n, F_{d,\gamma_d})$ be the randomized error of Monte Carlo. We know that for a specific f the error is bounded by $n^{-1/2} \cdot \|f\|_{L_2}$ and equality holds if $\text{INT}_d(f) = 0$. From this we easily obtain

$$e^{\text{MC}}(n, F_{d,\gamma_d}) = \left(\frac{\beta_d}{n}\right)^{1/2}.$$

Note that

$$e^{\text{MC}}(n, F_{d,\gamma_d}) \ge e^{\text{ran}}(0, F_{d,\gamma_d}) \quad \text{iff} \quad n \le \frac{\beta_d}{\alpha_d} + 1.$$

Hence, for such small n, Monte Carlo is even not better than the zero algorithm $A_0 = 0$. For such n, we can modify Monte Carlo and consider

$$A_c(f) = \frac{c}{n} \sum_{i=1}^{n} f(t_i)$$

with uniformly distributed and independent $t_i \in [0,1]^d$. Hence, we just re-scale Monte Carlo by multiplying by not yet prescribed c. One can obviously claim that there is really not much difference between A_c and Monte Carlo, although the proper choice of c may be helpful.

It is easy to check that the square of the randomized error of A_c for f is

$$e^{\text{ran}}(A_c, f)^2 = \left(1-c\right)^2 \text{INT}_d^2(f) + \frac{c^2}{n}\left[\text{INT}_d(f^2) - \text{INT}_d^2(f)\right].$$

From this we obtain

$$e^{\text{ran}}(A_c, f)^2 \le \max\left(\alpha_d\left[(1-c)^2 - c^2/n\right]_+, \beta_d c^2/n\right)\|f\|_d.$$

We choose c such that

$$\alpha_d\left[(1-c)^2 - c^2/n\right]_+ = \beta_d c^2/n$$

or

$$c = c_d = \frac{\sqrt{n\alpha_d}}{\sqrt{n\alpha_d} + \sqrt{\alpha_d + \beta_d}} = \frac{1}{1 + \sqrt{\frac{1}{n} + \frac{\beta_d}{n\alpha_d}}}.$$

For this c_d, we obtain the error bound

$$e^{\text{ran}}(A_{c_d}, F_{d,\gamma_d}) \le \frac{1}{\sqrt{n\beta_d^{-1}} + \sqrt{\alpha_d^{-1} + \beta_d^{-1}}}.$$

The error of A_c is always smaller than the initial error but for large n it tends to the error of Monte Carlo since c as a function of n tends to 1.

Assume now that we want to improve the initial error by a factor of $\varepsilon < 1$, and let $n^{\text{ran}}(\varepsilon, d)$ be the minimal number of randomized function values to achieve this goal for the normalized error criterion. Then, with the algorithm A_{c_d} from above, an easy calculation yields the upper bound

$$n^{\text{ran}}(\varepsilon, d) \le \left\lceil \frac{(1-\varepsilon)^2}{\varepsilon^2} \left(1 + \frac{\beta_d}{\alpha_d}\right) \right\rceil. \tag{17.24}$$

If instead we use Monte Carlo with $c = 1$ we need to perform

$$n^{\text{MC}}(\varepsilon, d) = \left\lceil \frac{1}{\varepsilon^2} \left(1 + \frac{\beta_d}{\alpha_d}\right) \right\rceil \tag{17.25}$$

randomized function values which is only slightly worse unless ε is close to 1. Now we prove a lower bound and establish optimality of A_{c_d} and Monte Carlo.

Theorem 17.14. *Consider multivariate integration over F_{d,γ_d} for the normalized error criterion. Then*

$$n^{\text{ran}}(\varepsilon, d) \ge \frac{(1-\varepsilon)^2(1+\varepsilon)^2}{4\varepsilon^2} \left(1 + \frac{\beta_d}{\alpha_d}\right) \quad \textit{for all } \varepsilon \le 1. \tag{17.26}$$

Hence, the algorithm A_{c_d} uses at most 4*, and Monte Carlo uses at most* $4/(1-\varepsilon_0^2)^2$ *times more randomized function values than needed. For Monte Carlo, we assume that* $\varepsilon \in (0, \varepsilon_0]$ *with* $\varepsilon_0 \in (0, 1)$.

Proof. For the class F_{d,γ_d} we divide the domain $[0,1]^d$ in N equal (in volume) disjoint parts and define f_i as a constant c in one part and zero otherwise. We have $f_i \in F_{d,\gamma_d}$ and we obtain $\|f_i\|_d = 1$ if we take $c = N/\sqrt{\alpha_d^{-1} + \beta_d^{-1}}$. Then $\text{INT}_d(f_i) = \eta$ with

$$\eta^2 = \frac{1}{\alpha_d^{-1} + N\beta_d^{-1}}.$$

From Lemma 17.10 we have the lower bound

$$\left[e^{\text{ran}}\left(n, F_{d,\gamma_d}\right)\right]^2 \ge \frac{N-n}{N} \cdot \frac{1}{\alpha_d^{-1} + N\beta_d^{-1}}.$$

We choose

$$N = 1 + \left\lfloor \frac{1-\varepsilon^2}{2\varepsilon^2} \left(1 + \frac{\beta_d}{\alpha_d}\right) \right\rfloor.$$

Then $e^{\text{ran}}(n, F_{d,\gamma_d}) \le \varepsilon\, e^{\text{ran}}(0, F_{d,\gamma_d})$ implies

$$n \ge N\left(1 - \varepsilon^2 \frac{\alpha_d^{-1} + N\beta_d^{-1}}{\alpha_d^{-1} + \beta_d^{-1}}\right).$$

Since

$$\frac{\alpha_d^{-1} + N\beta_d^{-1}}{\alpha_d^{-1} + \beta_d^{-1}} \le \frac{\alpha_d^{-1} + \beta_d^{-1} + \frac{1-\varepsilon^2}{2\varepsilon^2}(\beta_d^{-1} + \alpha_d^{-1})}{\alpha_d^{-1} + \beta_d^{-1}} = 1 + \frac{1-\varepsilon^2}{2\varepsilon^2},$$

and

$$1 - \varepsilon^2 \frac{\alpha_d^{-1} + N\beta_d^{-1}}{\alpha_d^{-1} + \beta_d^{-1}} \ge 1 - \varepsilon^2 \left(1 + \frac{1-\varepsilon^2}{2\varepsilon^2}\right) = \tfrac{1}{2}(1-\varepsilon^2),$$

we obtain

$$n \ge \tfrac{1}{2}(1-\varepsilon^2)N \ge \frac{(1-\varepsilon^2)^2}{4\varepsilon^2}\left(1 + \frac{\beta_d}{\alpha_d}\right).$$

Therefore

$$n^{\mathrm{ran}}(\varepsilon, d) \ge \frac{(1-\varepsilon)^2(1+\varepsilon)^2}{4\varepsilon^2}\left(1 + \frac{\beta_d}{\alpha_d}\right) \quad \text{for all } \varepsilon \le 1,$$

as claimed. The rest easily follows from the randomized error bounds for A_{c_d} and Monte Carlo. □

Theorem 17.14 states that A_{c_d} and Monte Carlo are almost optimal. Since ε_0 can be taken arbitrarily small, both algorithms minimize the number of randomized function values up to a factor 4. It is not clear if this factor can be improved and this leads us to our next problem.

Open Problem 79.

- Verify how much we can improve the quality of the algorithms A_{c_d} and Monte Carlo or how much we can improve the lower bound presented in Theorem 17.14.

Based on Theorem 17.14 it is easy to provide tractability conditions which we present in the following corollary.

Corollary 17.15. *Consider multivariate integration* INT $= \{\mathrm{INT}_d\}$ *for the spaces* $F_{d,\gamma}$ *in the randomized setting and for the normalized error criterion. Then*

- INT *is strongly polynomially tractable iff*

$$C := \sup_d \frac{\beta_d}{\alpha_d} < \infty.$$

Then

$$n^{\mathrm{ran}}(\varepsilon, d) \le \left\lceil (1 + C)\,\frac{(1-\varepsilon)^2}{\varepsilon^2} \right\rceil \quad \textit{for all } \varepsilon \in (0,1] \textit{ and } d \in \mathbb{N}.$$

- INT *is polynomially tractable iff there is a non-negative q such that*

$$C := \sup_d d^{-q} \frac{\beta_d}{\alpha_d} < \infty.$$

Then

$$n^{\rm ran}(\varepsilon, d) \le \left\lceil (1 + C\, d^q)\, \frac{(1-\varepsilon)^2}{\varepsilon^2} \right\rceil \quad \textit{for all } \varepsilon \in (0,1] \textit{ and } d \in \mathbb{N}.$$

- INT *is weakly tractable iff*

$$\lim_{d \to \infty} \frac{\ln\left(1 + \frac{\beta_d}{\alpha_d}\right)}{d} = 0.$$

Then

$$n^{\rm ran}(\varepsilon, d) = \exp\left(o(d)\right)\, \varepsilon^{-2}\, (1-\varepsilon)^2 \quad \textit{for all } \varepsilon \in (0,1] \textit{ and } d \in \mathbb{N}.$$

Observe that if $\beta_d \equiv 1$ and α_d approaches infinity then the unit ball of F_{d,γ_d} becomes the unit ball of $L_2([0,1]^d)$, and we have strong polynomial tractability. On the other hand, if $\alpha_d \equiv 1$ and β_d approaches infinity then the unit ball of F_{d,γ_d} becomes the unit ball of functions whose integrals in the absolute sense are at most 1. In this case, we may be in trouble depending on the speed of convergence of β_d to infinity. For instance, if $\beta_d = 2^{2^d}$ the problem is intractable, if $\beta_d = 2^d$ then the problem is weakly tractable but not polynomially tractable.

17.2 Importance Sampling

We first define what we mean by *importance sampling* and present a simple example to demonstrate its power. Then we describe results of Wasilkowski [328], Plaskota, Wasilkowski and Zhou [248], and Hinrichs [131], where importance sampling is used.

Let F_d be a subclass of $L_{2,\varrho_d}(D_d)$ with $\varrho_d > 0$ and $\int_{D_d} \varrho_d(x)\,{\rm d}x = 1$. For a weighted integral

$$\mathrm{INT}_{d,\varrho_d}(f) = \int_{D_d} f(x)\varrho_d(x)\,{\rm d}x \quad \text{for all } f \in F_d,$$

importance sampling has the general form

$$Q_{\omega_d,n}(f) = \frac{1}{n} \sum_{j=1}^{n} f(t_j) c_d(t_j),$$

with randomly and independently chosen t_j with distribution whose probability density is ω_d and $c_d = \varrho_d/\omega_d$. Clearly, the randomized error of $Q_{\omega_d,n}$ is

$$e^{\rm ran}(Q_{\omega_d,n}, F_d) = \frac{1}{\sqrt{n}} \sup_{f\in F_d} \left(\int_{D_d} f^2(t)\,\varrho_d^2(t)\,\frac{1}{\omega_d(t)}\,{\rm d}t \;-\; {\rm INT}^2_{d,\varrho_d}(f) \right)^{1/2}.$$

For $\omega_d = \varrho_d \equiv 1$ we obtain the standard Monte Carlo algorithm.

The main point of importance sampling is to find a probability density ω_d which minimizes or nearly minimizes the supremum above. In particular, we would like to find ω_d for which the randomized error of $Q_{\omega_d,n}$ does not depend exponentially on d.

We illustrate the power of importance sampling by the following example. Take now $D_d = [0,1]^d$ and $\varrho_d \equiv 1$, and consider the integration problem

$${\rm INT}_d(f) = \int_{[0,1]^d} f(x)\,{\rm d}x,$$

this time on a Hilbert space F_d with a scalar product

$$\langle f, g\rangle_{H_d} = \int_{[0,1]^d} f(x)g(x)\,h_d^{-1}(x)\,{\rm d}x.$$

To be specific, we take

$$h_d(x) = (e-1)^{-d}\exp\Big(\sum_{i=1}^d x_i\Big).$$

We then have ${\rm INT}_d(f) = \langle f, h_d\rangle_{H_d}$ and $\|h_d\|_{H_d} = 1$.

Of course, we may take the standard Monte Carlo algorithm that corresponds to $c_d = \omega_d \equiv 1$. For $f = h_d \in H_d$ with $\|h_d\|_{H_d} = 1$ we obtain the variance

$$\begin{aligned}\sigma^2(h_d) &= \int_{[0,1]^d} (h_d(x)-1)^2{\rm d}x\\ &= \int_{[0,1]^d} h_d(x)^2{\rm d}x - 1\\ &= \left(\frac{e+1}{2(e-1)}\right)^d - 1 \approx 1.08197^d.\end{aligned}$$

Hence, the problem is intractable for Monte Carlo.

However, we can consider *importance sampling* based on the measure with Lebesgue density h_d on $[0,1]^d$. Hence we write

$${\rm INT}_d(f) = \int_{[0,1]^d} \frac{f(x)}{h_d(x)}\cdot h_d(x)\,{\rm d}x = \int_{[0,1]^d} g(x)\cdot h_d(x)\,{\rm d}x$$

with $g = f/h_d$.

We use again n independently chosen sampling points in $[0,1]^d$, this time chosen with respect to the density h_d, and replace the integrand f by the integral of g. That is, $\omega_d = h_d$ and $c_d = 1/h_d$ and our algorithm $Q_{\omega,n}$ takes the form

$$Q_{\omega,n}(f) = \frac{1}{n}\sum_{j=1}^{n} f(t_j)\frac{1}{h_d(t_j)}.$$

Then, again, we obtain the standard error bound $n^{-1/2}$ for $\|g\|_{H_d} \le 1$, i.e., for

$$\int_{[0,1]^d} \frac{f(x)^2}{h_d(x)^2}\cdot h_d(x)\,\mathrm{d}x \le 1$$

or for $\|f\|_{H_d} \le 1$.

Hence, the problem is strongly polynomially tractable and we proved it by using importance sampling. Based on the known result that standard Monte Carlo is almost optimal for L_2 it is easy to prove that this algorithm is almost optimal.

This simple example demonstrates the power of importance sampling. In the successive subsections we report further examples, where importance sampling is successfully used.

17.2.1 Results for Sobolev spaces

As in Section 17.1.1.2, consider the space $H_{d,\gamma}$ of periodic functions with the reproducing kernel

$$\tilde{K}_{d,\gamma}(x,y) = \prod_{j=1}^{d}\left[1+\gamma_{d,j}(\min(x_j,y_j)-x_jy_j)\right] \quad \text{for all } x,y\in[0,1]^d.$$

This corresponds to the anchor $a = 0$ and product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}}\gamma_{d,j}$ with $\gamma_{d,j+1} \le \gamma_{d,j}$. We assume for simplicity that $\sup_d \gamma_{d,1} < \infty$.

Wasilkowski [328] studied the following generalization of Monte Carlo for multivariate integration

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \quad \text{for all } f\in H(\tilde{K}_d).$$

For $\alpha\in[0,1]$, let

$$\omega_{d,\alpha}(x) = \frac{\left[\tilde{K}_{d,\gamma}(x,x)\right]^{\alpha}}{\int_{[0,1]^d}\left[\tilde{K}_{d,\gamma}(y,y)\right]^{\alpha}\,\mathrm{d}y}$$

be a density function. Consider the algorithm

$$A_{n,\alpha}(f) = \frac{1}{n}\sum_{j=1}^{n}\frac{f(x_j)}{\omega_{d,\alpha}(x_j)}$$

with the randomized points $x_1, x_2, \dots, x_n$ which are independent and distributed according to the density $\omega_{d,\alpha}$. Note that for $\alpha = 0$, we have $\omega_{d,0} \equiv 1$, and $A_{n,0}$ becomes Monte Carlo. Hence, the main point is to select α to relax tractability conditions of Monte Carlo.

The randomized error of $A_{n,\alpha}$ for f is easily seen to be

$$\mathbb{E}\,[\mathrm{INT}_d(f) - A_{n,\alpha}(f)]^2 = \frac{1}{n}\left[\int_{[0,1]^d} \frac{f^2(x)}{\omega_{d,\alpha}(x)}\,\mathrm{d}x - \mathrm{INT}_d^2(f)\right].$$

Let

$$\Lambda_{d,\alpha} = \sup_{\|f\|_{H_{d,\gamma}} \le 1} \int_{[0,1]^d} \frac{f^2(x)}{\omega_{d,\alpha}(x)}\,\mathrm{d}x.$$

Then if we take F_d as the unit ball of $H(\tilde{K}_d)$ then the randomized error $e^{\mathrm{ran}}(A_{n,\alpha})$ is bounded by

$$\frac{1}{n^{1/2}}\left(\Lambda_{d,\alpha} - \|\mathrm{INT}_d\|^2\right)_+^{1/2} \le e^{\mathrm{ran}}(A_{n,\alpha}) \le \frac{1}{n^{1/2}}\,\Lambda_{d,\alpha}^{1/2}.$$

We know that $\|\mathrm{INT}_d\|^2 = \prod_{j=1}^d (1 + \gamma_{d,j}/12)$, and it is shown in Wasilkowski [328] that

$$\Lambda_{d,\alpha} = \prod_{j=1}^{d}\left[1 + \tfrac{1}{12}\gamma_{d,j} + \frac{(2\alpha-1)^2}{720}\gamma_{d,j}^2 + B(\alpha)\,\gamma_{d,j}^3 + \mathcal{O}(\gamma_{d,j}^4)\right],$$

where

$$B(\alpha) = \frac{\alpha^3}{180} - \frac{143\,\alpha^2}{21168} + \frac{139\,\alpha^3}{35280} + \frac{121}{60480}.$$

Consider first the absolute error criterion. Using the same analysis as in Section 17.1.1.2, it is easy to check that for any $\alpha \in [0,1]$, the algorithm $A_{n,\alpha}$ is strongly polynomially, polynomially or weakly tractable iff Monte Carlo is strongly polynomially, polynomially or weakly tractable. So in this case, the parameter α does *not* relax tractability conditions.

The situation is quite different for the normalized error criterion. Then we need to consider

$$\frac{\Lambda_{d,\alpha}}{\|\mathrm{INT}_d\|^2} = \prod_{j=1}^{d}\left(1 + \frac{(2\alpha-1)^2}{720}\,\gamma_{d,j}^2 + \mathcal{O}(\gamma_{d,j}^3)\right). \tag{17.27}$$

Hence for all $\alpha \ne \frac{1}{2}$, the last ratio depends on $\gamma_{d,j}^2$, and we have exactly the same tractability conditions for $A_{n,\alpha}$ as for Monte Carlo. But for $\alpha = \frac{1}{2}$ the term with $\gamma_{d,j}^2$ disappears and this leads to the following theorem that was proved by Wasilkowski [328] for the strong polynomial and polynomial parts. Obviously, the weak tractability part also easily follows from (17.27).

Theorem 17.16. *Consider multivariate integration for the normalized error criterion defined as in this subsection.*

The algorithm $A_{n,1/2}$ *is strongly polynomially tractable iff*

$$\sup_d \sum_{j=1}^d \gamma_{d,j}^3 < \infty,$$

is polynomially tractable iff

$$\sup_d \frac{\sum_{j=1}^d \gamma_{d,j}^3}{\ln d} < \infty,$$

is weakly tractable iff

$$\sup_d \frac{\sum_{j=1}^d \gamma_{d,j}^3}{d} = 0.$$

For Monte Carlo we have similar tractability conditions with the third power of $\gamma_{d,j}$ replaced by the second power. The relaxation of tractability conditions is important. For example, take the weights $\gamma_{d,j} = j^{-\beta}$ for a positive β. Then for $\beta \in (\frac{1}{3}, \frac{1}{2})$, the algorithm $A_{n,1/2}$ is strongly polynomially tractable and Monte Carlo is *not* polynomially but weakly tractable.

We now turn to the paper of Plaskota, Wasilkowski and Zhou [248] who consider, more generally, the weighted integration problem

$$\mathrm{INT}_{d,\varrho_d}(f) = \int_{D_d} f(x)\varrho_d(x)\,\mathrm{d}x,$$

where $\varrho_d > 0$ and $\int_{D_d} \varrho_d(x)\,\mathrm{d}x = 1$, i.e., ϱ is a probability density. They assume that the integrand is a function from a reproducing kernel Hilbert space $H(K_d)$ and that $K_d(x,x) > 0$ for almost all $x \in D_d$ and

$$C_d = \int_D \varrho_d(x) K_d(x,x)^{1/2}\,\mathrm{d}x < \infty.$$

They suggest to take the density

$$\omega_d(t) = \frac{\sqrt{K_d(t,t)}\,\varrho_d(t)}{C_d}$$

and study importance sampling

$$Q_{\omega_d,n}(f) = \frac{1}{n}\sum_{j=1}^n f(t_j)c_d(t_j),$$

with randomly and independently chosen t_j with distribution whose probability density is ω_d and $c_d = \varrho_d/\omega_d$.

Define an operator $\mathcal{S}\colon H(K_d) \to H(K_d)$ by

$$\mathcal{S}_d(f) = C_d \int_{D_d} f(t)\, \frac{\varrho_d(t)}{\sqrt{K_d(t,t)}}\, K_d(\cdot\,,t)\,\mathrm{d}t$$

and let $\lambda(\mathcal{S}_d)$ be the largest eigenvalue of $\mathcal{S}_d$. Plaskota, Wasilkowski and Zhou [248] prove the following result.

Theorem 17.17. *The randomized error of $Q_{\omega_d,n}$ on $H(K_d)$ is bounded by*

$$e^{\mathrm{ran}}(Q_{\omega_d,n}) \le \frac{\sqrt{\lambda(\mathcal{S}_d)}}{n}.$$

The proof follows from the standard error formula

$$\mathbb{E}\big(|\mathrm{INT}_{d,\varrho}(f) - Q_{\omega_d,n}(f)|^2\big) = \frac{C_d}{n} \int_{D_d} |f(x)|^2 \frac{\varrho_d(x)}{\sqrt{K_d(x,x)}}\,\mathrm{d}x - \frac{\mathrm{INT}_{d,\varrho}(f)^2}{n}.$$

One may show that

$$C_d \sup_{\|f\|_{H(K_d)} \le 1} \int_{D_d} |f(x)|^2 \frac{\varrho_d(x)}{\sqrt{K_d(x,x)}}\,\mathrm{d}x = \lambda(\mathcal{S}_d).$$

From this theorem one can deduce new tractability results for multivariate integration in the randomized setting. Tractability now depends on how $\lambda(\mathcal{S}_d)$ depends on d.

The approach of Plaskota, Wasilkowski and Zhou [248] leads, via importance sampling, to better algorithms and error bounds.

Open Problem 80.

- Consider the integration problem $\mathrm{INT}_{d,\varrho}$ on $H(K_d)$ as in the last theorem. Is it possible to modify the density

  $$\omega_d(t) = \frac{\sqrt{K_d(t,t)}\varrho_d(t)}{C_d}$$

  to prove even better error bounds?

We formulated this open problem in June 2009 and also discussed it with Hinrichs. Using deep results from the geometry of Banach spaces, such as the Pietsch Domination Theorem and the Little Grothendieck Theorem, Hinrichs [131] solved this open problem completely by proving the following result.

Theorem 17.18. *Let H be a Hilbert space of functions defined on D, $D \subseteq \mathbb{R}^d$, with reproducing kernel K. Let ϱ be a probability density on D such that the embedding $J\colon H \to L_1(\varrho)$ is a bounded operator. Assume that H has full support with respect to ϱ. Then there exists a density ω such that importance sampling $Q_{\omega,n}$ with ω for the approximation of*

$$\mathrm{INT}(f) = \int_D f(x)\varrho(x)\,\mathrm{d}x$$

has a randomized error bounded by

$$e^{\text{ran}}(Q_{\omega,n}) \le n^{-1/2} \cdot \left(\tfrac{1}{2}\pi\right)^{1/2} \|J\|_{H \to L_1(\varrho)}.$$

Full support with respect to ϱ means that there is no set with positive measure such that all functions from H vanish on it. Hinrichs [131] also proved the following theorem.

Theorem 17.19. *Let H be a reproducing kernel Hilbert space of functions with non-negative kernel K, i.e., $K(x,y) \ge 0$ for all x and y from D, and assume that the norm of* INT *is finite, i.e.,*

$$\|\text{INT}\| = \left(\int_D \int_D K(x,y)\varrho(x)\varrho(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} < \infty.$$

Assume that H has full support with respect to the measure $\varrho\,\mathrm{d}x$. Then there exists a density ω such that the randomized error of importance sampling with density ϱ is bounded by

$$e^{\text{ran}}(Q_{\omega,n}) \le n^{-1/2} \cdot \left(\tfrac{1}{2}\pi\right)^{1/2} \|\text{INT}\|.$$

Then it follows that all such problems are strongly polynomially tractable in the randomized setting with an ε^{-1} exponent at most 2 for the normalized error criterion since

$$n^{\text{ran}}(\varepsilon, d) \le \left\lceil \tfrac{1}{2}\pi\varepsilon^{-2}\right\rceil$$

for all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$. It should be noted that, in general, these results are non-constructive since we do not know how to construct ω from the kernel K and the density ϱ.[2]

17.3 Local Solution of the Laplace Equation

This section deals with the local solution of the Dirichlet problem for the Laplace equation. Using classical results of Muller [193] and Motoo [192] we show that this problem is polynomially tractable in the randomized setting for the absolute error criterion.

We start with two assumptions:

- $G \subseteq \mathbb{R}^d$ is an open and bounded set and $d \ge 2$,

[2]Added in proof: For tensor product spaces whose univariate reproducing kernel is decomposable and univariate integration is not trivial for the two spaces corresponding to decomposable parts, we have

$$e^{\text{ran}}(Q_n) \ge \lceil \varepsilon^{-2}/8 \rceil \quad \text{for all } \varepsilon \in (0,1) \text{ and } d \ge \frac{2\ln \varepsilon^{-1} - \ln 2}{\ln \alpha^{-1}},$$

where $\alpha \in [\frac{1}{2}, 1)$ depends on the particular space, as proved in [225]. In this case, the exponent of strong polynomial tractability is 2.

- $f: \partial G \to \mathbb{R}$ is continuous on the boundary ∂G of G.

By Δ we denote the Laplace operator

$$\Delta u(x) = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x) \quad \text{for all } x \in G,$$

where $u: G \to \mathbb{R}$ is a C^2-function. We want to solve the Dirichlet problem by finding a continuous function

$$u: \operatorname{cl} G \to \mathbb{R}$$

on the closure $\operatorname{cl} G$ of G such that

$$\begin{aligned} \Delta u &= 0 \quad \text{on } G, \\ u &= f \quad \text{on } \partial G. \end{aligned}$$

Kakutani [143], [144] proved that it is possible to obtain results about the existence and uniqueness of the solution of the Dirichlet problem using the Brownian motion, see also Karatzas and Shreve [147, Chapter 4, Section 5.7] and Øksendal [229, Chapter 9]. One may also use this approach to obtain randomized algorithms, see Sabelfeld and Shalimova [259] as well as Milstein and Tretyakov [188].

Let $(W_t^x)_{t\in[0,\infty)}$ be a d-dimensional Brownian motion starting at $x \in G$ and let P be its probability. We put

$$T^x = \inf\{t \in (0,\infty) \mid W_t^x \notin G\}$$

and obtain a random variable with values in $[0,\infty] := [0,\infty) \cup \{\infty\}$. Since the paths of the Brownian motion are continuous, we obtain

$$T^x = \inf\{t \in (0,\infty) \mid W_t^x \in \partial G\} \quad \text{for all } x \in G,$$

i.e., T^x is the hitting time of the boundary of G when we start at $x \in G$. It is easy to prove that for all $x \in \mathbb{R}^d$ we have

$$P(\{T^x < \infty\}) = 1.$$

For $x \in G$, we define on $\{T^x < \infty\}$

$$R^x = W_{T^x}^x$$

as the boundary point that is first hit by the process that starts at x. For $x \in \partial G$ we put $R^x = x$.

The distribution of R^x is called the *harmonic measure* on ∂G for the starting value $x \in \operatorname{cl} G$. Let

$$B_r^z = \{y \in \mathbb{R}^d \mid \|y - z\| < r\}$$

be the open Euclidean ball with radius $r > 0$ and center z and its boundary is the sphere

$$\partial B_r^z = \{y \in \mathbb{R}^d \mid \|y - z\| = r\}.$$

The boundary ∂G of G is compact, hence f is bounded and $f(R^x)$ is a random variable in L_2. We define

$$u^*(x) = \mathbb{E}(f(R^x)) \quad \text{for all } x \in \text{cl}\, G.$$

That is, $u^*(x)$ is the expectation of f with respect to the harmonic measure for the starting value at x. Clearly, $u^*(x) = f(x)$ for all $x \in \partial G$ so that u^* satisfies the boundary condition of the Dirichlet problem. We will see that u^* is, under some conditions, the unique solution of the Dirichlet problem.

Lemma 17.20. *The function u^* is C^∞ on G and harmonic.*

For the proof see Karatzas and Shreve [147, Section 4.2.A, 4.2.B].

Therefore u^* is a solution of the Dirichlet problem iff it is continuous for all $x \in \partial G$. It is also well known that every solution of the Dirichlet problem fulfills $u = u^*$, see Karatzas and Shreve [147, Proposition 4.2.7].

A point $x \in \partial G$ is called *regular* if

$$P(\{T^x > 0\}) = 0.$$

We add that convexity of G implies that all boundary points are regular. The following characterization of regular boundary points is important.

Lemma 17.21. *A point $x \in \partial G$ is regular iff*

$$\lim_{y \to x,\, y \in G} \mathbb{E}(g(R^y)) = g(x)$$

for all continuous mappings $g \colon \partial G \to \mathbb{R}$.

For the proof of all these statements see Karatzas and Shreve [147, Theorem 4.2.19, Theorem 4.2.12].

To summarize, and this is the starting point for randomized algorithms, the function u^* is the unique solution of the Dirichlet problem if every boundary point of G is regular. This is the case for Lipschitz domains and for convex domains. From now on we assume that all boundary points are regular.

We are ready to define our computational problem. For a fixed $x \in G$ we want to approximate the solution $u^*(x) = u(x)$ of the Dirichlet problem at x. Obviously, $u(x) = u(x, f)$ depends linearly on the boundary function of f. We assume that f belongs to a linear space F_d that will be specified later and will be a subset of continuous functions. That is why

$$I_d(f) = u(x, f) \quad \text{for all } f \in F_d$$

is a linear functional. We choose F_d such that I_d is also continuous. We stress that I_d is usually not a tensor product linear functional and therefore the previous results for such linear functionals do not apply.

We first consider the randomized setting for $I = \{I_d\}$ and later briefly mention what happens for the worst case setting. To simplify the presentation we assume that

- the function f is defined and continuous on $\operatorname{cl} G$.

To approximate

$$u(x) = \mathbb{E}(f(R^x))$$

it is natural to simulate $Y = f(R^x)$ several times and to use the mean value as an approximation. We approximate R^x by random variables $R^{x,\delta}$ with values in $\operatorname{cl} G$ that converge to R^x as $\delta \to 0$. We discretize the Brownian motion and study the *spherical process* that was developed by Muller [193].

For $z \in \operatorname{cl} G$, we define

$$r(z) = \inf\{\|z - y\| \mid y \in \partial G\}$$

and

$$S(z) = \partial B^z_{r(z)},$$

as the largest sphere in $\operatorname{cl} G$ with center z.

We define recursively

$$T^x_0 = 0, \quad \widetilde{W}^x_0 = x,$$

and

$$T^x_{k+1} = \inf\{t \in [T^x_k, T^x] \mid W^x_t \in S(\widetilde{W}^x_k)\}$$

and

$$\widetilde{W}^x_{k+1} = W^x_{T^x_{k+1}}.$$

Clearly,

$$\widetilde{W}^x_{k+1} \in S(\widetilde{W}^x_k) \subseteq \operatorname{cl} G.$$

The sequence $(\widetilde{W}^x_k)_{k \in \mathbb{N}_0}$ is called the *spherical process*. For $\delta > 0$, define the hitting time

$$k^\delta = \inf\{k \in \mathbb{N} \mid \widetilde{W}^x_k \in G^\delta\}$$

of the spherical process of the set

$$G^\delta = \{y \in G \mid r(y) \leq \delta\}$$

that is used as a stopping rule. Hence we consider

$$R^{x,\delta} = \widetilde{W}^x_{k^\delta}$$

on $\{k^\delta < \infty\}$. Then, with probability 1, we obtain $k^\delta < \infty$ and

$$\lim_{\delta \to 0} R^{x,\delta} = R^x.$$

To obtain an approximation of $u(x) = \mathbb{E}(Y)$ we replace $Y = f(R^x)$ by the random variable

$$Y^\delta = f(R^{x,\delta})$$

and define the randomized algorithm

$$M_n^\delta = \frac{1}{n}\sum_{i=1}^{n} Y_i^\delta$$

with independent copies $Y_1^\delta, Y_2^\delta, \dots, Y_n^\delta$ of Y^δ for the approximation of $u(x)$.

We present bounds for the error and the cost of M_n^δ due to Motoo [192], see also [194]. For the error bounds we need another assumption. Namely we assume that the class F_d of functions f is chosen such that there exists a positive L for which

- the solution u and the boundary data f satisfy

$$|u(y) - u(z)| + |f(y) - f(z)| \le L\,\|y - z\| \quad \text{for all } y, z \in \operatorname{cl} G. \tag{17.28}$$

Then the following error bound is valid.

Lemma 17.22.

$$e^{\mathrm{ran}}(M_n^\delta, f)^2 \le \frac{1}{n}\,\|f\|_\infty^2 + L^2\delta^2.$$

The cost of M_n^δ is given by n, the cost to compute the distances $r(z)$ and the expected number $\mathbb{E}(k^\delta)$ of steps of the spherical process to reach G^δ. For convex sets G this number was estimated by Motoo [192, p. 53]. In the non-trivial case, when $r(x) > \delta$, the bound is

$$\mathbb{E}(k^\delta) < (1 + \ln(r(x)/\delta)) \cdot 16\,d. \tag{17.29}$$

One may replace ln by $\ln_+ = \max(\ln, 0)$ and then this bound holds for all $\delta > 0$. The proof is based on the limit case which is the half-space $\{y \in \mathbb{R}^d \mid y_1 < r(x)\}$. Therefore we obtain an explicit bound on the cost of M_n^δ if we assume

- the set $G \subseteq \mathbb{R}^d$ is open, bounded, and convex and the cost to compute distances $r(z)$ for $z \in G$ are bounded by $c_{1,d}$.

We put

$$c_{2,d} = \sup_{z\in G} r(z)$$

and denote the cost of computing $f(R^{x,\delta})$ by $\mathbf{c}_d$. We obtain the following result.

Theorem 17.23. *The cost of M_n^δ is bounded by*

$$\operatorname{cost}(M_n^\delta, f) \le \left(\kappa\, d \cdot (c_{1,d} + d) \cdot \left(1 + \ln_+ \frac{c_{2,d}}{\delta}\right) + \mathbf{c}_d\right) n$$

with a constant $\kappa > 0$ that is independent of d, G, n, δ and x.

Proof. It is enough to consider the cost to simulate the random variable $Y^\delta = f(R^{x,\delta})$. For the simulation of $R^{x,\delta}$ we need k^δ steps of the spherical process. Each step needs the computation of a distance from the boundary, the simulation of the uniform distribution on a sphere and an addition. Since the cost for the random number generator is linear in d (one can use the Box Muller algorithm for the normal distribution and then a projection of the point on the sphere), we obtain

$$\text{cost}(R^{x,\delta}) \le (\kappa_1\, d + c_{1,d}) \cdot \mathbb{E}(k^\delta)$$

with a constant κ_1 independent on d, G, δ and x. From (17.29) we conclude

$$\text{cost}(R^{x,\delta}) \le \kappa_1 \cdot (d + c_{1,d}) \cdot (1 + \ln_+(c_{2,d}/\delta)) \cdot 16\, d.$$

Finally we need n calls of the oracle with cost $n\, \mathbf{c}_d$ to compute $f(R^{x,\delta})$. □

With the choice

$$\delta_n = n^{-1/2}$$

we finally consider the algorithms

$$M_n = M_n^{\delta_n}.$$

For our final result we assume the following.

- Let

$$c_{1,d} \le c\, d \quad \text{and} \quad \mathbf{c}_d \le c\, d^2,$$

  as well as

$$\|f\|_\infty \le c, \quad L \le c, \quad \text{and} \quad c_{2,d} \le c$$

  for some $c > 0$.

Then we have an explicit bound for the cost M_n when we want to have an error that is bounded by $\varepsilon > 0$.

Theorem 17.24. *Let*

$$n(\varepsilon) = \left\lceil \frac{2\, c^2}{\varepsilon^2} \right\rceil.$$

Then

$$e^{\text{ran}}(M_{n(\varepsilon)}, f) \le \varepsilon$$

and

$$\text{cost}(M_{n(\varepsilon)}, f) \le \kappa\, d^{\,2}\, \varepsilon^{-2}\, (1 + \ln_+(\varepsilon^{-1}))$$

with $\kappa > 0$ that only depends on c.

As always, let $n(\varepsilon, d)$ denote the minimal number of function values to solve the problem in the randomized setting for the absolute error criterion. Then

$$n(\varepsilon, d) \le n(\varepsilon) = \left\lceil \frac{2\, c^2}{\varepsilon^2} \right\rceil$$

and the problem is strongly polynomially tractable in the randomized setting for the absolute error criterion. Furthermore, its complexity increases at most quadratically in the dimension d.

We add in passing a word about the worst case setting. If we assume that f is a C^k-function then one can obtain at most $n^{-k/(d-1)}$ as the order of convergence, since we basically have to solve an integration problem over the boundary of G with respect to the harmonic measure. Hence the problem is certainly not polynomially tractable for C^k-functions in the worst case setting. Hence we have another example of a problem for which the polynomial intractability of the worst case setting is vanquished by switching to the randomized setting.

17.4 Notes and Remarks

NR 17.1:1. This section is mostly based on [281]. Subsection 17.1.6 is new and contains slightly improved lower bounds and a new example.

NR 17.2:1. The section on importance sampling is mainly based on Wasilkowski [328] and Plaskota, Wasilkowski, Zhou [248], again we added a new example at the beginning of this section. At the end of this section we present the recent result of Hinrichs [131].

NR 17.3:1. This section is based on the original papers of Muller [193] and Motoo [192], with minor modifications done in [194].

Chapter 18
Nonlinear Functionals

So far we studied in this volume tractability of linear functionals. In this chapter we study tractability of certain nonlinear functionals $S^{\mathrm{non}} = \{S_d\}$.

Quasi-linear multivariate problems will be thoroughly discussed in Volume III based on [341], [342] and Werschulz [339]. Quasi-linear problems are defined by a nonlinear operator $S_d(\cdot,\cdot)$ that depends linearly on the first argument and satisfies a Lipschitz condition with respect to both arguments. Both arguments are functions of d variables. Many computational problems of practical importance have this form. Examples include the solution of specific Dirichlet, Neumann, and Schrödinger problems. In Volume III we will show, under appropriate assumptions, that quasi-linear problems whose domain spaces are equipped with product or finite-order weights are polynomially tractable or strongly polynomially tractable in the worst case setting. In this chapter we study a couple of quasi-linear multivariate problems for which S_d is a nonlinear functional as well as a couple of multivariate problems that are not quasi-linear.

More precisely, we study tractability of the following nonlinear functionals:

1. *Integration with an unknown density function.*

2. *The local solution of Fredholm integral equations of the second kind.*

3. *Computation of fixed points.*

4. *Global optimization.*

5. *Computation of the volume.*

Only the first two problems are *quasi-linear*, hence these problems could be analyzed by applying a theory of quasi-linear problems. Since this theory will be presented in Volume III, here we do not use it. All five problems are analyzed by using a proof technique very much dependent on the particular problem.

As we shall see, the nonlinear functionals studied in this chapter will be defined on unweighted spaces. Nevertheless, we will show that some of them are polynomially or even strongly polynomially tractable. This holds, in particular, for Problems 1 and 2 in the randomized setting and for Problem 3 in the worst case setting. This is in a contrast to linear functionals that are usually intractable for unweighted spaces. Hence, some nonlinear functionals are easier than linear functionals. This shows that sometimes nonlinearity is a very fruitful property that makes the problem tractable. The reader should, however, keep in mind that this very much depends on the specific nonlinear problem and there is no general result saying that nonlinear problems are always easier than linear problems. There are seven open problems numbered from 81 to 87.

18.1 Integration with Unknown Density

In many applications one wants to compute an integral of the form

$$\int_\Omega f(x) \cdot c\, q(x)\, \mu(\mathrm{d}x) \tag{18.1}$$

with a density $c\,q$, where the number c is unknown and related to positive q by

$$\frac{1}{c} = \int_\Omega q(x)\, \mu(\mathrm{d}x),$$

for some known measure μ for which Ω is μ measurable and $\mu(\Omega) > 0$.

The numerical computation of the integral defining c^{-1} is often as hard as the original problem (18.1). Therefore it is desirable to have algorithms that are able to approximately compute (18.1) without knowing the normalizing number c based solely on n function values of f and q.

We assume that $\mathcal{F}(\Omega)$ is a class of real functions (f, q) defined on Ω from the space $L_2(\Omega, \mu)$ and $q > 0$. For such a class $\mathcal{F}(\Omega)$, we define the solution operator $S : \mathcal{F}(\Omega) \to \mathbb{R}$ by

$$S(f, q) = \frac{\int_\Omega f(x) \cdot q(x)\, \mu(\mathrm{d}x)}{\int_\Omega q(x)\, \mu(\mathrm{d}x)} \quad \text{for all } (f, q) \in \mathcal{F}(\Omega). \tag{18.2}$$

This *solution operator* is linear in f but not, in general, in q. Therefore S is a nonlinear functional.

If Ω is the d-dimensional unit ball with respect to the Euclidean norm and μ is the normalized Lebesgue measure, then we denote $\Omega = \Omega_d$, $\mathcal{F}(\Omega) = \mathcal{F}(\Omega_d)$ and $S = S_d$. Then we obtain

$$S^{\mathrm{non}} = \{S_d\}_{d \in \mathbb{N}}$$

and our goal is to study tractability of S^{non}.

We discuss algorithms for the approximate computation of $S(f, q)$. We start with a short discussion of deterministic algorithms for this problem and then discuss randomized algorithms.

18.1.1 Deterministic Algorithms and Quasi-Linearity

The following lemma shows that the problem is quasi-linear.

Lemma 18.1. *Assume that $f, \tilde{f} \in L_2(\Omega, \mu)$ and $q, \tilde{q} \in L_2(\Omega, \mu)$ with*

$$1 \le q, \tilde{q} \le C. \tag{18.3}$$

Then

$$|S(f, q) - S(\tilde{f}, \tilde{q})| \le \sqrt{C}\|f - \tilde{f}\|_2 + \|f\|_2 \cdot \|q - \tilde{q}\|_2 + \|f\|_2 \cdot \sqrt{C} \cdot \|q - \tilde{q}\|_1.$$

Proof. We denote $\langle f, g\rangle_{L_2(\Omega,\mu)}$ by $\langle f, g\rangle$ and $\|q\|_{L_1(\Omega,\mu)}$ by $\|q\|_1$. Then we can write

$$|S(f,q) - S(\tilde{f},\tilde{q})| = \left|\frac{\langle f,q\rangle}{\|q\|_1} - \frac{\langle \tilde{f},\tilde{q}\rangle}{\|\tilde{q}\|_1}\right| = \left|\frac{\langle f,q\rangle}{\|q\|_1} - \frac{\langle f,q\rangle}{\|\tilde{q}\|_1} + \frac{\langle f,q\rangle}{\|\tilde{q}\|_1} - \frac{\langle \tilde{f},\tilde{q}\rangle}{\|\tilde{q}\|_1}\right|.$$

Now we apply the triangle inequality and obtain the following estimates

$$\left|\frac{\langle f,q\rangle}{\|\tilde{q}\|_1} - \frac{\langle \tilde{f},\tilde{q}\rangle}{\|\tilde{q}\|_1}\right| \le \frac{\|\tilde{q}\|_2}{\|\tilde{q}\|_1}\|f - \tilde{f}\|_2 + \frac{\|f\|_2}{\|\tilde{q}\|_1}\|q - \tilde{q}\|_2$$

and

$$\left|\frac{\langle f,q\rangle}{\|q\|_1} - \frac{\langle f,q\rangle}{\|\tilde{q}\|_1}\right| \le \frac{\|f\|_2 \cdot \|q\|_2}{\|q\|_1 \cdot \|\tilde{q}\|_1}\|q - \tilde{q}\|_1.$$

From $\|q\|_2^2 \le \|q\|_1 \cdot \|q\|_\infty$ we have

$$\frac{\|q\|_2^2}{\|q\|_1^2} \le \frac{\|q\|_\infty}{\|q\|_1} \le C.$$

From these estimates, the lemma easily follows. □

At this point one could apply a theory of quasi-linear problems for $S^{\text{non}} = \{S_d\}$. For certain spaces of d-variate functions we will obtain polynomial and strongly polynomial tractability results.

However, there is a problem with this approach and hence we do not follow it in more detail. In many interesting applications, from Bayesian statistics and statistical physics, the number C in (18.3) is so huge that even for $d = 1$ it is not clear whether one can construct satisfactory algorithms based on the theory of quasi-linear problems. In fact, for huge C we would like to have also tractability with respect to C and we could only permit a polylogarithmic dependence on C. That is, if $n(\varepsilon, d, C)$ is the minimal number of function values of f and q from $\mathcal{F}(\Omega_d)$ to guarantee an ε-approximation for, say, the absolute error criterion in the worst case setting then for some non-negative p_1, p_2 and p_3 we would like to have

$$n(\varepsilon, d, C) = \mathcal{O}\big(\varepsilon^{-p_1}\, d^{\,p_2}\, [\ln C]^{p_3}\big) \tag{18.4}$$

for all $\varepsilon \in (0,1)$, $d \in \mathbb{N}$ and $C \ge C_0 > 1$ with the factor in the big $\mathcal{O}$ notation independent of ε^{-1}, d and C.

For the classes $\mathcal{F}_C(\Omega_d)$ defined later, we prove that $n(\varepsilon, d, C)$ depends polynomially on C even in the randomized setting. This means that (18.4) does not hold for the classes $\mathcal{F}_C(\Omega_d)$. We leave to the reader an open problem to identify classes $\mathcal{F}(\Omega_d)$ for which this holds.

Open Problem 81.

Consider $S^{\text{non}} = \{S_d\}$ defined as in this section in the worst case setting.

- Identify (interesting) classes $\mathcal{F}(\Omega_d)$ for which S^{non} is tractable in the sense of (18.4).

18.1.2 Randomized Algorithms

The Metropolis algorithm, or more accurately, the class of *Metropolis–Hastings algorithms* ranges among the most important algorithms in numerical analysis and scientific computation, see Beichl and Sullivan [12], [13], Liu [176], Randall [249] and Roberts and Rosenthal [253]. Here we consider arbitrary randomized algorithms A_n that use at most n function evaluations of f and q. Hence A_n is a mapping of the form

$$A_{n,\omega}(f,q) = \varphi_{n,\omega}\big(f(x_{1,\omega}), q(x_{1,\omega}), f(x_{2,\omega}), q(x_{2,\omega}), \dots, f(x_{n,\omega}), q(x_{n,\omega})\big)$$

for some random element ω, see Chapter 4 of Volume I for more details.

To have more general lower bounds, see Theorem 18.2, we may even allow that the number of function values is random and n is an upper bound for the expected number of function values for each f and q.

As we shall see, the lower bounds hold under very general assumptions concerning the available random number generator. Observe, however, that we cannot use a random number generator for the *target distribution* $\mu_q = q \cdot \mu / \|q\|_1$, since q is *not* completely known.

For the upper bounds we only study two algorithms, the (non-adaptive) *simple Monte Carlo algorithm* and the (adaptive) *Metropolis–Hastings algorithm*. The former can only be applied if a random number generator for μ on Ω is available. Thus there are natural situations when this algorithm cannot be used.

The Metropolis Hastings algorithm is based on a Markov chain. We use a ball walk as an example. Then we need a random number generator for the uniform distribution on a (Euclidean) ball. Thus the Metropolis Hasting algorithm can also be applied when a random number generator for μ on Ω is not available. For the ball walk on $\Omega \subseteq \mathbb{R}^d$, we need a "membership oracle" for Ω:

On input $x \in \mathbb{R}^d$ this oracle can decide with cost, say 1, whether $x \in \Omega$ or not. The randomized error of A_n is defined as in Chapter 4 of Volume I. That is, for $(f,q) \in \mathcal{F}(\Omega)$, we define

$$e(A_n,(f,q)) = \big(\mathbb{E}_\omega |S(f,q) - A_{n,\omega}(f,q)|^2\big)^{1/2},$$

where $\mathbb{E}_\omega$ means the expectation with respect to random ω. The randomized worst case error of A_n on the class $\mathcal{F}(\Omega)$ is

$$e(A_n,\mathcal{F}(\Omega)) = \sup_{(f,q)\in\mathcal{F}(\Omega)} e(A_n,(f,q)).$$

The nth minimal randomized error is

$$e_n(\mathcal{F}(\Omega)) = \inf_{A_n} e(A_n,\mathcal{F}(\Omega)).$$

We consider classes $\mathcal{F}(\Omega)$ that contain constant densities $q \equiv \text{constant} > 0$ and all f with $\|f\|_\infty \le 1$, i.e.,

$$\mathcal{F}_1(\Omega) = \big\{(f,q) \mid \sup_{x\in\Omega} |f(x)| \le 1 \ \text{ and } q \equiv \text{constant} > 0\big\} \subseteq \mathcal{F}(\Omega).$$

For the class $\mathcal{F}_1(\Omega)$, the problem (18.2) reduces to the classical integration problem for uniformly bounded functions, and it is well known that the nth minimal randomized error decreases at a rate $n^{-1/2}$. This holds under mild assumptions on the measure μ, for example, when μ does not have atoms.

We will only consider classes $\mathcal{F}(\Omega)$ for which $S(f,q) \in [-1,1]$ and the trivial algorithm $A_0 = 0$ has always error 1. Therefore we do not have to distinguish between the absolute and the normalized error criteria.

We give a short outline what will be done in this section. For the classes $\mathcal{F}_C(\Omega)$ and $\mathcal{F}^\alpha(\Omega)$, which will be introduced below, we easily obtain the optimal order

$$e_n(\mathcal{F}(\Omega)) = \Theta\big(n^{-1/2}\big).$$

Of course, this does not say anything about tractability since the dependence on d as well as on C and α is hidden in the big Θ notation. We will analyze how $e_n(\mathcal{F}(\Omega))$ depends on these parameters.

The classes $\mathcal{F}_C(\Omega)$, analyzed in Section 18.1.3, contain all densities q for which $\sup q / \inf q \le C$. We prove that the simple (non-adaptive) Monte Carlo algorithm is almost optimal, no sophisticated Markov chain Monte Carlo algorithm can be essentially better.

In typical applications we face huge C, for instance, $C = 10^{20}$. Theorem 18.2 states that we cannot decrease the error of optimal algorithms from 1 to $1/4$ even with a sample size $n = 10^{19}$. Hence the class $\mathcal{F}_C(\Omega)$ is so large that no algorithm can provide an acceptable error.

Thus we have to shrink the class $\mathcal{F}_C(\Omega)$ to “suitable and interesting” subclasses and study the question whether adaptive algorithms, such as the Metropolis algorithm, can significantly help in terms of the dependence on C.

We give a partially positive answer for the classes $\mathcal{F}^\alpha(\Omega_d)$. That is, $\Omega = \Omega_d$ is an Euclidean ball of $\mathbb{R}^d$ and μ is the normalized Lebesgue measure μ_Ω on Ω. The class $\mathcal{F}^\alpha(\Omega_d)$ contains log concave densities, where α is the Lipschitz constant of $\ln q$. We have

$$\mathcal{F}^\alpha(\Omega) \subseteq \mathcal{F}_C(\Omega)$$

if we take $\alpha = \ln C/2$. Hence even for huge C, the parameter α is relatively small. For $C = 10^{20}$ we have $\alpha = 45.358\ldots$.

For non-adaptive algorithms one gets similar lower bounds as for the classes $\mathcal{F}_C(\Omega)$, see [182] for details. An (adaptive) Metropolis algorithm, however, is much better. The main error estimate for this algorithm is given in Theorem 18.20.

18.1.3 Analysis for $\mathcal{F}_C(\Omega)$

Let μ be an arbitrary probability measure on a set Ω and consider the set

$$\mathcal{F}_C(\Omega) = \Big\{(f,q) \mid \|f\|_\infty \le 1,\ q > 0,\ \tfrac{q(x)}{q(y)} \le C \text{ for all } x,y \in \Omega\Big\}.$$

Obviously, we must assume that $C \geq 1$ since otherwise the class $\mathcal{F}_C(\Omega)$ would be empty. If $C = 1$ then q is constant and we have the ordinary integration problem since $S(f, q)$ does not depend on q and is just the integral of f. That is why, from now on we assume that $C > 1$.

In many applications the constant C is huge. We will prove that the complexity of the problem is linear in C. It follows from the next theorem that, in particular,

$$e(A_n, \mathcal{F}_C(\Omega)) \geq 1/6 \quad \text{for } n \approx C/4.$$

Therefore, for large C, the class $\mathcal{F}_C(\Omega)$ is too large. We have to look for smaller classes that contain many interesting pairs (f, q) and have significantly smaller complexity.

We first prove lower bounds for all non-adaptive and adaptive algorithms that use n evaluations of f and q.

Theorem 18.2. *Assume that we can partition Ω into $4n$ disjoint sets with equal measure. Then for any randomized algorithm A_n that uses at most n values of f and q on the average we have the lower bound*

$$e(A_n, \mathcal{F}_C(\Omega)) \geq \frac{\sqrt{2}}{12} \begin{cases} \sqrt{\frac{C}{4n}} & \text{if } 4n \geq C - 1, \\ \frac{3C}{C+4n-1} & \text{if } 4n < C - 1. \end{cases} \tag{18.5}$$

The lower bound will be obtained in two steps.

- We first construct a certain discrete probability measure on the class $\mathcal{F}_C(\Omega)$ so that the randomized error of A_n is lower bounded by the nth minimal average case error with respect to this measure. This approach is due to Bakhvalov [4].
- For the chosen prior on $\mathcal{F}_C(\Omega)$, we compute a lower bound on the nth minimal average case error.

To construct the prior, let $m = 4n$ and let $\Omega_1, \Omega_2, \dots, \Omega_m$ be the partition of Ω into sets of equal measure $1/m$. Let χ_{Ω_j} be the corresponding characteristic function of Ω_j for $j = 1, 2, \dots, m$. Furthermore, let

$$s = \left\lceil \frac{m}{C-1} \right\rceil.$$

Denote J_s^m as the set of all subsets of $\{1, 2, \dots, m\}$ of cardinality equal to s, and let $\mu_{m,s}$ be the equi-distribution on J_s^m, and let $\mathbb{E}_{m,s}$ denote the expectation with respect to the prior $\mu_{m,s}$. Let $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be independent and identically distributed random variables with

$$P(\varepsilon_j = -1) = P(\varepsilon_j = 1) = \tfrac{1}{2} \quad \text{for all } j = 1, 2, \dots, m.$$

The overall prior is the product probability on $J_s^m \times \{\pm 1\}^m$. For any realization $\omega = (I, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ we assign

$$f_\omega = \sum_{j \in I} \varepsilon_j \chi_{\Omega_j} \quad \text{and} \quad q_\omega = C \sum_{j \in I} \chi_{\Omega_j} + \sum_{j \notin I} \chi_{\Omega_j}.$$

The following observation is useful.

Lemma 18.3. *For any subset $N \subseteq \{1, 2, \dots, m\}$ of cardinality at most $2n$ we have*

$$\mathbb{E}_{m,s}\#(I \setminus N) \geq \frac{s}{2}.$$

Proof. Clearly, for any fixed $k \in \{1, 2, \dots, m\}$ we have $\mu_{m,s}(k \in I) = s/m$, thus

$$\mathbb{E}_{m,s}\#(I \setminus N) = \sum_{r \in N^c} \mathbb{E}_{m,s}\chi_I(r) = \#(N^c)\frac{s}{m} \geq \frac{s}{2},$$

where N^c denotes the complement of N. □

Proof of Theorem 18.2. Given the above prior, let us denote

$$e_{2n}^{\text{avg}}(\mathscr{F}_C(\Omega)) = \inf_{A_{2n}} \left(\mathbb{E}_{m,s}\mathbb{E}_\varepsilon \left|S(f, q) - A_{2n}(f, q)\right|^2\right)^{1/2}, \tag{18.6}$$

where the infimum is taken with respect to any, possibly adaptive, deterministic algorithm which uses at most $2n$ values of f and q.

Any randomized algorithm A_n uses with probability 1/2 at most $2n$ function values and hence we have, using Bakhvalov's argument, the relation

$$e(A_n, \mathscr{F}_C(\Omega)) \geq \tfrac{1}{2}\, e_{2n}^{\text{avg}}(\mathscr{F}_C(\Omega)). \tag{18.7}$$

We provide a lower bound for $e_{2n}^{\text{avg}}(\mathscr{F}_C(\Omega))^2$. To this end, note that for each realization (f_ω, q_ω) the integral $\int_\Omega q_\omega \, d\mu$ is constant. For $m \geq C - 1$, we can bound the integral by choosing the integer s such that

$$c_{m,s} = \int_\Omega q_\omega(x)\ \mu(\mathrm{d}x) = \frac{s\,C + (m - s)1}{m} \leq 3. \tag{18.8}$$

In the other case, when $m < C - 1$, we take $s = 1$ and obtain

$$c_{m,1} = \frac{C - 1 + m}{m}.$$

Now, to analyze the average case error, let A_{2n} be any (deterministic) algorithm, and let us assume that it uses function values from the set N of nodes. We have the decomposition

$$S(f_\omega, q_\omega) - A_{2n}(f_\omega, q_\omega) = \left(\frac{C}{m\, c_{m,s}} \sum_{j \in I \setminus N} \varepsilon_j\right) + \left(\frac{C}{m\, c_{m,s}} \sum_{j \in I \cap N} \varepsilon_j - A_{2n}(f_\omega, q_\omega)\right).$$

Given the set I, the random variables in the brackets are conditionally independent, thus uncorrelated. Hence we conclude that

$$\begin{aligned}\mathbb{E}_{m,s}\mathbb{E}_\varepsilon \left|S(f_\omega, q_\omega) - A_{2n}(f_\omega, q_\omega)\right|^2 &\geq \mathbb{E}_{m,s}\mathbb{E}_\varepsilon \left|\frac{C}{m\, c_{m,s}} \sum_{j \in I \setminus N} \varepsilon_j\right|^2 \\ &= \frac{C^2}{m^2 c_{m,s}^2}\mathbb{E}_{m,s}\#(J \setminus N) \geq \frac{C^2 s}{2m^2 c_{m,s}^2},\end{aligned}$$

by Lemma 18.3. In the case $m \geq C - 1$, we take an integer $s \in [m/C, 2m/(C-1))$. Note that such an integer exists since the length of the last interval is larger than 1. Then $c_{m,s} \leq 3$ and

$$\tfrac{1}{4}\,\mathbb{E}_{m,s}\,\mathbb{E}_{\varepsilon}\,|S(f,q) - A_{2n}(f,q)|^2 \geq \frac{C}{72n},$$

which in turn yields the first case bound in (18.5). In the other case $m < C - 1$, the value of $s = 1$ yields the second bound in (18.5). □

We now analyze an algorithm which we call the *simple Monte Carlo* algorithm. For random elements $X_1, X_2, \dots, X_n$ from Ω that are identically and independently distributed according to μ, define the algorithm by

$$A_n^{\text{simp}}(f,q) = \frac{\sum_{j=1}^n f(X_j)\,q(X_j)}{\sum_{j=1}^n q(X_j)}. \tag{18.9}$$

We will prove an upper bound for the randomized error of this algorithm, and we start with the following lemma.

Lemma 18.4. *For $q \in \mathcal{F}_C(\Omega)$, we have*

- $0 < \inf_{x\in\Omega} q(x) \leq \sup_{x\in\Omega} q(x) < \infty$.
- *For every probability measure κ on Ω we have $\|q\|_{L_2(\Omega,\kappa)} \leq \sqrt{C}\,\|q\|_{L_1(\Omega,\kappa)}$.*

Proof. To prove the first point, fix any $y_0 \in \Omega$. Then the assumption on q yields $q(x) \leq C q(y_0)$ for all $x \in \Omega$, and it proves that the supremum of q is finite. Reversing the roles of x and y, we have $0 < q(x_0) \leq C q(y)$ for all $y \in \Omega$, and it proves that the infimum of q is positive.

We turn to the second point. Note that both the assumption on q as well as the second point are invariant with respect to multiplication of q by a constant. Due to the first point, we may then assume that $1 \leq q(x) \leq C$ for all $x \in \Omega$. Using $1 \leq \int_\Omega q(x)\,\mu(\mathrm{d}x)$ we have

$$\int_\Omega q^2(x)\,\mu(\mathrm{d}x) \leq C \int_\Omega q(x)\,\mu(\mathrm{d}x) \leq C\left(\int_\Omega q(x)\,\mu(\mathrm{d}x)\right)^2.$$

This completes the proof of the second point and of the lemma. □

We turn to the error bound for the simple Monte Carlo algorithm.

Theorem 18.5. *For all $n \in \mathbb{N}$, we have*

$$e(A_n^{\text{simp}}, \mathcal{F}_C(\Omega)) \leq 2\min\left(1, \sqrt{\frac{2C}{n}}\right). \tag{18.10}$$

Proof. The upper bound 2 is trivial, it even holds deterministically. Fix any pair (f,q) from $\mathcal{F}_C(\Omega)$. For any sample $(X_1, X_2, \dots, X_n)$ and a function g from $L_2(\Omega,\mu)$ we denote the sample mean by

$$A_n^{\text{mean}}(g) = \frac{1}{n}\sum_{j=1}^{n} g(X_j).$$

As we know $e(A_n^{\text{mean}}, g) \le \|g\|_2\, n^{-1/2}$. Note that

$$A_n^{\text{simp}}(f,g) = \frac{A_n^{\text{mean}}(f\ g)}{A_n^{\text{mean}}(q)}.$$

Then we have

$$\begin{aligned}
&\left|S(f,q) - A_n^{\text{simp}}(f,q)\right| \\
&\le \left|S(f,q) - \frac{A_n^{\text{mean}}(fq)}{\int_\Omega q(x)\mu(\mathrm{d}x)}\right| + \left|\frac{A_n^{\text{mean}}(fq)}{\int_\Omega q(x)\mu(\mathrm{d}x)} - \frac{A_n^{\text{mean}}(fq)}{A_n^{\text{mean}}(q)}\right| \\
&\le \frac{\left(\left|\int_\Omega f(x)q(x)\mu(\mathrm{d}x) - A_n^{\text{mean}}(fq)\right| + \left|\frac{A_n^{\text{mean}}(fq)}{A_n^{\text{mean}}(q)}\right| \left|\int_\Omega q(x)\mu(\mathrm{d}x) - A_n^{\text{mean}}(q)\right|\right)}{\|q\|_1} \\
&\le \frac{1}{\|q\|_1}\Bigg(\left|\int_\Omega f(x)q(x)\mu(\mathrm{d}x) - A_n^{\text{mean}}(fq)\right| \\
&\qquad\qquad + \|f\|_\infty \left|\int_\Omega q(x)\mu(\mathrm{d}x) - A_n^{\text{mean}}(q)\right|\Bigg),
\end{aligned}$$

where we used $|A_n^{\text{mean}}(fq)/A_n^{\text{mean}}(q)| \le \|f\|_\infty$ since the numerator and denominator use the same sample. This and Lemma 18.4 yield the following error bound

$$\begin{aligned}
e(A_n^{\text{simp}},(f,q)) &\le \frac{\sqrt{2}}{\|q\|_1}\left[e(A_n^{\text{mean}}, fq) + \|f\|_\infty e(A_n^{\text{mean}}, q)\right] \\
&\le \frac{\sqrt{2}}{\|q\|_1\sqrt{n}}\left[\|fq\|_2 + \|f\|_\infty \|q\|_2\right] \le \frac{2\sqrt{2}\|f\|_\infty}{\sqrt{n}}\,\frac{\|q\|_2}{\|q\|_1} \le \frac{2\sqrt{2C}}{\sqrt{n}}.
\end{aligned}$$

The proof is completed by taking the supremum over $(f,q) \in \mathcal{F}_C(\Omega)$. □

Theorems 18.2 and 18.5 state that the nth minimal randomized error satisfies

$$e(n, \mathcal{F}_C(\Omega)) = \Theta\left(\sqrt{C/n}\right) \quad \text{as } n \to \infty.$$

This shows that the simple Monte Carlo algorithm is asymptotically optimal. However, for large C we must wait very long to see this decay of the nth minimal randomized error. For instance, take

$$C = 10^{20} \quad \text{and} \quad n = 10^{19}.$$

Then the second part of (18.5) yields that

$$e(n, \mathcal{F}_C(\Omega)) \geq \frac{\sqrt{2}}{4} \frac{1}{1.4 - 10^{-20}} = 0.2525\dots > \frac{1}{4}.$$

Hence for $\varepsilon < 1/4$ we have

$$n(\varepsilon, \mathcal{F}_C(\Omega)) > 10^{19}$$

which is astronomically large. This proves that indeed the class $\mathcal{F}_C(\Omega)$ is too large.

18.1.4 Log Concave Densities

The results of Section 18.1.3 clearly indicate that the class $\mathcal{F}_C(\Omega)$ is simply too large and hence one should look for smaller classes that still contain many interesting problems.

Randomized algorithms for problems where the target distribution is log concave proved to be important in many studies, we refer to Frieze, Kannan and Polson [70]. One of the main intrinsic features of such classes of distributions are *isoperimetric inequalities*, see Applegate and Kannan [1], Kannan, Lovász and Simonovits [145] and Vempala [318]. Here we study the nonlinear integration problem for densities from the class

$$\mathcal{R}^{\alpha}(\Omega_d) := \left\{ q \mid q > 0,\ \ln q \text{ concave},\ |\ln q(x) - \ln q(y)| \leq \alpha \|x - y\|_2 \right\},$$

where Ω_d is the d-dimensional unit ball with respect to the Euclidean norm and μ is the normalized Lebesgue measure on Ω_d. This class was studied in [182] together with integrands f that are square integrable with respect to the measure with density q. The main result of [182] was the construction of an algorithm A_n with an randomized error $e(A_n)$ such that

$$\lim_{n \to \infty} e(A_n)^2 \cdot n = \mathcal{O}\big(\max(d^2, d\,\alpha^2)\big)$$

with the factor in the big $\mathcal{O}$ notation independent of d and α. The authors of that paper believe that the problem is polynomially tractable but could verify it only asymptotically.

Here we present a result of Rudolf [257] for the class

$$\mathcal{F}^{\alpha}(\Omega_d) := \left\{ (f, q) \mid q \in \mathcal{R}^{\alpha}(\Omega_d) \text{ and } \|f\|_{\infty} \leq 1 \right\}.$$

Let $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$ be the minimal number of function values needed to compute an ε-approximation to $S_d(f, q)$ for the absolute error criterion in the randomized setting. Rudolf [257] proved the following theorem.

Theorem 18.6. *For the nonlinear integration problem S_d defined over $\mathcal{F}^{\alpha}(\Omega_d)$ we have*

$$n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d)) \leq (d + 1)\max(d + 1, \alpha^2)\left[64 \cdot 10^6\,\varepsilon^{-2} + \alpha\,1.28 \cdot 10^6 + 2\right]$$

for all $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$.

This means that $S^{\mathrm{non}} = \{S_d\}$ is polynomially tractable with an ε^{-1}- and a d-exponent at most 2*, and an α-exponent at most* 3.

We will prove Rudolf's theorem in the next section. Here, we comment on the estimate of the last theorem. There are some pluses and minuses in this estimate. The definite plus is that the bound is very concrete and there are no hanging factors in the big $\mathcal{O}$ notation. The minus is that ε^{-2} and α in the square bracket are multiplied by large numbers. In fact, such large numbers also occur in other papers in this area.

For instance, take $d = 9$, $\alpha \le 3$ and $\varepsilon = 10^{-2}$. Then $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$ is bounded roughly by $0.64 \cdot 10^{14}$ which is huge although much smaller then 10^{19} that we encountered in the previous subsection. In any case, if the numbers $64 \cdot 10^6$ and $1.28 \cdot 10^6$ are sharp then despite of polynomial tractability, the practical value of the bound on $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$ would be limited only to relatively large ε and relatively small d. However, it is not clear whether these factors are sharp. The hope is that a similar bound on $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$ can be proved with much smaller factors. In fact, it would be of great interest to find sharp lower bounds on $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$ and in this way to verify if large factors are indeed present. This is our next open problem.

Open Problem 82.

Consider the nonlinear integration problem $S^{\mathrm{non}} = \{S_d\}$ defined as in this subsection in the randomized setting.

- Improve Rudolf's bounds on $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$.
- In particular, check if the factors $64 \cdot 10^6$ and $1.28 \cdot 10^6$ can be significantly lowered.
- Find sharp lower bounds on $n(\varepsilon, \mathcal{F}^{\alpha}(\Omega_d))$.

18.1.5 Explicit Error Bounds for MCMC

In this section we present an explicit error bound of Rudolf [257] for MCMC (Markov Chain Monte Carlo) algorithms such as the Metropolis algorithm. We will be using the notions and terminology of MCMC, although we also try to define many of these notions to help the reader.

The goal is to approximate an integral of the form

$$S(f) = \int_{\Omega} f(x)\, \pi(\mathrm{d}x), \tag{18.11}$$

where Ω is a given set and π a probability measure of which we have only a limited knowledge. In our case, $\pi = \pi_q$ depends on q and we can only sample q. We generate a Markov chain $X_1, X_2, \dots$ with a transition kernel K, having π as its stationary distribution. After a certain *burn-in time* n_0, for a given function f we compute the approximation

$$A_{n,n_0}(f) = \frac{1}{n} \sum_{j=1}^{n} f(X_{j+n_0}).$$

The computation of $A_{n,n_0}(f)$ requires to compute n function values of f and $n_0 + n$ function values of q. Therefore the cost of A_{n,n_0} is proportional to $n + n_0$. As always, the individual error of A_{n,n_0} for a function f is measured in the mean square sense, i.e.,

$$e(A_{n,n_0}, f) = \left(\mathbb{E}\left|S(f) - A_{n,n_0}(f)\right|^2\right)^{1/2}.$$

We assume that $(\Omega, \mathcal{A})$ is a measurable space. Then we call $K\colon \Omega \times \mathcal{A} \to [0,1]$ a Markov kernel or a transition kernel if

- for each $x \in \Omega$ the mapping $A \in \mathcal{A} \mapsto K(x, A)$ induces a probability measure on Ω,
- for each $A \in \mathcal{A}$ the mapping $x \in \Omega \mapsto K(x, A)$ is an $\mathcal{A}$-measurable real function.

In addition,

$$\mathcal{M} = (\Omega, \mathcal{A}, \{K(x, \cdot) : x \in \Omega\})$$

is called the (associated) Markov scheme. Notation and much of the following analysis is taken from Lovász and Simonovits [178].

A Markov chain $X_1, X_2, \ldots$ is given through a Markov scheme $\mathcal{M}$ and a start distribution ν on Ω. The transition kernel $K(x, A)$ of the Markov chain describes the probability of getting from $x \in \Omega$ to $A \in \mathcal{A}$ in one step. We assume that π is the stationary distribution of the Markov chain, i.e.,

$$\pi(A) = \int_\Omega K(x, A)\pi(\mathrm{d}x) \quad \text{for all } A \in \mathcal{A}.$$

Another similar but stronger restriction on the chain is reversibility. A Markov scheme is *reversible* with respect to π if

$$\int_B K(x, A)\pi(\mathrm{d}x) = \int_A K(x, B)\pi(\mathrm{d}x) \quad \text{for all } A, B \in \mathcal{A}.$$

The next lemma is taken from Lovász and Simonovits [178]; we also give an idea of the proof.

Lemma 18.7. *Let $\mathcal{M}$ be a reversible Markov scheme and let $F\colon \Omega \times \Omega \to \mathbb{R}$ be integrable. Then*

$$\int_\Omega \int_\Omega F(x, y)\ K(x, \mathrm{d}y)\pi(\mathrm{d}x) = \int_\Omega \int_\Omega F(y, x)\ K(x, \mathrm{d}y)\pi(\mathrm{d}x). \tag{18.12}$$

Proof. The result is shown by using a standard technique of integration theory. Since the Markov scheme is reversible we have

$$\int_\Omega \int_\Omega I_{A\times B}(x, y)K(x, \mathrm{d}y)\pi(\mathrm{d}x) = \int_\Omega \int_\Omega I_{A\times B}(y, x)K(x, \mathrm{d}y)\pi(\mathrm{d}x)$$

for $A, B \in \mathcal{A}$. We obtain the equality of the integrals for an arbitrary set $C \in \mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A} \otimes \mathcal{A}$ is the product σ-algebra of $\mathcal{A}$ with itself. Then we consider the case of simple functions, and this case is quite straightforward. The next step is to obtain the equality for positive functions and after that extending the result to general integrable functions. □

Remark 18.8. If we have a Markov scheme, which is not necessarily reversible but has a stationary distribution, then

$$S(f) = \int_\Omega f(x)\pi(\mathrm{d}x) = \int_\Omega \int_\Omega f(y)K(x, \mathrm{d}y)\pi(\mathrm{d}x),$$

where $f : \Omega \to \mathbb{R}$ is integrable. This can be easily seen by using the same steps as in the proof of Lemma 18.7. □

By $K^n(x, \cdot)$ we denote the n-step transition probabilities. For $x \in \Omega$ and $A \in \mathcal{A}$, we have

$$K^n(x, A) = \int_\Omega K^{n-1}(y, A)K(x, \mathrm{d}y) = \int_\Omega K(y, A)K^{n-1}(x, \mathrm{d}y).$$

It is again a transition kernel of a Markov chain sharing the invariant distribution and reversibility with the original one. Thus Lemma 18.7 and Remark 18.8 also hold for the n-step transition probabilities, i.e.,

$$\int_\Omega \int_\Omega F(x, y)\, K^n(x, \mathrm{d}y)\, \pi(\mathrm{d}x) = \int_\Omega \int_\Omega F(y, x)\;K^n(x, \mathrm{d}y)\, \pi(\mathrm{d}x). \qquad (18.13)$$

For a Markov scheme $\mathcal{M}$, we define a nonnegative operator $P : L_\infty(\Omega, \pi) \to L_\infty(\Omega, \pi)$ by

$$(Pf)(x) = \int_\Omega f(y)K(x, \mathrm{d}y) \quad \text{for all } x \in \Omega.$$

(Nonnegative here means that $f \ge 0$ implies $Pf \ge 0$.) This operator is called Markov or transition operator of a Markov scheme $\mathcal{M}$ and describes the expected value of f after one step with the Markov chain from $x \in \Omega$. The expected value of f from $x \in \Omega$ after n-steps of the Markov chain is given as

$$(P^n f)(x) = \int_\Omega f(y)K^n(x, \mathrm{d}y).$$

Let us now consider P on the Hilbert space $L_2(\Omega, \pi)$ and let

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\, \pi(\mathrm{d}x) \quad \text{for all } f, g \in L_2(\Omega, \pi)$$

denote the canonical scalar product. Note that the function space is chosen according to the invariant measure π. Then using Lemma 18.7 we have

$$\langle f, f\rangle \pm \langle f, Pf\rangle = \frac{1}{2} \int_\Omega \int_\Omega (f(x) \pm f(y))^2 K(x, \mathrm{d}y)\pi(\mathrm{d}x) \ge 0. \qquad (18.14)$$

This implies that $\|P\|_{L_2 \to L_2} \leq 1$ and it is also easy to show that $\|P^n\|_{L_p \to L_p} \leq 1$ for all $1 \leq p \leq \infty$, see Baxter and Rosenthal [7, Lemma 1].

Let $X_1, X_2, \dots$ be a reversible Markov chain. The expectation of the chain with the starting distribution $\nu = \pi$ and the Markov kernel K of the scheme $\mathcal{M}$ is denoted by $\mathbb{E}_{\pi,K}$. Then for $f \in L_2(\Omega, \pi)$ we get

$$\begin{aligned} &\mathbb{E}_{\pi,K}(f(X_i)) = \mathbb{E}_{\pi,K}(f(X_1)) = \langle 1, f \rangle = S(f), \\ &\mathbb{E}_{\pi,K}(f(X_i)^2) = \mathbb{E}_{\pi,K}(f(X_1)^2) = \langle f, f \rangle = S(f^2), \\ &\mathbb{E}_{\pi,K}(f(X_i)f(X_j)) = \mathbb{E}_{\pi,K}(f(X_1)f(X_{|i-j|+1})) = \langle f, P^{|i-j|} f \rangle. \end{aligned} \tag{18.15}$$

The assumption that the initial distribution is stationary makes the calculation easy. In the general case, where the starting point is chosen by a given probability distribution ν, we obtain for $i \leq j$ and functions $f \in L_2(\Omega, \pi)$,

$$\begin{aligned} \mathbb{E}_{\nu,K}(f(X_i)) &= \int_\Omega P^i f(x) \nu(\mathrm{d}x), \\ \mathbb{E}_{\nu,K}(f(X_i)f(X_j)) &= \int_\Omega P^i (f(x) P^{j-i} f(x)) \nu(\mathrm{d}x). \end{aligned}$$

It is easy to verify by using (18.12) that P is self-adjoint as an operator acting on $L_2(\Omega, \pi)$.

A Markov scheme $\mathcal{M} = (\Omega, \mathcal{A}, \{K(x, \cdot) : x \in \Omega\})$ is called *lazy* if

$$K(x, \{x\}) \geq 1/2 \quad \text{for all } x \in \Omega.$$

This means that the chain stays at least with probability $1/2$ in the current state. The reason for this slowing down property is to deduce that the associated Markov operator P is positive semidefinite. Therefore we study only lazy chains. This is formalized in the next lemma.

Lemma 18.9. *Let $\mathcal{M}$ be a lazy, reversible Markov scheme. Then*

$$\langle Pf, f \rangle \geq 0 \quad \textit{for all } f \in L_2(\Omega, \pi). \tag{18.16}$$

Proof. We consider another Markov scheme

$$\widetilde{\mathcal{M}} = (\Omega, \mathcal{A}, \{\widetilde{K}(x, \cdot) : x \in \Omega\}),$$

where

$$\widetilde{K}(x, A) = 2K(x, A) - 1_A(x) \quad \text{for all } A \in \mathcal{A}.$$

To verify that $\widetilde{K}$ is again a transition kernel we need to show that $K(x, \{x\}) \geq 1/2$. The reversibility condition for $\widetilde{\mathcal{M}}$ holds since the scheme $\mathcal{M}$ is reversible. The Markov operator of $\widetilde{\mathcal{M}}$ is given by $\widetilde{P} = (2P - I)$, where I is the identity. Since we established reversibility of the new scheme, by applying Lemma 18.7 we obtain (18.14) for $\widetilde{P}$. Hence,

$$-\langle f, f \rangle \leq \langle (2P - I) f, f \rangle \leq \langle f, f \rangle$$

and therefore

$$\langle Pf, f\rangle = \tfrac{1}{2}\langle f, f\rangle + \tfrac{1}{2}\langle (2P - I)f, f\rangle \geq 0,$$

as claimed. □

We turn to the *conductance* of the Markov chain. For a Markov scheme

$$\mathcal{M} = (\Omega, \mathcal{A}, \{K(x, \cdot) : x \in \Omega\}),$$

which is not necessarily lazy, define the conductance as

$$\varphi(K, \pi) = \inf_{0<\pi(A)\leq 1/2} \frac{\int_A K(x, A^c)\pi(\mathrm{d}x)}{\pi(A)},$$

where π is a stationary distribution. The numerator of the conductance describes the probability of leaving A in one step, where the starting point is chosen by π. An important requirement for the next results is that the scheme has a positive conductance.

The following result will be needed and it is again from Lovász and Simonovits [178, Corollary 1.5, p. 372].

Lemma 18.10. *Let $\mathcal{M}$ be a lazy, reversible Markov scheme and let ν be the initial distribution. Furthermore we assume that the probability distribution ν has a bounded density function $\frac{d\nu}{d\pi}$ with respect to π. Then*

$$\left|\int_\Omega K^j(x, A)\,\nu(\mathrm{d}x) - \pi(A)\right| \leq \sqrt{\|\frac{d\nu}{d\pi}\|_\infty}\left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j \tag{18.17}$$

for $A \in \mathcal{A}$.

The left hand side of (18.17) can be transformed as follows

$$\begin{aligned}
&\int_\Omega K^j(x, A)\,\nu(\mathrm{d}x) - \pi(A)\\
&\quad= \int_\Omega \int_A K^j(x, \mathrm{d}y)\,\frac{d\nu}{d\pi}(x)\,\pi(\mathrm{d}x) - \pi(A)\\
&\underset{(18.13)}{=} \int_A \int_\Omega \frac{d\nu}{d\pi}(y)\,K^j(x, \mathrm{d}y)\,\pi(\mathrm{d}x)\\
&\quad= \int_A \int_\Omega \frac{d\nu}{d\pi}(y)\,(K^j(x, \mathrm{d}y) - \pi(\mathrm{d}y))\,\pi(\mathrm{d}x).
\end{aligned}$$

It is now clear that Lemma 18.10 for $A \in \mathcal{A}$ yields

$$\left|\int_A \int_\Omega \frac{d\nu}{d\pi}(y)\,\left[K^j(x, \mathrm{d}y) - \pi(\mathrm{d}y)\right]\,\pi(\mathrm{d}x)\right| \leq \sqrt{\|\frac{d\nu}{d\pi}\|_\infty}\left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j. \tag{18.18}$$

The right-hand side of (18.18) is a bound on the speed of convergence to stationarity of the Markov chain. It can be further estimated by

$$\sqrt{\|\frac{d\nu}{d\pi}\|_\infty}\left(1-\frac{\varphi(K,\pi)^2}{2}\right)^j \le \sqrt{\|\frac{d\nu}{d\pi}\|_\infty}\exp\left[-j\,\frac{\varphi(K,\pi)^2}{2}\right]. \tag{18.19}$$

To use the conductance we need a relation to the operator P. This will be given in the form of the Cheeger inequality. Define

$$L_2^0 = L_2^0(\Omega,\pi) = \{f \in L_2(\Omega,\pi) \mid S(f) = 0\}\,.$$

Lemma 18.11 (Cheeger's inequality). *Let $\mathcal{M}$ be a reversible Markov scheme with conductance $\varphi(K,\pi)$. Then*

$$\langle P^j g, g\rangle \le \left(1-\frac{\varphi(K,\pi)^2}{2}\right)^j \|g\|_2^2 \quad \text{for all } g \in L_2^0. \tag{18.20}$$

Proof. See Lovász and Simonovits [178, Corollary 1.8, p. 375]. □

For the next result, taken from Lovász and Simonovits [178, Theorem 1.9, p. 375], we assume that the starting point is chosen according to the stationary distribution. Then a burn-in time is not necessary.

Theorem 18.12. *Let $\mathcal{M}$ be a lazy, reversible Markov scheme with stationary distribution π, and let $X_1, X_2, \dots$ be a Markov chain generated by $\mathcal{M}$ with initial distribution π. Let*

$$S(f) = \int_\Omega f(x)\pi(\mathrm{d}x) \quad \text{for all } f \in L_2(\Omega,\pi),$$

and let

$$A_n(f) = A_{n,0}(f) = \frac{1}{n}\sum_{j=1}^n f(X_j) \quad \text{for all } f \in L_2(\Omega,\pi).$$

Then

$$e(A_n,f)^2 = \mathbb{E}_{\pi,K}\,|S(f)-A_n(f)|^2 \le \frac{4}{\varphi(K,\pi)^2\cdot n}\|f\|_2^2.$$

Proof. Let $g = f - S(f)$. Then $g \in L_2^0$. Using Lemma 18.9, Lemma 18.11 and $\|g\|_2 \le \|f\|_2$, we have

$$\mathbb{E}_{\pi,K}\,|S(f)-A_n(f)|^2 = \mathbb{E}_{\pi,K}\left|\frac{1}{n}\sum_{j=1}^n g(X_j)\right|^2$$
$$= \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}_{\pi,K}(g(X_j)g(X_i)) \underset{(18.15)}{=} \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}_{\pi,K}(g(X_1)g(X_{|i-j|+1})) =$$

$$
\begin{aligned}
&= \frac{1}{n^2}\Big(n\langle g, g\rangle + \sum_{k=1}^{n-1} 2(n-k)\langle P^k g, g\rangle\Big) \\
&\le \frac{1}{n^2}\sum_{k=0}^{n-1} 2(n-k)\langle P^k g, g\rangle \\
&\underset{(18.16)}{\le} \frac{2}{n}\sum_{k=0}^{\infty}\langle P^k g, g\rangle \\
&\underset{(18.20)}{\le} \frac{2}{n}\sum_{k=0}^{\infty}\left(1-\frac{\varphi(K,\pi)^2}{2}\right)^k \|g\|_2^2 = \frac{4}{\varphi(K,\pi)^2\cdot n}\|g\|_2^2 \le \frac{4}{\varphi(K,\pi)^2\cdot n}\|f\|_2^2,
\end{aligned}
$$

as claimed. Note that laziness of $\mathcal{M}$ is essentially used by applying $\langle P^k g, g\rangle \ge 0$ in the second inequality. □

Let us consider the more general case, where the initial distribution is not stationary. In the next statement, a relation is established between the error of starting with π and the error of starting not with the invariant distribution.

Lemma 18.13. *Let $\mathcal{M}$ be a reversible Markov scheme with stationary distribution π, let $X_1, X_2, \dots$ be a Markov chain generated by $\mathcal{M}$ with initial distribution ν. Let $\frac{d\nu}{d\pi}$ be a bounded density of ν with respect to π. Then for $g = f - S(f) \in L_2^0$ we get*

$$
\begin{aligned}
&\mathbb{E}_{\nu,K}\left|S(f) - A_{n,n_0}(f)\right|^2 \\
&= \mathbb{E}_{\pi,K}\left|S(f) - A_n(f)\right|^2 \\
&\quad + \frac{1}{n^2}\sum_{j=1}^{n}\int_\Omega\int_\Omega \frac{d\nu}{d\pi}(y)\big(K^{n_0+j}(x,\mathrm{d}y) - \pi(\mathrm{d}y)\big)\, g(x)^2\, \pi(\mathrm{d}x) \\
&\quad + \frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\int_\Omega\int_\Omega \frac{d\nu}{d\pi}(y)\, \big(K^{n_0+j}(x,\mathrm{d}y) - \pi(\mathrm{d}y)\big)\, g(x) P^{k-j} g(x)\, \pi(\mathrm{d}x).
\end{aligned} \tag{18.21}
$$

Proof. It is easy to see that

$$
\begin{aligned}
\mathbb{E}_{\nu,K}\left|S(f) - A_{n,n_0}(f)\right|^2 &= \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}_{\nu,K}(g(X_{n_0+j})g(X_{n_0+i})) \\
&= \frac{1}{n^2}\sum_{j=1}^{n}\int_\Omega P^{n_0+j} g(x)^2\, \nu(\mathrm{d}x) \\
&\quad + \frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\int_\Omega P^{n_0+j}(g(x)P^{k-j}g(x))\, \nu(\mathrm{d}x).
\end{aligned}
$$

For every function $h \in L_2(\Omega, \pi)$ and $i \in \mathbb{N}$, we apply (18.13) and conclude

$$\begin{aligned}
&\int_\Omega P^i h(x)\, \nu(\mathrm{d}x) \\
&= \int_\Omega \int_\Omega h(y)\, K^i(x, \mathrm{d}y)\, \frac{d\nu}{d\pi}(x)\, \pi(\mathrm{d}x) \\
&\underset{(18.13)}{=} \int_\Omega \int_\Omega \frac{d\nu}{d\pi}(y) K^i(x, \mathrm{d}y)\, h(x)\, \pi(\mathrm{d}x) \\
&= \int_\Omega h(x)\, \pi(\mathrm{d}x) + \int_\Omega \int_\Omega \frac{d\nu}{d\pi}(y) \left(K^i(x, \mathrm{d}y) - \pi(\mathrm{d}y)\right) h(x)\, \pi(\mathrm{d}x) \\
&\underset{(18.13)}{=} \int_\Omega P^i h(x) \pi(\mathrm{d}x) + \int_\Omega \int_\Omega \frac{d\nu}{d\pi}(y) \left(K^i(x, \mathrm{d}y) - \pi(\mathrm{d}y)\right) h(x)\, \pi(\mathrm{d}x).
\end{aligned}$$

Hence (18.21) follows, as claimed. □

The next result modifies the convergence property described in Lemma 18.10.

Lemma 18.14. *Let $\mathcal{M}$ be a lazy, reversible Markov scheme with stationary distribution π, and let ν be the initial distribution with bounded density $\frac{d\nu}{d\pi}$ with respect to π. Then for $h \in L_\infty(\Omega, \pi)$ and $j \in \mathbb{N}$, we have*

$$\begin{aligned}
&\left| \int_\Omega \int_\Omega \frac{d\nu}{d\pi}(y) \left(K^j(x, \mathrm{d}y) - \pi(\mathrm{d}y)\right) h(x)\, \pi(\mathrm{d}x) \right| \\
&\qquad \le 4\|h\|_\infty \sqrt{\left\|\frac{d\nu}{d\pi}\right\|_\infty} \left(1 - \frac{\varphi(K, \pi)^2}{2}\right)^j .
\end{aligned}$$

Proof. Define $p_j(x) = \int_\Omega \frac{d\nu}{d\pi}(y) \left(K^j(x, \mathrm{d}y) - \pi(\mathrm{d}y)\right)$. It is easy to see that the measurability of the density and the kernel carries over to p_j. Now we consider the positive and negative parts of the functions h and p_j. To formalize this we use

$$\begin{aligned}
\Omega_+^+ &:= \{x \in \Omega \mid p_j(x) \ge 0,\ h(x) \ge 0\}, \\
\Omega_-^+ &:= \{x \in \Omega \mid p_j(x) \ge 0,\ h(x) < 0\}, \\
\Omega_+^- &:= \{x \in \Omega \mid p_j(x) < 0,\ h(x) \ge 0\}, \\
\Omega_-^- &:= \{x \in \Omega \mid p_j(x) < 0,\ h(x) < 0\}.
\end{aligned}$$

These subsets of Ω are all included in the σ-algebra $\mathcal{A}$ since p_j and h are measurable functions. Applying (18.18), we obtain the following upper bound:

$$\begin{aligned}
&\left| \int_\Omega p_j(x) h(x)\, \pi(\mathrm{d}x) \right| \\
&\quad \le \left| \int_{\Omega_+^+} p_j(x) h(x)\, \pi(\mathrm{d}x) \right| + \left| \int_{\Omega_-^+} p_j(x) h(x)\, \pi(\mathrm{d}x) \right| \\
&\qquad + \left| \int_{\Omega_+^-} p_j(x) h(x)\, \pi(\mathrm{d}x) \right| + \left| \int_{\Omega_-^-} p_j(x) h(x)\, \pi(\mathrm{d}x) \right| \le
\end{aligned}$$

$$\leq \|h\|_\infty \left| \int_{\Omega_+^+} p_j(x)\,\pi(\mathrm{d}x) \right| + \|h\|_\infty \left| \int_{\Omega_-^+} p_j(x)\,\pi(\mathrm{d}x) \right|$$

$$+ \|h\|_\infty \left| \int_{\Omega_+^-} p_j(x)\,\pi(\mathrm{d}x) \right| + \|h\|_\infty \left| \int_{\Omega_-^-} p_j(x)\,\pi(\mathrm{d}x) \right|$$

$$\underset{(18.18)}{\leq} 4\|h\|_\infty \sqrt{\|\frac{d\nu}{d\pi}\|_\infty} \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j . \qquad \square$$

All results are now available to obtain the main error bound of Rudolf [257] for the MCMC method A_{n,n_0}.

Theorem 18.15. *Let $X_1, X_2, \ldots$ be a lazy, reversible Markov chain, defined by the scheme $\mathcal{M}$ and the initial distribution ν. Let the initial distribution have a bounded density $\frac{d\nu}{d\pi}$ with respect to π. Let*

$$A_{n,n_0}(f) = \frac{1}{n} \sum_{j=1}^{n} f(X_{n_0+j})$$

be the approximation of $S(f) = \int_\Omega f(x)\pi(\mathrm{d}x)$ for $f \in L_\infty(\Omega,\pi)$. Then

$$e(A_{n,n_0}, f) \leq \frac{2\sqrt{1 + 24\sqrt{\|\frac{d\nu}{d\pi}\|_\infty} \exp\left[-n_0 \frac{\varphi(K,\pi)^2}{2}\right]}}{\varphi(K,\pi) \cdot \sqrt{n}} \|f\|_\infty .$$

Proof. By Lemma 18.13 and Lemma 18.14 with $g = f - S(f)$, we have

$$\mathbb{E}_{\nu,K} \left|S(f) - A_{n,n_0}(f)\right|^2$$

$$\leq \mathbb{E}_{\pi,K} |S(f) - A_n(f)|^2 + \frac{4\|g\|_\infty^2}{n^2} \sum_{j=1}^{n} \sqrt{\|\frac{d\nu}{d\pi}\|_\infty} \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^{j+n_0}$$

$$+ \frac{8\|g\|_\infty^2}{n^2} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \|P^{k-j}\|_{L_\infty \to L_\infty} \sqrt{\|\frac{d\nu}{d\pi}\|_\infty} \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^{j+n_0} .$$

To simplify the notation we define

$$\varepsilon_0 := \sqrt{\|\frac{d\nu}{d\pi}\|_\infty} \exp\left[-n_0 \frac{\varphi(K,\pi)^2}{2}\right]. \tag{18.22}$$

From (18.19) and (18.22), we obtain

$$a := \mathbb{E}_{\nu,K} \left|S(f) - A_{n,n_0}(f)\right|^2$$

$$\leq \mathbb{E}_{\pi,K} |S(f) - A_n(f)|^2 + \frac{4\,\varepsilon_0 \|g\|_\infty^2}{n^2} \sum_{j=1}^{n} \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j$$

$$+ \frac{8\,\varepsilon_0 \|g\|_\infty^2}{n^2} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \|P^{k-j}\|_{L_\infty \to L_\infty} \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j .$$

Summing up the geometric series and estimating $\|P^i\|_{L_\infty \to L_\infty} \le 1$ for all $i \in \mathbb{N}$, we get

$$\begin{aligned} a &\le \mathbb{E}_{\pi,K}\, |S(f) - A_n(f)|^2 + \frac{8\,\varepsilon_0 \|g\|_\infty^2}{\varphi(K,\pi)^2 \cdot n^2} \\ &\quad + \frac{8\,\varepsilon_0 \|g\|_\infty^2}{n^2} \sum_{j=1}^{n-1} (n-j) \left(1 - \frac{\varphi(K,\pi)^2}{2}\right)^j \\ &\le \mathbb{E}_{\pi,K}\, |S(f) - A_n(f)|^2 + \frac{16\,\varepsilon_0 \|g\|_\infty^2}{\varphi(K,\pi)^2 \cdot n} + \frac{8\,\varepsilon_0 \|g\|_\infty^2}{\varphi(K,\pi)^2 \cdot n^2} \\ &\le \mathbb{E}_{\pi,K}\, |S(f) - A_n(f)|^2 + \frac{24\,\varepsilon_0 \|g\|_\infty^2}{\varphi(K,\pi)^2 \cdot n}. \end{aligned}$$

Applying Theorem 18.12 and using $\|f\|_2^2 \le \|f\|_\infty^2$ and $\|g\|_\infty^2 \le 4\|f\|_\infty^2$, we conclude the proof. □

A further estimate yields the following conclusion.

Corollary 18.16. *Let $X_1, X_2, \ldots$ be a lazy, reversible Markov chain. Let the initial distribution ν have a bounded density $\frac{d\nu}{d\pi}$ with respect to π. Let*

$$A_{n,n_0}(f) = \frac{1}{n} \sum_{j=1}^{n} f(X_{j+n_0}) \quad \textit{for all } f \in L_\infty(\Omega, \pi)$$

with a burn-in

$$n_0 \ge \frac{\ln\left(\|\frac{d\nu}{d\pi}\|_\infty\right)}{\varphi(K,\pi)^2}.$$

Then

$$e(A_{n,n_0}, f) \le \frac{10}{\varphi(K,\pi) \cdot \sqrt{n}} \|f\|_\infty. \tag{18.23}$$

If we denote by $\operatorname{cost}(f,\varepsilon)$ the number $n + n_0$ of time steps that are needed for an optimal algorithm to solve (18.11) to within an error ε, then we have

$$\operatorname{cost}(f,\varepsilon) \le \left\lceil \frac{\ln\left(\|\frac{d\nu}{d\pi}\|_\infty\right)}{\varphi(K,\pi)^2} \right\rceil + \left\lceil \frac{100\|f\|_\infty^2}{\varphi(K,\pi)^2 \cdot \varepsilon^2} \right\rceil.$$

This means that if we control the conductance of the underlying Markov chain, then we also control the error and the cost.

We are now ready to describe in more detail the algorithm of Rudolf. He studies a *Metropolis* algorithm based on a certain δ ball walk after a burn-in time. Let $\Omega \subseteq \mathbb{R}^d$ be a convex body and let

$$\mathcal{M} = (\Omega, \{Q(x,\cdot) : x \in \Omega\})$$

be a reversible Markov scheme with respect to a distribution μ. Here $(x, A) \mapsto Q(x, A)$ is the transition kernel, and A is Borel measurable. We want to simulate a distribution μ_q on Ω, which is defined by an unnormalized density q by

$$\mu_q(A) = \frac{\int_A q(x)\,\mu(\mathrm{d}x)}{\int_\Omega q(x)\,\mu(\mathrm{d}x)}. \tag{18.24}$$

The distribution μ_q is simulated by computing function values of q.

If we choose a starting point X_1 from a known distribution and take this as input into the algorithm, then we obtain the new Markov kernel

$$K_q(x, A) := \int_A \vartheta(x, y)\, Q(x, \mathrm{d}y) + 1_A(x)\left(1 - \int_\Omega \vartheta(x, y)\, Q(x, \mathrm{d}y)\right), \tag{18.25}$$

where

$$\vartheta(x, y) := \min\left(1, \frac{q(y)}{q(x)}\right)$$

is the *acceptance probability*, i.e., the probability that the move from x to y suggested by Q is really done. With probability $1 - \vartheta(x, y)$ we stay at x. Then the resulting Markov scheme $\mathcal{M}_q$ with kernel K_q is reversible with respect to μ_q.

Lemma 18.17. *If the Markov scheme $\mathcal{M}$ of the Metropolis algorithm is reversible with respect to a distribution μ, then the reversibility condition holds also for $\mathcal{M}_q$ with respect to μ_q.*

Proof. It is enough to show that

$$\int_A K_q(x, B)\,\mu_q(\mathrm{d}x) = \int_B K_q(x, A)\,\mu_q(\mathrm{d}x)$$

holds for disjoint sets A and B. Note that $\vartheta(y, x)q(y) = \vartheta(x, y)q(x)$ for $x, y \in \Omega$. Define $k := \int_\Omega q(x)\,\mu(\mathrm{d}x)$. Then

$$\begin{aligned}
\int_A K_q(x, B)\,\mu_q(\mathrm{d}x) &\underset{(18.25)}{=} \int_A \int_B \vartheta(x, y)\; Q(x, \mathrm{d}y)\,\mu_q(\mathrm{d}x)\\
&\underset{(18.24)}{=} \frac{1}{k}\int_A \int_B \vartheta(x, y)q(x)\; Q(x, \mathrm{d}y)\,\mu(\mathrm{d}x)\\
&= \frac{1}{k}\int_\Omega \int_\Omega \chi_A(x)\chi_B(y)\,\vartheta(x, y)q(x)\; Q(x, \mathrm{d}y)\,\mu(\mathrm{d}x)\\
&\underset{(18.12)}{=} \frac{1}{k}\int_\Omega \int_\Omega \chi_A(y)\chi_B(x)\,\vartheta(y, x)q(y)\; Q(x, \mathrm{d}y)\,\mu(\mathrm{d}x)\\
&= \frac{1}{k}\int_B \int_A \vartheta(x, y)q(x)\; Q(x, \mathrm{d}y)\,\mu(\mathrm{d}x)\\
&= \int_B K_q(x, A)\,\mu_q(\mathrm{d}x).
\end{aligned}$$

□

To apply the theory presented before, we also need the laziness property. To achieve this, we just have to flip a coin and stay at the current state with probability 1/2. Otherwise we do one step with the chain. Formally, this means that we consider

$$\bar{\mathcal{M}}_q = (\Omega, \{\bar{K}_q(x,\cdot) : x \in \Omega\}),$$

where

$$\bar{K}_q(x,A) := \tfrac{1}{2} K_q(x,A) + \tfrac{1}{2} 1_A(x).$$

This Markov scheme is lazy and reversible. To apply Theorem 18.15, we need to get a positive lower bound on its conductance. Therefore the following result is helpful.

Lemma 18.18. *Let $\mathcal{M} = (\Omega, \mathcal{A}, \{K(x,A) : x \in \Omega\})$ be an arbitrary reversible Markov scheme concerning π. The conductance of*

$$\bar{\mathcal{M}} = \left(\Omega, \mathcal{A}, \{\bar{K}(x,A) : x \in \Omega\}\right)$$

with $\bar{K}(x,A) = \frac{1}{2} K(x,A) + \frac{1}{2} I(x,A)$ is bounded from below by

$$\varphi(\bar{K},\pi) \geq \tfrac{1}{2} \varphi(K,\pi).$$

Proof. The result is obvious after taking the definition of the conductance into account. □

We come to a concrete Markov chain defined by a δ ball walk on the convex body Ω. This random walk was studied by many authors, see Lovász and Simonovits [178], Vempala [318] and [182].

The corresponding Markov scheme is $\mathcal{M}_\delta = (\Omega, \{Q_\delta(x,\cdot) : x \in \Omega\})$, where

$$Q_\delta(x,A) := \frac{\mathrm{vol}(B(x,\delta) \cap A)}{\mathrm{vol}(\delta B^d)} + \left(1 - \frac{\mathrm{vol}(B(x,\delta) \cap \Omega)}{\mathrm{vol}(\delta B^d)}\right) 1_A(x).$$

Here $B(x,\delta)$ denotes the ball of radius δ at $x \in \Omega$ and $\delta B^d := B(0,\delta)$. We choose $\delta \leq D$, where D is the diameter of Ω. It is easily seen that $\mathcal{M}_\delta$ is reversible with respect to the uniform distribution on Ω. By taking this ball walk as the kernel for the Metropolis algorithm we get $\mathcal{M}_{q,\delta} = (\Omega, \{K_{q,\delta}(x,\cdot) : x \in \Omega\})$ with

$$K_{q,\delta}(x,A) := \int_A \vartheta(x,y)\, Q_\delta(x,\mathrm{d}y) + 1_A(x) \left(1 - \int_\Omega \vartheta(x,y)\, Q_\delta(x,\mathrm{d}y)\right).$$

Now we assume that Ω is the d-dimensional (Euclidean) unit ball denoted by B^d. The following lower bound for the conductance is known, see [182, Corollary 1].

Lemma 18.19. *Let the Markov scheme*

$$\mathcal{M}_{q,\delta} = \left(B^d, \{K_{q,\delta}(x,\cdot) : x \in B^d\}\right)$$

be the Metropolis chain based on the ball walk $\mathcal{M}_\delta$, where $q \in \mathcal{R}^\alpha(B^d)$. Then for $\delta = \min\{1/\sqrt{d+1}, 1/\alpha\}$ we have

$$\varphi(K_{q,\delta}, \mu_q) \geq 0.0025 \frac{1}{\sqrt{d+1}} \min\left(\frac{1}{\sqrt{d+1}}, \frac{1}{\alpha}\right). \tag{18.26}$$

We stress that the geometry of the (Euclidean) unit ball is essential since the ball walk would get stuck with high probability in domains which have corners. Hence it is not clear how one could modify the results of this section if the ball is replaced by a cube or another convex body, see Open Problem 84.

Rudolf [257] studies the algorithm

$$A_{n,n_0}^{\delta}(f,q) = \frac{1}{n}\sum_{j=1}^{n} f(X_{j+n_0}).$$

The error bound of A_{n,n_0} is given in the following theorem.

Theorem 18.20. *Let $X_1, X_2, \ldots$ be the lazy Metropolis Markov chain which is based on a δ ball walk with $\delta = \min\left\{1/\sqrt{d+1}, 1/\alpha\right\}$. Then*

$$e(A_{n,n_0}, \mathcal{F}^{\alpha}(B^d)) \le 8000 \frac{\sqrt{d+1}\max\left\{\sqrt{d+1},\alpha\right\}}{\sqrt{n}},$$

where $n_0 \ge 1280000 \cdot \alpha(d+1)\max\left\{d+1,\alpha^2\right\}$.

Proof. We start the walk with the uniform distribution ν on the state space B^d. For ν and measurable $A \subseteq B^d$, we have

$$\nu(A) = \frac{\mathrm{vol}(A)}{\mathrm{vol}(B^d)} = \frac{1}{\mathrm{vol}(B^d)}\int_A \int_{B^d} \frac{q(y)}{q(x)}\,\mathrm{d}y\,\mu_q(\mathrm{d}x).$$

This implies that

$$\|\frac{d\nu}{d\mu_q}\|_\infty \le \exp(2\alpha) \quad \text{for all } q \in \mathcal{R}^{\alpha}(B^d).$$

Using this bound, the lower bound (18.26) for the conductance, and applying Lemma 18.18, Lemma 18.19 and (18.23), we obtain the error bound of Theorem 18.20. □

From Theorem 18.20 we easily obtain Theorem 18.6 by finding n for which the upper bound on $e(A_{n,n_0}, \mathcal{F}^{\alpha}(B^d))$ is at most ε. We also take n_0 as the smallest number allowed in Theorem 18.20 and keep in mind that each step of the burn-in time requires one evaluation of q.

We conclude this subsection with two open problems.

Open Problem 83.

- Observe that the densities $q \in \mathcal{R}^{\alpha}(\Omega)$ are unimodal, they have only one local maximum. Prove (in)tractability results for larger classes of densities that may have many maxima.

Open Problem 84.

- The results extend in a similar way to certain other families of convex sets $\Omega_d \subseteq \mathbb{R}^d$ for which the underlying ball walk Q_δ with a small δ has a positive lower bound for the conductance that is not too small. It is not clear, however, whether similar results can be proved for classes such as $\mathcal{F}^\alpha([0,1]^d)$ and we pose this as an open problem.

18.2 Integral Equations

We study the *local* solution of Fredholm integral equations of the second type

$$u(x) - \int_{[0,1]^d} q(x,t)\, u(t)\, \mathrm{d}t = f(x),$$

where $q\colon [0,1]^{2d} \to \mathbb{R}$ and $f\colon [0,1]^d \to \mathbb{R}$. We assume that both q and f are Lipschitz continuous as well as $\|q\|_\infty \le \gamma < 1$ and $\|f\|_\infty \le 1$. Under these conditions the solution u exists and is unique. We want to approximate the solution u only at one point s from $[0,1]^d$.

Let us describe the problem more formally. Define $D = [0,1]^d$ and let $C(D)$ be the space of continuous functions on D endowed with the supremum norm. Hence we assume that $f \in C(D)$ with $\|f\|_\infty \le 1$, and $g \in C(D^2)$ with $\|g\|_\infty \le \gamma$. The functions f and g are also Lipschitz continuous. We define the Lipschitz constant of a function $h\colon D^m \to \mathbb{R}$ as

$$\mathrm{Lip}(h) := \sup\biggl(\frac{|h(x)-h(y)|}{\|x-y\|_\infty} \;\Big|\; x, y \in D^m,\ x \ne y \biggr).$$

Fix $\gamma \in (0,1)$. We consider the following set of functions

$$F_d := \bigl\{ (f,q) \mid q \in C(D^2),\ \|q\|_\infty \le \gamma, \mathrm{Lip}(q) \le 1, \\ f \in C(D),\ \|f\|_\infty \le 1, \mathrm{Lip}(f) \le 1 \bigr\}.$$

To define the solution operator, we fix $s \in D$ and denote Id as the identity on the space $C(D)$. Furthermore for $q \in C(D^2)$, let T_q denote the Fredholm integral operator on $C(D)$, that is,

$$\bigl[T_q v\bigr](x) = \int_D q(x,t) v(t)\, \mathrm{d}t \quad \text{for all } v \in C(D).$$

Define the mapping $S_d\colon F_d \to \mathbb{R}$ by

$$S_d(f,q) := \bigl[(\mathrm{Id} - T_q)^{-1} f\bigr](s) = u(s)$$

as the solution of the Fredholm integral equation at s.

The mapping S_d is well defined since the operator norm of T_q satisfies

$$\|T_q\| \le \gamma < 1.$$

Hence, $(\mathrm{Id} - T_q)^{-1}$ exists and $\|(\mathrm{Id} - T_q)^{-1}\| \le 1/(1-\gamma)$. Therefore

$$S_d(f,q) \in \left[\frac{-1}{1-\gamma}, \frac{1}{1-\gamma}\right].$$

Using the constant functions $q \equiv \gamma$ and $f \equiv \alpha$ with $\alpha \in [-1, 1]$, we see that $S_d(f,g) = \alpha/(1-\gamma)$. This proves that any number from the interval above can be the solution. Hence, the problem S_d is scaled in the same way for any dimension d.

Obviously, S_d is linear in f and nonlinear in q. Therefore S_d is a nonlinear functional. It can be shown that it is quasi-linear, i.e., it satisfies a Lipschitz condition with respect to f and q but this property will not be used for our analysis.

By $S^{\mathrm{non}} = \{S_d\}$ we denote the sequence of the local solutions of Fredholm integral equations. We study tractability of S^{non} in the worst case and in two randomized settings. We prove that S^{non} is intractable and suffers from the curse of dimensionality in the worst case setting. Therefore we switch to the randomized setting. We briefly consider the standard randomized setting and then switch to the restricted randomized setting in which we assume that we can use only random bits. In both cases, we present polynomial tractability results.

18.2.1 Worst Case Setting

Let $n(\varepsilon, F_d)$ denote the minimal number of function values of f and q needed to compute an ε approximation to $u(s) = S_d(f,q)$ for the absolute error criterion in the worst case setting. It is known that

$$n(\varepsilon, F_d) = \Theta\big(\varepsilon^{-2d}\big)$$

with the factor in the big Θ notation independent of ε^{-1} but dependent on d. This is a result of Emelyanov and Ilin [67]. They also showed that the (total) complexity is of the same order. They use a "two-grid" algorithm to prove the upper bound and we stress that it was done already in 1967.

For small d, the result of Emelyanov and Ilin [67] is quite positive. However, for large d, the exponent of ε^{-1} is also large. As we know from Volume I, such a result contradicts polynomial tractability but weak tractability is not yet clear since it depends on the factors in the big Θ notation.

It is easy to show that the problem S^{non} suffers from the curse of dimensionality and is intractable. Indeed, it is enough to take $f \equiv 1$ and q independent of x, i.e., $q(x,t) = q(t)$ for all $x, t \in D$. Then the solution u is constant and

$$S_d(1,q) = \frac{1}{1 - \int_{[0,1]^d} q(t)\,\mathrm{d}t}.$$

Let $I_d(q) = \int_{[0,1]^d} q(t)\,\mathrm{d}t$ be the integral of q. It is easy to see that weak tractability of $S_1^{\text{non}} = \{S_d(1,\cdot)\}$ is equivalent to weak tractability of $I = \{I_d\}$. Indeed, note that $\|q\|_\infty \le \gamma$ implies that

$$S_d(1,q) \in \left[\frac{1}{1+\gamma}, \frac{1}{1-\gamma}\right].$$

Suppose that we have an algorithm A_n that approximates $S_d(1,\cdot)$ with the worst case error at most $\varepsilon < \frac{1}{2}(1+\gamma)^{-1}$. Consider the algorithm $B_n = 1 - A_n^{-1}$ for the approximation of I_d. Then

$$I_d(q) - B_n(q) = 1 - S_d(1,q)^{-1} - 1 + A_n^{-1}(q) = \frac{S_d(1,q) - A_n(q)}{A_n(q)\, S_d(1,q)}.$$

Since $|S_d(1,q) - A_n(q)| \le \varepsilon$, then

$$|A_n(q)| \ge S_d(1,q) - |S_d(1,q) - A_n(q)| \ge (1+\gamma)^{-1} - \varepsilon \ge \tfrac{1}{2}(1+\gamma)^{-1}.$$

Therefore

$$|I_d(q) - B_n(q)| \le 2(1+\gamma)^2 \varepsilon.$$

Hence, weak tractability of S_1^{non} implies weak tractability of I. Similarly, we can show the reverse implication.

We know that multivariate integration $I = \{I_d\}$ suffers from the curse of dimensionality and is intractable. This was shown in Example 1 of Chapter 3 in Volume I. Based on the result of Sukharev [291] we know that d-variate integration of Lipschitz functions requires roughly $(2\gamma\,\varepsilon)^{-d}/e$ function values so the curse of dimensionality is present. Hence, S^{non} also suffers from the curse of dimensionality and is intractable.

We add in passing that we also have intractability and the curse of dimensionality for the normalized error criterion in the worst case setting. This simply follows from the fact that the initial errors for both S_d and I_d do not depend on d and only depend on γ.

18.2.2 Restricted Randomized Setting

The curse of dimensionality in the worst case setting is a reason to switch to the randomized setting and to study randomized algorithms. Monte Carlo algorithms with the (dimension independent) rate ε^{-2} are well known, hence we have an upper bound for the complexity in the randomized setting of the form $C_d \cdot \varepsilon^{-2}$. Even the optimal rate of convergence is known, see Heinrich and Mathé [113], it is of the form

$$\mathrm{comp}^{\text{ran}}(\varepsilon, d) \approx C_d \cdot \varepsilon^{-2d/(d+1)} \tag{18.27}$$

for some unspecified positive factors C_d dependent on d.

In this section we deviate from Heinrich and Mathé [113] in two ways:

- We want to allow only a very restricted randomness, namely random bits or random elements from $\{0, 1\}$. This leads to the class of "coin tossing algorithms" or "restricted Monte Carlo algorithms", see [206]. We ask how many random bits, arithmetic operations, and function values of f and q are needed to achieve an error ε.

- We want to have explicit bounds, without factors C_d that depend in an uncontrolled way on d.

We construct a restricted Monte Carlo algorithm with error ε that uses roughly ε^{-2} function values and only $d \ln^2 \varepsilon$ random bits. The number of arithmetic operations is of the order $\varepsilon^{-2} + d \ln^2 \varepsilon$. In these bounds there are no factors that depend on d or ε. Hence, the cost of our algorithm increases only mildly with the dimension d since

$$\mathrm{comp}^{\mathrm{coin}}(\varepsilon, d) = \mathcal{O}\big(\varepsilon^{-2} + d\ \ln^2(\varepsilon^{-1} + 1)\big). \tag{18.28}$$

In particular, this means the problem S^{non} is polynomially tractable in this restricted randomized setting with an ε^{-1} exponent at most 2 and a d exponent at most 1. The proof of (18.28) is based on the results of [114] on the summation problem.

We now elaborate on the restricted randomized setting. We study the approximation of the solution operator S_d on the set F_d of problem elements by restricted randomized algorithms. For such algorithms we allow the operations of the real number model of computation with an oracle as explained in Chapter 4 of Volume I. In addition we also allow the instruction "choose a random bit", that is, "choose an element of $\{0, 1\}$ according to the equi-distribution", see [206] for a formal description of this model of computation.

Let $(\Omega, \mathcal{B}, P)$ be the countable infinite product of the probability space that corresponds to the coin tossing instruction. Let A be a restricted randomized algorithm. For each $\omega = (\omega_1, \omega_2, \omega_3, \dots) \in \Omega$, we define a (partial) mapping $A_\omega \colon F_d \to \mathbb{R}$ as follows. Given a problem element $(f, q) \in F$, we apply A to (f, q) taking, when necessary, ω_i as the ith random bit. If the algorithm terminates, we set $A_\omega(f, q)$ equal to the output of the algorithm and

$$e(S, A, (f, q), \omega) := |S(f, q) - A_\omega(f, q)|.$$

We define $\mathrm{cost}(A, (f, q), \omega)$ as the sum of the number of function evaluations, the number of used random bits ω_i, and the number of arithmetic operations needed to compute $A_\omega(f, q)$.

If the mappings $e(S, A, (f, q), .)$ and $\mathrm{cost}(A, (f, q), .)$ are defined almost everywhere on Ω and are measurable for each $(f, q) \in F_d$ then the quantities

$$e(S, A, F_d) := \sup_{(f,q)\in F_d} e(S, A, (f, q)) := \sup_{(f,q)\in F_d} \Big(\mathbb{E}\ e^2(S, A, (f, q), \cdot)\Big)^{1/2}$$

and

$$\mathrm{cost}(A, F_d) := \sup_{(f,q)\in F_d} \mathrm{cost}(A, (f, q)) := \sup_{(f,q)\in F_d} \mathbb{E}\ \mathrm{cost}(A, (f, q), \cdot)$$

are called the error and cost of A on F_d, respectively. Here, $\mathbb{E}$ denotes the expectation with respect to P.

We give a rough idea of our algorithm. For $(f,q) \in F_d$ and $m \in \mathbb{N}$, let the function $g_m^{(f,q)}\colon D^m \to \mathbb{R}$ be defined by

$$g_m^{(f,q)}(x) = q(s,x_1) \cdot \prod_{i=1}^{m-1} q(x_i,x_{i+1}) \cdot f(x_m) \quad \text{for all } x = [x_1,x_2,\dots,x_m] \in D^m.$$

Since $(\mathrm{Id} - T_q)^{-1} = \mathrm{Id} + \sum_{m=1}^{\infty} T_q^m$, we apply Neumann's series and obtain

$$S(f,q) = f(s) + \sum_{m=1}^{\infty} \int_{D^m} g_m^{(f,q)}(x)\,\mathrm{d}x. \tag{18.29}$$

We replace the infinite series of integrals by a finite series of sums and approximate each term by an algorithm from [114]. This approach results in the restricted randomized algorithm A_n whose precise definition we give below as well as the proof of the following theorem about its error and cost.

Theorem 18.21. *There are positive numbers c_1, c_2, c_3 depending only on γ such that*

$$e(S_d,A_n,F_d) \le c_1\, n^{-1/2} \quad \textit{and} \quad \mathrm{cost}(A_n,F_d) \le c_2\, n + c_3\, d \ln^2(n+1).$$

Hence, the cost of A_n increases at most linearly with the dimension d. In particular, we see again that the problem $S^{\mathrm{non}} = \{S_d\}$ is polynomially tractable in the restricted randomized setting. It is clear that (18.28) follows from Theorem 18.21.

First, we precisely define the restricted randomized algorithm A_n. We make use of tensor products Q_s^d of the midpoint rule Q_s^1, given by

$$Q_s^1(g) := \frac{1}{s}\sum_{i=1}^{s} g\Big(\frac{2i-1}{2s}\Big) \quad \text{for } g\colon [0,1] \to \mathbb{R}.$$

It is well known that Q_s^d is an optimal algorithm in the worst case setting for the approximation of the integration functional I_d over the class

$$F_{\mathrm{Lip}} = \{g \in C(D) \mid \mathrm{Lip}(g) \le 1\}.$$

We have

$$\sup_{f\in F_{\mathrm{Lip}}} |I_d(g) - Q_s^d(g)| = \frac{d}{2d+2}\, s^{-1}, \tag{18.30}$$

see Sukharev [291] as well as Example 1 of Chapter 3 in Volume I.

Furthermore, we use the restricted randomized algorithm $\widetilde{A}_n$ from [114]. This algorithm provides an approximation of the mean of a finite sequence of real numbers $\{h(i)\}_{i=0}^{N-1}$ in the following way.

Let P be a prime such that $P \in [N, 2N)$. As we know, due to Bertrand's Postulate or Chebyshev's theorem, such a prime exists. For the vector $h = \big[h(0), \dots, h(N-1)\big] \in \mathbb{R}^N$, define the vector

$$\tilde{h} = [\tilde{h}(0), \tilde{h}(1), \dots, \tilde{h}(P-1)] = \big[h(0), \dots, h(N-1), 0, \dots, 0\big] \in \mathbb{R}^P.$$

Then the algorithm $\widetilde{A}_n$ is of the form

$$\widetilde{A}_{n,\omega}(h) = \frac{P}{N\big(n + \sqrt{n(P-n)/(P-1)}\big)} \sum_{k=1}^{n} \tilde{h}(i_k^\omega),$$

where the i_k^ω are defined as follows. Choose

$$x^\omega \in \{0, 1, \dots, P-1\} \quad \text{and} \quad y^\omega \in \{1, 2, \dots, P-1\}$$

independently according to the respective uniform distributions and put

$$i_k^\omega := x^\omega + (k-1) \cdot y^\omega \mod P.$$

It turns out that $c \ln N$ random bits and arithmetic operations suffice to realize x^ω and y^ω on the average. Here, c is a positive constant not depending on N or P. In addition, the computation of the i_k^ω and of the corresponding sum requires $4n$ arithmetic operations[1] and the evaluation of n components of the vector $\tilde{h}$. The randomized error of $\widetilde{A}_n$ is bounded by

$$\mathbb{E}\left(\frac{1}{N}\sum_{i=0}^{N-1} h(i) - \widetilde{A}_n(h)\right)^2 \le \frac{2}{n}\,\frac{1}{N}\sum_{i=0}^{N-1} h^2(i), \tag{18.31}$$

see [114] for more details.

We now can define the restricted randomized algorithm A_n. For $n \ge 2$, let $s := \lceil n^{1/2} \rceil$ and

$$M := \lceil (\ln \gamma^{-1})^{-1} \ln s \rceil \quad \text{and} \quad n_m := \lceil \gamma^m n \rceil \quad \text{for all } m = 1, \dots, M.$$

Let $x_{s,m}^1, \dots, x_{s,m}^{s^{md}}$ denote the sample points of Q_s^{md}, and let the mapping $\Gamma_{s,m}\colon F_d \to \mathbb{R}^{s^{Md}}$ be given by

$$\Gamma_{s,m}(f,q) = \Big[\underbrace{g_m^{(f,q)}(x_{s,m}^1), \dots, g_m^{(f,q)}(x_{s,m}^{s^{md}}), \dots, g_m^{(f,q)}(x_{s,m}^1), \dots, g_m^{(f,q)}(x_{s,m}^{s^{md}})}_{s^{(M-m)d} \text{ times}}\Big].$$

Hence, the mean of the components of $\Gamma_{s,m}(f,q)$ is equal to $Q_s^{md}(g_m^{(f,q)})$. We finally set

$$A_{n,\omega}(f,q) = f(s) + \sum_{m=1}^{M} \widetilde{A}_{n_m,\omega}\big(\Gamma_{s,m}(f,q)\big).$$

[1] We assume here that the modulo operation is allowed and has unit cost.

We are ready to prove Theorem 18.21. We use positive numbers c_i' that depend on γ, but are independent of the dimension d and the point s. We first estimate the error of A_n. To this end, we introduce the mapping $\widetilde{A}_s\colon F_d \to \mathbb{R}$ by

$$\widetilde{A}_s(f,q) = f(s) + \sum_{m=1}^{M} Q_s^{md}(g_m^{(f,q)}).$$

For each $(f,q) \in F_d$, the triangle inequality yields

$$e\big(S_d, A_n, (f,q)\big) \le |S_d(f,q) - \widetilde{A}_s(f,q)| + \sum_{m=1}^{M} \Big(\mathbb{E}\big(Q_s^{md}(g_m^{(f,q)}) - \widetilde{A}_{n_m}(\Gamma_{s,m}(f,q))\big)^2\Big)^{1/2}. \tag{18.32}$$

The first term on the right-hand side corresponds to the error made by replacing the infinite series $\sum_{m=1}^{\infty} I_{md}(g_m^{(f,q)})$ with the finite series $\sum_{m=1}^{M} Q_s^{md}(g_m^{(f,q)})$, the second term corresponds to the respective error made by approximating the mean $Q_s^{md}(g_m^{(f,q)})$ by $\widetilde{A}_{n_m}$.

We estimate these errors using (18.30) and (18.31). This requires knowledge of the Lipschitz constants and of the norms of the $g_m^{(f,q)}$.

Lemma 18.22. *Let $(f,q) \in F_d$ and $m \in \mathbb{N}$. Then*

$$\mathrm{Lip}\big(g_m^{(f,q)}\big) \le (m+1)\gamma^{m-1} \quad \textit{and} \quad \|g_m^{(f,q)}\|_\infty \le \gamma^m.$$

Proof. The second inequality is an immediate consequence of the definition of $g_m^{(f,q)}$ and of the assumptions on the norms of q and f. To prove the first inequality, we take $(f,q) \in F_d$ and $m \in \{2,3,4,\ldots\}$, and consider the function

$$h_m^{(f,q)}\colon D^m \to \mathbb{R}, \quad x = [x_1, x_2, \ldots, x_m] \mapsto \prod_{i=1}^{m-1} q(x_i, x_{i+1}) \cdot f(x_m).$$

We show by induction that

$$\mathrm{Lip}\big(h_m^{(f,q)}\big) \le m\gamma^{m-2} \quad \text{for all } m \in \{2,3,4,\ldots\}. \tag{18.33}$$

Consider $m = 2$. The assumptions on q and f imply

$$\begin{aligned} |h_2^{(f,q)}(x_1,x_2) - h_2^{(f,q)}(x_1',x_2')| &= |q(x_1,x_2)f(x_2) - q(x_1',x_2')f(x_2')| \\ &\le |q(x_1,x_2)| \cdot |f(x_2) - f(x_2')| \\ &\quad + |q(x_1,x_2) - q(x_1',x_2')| \cdot |f(x_2')| \\ &\le \gamma \cdot \|x_2 - x_2'\|_\infty + \|[x_1,x_2] - [x_1',x_2']\|_\infty \cdot 1 \\ &\le 2\gamma^{2-2} \cdot \|[x_1,x_2] - [x_1',x_2']\|_\infty. \end{aligned}$$

Hence, (18.33) is valid for $m = 2$.

Suppose now that (18.33) holds for an integer $m \geq 2$. Using

$$\begin{aligned} a &:= h_{m+1}^{(f,q)}(x_1,\dots,x_{m+1}) - h_{m+1}^{(f,q)}(x_1',\dots,x_{m+1}') \\ &= q(x_1,x_2)\cdot\big(h_m^{(f,q)}(x_2,\dots,x_{m+1}) - h_m^{(f,q)}(x_2',\dots,x_{m+1}')\big) \\ &\quad + \big(q(x_1,x_2) - q(x_1',x_2')\big)\cdot h_m^{(f,q)}(x_2',\dots,x_{m+1}') \end{aligned}$$

we conclude that

$$\begin{aligned} |a| &\leq \gamma\cdot m\gamma^{m-2}\|[x_2,\dots,x_{m+1}] - [x_2',\dots,x_{m+1}']\|_\infty \\ &\quad + \|(x_1,x_2) - (x_1',x_2')\|_\infty\cdot\gamma^{m-1} \\ &\leq (m+1)\gamma^{(m+1)-2}\|[x_1,\dots,x_{m+1}] - [x_1',\dots,x_{m+1}']\|_\infty. \end{aligned}$$

Consequently, (18.33) holds also for $m+1$. Since $g_{m-1}^{(f,q)} = h_m^{(f,q)}(s,.)$, the first inequality follows from (18.33). The lemma is proved. □

We now estimate the terms on the right-hand side of (18.32).

Lemma 18.23.

(i) *There is a positive constant $c_1' = c_1'(\gamma)$ such that*

$$|S_d(f,q) - \widetilde{A}_s(f,q)| \leq c_1' s^{-1} \quad \text{for all } (f,q)\in F_d. \tag{18.34}$$

(ii) *For $(f,q)\in F_d$ and $m = 1,2,\dots,M$ we have*

$$\Big(\mathbb{E}\big(Q_s^{md}(g_m^{(f,q)}) - \widetilde{A}_{n_m}(\Gamma_{s,m}(f,q))\big)^2\Big)^{1/2} \leq \sqrt{2}\,n_m^{-1/2}\gamma^m.$$

Proof. Let $(f,q)\in F_d$. For $m\in\mathbb{N}$, using the bound for the Lipschitz constant of $g_m^{(f,q)}$ from Lemma 18.22 and the linearity of the functionals I_{md} and Q_s^{md} we conclude from (18.30) that

$$|I_{md}(g_m^{(f,q)}) - Q_s^{md}(g_m^{(f,q)})| \leq (m+1)\gamma^{m-1}\cdot\frac{md}{2md+2}s^{-1} \leq (m+1)\gamma^{m-1}s^{-1}.$$

Furthermore, we have $|I_{md}(g_m^{(f,q)})| \leq \gamma^m$ by the second inequality of Lemma 18.22. Consequently, we obtain

$$\begin{aligned} |S_d(f,q) - \widetilde{A}_s(f,q)| &\leq \sum_{m=1}^{M} |I_{md}(g_m^{(f,q)}) - Q_s^{md}(g_m^{(f,q)})| + \sum_{m=M+1}^{\infty} |I_{md}(g_m^{(f,q)})| \\ &\leq \sum_{m=1}^{\infty}(m+1)\gamma^{m-1}s^{-1} + \sum_{m=M+1}^{\infty}\gamma^m \\ &= \gamma^{-1}\Big(\frac{1}{(1-\gamma)^2} - 1\Big)s^{-1} + \frac{\gamma}{1-\gamma}\gamma^M. \end{aligned}$$

Since $\gamma^M \le s^{-1}$ by the definition of M, the point (i) follows. The second inequality of Lemma 18.22 and (18.31) yield the point (ii). □

Substituting the results of Lemma 18.23 into (18.32) we obtain

$$e\big(S_d, A_n, (f,q)\big) \le c_1' s^{-1} + \sqrt{2} \sum_{m=1}^{M} n_m^{-1/2} \cdot \gamma^m.$$

The definitions of the numbers n_m and s imply

$$\begin{aligned} e\big(S_d, A_n, (f,q)\big) &\le c_1' s^{-1} + \sqrt{2} \sum_{m=1}^{M} \gamma^{-m/2} n^{-1/2} \cdot \gamma^m \\ &\le c_1' s^{-1} + \sqrt{2}\, n^{-1/2} \sum_{m=0}^{\infty} \gamma^{m/2} \\ &\le c_2' n^{-1/2}. \end{aligned}$$

This establishes the error estimate of A_n.

We now prove the claim about the cost of A_n which is the sum of the information cost, the randomness cost and the arithmetic cost. We first consider the information cost. Since a function value of $g_m^{(f,q)}$ is made up of m function values of q and of one value of f, we have

$$\mathrm{cost}_{\mathrm{info}}(A_n, (f,q)) \le 1 + \sum_{m=1}^{M} (m+1) n_m.$$

From the definition of n_m, we obtain

$$\begin{aligned} \mathrm{cost}_{\mathrm{info}}(A_n, (f,q)) &\le 1 + \sum_{m=1}^{M} (m+1)(\gamma^m \cdot n + 1) \\ &\le 1 + n \sum_{m=1}^{\infty} (m+1)\gamma^m + \sum_{m=1}^{M} (m+1) \\ &= 1 + n \cdot \Big(\frac{1}{(1-\gamma)^2} - 1 \Big) + \frac{M(M+1)}{2} + M. \end{aligned}$$

Since $M^2 \le c_3' \ln^2 n \le c_3' n$, it follows that

$$\mathrm{cost}_{\mathrm{info}}(A_n, (f,q)) \le c_4' n.$$

Let us turn to the randomness cost. Due to the construction of A_n, we compute from a realization of x^ω and y^ω, the respective components of $\Gamma_{s,m}(g_m^{(f,q)})$ for $m = 1, 2, \ldots, M$. As already mentioned, $c \ln N$ random bits suffice to realize x^ω and y^ω on the average. Since $N = s^{Md}$ we get

$$\mathrm{cost}_{\mathrm{coin}}(A_n, (f,q)) \le c\, M\, d \ln s.$$

Since $M \leq c_5' \ln n$ and $\ln s \leq 2 \ln n$ we obtain

$$\mathrm{cost}_{\mathrm{coin}}(A_n, (f, q)) \leq c_6' \, d \, \ln^2 n.$$

We finally consider the arithmetic cost. As also already mentioned, $c \ln N$ arithmetic operations suffice for the realization of x^ω and y^ω on the average. For $m = 1, 2, \ldots, M$, the computation of the index i_k^ω and the corresponding components of $\Gamma_{s,m}(g_m^{(f,q)})$ as well as of their sum can be accomplished in $(3 + m + 1)n_m$ arithmetic operations. We also have to compute the sum of the $\widetilde{A}_{n_m,\omega}(\Gamma_{s,m}(g_m^{(f,q)}))$ and of $f(s)$. Hence, we obtain

$$\mathrm{cost}_{\mathrm{ari}}(A_n, (f, q)) \leq c \, M \, d \, \ln s \; + \sum_{m=1}^{M} (3 + m + 1)n_m + M.$$

The right hand side can be bounded similarly as above and we obtain

$$\mathrm{cost}_{\mathrm{ari}}(A_n, (f, q)) \leq c_7' \, n + c_6' \, d \, \ln^2 n.$$

Summing up these three cost estimates, we get

$$\mathrm{cost}(A_n, (f, q)) \leq c_8' \, n + c_9' \, d \, \ln^2 n.$$

This completes the proof of Theorem 18.21. □

18.3 Computation of Fixed Points

In this section we briefly discuss the approximate computation of a fixed point of a function f. We assume that $\Omega_d \subseteq \mathbb{R}^d$ is a convex and compact set and $f : \Omega_d \to \Omega_d$ is continuous. Then, by the fixed point theorem of Brouwer, f has at least one fixed point, i.e., the set

$$\alpha(f) = \{x \in \Omega_d \mid f(x) = x\} \subseteq \Omega_d$$

is non-empty. We use algorithms A_n based on n function values of f to approximate an arbitrary fixed point of f, and assume that $A_n(f) \in \Omega_d$.

We discuss two different error criteria. For the *root error criterion*, we measure the error of A_n by

$$e(A_n, f) = \mathrm{dist}(A_n(f), \alpha(f)) = \inf_{x \in \alpha(f)} \mathrm{dist}(A_n(f), x).$$

For dist we use the Euclidean metric in the space $\mathbb{R}^d$, i.e.,

$$\mathrm{dist}_2(x, y) = \Big(\sum_{j=1}^{d} |x_j - y_j|^2 \Big)^{1/2},$$

or the sup metric in the space $\mathbb{R}^d$, i.e.,

$$\mathrm{dist}_\infty(x, y) = \max_{i=1,2,\dots,d} |x_i - y_i|.$$

For the *residual error criterion*, we measure the error of A_n by

$$e(A_n, f) = \|f(A_n(f)) - A_n(f)\|$$

with the norm $\|\cdot\|$ that is either the norm in ℓ_2 or in ℓ_∞.

In general, these two error criteria can be quite different. It is possible that $A_n(f)$ is quite close to a fixed point but still $\|f(A_n(f)) - A_n(f)\|$ is large. On the other hand, it is also possible that the residual error is small and the root error is large. In the following we assume that f is Lipschitz continuous and then the residual error cannot be much larger than the root error.

For $L > 0$, we define the class

$$F_L(\Omega_d) = \{f : \Omega_d \to \Omega_d \mid \|f(x) - f(y)\| \le L \cdot \|x - y\|\}.$$

It is obvious that the complexity results depend on L.

We start with the case $L < 1$ and the root criteria. Then $F_L(\Omega_d)$ is a class of *contractions* and the fixed point $\alpha = \alpha(f) \in \Omega_d$ is uniquely defined. It is natural to study *simple iteration* and to apply the fixed point theorem of Banach. We first choose any $x_1 \in \Omega_d$, and let

$$A_n(f) = x_{n+1} = f(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Then

$$\|A_n(f) - \alpha(f)\| \le L^n \cdot \|x_1 - \alpha\|.$$

If Ω_d is the unit ball in $\mathbb{R}^d$ and x_1 the origin, then we obtain the error bound

$$e(A_n, F_L(\Omega_d)) \le L^n$$

which does not depend on d. To guarantee that the error is at most ε, we perform

$$n^{\mathrm{simp}}(\varepsilon, F_L(\Omega_d)) = \left\lceil \frac{\ln(1/\varepsilon)}{\ln(1/L)} \right\rceil \tag{18.35}$$

steps of the algorithm, which is the same as the number of function values used by the algorithm. In particular, the problem is *strongly polynomially tractable* if the diameter of Ω_d and the Lipschitz constant $L < 1$ do not depend on the dimension d. In fact, we also have strong T-tractability if

$$\limsup_{\varepsilon \to 0} \frac{\ln \ln \varepsilon^{-1}}{\ln T(\varepsilon^{-1}, 1)} < \infty.$$

This holds, in particular, for $T(x, y) = 1 + \ln x$.

All sounds very good but there can be, however, a problem. As long as L is not too close to 1, the number $n^{\rm simp}(\varepsilon, F_L(\Omega_d))$ is not very large even for very small ε. However, the number $n^{\rm simp}(\varepsilon, F_L(\Omega_d))$ blows up as L goes to 1. For $L = 1 - \delta$ and small positive δ we have

$$n^{\rm simp}(\varepsilon, F_L(\Omega_d)) = \frac{\ln \varepsilon^{-1}}{\delta} (1 + o(1)).$$

For many applications L is indeed very close to 1 and sometimes $L = L_d$ tends to 1 as d goes to infinity. Then d reappears and tractability issues must be again studied. Depending on how fast L_d goes to 1 we may still have polynomial or weak tractability. On the other hand, if L_d goes sufficiently fast to 1 we may have the curse of dimensionality and intractability. This leads us to the next open problem.

Open Problem 85.

Consider the fixed point problem for the class $F_{L_d}(\Omega_d)$ for the root criterion in the worst case setting with $L_d < 1$.

- Take Ω_d as the unit ball of $\mathbb{R}^d$ and find necessary and sufficient conditions on the speed of convergence of L_d to 1 to get various notions of tractability.
- Characterize all Ω_d and L_d for which we have, for instance, polynomial and strong polynomial tractability.

Equation (18.35) is sharp for the specific algorithm of simple iteration, but it is not clear whether this algorithm is optimal for the class $F_L(\Omega_d)$. Actually this depends on d and L. If d is relatively small and L is close to one, then an algorithm which is based on the Nemirovsky–Yudin–Shor construction of minimum volume circumscribed ellipsoids is much better than the simple iteration, see Huang, Khachiyan and Sikorski [137] and Sikorski [263] for details and further improvements. For this algorithm the number of function values is of order

$$n^{\rm NYS}(\varepsilon, F_L(\Omega)) = \mathcal{O}\left(d^2\left(\ln\frac{1}{\varepsilon} + \ln\frac{1}{1-L} + \ln d\right)\right).$$

Note that also $n^{\rm NYS}(\varepsilon, F_L(\Omega))$ blows up as L goes to 1 but much slower than before. Again for $L = 1 - \delta$ with a small positive δ we have

$$n^{\rm NYS}(\varepsilon, F_L(\Omega)) = \mathcal{O}\left(d^2\left(\ln\frac{1}{\varepsilon} + \ln\delta^{-1} + \ln d\right)\right).$$

On the other hand, based on the results of Nemirovsky [197] it was proved in [265] that simple iteration is almost optimal if

$$d \ge \frac{\ln 1/\varepsilon}{\ln 1/L}.$$

So far we reported results for the root criterion but it is clear that almost the same positive results hold also for the residual error as long as $L < 1$.

Now we discuss the case $L \geq 1$, again for the root criterion, and present a result of Tsay and Sikorski [314]. Consider the class

$$F_2 = \{f : [0,1]^2 \to [0,1]^2 \mid \|f(x) - f(y)\|_2 \leq \|x - y\|_2\}.$$

Then, for any information operator $N: F_2 \to \mathbb{R}^n$, $N(f) = [f(x_1), f(x_2), \dots, f(x_n)]$, there exist $g, h \in F_2$ such that $N(g) = N(h)$ and the distance of their fixed points is $\|\alpha(g) - \alpha(h)\|_2 = 1$. Consequently, no algorithm using finitely many function values can solve the problem with an error less than $\frac{1}{2}$. Hence the complexity is infinite and the fixed point problem is not only intractable but it is also unsolvable for $d = 2$ and $L = 1$ and $\varepsilon < \frac{1}{2}$. It is clear that the same is true for all $d \geq 2$ and $L \geq 1$.

We obtain a quite different result for $L \geq 1$ if we consider the residual error. It is easy to prove that the problem is at least *solvable* and that the number of function values needed for the class

$$F_d = \{f : [0,1]^d \to [0,1]^d \mid \|f(x) - f(y)\|_2 \leq L \cdot \|x - y\|_2\}$$

is at most of the order $(L/\varepsilon)^d$. It is enough to compute all function values on a grid and choose x_j that minimizes $\|f(x_i) - x_i\|_2$ on that grid. Of course this gives only an upper bound and it is not clear whether the problem is tractable or not. Actually it was proved in Huang, Khachiyan and Sikorski [137] that using the interior ellipsoid algorithm one can prove polynomial *tractability* for $L = 1$ and the upper bound

$$n(\varepsilon, F_d) = \mathcal{O}\left(d \ln(1/\varepsilon)\right).$$

We also have T-tractability for $T(x, y) = (1 + \ln x)\, y$ with the exponent 1.

The case $L > 1$ was studied by Hirsch, Papadimitriou and Vavasis [134] and more recently by Chen and Deng [31] for the class

$$\widetilde{F}_d = \{f : [0,1]^d \to [0,1]^d \mid \|f(x) - f(y)\|_\infty \leq L \cdot \|x - y\|_\infty\}.$$

For this class and $d \geq 2$, one needs at least $\left(\frac{CL}{\varepsilon d^2}\right)^{d-1}$ function values, where $C > 0$ does not depend on d, ε and $L > 1$. For a fixed d, this bound shows that polynomial tractability does not hold.

It is interesting to compare the last two results. For the 2-norm we have polynomial tractability whereas for the ∞-norm we have polynomial intractability. Hence, the choice of the norm is very essential and drastically changes the results.

18.4 Global Optimization

In this section we discuss two problems related to global optimization. We shall see that both problems, though they are nonlinear, are closely related to the linear L_∞-approximation problem.

More precisely, let Ω_d be an arbitrary nonempty set and let F_d be a convex and balanced set of bounded functions on Ω_d.

We consider the worst case error over the class F_d and discuss two different solution operators, S_d and $\widetilde{S}_d$, related to global optimization. The first one is

$$S_d(f) = \sup_{x \in \Omega_d} f(x),$$

i.e., we simply want to compute the supremum of a function.

In many applications we do not approximate the supremum but instead we want to compute a point $x \in \Omega_d$ such that $f(x)$ is close to $\sup f$. Hence we consider algorithms A_n with values in Ω_d that use at most n function value of f, and define the worst case error of A_n by

$$e(A_n) = \sup_{f \in F_d} \Big(\sup_{x \in \Omega_d} f(x) - f(A_n(f)) \Big).$$

We denote this problem by $\widetilde{S}_d$.

The linear L_∞-approximation is given by

$$\mathrm{App}_d \colon F_d \to L_\infty(\Omega_d),$$

where App_d denotes the identity (embedding) of F_d into $L_\infty(\Omega_d)$.

The following result is mainly from Wasilkowski [324], see also [205] for a minor modification.

Theorem 18.24. *Let F_d be a convex and symmetric subset of the space $C(\Omega_d)$ of continuous functions. Then*

$$\tfrac{1}{2} e^{\mathrm{wor}}(n, F_d, \mathrm{App}_d) \le e^{\mathrm{wor}}(n, F_d, S_d) \le e^{\mathrm{wor}}(n, F_d, \mathrm{App}_d)$$

and

$$e^{\mathrm{wor}}(n+1, F_d, \mathrm{App}_d) \le e^{\mathrm{wor}}(n, F_d, \widetilde{S}_d) \le 2 \cdot e^{\mathrm{wor}}(n, F_d, \mathrm{App}_d).$$

Due to this theorem, we can apply all positive and negative results on tractability for L_∞-approximation to obtain the same results for global optimization. The problem of L_∞-approximation will be studied in Volume III.

It is important to note that we deal here only with the information complexity. Due to general results, see Chapter 4, we often know for linear problems that the total complexity is basically the same as the information complexity. Such a result usually does not hold for nonlinear problems and it is possible that the total complexity is much higher than the information complexity.

We stress that there are other nonlinear problems that can be reduced to linear problems, see again Wasilkowski [324] and [205]. In this way we may obtain tractability results for these nonlinear problems by applying tractability results for linear problem.

We end this short section with two more open problems.

Open Problem 86.

- So far we only discussed the worst case setting. Not much is known for other settings. We believe that similar results hold for the randomized setting and pose this as an open problem.

We illustrate the open problem presented above by an example from [205, p. 57]. Consider the class of Lipschitz functions on $[0,1]^d$, i.e.,

$$|f(x) - f(y)| \le \max_{i=1,2,\dots,d} |x_i - y_i|.$$

For this class it is known that randomized algorithms A_n based on n function values can be better, as compared to optimal deterministic algorithms, at most by a factor of $2^{-1/2-1/d}$. This means that we get the same intractability results for approximation and global optimization in both the worst case and randomized settings.

The complexity of global optimization in the average case setting, for univariate functions, was thoroughly studied, see, e.g., Calvin [22].

Open Problem 87.

- We are not aware of any work on tractability for global optimization in the average case setting. We believe that this is a very interesting and difficult open problem.

18.5 Computation of the Volume

The approximation of volumes is an important computational problem and there are several different approaches in the literature. We mention only very few results and give a few pointers to the literature, see also Kannan, Lovász and Simonovits [146], Lovász [177], Lovász and Vempala [179], as well as [340].

Since we started this volume with discrepancy, we begin here with a related problem for convex bodies. For $d \in \mathbb{N}$, let B_d be the Euclidean ball in $\mathbb{R}^d$ with center 0 and radius 1. Let λ_d be the normalized Lebesgue measure in $\mathbb{R}^d$ such that $\lambda_d(B_d) = 1$.

Define F_d as the class of *convex* sets K from the unit ball B_d. The solution operator $S_d : F_d \to [0,1]$ is defined as

$$S_d(K) = \lambda_d(K) \quad \text{for all } K \in F_d.$$

That is, we want to approximate the volume of a convex set K. Note that S_d is a nonlinear functional. We call $S = \{S_d\}$ the *convex volume problem.*

As for discrepancy, we want to approximate $S_d(K)$ by taking n sample points $x_1, x_2, \dots, x_n$ from B_d and verify how many of them are from K. That is, the algorithm A_n is of the form

$$A_n(K) = \frac{1}{n} \sum_{i=1}^{n} 1_K(x_i). \tag{18.36}$$

Here, we assume that we can compute a *membership oracle*. That is, we can compute the characteristic function $1_K(x)$ for any K from F_d and any x from B_d. This tells us whether x belongs to K or it does not. This assumption is equivalent to computing function values of characteristic functions, and in this sense is equivalent to the use of standard information.

We first consider the worst case error. It is easy to see that the initial error is $\frac{1}{2}$ for every d, since we know a priori that $S(K) \in [0,1]$ and the best initial approximation is to take $\frac{1}{2}$. This means that the absolute and normalized error criteria are almost the same.

It is known that the worst case error of algorithms A_n of the form (18.36) is $\Omega(n^{-2/(d+1)})$, and it is known that this lower bound is optimal up to some power of $\ln n$, see Matoušek [184, p. 244]. We stress that the factor in the big Ω notation may depend on d. This problem is related to the numerical integration of convex functions $f : [0,1]^d \to \mathbb{R}$, where the optimal order of deterministic algorithms is $n^{-2/d}$, see [148].

This result means that we cannot achieve polynomial tractability if we use algorithms A_n of the form (18.36). One might hope that we can get much smaller error bounds by using more refined algorithms. Unfortunately, this is not the case. Indeed, the convex volume problem is *intractable* and suffers from the curse of dimensionality. This follows from the following result. There exists a positive c, independent of d and n, such that the (normalized) volume of the convex hull C_n of any points $x_1, x_2, \dots, x_n \in B_d$ is less than $n \cdot d^{-cd}$, see Lovász [177]. It is easy to conclude from this bound that the problem suffers the curse of dimensionality.

Indeed, if we choose arbitrary points x_j and obtain $1_K(x_j) = 1$ then K can be the convex hull C_n or the whole set B_d. Since an algorithm cannot distinguish between the volume of C_n and the volume of B_d, its worst case error is at least

$$\tfrac{1}{2}\big(1 - \mathrm{vol}(C_n)\big) \ge \tfrac{1}{2}\big(1 - n\, d^{-cd}\big).$$

Hence, if $n(\varepsilon, d)$ denotes the minimal number of membership oracle calls in the worst case setting then

$$n\big(\tfrac{1}{4}, d\big) \ge \tfrac{1}{2}\, d^{cd},$$

which shows the curse of dimensionality, as claimed.

We add in passing a related result from [340]. Assume that we want to approximate the volume of the set $g([0,1]^d) \subseteq \mathbb{R}^d$, where g is a smooth function from the class $C^s([0,1]^d)$, and we use standard information of g and the worst case setting. For $d \ge 2$ and $s \ge 2$, the optimal rate of convergence is $n^{-s/(d-1)}$. As we know, this contradicts polynomial tractability but weak tractability for this problem is open. For $s = 1$, we know bounds on optimal order of convergence. The lower bound is of order $n^{-1/(d-1)}$ and the upper bound is of order $n^{-2/((d-1)d)}$. Again this contradicts polynomial tractability. However, for $s = 1$, we also know that the problem is intractable and suffers from the curse of dimensionality. This follows from the fact that the problem is not easier than the $(d-1)$ dimensional integration problem for the class of Lipschitz functions, and the latter problem suffers from the curse of dimensionality by Sukharev's [291] result.

We now turn to the randomized setting and show that the curse of dimensionality of the convex volume problem is vanquished in this setting. If we take the standard Monte Carlo algorithm A_n^{MC} for which $x_1, x_2, \dots, x_n$ are independent random points distributed according to the uniform distribution on B_d, then we obtain the error bound

$$e(A_n^{\mathrm{MC}}) \le \frac{1}{2\sqrt{n}}.$$

Hence, there is no dependence on the dimension. This implies that the convex volume problem is strongly polynomially tractable with an exponent at most 2 in the randomized setting.

Due to this positive result one is inclined to ask for more: Is it possible to compute the volume of a convex body in polynomial (in d and ε^{-1}) time for the *relative* error?

To get the positive answer we must shrink a little the class F_d. Namely, for $r \in (0,1)$ we define $F_{d,r}$ as the class of convex bodies K from B_d for which $B_{d,r} \subseteq K$, where $B_{d,r}$ is the ball in $\mathbb{R}^d$ with center 0 and radius r. This means that we now assume that convex sets K cannot be too small and they always contain the ball $B_{d,r}$.

The relative error means that we approximate $S(K)$ by a randomized algorithm $A_{n,\omega}(K)$ such that

$$\frac{|S(K) - A_{n,\omega}(K)|}{S(K)}$$

should be at most ε on the average with respect to ω for all K from the class $F_{d,r}$.

The first polynomial time randomized algorithm for this problem was given by Dyer, Frieze and Kannan [63]. This result was improved several times and the main result of the paper by Lovász and Vempala [179] is the following.

The volume of a convex body K from $F_{d,r}$ can be approximated to within a relative error ε with probability $1 - \delta$ using

$$\mathcal{O}\Big(\frac{d^{\,4}}{\varepsilon^2} \ln^9 \frac{d}{\varepsilon\delta} + d^{\,4} \ln^8 \frac{d}{\delta} \ln r^{-1}\Big)$$

membership oracle calls. The algorithm uses Markov chain Monte Carlo and hence is related to results that were described in Section 18.1.

Integrating over δ, we find out that the minimal number $n(\varepsilon, d)$ of membership oracle calls in the randomized setting satisfies

$$n(\varepsilon, d) = \mathcal{O}\Big(\frac{d^{\,4}}{\varepsilon^2} \ln^9 \frac{d}{\varepsilon} + d^{\,4} \ln^8 d \ln r^{-1}\Big).$$

This means that the convex volume problem is polynomially tractable with an ε^{-1}-exponent at most 2 and a d-exponent at most 4 in the randomized setting.

18.6 Notes and Remarks

NR 18.1:1. Most of this section is based on Rudolf [257] and on [182]. Lemma 18.1 is taken from Master's Thesis of Rudolf [256]. Theorem 18.2 is a modified result from [182]; Theorems 18.20 and 18.15 are from Rudolf [257] and build on work of Lovász and Simonovits [178].

NR 18.2:1. Section 18.2.2 is based on [211]. We add in passing that the global solution of Fredholm equations (compute u, not only $u(s)$) is studied in Emelyanov, Ilin [67] and Heinrich [99].

NR 18.2.2:1. Monte Carlo methods that use only few random bits were also studied by Gao, Ye and Wang [72] and Pfeiffer [242], see [114] for a survey.

NR 18.3:1. In this section we only mention few results on the computation of fixed points taken from the book Sikorski [263] and from the recent survey paper Sikorski [264]. Both sources contain many more results and many references.

NR 18.4:1. Section 18.4 is mainly based on results of Wasilkowski [324], see also [205]. We mention some survey papers and books on optimization with an emphasis towards global optimization and towards complexity issues. They are Boender and Romeijn [16], Horst, Pardalos [136], Nemirovsky [198], Nemirovsky, Yudin [199], Nesterov, Nemirovskii [200], Pardalos, Romeijn [237], Vavasis [317], Zabinsky [361].

Chapter 19

Further Topics

So far we studied tractability of multivariate problems for which the space dimension d was finite but could be arbitrarily large. Moreover, all the considerations were based on a *classical computer*, i.e., all computations were performed classically using the *real number model*.

In this chapter we briefly discuss two-fold generalizations. In Sections 19.1, 19.2 and 19.3 we survey some results for *infinite-dimensional problems* such as *path integration* or *infinite-dimensional integration*, whereas in Section 19.4 we survey some results concerning computation performed on a *quantum computer*. There are three open problems numbered from 88 to 90.

19.1 Path Integration

Assume that $S\colon F \to G$, where F is a space of functions or functionals depending on infinitely many variables. That is, F is a space of real functions $f : X \to \mathbb{R}$ with $\dim(X) = \infty$. Let $n(\varepsilon, S)$ denote the information complexity (in various settings) which is the minimal number of function values needed to compute an ε approximation for the absolute or normalized error criterion. Since the error criterion can now change the error only by a factor, the tractability results are really the same for both the error criteria.

Similarly as before, we say S is *weakly tractable* if

$$\lim_{\varepsilon \to 0} \varepsilon \ln n(\varepsilon, S) = 0, \tag{19.1}$$

and S is called *polynomially tractable* if there are non-negative numbers C and p such that

$$n(\varepsilon, S) \le C\, \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1). \tag{19.2}$$

The exponent of polynomial tractability is defined as the infimum of p satisfying the bound above.

Many applications require approximate values of path integrals. The complexity of path integration was first studied in [330], and we survey some of the results of this and other papers.

Let μ be a zero mean Gaussian measure on a separable Banach space X of infinite dimension. An important example is the Wiener measure w on $C([0,1])$. We assume that the support of μ is infinite dimensional. We wish to approximate

$$S(f) = \int_X f(x)\, \mu(\mathrm{d}x)$$

for $f: X \to \mathbb{R}$ which is assumed to be integrable. As before, we assume that $f \in F$, where F is a given class representing our a priori information about functions. We consider deterministic algorithms as well as randomized algorithms. The first result of [330] is for the class

$$F^r = \{f: X \to \mathbb{R} \mid f^{(r)} \text{ cont. and } \|f^{(k)}(x)\| \le 1 \text{ for all } x \in X, k = 0, 1, \dots, r\}.$$

Here, $f^{(k)}$ denotes the kth Frechet derivative of f. We consider the worst case setting for deterministic algorithms based on function values. We now denote the minimal number $n(\varepsilon, S)$ of function values needed to approximate $S(f)$ to within ε by $n^{\text{wor}}(\varepsilon, S, F^r)$.

Theorem 19.1. *The path integration problem is polynomially intractable for F^r, i.e., the bound*

$$n^{\text{wor}}(\varepsilon, S, F^r) \le C\, \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1)$$

does not hold no matter how we choose C and p.

This result is, at least intuitively, easy to guess since we know that for finitely many variables d, the information complexity is of order $\varepsilon^{-d/r}$, and since for path integration d can be arbitrarily large there is no way to have polynomial tractability. It is not, however, clear whether the path integration problem is weakly tractable. This is our next open problem.

Open Problem 88.

- Find conditions on the zero-mean Gaussian measure μ under which the path integration problem is weakly tractable in the worst case setting for the class F^r.

More is known for $r = 1$ when we have Lipschitz functionals, i.e., when we consider the class

$$F_{\text{Lip}} = \{ f: X \to \mathbb{R} \mid |f(x) - f(y)| \le \|x - y\| \text{ for all } x \in X \}.$$

The *small ball function* is defined as

$$\varphi(\varepsilon) = -\log \mu(\{x \in X \mid \|x\| \le \varepsilon\}) \quad \text{for all } \varepsilon \in (0,1). \tag{19.3}$$

Suppose we have

$$\varphi(\varepsilon) = \Theta\left(\varepsilon^{-\alpha} \left(\log \varepsilon^{-1}\right)^{\beta}\right)$$

for some numbers $\alpha > 0$ and real β, as ε tends to zero. For the classical Wiener measure this holds for $\alpha = 2$ and $\beta = 0$, see the survey by Li and Shao [172].

The following result can be found in Creutzig, Dereich, Müller-Gronbach and Ritter [37], where the integration of Lipschitz functionals on a Banach space is studied in detail.

Theorem 19.2. *The information complexity of the path integration problem for F_{Lip} in the worst case setting is given by*

$$n^{\mathrm{wor}}(\varepsilon, S, F_{\mathrm{Lip}}) = \exp\big(\Theta\big(\varepsilon^{-\alpha}\,(\ln \varepsilon^{-1})^{\beta}\big)\big).$$

This result easily leads the following corollary.

Corollary 19.3. *Path integration is intractable in the worst case setting for Lipschitz functionals if $\alpha > 1$ or $\alpha = 1$ and $\beta > 0$. In particular, path integration is intractable for the Wiener measure. The problem is weakly tractable if $\alpha < 1$ or $\alpha = 1$ and $\beta < 0$.*

In any case, the path integration problem is not polynomially tractable for F^r. As always, there are different possibilities to obtain positive results:

- we may switch to the randomized setting and ask whether path integration is tractable in this setting,
- we may ask whether path integration is tractable even in the worst case setting for smaller classes of integrands.

Remark 19.4. So far we assumed the standard cost function, where each evaluation of the integrand f has a fixed cost. Although we mainly present results for this cost function, we should admit that this cost function is often too optimistic for integrands f that are defined on an infinite dimensional space X.

It is possible to propose a more realistic cost function for such integrands. Most algorithms for path integration are of the form which is called *variable subspace sampling* in Creutzig et al. [37]. Namely, we have an increasing sequence of finite dimensional subspaces $X_i \subseteq X$, and compute $f(x)$ for $x \in \bigcup_{i=1}^{\infty} X_i$. Then it seems reasonable to define the cost of evaluation $f(x)$ by $\inf\{\dim(X_i) \mid x \in X_i\}$, since x is given by this number of coordinates that have to be transferred to the oracle. The sequence of subspaces may be chosen arbitrarily, but it is fixed for a specific algorithm. More will be said on a variable cost in Section 19.2. □

We now discuss randomized algorithms for the classes F^r of integrands with finite smoothness. In the finite dimensional case of d, we know that the minimal randomized error with n function values is of order $n^{-r/d-1/2}$. This means that the order $n^{-1/2}$ of Monte Carlo can be improved only slightly if d is large. Therefore the following result of [330] is not surprising.

Theorem 19.5. *The path integration problem is polynomially tractable for F^r in the randomized setting since*

$$n^{\mathrm{ran}}(\varepsilon, S, F^r) = \mathcal{O}\big(\varepsilon^{-2}\big).$$

We also have

$$n^{\mathrm{ran}}(\varepsilon, S, F^r) = \Omega\big(\varepsilon^{-2-\delta}\big) \quad \textit{for all } \delta > 0.$$

More is known in the Lipschitz case where $r = 1$, see again Creutzig et al. [37].

Theorem 19.6. *The information complexity of the path integration problem for F_{Lip} in the randomized setting satisfies*

$$\liminf_{\varepsilon\to\infty} \frac{\ln n^{\text{ran}}(\varepsilon, S, F_{\text{Lip}})}{\ln \varepsilon^{-2}} = 1.$$

Variable subspace sampling with the cost function explained in Remark 19.4 is as powerful as full space sampling with the standard cost function iff α from (19.3) satisfies $0 < \alpha \le 2$.

Remark 19.7. It is very interesting that variable subspace sampling is often as powerful as full space sampling. To prove this result, Creutzig et al. [37] use *multilevel Monte Carlo* introduced by Heinrich [99], see also Heinrich [101] as well as Heinrich and Sindambiwe [116]. The authors of the last three papers studied global solutions of integral equations and parametric integration. A multilevel algorithm uses samples from finite-dimensional subspaces $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m$ with only a small proportion of samples taken from high-dimensional spaces. □

We now return to path integration in the worst case setting for deterministic algorithms. First we present a result of [330] where tractability is proved for a class of entire functions under the assumption that we can compute the derivatives of integrands at zero.

Let λ_k be the eigenvalues of the covariance operator C_μ of the zero-mean Gaussian measure μ, and let η_k from X be the normalized eigenvectors of C_μ, i.e., $C_\mu \eta_k = \lambda_k \eta_k$. Clearly $\sum_{k=1}^{\infty} \lambda_k < \infty$.

In what follows, the space H_∞ of entire functions will depend on a sequence of positive numbers β_k such that

$$\sup_k \lambda_k \beta_k < 1, \quad \sup_k \beta_k < \infty, \quad \text{and} \quad \lambda_k \beta_k \ge \lambda_{k+1}\beta_{k+1} \quad \text{for all } k.$$

Consider the sum-exponent $p_{\lambda\beta}$ of the sequence $\lambda\beta = \{\lambda_k \beta_k\}$ defined by

$$p_{\lambda\beta} = \inf\left\{p \mid \textstyle\sum_{k=1}^{\infty} (\lambda_k \beta_k)^p < \infty\right\}.$$

We clearly have $p_{\lambda\beta} \le 1$.

Let $\mathbb{N}_0^\infty$ denote the set of multi-indices $i = [i_1, i_2, \dots]$ with non-negative integers i_k such that $|i| = \sum_{k=1}^{\infty} i_k$ is finite.

The space H_∞ is a space of entire functions $f : X \to \mathbb{R}$ with inner product

$$\langle f, g\rangle = \sum_{i\in\mathbb{N}_0^\infty} \frac{\left[f^{(i)}(0)\prod_{k=1}^{\infty}\eta_k^{i_k}\right]\cdot\left[g^{(i)}(0)\prod_{k=1}^{\infty}\eta_k^{i_k}\right]}{\prod_{k=1}^{\infty} i_k!\,\beta_k^{i_k}}.$$

The following results are from [330].

Theorem 19.8. *Consider the path integration problem S on the unit ball F of H_∞. The information is restricted to function and derivative values at zero.*

Then S is polynomially tractable and its exponent $p(F)$ is bounded by

$$p(F) \le \frac{2p_{\lambda\beta}}{2-p_{\lambda\beta}}.$$

For $\lambda_k\beta_k = \Theta(k^{-r})$ with some $r > 1$, we have

$$p(F) = \frac{2p_{\lambda\beta}}{2-p_{\lambda\beta}} = \frac{2}{2r-1}.$$

Open Problem 89.

- For the class F of entire functions, algorithms and tractability are studied in [330] under the assumption that we can evaluate arbitrary derivatives of $f \in F$ at zero. Do we obtain similar results under the assumption that only function values of $f \in F$ can be computed?

Remark 19.9. Steinbauer [289] studies path integration for $X = C([0,1])$ with the Wiener measure w, i.e.,

$$S(f) = \int_X f(x)\, w(\mathrm{d}x).$$

He defines a class F of very smooth functions on X, for which the directional derivatives (in the directions of normalized Schauder functions that arise in the Lévy–Ciesielski decomposition) of even order are bounded by one. For $f \in F$, he defines approximations of $S(f)$ by

$$A_n(f) = \sum_{i=1}^{n} a_i f(x_i),$$

which are obtained by applying Smolyak's construction to a sequence of one-dimensional Gauss–Hermite quadrature formulas. He then shows that

$$|S(f) - A_n(f)| \le 2^{-1/4} n^{-1/4}.$$

Hence this path integration problem is polynomially tractable with an exponent at most 4.

Another class of functions with infinitely many variables is studied by Hickernell and Wang in [126]. □

Remark 19.10. Path integration is also important for non-Gaussian measures, and occurs in many applications, e.g., in financial mathematics. The distribution is often the solution of a stochastic differential equation. Complexity results can be found in Creutzig, Dereich, Müller-Gronbach and Ritter [37], Kloeden and Platen [149], as well as in Müller-Gronbach and Ritter [195]. □

Path integrals with respect to the Wiener measure give the solution to heat equations via the famous Feynman–Kac formula. That is why the approximate computation of

Feynman–Kac path integrals is of much interest and has been extensively studied. The aim is to approximate Wiener integrals of the form

$$S(v, V) = \int_{C([0,1])} v(x(t))\, H\left(\int_0^t V(x(s))\,\mathrm{d}s\right) w(\mathrm{d}x) \tag{19.4}$$

for some functions (v, V) from a class F, a fixed positive time t, and a fixed function H. Note that the formula (19.4) contains a Wiener integral but the operator S now only depends on v and V which are functions of a single variable. Clearly, S depends linearly on v and usually heavily non-linearly on V.

This problem is studied in [247] where the authors assume that H is a fixed entire function and that v and V are from a space F of functions defined over $\mathbb{R}$. The authors present a new deterministic algorithm and an explicit bound on its cost. For many important function classes F, the new algorithm is almost optimal. The algorithm requires precomputation of some real coefficients that are combinations of multivariate integrals with special weights. This precomputation is very difficult and so far limits the application of the new algorithm.

We illustrate some results of [247] by an example. In a typical case we have $H(x) = \exp(x)$ for $x \in \mathbb{R}$, and $v, V \in F := C^4([0,1])$. Known algorithms are usually randomized (Monte Carlo) algorithms. Suppose we want to compute $S(v, V)$ to within an error ε. Then the cost of the known Monte Carlo algorithms is of order $\varepsilon^{-2.5}$. In this situation, the new (almost optimal) deterministic algorithm has a cost of order roughly $\varepsilon^{-0.25}$, which is a dramatic improvement.

The work [247] was continued in the papers Kwas [161], Kwas and Li [162], as well as in Petras and Ritter [241]. The last paper studies the intrinsic difficulty of solving linear parabolic initial-value problems numerically at a single point.

All the papers mentioned before present algorithms relying on heavy precomputation. Hence one can say that these algorithms are only semi-constructive. This leads us to the next open problem.

Open Problem 90.

- Solve Feynman–Kac integrals by designing efficient algorithms without the need of precomputation.

19.2 Weighted Sobolev Space with $d = \infty$

We briefly report on integration for the weighted Sobolev space of functions depending on infinitely many variables recently studied in [156].

The weighted Sobolev space in question is a reproducing kernel Hilbert space defined in terms of its kernel. Let $\mathfrak{u}$ be a finite subset of $\mathbb{N}$ and let $\gamma = \{\gamma_{\mathfrak{u}}\}_{\mathfrak{u}\subseteq\mathbb{N},\,|\mathfrak{u}|<\infty}$ be a given sequence of non-negative weights such that

$$\gamma_{\emptyset} = 1 \quad \text{and} \quad \sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} < \infty.$$

Consider the kernel

$$K_\gamma(x,y) = 1 + \sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} \prod_{j\in\mathfrak{u}} \min(x_j, y_j) \quad \text{for all } x, y \in [0,1]^\infty.$$

Note that K_γ is well defined due to the assumption $\sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} < \infty$, and

$$K_\gamma(x,y) \in \Big[0, \sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}}\Big].$$

The Sobolev space $H(K_\gamma)$ has the inner product

$$\langle f, g\rangle_{H(K_\gamma)} = f(0)g(0) + \sum_{\mathfrak{u}:\,1\le|\mathfrak{u}|<\infty} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} f(x_{\mathfrak{u}};0) \frac{\partial^{|\mathfrak{u}|}}{\partial x_{\mathfrak{u}}} g(x_{\mathfrak{u}},0)\, \mathrm{d}x_u,$$

where $x_{\mathfrak{u}} = (x_j)_{j\in\mathfrak{u}}$ is a vector with $|\mathfrak{u}|$ components and $(x_{\mathfrak{u}};0)$ denotes the infinitely dimensional vector with components x_j for $j \in \mathfrak{u}$ and 0 otherwise. As always, for $\gamma_{\mathfrak{u}} = 0$ we assume that $\partial^{|\mathfrak{u}|}/\partial x_{\mathfrak{u}} f(x_{\mathfrak{u}};0) \equiv 0$ for all $x \in [0,1]^\infty$ and all $f \in H(K_\gamma)$, and interpret $0/0$ as 0.

Observe that for $\gamma_{\mathfrak{u}} = 0$ for all $\mathfrak{u} \not\subseteq \{1,2,\dots,d\}$, the space $H(K_\gamma)$ reduces to the standard Sobolev space for functions defined over $[0,1]^d$. For general weights with $\sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty|} \gamma_{\mathfrak{u}} < \infty$, the functions from $H(K_\gamma)$ may depend on infinitely many variables but with a decreasing importance of successive variables. For instance, take a linear function $f(x) = \sum_{j=1}^\infty \alpha_j x_j$. Then

$$f \in H(K_\gamma) \quad \text{iff} \quad \sum_{j=1}^\infty \frac{\alpha_j^2}{\gamma_{\{j\}}} < \infty.$$

Since $\sum_{j=1}^\infty \gamma_{\{j\}} \le \sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} < \infty$ then $\gamma_{\{j\}}$ must go to zero and this implies that α_j goes to zero as well.

For *product* weights, $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$ for some non-negative γ_j, we have

$$\sum_{\mathfrak{u}:\,|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}} = \prod_{j=1}^\infty (1+\gamma_j),$$

and the last product is finite iff $\sum_{j=1}^\infty \gamma_j < \infty$. In this case,

$$K_\gamma(x,y) = \prod_{j=1}^\infty \big(1 + \gamma_j \min(x_j, y_j)\big)$$

and $H(K_\gamma)$ is the infinite tensor product of the univariate reproducing kernel Sobolev spaces with the kernels $1 + \gamma_j \min(x_j, y_j)$.

For *finite-order* weights of order ω, i.e., $\gamma_{\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$, the space $H(K_\gamma)$ consists of functions that are infinite sums of functions depending on at most ω variables. We stress that in this case the number of finite-order weights is, in general, infinite for all $\omega \ge 1$.

It is easy to see that the integration problem

$$I_\infty(f) = \int_{[0,1]^\infty} f(x)\,\mathrm{d}x \quad \text{for all } f \in H(K_\gamma)$$

is well defined, and we have

$$I_\infty(f) = \langle f, h_\infty \rangle_{H(K_\gamma)} \quad \text{for all } f \in H(K_\gamma)$$

for

$$h_\infty(x) = 1 + \sum_{\mathfrak{u};\, 1 \le |\mathfrak{u}| < \infty} \gamma_{\mathfrak{u}} \prod_{j \in \mathfrak{u}} (x_j - \tfrac{1}{2} x_j^2),$$

and

$$\|I_\infty\| = \|h_\infty\|_{H(K_\gamma)} = \Big(1 + \sum_{\mathfrak{u}:\, 1 \le |\mathfrak{u}| < \infty} \gamma_{\mathfrak{u}} 3^{-|\mathfrak{u}|}\Big)^{1/2}.$$

We are ready to discuss algorithms for approximating I_∞. As in [156], we assume that we can compute $f(x)$ only for x with finitely many non-zero components, i.e., only for $x = (x_{\mathfrak{u}}; 0)$ for some finite $\mathfrak{u}$. Furthermore we assume that the cost of computing $f(x)$ depends on $|\mathfrak{u}|$ and is equal to $\$(|\mathfrak{u}|)$ for a given cost function $\$ \colon \{0, 1, \dots\} \to [1, \infty)$ that is non-decreasing. Typical examples of \$ are $\$(k) = [\max(1, k)]^s$ for some $s \ge 0$ and $\$(k) = r^k$ for some $r > 1$.

The idea to relate the cost of function evaluation to the number of active variables has been proposed in Creutzig, Dereich, Müller-Gronbach and Ritter [37], see also Hickernell, Müller-Gronbach, Niu and Ritter [120], and is called *variable subspace sampling*. We think that this is a very reasonable assumption, in particular for functions that depend on infinitely many variables, and we hope that this assumption leads to many new results in the near future.

As always for linear problems, we can restrict ourselves to linear algorithms that are now of the form

$$A_n(f) = \sum_{j=1}^{n} a_j\, f\big((x_j)_{\mathfrak{u}_j}; 0\big)$$

for some real numbers a_j and sample points $x_j \in [0, 1]^\infty$ as well as finite sets $\mathfrak{u}_j$. The cost of the algorithm A_n is now

$$\mathrm{cost}(A_n) = \sum_{j=1}^{n} \$(|\mathfrak{u}_j|).$$

The worst case error of A_n is defined in the usual way,

$$e(A_n, H(K_\gamma)) = \sup_{f:\, \|f\|_{H(K_\gamma)} \le 1} |I_\infty(f) - A_n(f)|,$$

and the ε-complexity $n(\varepsilon, H(K_\gamma))$ is now understood as the minimal cost among all algorithms with the worst case error at most ε,

$$n(\varepsilon, H(K_\gamma)) = \inf\{\, \mathrm{cost}(A_n) \mid e(A_n, H(K_\gamma)) \le \varepsilon\,\}.$$

Tractability of I_∞ is defined in terms of the behavior of $n(\varepsilon, H(K_\gamma))$. In particular, we have weak or polynomial tractability if (19.1) or (19.2) hold. However, the situation now is much more interesting than before for the finite dimension d since we must decide not only how to choose the sample points x_j but also how to choose the sets $\mathfrak{u}_j$ in order to minimize the cost under the condition that the worst case error is at most ε. The reader is referred to [156] for detailed analysis. Here we only want to mention two results from [156].

- Consider product weights with $\gamma_j = j^{-1}\,[\ln(1+j)]^{-\alpha}$ with $\alpha > 1$, and $\$(k) = [\max(1,k)]^s$ with $s > 0$. Then
  - I_∞ is intractable iff $\alpha \in (1, 3]$,
  - I_∞ is weakly tractable iff $\alpha > 3$.
  - I_∞ is not polynomially tractable.
- Consider finite-order weights of order ω, i.e., $\gamma_{\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$. Let $\{\mathfrak{u}_{\mathfrak{u}}\}_{\mathfrak{u}:\,|\mathfrak{u}|\le\omega} = \{\gamma_{\mathfrak{u}_j}\}$ such that $\gamma_{\mathfrak{u}_j} \ge \gamma_{\mathfrak{u}_{j+1}}$ for all j, and assume that

  $$\gamma_{\mathfrak{u}_j} = j^{-\beta} \quad \text{for some } \beta > 1.$$

  Then
  - Independently of the cost function \$, I_∞ is polynomially tractable with the exponent
    $$p \in \left[1, \max\left(1, \frac{2}{\beta-1}\right)\right].$$
    Hence, $p = 1$ for $\beta \ge 3$,
  - For $\$(k) = \Omega(k^s)$ with $s > 0$, I_∞ is polynomially tractable with the exponent
    $$p \in \left[\max\left(1, \frac{2\,\min(1, s/\omega)}{\beta-1}\right), \max\left(1, \frac{2}{\beta-1}\right)\right].$$
    Hence $p = \max(1, 2/(\beta-1))$ for $s \ge \omega$.

As we see, sometimes tractability results depend on the cost function and sometimes not. We also stress that for some cases we obtain the best results by using a variable subspace sampling and sometimes it is enough to sample from a fixed subspace. For finite-order weights, the decomposition formulas from [155] for multivariate functions are very useful. Again much more can be found in [156], also for other spaces.

19.3 The Result of Sobol

In 1974, Sobol [285] published a paper on integration for functions of infinitely many variables. He assumed that F is a set of functions $f : [0, 1]^{\mathbb{N}} \to \mathbb{R}$ such that

$$I_\infty(f) = \int_{[0,1]^{\mathbb{N}}} f(x)\,\mathrm{d}x$$

is well defined and the functions

$$f_{i,j}(x) = (x_i - x_j)^2 \quad \text{for all } x \in [0, 1]^{\mathbb{N}}$$

belong to F for all distinct i and j from $\mathbb{N}$.

He then proved that the worst case error of any QMC algorithm that uses n function values must be at least $\frac{1}{6}$ independently of how large n we take. Before we present an extension of his proof, we note that the result of Sobol means that the integration problem for the set F is unsolvable for QMC algorithms since

$$n^{\mathrm{QMC}}(\varepsilon) = \infty \quad \text{for all } \varepsilon \in \big(0, \tfrac{1}{6}\big),$$

where $n^{\mathrm{QMC}}(\varepsilon)$ denotes the minimal number of function values used for QMC algorithms with the worst case error at most ε for the absolute error criterion.

This result holds for the smallest set F given by

$$F^{\,\mathrm{small}} = \{f_{i,j} \mid i, j \in \mathbb{N},\ i \neq j\}.$$

However, note that

$$I_\infty(f) = \tfrac{1}{6} \quad \text{for all } f \in F^{\,\mathrm{small}}.$$

This means that the constant algorithm $A(f) \equiv \frac{1}{6}$ has zero error and the information complexity

$$n(\varepsilon, F^{\,\mathrm{small}}) = 0 \quad \text{for all } \varepsilon \in [0, 1].$$

Again, this shows that QMC algorithms are sometimes quite bad even for trivial integration problems.

Obviously, the result of Sobol applies for *all* sets F that contain $F^{\,\mathrm{small}}$. So let us now take $F^{\,\mathrm{conv}}$ as the absolute convex hull of $F^{\,\mathrm{small}}$, i.e.,

$$F^{\,\mathrm{conv}} = \big\{f \mid f = \textstyle\sum_{i<j} t_{i,j} f_{i,j},\ \sum_{i<j} |t_{i,j}| \le 1\big\}.$$

Note that

$$I_\infty(f) = \sum_{i<j} \tfrac{1}{6} t_{i,j}$$

is well defined and $I_\infty(f) \in [-\frac{1}{6}, \frac{1}{6}]$ for all $f \in F^{\,\mathrm{conv}}$.

By the result of Sobol, we know that all QMC algorithms are bad since their worst case error is at least $\frac{1}{6}$. But maybe other algorithms are much better as we saw for the set F^{small}? It turns out that this is *not* the case, and

$$n(\varepsilon, F^{\,\mathrm{conv}}) = \infty \quad \text{for all } \varepsilon \in (0, \mathrm{CRI}),$$

where CRI $= \frac{1}{6}$ for the absolute error criterion, and CRI $= 1$ for the normalized error criterion.

We now prove this result. First of all, we may restrict ourselves to nonadaptive information and linear algorithms since F^{conv} is balanced and convex, see Chapter 4 of Volume I. Hence, consider a linear algorithm that uses n function values

$$Q_n(f) = \sum_{k=1}^{n} a_k f(x_k)$$

for some real or complex a_k and sample points $x_k \in [0,1]^{\mathbb{N}}$. We now show that

$$e(Q_n) = \sup_{f \in F^{\text{conv}}} |I_\infty(f) - Q_n(f)| \geq \tfrac{1}{6};$$

modifying slightly Sobol's proof for $a_j = 1/n$.

For $m \in \mathbb{N}$, define the disjoint intervals $I_{m,j} = [(j-1)/m, j/m)$ for $j = 1, 2, \ldots, m-1$, and $I_{m,m}[(m-1)/m, 1]$.

Let $x_k = [x_{k,1}, x_{k,2}, \ldots]$. Then for all $x_{k,j}$ there is a unique $s_m(k,j)$ such that $x_{k,j} \in I_{m,s_m(k,j)}$. By

$$s_j = [s_m(1,j), s_m(2,j), \ldots, s_m(n,j)] \in \{1, 2, \ldots, m\}^n,$$

we denote the position of the jth components of the n sample points x_k. Since we have at most m^n such vectors, then if we take $d = m^n + 1$ then we find two identical vectors $s_i = s_j$ for distinct $i, j \leq d$. For these two indices i and j we have

$$|x_{k,i} - x_{k,j}| \leq m^{-1} \quad \text{for all } k = 1, 2, \ldots, n.$$

Then $f_{i,j} \in F^{\text{conv}}$, $I_\infty(f_{i,j}) = \frac{1}{6}$, and

$$|Q_n(f_{i,j})| = \left| \sum_{k=1}^{n} a_k (x_{k,i} - x_{k,j})^2 \right| \leq \frac{1}{m^2} \sum_{k=1}^{n} |a_k|.$$

Therefore

$$\big|I_\infty(f) - Q_n(f)\big| \geq \frac{1}{6} - \frac{1}{m^2} \sum_{k=1}^{n} |a_k|,$$

and this goes to $\frac{1}{6}$ as m approaches infinity. Since the initial error is $\frac{1}{6}$, this completes the proof.

19.4 Quantum Computation

So far, all the considerations were based on a *classical computer*, i.e., all computations were performed classically using the *real number model*. In particular, all the tractability results that we presented so far were based on this model of computation.

In this short section, we change the model of computation to *quantum computation*. We briefly present a mathematical model of quantum computation, and survey a few results based on this model. We illustrate quantum computation by the search algorithm of Grover that is a base of many quantum algorithms for continuous problems, in which we are primarily interested in. We conclude this section by discussing tractability for the quantum model of computation.

We should stress from the very beginning that we do not know whether quantum computers can be built in the near future. It is even not completely clear how we should formalize the model of computation and the cost function on a quantum computer. Since we are interested in continuous problems, the model proposed by Heinrich [100] plays a major role and we present several results under this model of quantum computation. We briefly report on other models of quantum computation in the Notes and Remarks of this section.

Quantum computers use the effects of quantum mechanics caused by *entangled states* that allow to break the *Bell inequalities*. All quantum algorithms are *randomized algorithms*, i.e., the output is a random variable. So it is natural to compare the quantum results with the results in the randomized setting. We are especially interested to know which problems can be solved significantly faster when a quantum computer is used instead of the classical one.

19.4.1 Model of Quantum Computation

We start with a 2-dimensional Hilbert space H_1 over the complex numbers $\mathbb{C}$. Let e_0 and e_1 be two orthonormal vectors from H_1. The space H_1 describes *quantum states* with one qubit (quantum bit) as unit vectors x,

$$x = \beta_0 e_0 + \beta_1 e_1 \quad \text{with } |\beta_0|^2 + |\beta_1|^2 = 1.$$

To describe quantum states with m qubits, we use the 2^m-dimensional tensor product

$$H_m = H_1 \otimes \cdots \otimes H_1$$

with m factors. An orthonormal basis in H_m is given by the 2^m vectors

$$b_\ell = e_{i_1} \otimes \cdots \otimes e_{i_m} \quad \text{for all } \ell = 0, 1, \ldots, 2^m - 1, \tag{19.5}$$

where $i_j \in \{0, 1\}$ is the $(m - j)$th binary bit of ℓ, i.e.,

$$\ell = \sum_{j=1}^{m} i_j \, 2^{m-j}.$$

The formally different objects $(i_1, i_2, \ldots, i_m)$ and ℓ or e_ℓ are often identified and are called *classical states*. They correspond to the 2^m possible states of m classical bits.

Every vector $x \in H_m$ has a unique representation

$$x = \sum_{\ell=0}^{2^m-1} \beta_\ell \, b_\ell \tag{19.6}$$

with $\beta_0, \beta_1, \ldots, \beta_{2^m-1} \in \mathbb{C}$, and

$$\|x\|^2 = \sum_{\ell=0}^{2^m-1} |\beta_\ell|^2.$$

Unit vectors x, so that $\|x\| = 1$, are called *quantum states* with m qubits. Every such state defines a probability measure on the set of classical states: the probability of e_ℓ is $|\beta_\ell|^2$.

A *quantum algorithm* is given by a finite sequence of certain unitary mappings

$$U_i \colon H_m \to H_m \quad \text{for } i = 1, 2, \ldots, r.$$

The computation starts with a classical state $b_k \in H_m$ as the input, and then the mappings U_i are applied. The mathematical result is the quantum state

$$x = U_r U_{r-1} \cdots U_1(b_k). \tag{19.7}$$

Hence, for quantum computation we can only multiply by unitary matrices (mappings). For m qubits these unitary matrices are $2^m \times 2^m$, so that even for moderate m they are huge. We only allow the following three kinds of unitary mappings:

- U only changes one quantum bit, i.e.,

  $$U(e_{i_1} \otimes \cdots \otimes e_{i_j} \otimes \cdots \otimes e_{i_m}) = e_{i_1} \otimes \cdots \otimes \widetilde{U}(e_{i_j}) \otimes \cdots \otimes e_{i_m}$$

  with an unitary mapping $\widetilde{U} \colon H_1 \to H_1$ and $j \in \{1, 2, \ldots, m\}$.

- U is a *controlled-not mapping*, i.e.,

  $$U(e_{i_1} \otimes \cdots \otimes e_{i_j} \otimes \cdots \otimes e_{i_k} \otimes \cdots \otimes e_{i_m}) = e_{i_1} \otimes \cdots \otimes e_{i_j} \otimes \cdots \otimes e_{i_k \oplus i_j} \otimes \cdots \otimes e_{i_m}$$

  with $j \neq k$. The symbol $\oplus$ denotes the addition modulo 2. If the jth quantum bit is in the state e_0, the kth state stays unchanged. Otherwise the kth quantum bit is changed from e_0 to e_1, or vice versa.

- U is a *quantum* query.

  The quantum query is a unitary mapping through which we transfer the input data of the problem that we want to solve. We will see examples of quantum queries later.

In the quantum model of computation, we assume that the cost of multiplying by each unitary matrix is taken as unity. Hence, the cost of a quantum algorithm of getting $x = U_r \cdots U_1\, b_k$ is r. Of course, one could compute the vector x by a classical computer. However, the cost would be then as large as $2r\, 4^m$. This means that as long as the number of qubits m is small then we can simulate quantum computation on a classical computer. Clearly, even for modest m there is no way that we can perform 4^m operations on a classical computer. This opens up a possibility that a quantum computer with sufficiently many qubits may significantly outperform a classical computer.

We stress that the output of a quantum algorithm is *not* x, but a classical state given by a physical measurement. This result is random and we obtain b_ℓ or $\ell \in \{0, 1, \ldots, 2^m - 1\}$ with the probability $|\beta_\ell|^2$, where $\beta_\ell = \langle x, b_\ell \rangle$ is the respective coefficient of the vector x in the orthonormal basis $\{b_\ell\}$.

Obviously, we can also perform operations on a classical computer. So knowing ℓ, we can compute $\varphi(\ell)$ classically for some mapping φ that maps $\{0, 1, \ldots, 2^m - 1\}$ to the target space of the solution elements. We can also repeat quantum and classical computations as many times as we wish.

From this short description, it should be clear that the model of quantum computation is at least as powerful as the model of classical computation. Furthermore, if we work with m qubits, then we can have at most 2^m different outputs of quantum computation, and therefore we can approximate the solution by at most 2^m elements $\varphi(\ell)$.

We illustrate quantum computation by a very simple quantum algorithm to realize the instruction "choose a random bit". Just take $m = 1$ and $W_1 \colon H_1 \to H_1$ defined by

$$W_1(e_i) = \frac{1}{\sqrt{2}}(e_0 + (-1)^i\, e_1), \quad \text{for } i = 0, 1,$$

the so-called *Walsh–Hadamard-transform*. In particular we obtain

$$W_1(e_0) = \frac{1}{\sqrt{2}} \cdot (e_0 + e_1),$$

hence the algorithm outputs 0 or 1 with probability 1/2 on input 0.

For $m \geq 1$ qubits, we define W_m as the m-fold tensor product of the Walsh–Hadamard-transform W_1, i.e.,

$$W_m(e_{i_1} \otimes \cdots \otimes e_{i_m}) = W_1(e_{i_1}) \otimes \cdots \otimes W_1(e_{i_m}).$$

Then

$$W_m(b_0) = \frac{1}{2^{m/2}} \sum_{\ell=0}^{2^m-1} b_\ell,$$

and we obtain each ℓ with the same probability 2^{-m}. Hence this algorithm produces 2^m random elements and can be seen as an ideal random number generator.

19.4.2 Grover's Search Algorithm

Consider the following search problem. Let

$$f : \{0, 1, \ldots, N-1\} \to \{0, 1\}$$

be a function such that $f(\ell) = 1$ for exactly one ℓ. Our problem is to search for this ℓ. Any deterministic or randomized algorithm that solves this problem with high probability, say 3/4, needs to compute a number of function values f proportional to N. The quantum algorithm of Grover, however, has a cost that is proportional to $N^{1/2} \cdot \ln N$. Hence we have (almost) a quadratic speed-up.

For simplicity we assume that $N = 2^m$. The algorithm of Grover works with m qubits and uses a *quantum query* $Q_f : H_m \to H_m$ for the evaluation of the function f. The unitary mapping Q_f is defined by

$$Q_f(b_\ell) = (-1)^{f(\ell)} \cdot b_\ell \quad \text{for } \ell = 0, 1, \ldots, 2^m.$$

In addition, we need the unitary operator Q_0 with

$$Q_0(b_0) = -b_0 \quad \text{and} \quad Q_0(b_\ell) = b_\ell \text{ for } \ell = 1, 2, \ldots, 2^m - 1.$$

Then the algorithm of Grover is given by

$$x = \left(-W_m Q_0 W_m Q_f\right)^k \left(W_m(b_0)\right). \tag{19.8}$$

It was proved by Boyer, Brassard, Høyer and Tapp [17] that the algorithm solves the search problem with probability $1 - 2^{-m}$ if we take

$$k = \lfloor \pi / 4\vartheta \rfloor \quad \text{with } \sin\vartheta = 2^{-m/2},$$

see also Nielsen and Chuang [203], Section 6.1. Clearly,

$$\vartheta \approx 2^{-m/2} \quad \text{and} \quad k \approx \frac{\pi}{4} N^{1/2} \text{ for large } N.$$

The cost of each iteration in (19.8) is roughly $\ln N$, hence the complete cost of Grover's algorithm is of the order $N^{1/2} \cdot \ln N$.

To prove this result, we need to show that x given by (19.8) satisfies

$$|\langle x, b_\ell \rangle|^2 \geq 1 - 2^{-m}.$$

That is, the classical initial state b_0 is transformed "almost" to the vector b_ℓ. The idea of the proof is the following. Let

$$z = \frac{1}{\sqrt{N-1}} \sum_{k \neq \ell} b_k$$

be the uniform superposition of all non-solutions. Then b_ℓ and z are orthogonal and already the first step of the algorithm maps b_0 into a linear combination of b_ℓ and z, namely we have

$$W_m(b_0) = s = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} b_k = \frac{1}{\sqrt{N}} b_\ell + \frac{\sqrt{N-1}}{\sqrt{N}} z,$$

or

$$W_m(b_0) = s = b_\ell \cdot \sin \alpha + z \cdot \cos \alpha,$$

where

$$\sin \alpha = \frac{1}{\sqrt{N}}$$

is the angle between s and z.

We always stay in this 2-dimensional space $\mathrm{span}(b_\ell, z)$. An iteration of the mappings $-W_m Q_0 W_m Q_f$ is a rotation with angle 2α of z in the direction of b_ℓ. After k iterations we obtain the vector

$$x = b_\ell \cdot \sin((2k+1)\alpha) + z \cdot \cos((2k+1)\alpha).$$

With the k from above, we see that we almost obtain $x_k = b_\ell$.

It is known that Grover's algorithm is almost optimal. To solve the search problem with probability at least 1/2, one needs a number of oracle calls that is proportional to $N^{1/2}$, see Nielsen and Chuang [203], Section 6.6.

19.4.3 Computation of Sums and Integrals

Quantum algorithms were designed for different problems of discrete mathematics as well as for certain continuous problems. An important problem, still discrete, is the summation problem. It is interesting that algorithms for this very simple problem can be used to obtain good quantum algorithms for many continuous problems such as integration, see also [114].

The problem is to approximate

$$S(f) = \frac{1}{N} \sum_{i=0}^{N-1} f(i)$$

for a Boolean function $f: \{0, 1, \dots, N-1\} \to \{0, 1\}$.

In the worst case setting, it is easy to prove that the nth minimal error is

$$\frac{(N-n)_+}{2N}.$$

In the randomized setting, the nth minimal error is bounded by

$$\frac{1}{2\sqrt{n}},$$

and this bound is basically sharp.

These well known error bounds can be significantly improved in the quantum setting. With n queries, one can achieve an error at most

$$\frac{3\pi}{4n} \text{ with probability at least } \frac{8}{\pi^2},$$

see Brassard, Høyer, Mosca and Tapp [18] and [112], [163]. This bound is also sharp as proved by Nayak and Wu [196].

The bounds for the summation problem can be used to obtain results for multivariate integration. Consider

$$S(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x$$

for integrands from the unit ball of $C^k([0,1]^d)$. Then the optimal rates of convergence are

$$n^{-(k/d+s)},$$

where

$$s = \begin{cases} 0 & \text{for the worst case setting,} \\ \frac{1}{2} & \text{for the randomized setting,} \\ 1 & \text{for the quantum setting.} \end{cases}$$

The result for the quantum setting was proved in [209], and greatly generalized by Heinrich [100], [102] for many other classes of functions. Path integration was studied in the quantum setting in [308].

19.4.4 Solution of PDEs

Heinrich [108] also studied boundary value problems for elliptic PDEs in the quantum setting, and compared the quantum results with his results from Heinrich [107] for the randomized setting. For comparison, we also mention some known results for the worst case setting, see Heinrich [107].

Assume we are given a second order elliptic problem $Lu = f$ in a smooth d-dimensional bounded domain Ω with homogeneous boundary conditions and right hand side $f \in C^r(\Omega)$. The problem is to compute the solution u on a d_1-dimensional sub-manifold using function values of f. With deterministic algorithms the optimal rate of convergence is

$$n^{-r/d}$$

and does not depend on d_1. Heinrich proved that the optimal rate is

$$n^{-\min\{(r+2)/d_1,\, r/d+1/2\}}$$

in the randomized setting, and

$$n^{-\min\{(r+2)/d_1,\, r/d+1\}}$$

in the quantum setting. The error is measured in the sup norm and some of the bounds are up to log terms.

19.4.5 Tractability in the Quantum Setting

There are more results on continuous problems in the quantum setting, and we refer to the surveys of Heinrich [106] and of Papageorgiou and Traub [233].

So far we know several continuous problems for which algorithms in the quantum setting have a faster rate of convergence than all classical deterministic and randomized algorithms. This is important since for problems with a relatively small number of variables the optimal rates of convergence are enough. In these cases, we really know the speedups between the quantum, randomized and worst case settings.

Tractability has *not* yet been studied in the quantum setting. The formal definition of tractability in this setting is obvious. If we fix the probability of success for quantum algorithms, we still have two tractability parameters ε^{-1} and d, and we can study different kinds of tractability as in other settings. If we prefer to have δ as a new parameter measuring the failure of quantum algorithms, and consider the case when δ goes to zero, then we have a similar situation as in the probabilistic setting. Then we can study different kinds of tractability in the quantum setting with respect to the three parameters ε^{-1}, d and δ^{-1}.

Nevertheless, even today it is possible to translate some existing results into tractability results. For instance, consider the multivariate integration problem described in Section 19.4.3 for $k = 1$. Then we know that

- in the worst case setting, this problem is intractable and suffers from the curse of dimensionality,

- in the randomized setting, this problem is strongly polynomially tractable with exponent 2.

  The upper bound is achieved even for the standard Monte Carlo algorithm, since the variances of functions in this case are uniformly bounded. The exponent of ε^{-1} cannot be smaller than two since asymptotically in n we know that the nth minimal error behaves like $n^{1/2+1/d}$, and therefore the information complexity behaves like $\varepsilon^{-2/(1+2/d)}$, and the exponent of ε^{-1} cannot be smaller than 2 for all $d \in \mathbb{N}$.

- in the quantum setting, this problem is strongly polynomially tractable with exponent 1.

  The upper bound follows from the quantum summation algorithms. The exponent ε^{-1} cannot be smaller than one since otherwise it would contradict the optimality of the quantum summation algorithms due to Nayak and Wu [196].

For this example, we see that the problem is strongly polynomially tractable in both the randomized and quantum setting, and that the exponent in the quantum setting is smaller than in the randomized setting. So, the quantum setting reduces the exponent by one. Similar conclusions may be drawn also for several other problems studied in this volume but we stop here. Instead, we want to finish this section by expressing our hope that tractability will be also thoroughly studied in the quantum setting.

19.5 Notes and Remarks

NR 19.1:1. Our short section on path integration gives only a survey of complexity results. For motivation and algorithms, the reader is referred to the papers we cited and the references in these papers. We also recommend the book of Egorov, Sobolevsky, Yanovich [66] that contains basic information on path integration.

NR 19.2:1. This section is based on [156]. For product weights, also $\gamma_j = j^{-\beta}$ with $\beta > 1$ and $\gamma_j = q^j$ with $q \in (0,1)$ are studied. For more general integration problems studied in [156], the domain of integration can be D^∞ with bounded or unbounded $D \subseteq \mathbb{R}$, the anchor point at which the inactive variables are fixed can be arbitrary as well as we may have weighted integration. In some cases, the underlying Hilbert space of the integration problem is *not* a reproducing kernel Hilbert space. This means that function evaluations are not allowed at some sample points. However, this happens only for sample points with infinitely many active variables that are never permitted due to the cost assumption.

NR 19.3:1. This section is based on Sobol [285].

NR 19.4:1. Further results on continuous problems in the quantum setting can be found in Bessen [15], Goćwin [86], Goćwin and Szczesny [87], Heinrich [104], [105], [106], [109], Kacewicz [140], [141], [142], Nielsen and Chuang [203], Papageorgiou [230], Papageorgiou and Traub [233] as well as [112], [163], [219].

NR 19.4:2. There are several papers studying continuous problems in the quantum setting with a modified model of quantum computation. For example, the so-called *power* quantum queries were studied in [234], [235]. Power queries are much more powerful than the usual quantum queries that are often called *bit* queries. They even allow to solve NP-complete problems in polynomial time in the quantum setting, see [235]. Obviously, today it is not clear if a future quantum computer will be able to implement power queries. There is also a modification of the quantum model of computation by allowing randomized bit queries, see [354]. For some continuous problems, randomized bit queries are much more powerful than deterministic bit queries.

Chapter 20

Summary: Uniform Integration for Three Sobolev Spaces

20.1 Introduction

The purpose of this chapter is to summarize and compare the results for linear functionals presented in this volume in the four settings:

- worst case,
- average case,
- probabilistic,
- randomized,

and for the three error criteria:

- absolute,
- normalized,
- relative.

We faced a tough decision which problem should we choose for the summary and comparison. On one hand, the problem should be computational important and well represent a huge class of linear functionals studied in this volume. On the other hand, the problem should be sufficiently easy to describe, so that we could focus on the results and not on technical details. We also wanted to have a problem for which we know many tractability results as well as several open issues to be hopefully resolved by our readers in the near future.

After some thought, we decided that this problem should be multivariate integration defined over the three standard (weighted) Sobolev spaces, which are related to discrepancy and to financial applications. These are the spaces from Appendix A of Volume I, which already appeared in different parts of Volume II. All three of them were studied in Chapter 17 for the randomized setting as the first, second and third Sobolev space. We will also use this terminology in this chapter. The first and third spaces are the unanchored Sobolev spaces. The second space is the anchored Sobolev space that also appeared quite often in other chapters dealing with the worst case setting. The second and third Sobolev spaces were studied in Section 16.9 of Chapter 16, where we showed that they are related to weighted Korobov spaces. All three of them were also studied in Section 11.6 of Chapter 11, where we characterized which unweighted tensor product functionals are tractable. The reader may notice that the first Sobolev space was not yet completely studied in the worst case setting, and we will partially fill this gap in this chapter. There is one open problem 91.

20.2 Preliminaries

We briefly remind the reader that the three weighted Sobolev space are reproducing kernel Hilbert spaces of real functions defined on $[0,1]^d$ whose first mixed derivatives are square integrable. They differ by the choice of the norm. We consider arbitrary weights given be a sequence

$$\gamma = \{\gamma_{d,\mathfrak{u}}\}_{\mathfrak{u}\subseteq[d],\ d\in\mathbb{N}} \quad \text{with } \gamma_{d,\emptyset} = 1.$$

To omit the trivial case, we always assume that at least one $\gamma_{d,\mathfrak{u}}$ is positive for non-empty $\mathfrak{u}$ for every $d \in \mathbb{N}$.

We first consider the worst case setting. We study $I = \{I_d\}$ for

$$I_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x \quad \text{for all } f \in H_d,$$

where H_d is the reproducing kernel Hilbert space with the following inner product:

• for the first Sobolev space

$$\langle f, g\rangle_{H_d} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^d} \frac{\partial^{|\mathfrak{u}|} f}{\partial x_{\mathfrak{u}}}(x)\, \frac{\partial^{|\mathfrak{u}|} g}{\partial x_{\mathfrak{u}}}(x)\,\mathrm{d}x,$$

• for the second Sobolev space

$$\langle f, g\rangle_{H_d} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial x_{\mathfrak{u}}}(x_{\mathfrak{u}}, a)\, \frac{\partial^{|\mathfrak{u}|} g}{\partial x_{\mathfrak{u}}}(x_{\mathfrak{u}}, a)\,\mathrm{d}x,$$

where $a \in [0,1]^d$,

• for the third Sobolev space

$$\langle f, g\rangle_{H_d} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{d-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial x_{\mathfrak{u}}}(x)\,\mathrm{d}x_{-\mathfrak{u}} \right) \cdot \left(\int_{[0,1]^{d-|\mathfrak{u}}} \frac{\partial^{|\mathfrak{u}|} g}{\partial x_{\mathfrak{u}}}(x)\,\mathrm{d}x_{-\mathfrak{u}} \right) \mathrm{d}x_{\mathfrak{u}}.$$

For the univariate case, $d = 1$, the inner products simplify to

$$\langle f, g\rangle_{H_1} = A(f,g) + \gamma_{1,\{1\}}^{-1} \int_0^1 f'(x)\, g'(x)\,\mathrm{d}x,$$

with three different choices of $A(f)$:

$$A(f,g) = \begin{cases} \int_0^1 f(x)\, g(x)\,\mathrm{d}x & \text{for the first Sobolev space,} \\ f(a)g(a) & \text{for the second Sobolev space,} \\ \int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x & \text{for the third Sobolev space.} \end{cases}$$

Let $K_{d,\gamma}$ be the reproducing kernel of one of the weighted Sobolev spaces. The form of $K_{d,\gamma}$ is especially intriguing for the first Sobolev space. For product weights, $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$, we have

$$K_{d,\gamma}(x,y) = \prod_{j=1}^{d} K_{\gamma_{d,j}}(x_j, y_j) \quad \text{for all } x, y \in [0,1]^d,$$

where

$$K_\gamma(x,y) = \frac{\sqrt{\gamma}}{\sinh \sqrt{\gamma}} \cosh\left(\sqrt{\gamma}(1-\max(x,y))\right) \cosh\left(\sqrt{\gamma}\,\min(x,y)\right),$$

see Thomas-Agnan [304] as well as Appendix A of Volume I.

For general weights, for $k \in \mathbb{N}^d$ we define

$$e_k(x) = \prod_{j=1}^{d} \cos\left(\pi(k_j-1)x_j\right) \quad \text{for all } x \in [0,1]^d.$$

The sequence $\{e_k\}_{k\in\mathbb{N}^d}$ is an orthogonal basis of the first weighted Sobolev space H_d and

$$\|e_k\|_{H_d} = 2^{-|\{j\in[d]\,|\,k_j>1\}|/2} \left(1 + \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} \left[\pi(k_j-1)\right]^2\right)^{1/2}.$$

The reproducing kernel takes now the form

$$K_{d,\gamma}(x,y) = \sum_{k\in\mathbb{N}^d} \frac{e_k(x)}{\|e_k\|_{H_d}} \frac{e_k(y)}{\|e_k\|_{H_d}} \quad \text{for all } x, y \in [0,1]^d,$$

see [343] and Appendix A of Volume I.

For the second Sobolev space, we have

$$K_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \frac{|x_j-a_j| + |y_j-a_j| - |x_j-y_j|}{2} \quad \text{for all } x, y \in [0,1]^d,$$

whereas for the third Sobolev space, we have

$$K_{d,\gamma}(x,y) = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} \frac{(x_j-y_j)^2 - |x_j-y_j| + \frac{1}{6} + 2(x_j-\frac{1}{2})(y_j-\frac{1}{2})}{2}$$

for all $x, y \in [0,1]^d$.

As we know, the squares of the initial errors of multivariate integration are

$$\|I_d\|^2 = \begin{cases} 1 & \text{for the first Sobolev space,} \\ \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} (a_j^2 - a_j + \frac{1}{3}) & \text{for the second Sobolev space,} \\ 1 & \text{for the third Sobolev space.} \end{cases}$$

We will sometimes use the notation $H_{d,j}$ and $K_{d,\gamma,j}$ to distinguish between the three Sobolev spaces and their reproducing kernels, and the index $j \in \{1,2,3\}$ will tell us which Sobolev space we have in mind. Clearly, the norm in the first space is at least as large as in the third space,

$$\|f\|_{H_{d,3}} \le \|f\|_{H_{d,1}} \quad \text{for all } f \in H_{d,1}.$$

This means that the unit ball of $H_{d,1}$ is a subset of the unit ball of $H_{d,3}$. This implies that multivariate integration over $H_{d,1}$ is at most as hard as over $H_{d,3}$ since the initial errors are the same in both spaces. Hence, all upper bounds that we presented in Section 16.9 for the space $H_{d,3}$ are also valid for the space $H_{d,1}$. Obviously, this also indicates that we need new lower bounds for the space $H_{d,1}$, and at this point we cannot rule out that lower bounds for $H_{d,1}$ may be significantly smaller than lower bounds for $H_{d,3}$.

We now turn to the average case and probabilistic setting. Then we extend the domain of I_d to the Banach space $F_d = C([0,1]^d)$ of continuous functions with the usual supremum norm. The space F_d is equipped with a Gaussian measure μ_d with zero mean whose covariance function is $K_{\mu_d} = K_{d,\gamma}$, where $K_{d,\gamma}$ is one of the reproducing kernels of the Sobolev space. It is well known that such a Gaussian measure exists, see Vakhania, Tarieladze and Chobanyan [316], p. 215. From Chapters 13 and 14 we know that the average case and probabilistic settings are related to the worst case setting for the space $H_{\mu_d} = H(K_{d,\gamma})$, that is, to one of the Sobolev spaces. Furthermore, we know how to translate the worst case results to get the results in the average case and probabilistic settings.

We specify tractability results for the three Sobolev spaces in the successive three sections. To simplify the presentation we only consider three specific families of weights.

- **CONS:** $\gamma_{d,\mathfrak{u}} = 1$ for all $\mathfrak{u} \subseteq [d]$ and all $d \in \mathbb{N}$.

  Of course, this corresponds to the unweighted case for which we usually have intractability and the curse of dimensionality. However, the reader will see some surprises even for these weights. In some cases, we also comment on "almost" constant weights, where we assume that

  $$\gamma_{d,\emptyset} = 1 \quad \text{and} \quad \gamma_{d,\mathfrak{u}} = c \quad \text{for all non-empty } \mathfrak{u} \subseteq [d] \text{ and all } d \in \mathbb{N},$$

  for some positive c.

- **PROD:** $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ with $\gamma_{d,j} = j^{-\beta}$ for some $\beta \ge 0$.

  This corresponds to product weights $\gamma_{d,j}$ independent of d, and we choose the specific form of $\gamma_{d,j} = j^{-\beta}$ to simplify the tractability statements. Much more is known for general product weights, but again we opt for simplicity in this chapter. Obviously, we get the constant weights for $\beta = 0$.

- **FINO:** $\gamma_{d,\mathfrak{u}} = 1$ for all $|\mathfrak{u}| \le \omega$, and $\gamma_{d,\mathfrak{u}} = 0$ for $|\mathfrak{u}| > \omega$. Here $\omega \ge 1$.

  This is a special choice of finite-order weights of order ω that makes the presentation easier. We studied general finite-order weights in most parts of this volume, and the reader can find many tractability results for general finite-order weights. In particular, we do not cover finite-diameter weights in this chapter.

We also restrict ourselves to three specific kinds of tractability, leaving the case of more general tractability functions to the reader. They are:

- **SPT:** Strong Polynomial Tractability,
- **PT:** Polynomial Tractability,
- **WT:** Weak Tractability.

We will present several tables with tractability results, and SPT, PT and WT will be used as the abbreviations for these three kinds of tractability.

There will be several more natural abbreviations. The settings and error criteria will be denoted as follows.

- **WOR:** Worst case setting,
- **AVG:** Average case setting,
- **PRO:** Probabilistic setting,
- **RAN:** Randomized setting,
- **ABS:** Absolute error,
- **NOR:** Normalized error,
- **REL:** Relative error.

For the first and third Sobolev spaces, the initial errors are 1, and the absolute and normalized error criteria coincide. This is simply indicated by

$$\textbf{ABS} = \textbf{NOR}.$$

Similarly, we know that the worst case and average case settings differ insignificantly, and we have the same tractability results as long as we consider strong polynomial, polynomial or weak tractability, see Theorem 13.2. We denote this by

$$\textbf{WOR} = \textbf{AVG}.$$

In the probabilistic setting, we have an additional tractability parameter δ. We will study the polynomial and logarithmic dependence on δ^{-1}. As we know, in the probabilistic setting we can have strong polynomial tractability or strong polylog tractability with respect to d and δ. By

$$\textbf{SPT}(d)\textbf{-P}, \quad \textbf{SPT}(\delta)\textbf{-P}, \quad \textbf{PT-P}, \quad \textbf{WT-P}$$

we mean strong polynomial with respect to d or δ, polynomial and weak tractability for polynomial dependence on δ^{-1}.

We denote the probabilistic setting with polynomial dependence on δ^{-1} by **PRO-P**. This means that if the information complexity in the probabilistic setting is bounded by

$$n^{\text{prob}}(\varepsilon, \delta, I_d) = \mathcal{O}\big(\varepsilon^{-1}\, d^q\, \delta^{-s}\big) \quad \text{for all } \varepsilon, \delta \in (0,1), \text{ and } d \in \mathbb{N},$$

then we have **SPT**(d)**-P** whenever $q = 0$, **SPT**(δ)**-P** whenever $s = 0$, and otherwise we have **PT-P**. Furthermore, if

$$\lim_{\varepsilon^{-1}+\delta^{-1}+d\to\infty} \frac{\ln n^{\text{prob}}(\varepsilon, \delta, I_d)}{\varepsilon^{-1} + \delta^{-1} + d} = 0,$$

then we have **WT-P**. Similarly, by

SPT(d)**-L**, **SPT**(δ)**-L**, **PT-L**, **WT-L**

we mean strong polynomial with respect to d or δ, polynomial and weak tractability for polynomial dependence on $1 + \ln \delta^{-1}$.

We denote the probabilistic setting with logarithmic dependence on δ^{-1} by **PRO-L**. The information complexity in the probabilistic setting now satisfies the bounds as above with δ^{-1} replaced by $1 + \ln \delta^{-1}$ or by $\ln \delta^{-1}$, if the extra 1 is not needed.

Some results do not depend on whether we permit polynomial or logarithmic dependence on δ^{-1}. In this case, we write

SPT(d)**-P/L**, **SPT**(δ)**-P/L**, **PT-P/L**, **WT-P/L**.

We hope that this elaborate notation is still clear to the reader. This notation will allow us to present tractability results in a very concise way. Finally, we add that we are mostly interested in tractability and the exponents of tractability will be only briefly mentioned in comments. For all three Sobolev spaces, the exponent of ε^{-1} must be at least 1 in the worst case setting, and at least $\frac{2}{3}$ in the randomized setting, since the information complexity is proportional to ε^{-1} or to $\varepsilon^{-2/3}$, respectively, even for the univariate case.

20.3 First Sobolev Space

For the first Sobolev space, the initial error of multivariate integration in the worst case setting is one, and therefore **ABS** = **NOR**. As already explained, we also have **WOR** = **AVG**. We are ready to present tractability results for this case.

We comment on Table 20.1. We begin with constant weights for which we claim that multivariate integration is intractable, and in fact, suffers from the curse of dimensionality. The reader cannot find this result in this volume since the first Sobolev space has not yet been considered. The proof of this result can be found in [280]. The proof technique is exactly the same as the proof technique that we explained in Chapters 11 and 12, and is based on the notion of decomposable kernels. The reproducing kernel of

Table 20.1. **WOR** = **AVG** and **ABS** = **NOR**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	YES	YES

the first Sobolev space is obviously not decomposable since it is always strictly positive. However, it has a decomposable part and the corresponding parts of the univariate representer of integration are non-zero, as shown in [280]. So we can use the results for kernels with decomposable parts, and show the curse of dimensionality.

Suppose now that we have "almost" constant weights, i.e., $\gamma_{d,\mathfrak{u}} = c$ for all non-empty $\mathfrak{u} \subseteq [d]$ and all $d \in \mathbb{N}$, where c is positive. For $c > 1$, multivariate integration is not easier than before and the initial error is unchanged. So the curse of dimensionality is still present. For $c < 1$, we make multivariate integration easier. However, it is easy to check that we can gain at most a factor of $c^{1/2}$. That is, the information complexity $n_c(\varepsilon, d)$ for c, and the information complexity $n(\varepsilon, d)$ for $c = 1$ are related by

$$n(c^{-1/2}\,\varepsilon, d) \leq n_c(\varepsilon, d) \leq n(\varepsilon, d).$$

Unfortunately, this cannot eliminate the curse of dimensionality. Hence, for all "almost" constant weights, multivariate integration suffers from the curse of dimensionality.

For product weights, we have strong polynomial tractability iff $\sum_{j=1}^{\infty} \gamma_j < \infty$. The last condition holds iff $\beta > 1$. Polynomial tractability holds iff $\sum_{j=1}^{d} \gamma_j$ is bounded by a multiple of $\ln d$, which holds iff $\beta \geq 1$. All of this can be found in Theorem 1 of [280] even for periodic subspaces of the first Sobolev space. The proof technique is exactly the same as the proof technique in Chapter 12. Weak tractability was not studied in [280] since the concept of weak tractability was formalized a few yeas later after the paper [280] had been published. In any case, using exactly the same reasoning as we did in Chapter 12 with estimates from [280] we check that weak tractability holds iff $\lim_d \sum_{j=1}^{d} \gamma_j / d = 0$, see also [85]. In our case, this is equivalent to $\beta > 0$. This proves the claims for the product weights.

We turn to finite-order weights. As we know, we have polynomial tractability for bounded finite-order weights, which obviously implies weak tractability. This explains two YES's in the last line of Table 20.1.

We now discuss strong polynomial tractability for finite-order weights. This problem has *not* yet been studied. In fact, we do not have sharp lower bounds for finite-order weights for general spaces. We presented sufficient conditions for strong polynomial tractability for finite-order weights, but these conditions are not satisfied for our finite-order weights, since all of them are 1.

This means that we must prove our claim that strong polynomial tractability does not hold for finite-order weights of order $\omega \geq 1$ and for the first Sobolev space.

We will show that for any $n \in \mathbb{N}$, the nth minimal worst case error is bounded roughly by 1 for large d. More precisely, we will prove that

$$\limsup_{d\to\infty} e(n,d) = 1 \quad \text{for all } n \in \mathbb{N},$$

which obviously contradicts strong polynomial tractability.

For an integer $m > n$ and $h = 1/m$, define the "hat"function

$$f_h(x) = \frac{1}{\left(1 + \frac{1}{12}\,h^2\right)^{1/2}} \begin{cases} x & \text{if } x \in \left[0, \frac{1}{2}h\right], \\ h - x & \text{if } x \in \left[\frac{1}{2}h, h\right], \\ 0 & \text{if } x \in [h, 1]. \end{cases}$$

By $H_{1,1}$ we mean the first Sobolev space for $d = 1$ with $\gamma_{1,\{1\}} = 1$. Clearly, $f_h \in H_{1,1}$ and

$$\|f_h\|_{H_{1,1}} = \sqrt{h} \quad \text{and} \quad I_1(f_h) = \frac{h^2}{4\left(1 + \frac{1}{12}h^2\right)^{1/2}}.$$

Define the shifted hat functions,

$$f_j(x) = f_h\big(x + (j-1)h\big) \quad \text{for all } j = 1, 2, \dots, m.$$

The support of f_j is $[(j-1)h, jh]$. They are orthogonal in $H_{1,1}$ and

$$\|f_j\|_{H_{1,1}} = \sqrt{h} \quad \text{and} \quad I_1(f_j) = I_1(f_h).$$

For $k = 1, 2, \dots, d$, we take

$$g_k(x) = \sum_{j=1}^{m} c_{k,j}\, f_j(x) \quad \text{for all } x \in [0,1],$$

where $c_{k,j} \in \{0, 1\}$. We will take n of the $c_{k,j}$ zero and the rest of them 1. Then

$$\|g_k\|^2_{H_{1,1}} = \frac{1}{m} \sum_{j=1}^{m} c_{k,j} = \frac{m-n}{m} \leq 1.$$

For a given positive integer s, we take

$$d = s\,m^2.$$

Finally, define

$$f(x_1, x_2, \dots, x_d) = \frac{g_1(x_1) + g_2(x_2) + \dots + g_d(x_d)}{\sqrt{c_m\, d}} \quad \text{for all } x_j \in [0,1],$$

where c_m is positive.

Clearly, $f \in H_{d,1}$ and we choose c_m such that $\|f\|_{H_{d,1}} = 1$. We now show that

$$c_m = 1 + \tfrac{1}{16}\, s\left(1 + o(1)\right) \quad \text{as } m \to \infty.$$

Indeed, we have

$$\begin{aligned} 1 = \|f\|^2_{H_{d,1}} &= \frac{1}{c_m\, d}\Big(\sum_{i,j=1}^{d} I_d(g_i g_j) + \sum_{j=1}^{d} \|g_j'\|^2_{L_2([0,1])}\Big) \\ &= \frac{1}{c_m\, d}\bigg((d^2 - d)\big[(m-n) I_1(f_h)\big]^2 + d\Big(1 - \frac{n}{m}\Big)\bigg) \\ &= \frac{1}{c_m}\left(1 + \tfrac{1}{16}\, s\left(1 + o(1)\right)\right), \end{aligned}$$

as claimed. Furthermore

$$\begin{aligned} I_d(f) &= \frac{1}{\sqrt{c_m\, d}}\left(\sum_{k=1}^{d}\sum_{j=1}^{m} c_{k,j}\right) I_1(f_h) \\ &= \tfrac{1}{4}\left(\frac{d}{c_m\, m^2}\right)^{1/2} \left(1 + o(1)\right) \\ &= \tfrac{1}{4}\left(\frac{s}{c_m}\right)^{1/2} \left(1 + o(1)\right) = \left(\frac{s}{s+16}\right)^{1/2} \left(1 + o(1)\right) \quad \text{as } m \to \infty. \end{aligned}$$

Take now an arbitrary algorithm A_n for the approximation of I_d. We can assume that A_n is linear and samples the function at points $x_1^*, x_2^*, \dots, x_n^*$ from $[0,1]^d$. Let $x_j^* = [x_{j,1}^*, x_{j,2}^*, \dots, x_{j,d}^*]$, i.e., $x_{j,k}^*$ is the kth component of x_j^*. We want to guarantee that

$$g_k(x_{j,k}^*) = 0 \quad \text{for all } j = 1, 2, \dots, n,$$

i.e., g_k vanishes at all the kth components of the sample points $x_1^*, x_2^*, \dots, x_n^*$. We can achieve this by taking exactly n of the $c_{k,\ell}$ zero. Namely, $c_{k,\ell} = 0$ if one of the $x_{j,k}^*$ belongs to the interval $[(\ell-1)h, \ell h]$. For such choice of the coefficients $c_{k,\ell}$, we have

$$f(x_j^*) = 0 \quad \text{for all } j = 1, 2, \dots, n.$$

As we know,

$$e^{\text{wor}}(A_n) \ge I_d(f) = \left(\frac{s}{s+16}\right)^{1/2} (1 + o(1)) \quad \text{as } m \to \infty.$$

Since A_n is arbitrary, the same bound holds for $e(n,d)$ with $d = sm^2$. Hence,

$$\limsup_{d\to\infty}\; e(n,d) \ge \left(\frac{s}{s+16}\right)^2.$$

This holds for arbitrary s, and since $e(n,d) \le e(0,d) = 1$, we finally conclude

$$\limsup_{d\to\infty} e(n,d) = 1.$$

This completes the proof, and justifies NO in the last row of Table 20.1.

Finally, we add a few words about the exponents of strong polynomial and polynomial tractability for product and finite-order weights. The ε^{-1} exponent is at most 2 if $\beta \ge 1$ or if we use finite-order weights. As we already mentioned, multivariate integration over the first Sobolev space is not harder than over the third Sobolev space. So we can use all estimates for the ε^{-1} exponent as well as for the d exponent from estimates that we presented in Section 16.9. They will also be presented in the section for the third Sobolev space. Nevertheless, it would be of interest to study the first Sobolev space directly and to verify whether it is possible to get better results than those obtained for the third Sobolev space.

We now briefly mention the worst and average case settings for the relative error. As we know, the information complexity in both settings is infinite,

$$n^{\mathrm{wor/avg}}(\varepsilon, I_d) = \infty \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

This means that multivariate integration is *unsolvable*, which is much worse than intractability or the curse of dimensionality. This explains Table 20.2 with all entries NO.

Table 20.2. **WOR** = **AVG** and **REL**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	NO	NO	NO
FINO	NO	NO	NO

We turn to the probabilistic setting. We first consider the absolute error criterion. We know that the probabilistic and worst case settings are related as explained in Chapter 13, and we have

$$n^{\mathrm{prob}}(\varepsilon, \delta, I_d) = n^{\mathrm{wor}}\left(\frac{\varepsilon(1+o(1))}{\sqrt{2\ln\,\delta^{-1}}}, I_d\right).$$

We allow polynomial or logarithmic dependence on δ^{-1}. This and Table 20.1 give us the summary of tractability results as follows.

We comment on Table 20.3. The reader may be surprised that strong polynomial tractability with respect to δ does not hold. This follows from the fact that for the

Table 20.3. **PRO** and **ABS**

	SPT(d)-P/L	SPT(δ)-P/L	PT-P/L	WT-P/L
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	NO	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	NO	YES	YES

absolute error criterion, we use the worst case results with ε replaced roughly by $\varepsilon/\sqrt{2 \ln \delta^{-1}}$, and since the information complexity in the worst case setting does depend on ε^{-1}, the information complexity in the probabilistic setting must depend on $\sqrt{\ln \delta^{-1}}$ even for the univariate case.

Although there is no difference between tractability results for polynomial and logarithmic dependence on δ^{-1}, their exponents are different. For polynomial dependence on δ^{-1}, the δ^{-1} exponent is arbitrarily small, whereas for logarithmic dependence on δ^{-1}, the δ^{-1} exponent is at least $\frac{1}{2}$ and at most 1. Furthermore, for product weights it goes to $\frac{1}{2}$ for large β, as it will be explained in the section for the third Sobolev space.

We now consider the probabilistic setting for the normalized error criterion. As we know from Chapter 13, we now have

$$n^{\text{prob}}(\varepsilon, \delta, I_d) = n^{\text{wor}}(\varepsilon, I_d),$$

and the information complexity does not depend on δ. That is why we have the following summary of tractability results.

Table 20.4. **PRO** and **NOR**

	SPT(d)-P/L	SPT(δ)-P/L	PT-P/L	WT-P/L
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	YES	YES	YES

We comment on Table 20.4. We first comment on strong polynomial tractability with respect to δ. This holds if we have

$$n^{\text{prob}}(\varepsilon, \delta, I_d) = \mathcal{O}\big(\varepsilon^{-p}\, d^{q}\big).$$

This is not the case for the constant weights, even if $\gamma_{d,\mathfrak{u}} \equiv c > 0$, and this explains the first NO in the column SPT(δ)-P/L. For product weights, polynomial tractability with respect to d holds iff $\beta \ge 1$. That is why we needed to put the same condition for strong polynomial tractability with respect to δ. For finite-order weights, we always have polynomial dependence on d, and this explains the last unconditional YES.

We now turn to the probabilistic setting for the relative error. As we know, again from Chapter 13, we now have

$$n^{\text{prob-rel}}(\varepsilon, \delta, I_d) = n^{\text{wor-nor}}\left(\tfrac{1}{2}\pi\varepsilon\delta(1+o(1), I_d\right). \tag{20.1}$$

So the parameter δ is as important as the parameter ε. In this case, the information complexity in the probabilistic setting depends polynomially on ε^{-1} as well as on δ^{-1} even for the univariate case. This means that tractability results do *not* hold if we only allow logarithmic dependence on δ^{-1}. That is, we have the summary in Table 20.5 given by all entries NO.

Table 20.5. **PRO-L** and **REL**

	SPT(d)-L	SPT(δ)-L	PT-L	WT-L
CONS	NO	NO	NO	NO
PROD	NO	NO	NO	NO
FINO	NO	NO	NO	NO

We now allow polynomial dependence on δ^{-1}. From (20.1), it is clear that we will obtain positive tractability results if we have tractability in the worst case setting with ε replaced by $\varepsilon\,\delta$. This also explains why we cannot have strong polynomial tractability with respect to δ. We summarize tractability results in Table 20.6.

Table 20.6. **PRO-P** and **REL**

	SPT(d)-P	SPT(δ)-P	PT-P	WT-P
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	NO	YES iff $\beta \ge 1$	YES iff $\beta > 0$
FINO	NO	NO	YES	YES

We add that the ε^{-1} and δ^{-1} exponents are the same and they are from the interval $[1,2]$. As already mentioned, they are bounded by the exponents for the third Sobolev class, respectively, and we will present these bounds later. They can be arbitrarily close to 1 if we take product weights with sufficiently large β. If we compare the last two tables, we see that polynomial dependence on δ^{-1} partially erased some but not all negative tractability results from the case when only logarithmic dependence on δ^{-1} is allowed.

We finally turn to the randomized setting. The initial error does not depend on the randomized setting since for $n = 0$ we do not use function values. Hence, the initial error equals to 1 also in the randomized setting. As before, this means that **ABS** = **NOR**.

The unit ball of the first Sobolev space is a subset of the space $L_2([0,1]^d)$. This implies that the variance of a function from the unit ball is bounded by one, and the standard Monte Carlo yields strong polynomial tractability with exponent at most 2 independently of the weights $\gamma_{d,\mathfrak{u}}$, see Corollary 17.3 of Chapter 17. Therefore we have the following positive tractability results.

Table 20.7. **RAN** and **ABS** = **NOR**

	SPT	PT	WT
CONS	YES	YES	YES
PROD	YES	YES	YES
FINO	YES	YES	YES

We also add that the exponent of strong polynomial tractability must be at least $\frac{2}{3}$ since it is known that the nth minimal errors for the univariate case are proportional to $n^{-3/2}$, see Bakhvalov [4]. Although strong polynomial tractability holds for all weights, probably we can obtain the exponent smaller than 2 only for some weights. It would be of interest to find conditions on, say, product weights for which this holds. This leads us to the next open problem.

Open Problem 91.

Consider multivariate integration for the first Sobolev space in the randomized setting for the absolute/normalized error criterion.

- Consider product weights $\gamma_{d,j} = j^{-\beta}$. Find all β, for which the exponent of strong polynomial tractability is less than 2, and all β for which the exponent attains its minimal value $\frac{2}{3}$.
- Characterize general weights for which the exponent of strong polynomial tractability is at most $p \in [\frac{2}{3}, 2)$.

We still have one case to consider. Namely, the randomized setting with the relative error criterion. Although this case has not been yet studied, it is relatively easy to show that multivariate integration is still unsolvable, as for the worst case and average case settings. In fact, even for $d = 1$, the minimal nth randomized error is one, and this holds for all n. This means that for the information complexity we obtain

$$n^{\text{ran-rel}}(\varepsilon, I_1) = \infty \quad \text{for all } \varepsilon \in (0,1).$$

The proof is as follows. As before, we use the hat functions f_j for $j = 1, 2, \dots, m$. For $k \in \mathbb{N}$ and $c = [c_1, c_2, \dots, c_m] \in \{-k, -k+1, \dots, k\}^m$, define

$$f_c(x) = \frac{1}{k} \sum_{j=1}^{m} c_j \, f_j(x) \quad \text{for all } x \in [0,1].$$

Then $f_c \in H_{1,1}$ and

$$\| f_c \|_{H_{1,1}}^2 = \frac{1}{k^2} \sum_{j=1}^{m} c_j^2 \| f_j \|_{H_{1,1}}^2 = \frac{1}{k^2\, m} \sum_{j=1}^{m} c_j^2 \le 1.$$

Hence, all f_c are in the unit ball of $H_{1,1}$, and obviously we have $(2k+1)^m$ of such functions. Finally we define

$$F_{m,k} = \{ f_c \mid c \in \{-k, -k+1, \dots, k\}^m \}$$

and equip $F_{m,k}$ with the uniform distribution such that every f_c occurs with probability $(2k+1)^{-m}$.

We now use Bakhvalov's approach and switch from the randomized setting to the average case setting. More precisely, for any randomized algorithm A_n that uses at most n function values, we have

$$e^{\text{ran}}(A_n) = \sup_{\|f\|_{H_{1,1}} \le 1} \mathbb{E}_\omega \frac{|I_1(f) - A_{n,\omega}(f)|}{|I_1(f)|} \ge \inf_{B_n} e^{\text{avg-rel}}(B_n),$$

where

$$e^{\text{avg-rel}}(B_n) = \frac{1}{(2k+1)^m} \sum_{f \in F_{m,k}} \frac{|I_1(f) - B_n(f)|}{|I_1(f)|}.$$

Here we take the infimum over all algorithms B_n that use at most n function values. For the relative error we adopt the convention that $0/0 = 0$.

Suppose first that $B_n(f) = 0$ for all $f \in F_{m,k}$. Then

$$e^{\text{avg-rel}}(B_n) = \frac{1}{(2k+1)^m} \left| \{ f \in F_{m,k} \mid I_1(f) \ne 0 \} \right|.$$

Note that $I_1(f_c) \ne 0$ iff $\sum_{j=1}^{m} c_j \ne 0$. We claim that at least $2k(2k+1)^{m-1}$ functions in $F_{m,k}$ have non-zero integral. Indeed, take arbitrary $c_1, c_2, \dots, c_{m-1}$, and consider the equation

$$\sum_{j=1}^{m} c_j = \sum_{j=1}^{m-1} c_j + c_m = 0$$

for $c_m \in \{-k, -k+1, \dots, k\}$. Then there is at most one solution for c_m. So for at least $2k$ choices of c_m, the integral is non-zero. Since c_j can take $2k+1$ different values for $j = 1, 2, \dots, m-1$, we have at least $2k(2k+1)^{m-1}$ choices of c with non-zero integral, as claimed. Therefore

$$e^{\text{avg-rel}}(0) \geq \frac{2k}{2k+1}.$$

Assume now that B_n is non-zero on $F_{m,k}$. Then there is

$$c^* = [c_1^*, c_2^*, \dots, c_m^*] \in \{-k, -k+1, \dots, k\}^m \ \text{ such that } \ B_n(f_{c^*}) \neq 0.$$

For this f_{c^*}, the algorithm B_n uses sample points $x_{1,c^*}, x_{2,c^*}, \dots, x_{n,c^*}$ from $[0,1]$. Then there exist indices $j_1, j_2, \dots, j_n$ such that

$$\{x_{1,c^*}, x_{2,c^*}, \dots, x_{n,c^*}\} \subseteq \bigcup_{i=1}^{n} \big[(j_i - 1)h, j_i h\big].$$

We have

$$f_{c^*}(x_{s,c^*}) = \frac{1}{k}\sum_{j=1}^{m} c_j^* f_j(x_{s,c^*}) = \frac{1}{k}\sum_{i=1}^{n} c_{j_i}^* f_{j_i}(x_{s,c^*})$$

for $s = 1, 2, \dots, n$.

Choose $m \geq 2n$. Then there exist n distinct indices $j_{n+1}, j_{n+2}, \dots, j_{2n}$ from $\{1, 2, \dots, m\} \setminus \{j_1, j_2, \dots, j_n\}$. Define the (fooling) function

$$g_{c^*} = \frac{1}{k}\Big(\sum_{i=1}^{n} c_{j_i}^* f_{j_i} - \sum_{i=1}^{n} c_{j_i}^* f_{j_{n+i}}\Big).$$

Note that $g_{c^*} \in F_{m,k}$ and $g_{c^*}(x_{s,c^*}) = f_{c^*}(x_{s,c^*})$ for all $s = 1, 2, \dots, n$. This implies that

$$B_n(f_{c^*}) = B_n(g_{c^*}) \neq 0.$$

Furthermore, we constructed g_{c^*} such that $I_1(g_{c^*}) = 0$, and therefore

$$e^{\text{avg-rel}}(B_n) \geq \frac{1}{(2k+1)^m}\,\frac{|I_1(g_{c^*}) - B_n(g_{c^*})|}{|I_1(g_{c^*})|} = \infty.$$

This proves that the randomized error of A_n satisfies

$$e^{\text{ran-rel}}(A_n) \geq \frac{2k}{2k+1},$$

and since k can be arbitrary large, we have $e^{\text{ran-rel}}(A_n) \ge 1$.

The same argument can be applied for randomized algorithms A_n with varying $n(f)$ and with the expected value of $n(f)$ bounded by n. Hence, no matter how large n may be, there is no way to guarantee that the error is less than one. Hence, $n^{\text{ran-rel}}(\varepsilon, I_1) = \infty$ as claimed.

As for the worst case and average case settings, the summary of tractability results for the randomized setting with the relative error is also given with all entries NO.

Table 20.8. **RAN** and **REL**

	SPT(d)-L	SPT(δ)-L	PT-L	WT-L
CONS	NO	NO	NO	NO
PROD	NO	NO	NO	NO
FINO	NO	NO	NO	NO

20.4 Second Sobolev Space

For the second Sobolev space, the initial error of multivariate integration is

$$\|I_d\| = \Big(1 + \sum_{\emptyset \ne \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} \big(a_j^2 - a_j + \tfrac{1}{3}\big)\Big)^{1/2},$$

where a is the anchor of the space.

Since the initial error is larger than one, the absolute and normalized error criteria are different. Furthermore, the absolute error criterion is more difficult. Indeed, as we shall see, some positive tractability results for the normalized error criterion do *not* hold for the absolute error criterion.

As already explained, we also have **WOR** $=$ **AVG** for the second Sobolev space. We are ready to present tractability results separately for the absolute and normalized error criteria.

We comment on Tables 20.9 and 20.10. We stress that tractability results do not depend on the anchor a. However, as we shall see the bounds on the exponents of tractability do depend on a.

The curse of dimensionality was proved in Chapter 11. Of course, it is enough to prove this only for the normalized error criterion. The curse is also present for the almost constant weights in which $\gamma_{d,\mathfrak{u}} = c > 0$ for all non-empty $\mathfrak{u} \subseteq [d]$.

Table 20.9. **WOR** = **AVG** and **ABS**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	YES	YES

Table 20.10. **WOR** = **AVG** and **NOR**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	YES	YES	YES

For product weights, strong polynomial tractability holds for the normalized error criterion iff $\beta > 1$. Hence, $\beta > 1$ is also necessary for strong polynomial tractability and the absolute error criterion. If $\beta > 1$ then the initial error is of order 1, and that is why this condition is also sufficient for the absolute error criterion. For both the absolute and normalized criterion, the exponent p of strong polynomial tractability is the same and $p \in [1, 2]$. In fact, it is easy to conclude from Theorem 16.21 of Chapter 16 that the exponent $p \leq \max(1, 2/\beta)$ and can be achieved by shifted lattice rules. Hence, for $\beta \geq 2$ the exponent $p = 1$, as for the univariate case.

Similarly, $\beta = 1$ yields polynomial tractability for both the absolute and normalized error criteria with an ε^{-1} exponent at most 2, and with different bounds on the d exponents depending on the anchor and the error criterion. For simplicity, we take the anchor a with the same components, i.e., $a_j = a \in [0, 1]$. Then the initial error is

$$\prod_{j=1}^{d} \left(1 + j^{-1}(a^2 - a + \tfrac{1}{3})\right)^{1/2} = \Theta\left(d^{\,(a^2 - a + 1/3)/2}\right).$$

We now discuss bounds on the d exponents. For the absolute error criterion, Remark 17.13 of Chapter 17 applies with $\alpha = 1$ and $\gamma_{d,1} = 1$, and states that

$$n^{\text{wor-abs}}(\varepsilon, d) = \Omega\left(\varepsilon^{-1}\, d^{\,(a^2 - a + 1/3)/2}\right) \quad \text{for all } \varepsilon \in (0, 1),\ d \in \mathbb{N}.$$

This means that the lower bound on the d exponent is $(a^2 - a + 1/3)/2$.

We obtain an upper bound on the d exponent from Theorem 16.21 of Chapter 16 with $\tau = \frac{1}{2}$, compare also with Theorem 16.5 and the comment after the proof for the corresponding Korobov space,

$$n^{\text{wor-abs}}(\varepsilon, d) = \mathcal{O}\big(\varepsilon^{-2}\, d^{\,a^2-a+1/2}\big) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Let q_a be the d exponent of polynomial tractability for the second Sobolev space with the anchor a. Then

$$\tfrac{1}{24} \le q_{a,L} := \tfrac{1}{2}\left(a^2 - a + \tfrac{1}{3}\right) \le q_a \le q_{a,U} := a^2 - a + \tfrac{1}{2} \le \tfrac{1}{2},$$

and

$$q_{a,U} - q_{a,L} = \tfrac{1}{2}\left(a^2 - a + \tfrac{2}{3}\right) \le \tfrac{1}{3}.$$

This means that we have pretty tight bounds on the d exponents. Furthermore, the d exponents are quite small. For example,

- for $a \in \{0,1\}$ we have
$$\tfrac{1}{6} \le q_a \le \tfrac{1}{2},$$
- for $a = \frac{1}{2}$ we have
$$\tfrac{1}{24} \le q_a \le \tfrac{1}{4}.$$

The reader may be afraid that the small d exponents are at the expense of huge factors in the estimates of $n^{\text{wor-abs}}(\varepsilon, d)$. This is fortunately *not* the case. Again from the same Theorem 16.21 of Chapter 16, we have

$$\begin{aligned} n^{\text{wor-abs}}(\varepsilon, d) &\le 4 + 2\,\varepsilon^{-2} \prod_{j=1}^{d} \left(1 + j^{-1}\left(a^2 - a + \tfrac{1}{2}\right)\right). \\ &\le +2\,\varepsilon^{-2} \exp\left(\left(a^2 - a + \tfrac{1}{2}\right) \sum_{j=1}^{d} j^{-1}\right). \end{aligned}$$

We now explain why we have the first term 4 and why we have the factor 2 in the second term. The error estimate of Theorem 16.21 has the form $C^{1/2}(n-1)^{-1/2}$ for the specific C and with a prime n. If we solve $C^{1/2}(n-1)^{-1/2} \le \varepsilon$ we obtain that

$$n \ge n^* := 1 + \left\lceil \varepsilon^{-2}\, C \right\rceil.$$

We can find a prime n in $[n^*, 2n^*]$, and therefore

$$n^{\text{wor-abs}}(\varepsilon, d) \le 2n^* \le 2(1 + 1 + \varepsilon^{-2}\, C) = 4 + 2\varepsilon^{-2} C.$$

That is why we have the extra 4 and the factor 2. Since

$$\sum_{j=1}^{d} j^{-1} \le \ln(d+1) + C_{\mathrm{Euler}}$$

with the Euler constant $C_{\mathrm{Euler}} = 0.5772\ldots$, and $2\exp(C_{\mathrm{Euler}}) = 3.5621\ldots$, we finally obtain

$$n^{\text{wor-abs}}(\varepsilon, d) \le 4 + 3.5622\,\varepsilon^{-2}\,(d+1)^{a^2-a+1/2} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N},$$

and everything is under control.

We now discuss the normalized error criterion; still for product weights with $\beta = 1$. In this case, we cannot apply Remark 17.13 and we do not know lower bounds on the d exponents. However, the upper bounds on the d exponents are even better, since

$$n^{\text{wor-nor}}(\varepsilon, d) = \mathcal{O}\big(\varepsilon^{-2}\, d^{\,1/6}\big).$$

Again, this follows from Theorem 16.21 of Chapter 16.

The d exponent is thus at most $\frac{1}{6}$, which is quite small. For $d = 360$ we have $d^{1/6} = 2.667\ldots$ and even for huge $d = 10^6$ we have $d^{1/6} = 10$.

We now check that the factors in the last big $\mathcal{O}$ notation are also harmless. We have

$$n^{\text{wor-nor}}(\varepsilon, d) \le 4 + 2\,\varepsilon^{-2} \prod_{j=1}^{d} \frac{1 + j^{-1}(a^2 - a + 1/2)}{1 + j^{-1}(a^2 - a + 1/3)}.$$

Note that

$$\prod_{j=1}^{d} \frac{1 + j^{-1}(a^2 - a + 1/2)}{1 + j^{-1}(a^2 - a + 1/3)} = \exp\Big(\sum_{j=1}^{d} \ln \frac{1 + (a^2 - a + 1/2)/j}{1 + (a^2 - a + 1/3)/j}\Big).$$

For $b_1 \ge b_2 \ge 0$, one can check that $\ln[(1 + b_1 x)/(1 + b_2 x)] \le (b_1 - b_2)x$ for all $x \in [0,1]$. Therefore

$$\begin{aligned}\prod_{j=1}^{d} \frac{1 + j^{-1}(a^2 - a + 1/2)}{1 + j^{-1}(a^2 - a + 1/3)} &\le \exp\Big(\frac{1}{6}\sum_{j=1}^{d} j^{-1}\Big) \\ &\le \exp\big(\tfrac{1}{6}\ln(d+1) + C_{\mathrm{Euler}}\big) \le \tfrac{1}{2}\, 3.5622\,(d+1)^{1/6}.\end{aligned}$$

Hence,

$$n^{\text{wor-nor}}(\varepsilon, d) \le 4 + 3.5622\,\varepsilon^{-2}\,(d+1)^{1/6},$$

and again everything is under control.

We turn to polynomially bounded finite-order weights. We now always have polynomial tractability again with possibly different d exponents for the absolute and normalized error criteria, see Theorem 16.25 of Chapter 16. Furthermore, we can have the ε^{-1} exponent sufficiently close to 1 at the expense of a larger d exponent. For example, for the normalized error criterion and for $\tau \in [1/2, 1)$, we have

$$n^{\text{wor-abs}}(\varepsilon, d) = \mathcal{O}\big(\varepsilon^{-1/\tau}\, d^{\,\omega(1-1/(2\tau))}\big)$$

with the factor in the big $\mathcal{O}$ notation independent of ε^{-1} and d but dependent on τ. In fact, we know that this factor must go to infinity as τ goes to 1. Ignoring this factor for a moment, we can say that the ε^{-1} exponent can be arbitrarily close to 1 with the d exponent always at most $\omega/2$. Furthermore, for $\tau = 1/2$ we have strong polynomial tractability.

It is natural to ask what are these factors and how fast they go to infinity as τ approaches 1. That is, we are looking for C_τ for which

$$n^{\text{wor}}(\varepsilon, d) \le 4 + 2\, C_\tau\, \varepsilon^{-1/\tau}\, d^{\,\omega(1-1/(2\tau))} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

We now find C_τ by using the explicit error bounds in Theorem 16.25 of Chapter 16. In this theorem, N_d denotes the number of non-zero finite-order weights. We know that $N_d \le 2d^{\,\omega}$, and that for large d we have roughly $N_d \approx d^{\,\omega}/\omega!$. Since we are mostly interested in large d, we simplify the analysis by taking $N_d = d^{\,\omega}/\omega!$. Then

$$C_\tau = \left(1 + 2\zeta(1/\tau)\left(\frac{\sqrt{6}}{\pi}\right)^{\omega}\right)\big[\,w!\,\big]^{-(1-1/(2\tau))}.$$

For $\tau = \frac{1}{2}$ we have $C_{1/2} = 3^{\omega}$. For τ going to 1, the Riemann zeta function blows up since $\zeta(1/\tau) = 1/(1-\tau)(1+o(1))$. Therefore

$$C_\tau = \left(\frac{2\sqrt{6}}{\pi(1-\tau)}\right)^{\omega} \frac{1}{\sqrt{\omega!}}\,\big(1 + o(1)\big).$$

It is up to the reader to judge for which τ and ω, the factor C_τ is nasty.

We stress that all bounds presented here for product and finite-order weights can be achieved by shifted lattice rules or by the QMC algorithms with the Niederreiter sequence, as explained in Section 16.9 of Chapter 16.

We claim in Table 20.9 that strong polynomial tractability does not hold for finite-order weights of order $\omega \ge 1$ for the absolute error criterion. We need to prove this claim. We proceed similarly as for the first Sobolev space. For the second Sobolev space with the anchor $a = [a_1, a_2, \dots, a_d]$ we show that

$$\limsup_{d\to\infty}\, e(n,d) = \infty \quad \text{for all } n \in \mathbb{N},$$

which contradicts strong polynomial tractability. To show this, take d functions f_j from $H_{1,2}$ such that $f_j(a_j) = 0$ and $\|f'\|_{L_2([0,1])} = 1$. Consider the function

$$f(x_1, x_2, \dots, x_d) = \frac{f_1(x_1) + f_2(x_2) + \cdots + f_d(x_d)}{\sqrt{d}} \quad \text{for all } x_j \in [0,1].$$

Then $f \in H_{d,2}$ and

$$\|f\|_{H_{d,2}}^2 = \frac{1}{d}\left(\|f_1'\|_{L_2([0,1])} + \|f_2'\|_{L_2([0,1])} + \cdots + \|f_d'\|_{L_2([0,1])}\right) = 1.$$

For an arbitrary algorithm A_n that uses sample points $x_1^*, x_2^*, \dots, x_n^*$, we can choose a function f_k that vanishes at the kth components of $x_{j,k}^*$ for $j = 1, 2, \dots, n$ as well as at a_j and

$$I_d(f_j) \ge \tfrac{1}{2}\, e(n+1, 1).$$

Obviously such a function exists since $e(n+1,1)$ is defined as the largest integral of univariate functions that vanish at $n+1$ optimally chosen points. Then

$$I_d(f) = \frac{d}{2\sqrt{d}} e(n+1, 1)$$

and goes to infinity since $e(n+1,1) = \Omega(n^{-1})$. Hence, we do not have strong polynomial tractability, and this justifies NO in the last row of Table 20.9.

We now briefly mention the relative error. In fact, we can consider simultaneously the relative error for the worst case, average case, randomized and probabilistic setting with logarithmic dependence on δ^{-1}. We claim that in all these settings,

$$n^{\text{wor/avg/ran/pro-l}}(\varepsilon, I_d) = \infty \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

We know that this holds in the worst case and average case settings. This also can be proved in the randomized setting with a slightly modified proof that we presented for the first Sobolev space. Obviously, this also holds in the probabilistic setting since the information complexity depends polynomially on δ^{-1}.

Again, this means that multivariate integration is *unsolvable*, which is much worse than intractability or the curse of dimensionality. This explains the following table with all entries NO.

Table 20.11. **WOR** = **AVG**, **RAN**, **PRO-L** and **REL**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	NO	NO	NO
FINO	NO	NO	NO

We now turn to the probabilistic setting. First we consider the absolute error criterion. We allow polynomial or logarithmic dependence on δ^{-1}. Similarly as for the first Sobolev space we have the following summary of tractability results.

Table 20.12. **PRO** and **ABS**

	SPT(d)-P/L	SPT(δ)-P/L	PT-P/L	WT-P/L
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	NO	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	NO	YES	YES

We comment on Table 20.12. All tractability results easily follow from tractability results presented in Table 20.9 with ε replaced roughly by $\varepsilon/\sqrt{2 \ln \delta^{-1}}$.

We turn to the probabilistic setting for the normalized error criterion. We now have the following summary of tractability results.

Table 20.13. **PRO** and **NOR**

	SPT(d)-P/L	SPT(δ)-P/L	PT-P/L	WT-P/L
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	YES	YES	YES	YES

We comment of Table 20.13. We now have strong polynomial tractability for finite-order weights. In fact, this holds for arbitrary finite-order weights, as shown in Theorem 16.23 of Chapter 16. All comments on the tractability exponents that we made for the worst case setting, are also now valid, again with ε replaced by $\varepsilon/\sqrt{2 \ln \delta^{-1}}$. In particular, this means that with logarithmic dependence on δ^{-1}, the δ^{-1} exponent is half of the ε^{-1} exponent. However, for polynomial dependence on δ^{-1}, the δ^{-1} exponent is arbitrarily small.

We have one more case for the probabilistic setting. Namely the relative error when we allow polynomial dependence on δ^{-1}. As we know this is the only case for which we can have some positive tractability results. They are summarized in the next table.

We briefly comment on Table 20.14. The results reported in this table follow from the results in Table 20.10, since the probabilistic setting for the relative error with polynomial dependence on δ^{-1} is equivalent to the worst case setting for the normalized case if we replace ε by $\frac{1}{2}\,\pi\,\varepsilon\,\delta(1 + o(1))$.

We finally turn to the randomized setting. Since the initial error is now larger

Table 20.14. **PRO-P** and **REL**

	SPT(d)-P	SPT(δ)-P	PT-P	WT-P
CONS	NO	NO	NO	NO
PROD	YES iff $\beta > 1$	NO	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	YES	NO	YES	YES

than 1, we need to consider the randomized setting for the absolute and normalized error criteria, and we already mentioned negative results for the relative error criterion. For the absolute error we have the following summary of tractability results.

Table 20.15. **RAN** and **ABS**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	YES	YES

We comment on Table 20.15. For constant weights, we may apply Theorem 17.11 and Corollary 17.12 of Chapter 17. Our problem is now indeed a tensor product problem and $\|\mathrm{INT}_1\| > 1$, as needed for Corollary 17.12 of Chapter 17. So we have the curse of dimensionality, and this explains the three NO's in the first row.

For product weights, it is enough to apply Theorem 17.6 of Chapter 17. This theorem states that the standard Monte Carlo yields

- strong polynomial tractability with exponent 2 if $\beta > 1$,
- polynomial tractability if $\beta = 1$, and
- weak tractability if $\beta > 0$.

The same necessary conditions for β for these three kinds of tractability follow from Remark 17.13.

For finite-order weights, Corollary 17.5 states that the standard Monte Carlo yields polynomial tractability with ε^{-1} exponent at most 2 and d exponent at most ω. This obviously implies weak tractability. Strong polynomial tractability does not hold, and this can be proven by modifying the proof presented in this section. Again, we bound the randomized error of any algorithm by

$$\sqrt{d}\, e^{\mathrm{ran}}(n+1,1),$$

and use the fact that $e^{\mathrm{ran}}(n+1,1) = \Omega(n^{-3/2})$.

We now turn to the randomized setting for the normalized error criterion. We remind the reader that for the second Sobolev space, the standard Monte Carlo algorithm yields strong polynomial tractability for product weights iff $\beta > \frac{1}{2}$, see Theorem 17.6 of Chapter 17. This may suggest that for constant weights, for which $\beta = 0$, we may be in trouble. That is why the reader may be surprised to see the following summary of tractability results.

Table 20.16. **RAN** and **NOR**

	SPT	PT	WT
CONS	YES	YES	YES
PROD	YES	YES	YES
FINO	YES	YES	YES

We comment on Table 20.16. So we even have strong polynomial tractability for the unweighted case. In fact, we claim that this positive result holds for *arbitrary* weights $\gamma_{d,\mathfrak{u}}$. Why? It is enough to apply the surprising result of Hinrichs [131], which was explained in Section 17.2 of Chapter 17. This result states that as long as the reproducing kernel is point-wise non-negative and the space $H_{d,2}$ has full support with respect to $\varrho = 1$ then

$$n^{\mathrm{rand}}(\varepsilon, d) \le \tfrac{1}{2}\,\pi\,\varepsilon^{-2} + 1.$$

In our case, these two assumptions hold. The reproducing kernel is not only point-wise non-negative but $K_{d,\gamma,2} \ge 1$. Furthermore this holds for arbitrary $\gamma_{d,\mathfrak{u}}$ and arbitrary anchor a. Clearly, $H_{d,2}$ has full support since the constant function $1 \in H_{d,2}$.

So we have strong polynomial tractability with exponent at most 2 for the second Sobolev space with arbitrary weights $\gamma_{d,\mathfrak{u}}$. Therefore we have all YES's in the last table. Also in this case, the ε^{-1} exponent must be at least $\frac{2}{3}$.

As in the Open Problem 91, we suspect that we can improve the bound on the exponent of strong polynomial tractability under some assumptions on the weights. The reader may also solve the analog of Open Problem 91 for the second Sobolev space.

20.5 Third Sobolev Space

For the third Sobolev space, the initial error of multivariate integration is again one, and therefore **ABS** = **NOR** as well **WOR** = **AVG**. The first table of tractability results is as follows.

Table 20.17. **WOR** = **AVG** and **ABS** = **NOR**

	SPT	PT	WT
CONS	NO	NO	NO
PROD	YES iff $\beta > 1$	YES iff $\beta \geq 1$	YES iff $\beta > 0$
FINO	NO	YES	YES

We comment on Table 20.17. As we know, multivariate integration for the third Sobolev space is not easier than for the first Sobolev space. Furthermore, the initial errors are the same. Therefore lower bounds for the first Sobolev space are also valid for the third Sobolev space. This means we can claim the curse of dimensionality for the constant and almost constant weights for the third Sobolev space, since the curse is present for the first Sobolev space.

For product weights, the conditions on β were proved in [280], again using the proof technique based on decomposable kernels. We now say a little more about tractability exponents based on Section 16.9 of Chapter 16. As for the second Sobolev space, for $\beta > 1$ we have strong polynomial tractability with exponent $p \in [1, 2]$. Theorem 16.21 of Chapter 16 states that $p \leq \max(1, 2/\beta)$. Hence, for $\beta \geq 2$ the exponent $p = 1$ reaches its minimal value.

For $\beta = 1$, we have polynomial tractability and proceeding exactly as before and using Theorem 16.21 of Chapter 16, we can show that

$$n^{\text{wor}}(\varepsilon, d) \leq 4 + 3.5622\, \varepsilon^{-2}\, d^{\,1/6}.$$

We turn to finite-order weights. Strong polynomial tractability does not hold for the third Sobolev space since it does not even hold for the first Sobolev space for which multivariate integration is not harder. As always, we have polynomial tractability for finite-order weights. From Theorem 16.25 of Chapter 16 we know that for any $\tau \in [1/2, 1)$ we have

$$n^{\text{wor}}(\varepsilon, d) = \mathcal{O}\big(\varepsilon^{-1/\tau}\, d^{\,\omega}\big)$$

with the factor in the big $\mathcal{O}$ notation independent of ε^{-1} and d but dependent on τ. So the ε^{-1} exponent can be arbitrarily close to 1 with the d exponent always at most ω. Furthermore, these bounds can be obtained by shifted lattice rules.

Proceeding as for the second Sobolev space and using Theorem 16.25 with $N_d \le 2d^{\,\omega}$, we can check that

$$n^{\rm wor}(\varepsilon,d) \le 4 + 2\,C_\tau\,\varepsilon^{-1/\tau}\,d^{\,\omega} \quad \text{for all } \varepsilon\in(0,1),\ d\in\mathbb{N}$$

with

$$C_\tau = 2\,\max\left(1, \frac{2\zeta(1/\tau)}{(\sqrt{2}\,\pi)^{1/\tau}}\right)^{\omega}.$$

For $\tau = \frac{1}{2}$ we have $C_{1/2} = 2$, whereas for τ tending to one, we have

$$C_\tau = \left(\frac{\sqrt{2}}{\pi(1-\tau)}\right)^{\omega}.$$

We now briefly mention the relative error. As before, for the worst case, average case, randomized and probabilistic setting with logarithmic dependence on δ^{-1}, the problem is unsolvable since the information complexity in all these settings is infinite,

$$n^{\rm wor/avg}(\varepsilon, I_d) = \infty \quad \text{for all } \varepsilon\in(0,1),\ d\in\mathbb{N}.$$

Again, for the randomized setting this result can be proved by a small modification of the proof presented for the first Sobolev space. We have the same table as Table 20.11.

We turn to the probabilistic setting. For the absolute/normalized, and relative error we have the same tractability tables as for the first Sobolev space, and we do not repeat them for the third Sobolev space. This follows directly from relations to the worst case setting. Obviously, the comments on tractability exponents are the same as for the worst case for the third Sobolev space with ε replaced by $\varepsilon/\sqrt{2\ln\delta^{-1}}$ for the absolute/normalized error criterion, and by $\frac{1}{2}\,\pi\,\varepsilon\,\delta(1+o(1))$ for the relative error criterion.

We turn now to the randomized setting for the absolute/normalized error criterion. We still have the same table of tractability results with all YES's but the proof is different. That is why we repeat this table and comment on it.

Table 20.18. **RAN** and **ABS** = **NOR**

	SPT	PT	WT
CONS	YES	YES	YES
PROD	YES	YES	YES
FINO	YES	YES	YES

For constant and product weights, the reproducing kernel of the third Sobolev space is

$$K_{d,\gamma,3}(x,y) = \prod_{j=1}^{d} \left(1 + j^{-\beta}\frac{(x_j-y_j)^2-(x_j-y_j)+\frac{1}{6}}{2}\right) \quad \text{for all } x,y \in [0,1]^d.$$

Since $(x_j-y_j)^2-(x_j-y_j)+\frac{1}{6} \in [-\frac{1}{12},\frac{1}{6}]$, the kernel is point-wise positive, and we can apply the result of Hinrichs [131] to claim strong polynomial tractability with exponent at most 2. This justifies the YES's in the first two rows of Table 20.18.

How about finite-order weights? Well, just now the reproducing kernel does not have to be point-wise non-negative. Indeed, take $\omega = 1$ and $x_j = \frac{1}{2}$ and $y_j = 0$ for $j = 1,2,\dots,d$. Then

$$K_{d,\gamma,3}(x,y) = 1 - \tfrac{1}{24}\,d.$$

For $d \ge 25$, the kernel takes negative values. Although we cannot now use the result of Hinrichs, it does not matter since the standard Monte Carlo does the job. Using Theorem 17.7 of Chapter 17 for the third Sobolev space with finite-order weights, we see that $n^{\mathrm{mc}}(\varepsilon, H_{d,\gamma,3}) \le \lceil \varepsilon^{-2} \rceil$. Hence, strong polynomial tractability holds with exponent at most 2 in this case, and we have three YES's in the last column of Table 20.18. As for the first Sobolev space, it would be of interest to find conditions on weights to improve the exponent of strong tractability. This means that it would be good to solve the analog of Open Problem 91 also for the third Sobolev space.

20.6 Notes and Remarks

NR 20.1:1. This chapter is based on the results obtained in this volume. However, the results that

- multivariate integration is not strongly polynomially tractable for finite-order weights in the worst case, and that
- multivariate integration is unsolvable in the randomized setting for the relative error

are new.

We remind the reader that we considered in this chapter the special finite-order weights, $\gamma_{d,\mathfrak{u}} = 1$ for all $|\mathfrak{u}| \le \omega$ and $\gamma_{d,\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega$, with $\omega \ge 1$. The results on these special finite-order weights can be generalized for arbitrary finite-order weights. This would lead us to necessary conditions on finite-order weights to get strong polynomial tractability. Then we could compare these necessary conditions with the sufficient conditions we already presented. In this way, we would see if they are sharp. We leave this problem to the reader.

Appendix D
List of Open Problems

Problems 1–30 are from Volume I, some of them have been already solved.

1. Integration and approximation for the classes $F_{d,r}$, Section 3.3.
2. Integration and approximation for the classes $F_{d,r(d)}$ and $F_{d,\infty}$, Section 3.3.
   •• Partially solved in [224].
3. Integration for a finite dimensional space F_d of trigonometric polynomials, Section 3.3. See Hinrichs and Vybíral [133] for more information.
4. Integration for weighted Korobov spaces, Section 3.3.
5. Approximation of C^∞-functions from the classes $F_{d,p}$, Section 3.3.
   •• Solved for $p = \infty$ in [224].
6. Construction of points with small star-discrepancy, Section 3.3.
7. On bounds for the star-discrepancy, Section 3.3.
8. Diagonal problems for C^r-functions from the class $F_{d,\gamma,r}$, Section 3.3.
9. Construction of good points for Gaussian integration for the isotropic Wiener measure, Section 3.3.
10. Tractability for approximation with folded Wiener sheet measures with increasing smoothness, Section 3.3.
11. Tractability for approximation with folded Wiener sheet measures with varying smoothness, Section 3.3.
12. Tractability for a modified error criterion, Section 3.3.
13. Tractability in the randomized setting for integration over weighted Sobolev spaces, Section 3.3.
    •• Partially solved by Hinrichs [131].
14. Tractability in the randomized setting for integration over periodic weighted Sobolev spaces, Section 3.3.
    •• Partially solved by Hinrichs [131].
15. Tractability in the randomized setting for general linear operators, Section 3.3.
16. On the power of adaption for linear problems, Section 4.2.1.

17. On the power of adaption for linear operators on convex sets, Section 4.2.1.

18. On the asymptotic optimality of linear algorithms for Sobolev embeddings for Λ^{std}, Section 4.2.4.
    •• Solved by Heinrich [110], for further results see Triebel [310], [311], [312].

19. On the existence of optimal measurable algorithms, Section 4.3.3.

20. On the power of adaption for linear problems in the randomized setting, Section 4.3.3.

21. On the (almost) optimality of linear algorithms for linear problems in the randomized setting, Section 4.3.3.

22. How good are linear randomized algorithms for linear problems?
    Section 4.3.3.

23. How good are linear randomized algorithms for linear problems defined over Hilbert spaces? Section 4.3.3.

24. On the optimality of measurable algorithms in the randomized setting,
    Section 4.3.3.

25. On Sobolev embeddings in the randomized setting, Section 4.3.3.
    •• Solved by Heinrich [110], [111].

26. Weak tractability of linear tensor product problems in the worst case setting with $\lambda_1 = 1$ and $\lambda_2 < 1$, Section 5.2.
    •• Solved by Papageorgiou and Petras [231].

27. Tractability of linear weighted tensor product problems for the absolute error criterion, Section 5.3.4.

28. Weak tractability for linear tensor product problems in the average case setting, Section 6.2.
    •• Solved by Papageorgiou and Petras [232].

29. Tractability of linear weighted product problems in the average case setting for the absolute error criterion, Section 6.3.

30. Weak tractability for linear weighted tensor product problems in the average case setting, Section 6.3.

31. Bounds for the exponent of the L_2 discrepancy, Section 9.2.2.

32. Construction of points with small L_2 discrepancy, Section 9.2.2

33. Bounds for the normalized L_2 discrepancy for equal weights, Section 9.2.3.

34. Bounds for the normalized L_2 discrepancy for optimal weights, Section 9.2.3.

35. Weighted B-discrepancy, Section 9.6.
    •• Solved by Gnewuch [80].

36. Exponent of strong tractability for the anchored Sobolev space, Section 9.7.

37. Construction of sample points, Section 9.7.

38. Tractability for the B-discrepancy, Section 9.7.

39. Tractability for the weighted B-discrepancy, Section 9.7.

40. Construction of points with small p star discrepancy, Section 9.8.1.

41. On the average p star discrepancy for shifted lattices, Section 9.8.1.

42. Construction of points with small star discrepancy, Section 9.9.

43. Arbitrary and positive quadrature formulas for Sobolev spaces, Section 10.5.3.

44. Tractability of integration for polynomials and C^∞ functions, Section 10.5.4.

45. Optimality of positive quadrature formulas for RKHS, Section 10.6.

46. Exponent of strong tractability for multivariate integration for a separable tensor product space, Section 10.7.7.

47. Exponent of strong tractability for multivariate integration for a separable tensor product space, Section 10.7.10.

48. Exponent of strong tractability for tensor product linear functionals with finite norms of h_1, Section 10.10.1.

49. Tractability of linear tensor product functionals with $\|I_1\| > 1$, Section 11.3.

50. Tractability of linear tensor product functionals with $\|I_1\| = 1$, Section 11.3.

51. Exponent of strong tractability for Gaussian integration, Section 11.4.1.

52. Tractability of weighted integration for constant weight, Section 11.4.1.

53. Exponent of strong tractability for an anchored Sobolev space, Section 11.4.2.

54. Strong polynomial tractability for the centered discrepancy, Section 11.4.3.

55. Intractability for functionals for a space of analytic functions, Section 11.5.4

56. Characterization of intractability of linear tensor product functionals for certain Sobolev spaces, Section 11.6.2.

57. Characterization of tractability of linear tensor product functionals for certain tensor product Sobolev spaces, Section 11.6.2.

58. Characterization of tractability of linear tensor product functionals for tensor product Sobolev spaces with $r \geq 2$, Section 11.6.2.

59. Generalization for problems with m dimensional D_1, $m > 1$, Section 11.6.2.

60. Tractability for order dependent weights, Section 12.3.3.

61. Exponent of strong polynomial tractability for weighted integration, Section 12.4.2.

62. Weighted integration with $r \geq 2$ and $k \geq 1$, Section 12.5.1.

63. Conditions on weights and anchor to obtain exponent 1, Section 12.6.2.

64. Conditions on finite-order weights, Section 12.6.4.

65. Conditions for T-tractability in the average and worst case settings, Section 13.4.

66. Weak-log tractability in the probabilistic setting, Section 14.4.

67. Weak tractability or T-tractability with the Smolyak algorithm, Section 15.2.5.

68. Strong tractability with the Smolyak algorithm for integration of smooth periodic functions, Section 15.2.6.

69. Strong tractability with the Smolyak algorithm for integration of smooth non-periodic functions, Section 15.2.7.

70. Conditions for general weights and the WTP algorithm for various kinds of tractability, Section 15.3.4.

71. Finite order weights for various kinds of tractability, Section 15.3.9.

72. On the sum and strong polynomial tractability exponents, Section 15.3.11.

73. Conditions on weights for weighted Korobov spaces, Section 16.4.3.

74. Conditions on weights for weighted Korobov spaces for T-tractability, Section 16.4.4

75. Cost of the CBC algorithm for order-dependent weights, Section 16.6.4.

76. Conditions for general weights for multivariate integration over weighted Korobov spaces, Section 16.8.

77. Shifted lattice rules for multivariate integration over the anchored or unanchored Sobolev spaces, Section 16.9.1.

78. Conditions on weights for Gaussian integration with Monte Carlo, Section 17.1.2.1.

79. Optimality of Monte Carlo for L_2 with different norms, Section 17.1.6.2.

80. Optimal densities for importance sampling, Section 17.2.1.
    •• Solved by Hinrichs [131].
81. Unknown density in the worst case setting, Section 18.1.1.
82. Unknown density for the class $\mathscr{F}^{\alpha}$, Section 18.1.4.
83. Unknown density with several maxima, Section 18.1.5.
84. Unknown density on general domains, Section 18.1.5.
85. Fixed point problem, Section 18.3.
86. Global optimization in the randomized setting, Section 18.4.
87. Global optimization in the average case setting, Section 18.4.
88. Weak tractability for path integration, Section 19.1.
89. Tractability of path integration for entire functions, Section 19.1.
90. Fast algorithms for Feynman–Kac integrals without precomputation, Section 19.1.
91. Randomized setting for multivariate integration, Section 20.3.

Appendix E
Errata for Volume I

The following typos or errors have been noted in Volume I of our book

Tractability of Multivariate Problems

1. page 22, line 14.

   $(\bar{h}_1 \bar{h}_2 \cdots \bar{h}_d)^{\alpha}$ should read $(\bar{h}_1 \bar{h}_2 \cdots \bar{h}_d)^{-\alpha}$.

2. page 25, line 7.

   "then" should read "than".

3. page 25, line 13.

   $n^{\mathrm{wor}}(\varepsilon, \mathrm{APP}_d, F_{d,p}, G_{d,m,p})$ should read $e^{\mathrm{wor}}(\varepsilon, \mathrm{APP}_d, F_{d,p}, G_{d,m,p})$.

4. page 26, line 10.

   $\|f\|_{G_{d,m,p}}$ should read $\|f_k\|_{G_{d,m,p}}$.

5. page 69 and 163

   We were not always consistent with the definition of CRI_d. On page 69, we have $\mathrm{CRI}_d =$ the initial error, and on page 163, we have $\mathrm{CRI}_d =$ the square of the initial error. The reader should use CRI_d as defined in the respective chapter.

6. page 157, line 1.

   "of Chapter 2" should read "of Chapter 4".

7. page 160, line -5.

   $\lceil C\, d^q \rceil + 1, \lceil C\, d^q \rceil + 1$ should read $\lceil C\, d^q \rceil + 1, \lceil C\, d^q \rceil + 2$.

8. page 164, line 13.

   $$\frac{\ln^2 j}{\ln^2(j-1)}\,\frac{1}{1-\sqrt{\beta}} \le \left(\frac{\ln 3}{\ln 2}\right)^2 (2+\sqrt{2})$$

   should read

   $$\frac{\ln^2 j}{\ln^2(j-1)}\,\frac{1}{(1-\sqrt{\beta})^2} \le \left(\frac{\ln 3}{\ln 2}\right)^2 (2+\sqrt{2})^2.$$

9. page 178, line -16.

   $n(\varepsilon, d)$ should read $\ln n(\varepsilon, d)$.

10. page 178, line -5.

$$\frac{n(\varepsilon,1)}{\varepsilon^{-1}+1} \quad \text{should read} \quad \frac{\ln n(\varepsilon,1)}{\varepsilon^{-1}+1}.$$

11. page 218, (5.29).

 $m_3(\varepsilon,d)$ in (5.29) should read $\ln m_3(\varepsilon,d)$.

12. page 265, line 4 of Section 6.3.

 L_j should read $L_j(f)$.

13. page 285, lines 8 and 10.

 $T((2\varepsilon)^{-1},d)$ should read $T(2\varepsilon^{-1},d)$.

14. page 334, line 4 of Theorem 8.25.

 The definition of a_i should be moved to three lines below. In this line, should be

$$\lim_{x\to 0} \frac{\ln f_i(x)}{x} = 0 \quad \text{for } i = 1,2.$$

15. page 342, line -6.

 "the survey" should read "the survey of".

16. page 345, line -6.

 $\beta_2 = (2\pi)^{-r}$ should read $\beta_2 = (2\pi)^{-2r}$.

17. page 345, line -4.

 $\mathfrak{u} \in [d]$ should read $\mathfrak{u} \subseteq [d]$.

18. page 346, line 5.

 $(1-\delta_{0,h_j}$ should read $(1-\delta_{0,h_j})$.

19. page 349, line 10.

 $2\pi h_j$ should read $2\pi\, \mathrm{i}\, h_j$.

20. page 352, line 8.

 $(1-\delta_{1,k_j})\sqrt{2}$ should read $(1-\delta_{1,k_j})2^{-1/2}$.

21. page 353, line -2.

 twice $K_\gamma(x,y)$ should read $K_\gamma(x,y)-1$.

22. page 355, the first line of (A.17)

 $\prod_{j\in\mathfrak{u}} \gamma_j^{-1}$ should read $\gamma_{d,\mathfrak{u}}^{-1}$.

23. page 357, line -5.

 $M^{-1}(x-\alpha,$ should read $M^{-1}(x-\alpha), .$

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