Encyclopedia of Mathematics and Its Applications 110

SPLINE FUNCTIONS ON TRIANGULATIONS

Ming-Jun Lai and Larry L. Schumaker

CAMBRIDGE

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITORIAL BOARD

P. Flajolet, M. Ismail, E. Lutwak

Volume 110

Spline Functions on Triangulations

Spline functions are universally recognized as highly effective tools in approximation theory, computer-aided geometric design, image analysis, and numerical analysis. The theory of univariate splines is well-known but this text is the first comprehensive treatment of the analogous multivariate theory.

A detailed mathematical treatment of polynomial splines on triangulations is presented, providing a basis for developing practical methods for using splines in numerous application areas. The treatment of the Bernstein-Bézier representation of polynomials will provide a valuable source for researchers and students in CAGD. Chapters on smooth macro-element spaces provide new tools to engineers and scientists for solving partial differential equations numerically. Workers in the geosciences will find the results on spherical splines on triangulations especially useful for approximation and data fitting on the sphere.

The book also includes a chapter on box splines, and four chapters on the latest research on trivariate splines.

This comprehensive book is ideal as a primary text for graduate courses in approximation theory, and as a source book for courses in computer-aided geometric design or in finite-element methods.

MING-JUN LAI is a Professor of Mathematics at the University of Georgia

LARRY SCHUMAKER is the Stevenson Professor of Mathematics at Vanderbilt University

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit http://www.cambridge.org/uk/series/sSeries.asp?code = EOM

68 R. Goodman and N. Wallach *Representations and Invariants of the Classical Groups*
69 T. Beth, D. Jungnickel, and H. Lenz *Design Theory I, 2nd edn*
70 A. Pietsch and J. Wenzel *Orthonormal Systems for Banach Space Geometry*
71 G. E. Andrews, R. Askey and R. Roy *Special Functions*
72 R. Ticciati *Quantum Field Theory for Mathematicians*
73 M. Stern *Semimodular Lattices*
74 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations I*
75 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations II*
76 A. A. Ivanov *Geometry of Sporadic Groups I*
77 A. Schinzel *Polynomials with Special Regard to Reducibility*
78 H. Lenz, T. Beth, and D. Jungnickel *Design Theory II, 2nd edn*
79 T. Palmer *Banach Algebras and the General Theory of* Algebras II*
80 O. Stormark *Lie's Structural Approach to PDE Systems*
81 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables*
82 J. P. Mayberry *The Foundations of Mathematics in the Theory of Sets*
83 C. Foias, O. Manley, R. Rosa and R. Temam *Navier–Stokes Equations and Turbulence*
84 B. Polster and G. Steinke *Geometries on Surfaces*
85 R. B. Paris and D. Kaminski *Asympiotics and Mellin–Barnes Integrals*
86 R. McEliece *The Theory of Information and Coding, 2nd edn*
87 B. Magurn *Algebraic Introduction to K-Theory*
88 T. Mora *Solving Polynomial Equation Systems I*
89 K. Bichteler *Stochastic Integration with Jumps*
90 M. Lothaire *Algebraic Combinatorics on Words*
91 A. A. Ivanov and S. V. Schpectorov *Geometry of Sporadic Groups II*
92 P. McMullen and E. Schulte *Abstract Regular Polytopes*
93 G. Giertz et al. *Continuous Lattices and Domains*
94 S. Finch *Mathematical Constants*
95 Y. Jabri *The Mountain Pass Theorem*
96 G. Gasper and M. Rahman *Basic Hypergeometric Series, 2nd edn*
97 M. C. Pedicchio and W. Tholen (eds.) *Categorical Foundations*
98 M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*
99 T. Mora *Solving Polynomial Equation Systems II*
100 E. Olivieri and M. Eulália Vares *Large Deviations and Metastability*
101 A. Kushner, V. Lychagin and V. Rubtsov *Contact Geometry and Nonlinear Differential Equations*
102 L. W. Beineke, R. J. Wilson, P. J. Cameron. (eds.) *Topics in Algebraic Graph Theory*
103 O. Staffans *Well-Posed Linear Systems*
104 J. M. Lewis. S. Lakshmivarahan and S. Dhall *Dynamic Data Assimilation*
105 M. Lothaire *Applied Combinatorics on Words*
106 A. Markoe *Analytic Tomography*
107 P. A. Martin *Multiple Scattering*
108 R. A. Brualdi *Combinatorial Matrix Classes*

Spline Functions on Triangulations

MING-JUN LAI

AND

LARRY L. SCHUMAKER

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521875929

© M. J. Lai and L. L. Schumaker 2007

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2007

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-87592-9 hardback

Cambridge University Press has no responsibility for
the persistence or accuracy of URLs for external or
third-party internet websites referred to in this publication,
and does not guarantee that any content on such
websites is, or will remain, accurate or appropriate.

Contents

Preface

The theory of univariate splines began its rapid development in the early sixties, resulting in several thousand research papers and a number of books. This development was largely over by 1980, and the bulk of what is known today was treated in the classic monographs of deBoor [Boo78] and Schumaker [Sch81]. Univariate splines have become an essential tool in a wide variety of application areas, and are by now a standard topic in numerical analysis books.

If 1960–1980 was the age of univariate splines, then the next twenty years can be regarded as the age of multivariate splines. Prior to 1980 there were some results for tensor-product splines, and engineers were using piecewise polynomials in two and three variables in the finite element method, but multivariate splines had attracted relatively little attention. Now we have an estimated 1500 papers on the subject.

The purpose of this book is to provide a comprehensive treatment of the theory of bivariate and trivariate polynomial splines defined on triangulations and tetrahedral partitions. We have been working on this book for more than ten years, and initially planned to include details on some of the most important applications, including for example CAGD, data fitting, surface compression, and numerical solution of partitial differential equations. But to keep the size of the book manageable, we have reluctantly decided to leave applications for another monograph.

For us, a multivariate spline is a function which is made up of pieces of polynomials defined on some partition $\triangle$ of a set Ω, and joined together to ensure some degree of global smoothness. We will focus primarily on the case where $\triangle$ is a triangulation of a planar region, a triangulation on the sphere, or a tetrahedral partition of a set Ω in $\mathbb{R}^3$.

The term "multivariate spline" has been used in the literature for other types of functions, see Remark 5.7 and the discussion in [Boo88] on "what is a multivariate spline?". Here we are following Schoenberg, who in 1966 discussed certain bivariate piecewise polynomials which he called splines. In particular, in the paper [CurS66] he and Curry examined certain analogs of the univariate B-spline. For some interesting correspondence involving these early developments, see the discussion in [Mic95].

As we shall see, multivariate polynomial splines have many of the same features which make the univariate splines such powerful tools for applications. In particular:

- splines are easy to work with computationally, and there are stable and efficient algorithms for evaluating their derivatives and integrals,
- there is a very convenient representation which provides a strong connection between the shape of a spline and its associated coefficients,

- splines are capable of approximating smooth functions well, and we can establish the exact relationship between the smoothness of a function and its order of approximation.

The book is organized as follows.

Chapter 1 is a self-contained treatment of bivariate polynomials. Of special interest here is the discussion of local approximation properties of polynomials. These are a form of Whitney theorems, and are the basis for our later treatment of the approximation power of bivariate spline spaces, as well as error bounds for various macro-element schemes. Interpolation with bivariate polynomials is also discussed here.

Chapter 2 deals with the Bernstein–Bézier representation of polynomials. This representation is the main tool for our theoretical developments, but is also critical for the efficient computational use of multivariate splines. In addition to introducing barycentric coordinates, Bernstein basis polynomials, and the B-form, we discuss derivatives, integrals, smoothness conditions, subdivision, degree raising, dual bases, quasi-interpolation, and the Bernstein operator. A thorough understanding of the notation and results of this chapter is an essential prerequisite to reading the rest of the book.

Chapter 3 should be of special interest to the computer-aided geometric design community as it contains a careful treatment of the connection between the shape of a polynomial surface patch and its associated set of B-coefficients. We discuss positivity, monotonicity, and convexity, as well as subdivision and degree raising as possible rendering schemes.

Chapter 4 introduces triangulations, and deals with their construction, storage, and combinatorics. Here we also discuss optimal triangulations and various refinement algorithms which are of particular importance for our later discussion of macro-element spaces, which are important tools for data fitting and the numerical solution of partial differential equations. This chapter also includes a discussion of triangulated quadrangulations.

Chapter 5 provides details on the Bernstein–Bézier approach to dealing with bivariate splines, and on methods for storing, evaluating, and rendering such splines. It also contains a discussion of the spline space $\mathcal{S}_d^0(\triangle)$, and introduces the important concept of minimal determining sets which is the key tool in discussing various properties of spline spaces, including dimensions, building local bases, and the construction of quasi-interpolation operators. The idea of nodal minimal determining sets is also introduced here. It is used heavily in Chapters 6–8 in the study of macro-element spaces.

Chapter 6 collects results on C^1 macro-element spaces associated with different splitting schemes. These spaces are particularly useful for applications since they have stable local bases and full approximation power.

Chapter 7 treats C^2 macro-element spaces associated with various splitting schemes. These spaces also have stable local bases and full approximation power.

Chapter 8 is concerned with families of C^r macro-element spaces based on various triangle splits including the Clough–Tocher, Powell–Sabin, and Powell–Sabin-12 splits, as well as certain splits based on quadrangulations. For each macro-element space we give both a stable local minimal determining set and a stable local nodal minimal determining set. In addition, for each element we construct a corresponding Hermite interpolation operator and give error bounds for it. The methods of this chapter and the previous two chapters have direct applications to scattered data fitting. Moreover, the macro-element spaces discussed here and in Chapters 6 and 7 can be used directly for the numerical solution of partial differential equations, and should be of special interest to the finite-element community.

Chapter 9 presents what is currently known about the dimension of bivariate spline spaces. For general triangulations and arbitrary smoothness r and degree d, we have to be satisfied with upper and lower bounds on dimension. However, exact dimension results are available for several important spaces including $\mathcal{S}_d^r(\triangle)$ for $d \geq 3r+2$. We also give results for fairly general superspline subspaces of $\mathcal{S}_d^r(\triangle)$. To get dimension statements for values of $d < 3r+2$, we have to restrict ourselves to special partitions. Here we give results for type-I and type-II partitions. In this chapter we also compute the generic dimension of the space $\mathcal{S}_3^1(\triangle)$. The problem of finding the dimension of $\mathcal{S}_3^1(\triangle)$ for arbitrary triangulations remains one of the most challenging open questions in bivariate spline theory.

Chapter 10 is devoted to the question of how well smooth functions can be approximated by bivariate splines on triangulations. In particular, we show that for $d \geq 3r+2$, the spaces $\mathcal{S}_d^r(\triangle)$ have full approximation order $d+1$, but have suboptimal approximation power for smaller d, and in fact for $d < (3r+2)/2$ have no approximation power at all.

Chapter 11 provides an explicit construction of stable local minimal determining sets for the space $\mathcal{S}_d^r(\triangle)$ for $d \geq 3r+2$ and for certain superspline subspaces of $\mathcal{S}_d^r(\triangle)$. These results ensure that at least for $d \geq 3r+2$, spline spaces on arbitrary triangulations are guaranteed to have full approximation power. The connection between stable local bases and local linear independence is also explored in this chapter.

Chapter 12 is devoted to a compact description of the theory of box splines as examples of polynomial spline spaces defined on special triangulations. Special emphasis is given to what we call type-I and type-II box splines. For more on box splines and related simplex splines, we recommend the survey articles of [DahM83] and [DaeL91], and the references therein. For a comprehensive monograph on box splines, see [BooHR93].

Chapter 13 contains a complete theory of certain spaces of splines defined on spherical triangulations introduced and studied extensively by Alfeld, Neamtu, and Schumaker. These splines are made up of pieces of trivariate homogeneous polynomials restricted to the sphere, and thus are actually piecewise spherical harmonics. The beauty of these spaces is the fact that the entire algebraic theory of bivariate splines can be carried over immediately. These spaces are valuable tools for fitting data and approximating functions defined on the sphere. In particular, there are spherical spline analogs of all of the bivariate macro-element spaces, which we expect will be useful numerical tools for the approximate solution of partial differential equations on the sphere.

Chapter 14 provides approximation results for spherical spline spaces. The key tool here is a certain radial projection mapping a spherical cap into a plane which is tangent to the sphere. The mapping provides a means of transferring results about bivariate splines to spherical splines.

Chapter 15 and the following three chapters are devoted to the theory of trivariate polynomial splines. This chapter lays the groundwork with a detailed discussion of trivariate polynomials paralleling our treatment of bivariate polynomials in Chapters 1 and 2. Of special importance is the discussion of trivariate Bernstein basis polynomials and the associated B-form of trivariate polynomials. A thorough understanding of the notation and results of this chapter is critical to the study of trivariate splines in the last three chapters of the book.

Chapter 16 can also be regarded as preparation for our treatment of trivariate splines. Here we introduce tetrahedral partitions, and discuss Euler relations, refinement methods, and properties of clusters.

Chapter 17 is our main chapter on trivariate splines defined over tetrahedral partitions. It contains all of the main features of our earlier development of the theory of bivariate splines, including minimal determining sets, stable local bases, dimension, and approximation power.

Chapter 18 is devoted to an exposition of the properties of several different C^1 and C^2 trivariate macro-element spaces which are suitable for trivariate data fitting and as approximating spaces for use in the finite element method. General C^r polynomial macro-element spaces are also treated.

Each chapter of the book includes a section with remarks, and a section with historical notes. We have collected most of the remarks in each chapter in a separate section at the end of the chapter with the aim of providing interesting and useful tangential information without interupting the flow of the book. We believe that historical notes are important to an understanding and appreciation of the development of this material, and we have made every effort to explain the history as accurately as possible.

We apologize in advance to anyone whose work has not been adequately acknowledged.

Having described the contents of the book, we now would like to say a few words about the list of references. Of the approximately 1500 papers on polynomial splines on triangulations and tetrahedral partitions, here we have listed only those works which are explicitly cited in this book. To find additional references, see the online bibliography of deBoor and Schumaker at www.math.vanderbilt.edu/∼schumake/splinebib.html.

To help the reader recognize both the authors and year of a citation, we have adopted a coding system which is based on the first three letters of the first author's name. When that is not unique, we add a fourth letter, and when there are co-authors, we add the first letter of the last name of each coauthor. Finally, each code word also includes the last two digits of the year of publication. Thus, for example, our joint paper on approximation by splines which appeared in 1998 is coded as [LaiS98].

Writing this book has been a long and arduous task. It began more than ten years ago when we realized that each of us had each prepared lecture notes on multivariate splines for classes that we were teaching at the time. Over the years, we have both taught the material again numerous times, and have benefited greatly from comments and suggestions from our many graduate students. We have also benefited from feedback from many of our colleagues. We do not attempt to list them all here, but we would like to mention especially Peter Alfeld, Oleg Davydov, Manfred von Golitschek, Tom Lyche, and Frank Zeilfelder, who not only frequently discussed the material in this book with us, but also commented on several drafts and provided corrections. Our special thanks go to Simon Foucart for his careful reading of the final manuscript. The first author would like to thank Charles Chui for introducing him to spline functions more than twenty years ago, and for his continued support.

Finally, we would like to thank our families for their patience over the years. This project has taken innumerable hours that could have been spent with them, and we gratefully acknowledge their support and understanding.

Bivariate Polynomials

In this chapter we discuss bivariate polynomials and their approximation and interpolation properties.

1.1. Introduction

Given a nonnegative integer d, throughout Chapters 1–12, we write $\mathcal{P}_d$ for the space of bivariate polynomials of degree d, i.e., the linear space of all real-valued functions of the form

$$p(x,y) := \sum_{0\le i+j\le d} c_{ij} x^i y^j, \tag{1.1}$$

where $\{c_{ij}\}_{0\le i+j\le d}$ are real numbers. It is easy to see that the monomials

$$\{x^i y^j\}_{0\le i+j\le d} \tag{1.2}$$

form a basis for $\mathcal{P}_d$. Indeed, to check that they are linearly independent, note that if

$$p(x,y) = \sum_{0\le i+j\le d} c_{ij} x^i y^j = 0, \qquad \text{all } (x,y) \in \mathbb{R}^2,$$

then $D_x^\alpha D_y^\beta p(0,0) = \alpha!\,\beta!\,c_{\alpha\beta} = 0$ for each $0 \le \alpha+\beta \le d$. Arranging the monomials in the lexicographical order

$$1,\quad \underbrace{x, y}, \quad \underbrace{x^2, xy, y^2}, \quad \ldots, \quad \underbrace{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d}, \tag{1.3}$$

we immediately see that the dimension of $\mathcal{P}_d$ is $1+2+\cdots+(d+1) = \binom{d+2}{2}$. In Chapter 2 we shall construct a different basis for $\mathcal{P}_d$ which is far more useful for our purposes.

1.2. Norms of Polynomials on Triangles

Given any domain Ω in $\mathbb{R}^2$, we define the usual ∞-norm of a function by

$$\|f\|_\Omega := \operatorname{ess\,sup}_{u\in\Omega} |f(u)|.$$

If f is continuous on Ω, we can replace the essential supremum by the maximum. For $1 \le q < \infty$, we define the usual q-norm by

$$\|f\|_{q,\Omega} := \Big[\int_\Omega |f(u)|^q \, du \Big]^{1/q}.$$

In the sequel we shall frequently work with norms of polynomials on triangles. In particular, we need the following result connecting the q-norms and the ∞-norms of polynomials.

Theorem 1.1. *Let T be a triangle, and let A_T be its area. Then for all $p \in \mathcal{P}_d$ and all $1 \le q < \infty$,*

$$A_T^{-1/q} \|p\|_{q,T} \le \|p\|_T \le K\, A_T^{-1/q} \|p\|_{q,T}, \tag{1.4}$$

where K is a constant depending only on d.

Proof: The first inequality follows immediately from the definition of the norms. To prove the second inequality, consider the standard triangle $\widetilde{T} = \{(x,y) : 0 \le x, y \le 1, x+y \le 1\}$. Since all norms on the finite dimensional space of polynomials are equivalent, it follows that

$$\|g\|_{\widetilde{T}} \le K\, \|g\|_{q,\widetilde{T}},$$

for all polynomials $g \in \mathcal{P}_d$, where K is a constant depending only on d. A change of variables maps any polynomial $p \in \mathcal{P}_d$ into a polynomial $g \in \mathcal{P}_d$ with $\|g\|_{\widetilde{T}} = \|p\|_T$ and $\|g\|_{q,\widetilde{T}} = A_T^{-1/q} \|p\|_{q,T}$, and the second inequality in (1.4) follows. □

1.3. Derivatives of Polynomials

If $p \in \mathcal{P}_d$, then its partial derivative $D_x^\alpha D_y^\beta p$ belongs to $\mathcal{P}_{d-\alpha-\beta}$. In particular, if p is written in the form (1.1), then

$$D_x^\alpha D_y^\beta p = \sum_{0 \le i+j \le d-\alpha-\beta} d_{ij} x^i y^j,$$

with

$$d_{ij} := \frac{(i+\alpha)\,!}{i\,!} \frac{(j+\beta)\,!}{j\,!} c_{i+\alpha, j+\beta},$$

for $0 \le i+j \le d-\alpha-\beta$.

In the following sections, we will need a version of the Markov inequality comparing the size of the derivative of a polynomial to the size of the polynomial itself, measured on a given triangle T. Let ρ_T be the radius of the largest disk contained in T.

Theorem 1.2. *There exists a constant K depending only on d such that for every polynomial $p \in \mathcal{P}_d$,*

$$\|D_x^\alpha D_y^\beta p\|_{q,T} \le \frac{K}{\rho_T^{\alpha+\beta}} \|p\|_{q,T}, \quad 0 \le \alpha + \beta \le d, \tag{1.5}$$

for all $1 \le q \le \infty$.

This theorem was proved for $q = \infty$ in [Coa66]. For a different proof, see [Wil74]. We give a proof for general $1 \le q \le \infty$ in Section 2.12.

We also need to work with directional derivatives. Given a vector $u := (u_x, u_y)$, the associated directional derivative of a function f is defined by

$$D_u f(x,y) = \frac{d}{dt} f\big(x + tu_x, y + tu_y)\big)\Big|_{t=0}. \tag{1.6}$$

It is well known from calculus that

$$D_u f(x,y) = u_x D_x f(x,y) + u_y D_y f(x,y).$$

This shows that the directional derivative of a polynomial of degree d is a polynomial of degree $d-1$.

1.4. Polynomial Approximation in the Maximum Norm

Our aim in this section is to give a bound on how well a smooth function defined on a set $\Omega \subset \mathbb{R}^2$ can be approximated by a polynomial of degree d, measured in the maximum norm. To describe the smoothness of f, we make use of the seminorms

$$|f|_{m+1,\Omega} := \max_{i+j=m+1} \|D_x^i D_y^j f\|_\Omega. \tag{1.7}$$

Let

$$|\Omega| := \max_{v,w \in \Omega} |v - w| \tag{1.8}$$

be the diameter of Ω.

Theorem 1.3. *Suppose that Ω is the closure of a convex domain in $\mathbb{R}^2$. Then for every $f \in C^{d+1}(\Omega)$, there exists a polynomial $p_f \in \mathcal{P}_d$ such that*

$$\|D_x^\alpha D_y^\beta (f - p_f)\|_\Omega \le K\, |\Omega|^{d+1-\alpha-\beta}\, |f|_{d+1,\Omega}, \tag{1.9}$$

for all $0 \le \alpha + \beta \le d$. The constant K depends only on d.

Proof: Let (x_c, y_c) be the center of the largest disk contained in Ω, and let

$$T_d f(x,y) := \sum_{0 \le i+j \le d} \frac{1}{i!\,j!} D_x^i D_y^j f(x_c, y_c)(x - x_c)^i (y - y_c)^j$$

be the Taylor expansion of f about the point (x_c, y_c) with remainder

$$f(x,y) - T_d f(x,y) = \sum_{i+j=d+1} \frac{1}{i!\,j!} D_x^i D_y^j f(\tilde{x}, \tilde{y})(x - x_c)^i (y - y_c)^j,$$

where $(\tilde{x}, \tilde{y})$ is some point on the line between (x,y) and (x_c, y_c). Taking the norm over Ω immediately gives (1.9) for $\alpha = \beta = 0$. The general result follows from the fact that for any $0 \le \alpha + \beta \le d$,

$$D_x^\alpha D_y^\beta\, T_d\, f = T_{d-\alpha-\beta}\, D_x^\alpha D_y^\beta\, f. \quad \square$$

Theorem 1.3 provides a bound on the distance of f to $\mathcal{P}_d$. In particular, if $f \in C^{d+1}(\Omega)$, then

$$d(f, \mathcal{P}_d)_\Omega := \inf_{p \in \mathcal{P}_d} \|f - p\|_\Omega = \mathcal{O}(|\Omega|^{d+1}). \tag{1.10}$$

It is known (see Remark 1.4) that (1.10) is best possible in the sense that no matter how smooth f may be, the exponent cannot be increased. For an analog of Theorem 1.3 which holds for nonconvex sets Ω, see Section 1.7.

1.5. Averaged Taylor Polynomials

Let $B := B(u_0, v_0, \rho) := \{(x, y) : (x - u_0)^2 + (y - v_0)^2 \le \rho^2\}$ be a disk in $\mathbb{R}^2$ of radius ρ with center (u_0, v_0). Let

$$g_B(u, v) := \begin{cases} ce^{-\rho^2/(\rho^2 - (u-u_0)^2 - (v-v_0)^2)}, & (u, v) \in B(u_0, v_0, \rho), \\ 0, & \text{otherwise}, \end{cases}$$

where c is chosen so that

$$\int_B g_B(u, v)\, du\, dv = 1.$$

Definition 1.4. *Given an integrable function $f \in L_1(B(x, y, \rho))$, let*

$$\begin{aligned} F_{d,B} f(x, y) := \sum_{0 \le i+j \le d} (-1)^{i+j} \frac{1}{i!\, j!} \int_{B(u_0, v_0, \rho)} & f(u, v) \\ & \times D_u^i D_v^j \left[(x - u)^i (y - v)^j g_B(u, v) \right] du\, dv. \end{aligned} \tag{1.11}$$

We call $F_{d,B} f$ the averaged Taylor polynomial of degree d relative to B associated with f.

Clearly, $F_{d,B} f$ is a bivariate polynomial of degree at most d. If $f \in C^d(\mathbb{R}^2)$, then integrating (1.11) by parts shows that

$$F_{d,B} f(x, y) = \int_{B(u_0, v_0, \rho)} T_{d,(u,v)} f(x, y)\, g_B(u, v)\, du\, dv, \tag{1.12}$$

where

$$T_{d,(u,v)} f(x, y) := \sum_{0 \le i+j \le d} \frac{1}{i!\, j!} D_u^i D_v^j f(u, v) (x - u)^i (y - v)^j$$

is the ordinary Taylor polynomial of degree d of f centered at (u, v). The following result makes use of the Sobolev spaces W_q^m defined in Section 1.6.

Lemma 1.5. *For every $0 \le \alpha + \beta \le d$ and $f \in W_1^{\alpha+\beta}(B(u_0, v_0, \rho))$,*

$$D_x^\alpha D_y^\beta F_{d,B} f = F_{d-\alpha-\beta,B}(D_x^\alpha D_y^\beta f).$$

Moreover, $p = F_{d,B}\, p$ for every polynomial $p \in \mathcal{P}_d$.

Proof: For the first assertion, we have

$$\begin{aligned} D_x^\alpha D_y^\beta F_{d,B} f(x,y) &= \int_{B(u_0,v_0,\rho)} D_x^\alpha D_y^\beta T_{d,(u,v)} f(x,y)\, g_B(u,v)\, du\, dv \\ &= \int_{B(u_0,v_0,\rho)} T_{d-\alpha-\beta,(u,v)} D_x^\alpha D_y^\beta f(x,y)\, g_B(u,v)\, du\, dv \\ &= F_{d-\alpha-\beta,B}(D_x^\alpha D_y^\beta f)(x,y). \end{aligned}$$

For the second assertion, we recall the following formula for the exact remainder of the classical Taylor polynomial:

$$\begin{aligned} &f(x,y) - T_{d,(u,v)} f(x,y) \\ &= (d+1) \sum_{\alpha+\beta=d+1} \frac{(x-u)^\alpha (y-v)^\beta}{\alpha!\, \beta!} \int_0^1 D_1^\alpha D_2^\beta f((x,y) + t(u-x, v-y))\, t^d\, dt. \end{aligned}$$

Here the differential operators D_1 and D_2 denote differentiation with respect to the first and second variables, respectively. This implies that

$$\begin{aligned} &f(x,y) - F_{d,B} f(x,y) \\ &= \int_{B(u_0,v_0,\rho)} \big[f(x,y) - T_{d,(u,v)} f(x,y)\big]\, g_B(u,v)\, du\, dv \\ &= \sum_{\alpha+\beta=d+1} \frac{d+1}{\alpha!\, \beta!} \int_{B(u_0,v_0,\rho)} \int_0^1 g_B(u,v)(x-u)^\alpha (y-v)^\beta \\ &\qquad \times D_1^\alpha D_2^\beta f((x,y) + t(u-x, v-y))\, t^d\, dt\, du\, dv, \end{aligned} \tag{1.13}$$

which immediately implies the second assertion. $\square$

Now suppose Ω is the closure of a convex domain in $\mathbb{R}^2$, and let B_Ω be the largest disk that can be inscribed in Ω. Then the associated averaged Taylor expansion F_{d,B_Ω} maps functions defined on B_Ω into polynomials of degree d. We now give a bound on the size of $F_{d,B_\Omega} f$ in terms of the size of f. The bound will depend on the shape of Ω, as measured by

$$\kappa_\Omega := \frac{|\Omega|}{\rho_\Omega}, \tag{1.14}$$

where ρ_Ω is the radius of B_Ω and $|\Omega|$ is the diameter of Ω as defined in (1.8). This ratio can be large if the set Ω is long and thin.

Lemma 1.6. *There exists a constant K depending only on d and κ_Ω such that*

$$||F_{d,B_\Omega} f||_{q,\Omega} \le K\, ||f||_{q,B_\Omega}, \tag{1.15}$$

for all $f \in L_q(B_\Omega)$ with $1 \le q \le \infty$.

Proof: It is easy to check that there exists a constant K_1 depending only on d such that

$$\|D_u^\alpha D_v^\beta g_{B_\Omega}\|_{L_\infty(\mathbb{R}^2)} \le \frac{K_1}{\rho_\Omega^{\alpha+\beta+2}},$$

for all nonnegative integers $\alpha, \beta \le d$. For fixed $(x,y) \in \Omega$, by the Leibniz formula and the definition of κ_Ω,

$$\begin{aligned}
&|D_u^\alpha D_v^\beta (x-u)^\alpha (y-v)^\beta g_{B_\Omega}(u,v)| \\
&\quad \le \sum_{\substack{i\le\alpha \\ j\le\beta}} \left[\binom{\alpha}{i}\binom{\beta}{j}\right]^2 i\,!\, j\,! \left|(x-u)^{\alpha-i}(y-v)^{\beta-j} D_u^{\alpha-i} D_v^{\beta-j} g_{B_\Omega}(u,v)\right| \\
&\quad \le \sum_{\substack{i\le\alpha \\ j\le\beta}} \left[\binom{\alpha}{i}^2\binom{\beta}{j}\right]^2 i\,!\, j\,!\, |\Omega|^{\alpha-i+\beta-j} \frac{K_1}{\rho_\Omega^{\alpha-i+\beta-j+2}} \le \frac{K_2}{\rho_\Omega^2},
\end{aligned}$$

for every $(u,v) \in \mathbb{R}^2$. Given $1 \le q < \infty$, let $1/q + 1/\tilde q = 1$. Then for all $f \in L_q(B_\Omega)$, we have

$$\begin{aligned}
&\|F_{d,B_\Omega} f\|_{q,\Omega} \\
&\quad \le \sum_{\alpha+\beta\le d} \frac{1}{\alpha\,!\,\beta\,!} \left\| \int_{B_\Omega} f(u,v) D_u^\alpha D_v^\beta \left[(x-u)^\alpha (y-v)^\beta g_{B_\Omega}(u,v)\right] du\,dv \right\|_{q,\Omega} \\
&\quad \le \sum_{\alpha+\beta\le d} \frac{1}{\alpha\,!\,\beta\,!} \Bigg\| \left(\int_{B_\Omega} |f(u,v)|^q\, du\,dv \right)^{1/q} \\
&\qquad\qquad \times \left(\int_{B_\Omega} |D_u^\alpha D_v^\beta (x-u)^\alpha (y-v)^\beta g_{B_\Omega}(u,v)|^{\tilde q}\, du\,dv \right)^{1/\tilde q} \Bigg\|_{q,\Omega} \\
&\quad \le \sum_{\alpha+\beta\le d} \frac{1}{\alpha\,!\,\beta\,!} ||f||_{q,B_\Omega} \left[\int_\Omega \left(\int_{B_\Omega} \left(K_2 \frac{1}{\rho_\Omega^2}\right)^{\tilde q} du\,dv \right)^{q/\tilde q} dx\,dy \right]^{1/q} \\
&\quad \le K_3 ||f||_{q,B_\Omega} \left[\left(\rho_\Omega^{-2\tilde q} \pi \rho_\Omega^2\right)^{q/\tilde q} |\Omega|^2 \right]^{1/q} \\
&\quad \le K_4 ||f||_{q,B_\Omega}.
\end{aligned}$$

Since K_4 depends only on d and κ_Ω, this completes the proof for $1 \le q < \infty$. The proof for $q = \infty$ is similar and simpler. $\square$

1.6. Polynomial Approximation in the q-Norm

In this section we give bounds on how well functions in Sobolev spaces can be approximated by polynomials, measured in a q-norm. Throughout this section we suppose that Ω is the closure of a convex domain in $\mathbb{R}^2$. We extend the results to the case where Ω is nonconvex in the following section. Suppose $1 \le q \le \infty$ and $0 \le d$. Then the associated Sobolev space is defined by

$$W_q^{d+1}(\Omega) := \{f : \|f\|_{d+1,q,\Omega} < \infty\},$$

where

$$\|f\|_{d+1,q,\Omega} := \begin{cases} \left(\displaystyle\sum_{k=0}^{d+1} |f|_{k,q,\Omega}^q \right)^{1/q}, & 1 \le q < \infty \\ \displaystyle\sum_{k=0}^{d+1} |f|_{k,\infty,\Omega}, & q = \infty, \end{cases}$$

with

$$|f|_{k,q,\Omega} := \begin{cases} \left(\displaystyle\sum_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{q,\Omega}^q \right)^{1/q}, & 1 \le q < \infty \\ \displaystyle\max_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{\infty,\Omega}, & q = \infty. \end{cases}$$

Suppose B_Ω is the largest closed disk that can be inscribed in Ω. Given a function f in $L_1(\Omega)$, let $F_{d,B_\Omega} f$ be the associated averaged Taylor polynomial defined in the previous section. We now give an error bound for how well $F_{d,B_\Omega} f$ approximates f on Ω. Let κ_Ω be as in (1.14).

Theorem 1.7. *Let Ω be the the closure of a convex domain in $\mathbb{R}^2$, and let $d \ge 0$. Then there exists a positive constant K depending only on d and κ_Ω such that for every $f \in W_q^{d+1}(\Omega)$ with $1 \le q \le \infty$,*

$$\|D_x^\alpha D_y^\beta (f - F_{d,B_\Omega} f)\|_{q,\Omega} \le K\,|\Omega|^{d+1-\alpha-\beta}\,|f|_{d+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le d$.

Proof: We need only prove

$$||f - F_{d,B_\Omega} f||_{q,\Omega} \le K\,|\Omega|^{d+1} |f|_{d+1,q,\Omega}, \tag{1.16}$$

since by Lemma 1.5,

$$||D_x^\alpha D_y^\beta (f - F_{d,B_\Omega} f)||_{q,\Omega} = ||D_x^\alpha D_y^\beta f - F_{d-\alpha-\beta,B_\Omega}(D_x^\alpha D_y^\beta f)||_{q,\Omega}.$$

To establish (1.16) we use the formula (1.13). We need an estimate of

$$\int_{B_\Omega} \int_0^1 g_{B_\Omega}(u,v)(x-u)^\alpha (y-v)^\beta D_1^\alpha D_2^\beta f((x,y) + t(u-x, v-y))\, t^d \,dt\,du\,dv$$

for every $(x, y) \in \Omega$. Let $(\mu, \nu) = (x, y) + t(u - x, v - y)$. Then $d\mu d\nu dt = t^2 du dv dt$. Let

$$G := \left\{(\mu, \nu, t) : t \in (0, 1],\ \left|\frac{(\mu, \nu) - (x, y)}{t} + (x - x_0, y - y_0)\right| \leq \rho_\Omega\right\},$$

where (x_0, y_0) is the center of the disk B_Ω. Then for $(u, v, t) \in B_\Omega \times (0, 1]$, $(\mu, \nu, t) \in G$. Since

$$\frac{\sqrt{(\mu - x)^2 + (\nu - y)^2}}{t} < \rho_\Omega + \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

we have

$$t_0(\mu, \nu) := \frac{\sqrt{(\mu - x)^2 + (\nu - y)^2}}{\rho_\Omega + \sqrt{(x - x_0)^2 + (y - y_0)^2}} < t.$$

Thus, letting χ_G be the characteristic function of G, we have

$$\begin{aligned}
&\int_{B_\Omega}\int_0^1 g_{B_\Omega}(u, v)(x - u)^\alpha (y - v)^\beta D_1^\alpha D_2^\beta f((x, y) + t(u - x, v - y))t^d \, dt\, du\, dv \\
&= \int_G g_{B_\Omega}\Big(\frac{(\mu - x, \nu - y)}{t} + (x, y)\Big)(x - \mu)^\alpha (y - \nu)^\beta D_1^\alpha D_2^\beta f(\mu, \nu) t^{-3}\, dt\, d\mu\, d\nu \\
&= \int_{\langle (x,y), B_\Omega\rangle} \Big[(x - \mu)^\alpha (y - \nu)^\beta D_\mu^\alpha D_\nu^\beta f(\mu, \nu) \\
&\qquad \times \int_0^1 \chi_G(\mu, \nu, t) g_{B_\Omega}((x, y) + (\mu - x, \nu - y)/t)t^{-3} dt\Big]\, d\mu\, d\nu,
\end{aligned}$$

where $\langle (x, y), B_\Omega\rangle$ denotes the convex hull of (x, y) and B_Ω. Note that

$$\begin{aligned}
&\Big|\int_0^1 \chi_G(\mu, \nu, t) g_{B_\Omega}((x, y) + (\mu - x, \nu - y)/t)t^{-3} dt\Big| \leq \frac{K_1}{\rho_\Omega^2}\int_{t_0(\mu,\nu)}^1 t^{-3} dt \\
&= \frac{K_1}{2\rho_\Omega^2}\left(\frac{\big(\rho_\Omega + \sqrt{(x - x_0)^2 + (y - y_0)^2}\big)^2}{(\mu - x)^2 + (\nu - y)^2} - 1\right) \\
&\leq \frac{K_1}{2}\left(1 + \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\rho_\Omega}\right)^2 \big((\mu - x)^2 + (\nu - y)^2\big)^{-1}.
\end{aligned}$$

By (1.14),

$$\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\rho_\Omega} \leq \kappa_\Omega.$$

Now suppose $1 \le q < \infty$ and $1/q + 1/\tilde{q} = 1$. Then

$$
\begin{aligned}
&||f - F_{d,B_\Omega} f||_{q,\Omega} \\
&\quad \le \frac{K_1}{2}(1+\kappa_\Omega)^2 \sum_{\alpha+\beta=d+1} \frac{(d+1)}{\alpha!\,\beta!} \Big\| \int_{\langle (x,y), B_\Omega \rangle} |D_\mu^\alpha D_\nu^\beta f(\mu,\nu)| \\
&\qquad\qquad\qquad \times [(x-\mu)^2 + (y-\nu)^2)]^{(d-1)/2} \, d\mu d\nu \Big\|_{q,\Omega} \\
&\quad \le K \sum_{\alpha+\beta=d+1} \frac{(d+1)}{\alpha!\,\beta!} \Big[\int_\Omega \Big(\int_\Omega |D_\mu^\alpha D_\nu^\beta f(\mu,\nu)| \\
&\qquad\qquad\qquad \times \; [(x-\mu)^2 + (y-\nu)^2]^{(d-1)/2} d\mu d\nu \Big)^q dx\, dy \Big]^{1/q} \\
&\quad \le K \sum_{\alpha+\beta=d+1} \Big[\int_\Omega ||D_\mu^\alpha D_\nu^\beta f||_{q,\Omega}^q \Big(\int_\Omega |\Omega|^{(d-1)\tilde{q}} d\mu d\nu \Big)^{q/\tilde{q}} dx\, dy \Big]^{1/q} \\
&\quad \le K \, |f|_{d+1,q,\Omega} \Big[\Big(|\Omega|^{(d-1)\tilde{q}+2} \Big)^{q/\tilde{q}} |\Omega|^2 \Big]^{1/q} \\
&\quad = K \, |\Omega|^{d+1} \, |f|_{d+1,q,\Omega}.
\end{aligned}
$$

where the constant $K := K_1(1+\kappa_\Omega)^2/2$ depends only on d and κ_Ω. This completes the proof for $1 \le q < \infty$. The proof for $q = \infty$ is similar and simpler. $\square$

Theorem 1.7 provides a bound on the L_q-distance of f to $\mathcal{P}_d$. In particular, if $f \in C^{d+1}(\Omega)$, then

$$
d(f, \mathcal{P}_d)_{q,\Omega} := \inf_{p \in \mathcal{P}_d} \|f - p\|_{q,\Omega} = \mathcal{O}(|\Omega|^{d+1}). \tag{1.17}
$$

It is known, see Remark 1.5, that (1.17) is best possible in the sense that no matter how smooth f may be, the exponent cannot be increased.

1.7. Approximation on Nonconvex Ω

In this section we extend Theorem 1.7 to the case where Ω is the closure of an arbitrary bounded (not necessarily convex) domain. We begin by recalling an extension theorem due to Stein [Ste70, p. 181] which applies to bounded domains with Lipschitz smooth boundaries.

Theorem 1.8. *Let Ω be a bounded domain with a Lipschitz smooth boundary. Then there exists a linear extension operator E extending functions from Ω to the convex hull $\mathrm{co}(\Omega)$ of Ω so that for all $f \in W_q^{d+1}(\Omega)$, $E(f)|_\Omega = f$, and*

$$
\|D_x^\alpha D_y^\beta E(f)\|_{q,\mathrm{co}(\Omega)} \le K \, \|D_x^\alpha D_y^\beta f\|_{q,\Omega},
$$

$1 \le q \le \infty$, and all $0 \le \alpha+\beta \le d+1$. The constant K depends only on d, q, and the Lipschitz constant of the boundary of Ω.

Here we are primarily interested in the case where Ω is a polygonal domain, i.e., a bounded domain with a polygonal boundary. In this case the Lipschitz constant of the boundary of Ω depends on the size of the smallest angle (which may be either inside or outside of Ω) between successive edges of the boundary of Ω.

Theorem 1.9. *Let Ω be the closure of a bounded domain in $\mathbb{R}^2$, and let B_Ω be the largest disk contained in $\mathrm{co}(\Omega)$, where $\mathrm{co}(\Omega)$ is the convex hull of Ω. Fix $d \ge 0$ and $1 \le q \le \infty$. Given $f \in W_q^{d+1}(\Omega)$, let $F_{d,B_\Omega} f$ be the associated averaged Taylor polynomial. Then*

$$\|D_x^\alpha D_y^\beta \big(f - F_{d,B_\Omega} f\big)\|_{q,\Omega} \le K\, |\Omega|^{d+1-\alpha-\beta}\, |f|_{d+1,q,\Omega}, \tag{1.18}$$

for all $0 \le \alpha+\beta \le d+1$. If Ω is convex, the constant K depends only on d and the shape parameter κ_Ω defined in (1.14). If Ω is not convex, K also depends on q and the Lipschitz constant of the boundary of Ω.

Proof: After extending f to $\mathrm{co}(\Omega)$ using Theorem 1.8, the result follows immediately from Theorem 1.7. □

1.8. Interpolation by Bivariate Polynomials

Since the linear space $\mathcal{P}_d$ of polynomials of degree d has dimension $n := \binom{d+2}{2}$, it is natural to try to use polynomials $p \in \mathcal{P}_d$ to interpolate prescribed real values $\{z_j\}_{j=1}^n$ at n given points $A := \{t_j\}_{j=1}^n$ in $\mathbb{R}^2$. However, as we shall see, this is not possible for all sets A.

Suppose $\{g_i\}_{i=1}^n$ is the set of monomials (1.2), arranged in the lexicographical order (1.3). Then the interpolation problem

$$p(t_i) = z_i, \qquad i = 1, \ldots, n, \tag{1.19}$$

amounts to finding coefficients $\{c_j\}_{j=1}^n$ such that

$$\sum_{j=1}^n c_j g_j(t_i) = z_i, \qquad i = 1, \ldots, n.$$

This system will have a unique solution whenever the matrix

$$M := \big[g_j(t_i)\big]_{i,j=1}^n \tag{1.20}$$

is nonsingular.

In contrast to the univariate case where interpolation at $d+1$ distinct points by a polynomial of degree d is always uniquely defined, in the bivariate case whether or not we can interpolate arbitrary values at the points in A depends on the *location* of the points. For some locations of the points in A, the matrix M in (1.20) is singular. For example, let $d = 1$, and suppose that A consists of three points $t_i = (x_i, y_i)$, $i = 1, 2, 3$, which lie on a straight line $y = ax + b$. Then

$$M = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & ax_1 + b \\ 1 & x_2 & ax_2 + b \\ 1 & x_3 & ax_3 + b \end{bmatrix},$$

which is clearly singular.

A set of points A is call poised with respect to $\mathcal{P}_d$ provided that the matrix M is nonsingular. The following theorem gives *sufficient* conditions on the locations of the points in a set A in order for it to be poised.

Theorem 1.10. *Given d, let $n := \binom{d+2}{2}$. Suppose that $A = \{t_i\}_{i=1}^n := \bigcup_{i=1}^{d+1} \{t_{ij}\}_{j=1}^i$ is a set of distinct points in $\mathbb{R}^2$ such that for some collection $\{L_i\}_{i=1}^{d+1}$ of distinct lines in the plane, for each $i = 1, \ldots, d+1$, the points $\{t_{ij}\}_{j=1}^i$ lie on L_i but not on $L_{i+1} \cup \cdots \cup L_{d+1}$. Then A is poised with respect to $\mathcal{P}_d$.*

Proof: We proceed by induction on d. If $d = 0$ the result is trivial. Now suppose the result holds for $d-1$. Let M be the matrix in (1.20), and let $c := (c_1, \ldots, c_n)$ be a solution of the system $Mc = 0$. To show that M is nonsingular, it suffices to show that c must be zero. This is equivalent to showing that if $p \in \mathcal{P}_d$ satisfies

$$p(t_i) = 0, \quad i = 1, \ldots, n, \tag{1.21}$$

then p is the identically zero polynomial. For each $i = 1, \ldots, d+1$, let $\alpha_i x + \beta_i y = \gamma_i$ be the equation of the line L_i.

Suppose now that p satisfies (1.21). Since p reduces to a univariate polynomial of degree d on L_{d+1} that vanishes at the $d+1$ distinct points $\{t_{d+1,j}\}_{j=1}^{d+1}$ on L_{d+1}, it follows that p vanishes identically on L_{d+1}, and so

$$p(x, y) = (\alpha_{d+1} x + \beta_{d+1} y - \gamma_{d+1})\, q(x, y),$$

where q is a polynomial of degree $d-1$. But now since none of the points in $\tilde{A} := \bigcup_{i=1}^d \{t_{ij}\}_{j=1}^i$ lie on L_{d+1}, we see that q vanishes at all of the points in $\tilde{A}$, and by the inductive hypothesis must be identically zero. But then $p \equiv 0$, and the proof is complete. $\square$

The following special case of Theorem 1.10 will be of interest later.

Theorem 1.11. *Let T be a triangle with vertices v_1, v_2, v_3, and let*

$$\xi_{ijk} := \frac{(iv_1 + jv_2 + kv_3)}{d}, \quad i+j+k=d. \tag{1.22}$$

Then the set of $\binom{d+2}{2}$ points $A := \{\xi_{ijk}\}_{i+j+k=d}$ is poised with respect to $\mathcal{P}_d$.

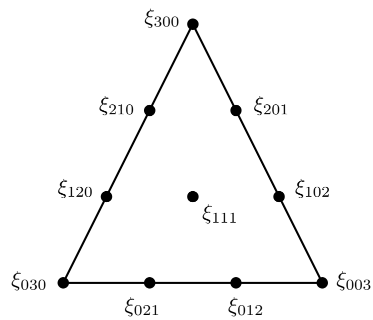


Fig. 1.1. The points in Theorem 1.11 for $d = 3$.

Figure 1.1 shows the locations of the points in Theorem 1.11 for $d = 3$. We now show how to explicitly express the polynomial that interpolates given data at the points ξ_{ijk} of (1.22) in terms of the Lagrange polynomials

$$p_{ijk}(v) := \prod_{\mu=0}^{i-1} \frac{a_\mu(v)}{a_\mu(\xi_{ijk})} \prod_{\nu=0}^{j-1} \frac{b_\nu(v)}{b_\nu(\xi_{ijk})} \prod_{\kappa=0}^{k-1} \frac{c_\kappa(v)}{c_\kappa(\xi_{ijk})}, \tag{1.23}$$

where a_μ, b_ν, c_κ are linear polynomials such that

$$\begin{aligned} a_\mu(v) &\sim \text{the line passing through the points } \xi_{\mu jk} \text{ with } \mu+j+k=d, \\ b_\nu(v) &\sim \text{the line passing through the points } \xi_{i\nu k} \text{ with } i+\nu+k=d, \\ c_\kappa(v) &\sim \text{the line passing through the points } \xi_{ij\kappa} \text{ with } i+j+\kappa=d. \end{aligned}$$

Here we are using the convention that a product appearing in (1.23) is defined to be 1 when the upper limit is negative. It is clear by construction that for all $i+j+k=d$, p_{ijk} is a polynomial of degree d that satisfies

$$p_{ijk}(\xi_{\nu\mu\kappa}) = \begin{cases} 1, & (\nu,\mu,\kappa) = (i,j,k), \\ 0, & \text{all other } \nu+\mu+\kappa=d. \end{cases} \tag{1.24}$$

This property of the p_{ijk} immediately implies that for every $\{z_{ijk}\}_{i+j+k=d}$, the unique polynomial of degree d that interpolates z_{ijk} at the points ξ_{ijk} is

$$p := \sum_{i+j+k=d} z_{ijk} p_{ijk}. \tag{1.25}$$

We now give a bound on how well this interpolating polynomial approximates smooth functions and their derivatives. Given a triangle T, let

$$\begin{aligned} \rho_{_T} &:= \text{the radius of the largest disk contained in } T, \\ |T| &:= \text{the length of the longest edge of } T, \\ \theta_{_T} &:= \text{the smallest angle in the triangle } T. \end{aligned} \tag{1.26}$$

By elementary trigonometry, it is easy to see that the associated shape parameter (1.14) satisfies

$$\kappa_{_T} := \frac{|T|}{\rho_{_T}} \le \frac{2}{\tan(\theta_{_T}/2)} \le \frac{2}{\sin(\theta_{_T}/2)}. \tag{1.27}$$

Theorem 1.12. *Let $\{\xi_{ijk}\}_{i+j+k=d}$ be the points defined in (1.22), and let $\{p_{ijk}\}_{i+j+k=d}$ be the corresponding Lagrange polynomials (1.23). Then there exists a constant K depending only on d and $\theta_{_T}$ such that for every $f \in C^{m+1}(T)$ with $0 \le m \le d$, the interpolating polynomial*

$$p_f := \sum_{i+j+k=d} f(\xi_{ijk})\, p_{ijk} \tag{1.28}$$

satisfies

$$\|D_x^\alpha D_y^\beta (f - p_f)\|_{_T} \le K\ |T|^{m+1-\alpha-\beta}\ |f|_{m+1,T}, \tag{1.29}$$

for all $0 \le \alpha + \beta \le m$.

Proof: Fix $(x,y) \in T$, and let $\xi_{ijk} = (\xi_{ijk}^x, \xi_{ijk}^y)$ for all $i+j+k=d$. Using the Taylor expansion of f about (x,y), we have

$$\begin{aligned} f(\xi_{ijk}) = &\sum_{0 \le \mu+\nu \le m} \frac{1}{\mu!\,\nu!} D_x^\mu D_y^\nu f(x,y)\,(\xi_{ijk}^x - x)^\mu (\xi_{ijk}^y - y)^\nu \\ &+ \sum_{\mu+\nu=m+1} \frac{1}{\mu!\,\nu!} D_x^\mu D_y^\nu f(\eta)\,(\xi_{ijk}^x - x)^\mu (\xi_{ijk}^y - y)^\nu, \end{aligned}$$

where $\eta := \eta(x,y,\xi_{ijk})$ is some point on the line from ξ_{ijk} to (x,y). Then for any nonnegative integers α and β with $\alpha + \beta \le m+1$, we have

$$\begin{aligned} D_x^\alpha D_y^\beta p_f(x,y) &= \sum_{i+j+k=d} f(\xi_{ijk})\, D_x^\alpha D_y^\beta p_{ijk}(x,y) = \\ \sum_{0 \le \mu+\nu \le m} \frac{1}{\mu!\,\nu!} &\sum_{i+j+k=d} D_x^\mu D_y^\nu f(x,y)\,(\xi_{ijk}^x - x)^\mu (\xi_{ijk}^y - y)^\nu\, D_x^\alpha D_y^\beta p_{ijk}(x,y) \\ + \sum_{\mu+\nu=m+1} \frac{1}{\mu!\,\nu!} &\sum_{i+j+k=d} D_x^\mu D_y^\nu f(\eta)\,(\xi_{ijk}^x - x)^\mu (\xi_{ijk}^y - y)^\nu\, D_x^\alpha D_y^\beta p_{ijk}(x,y). \end{aligned}$$

Now since interpolation reproduces polynomials up to degree d, we have

$$\begin{aligned}\sum_{i+j+k=d} & (\xi^x_{ijk} - x)^\mu (\xi^y_{ijk} - y)^\nu D^\alpha_x D^\beta_y p_{ijk}(x,y) \\ &= D^\alpha_u D^\beta_v \sum_{i+j+k=d} (\xi^x_{ijk} - x)^\mu (\xi^y_{ijk} - y)^\nu p_{ijk}(u,v)\Big|_{(u,v)=(x,y)} \\ &= D^\alpha_u D^\beta_v (u-x)^\mu (v-y)^\nu \Big|_{(u,v)=(x,y)} \\ &= \begin{cases} \alpha\,!\,\beta\,!\,, & \text{if } (\mu,\nu) = (\alpha,\beta), \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

This implies that the expression on the second line in the formula for $D^\alpha_x D^\beta_y p_f(x,y)$ reduces to $D^\alpha_x D^\beta_y f(x,y)$, and it follows that

$$\begin{aligned}\left|D^\alpha_x D^\beta_y [f(x,y) - p_f(x,y)]\right| = \Bigg| & \sum_{\mu+\nu=m+1} \frac{1}{\mu\,!\,\nu\,!} \sum_{i+j+k=d} D^\mu_x D^\nu_y f(\eta) \\ & \times (\xi^x_{ijk} - x)^\mu (\xi^y_{ijk} - y)^\nu D^\alpha_x D^\beta_y p_{ijk}(x,y)\Bigg| \\ & \le K_1 \, |T|^{m+1} \sum_{i+j+k=d} \|D^\alpha_x D^\beta_y p_{ijk}\|_T \; |f|_{m+1,T}.\end{aligned}$$

By Theorem 1.2,

$$\|D^\alpha_x D^\beta_y p_{ijk}\|_T \le \frac{K_2}{\rho_T^{\alpha+\beta}} \, \|p_{ijk}\|_T.$$

But by (1.23), it is easy to see that

$$\|p_{ijk}\|_T \le d^d, \qquad i+j+k=d.$$

Combining these facts shows that

$$\|D^\alpha_x D^\beta_y (f - p_f)\|_T \le K_3 \, \frac{|T|^{m+1}}{\rho_T^{\alpha+\beta}} \, |f|_{m+1,T}. \tag{1.30}$$

The bound (1.29) then follows immediately from (1.27). □

The exact size of the constant K appearing in Theorem 1.12 can be determined by a close inspection of the proof. For example, if $f \in C^2(T)$, then the linear polynomial p_f that interpolates at the vertices satisfies

$$\|f - p_f\|_T \le |T|^2 \, |f|_{2,T}. \tag{1.31}$$

1.9. Remarks

Remark 1.1. It is standard to refer to polynomials in the spaces $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \ldots$ as linear, quadratic, cubic, quartic, quintic, etc.

Remark 1.2. Given any index set $\Lambda = \{(i_\nu, j_\nu)\}_{\nu=1}^m$, we can define an associated linear space of polynomials $\mathcal{P}_\Lambda = \operatorname{span}\{x^i y^j\}_{(i,j)\in\Lambda}$, see Chapter 13 in [Sch81]. The space $\mathcal{P}_d$ of polynomials of degree d corresponds to the choice $\Lambda = \{(i,j) : 0 \le i + j \le d\}$. In the literature such polynomials are often referred to as being of total degree d. For simplicity, throughout this book we simply say that the polynomials in $\mathcal{P}_d$ are of degree d.

Remark 1.3. The choice $\Lambda = \{(i,j) : 0 \le i \le d, \;\; 0 \le j \le \tilde{d}\}$ leads to the class $\mathcal{P}_{d,\tilde{d}}$ of tensor polynomials of degree $(d, \tilde{d})$. For example, $\mathcal{P}_{1,1}$, $\mathcal{P}_{2,2}$, and $\mathcal{P}_{3,3}$ are the well-known bilinear, biquadratic, and bicubic polynomials. These classes of polynomials are most suitable for use on rectangular domains.

Remark 1.4. The exponent in the estimate (1.10) for the distance of a function $f \in C^{d+1}(\Omega)$ to the space of polynomials $\mathcal{P}_d$ is best possible in the sense that the exponent cannot be increased from $d+1$ to anything larger. To see this, let Ω be the triangle with vertices at $(0,0)$, $(h,0)$, $(h/2,h/2)$, and let $F := x^{d+1}/(d+1)!$. Then $|\Omega| = h$ and $|F|_{d+1,\Omega} = 1$. Now suppose there exists a polynomial p of degree d such that

$$\|F - p\|_\Omega \le K\, |\Omega|^{d+1+\epsilon}\, |F|_{d+1,\Omega} = K\, h^{d+1+\epsilon},$$

with some $\epsilon > 0$. But then

$$\begin{aligned} \left(\frac{h}{d+1}\right)^{d+1} &= |\triangle^{d+1} F(0,0)| = |\triangle^{d+1}(F-p)(0,0)| \\ &\le 2^{d+1}\|F-p\|_\Omega \le 2^{d+1} K\, h^{d+1+\epsilon}, \end{aligned}$$

where $\triangle$ is the forward difference operator with spacing $h/(d+1)$ operating on the x-variable. This is a contradiction for sufficiently small h.

Remark 1.5. The exponent in the estimate (1.17) for the distance of a function $f \in C^{d+1}(\Omega)$ to the space of polynomials $\mathcal{P}_d$ is also best possible in the sense that the exponent cannot be increased from $d+1$ to anything larger. To see this, fix $1 \le q < \infty$, and let Ω and F be as in the previous remark. Suppose there exists a polynomial p of degree d such that

$$\|F - p\|_{q,\Omega} \le K\, |\Omega|^{d+1+\epsilon}\, |F|_{d+1,q,\Omega} \le K\, h^{d+1+\epsilon}\, (h/2)^{2/q}$$

with some $\epsilon > 0$. Let $\widetilde{T}$ be the triangle with vertices $(0,0)$, $(h/2,0)$, and $(h/4,h/4)$. It is easy to see that for any $g \in C(\Omega)$,

$$\left\|g(\cdot + u(1,0))\right\|_{q,\widetilde{T}} \le \|g\|_{q,\Omega},$$

for all $0 \le u \le h/2$. Let $\triangle$ be the forward difference operator with spacing $t := h/(2(d+1))$ acting on the x-variable. Then

$$\begin{aligned}\Big(\frac{h}{2(d+1)}\Big)^{d+1}(h^2/16)^{1/q} &= \|\triangle^{d+1}F\|_{q,\widetilde{T}} = \|\triangle^{d+1}(F-p)\|_{q,\widetilde{T}} \\ &\le \sum_{i=0}^{d+1}\binom{d+1}{i}\|(F-p)(\cdot+it(1,0))\|_{q,\widetilde{T}} \\ &\le 2^{d+1}\|F-p\|_{q,\Omega} \le 2^{d+1}K\,h^{d+1+\epsilon}(h/2)^{2/q}.\end{aligned}$$

This is a contradiction for sufficiently small h.

Remark 1.6. Theorem 1.9 can be extended to work with different norms on the right- and left-hand sides. It can be shown by standard techniques (see e.g. [Sch81]) that there exists a constant K depending on the same quantities as in Theorem 1.9, such that if $f \in W_q^{m+1}(\Omega)$ for some $1 \le q \le \infty$, then

$$\|D_x^\alpha D_y^\beta(f - F_{m,B_\Omega}f)\|_{\tilde{q},\Omega} \le K\ |\Omega|^{m+1-\alpha-\beta+1/\tilde{q}-1/q}\ |f|_{m+1,q,\Omega}, \quad (1.32)$$

for all $0 \le \alpha+\beta \le m+1$ and all $1 \le \tilde{q} \le q \le \infty$.

Remark 1.7. A special version of Theorem 1.10 was originally established in [ChunY77]. An explicit formula for the determinant of the matrix M in (1.20) in the special case corresponding to Theorem 1.10 can be found in [ChuL87a]. To state this formula, let $m_k := \binom{k+2}{2}$ for $k = 0,\dots,d$, and let $\operatorname{dist}(u,v)$ and $\operatorname{dist}(w,L_k)$ denote the distance between two points u and v, and between a point w and a line L_k, respectively. Then

$$\det M = \pm \prod_{k=1}^{d}\Big[\prod_{m_{k-1}+1\le i<j\le m_k}\operatorname{dist}(t_j,t_i)\prod_{p=1}^{m_{k-1}}\operatorname{dist}(t_p,L_k)\Big].$$

Remark 1.8. The sufficient conditions of Theorem 1.10 are quite restrictive. It is easy to see that “almost all” sets of $\binom{d+2}{2}$ distinct points in $\mathbb{R}^2$ are poised for interpolation by $\mathcal{P}_d$. Indeed, the determinant D of the matrix M appearing in (1.20) is a polynomial function in the $2n$ variables $x_1, y_1, \dots, x_n, y_n$, where $t_i = (x_i, y_i)$ are the interpolation nodes. Since D is nontrivial (we know there are some point sets where it does not vanish), a set of points can be nonpoised only if it lies in the zero set of D. But since D is a polynomial, this is a set of measure zero in $\mathbb{R}^{2n}$.

Remark 1.9. In the finite-element literature, the Bramble–Hilbert lemma [BraH71] has often been used to give error bounds for approximation processes which reproduce polynomials locally. Here we do not use this lemma, and instead have constructed more explicit quasi-interpolation methods. This has the advantage that it gives more information on the nature of the constants involved.

1.10. Historical Notes

Polynomials have played a key role in approximation theory and numerical analysis for hundreds of years, and there is no need to try to trace their history here. However, we make a few remarks about the specific results of this chapter.

The Markov inequality is well known for univariate polynomials, but seems to be less well known for bivariate polynomials. For the maximum norm two different proofs can be found in [Coa66, Wil74]. In Section 2.12 we give a completely different proof of the inequality for general q-norms. It is based on our paper [LaiS98].

The approximation power of polynomials is a major topic in approximation theory. However, most of the results in the classical literature express the order of approximation in terms of the degree d of the polynomials (so-called Jackson theorems) rather than in terms of the size of the domain (so-called Whitney theorems). Whitney type results on triangles and clusters of triangles are essential to giving error bounds for spline approximation and for the finite-element method. We have not attempted to trace the history of such theorems, but two early references are [Coa66, CiaR72]. Our proof of Theorem 1.7 and the construction of averaged Taylor polynomials follows [BreS94].

Sobolev spaces play a major role in approximation. For a detailed treatment, see [Ada75].

Interpolation by univariate polynomials is well understood, and is treated in every numerical analysis text. In the univariate case, every set of $d + 1$ distinct nodes is poised for interpolation by polynomials of degree d. The multivariate problem is much more difficult. Theorem 1.10 is a more general version of a result of Chung–Yao [ChunY77]. More explicit results on the determinant of the corresponding collocation matrix were given in [ChuL87a], see also [Bos91]. For error bounds for Chung–Yao interpolation, see [Boo97].

Bernstein–Bézier Methods for Bivariate Polynomials

In this chapter we show that a bivariate polynomial can be written in an especially convenient form in terms of the barycentric coordinates associated with a triangle. We then discuss some properties of this form, and show how to use it in practice to efficiently evaluate polynomials, their derivatives, integrals, and inner products. In addition, we discuss the Markov inequality, smoothness conditions for polynomials on adjoining triangles, subdivision, degree raising, dual bases, and the Bernstein approximation operator.

2.1. Barycentric Coordinates

In this section we discuss barycentric coordinates, which as we shall see, are much more useful than the usual Cartesian coordinates for working with polynomials on triangles. Suppose T is a nondegenerate triangle (one with nonzero area) in $\mathbb{R}^2$ with vertices

$$v_i := (x_i, y_i), \quad i = 1, 2, 3.$$

It will be convenient to write $T := \langle v_1, v_2, v_3 \rangle$. Throughout this book we will assume that the vertices are numbered in counter-clockwise order.

Lemma 2.1. *Every point $v := (x, y) \in \mathbb{R}^2$ has a unique representation in the form*

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3, \tag{2.1}$$

with

$$1 = b_1 + b_2 + b_3. \tag{2.2}$$

The numbers b_1, b_2, b_3 are called the barycentric coordinates of the point v relative to the triangle T.

Proof: In matrix form, equations (2.1) and (2.2) become

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \tag{2.3}$$

The area of T is given by

$$A_T = \frac{1}{2}\det(M), \tag{2.4}$$

where M is the matrix in (2.3). A_T is positive whenever T is nondegenerate and its vertices are numbered in counter-clockwise order. Thus, (2.3) is a nonsingular system, and by Cramer's rule

$$b_1 = \frac{1}{2A_T}\det\begin{bmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{bmatrix}, \tag{2.5}$$

with similar expressions for b_2 and b_3. □

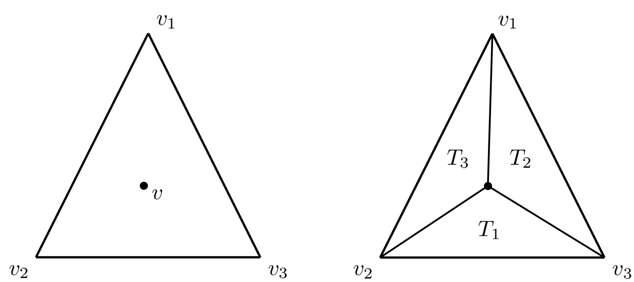


Fig. 2.1. Barycentric coordinates of a point v in a triangle T.

Barycentric coordinates have an interesting geometric interpretation. Given $v \in T$, let $T_1 := \langle v, v_2, v_3 \rangle$, $T_2 := \langle v, v_3, v_1 \rangle$, and $T_3 := \langle v, v_1, v_2 \rangle$, see Figure 2.1. Then the barycentric coordinates of v relative to T are given by

$$b_i = \frac{A_{T_i}}{A_T}, \qquad i = 1, 2, 3. \tag{2.6}$$

For this reason, barycentric coordinates are also referred to as areal coordinates.

It is clear from their definition that the barycentric coordinates of a point $v = (x, y)$ vary as we move the point v, i.e., the b_i are functions of v. More specifically, we have the following result.

Lemma 2.2. *For each $i = 1, 2, 3$, the function b_i is a linear polynomial in x and y which assumes the value 1 at the vertex v_i and vanishes at all points on the edge of T opposite to v_i.*

Proof: Expanding (2.5) by the first column, we get

$$b_1 = \frac{(x_2y_3 - y_2x_3) - x(y_3 - y_2) + y(x_3 - x_2)}{2A_T},$$

which shows that b_1 is a linear polynomial. The other two assertions about b_1 follow directly from (2.5). The proofs for b_2 and b_3 are similar. □

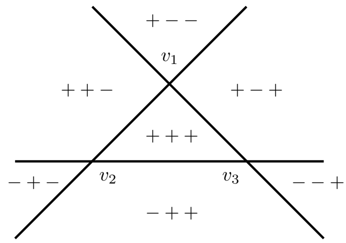


Fig. 2.2. Subregions of $\mathbb{R}^2$ defined by the signs of the barycentric coordinates.

We now show that the location of a point $v \in \mathbb{R}^2$ relative to a triangle T is indicated by the signs of its barycentric coordinates.

Theorem 2.3. *Given a triangle T, let $\{\Omega_i\}_{i=0}^6$ be the interiors of the seven regions obtained by extending the edges of T indefinitely, see Figure 2.2. Then for each fixed $0 \le i \le 6$, the signs of each of the barycentric coordinates b_1, b_2, b_3 are the same for all points $v \in \Omega_i$. In particular, a point v lies in the interior of T if and only if all three of its barycentric coordinates are positive.*

Proof: As noted above, each b_i is a linear polynomial which vanishes on the line containing the edge opposite to v_i. Thus it has one sign on one side of the line, and the opposite sign on the other side. The fact that $b_i > 0$ for points $v \in T$ follows from (2.6). $\square$

2.2. Bernstein Basis Polynomials

Throughout this section let T be a fixed triangle, and for each $v = (x, y) \in \mathbb{R}^2$, let b_1, b_2, b_3 be its barycentric coordinates. Given nonnegative integers i, j, k summing to d, let

$$B_{ijk}^d := \frac{d!}{i!\, j!\, k!} b_1^i b_2^j b_3^k. \tag{2.7}$$

Since, as observed in the previous section, each of the b_i is a linear polynomial in x and y, it follows that $B_{ijk}^d(x, y)$ is a polynomial of degree d. We call these polynomials the Bernstein basis polynomials of degree d relative to T. At times it will be convenient to allow negative subscripts, in which case we define B_{ijk}^d to be identically zero by convention. Often we shall write $B_{ijk}^d(v)$ using v as the argument rather than (x, y).

Bernstein basis polynomials have many nice properties. For example, it follows from the trinomial expansion

$$1 = (b_1 + b_2 + b_3)^d = \sum_{i+j+k=d} \frac{d!}{i!\, j!\, k!} b_1^i b_2^j b_3^k \tag{2.8}$$

that the B^d_{ijk} form a partition of unity, i.e.,

$$\sum_{i+j+k=d} B^d_{ijk}(v) \equiv 1, \qquad \text{for all } v \in \mathbb{R}^2. \tag{2.9}$$

Combining this with Theorem 2.3 we see that

$$0 \le B^d_{ijk}(v) \le 1, \qquad \text{for all } v \text{ in the triangle } T. \tag{2.10}$$

We now show that the set of Bernstein basis polynomials forms a basis for $\mathcal{P}_d$.

Theorem 2.4. *The set*

$$\mathcal{B}^d := \{B^d_{ijk}\}_{i+j+k=d} \tag{2.11}$$

of Bernstein basis polynomials is a basis for the space of polynomials $\mathcal{P}_d$.

Proof: Since the number of Bernstein basis polynomials in $\mathcal{B}^d$ is equal to the dimension $\binom{d+2}{2}$ of $\mathcal{P}_d$, we need only establish that all of the polynomials $x^\nu y^\mu$ for $0 \le \nu+\mu \le d$ are in the span of $\mathcal{B}^d$. The partition of unity property (2.9) shows that $1 \in \operatorname{span}(\mathcal{B}^d)$. Now by (2.1),

$$\begin{bmatrix} x \\ y \end{bmatrix} = b_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + b_3 \begin{bmatrix} x_3 \\ y_3 \end{bmatrix},$$

and using (2.9) with $d-1$, we have

$$\begin{aligned} x &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\ &= (b_1 x_1 + b_2 x_2 + b_3 x_3)\Big(\sum_{i+j+k=d-1} B^{d-1}_{ijk}(x,y) \Big) \\ &= \sum_{i+j+k=d} \frac{1}{d}\big(ix_1 + jx_2 + kx_3\big) B^d_{ijk}(x,y). \end{aligned} \tag{2.12}$$

The analogous relation for y is

$$y = \sum_{i+j+k=d} \frac{1}{d}\big(iy_1 + jy_2 + ky_3\big) B^d_{ijk}(x,y).$$

This shows that both x and y are in $\operatorname{span}(\mathcal{B}^d)$.

We now proceed by induction. Assuming the theorem holds for polynomials of degree $d-1$, we know that

$$x^{\nu-1} y^\mu = \sum_{i+j+k=d-1} c_{ijk} B^{d-1}_{ijk}(x,y)$$

for some coefficients c_{ijk}. But then

$$x^\nu y^\mu = (b_1 x_1 + b_2 x_2 + b_3 x_3) \sum_{i+j+k=d-1} c_{ijk} B^{d-1}_{ijk}(x,y).$$

Multiplying these sums out and collecting terms, we have

$$x^\nu y^\mu = \sum_{i+j+k=d} d_{ijk} B^d_{ijk}(x,y)$$

for some constants d_{ijk}. This completes the proof. $\square$

We conclude this section by showing that the Bernstein basis function B^d_{ijk} has a unique maximum at the point $\xi_{ijk} := (iv_1 + jv_2 + kv_3)/d$.

Theorem 2.5. *For any $i+j+k=d$,*

$$B^d_{ijk}(v) < B^d_{ijk}(\xi_{ijk}), \qquad v \in T,\ v \neq \xi_{ijk}.$$

Proof: We consider first $i,j,k > 0$, in which case ξ_{ijk} is in the interior of T and B^d_{ijk} vanishes on all three edges of T. In order to identify an extreme point of B^d_{ijk}, we need to compute its derivative in two different directions. Suppose the vertices of T are v_1, v_2, v_3. For any $v \in T$, applying Lemma 2.11 below, we get

$$D_{v_1 - v_2} B^d_{ijk}(v) = B^d_{ijk}(v) \left(\frac{i}{b_1} - \frac{j}{b_2} \right),$$

$$D_{v_1 - v_3} B^d_{ijk}(v) = B^d_{ijk}(v) \left(\frac{i}{b_1} - \frac{k}{b_3} \right).$$

Since $B^d_{ijk}(v) > 0$, these two expressions vanish simultaneously if and only if

$$ib_2 - jb_1 = 0,$$

$$i(1 - b_1 - b_2) - kb_1 = 0.$$

The unique solution of this system is $(b_1, b_2, b_3) = (i/d, j/d, k/d)$, which are the barycentric coordinates of ξ_{ijk}. It follows that B^d_{ijk} takes a unique maximum at ξ_{ijk}. The proof for other choices of i,j,k is similar. $\square$

2.3. The B-form

In view of Theorem 2.4, given any triangle T, every polynomial p of degree d can be written uniquely in the form

$$p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}, \tag{2.13}$$

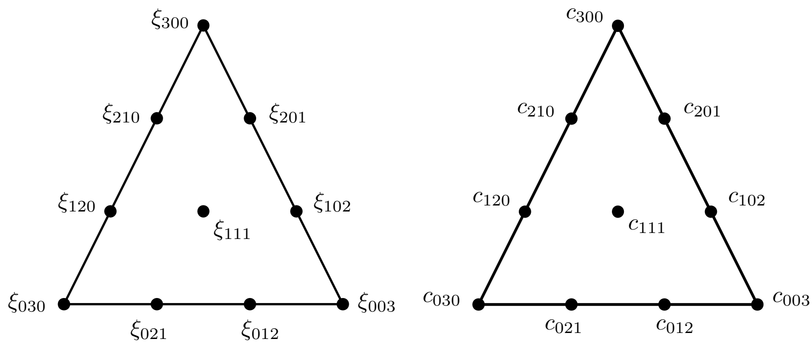


Fig. 2.3. Domain points ξ_{ijk} and B-coefficients c_{ijk} of a cubic polynomial.

where B^d_{ijk} are the Bernstein basis polynomials associated with T. We refer to the representation (2.13) as the B-form of p relative to T. We call the c_{ijk} the B-coefficients of p, and define the associated set of domain points to be

$$\mathcal{D}_{d,T} := \{\xi_{ijk} := (iv_1 + jv_2 + kv_3)/d\}_{i+j+k=d}. \tag{2.14}$$

For later use, we need to agree on an ordering for the $\binom{d+2}{2}$ coefficients in (2.13). In the sequel we shall always assume that they are in lexicographical order. This means that $c_{\nu\mu\kappa}$ comes before c_{ijk} provided $\nu > i$, or if $\nu = i$, then $\mu > j$, or if $\nu = i$ and $\mu = j$, then $\kappa > k$. Thus, for example, for $d = 3$ the order is

$$c_{300},\ \underbrace{c_{210},\ c_{201}},\ \underbrace{c_{120},\ c_{111},\ c_{102}},\ \underbrace{c_{030},\ c_{021},\ c_{012},\ c_{003}}.$$

We can now think of the coefficients of a polynomial written in the B-form (2.13) as *components* of a vector c. We already encountered the domain points in Theorem 1.11, where we noticed that they are uniformly spaced in T. Figure 2.3 (left) shows the locations of the domain points ξ_{ijk} for $d = 3$. In Figure 2.3 (right) we have placed each coefficient c_{ijk} next to its corresponding domain point ξ_{ijk} to indicate the association. At times we will write B^d_ξ to stand for B^d_{ijk} when $\xi = \xi_{ijk}$.

For later use we introduce two additional pieces of notation. Given $0 \le m \le d$, we refer to the set of domain points $R^T_m(v_1) := \{\xi_{d-m,j,m-j}\}_{j=0}^m$ as the ring of radius m around the vertex v_1. We refer to the set $D^T_m(v_1) := \cup_{n=0}^m R^T_n(v_1)$ as the disk of radius m around the vertex v_1. The rings and disks around v_2 and v_3 are defined similarly.

2.4. Stability of the B-form Representation

In this section we show that the set $\mathcal{B}^d$ of Bernstein basis polynomials is a stable basis for $\mathcal{P}_d$ in the sense that for any polynomial p written in the B-form (2.13), its size measured in the uniform norm is comparable to the size of its coefficient vector c measured by

$$\|c\|_\infty := \max_{i+j+k=d} |c_{ijk}|. \tag{2.15}$$

Let $\{g_1, \ldots, g_n\}$ be the Bernstein basis polynomials of degree d, arranged in lexicographical order, and let $\{t_1, \ldots, t_n\}$ be the associated domain points (2.14) arranged in the same order, where $n := \binom{d+2}{2}$. Theorem 2.4 together with Theorem 1.11 imply that the matrix

$$M := \left[g_j(t_i)\right]_{i,j=1}^n \tag{2.16}$$

is nonsingular. Let

$$K := \|M^{-1}\|_\infty, \tag{2.17}$$

where $\|M\|_\infty := \max\{\|Mc\|_\infty / \|c\|_\infty : c \neq 0\}$ is the usual infinity matrix norm. Since the entries of M depend only on the barycentric coordinates of the points ξ_{ijk}, it follows that both M and K depend only on d.

Theorem 2.6. *Let p be a polynomial written in the B-form (2.13) with coefficient vector c. Then*

$$\frac{\|c\|_\infty}{K} \leq \|p\|_T \leq \|c\|_\infty, \tag{2.18}$$

where K is the constant in (2.17) and depends only on d.

Proof: We can compute the coefficient vector c (whose components are the c_{ijk} in lexicographical order) from the system of equations $Mc = r$, where $r := (p(t_1), \ldots, p(t_n))^T$. But then

$$\|c\|_\infty = \|M^{-1}r\|_\infty \leq \|M^{-1}\|_\infty \, \|r\|_\infty.$$

Since $\|r\|_\infty \leq \|p\|_T$, this establishes the first inequality in (2.18). The second inequality follows immediately from the nonnegativity (2.10) and the partition of unity property (2.9) of the Bernstein basis polynomials. □

We now extend this stability result to the q-norms.

Theorem 2.7. *Given $1 \leq q < \infty$, there exists a constant $K > 0$ depending only on d such that if p is a polynomial written in the B-form (2.13), then*

$$\frac{A_T^{1/q}}{K} \|c\|_q \leq \|p\|_{q,T} \leq A_T^{1/q} \, \|c\|_q. \tag{2.19}$$

Proof: By Theorem 2.6, $\|p\|_T \le \|c\|_\infty \le K_1\|p\|_T$. Combining this with (1.4) and $\|c\|_q^q \le \binom{d+2}{2}\|c\|_\infty^q$ yields the first inequality. Combining it with (1.4) and $\|c\|_\infty \le \|c\|_q$ yields the second inequality. □

2.5. The de Casteljau Algorithm

It is clear from Theorem 2.4 that to *store* a polynomial written in the B-form (2.13), we need only store its coefficient vector c. We now present an efficient and stable algorithm for *evaluating* p at a given point $v := (x, y)$. The algorithm is based on the simple recurrence relation

$$B^d_{ijk} = b_1 B^{d-1}_{i-1,j,k} + b_2 B^{d-1}_{i,j-1,k} + b_3 B^{d-1}_{i,j,k-1}, \qquad \text{all } i+j+k=d, \quad (2.20)$$

which is an immediate consequence of the definition of B^d_{ijk}. Here we are using the convention agreed to in Section 2.2 whereby expressions with negative subscripts are to be considered to be zero. Thus, for example,

$$B^d_{d00} = b_1 B^{d-1}_{d-1,0,0} + b_2 B^{d-1}_{d,-1,0} + b_3 B^{d-1}_{d,0,-1} = b_1 B^{d-1}_{d-1,0,0}.$$

Theorem 2.8. *Let p be a polynomial written in the B-form (2.13) with coefficients*

$$c^{(0)}_{ijk} := c_{ijk}, \qquad i+j+k=d. \quad (2.21)$$

Suppose v has barycentric coordinates $b := (b_1, b_2, b_3)$, and for all $\ell = 1, \ldots, d$, let

$$c^{(\ell)}_{ijk} := b_1 c^{(\ell-1)}_{i+1,j,k} + b_2 c^{(\ell-1)}_{i,j+1,k} + b_3 c^{(\ell-1)}_{i,j,k+1}, \quad (2.22)$$

for $i+j+k = d-\ell$. Then

$$p(v) = \sum_{i+j+k=d-\ell} c^{(\ell)}_{ijk} B^{d-\ell}_{ijk}(v), \quad (2.23)$$

for all $0 \le \ell \le d$. In particular,

$$p(v) = c^{(d)}_{000}. \quad (2.24)$$

Proof: We proceed by induction on ℓ. Equation (2.23) is trivially true for $\ell = 0$. Now assume it holds for $\ell - 1$. Then using the recursion relation (2.20) for Bernstein basis polynomials of degree $d - \ell + 1$, we have

$$\begin{aligned} p(v) &= \sum_{i+j+k=d-\ell+1} c^{(\ell-1)}_{ijk} B^{d-\ell+1}_{ijk}(v) \\ &= \sum_{i+j+k=d-\ell+1} c^{(\ell-1)}_{ijk} \big[b_1 B^{d-\ell}_{i-1,j,k}(v) + b_2 B^{d-\ell}_{i,j-1,k}(v) + b_3 B^{d-\ell}_{i,j,k-1}(v) \big]. \end{aligned}$$

This can be split into three sums. The first sum can be rewritten as

$$\sum_{i+j+k=d-\ell+1,\ i\geq 1} b_1 c_{ijk}^{(\ell-1)} B_{i-1,j,k}^{d-\ell}(v) = \sum_{i+j+k=d-\ell} b_1 c_{i+1,j,k}^{(\ell-1)} B_{ijk}^{d-\ell}(v),$$

with similar formulae for the other two sums. Combining them leads to (2.23). Finally, for $\ell = d$, (2.23) reduces to (2.24) in view of the fact that the only Bernstein polynomial of degree zero is $B_{000}^0 \equiv 1$. □

Theorem 2.8 immediately leads to an algorithm for evaluating a polynomial p in the B-form (2.13).

Algorithm 2.9. *(de Casteljau)*
For $\ell = 1, \ldots, d$
For all $i + j + k = d - \ell$

$$c_{ijk}^{(\ell)} := b_1 c_{i+1,j,k}^{(\ell-1)} + b_2 c_{i,j+1,k}^{(\ell-1)} + b_3 c_{i,j,k+1}^{(\ell-1)}$$

Discussion: By Theorem 2.8 the value of $p(v)$ is given by $c_{000}^{(d)}$. The operation count (multiplications and divisions) for this algorithm is $(d^3 + 3d^2 + 2d)/2 = 3[\binom{d+1}{2} + \binom{d}{2} + \cdots + \binom{2}{2}]$. There is a simplified version of this algorithm if v falls on an edge of T, see Remark 2.5. □

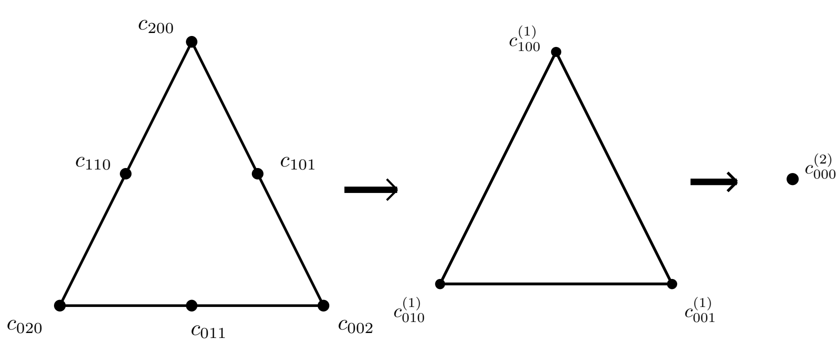


Fig. 2.4. The intermediate coefficients produced by the de Casteljau algorithm.

Figure 2.4 illustrates the progress of the algorithm in the case $d = 2$. It produces a sequence of coefficient vectors $c^{(\ell)}$ of length $m_\ell := \binom{d-\ell+2}{2}$ whose components are obtained from the coefficients at the previous level by taking combinations of three neighboring coefficients at a time, using the weights (b_1, b_2, b_3). For points $v \in T$, this algorithm is numerically very stable, since in this case the barycentric coordinates b_1, b_2, b_3 are all nonnegative and add to one, and so each step of the algorithm involves taking a convex combination of previously computed quantities.

The de Casteljau algorithm has several very important consequences, some of which we will explore in more detail below. It is of interest to note that the intermediate coefficients $c_{ijk}^{(\ell)}$ produced by the de Casteljau algorithm are in fact polynomials of degree ℓ in v. For example,

$$c_{ijk}^{(1)} = c_{i+1,j,k} b_1 + c_{i,j+1,k} b_2 + c_{i,j,k+1} b_3, \qquad i+j+k = d-1, \tag{2.25}$$

are linear polynomials since b_1, b_2, and b_3 are each linear polynomials in v. More explicitly, we have the following result.

Theorem 2.10. *The coefficients in the de Casteljau algorithm and in equation (2.23) are given by*

$$c_{ijk}^{(\ell)} = \sum_{\nu+\mu+\kappa=\ell} c_{i+\nu,j+\mu,k+\kappa} B_{\nu\mu\kappa}^{\ell}(v), \qquad i+j+k = d-\ell. \tag{2.26}$$

Proof: We first rewrite (2.25) as

$$c_{ijk}^{(1)} = (b_1 E_1 + b_2 E_2 + b_3 E_3) c_{ijk}\,,$$

where $E_1 c_{ijk} = c_{i+1,j,k}$, $E_2 c_{ijk} = c_{i,j+1,k}$ and $E_3 c_{ijk} = c_{i,j,k+1}$. With this notation, we can write the formula in Algorithm 2.9 as

$$c_{ijk}^{(\ell)} = (b_1 E_1 + b_2 E_2 + b_3 E_3) c_{ijk}^{(\ell-1)}.$$

We now use the formula again for $c_{ijk}^{(\ell-1)}$ and repeat the process $\ell-1$ times. Using a minor variant of the trinomial expansion (2.8), this leads to

$$\begin{aligned} c_{ijk}^{(\ell)} &= (b_1 E_1 + b_2 E_2 + b_3 E_3)^{\ell} c_{ijk} = \sum_{\nu+\mu+\kappa=\ell} B_{\nu\mu\kappa}^{\ell}(v) E_1^{\nu} E_2^{\mu} E_3^{\kappa} c_{ijk} \\ &= \sum_{\nu+\mu+\kappa=\ell} c_{i+\nu,j+\mu,k+\kappa} B_{\nu\mu\kappa}^{\ell}(v). \quad \square \end{aligned}$$

2.6. Directional Derivatives

Suppose u is a vector in $\mathbb{R}^2$. Then for any differentiable function f, we define its directional derivative at v with respect to u to be

$$D_u f(v) := \frac{d}{dt} f\big(v + tu\big)\Big|_{t=0}. \tag{2.27}$$

In this section we present formulae for directional derivatives of a polynomial written in B-form.

In working with directional derivatives, it is important to pay attention to the difference between u and v appearing in (2.27): v is a point in $\mathbb{R}^2$, and u is a vector. Each point $v := (v_x, v_y)$ in $\mathbb{R}^2$ is uniquely defined by its barycentric coordinates (b_1, b_2, b_3). Each vector u is also uniquely described by a triple (a_1, a_2, a_3), namely

$$a_i := \alpha_i - \beta_i, \qquad i = 1, 2, 3,$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are the barycentric coordinates of two points w and $\widetilde{w}$ such that $u := w - \widetilde{w}$. Note that the barycentric coordinates b_1, b_2, b_3 of v sum to 1, while the a_1, a_2, a_3 describing u sum to 0. We call the triple (a_1, a_2, a_3) the directional coordinates of u.

Lemma 2.11. *Suppose u is a vector whose directional coordinates are (a_1, a_2, a_3). Then for any $i + j + k = d$,*

$$D_u B^d_{ijk}(v) = d\big[a_1 B^{d-1}_{i-1,j,k}(v) + a_2 B^{d-1}_{i,j-1,k}(v) + a_3 B^{d-1}_{i,j,k-1}(v)\big]. \qquad (2.28)$$

Proof: Suppose $b_i := b_i(v)$, $i = 1, 2, 3$, are the barycentric coordinates of the point v. Then the barycentric coordinates of the point $v + tu$ are

$$\big(b_1 + ta_1, b_2 + ta_2, b_3 + ta_3\big),$$

and thus

$$B^d_{ijk}(v + tu) = \frac{d\,!}{i\,!\; j\,!\; k\,!}\big[(b_1 + ta_1)^i (b_2 + ta_2)^j (b_3 + ta_3)^k\big].$$

Now differentiating with respect to t and evaluating at $t = 0$, we get

$$D_u B^d_{ijk}(v) = \frac{d\,!}{i\,!\; j\,!\; k\,!}\big[i b_1^{i-1} a_1 b_2^j b_3^k + b_1^i j b_2^{j-1} a_2 b_3^k + b_1^i b_2^j k b_3^{k-1} a_3\big],$$

which immediately gives (2.28). □

In using (2.28), it is important to keep in mind our *convention* that Bernstein basis polynomials with negative subscripts are taken to be identically zero.

We also note that in defining the directional derivative D_u, we did *not* assume that u is a unit vector. In fact, the value of $D_u p$ will in general depend on the *length* of u relative to the size of T as well as its *direction.* For a specific example, suppose T has vertices v_1, v_2, v_3 and let $w = (v_1 + v_2)/2$. Then the direction vector $u := v_2 - v_1$ corresponds to the triple $(-1, 1, 0)$, while the direction vector $\tilde{u} := w - v_1$ corresponds to the triple $(-.5, .5, 0)$. Thus, (2.28) will give different results for the two direction vectors u and $\tilde{u}$, even though they point in the same direction. For example, $D_u B^1_{010}(v) = B^0_{000}(v) = 1$ while $D_{\tilde{u}} B^1_{010}(v) = .5 B^0_{000}(v) = .5$, for all $v \in T$.

We are now ready to give a formula for the derivative of a general polynomial in B-form.

Theorem 2.12. *Let p be a polynomial written in the B-form (2.13) relative to a triangle T, and let u be a direction vector described by the triple $a := (a_1, a_2, a_3)$. Then the directional derivative at v of p in the direction u is given by*

$$D_u p(v) = d \sum_{i+j+k=d-1} c_{ijk}^{(1)}(a) B_{ijk}^{d-1}(v), \tag{2.29}$$

where $c_{ijk}^{(1)}(a)$ are the quantities arising in the first step of the de Casteljau algorithm based on the triple a.

Proof: Applying D_u to (2.13), we get

$$D_u p(v) = \sum_{i+j+k=d} c_{ijk} D_u B_{ijk}^d(v).$$

Inserting (2.28), collecting the coefficients of $B_{ijk}^{d-1}(v)$ for $i+j+k=d-1$, and taking account of (2.25) gives (2.29). □

Equation (2.29) gives a simple formula for the coefficients of the first derivative of a polynomial p written in B-form: they are just d times the quantities $c_{ijk}^{(1)}(a)$ obtained in the first step of the de Casteljau algorithm using the triple a. Thus, to evaluate $D_u p$ at a point v with barycentric coordinates $b = (b_1, b_2, b_3)$ using the de Casteljau algorithm, we simply apply one step of the algorithm using a, followed by $d-1$ steps using b.

We now give a formula for higher-order directional derivatives of p.

Theorem 2.13. *Let $1 \le m \le d$, and suppose we are given a set $u_1, \ldots, u_m$ of m directions described by the triples*

$$a^{(i)} := (a_1^{(i)}, a_2^{(i)}, a_3^{(i)}), \qquad \textit{with } a_1^{(i)} + a_2^{(i)} + a_3^{(i)} = 0,$$

for $i = 1, \ldots, m$. Then

$$D_{u_m} \cdots D_{u_1} p(v) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c_{ijk}^{(m)}(a^{(1)}, \ldots, a^{(m)}) B_{ijk}^{d-m}(v), \tag{2.30}$$

where $c_{ijk}^{(m)}(a^{(1)}, \ldots, a^{(m)})$ are the quantities obtained after carrying out m steps of the de Casteljau algorithm using $a^{(1)}, \ldots, a^{(m)}$ in order.

Proof: The result follows by applying Theorem 2.12 repeatedly. □

Equation (2.30) reaffirms that the m-th mixed directional derivative of a polynomial p is a polynomial of degree $d-m$. To evaluate $D_{u_m} \cdots D_{u_1} p(v)$ at a point v with barycentric coordinates $b := (b_1, b_2, b_3)$, we simply apply the de Casteljau algorithm to the coefficient vector of p using the triples

$a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ in order, and then follow these m steps with an additional $d-m$ steps using the triple b of barycentric coordinates of v.

Since polynomials are infinitely differentiable functions, the order in which the derivatives are taken in (2.30) does not matter. The following lemma verifies this directly by showing that if we apply the de Casteljau algorithm with two different triples, we get the same result no matter which one we use first.

Lemma 2.14. *Suppose $c^{(2)}(a,b)$ is the vector of coefficients obtained by applying the de Casteljau algorithm to a vector c, first using the triple $a := (a_1, a_2, a_3)$ and then using the triple $b := (b_1, b_2, b_3)$. Let $c^{(2)}(b,a)$ be the result obtained if we first use b and then a. Then*

$$c^{(2)}(a,b) = c^{(2)}(b,a).$$

Proof: It suffices to focus on what happens to a typical triangle of six coefficients in the array c. We denote them by c_{ijk} with $i+j+k=2$. Applying the de Casteljau algorithm first with a and then with b to these six coefficients, we get

$$\begin{aligned} b_1(a_1c_{200} + a_2c_{110} + a_3c_{101}) + b_2(a_1c_{110} + a_2c_{020} + a_3c_{011}) \\ + b_3(a_1c_{101} + a_2c_{011} + a_3c_{002}). \end{aligned}$$

Reversing the use of a and b, we get

$$\begin{aligned} a_1(b_1c_{200} + b_2c_{110} + b_3c_{101}) + a_2(b_1c_{110} + b_2c_{020} + b_3c_{011}) \\ + a_3(b_1c_{101} + b_2c_{011} + b_3c_{002}). \end{aligned}$$

Since these two expressions are equal, the result follows. $\square$

We can now establish the following higher-order version of (2.29) along with its dual form.

Theorem 2.15. *Let p be a polynomial written in B-form, and suppose u is a direction vector described by a triple a as in Theorem 2.12. Then for any $1 \le m \le d$,*

$$D_u^m p(v) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c_{ijk}^{(m)}(a) B_{ijk}^{d-m}(v), \tag{2.31}$$

where $c_{ijk}^{(m)}(a)$ are the quantities obtained after m steps of the de Casteljau algorithm applied to the coefficients of p using the triple a. In addition,

$$D_u^m p(v) = \frac{d!}{(d-m)!} \sum_{i+j+k=m} c_{ijk}^{(d-m)}(b) B_{ijk}^{m}(u), \tag{2.32}$$

where $b := b(v) := (b_1, b_2, b_3)$ *is the triple of barycentric coordinates of* v.

Proof: The first formula is just a specialization of (2.30). The second one follows from Lemma 2.14 and Theorem 2.8. $\square$

This result says that to evaluate $D_u^m p(v)$, we can apply the de Casteljau algorithm in two different ways. We can either carry out m steps of the algorithm with the triple a describing u followed by $d-m$ steps with the triple b of barycentric coordinates of v, or we can reverse the process and carry out $d-m$ steps of the algorithm with b followed by m steps with a.

It is clear from (2.31) that the m-th order derivative restricted to the edge $\langle v_2, v_3 \rangle$ opposite to v_1 depends only on the coefficients c_{ijk} of p with $0 \le i \le m$. Referring to Figure 2.3, these are the coefficients which are in the bottom $m+1$ rows of the array of coefficients. In particular, the first derivative depends only on the coefficients in the bottom two rows.

For later purposes, we now present a formula for the directional derivative in a direction u defined by the difference of two of the vertices of T.

Theorem 2.16. *Let* p *be as in (2.13). Then for any* $1 \le m \le d$, *the* m*-th order directional derivative of* p *in the direction* $u = v_2 - v_1$ *is given by (2.31) with*

$$c_{ijk}^{(m)} := \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell c_{i+\ell, j+m-\ell, k}, \quad i+j+k = d-m. \tag{2.33}$$

Proof: For this direction we have $a = (-1, 1, 0)$, and so by (2.26),

$$c_{ijk}^{(m)}(a) = \sum_{\nu+\mu=m} \frac{m!\,(-1)^\nu}{\nu!\,\mu!} c_{i+\nu, j+\mu, k}, \quad i+j+k = d-m,$$

which immediately reduces to (2.33). $\square$

The expression (2.33) is just the m-th forward difference of $m+1$ consecutive coefficients corresponding to domain points lying on a line parallel to the left edge of the triangular tableau of coefficients of p. Similar formulae hold for the derivatives in the directions $v_3 - v_1$ and $v_3 - v_2$.

2.7. Derivatives at a Vertex

In this section we investigate the connection between the derivatives of a polynomial p at a vertex and its B-coefficients corresponding to domain points in a disk (see page 23) around that vertex.

The following lemma is easily proved by double induction. A set M of pairs of nonnegative integers is called a lower set if for any $(m, n) \in M$, all pairs of the form (i, j) with $0 \le i \le m$ and $0 \le j \le n$ also belong to M.

Lemma 2.17. *Suppose that M is a lower set, and that*

$$f(m,n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} (-1)^{i+j} g(i,j), \quad \text{all } (m,n) \in M.$$

Then

$$g(i,j) = \sum_{m=0}^{i} \sum_{n=0}^{j} \binom{i}{m} \binom{j}{n} (-1)^{m+n} f(m,n), \quad \text{all } (i,j) \in M.$$

Theorem 2.18. *Let $T := \langle v_1, v_2, v_3 \rangle$, and fix $0 \le \rho \le d$. Then the set of derivatives $\{D^m_{v_2-v_1} D^n_{v_3-v_1} p(v_1)\}_{m+n\le\rho}$ can be computed from the set of coefficients of p corresponding to domain points lying in the disk $D^T_\rho(v_1)$ and vice versa. Moreover,*

$$|D^m_{v_2-v_1} D^n_{v_3-v_1} p(v_1)| \le \frac{2^{m+n}\, d\,!}{(d-m-n)\,!} \max_{\xi \in D^T_{m+n}(v_1)} |c_\xi|, \quad \text{all } m+n \le \rho, \tag{2.34}$$

and

$$|c_\xi| \le 2^\rho \max_{m+n\le\rho} |D^m_{v_2-v_1} D^n_{v_3-v_1} p(v_1)|, \quad \text{all } \xi \in D^T_\rho(v_1). \tag{2.35}$$

Similar assertions hold for v_2 and v_3.

Proof: Evaluating (2.30) at v_1, we get

$$\begin{aligned} &D^m_{v_2-v_1} D^n_{v_3-v_1} p(v_1) \\ &\qquad = \frac{(-1)^{m+n} d\,!}{(d-m-n)\,!} \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} (-1)^{i+j} c_{d-i-j,i,j}, \end{aligned} \tag{2.36}$$

for any $0 \le m+n \le d$. All of the coefficients in (2.36) lie in the disk $D^T_{m+n}(v_1)$, and (2.34) follows immediately.

To establish the converse, we apply Lemma 2.17 to (2.36) to get

$$c_{d-i-j,i,j} = \sum_{m=0}^{i} \sum_{n=0}^{j} \binom{i}{m} \binom{j}{n} \frac{(d-m-n)\,!}{d\,!} D^m_{v_2-v_1} D^n_{v_3-v_1} p(v_1), \tag{2.37}$$

which immediately gives (2.35). $\square$

We conclude this section with a useful variant of Theorem 2.18. As before, given a triangle T, we write ρ_T for the radius of the largest disk that can be inscribed in T.

Theorem 2.19. *Let $T := \langle v_1, v_2, v_3 \rangle$, and fix $0 \le \rho \le d$ and $1 \le i \le 3$. Then the set of derivatives $\{D_x^m D_y^n p(v_i)\}_{m+n \le \rho}$ can be computed from the set of coefficients of p corresponding to domain points lying in the disk $D_\rho^T(v_i)$, and vice versa. Moreover, for each $1 \le i \le 3$,*

$$|D_x^m D_y^n p(v_i)| \le \frac{d\,!}{(d-m-n)\,!}\, \rho_T^{-(m+n)} \max_{\xi \in D_{m+n}^T(v_i)} |c_\xi|, \quad \textit{all } m+n \le \rho, \tag{2.38}$$

and

$$|c_\xi| \le 2^\rho \sum_{\nu=0}^{\rho} 2^\nu |T|^\nu \max_{m+n=\nu} |D_x^m D_y^n p(v_i)|, \qquad \textit{all } \xi \in D_\rho^T(v_i). \tag{2.39}$$

Proof: Suppose $v_i := (x_i, y_i)$ for $i = 1, 2, 3$. We first establish (2.38) for $i = 1$. The other vertices can be treated in the same way. First we note that the unit direction vector pointing in the direction of the x-axis has directional coordinates $(a_1, a_2, a_3) := (y_2 - y_3, y_3 - y_1, y_1 - y_2)/2A_T$, where A_T is the area of T. Similarly, the unit direction vector pointing in the direction of the y-axis has directional coordinates $(\tilde a_1, \tilde a_2, \tilde a_3) := (x_3 - x_2, x_1 - x_3, x_2 - x_1)/2A_T$. Using the fact that the area of a triangle is equal to its perimeter times $\rho_T/2$, it follows that

$$\begin{aligned} \frac{\rho_T}{2}\big(|y_2 - y_3| + |y_3 - y_1| + |y_1 - y_2|\big) &\le A_T, \\ \frac{\rho_T}{2}\big(|x_2 - x_3| + |x_3 - x_1| + |x_1 - x_2|\big) &\le A_T. \end{aligned} \tag{2.40}$$

This implies

$$\begin{aligned} |a_1| + |a_2| + |a_3| &\le \frac{1}{\rho_T}, \\ |\tilde a_1| + |\tilde a_2| + |\tilde a_3| &\le \frac{1}{\rho_T}. \end{aligned} \tag{2.41}$$

By Theorem 2.13 we can compute the coefficients of $D_x^m D_y^n p$ by carrying out $m + n$ steps of the de Casteljau algorithm starting with the coefficients of p. By (2.41) the coefficients obtained in step ℓ are at most ρ_T^{-1} as large as those in step $\ell - 1$, and (2.38) follows.

To prove (2.39), we use the elementary identities

$$\begin{aligned} D_{v_2 - v_1} p(v_1) &= (x_2 - x_1) D_x p(v_1) + (y_2 - y_1) D_y p(v_1), \\ D_{v_3 - v_1} p(v_1) &= (x_3 - x_1) D_x p(v_1) + (y_3 - y_1) D_y p(v_1). \end{aligned}$$

Then (2.39) follows directly from (2.35). $\square$

2.8. Cross Derivatives

Let $T := \langle v_1, v_2, v_3 \rangle$, and suppose u is some vector which is not parallel to the edge $e := \langle v_2, v_3 \rangle$. Then we say that the directional derivative D_u is a cross derivative to e. At times it may be convenient to take u to be perpendicular to e, but we do not require this in general. Also, we do not require that u be a unit vector. In this section we explore the connection between the B-coefficients of a polynomial p defined on T and its cross derivatives evaluated at points v lying on the edge e.

First we examine the formulae for computing cross derivatives from the B-coefficients $\{c_{ijk}\}_{i+j+k=d}$ of p. Suppose $a := (a_1, a_2, a_3)$ is the triple of directional coordinates of u. Then by Theorem 2.15,

$$D_u^m p(v) = \frac{d\,!}{(d-m)\,!} \sum_{j+k=d-m} c_{0jk}^{(m)}(a) B_{0jk}^{d-m}(v), \tag{2.42}$$

where $c_{0jk}^{(m)}(a)$ are obtained from the c_{ijk} by applying the de Casteljau algorithm using the vector a. Examining this algorithm, we see that the $c_{0jk}^{(m)}$ involve only coefficients c_{ijk} with $0 \le i \le m$. These coefficients correspond to domain points lying on e or on the next m rows parallel to e. Since it is based on the de Casteljau algorithm, it is clear that the computation of cross derivatives along an edge from the coefficients of p is a stable process.

We now turn to the converse. Suppose we know all of the coefficients $\{c_{ijk}\}_{0 \le i \le m-1}$ of p. These lie on e and in the first $m-1$ rows parallel to e. We now show how to compute the coefficients $\{c_{mjk}\}_{j+k=d-m}$ from values of the cross derivatives $D_u^m p(v)$ at $d-m+1$ points along the edge e.

Lemma 2.20. *Suppose the coefficients $\{c_{ijk}\}_{0 \le i \le m-1}$ of the polynomial p are known. Suppose we are also given $r := (D_u^m p(\eta_0), \ldots, D_u^m p(\eta_{d-m}))^T$ for some distinct points $\eta_0, \ldots, \eta_{d-m}$ in the interior of $e := \langle v_2, v_3 \rangle$. Then the coefficients $c := (c_{m,d-m,0}, \ldots, c_{m,0,d-m})^T$ can be uniquely computed from r and $\{c_{ijk}\}_{0 \le i \le m-1}$. Moreover, if we choose $\eta_i := v_2 + (i+1)(v_3 - v_2)/(d-m+2)$ for $i = 0, \ldots, d-m$, then the computation is stable in the sense that*

$$\|c\|_\infty \le K \left[\|r\|_\infty + \max_{0 \le i \le m-1} |c_{ijk}| \right], \tag{2.43}$$

where K is a constant depending only on m, d, and the vector a of directional coordinates of u.

Proof: Evaluating (2.42) at $\eta_0, \ldots, \eta_{d-m}$ leads to the system of equations

$$Mc = r,$$

for the vector $c := (c_{0,d-m,0}^{(m)}, \ldots, c_{0,0,d-m}^{(m)})^T$ of coefficients in (2.42), where M is the $(d-m+1) \times (d-m+1)$ matrix

$$M := \frac{d\,!}{(d-m)\,!} \left[B_{0,d-m-j,j}^{d-m}(\eta_i) \right]_{i,j=0}^{d-m}.$$

This matrix contains the values of the univariate Bernstein basis polynomials (cf. Remark 2.4), and is thus nonsingular for any distinct $\eta_0, \ldots, \eta_{d-m}$. For equally spaced η_i, it depends only on m and d. Thus,

$$\|c\|_\infty \le \left\|M^{-1}\right\|_\infty \|r\|_\infty.$$

But by Theorem 2.10, for each $0 \le j \le d-m$,

$$\begin{aligned} c^{(m)}_{0,d-m-j,j}(a) &= \sum_{\nu+\mu+\kappa=m} c_{\nu,d-m-j+\mu,j+\kappa} B^m_{\nu\mu\kappa}(a) \\ &= c_{m,d-m-j,j} \;+ \sum_{\substack{\nu+\mu+\kappa=m \\ \nu\ne m}} c_{\nu,d-m-j+\mu,j+\kappa} B^m_{\nu\mu\kappa}(a). \end{aligned}$$

Since all of the coefficients in the last sum are known, we can solve for $c_{m,d-m-j,j}$, and (2.43) follows. $\square$

In practice we often know the coefficients of p not only for domain points in the first $m-1$ rows parallel to e, but also for domain points in both of the disks $D^T_\rho(v_2)$ and $D^T_\rho(v_3)$ for some $m \le \rho < d/2$. In this case we have the following variant of Lemma 2.20.

Lemma 2.21. *Let $m \le \rho < d/2$. Suppose we are given the coefficients*

$$\mathcal{C} := \bigcup_{i=0}^{m-1} \{c_{ijk}\}_{j+k=d-i} \cup \{c_\xi\}_{\xi\in D_\rho(v_2)} \cup \{c_\xi\}_{\xi\in D_\rho(v_3)}$$

of the polynomial p. Suppose for distinct points $\eta_{\rho-m+1}, \ldots, \eta_{d-\rho-1}$ on the edge $e := \langle v_2, v_3\rangle$, we also know $r := (D^m_u p(\eta_{\rho-m+1}), \ldots, D^m_u p(\eta_{d-\rho-1}))^T$. Then $c := (c_{m,d-\rho-1,\rho-m+1}, \ldots, c_{m,\rho-m+1,d-\rho-1})^T$ can be uniquely computed from the known coefficients $\mathcal{C}$ and the vector r. Moreover, if we choose $\eta_i := v_2+(\rho-m+i)(v_3-v_2)/(d-2\rho+m)$ for $i = \rho-m+1, \ldots, d-\rho-1$, then the computation is stable in the sense that

$$\|c\|_\infty \le K\left[\|r\|_\infty + \max_{c_{ijk}\in\mathcal{C}} |c_{ijk}|\right], \tag{2.44}$$

where K is a constant depending only on m, d, and the directional coordinates a of u.

Proof: Evaluating (2.42) at $\eta_{\rho-m+1}, \ldots, \eta_{d-\rho-1}$ leads to the system of equations

$$Mc = r,$$

for the vector $c^{(m)} := (c^{(m)}_{0,d-\rho-1,\rho-m+1}, \ldots, c^{(m)}_{0,\rho-m+1,d-\rho-1})^T$ of coefficients in (2.42), where M is the matrix

$$M := \frac{d!}{(d-m)!}\left[B^{d-m}_{0,d-m-j,j}(\eta_i)\right]^{d-\rho-1}_{i,j=\rho-m+1}.$$

As in the previous theorem, this matrix corresponds to the values of univariate Bernstein polynomials, and thus is nonsingular for any distinct η_i. For equally spaced η_i, it depends only on m and d. Thus,

$$\|c^{(m)}\|_\infty \le \left\|M^{-1}\right\|_\infty \|r\|_\infty.$$

But by Theorem 2.10, for each $1 \le j \le d-2\rho+m-1$,

$$\begin{aligned} c^{(m)}_{0,d-\rho-j,\rho-m+j}(a) &= \sum_{\nu+\mu+\kappa=m} c_{\nu,d-\rho-j+\mu,\rho-m+j+\kappa} B^m_{\nu\mu\kappa}(a) \\ &= c_{m,d-\rho-j,\rho-m+j} + \sum_{\substack{\nu+\mu+\kappa=m \\ \nu\neq m}} c_{\nu,d-\rho-j+\mu,\rho-m+j+\kappa} B^m_{\nu\mu\kappa}(a). \end{aligned}$$

Since all of the coefficients in the last sum are known, we can solve for $c_{m,d-\rho-j,\rho-m+j}$, and (2.44) follows. □

2.9. Computing Coefficients by Interpolation

Theorem 1.11 shows that if we are given the values of a polynomial at the set of $\binom{d+2}{2}$ domain points $\mathcal{D}_{d,T} = \{\xi_{ijk}\}_{i+j+k=d}$ associated with a triangle T, then we can uniquely compute the B-coefficients of p. For later use in the study of macro-elements, we need a generalization of this result where some of the B-coefficients are already known, and the rest are to be determined by interpolation at an appropriate subset of $\mathcal{D}_{d,T}$. In this regard, we have the following conjecture of the second author, see Remark 2.7.

Conjecture 2.22. *Given d and a triangle $T := \langle v_1, v_2, v_3\rangle$, let Γ be an arbitrary subset of $\mathcal{D}_{d,T}$. Then the matrix*

$$M := \left[B^d_\eta(\xi)\right]_{\xi,\eta\in\Gamma} \tag{2.45}$$

is nonsingular. Thus, for any real numbers $\{z_\xi\}_{\xi\in\Gamma}$, there is a unique $p := \sum_{\eta\in\Gamma} c_\eta B^d_\eta$ such that $p(\xi) = z_\xi$ for all $\xi \in \Gamma$.

Discussion: Note that the matrix M does not depend on the size or shape of the triangle T since the entries are in terms of barycentric coordinates. The determinant of M is also independent of the order assigned to the elements of Γ, as long as we use the same order for both the rows and columns. We may assume they are in lexicographical order, see page 23. It has also been conjectured that this determinant is always positive. □

It is easy to check directly that this conjecture is valid for $d \le 3$. It has also been verified numerically for $d \le 7$, but as yet, has not been established for general d except for special classes of Γ. It is trivially true for $\Gamma := \mathcal{D}_{d,T}$ by Theorem 1.11. We now establish the conjecture for two special cases which will be needed later in the book. Suppose $T := \langle v_1, v_2, v_3\rangle$.

Lemma 2.23. *Let $\Gamma := \mathcal{D}_{d,T} \setminus \{\xi_{d00}, \xi_{0d0}, \xi_{00d}\}$. This set has cardinality $n := \binom{d+2}{2} - 3$, and the matrix (2.45) is nonsingular.*

Proof: The matrix $M_0 := \left[B^d_\xi(\eta)\right]_{\xi,\eta\in\mathcal{D}_{d,T}}$ is nonsingular by Theorem 1.11. Now by the properties of the Bernstein basis polynomials, the column of M_0 corresponding to B^d_{d00} has all zero entries except in the row corresponding to ξ_{d00} where $B^d_{d00}(\xi_{d00}) = 1$. Similarly, the column corresponding to B^d_{0d0} has all zero entries except for $B^d_{0d0}(\xi_{0d0}) = 1$, while the column corresponding to B^d_{00d} has all zero entries except for $B^d_{00d}(\xi_{00d}) = 1$. It follows that $\det\,[B^d_\xi(\eta)]_{\xi,\eta\in\Gamma} = \pm\det M_0$, and the claim follows. □

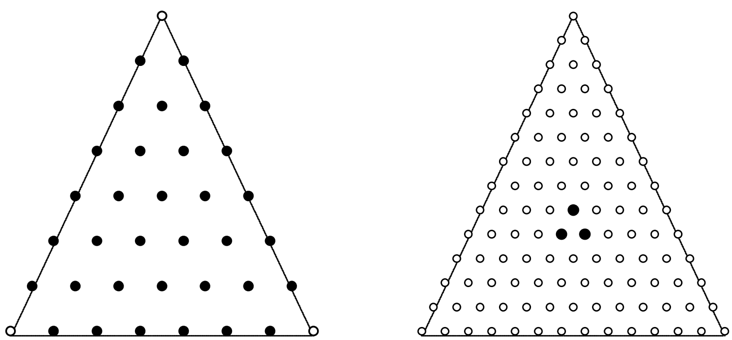


Fig. 2.5. The sets Γ of Examples 2.24 (left) and 2.26 (right).

Example 2.24. Let $d = 7$ and let Γ be as in Lemma 2.23. The 33 points in this set are marked with a black dot in Figure 2.5 (left).

Lemma 2.25. *Let $\Gamma := \mathcal{D}_{d,T} \setminus \{\xi_{ijk} : i \ge m_1,\ j \ge m_2,\ k \ge m_3\}$ for some $m_1, m_2, m_3 \ge 0$ with $m := m_1 + m_2 + m_3 < d$. Then the matrix (2.45) is nonsingular.*

Proof: In this case the set Γ is just the set of domain points such that for each $i = 1, 2, 3$, their distance to the edge $\langle v_{i+1}, v_{i+2}\rangle$ of T is at least m_i. This set has cardinality $n := \binom{d-m+2}{2}$. After multiplying the columns of M by appropriate ratios of factorials, and removing common factors of the form $(\frac{\nu}{d})^{m_1}(\frac{\mu}{d})^{m_2}(\frac{\kappa}{d})^{m_3}$ from each row of M, we find that $M = a\widetilde{M}$, where a is a nonzero constant depending on m and d, and where $\widetilde{M}$ is the $n \times n$ matrix with entries $B^{d-m}_{ijk}(\xi_{\nu\mu\kappa})$ where $i + j + k = d - m$ and (ν, μ, κ) runs over Γ. Now the set of points $\{\xi_{\nu\mu\kappa}\}$ satisfy the conditions of Theorem 1.10, and it follows that $\widetilde{M}$ is nonsingular. □

Example 2.26. Let $d = 13$ and suppose Γ is as in Lemma 2.25 with $m_1 = m_2 = m_3 = 4$.

Discussion: The three points in Γ are marked with a black dot in Figure 2.5 (right), and the matrix is

$$M := \begin{bmatrix} B^{13}_{544}(\xi_{544}) & B^{13}_{454}(\xi_{544}) & B^{13}_{445}(\xi_{544}) \\ B^{13}_{544}(\xi_{454}) & B^{13}_{454}(\xi_{454}) & B^{13}_{445}(\xi_{454}) \\ B^{13}_{544}(\xi_{445}) & B^{13}_{454}(\xi_{445}) & B^{13}_{445}(\xi_{445}) \end{bmatrix}.$$

Since $\xi^T_{ijk} = (\frac{i}{13}, \frac{j}{13}, \frac{k}{13})$, we can factor $5^4 4^4 4^4/13^{13}$ out of each row, and after factoring out the factorials in each column, we obtain

$$M = a \begin{bmatrix} 5 & 4 & 4 \\ 4 & 5 & 4 \\ 4 & 4 & 5 \end{bmatrix}$$

for some nonzero constant a. This is clearly nonsingular. $\square$

For later use in the study of macro-elements, it is important to know that the computation of coefficients by interpolation as described in Conjecture 2.22 is stable in the sense that the computed coefficients are not too large compared to the known coefficients and the given data.

Theorem 2.27. *Let Γ be a subset of $\mathcal{D}_{d,T}$ such that the matrix in (2.45) is nonsingular. Suppose all of the B-coefficients of the polynomial $p \in \mathcal{P}_d$ are known except for those with subscripts in the set Γ. Then given any $\{z_{ijk}\}_{(i,j,k)\in\Gamma}$, there exists a unique set of coefficients $\{c_{ijk}\}_{(i,j,k)\in\Gamma}$ such that*

$$p(\xi_{\nu\mu\kappa}) = z_{\nu\mu\kappa}, \qquad \text{all } (\nu,\mu,\kappa) \in \Gamma.$$

Moreover, there exists a constant K depending only on m and d such that

$$|c_{ijk}| \le K \Big[\max_{(i,j,k)\in\Gamma} |z_{ijk}| + \max_{(i,j,k)\notin\Gamma} |c_{ijk}| \Big] \tag{2.46}$$

for all $(i,j,k) \in \Gamma$.

Proof: The interpolation conditions lead to a nonsingular system of equations for the unknown coefficients with matrix M as in (2.45). But then (2.46) holds with $K = \|M^{-1}\|_\infty$. $\square$

2.10. Conditions for Smooth Joins of Polynomials

In preparation for our study of spline spaces, we now give conditions for a smooth join between two polynomials on adjoining triangles.

Theorem 2.28. *Let* $T := \langle v_1, v_2, v_3 \rangle$ *and* $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ *be triangles sharing the edge* $e := \langle v_2, v_3 \rangle$. *Let*

$$p(v) := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v) \tag{2.47}$$

and

$$\tilde{p}(v) := \sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{B}^d_{ijk}(v), \tag{2.48}$$

where $\{B^d_{ijk}\}$ *and* $\{\tilde{B}^d_{ijk}\}$ *are the Bernstein basis polynomials associated with* T *and* $\widetilde{T}$, *respectively. Suppose* u *is any direction not parallel to* e. *Then*

$$D^n_u p(v) = D^n_u \tilde{p}(v), \qquad \text{all } v \in e \ \text{ and } \ n = 0, \ldots, r, \tag{2.49}$$

if and only if

$$\tilde{c}_{njk} = \sum_{\nu+\mu+\kappa=n} c_{\nu, k+\mu, j+\kappa} B^n_{\nu\mu\kappa}(v_4), \qquad \begin{array}{c} j+k = d-n \\ n = 0, \ldots, r. \end{array} \tag{2.50}$$

Proof: Since p and $\tilde{p}$ reduce to univariate polynomials along e, it is clear that they join continuously along e if and only if

$$\tilde{c}_{0jk} = c_{0kj}, \qquad j+k = d, \tag{2.51}$$

which is the case $r = 0$. To show the result for $r > 0$, we first note that (2.49) holds if and only if it holds for the direction $u = v_4 - v_2$. This is because all derivatives of p and $\tilde{p}$ corresponding to the direction $v_3 - v_2$ agree at every point on e, and derivatives in all other directions will be linear combinations of D_u and $D_{v_3 - v_2}$.

Let $b = (b_1, b_2, b_3)$ be the barycentric coordinates of v_4 relative to T. Then the directional coordinates (see page 28) of u relative to the triangles T and $\widetilde{T}$ are $a = (b_1, b_2 - 1, b_3)$ and $\tilde{a} := (1, 0, -1)$, respectively. By Theorem 2.15, for each $0 \le n \le r$,

$$D^n_u p|_e = \frac{d!}{(d-n)!} \sum_{j+k=d-n} c^{(n)}_{0jk}(a) B^{d-n}_{0jk},$$

$$D^n_u \tilde{p}|_e = \frac{d!}{(d-n)!} \sum_{j+k=d-n} \tilde{c}^{(n)}_{0jk}(\tilde{a}) \tilde{B}^{d-n}_{0jk},$$

where $c^{(n)}_{ijk}(a)$ and $\tilde{c}^{(n)}_{ijk}(a)$ are the coefficients obtained by applying n steps of the de Casteljau algorithm to $\{c_{ijk}\}$ and $\{\tilde{c}_{ijk}\}$ using a and $\tilde{a}$, respectively.

Since for points v on e, $\tilde{B}^{d-n}_{0jk}(v) = B^{d-n}_{0kj}(v)$, it follows that (2.49) holds if and only if

$$\tilde{c}^{(n)}_{0jk}(\tilde{a}) = c^{(n)}_{0kj}(a), \qquad j+k = d-n, \quad n = 0,\ldots,r. \tag{2.52}$$

Arguing as in the proof of Theorem 2.16, we have

$$\tilde{c}^{(n)}_{0jk}(\tilde{a}) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \tilde{c}_{m,j,d-j-m} \quad j+k = d-n.$$

On the other hand, following the proof of Theorem 2.10 leads to

$$\begin{aligned}
c^{(n)}_{0kj}(a) &= (b_1E_1 + (b_1-1)E_2 + b_3E_3)^n c_{0kj} \\
&= (b_1E_1 + b_2E_2 + b_3E_3 - E_2)^n c_{0kj} \\
&= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (b_1E_1 + b_2E_2 + b_3E_3)^m c_{0,k+n-m,j} \\
&= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} c^{(m)}_{0,d-j-m,j}(b), \quad j+k = d-n.
\end{aligned}$$

It follows that (2.52) holds if and only if

$$\tilde{c}_{n,j,d-j-n} = c^{(n)}_{0,d-j-n,j}(b), \qquad j = 0,\ldots,n, \quad n = 0,\ldots,r,$$

which is equivalent to (2.50). $\square$

The conditions (2.50) can be interpreted geometrically if we define the control points associated with the polynomial p in (2.47) to be the points $C_{ijk} := (\xi^T_{ijk}, c_{ijk})$ in $\mathbb{R}^3$ with a similar definition for the control points $\tilde{C}_{ijk}$ associated with the polynomial $\tilde{p}$. Then the condition (2.51) for C^0 continuity says that the control points of p and $\tilde{p}$ associated with domain points along the edge e must agree. The condition for C^1 smoothness across the edge is that (2.51) holds along with

$$\tilde{c}_{1jk} = b_1 c_{1,k,j} + b_2 c_{0,k+1,j} + b_3 c_{0,k,j+1}, \qquad j+k = d-1. \tag{2.53}$$

Since

$$\tilde{\xi}_{1jk} = b_1 \xi_{1,j,k} + b_2 \xi_{0,k+1,j} + b_3 \xi_{0,k,j+1}, \qquad j+k = d-1,$$

we can rewrite (2.53) in terms of control points as

$$\tilde{C}_{1jk} = b_1 C_{1,k,j} + b_2 C_{0,k+1,j} + b_3 C_{0,k,j+1}, \qquad j+k = d-1. \tag{2.54}$$

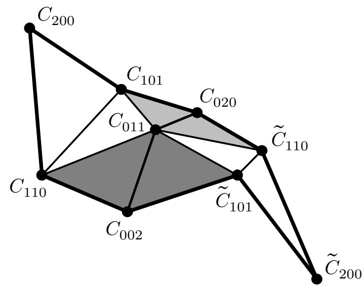


Fig. 2.6. Geometric interpretation of C^1 smoothness conditions.

These conditions hold if and only if for each $j + k = d - 1$, there is a plane which contains the four control points in (2.54). Figure 2.6 illustrates the case $d = 2$. It is also possible to interpret C^r smoothness conditions geometrically, see [Lai97] and [Kas98].

2.11. Computing Coefficients from Smoothness

In this section we explore how smoothness conditions between two adjoining polynomials can be used to compute some B-coefficients from others. Suppose that $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ are two adjoining triangles which share the edge $e := \langle v_2, v_3 \rangle$. Let p and $\tilde{p}$ be two polynomials of degree d with B-coefficients c_{ijk} and $\tilde{c}_{ijk}$ relative to T and $\widetilde{T}$, respectively, see (2.47) and (2.48). Throughout this section we suppose that p and $\tilde{p}$ join with C^r continuity across the edge e.

First, suppose we know all of the B-coefficients of p. Then clearly (2.50) can be used to compute the B-coefficients of $\tilde{p}$ corresponding to the domain points in the first r rows parallel to the edge e. The following lemma shows that this is a stable process.

Lemma 2.29. *Suppose that p and $\tilde{p}$ are polynomials on T and $\widetilde{T}$ which join with C^r smoothness across a common edge e as described in Theorem 2.28. Suppose the coefficients $\{c_{ijk}\}_{0 \le i \le r}$ of p are known, and that $C := \max_{0 \le i \le r} |c_{ijk}|$. Then the coefficients $\{\tilde{c}_{mjk}\}_{m \le r}$ of $\tilde{p}$ can be computed from (2.50), and are bounded by KC, where K is a constant depending only on the smallest angle $\theta_\triangle$ in the triangulation $\triangle := \{T, \widetilde{T}\}$.*

Proof: Suppose that (b_1, b_2, b_3) are the barycentric coordinates of v_4 with respect to T. Each of them is a ratio of the areas of two triangles which share a common edge. Now the area of the triangle T with edges e and $\tilde{e}$ separated by an angle θ is given by $A_T = \frac{1}{2}|e||\tilde{e}| \sin\theta$. By (4.3), the edges of T and of $\widetilde{T}$ are of comparable size with a constant depending only on $\theta_\triangle$. It follows that $|b_1|, |b_2|, |b_3|$ are bounded by a constant depending only on $\theta_\triangle$, and the result follows. $\square$

There is another important way in which the smoothness conditions can be used to compute coefficients of a spline from given coefficients.

Lemma 2.30. *Let $0 \le \ell+1 \le q, \tilde q$ and $q+\tilde q-\ell \le m \le d$. Suppose p and $\tilde p$ are as in Lemma 2.29, where the vertices v_1, v_2, v_4 of T and $\widetilde T$ are not collinear. Suppose that we know all coefficients c_{ijk} and $\tilde c_{ijk}$ of the polynomials p and $\tilde p$ corresponding to domain points in $D^T_{m-1}(v_2) \cup D^{\widetilde T}_{m-1}(v_2)$. In addition suppose that we also know the coefficients of p and $\tilde p$ on the ring $R_m(v_2)$ within a distance $q+\tilde q-\ell$ of $e := \langle v_2, v_3\rangle$, except for the coefficients*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \qquad && \nu = \ell+1,\ldots,q, \\ \tilde c_\nu &:= \tilde c_{\nu,m-\nu,d-m}, \qquad && \nu = \ell+1,\ldots,\tilde q. \end{aligned} \tag{2.55}$$

Then these coefficients are uniquely determined by the smoothness conditions

$$\tilde c_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(v_4), \qquad \ell+1 \le n \le q+\tilde q-\ell. \tag{2.56}$$

Proof: Let $c := (c_{\ell+1},\ldots,c_q,\tilde c_{\ell+1},\ldots,\tilde c_{\tilde q})^T$, and let $\alpha_1,\alpha_2,\alpha_3$ be the barycentric coordinates of v_4 relative to T, i.e., $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. By the noncollinearity assumption, α_1 and α_3 cannot be zero. Now (2.56) can be written in the form

$$Mc = b, \tag{2.57}$$

with

$$M := \begin{bmatrix} A & -I \\ B & O \end{bmatrix},$$

where I is the $(\tilde q-\ell)\times(\tilde q-\ell)$ identity matrix, O is the $(q-\ell)\times(\tilde q-\ell)$ zero matrix,

$$A_{ij} := \binom{\ell+i}{\ell+j} \alpha_1^{\ell+j}\alpha_3^{i-j}, \qquad \begin{aligned} i &= 1,\ldots,\tilde q-\ell, \\ j &= 1,\ldots,q-\ell, \end{aligned}$$

and

$$B_{ij} := \binom{\tilde q+i}{\ell+j} \alpha_1^{\ell+j}\alpha_3^{\tilde q-\ell+i-j}, \qquad i,j = 1,\ldots,q-\ell.$$

The right-hand side of (2.57) is given by

$$b_\nu = \begin{cases} -a_\nu, & 1 \le \nu \le \tilde q-\ell, \\ \tilde c_{\ell+\nu} - a_\nu, & \tilde q-\ell+1 \le \nu \le q+\tilde q-2\ell, \end{cases}$$

where

$$a_\nu := \sum_{i+j+k=\ell+\nu}{}' c_{i,j+d-r+\nu+\ell-n,k+r-\nu-\ell} B^{\ell+\nu}_{ijk}(v_4).$$

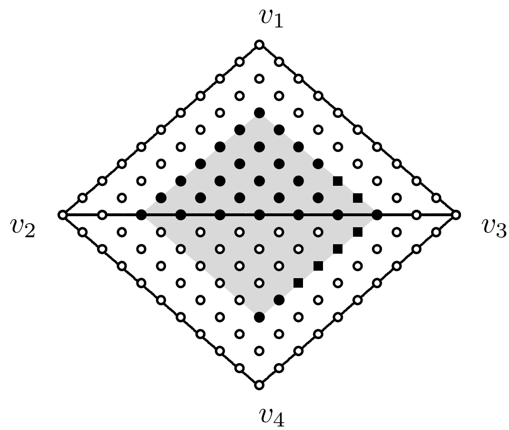


Fig. 2.7. Use of Lemma 2.30 in Example 2.31.

Here the prime on the sum means that the sum is taken over all i, j, k such that $c_{i,j+d-r+\nu+\ell-n,k+r-\nu-\ell}$ is not one of the coefficients defined in (2.55).

By the block structure, to prove that M is nonsingular, it suffices to examine B. Let $\tilde{B}$ be the matrix obtained by factoring the nonzero term $\alpha_1^{\ell+j}/(\ell+j)!$ from the j-th column of B for each $j = 1, \ldots, q-\ell$. We note that the matrix $\tilde{B}$ is the Gram matrix corresponding to the functions $\{x^{\tilde{q}+i}\}_{i=1}^{q-\ell}$, and the linear functionals $\{\epsilon_{\alpha_3} D^{\ell+j}\}_{j=1}^{q-\ell}$, where ϵ_{α_3} is point evaluation at α_3. Now if $\det(\tilde{B})$ were zero, there would exist a nontrivial polynomial $f := \sum_{i=1}^{q-\ell} a_i x^{\tilde{q}+i}$ satisfying $D^{\ell+j} f(\alpha_3) = 0$, for $j = 1, \ldots, q-\ell$. But then $g := D^{\ell+1} f$ would be a nontrivial polynomial of degree $q+\tilde{q}-2\ell-1$ which vanishes $\tilde{q}-\ell$ times at 0 and $q-\ell$ times at α_3. This is impossible, and we conclude that C cannot be zero. $\square$

Lemma 2.30 cannot be used when the points v_1, v_2, v_4 are collinear. In this case we say that the edge $e := \langle v_2, v_3 \rangle$ is degenerate. It should also not be used when e is near-degenerate, i.e., when α_3 becomes very small, since in this case the computation is not stable, see Section 10.3.1.

Example 2.31. Let $d = 10$, $m = r = 8$, $q = 2$, $\tilde{q} = 4$, and $\ell = 0$ in Lemma 2.30.

Discussion: Figure 2.7 shows the domain points of two adjoining triangles. We assume that we know the coefficients of p and $\tilde{p}$ corresponding to points marked with dots, and that we want to compute the coefficients c_{127}, c_{226} of p and $\tilde{c}_{172}$, $\tilde{c}_{262}$, $\tilde{c}_{352}$, $\tilde{c}_{442}$ of $\tilde{p}$ which correspond to the domain points marked with squares. These six coefficients lie on $R_8(v_2)$, and by the lemma, are uniquely determined by the $C^1, \ldots, C^6$ smoothness conditions listed in (2.56). We have shaded the support of this set of smoothness conditions. Note that it is not necessary to know all the coefficients of p and $\tilde{p}$

corresponding to domain points in the disk $D_7(v_2)$, just those involved in these six smoothness conditions. $\square$

2.12. The Markov Inequality on Triangles

In this section we prove the Markov inequality for bivariate polynomials.

Theorem 2.32. *Let $T := \langle v_1, v_2, v_3 \rangle$ be a triangle, and fix $1 \le q \le \infty$. Then there exists a constant K depending only on d such that for any polynomial $p \in \mathcal{P}_d$ and any nonnegative integers α and β with $0 \le \alpha + \beta \le d$,*

$$\|D_x^\alpha D_y^\beta p\|_{q,T} \le \frac{K}{\rho_T^{\alpha+\beta}} \|p\|_{q,T}, \tag{2.58}$$

where ρ_T denotes the radius of the largest circle inscribed in T.

Proof: As observed in the proof of Theorem 2.19, the unit vector u pointing in the direction of the x-axis has barycentric coordinates $(y_2-y_3, y_3-y_1, y_1-y_2)/2A_T$, where A_T is the area of T. Thus, if $p := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}$, then by (2.29),

$$D_x p = \frac{d}{2A_T} \sum_{i+j+k=d-1} \bigl[(y_2 - y_3) c_{i+1,j,k} + (y_3 - y_1) c_{i,j+1,k} \\ + (y_1 - y_2) c_{i,j,k+1}\bigr] B^{d-1}_{ijk}.$$

Since the B^{d-1}_{ijk} form a partition of unity, it follows that

$$\|D_x p\|_{\infty,T} \le \frac{d\,\|c\|_\infty (|y_2 - y_3| + |y_3 - y_1| + |y_1 - y_2|)}{2A_T}.$$

Then by (2.40),

$$\|D_x p\|_{\infty,T} \le \frac{d\,\|c\|_\infty}{\rho_T}.$$

Now (2.58) follows for $\alpha = 1$, $\beta = 0$, and $q = \infty$ from the stability result of Theorem 2.6. To prove (2.58) for general $1 \le q \le \infty$, we combine $\|c\|_\infty \le \|c\|_q$ with Theorems 1.1 and 2.7 to get

$$\|D_x p\|_{q,T} \le A_T^{1/q} \|D_x p\|_{\infty,T} \le \frac{d A_T^{1/q}}{\rho_T} \|c\|_\infty \le \frac{d A_T^{1/q}}{\rho_T} \|c\|_q \le \frac{dK}{\rho_T} \|p\|_{q,T},$$

where K is the constant in (2.19). The same proof can be used to estimate $\|D_y p\|_{q,T}$, and the general result follows by induction. $\square$

2.13. Integrals and Inner Products of B-Polynomials

In this section we give explicit formulae for integrals and inner products of polynomials in B-form.

Theorem 2.33. *Given a triangle T, let p be a polynomial of degree d written in the B-form (2.13). Then*

$$\int_T p(x,y)\,dx\,dy = \frac{A_T}{\binom{d+2}{2}} \sum_{i+j+k=d} c_{ijk}, \tag{2.59}$$

where A_T is the area of T.

Proof: Using (2.3), it is easy to see that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix},$$

where by (2.4), the value of the corresponding determinant is $2A_T$. Then using the fact that

$$\int_0^1 x^i (1-x)^j\,dx = \frac{i!\,j!}{(i+j+1)!},$$

we get

$$\begin{aligned}
&\frac{i!\,j!\,k!}{d!} \int_T B^d_{ijk}(x,y)\,dx\,dy \\
&= \int_T b_1^i b_2^j (1-b_1-b_2)^k\,dx\,dy = 2A_T \int_0^1 \int_0^{1-b_1} b_1^i b_2^j (1-b_1-b_2)^k\,db_2\,db_1 \\
&= 2A_T \int_0^1 b_1^i (1-b_1)^{j+k+1} \int_0^{1-b_1} \Big(\frac{b_2}{1-b_1}\Big)^j \Big(1 - \frac{b_2}{1-b_1}\Big)^k \frac{db_2}{(1-b_1)}\,db_1 \\
&= 2A_T \int_0^1 u^i (1-u)^{j+k+1}\,du \int_0^1 t^j (1-t)^k\,dt \\
&= 2A_T \frac{i!\,(j+k+1)!}{(i+j+k+2)!} \frac{j!\,k!}{(j+k+1)!},
\end{aligned}$$

which immediately implies

$$\int_T B^d_{ijk}(x,y)\,dxdy = \frac{2A_T}{(d+2)(d+1)} = \frac{A_T}{\binom{d+2}{2}}, \tag{2.60}$$

for all $i+j+k=d$. Then (2.59) follows by integrating (2.13) term by term. □

Theorem 2.33 leads directly to an explicit formula for the inner product of any two Bernstein basis polynomials. First, we note that

$$B^d_{ijk} B^d_{\nu\mu\kappa} = \frac{\binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}}{\binom{2d}{d}} B^{2d}_{i+\nu,j+\mu,k+\kappa}. \tag{2.61}$$

Using (2.60), we immediately get the following inner product formula.

Theorem 2.34.

$$\int_T B^d_{ijk}(x,y)\,B^d_{\nu\mu\kappa}(x,y)\,dxdy = \frac{\binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}A_T}{\binom{2d}{d}\binom{2d+2}{2}}. \tag{2.62}$$

Theorem 2.34 implies that for any two polynomials

$$p := \sum_{i+j+k=d} c_{ijk} B^d_{ijk} \quad \text{and} \quad q := \sum_{\nu+\mu+\kappa=d} \tilde{c}_{\nu\mu\kappa}\, B^d_{\nu\mu\kappa},$$

the inner product of p and q is given by

$$\int_T p(x,y)\,q(x,y)\,dx\,dy = \frac{A_T}{\binom{2d}{d}\binom{2d+2}{2}} \sum_{\substack{i+j+k=d\\ \nu+\mu+\kappa=d}} c_{ijk}\tilde{c}_{\nu\mu\kappa}\binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}.$$

This inner product can be written in the form

$$\int_T p(x,y)\,q(x,y)\,dx\,dy = \frac{A_T}{\binom{2d}{d}\binom{2d+2}{2}}\, c^T G\, \tilde{c}, \tag{2.63}$$

where c and $\tilde{c}$ are the vectors of coefficients of p and q, respectively (in lexicographical order), and G is a $\binom{d+2}{2}$ square matrix.

Example 2.35. When $d = 1$,

$$G = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 2.36. When $d = 2$,

$$G = \begin{bmatrix} 6 & 3 & 3 & 1 & 1 & 1 \\ 3 & 4 & 2 & 3 & 2 & 1 \\ 3 & 2 & 4 & 1 & 2 & 3 \\ 1 & 3 & 1 & 6 & 3 & 1 \\ 1 & 2 & 2 & 3 & 4 & 3 \\ 1 & 1 & 3 & 1 & 3 & 6 \end{bmatrix}.$$

Example 2.37. When $d = 3$,

$$G = \begin{bmatrix} 20 & 10 & 10 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \\ 10 & 12 & 6 & 9 & 6 & 3 & 4 & 3 & 2 & 1 \\ 10 & 6 & 12 & 3 & 6 & 9 & 1 & 2 & 3 & 4 \\ 4 & 9 & 3 & 12 & 6 & 2 & 10 & 6 & 3 & 1 \\ 4 & 6 & 6 & 6 & 8 & 6 & 4 & 6 & 6 & 4 \\ 4 & 3 & 9 & 2 & 6 & 12 & 1 & 3 & 6 & 10 \\ 1 & 4 & 1 & 10 & 4 & 1 & 20 & 10 & 4 & 1 \\ 1 & 3 & 2 & 6 & 6 & 3 & 10 & 12 & 9 & 4 \\ 1 & 2 & 3 & 3 & 6 & 6 & 4 & 9 & 12 & 10 \\ 1 & 1 & 4 & 1 & 4 & 10 & 1 & 4 & 10 & 20 \end{bmatrix}.$$

2.14. Subdivision

Suppose that p is a polynomial written in B-form on a triangle T with vertices v_1, v_2, and v_3. Then, clearly any point w in the interior of T splits T into three subtriangles

$$T_1 := \langle w, v_2, v_3 \rangle, \quad T_2 := \langle w, v_3, v_1 \rangle, \quad T_3 := \langle w, v_1, v_2 \rangle,$$

see Figure 2.1 (right). In this section we show how to write p in B-form on each of the subtriangles.

Theorem 2.38. *Given a triangle T, let w be a point in the interior of T with barycentric coordinates $a := (a_1, a_2, a_3)$. For each $\ell = 1, 2, 3$, let $B_{ijk}^{T_\ell, d}$ be the Bernstein basis polynomials associated with $T_\ell := \langle w, v_{\ell+1}, v_{\ell+2} \rangle$. Then for any polynomial p with B-coefficients as in (2.13),*

$$p(v) = \begin{cases} \displaystyle\sum_{i+j+k=d} c_{0jk}^{(i)} B_{ijk}^{T_1,d}(v), & v \in T_1, \\ \displaystyle\sum_{i+j+k=d} c_{i0k}^{(j)} B_{ijk}^{T_2,d}(v), & v \in T_2, \\ \displaystyle\sum_{i+j+k=d} c_{ij0}^{(k)} B_{ijk}^{T_3,d}(v), & v \in T_3, \end{cases}$$

where $c_{ijk}^{(\nu)} := c_{ijk}^{(\nu)}(a)$ are the quantities obtained in the ν-th step of the de Casteljau algorithm based on the triple a, starting with the coefficients $c_{ijk}^{(0)} := c_{ijk}$.

Proof: Given $v \in T_1$, let $\tilde{b}_1(v), \tilde{b}_2(v), \tilde{b}_3(v)$ be the barycentric coordinates of v relative to the triangle T_1. Substituting $w = a_1 v_1 + a_2 v_2 + a_3 v_3$ in $v = \tilde{b}_1 w + \tilde{b}_2 v_2 + \tilde{b}_3 v_3$, we get

$$v = \tilde{b}_1 a_1 v_1 + (\tilde{b}_1 a_2 + \tilde{b}_2) v_2 + (\tilde{b}_1 a_3 + \tilde{b}_3) v_2.$$

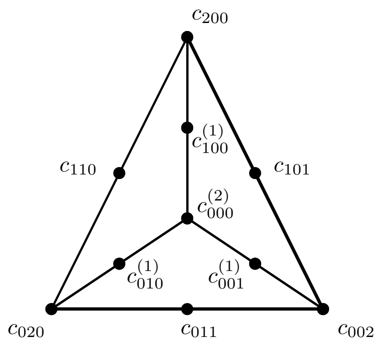


Fig. 2.8. Subdivision coefficients.

Thus,

$$B^{T,d}_{\nu\beta\gamma}(v) = \frac{d!\,(\tilde b_1 a_1)^\nu(\tilde b_1 a_2+\tilde b_2)^\beta(\tilde b_1 a_3+\tilde b_3)^\gamma}{\nu!\,\beta!\,\gamma!}$$
$$= \sum_{\mu=0}^{\beta}\sum_{\kappa=0}^{\gamma} B^{T_1,d}_{\nu+\mu+\kappa,\beta-\mu,\gamma-\kappa}(v)B^{T,\nu+\mu+\kappa}_{\nu\mu\kappa}(w),$$

for all $\nu+\beta+\gamma=d$. Now substituting this in

$$p(v) = \sum_{\nu+\beta+\gamma=d} c_{\nu\beta\gamma}B^{T,d}_{\nu\beta\gamma}(v),$$

we get

$$p(v) = \sum_{\nu+\beta+\gamma=d} c_{\nu\beta\gamma}\sum_{\mu=0}^{\beta}\sum_{\kappa=0}^{\gamma} B^{T_1,d}_{\nu+\mu+\kappa,\beta-\mu,\gamma-\kappa}(v)B^{T,\nu+\mu+\kappa}_{\nu\mu\kappa}(w).$$

Choosing $\beta=j+\mu$, $\gamma=k+\kappa$, and $\nu+\mu+\kappa=i$, we see that the coefficient of $B^{T_1,d}_{ijk}$ in this expansion is

$$\sum_{\nu+\mu+\kappa=i} c_{\nu,j+\mu,k+\kappa}B^{T,i}_{\nu\mu\kappa}(w),$$

which is just $c^{(i)}_{0jk}$ by (2.26). A similar proof works if v is in one of the other two triangles T_2 and T_3. □

One can visualize the coefficients of p in subdivided form as a pyramid of coefficients formed by stacking up the intermediate results in the de Casteljau algorithm as shown in Figure 2.8 for the case $d=3$.

The formulae in Theorem 2.38 remain valid in case w lies on the interior of one of the edges of T. In this case T is split into just two triangles rather than three, and the amount of computation to find the new coefficients is significantly reduced. It can be made even smaller if w is chosen to be the midpoint of an edge, say $w = (v_2 + v_3)/2$. Then the barycentric coordinates of w are $(0, 1/2, 1/2)$, and no multiplications are needed to find the new coefficients, just division by 2 (which is a binary shift operator). This observation leads to a highly efficient algorithm (see Method 3 in Section 3.8) for evaluating a polynomial in B-form at a large number of points in T for the purposes of displaying the associated surface.

2.15. Degree Raising

It is clear that a polynomial of degree d can also be regarded as a polynomial of any degree $\tilde{d} > d$. In this section we show how to find the B-coefficients of the degree-raised polynomial.

Theorem 2.39. *Let p be a polynomial of degree d defined on a triangle T written in the B-form (2.13). Let $c_{ijk}^{[d]} := c_{ijk}$ be its coefficients. Then*

$$p = \sum_{i+j+k=d+1} c_{ijk}^{[d+1]} B_{ijk}^{d+1}, \tag{2.64}$$

where B_{ijk}^{d+1} are the Bernstein basis polynomials of degree $d+1$ associated with T, and where

$$c_{ijk}^{[d+1]} := \frac{i c_{i-1,j,k}^{[d]} + j c_{i,j-1,k}^{[d]} + k c_{i,j,k-1}^{[d]}}{d+1}, \tag{2.65}$$

for $i + j + k = d + 1$. Here coefficients with negative subscripts are taken to be zero.

Proof: Multiplying both sides of the equation (2.13) by $1 = b_1 + b_2 + b_3$, we get

$$p = \sum_{i+j+k=d} c_{ijk}^{[d]} \frac{d!}{i!\,j!\,k!} b_1^i b_2^j b_3^k (b_1 + b_2 + b_3).$$

Then multiplying out and collecting terms, we get (2.65). □

To raise the degree of a polynomial by more than one, we simply repeat the above process.

2.16. Dual Bases for the Bernstein Basis Polynomials

A set of linear functionals $\Lambda^d = \{\lambda_{ijk}\}_{i+j+k=d}$ defined on $\mathcal{P}_d$ with the property

$$\lambda_{\nu\mu\kappa} B_{ijk} = \delta_{ijk,\nu\mu\kappa}, \qquad \text{all } i+j+k=d \text{ and } \nu+\mu+\kappa=d, \tag{2.66}$$

is called a dual basis for $\mathcal{B}^d$. In this section we explicitly construct a set of linear functionals defined on $C(T)$ that form a dual basis for $\mathcal{B}^d$, and a second set defined on $C^d(T)$ that also form a dual basis.

We use the notation of Section 2.4. In particular, with $m := \binom{d+2}{2}$, let $\{g_1, \ldots, g_m\}$ be the Bernstein basis polynomials $\{B_{ijk}\}_{i+j+k=d}$ arranged in lexicographical order, and let $\{t_1, \ldots, t_m\}$ be the associated domain points $\{\xi_{ijk}\}_{i+j+k=d}$ arranged in the same order. Let M be the nonsingular matrix (2.16). Now given $i + j + k = d$, let r^{ijk} be the vector of length m with zeros in all positions except for 1 in position

$$\ell := \binom{d-i+1}{2} + d - i - j + 1,$$

which is the index such that $t_\ell = \xi_{ijk}$. Let $a^{ijk} := (a_1^{ijk}, \ldots, a_m^{ijk})^T$ be the solution of the system

$$M a^{ijk} = r^{ijk}. \tag{2.67}$$

Define

$$\lambda_{ijk} p := \sum_{\nu=1}^{m} a_\nu^{ijk} p(t_\nu).$$

Theorem 2.40. *The linear functionals $\{\lambda_{ijk}\}_{i+j+k=d}$ form a dual basis for $\mathcal{B}^d$.*

Proof: The equations (2.67) assert that

$$\lambda_{\nu\mu\kappa} B_{ijk}^d = \begin{cases} 0, & (\nu,\mu,\kappa) \neq (i,j,k), \\ 1, & (\nu,\mu,\kappa) = (i,j,k), \end{cases}$$

for all $\nu + \mu + \kappa = d$, and the duality follows. $\square$

The dual functionals λ_{ijk} constructed in Theorem 2.40 are defined for all continuous functions on T. The following lemma shows that their norms are uniformly bounded.

Lemma 2.41. *There exists a constant K depending only on d such that*

$$|\lambda_{ijk} f| \leq K \|f\|_T, \qquad \text{all } i + j + k = d, \tag{2.68}$$

for all $f \in C(T)$.

Proof: By (2.67), $|a_\nu^{ijk}| \leq \|M^{-1}\|_\infty$ for all $i + j + k = d$. But then

$$|\lambda_{ijk} f| \leq \sum_{\nu=1}^{m} |a_\nu^{ijk}| \; |f(t_\nu)| \leq \|f\|_T \sum_{\nu=1}^{m} |a_\nu^{ijk}| \leq K \|f\|_T,$$

where $K := \binom{d+2}{2} \|M^{-1}\|_\infty$. $\square$

The linear functionals of Theorem 2.40 are based on point evaluations and are defined for all continuous functions. In the following theorem we give a set of linear functionals defined in terms of derivatives which also gives a dual basis for $\mathcal{P}_d$.

Theorem 2.42. *The linear functionals*

$$\widetilde{\lambda}_{d-i-j,i,j}\, p := \sum_{m=0}^{i}\sum_{n=0}^{j}\binom{i}{m}\binom{j}{n}\frac{(d-m-n)!}{d!}D_{v_2-v_1}^{m}D_{v_3-v_1}^{n}p(v_1),$$

$0 \le i+j \le d$, *form a dual basis for* $\mathcal{B}^d$.

Proof: It is clear from (2.37) that for any polynomial p of degree d, $\widetilde{\lambda}_{d-i-j,i,j}\, p$ gives its B-coefficient $c_{d-i-j,i,j}$. But this immediately implies

$$\widetilde{\lambda}_{\nu\mu\kappa} B_{ijk} = \delta_{ijk,\nu\mu\kappa}, \qquad \text{all } i+j+k=d \quad \text{and} \quad \nu+\mu+\kappa=d. \quad \square$$

2.17. A Quasi-interpolant

In Theorem 1.12 we showed that any function in $C^{d+1}(T)$ on a triangle T is approximated to order $\mathcal{O}(|T|^{d+1})$ by the polynomial p of degree d which interpolates f at the set of domain points $\mathcal{D}_{d,T} := \{\xi_{ijk}\}_{i+j+k=d}$ described in (2.14), where $|T|$ is the diameter of the triangle T. In this section we show how to construct another polynomial with the same approximation order.

Let $\{\lambda_{ijk}\}_{i+j+k=d}$ be the dual linear functionals to the Bernstein basis polynomials $\{B_{ijk}^d\}_{i+j+k=d}$ of Theorem 2.40. Recall that $\lambda_{ijk} f$ is a linear combination of the values of f at the points of $\mathcal{D}_{d,T}$. For any function f defined on $\mathcal{D}_{d,T}$, let

$$Qf := \sum_{i+j+k=d} (\lambda_{ijk} f)\; B_{ijk}^d. \tag{2.69}$$

We can think of Q as an *operator* mapping functions defined on $\mathcal{D}_{d,T}$ into the space $\mathcal{P}_d$ of polynomials of degree d. It is customary to call this type of operator a quasi-interpolation operator.

Theorem 2.43. *Q is a projector mapping $C(T)$ onto $\mathcal{P}_d$. Moreover,*

$$\|Qf\|_T \le K\, \|f\|_T, \tag{2.70}$$

for all $f \in C(T)$, where K depends only on d.

Proof: Suppose $p := \sum_{i+j+k=d} c_{ijk} B_{ijk}^d$. Then by the duality (2.66), $\lambda_{ijk}\, p = c_{ijk}$ for all $i+j+k=d$, and so

$$Qp = \sum_{i+j+k=d} (\lambda_{ijk}\, p) B_{ijk}^d = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d = p,$$

which establishes that $Qp = p$ for all $p \in \mathcal{P}_d$. To show that Q is bounded on $C(T)$, we use the bound (2.68) on the λ_{ijk} and the fact that the Bernstein basis polynomials form a partition of unity as in (2.9). Then for any $v \in T$,

$$\begin{aligned} |Qf(v)| &\le \sum_{i+j+k=d} |\lambda_{ijk} f| \, B^d_{ijk}(v) \\ &\le K \,\|f\|_T \sum_{i+j+k=d} B^d_{ijk}(v) = K \,\|f\|_T, \end{aligned}$$

which immediately implies (2.70) with the constant in (2.68). $\square$

Corollary 2.44. *For any $f \in C(T)$,*

$$\|f - Qf\|_T \le (1 + K)\; d(f, \mathcal{P}_d)_T, \tag{2.71}$$

where

$$d(f, \mathcal{P}_d)_T := \inf_{p \in \mathcal{P}_d} \|f - p\|_T$$

is the distance of f from the space of polynomials $\mathcal{P}_d$ *measured in the uniform norm on T.*

Proof: Since $Qp = p$ for any $p \in \mathcal{P}_d$,

$$\begin{aligned} \|f - Qf\|_T &\le \|f - p\|_T + \|p - Qp\|_T + \|Q(p - f)\|_T \\ &\le (1 + \|Q\|)\, \|f - p\|_T, \end{aligned}$$

where $\|Q\|$ is the usual operator norm defined by $\|Q\| := \sup\{\|Qf\|/\|f\| : f \ne 0\}$. Then (2.71) follows from (2.70) since p is arbitrary. $\square$

Combining (2.71) with Theorem 1.3, it follows that

$$\|f - Qf\|_T \le K\, |T|^{m+1}\, |f|_{m+1,T},$$

for all $f \in C^{m+1}(T)$ with $0 \le m \le d$. Here $|f|_{m+1,T}$ is the seminorm of f defined in (1.7), and K is a constant depending only on d.

2.18. The Bernstein Approximation Operator

Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, let $\{\xi_{ijk}\}_{i+j+k=d}$ be the corresponding domain points. Then for any function $f \in C(T)$, we can define a mapping B_d from $C(T)$ into $\mathcal{P}_d$ by

$$B_d f := \sum_{i+j+k=d} f(\xi_{ijk}) B^d_{ijk}. \tag{2.72}$$

B_d is called the Bernstein operator, and $B_d f$ is called the Bernstein polynomial associated with f. It has been studied extensively in the literature. We do not have space here to present all of its interesting properties.

B_d is a quasi-interpolation operator which is much simpler than the Q defined in (2.69), and which has certain nice properties. For example B_d is a positive or monotone operator in the sense that if $f(v) \geq 0$ on T, then $B_d f(v) \geq 0$ on T. On the other hand, it is not a projector, and does not produce as good an approximation as Q for functions that are smoother than $C^2(T)$, see Remark 2.11. Let

$$\begin{aligned}\omega(f,h) := \max\{|f(x,y) - f(\tilde{x},\tilde{y})| :\ (x,y),\ (\tilde{x},\tilde{y}) \in T,\\ |x-\tilde{x}|^2 + |y-\tilde{y}|^2 \leq h^2\}\end{aligned} \tag{2.73}$$

be the modulus of continuity of f relative to the triangle T.

Theorem 2.45. *For any d,*

$$\|f - B_d f\|_T \leq \begin{cases} \omega(f,|T|), & \text{if } f \in C(T), \\ |T|\,|f|_{1,T}, & \text{if } f \in C^1(T), \\ \frac{|T|^2}{d}|f|_{2,T}, & \text{if } f \in C^2(T). \end{cases}$$

Proof: Using the partition of unity property (2.9),

$$B_d f(v) - f(v) = \sum_{i+j+k=d} \big[f(\xi_{ijk}) - f(v)\big] B^d_{ijk}(v),$$

which implies the result for $f \in C(T)$. The result for $f \in C^1(T)$ follows immediately from the fact that for any such function,

$$\omega(f,h) \leq h|f|_{1,T}. \tag{2.74}$$

We now prove the result for $f \in C^2(T)$. For any $i+j+k=d$,

$$f(\xi_{ijk}) - f(v) = D_{ijk} f(v) + \frac{1}{2} D^2_{ijk} f\big(v + \eta_{ijk}(\xi_{ijk} - v)\big),$$

for some $\eta_{ijk} \in [0,1]$, where $D_{ijk} := D_{\xi_{ijk}-v}$ is the directional derivative. Thus,

$$\begin{aligned} B_d f(v) - f(v) &= \sum_{i+j+k=d} [f(\xi_{ijk}) - f(v)] B^d_{ijk}(v) \\ &= \sum_{i+j+k=d} D_{ijk} f(v) B^d_{ijk}(v) + \frac{1}{2} \sum_{i+j+k=d} D^2_{ijk} f\big(v + \eta_{ijk}(\xi_{ijk} - v)\big) B^d_{ijk}(v). \end{aligned}$$

Suppose $v = (x, y)$. In the proof of Theorem 2.4 we showed that

$$\begin{aligned} x &= \sum_{i+j+k=d} \xi^x_{ijk} B^d_{ijk}(v), \\ y &= \sum_{i+j+k=d} \xi^y_{ijk} B^d_{ijk}(v), \end{aligned} \tag{2.75}$$

where $\xi_{ijk} := (\xi^x_{ijk}, \xi^y_{ijk})$. Since

$$D_{ijk} f(v) = (\xi^x_{ijk} - x) D_x f(v) + (\xi^y_{ijk} - y) D_y f(v),$$

it follows that

$$\begin{aligned} &\sum_{i+j+k=d} D_{ijk} f(v) B^d_{ijk}(v) \\ &= D_x f(v) \sum_{i+j+k=d} (\xi^x_{ijk} - x) B^d_{ijk}(v) + D_y f(v) \sum_{i+j+k=d} (\xi^y_{ijk} - y) B^d_{ijk}(v) = 0. \end{aligned}$$

Now

$$|D^2_{ijk} f(v + \eta_{ijk}(\xi_{ijk} - v))| \le 2 |v - \xi_{ijk}|^2 |f|_{2,T},$$

and thus

$$|f(v) - B_d f(v)| \le |f|_{2,T} \sum_{i+j+k=d} |v - \xi_{ijk}|^2 B^d_{ijk}(v).$$

To complete the proof, we apply the following lemma. □

Lemma 2.46. *For any $v \in T$,*

$$\sum_{i+j+k=d} |v - \xi_{ijk}|^2 B^d_{ijk}(v) \le \frac{|T|^2}{d}. \tag{2.76}$$

Proof: Let b_1, b_2, b_3 be the barycentric coordinates of $v := (x, y)$ relative to T. An elementary calculation shows that

$$\begin{aligned} \sum_{i+j+k=d} ij B^d_{ijk}(x,y) &= d(d-1) b_1 b_2, \\ \sum_{i+j+k=d} ik B^d_{ijk}(x,y) &= d(d-1) b_1 b_3, \\ \sum_{i+j+k=d} jk B^d_{ijk}(x,y) &= d(d-1) b_2 b_3, \\ \sum_{i+j+k=d} i^2 B^d_{ijk}(x,y) &= d(d-1) b_1^2 + d b_1, \\ \sum_{i+j+k=d} j^2 B^d_{ijk}(x,y) &= d(d-1) b_2^2 + d b_2, \\ \sum_{i+j+k=d} k^2 B^d_{ijk}(x,y) &= d(d-1) b_3^2 + d b_3. \end{aligned} \tag{2.77}$$

Then

$$\sum_{i+j+k=d} |v-\xi_{ijk}|^2 B^d_{ijk}(v) = \sum_{i+j+k=d} \big[(\xi^x_{ijk}-x)^2 + (\xi^y_{ijk}-y)^2\big] B^d_{ijk}(v)$$
$$= \sum_{i+j+k=d} \Big[\frac{i}{d}(x_1-x) + \frac{j}{d}(x_2-x) + \frac{k}{d}(x_3-x)\Big]^2 B^d_{ijk}(v)$$
$$+ \sum_{i+j+k=d} \Big[\frac{i}{d}(y_1-y) + \frac{j}{d}(y_2-y) + \frac{k}{d}(y_3-y)\Big]^2 B^d_{ijk}(v).$$

Using (2.77) and the identities $1 = b_1+b_2+b_3$ and $x = b_1x_1+b_2x_2+b_3x_3$, we have

$$\sum_{i+j+k=d} \Big[\frac{i}{d}(x_1-x) + \frac{j}{d}(x_2-x) + \frac{k}{d}(x_3-x)\Big]^2 B^d_{ijk}(v)$$
$$= \frac{d(d-1)}{d^2}[b_1(x_1-x)+b_2(x_2-x)+b_3(x_3-x)]^2$$
$$+ \frac{1}{d}[b_1(x_1-x)^2+b_2(x_2-x)^2+b_3(x_3-x)^2]$$
$$= \frac{1}{d}[b_1(x_1-x)^2+b_2(x_2-x)^2+b_3(x_3-x)^2].$$

Similarly, using $y = b_1y_1+b_2y_2+b_3y_3$, we have

$$\sum_{i+j+k=d} \Big[\frac{i}{d}(y_1-y) + \frac{j}{d}(y_2-y) + \frac{k}{d}(y_3-y)\Big]^2 B^d_{ijk}(v)$$
$$= \frac{1}{d}[b_1(y_1-y)^2+b_2(y_2-y)^2+b_3(y_3-y)^2].$$

Combining these inequalities gives (2.76). □

Theorem 2.45 gives precise information on how quickly $B_d f$ converges to f as $|T| \to 0$. Our next theorem describes the rate of convergence as $d \to \infty$.

Theorem 2.47. *For any d,*

$$\|f - B_d f\|_T \le \begin{cases} (1+|T|^2)\,\omega(f, \frac{1}{\sqrt{d}}), & \text{if } f \in C(T), \\ \frac{(1+|T|^2)}{\sqrt{d}}|f|_{1,T}, & \text{if } f \in C^1(T), \\ \frac{|T|^2}{d}|f|_{2,T}, & \text{if } f \in C^2(T). \end{cases} \tag{2.78}$$

Proof: We begin by proving the result for $f \in C(T)$. Given $h > 0$ and points $v := (x,y), \tilde{v} := (\tilde{x},\tilde{y}) \in T$, let $m := m(v,\tilde{v},h)$ be the integer

$$m := \Bigg\lfloor \frac{\big(|x-\tilde{x}|^2 + |y-\tilde{y}|^2\big)^{\frac{1}{2}}}{h} \Bigg\rfloor.$$

Then
$$|f(v) - f(\tilde{v})| \le (1+m)\omega(f,h),$$
and using Lemma 2.46, it follows that
$$\begin{aligned}
|f(v) - B_d f(v)| &\le \sum_{i+j+k=d} |f(v) - f(\xi_{ijk})| B^d_{ijk}(v) \\
&\le \omega(f,h) \sum_{i+j+k=d} [1 + m(v, \xi_{ijk}, h)] B^d_{ijk}(v) \\
&\le \omega(f,h) \sum_{i+j+k=d} \left[1 + \frac{|x - \xi^x_{ijk}|^2 + |y - \xi^y_{ijk}|^2}{h^2} \right] B^d_{ijk}(v) \\
&\le \omega(f,h) \big[1 + \frac{1}{dh^2} |T|^2 \big].
\end{aligned}$$
Letting $h = 1/\sqrt{d}$ implies the first inequality in (2.78). The result for $f \in C^1(T)$ follows immediately from (2.74). Finally, the result for $f \in C^2(T)$ was already established in Theorem 2.45. □

We conclude this section with an interesting result about the behavior of the Bernstein polynomials $B_d f$ in the case where f is convex. The proof is based on degree raising. Recall that f is convex on T if and only if for all m,
$$\sum_{i=1}^m \alpha_i f(v_i) \ge f\left(\sum_{i=1}^m \alpha_i v_i \right)$$
for any points $v_i \in T$ and any $\alpha_i \ge 0$ with $\sum_{i=1}^m \alpha_i = 1$.

Theorem 2.48. *Suppose f is convex on the triangle T. Then*
$$B_1 f(v) \ge B_2 f(v) \ge \cdots \ge B_d f(v) \ge \cdots \ge f(v), \qquad \text{all } v \in T.$$

Proof: Using equations (2.75), it is clear that
$$B_d f(v) = \sum_{i+j+k=d} f(\xi^d_{ijk}) B^d_{ijk}(v) \ge f\Big(\sum_{i+j+k=d} \xi^d_{ijk} B^d_{ijk}(v) \Big) = f(v),$$
where we now write the superscript on ξ^d_{ijk} to emphasize that they are domain points associated with polynomials of degree d. This implies the last inequality in the statement of the theorem.

To complete the proof, we now show that $B_{d+1} f(v) \le B_d f(v)$ for all $v \in T$ and all d. First, it is easy to check that
$$\xi^{d+1}_{ijk} = \frac{i\xi^d_{i-1,j,k} + j\xi^d_{i,j-1,k} + k\xi^d_{i,j,k-1}}{d+1}, \quad \text{all } i+j+k = d+1.$$

Since f is convex,

$$f(\xi_{ijk}^{d+1}) \le \frac{if(\xi_{i-1,j,k}^d) + jf(\xi_{i,j-1,k}^d) + kf(\xi_{i,j,k-1}^d)}{d+1},$$

for all $i+j+k=d+1$. Now applying the degree-raising formula (2.65), we have

$$\begin{aligned} B_{d+1}f - B_d f &= \sum_{i+j+k=d+1} f(\xi_{ijk}^{d+1}) B_{ijk}^{d+1} - \sum_{\nu+\mu+\kappa=d} f(\xi_{\nu\mu\kappa}^d) B_{\nu\mu\kappa}^d \\ &= \sum_{i+j+k=d+1} \left[f(\xi_{ijk}^{d+1}) - \frac{if(\xi_{i-1,j,k}^d) + jf(\xi_{i,j-1,k}^d) + kf(\xi_{i,j,k-1}^d)}{d+1} \right] B_{ijk}^{d+1}, \end{aligned}$$

which is less than or equal to zero. This completes the proof. $\square$

2.19. Remarks

Remark 2.1. It is clear from their interpretation in terms of areas of triangles that the barycentric coordinates of a point v in a triangle T do not depend on the orientation or location of the triangle in the plane. Actually, more is true: barycentric coordinates are affine invariant. In particular, let ϕ be the map taking points $(x, y) \in \mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$\phi \begin{bmatrix} x \\ y \end{bmatrix} := A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

where A is a 2×2 matrix. Maps of this form are called affine maps, and include both translation and rotation as special cases. Suppose the barycentric coordinates of the point $v := (x, y)^T$ relative to the triangle $T := \langle v_1, v_2, v_3 \rangle$ are (b_1, b_2, b_3). Then

$$\begin{aligned} \phi\Big(\sum_{i=1}^3 b_i v_i\Big) &= A \sum_{i=1}^3 b_i v_i + v_0 = \sum_{i=1}^3 b_i A v_i + \sum_{i=1}^3 b_i v_0 \\ &= \sum_{i=1}^3 b_i (A v_i + v_0) = \sum_{i=1}^3 b_i \phi(v_i), \end{aligned}$$

where we are now thinking of v, $v_0 := (x_0, y_0)^T$, and the v_i as 2-vectors. This shows that the barycentric coordinates of $\phi(v)$ relative to $\phi(T)$ are the same as those of v relative to T.

Remark 2.2. Many authors use multi-indices in working with the B-form, see e.g. [Boo87, Far88, HosL93]. However, in this book we have chosen to write all subscripts explicitly because we think it enhances understanding.

Remark 2.3. It is easy to prove that

$$b_1 = \frac{1}{d} \sum_{i+j+k=d} i B^d_{ijk}.$$

A similar formula holds for b_2 and b_3, and it follows that the B-coefficients of these functions are

$$c_{ijk} := \begin{cases} i/d, & \text{for } b_1, \\ j/d, & \text{for } b_2, \\ k/d, & \text{for } b_3, \end{cases} \qquad i+j+k=d.$$

Remark 2.4. It is clear that if we restrict a bivariate polynomial to a line, we get a univariate polynomial. The situation where we restrict a polynomial p in the B-form (2.13) to an edge of the associated triangle T is especially interesting. Suppose the vertices of T are v_1, v_2, v_3, and that we restrict p to the edge $\langle v_2, v_3 \rangle$ where the barycentric coordinate function $b_1(v)$ is identically zero. Then

$$p(v) = \sum_{j+k=d} c_{0jk} \frac{d\,!}{j\,!\,k\,!} b_2^j b_3^k = \sum_{j=0}^{d} c_{0,j,d-j} \frac{d\,!}{j\,!\,(d-j)\,!} b_2^j (1-b_2)^{d-j}. \tag{2.79}$$

This is just a sum of univariate Bernstein basis polynomials, which in standard notation are defined on the interval $[0,l]$ by

$$\phi^d_j(t) := \frac{d\,!}{j\,!\,(d-j)\,!} \Big(\frac{t}{l}\Big)^j \Big(\frac{l-t}{l}\Big)^{d-j}, \qquad j=0,\ldots,d, \tag{2.80}$$

for $0 \le t \le l$. Univariate Bernstein basis polynomials have been extensively studied in approximation theory and more recently in CAGD where they are extremely useful for describing curves both in functional form and in parametric form. For details and references, see [Lor53, GonM83, GonM86, Far88, HosL93, Gal00, CohRE01, FarHK02, PraBP02].

Remark 2.5. Suppose $T := \langle v_1, v_2, v_3 \rangle$ is a triangle, and let p be a bivariate polynomial in the B-form (2.13). Then in view of the previous remark, if a point v falls on the edge $\langle v_2, v_3 \rangle$ of T, we can compute $p(v)$ by evaluating the univariate polynomial in (2.79). In this case, we can use the following simplified version of the de Casteljau Algorithm 2.9. Let $c^{(0)}_{0jk} := c_{0jk}$ for all $j+k=d$, and let $(0, b_2, b_3)$ be the barycentric coordinates of v.

Algorithm 2.49. *(Univariate de Casteljau)*

For $\ell = 1, \ldots, d$
For all $j+k = d-\ell$
$c^{(\ell)}_{0jk} := b_2\, c^{(\ell-1)}_{0,j+1,k} + b_3\, c^{(\ell-1)}_{0,j,k+1}$

By Theorem 2.8 the value of $p(v)$ is given by $c^{(d)}_{000}$. The operation count (multiplications and divisions) for this algorithm is $d^2 + d$ as compared to $(d^3 + 3d^2 + 2d)/2$ for Algorithm 2.9.

Remark 2.6. Any basis for the space of polynomials can be orthogonalized with respect to the usual L_2 inner product on a fixed triangle, but depending on what basis we start with, there may not be nice expressions for the resulting orthogonal polynomials. For a recursive construction where the orthogonal polynomials are expressed in B-form, see [FaroGS03].

Remark 2.7. Conjecture 2.22 was formulated in 2003 by the second author. It was stated formally for the first time in [AlfS03].

Remark 2.8. Suppose p is a bivariate polynomial of degree d. Then clearly p can also be considered as a tensor-product polynomial of degree at most d in x and y. Now suppose we are given the B-form of p relative to a triangle T, and suppose H is a rectangular subset of T. Explicit formula for the control points of the tensor-product representation of p in terms of the control points of the B-form of p can be found in [Las02]. To get a tensor-product representation of p on all of T, one can split T into three rectangular subregions, see [Hu96, LinW91].

Remark 2.9. Subdivision of a polynomial in B-form as discussed in Section 2.14 can also be explained in terms of so-called blossoms or polar forms. For details and references, see [Sei89].

Remark 2.10. Since for any $f \in C(T)$, $\omega(f;t)$ goes to zero as t goes to zero, Theorem 2.47 implies that $B_d f$ converges uniformly to f as $d \to \infty$. This establishes the celebrated Weierstrass Approximation Theorem, which asserts that any continuous function can be approximated to arbitrary accuracy by a polynomial of sufficiently high degree.

Remark 2.11. Since the Bernstein operator B_d introduced in (2.72) is a positive operator, it is saturated. This means that no matter how smooth f may be, we cannot get a rate of convergence which exceeds $\frac{1}{d}$, which is the rate given in Theorem 2.47 for $f \in C^2(T)$. For more on saturation, see [Lor53]. See also [Lai92a] for the asymptotic expansion of $f - B_d$.

Remark 2.12. In [SchV86] it was suggested that for practical computations, there may be some advantage in replacing the B-form by a representation which uses renormalized Bernstein basis polynomials where the factorials in (2.7) are dispensed with. This leads to an evaluation algorithm which is more efficient than the de Casteljau algorithm. However, a recent comparison of the representations indicates that the B-form is somewhat more stable [MaiP06a].

Remark 2.13. Given a triangle T, let b_1, b_2, b_3 be the linear polynomials describing the barycentric coordinates of points in T. Then we can define the associated trigonometric Bernstein basis functions of degree d as $TB^d_{ijk} := \sin^i(b_1)\sin^j(b_2)\sin^k(b_3)/\sin^d(1)$ for $i + j + k = d$. Clearly these functions reduce to trigonometric polynomials of degree d along any line cutting through T, and in particular along the edges. It was shown in

[Wal97] that these functions have some of the properties of the Bernstein basis polynomials B^d_{ijk} discussed in this chapter, including the partition of unity property.

2.20. Historical Notes

There is a rich literature on the univariate Bernstein polynomials introduced by Bernstein in 1912, see [Lor53] or any of the standard approximation theory books. For a more recent list of references on Bernstein polynomials and related matters, see [GonM83, GonM86]. The use of univariate Bernstein polynomials in representing curves for CAGD purposes seems to have been implicit in the work P. Bézier (an engineer who led a design laboratory at Renault). For an account of Bézier's work at Renault, see Chapter 1 of [Far88], which also discusses work carried out at General Motors and at Citroen in the early 1960's. For a more detailed biography of Bézier, see [LauS01]. While Bézier was working at Renault in the early 1960's, P. de Casteljau led a similar design laboratory at Citroen. In 1959 he had published a technical report [Cas59] (in French) which contains a version of his algorithm for curves. The bivariate analog contained in Algorithm 2.9 can be found in [Cas63], see [BoeM99] for the history and photocopies of the relevant pages from [Cas59, Cas63]. de Casteljau's work seems not to have been known in the mathematical world until W. Boehm discovered these technical reports in 1975.

Bivariate Bernstein basis polynomials associated with a triangle were studied in the approximation theory literature in the early 1950's, but did not attract a lot of attention at the time, see [Lor53, Sta59] and also [Sta80]. Barycentric coordinates relative to a triangle were used much earlier, and can be found already in the work of Moebius in 1827, see his collected works [Moe86].

The study of bivariate Bernstein basis polynomials as a tool for representing surface patches in a CAGD context seems to have begun in the mid 1970's, see [Far77, Far79, Sab77]. The expansion of a bivariate polynomial in terms of Bernstein basis polynomials became known as the Bernstein–Bézier representation, see [BoeFK84], and is still called that in much of the literature. Here we have adopted the suggestion of de Boor [Boo87] to call it the B-form to honor both Bernstein and Bézier. The stability of the Bernstein basis polynomials in the univariate setting was studied in [FaroG96]. We could not find a reference for the stability of the bivariate basis described in Theorem 2.6. Formulae for computing directional derivatives of polynomials written in B-form can be found in Sabin's thesis [Sab77].

The discovery that C^1 smoothness across the common edge between two polynomials defined on adjoining triangles can be characterized by very simple linear conditions on the coefficients was a major step forward in making the B-form an essential tool in studying multivariate splines.

The C^1 case was discussed in [Sab77], while the geometric meaning of the conditions was first explained in [Far82]. The smoothness conditions for C^r joins between two polynomials in B-form were introduced in [Far80], see also [Far86]. The geometric meaning of C^r smoothness conditions was explored in [Lai97] and [Kas98]. Alternative smoothness conditions in terms of the angles of the triangles involved were described in [Hon95].

Various versions of Lemma 2.30 on the use of smoothness conditions to compute certain coefficients of a pair of polynomials on adjoining triangles seem to have been discovered independently by several authors, see [BooH88, IbrS91]. The result in the general form presented here comes from [AlfS02a].

Conjecture 2.22 on interpolation by restricted sets of Bernstein basis polynomials was formulated by the second author in 2003. A formal statement can be found in [AlfS05a]. The results for the special choice of Γ in Lemma 2.25 are new. The Markov inequality (2.58) for bivariate polynomials was established for the maximum norm in [Coa66], and also in [Wil74] by an argument based on the univariate result. Our proof of Theorem 2.32, which establishes the inequality for general q-norms, is based on the B-form, and comes from our paper [LaiS98].

The simple formulae (2.60) for the integrals of the Bernstein basis polynomials may have been derived by probabilists, but we have not found a reference. These formulae are mentioned in the survey [BoeFK84], and also show up in [Far88]. The formulae for inner products in Theorem 2.34 can be found in [ChuL90a].

The observation that the de Casteljau algorithm can be used to find the coefficients of a subdivided B-polynomial can be found in [Far80], see also [Gol83, Pra84]. The formulae of Theorem 2.39 for degree raising of B-polynomials seems to have been known even earlier, see [Far79] where it was also noted that the sequence of degree-raised surfaces converges to the corresponding polynomial surface.

Dual bases for the Bernstein basis polynomials were constructed independently by several authors, see [ZhaS88, Lai91, Wu93]. The results presented in Section 2.16 follow [Lai91]. The use of quasi-interpolants to establish approximation results is well established in approximation theory. They were studied in the context of univariate splines in [Boo68, Boo73, BooF73, LycS75]. For work in the bivariate setting, see [Boo90, ChuL90b], and references therein. For a discussion of how to get bivariate approximation results without using quasi-interpolants, see [Boo92].

The univariate analog of the Bernstein approximation operator discussed in Section 2.18 is well known in classical approximation theory, see [Lor53]. For some asymptotic expansions connected with the bivariate operator, see [FenK92] and [Lai92a]. Bernstein basis functions and the Bernstein polynomial operator have several other interesting properties which we do not have space to discuss here.

B-Patches

In this chapter we discuss properties of surface patches associated with polynomials defined on a triangle. In particular, we show that their shapes are closely related to properties of two useful control structures – control nets and control surfaces. In addition to discussing positivity, monotonicity, and convexity, we show how interpolation, degree raising, and subdivision can be used to efficiently render surface patches.

3.1. Control Nets and Control Surfaces

Given a function f defined on a triangle T, let

$$\mathcal{G}_f := \{(x, y, f(x, y)) : (x, y) \in T\}$$

be its graph. $\mathcal{G}_f$ is a surface lying in $\mathbb{R}^3$ which we call the **surface patch** associated with f. We call $\mathcal{G}_p$ a **B-patch** when it corresponds to a B-polynomial

$$p := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}. \tag{3.1}$$

There is another surface associated with p which will play an important role in this chapter. Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, let

$$\mathcal{D}_{d,T} := \Big\{ \xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} \Big\}_{i+j+k=d} \tag{3.2}$$

be the corresponding set of domain points. Let $\triangle_{T,d}$ be the triangulation of T obtained by connecting neighboring points ξ_{ijk} of $\mathcal{D}_{d,T}$, where two domain points ξ_{ijk} and $\xi_{\nu\mu\kappa}$ are considered to be neighbors provided

$$|i - \nu| + |j - \mu| + |k - \kappa| = 2.$$

The triangulation $\triangle_{T,d}$ consists of $\binom{d+1}{2} + \binom{d}{2}$ congruent triangles

$$\begin{aligned} T_{ijk} &:= \langle \xi_{i+1,j,k}, \xi_{i,j+1,k}, \xi_{i,j,k+1} \rangle, \qquad i + j + k = d - 1, \\ \widetilde{T}_{ijk} &:= \langle \xi_{i,j+1,k+1}, \xi_{i+1,j,k+1}, \xi_{i+1,j+1,k} \rangle, \qquad i + j + k = d - 2, \end{aligned} \tag{3.3}$$

see Figure 3.1. Given a B-polynomial (3.1), let

$$s_p(v) := \begin{cases} s_{ijk}(v), & \text{for } v \in T_{ijk}, \ \ i + j + k = d - 1, \\ \tilde{s}_{ijk}(v), & \text{for } v \in \widetilde{T}_{ijk}, \ \ i + j + k = d - 2, \end{cases}$$

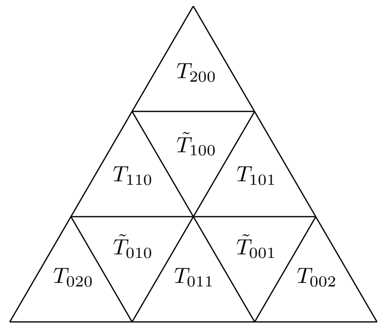


Fig. 3.1. The triangulation $\triangle_{T,3}$ of a triangle T.

where s_{ijk} is the linear polynomial which interpolates the values $\{c_{i+1,j,k}, c_{i,j+1,k}, c_{i,j,k+1}\}$ at the vertices of T_{ijk}, and $\tilde{s}_{ijk}$ is the linear polynomial which interpolates the values$\{c_{i,j+1,k+1}, c_{i+1,j,k+1}, c_{i+1,j+1,k}\}$ at the vertices of $\widetilde{T}_{ijk}$. Let $\mathcal{C}_p := \mathcal{G}_{s_p}$ be the surface patch associated with s_p. We call it the control surface associated with p.

By definition, s_p is a continuous piecewise linear function, i.e., a C^0 linear spline. The control surface $\mathcal{C}_p$ is the union of the triangular facets

$$\mathcal{T}_{ijk} := \langle C_{i+1,j,k}, C_{i,j+1,k}, C_{i,j,k+1} \rangle, \qquad i+j+k = d-1$$

and

$$\widetilde{\mathcal{T}}_{ijk} := \langle C_{i,j+1,k+1}, C_{i+1,j,k+1}, C_{i+1,j+1,k} \rangle, \qquad i+j+k = d-2,$$

where

$$C_{ijk} := (\xi_{ijk}, c_{ijk}), \qquad i+j+k = d.$$

These triangles lie in $\mathbb{R}^3$ and are in one-to-one correspondence with the triangles T_{ijk} and $\widetilde{T}_{ijk}$ which make up $\triangle_{T,d}$.

The union of the edges of $\mathcal{C}_p$ is a wireframe object $\mathcal{N}_p$ which is called the control net associated with p. The vertices $\{C_{ijk}\}_{i+j+k=d}$ of $\mathcal{N}_p$ are called the control points of p. For a typical patch and its control net, see Figure 3.2.

Clearly, the shape of a B-patch $\mathcal{G}_p$ depends on the choice of the coefficients $\{c_{ijk}\}_{i+j+k=d}$ of p or, equivalently, on the locations of the control points $\{C_{ijk}\}_{i+j+k=d}$ in $\mathbb{R}^3$. Our aim in this chapter is to show that there is a close geometric relationship between a B-patch and its control points, control net, and control surface. This connection can be employed as a design tool. The idea is to plot both the patch and its control net (or control surface) on a display screen. Then we can interactively edit the shape of the patch by adjusting the control points, which amounts to adjusting the values of the coefficients c_{ijk} of the polynomial p.

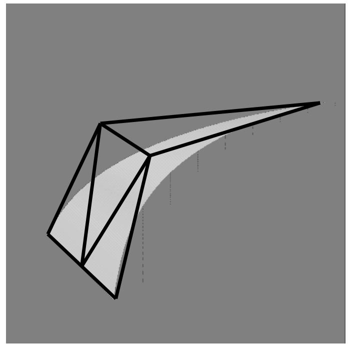


Fig. 3.2. A quadratic patch $\mathcal{G}_p$ and its associated control net $\mathcal{N}_p$.

Since all of the Bernstein basis polynomials are nonnegative on T, if we move a control point C_{ijk} *up or down* (i.e., increase or decrease the value of c_{ijk}), the B-patch $\mathcal{G}_p$ always moves in the *same direction.* Since B^d_{ijk} takes its maximum at ξ_{ijk} by Theorem 2.5, it follows that when we move the control point $C_{ijk} := (\xi_{ijk}, c_{ijk})$, the largest change in the surface takes place for points in T near ξ_{ijk}. The following theorem is also helpful for adjusting the shape of a B-patch.

Theorem 3.1. *Let $\mathcal{G}_p$ be the B-patch associated with a polynomial p. Then:*

1) *$\mathcal{G}_p$ passes through the points $C_{d00} := (v_1, c_{d00})$, $C_{0d0} := (v_2, c_{0d0})$, and $C_{00d} := (v_3, c_{00d})$, which are the corner vertices of the control net $\mathcal{N}_p$.*
2) *The intersection of $\mathcal{G}_p$ with the vertical plane containing the edge $\langle v_1, v_2\rangle$ is a curve whose tangent vector at the point C_{d00} points in the same direction as the vector $C_{d-1,1,0} - C_{d00}$.*

The analogous assertions hold for the other edges and vertices.

Proof: The fact that $\mathcal{G}_p$ passes through the point (v_1, c_{d00}) follows immediately from the fact that the only Bernstein polynomial with nonzero value at v_1 is B^d_{d00}. A similar proof works at v_2 and v_3. To establish 2), let $P(t) := \sum_{i+j+k=d} C_{ijk} B^d_{ijk}|_e$. where $e := \langle v_1, v_2\rangle$. Suppose l is the length of e. Then we can write

$$P(t) = \sum_{i=0}^{d} C_{d-i,i,0}\, \phi^d_i(t)$$

in terms of the univariate Bernstein polynomials (2.80) defined on $[0, l]$. This is a parametric curve in $\mathbb{R}^3$ with

$$P'(0) = \frac{d\,[C_{d-1,1,0} - C_{d00}]}{l}. \quad \square$$

3.2. The Convex Hull Property

In this section we establish an important geometric connection between the surface patch $\mathcal{G}_p$ associated with a polynomial p, its control net $\mathcal{N}_p$, and its control surface $\mathcal{C}_p$.

Theorem 3.2. *The B-patch $\mathcal{G}_p$ lies in the convex hull of its control net $\mathcal{N}_p$ or control surface $\mathcal{C}_p$.*

Proof: The convex hull of $\mathcal{N}_p$ is the same as the convex hull of $\mathcal{C}_p$. Fix $(x, y) \in T$ with barycentric coordinates b_1, b_2, b_3. Then in view of (2.75), points on the surface $\mathcal{G}_p$ can be written as

$$P(x,y) := \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} = \sum_{i+j+k=d} \begin{bmatrix} \xi^x_{ijk} \\ \xi^y_{ijk} \\ c_{ijk} \end{bmatrix} B^d_{ijk}(x,y) = \sum_{i+j+k=d} C_{ijk} B^d_{ijk}(x,y), \tag{3.4}$$

where $\xi_{ijk} := (\xi^x_{ijk}, \xi^y_{ijk})$ and $C_{ijk} := (\xi^x_{ijk}, \xi^y_{ijk}, c_{ijk})$ are the control points. Since the B^d_{ijk} are nonnegative and sum to one, this simply states that the point $P(x, y)$ is a *convex combination* of the control points, and hence lies in the convex hull of the control net $\mathcal{N}_p$. $\square$

There is a more geometric proof of Theorem 3.2 which is quite instructive. For each $\ell = 0, \ldots, d$, let

$$C^{(\ell)}_{ijk} := (\xi^\ell_{ijk}, c^\ell_{ijk}), \qquad i + j + k = d - \ell,$$

be the intermediate values obtained from the de Casteljau algorithm starting with $C^{(0)}_{ijk} := C_{ijk} = (\xi_{ijk}, c_{ijk})$ for $i + j + k = d$. Then each of the points $C^{(\ell)}_{ijk}$ lies on the triangular facet

$$\mathcal{T}^{(\ell)}_{ijk} := \langle C^{(\ell-1)}_{i+1,j,k}, C^{(\ell-1)}_{i,j+1,k}, C^{(\ell-1)}_{i,j,k+1} \rangle$$

formed from the three control points at the previous level, and hence remains in the convex hull of $\mathcal{N}_p$. Now by Theorem 2.8 and (3.4),

$$\xi^{(d)}_{000} = \sum_{i+j+k=d} \xi_{ijk} B^d_{ijk}(x,y) = (x,y),$$

and thus the final point $C^{(d)}_{000} = (\xi^{(d)}_{000}, c^{(d)}_{000}) = (x, y, p(x, y))$ also lies in the convex hull of $\mathcal{N}_p$.

3.3. Positivity of B-patches

In the next three sections we examine to what extent properties of the coefficients of a polynomial in B-form (or equivalently of its control net) determine the *shape* of the corresponding surface patch. In this section we discuss positivity. The following theorem gives a simple *sufficient* condition on the coefficients of a polynomial p in B-form to ensure that the corresponding surface patch is positive or nonnegative.

Theorem 3.3. *If all of the coefficients* $\{c_{ijk}\}_{i+j+k=d}$ *of* p *are positive (nonnegative), then* p *is positive (nonnegative) on* T.

Proof: The statement about nonnegativity follows immediately from the fact that each of the Bernstein basis polynomials B^d_{ijk} is nonnegative on T. The statement about positivity follows from the fact that at each point, at least one basis polynomial is strictly positive since their sum is one. □

In view of the fact that $\mathcal{C}_p$ consists of triangular facets, it is clear that $\mathcal{C}_p$ is positive (nonnegative) if and only if the coefficients $\{c_{ijk}\}_{i+j+k=d}$ of p are positive (nonnegative). But then by the convex hull property, the B-patch $\mathcal{G}_p$ will also be positive (nonnegative) under the same conditions. We can restate Theorem 3.3 in a more geometric form.

Theorem 3.4. *Suppose* $\mathcal{C}_p$ *is a control surface associated with a polynomial* p. *Then the corresponding B-patch* $\mathcal{G}_p$ *is positive (nonnegative) if* $\mathcal{C}_p$ *is positive (nonnegative).*

Theorem 3.3 gives only sufficient conditions for positivity (or nonnegativity) of p. It is easy to see that for $d = 1$, these conditions are also necessary. However, this is no longer the case for $d \geq 2$. A polynomial p can be positive on T even if some of its coefficients are negative (in which case the associated control surface also becomes negative). Here is an explicit example.

Example 3.5. Consider the quadratic polynomial p whose coefficients in B-form are $\{0, 0, 0, 1, -1, 1\}$, in lexicographical order.

Discussion: Let b_1, b_2, b_3 be the barycentric coordinate functions associated with T. Then $p = b_2^2 - 2b_2b_3 + b_3^2 = (b_2 - b_3)^2$ is a nonnegative polynomial. The minimum value of p in T is zero, and occurs at all points on the line defined by $b_2 = b_3$. If we increase all coefficients of p by .5, the resulting polynomial q still has one negative coefficient, but is strictly positive on T. □

For $d > 1$, it seems to be a difficult problem to give necessary and sufficient conditions on the B-form coefficients of a polynomial p for it to be positive (nonnegative) on T. There is, however, a characterization for the case $d = 2$. First we note that a quadratic polynomial p in B-form can be written as $p = b^T A\, b$, where $b = (b_1, b_2, b_3)^T$, and

$$A := \begin{bmatrix} c_{200} & c_{110} & c_{101} \\ c_{110} & c_{020} & c_{011} \\ c_{101} & c_{011} & c_{002} \end{bmatrix}.$$

We also note that the restriction of p to the edge $\langle v_1, v_2 \rangle$ is

$$p = c_{200}\, b_1^2 + 2c_{110}\, b_1 b_2 + c_{020}\, b_2^2, \tag{3.5}$$

with similar formulae for the other two edges of T.

Theorem 3.6. *A quadratic polynomial p is nonnegative on T if and only if*

$$
\begin{aligned}
&1)\quad c_{200} \ge 0, \quad c_{020} \ge 0, \quad c_{002} \ge 0,\\
&2)\quad c_{110} \ge -\sqrt{c_{200}c_{020}}, \quad c_{101} \ge -\sqrt{c_{200}c_{002}}, \quad c_{011} \ge -\sqrt{c_{002}c_{020}},\\
&3a)\quad \det(A) \ge 0, \text{ or}\\
&3b)\quad c_{011}\sqrt{c_{200}} + c_{101}\sqrt{c_{020}} + c_{110}\sqrt{c_{002}} + \sqrt{c_{200}c_{020}c_{002}} \ge 0.
\end{aligned}
$$

The polynomial p is positive on T if and only if these inequalities are strict.

Proof: We first prove the *sufficiency*. Let $\mathcal{C} := \{c_{110}, c_{101}, c_{011}\}$. Suppose that conditions 1), 2), 3a) hold. If all coefficients in $\mathcal{C}$ are nonnegative, then it is clear that p is nonnegative on T. If any one of the coefficients in $\mathcal{C}$ is negative, say c_{110}, then conditions 1), 2), 3a) imply that all principal minors of A are nonnegative, and thus it is a symmetric nonnegative definite matrix. But then it is clear from $p = b^T Ab$ that $p(v) \ge 0$ not only in T, but even for all points in $\mathbb{R}^2$. Now suppose conditions 1), 2), 3b) hold. We examine three cases.

Case S1. (Exactly one of the coefficients in $\mathcal{C}$ is negative.) Suppose $c_{110} < 0$ while $c_{101}, c_{011} \ge 0$, and let $\widehat{p}(v) := c_{200}b_1^2 + 2c_{110}b_1b_2 + c_{020}b_2^2$. Clearly, $p(v) \ge \widehat{p}(v)$. Now since the coefficients of $\widehat{p}$ satisfy conditions 1), 2), 3a), we conclude that $p(v) \ge 0$.

Case S2. (Two of the coefficients in $\mathcal{C}$ are negative.) Suppose $c_{110}, c_{101} < 0$ while $c_{011} \ge 0$. Then condition 3b) implies that there exists $-\sqrt{c_{002}c_{020}} \le \widehat{c}_{011} \le c_{011}$ such that

$$\widehat{c}_{011}\sqrt{c_{200}} + c_{101}\sqrt{c_{020}} + c_{110}\sqrt{c_{002}} + \sqrt{c_{200}c_{020}c_{002}} = 0.$$

Clearly, $\widehat{p}(v) := p(v) - 2c_{011}b_2b_3 + 2\widehat{c}_{011}b_2b_3 \le p(v)$. It is easy to check that the coefficients of $\widehat{p}$ satisfy the conditions 1) and 2). We now check that they satisfy 3a). Let $\widehat{A}$ be the coefficient matrix associated with $\widehat{p}$. It is a simple calculation using condition 2) to check that

$$
\begin{aligned}
\det(\widehat{A}) &= c_{200}c_{020}c_{002} + 2c_{110}c_{101}\widehat{c}_{011} - c_{110}^2c_{002} - c_{101}^2c_{020} - (\widehat{c}_{011})^2c_{200}\\
&= 2(\sqrt{c_{200}c_{020}} + c_{110})(\sqrt{c_{200}c_{002}} + c_{101})(\sqrt{c_{020}c_{002}} + \widehat{c}_{011})\\
&\qquad - (\widehat{c}_{011}\sqrt{c_{200}} + c_{101}\sqrt{c_{020}} + c_{110}\sqrt{c_{002}} + \sqrt{c_{200}c_{020}c_{002}})^2\\
&= 2(\sqrt{c_{200}c_{020}} + c_{110})(\sqrt{c_{200}c_{002}} + c_{101})(\sqrt{c_{020}c_{002}} + \widehat{c}_{011}) \ge 0.
\end{aligned}
$$

This shows $\widehat{p}$ and thus p is nonnegative.

Case S3. (All of the coefficients in $\mathcal{C}$ are negative.) Here $c_{200}, c_{020}, c_{002}$ must all be positive since otherwise condition 2) could not hold. Let

$$\lambda_1 = \sqrt{c_{200}}\, b_1, \quad \lambda_2 = \sqrt{c_{020}}\, b_2, \quad \lambda_3 = \sqrt{c_{002}}\, b_3.$$

This implies that $p(v)$ is given by

$$\begin{aligned}
&\lambda_1^2+\lambda_2^2+\lambda_3^2+2\frac{c_{110}}{\sqrt{c_{200}c_{020}}}\lambda_1\lambda_2+2\frac{c_{101}}{\sqrt{c_{200}c_{002}}}\lambda_1\lambda_3+2\frac{c_{011}}{\sqrt{c_{020}c_{002}}}\lambda_2\lambda_3\\
&\geq \lambda_1^2+\lambda_2^2+\lambda_3^2+\frac{c_{110}}{\sqrt{c_{200}c_{020}}}(\lambda_1^2+\lambda_2^2)+\frac{c_{101}}{\sqrt{c_{200}c_{002}}}(\lambda_1^2+\lambda_3^2)\\
&\quad+\frac{c_{011}}{\sqrt{c_{020}c_{002}}}(\lambda_2^2+\lambda_3^2)\\
&=\lambda_1^2\left(1+\frac{c_{110}}{\sqrt{c_{200}c_{020}}}+\frac{c_{101}}{\sqrt{c_{200}c_{002}}}\right)+\lambda_2^2\left(1+\frac{c_{110}}{\sqrt{c_{200}c_{020}}}+\frac{c_{011}}{\sqrt{c_{020}c_{002}}}\right)\\
&\quad+\lambda_3^2\left(1+\frac{c_{011}}{\sqrt{c_{020}c_{002}}}+\frac{c_{101}}{\sqrt{c_{200}c_{002}}}\right)\\
&=\frac{\lambda_1^2}{\sqrt{c_{200}c_{002}c_{020}}}\left(\sqrt{c_{200}c_{002}c_{020}}+c_{110}\sqrt{c_{002}}+c_{101}\sqrt{c_{020}}\right)\\
&\quad+\frac{\lambda_2^2}{\sqrt{c_{200}c_{002}c_{020}}}\left(\sqrt{c_{200}c_{002}c_{020}}+c_{110}\sqrt{c_{002}}+c_{011}\sqrt{c_{200}}\right)\\
&\quad+\frac{\lambda_3^2}{\sqrt{c_{200}c_{002}c_{020}}}\left(\sqrt{c_{200}c_{002}c_{020}}+c_{101}\sqrt{c_{020}}+c_{011}\sqrt{c_{200}}\right)\geq 0,
\end{aligned}$$

where we have used 3b) and the inequality $2xy \leq x^2+y^2$.

This completes the proof of sufficiency. We turn now to *necessity.* Since the c_{200}, c_{020} and c_{002} are the values of p at the vertices, condition 1) is clearly necessary. To show the necessity of 2), we choose $c_{110} = -\sqrt{c_{200}c_{020}}$. Then evaluating the resulting polynomial (3.5) at the point whose barycentric coordinates are

$$\Big(\frac{\sqrt{c_{020}}}{\sqrt{c_{200}}+\sqrt{c_{020}}}, \frac{\sqrt{c_{200}}}{\sqrt{c_{200}}+\sqrt{c_{020}}}, 0\Big),$$

we get zero. Thus, if c_{110} were any smaller, we would get negative values on the edge $\langle v_1, v_2\rangle$. A similar argument establishes the other inequalities in condition 2).

It remains to establish the necessity of condition 3). We may assume that conditions 1) and 2) hold. There are two cases.

Case N1. (One of the coefficients $c_{200}, c_{020}, c_{002}$ is zero.) Say $c_{200} = 0$. Then condition 2) implies that $c_{110} \geq 0$ and $c_{101} \geq 0$ which implies condition 3b) must hold in this case.

Case N2. (All of the coefficients $c_{200}, c_{020}, c_{002}$ are positive.) We now assume that condition 3b) does not hold, and show that 3a) does. Let

$$w_1 = \frac{c_{110}}{\sqrt{c_{020}c_{200}}}, \qquad w_2 = \frac{c_{101}}{\sqrt{c_{200}c_{002}}}, \qquad w_3 = \frac{c_{011}}{\sqrt{c_{020}c_{002}}}.$$

By condition 2), $w_i \geq -1$ for $i = 1, 2, 3$. Since we are assuming 3b) fails, it follows that $1 + w_1 + w_2 + w_3 < 0$, and so $w_i < 1$, $i = 1, 2, 3$, and $w_1 w_2 - w_3$, $w_1 w_3 - w_2$, and $w_2 w_3 - w_1$ are all positive. Now let

$$D := \begin{bmatrix} 1/\sqrt{c_{200}} & 0 & 0 \\ 0 & 1/\sqrt{c_{020}} & 0 \\ 0 & 0 & 1/\sqrt{c_{002}} \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & w_1 & w_2 \\ w_1 & 1 & w_3 \\ w_2 & w_3 & 1 \end{bmatrix}.$$

Then $A = D^{-1} B D^{-1}$ and $\phi := \det(B) = \dfrac{\det(A)}{c_{200} c_{002} c_{020}}$. To complete the proof, it suffices to show that $\phi \geq 0$.

Clearly, $CB = \phi I_3$ where I_3 is the 3×3 identity matrix, and

$$C := \begin{bmatrix} 1 - w_3^2 & w_2 w_3 - w_1 & w_1 w_3 - w_2 \\ w_2 w_3 - w_1 & 1 - w_2^2 & w_1 w_2 - w_3 \\ w_1 w_3 - w_2 & w_1 w_2 - w_3 & 1 - w_1^2 \end{bmatrix}.$$

Note that all entries of C and thus of DCD are nonnegative. Now let v_0 be a point in T whose barycentric coordinates are

$$b := (b_1, b_2, b_3) = \frac{uDCD}{g},$$

where $u = (1, 1, 1)$ and $g := uDCDu^T$. Then we have

$$p(v_0) = b A b^T = \frac{uDCDD^{-1}BD^{-1}DCDu^T}{g^2} = \frac{\phi}{g} \geq 0.$$

The proof that p is positive when all the inequalities in 1)–3) are strict is similar. $\square$

Although no analogous characterization is known for the positivity of cubic polynomials, it is possible to give a similar sufficient condition. Let

$$A_1 := \begin{bmatrix} c_{300} & c_{210} & c_{201} \\ c_{210} & c_{120} & c_{111} \\ c_{201} & c_{111} & c_{102} \end{bmatrix}, \qquad A_2 := \begin{bmatrix} c_{210} & c_{120} & c_{111} \\ c_{120} & c_{030} & c_{021} \\ c_{111} & c_{021} & c_{012} \end{bmatrix},$$

and

$$A_3 := \begin{bmatrix} c_{201} & c_{111} & c_{102} \\ c_{111} & c_{021} & c_{012} \\ c_{102} & c_{012} & c_{003} \end{bmatrix}.$$

Theorem 3.7. *Suppose the B-coefficients of a cubic polynomial p satisfy*

1) $c_{300} \geq 0,\ \ c_{030} \geq 0,\ \ c_{003} \geq 0,$
2) $c_{300}c_{120} \geq c_{210}^2,\ \ c_{030}c_{012} \geq c_{021}^2,\ \ c_{003}c_{201} \geq c_{102}^2,$
3) $\det(A_1) \geq 0,\ \ \det(A_2) \geq 0,\ \ \det(A_3) \geq 0.$

Then p is nonnegative on T. Moreover, p is positive on T provided these inequalities are strict.

Proof: Given $v \in T$, let $b := (b_1, b_2, b_3)^T$ be its barycentric coordinates. Then

$$p(v) = b_1 b^T A_1 b + b_2 b^T A_2 b + b_3 b^T A_3 b.$$

Now the hypotheses imply that the leading principal minors of A_1 are nonnegative, and thus it is a symmetric nonnegative definite matrix. Similarly, the hypotheses imply that the leading principal minors of the matrix $\tilde{A}_2$ obtained from A_2 by permuting rows and columns to the order $\{2,3,1\}$ are nonnegative. It follows that $\tilde{A}_2$ and thus also A_2 is symmetric and nonnegative definite. The proof for A_3 is similar, and the result follows. □

We conclude this section with a theorem which gives a *necessary and sufficient condition* for a patch $\mathcal{G}_p$ to be positive. Let $\mathcal{C}_p^{(\ell)}$ be the control surface associated with the B-form coefficients of p after degree raising ℓ times.

Theorem 3.8. *The patch $\mathcal{G}_p$ is positive on T if and only if the control surface $\mathcal{C}_p^{(\ell)}$ is positive for some sufficiently large ℓ.*

Proof: By Theorem 3.4, if $\mathcal{C}_p^{(\ell)}$ is positive, then $\mathcal{G}_p$ is positive. The converse follows from the fact established in Theorem 3.23 below that the sequence of control surfaces $\mathcal{C}_p^{(\ell)}$ converges to $\mathcal{G}_p$ uniformly on T as $\ell \to \infty$. □

3.4. Monotonicity of B-patches

We begin by defining monotonicity of a bivariate function.

Definition 3.9. *Let u be a vector in $\mathbb{R}^2$. We say that a function f defined on T is* monotone increasing in the direction u *provided that $f(w_2) \geq f(w_1)$ for all points w_1, w_2 in T such that the vector $w_2 - w_1$ points in the same direction as u. If f has a directional derivative in the direction u, this is equivalent to $D_u f(v) \geq 0$ for all $v \in T$.*

If f is monotone increasing in the direction u, then intersecting the surface patch $\mathcal{G}_f$ with a vertical plane containing the direction vector u, we get a curve which is monotone increasing as we move along the line in T described by u.

The concepts of strictly monotone increasing and (strictly) monotone decreasing are defined in the analogous way. In the remainder of this section we shall deal only with the monotone increasing case, as the other cases can be handled in the same way.

Theorem 3.10. *Let u be a direction vector corresponding to the triple $a := (a_1, a_2, a_3)$ with $a_1 + a_2 + a_3 = 0$ as described in Section 2.6. Suppose p is a polynomial of degree d whose B-coefficients satisfy*

$$c^{(1)}_{ijk} := a_1 c_{i+1,j,k} + a_2 c_{i,j+1,k} + a_3 c_{i,j,k+1} \geq 0, \qquad i + j + k = d - 1. \tag{3.6}$$

Then p and its associated B-patch $\mathcal{G}_p$ are monotone increasing in the direction u.

Proof: The $c^{(1)}_{ijk}$ are just the quantities arising in the first step of the de Casteljau algorithm based on a. By Theorem 2.12, they are the coefficients of $\frac{1}{d} D_u p$, which is a polynomial of degree $d-1$. But then Theorem 3.3 implies that $D_u p(v) \geq 0$ for all $v \in T$. □

Corollary 3.11. *If the control surface $\mathcal{C}_p$ associated with a polynomial p is monotone increasing in a direction u, then so is p.*

Proof: For each $i+j+k = d-1$, $c^{(1)}_{ijk}$ can be interpreted as the value of the directional derivative $D_u q_{ijk}$ of the linear polynomial q_{ijk} corresponding to the control surface $\mathcal{C}_p$ on the subtriangle T_{ijk} defined in (3.3). Thus the assumption that $\mathcal{C}_p$ is monotone increasing in the direction u implies that all of the $c^{(1)}_{ijk} \geq 0$, and Theorem 3.10 applies. □

Theorem 3.10 and Corollary 3.11 give *sufficient* conditions for a B-patch to be monotone in a given direction. However, they are not *necessary* since, as we saw in Example 3.5, the surface $D_u p$ can be positive without the conditions (3.6) being satisfied.

By Theorem 3.10, we can make a polynomial p be monotone in the direction u by forcing its coefficients to satisfy the *linear side conditions* (3.6). To make p be monotone in two different directions, say u_1 and u_2, we can enforce two sets of such conditions. If this is done, then p will in fact be monotone for all directions u in the cone

$$\{u = \alpha_1 u_1 + \alpha_2 u_2 : \alpha_1, \alpha_2 \geq 0\}.$$

Indeed, for any such vector u,

$$D_u p = \alpha_1 D_{u_1} p + \alpha_2 D_{u_2} p,$$

and so if the derivatives on the right are both positive, so is the one on the left.

For the remainder of this section we consider the special case where the direction of interest is aligned with an edge of the triangle T. In this case the conditions (3.6) can be written in a convenient form involving certain coefficient differences. Given $\nu, \mu \in \{1,2,3\}$, we define

$$\Delta_{\nu\mu} c_{ijk} := E_\nu c_{ijk} - E_\mu c_{ijk}, \qquad i+j+k = d-1,$$

where

$$E_\nu c_{ijk} := \begin{cases} c_{i+1,j,k}, & \text{if } \nu = 1, \\ c_{i,j+1,k}, & \text{if } \nu = 2, \\ c_{i,j,k+1}, & \text{if } \nu = 3. \end{cases}$$

It follows immediately from the definition that

$$\begin{aligned} \Delta_{\nu\mu} &= -\Delta_{\mu\nu}, \\ \Delta_{\nu\mu} &= \Delta_{\nu\alpha} - \Delta_{\mu\alpha}, \\ \Delta_{\nu\mu}\Delta_{\alpha\beta} &= \Delta_{\alpha\beta}\Delta_{\nu\mu}, \end{aligned}$$

for all ν, μ, α, β.

Theorem 3.12. *Let $u = v_2 - v_1$, and suppose the coefficients of p are such that*

$$\Delta_{21} c_{ijk} = c_{i,j+1,0} - c_{i+1,j,0} \geq 0, \qquad i+j+k = d-1.$$

Then p is monotone increasing in the direction u.

Proof: The assertion is an immediate corollary of Theorem 3.10, since the direction vector u is described by the triple $a = (-1, 1, 0)$. $\square$

3.5. Convexity of B-patches

A function f defined on a triangle T is said to be convex in the direction u provided

$$\frac{f(w_3) - f(w_2)}{|w_3 - w_2|} \geq \frac{f(w_2) - f(w_1)}{|w_2 - w_1|},$$

for all ordered sets of points w_1, w_2, w_3 in T lying on a line pointing in the direction of u. We say that f is convex on T provided it is convex in all directions.

As is well known from calculus, if f has two derivatives in the direction u, then this definition of convexity in the direction u is equivalent to

$$D_u^2 f(v) \geq 0, \qquad \text{all } v \in T.$$

If the function f is convex in the direction u, then the intersection of the surface patch $\mathcal{G}_f$ associated with f with a vertical plane containing u is a curve which is convex as we move in the direction described by u.

We now present a sufficient condition on the B-coefficients of a polynomial p for it to be convex in the direction of one of the edges of T.

Theorem 3.13. *The polynomial p is convex in the direction $u := v_2 - v_1$ provided that*

$$\Delta_{21}^2 c_{ijk} = c_{i,j+2,k} - 2c_{i+1,j+1,k} + c_{i+2,j,k} \geq 0, \quad i+j+k = d-2. \tag{3.7}$$

Proof: The direction u is described by the triple $a = (-1, 1, 0)$. This implies that the B-coefficients of $\frac{1}{d(d-1)} D_u^2 p$ in (2.42) are $c_{ijk}^{(2)} := \Delta_{21}^2 c_{ijk}$, and it follows from (3.7) that the second derivative of p in the direction u is positive at all points in T. $\square$

Conditions for convexity in an arbitrary direction u are somewhat more complicated.

Theorem 3.14. *Suppose*

$$u := \eta_2(v_2 - v_1) + \eta_3(v_3 - v_1), \tag{3.8}$$

for some real numbers η_2 and η_3. Then p is convex in the direction u provided that

$$(\eta_2, \eta_3)\, A_{ijk} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix} \geq 0, \qquad i+j+k = d-2, \tag{3.9}$$

where

$$A_{ijk} := \begin{bmatrix} \Delta_{21}^2 c_{ijk} & \Delta_{21}\Delta_{31} c_{ijk} \\ \Delta_{21}\Delta_{31} c_{ijk} & \Delta_{31}^2 c_{ijk} \end{bmatrix}. \tag{3.10}$$

Proof: In this case,

$$\begin{aligned} D_u^2 p &= d(d-1) \sum_{i+j+k=d-2} \Big[\Big(\sum_{m=2}^{3} \eta_m \Delta_{m,1} \Big)^2 c_{ijk} \Big] B_{ijk}^{d-2} \\ &= d(d-1) \sum_{i+j+k=d-2} \Big[\sum_{m=2}^{3} \sum_{n=2}^{3} \eta_m \eta_n \Delta_{m,1} \Delta_{n,1} c_{ijk} \Big] B_{ijk}^{d-2} \\ &= d(d-1) \sum_{i+j+k=d-2} (\eta_2, \eta_3)\, A_{ijk} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix} B_{ijk}^{d-2}, \end{aligned} \tag{3.11}$$

and the condition (3.9) is clearly sufficient. $\square$

We can now use Theorem 3.14 to give sufficient conditions for p to be convex in *all* directions.

Theorem 3.15. *Suppose each of the matrices A_{ijk} in (3.10) is nonnegative definite. Then p is convex on T. This condition is also necessary when $d = 2$ and $d = 3$.*

Proof: The sufficiency is obvious, since if the A_{ijk} are nonnegative definite, then the coefficients of B^{d-2}_{ijk} in (3.11) will be nonnegative for all choices of (η_2, η_3), and hence $D^2_u p(v) \ge 0$ for all directions u and all $v \in T$. We now prove the necessity for $d = 2$. In this case $D^2_u p$ has the constant value

$$(\eta_2, \eta_3)\, A_{000} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix}.$$

This is nonnegative for all choices of (η_2, η_3) only if A_{000} is nonnegative definite. When $d = 3$, $D^2_u p$ is a linear polynomial which interpolates the three values

$$(\eta_2, \eta_3)\, A_{100} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix}, \quad (\eta_2, \eta_3)\, A_{010} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix}, \quad (\eta_2, \eta_3)\, A_{001} \begin{bmatrix} \eta_2 \\ \eta_3 \end{bmatrix},$$

at the three vertices v_1, v_2, v_3 of T, respectively. Thus, $D^2_u p(v) \ge 0$ for all $v \in T$ can only hold if all three of these quantities are nonnegative. Since this has to hold for all directions u, and thus for all (η_2, η_3), we conclude that the matrices $A_{100}, A_{010}, A_{001}$ must be nonnegative definite. □

We now translate Theorem 3.15 into conditions directly on the coefficients.

Theorem 3.16. *Suppose the B-coefficients of a polynomial p satisfy*

$$\Delta^2_{21} c_{ijk} \ge |\Delta_{21}\Delta_{31} c_{ijk}| \tag{3.12}$$

and

$$\Delta^2_{31} c_{ijk} \ge |\Delta_{21}\Delta_{31} c_{ijk}|, \tag{3.13}$$

for all $i + j + k = d - 2$. Then p is convex on T.

Proof: A 2×2 symmetric matrix is nonnegative definite if and only if its principal minors are nonnegative. Thus, the matrix A_{ijk} in (3.10) is nonnegative definite if and only if

$$\Delta^2_{21} c_{ijk} \ge 0, \tag{3.14}$$

$$\Delta^2_{21} c_{ijk}\, \Delta^2_{31} c_{ijk} - (\Delta_{21}\Delta_{31} c_{ijk})^2 \ge 0. \tag{3.15}$$

Clearly, (3.12)–(3.13) imply these conditions, and the result follows. □

Using the simple relations

$$\begin{aligned} \Delta^2_{21} c_{ijk} &= \Delta_{21}\Delta_{23} c_{ijk} + \Delta_{21}\Delta_{31} c_{ijk}, \\ \Delta^2_{31} c_{ijk} &= \Delta_{31}\Delta_{32} c_{ijk} + \Delta_{31}\Delta_{21} c_{ijk}, \end{aligned} \tag{3.16}$$

we can rewrite the two conditions in Theorem 3.16 as the following four linear conditions:

$$\Delta_{21}\Delta_{23}c_{ijk} \ge 0, \quad \Delta_{21}\Delta_{23}c_{ijk} + 2\Delta_{21}\Delta_{31}c_{ijk} \ge 0,$$
$$\Delta_{31}\Delta_{32}c_{ijk} \ge 0, \quad \Delta_{31}\Delta_{32}c_{ijk} + 2\Delta_{21}\Delta_{31}c_{ijk} \ge 0.$$

In the following theorem we weaken these conditions even further.

Theorem 3.17. *Suppose the B-coefficients of a polynomial p satisfy*

$$\begin{aligned}
\Delta_{21}\Delta_{23}c_{ijk} + 2\Delta_{21}\Delta_{31}c_{ijk} &\ge 0,\\
2\Delta_{21}\Delta_{23}c_{ijk} + \Delta_{21}\Delta_{31}c_{ijk} &\ge 0,\\
\Delta_{31}\Delta_{32}c_{ijk} + 2\Delta_{21}\Delta_{31}c_{ijk} &\ge 0,\\
2\Delta_{31}\Delta_{32}c_{ijk} + \Delta_{21}\Delta_{31}c_{ijk} &\ge 0,\\
\Delta_{21}\Delta_{23}c_{ijk} + 2\Delta_{31}\Delta_{32}c_{ijk} &\ge 0,\\
2\Delta_{21}\Delta_{23}c_{ijk} + \Delta_{31}\Delta_{32}c_{ijk} &\ge 0,
\end{aligned} \tag{3.17}$$

for all $i + j + k = n - 2$. Then p is convex.

Proof: We claim that conditions (3.17) imply

$$\begin{aligned}
\Delta_{21}^2 c_{ijk} &\ge \frac{|\Delta_{21}\Delta_{31}c_{ijk}|}{2},\\
\Delta_{31}^2 c_{ijk} &\ge \frac{|\Delta_{21}\Delta_{31}c_{ijk}|}{2},\\
2\Delta_{21}^2 c_{ijk} + \Delta_{31}^2 c_{ijk} &\ge 3|\Delta_{21}\Delta_{31}c_{ijk}|,\\
\Delta_{21}^2 c_{ijk} + 2\Delta_{31}^2 c_{ijk} &\ge 3|\Delta_{21}\Delta_{31}c_{ijk}|.
\end{aligned} \tag{3.18}$$

Suppose $\Delta_{21}\Delta_{31}c_{ijk} \ge 0$. Then by (3.16) and the first inequality in (3.17),

$$\Delta_{21}^2 c_{ijk} = \Delta_{21}\Delta_{23}c_{ijk} + \Delta_{21}\Delta_{31}c_{ijk} \ge \frac{\Delta_{21}\Delta_{31}c_{ijk}}{2}.$$

On the other hand if $\Delta_{21}\Delta_{31}c_{ijk} < 0$, then by the second inequality in (3.17),

$$\Delta_{21}^2 c_{ijk} = \Delta_{21}\Delta_{23}c_{ijk} + \Delta_{21}\Delta_{31}c_{ijk} \ge \Delta_{21}\Delta_{23}c_{ijk} \ge \frac{-\Delta_{21}\Delta_{31}c_{ijk}}{2}.$$

These two inequalities imply the first inequality in (3.18). The other inequalities in (3.18) follow in a similar way.

We now show that the inequalities (3.18) imply the nonnegative definiteness of the matrix A_{ijk} in (3.10). Note that $\Delta_{21}^2 c_{ijk} \ge 0$. Suppose that $\Delta_{21}^2 c_{ijk} \ge \Delta_{31}^2 c_{ijk}$. Then if $\Delta_{31}^2 c_{ijk} \ge |\Delta_{21}\Delta_{31}c_{ijk}|$, we are done since

$$\Delta_{21}^2 c_{ijk}\Delta_{31}^2 c_{ijk} \ge (\Delta_{31}^2 c_{ijk})^2 \ge |\Delta_{21}\Delta_{31}c_{ijk}|^2,$$

that is, $\det(A_{ijk}) \geq 0$. Hence, A_{ijk} is nonnegative definite. On the other hand, if $\Delta_{31}^2 c_{ijk} < |\Delta_{21}\Delta_{31}c_{ijk}|$, then (3.18) implies

$$\begin{aligned}\det(A_{ijk}) &= \Delta_{21}^2 c_{ijk}\Delta_{31}^2 c_{ijk} - |\Delta_{21}\Delta_{31}c_{ijk}|^2\\ &= (\Delta_{21}^2 c_{ijk} + 2\Delta_{31}^2 c_{ijk} - 3|\Delta_{21}\Delta_{31}c_{ijk}|)\Delta_{31}^2 c_{ijk}\\ &\quad + 2(|\Delta_{21}\Delta_{31}c_{ijk}| - \Delta_{31}^2 c_{ijk})\Big(\Delta_{31}^2 c_{ijk} - \frac{|\Delta_{21}\Delta_{31}c_{ijk}|}{2}\Big) \geq 0,\end{aligned}$$

which again implies that A_{ijk} is nonnegative definite. The case $\Delta_{31}^2 c_{ijk} \geq \Delta_{21}^2 c_{ijk}$ is similar. $\square$

In the remainder of this section we explore the connection between convexity of a B-patch and convexity of its control surface.

Theorem 3.18. *If the control surface $\mathcal{C}_p$ associated with a polynomial p is convex, then p is also convex.*

Proof: First we show that (3.14) together with the conditions

$$\begin{aligned}&\Delta_{21}\Delta_{31}c_{ijk} \geq 0,\\ &\Delta_{12}\Delta_{32}c_{ijk} \geq 0, \qquad i+j+k = d-2,\\ &\Delta_{13}\Delta_{23}c_{ijk} \geq 0,\end{aligned} \tag{3.19}$$

imply that p is convex. Indeed, assuming these conditions, it follows that

$$\begin{aligned}\Delta_{21}^2 c_{ijk} - \Delta_{21}\Delta_{31}c_{ijk} &= \Delta_{21}(\Delta_{21} - \Delta_{31})c_{ijk}\\ &= \Delta_{21}\Delta_{23}c_{ijk} = \Delta_{12}\Delta_{32}c_{ijk} \geq 0.\end{aligned}$$

But then

$$\Delta_{21}^2 c_{ijk} \geq \Delta_{21}\Delta_{31}c_{ijk} = |\Delta_{21}\Delta_{31}c_{ijk}|.$$

A similar argument shows that $\Delta_{31}^2 c_{ijk} \geq |\Delta_{21}\Delta_{31}c_{ijk}|$, and Theorem 3.16 applies.

Now we show that if $\mathcal{C}_p$ is convex, then (3.14) and (3.19) must hold. The fact that $\mathcal{C}_p$ is convex along the edge corresponding to $\langle v_1, v_2\rangle$ immediately implies (3.14). The condition

$$\Delta_{21}\Delta_{31}c_{ijk} \geq 0$$

can be written as

$$c_{i,j+1,k+1} + c_{i+2,j,k} - c_{i+1,j+1,k} - c_{i+1,j,k+1} \geq 0.$$

This states that the part of the control surface consisting of the two triangles

$$\begin{aligned}\mathcal{T}_{i+1,j,k} &= \langle C_{i+2,j,k}, C_{i+1,j+1,k}, C_{i+1,j,k+1}\rangle,\\ \widetilde{\mathcal{T}}_{ijk} &= \langle C_{i,j+1,k+1}, C_{i+1,j,k+1}, C_{i+1,j+1,k}\rangle\end{aligned}$$

sharing the edge $\langle C_{i+1,j+1,k}, C_{i+1,j,k+1}\rangle$ is convex. Similarly, the second and third conditions in (3.19) assert that certain other adjoining pairs of triangles form a convex part of the control surface. $\square$

The converse of Theorem 3.18 does not hold, i.e., there exists a convex surface patch $\mathcal{G}_p$ whose control surface $\mathcal{C}_p$ is not convex.

Example 3.19. Let p be the cubic polynomial with B-coefficients $\{.2, 0, 0, 0, -.4, 0, .2, 0, 0, .2\}$.

Discussion: It is easy to check that the hypotheses of Theorem 3.16 are satisfied, and we conclude that p is convex. On the other hand, since the value of $\Delta_{21}\Delta_{31}c_{100}$ is negative, it follows from the proof of Theorem 3.18 that the control surface $\mathcal{C}_p$ is not convex. $\square$

3.6. Control Surfaces and Subdivision

We have shown that the control surface $\mathcal{C}_p$ associated with a B-polynomial p defined on a triangle T does a good job of modeling the shape of the corresponding B-patch $\mathcal{G}_p$. In this section we give a bound on the difference between these two surfaces. In addition, we show how to construct a sequence of control surfaces which converges uniformly to $\mathcal{G}_p$.

Since $\mathcal{G}_p$ and $\mathcal{C}_p$ are graphs of bivariate functions defined on the triangle T, we define $\|\mathcal{G}_p - \mathcal{C}_p\|_T$ to be the maximum distance between the two surfaces measured in the direction of the z-axis. It follows that

$$\|\mathcal{G}_p - \mathcal{C}_p\|_T = \|p - s_p\|_T,$$

where s_p is the C^0 linear spline on $\triangle_{T,d}$ which defines $\mathcal{C}_p$.

Theorem 3.20. *There exists a constant K depending only on d such that for every polynomial p of degree d,*

$$\|p - s_p\|_T \le K\,|T|^2\,|p|_{2,T}. \tag{3.20}$$

Proof: By the triangle inequality,

$$\|p - s_p\|_T \le \|p - s^*\|_T + \|s^* - s_p\|_T, \tag{3.21}$$

where s^* is the C^0 linear spline on $\triangle_{T,d}$ that interpolates p at the points $\{\xi_{ijk}\}_{i+j+k=d}$. We now consider the norm of $p - s^*$ on subtriangles of the triangulation $\triangle_{T,d}$ defined in Section 3.1. Since each of these subtriangles has diameter $|T|/d$, applying (1.31) and taking the maximum over all such subtriangles, we get

$$\|p - s^*\|_T \le \frac{|T|^2}{d^2}\,|p|_{2,T}.$$

Both s_p and s^* are C^0 linear splines on $\triangle_{T,d}$, where $s_p(\xi_{ijk}) = c_{ijk}$ and $s^*(\xi_{ijk}) = p(\xi_{ijk})$. It follows that

$$\|s^* - s_p\|_T \le \max_{i+j+k=d} |p(\xi_{ijk}) - c_{ijk}|.$$

The Bernstein polynomial (2.72) associated with p is $B_d p = \sum p(\xi_{ijk}) B^d_{ijk}$. Applying Theorem 2.6 (the stability of the Bernstein basis polynomials) and Theorem 2.45 (on the approximation power of the Bernstein polynomial operator), we get

$$\max_{i+j+k=d} |p(\xi_{ijk}) - c_{ijk}| \le K_1 \|B_d p - p\|_T \le \frac{K_1}{d} \; |T|^2 \; |p|_{2,T},$$

where K_1 is the constant in Theorem 2.6. Combining these inequalities yields the desired result. □

We now show how to create a sequence of control surfaces which converge to $\mathcal{G}_p$. Suppose we apply the subdivision method of Section 2.14 to partition T into subtriangles $\{T_i\}_{i=1}^n$. Suppose that $\{p_i\}_{i=1}^n$ are the associated polynomials such that

$$p(v) = \begin{cases} p_1(v), & v \in T_1 \\ \vdots & \\ p_n(v), & v \in T_n. \end{cases}$$

For each $i = 1, \ldots, n$, let s_{p_i} be the linear splines defining the control surfaces associated with p_i, and let

$$s_p^{(1)}(v) := \begin{cases} s_{p_1}(v), & v \in T_1 \\ \vdots & \\ s_{p_n}(v), & v \in T_n. \end{cases}$$

Then $s_p^{(1)}$ is a continuous linear spline, and its graph $\mathcal{C}_p^{(1)}$ can be regarded as an alternate control surface for the B-patch $\mathcal{G}_p$. Clearly,

$$\|\mathcal{G}_p - \mathcal{C}_p^{(1)}\|_T = \|p - s_p^{(1)}\|_T = \max_i \|p_i - s_{p_i}\|_{T_i}.$$

Applying Theorem 3.20 to each of the pieces p_i of p, we get the bound

$$\|p - s_p^{(1)}\|_T \le K\big(\max_i |T_i|\big)^2 |p|_{2,T},$$

where K is as in (3.20). Thus, $s_p^{(1)}$ will be closer to p than s_p provided

$$\max_i |T_i| < |T|.$$

Theorem 3.21. *Suppose $\mathcal{A}$ is a subdivision algorithm which splits a given triangle T into subtriangles such that the maximum diameter of the subtriangles is at most α times the diameter of T, where $\alpha < 1$. Let $\mathcal{C}_p^{(m)}$ be the control surface obtained after m applications of the algorithm. Then*

$$\|\mathcal{G}_p - \mathcal{C}_p^{(m)}\|_T \le K\, \alpha^{2m}\, |T|^2\, |p|_{2,T},$$

where K is the constant in Theorem 3.20 and depends only on d.

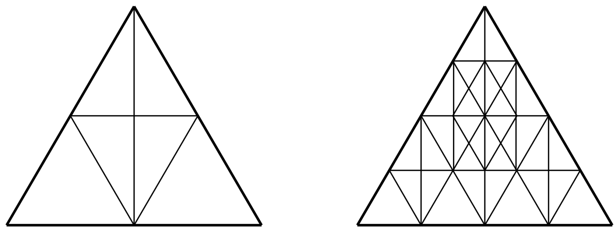


Fig. 3.3. Two levels of six-refinement of a triangle.

This result shows that the sequence $\mathcal{C}_p^{(i)}$ converges uniformly to $\mathcal{G}_p$ at a geometric rate. To apply Theorem 3.21 in practice, we need a convenient algorithm to subdivide a B-patch so that the sizes of the subtriangles are at most α times as large as the original triangle. This can be done in many ways. The following algorithm achieves a factor of $\alpha = 1/2$, and is based on performing the basic subdivision step six times.

Algorithm 3.22. (Six-refinement of $T = \langle v_1, v_2, v_3 \rangle$)

1) *Subdivide T into two triangles using the midpoint η of the side $\langle v_2, v_3 \rangle$.*
2) *Subdivide each of these triangles into two parts using the midpoints of the sides $\langle v_1, v_2 \rangle$ and $\langle v_1, v_3 \rangle$.*
3) *Subdivide the two triangles which have one vertex at v_1 by using the midpoint of the edge $\langle v_1, \eta \rangle$.*

This algorithm is based on subdividing at midpoints of edges. It creates six subtriangles. Given a B-polynomial defined on T, we can use the de Casteljau algorithm to create the B-coefficients of the polynomial pieces corresponding to the subtriangles obtained in each step. This is particularly efficient since each step involves splitting on the midpoint of an edge, and so the triple used in the de Casteljau algorithm is always a permutation of $(.5, .5, 0)$.

Figure 3.3 (left) shows the result of performing one cycle of this refinement process. Figure 3.3 (right) shows the result after applying it again to each of the six subtriangles obtained in the first cycle (giving 36 triangles).

3.7. Control Surfaces and Degree Raising

In this section we show that it is also possible to use degree raising to create a sequence of control surfaces which converge uniformly to a B-patch. Given a polynomial p in B-form (3.1), after ℓ degree raising steps, we can write p as

$$p^{[d+\ell]} := \sum_{i+j+k=d+\ell} c_{ijk}^{[d+\ell]} B_{ijk}^{d+\ell}.$$

For each ℓ, let $\mathcal{C}^{[d+\ell]}$ be the control surface associated with $p^{[d+\ell]}$. It is the graph of a linear spline $s^{[d+\ell]}$ defined on the triangulation $\triangle_{T,d+\ell}$ associated with the set of domain points

$$\mathcal{D}_{T,d+\ell} := \Big\{ \xi_{ijk}^{[d+\ell]} = \frac{(iv_1 + jv_2 + kv_3)}{d+\ell} \Big\}_{i+j+k=d+\ell}, \tag{3.22}$$

where v_1, v_2, v_3 are the vertices of T. The following theorem shows that the control surfaces $\mathcal{C}^{[d+\ell]}$ converge uniformly to the B-patch $\mathcal{G}_p$ as $\ell \to \infty$.

Theorem 3.23. *There exists a constant K depending only on the size of T such that*

$$\|\mathcal{G}_p - \mathcal{C}^{[d+\ell]}\|_T = \|p - s^{[d+\ell]}\|_T \le \frac{K}{(d+\ell)} |p|_{2,T}, \tag{3.23}$$

for all $\ell \ge 0$.

Proof: By the triangle inequality,

$$\|p - s^{[d+\ell]}\|_T \le \|p - \tilde{s}^{[d+\ell]}\|_T + \|\tilde{s}^{[d+\ell]} - s^{[d+\ell]}\|_T, \tag{3.24}$$

where $\tilde{s}^{[d+\ell]}$ is the linear spline describing the control surface defined by the Bernstein polynomial

$$B_{d+\ell}\, p := \sum_{i+j+k=d+\ell} p\big(\xi_{ijk}^{[d+\ell]}\big) B_{ijk}^{d+\ell} \tag{3.25}$$

associated with p. Since $\tilde{s}^{[d+\ell]}$ interpolates p at the points $\xi_{ijk}^{[d+\ell]}$, it follows from (1.31) that

$$\|p - \tilde{s}^{[d+\ell]}\|_T \le \Big(\frac{|T|}{d+\ell}\Big)^2 \; |p|_{2,T}.$$

It remains to bound the second term in (3.24).

Since $s^{[d+\ell]}$ and $\tilde{s}^{[d+\ell]}$ are linear splines on the triangulation $\triangle_{d+\ell,T}$, their difference is bounded by the maximum of the differences at the domain points $\xi_{ijk}^{[d+\ell]}$. But $s^{[d+\ell]}(\xi_{ijk}^{[d+\ell]}) = c_{ijk}^{[d+\ell]}$ while $\tilde{s}^{[d+\ell]}\big(\xi_{ijk}^{[d+\ell]}\big) = p\big(\xi_{ijk}^{[d+\ell]}\big)$, and thus

$$\|s^{[d+\ell]} - \tilde{s}^{[d+\ell]}\|_T \le \max_{i+j+k=d+\ell} \big|p\big(\xi_{ijk}^{[d+\ell]}\big) - c_{ijk}^{[d+\ell]}\big|.$$

Taking account of (3.25), Theorem 2.6 (the stability of the Bernstein basis polynomials) and Theorem 2.45 (on the approximation power of the Bernstein polynomial operator), we get

$$\max_{i+j+k=d} \big|p(\xi_{ijk}^{[d+\ell]}) - c_{ijk}\big| \le K_1 \,\|B_{d+\ell}\, p - p\|_T \le \frac{K_1}{d+\ell} \; |T|^2 \; |p|_{2,T},$$

where K_1 is the constant in Theorem 2.6. Combining the above inequalities yields (3.23). □

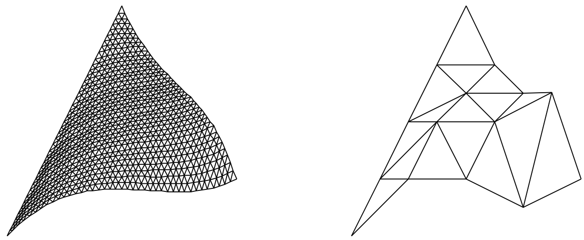


Fig. 3.4. The B-patch of Example 3.24 and its control surface.

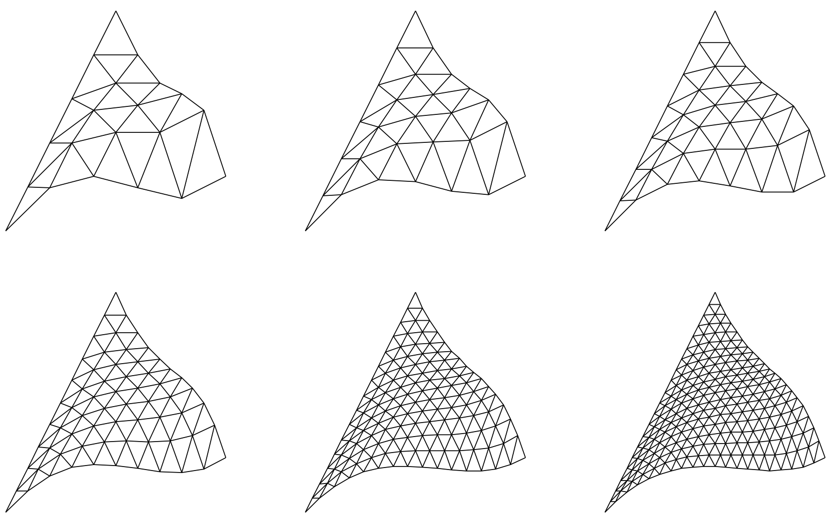


Fig. 3.5. The control surfaces corresponding to degrees 5, 6, 7, 10, 15, and 20.

Example 3.24. Let p be the polynomial of degree 4 with the following B-coefficients:

$$\begin{array}{ccccc} 3 & & & & \\ 2.5 & 2.5 & & & \\ 2.5 & 2.5 & 2 & & \\ 3 & 2.5 & 2.5 & 1.5 & \\ 2 & 1.5 & 2 & 2 & 1 \end{array}$$

Figure 3.4 shows the associated surface and control net. The control nets corresponding to degrees 5, 6, 7, 10, 15, and 20 are shown in Figure 3.5.

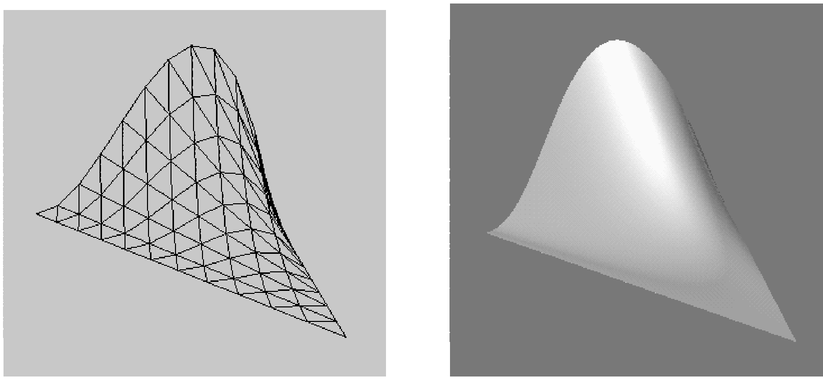


Fig. 3.6. Wire-frame and shaded renderings.

3.8. Rendering a B-Patch

The process of creating an image of a surface on a display is called rendering. In this section we discuss several approaches to rendering B-patches associated with a given polynomial p. Depending on the type of computing facilities available, there are two basic ways to render surfaces:

1) Display a wire frame model of the surface,

2) Display a colored and shaded image of the surface, assuming that it is illuminated by some light source(s).

Figure 3.6 shows a typical B-patch rendered in both ways. Usually, the shaded image gives a better "feel" for the shape of the surface, particularly if we are working on a computer which permits rotating the object in space so that we can observe how the light is reflected off the surface.

Both images in Figure 3.6 were produced with standard available software which is capable of rendering faceted surfaces in $\mathbb{R}^3$. While these software products are typically capable of working with facets with four or more vertices, for our purposes it is natural to work with triangular facets which are associated with the graph of a C^0 linear spline s defined over a subtriangulation of the domain triangle T. In practice, the B-patch surfaces $\mathcal{G}_p$ we want to render are not of this form, and so the first step in the rendering process is to create a C^0 linear spline s (on a fairly fine triangulation $\triangle_T$ of T) which approximates p. This can be done in several ways. We discuss the following four methods:

1) Choose s to be the control surface of p.

2) Choose s to be the control surface of a degree-raised version of p.

3) Choose s to be the control surface of a subdivided version of p.
4) Choose s to interpolate p at the vertices of some triangulation of T.

Method 1. Using the control surface of p. This is the method of choice if the control surface $\mathcal{C}_p$ is sufficiently close to the B-patch $\mathcal{G}_p$ and if it has sufficiently small facets to give a smooth looking rendering. This is usually not the case in practice, since we typically work with low degree B-patches, and the size of a typical facet of $\mathcal{C}_p$ is approximately $|T|/d$ (see Figure 3.4).

Method 2. Rendering by Degree Raising. In order to get a faceted surface associated with p which has smaller facets and which is closer to the B-patch $\mathcal{G}_p$, we can take the control surface $\mathcal{C}_p^{[d+\ell]}$ associated with the B-form obtained after applying ℓ steps of degree raising to p. This surface has facets of size approximately $|T|/(d+\ell)$, and Theorem 3.23 assures that as $\ell \to \infty$, the control surfaces $\mathcal{C}_p^{\ell}$ approach the B-patch $\mathcal{G}_p$. The inequality (3.23) shows that the order of convergence is $\mathcal{O}\big(\frac{1}{d+\ell}\big)$.

Method 3. Rendering by Subdivision. Here we use the control surfaces $C_p^{(\ell)}$ of Theorem 3.21. By the theorem, these surfaces converge to the B-patch with order $\alpha^{2\ell}$, where ℓ is the number of cycles of subdivision performed. Using the six-refinement method of Algorithm 3.22, we have $\alpha = 1/2$, which gives a convergence order of $\mathcal{O}\big(\frac{1}{2}\big)^{2\ell}$.

Method 4. Rendering by Interpolation. Instead of working with control surfaces, we can create a C^0 linear spline approximating p by interpolation. Indeed, given a uniform triangulation $\triangle_{T,\ell}$ of T (see Figure 3.1), by Theorem 1.11 there is a unique C^0 linear spline defined on $\triangle_{T,\ell}$ that interpolates p at the vertices of $\triangle_{T,\ell}$. It suffices to compute the values of p at these vertices since they uniquely determine the triangular pieces of s. These values can be efficiently computed using the de Casteljau algorithm. Theorem 1.3 implies that

$$\|p - s\|_T \le K\, \frac{|T|^2}{\ell^2}\, |p|_{2,T},$$

and so here the convergence order is $\mathcal{O}\big(\frac{1}{\ell^2}\big)$.

Comparison of Methods. To decide which of these methods is best in practice, we have to carry out an operation count for each method. The geometric convergence of the subdivision method looks impressive compared to the linear convergence of degree raising, and the quadratic convergence of interpolation, but we should keep in mind that the amount of calculation for the interpolation method grows linearly with ℓ, while for the subdivision method it grows much faster.

3.9. Parametric Patches

The B-patches discussed in this chapter are the graphs of bivariate functions on a triangle T. To get true 3D objects, we can use B-polynomials to form parametric patches. In particular, given a triangle T and points $\boldsymbol{c}_{ijk} \in \mathbb{R}^3$, we can define the associated parametric patch to be the surface consisting of the points $\{S(v) : v \in T\}$, where

$$S(v) := \sum_{i+j+k=d} \boldsymbol{c}_{ijk} B^d_{ijk}(v).$$

To get an associated control surface, we can connect neighboring points $\boldsymbol{c}_{ijk}$ with each other in the same way as was done in Section 3.1. This again gives a faceted surface which closely models the shape of the patch. For details, see any of the standard CAGD books such as [Far88, HosL93, Gal00, PraBP02,, CohRE01, FarHK02].

3.10. Remarks

Remark 3.1. In Section 3.5 we have discussed convexity of B-polynomials and B-patches. We can extend all of the result there to the case of strict convexity by simply replacing "greater than or equal" by "greater than" everywhere.

Remark 3.2. Several authors have worked on the problem of giving estimates for the difference between a B-patch and its corresponding control surface, see [Dah86, PraK94, NaiPL99, Rei00]. The last paper gives both pointwise estimates as well as bounds in the p-norms for all $1 \le p \le \infty$. The bounds are in terms of certain second differences of the control points, and are shown to have best possible constants.

3.11. Historical Notes

The idea of adjusting the shape of a curve or surface patch by adjusting the vertices of a control polygon or control net goes back to the early work of de Casteljau and Bézier in the automobile industry, see the discussion of their work in Section 2.20. Control nets and control surfaces as defined in Section 3.1 appear in the mathematical literature in the dissertations [Far77, Far79] and [Sab77], and later in the journal publications [ChaW81, Far82, BoeFK84].

The question of nonnegativity and positivity of a B-patch associated with a Bernstein polynomial on a triangle was addressed in [ChaW81, WangL88, He97]. These papers focused on sufficient conditions with no matching necessary conditions. The if and only if conditions of Theorem 3.6 in the quadratic case were obtained in the late 1980's by E. Nadler, but were not published until [Nad93]. Alternative proofs were given in [MicP89, ChaS94]. Here we follow the proof of [MicP89].

We could not find much in the literature on monotonicity of B-patches, but there have been a lot of papers written on the convexity of B-patches, starting with [ChaD84]. A geometric interpretation of their result can be found in [BarnW84]. Additional results on convexity can be found in [ChaF84, ChaH85, ChaSu85, DahM88, WangL88, Gra89, Dah91, FenK91, GreZ91, Pra92, Lai93, He95, CarFP97, He97]. Theorems 3.15 and 3.16 are contained in [Lai93], which we have followed here. An improvement of Theorem 3.16 is given in [CarFP97]. The proof of Theorem 3.17 is new.

Connections between convexity and subdivision were addressed in [Goo91, FenCZ94, GooP95]. Necessary and sufficient conditions for a weaker convexity condition called axial convexity can be found in [Sau91].

The fact that the control surfaces obtained by repeated subdivision converge to the B-patch seems to have been a kind of folk theorem in the CAGD community. For a proof of this fact along with the rate of convergence, see [Dah86, Dah90].

The convergence of degree raising was observed in [Far79]. A proof showing quadratic convergence was presented in [Fen87]. Our proof here uses techniques introduced in [CohS85] for the univariate case. For a completely different proof, see [PraK94]. A related variation diminishing property was established in [Goo87].

Triangulations and Quadrangulations

In this chapter we discuss various properties of triangulations and quadrangulations, including how to store, construct, and refine them.

4.1. Properties of Triangles

Before defining triangulations in the next section, we briefly review some facts about triangles. Suppose we are given three noncollinear points v_1, v_2, v_3 in $\mathbb{R}^2$. Then the convex hull of these points form a triangle which we write as $T := \langle v_1, v_2, v_3 \rangle$. We call the points $v_i := (x_i, y_i)$ the vertices of T, and denote the three edges of T by $\langle v_1, v_2, \rangle$, $\langle v_2, v_3 \rangle$, and $\langle v_3, v_1 \rangle$. The area of T is given by

$$A_T = \frac{1}{2} \det(M), \tag{4.1}$$

where

$$M = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Note that the area of a triangle is positive if the vertices of T are listed in *counterclockwise order*, and is negative otherwise. Throughout this book we shall always list the vertices of triangles in counterclockwise order unless otherwise specified.

We now introduce some ways to measure the size and shape of a triangle.

Definition 4.1. *Given a triangle T, we write $|T|$ for the length of its longest edge, and ρ_T for the radius of the largest disk that can be inscribed in T. The center of this disk is called the* incenter *of T, and ρ_T is called the* inradius *of T. We call the ratio $\kappa_T := |T|/\rho_T$ the* shape parameter *of T.*

For an equilateral triangle, $\kappa_T = 2\sqrt{3}$. Any other triangle has a larger shape parameter. Another way to measure the shape of a triangle is in terms of its angles. Let θ_T be the smallest angle in T. We now show that the size of θ_T can be used to bound the shape parameter κ_T.

Lemma 4.2. *For any triangle T,*

$$\frac{1}{\tan(\theta_T/2)} \le \kappa_T \le \frac{2}{\tan(\theta_T/2)} \le \frac{2}{\sin(\theta_T/2)}. \tag{4.2}$$

Proof: Let v be a vertex with angle θ_T, and let e be an attached edge. The line connecting the incenter of T to v must bisect the angle at that vertex. Thus, $\rho_T/|e| \le \tan(\theta_T/2)$, which immediately implies the first inequality. The second inequality was proved in (1.27). $\square$

It is also useful to have a bound on the ratio of the lengths of any two edges e and $\tilde{e}$ of a triangle T. The law of sines immediately gives

$$\frac{|e|}{|\tilde{e}|} \le \frac{1}{\sin\theta_T}. \tag{4.3}$$

4.2. Triangulations

Definition 4.3. *A collection $\triangle := \{T_1, \dots, T_N\}$ of triangles in the plane is called a* triangulation *of $\Omega = \bigcup_{i=1}^N T_i$ provided that if a pair of triangles in $\triangle$ intersect, then their intersection is either a common vertex or a common edge.*

This definition allows quite general triangulations. For example, $\triangle$ may consist of two triangles which are completely separated, or it may consist of two triangles which touch only at a vertex. Moreover, the definition also allows triangulations of domains Ω with one or more holes as shown in Figure 4.1. Such triangulations arise frequently in the finite-element method for solving partial differential equations. The configuration of triangles in Figure 4.2 is not a triangulation.

Definition 4.4. *The vertices of the triangles of $\triangle$ are called the vertices of the triangulation $\triangle$. If a vertex v is a boundary point of Ω, we say that it is a* boundary vertex. *Otherwise, we call it an* interior vertex. *Similarly, the edges of the triangles of $\triangle$ are called the edges of the triangulation $\triangle$. If an edge e lies on the boundary of Ω, we say that it is a* boundary edge. *Otherwise, we say it is an* interior edge. *We denote the sets of interior and boundary vertices of $\triangle$ by $\mathcal{V}_I$ and $\mathcal{V}_B$, respectively. Similarly, we write $\mathcal{E}_I$ and $\mathcal{E}_B$ for the sets of interior and boundary edges of $\triangle$, respectively.*

Given a triangulation $\triangle$, we shall often need to work with certain subtriangulations called stars.

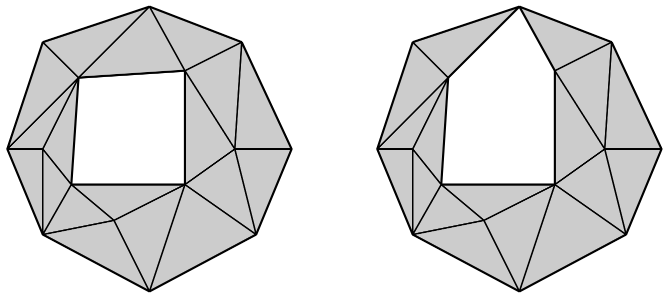


Fig. 4.1. Two triangulations.

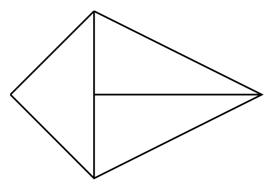


Fig. 4.2. An example of a set of triangles which do not form a triangulation.

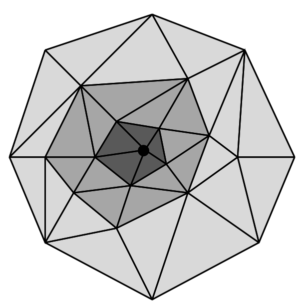


Fig. 4.3. star(v) (dark grey) and star$^2(v)$ (medium and dark grey).

Definition 4.5. *If v is a vertex of a triangulation $\triangle$, then we define the* star of v, *which we denote by $star(v) := star^1(v)$, to be the set of all triangles in $\triangle$ which share the vertex v. We define $star^i(v)$ inductively for $i > 1$ to be the set of all triangles in $\triangle$ which have a nonempty intersection with some triangle in $star^{i-1}(v)$. Similarly, we define $star^0(T) := T$, and $star^j(T) := \bigcup\{star(v) : v \in star^{j-1}(T)\}$ for all $j \geq 1$.*

By definition, $\mathrm{star}^j(v)$ and $\mathrm{star}^j(T)$ are sets of triangles. However, in the sequel we will also use this notation for the corresponding sets of points in $\mathbb{R}^2$. We illustrate this concept in Figure 4.3 where v is marked with a black dot, star(v) is the set of points in dark grey, while star$^2(v)$ is the set of points shown in either dark or medium grey.

4.3. Regular Triangulations

We emphasize that most of the results of this book hold for general triangulations as described in Definition 4.3. However, some of our results require triangulations with more structure.

Definition 4.6. *We say that a triangulation $\triangle$ is* shellable *provided it consists of a single triangle, or if it can be obtained from a shellable triangulation $\widetilde{\triangle}$ by adding a triangle T that intersects $\widetilde{\triangle}$ precisely along either one or two edges.*

Not all triangulations are shellable. For example, a triangulation consisting of two triangles touching only at a vertex is clearly not shellable. The triangulations in Figure 4.1 are also not shellable, since no matter how we try to build them by adding one triangle at a time, eventually we will get two triangles which touch only at a vertex.

Definition 4.7. *We say that a triangulation $\triangle$ is* regular *provided*

1) *$\triangle$ is shellable, or*
2) *it can be obtained from a shellable triangulation $\widetilde{\triangle}$ by removing one or more shellable subtriangulations, all of whose vertices are interior vertices of $\widetilde{\triangle}$.*

The following result follows immediately from the above definition.

Lemma 4.8. *Suppose $\triangle$ is a regular triangulation. Then for every vertex v of $\triangle$, star(v) is a shellable subtriangulation of $\triangle$.*

Lemma 4.8 can be used to test whether a given triangulation $\triangle$ is regular. For example, we claim that the triangulation in Figure 4.1 (right) is not regular. Indeed, for the vertex v at the top, star(v) is not shellable since it consists of two triangles touching only at v. This triangulation can be constructed from a shellable triangulation $\widetilde{\triangle}$ by removing a shellable subtriangulation, but one with a vertex on the boundary of $\widetilde{\triangle}$.

4.4. Euler Relations

For later use we now present some formulae connecting the numbers of vertices, edges, and triangles in a given triangulation $\triangle$. This is one of the situations where we need to restrict ourselves to regular triangulations since the results do not hold for general triangulations. Let

$$
\begin{aligned}
V_I &:= \text{number of interior vertices}, \\
V_B &:= \text{number of boundary vertices}, \\
V &:= \text{total number of vertices}, \\
E_I &:= \text{number of interior edges}, \\
E_B &:= \text{number of boundary edges}, \\
E &:= \text{total number of edges} \\
N &:= \text{number of triangles}, \\
H &:= \text{number of holes}.
\end{aligned}
\tag{4.4}
$$

We first give a result for shellable triangulations.

Theorem 4.9. *Suppose $\triangle$ is a shellable triangulation, and that in building $\triangle$, for each $i = 1, 2$, the number of times that we add a triangle which touches on i edges is α_i. Then*

1) $N = 1 + \alpha_1 + \alpha_2$,
2) $E_I = \alpha_1 + 2\alpha_2$,
3) $E_B = \alpha_1 - \alpha_2 + 3$,
4) $V_I = \alpha_2$,
5) $V_B = \alpha_1 - \alpha_2 + 3$.

Proof: The proof is just a simple matter of counting. To get formula 1), we start with one triangle, and note that α_i is the number of times that we add a triangle touching on i edges, so the total number of triangles added is $N = 1 + \alpha_1 + \alpha_2$. To establish formula 2), we note that each time we add a triangle to an existing shellable triangulation that touches on i edges, the number of interior edges is increased by i. We conclude that the total number of interior edges is given by $E_I = \alpha_1 + 2\alpha_2$. The proofs of the other formulae are similar. $\square$

The formulae in Theorem 4.9 can be combined in various ways to yield relationships between the number of vertices and edges of a shellable triangulation. We give several typical such relationships in the following theorem.

Theorem 4.10. *Suppose $\triangle$ is a shellable triangulation. Then*

$$
\begin{aligned}
E_B &= V_B, \\
E_I &= 3V_I + V_B - 3, \\
N &= 2V_I + V_B - 2.
\end{aligned}
\tag{4.5}
$$

We now extend this result to the class of regular triangulations.

Theorem 4.11. *Suppose $\triangle$ is a regular triangulation of a domain Ω with H holes. Then*

$$\begin{aligned} E_B &= V_B, \\ E_I &= 3V_I + V_B + 3H - 3, \\ N &= 2V_I + V_B + 2H - 2. \end{aligned} \tag{4.6}$$

Proof: We proceed by induction on H. If $\triangle$ has no holes, then it is shellable, and the result follows from Theorem 4.10. Now suppose $\triangle$ has H holes, and that it can be obtained from a triangulation $\widetilde{\triangle}$ with $H-1$ holes by removing a shellable subtriangulation $\widehat{\triangle}$ of $\widehat{N}$ triangles. Let $\widehat{V}_B$ and $\widetilde{V}_B$ be the number of boundary vertices of $\widehat{\triangle}$ and $\widetilde{\triangle}$, respectively. Now with analogous notation for the other quantities, we have

$$V_B = \widetilde{V}_B + \widehat{V}_B, \qquad V_I = \widetilde{V}_I - \widehat{V}_I - \widehat{V}_B,$$
$$E_B = \widetilde{E}_B + \widehat{E}_B, \qquad E_I = \widetilde{E}_I - \widehat{E}_I - \widehat{E}_B.$$

Then combining the inductive hypothesis with the formulae $\widehat{V}_B = \widehat{E}_B$, $\widehat{E}_I = 3\widehat{V}_I + \widehat{V}_B - 3$ and $\widehat{N} = 2\widehat{V}_I + \widehat{V}_B - 2$ immediately gives (4.6) for $\triangle$. $\square$

The formulae in Theorems 4.10 and 4.11 do not hold for general triangulations. In particular they fail for a triangulation consisting of two triangles touching at a vertex. They also fail for the triangulation shown in Figure 4.1 (right). Combining the formulae in (4.6) leads to the classical Euler formula $N - E + V = 1 - H$. This formula is valid for general connected triangulations, including those in Figure 4.1.

4.5. Storing Triangulations

In order to be able to use triangulations in practice, we need a way to store them in a computer. This is a matter of choosing an appropriate data structure, and is important in practice since we often have to deal with very large triangulations involving many thousands of triangles.

Clearly, the first step in storing a triangulation is to store the locations of the vertices $v_i = (x_i, y_i)$, $i = 1, \ldots, V$. This involves storing the $2V$ real numbers x_i and y_i. Next we have to store information about how the vertices of a triangulation are connected. We discuss several different approaches.

A Triangle List. In this approach we describe each triangle in the triangulation with a triple of integers $(\alpha_i, \beta_i, \gamma_i)$ chosen so that the i-th triangle is $T_i = \langle v_{\alpha_i}, v_{\beta_i}, v_{\gamma_i} \rangle$. This storage scheme requires a total of $3N$ integers. For regular triangulations, this is equal to $6V_I + 3V_B + 6H - 6$.

An Edge List. In this approach we describe each edge in the triangulation with a pair of integers (ν_i, μ_i) such that the i-th edge in the triangulation is given by $e_i := \langle v_{\nu_i}, v_{\mu_i} \rangle$. This scheme requires $2E$ integers, which is equal to $6V_I + 4V_B + 6H - 6$ for regular triangulations. More information is required to describe general regular triangulations, and in particular triangulations with holes.

An Adjacency List. In this approach, for each vertex v_i we store a list of integers describing which other vertices are attached to v_i in the triangulation. This can be done most conveniently using a pair of linked lists. The first list contains integers $m_1, \ldots, m_V$, where m_i is the number of vertices attached to v_i. The second list contains V blocks of integers, where the i-th block contains the list of subscripts of vertices that are attached to v_i. In practice it is most convenient to arrange the integers in such a block so that the corresponding vertices are connected in *counterclockwise order* around v_i. If v_i is a boundary vertex, this uniquely determines the first integer in the block, but if v_i is an interior vertex, we can start with an arbitrary vertex attached to v_i. For shellable triangulations, this scheme requires V integers for the first list, and $2E = 6V_I + 4V_B - 6$ integers for the second list. More information is required to describe general regular triangulations, and in particular triangulations with holes.

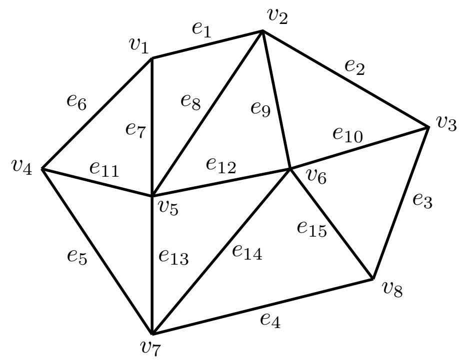
Since the amount of storage required in all three cases is essentially the same, the choice of data structure to use in a given application will depend on other factors. In many applications it is useful to work with all three data structures simultaneously. Figure 4.4 shows a typical triangulation with the associated lists.

In some applications, we need even more information about the connectivity of a triangulation. The following additional lists are often useful:

Triangle Edge List. This list consists of N integer triples (k_i^1, k_i^2, k_i^3) with the property that for a given triangle T_j, $e_{k_j^1}$, $e_{k_j^2}$, and $e_{k_j^3}$ are the edges (in the above edge list) which make up the edges of T_j. We can assume that as we go around the boundary of T_j in counterclockwise order we encounter the edges $e_{k_j^1}$, $e_{k_j^2}$, and $e_{k_j^3}$ in order, where $e_{k_j^1}$ follows the first vertex v_{α_j}.

Neighboring Triangle List. This list consists of N integer triples (n_i^1, n_i^2, n_i^3) with the property that for a given triangle T_j, the three triangles $T_{n_j^1}$, $T_{n_j^2}$, and $T_{n_j^3}$ are the triangles attached to T_j across the edges $e_{k_j^1}$, $e_{k_j^2}$, and $e_{k_j^3}$, respectively. A zero value for an n_i^j indicates a boundary edge.

Neighbor lists are very useful in enforcing smoothness conditions across edges, and are also useful in making evaluation of a piecewise surface more efficient. In Figure 4.4, the vertices of the triangulation are numbered as



Edges:

i	ν_i, μ_i
1	1,2
2	2,3
3	3,8
4	7,8
5	4,7
6	1,4
7	1,5
8	2,5
9	2,6
10	3,6
11	4,5
12	5,6
13	5,7
14	6,7
15	6,8

Triangles:

i	$\alpha_i, \beta_i, \gamma_i$	e_i^1, e_i^2, e_i^3	n_i^1, n_i^2, n_i^3
1	1,4,5	6,11,7	0,5,2
2	1,5,2	7,8,1	1,3,0
3	2,5,6	8,12,9	2,6,4
4	2,6,3	9,10,2	3,8,0
5	4,7,5	5,13,11	0,6,1
6	5,7,6	13,14,12	5,7,3
7	6,7,8	14,4,15	6,0,8
8	3,6,8	10,15,3	4,7,0

Adjacency Lists:

3,4,3,3,5,5,4,3
4,5,2, 1,5,6,3, 2,6,8, 7,5,1, 6,2,1,4,7, 2,5,7,8,3, 8,6,5,4, 3,6,7

Fig. 4.4. A triangulation and its associated lists.

1 to 8, and are labeled as $v_1, \ldots, v_8$ in the figure. The 15 edges are labeled as $e_1, \ldots, e_{15}$. For example, the eighth edge connects vertex 2 with vertex 5. Looking in the triangle table, we see that the third triangle has vertices 2,5,6. The associated edges are 8,12,9, and the triangles lying on the other sides of these edges are numbered 2,6,4. If an edge is a boundary edge there is no triangle on the other side of the edge – this case is marked with 0. The first adjacency list tells how many vertices are attached to each of the vertices. For example, there are three vertices attached to vertex number 1. Looking in the second list, we see that these are 4,5,2 (in counterclockwise order). Similarly, there are four vertices attached to vertex number 2. These are the next four numbers in the second list, namely, 1,5,6,3.

4.6. Constructing Triangulations

Devising robust computer algorithms for constructing triangulations is an important problem in computational geometry. No one algorithm is appropriate in all cases since the starting information and ultimate use for the triangulation both depend on the application. There are two rather different situations.

Triangulating a Given Set of Vertices. This situation arises frequently in data fitting, where the given vertices are often the points in the domain at which measurements have been taken. In practice, we may be given the following information:

A. *Vertices Only.* In this case we are given a set of points $v_1, \ldots, v_V$ which are to serve as the vertices of the triangulation, and the aim is to construct a triangulation $\triangle$ of the convex hull of the vertices. There are, of course, many such triangulations, and so in practice we need some criterion to help select a suitable triangulation.

B. *A List of Boundary Vertices.* In some applications we may not want the triangulation to cover the entire convex hull of the vertices, but have a certain boundary shape in mind. Assuming we want a triangulation without holes, we can describe the boundary with a list of integers $j_1, \ldots, j_m$ such that $v_{j_1}, \ldots, v_{j_m}$ should be the boundary vertices in counterclockwise order. If we want to include holes, we would need additional lists of boundary vertices for each hole.

C. *Some Prescribed Interior Edges.* In some applications, for example in modeling surfaces with faults, it is useful to ensure that the triangulation to be constructed includes certain interior edges. This can be accomplished by giving a list of the endpoints of edges which must be included in $\triangle$.

Grid Generation. In solving boundary-value problems involving partial differential equations, the aim is to construct a suitable space of (smooth) piecewise polynomials in which to seek a solution. The choice of triangulation in this case is dictated by a variety of factors, including the following:

1) The associated space of splines should be capable of approximating the solution well – this may mean that in certain areas (for example near corners where a crack can occur in a material) we need more triangles than in other areas. These are often called graded meshes.

2) Assuming the boundary is polygonal (or has been approximated by a polygon), we want to include all corner points of the boundary in the list of vertices.

The problem of grid generation for use in solving boundary-value problems is an interesting and difficult problem. It is beyond the scope of this book, but is treated in many papers and other books.

We conclude this section with a simple algorithm for constructing a triangulation of the convex hull of a set $\mathcal{V}$ of vertices. However, it is not particularly useful in practice, since it often produces unsatisfactory triangulations. More useful algorithms will be discussed in Section 4.12.

If $\triangle$ is a triangulation and v is a point in $\mathbb{R}^2$, we say that a vertex w of $\triangle$ is visible to v provided it is possible to draw a line from v to w which does not cross any of the edges of $\triangle$.

Algorithm 4.12. (Vertex Insertion Algorithm)
Let $\mathcal{V} = \{v_i\}_{i=1}^V$ be a given set of points in $\mathbb{R}^2$, and let Ω be the convex hull of $\mathcal{V}$.

1) *Connect v_1, v_2, v_3 to form an initial triangulation $\triangle^{(0)}$ consisting of one triangle.*
2) *For $i = 1$ step 1 until $n - 3$ do*
   a) *If v_i is strictly inside some triangle T of $\triangle^{(i-1)}$, connect v_i to the three vertices of T. If v_i is on an edge e of a triangle T in $\triangle^{(i-1)}$, connect it to the opposite vertex of all triangles sharing edge e.*
   b) *Otherwise, connect v_i to all of the vertices of $\triangle^{(i-1)}$ which are visible to v_i.*
   c) *Define $\triangle^{(i)}$ to be the new triangulation.*

This algorithm starts with a single triangle, and then adds additional triangles in each step of the loop. At the i-th step we have a triangulation $\triangle^{(i)}$ with vertices in the set $\mathcal{V}_i := \{v_1, \dots, v_i\}$, and the union of the associated triangles will be the convex hull of $\mathcal{V}_i$.

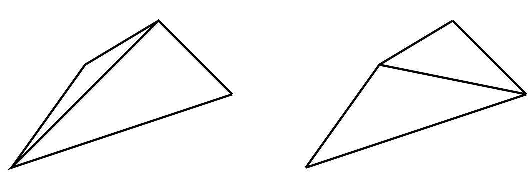


Fig. 4.5. Two different triangulations with the same set of vertices.

The triangulations in Figure 4.5 were both created by Algorithm 4.12, but with a different numbering of the vertices. This example clearly shows that the resulting triangulation is very sensitive to the way in which the vertices are numbered. For most purposes, the triangulation on the right is much better than the one on the left. This suggests that we should introduce some criterion to distinguish between triangulations, and then use it to choose a "best" triangulation. We discuss this idea in Sections 4.9 through 4.12.

4.7. Clusters of Triangles

In dealing with spline spaces on triangulations, we often have to work with small clusters of triangles contained in a given triangulation $\triangle$. We say that a collection $\mathcal{T}$ of triangles in $\triangle$ is an ℓ-cluster of triangles provided there is a vertex v in $\triangle$ such that all of the triangles in $\mathcal{T}$ are contained in $\mathrm{star}^\ell(v)$.

In this section we establish several useful properties of clusters of triangles. Our first result gives a bound on how many triangles there can be in an ℓ-cluster in terms of the smallest angle in the triangles of $\mathcal{T}$. For each triangle $T \in \mathcal{T}$, let θ_T be the smallest angle in T. Let

$$\theta_{\mathcal{T}} := \min_{T \in \mathcal{T}} \theta_T.$$

Lemma 4.13. *Given an ℓ-cluster $\mathcal{T}$ of N triangles, let $a := 2\pi/\theta_{\mathcal{T}}$. Then*

$$N \le \sum_{j=1}^{\ell} a^j. \tag{4.7}$$

Proof: It suffices to prove the result when $\mathcal{T}$ is $\mathrm{star}^\ell(v)$ for some vertex v. We first consider the case where $\ell = 1$. Then there are N triangles attached to v. Clearly $N\theta_{\mathcal{T}} \le 2\pi$, and so $N \le a$, which establishes (4.7) in this case. Now suppose $\ell > 1$. We say that a vertex w is at *level j with respect to v* if the shortest path from v to w involves j edges of $\mathcal{T}$. By the result for $\mathrm{star}(v)$, we know that there are at most a vertices at level 1. Each of these vertices can be surrounded by at most a triangles, and we conclude that a 2-cluster can contain at most $a + a^2$ triangles, and at most a^2 vertices at level 2. The result now follows by induction. $\square$

The bound (4.7) in Lemma 4.13 can be improved by separating the cases where ℓ is even and odd, and counting only triangles surrounding vertices on every other level, see Remark 4.4. Recall that for any closed set Ω in $\mathbb{R}^2$, we define its diameter by

$$|\Omega| := \max_{v,w \in \Omega} |v - w|. \tag{4.8}$$

The diameter of a triangle is just the length of its longest edge. Our next result gives a bound on the relative sizes of edges and areas of triangles in a cluster.

Lemma 4.14. *Given an ℓ-cluster $\mathcal{T}$, let $\theta_{\mathcal{T}}$ be the smallest angle in $\mathcal{T}$, and let $b := 1/\sin\theta_{\mathcal{T}}$ and $n := \lceil 2(2\ell-1)\pi/\theta_{\mathcal{T}} + 2\rceil$. Then for any two triangles T and $\widetilde{T}$ in $\mathcal{T}$,*

$$\frac{|T|}{|\widetilde{T}|} \le b^n, \tag{4.9}$$

$$\frac{A_T}{A_{\widetilde{T}}} \le b^{2n+1}. \tag{4.10}$$

Proof: Let e_1 and e_2 be two edges of $\mathcal{T}$, and for each $i = 1, 2$, let w_i be a vertex on e_i. Then w_1 and w_2 are connected by a path of at most 2ℓ edges which pass through at most $2\ell - 1$ interior vertices of $\mathcal{T}$. By Lemma 4.13 each of these has at most $a := 2\pi/\theta_{\mathcal{T}}$ edges attached to it. By (4.3) the ratio of any two edges sharing a vertex is bounded by b. Now we can compare e_1 and e_2 by performing at most $(2\ell-1)a$ comparisons around interior vertices, plus an additional two which are needed if e_1 and e_2 are not connected to any interior vertices. This gives (4.9). To prove (4.10), we note that for any $T, \widetilde{T} \in \mathcal{T}$, $A_T \le |e_1||e_2|/2$, where e_1, e_2 are edges of T, while $A_{\widetilde{T}} \ge \sin(\theta_{\mathcal{T}})|e_3||e_4|/2$, where e_3, e_4 are edges of $\widetilde{T}$. Thus, (4.10) follows from (4.9). □

Given an ℓ-cluster $\mathcal{T}$, let $\Omega_{\mathcal{T}}$ be the union of the triangles in $\mathcal{T}$. Let $|\mathcal{T}|$ be the length of the longest edge in $\mathcal{T}$. Then clearly

$$|\Omega_{\mathcal{T}}| \le 2\ell|\mathcal{T}|. \tag{4.11}$$

Moreover, if $\rho_{\mathcal{T}}$ is the smallest inradius of the triangles in $\mathcal{T}$, then combining Lemmas 4.2 and 4.14 we have

$$\frac{|\Omega_{\mathcal{T}}|}{\rho_{\mathcal{T}}} \le K, \tag{4.12}$$

where K is a constant depending only on ℓ and $\theta_{\mathcal{T}}$.

4.8. Refinements of Triangulations

Suppose that $\triangle$ and $\triangle_R$ are triangulations of a set Ω.

Definition 4.15. *We say that $\triangle_R$ is a* refinement *of $\triangle$ provided*

1) *every vertex of $\triangle$ is a vertex of $\triangle_R$,*
2) *every triangle $t \in \triangle_R$ is a subtriangle of some triangle T in $\triangle$.*

When $\triangle_R$ is a refinement of $\triangle$, we call $\triangle$ the coarser triangulation and $\triangle_R$ the finer triangulation. In this case we also say that the two triangulations are nested.

There are many ways to refine a given triangulation. In practice we are often interested in systematic refinement algorithms in which a given refinement scheme is applied to every triangle in $\triangle$. In the following subsections we describe several refinement schemes which we will use in Chapters 6–8 in the construction of certain macro-element spaces.

4.8.1 Clough–Tocher Refinement

Definition 4.16. *Let $T = \langle v_1, v_2, v_3\rangle$ be a triangle, and let $v_{_T} := (v_1 + v_2 + v_3)/3$ be the* barycenter *of T. If we connect $v_{_T}$ to each of the vertices of T, then T is split into three triangles. We call this the* Clough–Tocher split *$T_{_{CT}}$ of T. If we apply this splitting operation to each of the triangles of a triangulation $\triangle$, we call the resulting triangulation $\triangle_{CT}$ the* Clough–Tocher refinement *of $\triangle$.*

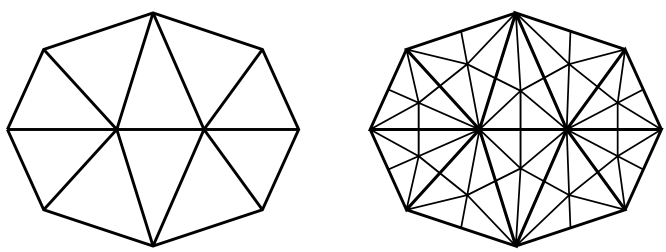


Fig. 4.6. A triangulation and its Clough–Tocher refinement.

Figure 4.6 shows a typical triangulation and its Clough–Tocher refinement. The following lemma provides some useful information on the shape of the subtriangles in a Clough–Tocher split of a given triangle. This result will be useful later in our study of macro-element spaces based on Clough–Tocher splits.

Lemma 4.17. *Let $T_{_{CT}}$ be the Clough–Tocher split of a triangle T, and let θ_{CT} and θ_T be the smallest angles in $T_{_{CT}}$ and T, respectively. Then $\theta_{CT} \ge 2\theta_T/3\pi$.*

Proof: Suppose $T = \langle v_1, v_2, v_3\rangle$. For each $i = 1, 2, 3$, let β_i be the angle of T at v_i, and let l_i be the length of the edge e_i of T opposite v_i. A simple computation shows that for each $i = 1, 2, 3$, if we extend the line $\langle v_i, v_{_T}\rangle$, then it intersects the edge e_i at the midpoint w_i of e_i. It is clear that for each subtriangle $T_i := \langle v_{_T}, v_{i+1}, v_{i+2}\rangle$ of $T_{_{CT}}$, the angle at $v_{_T}$ is larger than β_i. We now show that the angle α of T_1 at v_2 exceeds $2\theta_T/3\pi$. The proof for the other angles in $T_{_{CT}}$ is similar. The proof now divides into two cases.

Case 1: $l_3 \geq l_2$. Since the area of $\langle v_2, v_3, w_2 \rangle$ is exactly one-half the area of T, it follows that

$$\frac{1}{2} l_1 l_m \sin\alpha = \frac{1}{4} l_1 l_3 \sin\beta_2,$$

where l_m is the length of the edge $\langle v_2, w_2 \rangle$. Since $l_3 + l_2/2 > l_m$, it follows that

$$\sin\alpha \geq \frac{l_3}{2(l_3 + l_2/2)} \sin\beta_2 \geq \frac{l_3}{2(l_3 + l_3/2)} \sin\beta_2 \geq \frac{1}{3} \sin\theta_T,$$

and the result follows from the fact that, in general, $2\theta/\pi \leq \sin\theta \leq \theta$ for any $\theta \leq \pi/2$.

Case 2: $l_2 > l_3$. In this case

$$\frac{1}{2} l_1 l_m \sin\alpha = \frac{1}{4} l_1 l_2 \sin\beta_3.$$

Thus,

$$\sin\alpha \geq \frac{l_2}{2(l_3 + l_2/2)} \sin\beta_3 \geq \frac{l_2}{2(l_2 + l_2/2)} \sin\beta_3 \geq \frac{1}{3} \sin\theta_T,$$

and the result follows as before. □

4.8.2 Powell–Sabin Refinement

Definition 4.18. *Let $\triangle$ be a triangulation, and suppose that for each triangle T, u_T denotes its incenter. For each triangle T, connect u_T to each of the three vertices of T. Connect u_T and $u_{\widetilde{T}}$ whenever the triangles T and $\widetilde{T}$ share a common edge. In addition, connect the middle of each boundary edge to the incenter of the associated triangle. We call the resulting triangulation $\triangle_{PS}$ the* Powell–Sabin refinement *of $\triangle$.*

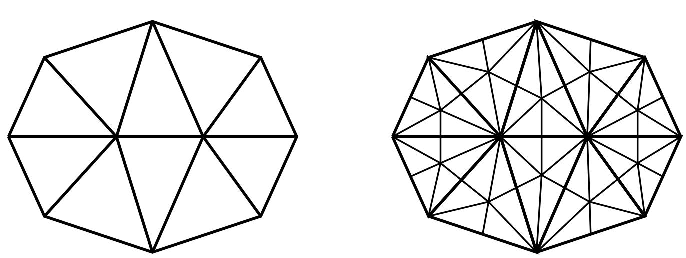


Fig. 4.7. A triangulation and its Powell–Sabin refinement.

Figure 4.7 shows a typical triangulation and its Powell–Sabin refinement. Note that each triangle in the original triangulation is split into six subtriangles. It is not immediately clear that the Powell–Sabin refinement is well defined for arbitrary triangulations, since we need the line segment connecting the incenters of two adjoining triangles to intersect their common edge at an interior point of that edge. The following lemma ensures that this always happens.

Lemma 4.19. *Suppose $T = \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} = \langle v_4, v_3, v_2 \rangle$ are two triangles sharing the edge $e = \langle v_2, v_3 \rangle$, and that u_T and $u_{\widetilde{T}}$ are the corresponding incenters. Then the line joining u_T and $u_{\widetilde{T}}$ intersects e at some point strictly between v_2 and v_3.*

Proof: Since u_T is the incenter of T, the line through u_T and perpendicular to e intersects e at some point w strictly between v_2 and v_3. The same holds for the analogous point $\widetilde{w}$ associated with $\widetilde{T}$. Now the line from u_T to $u_{\widetilde{T}}$ must cross e at some point w_e lying between w and $\widetilde{w}$, and the proof is complete. $\square$

The following result gives information on the shape of the subtriangles arising in the Powell–Sabin split of a given triangle. It will be useful later in our study of macro-elements based on the Powell–Sabin split of a triangle.

Lemma 4.20. *Suppose $\triangle_{PS}$ is the Powell–Sabin refinement of a given triangulation $\triangle$. Then*

$$\theta_{PS} \geq \frac{1}{4\pi} \theta_\triangle \sin \theta_\triangle, \tag{4.13}$$

where θ_{PS} is the smallest angle in $\triangle_{PS}$, and $\theta_\triangle$ is the smallest angle in $\triangle$. For each edge $e := \langle v_1, v_2 \rangle$, let w_e be the split point on the edge e. Moreover, there exist constants $0 < K_1 \leq K_2$ depending only on $\theta_\triangle$ such that

$$0 < K_1 \leq \frac{h_1}{h_2} \leq K_2, \tag{4.14}$$

where $h_1 = |\langle v_1, w_e \rangle|$ and $h_2 = |\langle w_e, v_2 \rangle|$.

Proof: Suppose $T := \langle v_1, v_2, v_3 \rangle$ is a triangle in $\triangle$ with angles α_i at the vertices v_i, and let u be the incenter of T. To prove (4.13), it suffices to consider just one of the subtriangles in the Powell–Sabin refinement, say $t := \langle u, v_1, w_e \rangle$, where $e := \langle v_1, v_2 \rangle$. Since the angle of t at v_1 is $\alpha_1/2$, while the angle at w_e is bounded below by $\alpha_2/2$, we conclude that these two angles are bounded below by $\theta_\triangle/2$. Let β be the angle of t at u, and let γ be the angle of the triangle $\langle u, w_e, v_2 \rangle$ at u. We need to get a lower bound on these two angles. We consider only the case where w_e is obtained by joining incenters of neighboring triangles. The case where it is the midpoint of the edge e is similar. Let $\widetilde{T} := \langle v_1, \tilde{v}_3, v_2 \rangle$ be a triangle in $\triangle$ sharing the edge e, see Figure 4.8, and let $\tilde{u}$ be its incenter. Let $\tilde{\alpha}_i$ be the angles of $\widetilde{T}$

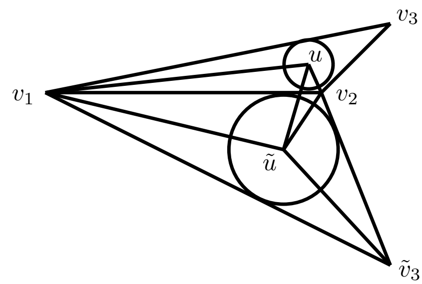


Fig. 4.8. The geometry of incenters.

at its vertices. Now consider the triangle $\langle u, v_1, \tilde{u}\rangle$ with angle $(\alpha_1 + \tilde{\alpha}_1)/2$ at v_1. Using the law of sines in this triangle, we have

$$\beta \geq \sin\beta = \frac{\tilde{r}\sin((\alpha_1 + \tilde{\alpha}_1)/2)}{l + \tilde{l}},$$

where $\tilde{r} := |\langle v_1, \tilde{u}\rangle|$, $l := |\langle u, w_e\rangle|$, and $\tilde{l} := |\langle \tilde{u}, w_e\rangle|$. A simple argument shows that $\alpha_1 + \alpha_2 \leq \pi - \theta_\triangle$ which implies that $\beta + \gamma > \pi$, which in turn implies $\ell \leq |e|$. A similar argument shows that $\tilde{\ell} \leq |e|$. On the other hand, $\tilde{r} \geq |e|\tan(\theta_\triangle/2)/2 \geq |e|\theta_\triangle/2\pi$, and (4.13) follows.

We turn now to the proof of (4.14). By the law of sines,

$$\frac{\sin\beta}{h_1} = \frac{\sin(\alpha_1/2)}{l}, \qquad \frac{\sin\gamma}{h_2} = \frac{\sin(\alpha_2/2)}{l}.$$

This implies

$$\frac{h_1}{h_2} = \frac{\sin(\beta)\sin(\alpha_2/2)}{\sin(\gamma)\sin(\alpha_1/2)},$$

and the upper bound in (4.14) follows with a constant depending only on $\theta_\triangle$. The lower bound can be established by inverting this equation. $\square$

4.8.3 Powell–Sabin-12 Refinement

Definition 4.21. *Given a triangulation $\triangle$, let $\triangle_{CT}$ be its Clough–Tocher refinement based on splitting each triangle about its barycenter v_T. Now for each triangle T, connect v_T to the midpoints w_1^T, w_2^T, w_3^T of the edges of T, where w_i^T is opposite v_i. Also connect w_i^T to w_{i+1}^T, where w_4^T is identified with w_1^T. Then we call the resulting triangulation $\triangle_{PS12}$ the* Powell–Sabin-12 refinement *of $\triangle$.*

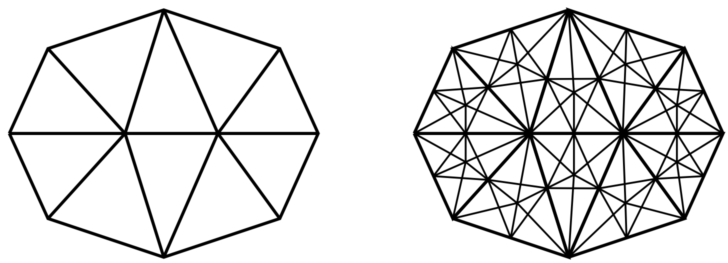


Fig. 4.9. A triangulation and its Powell–Sabin-12 refinement.

Figure 4.9 shows a typical triangulation and its Powell–Sabin-12 refinement. Each triangle T of $\triangle$ is split into 12 subtriangles. Note that, in contrast to the Powell–Sabin refinement, the split points on the edges of triangles of $\triangle$ are at the midpoints of the edges. As a result, this refinement has shape properties which are as good as the Clough–Tocher refinement, and much better than the Powell–Sabin refinement. The following result provides a lower bound on the size of the angles in a Powell–Sabin-12 refinement of a triangle.

Lemma 4.22. *Let T_{PS12} be the Powell–Sabin-12 split of a triangle T, and let θ_{PS12} and θ_T be the smallest angles in T_{PS12} and T, respectively. Then $\theta_{PS12} \geq 2\theta_T/3\pi$.*

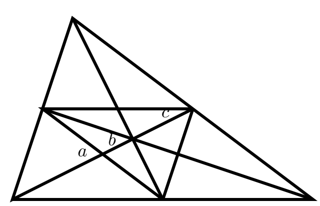


Fig. 4.10. Angles in the Powell–Sabin-12 Split.

Proof: Given $T := \langle v_1, v_2, v_3 \rangle$, let $\triangle_4$ be the split of T into four subtriangles which is obtained by connecting the midpoints of each edge of T. Then each of these four triangles is similar to T, and hence the smallest angle in $\triangle_4$ is the same as the smallest angle in T. Now let v_T be the barycenter of T. A simple computation shows that v_T is also the barycenter of the middle triangle of $\triangle_4$. It follows from Lemma 4.17 that the bound holds for all angles of T_{PS12} except for those of the type marked with a, b, c in Figure 4.10. But $a > b > c$, and so it suffices to consider c. But since the upper edge attached to the angle c is parallel to the bottom edge of T, we see that c is the same as the angle at the lower left in the figure, which by Lemma 4.17 is at least $2\theta_T/3\pi$. $\square$

4.8.4 Uniform Refinement

Algorithm 4.23. (Uniform Refinement). *Let $\triangle$ be a given triangulation. Let $\triangle_u$ be the triangulation which is obtained by splitting each triangle $T \in \triangle$ into four subtriangles by connecting the midpoints of the edges of T with straight lines.*

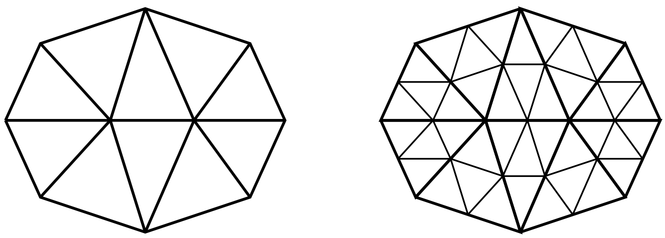


Fig. 4.11. A triangulation and its uniform refinement.

Figure 4.11 shows a triangulation $\triangle$ and its uniform refinement. It is clear that the refined triangulation contains four times as many triangles as $\triangle$. Moreover, it is also clear that for each $T \in \triangle$, the four new triangles which cover T are all similar to T and thus have the same smallest angle as T. We conclude that the smallest angle in $\triangle_u$ is the same as the smallest angle in $\triangle$.

4.9. Optimal Triangulations

Suppose Ω is a polygonal domain, and let $\mathcal{V}$ be a set of points in Ω which includes all the vertices of the polygon forming the boundary of Ω. Suppose $\mathrm{Tri}\,(\Omega, \mathcal{V})$ is the set of all possible triangulations of Ω with vertices $\mathcal{V}$. Here we are allowing Ω to be nonconvex and to have holes in it. In this section we introduce a way of comparing triangulations in $\mathrm{Tri}\,(\Omega, \mathcal{V})$ with each other, and then use it to define optimal triangulations.

For simplicity, we focus on the case where we assign a single real number $a(\triangle)$ to each triangulation $\triangle \in \mathrm{Tri}\,(\Omega, \mathcal{V})$, and seek to maximize this number. For example, we can take $a(\triangle)$ to be the smallest angle among all the triangles in $\triangle$. For other choices of criteria, including vector criteria, see Remark 4.9.

Definition 4.24. *A triangulation $\triangle^* \in \mathit{Tri}\,(\Omega, \mathcal{V})$ is called an* optimal triangulation with respect to the criterion *a provided there is no $\triangle \in \mathit{Tri}\,(\Omega, \mathcal{V})$ with $a(\triangle) > a(\triangle^*)$.*

The following questions immediately arise:

1) Does there exist an optimal triangulation?
2) Is there a unique optimal triangulation?

3) How can we characterize an optimal triangulation?
4) How can we construct an optimal triangulation?

The first question is easy. Since $\mathrm{Tri}(\Omega, \mathcal{V})$ contains a finite number of triangulations, it is clear that there always exists at least one optimal triangulation. We give an example in the next section to show that optimal triangulations need not be unique. In general, it is difficult to characterize optimal triangulations, or even to recognize if a particular triangulation is optimal. This makes the design of algorithms to find optimal triangulations difficult. But as we shall see in the next section, if we choose $a(\triangle)$ to measure the minimum angle in $\triangle$, then there is a simple characterization which leads to practical algorithms for constructing the corresponding optimal triangulation.

4.10. Maxmin-angle Triangulations

In this section we examine optimal triangulations which are based on avoiding triangles with small angles, i.e., we seek to maximize the smallest angle $a(\triangle)$ in $\triangle$. We call a triangulation which is optimal with respect to this criterion a maxmin-angle triangulation. We emphasize that throughout this section we allow Ω to be nonconvex and to have holes.

To begin our study of maxmin-angle triangulations, we first examine the case where Ω is a quadrilateral whose vertices lie on a circle.

Lemma 4.25. *Suppose Q is a quadrilateral whose vertices v_1, v_2, v_3, v_4 lie on a circle, and let $\triangle$ and $\widetilde{\triangle}$ be the two possible triangulations of Q. Then $a(\triangle) = a(\widetilde{\triangle})$, i.e., both triangulations are optimal with respect to the maxmin-angle criterion.*

Proof: Suppose $v_1, \ldots, v_4$ are the vertices of Q in counterclockwise order as shown in Figure 4.12. Then the claim follows immediately from the fact that if u_1, u_2, u_3 are three points on a circle, then the angle between $\langle u_2, u_1 \rangle$ and $\langle u_2, u_3 \rangle$ is equal to one half the length of the subtended arc $\overline{u_1 u_3}$. $\square$

The situation where the vertices fall on a circle as in Lemma 4.25 is called the neutral case. We refer to the diagonal edges in Figure 4.12 as neutral edges. The situation is different if not all four points lie on a circle.

Lemma 4.26. *Suppose Q is a strictly convex quadrilateral with vertices v_1, v_2, v_3, v_4 in counterclockwise order as in Figure 4.13. Let C be the circle passing through v_1, v_2, v_3. If v_4 is inside C, then the maxmin-angle triangulation is the one that contains the edge $\langle v_2, v_4 \rangle$. If v_4 is outside C, then the maxmin-angle triangulation is the one that contains the edge $\langle v_1, v_3 \rangle$.*

Proof: Suppose v_4 is inside C, and label the angles as in Figure 4.13. Since $\theta_2 < \theta_4$ and $\theta_5 < \theta_1$, it follows that neither θ_1 nor θ_4 can be the smallest angle in the triangulation in Figure 4.13 (left). Similarly, neither $\widetilde{\theta}_1$ nor $\widetilde{\theta}_4$

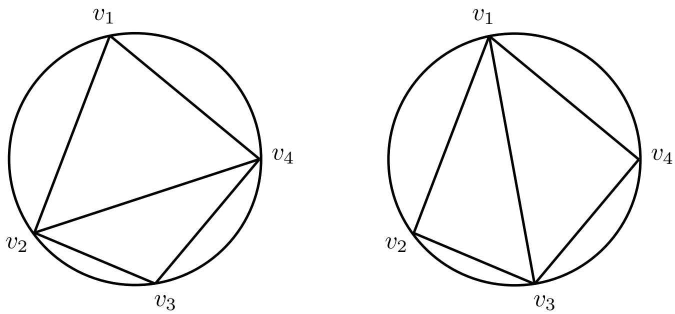


Fig. 4.12. Two triangulations of four points on a circle.

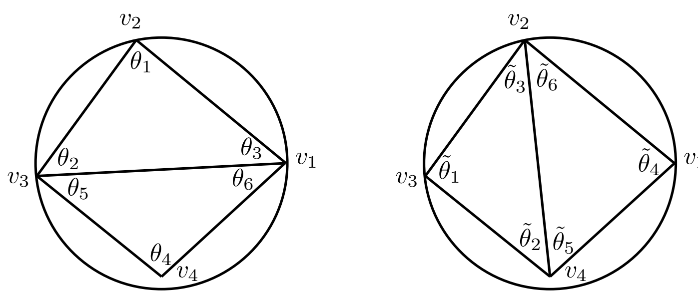


Fig. 4.13. Two triangulations of four points not on a circle.

can be the smallest angle in the triangulation in Figure 4.13 (right). Let w be the intersection with C of the extension of the edge $\langle v_2, v_4 \rangle$. Let $\{\widetilde{\theta}'_i\}$ be the angles in the triangles $\langle v_2, v_3, w \rangle$ and $\langle v_2, w, v_1 \rangle$, numbered in the same way as the $\widetilde{\theta}_i$. Comparing arcs, we see that $\widetilde{\theta}_2 > \widetilde{\theta}'_2 = \theta_3$, $\widetilde{\theta}_3 > \theta_6$, $\widetilde{\theta}_5 > \widetilde{\theta}'_5 = \theta_2$, and $\widetilde{\theta}_6 > \theta_5$. It follows that in this case the triangulation on the right is the maxmin-angle triangulation. The case when v_4 is outside of C is similar. $\square$

Lemma 4.26 shows that to find the maxmin-angle triangulation of a strictly convex quadrangulation, we can first triangulate with an arbitrary pair of triangles T and $\widetilde{T}$. We then construct the circumcircle around T, i.e., the unique circle which passes through the three vertices of T. Next we check to see if the fourth vertex is inside C, i.e., if it lies in the interior of the corresponding disk. If it is inside, we can swap the diagonal of the quadrilateral $Q := T \cup \widetilde{T}$ to get a triangulation $\widetilde{\triangle}$ with $a(\widetilde{\triangle}) > a(\triangle)$. This observation suggests that to construct a maxmin-angle triangulation of a domain Ω with an arbitrary set of given vertices $\mathcal{V}$, we could start with any triangulation $\triangle^{(0)}$ and adjust its edges by swapping. We shall see later that this actually works. Here is the basic swap algorithm.

Algorithm 4.27. (Swap Algorithm) *Let $\triangle^{(0)}$ be an arbitrary triangulation of a set Ω with vertices $\mathcal{V}$. Set $m = 0$.*

Do until no longer possible:

> *Let $\mathcal{Q}$ be the set of all strictly convex quadrilaterals Q in $\triangle^{(m)}$ such that swapping the diagonal of Q would increase the minimal angle in the two triangles making up Q. If $\mathcal{Q}$ is empty, stop. Otherwise, choose a $Q^* \in \mathcal{Q}$ corresponding to the smallest minimal angle among all pairs of triangles in the quadrilaterals Q of $\mathcal{Q}$. Swap the diagonal of Q^*, increase m by one, and let $\triangle^{(m)}$ be the new triangulation.*

Since $a(\triangle^{(m)})$ does not necessarily increase in each step, to see that this algorithm always terminates after a finite number of steps, we now define a *vector measure* of the quality of a triangulation. Associated with a triangulation $\triangle = \{T_i\}_{i=1}^N$, we define $\alpha(\triangle) := (\alpha_1, \ldots, \alpha_N) := (a(T_1), \ldots, a(T_N))$, where for each triangle $T \in \triangle$, $a(T)$ is the minimum angle in T, and where in forming $\alpha(\triangle)$, the triangles of $\triangle$ have been ordered so that $a(T_1) \le a(T_2) \le \cdots \le a(T_N)$. If $\tilde{\alpha}$ is the analogous vector corresponding to an alternative triangulation $\widetilde{\triangle}$, then we define $\alpha < \tilde{\alpha}$ provided that for some $1 \le m \le N$, $\alpha_i \le \tilde{\alpha}_i$ for $i = 1, \ldots, m-1$ and $\alpha_m < \tilde{\alpha}_m$. Now in terms of this order, it is easy to see that $\alpha(\triangle^{(m)}) < \alpha(\triangle^{(m+1)})$, and since there are only a finite number of triangulations in $\mathrm{Tri}\,(\Omega, \mathcal{V})$, the algorithm must stop after a finite number of steps.

Definition 4.28. *We say that a triangulation $\triangle$ is a* locally optimized triangulation with respect to the maxmin-angle criterion *provided that for every strictly convex quadrilateral Q in $\triangle$, swapping its diagonal would not increase the minimal angle in the two triangles making up Q.*

Algorithm 4.27 can be applied to any triangulation to create a locally optimized triangulation. We now show that any triangulation that is locally optimized with respect to the maxmin-angle criterion must be a maxmin-angle triangulation. Figure 4.14 shows that the converse is not true, i.e., there exist triangulations that are optimal with respect to the maxmin-angle criterion, but are not locally optimized.

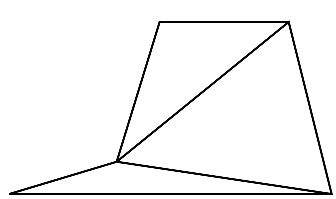


Fig. 4.14. A maxmin-angle triangulation which is not locally optimized.

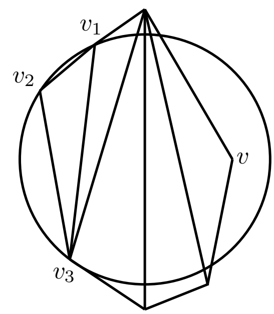


Fig. 4.15. Triangles in the proof of Lemma 4.29.

Given a triangle T in a triangulation $\triangle$ of a domain Ω and a vertex v of $\triangle$, we say that v is internally connected to T provided there is a line segment joining v and T such that all points on the line segment (except possibly for v) lie in the interior of Ω.

Lemma 4.29. *Suppose the triangulation $\triangle$ is locally optimized with respect to the maxmin-angle criterion. Given any triangle T in $\triangle$, let C be its circumcircle. Then no vertex v of $\triangle$ that is internally connected to T can lie in the interior of C.*

Proof: Suppose $T := \langle v_1, v_2, v_3 \rangle$, and that the vertex v is internally connected to T and lies in the interior of the circumcircle C around T. If there are several such vertices, we may assume that v is one which is closest to the edge $e := \langle v_1, v_3 \rangle$, see Figure 4.15. Since v is internally connected to T, there must be a collection of triangles $T_1, \ldots, T_k$ of $\triangle$ lying between v and T. Referring to the figure, suppose these are numbered from left to right, with T_1 sharing the edge e and T_k sharing the vertex v. Now since $\triangle$ is locally optimized, none of the vertices of T_1 can lie in the interior of C. Now consider the circumcircle C_1 around T_1. Since two distinct circles can intersect in at most two points, we see that v is in the interior of C_1. By the local optimality, none of the vertices of T_2 can lie in the interior of C_1. Repeating this argument, we find that v must lie in the interior of the circumcircle around T_{k-1}. But this contradicts the assumption that $\triangle$ is locally optimized. $\square$

If Ω is convex, then it is easy to see that the condition that v be internally connected to T in Lemma 4.29 is automatically satisfied. However, for general nonconvex domains Ω, the condition cannot be dropped. Indeed, Figure 4.16 shows a locally optimized triangulation of a nonconvex Ω which contains a triangle T whose circumcircle contains another vertex v. Of course, in this example v is not internally connected to T. We now establish a variant of Lemma 4.29 concerning points on the circumcircle.

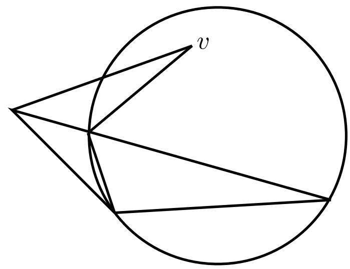


Fig. 4.16. A circumcircle in a locally optimized triangulation can contain other vertices.

Lemma 4.30. *Suppose the triangulation $\triangle$ is locally optimized with respect to the maxmin-angle criterion. Given a triangle $T \in \triangle$, let C be its circumcircle, and suppose there exists some vertex $v \notin T$ that lies on C and is internally connected to an edge e of T. Then e must be a neutral edge of $\triangle$.*

Proof: Since v is internally connected to T, there must be a collection of triangles $T_1, \ldots, T_k$ of $\triangle$ lying between v and T. Suppose $T := \langle v_1, v_2, v_3 \rangle$ and $T_1 := \langle v_1, v_3, v_4 \rangle$. Since $\triangle$ is locally optimized, v_4 cannot lie inside of C. If v_4 lies on C, then e is a neutral edge, and we are done. If v_4 is outside C, then arguing in the same way as in the proof of Lemma 4.29 leads to a contradiction of the assumption that $\triangle$ is locally optimized. $\square$

We now show that except for possibly some neutral edges, any two locally optimized triangulations must be the same.

Theorem 4.31. *Suppose $\triangle$ and $\widetilde{\triangle}$ are two triangulations of a domain Ω corresponding to the same set of vertices $\mathcal{V}$, and that both $\triangle$ and $\widetilde{\triangle}$ are locally optimized with respect to the maxmin-angle criterion. Suppose e is an interior edge of $\triangle$, but e is not a neutral edge. Then e must be an interior edge of the triangulation $\widetilde{\triangle}$.*

Proof: Let e be an interior edge of $\triangle$ that is not a neutral edge of $\triangle$, and suppose that e is not an interior edge of $\widetilde{\triangle}$. We may suppose $e := \langle v_2, v_3 \rangle$ and $T := \langle v_1, v_2, v_3 \rangle$ as shown in Figure 4.17. Then since the triangles of $\widetilde{\triangle}$ cover the same domain Ω as $\triangle$, there must be some triangle $\widetilde{T} := \langle v_3, v_4, v_5 \rangle$ in $\widetilde{\triangle}$ such that a part of e near v_3 is in the interior of $\widetilde{T}$. Note that v_4 may be equal to v_1, or it may be on the circumcircle C around T, but it cannot be inside C since then it would be internally connected to T which is impossible by Lemma 4.29. By the lemma, v_5 cannot lie inside C, and by Lemma 4.30, it cannot lie on C.

Now consider the circumcircle $\widetilde{C}$ around $\widetilde{T}$ which passes through v_3, v_4, v_5. Since v_4 is not inside C and v_5 is outside C, it follows that v_2 must be in

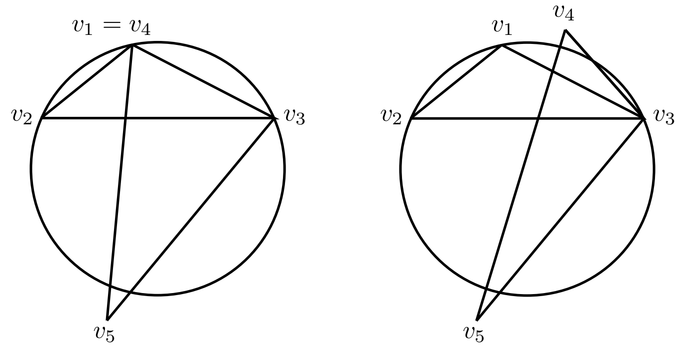


Fig. 4.17. Two of the possible configurations in the proof of Theorem 4.31.

the interior of $\tilde{C}$. But v_2 is internally connected to $\widetilde{T}$, and by Lemma 4.29 this is a contradiction of the local optimality of $\widetilde{\triangle}$. We conclude that e must also be an interior edge of $\widetilde{\triangle}$. $\square$

We can now prove our main result about triangulations that are locally optimized with respect to the maxmin-angle criterion.

Theorem 4.32. *Suppose $\triangle$ is any triangulation that is locally optimized with respect to the maxmin-angle criterion. Then $\triangle$ is a maxmin-angle triangulation.*

Proof: Let $\widetilde{\triangle}$ be any maxmin-angle triangulation of the same domain Ω and with the same vertices as $\triangle$. Then we can apply the swap algorithm to locally optimize $\widetilde{\triangle}$. But then by Theorem 4.31, $\triangle$ and $\widetilde{\triangle}$ are the same up to neutral edges, and it follows that $\triangle$ is also a maxmin-angle triangulation, i.e., $a(\triangle) = a(\widetilde{\triangle})$. $\square$

4.11. Delaunay Triangulations

There is a close connection between maxmin-angle triangulations and the classical Delaunay triangulations.

Definition 4.33. *Suppose Ω is a convex polygonal domain. Then a triangulation $\triangle$ of Ω is called a* Delaunay triangulation *provided that for every triangle T in $\triangle$, there is no vertex $v \notin T$ inside the circumcircle around T.*

It follows from the results of the previous section that a triangulation of a convex domain Ω is locally optimized with respect to the maxmin-angle criterion if and only if it is a Delaunay triangulation. Thus, for convex Ω, any algorithm for constructing a Delaunay triangulation can be used to create a maxmin-angle triangulation. To deal with nonconvex Ω, we introduce the following natural extension of Definition 4.33.

Definition 4.34. *Suppose Ω is a nonconvex domain. Then a triangulation $\triangle$ of Ω is called a* Delaunay triangulation *provided that for every triangle T in $\triangle$, no vertex $v \notin T$ that is internally connected to T can lie inside the circumcircle around T.*

With this definition, it follows from the results of the previous section that for a general domain Ω, a triangulation is locally optimized with respect to the maxmin-angle criterion if and only if it is a Delaunay triangulation.

4.12. Constructing Delaunay Triangulations

The problem of designing algorithms for constructing Delaunay triangulations, or equivalently maxmin-angle triangulations, has received a great deal of attention in the computational geometry community, and there are several competing algorithms available. Here we briefly discuss two.

Method 4.35. (Swap Algorithm)

1) Construct an initial triangulation $\triangle$ using any convenient method, for example, Algorithm 4.12.
2) Apply the swapping Algorithm 4.27 to locally optimize $\triangle$.

Method 4.36. (Modified Vertex Insertion Algorithm)

Add the following step to Algorithm 4.12:

3) Put the edges opposite to v_i into a stack and check them one by one to see if they can be swapped to improve the triangulation. If an edge is swapped, put the new edges opposite to v_i onto the stack. Continue swapping until the stack is empty.

Theorem 4.31 can be used to show that both of these methods produce a maxmin-angle triangulation. Both algorithms have a worst-case complexity of $\mathcal{O}(V^2)$, where V is the number of vertices in $\mathcal{V}$. This can become prohibitive for very large vertex sets, but both methods are quite usable for moderate numbers of vertices. For a Fortran implementation of the second method, see [Ren84].

Because of the high order of complexity of these algorithms, the problem of finding more efficient algorithms for creating Delaunay triangulations has attracted a lot of attention. Algorithms that achieve the minimal worst-case complexity of $\mathcal{O}(V \log V)$ are based on a divide and conquer strategy, see [PreS85, Ede87]. For an efficient C implementation of this idea, see [She02].

4.13. Type-I and Type-II Triangulations

Suppose

$$x_0 < x_1 < \cdots < x_k < x_{k+1},$$
$$y_0 < y_1 < \cdots < y_l < y_{l+1}.$$

Then

$$\mathcal{R}_{k,l} := \{H_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]\}_{i=0,j=0}^{k \quad \ell}$$

provides a rectangular partition of the rectangle $H := [x_0, x_{k+1}] \times [y_0, y_{l+1}]$. There are two triangulations associated with this partition which are of special interest.

Definition 4.37. *The triangulation $\triangle_I$ obtained by drawing in the north-east diagonals in all subrectangles H_{ij} of $\mathcal{R}_{k,l}$ is called a* type-I triangulation. *If both the $\{x_i\}$ and the $\{y_j\}$ are uniformly spaced, then we call $\triangle_I$ a* uniform type-I triangulation. *It is also called a* three-direction mesh.

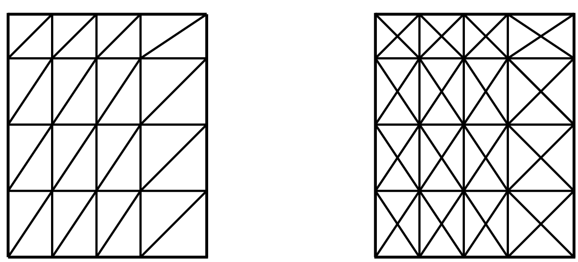


Fig. 4.18. Nonuniform Type-I and Type-II Triangulations.

Figure 4.18 (left) shows a typical nonuniform type-I triangulation. It is easy to see that with the notation of (4.4), the numbers of vertices, edges, and triangles are given by

$$\begin{aligned}
V_I &= kl, \\
V_B &= 2(k+l)+4, \\
E_I &= 3kl + 2(k+l) + 1, \\
E_B &= 2(k+l)+4, \\
N &= 2(k+1)(l+1).
\end{aligned}$$

Definition 4.38. *The triangulation $\triangle_{\text{II}}$ obtained by drawing in both diagonals in all subrectangles H_{ij} of $\mathcal{R}_{k,l}$ is called a* type-II triangulation. *If both the $\{x_i\}$ and the $\{y_j\}$ are uniformly spaced, then we call $\triangle_{\text{II}}$ a* uniform type-II triangulation. *It is also called a* four-direction mesh.

Figure 4.18 (right) shows a typical nonuniform type-II triangulation. With the notation of (4.4), the numbers of vertices, edges, and triangles are

$$\begin{aligned} V_I &= 2kl + (k + l) + 1, \\ V_B &= 2(k + l) + 4, \\ E_I &= 6kl + 5(k + l) + 4, \\ E_B &= 2(k + l) + 4, \\ N &= 4(k + 1)(l + 1). \end{aligned}$$

4.14. Quadrangulations

Suppose $\{v_i := (x_i, y_i)\}_{i=1}^4$ are four points in $\mathbb{R}^2$. Then we say that the convex hull $Q := \langle v_1, v_2, v_3, v_4 \rangle$ is a quadrilateral. We say that Q is strictly convex provided the angles at its four vertices are all less than π.

Definition 4.39. *A set* $\diamondsuit := \{Q_i\}_{i=1}^n$ *of quadrilaterals is called a* quadrangulation *of* $\Omega := \cup_{i=1}^n Q_i$ *provided the intersection of any two quadrilaterals is either empty, a single point, or a common edge. We say that* $\diamondsuit$ *is* strictly convex *provided every quadrilateral in* $\diamondsuit$ *is strictly convex.*

This definition allows quite general quadrangulations. For example, $\diamondsuit$ may consist of two quadrilaterals which are completely separated, or it may consist of two quadrilaterals which touch only at a vertex.

Definition 4.40. *We say that a quadrangulation* $\diamondsuit$ *is* regular *provided that for each vertex* v*, the set* $\{Q_1, \ldots, Q_m\}$ *of quadrilaterals sharing the vertex* v *can be ordered so that* Q_{i+1} *shares at least one edge with* Q_i *for* $i = 1, \ldots, m - 1$.

Regular quadrangulations can cover domains Ω with one or more holes. Such partitions are important in the finite-element method for solving partial differential equations. A simple example of a nonregular quadrangulation is one where two quadrilaterals touch only at a vertex. We now examine the relationship between the number of vertices, edges, and quadrilaterals in a regular quadrangulation. Let V_I^Q and V_B^Q be the number of interior and boundary vertices of $\diamondsuit$, respectively.

Lemma 4.41. *For any regular quadrangulation* $\diamondsuit$ *with no holes,*

$$\begin{aligned} N_Q &:= \text{number of quadrilaterals in } \diamondsuit = (V_B^Q + 2V_I^Q - 2)/2, \\ E_I^Q &:= \text{number of interior edges of } \diamondsuit = (V_B^Q + 4V_I^Q - 4)/2, \\ E_B^Q &:= \text{number of boundary edges of } \diamondsuit = V_B^Q. \end{aligned} \tag{4.15}$$

Proof: The formulae can be verified by straightforward induction on the number of quadrilaterals in $\diamondsuit$. $\square$

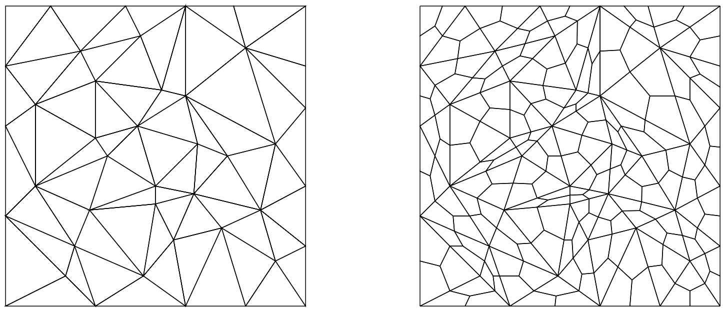


Fig. 4.19. A triangulation and the quadrangulation produced by Method 4.42.

It is clear from the first formula in this lemma that quadrangulations exist only when V_B^Q is even. The problem of constructing quadrangulations associated with a given set of vertices has been studied much less than the corresponding problem for triangulations, see Remark 4.13. It seems there are no known algorithms that are guaranteed to produce strictly convex quadrangulations, even if Ω is strictly convex. On the other hand, it is possible to convert an arbitrary triangulation into a strictly convex quadrangulation. We now present two methods for carrying out this conversion.

Method 4.42. *Suppose $\triangle$ is a regular triangulation of a polygon Ω without holes. Then subdivide each triangle $T \in \triangle$ by connecting its barycenter to the midpoints of its edges.*

A typical example of the use of this method is shown in Figure 4.19. Each quadrilateral in $\diamondsuit$ has one vertex at a barycenter of a triangle $T \in \triangle$, one vertex at a vertex of T, and two vertices at midpoints of edges of T.

Method 4.43. *Suppose $\triangle$ is a regular triangulation without holes of a polygon Ω. Let $v_1, \ldots, v_n$ be the boundary vertices of $\triangle$, numbered in clockwise order.*

1) *For each triangle $T \in \triangle$, subdivide T into three subtriangles by connecting its barycenter to its three vertices.*
2) *Remove all interior edges of $\triangle$.*
3) *For each boundary vertex v of $\triangle$ and each triangle T attached to v, remove the edge $\langle v, w\rangle$, where w is the barycenter of T.*
4) *For each boundary edge e of $\triangle$, let p be its midpoint, and let w be the barycenter of the triangle of $\triangle$ with edge e. Insert the edge $\langle p, w\rangle$.*
5) *For each boundary vertex v_i, $1 \le i \le n$, suppose $T_1, \ldots, T_m$ are the triangles of $\triangle$ that share the vertex v_i, numbered in counterclockwise*

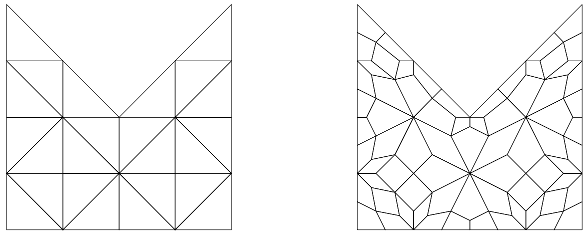


Fig. 4.20. A triangulation and the quadrangulation produced by Method 4.43.

order around v_i. Let $w_1, \ldots, w_m$ be their barycenters. If $m = 1$, do nothing further. If $m > 1$, do the following. For each $j = 1, \ldots, m-1$, choose a point u_j on $e_j := \langle v_i, \tilde{v}_j \rangle$, where e_j was the edge of $\triangle$ shared by T_j and T_{j+1}. Choose the point u_j so that it is closer to v_i than to $\tilde{v}_j$, and such that the angle between $\langle u_j, w_j \rangle$ and $\langle u_j, w_{j+1} \rangle$ is less than π. Add the edges $\langle u_j, v_i \rangle$, $\langle u_j, w_j \rangle$, and $\langle u_j, w_{j+1} \rangle$ for $j = 1, \ldots, m-1$.

The points u_j in step 5) can always be selected to satisfy the required angle condition. Indeed, the angle between $\langle v_i, w_j \rangle$ and $\langle v_i, w_{j+1} \rangle$ is less than π, and thus if u_j is sufficiently close to v_i, the angle between $\langle u_j, w_j \rangle$ and $\langle u_j, w_{j+1} \rangle$ will also be less than π. This ensures that the algorithm is well defined.

Figure 4.20 shows an example of the use of Method 4.43 to convert a given triangulation to a quadrangulation. Both Methods 4.42 and 4.43 can be extended to work with regular triangulations with holes.

Theorem 4.44. *The quadrangulation produced by Methods 4.42 and 4.43 are strictly convex.*

Proof: Consider first Method 4.42. Let T be a triangle in $\triangle$ with barycenter w. Then w must lie inside the subtriangle of T formed by connecting the midpoints of the edges of T since every point on a line connecting midpoints has one barycentric coordinate equal to $1/2$, while the barycentric coordinates of w are $(1/3, 1/3, 1/3)$. This ensures that each of the three quadrilaterals inserted into T is strictly convex since all four of its interior angles are less than π.

Now consider Method 4.43. This method creates one quadrilateral associated with each interior edge $e := \langle v_1, v_2 \rangle$ of $\triangle$. If neither vertex of e is on the boundary of Ω, then we get the same quadrilateral as in Method 4.42. If v_1 is on the boundary, then instead of the quadrilateral of Method 4.42, we get a slightly squashed quadrilateral where v_1 is replaced by a point u_1 on e that was chosen so that the interior angle at u_1 is less than π. If both v_1 and v_2 are on the boundary of Ω, then instead of the quadrilateral of Method 4.42, we get a slightly squashed quadrilateral where v_1, v_2 are replaced by points u_1, u_2 on e chosen so that the interior angles at those points are less than π. It remains to consider quadrilaterals attached to boundary vertices of $\triangle$. They are strictly convex since we used midpoints of boundary edges in step 4), and in step 5) we chose the points u_j closer to v_i than to $\tilde{v}_i$. $\square$

We now give formulae for the number of points inserted and the number of quadrilaterals produced by Methods 4.42 and 4.43.

Theorem 4.45. *Let $\triangle$ be a regular triangulation without holes. Let V_I and V_B be the number of interior and boundary vertices, respectively. Let E_0 be the number of interior edges of $\triangle$ where neither end is a boundary vertex of $\triangle$. Similarly, let E_1, E_2 be the number of interior edges of $\triangle$ with one or two ends on the boundary of $\triangle$, respectively. Let I_1, I_2 be the number of inserted points for Methods 4.42 and 4.43, and let N_1, N_2 be the number of quadrilaterals created. Then*

$$\begin{aligned}
I_1 &= 2V_I + 2V_B + E_0 + E_1 + E_2 - 2, \\
N_1 &= 3V_I + 2V_B + E_0 + E_1 + E_2 - 3, \\
I_2 &= 2V_I + 2V_B + E_1 + 2E_2 - 2, \\
N_2 &= 3V_I + 2V_B + E_1 + 2E_2 - 3.
\end{aligned}$$

Proof: Consider first Method 4.42. We add the barycenter of each triangle and the midpoint of each edge of $\triangle$. By the Euler relations (4.5), there are $2V_I + V_B - 2$ triangles and V_B boundary edges. By Lemma 4.41 the number of quadrilaterals in the quadrangulation $\diamondsuit$ is $(V_B^Q - 2)/2 + V_I^Q$, where V_B^Q and V_I^Q are the number of boundary and interior vertices of $\diamondsuit$, respectively. Clearly, we have $V_B^Q = 2V_B$, and it is easy to check that $V_I^Q = V_I + N_T + E_0 + E_1 + E_2$, where N_T is the number of triangles in $\triangle$. The formula for N_1 follows. It is also obvious from the construction that $N_1 = 3N_T$. Now consider Method 4.43. The only difference in the insertion count is that no points are inserted on interior edges that do not have any vertex on the boundary, one point is inserted on interior edges that have one vertex on the boundary, and two points are inserted on interior edges that have two vertices on the boundary. For the quadrilateral count, the difference is that now $V_I^Q = V_I + N_T + E_1 + 2E_2$. The formula for N_2 follows. $\square$

When E_0 is large compared to E_2, Method 4.43 inserts many fewer points and creates many fewer quadrilaterals than Method 4.42.

Definition 4.46. *Given a quadrangulation $\diamondsuit$, let $\diamondsuit_R$ be the quadrangulation obtained by connecting the midpoints of the four edges of each quadrilateral to the intersection of its two diagonals. We call $\diamondsuit_R$ the* uniform refinement *of $\diamondsuit$.*

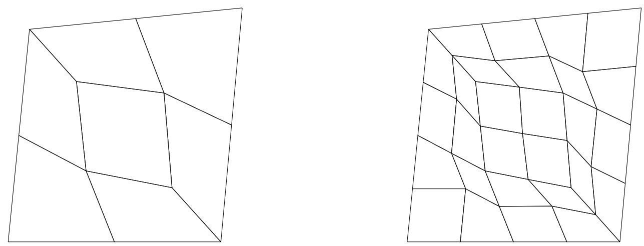


Fig. 4.21. A quadrangulation and its uniform refinement.

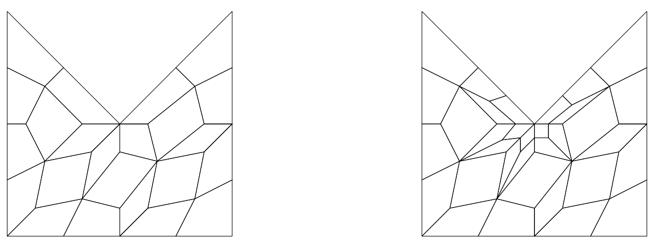


Fig. 4.22. A quadrangulation and its local refinement at a re-entrant corner.

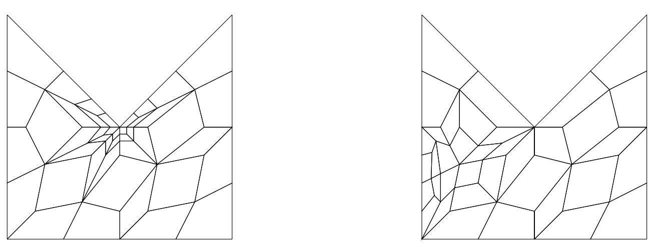


Fig. 4.23. Two steps of refinement at a re-entrant corner and along a crack.

Figure 4.21 shows a typical quadrangulation and its uniform refinement. Note that this definition is a bit different than uniform refinement of triangulations where we simply connected the midpoints of the edges of each triangle. We could do that here with quadrilaterals, but the resulting refined quadrangulation is not as useful as the one we have defined.

Uniform refinement is useful for constructing sequences of nested quadrangulations for use in multilevel methods for partial differential equations and for multiresolution approximation, see Section 4.16. We also note that refinement can also be done locally as shown in Figures 4.22 and 4.23. Local refinement is important in solving boundary value problems where there are singularities due to re-entrant corners and cracks in the domain Ω.

4.15. Triangulated Quadrangulations

Suppose $\diamondsuit$ is a strictly convex quadrangulation. We now discuss a natural triangulation associated with $\diamondsuit$.

Definition 4.47. *Given a strictly convex quadrangulation $\diamondsuit$, we define the* induced triangulation $\diamondplus$ *associated with $\diamondsuit$ to be the triangulation obtained by drawing in both diagonals in each quadrilateral in $\diamondsuit$.*

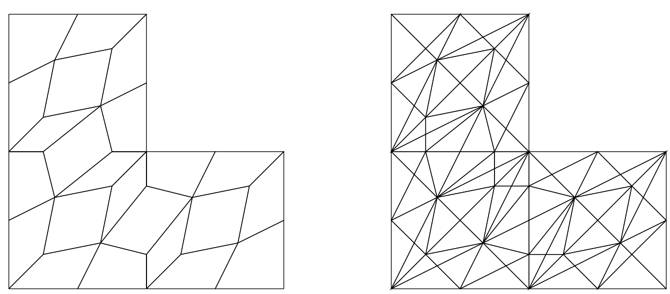


Fig. 4.24. A quadrangulation and its induced triangulation.

Figure 4.24 shows a typical strictly convex quadrangulation and its induced triangulation. It is natural to expect that the smallest angle $\theta_\triangle$ in the induced triangulation will not be too small compared to the smallest angle $\theta_\diamondsuit$ in the quadrangulation $\diamondsuit$. However, this is not the case as shown in the following example.

Example 4.48. Let Q be the quadrilateral with vertices $(0, 0), (1, 0), (1, 1)$, and $(0, \epsilon)$.
Discussion: Clearly, for all $0 < \epsilon \leq 1$, the smallest angle in Q is at least $\pi/4$. On the other hand, as ϵ goes to zero, the smallest angle in the induced triangulation becomes arbitrarily small. $\square$

This example shows that in working with triangulated quadrangulations, some care is needed in the choice of $\diamondsuit$ to make sure that the smallest angles in the induced triangulation is not too small. This means we cannot start with an arbitrary strictly convex quadrangulation. In this regard, it is natural to ask what happens if we start with a triangulation $\triangle$ and construct $\diamondsuit$ by one of the methods discussed in Section 4.14. The following two results follow immediately upon examining Figures 4.19 and 4.20.

Theorem 4.49. *Suppose $\Diamond$ is the quadrangulation obtained from a given triangulation $\triangle$ by applying Method 4.42. Then the associated induced triangulation $\diamondplus$ is just the Powell–Sabin-12 refinement of $\triangle$.*

Theorem 4.50. *Suppose $\Diamond$ is the quadrangulation obtained from a given triangulation $\triangle$ by applying Method 4.43. Then the associated induced triangulation $\diamondplus$ is just the Powell–Sabin refinement of $\triangle$.*

Since refinement is important for applications, we devote the remainder of this section to a discussion of what happens to the angles in the induced triangulations associated with a sequence of quadrangulations obtained from a given quadrangulation by uniform refinement as discussed in the previous section.

Theorem 4.51. *Suppose $\Diamond_0, \Diamond_1, \ldots$ is a sequence of quadrangulations, where $\Diamond_{m+1}$ is obtained from $\Diamond_m$ by uniform refinement. Let $\diamondplus_0, \diamondplus_1, \ldots$ be the corresponding induced triangulations, and let θ_m be the smallest angle in $\diamondplus_m$ for all $m \geq 0$. Then there exists a constant $0 < K < 1$ depending only on θ_0 such that*

$$\theta_m \geq K\theta_0, \qquad \textit{all } m > 0. \tag{4.16}$$

Proof: It is not hard to see that $\theta_m = \theta_1$ for all $m > 1$. Thus, we need only prove the inequality (4.16) holds for θ_1. Let $Q := \langle v_1, v_2, v_3, v_4 \rangle$ be a quadrilateral in $\Diamond_0$, and let v_Q be the point where the diagonals of Q intersect. Then Q is divided into four triangles with angles $\alpha_1, \ldots, \alpha_8$ and a_Q, b_Q as shown in Figure 4.25.

Without loss of generality we may assume $a_Q \leq \pi/2 \leq b_Q$. Suppose $m_1, \ldots, m_4$ are the midpoints of the sides of Q. After refinement, Q is subdivided into 16 triangles, and as shown in Figure 4.26, many of the angles are of exactly the same size as in the original triangulation of Q. In fact the only new angles are $\beta_1, \ldots, \beta_8$. We now show that

$$\beta_i \geq K_Q\,\theta_0, \qquad i = 1, \ldots, 8,$$

where

$$K_Q := \frac{2}{\pi} \sin(a_Q/2).$$

Since $a_Q \leq \pi/2$ and the line $\langle m_1, m_4 \rangle$ bisects the line $\langle v_1, v_Q \rangle$, it follows that $\tan \beta_1 \geq \tan \alpha_1$ and so $\beta_1 \geq \alpha_1$. A similar argument shows that $\beta_4 \geq \alpha_4$, $\beta_5 \geq \alpha_5$, and $\beta_8 \geq \alpha_8$. We now examine β_2 and β_3 and consider only the case $\beta_2 \leq \beta_3$ as the alternative case is very similar. This implies that $\alpha_2 \leq \pi/2$, and also $\beta_3 \geq a_Q/2$ since $\beta_2 + \beta_3 = a_Q$. Clearly, the edges

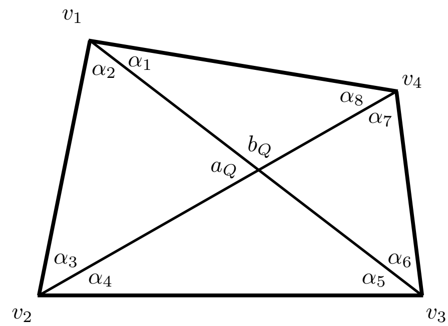


Fig. 4.25. Angles in a triangulated quadrilateral.

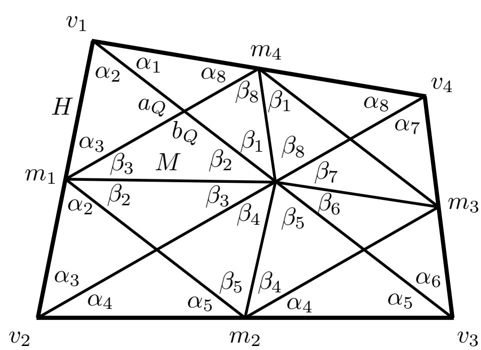


Fig. 4.26. Angles in a refined quadrilateral.

$\langle v_1, m_1\rangle$ and $\langle m_1, v_2\rangle$ have a common length which we denote by H. Let M denote the length of the edge $\langle m_1, v_Q\rangle$. Then by the law of sines,

$$\frac{\sin\beta_2}{H} = \frac{\sin\alpha_2}{M}, \qquad \frac{\sin\beta_3}{H} = \frac{\sin\alpha_3}{M}.$$

Combining these identities with $2x/\pi \le \sin x \le x$ for $0 \le x \le \pi/2$, we conclude that

$$\beta_2 \ge \sin\beta_2 = \frac{\sin\beta_3\,\sin\alpha_2}{\sin\alpha_3} \ge \sin\left(\frac{a_Q}{2}\right)\sin(\theta_0) \ge \frac{2}{\pi}\sin\left(\frac{a_Q}{2}\right)\theta_0 = K_Q\,\theta_0.$$

But then $\beta_3 \ge \beta_2$ is also greater than or equal to $K_Q\,\theta_0$. A similar argument applies to β_6, β_7. We conclude that the smallest angle in the 16 triangles in the refinement of Q is at least $K_Q\,\theta_0$. Now taking the minimum of K_Q over all Q in $\diamondsuit_0$, it follows that $\theta_1 \ge K\theta_0$, where

$$K := \frac{2}{\pi}\sin\left(\frac{\bar{a}}{2}\right), \qquad \bar{a} := \min_{Q\in\diamondsuit} a_Q. \quad \square$$

4.16. Nested Sequences of Triangulations

A sequence of triangulations $\triangle_0, \triangle_1, \triangle_2, \ldots$, such that $\triangle_n$ is a refinement of $\triangle_{n-1}$ for each n is called a nested sequence of triangulations. Spline spaces built on nested sequences of triangulations are useful for creating multi-level methods for the numerical solution of partial differential equations, see [Osw88, Bra97]. They can also be used for multiresolution approximation and the construction of wavelet spaces which are useful for image and surface compression, see Remark 5.9. For applications, it is important to have nested sequences of triangulations with the following two properties:

1) $|\triangle_n| \to 0$ as $n \to \infty$,
2) for some fixed constant K, the smallest angle θ_n in $\triangle_n$ satisfies $\theta_n \geq K\theta_1$ for all $n \geq 2$.

In Section 4.8 we have described several methods for constructing a refinement of a given triangulation. Any of these methods can be used to create a nested sequence of triangulations by starting with an arbitrary triangulation $\triangle_1$, and repeatedly applying the refinement method to create $\triangle_2$, $\triangle_3$, etc. However, only some of the methods can be used to create sequences of triangulations with properties 1) and 2).

Method 4.52. (Uniform Refinement) In view of the discussion in Section 4.8.4, it is clear that starting with an arbitrary triangulation $\triangle_1$ and using uniform refinement to create $\triangle_2, \triangle_3, \ldots$, we get a sequence of triangulations satisfying both 1) and 2). In fact, the smallest angle θ_n in $\triangle_n$ satisifes $\theta_n = \theta_1$ for all n.

Method 4.53. (Clough–Tocher Refinement) The sequence of triangulations resulting from repeatedly applying Clough–Tocher refinement to a given triangulation $\triangle_1$ satisfies 1). By Theorem 4.17, $\theta_n \geq (2/3\pi)^n\theta_1$. Since for any initial triangulation $\triangle_1$, θ_n goes to zero as $n \to \infty$, it follows that 2) is not satisfied.

Method 4.54. (Powell–Sabin Refinement) The sequence of triangulations resulting from repeatedly applying Powell–Sabin refinement to a given triangulation $\triangle_1$ satisfies 1). Lemma 4.20 shows that the size of θ_n can be bounded below by $\theta_{n-1}\sin(\theta_{n-1})/4\pi$. But for any initial triangulation $\triangle_1$, it is easy to see that θ_n goes to zero as $n \to \infty$, so 2) is not satisfied.

Method 4.55. (Powell–Sabin-12 Refinement). The sequence of triangulations resulting from repeatedly applying Powell–Sabin-12 refinement to a given triangulation $\triangle_1$ will also fail to satisfy 2). However, there is a way to get a sequence of Powell–Sabin-12 triangulations that does satisfy both 1) and 2). We proceed as follows. Given an initial triangulation $\triangle_1$, let $\widetilde{\triangle}_2, \widetilde{\triangle}_3, \ldots$ be the the sequence of triangulations obtained by repeated uniform refinement. Then for all n, the smallest angle $\widetilde{\theta}_n$ in $\widetilde{\triangle}_n$ is the same as the smallest angle in $\triangle_1$. Now let $\triangle_n$ be the result of applying the Powell–Sabin-12 split to each triangle in $\widetilde{\triangle}_n$. Clearly this sequence satisfies 1). Moreover, in view of Theorem 4.22, $\theta_n \geq 2\tilde{\theta}_n/3\pi = 2\theta_1/3\pi$, so 2) is also satisfied.

Method 4.56. (Triangulated Quadrangulations with Uniform Refinement) Suppose $\diamondsuit_0, \diamondsuit_1, \ldots$ is a sequence of quadrangulations, where $\diamondsuit_{n+1}$ is obtained from $\diamondsuit_n$ by uniform refinement as described in Definition 4.46. Let $\boxtimes_0, \boxtimes_1, \ldots$ be the corresponding induced triangulations, Clearly, this sequence of triangulations satisfies 1). Moreover, by Theorem 4.51, it also satisfies 2).

Method 4.57. (Six Refinement) The six-refinement method described in Algorithm 3.22 can also be used to create a sequence of triangulations satisfying 1), but as can be seen from Figure 3.3, the size of the smallest angle in $\triangle_n$ will go to zero as $n \to \infty$.

4.17. Remarks

Remark 4.1. The term *regular triangulation* has been used extensively in mathematics, computer science, and engineering, but unfortunately, with many different meanings. In some papers, it refers to a uniform type-I triangulation. In computer graphics it is used for triangulations which may involve triangles of different sizes, but which all have the same shape, see e.g. [BalKV99]. In [Far82] it refers to a triangulation whose interior vertices have degree 6 and whose boundary vertices have degree 4. In much of the spline literature it is used for very general triangulations as described in Definition 4.3, see [AlfS87, AlfPS87a–AlfPS87c] for some early examples. In the computational geometry literature, it is a generalization of the Delaunay triangulation which is obtained as the dual of a Voronoi diagram where each point is assigned a weight, see e.g. [VigNC02] and references therein. On the other hand, in the finite-element literature, it is used to describe a property of families of triangulations which we describe in the next remark.

Remark 4.2. Suppose $\mathcal{F}$ is a family of triangulations of a polygonal domain Ω. Then $\mathcal{F}$ is called quasi-uniform or quasi-regular provided that for every triangulation $\triangle$ in $\mathcal{F}$, all triangles in $\triangle$ have comparable sizes in the sense that

$$\frac{|T|}{\rho_T} \leq C < \infty, \qquad \text{all triangles } T \in \triangle,$$

where ρ_T is the inradius of T. For example, the class of all triangulations whose smallest angles are bounded away from zero by a positive constant have this property, see Lemma 4.2. Sequences of triangulations $\triangle_n$ that are quasi-uniform and also have the property that $|\triangle_n| \to 0$ as $n \to \infty$ have been called regular, see [CiaR72, Cia78a].

Remark 4.3. In defining a triangulation of a set Ω, we have required that the set Ω have a polygonal boundary. For much of what we want to do, we could allow Ω to have a curved boundary, in which case triangles near the boundary could have one or more curved edges.

Remark 4.4. It is possible to improve the bounds in Lemma 4.13 by counting triangles on every other level, see [LaiS98]. This gives

$$N \le \begin{cases} \displaystyle\sum_{\nu=0}^{k} a^{2\nu+1}, & \ell = 2k+1, \\ \displaystyle\sum_{\nu=1}^{k} a^{2\nu}, & \ell = 2k. \end{cases}$$

Remark 4.5. Suppose the sides of a triangle T are of length a, b, c, and that the angles opposite these sides are θ_a, θ_b, and θ_c, respectively. Then the inradius of T is given by

$$\begin{aligned} \rho_T &= \frac{1}{2}\left[\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}\right]^{\frac{1}{2}} = \frac{2A_T}{a+b+c} \\ &= 4R\sin\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_c}{2}\right), \end{aligned}$$

where A_T is the area of T and R is the circumradius of T.

Remark 4.6. Theorem 4.10 can also be proved directly by induction. Clearly, the formulae in (4.5) hold for a single triangle. Now suppose we add a triangle to an existing shellable triangulation $\widetilde{\triangle}$, and that it touches $\widetilde{\triangle}$ along k edges with $k = 1, 2$. Let δV_B be the change in the number of boundary vertices after adding T. Then, with analogous notation for the other quantities, we have the following table:

k	δV_B	δV_I	δE_B	δE_I
1	1	0	1	1
2	-1	1	-1	2

The desired formulae follow.

Remark 4.7. In Definition 4.24 we have chosen to call a triangulation $\triangle^*$ optimal provided $a(\triangle^*)$ is *maximal*. Alternatively, we could have chosen to call $\triangle^*$ optimal if $a(\triangle^*)$ is *minimal*. We have chosen this definition since in Section 4.10 we seek to maximize minimum angles. In any case, minimization problems can always be converted to maximization problems.

Remark 4.8. It is by no means obvious that any triangulation of the convex hull of a given set of vertices $\mathcal{V}$ can be obtained from any other triangulation by a series of swaps. That this is possible was established in [Law72]. The fact that the same thing holds for triangulations of a nonconvex domain Ω was established more recently in [DynGR93].

Remark 4.9. Many other criteria have been suggested for comparing triangulations with each other besides the maxmin-angle criterion, although none of them lead to as elegant a theory as we have for the Delaunay triangulations. For example, one could try to minimize the maximum angle in the triangulation, leading to a so-called minmax-angle triangulation. This criterion does not have many of the nice properties of maxmin. For example, locally optimized triangulations are not necessarily optimal. For an explicit example due to G. Nielson in 1987, see [Sch93a]. For a surveys of some of the other criteria in the literature, see [Sch87, Sch93b].

Remark 4.10. It should be noted that none of the triangulation methods discussed here are affine invariant. Thus, if we change the scale used for the x and y axes, we may get an entirely different triangulation. The question of how to build affine invariant triangulations has been discussed in [Nie93].

Remark 4.11. There is a close connection between Delaunay triangulations and classical Voronoi tilings which are also called Thiessen tesselations and Dirichlet tesselations. Suppose $\mathcal{V} := \{v_i\}_{i=1}^V$ is a set of points in the plane. Then for each $1 \le i \le V$, the set

$$R_i := \{v \in \mathbb{R}^2 : d(v, v_i) \le d(v, v_j), \text{ all } 1 \le j \le V\}$$

is called the Voronoi tile associated with v_i. R_i is just the set of all points in the plane which are at least as close to v_i as they are to any other point in $\mathcal{V}$. In some applications it is called the region of influence of v_i. R_i are closed (possibly unbounded) sets, and their union provides a tiling of all of $\mathbb{R}^2$. Now v_i and v_j are called strong neighbors provided that their Voronoi regions R_i and R_j intersect in a nontrivial line segment. It is known that a Delaunay triangulation $\triangle$ is the dual of the Voronoi tiling in the sense that two vertices v_i and v_j are connected to form an edge of $\triangle$ if and only if they are strong neighbors.

Remark 4.12. The standard definition of a Delaunay triangulation works only with convex Ω. Definition 4.34 is an extension to nonconvex Ω. The key point is that the circle criterion of Definition 4.33 is not required to be satisfied by some of the triangles (near the boundary).

Remark 4.13. In the mid 1990's we asked people in computational geometry about constructing quadrangulations with given vertices. This question led to the paper [BoseT97], which contains an algorithm that creates a quadrangulation of the convex hull Ω of the points whenever the boundary of Ω contains an even number of points. The algorithm has complexity $\mathcal{O}(n\log(n))$, but has a serious defect for our purposes since in general the quadrilaterals obtained need not be strictly convex. It can also lead to quadrilaterals with very small angles.

Remark 4.14. Several authors have considered the problem of converting a given triangulation to a quadrangulation, see [Hei83, JohSK91, RamRT98, LaiS99, BoseRTT02]. These algorithms generally require adding additional vertices, and do not always produce strictly convex quadrangulations. The algorithms based on Methods 4.42 and 4.43 do guarantee strict convexity of all quadrilaterals. They come from [LaiS99], although we have reformulated Method 4.43 to make its operation clearer.

Remark 4.15. It is possible to define spline spaces on more general partitions. Let Ω be a simply connected polygonal domain. Suppose that Ω is the union of polygons $P_1, \ldots, P_n$ such that no vertex of a polygon lies in the interior of an edge of another polygon. Such a partition is called a rectilinear partition. In the special case that the partition is formed by drawing straight lines connecting pairs of boundary points of Ω, the partition is called a cross-cut partition.

4.18. Historical Notes

Piecewise polynomial functions defined on triangulated domains have been used in mathematics and engineering for a long time. They play a major role in data fitting, and are the basic tool for most finite-element methods. However, the question of exactly what constitutes a triangulation has often been glossed over in the literature, and there is some confusion in the literature about what is an acceptable triangulation for a given purpose. Such triangulaions are often called regular, but with completely different meanings. Thus, we have taken extra care to be very precise about what we mean by a triangulation and by a regular triangulation.

It is essential for finite-element applications to allow triangulations with holes, so it is with this mind that we have formulated Definition 4.3. Many papers use some form of this definition, but often with some additional restrictions such as simple connectedness. The idea of a shellable triangulation seems to have been introduced first in topology, where there are many papers on shellability of polytopes, see e.g. [Rud58] and the books [Rus73, Zie95]. We have not seen the concept used in the spline literature in connection with planar triangulations. Our definition of regular triangulation is also nonstandard, cf. Remark 4.1, but we believe it captures the properties we need in working with splines on triangulations.

Formulae connecting the numbers of edges and vertices in a triangulation have been known at least since Euler. The formulae in Theorem 4.11 were stated in [EwiFG70] without any restrictions on the triangulation. The "proof" was based on an inductive argument, see Remark 4.6, which is clearly not valid for arbitrary triangulations. By restricting ourselves to regular triangulations, we have given the first rigorous proof.

The problem of constructing and storing triangulations also has a long history, and is now a well established part of computational geometry, see e.g. the books [PreS85, Ede87, Ede01]. There are two rather different situations. In the first, one is given a set of points which are to serve as the vertices of the triangulation. In some cases one also stipulates the boundary, or that certain other edges should be present. Most of the algorithms for solving this problem deliver an associated Delaunay triangulation. The second situation arises frequently in applications of the finite-element method to solving partial differential equations. In this case we are usually given a domain Ω (which may have holes) with a polygonal boundary, and we would like to construct a triangulation whose boundary agrees with the boundary of Ω. In the FEM literature this is usually called grid generation, and the aim is to make sure that the triangles in $\triangle$ have good shape. Frequently, it is important to have smaller triangles in certain parts of Ω than in others. For details, references, and access to state-of-the-art C-code for solving both problems, see [She02]. The results on clusters of triangles in Section 4.7 are taken from our paper [LaiS98].

The Clough–Tocher refinement discussed in Section 4.8.1 was introduced in [CloT65] in connection with the definition of a macro-element. Sometimes it is referred to as the Hsieh–Clough–Tocher split [Cia78b], and is well known in the finite-element literature, see [Cia78a, BreS94, Bra97]. Having lower bounds on the smallest angles in the subtriangles is important for proving error bounds for an associated Hermite interpolation method and for certain finite-element applications. We could not find explicit bounds in the literature. Our Lemma 4.17 seems to be new.

The Powell–Sabin refinement discussed in Section 4.8.2 was introduced in [PowS77] for the purpose of constructing a C^1 macro-element. This element has been used both in approximation theory and in the numerical solution of partial differential equations. It was observed in the literature that if the refinement is constructed using incenters, then the refinement is at least well defined in the sense that the lines connecting the incenters of two adjoining macro-triangles cross the common edge at an interior point. However, we could not find explicit bounds on the ratio of the lengths of the subedges introduced by the split, or on the smallest angles in the resulting subtriangles. Our Lemma 4.20 is new. The Powell–Sabin-12 refinement discussed in Section 4.8.3 was also introduced in [PowS77], but nothing was said about the shape of the subtriangles. Lemma 4.22 seems to be new.

The idea of swapping edges in a triangulation to improve it was discussed already in [Law72], but is probably much older. Lawson introduced vector measures on the quality of a triangulation, and distinguished between locally optimized and optimal triangulations. He studied maxmin-angle triangulations and their connections to Delaunay triangulations, see also [Law86]. Our treatment of maxmin-angle triangulations basically follows this early work, although we give a different proof of Theorem 4.32, and make use of the concept of internally connected.

Thiessen tesselations are almost 100 years old, and Delaunay's original work was published in 1934, see the books [PreS85, Ede87, Ede01]. Many authors have given algorithms for constructing Delaunay triangulations (for some references, see [Sch87, Sch93b] and the above books. Many of these authors claimed that their algorithms had a worst-case complexity of $\mathcal{O}(n \log(n))$, but as shown in [Sch87], the claims were often incorrect. Correct $\mathcal{O}(n \log(n)$ algorithms have been devised based on a divide-and-conquer strategy, see [PreS85, Ede87, Ede01]. For an efficient modern C implementation, see [She02]. Swapping edges is a useful technique for adjusting triangulations with other criteria (such as goodness of fit), see [DynGR93, Sch93b] and [Sch93a] where it is combined with simulated annealing methods.

Triangulated quadrangulations were used by Fraeijs de Veubeke [Fra68] and Sander [San64] to construct certain macro-elements to be used in solving plate bending problems, see also [CiavN74]. The idea of using triangulated quadrangulations was not taken up by spline researchers until more recently, see [Lai96a, LaiS97, LaiS99]. As with triangulations, we have tried to be careful with our definitions of quadrangulations and triangulated quadrangulations. The algorithms for converting triangulations to quadrangulations described in Section 4.14 come from our paper [LaiS99], although Method 4.43 as presented here is a reformulation of the original method. The key fact in the proof of Theorem 4.51 that the angles in a sequence of triangulated quadrangulations that have been subjected to uniform refinement do not change after the first stage of refinement was observed in [DahOS94].

Bernstein–Bézier Methods for Spline Spaces

In this chapter we introduce a key tool for both the practical and theoretical treatment of spline spaces: the Bernstein–Bézier representation. In addition, we show how it can be used to study various aspects of spline spaces such as dimension, construction of stable local bases, and approximation power.

5.1. The B-form Representation of Splines

As shown in Chapter 2, it is very convenient to represent bivariate polynomials in Bernstein–Bézier (B-) form. In this section we show how this idea can be extended to represent splines. Let $\triangle = \{T_i\}_{i=1}^N$ be a regular triangulation of a polygonal set $\Omega \subseteq \mathbb{R}^2$. Then given a positive integer d, we define the corresponding set of domain points to be the set

$$\mathcal{D}_{d,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{d,T}, \tag{5.1}$$

where $\mathcal{D}_{d,T}$ is the set of domain points (2.14) associated with T. Domain points of $\mathcal{D}_{d,T}$ on edges shared by two triangles are included just once in $\mathcal{D}_{d,\triangle}$. Figure 5.1 shows the set $\mathcal{D}_{2,\triangle}$ for a typical triangulation $\triangle$.

We now show how to use the domain points to parametrize the linear space of C^0 splines of degree d, defined as

$$\mathcal{S}_d^0(\triangle) := \{s \in C^0(\Omega) : s|_{T_i} \in \mathcal{P}_d, \quad i = 1, \ldots, N\}. \tag{5.2}$$

Given $s \in \mathcal{S}_d^0(\triangle)$, for each triangle $T \in \triangle$, we know by Theorem 2.4 that there exists a unique set of coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}}$ such that

$$s|_T = \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^{T,d}, \tag{5.3}$$

where $B_\xi^{T,d}$ are the Bernstein basis polynomials of degree d associated with the triangle T. Since s is continuous, if ξ lies on an edge between two different triangles T and $\widetilde{T}$, then the coefficients c_ξ for $s|_T$ and $s|_{\widetilde{T}}$ are the same. This shows that for each spline $s \in \mathcal{S}_d^0(\triangle)$, there is a unique associated set of coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. We call these the B-coefficients of s. Since the converse also holds, i.e., given any $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ there is a unique spline $s \in \mathcal{S}_d^0(\triangle)$ defined by (5.3), we have the following result.

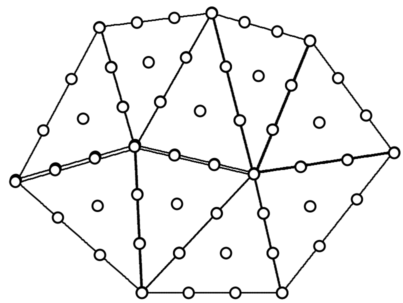


Fig. 5.1. The set $\mathcal{D}_{2,\triangle}$ of domain points associated with a triangulation $\triangle$.

Theorem 5.1. *Every spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely defined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$.*

Often it will be more convenient to work with a *vector of coefficients* rather than a *set of coefficients*. This requires that the coefficients be assigned some order. One convenient way to do this is to assume that the triangles and edges of $\triangle$ are oriented, i.e., for each triangle $T := \langle v_1, v_2, v_3\rangle$ we designate a first vertex v_1 and number the remaining vertices in counterclockwise order. For each edge $e := \langle u_1, u_2\rangle$, we associate a pair of vertices, where we think of the edge as pointing from u_1 to u_2. If we adopt these conventions, then one simple way to order the domain points and associated B-coefficients of a spline in $\mathcal{S}_d^0(\triangle)$ is to list the points at the vertices first, then all points on the oriented edges, and finally the points in the interior of each triangle in the lexicographical order described in Section 2.3. This is the order we assume whenever we talk about a vector of B-coefficients.

5.2. Storing, Evaluating and Rendering Splines

In view of the discussion in the previous section, to store a spline $s \in \mathcal{S}_d^0(\triangle)$ in a digital computer, it suffices to store its associated coefficient vector c, assuming we have already stored information describing the triangulation $\triangle$. To evaluate s at a given point v, we may apply the following algorithm.

Algorithm 5.2. (Compute $s(v)$)

1) *Find a triangle T which contains v.*
2) *Extract the B-coefficients of $s|_T$ from c.*
3) *Compute the barycentric coordinates $b := (b_1, b_2, b_3)$ of v relative to T.*
4) *Use the de Casteljau algorithm to compute $s(v)$.*

Discussion: Step 1) requires searching through the triangles of $\triangle$. For some suggestions on how to make this process efficient, see Remark 5.2.

Note that if v falls at a vertex, then $s(v)$ is just the value of the coefficient associated with v, and steps 3) and 4) can be skipped. If v falls in the interior of an edge, it will be in two different triangles. However, because of continuity, it doesn't matter which of the two triangles we use to evaluate $s(v)$. In this case we could also find $s(v)$ by evaluating a univariate polynomial using a simplified version of the de Casteljau algorithm, see Remark 2.5. $\square$

We have the analogous algorithm for evaluating an m-th order directional derivative defined by a set of directions $u_1, \ldots, u_m$ with directional coordinates $a_1, \ldots, a_m$.

Algorithm 5.3. (Compute $D_{u_m,\ldots,u_1} s(v)$)

1) *Find a triangle T which contains v.*
2) *Find the B-coefficients of the polynomial $s|_T$.*
3) *Compute the barycentric coordinates $b := (b_1, b_2, b_3)$ of v.*
4) *Compute the coefficients of $D_{u_m,\ldots,u_1} s(v)$ by applying m steps of the de Casteljau algorithm using the triples $a_1, \ldots, a_m$ successively.*
5) *Compute $D_{u_m,\ldots,u_1} s(v)$ by applying $d - m$ steps of the de Casteljau algorithm using the triple b.*

This algorithm is based on the expansion (2.30) of $D_{u_m,\ldots,u_1} s(v)$ in terms of Bernstein basis polynomials of degree $d - m$. The combination of steps 4) and 5) amounts to carrying out a total of d steps of the de Casteljau algorithm, and thus the operation count for computing a derivative is the same as for computing the value of the spline itself, namely, $(d^3+3d^2+2d)/2$.

Since a spline is a piecewise polynomial, the evaluation of integrals and inner products of splines over a triangle or collection of triangles is straightforward using the formulae in Section 2.13. Similarly, to render a spline we simply apply the methods of Section 3.8 to each polynomial piece.

5.3. Control Surfaces and the Shape of Spline Surfaces

Given a spline $s \in \mathcal{S}_d^0(\triangle)$, we define the associated spline surface to be the graph of s as a bivariate function, i.e., the set

$$\mathcal{G}_s := \{(x, y, s(x, y)) : (x, y) \in \Omega\}.$$

We define the control surface $\mathcal{C}_s$ as

$$\mathcal{C}_s := \bigcup_{T \in \triangle} \mathcal{C}_{s|_T},$$

where $\mathcal{C}_{s|_T}$ are the control surfaces associated with the polynomials $s|_T$. Each $\mathcal{C}_{s|_T}$ is a continuous surface made up of triangular facets, and is the

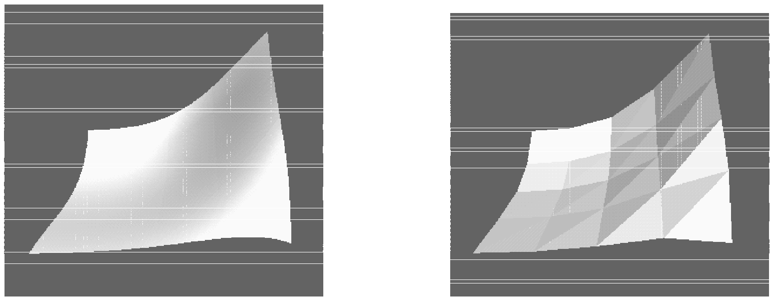


Fig. 5.2. A spline surface and its associated control surface.

graph of a C^0 linear spline over the triangulation $\triangle_{d,T}$ obtained from T by connecting neighboring domain points as described in Section 3.1. We refer to the points $\{(\xi, c_\xi)\}_{\xi \in \mathcal{D}_{d,\triangle}}$ as the control points for s.

It is clear that $\mathcal{C}_s$ is also a continuous surface made up of triangular facets, and in fact we can think of $\mathcal{C}_s$ as being the graph of a spline in $\mathcal{S}_1^0(\triangle_d)$, where $\triangle_d$ is the triangulation obtained from $\triangle$ by splitting each triangle T in $\triangle$ into $d^2 = \binom{d+1}{2} + \binom{d}{2}$ triangles as in (3.3). If we work only with the edges of the triangular facets making up $\mathcal{C}_s$, we get a wireframe object which we define to be the control net associated with s. Figure 5.2 shows a typical spline surface and its associated control surface.

The results of Chapter 3 immediately imply a close connection between the shape of the control surface $\mathcal{C}_s$ of a spline s and the shape of the corresponding spline surface $\mathcal{G}_s$. In particular, by the results of Chapter 3:

1) $\mathcal{G}_s$ lies in the convex hull of the control surface $\mathcal{C}_s$.
2) If the control surface $\mathcal{C}_s$ is nonnegative (positive) on Ω, then so is $\mathcal{G}_s$.
3) If $\mathcal{C}_s$ is monotone in some direction u, then so is $\mathcal{G}_s$.

For B-patches, the convexity of the control surface in some direction implies the convexity of the polynomial in that direction. This property does not extend to surface patches associated with splines in $\mathcal{S}_d^0(\triangle)$, since such splines are only continuous. As in the B-patch case, the shape of a spline $s \in \mathcal{S}_d^0(\triangle)$ can also be specified directly in terms of its coefficients using the results of Chapter 3.

5.4. Dimension and a Local Basis for $\mathcal{S}_d^0(\triangle)$

Since the linear space $\mathcal{S}_d^0(\triangle)$ is in one-to-one correspondence with the set $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$, it is clear that the dimension of $\mathcal{S}_d^0(\triangle)$ is equal to the cardinality of $\mathcal{D}_{d,\triangle}$. A simple count gives us the following theorem.

Theorem 5.4. *For any triangulation* $\triangle$,

$$\dim \mathcal{S}_d^0(\triangle) = \#\mathcal{D}_{d,\triangle} = V + (d-1)E + \binom{d-1}{2} N, \tag{5.4}$$

where V, E, *and* N *are the number of vertices, edges, and triangles in* $\triangle$.

We now construct locally supported basis functions in $\mathcal{S}_d^0(\triangle)$ which are analogs of the classical B-splines. For each $\xi \in \mathcal{D}_{d,\triangle}$, let ψ_ξ be the spline in $\mathcal{S}_d^0(\triangle)$ that satisfies

$$\gamma_\eta \psi_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle},$$

where γ_η is a linear functional that picks off the coefficient associated with the domain point η. It is possible to give an explicit construction of γ_η based on the results of Section 2.16, but here we do not need it in explicit form. Note that ψ_ξ has all zero coefficients except for $c_\xi = 1$.

Since for each triangle the associated Bernstein basis polynomials are nonnegative, it follows immediately that

$$\psi_\xi(v) \geq 0, \qquad \text{all } v \in \Omega.$$

Since ψ_ξ is identically zero on all triangles which do not contain ξ, we see that the support of ψ_ξ is as follows:

1) A single triangle T, if ξ is in the interior of T.
2) $T \cup \widetilde{T}$, if ξ is on the edge between the triangles T and $\widetilde{T}$.
3) The union of all triangles sharing the vertex v, if $\xi = v$.

Theorem 5.5. *The set of splines* $\mathcal{B} := \{\psi_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ *forms a basis for* $\mathcal{S}_d^0(\triangle)$ *which provides a partition of unity on* Ω, *i.e.*,

$$\sum_{\xi \in \mathcal{D}_{d,\triangle}} \psi_\xi(v) \equiv 1, \qquad \textit{all } v \in \Omega.$$

Proof: Since $\dim \mathcal{S}_d^0(\triangle) = \#\mathcal{D}_{d,\triangle}$, to show $\mathcal{B}$ is a basis it suffices to show that the ψ_ξ are linearly independent. Suppose that

$$s := \sum_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi \, \psi_\xi \equiv 0, \qquad \text{on } \Omega.$$

Then for every triangle $T \in \triangle$, the restriction $s|_T$ is a polynomial of degree d which is identically 0, and so all the coefficients in its B-form must vanish. This means that $c_\eta = 0$ for all $\eta \in \mathcal{D}_{d,T}$. Since this holds for all $T \in \triangle$, we have shown that all coefficients must be 0. The partition of unity assertion follows immediately from the corresponding property (2.9) of Bernstein basis polynomials. $\square$

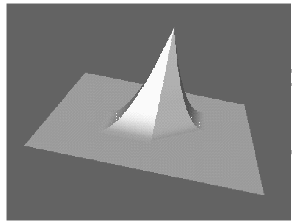


Fig. 5.3. A typical basis function $\psi_\xi \in \mathcal{S}_3^0(\triangle)$.

For linear splines (the case $d = 1$), each basis function is a hat function with support on the star of a vertex v. For $d = 2$, we also get some splines with supports on two neighboring triangles, and for $d \geq 3$, some splines with supports on a single triangle. Figure 5.3 shows a typical basis spline ψ_ξ in the cubic spline space $\mathcal{S}_3^0(\triangle)$ with support on the star of a vertex.

5.5. Spaces of Smooth Splines

In practice we usually need to work with spaces of splines that have some additional smoothness beyond C^0 continuity. In this section we introduce some notation for dealing with spaces of smooth splines.

5.5.1 Supersplines

Given $0 \leq r \leq d$ and a triangulation $\triangle$, we write

$$\mathcal{S}_d^r(\triangle) := \mathcal{S}_d^0(\triangle) \cap C^r(\Omega)$$

for the space of C^r continuous splines of degree d. For many applications it is useful to work with splines that have enhanced smoothness at certain vertices. We say that a spline $s \in \mathcal{S}_d^0(\triangle)$ is C^ρ smooth at v provided that all of the polynomials $s|_T$ such that T is a triangle with vertex at v have common derivatives up to order ρ at the point v. In this case we write $s \in C^\rho(v)$.

Definition 5.6. *Let $\mathcal{V} := \{v_1, \ldots, v_V\}$ be the set of vertices of $\triangle$, and let $0 \leq r \leq \rho \leq d$. Then we define the associated* space of supersplines *to be*

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \ s \in C^\rho(v) \quad \text{for all } v \in \mathcal{V}\}. \tag{5.5}$$

The splines in the space (5.5) have the same supersmoothness ρ at each vertex of $\triangle$. To get a more general space, let $\rho := \{\rho_v\}_{v \in \mathcal{V}}$ with $r \leq \rho_v \leq d$ for each $v \in \mathcal{V}$, and define the associated space of supersplines to be

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \ s \in C^{\rho_v}(v) \quad \text{for all } v \in \mathcal{V}\}. \tag{5.6}$$

In the next section we define even more general superspline spaces with variable smoothness across the edges as well as at the vertices.

5.5.2 Smoothness Conditions Across Edges

In Theorem 2.28 we gave conditions for two polynomials defined on adjoining triangles to join together with C^r smoothness. Those conditions were defined in terms of B-coefficients. For convenience, we now define certain associated linear functionals.

Suppose $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ are two triangles sharing an interior edge $e := \langle v_2, v_3 \rangle$ of $\triangle$. Fix $0 \le n \le j \le d$. Then for any spline $s \in \mathcal{S}_d^0(\triangle)$, let

$$\tau_{j,e}^n s := c_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} \tilde{c}_{\nu,j-n+\mu,d-j+\kappa} \tilde{B}_{\nu\mu\kappa}^n(v_1), \tag{5.7}$$

where $\{c_{ijk}\}_{i+j+k=d}$ and $\{\tilde{c}_{ijk}\}_{i+j+k=d}$ are the B-coefficients of $s|_T$ and $s|_{\widetilde{T}}$, respectively, and $\tilde{B}_{\nu\mu\kappa}^n$ are the Bernstein basis polynomials of degree n associated with the triangle $\widetilde{T}$. We call $\tau_{j,e}^n$ a smoothness functional of order n, and refer to $\xi_{n,d-j,j-n}^T$ as the tip of $\tau_{j,e}^n$.

Definition 5.7. *Given a set $\mathcal{T}$ of linear functionals of the form (5.7) associated with oriented edges of $\triangle$, we define the corresponding* space of smooth splines *as*

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) := \{s \in \mathcal{S}_d^0(\triangle) : \tau s = 0 \text{ for all } \tau \in \mathcal{T}\}. \tag{5.8}$$

In describing spaces of splines via sets $\mathcal{T}$ of smoothness functionals across edges, it is important to keep in mind that since smoothness conditions across different edges of a triangulation may link together, what appear to be independent smoothness conditions may in fact be redundant. This typically happens when $\triangle$ contains an interior vertex v, as shown in the following example.

Example 5.8. Let $\triangle$ consist of three triangles $T_1 := \langle v, v_1, v_2 \rangle$, $T_2 := \langle v, v_2, v_3 \rangle$, and $T_3 := \langle v, v_3, v_1 \rangle$ sharing a common vertex v. Then if $s \in \mathcal{S}_1^0(\triangle)$ satisfies the C^1 smoothness condition across one of the edges $e_i := \langle v, v_i \rangle$, $i = 1, 2, 3$, then it automatically satisfies the C^1 smoothness conditions across the other two.

Discussion: Let c_v be the coefficient of s corresponding to the domain point v, and for each $i = 1, 2, 3$, let c_i be the coefficient of s corresponding to the domain point v_i. Suppose $v_1 = r_1 v + s_1 v_2 + t_1 v_3$. Then the C^1 condition across e_3 can be written as $c_1 = r_1 c_v + s_1 c_2 + t_1 c_3$. But these two equations imply

$$\begin{aligned} v_2 &= \frac{-r_1}{s_1} v - \frac{t_1}{s_1} v_3 + \frac{1}{s_1} v_1, \\ c_2 &= \frac{-r_1}{s_1} c_v - \frac{t_1}{s_1} c_3 + \frac{1}{s_1} c_1, \end{aligned}$$

and

$$v_3 = \frac{-r_1}{t_1} v + \frac{1}{t_1} v_1 - \frac{s_1}{t_1} v_2,$$
$$c_3 = \frac{-r_1}{t_1} c_v + \frac{1}{t_1} c_1 - \frac{s_1}{t_1} c_2,$$

which says that the C^1 smoothness conditions across e_1 and e_2 are also satisfied. □

Example 5.8 can also be explained geometrically. By the geometric interpretation of C^1 smoothness conditions across edges, a spline s will satisfy all three C^1 smoothness conditions if and only if all of the control points of s associated with domain points in the disk $\mathcal{D}_1(v)$ are coplanar. But this will happen as soon as we enforce the C^1 condition across just one edge.

5.5.3 Smoothness at a Vertex

By Theorem 2.28, a spline in $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ will be C^r smooth across e whenever $\mathcal{T}$ includes all of the linear functionals $\{\tau_{j,e}^n\}_{j=n}^d$, for $n = 1, \ldots, r$. The following lemma shows that the functionals defined in (5.7) can also be used to describe smoothness at a vertex. Thus, the superspline spaces introduced in Section 5.5.1 can be written in the form $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ with an appropriate choice of $\mathcal{T}$.

Lemma 5.9. *Let $s \in \mathcal{S}_d^0(\triangle)$. Suppose v is a vertex of $\triangle$, and that $e_1, \ldots, e_m$ are the interior edges of $\triangle$ attached to v. Suppose in addition that for $i = 1, \ldots, m$,*

$$\tau_{j,e_i}^n s = 0, \qquad n \le j \le \rho \ \text{ and } \ 1 \le n \le \rho. \tag{5.9}$$

Then $s \in C^\rho(v)$.

Proof: Given $s \in \mathcal{S}_d^0(\triangle)$, we can regard its B-coefficients associated with domain points lying in the disk $D_\rho(v)$ as the coefficients of a spline g in $\mathcal{S}_\rho^0(\triangle)$. Now $s \in C^\rho(v)$ if and only if g reduces to a single polynomial. But g reduces to a polynomial if and only if (5.9) holds. □

Example 5.8 shows that in general we can force a spline in $\mathcal{S}_d^0(\triangle)$ to belong to $C^\rho(v)$ by enforcing only a part of the smoothness conditions (5.9), since some of them are redundant.

Lemma 5.10. *Suppose $s \in \mathcal{S}_d^0(\triangle) \cap C^\rho(v)$ for some vertex v, and let $T_1, \ldots, T_m$ be the triangles of $\triangle$ sharing the vertex v. Then we can set the coefficients of s corresponding to domain points in $D_\rho(v) \cap T_1$ to arbitrary values, and the coefficients corresponding to all other domain points in $D_\rho(v)$ will be consistently determined by smoothness conditions.*

Proof: As observed in the proof of Lemma 5.9, we can regard the coefficients of s corresponding to domain points in the disk $D_\rho(v)$ as being coefficients of a polynomial g of degree d. By the results of Section 2.2, we can set the coefficients corresponding to domain points in $D_\rho(v) \cap T_1$ to arbitrary values. But then by the results of Section 2.7, this determines the derivatives $\{D_x^\alpha D_y^\beta s(v)\}_{0 \le \alpha+\beta \le \rho}$, which in turn uniquely determines the coefficients of g restricted to each of the other triangles $T_2, \ldots, T_m$. $\square$

5.5.4 The Matrix Form of Smoothness Conditions

Since smoothness conditions on a spline $s \in \mathcal{S}_d^0(\triangle)$ are just linear conditions on the vector c of B-coefficients of s, it is clear that for any given set $\mathcal{T}$ of smoothness conditions, there is a matrix $A := A_{\mathcal{T}}$ such that

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) = \{s \in \mathcal{S}_d^0(\triangle) : \, Ac = 0\}. \tag{5.10}$$

Clearly, the matrix A is of size $m \times n$, where m is the number of smoothness conditions in $\mathcal{T}$, and n is the dimension of $\mathcal{S}_d^0(\triangle)$. It is also clear that A is quite sparse since a typical C^r smoothness condition across an edge involves only $\binom{r+2}{2} + 1$ coefficients. Thus, for example, a C^1 condition involves only four coefficients, so the corresponding row in the matrix A has only four entries that can be nonzero.

Theorem 5.11. *Let $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ be the space of smooth splines defined in (5.10) corresponding to a matrix A. Then the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ is equal to $n - k$, where k is the rank of A.*

Proof: If c is a vector satisfying $Ac = 0$, then the corresponding spline in $\mathcal{S}_d^0(\triangle)$ with B-coefficient vector c clearly belongs to $\mathcal{S}_d^0(\triangle)$. But the number of linear independent solutions of $Ac = 0$ is exactly $n - k$. $\square$

Given a specific triangulation $\triangle$ and smoothness set $\mathcal{T}$, this theorem can be used to compute the dimension of the corresponding spline space numerically. However, it is not of much use in finding general formulae for dimensions of spline spaces because of the difficulty in identifying the rank of the matrix A, due to the redundancies mentioned above.

5.6. Minimal Determining Sets

Theorem 5.1 shows that every spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. Suppose now that $\mathcal{S} := \mathcal{S}_d^{\mathcal{T}}(\triangle)$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$ that is defined by enforcing some set of smoothness conditions $\mathcal{T}$ across the edges of a triangulation $\triangle$ as described in the previous section. Then for a spline $s \in \mathcal{S}_d^0(\triangle)$ to be in $\mathcal{S}$, its set of B-coefficients must satisfy the linear side conditions in (5.10). This means that we cannot assign arbitrary values to every coefficient of s. Instead, we can only assign values to certain coefficients, and all remaining coefficients will be determined by the smoothness conditions. We now pursue this idea in more detail.

Definition 5.12. *Suppose $\Gamma \subseteq \mathcal{D}_{d,\triangle}$ is such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \Gamma$, then $s \equiv 0$. Then we say that Γ is a* determining set for $\mathcal{S}$. *If $\mathcal{M}$ is a determining set for a spline space $\mathcal{S}$ and $\mathcal{M}$ has the smallest cardinality among all possible determining sets for $\mathcal{S}$, then we call $\mathcal{M}$ a* minimal determining set (MDS) *for $\mathcal{S}$.*

Clearly, for any spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, the set of domain points $\mathcal{D}_{d,\triangle}$ is always a determining set for $\mathcal{S}$. But for any spline space $\mathcal{S}$ satisfying at least one smoothness condition, there will be determining sets with fewer points than the number of points in $\mathcal{D}_{d,\triangle}$. In general, there will be more than one minimal determining set corresponding to a given spline space $\mathcal{S}$.

Theorem 5.13. *Suppose $\mathcal{S}$ is an m-dimensional linear subspace of $\mathcal{S}_d^0(\triangle)$, and suppose that Γ is a determining set for $\mathcal{S}$. Then $m \le \#\Gamma$. Moreover, if $\mathcal{M}$ is a determining set for $\mathcal{S}$ with $\#\mathcal{M} = m$, then $\mathcal{M}$ is minimal.*

Proof: For each $\xi \in \mathcal{D}_{d,\triangle}$, let γ_ξ be a linear functional defined on $\mathcal{S}_d^0(\triangle)$ such that $\gamma_\xi s$ is the B-coefficient of s associated with the domain point ξ. Let $B_1, \ldots, B_m$ be a basis for $\mathcal{S}$. Now suppose Γ is a determining set for $\mathcal{S}$ with $\#\Gamma < m$. This implies that there exists at least one nontrivial solution of the homogeneous system

$$\sum_{j=1}^{m} a_j \gamma_\xi B_j = 0, \quad \xi \in \Gamma.$$

But this contradicts the linear independence of the $B_1, \ldots, B_m$, and so $m \le \#\Gamma$ for any determining set. It follows that if $\mathcal{M}$ is a determining set of cardinality m, it must be minimal. $\square$

In general, it is a nontrivial task to construct minimal determining sets for spline spaces. Indeed, in practice we often don't know the dimension of $\mathcal{S}$, and so don't even know how many domain points to put in a minimal determining set.

Definition 5.14. *If $\mathcal{M}$ is a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, we say that it is* consistent *provided that if we fix the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \mathcal{S}$, then all of the coefficients of s are determined, and all smoothness conditions defining $\mathcal{S}$ are satisfied with these coefficients.*

The following theorem provides an important tool for constructing minimal determining sets.

Theorem 5.15. *Suppose $\mathcal{M}$ is a consistent determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then $\mathcal{M}$ is minimal.*

Proof: Since $\mathcal{M}$ is consistent, for each $\xi \in \mathcal{M}$ we can construct a spline $s_\xi \in \mathcal{S}$ with $c_\xi = 1$ and $c_\eta = 0$ for all $\eta \in \mathcal{M}$. The spline s_ξ may have other nonzero coefficients, but all smoothness conditions are satisfied. Now let $\{\gamma_\xi\}_{\xi\in\mathcal{M}}$ be linear functionals such that γ_ξ applied to s gives the coefficient c_ξ. Using these functionals, it is easy to see that the splines $\{s_\xi\}_{\xi\in\mathcal{M}}$ are linearly independent, and thus $m := \dim \mathcal{S} \ge \#\mathcal{M}$. Now since $\mathcal{M}$ is a determining set, Theorem 5.13 implies $m \le \#\mathcal{M}$, and we conclude that $m = \#\mathcal{M}$. But then the second part of Theorem 5.13 gives us that $\mathcal{M}$ is a minimal determining set. $\square$

The proof of Theorem 5.15 shows how to construct a basis for $\mathcal{S}$ whenever we have a consistent determining set $\mathcal{M}$ for $\mathcal{S}$. We explore this observation in more detail in Section 5.8.

Given a minimal determining set $\mathcal{M}$ for $\mathcal{S}$, suppose we assign values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$. Then for every $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$, the coefficient c_η can be computed from the known coefficients by using the smoothness conditions. We say that c_η *depends on* c_ξ, $\xi \in \mathcal{M}$, if changing the value of c_ξ also causes the value of c_η to change. We write

$$\Gamma_\eta := \{\xi \in \mathcal{M} : c_\eta \text{ depends on } c_\xi\}. \tag{5.11}$$

Definition 5.16. *Suppose $\mathcal{M}$ is a minimal determining set for a linear space of splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. We say that $\mathcal{M}$ is* local *provided there exists an integer ℓ not depending on $\triangle$ such that*

$$\Gamma_\eta \subseteq \mathit{star}^\ell(T_\eta), \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}, \tag{5.12}$$

where T_η is a triangle containing η. We say that $\mathcal{M}$ is stable *provided there exists a constant K depending only on ℓ and the smallest angle $\theta_\triangle$ in the triangulation $\triangle$ such that*

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}. \tag{5.13}$$

We shall show in Chapters 6–8 that a wide variety of macro-element spaces (see Section 5.10 for a precise definition) have stable local minimal determining sets. In addition, in Chapter 11 we show that for all $d \ge 3r+2$, the spline space $\mathcal{S}_d^r(\triangle)$ as well as the superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ defined in (5.5) also have stable local minimal determining sets. On the other hand, there are many examples of spline spaces which do not admit stable local minimal determining sets. These include the space $\mathcal{S}_3^1(\triangle)$, even if $\triangle$ is a uniform type-I partition, see Remark 5.3.

5.7. Approximation Power of Spline Spaces

Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space associated with a triangulation $\triangle$ of a domain Ω, and suppose $\mathcal{M}$ is a stable local minimal determining set for

$\mathcal{S}$. In this section we show how to construct an explicit linear operator Q, called a quasi-interpolation operator, mapping $L_1(\Omega)$ into $\mathcal{S}$ which provides optimal order (see Remark 5.4) approximations of smooth functions on Ω.

For each domain point ξ in $\mathcal{D}_{d,\triangle}$, let T_ξ be a triangle containing ξ. In addition, let γ_ξ be a linear functional that for any spline $s \in \mathcal{S}_d^0(\triangle)$, picks off the B-coefficient c_ξ corresponding to ξ. An explicit construction of such γ_ξ can be found in Section 2.16, but here we do not need them in explicit form. Given $f \in L_1(\Omega)$, let

$$c_\xi = \gamma_\xi(F_\xi f), \qquad \xi \in \mathcal{M}, \tag{5.14}$$

where $F_\xi f$ is the averaged Taylor polynomial of degree d associated with f based on the largest disk contained in T_ξ, see (1.11). Then we define Qf to be the spline in $\mathcal{S}$ whose coefficients are c_ξ for $\xi \in \mathcal{M}$, and whose remaining coefficients are determined from smoothness conditions.

For each $f \in L_1(\Omega)$, the spline Qf is well defined since $\mathcal{M}$ is a minimal determining set, and thus all of its coefficients $\{c_\eta\}_{\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}}$ can be computed as linear combinations of those listed in (5.14). In particular, suppose $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$ lies in a triangle $T := T_\eta$. Then since $\mathcal{M}$ is a stable local MDS, this linear combination involves only coefficients associated with domain points in a set $\Gamma_\eta \subseteq \mathcal{M}$ lying in $\Omega_T := \mathrm{star}^\ell(T)$, where ℓ is the integer constant in (5.12). The stability of $\mathcal{M}$ guarantees

$$|c_\eta| \le K_0 \max_{\xi \in \Gamma_\eta} |c_\xi|, \tag{5.15}$$

where K_0 depends only on ℓ and the smallest angle in Ω_T.

Theorem 5.17. *The operator Q is a linear projector mapping $L_1(\Omega)$ onto $\mathcal{S}$. Moreover, there exists a constant K such that for all $1 \le q \le \infty$, all triangles $T \in \triangle$, and all $f \in L_q(\Omega_T)$,*

$$\|Qf\|_{q,T} \le K\, \|f\|_{q,\Omega_T}, \tag{5.16}$$

where K depends only on d, ℓ, and the smallest angle in Ω_T.

Proof: It is clear from the definition that Q is a linear operator. The claim that $Qs = s$ for all splines $s \in \mathcal{S}$ follows from the fact that $F_\xi p = p$ for all $p \in \mathcal{P}_d$. We now establish (5.16) in the case $1 \le q < \infty$. The case $q = \infty$ is similar and simpler. Given $\xi \in \mathcal{M}$, let T_ξ be the triangle containing ξ used to define F_ξ. Then by Theorem 2.7 and Lemma 1.6,

$$|c_\xi| = |\gamma_\xi(F_\xi f)| \le \frac{K_1}{A_{T_\xi}^{1/q}} \|F_\xi f\|_{q,T_\xi} \le \frac{K_1 K_2}{A_{T_\xi}^{1/q}} \|f\|_{q,T_\xi},$$

where K_1 is the constant appearing in (2.19), and K_2 is the constant appearing in (1.15). It depends only on d and the shape parameter of T_ξ, which in turn depends only on the smallest angle in T_ξ by Lemma 4.2. Now fix a triangle T. Then by (5.15), for all $\eta \in T$,

$$|c_\eta| \le \frac{K_0 K_1 K_2}{A_{min}^{1/q}} \|f\|_{q,\Omega_T},$$

where A_{min} is the area of the smallest triangle in Ω_T. Using the nonnegativity of the Bernstein basis polynomials and the fact that they form a partition of unity, this immediately implies

$$\|Qf\|_{q,T} = \left[\int_T \Big| \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^T \Big|^q \right]^{1/q} \le K_0 K_1 K_2 \frac{A_T^{1/q}}{A_{min}^{1/q}} \|f\|_{q,\Omega_T}.$$

Finally applying (4.10), which says that all triangles in Ω_T have comparable areas with a constant depending only on the smallest angle in Ω_T, we get (5.16). $\square$

We can now give a local approximation result for Q.

Theorem 5.18. *Given a triangle T in $\triangle$, let Ω_T be as in Theorem 5.17. Then for every $f \in W_q^{m+1}(\Omega_T)$ with $0 \le m \le d$ and $1 \le q \le \infty$,*

$$\left\| D_x^\alpha D_y^\beta (f - Qf) \right\|_{q,T} \le K \, |T|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega_T}, \tag{5.17}$$

for all $0 \le \alpha + \beta \le m$. If Ω_T is convex, then K depends only on d, ℓ, and the smallest angle in Ω_T. If Ω_T is not convex, then K also depends on the Lipschitz constant of the boundary of Ω_T.

Proof: Fix T and $0 \le \alpha + \beta \le m$. By Theorem 1.9, there exists a polynomial $p \in \mathcal{P}_m$ depending on f so that

$$\left\| D_x^i D_y^j (f - p) \right\|_{q,\Omega_T} \le K_1 \, |\Omega_T|^{m+1-i-j} \, |f|_{m+1,q,\Omega_T}, \tag{5.18}$$

for all $0 \le i + j \le m$. If Ω_T is convex, then K_1 is a constant depending on m and the shape parameter $\kappa_{\Omega_T} := |\Omega_T|/\rho_{\Omega_T}$ of Ω_T, where $|\Omega_T|$ is the diameter of Ω_T and ρ_{Ω_T} is the radius of the largest disk contained in Ω_T. If Ω_T is nonconvex, then K_1 also depends on the Lipschitz constant of the boundary of Ω_T. Now by (4.11) and Lemma 4.14, $|\Omega_T| \le K_2 |T|$, where K_2 is a constant depending on ℓ and the smallest angle in Ω_T. It follows that $|\Omega_T|/\rho_{\Omega_T} \le K_2 |T|/\rho_T$, where ρ_T is the radius of the largest disk contained in T. Thus, the shape parameter κ_{Ω_T} of Ω_T is bounded by K_2 times the shape parameter κ_T of T, which in view of Lemma 4.2 is bounded by a constant depending on the smallest angle in Ω_T. Since Q reproduces polynomials of degree d, we have

$$\|D_x^\alpha D_y^\beta (f - Qf)\|_{q,T} \le \|D_x^\alpha D_y^\beta (f - p)\|_{q,T} + \|D_x^\alpha D_y^\beta Q(f - p)\|_{q,T}.$$

In view of (5.18), it suffices to estimate the second term. For any function g, the restriction of Qg to T is a polynomial of degree d. Thus, using the Markov inequality (1.5) along with (5.16) and (5.18) for $i = j = 0$, it follows that

$$\begin{aligned}
\|D_x^\alpha D_y^\beta Q(f-p)\|_{q,T} &\le \frac{K_3}{\rho_T^{\alpha+\beta}} \|Q(f-p)\|_{q,T} \\
&\le \frac{K_4}{\rho_T^{\alpha+\beta}} \|f-p\|_{q,\Omega_T} \\
&\le K_5 \, \frac{|\Omega_T|^{m+1}}{\rho_T^{\alpha+\beta}} \, |f-p|_{m+1,q,\Omega_T} \\
&\le K_5 \, K_2^{m+1} \, \kappa_T^{\alpha+\beta} \, |T|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega_T}.
\end{aligned}$$

Combining the above inequalities, we get (5.18). □

We can now give a global version of this approximation result. For any triangulation $\triangle$, we define its mesh size $|\triangle|$ to be the length of the longest edge in the triangulation.

Theorem 5.19. *For every $f \in W_q^{m+1}(\Omega)$ with $0 \le m \le d$ and $1 \le q \le \infty$,*

$$\|D_x^\alpha D_y^\beta (f - Qf)\|_{q,\Omega} \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega}, \tag{5.19}$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on d, ℓ, and the smallest angle in the triangulation $\triangle$. If Ω is nonconvex, it also depends on the Lipschitz constant of the boundary of Ω.

Proof: For $q = \infty$, (5.19) follows immediately from (5.17) by taking the maximum over all triangles T in $\triangle$. To get the result for $q < \infty$, we take the q-th power of both sides of (5.17) and sum over all triangles in $\triangle$. Since Ω_T contains other triangles besides T, some triangles will appear more than once in the sum on the right. However, a triangle $\widetilde{T}$ appears in the sum on the right only if it lies in $\mathrm{star}^\ell(T)$ for some triangle $T \in \triangle$. Lemma 4.13 asserts that for each T, the number of triangles in $\mathrm{star}^\ell(T)$ is bounded by a constant K_1 depending only on the smallest angle in $\triangle$, and (5.19) follows. □

In Theorem 5.19 we have made no assumption about the connection between the order of the derivative being taken on the left-hand side of (5.19) and the smoothness of the spline Qf. In particular, for points u on the edges of the triangles of the underlying triangulation $\triangle$, $D_x^\alpha D_y^\beta Qf(u)$ may not even exist for some values of α, β. However, since the q-norm of a function only depends on values of the function almost everywhere, we can ignore such points.

Theorem 5.19 shows that spaces of splines with stable local minimal determining sets provide optimal order approximation of smooth functions, see Remark 5.4. All of the macro-element spaces in Chapters 6–8 have this property. In Section 10.3 we show that for all $d \geq 3r+2$, the space $\mathcal{S}_d^r(\triangle)$ has a stable local minimal determining set and thus has optimal approximation power. We also show in Section 10.4 that when $d < 3r+2$, $\mathcal{S}_d^r(\triangle)$ does not have optimal approximation power, and thus cannot have a stable local minimal determining set.

The problem of determining the approximation power for spline spaces without a stable local MDS is nontrivial. For some results on splines on type-I partitions, see Section 10.4.

5.8. Stable Local Bases

In this section we show how minimal determining sets can be used to construct bases for spline spaces. This construction is mostly for theoretical completeness. In computations with splines, it is almost always more convenient to work directly with the B-representation rather than with any basis. In fact, even the approximation results of the previous section were established without reference to any basis for the spline spaces.

Theorem 5.20. *Suppose $\mathcal{M}$ is a minimal determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then for each $\xi \in \mathcal{M}$, there is a unique spline $\psi_\xi \in \mathcal{S}$ such that*

$$\gamma_\eta \psi_\xi = \delta_{\eta,\xi}, \qquad \textit{all } \eta \in \mathcal{M}. \tag{5.20}$$

*Moreover, $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a basis for $\mathcal{S}$, which we refer to as the $\mathcal{M}$-*basis *of $\mathcal{S}$.*

Proof: By Theorem 5.13, the cardinality of $\mathcal{M}$ is equal to the dimension of $\mathcal{S}$, and so it suffices to show that the splines ψ_ξ are linearly independent. But this follows immediately from (5.20). □

If $\mathcal{M}$ is a *stable local* minimal determining set for $\mathcal{S}$, then the corresponding $\mathcal{M}$-basis has an important stability property.

Theorem 5.21. *Suppose $\mathcal{M}$ is a stable local minimal determining set for the linear space of splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, and let $\Psi := \{\psi_\xi\}_{\xi\in\mathcal{M}}$ be the set of basis functions described in Theorem 5.20. Then Ψ is a* stable local basis *for $\mathcal{S}$ in the sense that for all $\xi \in \mathcal{M}$:*

1) $\|\psi_\xi\|_\Omega \leq K$,
2) $\sigma(\psi_\xi) := \mathit{supp}\, \psi_\xi \subseteq \mathit{star}^\ell(T_\xi)$, *where T_ξ is a triangle containing ξ.*

Here ℓ is the integer constant in (5.12), and the constant K depends only on ℓ and the smallest angle in $\triangle$.

Proof: Let ℓ and K be the constants appearing in Definition 5.16 of the stability of $\mathcal{M}$. Fix $\xi \in \mathcal{M}$. Then all of the B-coefficients of ψ_ξ corresponding to $\eta \in \mathcal{M}$ are zero except for $c_\xi = 1$. All remaining B-coefficients of

ψ_ξ can be computed from those in $\mathcal{M}$. Since $\mathcal{M}$ is local, we see that all computed coefficients c_η with η outside of $\mathrm{star}^\ell(T_\xi)$ must be zero, where T_ξ is a triangle containing ξ. This establishes 2). Now by the stability of $\mathcal{M}$, all computed coefficients satisfy $|c_\eta| \le K$. Finally, using the nonnegativity of the Bernstein basis polynomials and the fact that they form a partition of unity, we get $\|\psi_\xi\|_\Omega \le K$. $\square$

We conclude this section by showing that for any $1 \le q < \infty$, the $\mathcal{M}$-basis for $\mathcal{S}$ can be renormed to form a basis for $\mathcal{S}$ that is stable in the q-norm.

Theorem 5.22. *Let $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ be the $\mathcal{M}$-basis for $\mathcal{S}$ associated with a stable local minimal determining set $\mathcal{M}$. Given $1 \le q \le \infty$, let $\Psi_q := \{\psi_{\xi,q} := (A_{T_\xi})^{-1/q}\psi_\xi\}_{\xi\in\mathcal{M}}$, where for each ξ, T_ξ is a triangle containing ξ and A_{T_ξ} is its area. Then Ψ_q forms a q-stable basis for $\mathcal{S}$ in the sense that there exist constants C_1 and C_2 depending only on d, ℓ, and the smallest angle $\theta_\triangle$ in $\triangle$ such that*

$$C_1\,\|a\|_q \;\le\; \Big\|\sum_{\xi\in\mathcal{M}} a_\xi \psi_{\xi,q}\Big\|_{q,\Omega} \;\le\; C_2\,\|a\|_q, \tag{5.21}$$

for all choices of the coefficient vector $a = (a_\xi)_{\xi\in\mathcal{M}}$.

Proof: We consider the case $1 \le q < \infty$ as the case $q = \infty$ is similar (and simpler). We begin by establishing the first inequality in (5.21). Suppose $s = \sum_{\xi\in\mathcal{M}} a_\xi\psi_{\xi,q}$. Fix a triangle T, and let $\xi \in T\cap\mathcal{M}$. Then the corresponding B-coefficient of $s|_T$ is $a_\xi(A_T)^{-1/q}$, and by Theorem 2.7,

$$\sum_{\xi\in T\cap\mathcal{M}} |a_\xi|^q \le K_5^q\,\|s|_T\|_{q,T}^q,$$

where K_5 is the constant in that theorem. Summing over all T, we get

$$\sum_{\xi\in\mathcal{M}} |a_\xi|^q \le K_5^q\,\|s\|_{q,\Omega}^q.$$

To prove the second inequality, we need more notation. Given a triangle T, let

$$\Sigma_T := \{\xi : T \subseteq \sigma(\psi_\xi)\}, \tag{5.22}$$

where $\sigma(\psi_\xi)$ denotes the support of ψ_ξ. By the localness of the ψ_ξ, $\sigma(\psi_\xi) \subseteq \mathrm{star}^\ell(T_\xi)$. Let $U_T := \bigcup_{\xi\in\Sigma_T}\sigma(\psi_\xi)$. For $\xi\in\Sigma_T$, let T_ξ be a triangle that contains ξ. Then T_ξ and T both lie in the cluster U_T, and Lemma 4.14 implies that $K_1 := \max_{T\in\triangle}\max_{\xi\in\Sigma_T} A_T/A_{T_\xi}$ is finite and depends only on ℓ and $\theta_\triangle$. Now

$$\begin{aligned}\int_T |s|^q = \int_T \Big|\sum_{\xi\in\Sigma_T} a_\xi(A_{T_\xi})^{-1/q}\psi_\xi\Big|^q &\le \max_{\xi\in\Sigma_T}\|\psi_\xi\|_T^q\,\max_{\xi\in\Sigma_T}\frac{A_T}{A_{T_\xi}}\Big(\sum_{\xi\in\Sigma_T}|a_\xi|\Big)^q \\ &\le K_1K_2^q\,K_3^{q-1}\sum_{\xi\in\Sigma_T}|a_\xi|^q,\end{aligned}$$

where K_2 is the constant in Theorem 5.21 and $K_3 := \#\Sigma_T$. We now sum over all triangles T in $\triangle$. A given a_ξ can appear more than once on the right-hand side. In fact, the number of times it appears is equal to the number of triangles in $\sigma(\psi_\xi)$, which by Lemma 4.13 is bounded by a constant K_4 depending only on ℓ and $\theta_\triangle$. It follows that

$$\|s\|_q^q = \sum_{T\in\triangle} \int_T |s|^q \le K_1\, K_2^q\, K_3^{q-1} K_4\, \|a\|_q^q,$$

and the proof of the second inequality in (5.21) is complete. □

5.9. Nodal Minimal Determining Sets

So far, the emphasis in this chapter has been on parametrizing spline spaces $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ using the Bernstein–Bézier form, i.e., in terms of B-coefficients. However, in the finite-element literature it is common to parametrize spline spaces in terms of so-called nodal parameters, also called degrees of freedom. To define them, we consider linear functionals of the form

$$\lambda := \varepsilon_t \sum_{\alpha+\beta=m} a_{\alpha,\beta} D_x^\alpha D_y^\beta, \tag{5.23}$$

where ε_t denotes point evaluation at the point t. We refer to the point t as the carrier of λ.

Definition 5.23. *Suppose $\mathcal{N} = \{\lambda_i\}_{i=1}^n$ is a set of linear functionals of the form (5.23). We call $\mathcal{N}$ a* nodal determining set (NDS) *for $\mathcal{S}$ provided that $\lambda s = 0$ for all $\lambda \in \mathcal{N}$ implies $s \equiv 0$. We call $\mathcal{N}$ a* nodal minimal determining set (NMDS) *for $\mathcal{S}$ if there is no nodal determining set with fewer elements.*

Following the proof of Theorem 5.13, it is easy to see that a nodal determining set $\mathcal{N}$ for $\mathcal{S}$ is minimal if and only if $\#\mathcal{N} = \dim \mathcal{S}$. In this case we can assign arbitrary values to $\{\lambda s\}_{\lambda\in\mathcal{N}}$, and all coefficients of s will be uniquely and consistently determined. We give a simple example to illustrate this concept.

Example 5.24. Given a triangulation $\triangle$, let $\mathcal{V}$ be the set of vertices of $\triangle$, and let $\mathcal{E}$ be the set of edges of $\triangle$. For each edge $e \in \mathcal{E}$, let u_e be the midpoint of e. Then $\mathcal{N} := \{\varepsilon_v\}_{v\in\mathcal{V}} \ \cup\ \{\varepsilon_{u_e}\}_{e\in\mathcal{E}}$, is a nodal minimal determining set for $\mathcal{S}_2^0(\triangle)$.

Discussion: Suppose $s \in \mathcal{S}_2^0(\triangle)$ and that we are given the values $\{s(v)\}_{v\in\mathcal{V}}$ and $\{s(u_e)\}_{e\in\mathcal{E}}$. For each $v \in \mathcal{V}$, the value $s(v)$ uniquely determines the B-coefficient associated with the domain point at v. Moreover, for each edge $e := \langle v_1, v_2\rangle$, the B-coefficient associated with the domain point $\xi = u_e$ is easily seen to be $c_\xi = 4s(u_e) - s(v_1) - s(v_2)$. □

If $\mathcal{N} := \{\lambda_i\}_{i=1}^n$ is a nodal minimal determining set for a spline space $\mathcal{S}$, then for each $1 \le i \le n$, there is a unique spline $\phi_i \in \mathcal{S}$ such that

$$\lambda_j \phi_i = \delta_{ij}, \qquad j = 1, \ldots, n. \tag{5.24}$$

Since the splines $\Phi := \{\phi_i\}_{i=1}^n$ are clearly linearly independent, it follows that Φ is a basis for $\mathcal{S}$. We refer to it as the $\mathcal{N}$-basis for $\mathcal{S}$.

We now show how the basis Φ can be used to give a formal solution to a certain Hermite interpolation problem. Suppose f is sufficiently differentiable so that all of the values $\lambda_i f$ can be computed, and set

$$Hf := \sum_{i=1}^n (\lambda_i f)\, \phi_i. \tag{5.25}$$

Then it follows immediately from (5.24) that $s := Hf$ satisfies

$$\lambda_j s = \lambda_j f, \qquad j = 1, \ldots, n. \tag{5.26}$$

It follows that the Hermite interpolation operator H reproduces all splines in $\mathcal{S}$, i.e.,

$$Hs = s, \qquad \text{all } s \in \mathcal{S}.$$

We emphasize that in practice, we can usually solve the Hermite interpolation problem (5.26) without actually constructing any basis. This will be illustrated in Chapters 6–8.

Definition 5.25. *Suppose $\mathcal{N}$ is a nodal minimal determining set for a linear space of splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, and suppose the order of the highest derivative involved in the linear functionals in $\mathcal{N}$ is $\bar{m}$. We say that $\mathcal{N}$ is* local *provided that there exists an integer ℓ not depending on $\triangle$ such that for every $s \in \mathcal{S}$, $T \in \triangle$, and $\xi \in \mathcal{D}_T$, the coefficient c_ξ of s can be computed from nodal data at points in $\Omega_T := \mathrm{star}^\ell(T)$. We say that $\mathcal{N}$ is* stable *provided that there exists a constant K depending only on ℓ and the smallest angle in $\triangle$ such that for every $s \in \mathcal{S}$, $T \in \triangle$, and $\xi \in \mathcal{D}_{d,T}$,*

$$|c_\xi| \le K \sum_{\nu=0}^{\bar{m}} |T|^\nu |s|_{\nu,\Omega_T}. \tag{5.27}$$

We shall show in Chapters 6–8 that a wide variety of macro-element spaces have stable local nodal minimal determining sets. We now give an error bound for Hermite interpolation with a spline space $\mathcal{S}$ possessing a stable local NMDS.

Theorem 5.26. *Suppose $\mathcal{N}$ is a stable local nodal minimal determining set for a spline space $\mathcal{S}$, and let ℓ and $\bar{m}$ be the constants in Definition 5.25. Let H be the associated Hermite interpolation operator H defined in (5.25).*

Given a triangle T in $\triangle$, let $\Omega_T := \mathrm{star}^\ell(T)$. Then for every $f \in C^{m+1}(\Omega_T)$ with $\bar{m} \le m \le d$,

$$\|D_x^\alpha D_y^\beta (f - Hf)\|_T \le K\,|T|^{m+1-\alpha-\beta}\,|f|_{m+1,\Omega_T}, \tag{5.28}$$

for all $0 \le \alpha+\beta \le m$. If Ω_T is convex, K depends only on d, ℓ, and the smallest angle in the triangles of Ω_T. If Ω_T is not convex, then K also depends on the Lipschitz constant of the boundary of Ω_T.

Proof: Fix T. By Theorem 1.9, there exists a polynomial $p := p_f \in \mathcal{P}_m$ so that

$$\|D_x^i D_y^j (f-p)\|_{\Omega_T} \le K_1\,|\Omega_T|^{m+1-i-j}\,|f|_{m+1,\Omega_T}, \tag{5.29}$$

for all $0 \le i+j \le m$. If Ω_T is convex, then K_1 is a constant depending on m and the shape parameter κ_{Ω_T} of Ω_T. If Ω_T is nonconvex, then K_1 also depends on the Lipschitz constant of the boundary of Ω_T. As shown in the proof of Theorem 5.18, κ_{Ω_T} is bounded by a constant depending only on ℓ and the smallest angle in the triangles of Ω_T. By (4.11) and Lemma 4.14, $|\Omega_T| \le K_2|T|$, where K_2 is also a constant depending on ℓ and the smallest angle in Ω_T.

Now fix $0 \le \alpha+\beta \le m$. Then by the linearity of H, and the fact that it reproduces polynomials of degree d,

$$\|D_x^\alpha D_y^\beta (f-Hf)\|_T \le \|D_x^\alpha D_y^\beta (f-p)\|_T + \|D_x^\alpha D_y^\beta H(f-p)\|_T.$$

In view of (5.29), it suffices to estimate the second term. Since the Bernstein basis polynomials are nonnegative and form a partition of unity, (5.27) implies

$$\|H(f-p)\|_T \le K_3 \sum_{\nu=0}^{\bar{m}} |T|^\nu |f-p|_{\nu,\Omega_T}.$$

Using the Markov inequality (1.5) and (5.29), it follows that

$$\begin{aligned}
\|D_x^\alpha D_y^\beta H(f-p)\|_T &\le \frac{K_4}{\rho_T^{\alpha+\beta}}\,\|H(f-p)\|_T \le \frac{K_3K_4}{\rho_T^{\alpha+\beta}} \sum_{\nu=0}^{\bar{m}} |T|^\nu\,|f-p|_{\nu,\Omega_T} \\
&\le \frac{K_5}{\rho_T^{\alpha+\beta}}\,|\Omega_T|^{m+1}|f|_{m+1,\Omega_T} \\
&\le K_5\,K_2^{m+1}\,\kappa_T^{\alpha+\beta}\,|T|^{m+1-\alpha-\beta}\,|f|_{m+1,\Omega_T}.
\end{aligned}$$

Combining the above inequalities, we get (5.28). $\square$

The global version of (5.28) also holds with T and Ω_T replaced by Ω, as is easily seen by taking the maximum over all triangles T in $\triangle$.

5.10. Macro-element Spaces

In the sequel we shall often work with spaces of splines that are defined on triangulations $\triangle_R$ which are obtained from a given triangulation $\triangle$ by applying some refinement process to each of the triangles T in $\triangle$. For example, we will be especially interested in the case where $\triangle_R$ is obtained by applying Powell–Sabin or Clough–Tocher splits to each of the triangles in $\triangle$.

Let $\mathcal{N}$ be a nodal minimal determining set for a space of splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle_R)$. For each triangle $T \in \triangle$, we define

$$\mathcal{N}_T := \{\lambda \in \mathcal{N} : \text{ the carrier of } \lambda \text{ is contained in } T\}. \tag{5.30}$$

Definition 5.27. *We call $\mathcal{S}$ a* macro-element space *provided that there is a nodal minimal determining set $\mathcal{N}$ for $\mathcal{S}$ such that for all triangles $T \in \triangle$, $s|_T$ is uniquely determined from the values $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$.*

This definition asserts that if $\mathcal{S}$ is a macro-element space with nodal determining set $\mathcal{N}$, then the Hermite-interpolating spline s defined in the previous section can be computed one triangle at a time, and in particular, $s|_T$ can be computed from data at points in T. We give several examples of macro-element spaces in Chapters 6–8.

Constructing macro-element spaces is nontrivial, and depending on the nature of the type of split used, usually requires working with supersplines. The following result shows why.

Theorem 5.28. *Suppose that $T := \langle v_1, v_2, v_3 \rangle$ is a triangle, and that T_R is a refinement such that there are $n \geq 0$ interior edges connected to the vertex v_1. Let s be a spline of degree d and smoothness r defined on T_R. Then the cross derivatives of s up to order r on the edges $e := \langle v_1, v_2 \rangle$ and $\tilde{e} := \langle v_1, v_3 \rangle$ can be specified independently only if we require $s \in C^\rho(v_1)$, with*

$$\rho \geq \left\lceil \frac{(n+2)r - n}{n+1} \right\rceil. \tag{5.31}$$

Proof: Consider the coefficients of s corresponding to domain points on a typical ring $R_{\rho+1}(v_1)$. There are $(n+1)(\rho+1)+1$ such coefficients. As we cross each interior edge attached to v_1 they are subject to r conditions. Now specifying cross derivatives up to order r across the edge e determines the $r+1$ coefficients of s corresponding to domain points on $R_{\rho+1}(v_1)$ within distance r of e. Similarly, specifying cross derivatives up to order r across the edge $\tilde{e}$ determines the $r+1$ coefficients of s corresponding to domain points on $R_{\rho+1}(v_1)$ within distance r of $\tilde{e}$. It follows that a necessary condition for the number of undetermined coefficients on $R_{\rho+1}(v_1)$ to be at least equal to the number of conditions is $(n+1)(\rho+1)+1-2(r+1) \geq nr$, which leads immediately to (5.31). $\square$

5.11. Remarks

Remark 5.1. Throughout this chapter (and the book), we are using the B-form described in Section 5.1 as the key tool for dealing with bivariate spline spaces. As described, this works only for spline spaces that are at least in $C^0(\Omega)$. Of course, it is also possible to parametrize the space $\mathcal{PP}_d(\triangle)$ of all piecewise polynomials of degree d defined on a given triangulation $\triangle$ using B-coefficients. Indeed, we simply associate each function in $\mathcal{PP}(\triangle)$ with the union of the sets of B-coefficients of the individual pieces of s. For each interior edge e of $\triangle$, there will be two different coefficients associated with each domain point in the interior of e. If v is a boundary vertex of $\triangle$ of degree m, then there will be $m-1$ different coefficients associated with the domain point at v, while if v is an interior vertex of degree m, then there will be m different coefficients associated with v.

Remark 5.2. In order to evaluate a spline defined on a triangulation containing N triangles, we first have to find the triangle that contains the point v of interest. One way to test whether v lies in a given triangle T is to compute its barycentric coordinates relative to that triangle and check whether they are all nonnegative. In the worst case this requires $\mathcal{O}(N)$ computations, which can be very expensive if N is large. In practice we often have to evaluate at many points. If these points are numbered so that two successive points are near each other, we can greatly reduce the amount of calculation required to locate the triangle containing v_i if we take account of which triangle contained v_{i-1}. The neighboring triangle list discussed in Section 4.5 can be very helpful for this purpose.

Remark 5.3. It is not hard to give examples of spaces that do not have stable local minimal determining sets. For example, the space $\mathcal{S}_3^1(\triangle)$ does not have one, even if $\triangle$ is a uniform type-I triangulation. Indeed, if it did, then by Theorem 5.19 it would approximate functions in $W_\infty^4(\Omega)$ to order $\mathcal{O}(|\triangle|^4)$. But it is shown in Theorem 10.25 that $\mathcal{S}_3^1(\triangle)$ has approximation power at most $\mathcal{O}(|\triangle|^3)$.

Remark 5.4. Given any linear subspace X of $L_q(\Omega)$, we write $d(f,X)_q := \inf_{g\in X}\|f-g\|_q$ for the usual distance of f to X. Suppose $\mathcal{S}\subseteq\mathcal{S}_d^0(\triangle)$ is a spline space associated with a triangulation $\triangle$ of a domain Ω. Then we say that $\mathcal{S}$ has optimal order approximation power with respect to the q-norm provided that there exists a constant K depending only on the smallest angle in $\triangle$ such that for every sufficiently smooth function f,

$$d(f,\mathcal{S})_q \le K\, d(f,\mathcal{PP}_d(\triangle))_q,$$

where $\mathcal{PP}_d(\triangle)$ is the space of piecewise polynomials on $\triangle$. Another way to say this is that there exists a spline $s\in\mathcal{S}$ such that on each triangle T, the order of approximation of f by s is the same as the order of approximation of f by a polynomial of degree d. As shown in Remarks 1.4 and 1.5, unless $f \in \mathcal{P}_d$, no matter how smooth f may be, $d(f, \mathcal{P}_d)_q = \mathcal{O}(|T|^{d+1+\epsilon})$ does not hold with $\epsilon > 0$.

Remark 5.5. In Theorem 5.5 we have constructed a basis for the space of splines $\mathcal{S}_d^0(\triangle)$. However, in practice there is no need to actually work with these basis functions (which would involve finding and storing the coefficients of each basis function), since as shown in Section 5.2, we can store and manipulate a spline in $\mathcal{S}_d^0(\triangle)$ by working directly with its set of B-coefficients. This same observation holds for smoother spline spaces and spaces of supersplines. Even when we have explicit bases for such spaces, it is generally more efficient to store and work with B-coefficients rather than the coefficients of an expansion in terms of basis functions.

Remark 5.6. Peter Alfeld has written a Java program that is capable of numerically computing the dimension of bivariate spline spaces. It is also useful for finding minimal determining sets $\mathcal{M}$, and even outputs the formulae for computing all coefficients associated with domain points not in $\mathcal{M}$. The program uses residual arithmetic, and is described in some detail in [Alf00]. It can be downloaded from Alfeld's website at the University of Utah.

Remark 5.7. The classical univariate natural cubic spline is a piecewise cubic polynomial that is C^2 globally. It was recognized in 1957 by Holliday that it also has the remarkable property that among all smooth functions that interpolate a given set of values $\{y_i\}_{i=1}^n$ at points $x_1 < x_2 < \cdots < x_n$, the natural spline interpolant minimizes the expression $E := \int_{x_1}^{x_n} [f''(t)]^2\, dt$. This suggests two ways to extend splines to the bivariate setting: 1) look at piecewise polynomials with some smoothness between pieces, or 2) look for functions that minimize an appropriate bivariate analog of E. It turns out that with natural choices for E, bivariate splines defined as solutions of minimization problems are not piecewise polynomials. For a detailed treatment of multivariate splines from this perspective, see the books [Wah90, BezV01, ArcCT04]. The variational approach to splines can also be carried out on the sphere, see [FreeGS98].

Remark 5.8. Suppose $\triangle_0, \triangle_1, \triangle_2, \ldots,$ is a sequence of triangulations such that $\triangle_n$ is a refinement of $\triangle_{n-1}$ for each n. Suppose $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots,$ is a sequence of spline spaces defined on these triangulations. If $\mathcal{S}_{n-1} \subset \mathcal{S}_n$ for each n, then we say that the sequence of spline spaces is nested. Nested sequences of splines play an important role in the numerical solution of partial differential equations, see [Osw88, DahOS94, LaiW96]. They have also been used for surface compression in [HonS04]. The spaces $\mathcal{S}_d^r(\triangle_n)$ are nested for any choice of $0 \le r < d$. However, many other sequences of spline spaces are not nested. For example, the superspline spaces of Definition 5.6 are not nested since a given spline $s \in \mathcal{S}_d^{r,\rho}(\triangle_{n-1})$ will not have the needed supersmoothness to belong to $\mathcal{S}_d^{r,\rho}(\triangle_n)$. Most of the finite-element spaces to be discussed in Chapters 6–8 are superspline spaces, and thus are also not nested, see Remarks 6.4, 7.6, and 8.7. For a construction of nested sequences of special superspline spaces, see [Dav02a].

Remark 5.9. Suppose $\mathcal{S}(\triangle)$ is a spline space defined on a triangulation $\triangle$ of a domain Ω. Let $\triangle_R$ be a refinement of $\triangle$, and suppose $\mathcal{S}(\triangle_R)$ is an associated spline space with $\mathcal{S}(\triangle) \subset \mathcal{S}(\triangle_R)$. Let $\Phi := \{\phi_i\}_{i=1}^n$ be a basis for $\mathcal{S}(\triangle)$, and let $\Psi := \{\psi\}_{i=1}^m$ be a linearly independent set of splines in $\mathcal{S}(\triangle_R)$ such that $W(\triangle_R) :=$ span Ψ is the orthogonal complement of $\mathcal{S}(\triangle)$ in $\mathcal{S}(\triangle_R)$ with respect to the usual L_2 inner product on Ω. We can write this as $\mathcal{S}(\triangle_R) = \mathcal{S}(\triangle) \oplus \mathcal{W}(\triangle_R)$. The space $\mathcal{W}$ is called a wavelet space, and the functions ψ_i are called prewavelets. Then $\Phi \cup \Psi$ is a basis for $\mathcal{S}(\triangle_R)$, and every function in the finer space $\mathcal{S}(\triangle_R)$ can be written as a unique combination of the functions in $\Phi \cup \Psi$. Those in Φ belong to the coarse spline space, while those in Ψ belong to the fine spline space. We can consider such an expansion as a multiresolution expansion, where terms involving the basis functions in Φ provide an initial approximation, while those involving basis functions in Ψ provide detail. There is an extensive theory of univariate spline wavelets. In the bivariate case most of the results deal with box spline wavelets, see Remark 12.11. For arbitrary triangulations, construction of wavelet spaces is much more difficult. Results for the spline space $\mathcal{S}_1^0(\triangle)$ using uniform refinement to get $\triangle_R$ can be found in [KotO95, FloQ98, FloQ99]. In this case there is one prewavelet associated with the midpoint of each edge $e := \langle u, v \rangle$ of $\triangle$. It has support on $\mathrm{star}(u) \cup \mathrm{star}(v)$ with respect to the triangulation $\triangle_R$.

5.12. Historical Notes

Farin was the first to propose using the B-form to study bivariate splines on triangulations in his dissertation [Far79]. It was next used in [BarnF81], where explicit formulae for the B-coefficients of a C^1 quintic spline interpolating certain Hermite data were given, see the macro-element in Section 6.1. Its use seems to have been well established by the time of the survey [BoeFK84]. In [Sabl85a] it was used to construct locally supported splines. The B-form is also the key tool in the work of Alfeld, Piper, and Schumaker [AlfS87, AlfPS87a – AlfP87c] on the dimension of bivariate spline spaces. The concept of minimal determining sets was first introduced in [AlfS87] as a means to compute the dimension of $\mathcal{S}_d^r(\triangle)$ for $d \geq 4r+1$, and was later used in [AlfPS87b] to construct minimally supported bases for the same spaces. Although used previously in the finite-element community, the idea of nodal determining sets was first formalized in [DavS00a] in the study of the approximation power of certain C^1 spline spaces.

The concept of stable local minimal determining sets appears for the first time in [DavS02]. There it was defined to mean that the associated dual basis is stable and local as described in Theorem 5.21. Our Definition 5.16 does not refer to a basis at all. The idea of bypassing bases altogether in discussing stability is due to [AlfS02a, AlfS02b]. This definition of stable MDS has the advantage that we can establish approximation results without having to construct basis functions. The proof that a stable local basis in the max norm can be renormed to create a stable local basis in the q-norm as in Theorem 5.22 comes from our paper [LaiS98]. The construction of stable local minimal determining sets, stable local nodal minimal determining sets, and stable local bases for specific spline spaces will be discussed in detail in Chapters 6– 8, 10.

Superspline spaces first entered the literature in [ChuL90b], where the space (5.5) was studied for the special choice $\rho := r+\lfloor(r+1)/2\rfloor$. The spaces (5.5) with arbitrary $r \le \rho \le d$ were introduced in [Sch89], where their connection to classical finite-element spaces was explored. The superspline spaces in (5.6) with varying smoothness at the vertices were defined and studied in [IbrS91]. The general superspline spaces (5.8) were introduced in [AlfS03], where the smoothness functionals (5.7) were also defined.

In Section 5.7 we give an explicit bound for how well a space of splines with a stable local MDS approximates smooth functions and their derivatives. The key ingredients needed to establish Theorems 5.18 and 5.19 are the stability of the B-form, the partition of unity property of Bernstein basis polynomials, the Markov inequality, and a Whitney approximation theorem for polynomial approximation. These ingredients are also used to prove Theorem 5.26 for Hermite interpolation with spline spaces with a local stable NMDS. This approach to error bounds for spline interpolation and approximation was introduced in [NurRSZ04], and later exploited in [SchS04, SchS05, AlfS05a–AlfS05c]. Earlier, such theorems were proved in the spline literature by constructing quasi-interpolants based on stable local bases, see e.g. [LaiS98]. The approach here does not make use of bases at all. For more on the history of approximation results, see Section 10.6.

The fact that supersmoothness is needed at the vertices of a triangle T in order to construct a macro-element space based on a split of T was recognized in several early papers, but the formula for the minimal smoothness needed given in Theorem 5.28 seems to appear first in [Sch84a].

C^1 Macro-element Spaces

In this chapter we discuss several of the most useful C^1 macro-element spaces. For each of the spaces, we give both a stable local minimal determining set and a stable local nodal determining set, and show that the space has full approximation power.

6.1. A C^1 Polynomial Macro-element Space

Let $\triangle$ be a triangulation of a domain Ω, and let $\mathcal{V}$ be its set of vertices. In this section we discuss the superspline space

$$\mathcal{S}_5^{1,2}(\triangle) := \{s \in \mathcal{S}_5^1(\triangle) : s \in C^2(v) \text{ for all } v \in \mathcal{V}\}.$$

To state our first theorem we need some additional notation. For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v, and let $\mathcal{M}_v := D_2(v) \cap T_v$, where $D_2(v)$ is the set of domain points in the disk of radius 2 around v. For each edge $e := \langle v_2, v_3 \rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3 \rangle$ be some triangle containing e, and let $\mathcal{M}_e := \{\xi_{122}^{T_e}\}$. Let $\mathcal{E}$ be the set of edges of $\triangle$, and suppose that N, E, and V are the number of triangles, edges, and vertices of $\triangle$, respectively.

Theorem 6.1. *dim* $\mathcal{S}_5^{1,2}(\triangle) = 6V + E$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: To show that $\mathcal{M}$ is a minimal determining set, we make use of Theorem 5.15. Thus, we need to show that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_5^{1,2}(\triangle)$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined. First, for each $v \in \mathcal{V}$, we set the coefficients of $s \in \mathcal{S}_5^{1,2}(\triangle)$ corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^2 supersmoothness at v, all coefficients corresponding to domain points in $D_2(v)$ are consistently determined by Lemma 5.10. So far we have determined all of the coefficients $\{c_\xi\}_{\xi \in D}$, where $D := \bigcup_{v \in \mathcal{V}} D_2(v)$. Since the disks $D_2(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

For each edge e of $\triangle$, we now choose $c_{122}^{T_e}$. If $e := \langle v_2, v_3 \rangle$ is an interior edge of $\triangle$ which is shared by two triangles $T_e := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T}_e :=$

$\langle v_4, v_3, v_2 \rangle$, then we can use the C^1 smoothness across e to determine $c_{122}^{\widetilde{T}_e}$. We claim that no inconsistencies can arise in this way. To see this, for each edge $e := \langle u, v \rangle$, let $E_1(e) := \{\eta :\ \mathrm{dist}(\eta, e) \le 1,\ \eta \notin D_1(u) \cup D_1(v)\}$. Then coefficients associated with $E_1(e)$ do not enter any smoothness conditions involving coefficients associated with sets $E_1(\tilde{e})$ for other edges $\tilde{e}$. We have shown that $\mathcal{M}$ is a consistent determining set, and thus by Theorem 5.15 is a minimal determining set. By Theorem 5.13, the dimension of $\mathcal{S}_5^{1,2}(\triangle)$ is equal to the cardinality of $\mathcal{M}$, which is clearly $6V + E$.

We now check that $\mathcal{M}$ is local in the sense of Definition 5.16. Suppose $\eta \notin \mathcal{M}$ lies in T_η. If $\eta \in D_2(v)$ for some vertex v, then c_η depends on the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_v}$. Thus, the set Γ_η in Definition 5.16 is just $\mathcal{M}_v$, which is contained in $\mathrm{star}(v) \subset \mathrm{star}(T_\eta)$. Now suppose $\eta \in E_1(e)$ for some edge $e := \langle u, v \rangle$. Then c_η depends on the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v}$, and thus $\Gamma_\eta \subset \mathrm{star}(T_\eta)$.

We claim that $\mathcal{M}$ is also stable as defined in Definition 5.16. Indeed, all coefficients corresponding to $\eta \notin \mathcal{M}$ can be computed directly from smoothness conditions, which by Lemma 2.29 is a stable process. $\square$

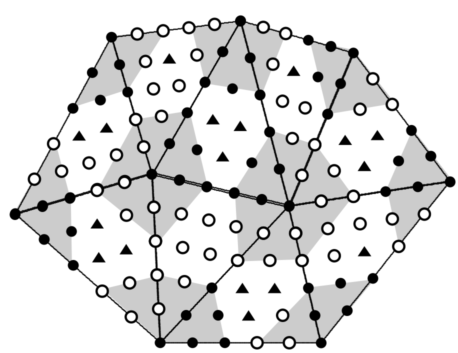


Fig. 6.1. A minimal determining set for $\mathcal{S}_5^{1,2}(\triangle)$.

The freedom in choosing the triangles T_v and T_e in the definition of $\mathcal{M}$ implies that there are many different minimal determining sets similar to the one described in the theorem. Figure 6.1 shows one example of a minimal determining set of this type for a triangulation with $V = 9$ and $E = 17$. By Theorem 6.1, $\dim \mathcal{S}_5^{1,2}(\triangle) = 6 \times 9 + 17 = 71$. In the figure we have marked the domain points in the sets $\mathcal{M}_v$ with black dots, and those in the sets $\mathcal{M}_e$ with triangles.

Since it has a stable local MDS, we can now apply Theorem 5.19 to show that $\mathcal{S}_5^{1,2}(\triangle)$ has full approximation power.

Theorem 6.2. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 5$, there exists a spline $s_f \in \mathcal{S}_5^{1,2}(\triangle)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set $\mathcal{N}$ for $\mathcal{S}_5^{1,2}(\triangle)$, and then use it to construct a Hermite interpolating spline. To describe $\mathcal{N}$, we need some additional notation. We assign an orientation to each edge $e := \langle u, v \rangle$ of $\triangle$, and let u_e be the unit vector corresponding to rotating e ninety degrees in a counterclockwise direction. We write $\eta_e := (u+v)/2$ for the midpoint of e, and D_{u_e} for the directional derivative associated with u_e. Finally, let ε_t be the point evaluation functional defined by $\varepsilon_t f = f(t)$.

Theorem 6.3. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_5^{1,2}(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 2}$,
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}\}$.

Proof: The cardinality of $\mathcal{N}$ is $6V + E$. Since we already know from Theorem 6.1 that the dimension of $\mathcal{S}_5^{1,2}(\triangle)$ is $6V + E$, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show that if $s \in \mathcal{S}_5^{1,2}(\triangle)$, then all of its coefficients are determined by the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$. First, for every vertex v of $\triangle$, we can compute the coefficients $\{c_\xi\}_{\xi \in D_2(v)}$ directly from the derivative information at v. For a typical triangle $T := \langle v_1, v_2, v_3 \rangle$,

$$\begin{aligned}
c_{500}^T &= s(v_1), \\
c_{410}^T &= [h_2 s_x(v_1) + \tilde{h}_2 s_y(v_1)]/5 + s(v_1), \\
c_{401}^T &= [h_3 s_x(v_1) + \tilde{h}_3 s_y(v_1)]/5 + s(v_1), \\
c_{320}^T &= [h_2^2 s_{xx}(v_1) + 2h_2\tilde{h}_2 s_{xy}(v_1) + \tilde{h}_2^2 s_{yy}(v_1)]/20 + 2c_{410}^T - s(v_1), \\
c_{311}^T &= [h_2 h_3 s_{xx}(v_1) + (h_2\tilde{h}_3 + h_3\tilde{h}_2) s_{xy}(v_1) + \tilde{h}_2\tilde{h}_3 s_{yy}(v_1)]/20 \\
&\qquad + c_{401}^T + c_{410}^T - s(v_1), \\
c_{302}^T &= [h_3^2 s_{xx}(v_1) + 2h_3\tilde{h}_3 s_{xy}(v_1) + \tilde{h}_3^2 s_{yy}(v_1)]/20 + 2c_{401}^T - s(v_1),
\end{aligned}$$

where $h_i := x_i - x_1$ and $\tilde{h}_i := y_i - y_1$ for $i = 2, 3$. Similar formulae hold at v_2 and v_3. We now use Lemma 2.21 to derive a formula for c_{122}^T.

For $e = \langle v_2, v_3 \rangle$, suppose (a_1, a_2, a_3) are the directional coordinates of u_e relative to T. Then

$$\begin{aligned} c_{122}^T = \frac{16}{30a_1} D_{u_e} s(\eta_e) - \frac{1}{6}[c_{140}^T + 4c_{131}^T + 4c_{113}^T + c_{104}^T] \\ - \frac{a_2}{6a_1}\left[c_{050}^T + 4c_{041}^T + 6c_{032}^T + 4c_{023}^T + c_{014}^T\right] \\ - \frac{a_3}{6a_1}\left[c_{041}^T + 4c_{032}^T + 6c_{023}^T + 4c_{014}^T + c_{005}^T\right], \end{aligned}$$

where $\eta_e = (v_2 + v_3)/2$. Analogous formulae hold for c_{212}^T and c_{221}^T. Since we have now shown how to compute every coefficient of s from the nodal data, the proof that $\mathcal{N}$ is a NMDS is complete. The above formulae show that for every T in $\triangle$ and every $\xi \in \mathcal{D}_{5,T}$,

$$|c_\xi^T| \le K_1 \sum_{\nu=0}^{2} |T|^\nu |f|_{\nu,T},$$

where K_1 is a constant depending only on the smallest angle in $\triangle$. This establishes that $\mathcal{N}$ is local and stable. □

The proof of Theorem 6.3 shows that for each triangle T in $\triangle$, the coefficients of $s|_T$ can be computed from values of s and its derivatives at points in T. Thus, $\mathcal{S}_5^{1,2}(\triangle)$ is a macro-element space as defined in Definition 5.27. Theorem 6.3 also shows that for every function $f \in C^2(\Omega)$, there is a unique spline $s \in \mathcal{S}_5^{1,2}(\triangle)$ solving the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha + \beta \le 2, \\ D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E}. \end{aligned}$$

This defines a linear projector $\mathcal{I}_P^1$ mapping $C^2(\Omega)$ onto the superspline space $\mathcal{S}_5^{1,2}(\triangle)$, and in particular, $\mathcal{I}_P^1$ reproduces polynomials of degree five. We can now apply Theorem 5.26 to get an error bound for this interpolation operator.

Theorem 6.4. *For every $f \in C^{m+1}(\Omega)$ with $1 \le m \le 5$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_P^1 f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Proof: Theorem 5.26 implies

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_P^1 f)\|_T \le K_1\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,T}.$$

Taking the maximum over all triangles in $\triangle$ immediately implies the global result. $\square$

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_5^{1,2}(\triangle)$ described in Theorem 6.1. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_5^{1,2}(\triangle)$. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles containing e.

The $\mathcal{N}$-basis defined in (5.24) associated with the nodal MDS of Theorem 6.3 provides a different stable local basis for $\mathcal{S}_5^{1,2}(\triangle)$.

6.2. A C^1 Clough–Tocher Macro-element Space

Given a triangulation $\triangle$ of a domain Ω, let $\triangle_{CT}$ be the corresponding Clough–Tocher refinement of $\triangle$ where for each triangle T, the split point v_T is chosen to be the barycenter of T, see Definition 4.16. We refer to the triangles of $\triangle$ as macro-triangles, and to the triangles of $\triangle_{CT}$ as micro-triangles. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$. Let V and E be the number of vertices and edges of $\triangle$, respectively.

For each $v \in \mathcal{V}$, let T_v be some triangle of $\triangle_{CT}$ with vertex at v, and let $\mathcal{M}_v := D_1(v) \cap T_v$. For each edge e of $\triangle$, let T_e be some triangle of $\triangle_{CT}$ containing that edge, and let $\mathcal{M}_e := \{\xi_{111}^{T_e}\}$.

Theorem 6.5. *dim* $\mathcal{S}_3^1(\triangle_{CT}) = 3V + E$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: We use Theorem 5.15 to show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$. We need to show that the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_3^1(\triangle_{CT})$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined. First, for each $v \in \mathcal{V}$, we set the coefficients of s corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^1 smoothness at v, all coefficients corresponding to domain points in $D_1(v)$ are consistently determined by Lemma 5.10. So far we have determined all of the coefficients $\{c_\xi\}_{\xi\in D}$, where $D := \bigcup_{v\in\mathcal{V}} D_1(v)$. Since the disks $D_1(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

For each edge e of $\triangle$, we now choose $c_{111}^{T_e}$. If $e := \langle v_2, v_3\rangle$ is an interior edge of $\triangle$ which is shared by two triangles $T_e := \langle v_1, v_2, v_3\rangle$ and

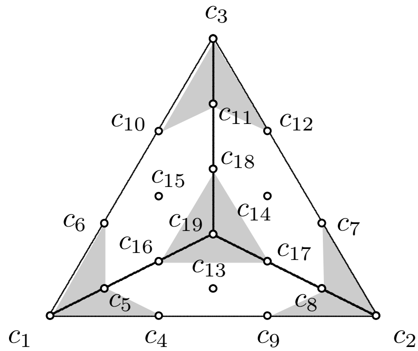


Fig. 6.2. B-coefficients of $s|_T$.

$\widetilde{T}_e := \langle v_4, v_3, v_2 \rangle$, then we can use the C^1 smoothness across e to determine the corresponding coefficient $c_{111}^{\widetilde{T}_e}$.

Given a macro-triangle $T := \langle v_1, v_2, v_3 \rangle$ in $\triangle$, suppose the coefficients of $s|_T$ are numbered as in Figure 6.2. Suppose we already know the coefficients $c_1, \ldots, c_{15}$. Then using the C^1 smoothness across the edges $e_i := \langle v_T, v_i \rangle$, we get

$$
\begin{aligned}
c_{16} &= (c_{15} + c_5 + c_{13})/3, \\
c_{17} &= (c_{13} + c_8 + c_{14})/3, \\
c_{18} &= (c_{14} + c_{11} + c_{15})/3, \\
c_{19} &= (c_{18} + c_{16} + c_{17})/3.
\end{aligned}
\tag{6.1}
$$

It is easy to check that the remaining C^1 smoothness conditions across the interior edges e_i of the Clough–Tocher split of T are satisfied.

We now see that no inconsistencies can arise in setting the coefficients in the sets $\mathcal{M}_e$, even though coefficients in two different such sets may be connected by smoothness conditions across the interior edges of the splits. We have shown that if we fix the coefficients of a spline s corresponding to domain points in the set $\mathcal{M}$, then all of the coefficients of s are consistently determined. It follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set, and by Theorem 5.13, the dimension of $\mathcal{S}_3^1(\triangle_{CT})$ is equal to the cardinality of $\mathcal{M}$, which is clearly $3V + E$.

We now claim that the MDS $\mathcal{M}$ is local. Suppose $\eta \notin \mathcal{M}$ lies in T_η. If $\eta \in D_1(v)$ for some vertex v, then the corresponding set Γ_η of Definition 5.16 is just $\mathcal{M}_v$ which is contained in $\operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. For each edge $e := \langle u, v \rangle$ of $\triangle$, let $E_1(e) := \{\eta : \operatorname{dist}(\eta, e) \le 1,\, \eta \notin D_1(u) \cup D_1(v)\}$. Now if $\eta \in E_1(e)$, then as in the proof of Theorem 6.1, $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \operatorname{star}(T_\eta)$. Finally, if η lies inside a macro-triangle

$T_\eta := \langle v_1, v_2, v_3 \rangle$ with edges e_1, e_2, e_3, but not in any of the above sets, then $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \cup \mathcal{M}_{e_1} \cup \mathcal{M}_{e_2} \cup \mathcal{M}_{e_3} \subset \operatorname{star}(T_\eta)$.

Since all coefficients associated with domain points $\eta \notin \mathcal{M}$ are computed directly from smoothness conditions, Lemma 2.29 ensures that these computations are stable as defined in Definition 5.16, with a constant depending only on the smallest angle in $\triangle_{CT}$. $\square$

The constant in the stability of $\mathcal{M}$ in Theorem 6.5 depends on the smallest angle in the triangulation $\triangle_{CT}$. By Lemma 4.17 this angle is bounded below by a constant times the smallest angle in $\triangle$.

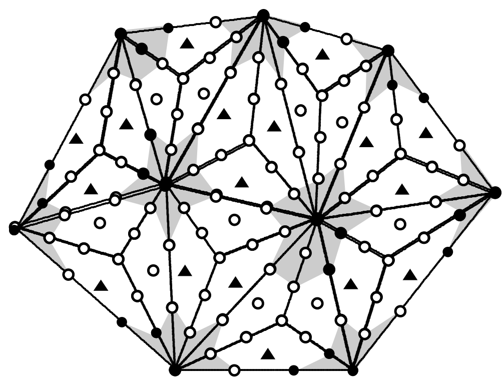


Fig. 6.3. A minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$.

Figure 6.3 shows a minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$ of the type described in Theorem 6.5 for a triangulation with $V = 9$ and $E = 17$. By Theorem 6.5, $\dim \mathcal{S}_3^1(\triangle_{CT}) = 3 \times 9 + 17 = 44$. In the figure, the points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles.

Since $\mathcal{S}_3^1(\triangle_{CT})$ has a stable local minimal determining set, we can now apply Theorem 5.19 to conclude that it has full approximation power.

Theorem 6.6. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 3$, there exists a spline $s_f \in \mathcal{S}_3^1(\triangle_{CT})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$. As in Theorem 6.3, for each edge $e := \langle u, v \rangle$ of $\triangle$, let $\eta_e := (u+v)/2$ be the midpoint of e, and let D_{u_e} be the directional derivative associated with the unit vector u_e corresponding to rotating e ninety degrees in a counterclockwise direction. Let ε_t denote point evaluation at t.

Theorem 6.7. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\nu D_y^\mu)\}_{0 \le \nu+\mu \le 1}$,
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}\}$.

Proof: The cardinality of $\mathcal{N}$ is $3V + E$, which by Theorem 6.5 is the dimension of $\mathcal{S}_3^1(\triangle_{CT})$. Thus, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show if $s \in \mathcal{S}_3^1(\triangle_{CT})$, then the data $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determine all coefficients of s. Let T be a triangle of $\triangle$, and let (x_c, y_c) be its barycenter. Referring to Figure 6.2 and using the formulae of Section 2.7, we have

$$\begin{aligned}
c_1 &= s(v_1), \\
c_2 &= s(v_2), \\
c_3 &= s(v_3), \\
c_4 &= [(x_2 - x_1)s_x(v_1) + (y_2 - y_1)s_y(v_1)]/3 + s(v_1), \\
c_5 &= [(x_c - x_1)s_x(v_1) + (y_c - y_1)s_y(v_1)]/3 + s(v_1), \\
c_6 &= [(x_3 - x_1)s_x(v_1) + (y_3 - y_1)s_y(v_1)]/3 + s(v_1), \\
c_7 &= [(x_3 - x_2)s_x(v_2) + (y_3 - y_2)s_y(v_2)]/3 + s(v_2), \\
c_8 &= [(x_c - x_2)s_x(v_2) + (y_c - y_2)s_y(v_2)]/3 + s(v_2), \\
c_9 &= [(x_1 - x_2)s_x(v_2) + (y_1 - y_2)s_y(v_2)]/3 + s(v_2), \\
c_{10} &= [(x_1 - x_3)s_x(v_3) + (y_1 - y_3)s_y(v_3)]/3 + s(v_3), \\
c_{11} &= [(x_c - x_3)s_x(v_3) + (y_c - y_3)s_y(v_3)]/3 + s(v_3), \\
c_{12} &= [(x_2 - x_3)s_x(v_3) + (y_2 - y_3)s_y(v_3)]/3 + s(v_3).
\end{aligned} \tag{6.2}$$

We now use Lemma 2.21 to find a formula for c_{13}. For $e := \langle v_1, v_2 \rangle$, suppose (a_1, a_2, a_3) are the directional coordinates of u_e. Then

$$c_{13} = \frac{4}{6a_3} D_{u_e} s(\eta_e) - \frac{1}{2}(c_5 + c_8) - \frac{a_1}{2a_3}(c_1 + 2c_4 + c_9) - \frac{a_2}{2a_3}(c_4 + 2c_9 + c_2).$$

Similar formulae hold for c_{14} and c_{15}. Finally, $c_{16}, \ldots, c_{19}$ can be computed from (6.1). Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 1$. $\square$

For each triangle T in $\triangle$, $s|_T$ is determined by the data involving evaluation at points in T. Thus, the coefficients of s can be computed locally, one triangle at a time, and so $\mathcal{S}_3^1(\triangle_{CT})$ is a macro-element space.

Theorem 6.7 shows that for every function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}_3^1(\triangle_{CT})$ solving the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 1, \\ D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E}. \end{aligned}$$

This defines a linear projector $\mathcal{I}_{CT}^1$ mapping $C^1(\Omega)$ onto the spline space $\mathcal{S}_3^1(\triangle_{CT})$, and in particular, $\mathcal{I}_{CT}^1$ reproduces polynomials of degree three. We can now apply Theorem 5.26 to get an error bound for this interpolation operator.

Theorem 6.8. *For every $f \in C^{m+1}(\Omega)$ with $0 \le m \le 3$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_{CT}^1 f)\|_\Omega \le K\,|\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega},$$

for all $0 \le \alpha+\beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_3^1(\triangle_{CT})$ described in Theorem 6.5. By Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_3^1(\triangle_{CT})$. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles containing e.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 6.7 provides a different stable local basis for $\mathcal{S}_3^1(\triangle_{CT})$.

6.3. A C^1 Powell–Sabin Macro-element Space

Given a triangulation $\triangle$ of the domain Ω, let $\triangle_{PS}$ be the corresponding Powell–Sabin refinement as described in Definition 4.18 based on the incenters of the triangles of $\triangle$. Let $\mathcal{V}$ be the set of vertices of $\triangle$, and let V be its cardinality. For each $v \in \mathcal{V}$, let T_v be some triangle of $\triangle_{PS}$ with vertex at v, and let $\mathcal{M}_v := D_1(v) \cap T_v$.

Theorem 6.9. *dim $\mathcal{S}_2^1(\triangle_{PS}) = 3V$, and the set $\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v$ is a stable local minimal determining set.*

Proof: To show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$ we make use of Theorem 5.15. We need to show that the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_2^1(\triangle_{PS})$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined. First, for each $v \in \mathcal{V}$, we fix the coefficients of s corresponding to the domain points in $\mathcal{M}_v$. Then

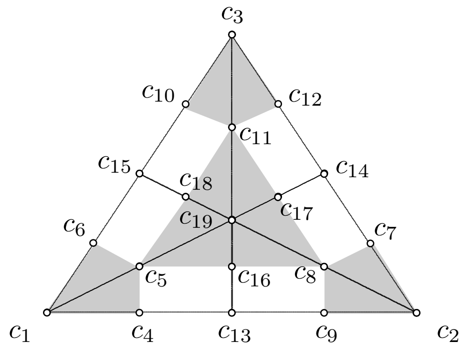


Fig. 6.4. Coefficients of $s|_T$.

in view of the C^1 smoothness at v, all coefficients corresponding to domain points in $D_1(v)$ are consistently determined by Lemma 5.10. So far we have determined all of the coefficients $\{c_\xi\}_{\xi\in D}$, where $D := \bigcup_{v\in\mathcal{V}} D_1(v)$. Since the disks $D_1(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

Now consider a macro-triangle T, and suppose the coefficients of $s|_T$ are numbered as $c_1,\ldots,c_{19}$ as in Figure 6.4. Suppose we know the values of $c_1,\ldots,c_{12}$. Let v_T be the incenter of of T, and let w_1,w_2,w_3 be the vertices located on the edges of T. More specifically, suppose $w_i := r_i v_i + s_i v_{i+1}$, for $i=1,2,3$, where we identify $v_4 \equiv v_1$. Then using C^1 smoothness conditions across the edges $e_i := \langle v_T, w_i\rangle$, we get

$$\begin{aligned} c_{13} &= r_1 c_4 + s_1 c_9, \\ c_{14} &= r_2 c_7 + s_2 c_{12}, \\ c_{15} &= r_3 c_{10} + s_3 c_6, \\ c_{16} &= r_1 c_5 + s_1 c_8, \\ c_{17} &= r_2 c_8 + s_2 c_{11}, \\ c_{18} &= r_3 c_{11} + s_3 c_5. \end{aligned} \tag{6.3}$$

The C^1 smoothness at v_T implies that the control points of s associated with domain points in the disk $D_1(v_T)$ must lie on a plane, and thus

$$c_{19} = a_1 c_5 + a_2 c_8 + a_3 c_{11}, \tag{6.4}$$

where (a_1,a_2,a_3) are the barycentric coordinates of the point v_T. We have now determined all coefficients of s in such a way that there are no inconsistencies in smoothness conditions across interior edges of the macro-triangles.

However, it remains to check for consistency across the boundary edges of the macro-triangles. Suppose $T := \langle v_1, v_2, v_3\rangle$ is a macro-triangle, and

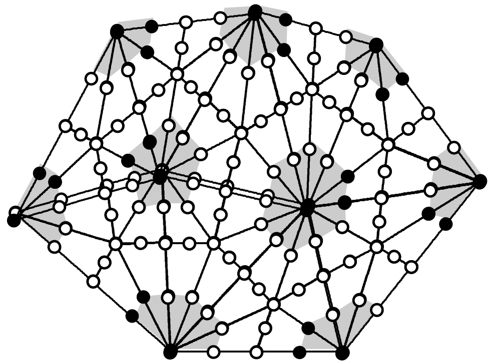


Fig. 6.5. A minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$.

that $\widetilde{T}$ is a triangle which shares the edge $e := \langle v_1, v_2 \rangle$ with T. Recall that the Powell–Sabin refinement is constructed so that the the point w_1 lies on the line from v_T to $v_{\tilde{T}}$. Let $\tilde{c}_5, \tilde{c}_{16}, \tilde{c}_8$ be the coefficients of $s|_{\widetilde{T}}$ located across the edge e from c_5, c_{16}, c_8. Then by construction, the C^1 conditions connecting $c_5, c_4, \tilde{c}_5$ and $c_8, c_9, \tilde{c}_8$ are automatically satisfied since these coefficients lie in the disks $D_1(v_1)$ and $D_1(v_2)$, respectively. Let (α, β, γ) be the barycentric coordinates of $v_{\tilde{T}}$ relative to $\langle v_T, v_1, w_1 \rangle$. Thus, $v_{\tilde{T}} = \alpha v_T + \gamma w_1$. Now we know that $\tilde{c}_5 = \alpha c_5 + \gamma c_4$ and $\tilde{c}_8 = \alpha c_8 + \gamma c_9$. But then

$$\tilde{c}_{16} = r_1 \tilde{c}_5 + s_1 \tilde{c}_8 = r_1(\alpha c_5 + \gamma c_4) + s_1(\alpha c_8 + \gamma c_9) = \alpha c_{16} + \gamma c_{13},$$

which shows that the middle C^1 condition across e is also satisfied. It now follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$, and by Theorem 5.13, the dimension of $\mathcal{S}_2^1(\triangle_{PS})$ is equal to the cardinality of $\mathcal{M}$, which is clearly $3V$.

We now check that $\mathcal{M}$ is local. Suppose $\eta \notin \mathcal{M}$ lies in T_η. If $\eta \in D_1(v)$ for some vertex of $\triangle$, then clearly the set Γ_η of Definition 5.16 is $\Gamma_\eta = \mathcal{M}_v \subset \operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. If η lies in a macro-triangle $T_\eta := \langle v_1, v_2, v_3 \rangle$ but not in any of the disks $D_1(v_i)$, then $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \subset \operatorname{star}(T_\eta)$. The computation of each coefficient associated with a domain point not in $\mathcal{M}$ is stable since it is based on smoothness conditions, as can be seen from the above formulae. $\square$

The constant in the stability of the MDS in Theorem 6.9 depends on the smallest angle in the triangulation $\triangle_{PS}$. By Lemma 4.20, this angle is bounded below by a constant times the smallest angle in $\triangle$ itself.

Figure 6.5 shows an example of a minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$ of the type described in Theorem 6.9 for a triangulation with $V = 9$ and $E = 17$. By Theorem 6.9, dim $\mathcal{S}_2^1(\triangle_{PS}) = 3 \times 9 = 27$. The points in $\mathcal{M}$ are marked with black dots in the figure.

Since $\mathcal{S}_2^1(\triangle_{PS})$ has a stable local minimal determining set, we can now apply Theorem 5.19 to conclude that it has full approximation power.

Theorem 6.10. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 2$, there exists a spline $s_f \in \mathcal{S}_2^1(\triangle_{PS})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$ and use it to solve a Hermite interpolation problem. We write ε_t for point evaluation at t.

Theorem 6.11. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 1}$$

is a stable local nodal minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$.

Proof: It is obvious that the cardinality of $\mathcal{N}$ is $3V$, and thus it suffices to show if $s \in \mathcal{S}_2^1(\triangle_{PS})$, then the data $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determines all coefficients of s. Let T_{PS} be the Powell–Sabin split of a macro triangle $T := \langle v_1, v_2, v_3\rangle$ in $\triangle$, and let (x_c, y_c) be the incenter of T. Then referring to Figure 6.4 and using the formulae of Section 2.7, we have

$$\begin{aligned}
c_1 &= s(v_1),\\
c_2 &= s(v_2),\\
c_3 &= s(v_3),\\
c_4 &= [(\hat{x}_1 - x_1)s_x(v_1) + (\hat{y}_1 - y_1)s_y(v_1)]/2 + s(v_1),\\
c_5 &= [(x_c - x_1)s_x(v_1) + (y_c - y_1)s_y(v_1)]/2 + s(v_1),\\
c_6 &= [(\hat{x}_3 - x_1)s_x(v_1) + (\hat{y}_3 - y_1)s_y(v_1)]/2 + s(v_1),\\
c_7 &= [(\hat{x}_2 - x_2)s_x(v_2) + (\hat{y}_2 - y_2)s_y(v_2)]/2 + s(v_2),\\
c_8 &= [(x_c - x_2)s_x(v_2) + (y_c - y_2)s_y(v_2)]/2 + s(v_2),\\
c_9 &= [(\hat{x}_1 - x_2)s_x(v_2) + (\hat{y}_1 - y_2)s_y(v_2)]/2 + s(v_2),\\
c_{10} &= [(\hat{x}_3 - x_3)s_x(v_3) + (\hat{y}_3 - y_3)s_y(v_3)]/2 + s(v_3),\\
c_{11} &= [(x_c - x_3)s_x(v_3) + (y_c - y_3)s_y(v_3)]/2 + s(v_3),\\
c_{12} &= [(\hat{x}_2 - x_3)s_x(v_3) + (\hat{y}_2 - y_3)s_y(v_3)]/2 + s(v_3).
\end{aligned} \tag{6.5}$$

where $w_i := (\hat{x}_i, \hat{y}_i)$ are the points on the edges of T as in the proof of Theorem 6.9. The coefficients $c_{13}, \ldots, c_{19}$ can now be computed from (6.3) and (6.4). Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 1$. $\square$

For each triangle T in $\triangle$, the set of data involving evaluation at points in T determines $s|_T$, i.e., the coefficients of s can be computed locally, one triangle at a time. This shows that $\mathcal{S}_2^1(\triangle_{PS})$ is a macro-element space.

Theorem 6.11 shows that for every function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}_2^1(\triangle_{PS})$ solving the Hermite interpolation problem

$$D_x^\alpha D_y^\beta s(v) = D_x^\alpha D_y^\beta f(v), \qquad \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha + \beta \le 1.$$

This defines a linear projector $\mathcal{I}_{PS}^1$ mapping $C^1(\Omega)$ onto the spline space $\mathcal{S}_2^1(\triangle_{PS})$, and in particular $\mathcal{I}_{PS}^1$ reproduces polynomials of degree two. We can now apply Theorem 5.26 to get an error bound for this interpolation operator.

Theorem 6.12. *For every $f \in C^{m+1}(\Omega)$ with $0 \le m \le 2$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_{PS}^1 f)\|_\Omega \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2^1(\triangle_{PS})$ described in Theorem 6.9. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_2^1(\triangle_{PS})$. In particular, if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\mathrm{star}(v)$.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 6.11 provides a different stable local basis for $\mathcal{S}_2^1(\triangle_{PS})$.

6.4. A C^1 Powell–Sabin-12 Macro-element Space

Given a triangulation $\triangle$ of a domain Ω, let $\triangle_{PS12}$ be the corresponding Powell–Sabin-12 refinement as described in Definition 4.21 based on the barycenters v_T of the triangles of $\triangle$. Let $\mathcal{V}$ be the set of vertices of $\triangle$, and let V be its cardinality. Similarly, let $\mathcal{E}$ be the set of edges of $\triangle$, and let E be its cardinality.

For each $v \in \mathcal{V}$, let T_v be some triangle of $\triangle_{PS12}$ with vertex at v, and let $\mathcal{M}_v := D_1(v) \cap T_v$. For each edge e of $\triangle$, let w_e be the midpoint of e, and let $\xi_e := (v_T + w_e)/2$, where v_T is the barycenter of some triangle $T \in \triangle$ containing e. Let $\mathcal{M}_e := \{\xi_e\}$.

Theorem 6.13. *dim* $\mathcal{S}_2^1(\triangle_{PS12}) = 3V + E$ *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: The proof is very similar to the proof of Theorem 6.9, and thus we can be brief. First, for each $v \in \mathcal{V}$, we fix the coefficients of $s \in \mathcal{S}_2^1(\triangle_{PS12})$ corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^1 smoothness at v, by Lemma 5.10, coefficients corresponding to all domain points in $D_1(v)$ are consistently determined. So far we have determined all of the coefficients $\{c_\xi\}_{\xi\in D}$, where $D := \bigcup_{v\in\mathcal{V}} D_1(v)$. Since the disks $D_1(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

For each edge e of $\triangle$, we now set the B-coefficient of s corresponding to the domain point in $\mathcal{M}_e$. At this point we know the coefficients corresponding to three domain points on $R_1(w_e)$. Since C^1 smoothness at w_e is equivalent to all of the control points associated with domain points in $D_1(w_e)$ being collinear, this consistently determines all of these coefficients. Note that all smoothness conditions across edges of $\triangle$ are satisfied by the coefficients we have chosen so far.

Now fix a triangle $T \in \triangle$, and consider the triangle $\hat{T} := \langle w_1, w_2, w_3\rangle$, where w_i are the midpoints of the edges of T. The Powell–Sabin-12 split of T creates a Powell–Sabin-6 split of $\hat{T}$, see Figure 6.6. Now we can appeal to the proof of Theorem 6.9 to see that all coefficients corresponding to domain points lying in $\hat{T}$ are consistently determined. It now follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2^1(\triangle_{PS12})$, and by Theorem 5.13, the dimension of $\mathcal{S}_2^1(\triangle_{PS12})$ is equal to the cardinality of $\mathcal{M}$, which is clearly $3V + E$.

We now verify that $\mathcal{M}$ is local. Suppose $\eta \notin \mathcal{M}$ lies in a triangle T_η. If $\eta \in D_1(v)$ for some vertex v, then the corresponding set Γ_η of Definition 5.16 is just $\mathcal{M}_v \subset \mathrm{star}(T_\eta)$. If $\eta \in D_1(w_e)$ for some edge $\langle u, v\rangle$, then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \mathrm{star}(T_\eta)$. Finally, if η lies in a macro-triangle $T_\eta := \langle v_1, v_2, v_3\rangle$ with edges e_1, e_2, e_3, but not in any of the disks $D_1(v_i)$ or $D_1(w_{e_i})$, then $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \cup \mathcal{M}_{e_1} \cup \mathcal{M}_{e_2} \cup \mathcal{M}_{e_3} \subset \mathrm{star}(T_\eta)$. The computation of each coefficient associated with a domain point not in $\mathcal{M}$ is stable since it is based on smoothness conditions. $\square$

The constant in the stability of the MDS in Theorem 6.13 depends on the smallest angle in the triangulation $\triangle_{PS12}$. By Lemma 4.22, this angle is bounded below by a constant times the smallest angle in $\triangle$ itself.

Figure 6.6 shows an example of a minimal determining set for the space $\mathcal{S}_2^1(\triangle_{PS12})$ of the type described in Theorem 6.9 for a triangulation with $V = 9$ and $E = 17$. By Theorem 6.9, dim $\mathcal{S}_2^1(\triangle_{PS12}) = 3 \times 9 + 17 = 44$.

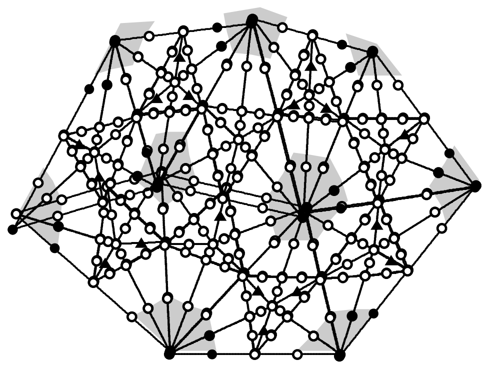


Fig. 6.6. A minimal determining set for $\mathcal{S}_2^1(\triangle_{PS12})$.

Points in the sets $\mathcal{M}_v$ are marked with black dots in the figure, while those in the sets $\mathcal{M}_e$ are marked with triangles.

Since $\mathcal{S}_2^1(\triangle_{PS12})$ has a stable local minimal determining set, we can now apply Theorem 5.19 to see that $\mathcal{S}_2^1(\triangle_{PS12})$ has full approximation power.

Theorem 6.14. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 2$, there exists a spline $s_f \in \mathcal{S}_2^1(\triangle_{PS12})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set for $\mathcal{S}_2^1(\triangle_{PS12})$ and use it to solve a Hermite interpolation problem. For each edge e of $\triangle$, let w_e be the split point on e, and let D_{u_e} be the unit derivative in the direction u_e corresponding to rotating e ninety degrees in the counterclockwise direction. Let ε_t denote point evaluation at t.

Theorem 6.15. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e,$$

is a stable local nodal minimal determining set for $\mathcal{S}_2^1(\triangle_{PS12})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 1}$,
2) $\mathcal{N}_e := \{\varepsilon_{w_e} D_{u_e}\}$.

Proof: It is obvious that the cardinality of $\mathcal{N}$ is $3V + E$, and thus it suffices to show if $s \in \mathcal{S}_2^1(\triangle_{PS12})$, then the data $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determine all coefficients of s. As in the proof of Theorem 6.11, for each vertex v of $\triangle$, the data corresponding to $\mathcal{N}_v$ determines all coefficients of s corresponding to the disk $D_1(v)$. By the proof of Theorem 6.13, the coefficients of s corresponding to the points w_e on the edges of $\triangle$ are determined. For each point w_e, using the value of $D_{u_e}s(w_e)$, we can compute all coefficients of s corresponding to domain points in the disk $D_1(w_e)$. The remaining coefficients of s can be computed as in the proof of Theorem 6.13. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 1$. $\square$

Theorem 6.15 shows that for every function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}_2^1(\triangle_{PS12})$ that solves the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 1, \\ D_{u_e}s(w_e) &= D_{u_e}f(w_e), && \text{all } e \in \mathcal{E}. \end{aligned}$$

This defines a linear projector $\mathcal{I}^1_{PS12}$ mapping $C^1(\Omega)$ onto the spline space $\mathcal{S}_2^1(\triangle_{PS12})$, and in particular, $\mathcal{I}^1_{PS12}$ reproduces polynomials of degree two. We can now apply Theorem 5.26 to get an error bound for this interpolation operator.

Theorem 6.16. *For every $f \in C^{m+1}(\Omega)$ with $0 \le m \le 2$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}^1_{PS12} f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha+\beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2^1(\triangle_{PS12})$ described in Theorem 6.13. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_2^1(\triangle_{PS12})$. In particular, if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle_{PS12}$, then the support of ψ_ξ lies in $\mathrm{star}(v)$. If $\xi \in \mathcal{M}_e$ for some edge e, then the support of ψ_ξ lies in the union of the triangles containing e. The $\mathcal{N}$-basis of (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 6.15 provides an alternative stable local basis for $\mathcal{S}_2^1(\triangle_{PS12})$.

6.5. A C^1 Quadrilateral Macro-element Space

Let $\diamondsuit$ be a strictly convex quadrangulation, and let $\diamondplus$ be the triangulation obtained from $\diamondsuit$ by drawing in the diagonals of each quadrilateral. In this section we discuss the cubic spline space $\mathcal{S}_3^1(\diamondplus)$. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\diamondsuit$, and let V and E be the cardinalities of $\mathcal{V}$ and $\mathcal{E}$, respectively.

For each $v \in \mathcal{V}$, let T_v be the triangle in $\diamondplus$ with vertex at v which has the largest shape parameter (see Definition 4.1) among all triangles sharing the vertex v. Let $\mathcal{M}_v := D_1(v) \cap T_v$. For each edge e of $\diamondsuit$, let T_e be some triangle in $\diamondplus$ containing that edge, and let $\mathcal{M}_e := \{\xi^{T_e}_{111}\}$.

Theorem 6.17. *dim* $\mathcal{S}_3^1(\diamondplus) = 3V + E$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: We use Theorem 5.15 to show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_3^1(\diamondplus)$. We need to show that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_3^1(\diamondplus)$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined. First, for each $v \in \mathcal{V}$, we fix the coefficients of s corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^1 smoothness at v, by Lemma 5.10 all coefficients corresponding to domain points in $D_1(v)$ are consistently determined. So far we have determined all of the coefficients $\{c_\xi\}_{\xi \in D}$, where $D := \bigcup_{v \in \mathcal{V}} D_1(v)$. Since the disks $D_1(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

For each edge e of $\Diamond$, we now fix $c_{111}^{T_e}$. If $e := \langle v_2, v_3 \rangle$ is an interior edge of $\diamondplus$ which is shared by two triangles $T_e := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T}_e := \langle v_4, v_3, v_2 \rangle$ of $\diamondplus$, then we can use the C^1 smoothness across e to determine the corresponding coefficient $c_{111}^{\widetilde{T}_e}$.

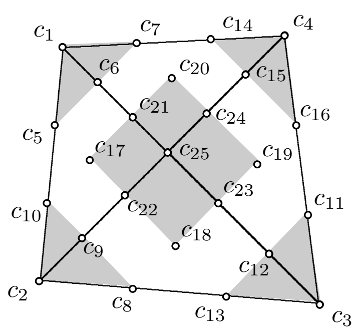


Fig. 6.7. Coefficients of $s|_Q$.

We now show that for each macro-quadrilateral $Q := \langle v_1, v_2, v_3, v_4 \rangle$, the coefficients corresponding to the remaining domain points in Q are consistently determined. Suppose we number the coefficients of $s|_Q$ as shown in Figure 6.7, and suppose the intersection v_Q of the two diagonals of Q is given by $v_Q = r_1 v_1 + s_1 v_3 = r_2 v_4 + s_2 v_2$. Then applying the C^1 smoothness conditions, we get

$$\begin{aligned} c_{21} &= r_2 c_{20} + s_2 c_{17}, \\ c_{22} &= r_1 c_{17} + s_1 c_{18}, \\ c_{23} &= r_2 c_{19} + s_2 c_{18}, \\ c_{24} &= r_1 c_{20} + s_1 c_{19}. \end{aligned} \tag{6.6}$$

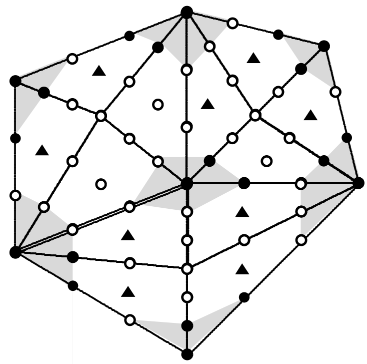


Fig. 6.8. A minimal determining set for $\mathcal{S}_3^1(\diamondplus)$.

Finally, the coefficient c_{25} can be computed from the C^1 smoothness condition at v_Q on either diagonal. Since using either condition leads to

$$c_{25} = r_1 r_2 c_{20} + s_1 r_2 c_{19} + r_1 s_2 c_{17} + s_1 s_2 c_{18}, \tag{6.7}$$

we see that c_{25} is also consistently determined. We have shown that all of the coefficients of $s \in \mathcal{S}_3^1(\diamondplus)$ are consistently determined once we fix the coefficients corresponding to $\mathcal{M}$, and Theorem 5.15 implies that $\mathcal{M}$ is a minimal determining set. Moreover, by Theorem 5.13, the dimension of $\mathcal{S}_3^1(\triangle_{CT})$ is equal to the cardinality of $\mathcal{M}$, which is clearly $3V + E$.

We now check that $\mathcal{M}$ is local. Suppose $\eta \notin \mathcal{M}$ is a domain point in a quadrilateral Q of $\diamondplus$. Then c_η is determined from the coefficients associated with points in Q or in triangles neighboring Q, i.e., the set Γ_η in Definition 5.16 is a subset of $\mathrm{star}(Q)$. Concerning stability, we note that by Theorem 2.19, the computation of coefficients in $D_1(v)$ from $\mathcal{M}_v$ is stable due to our choice of the triangle T_v defining $\mathcal{M}_v$. All other coefficients are obtained directly from smoothness conditions via the above formulae, which by Theorem 2.29 is stable. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 1$. $\square$

The choice of the triangles T_v used to define the $\mathcal{M}_v$ in the MDS of Theorem 6.17 ensures that the constant of stability of $\mathcal{M}$ does not depend on the smallest angle in $\diamondplus$ but only on the smallest angle in $\diamond$. This is important since as we saw in Example 4.48, angles in $\diamondplus$ can be small even when the angles in $\diamond$ are not.

Figure 6.8 shows an example of a minimal determining set for $\mathcal{S}_3^1(\diamondplus)$ of the type given in Theorem 6.17 for a triangulation with $V = 7$ and $E = 9$. Domain points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles.

Since $\mathcal{S}_3^1(\diamondplus)$ has a stable local minimal determining set, we can now apply Theorem 5.19 to show that it has full approximation power.

Theorem 6.18. *For every $f \in W_p^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 3$, there exists a spline $s_f \in \mathcal{S}_3^1(\diamondplus)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\diamondplus|^{m+1-\alpha-\beta}\,|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\diamondplus$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_3^1(\diamondplus)$, and then use it to solve a Hermite interpolation problem. For each edge $e := \langle u, v\rangle$ of $\Diamond$, let $w_e := (u+v)/2$ be the midpoint of e, and let D_{u_e} be the cross-boundary derivative associated with the unit vector u_e corresponding to rotating e ninety degrees in a counterclockwise direction.

Theorem 6.19. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e$$

is a stable local nodal determining set for $\mathcal{S}_3^1(\diamondplus)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\nu D_y^\mu)\}_{0\le\nu+\mu\le 1}$
2) $\mathcal{N}_e := \{\varepsilon_{w_e} D_{u_e}\}$.

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $3V + E$. Since we already know from Theorem 6.17 that the dimension of $\mathcal{S}_3^1(\diamondplus)$ is $3V+E$, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show if $s \in \mathcal{S}_3^1(\diamondplus)$, then the data $\{\lambda s\}_{\lambda\in\mathcal{N}}$ determines all coefficients of s. Let Q be a quadrilateral of $\Diamond$. Then referring to Figure 6.7 and using the formulae in Section 2.7, it is easy to see that

$$\begin{aligned}
c_1 &= s(v_1),\\
c_2 &= s(v_2),\\
c_3 &= s(v_3),\\
c_4 &= s(v_4),\\
c_5 &= [(x_2 - x_1)s_x(v_1) + (y_2 - y_1)s_y(v_1)]/3 + s(v_1),\\
c_6 &= [(x_Q - x_1)s_x(v_1) + (y_Q - y_1)s_y(v_1)]/3 + s(v_1),\\
c_7 &= [(x_4 - x_1)s_x(v_1) + (y_4 - y_1)s_y(v_1)]/3 + s(v_1),\\
c_8 &= [(x_3 - x_2)s_x(v_2) + (y_3 - y_2)s_y(v_2)]/3 + s(v_2),\\
c_9 &= [(x_Q - x_2)s_x(v_2) + (y_Q - y_2)s_y(v_2)]/3 + s(v_2),\\
c_{10} &= [(x_1 - x_2)s_x(v_2) + (y_1 - y_2)s_y(v_2)]/3 + s(v_2),
\end{aligned}$$

and

$$
\begin{aligned}
c_{11} &= [(x_4 - x_3)s_x(v_3) + (y_4 - y_3)s_y(v_3)]/3 + s(v_3),\\
c_{12} &= [(x_Q - x_3)s_x(v_3) + (y_Q - y_3)s_y(v_3)]/3 + s(v_3),\\
c_{13} &= [(x_2 - x_3)s_x(v_3) + (y_2 - y_3)s_y(v_3)]/3 + s(v_3),\\
c_{14} &= [(x_1 - x_4)s_x(v_4) + (y_1 - y_4)s_y(v_4)]/3 + s(v_4),\\
c_{15} &= [(x_Q - x_4)s_x(v_4) + (y_Q - y_4)s_y(v_4)]/3 + s(v_4),\\
c_{16} &= [(x_3 - x_4)s_x(v_4) + (y_3 - y_4)s_y(v_4)]/3 + s(v_4),
\end{aligned}
$$

where $(x_Q, y_Q) := v_Q$. The coefficients $c_{17}, c_{18}, c_{19}, c_{20}$ can now be computed from cross-boundary information, see the proof of Theorem 6.7 for analogous computations for the Clough–Tocher macro-elements. We compute the coefficients $c_{21}, \ldots, c_{24}$ from (6.6). Finally, we compute the coefficient c_{25} from (6.7). Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 1$. $\square$

For each quadrilateral Q in $\diamondsuit$, the linear functionals involving evaluation at points in Q determine $s|_Q$, i.e., the coefficients of s can be computed locally, one quadrilateral at a time. Theorem 6.19 shows that for every function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}^1_3(\diamondplus)$ that solves the Hermite interpolation problem

$$
\begin{aligned}
D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), &\qquad& \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha + \beta \le 1,\\
D_{u_e} s(w_e) &= D_{u_e} f(w_e), && \text{all } e \in \mathcal{E}.
\end{aligned}
$$

This defines a linear projector $\mathcal{I}^1_Q$ mapping $C^1(\Omega)$ onto the spline space $\mathcal{S}^1_3(\diamondplus)$, and in particular $\mathcal{I}^1_Q$ reproduces polynomials of degree three. We can now apply Theorem 5.26 to get an error bound for this interpolation operator.

Theorem 6.20. *For every $f \in C^{m+1}(\Omega)$ with $0 \le m \le 3$,*

$$
\|D_x^\alpha D_y^\beta (f - \mathcal{I}^1_Q f)\|_\Omega \le K\, |\diamondplus|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},
$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\diamondplus$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}^1_3(\diamondplus)$ described in Theorem 6.17. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}^1_3(\diamondplus)$, and

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\diamondsuit$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\diamondsuit$, then the support of ψ_ξ is contained in the union of the triangles containing e.

The $\mathcal{N}$-basis defined in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 6.19 provides an alternative stable local basis for $\mathcal{S}_3^1(\diamondplus)$.

6.6. Comparison of C^1 Macro-element Spaces

As a guide to comparing the various C^1 macro-element spaces discussed in this chapter, in Table 6.1 we list the following information:

- $d :=$ the degree of the spline,
- $n_{tri} :=$ the number of subtriangles in each macro-element,
- $n_{der} :=$ the maximum derivative needed to compute the interpolant,
- $n_{dim} :=$ the dimension of the spline space,
- $n_{coef} :=$ the dimension of the space of continuous splines on the same triangulation.

These quantities are of interest for a variety of reasons:

1) In general we would like to work with low degree splines. They involve fewer coefficients, and have less tendency to oscillate.
2) For the purposes of evaluation, it is more efficient to work with macro-elements with fewer subtriangles as less effort is required to locate the triangle which contains a given point v of interest.
3) In practice we often have to estimate derivatives. Thus, methods which require fewer derivatives may have some advantages. In addition, the error bounds are valid for less smooth functions.
4) The dimension of the spline space measures its complexity.
5) The quantity n_{coef} measures the complexity of storing and working with a spline as a C^0 spline.

Space	d	n_{tri}	n_{der}	n_{dim}	n_{coef}
$\mathcal{S}_5^{1,2}(\triangle)$	5	1	2	$6V+E$	$25V_I+15V_B-24$
$\mathcal{S}_3^1(\triangle_{CT})$	3	3	1	$3V+E$	$27V_I+15V_B-26$
$\mathcal{S}_2^1(\triangle_{PS})$	2	6	1	$3V$	$24V_I+14V_B-23$
$\mathcal{S}_2^1(\triangle_{PS12})$	2	12	1	$3V+E$	$48V_I+26V_B-47$
$\mathcal{S}_3^1(\diamondplus)$	3	4	1	$3V^Q+E^Q$	$18V_I^Q+21V_B^Q/2-17$

Tab. 6.1. A comparison of C^1 macro-elements.

For the triangle-based elements, $V = V_I + V_B$ and $E = E_I + E_B$ denote the number of vertices and edges in the original triangulation $\triangle$, before any splitting. For the quadrangulation-based element, $V^Q = V_I^Q + V_B^Q$ and $E^Q = E_I^Q + E_B^Q$ denote the number of vertices and edges in the quadrangulation. The table suggests that the most efficient space may be $\mathcal{S}_5^{1,2}(\triangle)$

since its complexity is not much higher than the other spaces, but it has the highest approximation power. The space $\mathcal{S}_3^1(\diamondplus)$ is also a good choice when it is convenient to work with quadrangulations.

6.7. Remarks

Remark 6.1. We have called the space discussed in Section 6.1 the C^1 polynomial macro-element space, even though it is constructed on a triangulation $\triangle$ that has not been refined. On each triangle of $\triangle$, a spline in the space reduces to a polynomial of degree five. The space still fits into the framework of Definition 5.27.

Remark 6.2. It often happens that macro-element spline spaces have unexpected supersmoothness at interior vertices. For example, for the C^1 Clough–Tocher macro-element space discussed in Section 6.2, it is easy to see that any $s \in \mathcal{S}_3^1(\triangle_{CT})$ is automatically C^2 at the split point v_T for all $T \in \triangle$. Fix $T := \langle v_1, v_2, v_3 \rangle$, and suppose the coefficients of $s|_T$ are as in Figure 6.2. Note that the barycentric coordinates of v_2 with respect to the triangle $\langle v_T, v_3, v_1 \rangle$ are $(3, -1, 1)$. Combining the equations in (6.1) leads to

$$c_8 = 9c_{19} - 6c_{16} - 6c_{18} + c_{11} + 2c_{15} + c_5,$$

which is just the C^2 smoothness conditions across the edge $\langle v_T, v_2 \rangle$ needed for $s \in C^2(v_T)$. A similar argument show that the C^2 smoothness conditions across the other edges needed for $s \in C^2(v_T)$ are also satisfied.

Remark 6.3. A C^1 cubic macro-element space was constructed in [Gao92] based on the so-called Morgan-Scott split, which splits a triangle into seven subtriangles, see Figure 9.3. The dimension of this macro-element space is $3V+E+4N$, where V, E, N are the numbers of vertices, edges, and triangles in the original triangulation. We do not give a detailed description of this macro-element space here since it has several disadvantages compared to the macro-element space $\mathcal{S}_3^1(\triangle_{CT})$ based on the Clough–Tocher split. In particular, as shown in Section 6.2, the dimension of $\mathcal{S}_3^1(\triangle_{CT})$ is smaller, namely $3V + E$, and the Clough–Tocher split uses only three subtriangles.

Remark 6.4. As pointed out in Remark 5.8, nested sequences of spline spaces are important for applications. Except for the polynomial macro-element space $\mathcal{S}_5^{1,2}(\triangle)$ discussed in Section 6.1, all of the macro-element spaces in this chapter can be used to build nested sequences of spline spaces. However, to be of real use in applications, it is important that the underlying sequence of triangulations $\triangle_1, \triangle_2, \ldots$ be such that $\triangle_n \to 0$ as $n \to \infty$ while the smallest angle in $\triangle_n$ is bounded below by a constant times the smallest angle in $\triangle_1$. In Section 4.16 we discussed which standard refinement methods produce sequences of triangulations with these properties.

Taking account of this requirement, only two of the macro-element spaces of this chapter lead to useful nested sequences:

1) the sequence $\mathcal{S}_2^1(\triangle_n)$, where $\triangle_1, \triangle_2, \ldots$ is the sequence of nested Powell–Sabin-12 triangulations in Method 4.55.
2) the sequence $\mathcal{S}_3^1(\diamondplus_n)$, where $\diamondplus_1, \diamondplus_2, \ldots$ is the sequence of nested induced triangulations associated with quadrangulations described in Method 4.56.

6.8. Historical Notes

The superspline space $\mathcal{S}_5^{1,2}(\triangle)$ discussed in Section 6.1 is the macro-element space which arises if one tries to find nodal data to describe a quintic polynomial on a single triangle. It was introduced in the finite-element literature (without using Bernstein–Bézier techniques) in [Zla68, Zen70], and is probably the most-cited C^1 macro-element in modern finite-element books. It is sometimes called the Argyris element, see [Cia78b]. An explicit construction of a nodal basis for this space was given in [MorS75], see also the construction of vertex splines in [ChuL85]. The description of the minimal determining set $\mathcal{M}$ given in Theorem 6.1 appeared later in [Sch89] which was written to identify classical finite-element spaces used by engineers as certain superspline spaces.

The C^1 Clough–Tocher macro-element was introduced in nodal form in [CloT65]. A condensed version of this element was later described in B-form in [BarnF81].

The macro-element space $\mathcal{S}_2^1(\triangle_{PS})$ discussed in Section 6.3 was studied first by Powell–Sabin [PowS77] in nodal form. The space has been heavily used in applications, and it was generally believed that they had full approximation power, but we could not find a rigorous proof of this in the literature since the question of how small the sides and angles in the split triangles can become does not seem to have been addressed. We have corrected this by providing the needed bounds in Lemma 4.20, allowing us to give a rigorous proof of the approximation power in Theorem 6.10.

The macro-element space $\mathcal{S}_2^1(\triangle_{PS12})$ in Section 6.4 was also introduced in [PowS77] in nodal form. The space was later treated in [ChuH90a], where explicit formulae for the B-coefficients in terms of the nodal data can be found. The authors go on to construct a basis using generalized vertex splines and then use it to show that the space has full approximation power.

The macro-element space $\mathcal{S}_3^1(\diamondplus)$ described in Section 6.5 was studied independently by Fraeijs de Veubeke [Fra68] and Sander [San64], see also [CiavN74]. It is often called the FVS element. The approximation power of the FVS spaces in the L_2 norm was determined in [CiavN74]. For these spaces, locally supported bases and the approximation power in the L_∞ norm were investigated in [Lai95] and [Lai96a].

C^2 Macro-element Spaces

In this chapter we discuss several of the most useful C^2 macro-element spaces. For each of the spaces, we give both a stable local minimal determining set and a stable local nodal determining set, and show that the space has full approximation power.

7.1. A C^2 Polynomial Macro-element Space

Let $\triangle$ be a triangulation of a domain Ω with vertices $\mathcal{V}$. In this section we consider the superspline space

$$\mathcal{S}_9^{2,4}(\triangle) := \{s \in \mathcal{S}_9^2(\triangle) : s \in C^4(v) \text{ for all } v \in \mathcal{V}\}.$$

Let $\mathcal{E}$ be the set of edges of $\triangle$, and suppose N, E, and V are the numbers of triangles, edges, and vertices of $\triangle$, respectively.

For each $v \in \mathcal{V}$, let T_v be a triangle with vertex at v, and let $\mathcal{M}_v := D_4(v) \cap T_v$. For each edge $e := \langle v_2, v_3 \rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3 \rangle$ be a triangle containing e, and let $\mathcal{M}_e := \{\xi_{144}^{T_e}, \xi_{234}^{T_e}, \xi_{243}^{T_e}\}$. For each triangle T of $\triangle$, let $\mathcal{M}_T := \{\xi_{333}^T\}$.

Theorem 7.1. *dim* $\mathcal{S}_9^{2,4}(\triangle) = 15V + 3E + N$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e \cup \bigcup_{T \in \triangle} \mathcal{M}_T$$

is a stable local minimal determining set.

Proof: We use Theorem 5.15 to show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$. We need to show that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_9^{2,4}(\triangle)$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined. First, for each $v \in \mathcal{V}$, we set the coefficients of s corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^4 supersmoothness at v, by Lemma 5.10, all coefficients corresponding to domain points in $D_4(v)$ are consistently determined. So far we have determined all of the coefficients $\{c_\xi\}_{\xi \in D}$, where $D := \bigcup_{v \in \mathcal{V}} D_4(v)$. Since the disks $D_4(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

For each edge e of $\triangle$, we now fix $\{c_\xi\}_{\xi \in \mathcal{M}_e}$. If $e := \langle u, v \rangle$ is a boundary edge of $\triangle$, this determines all coefficients in the set $E_2(e) :=$

$\{\eta : \mathrm{dist}(\eta, e) \le 2,\ \eta \notin D_4(u) \cup D_4(v)\}$. If e is an interior edge of $\triangle$, then the unset coefficients in $E_2(e)$ can be computed from the C^2 smoothness across e. No inconsistencies can arise in this way since the coefficients associated with $E_2(e)$ do not enter any smoothness conditions involving coefficients associated with sets $E_2(\tilde{e})$ for other edges $\tilde{e}$. Finally, for each triangle T, we fix the coefficient c^T_{333}. It does not enter any smoothness conditions. We have shown that $\mathcal{M}$ is a consistent determining set, and thus by Theorem 5.15 is a minimal determining set. By Theorem 5.13, the dimension of $\mathcal{S}_9^{2,4}(\triangle)$ is equal to the cardinality of $\mathcal{M}$, which is clearly $15V + 3E + N$.

We now check that $\mathcal{M}$ is local in the sense of Definition 5.16. Suppose $\eta \notin \mathcal{M}$ lies in T_η. If $\eta \in D_4(v)$ for some vertex v, then c_η depends on the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_v}$. Thus, the set Γ_η in Definition 5.16 is just $\mathcal{M}_v \subset \mathrm{star}(v) \subset \mathrm{star}(T_\eta)$. Now suppose $\eta \in E_2(e)$ for some edge $e := \langle u, v \rangle$. Then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \mathrm{star}(T_\eta)$.

We claim that $\mathcal{M}$ is also stable as defined in Definition 5.16. Indeed, all coefficients corresponding to $\eta \notin \mathcal{M}$ are computed directly from smoothness conditions, which by Lemma 2.29 is a stable process. □

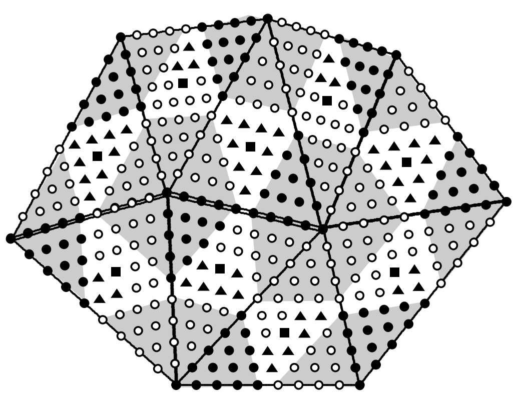


Fig. 7.1. A minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$.

Figure 7.1 shows an example of a minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$ of the type given in Theorem 7.1 for a triangulation with $V = 9$, $E = 17$, and $N = 9$. By the theorem, $\dim \mathcal{S}_9^{2,4}(\triangle) = 15 \cdot 9 + 3 \cdot 17 + 9 = 195$. Domain points in the sets $\mathcal{M}_v$ are marked with black dots. Those in the sets $\mathcal{M}_e$ are marked with triangles, while those in the sets $\mathcal{M}_T$ are marked with squares.

Since $\mathcal{S}_9^{2,4}(\triangle)$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

Theorem 7.2. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 9$, there exists a spline $s_f \in \mathcal{S}_9^{2,4}(\triangle)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$. We follow the notation introduced in Section 6.1. In particular, we assign an orientation to each edge $e := \langle u, v \rangle$ of $\triangle$, and let u_e be the unit vector corresponding to rotating e ninety degrees in the counterclockwise direction. We write $\eta_e := (u+v)/2$ for the midpoint of e, and D_{u_e} for the cross-boundary derivative associated with u_e. In addition, let $\eta_{1,e} := (2u+v)/3$, $\eta_{2,e} := (u+2v)/3$. Given a triangle T, let v_T be its barycenter. We write ε_t for the point evaluation functional at t.

Theorem 7.3. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{T \in \triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 4}$ *for each $v \in \mathcal{V}$,*
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}, \varepsilon_{\eta_{1,e}} D_{u_e}^2, \varepsilon_{\eta_{2,e}} D_{u_e}^2\}$ *for each edge $e \in \mathcal{E}$,*
3) $\mathcal{N}_T := \varepsilon_{v_T}$ *for each triangle T in $\triangle$.*

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $15V + 3E + N$. Since we already know from Theorem 7.1 that the dimension of $\mathcal{S}_9^{2,4}(\triangle)$ is $15V+3E+N$, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show that if $s \in \mathcal{S}_9^{2,4}(\triangle)$, then all of its coefficients are determined by the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$. First, for every vertex v of $\triangle$, we use the formulae (2.37) to uniquely determine all of the coefficients c_η of s corresponding to domain points $\eta \in D_4(v)$. Using Lemma 2.21, we can then compute the coefficients of the form $c_{144}^T, c_{234}^T, c_{243}^T$ from the cross-boundary information associated with the sets $\mathcal{N}_e$. Finally, for each triangle T, the coefficient c_{333}^T can be computed from $s(v_T)$ by Lemma 2.25. Examining these computations shows that there exists a constant K_1 depending only on the smallest angle in $\triangle$ such that for any $T \in \triangle$ and $\xi \in \mathcal{D}_{9,T}$,

$$|c_\xi^T| \le K_1 \sum_{\nu=0}^{4} |T|^\nu\, |f|_{\nu,T}.$$

This proves that $\mathcal{N}$ is local and stable. $\square$

For each triangle T in $\triangle$, the set of data involving evaluation at points in T uniquely determines $s|_T$, i.e., the coefficients of $s \in \mathcal{S}_9^{2,4}(\triangle)$ can be computed locally, one triangle at a time. Thus, $\mathcal{S}_9^{2,4}(\triangle)$ is a macro-element space in the sense of Definition 5.27. Theorem 7.3 shows that for any function $f \in C^4(\Omega)$, there is a unique spline $s \in \mathcal{S}_9^{2,4}(\triangle)$ solving the Hermite interpolation problem

$$\begin{aligned}
D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 4,\\
D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E},\\
D_{u_e}^2 s(\eta_{\nu,e}) &= D_{u_e}^2 f(\eta_{\nu,e}), && \text{all } e \in \mathcal{E} \text{ and } \nu = 1,2,\\
s(v_T) &= f(v_T), && \text{all triangles } T \text{ in } \triangle.
\end{aligned}$$

This defines a linear projector $\mathcal{I}_P^2$ mapping $C^4(\Omega)$ onto the superspline space $\mathcal{S}_9^{2,4}(\triangle)$, and in particular $\mathcal{I}_P^2$ reproduces polynomials of degree nine. We now apply Theorem 5.26 to get the following error bound for this interpolation operator.

Theorem 7.4. *For every $f \in C^{m+1}(\Omega)$ with $3 \le m \le 9$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_P^2 f)\|_\Omega \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,\Omega},$$

for all $0 \le \alpha+\beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Proof: Since the set $\mathcal{N}$ in Theorem 7.3 is a stable local NMDS, it follow from Theorem 5.26 that

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_P^2 f)\|_T \le K_2\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,T}.$$

Taking the maximum over all triangles immediately gives the global result. $\square$

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_9^{2,4}(\triangle)$ described in Theorem 7.1. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_9^{2,4}(\triangle)$. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles sharing e,
3) if $\xi \in \mathcal{M}_T$ for some triangle T, then the support of ψ_ξ is contained in T.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.3 provides a different stable local basis for $\mathcal{S}_9^{2,4}(\triangle)$.

7.2. A C^2 Clough–Tocher Macro-element Space

Suppose $\triangle$ is a triangulation of the domain Ω, and let $\triangle_{CT}$ be the corresponding Clough–Tocher refinement of $\triangle$ as described in Definition 4.16. Let $\mathcal{V}$ be the set of vertices of $\triangle$, and let $\mathcal{V}_c := \{v_T\}$ be the set of barycenters which are introduced to form the Clough–Tocher splits. Let $\mathcal{E}$ be the set of edges of $\triangle$.

For each $T := \langle v_1, v_2, v_3 \rangle$ in $\triangle$, let $e_T := \langle v_1, v_T \rangle$, and let τ^5_{5,e_T} be the C^5 smoothness condition across e_T defined in (5.7). In this section we discuss the superspline space

$$\begin{aligned} \mathcal{S}_2(\triangle_{CT}) := \{s \in \mathcal{S}^2_7(\triangle_{CT}) :\ & s \in C^3(v), \text{ all } v \in \mathcal{V}, \\ & s \in C^6(v), \text{ all } v \in \mathcal{V}_c, \\ & \tau^5_{5,e_T} s = 0, \text{ all } T \in \triangle\}. \end{aligned}$$

Let V and E denote the numbers of vertices and edges of $\triangle$, respectively. For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v, and let $\mathcal{M}_v := D_3(v) \cap T_v$. For each edge $e := \langle v_2, v_3 \rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3 \rangle$ be a triangle containing e, and let $\mathcal{M}_e := \{\xi^{T_e}_{133}, \xi^{T_e}_{223}, \xi^{T_e}_{232}\}$.

Theorem 7.5. *dim* $\mathcal{S}_2(\triangle_{CT}) = 10V + 3E$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: The proof of this result is different from the proof of Theorem 7.1 and of similar results in Chapter 6. Here we have to be careful, since with the extra smoothness defining $\mathcal{S}_2(\triangle_{CT})$, it is not easy to see whether a determining set is consistent and thus minimal.

We deal first with the case where $\triangle$ is just a single triangle T to which the Clough–Tocher split has been applied, see Figure 7.2. Let T_{CT} be the Clough–Tocher refinement of T. We first show that in this case $\mathcal{M}$ is a determining set. For each vertex v of T, we fix the coefficients of $s \in \mathcal{S}_2(T_{CT})$ corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^3 supersmoothness at v, Lemma 5.10 shows that all coefficients corresponding to domain points in $D_3(v)$ are determined. Next for each edge e of T, we fix the coefficients corresponding to the set $\mathcal{M}_e$. We claim that the remaining unset coefficients are determined by smoothness conditions. First, for each $i = 1, 2, 3$, we use Lemma 2.30 to compute the three unset coefficients corresponding to domain points on each of the rings $R_4(v_i)$. We can use the lemma since the C^6 smoothness at v_T gives us three smoothness conditions on each such ring. We now use the lemma to determine the five unset coefficients corresponding to domain points on the ring $R_5(v_1)$. The C^6

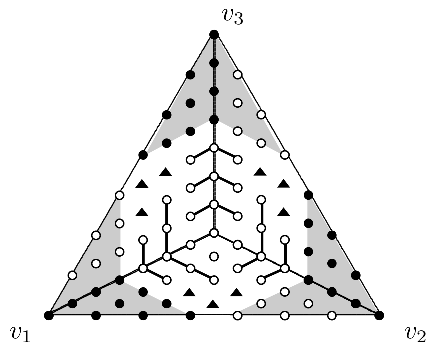


Fig. 7.2. Computing the coefficients for a Clough–Tocher macro-triangle.

smoothness at v_T provides four smoothness conditions on this ring, while the special smoothness condition $\tau^5_{5,e_T} s = 0$ gives us a fifth condition, see Figure 7.2. Applying the lemma again, we get the four remaining undetermined coefficients corresponding to domain points on the ring $R_5(v_2)$ and the three coefficients on $R_5(v_3)$. The remaining coefficients can then be determined by additional applications of the lemma. Since we have now shown that all coefficients of s are determined, it follows that $\mathcal{M}$ is a determining set for $\mathcal{S}_2(T_{CT})$.

To prove that $\mathcal{M}$ is minimal, we now show that $\#\mathcal{M} = \dim \mathcal{S}_2(T_{CT})$. By Theorem 9.7, the dimension of $\mathcal{S}^2_7(T_{CT}) \cap C^6(v_T)$ is 43. Now $\mathcal{S}_2(T_{CT})$ is the subspace that satisfies four additional smoothness conditions, namely the condition $\tau^5_{5,e_T} s = 0$ along with three additional conditions to get C^3 smoothness at the vertices of T. Thus,

$$39 = \dim \mathcal{S}^1_7(T_{CT}) \cap C^6(v_T) - 4 \leq \dim \mathcal{S}_2(T_{CT}) \leq \#\mathcal{M} = 39,$$

where we have used Theorem 5.13 to get the last inequality. This implies that $\dim \mathcal{S}_2(T_{CT}) = \#\mathcal{M} = 39$. By that theorem, $\mathcal{M}$ is an MDS for $\mathcal{S}_2(T_{CT})$.

We now return to the case where $\triangle$ is arbitrary. We claim that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_2(\triangle_{CT})$. First, for each vertex v we fix the coefficients of a spline $s \in \mathcal{S}_2(\triangle_{CT})$ for all domain points in $\mathcal{M}_v$. Then in view of the C^3 smoothness at v, by Lemma 5.10 all coefficients of s corresponding to domain points in $D_3(v)$ are consistently determined. So far we have determined all of the coefficients $\{c_\xi\}_{\xi \in D}$, where $D := \bigcup_{v \in \mathcal{V}} D_3(v)$. Since the disks $D_3(v)$ do not overlap, it follows that all smoothness conditions that involve only these coefficients are satisfied.

Now for each edge $e := \langle u, v \rangle$ of $\triangle$ we fix the coefficients corresponding to $\mathcal{M}_e$. If e is a boundary edge of $\triangle$, this determines all coefficients corresponding to $E_2(e) := \{\eta : \operatorname{dist}(\eta, e) \leq 2,\ \eta \notin D_3(u) \cup D_3(v)\}$. If e

is an interior edge of $\triangle$, then we can use the C^2 smoothness across e to determine the remaining unset coefficients of s corresponding to domain points in $E_2(e)$. Next, for each macro-triangle T, we see that the coefficients corresponding to the remaining domain points in T are consistently determined by our previous arguments for a single macro-triangle. This shows that no inconsistencies can arise in setting the coefficients in the sets $\mathcal{M}_e$, even though coefficients in two different such sets may be connected by smoothness conditions across the interior edges of T_{CT}.

We have shown that if we fix the coefficients of a spline s corresponding to domain points in the set $\mathcal{M}$, then all of the coefficients of s are consistently determined. It follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set, and by Theorem 5.13, the dimension of $\mathcal{S}_2(\triangle_{CT})$ is equal to the cardinality of $\mathcal{M}$, which is clearly $10V + 3E$.

We now check that $\mathcal{M}$ is local in the sense of Definition 5.16. Suppose that $\eta \notin \mathcal{M}$ lies in T_η. If $\eta \in D_3(v)$ for some vertex v, then the set Γ_η in the definition is just $\mathcal{M}_v \subset \mathrm{star}(v) \subset \mathrm{star}(T_\eta)$. Now suppose $\eta \in E_2(e)$ for some edge $e := \langle u, v \rangle$. In this case $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \mathrm{star}(T_\eta)$. Finally, if η is one of the remaining domain points in a macro-triangle $T_\eta := \langle v_1, v_2, v_3 \rangle$ with edges e_1, e_2, e_3, then $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \cup \mathcal{M}_{e_1} \cup \mathcal{M}_{e_2} \cup \mathcal{M}_{e_3} \subset \mathrm{star}(T_\eta)$.

The stability of $\mathcal{M}$ follows from the fact that all computed coefficients were obtained from smoothness conditions using either Lemma 2.29 or Lemma 2.30. □

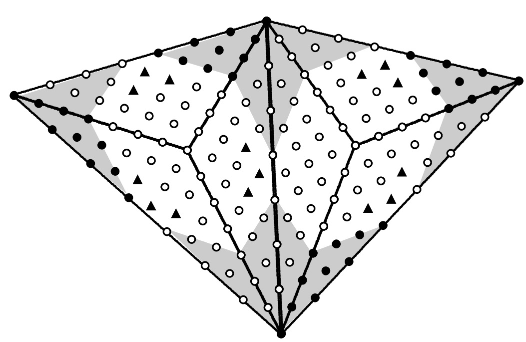


Fig. 7.3. A minimal determining set for $\mathcal{S}_2(\triangle_{CT})$.

The constant in the stability of $\mathcal{M}$ in Theorem 7.5 depends on the smallest angle in the triangulation $\triangle_{CT}$. By Lemma 4.17 this angle is bounded below by a constant times the smallest angle in $\triangle$. Figure 7.3 shows the minimal determining set $\mathcal{M}$ for a triangulation with $V = 4$ and $E = 5$. This gives $\dim \mathcal{S}_2(\triangle_{CT}) = 10 \cdot 4 + 3 \cdot 5 = 55$. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles.

Since $\mathcal{S}_2(\triangle_{CT})$ has a stable local MDS, we can now apply Theorem 5.19 to show that it has full approximation power.

Theorem 7.6. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 7$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_{CT})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_2(\triangle_{CT})$. For each edge $e := \langle u, v \rangle$ of $\triangle$, let $\eta_e := (u + v)/2$ be the midpoint of e, and let D_{u_e} be the cross-boundary derivative associated with a unit vector u_e corresponding to rotating e ninety degrees in a counterclockwise direction. In addition, let $\eta_{1,e} := (2u + v)/3$ and $\eta_{2,e} := (u + 2v)/3$. We write ε_t for the point evaluation functional at t.

Theorem 7.7. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_{CT})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 3}$ *for each vertex $v \in \mathcal{V}$,*
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}, \varepsilon_{\eta_{1,e}} D_{u_e}^2, \varepsilon_{\eta_{2,e}} D_{u_e}^2\}$ *for each edge $e \in \mathcal{E}$.*

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $10V + 3E$, which by Theorem 7.5 is also the dimension of $\mathcal{S}_2(\triangle_{CT})$. Let $\mathcal{M}$ be the MDS in Theorem 7.5. Then to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show that if $s \in \mathcal{S}_2(\triangle_{CT})$, then all of its coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ are determined by the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$. First, for every vertex v of $\triangle$, we use the formulae (2.37) to uniquely determine all of the coefficients c_ξ of s corresponding to domain points in $\mathcal{M}_v$. Using Lemma 2.21, we can then compute the coefficients corresponding to domain points in $\mathcal{M}_e$ from the cross-boundary information associated with $\mathcal{N}_e$. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 3$. □

Since for each triangle T in $\triangle$, $s|_T$ can be computed locally, one triangle at a time, it follows that $\mathcal{S}_2(\triangle_{CT})$ is a macro-element space in the sense of Definition 5.27. Theorem 7.7 shows that for any function $f \in C^3(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_{CT})$ solving the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha + \beta \le 3, \\ D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E}, \\ D_{u_e}^2 s(\eta_{\nu,e}) &= D_{u_e}^2 f(\eta_{\nu,e}), && \text{all } e \in \mathcal{E} \text{ and } \nu = 1, 2. \end{aligned}$$

This defines a linear projector $\mathcal{I}^2_{CT}$ mapping $C^3(\Omega)$ onto the superspline space $\mathcal{S}_2(\triangle_{CT})$, and in particular, $\mathcal{I}^2_{CT}$ reproduces polynomials of degree seven. We now apply Theorem 5.26 to get the following error bound for this interpolation operator.

Theorem 7.8. *For every $f \in C^{m+1}(\Omega)$ with $2 \le m \le 7$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}^2_{CT} f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2(\triangle_{CT})$ described in Theorem 7.5. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_2(\triangle_{CT})$, and

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles containing e.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.7 provides a different stable local basis for $\mathcal{S}_2(\triangle_{CT})$.

7.3. A C^2 Powell–Sabin Macro-element Space

Given a triangulation $\triangle$ of the domain Ω, let $\triangle_{PS}$ be the corresponding Powell–Sabin refinement of $\triangle$ as described in Definition 4.18. Let $\mathcal{V}$ be the set of vertices of $\triangle$, and let $\mathcal{V}_c$ be the set of incenters v_T introduced in the refinement process.

For each triangle $T := \langle v_1, v_2, v_3\rangle$ in $\triangle$, let $e_T := \langle v_1, v_T\rangle$ be one of the edges forming the Powell–Sabin split of T. Note that e_T is oriented so that it points toward v_T. Let τ^3_{4,e_T} be the special smoothness condition associated with e_T defined in (5.7). For each triangle T, there are six edges that are introduced to form the Powell–Sabin split of T, and three of them are not connected to the vertices of T. Let $\mathcal{E}_T$ be the set of all such edges. In this section we work with the spline space

$$\begin{aligned}\mathcal{S}_2(\triangle_{PS}) := \{s \in \mathcal{S}^2_5(\triangle_{PS}) :\ & s \in C^3(v),\ \text{all } v \in \mathcal{V}\cup\mathcal{V}_c,\\ & s \in C^3(e),\ \text{all } e \in \mathcal{E}_T,\\ & \tau^3_{4,e_T} s = 0,\ \text{all } T \in \triangle\}.\end{aligned}$$

Let V be the number of vertices of $\triangle$. For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v, and let $\mathcal{M}_v := D_3(v) \cap T_v$.

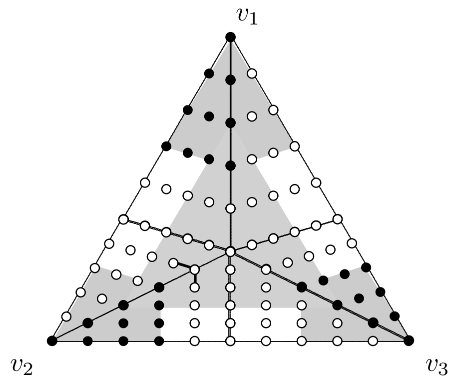


Fig. 7.4. Computing the coefficients of a C^2 Powell–Sabin macro-element.

Theorem 7.9. *dim* $\mathcal{S}_2(\triangle_{PS}) = 10\,V$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v$$

is a stable local minimal determining set.

Proof: As in the proof of Theorem 7.5, we begin by dealing with the case where $\triangle$ consists of a single triangle $T = \langle v_1, v_2, v_3 \rangle$. Let T_{PS} be the Powell–Sabin split of T, see Figure 7.4, where we have marked points in $\mathcal{M}$ with black dots. For each edge e of T, let w_e be the vertex of T_{PS} lying in the interior of e. Suppose now that we fix the coefficients of $s \in \mathcal{S}_2(T_{PS})$ corresponding to domain points in $\mathcal{M}$. Then by the C^3 supersmoothness at the vertices, all coefficients corresponding to domain points in the disks $D_3(v_i)$ are determined by Lemma 5.10.

We now show that the coefficients of s corresponding to the remaining domain points in T are determined from smoothness conditions. For each edge $e := \langle u, v \rangle$ of T, we use the C^3 smoothness across the edge $\langle v_T, w_e \rangle$ to determine the unset coefficients corresponding to domain points in the set $E_2(e) := \{\xi : \operatorname{dist}(\xi, e) \le 2,\ \xi \notin D_3(u) \cup D_3(v)\}$. Now taking account of the smoothness condition $\tau^3_{4,e_T} s = 0$, we can use Lemma 2.30 to compute the unset coefficients corresponding to domain points on the ring $R_4(v_1)$. Using the lemma again, we can compute the remaining undetermined coefficients corresponding to domain points at a distance three from $\langle v_1, v_2 \rangle$ and from $\langle v_2, v_3 \rangle$, followed by the remaining undetermined coefficients corresponding to domain points on the rings $R_4(v_2)$ and $R_4(v_3)$. At this point we have determined the coefficients of s at three domain points on the ring $R_1(v_T)$, which in turn determines all coefficients corresponding to $D_1(v_T)$. We have shown that $\mathcal{M}$ is a determining set for $\mathcal{S}_2(T_{PS})$.

To prove that $\mathcal{M}$ is an MDS for $\mathcal{S}_2(T_{PS})$, we show that $\#\mathcal{M} = \dim \mathcal{S}_2(T_{PS})$. First, we observe that by Theorem 9.7, the dimension of the superspline space $\mathcal{S}_5^2(T_{PS}) \cap C^3(v_T)$ is 40. To force a spline in $\mathcal{S}_5^2(T_{PS}) \cap C^3(v_T)$ to be in $\mathcal{S}_2(T_{PS})$ we must enforce nine C^3 smoothness conditions across the edges in $\mathcal{E}_T$, along with the special smoothness condition $\tau_{4,e_T}^3 s = 0$. It follows that

$$30 = \dim \mathcal{S}_5^2(T_{PS}) \cap C^3(v_T) - 10 \le \dim \mathcal{S}_2(T_{PS}) \le \#\mathcal{M} = 30.$$

This implies that $\dim \mathcal{S}_2(T_{PS}) = \#\mathcal{M} = 30$, and thus $\mathcal{M}$ is an MDS for $\mathcal{S}_2(T_{PS})$.

We return now to the case where $\triangle_{PS}$ is the Powell–Sabin refinement of an arbitrary triangulation $\triangle$. Suppose we set the coefficients of a spline $s \in \mathcal{S}_2(\triangle_{PS})$ corresponding to all domain points in $\mathcal{M}$. This consistently determines the coefficients of s for all domain points in the disks $D_3(v)$ for $v \in \mathcal{V}$. Now by the above argument, for each macro-triangle T the remaining coefficients associated with domain points in T are determined in such a way that all smoothness conditions across interior edges of T_{PS} are satisfied. To verify the consistency of $\mathcal{M}$, we now have to check that the smoothness conditions across edges e of each macro-triangle are satisfied. These conditions involve the coefficients corresponding to domain points in the sets $E_2(e)$. Suppose T and $\widetilde{T}$ are neighboring triangles sharing an edge e, and let v_T and $v_{\tilde{T}}$ be their incenters. Then using the fact that the edges $\langle v_T, w_e \rangle$ and $\langle v_{\tilde{T}}, w_e \rangle$ lie on the line from v_T to $v_{\tilde{T}}$, it is easy to check that all C^1 and C^2 smoothness conditions across e are satisfied, see the proof of Theorem 6.9.

We have shown that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_2(\triangle_{PS})$, and it follows from Theorem 5.15 that $\mathcal{M}$ is an MDS for $\mathcal{S}_2(\triangle_{PS})$. In addition, we conclude from Theorem 5.13 that the dimension of $\mathcal{S}_2(\triangle_{PS})$ is equal to the cardinality of $\mathcal{M}$, which is $10V$.

To see that $\mathcal{M}$ is local, suppose $\eta \notin \mathcal{M}$ lies in the triangle T_η. If $\eta \in D_3(v)$ for some vertex v, then the set Γ_η in Definition 5.16 is just $\mathcal{M}_v \subset \text{star}(v) \subset \text{star}(T_\eta)$. Now suppose η lies in a triangle $T_\eta := \langle v_1, v_2, v_3 \rangle$, but not in any of the disks $D_3(v_i)$, $i = 1, 2, 3$. In this case $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \subset \text{star}(T_\eta)$. The stability of $\mathcal{M}$ follows from the fact that all coefficients corresponding to domain points not in $\mathcal{M}$ are computed from smoothness conditions using Lemmas 2.29 and 2.30. $\square$

The constant in the stability of $\mathcal{M}$ in Theorem 7.9 depends on the smallest angle in the triangulation $\triangle_{PS}$. By Lemma 4.20 this angle is bounded below by a constant times the smallest angle in $\triangle$.

Figure 7.5 shows the minimal determining set for $\mathcal{S}_2(\triangle_{PS})$ for a triangulation consisting of two macro-triangles. The points in $\mathcal{M}$ are marked with black dots. Since $\mathcal{S}_2(\triangle_{PS})$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

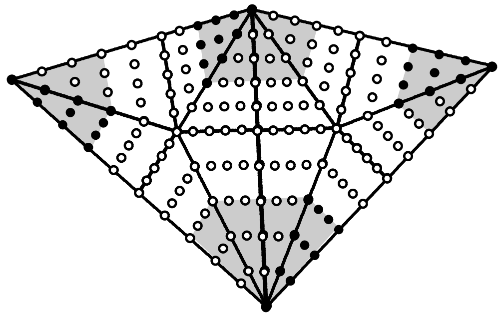


Fig. 7.5. A minimal determining set for $\mathcal{S}_2(\triangle_{PS})$.

Theorem 7.10. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 5$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_{PS})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\;|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_2(\triangle_{PS})$. Let ε_t denote point evaluation at t.

Theorem 7.11. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le 3}$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_{PS})$.

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $10V$. Since we already know from Theorem 7.9 that the dimension of $\mathcal{S}_2(\triangle_{PS})$ is $10V$, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show that if $s \in \mathcal{S}_2(\triangle_{PS})$, then all of its coefficients corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 7.9 are determined by the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$. This is clear, since for every vertex v of $\triangle$, we can use the formulae (2.37) to uniquely determine all of the coefficients c_ξ of s corresponding to domain points $\xi \in \mathcal{M}_v$. $\square$

For each triangle T in $\triangle$, the set of data involving evaluation at points in T uniquely determines $s|_T$, i.e., the coefficients of $s \in \mathcal{S}_2(\triangle_{PS})$ can be computed locally, one triangle at a time. Thus, $\mathcal{S}_2(\triangle_{PS})$ is a macro-element space in the sense of Definition 5.27.

Theorem 7.11 shows that for any function $f \in C^3(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_{PS})$ solving the Hermite interpolation problem

$$D_x^\alpha D_y^\beta s(v) = D_x^\alpha D_y^\beta f(v), \qquad \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha + \beta \le 3.$$

This defines a linear projector $\mathcal{I}_{PS}^2$ mapping $C^3(\Omega)$ onto the superspline space $\mathcal{S}_2(\triangle_{PS})$, and in particular, $\mathcal{I}_{PS}^2$ reproduces polynomials of degree five. We now apply Theorem 5.26 to get the following error bound for this interpolation operator.

Theorem 7.12. *For every $f \in C^{m+1}(\Omega)$ with $2 \le m \le 5$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_{PS}^2 f)\|_\Omega \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2(\triangle_{PS})$. described in Theorem 7.9. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_2(\triangle_{PS})$, where for each $v \in \mathcal{V}$ and each $\xi \in \mathcal{M}_v$, the support of ψ_ξ is contained in $\operatorname{star}(v)$. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.11 provides a different stable local basis for $\mathcal{S}_2(\triangle_{PS})$.

7.4. A C^2 Wang Macro-element Space

Given a triangulation $\triangle$, suppose $\triangle_W$ is the triangulation obtained by splitting each triangle $T := \langle v_1, v_2, v_3 \rangle$ into seven subtriangles as shown in Figure 7.6 (left) based on the interior vertices

$$w_1^T := \frac{4v_1 + 2v_2 + v_3}{7}, \quad w_2^T := \frac{v_1 + 4v_2 + 2v_3}{7}, \quad w_3^T := \frac{2v_1 + v_2 + 4v_3}{7}.$$

For each triangle T in $\triangle$, let $T^* := \langle w_1^T, w_2^T, w_3^T \rangle$ be the the center triangle in the split of T. In this section we consider the space

$$\mathcal{S}_2(\triangle_W) := \{s \in \mathcal{S}_5^2(\triangle_W) : \, s \text{ is } C^3 \text{ across the edges of } T^*, \text{ all } T \in \triangle\}.$$

Let V and E denote the numbers of vertices and edges of $\triangle$, respectively. For each vertex v of $\triangle$, let T_v be a triangle of $\triangle_W$ attached to v which has the largest shape parameter (see Definition 4.1) among all triangles sharing the vertex v, and let $\mathcal{M}_v := D_2(v) \cap T_v$. For each edge $e := \langle v_2, v_3 \rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3 \rangle$ be a triangle of $\triangle_W$ containing e, and let $\mathcal{M}_e := \{\xi_{122}^{T_e}, \xi_{221}^{T_e}, \xi_{212}^{T_e}\}$.

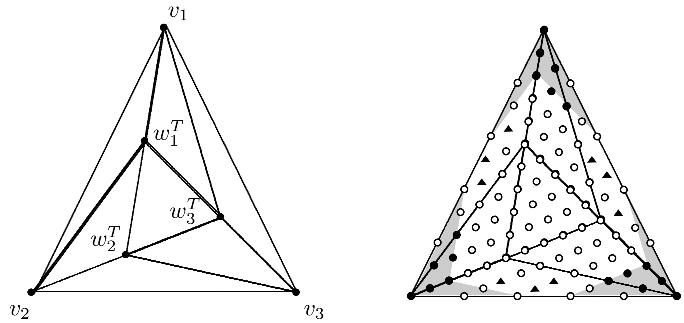


Fig. 7.6. A minimal determining set for $\mathcal{S}_2(\triangle_W)$.

Theorem 7.13. *dim* $\mathcal{S}_2(\triangle_W) = 6V + 3E$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: Suppose we fix the coefficients of a spline $s \in \mathcal{S}_2(\triangle_W)$ corresponding to $\mathcal{M}_v$ for each vertex v of $\triangle$. Then by Lemma 5.10, the coefficients of s are determined in the disks $D_2(v)$. Now if we fix the coefficients of s corresponding to $\mathcal{M}_e$ for each edge $e := \langle u, v\rangle$ of $\triangle$, then using the smoothness across the edges, we see that all coefficients corresponding to domain points in $E_2(e) := \{\xi : \operatorname{dist}(\xi, e) \le 2,\ \xi \notin D_2(u) \cup D_2(v)\}$ are determined. It is not clear a priori that these coefficients can be set consistently, but using the Java program described in Remark 5.6, it can be checked that for each macro-triangle $T := \langle v_1, v_2, v_3\rangle$, all unset coefficients corresponding to domain points lying in T are consistently determined. In particular, for each such domain point η, the program gives an explicit formula for c_η as a linear combination of the coefficients associated with domain points in the sets $\mathcal{M}_{v_i}$ and $\mathcal{M}_{e_i}$, where e_1, e_2, e_3 are the edges of T. The weights in these linear combinations are all fixed numbers, independent of the size and shape of T. It follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2(\triangle_W)$. By Theorem 5.13, its dimension is equal to $\#\mathcal{M} = 6V + 3E$.

To see that $\mathcal{M}$ is local, suppose $\eta \notin \mathcal{M}$ lies in a triangle T_η. If $\eta \in D_2(v)$, then the set Γ_η of Definition 5.16 is just $\mathcal{M}_v \subset \operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. If $\eta \in E_2(e)$ for some edge $e := \langle u, v\rangle$, then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \operatorname{star}(T_\eta)$. For any other η lying in a macro-triangle $T_\eta := \langle v_1, v_2, v_3\rangle$, Γ_η is the union of the sets $\mathcal{M}_{v_i}$ and $\mathcal{M}_{e_i}$ for $i = 1, 2, 3$, which is contained in $\operatorname{star}(T_\eta)$. The stability of the computation of coefficients in the disks $D_2(v)$ follows from our choice of T_v. The stability of the computation of other coefficients associated with domain points in a triangle T follows from the fact that they involve fixed formulae which do not vary with the size or shape of the triangle. $\square$

Figure 7.6 (right) shows the minimal determining set for the space $\mathcal{S}_5^2(T_W)$ corresponding to the split T_W of a single macro-triangle. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles.

Since $\mathcal{S}_2(\triangle_W)$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

Theorem 7.14. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 5$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_W)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_2(\triangle_W)$. For each edge $e := \langle u, v\rangle$ of $\triangle$, let D_{u_e} be the cross-boundary derivative associated with the unit vector u_e corresponding to rotating e ninety degrees in the counterclockwise direction. Let $\eta_e := (u+v)/2$ be the midpoint of e, and let $\eta_{1,e} := (2u+v)/3$ and $\eta_{2,e} := (u+2v)/3$. Let ε_t be the point evaluation functional at t.

Theorem 7.15. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_W)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\nu D_y^\mu\}_{0 \le \nu+\mu \le 2}$, *for all vertices $v \in \mathcal{V}$,*
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}, \varepsilon_{\eta_{1,e}} D_{u_e}^2, \varepsilon_{\eta_{2,e}} D_{u_e}^2\}$, *for all edges e of $\triangle$.*

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $6V + 3E$, and that setting the nodal data in $\mathcal{N}$ for a spline $s \in \mathcal{S}_2(\triangle_W)$ determines the coefficients of s corresponding to the MDS $\mathcal{M}$ of Theorem 7.13. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 2$. $\square$

Since for each triangle T in $\triangle$, $s|_T$ can be computed locally, one triangle at a time, it follows that $\mathcal{S}_2(\triangle_W)$ is a macro-element space in the sense of Definition 5.27. By Theorem 7.7, for any function $f \in C^2(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_W)$ solving the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 2, \\ D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E}, \\ D_{u_e}^2 s(\eta_{\nu,e}) &= D_{u_e}^2 f(\eta_{\nu,e}), && \text{all } e \in \mathcal{E} \text{ and } \nu = 1, 2. \end{aligned}$$

This defines a linear projector $\mathcal{I}_W^2$ mapping $C^2(\Omega)$ onto $\mathcal{S}_2(\triangle_W)$, and in particular $\mathcal{I}_W^2$ reproduces polynomials of degree five. Applying Theorem 5.26, we get the following error bound for this interpolation operator.

Theorem 7.16. *For every $f \in C^{m+1}(\Omega)$ with $1 \le m \le 5$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_W^2 f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2(\triangle_W)$ described in Theorem 7.13. By Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_2(\triangle_W)$, and

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles sharing e.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.15 provides a different stable local basis for $\mathcal{S}_2(\triangle_W)$.

7.5. A C^2 Double Clough–Tocher Macro-element

Given a triangulation $\triangle$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$, let $\triangle_{DCT}$ be the triangulation that is obtained by applying the Clough–Tocher split to each triangle of $\triangle$ and then applying the same split again to each resulting triangle. We call $\triangle_{DCT}$ the double Clough–Tocher refinement of $\triangle$. Figure 7.7 shows the double Clough–Tocher split of a single triangle.

Given a triangle $T := \langle v_1, v_2, v_3 \rangle$ of $\triangle$, let v_T be its barycenter, and for each $i = 1, 2, 3$, let v_i^T be the barycenter of the triangle $T_i := \langle v_T, v_i, v_{i+1} \rangle$, where v_4 is identified with v_1. Let $e_i^T := \langle v_i, v_T \rangle$ for $i = 1, 2, 3$. In this section we discuss the superspline space

$$\begin{aligned} \mathcal{S}_2(\triangle_{DCT}) := \{ s \in \mathcal{S}_5^2(\triangle_{DCT}) : \text{ for all } T \in \triangle,\ s \in C^3(v_T) \text{ and} \\ s \in C^4(v_i^T) \cap C^3(e_i^T) \text{ for } i = 1, 2, 3 \}. \end{aligned}$$

Let V and E be the numbers of vertices and edges in $\triangle$, respectively. For each vertex v of $\triangle$, let T_v be some triangle attached to v, and let $\mathcal{M}_v := D_2(v) \cap T_v$. For each edge e of $\triangle$, let T_e be some triangle in $\triangle_{DCT}$ containing the edge e, and let $\mathcal{M}_e := \{\xi_{122}^{T_e}, \xi_{212}^{T_e}, \xi_{221}^{T_e}\}$.

Theorem 7.17. *dim $\mathcal{S}_2(\triangle_{DCT}) = 6V + 3E$, and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

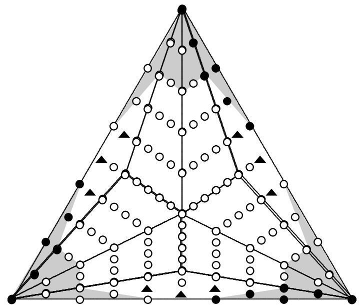


Fig. 7.7. A minimal determining set for $\mathcal{S}_2(T_{DCT})$.

Proof: Let $s \in \mathcal{S}_2(\triangle_{DCT})$, and suppose we have fixed its coefficients corresponding to the sets $\mathcal{M}_v$ for all vertices $v \in \mathcal{V}$. Then by Lemma 5.10 all coefficients associated with domain points in the disks $D_2(v)$ are consistently determined. Next, for each edge $e := \langle u, v\rangle$ of $\triangle$, we set the coefficients of s corresponding to $\mathcal{M}_e$. Then using the C^2 smoothness across interior edges, we see that all coefficients of s are determined for domain points in the sets $E_2(e) := \{\xi : \operatorname{dist}(\xi, e) \leq 2,\ \xi \notin D_2(u) \cup D_2(v)\}$. Now let T be a macro-triangle, and let T_{DCT} be its corresponding double Clough–Tocher split. Then using the Java program described in Remark 5.6, it can be checked that the remaining coefficients of s corresponding to domain points lying in T are consistently determined. The program gives explicit formulae for computing such coefficients as linear combinations of the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$. The weights in these linear combinations are fixed numbers, independent of the size and shape of T. It follows from Theorem 5.15 that $\mathcal{M}$ is an MDS, and from Theorem 5.13 that the dimension of $\mathcal{S}_2(\triangle_{DCT})$ is $6V + 3E$.

To see that $\mathcal{M}$ is local, suppose $\eta \notin \mathcal{M}$ lies in the triangle T_η. If $\eta \in D_2(v)$, then the set Γ_η of Definition 5.16 is just $\mathcal{M}_v \subset \operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. If $\eta \in E_2(e)$ for some edge $e := \langle u, v\rangle$, then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \operatorname{star}(T_\eta)$. For any other η lying in a macro-triangle $T_\eta := \langle v_1, v_2, v_3\rangle$, Γ_η is the union of the sets $\mathcal{M}_{v_i}$ and $\mathcal{M}_{e_i}$ for $i = 1, 2, 3$, which is contained in $\operatorname{star}(T_\eta)$. The stability of the computation of coefficients in the disks $D_2(v)$ follows from Lemma 5.10. The stability of the computation of other coefficients associated with domain points in a triangle T follows from the fact that they involve fixed formulae which do not vary with the size or shape of the triangle. □

The constant in the stability of $\mathcal{M}$ in Theorem 7.17 depends on the smallest angle in the triangulation $\triangle_{DCT}$. By Lemma 4.17 this angle is bounded below by a constant times the smallest angle in $\triangle$. Figure 7.7 shows a minimal determining set for the space $\mathcal{S}(T_{DCT})$ corresponding to the split T_{DCT} of a single macro-triangle. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles. Since $\mathcal{S}_2(\triangle_{DCT})$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

Theorem 7.18. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 5$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_{DCT})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_2(\triangle_{DCT})$. For each edge e of $\triangle$, we let D_{u_e} be the cross-boundary derivative associated with the unit vector u_e corresponding to rotating e ninety degrees in a clockwise direction. In addition, if $e := \langle u, v\rangle$, we write $\eta_e := (u+v)/2$, $\eta_{e,1} := (2u+v)/3$, and $\eta_{e,2} := (u+2v)/3$. Let ε_t be the point evaluation functional at t.

Theorem 7.19. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_{DCT})$, where

1) *$\mathcal{N}_v := \{\varepsilon_v D_x^\nu D_y^\mu\}_{0\le\nu+\mu\le 2}$, for all vertices $v \in \mathcal{V}$,*
2) *$\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}, \varepsilon_{\eta_{1,e}} D_{u_e}^2, \varepsilon_{\eta_{2,e}} D_{u_e}^2\}$, for all edges $e \in \mathcal{E}$.*

Proof: It is easy to check that all of the coefficients of s corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 7.17 can be computed from the values $\{\lambda s\}_{\lambda\in\mathcal{N}}$. Since the cardinality of $\mathcal{N}$ is equal to the dimension of $\mathcal{S}_2(\triangle_{DCT})$, it follows that $\mathcal{N}$ is a nodal MDS. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 2$. $\square$

Since for each triangle T in $\triangle$, $s|_T$ can be computed locally, one triangle at a time, it follows that $\mathcal{S}_2(\triangle_{DCT})$ is a macro-element space in the sense of Definition 5.27. Theorem 7.19 shows that for any function $f \in C^2(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_{DCT})$ solving the Hermite interpolation problem

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 2,\\ D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E},\\ D_{u_e}^2 s(\eta_{\nu,e}) &= D_{u_e}^2 f(\eta_{\nu,e}), && \text{all } e \in \mathcal{E} \text{ and } \nu = 1,2. \end{aligned}$$

This defines a linear projector $\mathcal{I}_{DCT}^2$ mapping $C^2(\Omega)$ onto $\mathcal{S}_2(\triangle_{DCT})$. Since this interpolation operator reproduces polynomials of degree five, applying Theorem 5.26, we get the following error bound.

Theorem 7.20. *For every $f \in C^{m+1}(\Omega)$ with $1 \le m \le 5$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}^2_{DCT} f)\|_\Omega \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,\Omega},$$

for all $0 \le \alpha+\beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS described in Theorem 7.17. Then by Theorem 5.21, the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}_2(\triangle_{DCT})$. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\operatorname{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the triangles sharing e.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.19 provides a different stable local basis for $\mathcal{S}_2(\triangle_{DCT})$.

7.6. A C^2 Quadrilateral Macro-element Space

Let $\diamondsuit$ be a strictly convex quadrangulation of a domain Ω, and let $\diamondplus$ be the triangulation obtained from $\diamondsuit$ be drawing in the diagonals of each quadrilateral. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\diamondsuit$, respectively. Let $\mathcal{V}_c$ be the set of points where the inserted diagonals intersect, and for each quadrilateral, let $\mathcal{E}_Q$ be the set of four edges inserted in Q. Finally, for each Q, let e_Q be one of the edges in $\mathcal{E}_Q$, and let τ^4_{5,e_Q} and τ^5_{5,e_Q} be the special smoothness conditions defined in (5.7). In this section we discuss the superspline space

$$\begin{aligned}\mathcal{S}_2(\diamondplus) := \{s \in \mathcal{S}^2_7(\diamondplus) :\ & s \in C^3(v),\ \text{all } v \in \mathcal{V},\\ & s \in C^4(v),\ \text{all } v \in \mathcal{V}_c,\\ & s \in C^3(e),\ \text{all } e \in \mathcal{E}_Q,\\ & \tau^4_{5,e_Q} s = \tau^5_{5,e_Q} s = 0,\ \text{all } Q \in \diamondsuit\}.\end{aligned}$$

Note that although we are defining a C^2 macro-element space, we are requiring splines in $\mathcal{S}_2(\diamondplus)$ to be C^3 on each quadrilateral.

Let V and E be the number of vertices and edges of $\diamondsuit$, respectively. For each $v \in \mathcal{V}$, let T_v be the triangle in $\diamondplus$ with vertex at v which has the largest shape parameter (see Definition 4.1) among all triangles sharing the vertex v. Let $\mathcal{M}_v := D_3(v) \cap T_v$. For each edge e of $\triangle$, let T_e be some triangle in $\diamondplus$ containing that edge, and let $\mathcal{M}_e := \{\xi^{T_e}_{133}, \xi^{T_e}_{223}, \xi^{T_e}_{232}\}$.

Theorem 7.21. *dim* $\mathcal{S}_2(\diamondplus) = 10V + 3E$, *and*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: We first consider the case where $\diamondsuit$ consists of a single quadrilateral $Q := \langle v_1, v_2, v_3, v_4 \rangle$. Let $\triangle_Q$ be the triangulation of Q obtained by inserting the diagonals, and let $v_{_Q}$ be the center point, see Figure 7.8. We may suppose $e_{_Q} = \langle v_1, v_{_Q} \rangle$. We first show that $\mathcal{M}$ is a determining set for $\mathcal{S}_2(\triangle_Q)$. For each vertex $v \in \mathcal{V}$, setting the coefficients of $s \in \mathcal{S}_2(\triangle_Q)$ corresponding to the domain points in $\mathcal{M}_v$ determines all coefficients corresponding to $D_3(v)$, see Lemma 5.10. We now show how to use the smoothness conditions to compute the remaining coefficients. First we use Lemma 2.30 to compute the unset coefficients corresponding to domain points on the rings $R_4(v_i)$ for $i = 1, 2, 3, 4$. Then making use of the two special smoothness conditions across $e_{_Q}$, we use Lemma 2.30 to compute the remaining five unset coefficients of s corresponding to domain points on $R_5(v_1)$. Next we use the C^4 smoothness at $v_{_Q}$ and Lemma 2.30 to compute the four remaining unset coefficients on $R_7(v_2)$, followed by the four remaining unset coefficients on $R_7(v_1)$. The rest of the coefficients can then be computed using the lemma four more times.

We have shown that $\mathcal{M}$ is a determining set for $\mathcal{S}_2(\triangle_Q)$. To see that it is minimal, we first observe that the space $\mathcal{S}_2(\triangle_Q)$ is the subspace of $\mathcal{S}_7^3(\triangle_Q) \cap C^4(v_{_Q})$ that satisfies the two additional smoothness conditions described by the τ's. By Theorem 9.7, dim $\mathcal{S}_7^3(\triangle_Q) \cap C^4(v_{_Q}) = 54$, and it follows that

$$52 = \dim \mathcal{S}_7^3(\triangle_Q) \cap C^4(v_{_Q}) - 2 \leq \dim \mathcal{S}_2(\triangle_Q) \leq \#\mathcal{M} = 52.$$

But then dim $\mathcal{S}_2(\triangle_Q) = \#\mathcal{M} = 52$, and Theorem 5.13 implies that $\mathcal{M}$ is an MDS for $\mathcal{S}_2(\triangle_Q)$.

To get the result for arbitrary triangulations, we show that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_2(\diamondplus)$. For each vertex $v \in \mathcal{V}$, we can fix the coefficients corresponding to $\mathcal{M}_v$, and by Lemma 5.10 it follows that the coefficients corresponding to domain points in $D_3(v)$ are consistently determined. For each edge e of $\diamondsuit$, we now choose the coefficients corresponding to $\mathcal{M}_e$. If $e := \langle u, v \rangle$ is an interior edge of a quadrilateral in $\diamondsuit$, using the C^2 smoothness across e, it is easy to see that choosing the coefficients corresponding to $\mathcal{M}_e$ consistently determines all coefficients in $E_2(e) := \{\xi : \text{dist}(\xi, e) \leq 2,\ \xi \notin D_3(u) \cup D_3(v)\}$. Coefficients in two different sets $E_2(e)$ and $E_2(\tilde{e})$ may be connected by smoothness conditions in the interior of a quadrilateral $Q := \langle v_1, v_2, v_3, v_4 \rangle$. But we have shown above that coefficients corresponding to domain points inside Q are consistently determined from those corresponding to $D_3(v_i)$ and $E_2(e_i)$ for $i = 1, 2, 3, 4$, where e_i are the edges of Q. We have shown that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_2(\diamondplus)$, and by Theorem 5.15 it follows that $\mathcal{M}$ is minimal. Theorem 5.13 shows that the dimension of $\mathcal{S}_2(\diamondplus)$ is equal to the cardinality of $\mathcal{M}$, which is $10V + 3E$.

We now check that $\mathcal{M}$ is local. Suppose $\eta \notin \mathcal{M}$ lies in the triangle T_η. If $\eta \in D_3(v)$ for some vertex of $\Diamond$, then clearly the set Γ_v of Definition 5.16 is just $\mathcal{M}_v \subset \mathrm{star}(v) \subset \mathrm{star}(T_\eta)$. If $\eta \in E_2(e)$ for some edge $e := \langle u, v \rangle$ of $\Diamond$, then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \mathrm{star}(T_\eta)$. If η is a remaining point in some quadrilateral Q, then Γ_η is the union of the sets $\mathcal{M}_{v_i}$ and $\mathcal{M}_{e_i}$ for $i = 1, 2, 3, 4$, which is a subset of $\mathrm{star}(Q)$. To check the stability of $\mathcal{M}$, we first note that for each vertex $v \in \mathcal{V}$, the computation of the coefficients associated with domain points in $D_3(v)$ is stable due to our choice of the triangle T_v in which to choose the points of $\mathcal{M}_v$. The stability of the remaining computations follows from the fact that they are based on Lemma 2.30. $\square$

The choice of the sets $\mathcal{M}_v$ in the MDS of Theorem 7.21 ensures that the constant of stability for $\mathcal{M}$ does not depend on the smallest angle in $\diamondplus$ but only on the smallest angle in $\Diamond$. This is important since as we saw in Example 4.48, angles in $\diamondplus$ can be small even when the angles in $\Diamond$ are not.

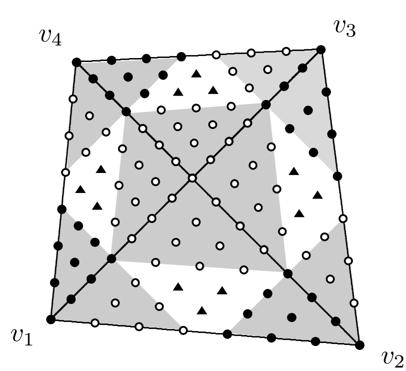


Fig. 7.8. A minimal determining set for $\mathcal{S}_2(\triangle_Q)$.

Figure 7.8 shows the MDS for the space $\mathcal{S}_2(\triangle_Q)$. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles. Since $\mathcal{S}_2(\diamondplus)$ has a stable local MDS, we can now apply Theorem 5.19 to show that it has full approximation power.

Theorem 7.22. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 7$, there exists a spline $s_f \in \mathcal{S}_2(\diamondplus)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K \,|\diamondplus|^{m+1-\alpha-\beta}\, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\Diamond$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_2(\diamondplus)$. For each edge $e := \langle u, v\rangle$ of $\triangle$, let $\eta_e := (u+v)/2$, $\eta_{1,e} := (2u+v)/2$, $\eta_{2,e} := (u+2v)/2$ and let D_{u_e} be the cross-boundary derivative associated with a unit vector u_e corresponding to rotating e ninety degrees in the counterclockwise direction. Let ε_t denote point evaluation at t.

Theorem 7.23. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\diamondplus)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\nu D_y^\mu)\}_{0 \le \nu+\mu \le 3}$ *for all vertices $v \in \mathcal{V}$,*
2) $\mathcal{N}_e := \{\varepsilon_{\eta_e} D_{u_e}, \varepsilon_{\eta_{1,e}} D_{u_e}^2, \varepsilon_{\eta_{2,e}} D_{u_e}^2\}$ *for all edges e of $\Diamond$.*

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is $10V + 3E$. Since we already know from Theorem 7.21 that the dimension of $\mathcal{S}_2(\diamondplus)$ is $10V+3E$, to show that $\mathcal{N}$ is a nodal minimal determining set, it suffices to show that if $s \in \mathcal{S}_2(\diamondplus)$, then the data in $\mathcal{N}$ determine all coefficients of s corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 7.21. This follows by the same arguments as in the proof of Theorem 7.11. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} = 3$. $\square$

Note that for each quadrilateral Q in $\Diamond$, the linear functionals involving evaluation at points in Q uniquely determine $s|_Q$, i.e., the coefficients of s can be computed locally, one quadrilateral at a time. Thus, $\mathcal{S}_2(\diamondplus)$ is a macro-element space. Theorem 7.21 shows that for any function $f \in C^3(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\diamondplus)$ that solves the Hermite interpolation problem

$$\begin{aligned}
D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } v \in \mathcal{V} \text{ and } 0 \le \alpha+\beta \le 3,\\
D_{u_e} s(\eta_e) &= D_{u_e} f(\eta_e), && \text{all } e \in \mathcal{E},\\
D_{u_e}^2 s(\eta_{\nu,e}) &= D_{u_e}^2 f(\eta_{\nu,e}), && \text{all } e \in \mathcal{E} \text{ and } \nu = 1,2.
\end{aligned}$$

This defines a linear projector $\mathcal{I}_Q^2$ mapping $C^3(\Omega)$ onto the spline space $\mathcal{S}_2(\diamondplus)$, and in particular $\mathcal{I}_Q^2$ reproduces polynomials of degree seven. We now apply Theorem 5.26 to get the following error bound for this interpolation operator.

Theorem 7.24. *For every $f \in C^{m+1}(\Omega)$ with $2 \le m \le 7$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_Q^2 f)\|_\Omega \le K\, |\diamondplus|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on the smallest angle in $\Diamond$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the stable local minimal determining set for $\mathcal{S}_2(\diamondplus)$ described in Theorem 7.21. By Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_2(\diamondplus)$. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\Diamond$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\Diamond$, then the support of ψ_ξ is contained in the union of the triangles sharing the edge e.

The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 7.23 provides a different stable local basis for $\mathcal{S}_2(\diamondplus)$.

7.7. Comparison of C^2 Macro-element Spaces

As a guide to comparing the various C^2 macro-element spaces discussed above, in Table 7.1 we list the following information:

- $d :=$ the degree of the spline,
- $n_{tri} :=$ the number of subtriangles involved in the split of a single macro-triangle,
- $n_{der} :=$ the maximum derivative needed to compute the interpolant,
- $n_{dim} :=$ the dimension of the spline space,
- $n_{coef} :=$ the dimension of the space of continuous splines on the same triangulation.

Space	d	n_{tri}	n_{der}	n_{dim}	n_{coef}
$\mathcal{S}_9^{2,4}(\triangle)$	9	1	4	$15V + 3E + N$	$81V_I + 45V_B - 80$
$\mathcal{S}_2(\triangle_{CT})$	7	3	3	$10V + 3E$	$147V_I + 77V_B - 146$
$\mathcal{S}_2(\triangle_{PS})$	5	6	3	$10V$	$150V_I + 80V_B - 149$
$\mathcal{S}_2(\triangle_{DCT})$	5	9	2	$6V + 3E$	$225V_I + 115V_B - 224$
$\mathcal{S}_2(\triangle_W)$	5	7	2	$6V + 3E$	$175V_I + 90V_B - 174$
$\mathcal{S}_2(\diamondplus)$	7	4	3	$10V^Q + 3E^Q$	$98V_I^Q + 105V_B^Q/2 - 97$

Tab. 7.1. A comparison of C^2 macro-elements.

These quantities are of interest for a variety of reasons:

1) In general we would like to work with low degree splines. They involve fewer coefficients, and have less tendency to oscillate.
2) For the purposes of evaluation, it is more efficient to work with macro-elements with fewer subtriangles as less effort is required to locate the triangle which contains a given point v of interest.
3) Since in practice we often have to estimate derivatives, methods which require fewer derivatives may have some advantages. In addition, the error bounds are valid for less smooth functions.
4) The dimension of the spline space measures its complexity.
5) The quantity n_{coef} measures the complexity of storing and working with a spline considering it as a C^0 spline.

For the triangle-based elements, $V = V_I + V_B$ and $E = E_I + E_B$ denote the number of vertices and edges in the original triangulation $\triangle$, before any splitting. For the quadrangulation-based element, $V^Q = V_I^Q + V_B^Q$ and $E^Q = E_I^Q + E_B^Q$ denote the number of vertices and edges in the quadrangulation. The table shows that the spaces $\mathcal{S}_2(\triangle_W)$ and $\mathcal{S}_2(\triangle_{DCT})$ have the advantage that their nodal minimal determining sets include derivatives up to order 2 at the vertices rather than the derivatives up to order 3 or 4 needed for the other spaces. The space $\mathcal{S}_2(\triangle_W)$ involves fewer subtriangles and fewer coefficients than $\mathcal{S}_2(\triangle_{DCT})$.

Comparing the complexity of the triangle-based methods with the quadrilateral-based method is complicated by the fact that if we have a triangulation and quadrangulation based on the same set of vertices, then $V = V^Q$ but E^Q is generally smaller than E. For example, if we start with n^2 vertices on a rectangular grid, then for the associated type-I triangulation we have $E = (3n-1)(n-1)$ while $E_Q = 2n(n-1)$. This gives $\dim \mathcal{S}_2(\diamondplus) = 16n^2 - 6n$ while $\dim \mathcal{S}_2(\triangle_W) = 15n^2 - 12n + 3$.

7.8. Remarks

Remark 7.1. It follows from Theorem 5.28 that it is impossible to construct C^2 macro-elements based the Clough–Tocher split using splines of degree lower than seven, cf. [LaiS01]. Similarly, it is impossible to construct C^2 macro-elements based the Powell–Sabin split using splines of degree lower than five, c.f. [LaiS03]. Thus, the elements presented in Sections 7.2 and 7.3 are optimal in this sense. We have not described the macro-elements in [LaiS01, LaiS03] in this chapter, but they are contained in the general C^r families of elements discussed in Sections 8.2 and 8.4.

Remark 7.2. The C^2 macro-elements of Sections 7.2 and 7.3 based on the Clough–Tocher and Powell–Sabin splits were introduced in [AlfS02a] and [AlfS02b], where it was shown that these elements are optimal in the sense that they have the least number of degrees of freedom among all possible C^2 macro-elements based on these splits.

Remark 7.3. For a C^2 macro-element based on the Powell–Sabin-12 split and using splines of degree seven, see [SchS06]. Since the C^2 macro-element of Section 7.3 defined on the usual Powell–Sabin-6 split is also of degree seven, the two elements can be used together. For a recent wavelet application where this is done, see [JiaLiu06].

Remark 7.4. While it is impossible to create a C^2 macro-element space of degree six based on triangulated quadrilaterals, it is possible to use C^2 sixth degree splines for interpolation and approximation purposes as shown in [LaiS97], see also [Gao93].

Remark 7.5. It is possible to give a rigorous mathematical proof of Theorem 7.13 using standard Bernstein–Bézier arguments, i.e., without depending on Alfeld's Java program, see [Wang92]. However, we have not seen such a proof for Theorem 7.17.

Remark 7.6. As pointed out in Remark 5.8, nested sequences of spline spaces are important for applications. All of the macro-element spaces discussed in this chapter are superspline spaces, and thus none of them is suitable for building nested sequences of spline spaces.

7.9. Historical Notes

The C^2 polynomial element discussed in Section 7.1 was introduced in [Zen74] in nodal form. The method was later treated in B-form in [Whe86], where explicit formulae for the B-coefficients in terms of the nodal data can be found.

The first C^2 element to be based on a split triangle uses the double Clough–Tocher split, see [Alf84a]. This element is based on splines of degree five, and is a condensed version of the macro-element presented in Section 7.5. The noncondensed version was introduced by Alfeld in [Alf00] as a result of experimentation with his Java program (see Remark 5.6). Our mathematical treatment of this element and its approximation power are new.

Macro-elements based on the Clough–Tocher and Powell–Sabin splits of a triangle were developed by several authors [Sabl85b, Sabl87, LagS89a, LagS89b, LagS93, LagS94, Lai96b, LaiS01, AlfS02a, AlfS02b, LaiS03]. The discussion in Section 7.2 of the C^2 Clough–Tocher macro-element follows [AlfS02a]. Our treatment of the C^2 Powell–Sabin macro-element in Section 7.3 is based on [AlfS02b]. The macro-element space described in Section 7.4 was introduced in [Wang92].

The C^2 macro-element space in Section 7.6 based on a triangulated quadrangulation is taken from [LaiS02]. For earlier work on macro-elements based on triangulated quadrangulations, see [LagS89a, LagS95].

C^{r} Macro-element Spaces

In this chapter we discuss families of C^r macro-element spaces for all $r \geq 1$. The spaces in Chapters 6 and 7 are included as special cases for $r = 1, 2$. For each space, we give a stable local minimal determining set and a stable local nodal minimal determining set, and show that the space has full approximation power.

8.1. Polynomial Macro-element Spaces

Suppose $\triangle$ is a triangulation of a domain Ω, and let $\mathcal{V}$ be its set of vertices. In this section we discuss properties of the superspline space

$$\mathcal{S}^{r,2r}_{4r+1}(\triangle) := \{s \in \mathcal{S}^r_{4r+1}(\triangle) : \, s \in C^{2r}(v), \text{ all } v \in \mathcal{V}\}.$$

Let $\mathcal{E}$ be the set of edges of $\triangle$, and suppose V, E, and N are the numbers of vertices, edges, and triangles of $\triangle$, respectively.

For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v, and let $\mathcal{M}_v := D_{2r}(v) \cap T_v$. For each edge $e := \langle v_2, v_3 \rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3 \rangle$ be some triangle containing e, and let $\mathcal{M}_e := \{\xi^{T_e}_{ijk} : 1 \leq i \leq r \text{ and } j,k \leq 2r\}$. Finally, for each triangle $T \in \triangle$, let $\mathcal{M}_T := \{\xi^T_{ijk} : \, i,j,k > r\}$.

Theorem 8.1. *For all $r \geq 1$,*

$$\mathit{dim}\; \mathcal{S}^{r,2r}_{4r+1}(\triangle) = \binom{2r+2}{2} V + \binom{r+1}{2} E + \binom{r}{2} N,$$

and

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e \cup \bigcup_{T \in \triangle} \mathcal{M}_T$$

is a stable local minimal determining set.

Proof: We use Theorem 5.15 to show that $\mathcal{M}$ is an MDS for $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$. We need to show that $\mathcal{M}$ is a consistent determining set. First, for each $v \in \mathcal{V}$, suppose we fix the coefficients of s corresponding to the domain points in $\mathcal{M}_v$. Then in view of the C^{2r} supersmoothness at v, all domain points in $D_{2r}(v)$ are consistently determined by Lemma 5.10. Since the disks $D_{2r}(v)$ do not overlap and none of the other coefficients have yet been assigned, so far we have not violated any smoothness condition. Now for each edge $e := \langle u, v \rangle$ in $\mathcal{E}$, we fix the coefficients of s corresponding to domain points in $\mathcal{M}_e$. This determines all coefficients of s corresponding to domain points in the set $E_r(e) := \{\xi : \operatorname{dist}(\xi, e) \le r,\ \xi \notin D_{2r}(u) \cup D_{2r}(v)\}$, where we use the C^r smoothness across e if e is an interior edge. Since coefficients corresponding to two different sets $E_r(e)$ and $E_r(\tilde{e})$ are not connected by smoothness conditions, no inconsistencies can be introduced in this step. Finally, for each triangle T, we can set the coefficients of s corresponding to $\mathcal{M}_T$ to arbitrary values since these coefficients do not enter into any smoothness conditions. We have shown that $\mathcal{M}$ is a consistent determining set, and thus by Theorem 5.15, it is a minimal determining set. By Theorem 5.13, the dimension of $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$ is equal to the cardinality of $\mathcal{M}$. This is easily seen to be given by the formula in the statement of the theorem.

We now show that $\mathcal{M}$ is local in the sense of Definition 5.16. Suppose $\eta \notin \mathcal{M}$ lies in the triangle T_η. If $\eta \in D_{2r}(v)$ for some vertex v, then clearly the set Γ_η in the definition is just $\mathcal{M}_v \subset \operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. If $\eta \in E_r(e)$ for some edge $e := \langle u, v\rangle$, then $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \operatorname{star}(T_\eta)$. The stability of $\mathcal{M}$ follows from the fact that all unset coefficients of s can be computed directly from smoothness conditions, see Lemma 2.29. $\square$

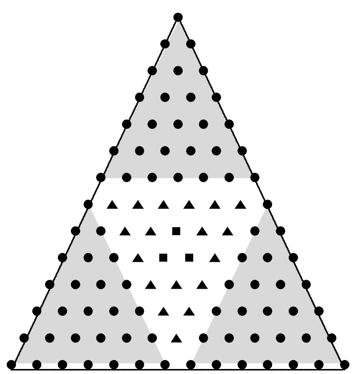


Fig. 8.1. A minimal determining set for $\mathcal{S}^{3,6}_{13}(T)$.

The family of spaces $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$ includes the spaces $\mathcal{S}^{1,2}_{5}(\triangle)$ and $\mathcal{S}^{2,4}_{9}(\triangle)$ discussed in Sections 6.1 and 7.1. Figure 8.1 shows $\mathcal{M}$ for the space $\mathcal{S}^{3,6}_{13}(T)$ defined on a single triangle T. Points in the sets $\mathcal{M}_v, \mathcal{M}_e, \mathcal{M}_T$ are shown with dots, triangles, and squares, respectively. Table 8.1 lists the dimensions of $\mathcal{S}^{r,2r}_{4r+1}(T)$ for $r = 1, \ldots, 6$.

Since the set $\mathcal{M}$ in Theorem 8.1 is a stable local MDS, Theorem 5.19 implies the following result which shows that $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$ has full approximation power.

r	$2r$	$4r+1$	dim
1	2	5	21
2	4	9	55
3	6	13	105
4	8	17	171
5	10	21	253
6	12	25	351

Tab. 8.1. Dimensions of the spaces $\mathcal{S}^{r,2r}_{4r+1}(T)$.

Theorem 8.2. *For every $f \in W^{m+1}_q(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 4r+1$, there exists a spline $s_f \in \mathcal{S}^{r,2r}_{4r+1}(\triangle)$ such that*

$$\|D^\alpha_x D^\beta_y (f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$. We follow the notation introduced in Section 6.1. In particular, we assign an orientation to each edge $e := \langle u, v\rangle$ of $\triangle$, and let u_e be the unit vector corresponding to rotating e ninety degrees in the counterclockwise direction. We write D_{u_e} for the cross-boundary derivative associated with u_e. In addition, for each $1 \le i \le r$, let

$$\eta^i_{j,e} := \frac{(i+1-j)u + jv}{i+1}, \qquad j = 1, \ldots, i. \tag{8.1}$$

Finally, let ε_t be the point evaluation functional at the point t.

Theorem 8.3. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{T \in \triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha_x D^\beta_y\}_{0 \le \alpha+\beta \le 2r}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^r \{\varepsilon_{\eta^i_{j,e}} D^i_{u_e}\}_{j=1}^i$,
3) $\mathcal{N}_T := \{\varepsilon_{\xi^T_{ijk}}\}_{i,j,k>r}$.

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is equal to the dimension of $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$ as given in Theorem 8.1. Thus, it suffices to show that the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determine the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of $s \in \mathcal{S}_{4r+1}^{r,2r}(\triangle)$, where $\mathcal{M}$ is the MDS in Theorem 8.1. For each vertex v, we can use the formulae (2.37) to determine the coefficients c_ξ corresponding to $\xi \in \mathcal{M}_v$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$. For each edge e, we can use Lemma 2.21 to determine the coefficients c_ξ corresponding to $\xi \in \mathcal{M}_e$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_e}$. Finally, for each triangle $T \in \triangle$, we can use Lemma 2.25 to compute the coefficients corresponding to $\xi \in \mathcal{M}_T$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$. To establish that $\mathcal{N}$ is local and stable, we note that there exists a constant K_1 depending only on the smallest angle in $\triangle$ such that for all $\xi \in T$,

$$|c_\xi| \leq K_1 \sum_{\nu=0}^{2r} |T|^\nu \, |f|_{\nu,T}. \quad \square$$

Theorem 8.3 shows that $\mathcal{S}_{4r+1}^{r,2r}(\triangle)$ is a macro-element space in the sense of Definition 5.27. In particular, for each macro-triangle $T \in \triangle$, all coefficients of $s|_T$ can be computed from the values of s and its derivatives at points in T. The theorem also shows that for any function $f \in C^{2r}(\Omega)$, there is a unique spline $s \in \mathcal{S}_{4r+1}^{r,2r}(\triangle)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently,

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } 0 \leq \alpha+\beta \leq 2r \text{ and } v \in \mathcal{V}, \\ D_{u_e}^i s(\eta_{j,e}^i) &= D_{u_e}^i f(\eta_{j,e}^i), && \text{all } j = 1,\ldots,i, \ 1 \leq i \leq r, \text{ and } e \in \mathcal{E}, \\ s(\xi_{ijk}^T) &= \ \ f(\xi_{ijk}^T), && \text{all } i,j,k > r \text{ and } T \in \triangle. \end{aligned}$$

This defines a linear projector $\mathcal{I}_P^r$ mapping $C^{2r}(\Omega)$ onto $\mathcal{S}_{4r+1}^{r,2r}(\triangle)$, and in particular, $\mathcal{I}_P^r$ reproduces polynomials of degree $4r+1$. We can now apply Theorem 5.26 to establish the following error bound.

Theorem 8.4. *For every $f \in C^{m+1}(\Omega)$ with $2r-1 \leq m \leq 4r+1$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_P^r f)\|_\Omega \leq K \, |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega},$$

for all $0 \leq \alpha+\beta \leq m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Proof: Fix $T \in \triangle$. Since $\mathcal{N}$ is a stable local NMDS, it follows from Theorem 5.26 that

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_r f)\|_T \leq K_2 \, |T|^{m+1-\alpha-\beta} \, |f|_{m+1,T},$$

and taking the maximum over $T \in \triangle$ immediately gives the global result. $\square$

Suppose $\mathcal{M}$ is the MDS of Theorem 8.1. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v, \\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e, \\ T, & \text{if } \xi \in \mathcal{M}_T \text{ for some triangle } T. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge.

The $\mathcal{N}$-basis (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.3 provides a different stable local basis for $\mathcal{S}^{r,2r}_{4r+1}(\triangle)$.

8.2. Clough–Tocher Macro-element Spaces

Given a triangulation $\triangle$ of a domain Ω, let $\triangle_{CT}$ be the corresponding Clough–Tocher refinement of $\triangle$ as described in Definition 4.16. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$, respectively, and let $\mathcal{V}_c := \{v_T\}_{T\in\triangle}$ be the set of barycenters used to form $\triangle_{CT}$. In this section we shall work with the following C^r Clough–Tocher macro-element space:

$$\begin{aligned} \mathcal{S}_r(\triangle_{CT}) := \{s \in \mathcal{S}^r_d(\triangle_{CT}) :\ & s \in C^\rho(v), \ \text{all } v \in \mathcal{V}, \\ & s \in C^\mu(v), \ \text{all } v \in \mathcal{V}_c\}, \end{aligned}$$

where

$$(\rho,\mu,d) := \begin{cases} (3\ell, 5\ell+1, 6\ell+1), & \text{if } r = 2\ell, \\ (3\ell+1, 5\ell+2, 6\ell+3), & \text{if } r = 2\ell+1. \end{cases} \tag{8.2}$$

Let V, E, and N be the numbers of vertices, edges, and triangles of $\triangle$, respectively. For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v, and let $\mathcal{M}_v := D_\rho(v) \cap T_v$. For each edge $e := \langle v_2, v_3\rangle$ of $\triangle$, let $T_e := \langle v_1, v_2, v_3\rangle$ be some triangle containing that edge, and let $\mathcal{M}_e := \{\xi^{T_e}_{ijk} : 1 \le i \le r, \text{ and } j,k \le d-\rho-1\}$. Finally, for each macro-triangle T, let $\hat{T}$ be one of the subtriangles of T, and let $\mathcal{M}_T := D^{\hat{T}}_{r-2}(v_T)$.

Theorem 8.5. *For all $r \ge 1$,*

$$\dim \mathcal{S}_r(\triangle_{CT}) = \binom{\rho+2}{2} V + \binom{r+1}{2} E + \binom{r}{2} N,$$

and

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set.

Proof: We begin by considering the case where $\triangle$ consists of the Clough–Tocher split T_{CT} of a single triangle T. We first show that in this case $\mathcal{M}$ is a determining set for $\mathcal{S}_r(T_{CT})$. Let $s \in \mathcal{S}_r(T_{CT})$ and suppose we fix its coefficients corresponding to $\mathcal{M}$. Then for each vertex v of T, using the C^ρ supersmoothness at v, we can use Lemma 5.10 to compute all coefficients of s corresponding to domain points in the disk $D_\rho(v)$. Now for each edge $e := \langle u, v\rangle$ of T, by the definition of $\mathcal{M}_e$, we have fixed all remaining coefficients of s corresponding to domain points in the set $E_r(e) := \{\xi : \operatorname{dist}(\xi, e) \le r \text{ and } \xi \notin D_\rho(u) \cup D_\rho(v)\}$.

We now claim that all remaining coefficients of s are determined, i.e., those corresponding to domain points in the disk $D_\mu(v_T)$. We can regard these coefficients as the coefficients of a spline defined on the triangle $\widetilde{T} := \langle u_1, u_2, u_3\rangle$, where $u_i := (\mu v_i + (d-\mu)v_T)/d$, $i = 1, 2, 3$. By the C^μ supersmoothness at v_T, this spline can also be viewed as a polynomial p of degree μ which has been subjected to subdivision using the split point v_T. It suffices to show that $p \equiv 0$ under the assumption that all coefficients corresponding to $\mathcal{M}$ have been set to zero. In this case, the coefficients of p corresponding to domain points that lie within a distance ℓ of the edges of $\widetilde{T}$ are zero, and so p and its cross derivatives up to order ℓ vanish on each of the three edges of $\widetilde{T}$. By Bezout's theorem, it follows that $p = a_1^{\ell+1} a_2^{\ell+1} a_3^{\ell+1} q$, where a_i is a linear polynomial that vanishes on the i-th edge of $\widetilde{T}$, and q is a polynomial of degree $\mu - 3(\ell+1) = r-2$. Now setting the coefficients of s corresponding to $\mathcal{M}_T$ to zero implies that $D_x^\alpha D_y^\beta s(v_T) = D_x^\alpha D_y^\beta p(v_T) = 0$ for $0 \le \alpha+\beta \le r-2$. It follows that $q \equiv 0$, and thus also $p \equiv 0$.

We have shown that $\mathcal{M}$ is a determining set for $\mathcal{S}_r(T_{CT})$. We now show that this determining set is minimal. We distinguish the cases where r is even and odd. Suppose $r = 2\ell$. To show that $\mathcal{M}$ is minimal, we show that its cardinality is equal to the dimension of $\mathcal{S}_{2\ell}(T_{CT})$. It is easy to check that in this case

$$\#\mathcal{M} = \frac{43\ell^2 + 31\ell + 6}{2}.$$

Now consider the superspline space $\mathcal{S}_{6\ell+1}^{2\ell}(T_{CT}) \cap C^{5\ell+1}(v_T)$. By Theorem 9.7, the dimension of this space is $(46\ell^2 + 34\ell + 6)/2$. Our space $\mathcal{S}_{2\ell}(T_{CT})$ is the subspace which belongs to $C^{3\ell}(v_i)$ for $i = 1, 2, 3$. Enforcing this supersmoothness requires an additional $3(\ell^2+\ell)/2$ conditions, and thus

$$\frac{46\ell^2 + 34\ell + 6}{2} - \frac{3(\ell^2+\ell)}{2} \le \dim \mathcal{S}_{2\ell}(T_{CT}) \le \frac{43\ell^2 + 31\ell + 6}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_{2\ell}(T_{CT})$. It follows from Theorem 5.13 that $\mathcal{M}$ is an MDS for $\mathcal{S}_{2\ell}(T_{CT})$. The proof that $\mathcal{M}$ is an MDS for $\mathcal{S}_r(T_{CT})$

in the case $r = 2\ell + 1$ is very similar. In this case,

$$\#\mathcal{M} = \frac{43\ell^2 + 65\ell + 24}{2},$$

and $\mathcal{S}_{2\ell+1}(T_{CT})$ is the subspace of $\mathcal{S}^{2\ell+1}_{6\ell+3}(T_{CT}) \cap C^{5\ell+2}(v_T)$ which belongs to $C^{3\ell+1}(v_i)$ for $i = 1, 2, 3$. Enforcing this supersmoothness requires an additional $3(\ell^2 + \ell)/2$ conditions, and thus

$$\frac{46\ell^2 + 68\ell + 24}{2} - \frac{3(\ell^2 + \ell)}{2} \le \dim \mathcal{S}_{2\ell+1}(T_{CT}) \le \frac{43\ell^2 + 65\ell + 24}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_{2\ell+1}(T_{CT})$. This proves $\mathcal{M}$ is an MDS for $\mathcal{S}_{2\ell+1}(T_{CT})$.

We now return to the case where $\triangle_{CT}$ is the Clough–Tocher refinement of an arbitrary triangulation $\triangle$. Suppose we have assigned values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \mathcal{S}_r(\triangle_{CT})$. Then for each $v \in \mathcal{V}$, in view of the C^ρ supersmoothness at v, the coefficients of s corresponding to domain points in $D_\rho(v)$ are consistently determined by Lemma 5.10. Moreover, for each edge e, all of the coefficients associated with domain points in $E_r(e) := \{\xi : \operatorname{dist}(\xi, e) \le r,\ \xi \notin D_\rho(u) \cup D_\rho(v)\}$ are also determined. Coefficients in two different such sets $E_r(e)$ and $E_r(\tilde{e})$ may be connected by smoothness conditions through the interior of a macro-triangle, and so it is not obvious a priori that all smoothness conditions are satisfied. But as shown above, for any triangle $T := \langle v_1, v_2, v_3\rangle$ with edges e_1, e_2, e_3, we can choose the coefficients in the sets $D_\rho(v_i)$, $E_r(e_i)$, and $\mathcal{M}_T$ arbitrarily, and still satisfy all smoothness conditions inside T.

We have shown that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_r(\triangle_{CT})$, and it follows from Theorem 5.15 that it is an MDS. Theorem 5.13 then implies that the dimension of $\mathcal{S}_r(\triangle_{CT})$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be given by the formula in the statement of the theorem.

We now show that $\mathcal{M}$ is local in the sense of Definition 5.16. Suppose $\eta \notin \mathcal{M}$ lies in the triangle T_η. If $\eta \in D_\rho(v)$ for some vertex, then the set Γ_η in the definition is just $\mathcal{M}_v \subset \operatorname{star}(v) \subset \operatorname{star}(T_\eta)$. Now suppose $\eta \in E_r(e)$ for some edge $e := \langle u, v\rangle$. In this case $\Gamma_\eta = \mathcal{M}_u \cup \mathcal{M}_v \cup \mathcal{M}_e \subset \operatorname{star}(T_\eta)$. Finally, if η is one of the remaining domain points in a macro-triangle $T_\eta := \langle v_1, v_2, v_3\rangle$ with edges e_1, e_2, e_3, then $\Gamma_\eta = \mathcal{M}_{v_1} \cup \mathcal{M}_{v_2} \cup \mathcal{M}_{v_3} \cup \mathcal{M}_{e_1} \cup \mathcal{M}_{e_2} \cup \mathcal{M}_{e_3} \cup \mathcal{M}_T \subset \operatorname{star}(T_\eta)$. All of the computations mentioned above are stable with a constant depending on the smallest angle in $\triangle_{CT}$, which by Lemma 4.17 is bounded below by a constant times the smallest angle in $\triangle$. $\square$

The space $\mathcal{S}_1(\triangle_{CT})$ is exactly the same as the space $\mathcal{S}^1_3(\triangle_{CT})$ discussed in Section 6.2. However, the space $\mathcal{S}_2(\triangle_{CT})$ is not the same as the C^2

macro-element space discussed in Section 7.2. That space is an example of a different family of macro-element spaces to be discussed in Section 8.3.

For the Clough–Tocher split T_{CT} of a single triangle, Table 8.2 shows the values of r, ρ, μ, d and dim $\mathcal{S}_r(T_{CT})$ for $r \le 8$. Figure 8.2 shows the minimal determining sets for $r = 3, \ldots, 6$, where the points in $\mathcal{M}_v$, $\mathcal{M}_e$, and $\mathcal{M}_T$ are marked with dots, triangles, and diamonds, respectively. To show the supersmoothness, we have shaded the disks $D_\rho(v_i)$ and $D_\mu(v_T)$.

Since $\mathcal{S}_r(\triangle_{CT})$ has a stable local MDS, we can use Theorem 5.19 to prove that it has full approximation power.

Theorem 8.6. *Given $r \ge 0$, let d be as in (8.2). Then for every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le d$, there exists a spline $s_f \in \mathcal{S}_r(\triangle_{CT})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_r(\triangle_{CT})$. For each edge $e := \langle v, u \rangle$ of $\triangle$, let D_{u_e} be the cross-boundary derivative associated with the unit vector u_e obtained by rotating e ninety degrees in the counterclockwise direction. Let $\eta_{j,e}^i$ be points on e defined as in (8.1) for $i = 1, \ldots, r$. For each triangle T, let v_T be its barycenter, and for any point t, let ε_t be point evaluation at t.

Theorem 8.7. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{T \in \triangle} \mathcal{N}_T \tag{8.3}$$

is a stable local nodal minimal determining set for $\mathcal{S}_r(\triangle_{CT})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le \rho}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^r \{\varepsilon_{\eta_{j,e}^i} D_{u_e}^i\}_{j=1}^i$,
3) $\mathcal{N}_T := \{\varepsilon_{v_T} D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le r-2}$.

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is equal to the dimension of $\mathcal{S}_r(\triangle_{CT})$ as given in Theorem 8.5. Thus, it suffices to show that the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determine the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ for $s \in \mathcal{S}_r(\triangle_{CT})$, where $\mathcal{M}$ is the MDS in Theorem 8.5. First, for every vertex v of $\triangle$, we can use the formulae (2.37) to determine the coefficients of s corresponding to domain points in $\mathcal{M}_v$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$. Similarly, for all $e \in \mathcal{E}$, Lemma 2.21 can be used to compute all coefficients associated with $\mathcal{M}_e$ from the data $\{\lambda\}_{\lambda \in \mathcal{N}_e}$. Finally, for each triangle $T \in \triangle$, we combine

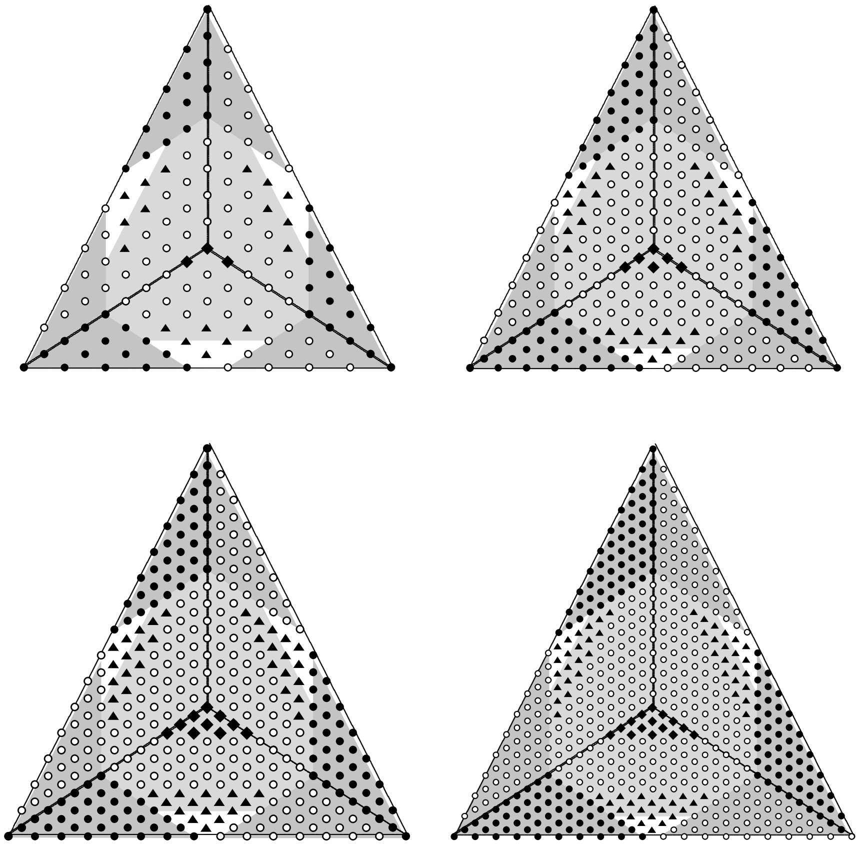


Fig. 8.2. Minimal determining sets for $\mathcal{S}_r(T_{CT})$ for $r = 3, 4, 5, 6$.

r	ρ	μ	d	dim
1	1	2	3	12
2	3	6	7	40
3	4	7	9	66
4	6	11	13	120
5	7	12	15	163
6	9	16	19	243
7	10	17	21	303
8	12	21	25	409

Tab. 8.2. Dimensions of the spaces $\mathcal{S}_r(T_{CT})$.

Lemma 2.27 with Lemma 2.25 to compute the coefficients corresponding to $\mathcal{M}_T$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$. By construction, it is easy to see that if $\xi \in \mathcal{M} \cap T$, then

$$|c_\xi| \le K_1 \sum_{\nu=0}^{\rho} |T|^\nu |f|_{\nu,T}.$$

This shows that $\mathcal{N}$ is local and stable. $\square$

Theorem 8.7 shows that $\mathcal{S}_r(\triangle_{CT})$ is a macro-element space in the sense of Definition 5.27. In particular, for each macro-triangle $T \in \triangle$, all coefficients of $s|_T$ can be computed from the values of s and its derivatives at points in T. The theorem also shows that for any function $f \in C^{2m}(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\triangle_{CT})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently,

$$\begin{aligned}
D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } 0 \le \alpha + \beta \le \rho \text{ and } v \in \mathcal{V},\\
D_{u_e}^i s(\eta_{j,e}^i) &= D_{u_e}^i f(\eta_{j,e}^i), && \text{all } j = 1, \dots, i, \ \ 1 \le i \le r, \text{ and } e \in \mathcal{E},\\
D_x^\alpha D_y^\beta s(v_T) &= D_x^\alpha D_y^\beta f(v_T), && \text{all } 0 \le \alpha + \beta \le r-2 \text{ and } T \in \triangle.
\end{aligned}$$

This defines a linear projector $\mathcal{I}_{CT}^r$ mapping $C^\rho(\Omega)$ onto $\mathcal{S}_r(\triangle_{CT})$. In particular, $\mathcal{I}_{CT}^r$ reproduces polynomials of degree d, where d is as in (8.2). We can now apply Theorem 5.26 to establish the following result whose proof is similar to the proof of Theorem 8.4.

Theorem 8.8. *For every $f \in C^{m+1}(\Omega)$ with $\rho - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_{CT}^r f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS of Theorem 8.5. Then by Theorem 5.21 the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_r(\triangle_{CT})$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v,\\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e,\\ T, & \text{if } \xi \in \mathcal{M}_T \text{ for some triangle } T. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.7 provides a different stable local basis for $\mathcal{S}_r(\triangle_{CT})$.

8.3. CT Spaces with Natural Degrees of Freedom

In this section we define a family of subspaces $\widetilde{\mathcal{S}}_r(\triangle_{CT})$ of the spaces $\mathcal{S}_r(\triangle_{CT})$ of the previous section which have fewer degrees of freedom, but the same approximation power. In addition, these new spaces have the advantage that they are parametrized by natural degrees of freedom, i.e., they have a nodal minimal determining set which involves evaluation only at points on the edges of $\triangle$.

For each $r \geq 1$, let ℓ, d, ρ, μ be as in the previous section. To define the spaces of interest in this section, we make use of the smoothness functionals introduced in Section 5.5.2. For each triangle $T := \langle v_1, v_2, v_3 \rangle$ in $\triangle$, let $\hat{e}_i := \langle v_i, v_T \rangle$ for $i = 1, 2, 3$, be the interior edges forming the Clough–Tocher split of T. Let

$$\mathcal{T}_{T,1} := \bigcup_{i=1}^{r-\ell-1} \{\tau_{\rho+i+1,\hat{e}_1}^{\rho-r+\ell+i+j+1}\}_{j=1}^{i} \cup \bigcup_{i=1}^{\ell} \{\tau_{d-2\ell+i-1,\hat{e}_1}^{\rho+i+j}\}_{j=1}^{\ell-i+1},$$

$$\mathcal{T}_{T,2} := \bigcup_{i=1}^{r-\ell-1} \{\tau_{\rho+i+1,\hat{e}_2}^{\rho-r+\ell+i+j+1}\}_{j=1}^{i} \cup \bigcup_{i=1}^{\ell-1} \{\tau_{d-2\ell+i-1,\hat{e}_2}^{\rho+i+j}\}_{j=1}^{\ell-i},$$

and let

$$\widetilde{\mathcal{S}}_r(\triangle_{CT}) := \{s \in \mathcal{S}_r(\triangle_{CT}) : \tau s = 0, \text{ all } \tau \in \mathcal{T}_T \text{ and all } T \in \triangle \}$$

with $\mathcal{T}_T := \mathcal{T}_{T,1} \cup \mathcal{T}_{T,2}$. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$, and let V and E be their cardinalities, respectively.

Theorem 8.9. *For all $r \geq 1$,*

$$\dim \widetilde{\mathcal{S}}_r(\triangle_{CT}) = \binom{\rho+2}{2} V + \binom{r+1}{2} E,$$

and

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set, where $\mathcal{M}_v$ and $\mathcal{M}_e$ are as in Theorem 8.5.

Proof: We begin by considering the case where $\triangle$ consists of the Clough–Tocher split T_{CT} of a single triangle $T := \langle v_1, v_2, v_3 \rangle$ into three subtriangles $T^{[i]} := \langle v_T, v_i, v_{i+1} \rangle$, $i = 1, 2, 3$. We first deal with the case where $r = 2\ell$, and begin by showing that $\mathcal{M}$ is a determining set for $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$. We will show later that $\mathcal{M}$ is minimal.

Suppose that we set the coefficients c_ξ of $s \in \widetilde{\mathcal{S}}_{2\ell}(T_{CT})$ for all $\xi \in \mathcal{M}$. Then by the C^ρ supersmoothness at the vertices of T, we can compute all coefficients of s corresponding to domain points in the disks $D_\rho(v_i)$, $i = 1, 2, 3$.

Next, we use Lemma 2.30 to solve for the undetermined coefficients corresponding to domain points on the rings $R_{3\ell+i}(v_1)$ and $R_{3\ell+i}(v_2)$ for $i = 1, \ldots, \ell$. On each ring this involves solving a nonsingular system of $2(\ell + i) - 1$ linear equations. Note that the spline satisfies all of the smoothness conditions required for the lemma, since either they are already implicit in the supersmoothness of the space, or have been explicitly enforced in the definition of $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$.

Continuing, we now compute undetermined coefficients on the ring $R_{4\ell+1}(v_1)$. This involves solving a $(4\ell + 1) \times (4\ell + 1)$ system. Then we do the ring $R_{4\ell+1}(v_2)$ which involves solving a $4\ell \times 4\ell$ system since $R_{4\ell+1}(v_1)$ and $R_{4\ell+1}(v_2)$ overlap in one point. We continue alternating between rings around v_1 and v_2. In particular, for each $i = 2, \ldots, \ell$ we do the ring $R_{4\ell+i}(v_1)$ followed by the ring $R_{4\ell+i}(v_2)$. The first of these involves solving a $(4\ell+1) \times (4\ell+1)$ system, and the second involves solving a $4\ell \times 4\ell$ system.

Next we successively compute undetermined coefficients on each of the rings $R_{3\ell+i}(v_3)$ for $i = 1, \ldots, 3\ell + 1$. Each of these involves solving a $(2\ell + 1) \times (2\ell + 1)$ system. Finally, the remaining coefficients in $T^{[1]}$ can be computed from the smoothness conditions across the edge $\hat{e}_1$. We have now computed all coefficients of s, and so $\mathcal{M}$ is a determining set for $\widetilde{\mathcal{S}}_r(T_{CT})$. To show that $\mathcal{M}$ is a minimal determining set for $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$, we show that its cardinality is equal to the dimension of $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$. It is easy to check that

$$\#\mathcal{M} = \frac{39\ell^2 + 33\ell + 6}{2}.$$

Now consider the superspline space $\mathcal{S}^{2\ell}_{6\ell+1}(T_{CT}) \cap C^{5\ell+1}(v_T)$. By Theorem 9.7, the dimension of this space is $(46\ell^2 + 34\ell + 6)/2$. Our space $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$ is the subspace which satisfies the $2\ell^2 - \ell$ special conditions and the supersmoothness $C^{3\ell}(v_i)$ for $i = 1, 2, 3$. Enforcing this supersmoothness at v_1, v_2, v_3 requires an additional $3(\ell^2 + \ell)/2$ conditions, and thus

$$\frac{46\ell^2 + 34\ell + 6}{2} - \frac{4\ell^2 - 2\ell}{2} - \frac{3(\ell^2 + \ell)}{2} \le \dim \widetilde{\mathcal{S}}_{2\ell}(T_{CT}) \le \frac{39\ell^2 + 33\ell + 6}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$. This proves $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_{2\ell}(T_{CT})$.

The proof that $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_r(T_{CT})$ in the case $r = 2\ell + 1$ is very similar. In this case,

$$\#\mathcal{M} = \frac{39\ell^2 + 63\ell + 24}{2},$$

and $\widetilde{\mathcal{S}}_{2\ell+1}(T_{CT})$ is the subspace of $\mathcal{S}^{2\ell+1}_{6\ell+3}(T_{CT}) \cap C^{5\ell+2}(v_T)$ which satisfies $2\ell^2+\ell$ special conditions along with the supersmoothness $C^{3\ell+1}(v_i)$ for $i=1,2,3$. Enforcing this supersmoothness requires an additional $3(\ell^2+\ell)/2$ conditions, and thus

$$\begin{aligned}\frac{46\ell^2+68\ell+24}{2} - \frac{4\ell^2+2\ell}{2} - \frac{3(\ell^2+\ell)}{2} &\le \dim \widetilde{\mathcal{S}}_{2\ell+1}(T_{CT}) \\ &\le \frac{39\ell^2+63\ell+24}{2}.\end{aligned}$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\widetilde{\mathcal{S}}_{2\ell+1}(T_{CT})$. This proves $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_{2\ell+1}(T_{CT})$.

We now return to the case where $\triangle_{CT}$ is the Clough–Tocher refinement of an arbitrary triangulation $\triangle$. Suppose we have assigned values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \widetilde{\mathcal{S}}_r(\triangle_{CT})$. Then for each $v \in \mathcal{V}$, in view of the C^ρ supersmoothness at v, the coefficients of s corresponding to domain points in $D_\rho(v)$ are consistently determined by Lemma 5.10. Moreover, for each edge $e := \langle u, v\rangle$, all of the coefficients associated with domain points in $E_r(e) := \{\xi : \operatorname{dist}(\xi, e) \le r \text{ and } \xi \notin D_\rho(u) \cup D_\rho(v)\}$ are also determined. Coefficients in two different such sets $E_r(e)$ and $E_r(\tilde{e})$ may be connected by smoothness conditions through the interior of a macro-triangle, and so it is not obvious a priori that all smoothness conditions are satisfied. But as shown above, for any triangle $T := \langle v_1, v_2, v_3\rangle$ with edges e_1, e_2, e_3, we can choose the coefficients in the sets $D_\rho(v_i)$ and $E_r(e_i)$ arbitrarily, and still satisfy all smoothness conditions inside T.

We have shown that $\mathcal{M}$ is a consistent determining set for $\widetilde{\mathcal{S}}_r(\triangle_{CT})$, and it follows from Theorem 5.15 that it is an MDS. Theorem 5.13 then implies that the dimension of $\widetilde{\mathcal{S}}_r(\triangle_{CT})$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be given by the formula stated in the theorem.

The proof that $\mathcal{M}$ is local and stable follows along the same lines as the proof of Theorem 8.5, except that here we also have to take account of the stability of the computations involving Theorem 2.30. $\square$

Figure 8.3 illustrates the proof of Theorem 8.9 for a single macro-triangle in the cases $r = 3, 4$. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles. The tips of the extra smoothness conditions are marked with brackets. The steps of the computation of coefficients associated with interior domain points are indicated by straight lines drawn through the points. Table 8.3 lists the dimensions of the spaces $\widetilde{\mathcal{S}}_r(T_{CT})$ for $1 \le r \le 8$.

Since $\widetilde{\mathcal{S}}_r(\triangle_{CT})$ has a stable local MDS, we can apply Theorem 5.19 to conclude that $\widetilde{\mathcal{S}}_r(\triangle_{CT})$ has full approximation power.

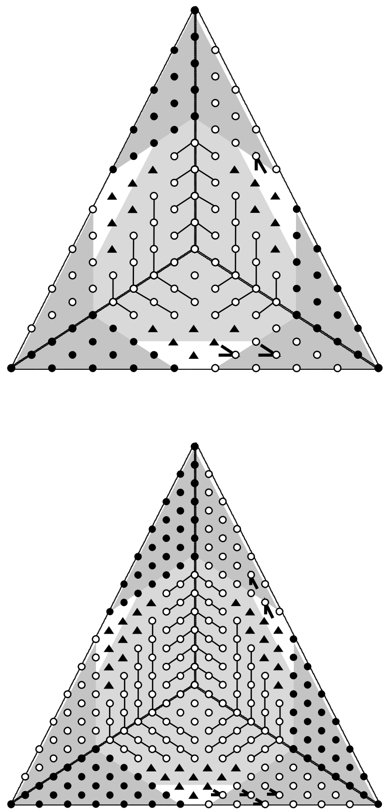


Fig. 8.3. Minimal determining sets for $\widetilde{\mathcal{S}}_r(T_{CT})$ for $r = 3, 4$.

r	ρ	μ	d	dim
1	1	2	3	12
2	3	6	7	39
3	4	7	9	63
4	6	11	13	114
5	7	12	15	153
6	9	16	19	228
7	10	17	21	283
8	12	21	25	381

Tab. 8.3. Dimensions of the spaces $\widetilde{\mathcal{S}}_r(T_{CT})$.

Theorem 8.10. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le d$, there exists a spline $s_f \in \widetilde{\mathcal{S}}_r(\triangle_{CT})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\,|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

The following theorem can be proved in the same way as Theorem 8.7.

Theorem 8.11. *Let*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e,$$

where $\mathcal{N}_v$ and $\mathcal{N}_e$ are as in Theorem 8.7. Then $\mathcal{N}$ is a stable local nodal minimal determining set for $\widetilde{\mathcal{S}}_r(\triangle_{CT})$.

Theorem 8.11 shows that $\widetilde{\mathcal{S}}_r(\triangle_{CT})$ is a macro-element space in the sense of Definition 5.27. In particular, for each macro-triangle $T \in \triangle$, all coefficients of $s|_T$ can be computed from the values of s and its derivatives at points in T. The theorem also shows that for any function $f \in C^\rho(\Omega)$, there is a unique spline $s \in \widetilde{\mathcal{S}}_r(\triangle_{CT})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently,

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } 0 \le \alpha+\beta \le \rho \text{ and } v \in \mathcal{V}, \\ D_{u_e}^i s(\eta_{j,e}^i) &= D_{u_e}^i f(\eta_{j,e}^i), && \text{all } j = 1,\ldots,i,\ \ 1 \le i \le r, \text{ and } e \in \mathcal{E}. \end{aligned}$$

This defines a linear projector $\widetilde{\mathcal{I}}_{CT}^r$ mapping $C^\rho(\Omega)$ onto $\widetilde{\mathcal{S}}_r(\triangle_{CT})$. In particular, $\widetilde{\mathcal{I}}_{CT}^r$ reproduces polynomials of degree d, where d is as in (8.2). We can now apply Theorem 5.26 to establish the following error bound.

Theorem 8.12. *For every function $f \in C^{m+1}(\Omega)$ with $\rho - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \widetilde{\mathcal{I}}_{CT}^r f)\|_\Omega \le K\,|\triangle|^{m+1-\alpha-\beta}|f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS of Theorem 8.9. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\widetilde{\mathcal{S}}_r(\triangle_{CT})$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v, \\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.11 provides a different stable local basis for $\widetilde{\mathcal{S}}_r(\triangle_{CT})$.

8.4. Powell–Sabin Macro-element Spaces

In this section we discuss a family of C^r splines which is based on Powell–Sabin refinements of triangulations. Let $\triangle$ be a triangulation of a domain Ω, and let $\triangle_{PS}$ be the corresponding Powell–Sabin refinement of $\triangle$ as described in Definition 4.18. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$. Let $\mathcal{V}_c := \{v_T\}_{T\in\triangle}$ be the set of incenters which are used to form the Powell–Sabin splits, and let $\mathcal{W}$ be the set of vertices introduced in the interiors of edges of $\triangle$. Let $\widetilde{\mathcal{E}}$ be the set of edges e of $\triangle_{PS}$ of the form $e := \langle w, v_T\rangle$, where $w \in \mathcal{W}$. Given $r > 0$, let d, ρ, μ be as in Table 8.4. Then we define the C^r Powell–Sabin macro-element space to be

$$\begin{aligned} \mathcal{S}_r(\triangle_{PS}) := \{s \in \mathcal{S}^r_d(\triangle_{PS}) :\ & s \in C^\rho(v) \text{ all } v \in \mathcal{V}, \\ & s \in C^\mu(v), \text{ all } v \in \mathcal{V}_c, \\ & s \in C^\mu(e), \text{ all } e \in \widetilde{\mathcal{E}}\}. \end{aligned} \tag{8.4}$$

r	d	ρ	μ	m	n	δ
$4\ell+1$	$9\ell+2$	$6\ell+1$	$6\ell+1$	2ℓ	ℓ	0
$4\ell+2$	$9\ell+5$	$6\ell+3$	$6\ell+3$	$2\ell+1$	ℓ	$3\ell+1$
$4\ell+3$	$9\ell+7$	$6\ell+4$	$6\ell+5$	$2\ell+1$	$\ell+1$	0
$4\ell+4$	$9\ell+10$	$6\ell+6$	$6\ell+7$	$2\ell+2$	$\ell+1$	$3\ell+3$

Tab. 8.4. Parameters for the C^r Powell–Sabin macro-element space.

Let V, E, and N be the numbers of vertices, edges, and triangles of $\triangle$, respectively. For each $v \in \mathcal{V}$, let T_v be some triangle with vertex at v. For each edge e of $\triangle$, let $T_e := \langle v_T, u, w_e\rangle$, where u is a vertex and v_T is the incenter of some triangle $T \in \triangle$ containing e. For each triangle $T \in \triangle$, let $\mathcal{E}_T$ be the set of edges of T. Let $r, d, \rho, \mu, m, n, \delta$ be as in Table 8.4.

Theorem 8.13. *For all* $r \ge 1$,

$$\dim \mathcal{S}_r(\triangle_{PS}) = \binom{\rho+2}{2}V + \binom{n+1}{2}E + \Big[3\binom{n}{2} + \delta\Big]N.$$

Moreover, the set

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{T\in\triangle} (\mathcal{M}_T^1 \cup \mathcal{M}_T^2 \cup \mathcal{M}_T^3)$$

is a stable local minimal determining set for $\mathcal{S}_r(\triangle_{PS})$, *where*

1) $\mathcal{M}_v := D_\rho(v) \cap T_v$,
2) $\mathcal{M}_e := \bigcup_{j=d-\mu+1}^{r} \{\xi^{T_e}_{j,k,d-j-k}\}_{k=0}^{j+\mu-d-1}$,
3) $\mathcal{M}_T^1 := \bigcup_{e\in\mathcal{E}_T} \bigcup_{j=r+1}^{d-2m-2} \{\xi^{T_e}_{j,k,d-j-k}\}_{k=0}^{d-2m-2-j}$,
4) $\mathcal{M}_T^2 := \bigcup_{i=1}^{3} \Big\{\dfrac{jv_T + (d-j)v_i}{d}\Big\}_{j=\rho+1}^{d-m-1}$, *if* r *is even,*
5) $\mathcal{M}_T^3 := \{v_T\}$, *if* $r = 2 \,(\mathrm{mod}\ 4)$.

Proof: We begin by considering the case where $\triangle$ consists of the Powell–Sabin split T_{PS} of a single triangle $T := \langle v_1, v_2, v_3\rangle$. We first show that $\mathcal{M}$ is a determining set for $\mathcal{S}_r(T_{PS})$. We prove later that it is minimal. Let $s \in \mathcal{S}_r(T_{PS})$, and suppose we fix its coefficients corresponding to $\mathcal{M}$. For each vertex v of $\triangle$, by the C^ρ supersmoothness at v, we can use Lemma 5.10 to compute all coefficients of s corresponding to domain points in the disk $D_\rho(v)$.

The analysis now divides into four cases depending on r. We begin with the proof for $r = 4\ell+1$. For each edge $e := \langle u, v\rangle$ of T, we compute the coefficients associated with domain points in the set $E_r(e) := \{\xi : \operatorname{dist}(\xi, e) \le r,\ \xi \notin D_\rho(u) \cup D_\rho(v)\}$ using Lemma 2.30 and the C^μ smoothness across the edge $\langle w_e, v_T\rangle$, where w_e is the vertex in the interior of e. At this point we can use the C^r smoothness across the edges $\langle v_i, v_T\rangle$ to compute the coefficients on the rings $R_{\rho+1}(v_i)$ for $i = 1,2,3$. Then for each edge e of T, we use Lemma 2.30 to compute the remaining coefficients corresponding to domain points in the row of points with distance $r+1$ to e. For each $j = 2, \ldots, d-m-1$, we repeat this process, first doing the rings $R_{\rho+j}(v_i)$, then the rows of domain points at distance $r+j$ from the edges of T.

We have now shown that $\mathcal{M}$ is a determining set for $\mathcal{S}_{4\ell+1}(T_{PS})$. To show that it is a minimal determining set for $\mathcal{S}_{4\ell+1}(T_{PS})$, we show that its cardinality is equal to the dimension of $\mathcal{S}_{4\ell+1}(T_{PS})$. It is easy to check that

$$\#\mathcal{M} = 57\ell^2 + 45\ell + 9.$$

By Theorem 9.7,

$$\dim \mathcal{S}_d^r(T_{PS}) \cap C^\mu(v_T) = \binom{\mu+2}{2} + 6\left[\binom{d-r+1}{2} - \binom{\mu-r+1}{2}\right],$$

which in this case reduces to $81\ell^2+54\ell+9$. Enforcing the C^ρ supersmoothness at the vertices requires $3\binom{2\ell+1}{2}$ special conditions, and enforcing C^μ supersmoothness across the edges $\langle v_T, w_i\rangle$ requires $3(\mu-r)(d-\mu)$ special conditions. Thus,

$$81\ell^2+54\ell+9-3(2\ell^2+\ell)-3(6\ell^2+2\ell) \le \dim \mathcal{S}_{4\ell+1}(T_{PS}) \le 57\ell^2+45\ell+9.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_{4\ell+1}(T_{PS})$. This proves $\mathcal{M}$ is an MDS for $\mathcal{S}_r(T_{PS})$ in the case $r = 4\ell+1$.

A similar proof shows that $\mathcal{M}$ is an MDS for $\mathcal{S}_r(T_{PS})$ in the other three cases, where

$$\dim \mathcal{S}_r(T_{PS}) = \begin{cases} 57\ell^2+84\ell+31, & r = 4\ell+2, \\ 57\ell^2+105\ell+48, & r = 4\ell+3, \\ 57\ell^2+144\ell+90, & r = 4\ell+4. \end{cases}$$

We now return to the general case where $\triangle_{PS}$ is the Powell–Sabin split of an arbitrary triangulation $\triangle$. Suppose we have assigned values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_r(\triangle_{PS})$. As before, we use the C^ρ supersmoothness at each vertex v coupled with Lemma 5.10 to compute the coefficients of s corresponding to the disk $D_\rho(v)$. For each interior edge e, we have already set the coefficients corresponding to $\mathcal{M}_e$. Then using the C^r smoothness conditions across e, we can compute the coefficients corresponding to domain points in the analogous set $\widetilde{\mathcal{M}}_e$ in the other triangle sharing the edge e.

For each triangle T, we now appeal to the above argument to show that all coefficients of $s|_T$ are determined while satisfying all smoothness conditions across the interior edges of T. It remains to check that if T and $\widetilde{T}$ are neighboring triangles sharing an edge e, then all C^r smoothness conditions across e are satisfied. This follows from the fact that the Powell–Sabin refinement is formed by connecting the incenters of T and $\widetilde{T}$ with a straight line, cf. the proof of Theorem 6.9. We conclude that $\mathcal{M}$ is a consistent MDS for $\mathcal{S}_r(\triangle_{PS})$, and thus by Theorem 5.15 is a minimal determining set. Theorem 5.13 then implies that the dimension of $\mathcal{S}_r(\triangle_{PS})$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be given by the formula in the statement of the theorem.

The fact that $\mathcal{M}$ is local follows from the observation that if $\eta \notin \mathcal{M}$ is a domain point in a triangle T_η, then c_η can be determined from coefficients associated with domain points $\xi \in \mathcal{M}$ lying in triangles touching

T_η. It follows that the set Γ_η in Definition 5.16 is contained in $\mathrm{star}(T_\eta)$. The stability of $\mathcal{M}$ follows from the fact that all coefficients $\{c_\eta\}_{\eta\notin\mathcal{M}}$ are computed from smoothness conditions using Lemmas 2.29 and 2.30. The constant of stability depends on the smallest angle in $\triangle_{PS}$. As shown in Lemma 4.20, this is bounded below by the smallest angle in $\triangle$. $\square$

The family $\mathcal{S}_r(\triangle_{PS})$ includes the space $\mathcal{S}_1(\triangle_{PS})$ discussed in Section 6.3. However, it does not include the space $\mathcal{S}_2(\triangle_{PS})$ discussed in Section 7.3, as that space involves an extra smoothness condition. It belongs to a family of Powell–Sabin macro-element spaces with natural degrees of freedom to be discussed in the following section. For the Powell–Sabin split T_{PS} of a single triangle, Table 8.5 shows the values of r, ρ, μ, d and $\dim \mathcal{S}_r(T_{PS})$ for $1 \le r \le 10$. Figure 8.4 shows the minimal determining sets for $r = 3, 4, 5, 6$, where the points in $\mathcal{M}_v$, $\mathcal{M}_e$, and $\mathcal{M}_T$ are marked with black dots, triangles, and squares, respectively. To help understand the supersmoothness, we have shaded the disks $D_\rho(v_i)$ and $D_\mu(v_T)$.

Since $\mathcal{S}_r(\triangle_{PS})$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

Theorem 8.14. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le d$, there exists a spline $s_f \in \mathcal{S}_r(\triangle_{PS})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-\alpha-\beta}\;|f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now describe a nodal minimal determining set for $\mathcal{S}_r(\triangle_{PS})$. Let r, d, ρ, μ, m, n be as in Table 8.4. For each edge e of $\triangle$, let D_{u_e} be the cross-boundary derivative associated with a unit vector u_e corresponding to rotating e by ninety degrees in a counterclockwise direction, and let w_e be the vertex of $\triangle_{PS}$ on e. For each triangle $T \in \triangle$, let v_T be its incenter, and let $\mathcal{E}_T$ be its set of edges. Let ε_t be the point evaluation functional which produces the value of a function at t.

Theorem 8.15. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{T\in\triangle} (\mathcal{N}_T^1 \cup \mathcal{N}_T^2 \cup \mathcal{N}_T^3)$$

is a stable local nodal minimal determining set for $\mathcal{S}_r(\triangle_{PS})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0\le\alpha+\beta\le\rho}$,

2) $\mathcal{N}_e := \displaystyle\bigcup_{j=d-\mu+1}^{r} \{\varepsilon_{w_e} D^j_{\langle w_e, v_T\rangle} D_e^k\}_{k=0}^{j+\mu-d-1}$,

3) $\mathcal{N}_T^1 := \displaystyle\bigcup_{e\in\mathcal{E}_T} \bigcup_{j=r+1}^{d-2m-2} \{\varepsilon_{w_e} D^j_{\langle w_e, v_T\rangle} D_{u_e}^k\}_{k=0}^{d-2m-2-j}$,

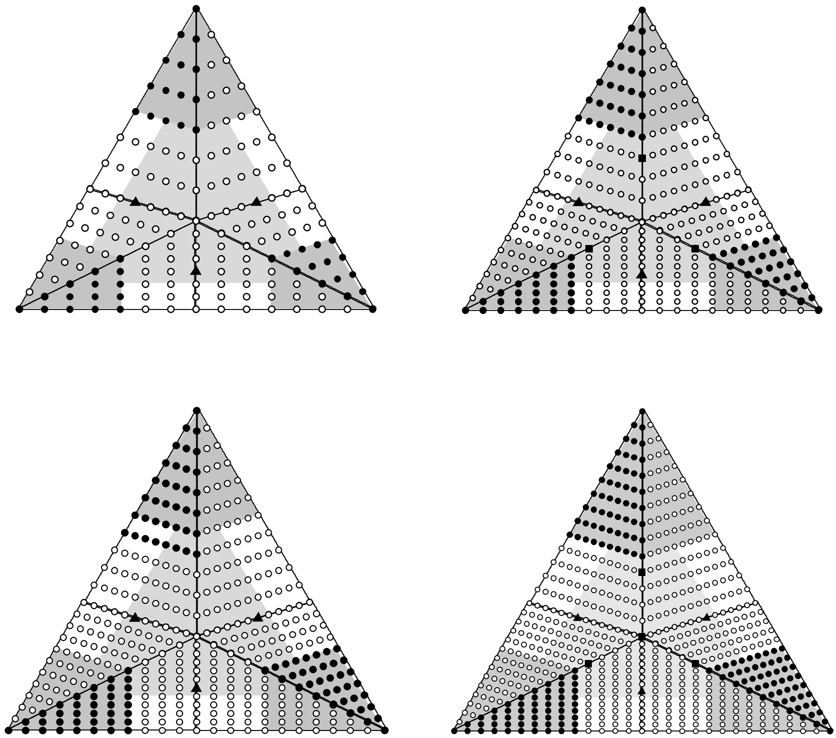


Fig. 8.4. Minimal determining sets for $\mathcal{S}_r(T_{PS})$ for $r = 3, 4, 5, 6$.

r	ρ	μ	d	dim
1	1	2	2	9
2	3	3	5	31
3	4	5	7	48
4	6	7	10	90
5	7	7	11	111
6	9	9	14	172
7	10	11	16	210
8	12	13	19	291
9	13	13	20	327
10	15	15	23	427

Tab. 8.5. Dimensions of the spaces $\mathcal{S}_r(T_{PS})$.

4) $\mathcal{N}_T^2 := \bigcup_{i=1}^{3} \{\varepsilon_{v_i} D^j_{\langle v_i, v_T \rangle}\}_{j=\rho+1}^{d-m-1}$, *if r is even,*

5) $\mathcal{N}_T^3 := \{\varepsilon_{v_T}\}$, *if $r = 2 \pmod 4$.*

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is equal to the dimension of $\mathcal{S}_r(\triangle_{PS})$ as given in Theorem 8.13. Thus, it suffices to show that the values of $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determine the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of $s \in \mathcal{S}_r(\triangle_{PS})$, where $\mathcal{M}$ is the MDS in that theorem. For each vertex v of $\triangle$, we can use the formulae (2.37) to determine all of the coefficients c_ξ of s corresponding to domain points $\xi \in \mathcal{M}_v$ from the derivatives of s corresponding to $\mathcal{N}_v$. The formulae can also be used to compute all coefficients corresponding to $\mathcal{M}_e, \mathcal{M}_T^1, \mathcal{M}_T^2, \mathcal{M}_T^3$ from the data associated with $\mathcal{N}_e, \mathcal{N}_T^1, \mathcal{N}_T^2, \mathcal{N}_T^3$. Concerning stability, it is easy to check that (5.27) holds with $\bar{m} := \rho$ if r is odd, and $\bar{m} := d - m - 1$ if r is even. $\square$

Theorem 8.15 shows that $\mathcal{S}_r(\triangle_{PS})$ is a macro-element space in the sense of Definition 5.27. In particular, for each macro-triangle $T \in \triangle$, all coefficients of $s|_T$ can be computed from the values of s and its derivatives at points in T. Then for any function $f \in C^{\bar{m}}(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\triangle_{PS})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}.$$

This defines a linear projector $\mathcal{I}_{PS}^r$ mapping $C^{\bar{m}}(\Omega)$ onto $\mathcal{S}_r(\triangle_{PS})$. In particular, $\mathcal{I}_{PS}^r$ reproduces polynomials of degree d, where d is as in Table 8.4. We can now apply Theorem 5.26 as in the proof of Theorem 8.8 to establish the following error bound for this interpolation operator.

Theorem 8.16. *For every function $f \in C^{m+1}(\Omega)$ with $\bar{m} - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_{PS}^r f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta}\, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS of Theorem 8.13. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_r(\triangle_{PS})$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v \text{ of } \triangle, \\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e \text{ of } \triangle, \\ T, & \text{if } \xi \in \mathcal{M}_T^1 \cup \mathcal{M}_T^2 \cup \mathcal{M}_T^3 \text{ for some triangle } T \in \triangle. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.15 provides an alternative local basis for $\mathcal{S}_r(\triangle_{PS})$.

8.5. PS Spaces with Natural Degrees of Freedom

In this section we discuss a family of C^r macro-element spaces defined on Powell–Sabin triangulations which have fewer degrees of freedom than the spaces described in Section 8.4, but have the same approximation power. More importantly, the spaces here have nodal bases which involve only natural degrees of freedom.

To define the spaces of interest, we need to distinguish eight cases based on the value of r mod 8. For each $\ell \geq 0$, we define parameters as in the following table.

r	ρ	μ	d	n_1	n_2	n_3
8ℓ	12ℓ	$12\ell+1$	$18\ell+1$	ℓ	2ℓ	$4\ell+3$
$8\ell+1$	$12\ell+1$	$12\ell+1$	$18\ell+2$	ℓ	2ℓ	$4\ell+2$
$8\ell+2$	$12\ell+3$	$12\ell+3$	$18\ell+5$	ℓ	$2\ell+1$	$4\ell+3$
$8\ell+3$	$12\ell+4$	$12\ell+5$	$18\ell+7$	$\ell+1$	$2\ell+1$	$4\ell+4$
$8\ell+4$	$12\ell+6$	$12\ell+7$	$18\ell+10$	$\ell+1$	$2\ell+1$	$4\ell+5$
$8\ell+5$	$12\ell+7$	$12\ell+7$	$18\ell+11$	$\ell+1$	$2\ell+1$	$4\ell+4$
$8\ell+6$	$12\ell+9$	$12\ell+9$	$18\ell+14$	$\ell+1$	$2\ell+2$	$4\ell+5$
$8\ell+7$	$12\ell+10$	$12\ell+11$	$18\ell+16$	$\ell+1$	$2\ell+2$	$4\ell+6$

Tab. 8.6. Parameters for the C^r Powell–Sabin macro-element space (8.5).

As in the previous section, we write $\mathcal{V}$ and $\mathcal{E}$ for the sets of vertices and edges of $\triangle$, $\mathcal{V}_c := \{v_T\}_{T\in\triangle}$ for the set of incenters used to form the Powell–Sabin refinement, and $\mathcal{W} := \{w_e\}_{e\in\mathcal{E}}$ for the set of vertices inserted in the interior of edges of $\triangle$. For each triangle $T := \langle v_1, v_2, v_3\rangle$ in $\triangle$, let $e_k := \langle v_k, v_T\rangle$ for $k = 1, 2, 3$, and let

$$\mathcal{T}_{T,1} := \bigcup_{j=1}^{n_1} \{\tau^{r+i}_{\rho+j,e_1}, \tau^{r+i}_{\rho+j,e_2}, \tau^{r+i}_{\rho+j,e_3}\}_{i=1}^{m_j},$$

where

$$m_j = \begin{cases} 2j-1, & \text{if } r \text{ is even},\\ 2j-2, & \text{if } r \text{ is odd}.\end{cases}$$

In addition, let

$$\mathcal{T}_{T,2} := \bigcup_{j=n_1+1}^{n_2} \Big(\{\tau^{r+i}_{\rho+j,e_1}\}_{i=1}^{n_3-2j} \cup \{\tau^{r+i}_{\rho+j,e_2}\}_{i=1}^{n_3-2j-1} \cup \{\tau^{r+i}_{\rho+j,e_3}\}_{i=1}^{n_3-2j-2}\Big).$$

and set $\mathcal{T} := \mathcal{T}_{T,1} \cup \mathcal{T}_{T,2}$. The space of interest in this section is

$$\widetilde{\mathcal{S}}_r(\triangle_{PS}) := \{s \in \mathcal{S}_r(\triangle_{PS}) : \tau s = 0, \text{ all } \tau \in \mathcal{T}_T \text{ and } T \in \triangle\}, \tag{8.5}$$

where $\mathcal{S}_r(\triangle_{PS})$ is the Powell–Sabin space in Section 8.4. Let V and E be the number of vertices and edges of $\triangle$.

Theorem 8.17. *For all $r \geq 1$,*

$$\dim \widetilde{\mathcal{S}}_r(\triangle_{PS}) = \binom{\rho+2}{2}V + \binom{r+\mu-d+1}{2}E.$$

Moreover, the set

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a minimal determining set, where $\mathcal{M}_v$ and $\mathcal{M}_e$ are as in Theorem 8.13.

Proof: The proof is similar to the proof of Theorem 8.13. We begin by considering the case where $\triangle$ consists of the Powell–Sabin split T_{PS} of a single triangle $T := \langle v_1, v_2, v_3\rangle$. We first show that $\mathcal{M}$ is a determining set for $\widetilde{\mathcal{S}}_r(T_{PS})$, and show that it is minimal later. Let $s \in \widetilde{\mathcal{S}}_r(T_{PS})$, and suppose we fix its coefficients corresponding to $\mathcal{M}$. For each vertex $v \in \mathcal{V}$, coupling the C^ρ supersmoothness at v with Lemma 5.10, we can compute all coefficients of s corresponding to domain points in the disk $D_\rho(v)$.

Let $e_i := \langle v_i, v_{i+1}\rangle$, $i = 1, 2, 3$, be the three edges of T, and write $E_{j,i}$ for the set of domain points ξ of s with $\operatorname{dist}(\xi, e_i) = j$. Then by the C^μ smoothness across the edges $\langle w, v_T\rangle$ with $w \in \mathcal{W}$, we can use Lemma 2.30 to compute the μ undetermined coefficients corresponding to the domain points in $E_{j,i}$ for $j = 0, \ldots, r$ and $i = 1, 2, 3$. Next we compute the $2(\rho - r + j - 1) + 1$ undetermined coefficients corresponding to the rings $R_{\rho+j}(v_i)$ for $1 \leq j \leq \rho$ and $i = 1, 2, 3$. The computation of the remaining unknown coefficients of s divides into six cases.

Case 1: $r = 8\ell, 8\ell + 1$. We perform a cycle of computations. For each $j = n_1 + 1, \ldots, n_2$:

a) compute the $2(\rho - r + j - 1) + 1$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_1)$,
b) compute the $d - r + 2n_1 - 2j + 1$ remaining coefficients corresponding to domain points in the rows $E_{r+2(j-n_1)-1,1}$ and $E_{r+2(j-n_1)-1,3}$,
c) compute the $2(\rho - r + j - 1)$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_2)$,
d) compute the $d - r + 2n_1 - 2j + 1$ remaining coefficients corresponding to domain points in the row $E_{r+2(j-n_1)-1,2}$,
e) compute the $2(\rho - r + j - 1) - 1$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_3)$,

f) if $j < n_2$, compute the $d - r + 2n_1 - 2j$ remaining coefficients corresponding to domain points in the rows $E_{r+2(j-n_1),i}$ for $i = 1, 2, 3$.

At this point the only remaining unknown coefficients correspond to domain points inside the disk $D_\mu(v_T)$. In view of the C^μ supersmoothness at v_T, we can consider these coefficients to be those of a polynomial p of degree $\mu = 12\ell + 1$ on a triangle $\widetilde{T} := \langle u_1, u_2, u_3 \rangle$ which has been subjected to a Powell–Sabin split, where $u_k := (\mu v_k + (d - \mu) v_T)/d$, $k = 1, 2, 3$. Since we have already computed all coefficients of s corresponding to domain points in the disks $D_{\rho+n_2}(v_i)$, this gives us all coefficients of p corresponding to domain points in the disks $D_{8\ell}(u_i)$ for $i = 1, 2, 3$. These determine all coefficients of p, and thus all remaining coefficients of s.

Case 2: $r = 8\ell + 2$. In this case we do the cycle of computations of Case 1 for $j = n_1 + 1, \ldots, n_2 - 1$. Next we compute the $r + 1$ unknown coefficients corresponding to domain points on the ring $R_{\rho+n_2}(v_1)$. This gives us all of the coefficients of a polynomial p of degree $\mu = 12\ell + 3$ in the disks $D_{8\ell+2}(u_1)$ and $D_{8\ell+1}(u_i)$ for $i = 2, 3$. These determine all coefficients of p, and hence all remaining coefficients of s.

Case 3: $r = 8\ell + 3$. If $\ell > 0$, we first compute the $d - r - 1$ unknown coefficients corresponding to domain points in each of the rows $E_{r+1,i}$ for $i = 1, 2, 3$. We then do the following cycle of computations. For each $j = n_1 + 1, \ldots, n_2 - 1$:

a) compute the $2(\rho - r + j - 1) + 1$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_1)$,
b) compute the $d - r + 2n_1 - 2j + 1$ remaining coefficients corresponding to domain points in the rows $E_{r+2(j-n_1),1}$ and $E_{r+2(j-n_1),3}$,
c) compute the $2(\rho - r + j - 1)$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_2)$,
d) compute the $d - r + 2n_1 - 2j + 1$ remaining coefficients corresponding to domain points in the row $E_{r+2(j-n_1),2}$,
e) compute the $2(\rho - r + j - 1) - 1$ remaining coefficients corresponding to domain points on the ring $R_{\rho+j}(v_3)$,
f) if $j < n_2$, compute the $d - r + 2n_1 - 2j$ remaining coefficients corresponding to domain points in the rows $E_{r+2(j-p)+1,i}$, $i = 1, 2, 3$.

To complete the computation, we find the unknown coefficients corresponding to domain points on the ring $R_{\rho+n_2}(v_1)$, on the edge $E_{r+2(n_2-n_1)-1,1}$ if $n_2 > n_1$, and on $R_{\rho+n_2}(v_2)$. This gives us all coefficients of the degree $\mu = 12\ell + 5$ polynomial p in the disks $D_{8\ell+3}(u_i)$, $i = 1, 2$, and $D_{8\ell+2}(u_3)$. These determine all coefficients of p, and thus all remaining coefficients of s.

Case 4: $r = 8\ell + 4, r = 8\ell + 5$. We proceed as in Case 3, except now we do the cycles for all $j = n_1 + 1, \ldots, n_2$. This gives us all coefficients of the degree $\mu = 12\ell + 7$ polynomial p in the disks $D_{8\ell+4}(u_i)$, $i = 1, 2, 3$. These determine all coefficients of p, and hence all remaining coefficients of s.

Case 5: $r = 8\ell+6$. We first compute the unknown coefficients corresponding to domain points in the rows $E_{r+1,i}$ for $i = 1, 2, 3$, and then perform the cycles as in Case 3 for $j = n_1+1, \ldots, n_2-1$. To complete the computation, we then compute the remaining coefficients corresponding to domain points on the ring $R_{\rho+n_2}(v_1)$. This gives us all coefficients of the degree $\mu = 12\ell + 9$ polynomial p in the disks $D_{8\ell+6}(v_1)$ and $D_{8\ell+5}(u_i)$, $i = 2, 3$. These determine all coefficients of p, and thus all remaining coefficients of s.

Case 6: $r = 8\ell + 7$. We begin by doing the cycles of Case 1 for $j = n_1 + 1, \ldots, n_2 - 1$. Then we compute the unknown coefficients corresponding to domain points on $R_{\rho+n_2}(v_1)$, on the edge $E_{r+2(n_2-n_1)-1,1}$, and the ring $R_{\rho+n_2}(v_2)$. This gives us all coefficients of the degree $\mu = 12\ell+11$ polynomial p in the disks $D_{8\ell+7}(v_i)$, $i = 1, 2$, and $D_{8\ell+6}(u_3)$. These determine all coefficients of p, and hence all remaining coefficients of s.

This completes the proof that $\mathcal{M}$ is a determining set for $\widetilde{\mathcal{S}}_r(T_{PS})$. We now show that it is minimal. It is easy to check that

$$\#\mathcal{M} = 3\binom{\rho+2}{2} + 3\binom{r+\mu-d+1}{2},$$

which reduces to the numbers in the last column of Table 8.7. Now for each r, the spline space $\widetilde{\mathcal{S}}_r(T_{PS})$ is the subspace of the spline space $\mathcal{S}_r(T_{PS})$ satisfying the smoothness conditions corresponding to the smoothness functionals in the set $\mathcal{T}_T$, whose cardinality is equal to the number κ given in Table 8.7. Now

$$\dim \widetilde{\mathcal{S}}_r(T_{PS}) \geq \dim \mathcal{S}_r(T_{PS}) - \kappa,$$

which reduces to $\#\mathcal{M}$ for all r, and we conclude that $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_r(T_{PS})$ and $\dim \widetilde{\mathcal{S}}_r(T_{PS}) = \#\mathcal{M}$.

r	$\dim \mathcal{S}_r(T_{PS})$	κ	$\dim \widetilde{\mathcal{S}}_r(T_{PS})$
8ℓ	$228\ell^2 + 60\ell + 3$	$6\ell^2 + 3\ell$	$222\ell^2 + 57\ell + 3$
$8\ell+1$	$228\ell^2 + 90\ell + 9$	$6\ell^2 - 3\ell$	$222\ell^2 + 93\ell + 9$
$8\ell+2$	$228\ell^2 + 168\ell + 31$	$6\ell^2 + 3\ell + 1$	$222\ell^2 + 165\ell + 30$
$8\ell+3$	$228\ell^2 + 210\ell + 48$	$6\ell^2 + 3\ell$	$222\ell^2 + 207\ell + 48$
$8\ell+4$	$228\ell^2 + 288\ell + 90$	$6\ell^2 + 9\ell + 3$	$222\ell^2 + 279\ell + 87$
$8\ell+5$	$228\ell^2 + 318\ell + 111$	$6\ell^2 + 3\ell$	$222\ell^2 + 315\ell + 111$
$8\ell+6$	$228\ell^2 + 396\ell + 172$	$6\ell^2 + 9\ell + 4$	$222\ell^2 + 387\ell + 168$
$8\ell+7$	$228\ell^2 + 438\ell + 210$	$6\ell^2 + 9\ell + 3$	$222\ell^2 + 429\ell + 207$

Tab. 8.7. Dimension of $\widetilde{\mathcal{S}}_r(T_{PS})$.

Now suppose $\triangle_{PS}$ is the Powell–Sabin refinement of an arbitrary triangulation $\triangle$. Then we can use our results above for a single triangle to show that $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_r(\triangle_{PS})$ in the same way as in Theorem 8.13. By Theorem 5.13, the dimension of $\widetilde{\mathcal{S}}_r(\triangle_{PS})$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be equal to the formula in the statement of the theorem. The proof that $\mathcal{M}$ is local and stable is also the same as in Theorem 8.13. $\square$

For the Powell–Sabin split T_{PS} of a single triangle, Figure 8.5 shows the minimal determining sets for $r = 3, 4, 5, 6$, where the points in $\mathcal{M}_v$ and $\mathcal{M}_e$ are marked with black dots and triangles, respectively. To help understand the supersmoothness, we have shaded the disks $D_\rho(v_i)$ and $D_\mu(v_T)$. The tips of the special smoothness conditions are indicated with brackets, and the lines drawn through domain points show some of the computations using Lemma 2.30. Table 8.8 shows the values of r, ρ, μ, d and $\dim \widetilde{\mathcal{S}}_r(T_{PS})$ for $1 \leq r \leq 10$. Since $\widetilde{\mathcal{S}}_r(\triangle_{PS})$ has a stable local MDS, we can apply Theorem 5.19 to show that it has full approximation power.

Theorem 8.18. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \leq q \leq \infty$ and $0 \leq m \leq d$, there exists a spline $s_f \in \widetilde{\mathcal{S}}_r(\triangle_{PS})$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \leq K\,|\triangle|^{m+1-\alpha-\beta}\;|f|_{m+1,q,\Omega},$$

for all $0 \leq \alpha + \beta \leq m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

The following theorem can be proved in the same way as Theorem 8.15.

Theorem 8.19. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \;\cup\; \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\widetilde{\mathcal{S}}_r(\triangle_{PS})$, where $\mathcal{N}_v$ and $\mathcal{N}_e$ are as in Theorem 8.15.

Theorem 8.19 shows that $\widetilde{\mathcal{S}}_r(\triangle_{PS})$ is a macro-element space in the sense of Definition 5.27. In particular, for each macro-triangle $T \in \triangle$, all coefficients of $s|_T$ can be computed from the values of s and its derivatives at points in T. The theorem also shows that for any function $f \in C^\rho(\Omega)$, there is a unique spline $s \in \widetilde{\mathcal{S}}_r(\triangle_{PS})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}.$$

This defines a linear projector $\widetilde{\mathcal{I}}_{PS}^{\,r}$ mapping $C^\rho(\Omega)$ onto $\widetilde{\mathcal{S}}_r(\triangle_{PS})$. In particular, $\widetilde{\mathcal{I}}_{PS}^{\,r}$ reproduces polynomials of degree d, where d is as in Table 8.6. We can now apply Theorem 5.26 as in the proof of Theorem 8.8 to establish the following error bound for this interpolation operator.

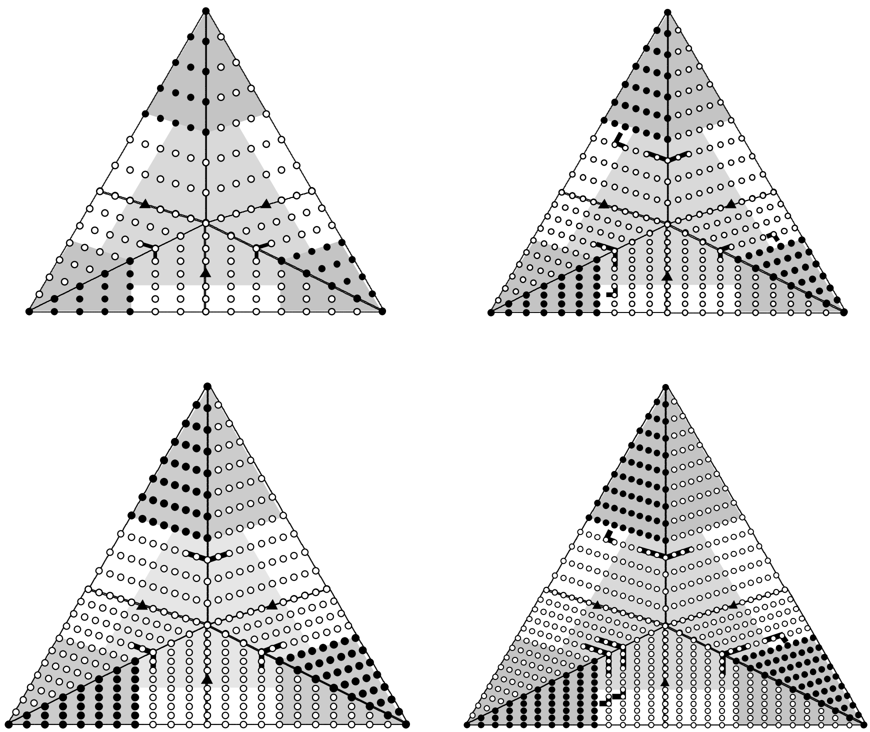


Fig. 8.5. Minimal determining sets for $\widetilde{\mathcal{S}}_r(T_{PS})$ for $r = 3, 4, 5, 6$.

r	ρ	μ	d	dim
1	1	2	2	9
2	3	3	5	30
3	4	5	7	48
4	6	7	10	87
5	7	7	11	111
6	9	9	14	168
7	10	11	16	207
8	12	13	19	282
9	13	13	20	324
10	15	15	23	417

Tab. 8.8. The dimension of $\widetilde{\mathcal{S}}_r(T_{PS})$.

Theorem 8.20. *For every function $f \in C^{m+1}(\Omega)$ with $\rho - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \widetilde{\mathcal{I}}_{PS}^{\,r} f)\|_\Omega \le K\, |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS of Theorem 8.17. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\widetilde{\mathcal{S}}_r(\triangle_{PS})$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v, \\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e \text{ of } \triangle. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.19 provides a different stable local basis for $\widetilde{\mathcal{S}}_r(\triangle_{PS})$.

8.6. Quadrilateral Macro-element Spaces

Suppose $\diamondsuit$ is a quadrangulation of a domain Ω, and let $\boxtimes$ be the induced triangulation obtained by inserting both diagonals in each quadrilateral Q in $\diamondsuit$, see Section 4.15. We denote the sets of vertices and edges of $\diamondsuit$ by $\mathcal{V}$ and $\mathcal{E}$, and let $\mathcal{V}_c := \{v_Q\}_{Q \in \diamondsuit}$ be the set of vertices of $\boxtimes$ which are introduced to form the splits. To define the spline spaces of interest in this section, we first introduce some parameters which depend on whether r is even or odd, see Table 8.9.

r	$\tilde{r}$	ρ	μ	d
2ℓ	$2\ell+1$	3ℓ	4ℓ	$6\ell+1$
$2\ell+1$	$2\ell+1$	$3\ell+1$	$4\ell+1$	$6\ell+3$

Tab. 8.9. Parameters for the C^r quadrilateral macro-element space (8.6).

We also need some special smoothness conditions. For each quadrilateral $Q := \langle v_1, v_2, v_3, v_4 \rangle$ in $\diamondsuit$, let $\mathcal{E}_Q := \{e_k := \langle v_k, v_{\scriptscriptstyle T} \rangle\}_{k=1}^4$ be the set of interior edges forming the split of Q. Let

$$\mathcal{T}_Q := \begin{cases} \displaystyle\bigcup_{k=1}^{4} \bigcup_{i=1}^{\ell-1} \{\tau_{\rho+i+1,e_k}^{r+j+1}\}_{j=1}^{2i} \cup \{\tau_{\mu+1,e_1}^{r+j+1}\}_{j=1}^{2\ell}, & \text{if } r = 2\ell, \\ \displaystyle\bigcup_{k=1}^{4} \bigcup_{i=1}^{\ell-1} \{\tau_{\rho+i,e_k}^{r+j}\}_{j=1}^{2i} \cup \bigcup_{k=1}^{3} \{\tau_{\mu+1,e_k}^{r+j}\}_{j=1}^{2\ell}, & \text{if } r = 2\ell+1. \end{cases}$$

We now define the C^r quadrilateral macro-element spaces by

$$\begin{aligned}\mathcal{S}_r(\diamondplus) := \{s \in \mathcal{S}_d^r(\diamondplus) :\ & s \in C^\rho(v), \ \text{ all } v \in \mathcal{V}, \\ & s \in C^\mu(v), \ \text{ all } v \in \mathcal{V}_c, \\ & s \in C^{\tilde{r}}(e), \ \text{ all } e \in \mathcal{E}_Q \text{ and } Q \in \Diamond, \\ & \tau s = 0, \ \text{ all } \tau \in \mathcal{T}_Q \text{ and } Q \in \Diamond\}.\end{aligned} \tag{8.6}$$

For each vertex $v \in \mathcal{V}$, let T_v be the triangle in $\diamondplus$ with vertex at v which has the largest shape parameter (see Definition 4.1) among all triangles sharing the vertex v. Let $\mathcal{M}_v := D_\rho(v) \cap T_v$. For each edge $e := \langle u, v\rangle$ of $\triangle$, let T_e be some triangle in $\diamondplus$ containing the edge e, and let $\mathcal{M}_e := \{\xi \in \mathcal{D}_{T_e,d} : \operatorname{dist}(\xi, e) \le r,\ \xi \notin D_\rho(u) \cup D_\rho(v)\}$. Let V and E be the number of vertices and edges of $\Diamond$.

Theorem 8.21. *For all $r \ge 1$,*

$$\dim \mathcal{S}_r(\diamondplus) = \binom{\rho+2}{2} V + \binom{r+1}{2} E,$$

and

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set.

Proof: We begin by considering the case where $\triangle_Q$ is the triangulation associated with a single quadrilateral $Q := \langle v_1, v_2, v_3, v_4\rangle$ divided into four subtriangles $T^{[i]} := \langle v_Q, v_i, v_{i+1}\rangle$, $i = 1, 2, 3, 4$. We first show that $\mathcal{M}$ is a determining set for $\mathcal{S}_r(\triangle_Q)$. We show that it is minimal later. Suppose we fix the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_r(\triangle_Q)$. Then for each vertex v_i of Q, in view of the C^ρ supersmoothness at v_i, by Lemma 5.10 we can compute all coefficients of s corresponding to domain points in the disk $D_\rho(v_i)$. The analysis now divides into two cases.

Suppose that $r = 2\ell$. Using Lemma 2.30, we now compute the undetermined coefficients of s corresponding to domain points on the rings $R_{\rho+i}(v_\nu)$ for $i = 1, \ldots, \ell$ and $\nu = 1, 2, 3, 4$. Note that the spline satisfies all of the smoothness conditions required for the lemma, since either they are already implicit in the supersmoothness of the space, or have been explicitly enforced in the definition of $\mathcal{S}_r(\triangle_Q)$. Using the lemma again, we compute the remaining coefficients corresponding to domain points on $R_{\mu+1}(v_1)$.

We now carry out a sequence of calculations. First we compute the 4ℓ remaining coefficients corresponding to points in the set E_0, where in general, E_i is the set of domain points in $T^{[1]} \cup T^{[2]}$ at a distance i from the edge $\langle v_1, v_3\rangle$. Then we compute the 4ℓ remaining coefficients corresponding to points in the set $\widetilde{E}_0$, where $\widetilde{E}_i$ is the set of domain points in $T^{[2]} \cup T^{[3]}$ at a distance i from the edge $\langle v_2, v_4\rangle$. The remaining coefficients in $T^{[1]} \cup$

$T^{[2]} \cup T^{[3]}$ are computed by alternately working on the sets E_i and $\widetilde{E}_i$ for $i = 1, \ldots, r$. Finally, we compute the remaining coefficients in $T^{[4]}$ from the C^r smoothness conditions.

We have shown that all coefficients of s are determined by those corresponding to the domain points in the set $\mathcal{M}$. This shows that $\mathcal{M}$ is a determining set for $\mathcal{S}_r(\triangle_Q)$. We now show that it is minimal. By Theorem 9.7, the dimension of the superspline space $\mathcal{S}^{2\ell+1}_{6\ell+1}(\triangle_Q) \cap C^{4\ell}(v_Q)$ is $32\ell^2 + 18\ell + 4$. Our space $\mathcal{S}_{2\ell}(\triangle_Q)$ is the subspace which satisfies the $4\ell^2 - 2\ell$ special conditions and the supersmoothness $C^{3\ell}(v_i)$ for $i = 1, 2, 3, 4$. Enforcing the supersmoothness requires an additional $2\ell^2 - 2\ell$ conditions. Thus,

$$\begin{aligned}(32\ell^2 + &18\ell + 4) - (4\ell^2 - 2\ell) - (2\ell^2 - 2\ell) \\ &\le \dim \mathcal{S}_{2\ell}(\triangle_Q) \le \#\mathcal{M} = 26\ell^2 + 22\ell + 4.\end{aligned}$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_r(\triangle_Q)$, and thus $\mathcal{M}$ is an MDS.

The proof that $\mathcal{M}$ is an MDS for $\mathcal{S}_r(\triangle_Q)$ in the case $r = 2\ell + 1$ is very similar. By Theorem 9.7, the dimension of the superspline space $\mathcal{S}^{2\ell+1}_{6\ell+3}(\triangle_Q) \cap C^{4\ell+1}(v_Q)$ is $32\ell^2 + 46\ell + 16$. Now $\mathcal{S}_{2\ell+1}(\triangle_Q)$ is the subspace that satisfies the $4\ell^2 + 2\ell$ special conditions along with the supersmoothness $C^{3\ell+1}(v_i)$ for $i = 1, 2, 3, 4$. Enforcing this supersmoothness requires an additional $2\ell^2 + 2\ell$ conditions. It follows that

$$\begin{aligned}(32\ell^2 + &46\ell + 16) - (4\ell^2 + 2\ell) - (2\ell^2 + 2\ell) \\ &\le \dim \mathcal{S}_{2\ell+1}(\triangle_Q) \le \#\mathcal{M} = 26\ell^2 + 42\ell + 16.\end{aligned}$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_{2\ell+1}(\triangle_Q)$. This proves $\mathcal{M}$ is an MDS for $\mathcal{S}_r(\triangle_Q)$ in the case $r = 2\ell + 1$.

Now suppose $\diamondsuit$ is the triangulation associated with a general quadrilateral partition $\diamondsuit$. Suppose we have assigned values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_r(\diamondsuit)$. Then for each vertex v of $\diamondsuit$, by the C^ρ supersmoothness at v, Lemma 5.10 shows that the coefficients of s corresponding to the disk $D_\rho(v)$ are consistently determined. For each interior edge e of $\diamondsuit$, we have chosen the coefficients corresponding to domain points in $\mathcal{M}_e$. Then using the C^r smoothness across that edge, we can compute the coefficients of s corresponding to the domain points in the analogous set $\widetilde{\mathcal{M}}_e$ in the other triangle $\widetilde{T}_e$ in $\diamondsuit$ sharing the edge e. Then by the above arguments, for each quadrilateral Q in $\diamondsuit$, $s|_Q$ is determined on Q in such a way that all smoothness conditions across interior edges of Q are satisfied. It follows that $\mathcal{M}$ is a consistent determining set for $\mathcal{S}_r(\diamondsuit)$. Theorem 5.15 then implies that $\mathcal{M}$ is an MDS for $\widetilde{\mathcal{S}}_r(\diamondsuit)$. By Theorem 5.13, the dimension of $\mathcal{S}_r(\diamondsuit)$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be given by the formula in the statement of the theorem.

The proof that $\mathcal{M}$ is local and stable is similar to the proof of Theorem 8.9. For stability, the choice of the triangle T_v used to define $\mathcal{M}_v$ is critical to assuring the stability of the computation of coefficients associated with domain points in $D_\rho(v)$. $\square$

The choice of $\mathcal{M}_v$ for the MDS of Theorem 8.21 ensures that the constant of stability for $\mathcal{M}$ does not depend on the smallest angle in $\diamondplus$ but only on the smallest angle in $\Diamond$. This is important since as we saw in Example 4.48, angles in $\diamondplus$ can be small even when the angles in $\Diamond$ are not.

The family $\mathcal{S}_r(\triangle_Q)$ contains the spaces $\mathcal{S}_1(\triangle_Q)$ and $\mathcal{S}_2(\triangle_Q)$ discussed in Sections 6.5 and 7.6. Figure 8.6 illustrates Theorem 8.21 for a single macro-quadrilateral in the cases $r = 3, 4$. Points in the sets $\mathcal{M}_v$ are marked with black dots, while those in the sets $\mathcal{M}_e$ are marked with triangles. Table 8.10 lists the dimensions of the spaces $\mathcal{S}_r(\triangle_Q)$ for $1 \le r \le 8$.

Since $\mathcal{S}_r(\diamondplus)$ has a stable local MDS, we can apply Theorem 5.19 to conclude that $\mathcal{S}_r(\diamondplus)$ has full approximation power.

Theorem 8.22. *For every $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le d$, there exists a spline $s_f \in \mathcal{S}_r(\diamondplus)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{q,\Omega} \le K\, |\diamondplus|^{m+1-\alpha-\beta}\, |f|_{m+1,q,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\Diamond$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

For each edge $e := \langle u, v\rangle$ of $\Diamond$, let $\eta_{j,e}^i$ be the points defined in (8.1), and let u_e be the unit vector corresponding to rotating e ninety degrees in the counterclockwise direction. We write D_{u_e} for the cross-boundary derivative associated with u_e. Let ε_t denote point evaluation at t. The following theorem can be proved in the same way as Theorem 8.7.

Theorem 8.23. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_r(\diamondplus)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le \rho}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^r \{\varepsilon_{\eta_{j,e}^i} D_{u_e}^i\}_{j=1}^i$.

The space $\mathcal{S}_r(\diamondplus)$ can be regarded as a macro-element space since for each quadrilateral Q in $\Diamond$, we can compute the B-coefficients of $s|_Q$ from values of s and its derivatives at points in Q. Theorem 8.23 shows that

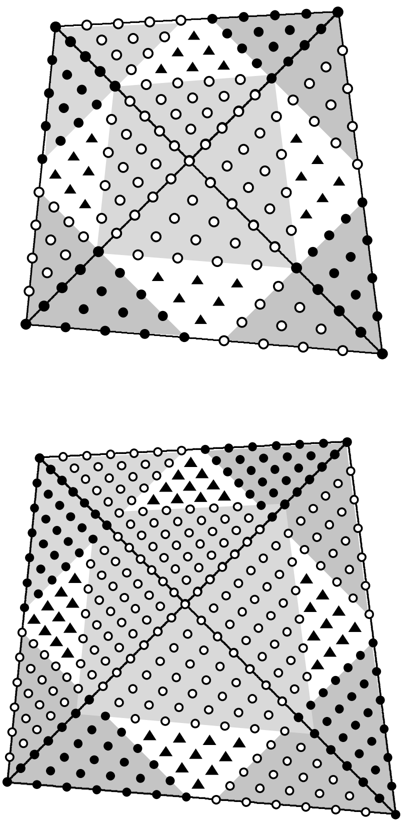


Fig. 8.6. Minimal determining sets for $\mathcal{S}_3(\triangle_Q)$ and $\mathcal{S}_4(\triangle_Q)$.

r	ρ	μ	d	dim
1	1	2	3	16
2	3	4	7	52
3	4	5	9	84
4	6	8	13	152
5	7	9	15	204
6	9	12	19	304
7	10	13	21	376
8	12	16	25	508

Tab. 8.10. Dimensions of the spaces $\mathcal{S}_r(\triangle_Q)$.

for any function $f \in C^\rho(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\diamondplus)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently,

$$\begin{aligned} D_x^\alpha D_y^\beta s(v) &= D_x^\alpha D_y^\beta f(v), && \text{all } 0 \le \alpha + \beta \le \rho \text{ and } v \in \mathcal{V}, \\ D_{u_e}^i s(\eta_{j,e}^i) &= D_{u_e}^i f(\eta_{j,e}^i), && \text{all } j = 1, \ldots, i, \ \ 1 \le i \le r, \text{ and } e \in \mathcal{E}. \end{aligned}$$

This defines a linear projector $\mathcal{I}_Q^r$ mapping $C^\rho(\Omega)$ onto $\mathcal{S}_r(\diamondplus)$. In particular, $\mathcal{I}_Q^r$ reproduces polynomials of degree d, where d is given in Table 8.9. We can now apply Theorem 5.26 to establish the following error bound for this interpolation operator.

Theorem 8.24. *For every function $f \in C^{m+1}(\Omega)$ with $\rho - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}_Q^r f)\|_\Omega \le K \, |\diamondplus|^{m+1-\alpha-\beta} \, |f|_{m+1,\Omega},$$

for all $0 \le \alpha + \beta \le m$. If Ω is convex, then the constant K depends only on r and the smallest angle in $\lozenge$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Suppose $\mathcal{M}$ is the MDS of Theorem 8.21. Then by Theorem 5.21, the corresponding $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}_r(\diamondplus)$, where

$$\operatorname{supp}(\psi_\xi) \subseteq \begin{cases} \operatorname{star}(v), & \text{if } \xi \in \mathcal{M}_v \text{ for some vertex } v, \\ T_e \cup \widetilde{T}_e, & \text{if } \xi \in \mathcal{M}_e \text{ for some edge } e. \end{cases}$$

Here T_e is the triangle associated with the edge e, and $\widetilde{T}_e$ is the second triangle containing e if e is an interior edge. The $\mathcal{N}$-basis in (5.24) associated with the nodal MDS $\mathcal{N}$ of Theorem 8.23 provides a different stable local basis for $\mathcal{S}_r(\diamondplus)$.

8.7. Remarks

Remark 8.1. It is easy to see that it is impossible to construct C^r polynomial macro-elements using splines of degree less than $4r + 1$. Indeed, by Theorem 5.28, if T is a triangle, then in order to avoid incompatibilities, we have to work with splines with supersmoothness at least $C^{2r}(v)$ at the vertices v of T. This means that in order to construct a polynomial macro-element on T, we have to include the disks $D_{2r}(v)$ in the minimal determining set. In order to ensure that the corresponding coefficients can be set independently, we have to be sure that the disks do not overlap, and it follows that we must work with splines of degree at least $4r + 1$.

Remark 8.2. In Theorem 8.3 it is possible to replace the set $\mathcal{N}_T$ defined there by a set of derivatives at single point in T, say the barycenter v_T. In particular, we can take $\mathcal{N}_T := \{\epsilon_{v_T} D_x^\nu D_y^\mu\}_{\nu+\mu\leq r-2}$. This is the more traditional choice, although from the standpoint of Hermite interpolation, it is generally easier to get data on function values than on derivatives.

Remark 8.3. It was shown in [LaiS01] that it is impossible to construct C^r macro-elements based the Clough–Tocher split using splines of lower degree than $6\ell+1$ if $r=2\ell$, or $6\ell+3$ if $r=2\ell+1$. To see this, suppose T_{CT} is the Clough–Tocher split of a triangle T. Then there is exactly one interior edge connected to each vertex of T. Thus, by Theorem 5.28, to build a C^r macro-element based on the Clough–Tocher split, we need to enforce at least C^ρ supersmoothness at each vertex of T, where

$$\rho \geq \left\lceil \frac{3r-1}{2} \right\rceil = \begin{cases} 3\ell, & \text{if } r=2\ell, \\ 3\ell+1, & \text{if } r=2\ell+1. \end{cases} \tag{8.7}$$

This means that in order to construct a macro-element on the Clough–Tocher split, we have to include the disks $D_\rho(v)$ in the minimal determining set. Since these disks are not allowed to overlap, the claim follows.

Remark 8.4. It was shown in [LaiS03] that it is impossible to construct C^r macro-elements based the Powell–Sabin split using splines of lower degree than used in Sections 8.4 and 8.5. To see this, we first observe that as for the Clough–Tocher split discussed in Remark 8.3, we need to enforce C^ρ supersmoothness at each of the vertices of the macro-triangle. As before, this means that we have to include the disks $D_\rho(v)$ in the minimal determining set. Any two such disks are in two different micro-triangles, i.e., they are separated by an edge of T_{PS}. This leads to the degrees used in Sections 8.4 and 8.5.

Remark 8.5. It was shown in [AlfS02a] and [AlfS02b] that the macro-elements constructed in Sections 8.3 and 8.5 on the Clough–Tocher and Powell–Sabin splits, respectively, have the minimal number of degrees of freedom possible for such macro-elements. The reason is that these elements are based only on natural degrees of freedom (i.e., nodal data which involve evaluation at points on the boundary), and none of this data can be eliminated.

Remark 8.6. Classical macro-elements are superspline spaces defined on a macro-triangle T that has been subdivided into micro-triangles. A spline in the space is then typically determined by nodal data at the vertices of T, at certain points along the edges of T, and in some cases at additional points in the interior of T. Recently it has been shown [RayS05] that it is possible to define C^r macro-elements based on Clough–Tocher splits that require data on only two of the sides of the macro-triangle. Such elements are useful for finite-element computations, and also for solving interpolation and fitting problems. An interesting application is to the problem of filling an n-sided hole in a surface.

Remark 8.7. As pointed out in Remark 5.8, nested sequences of spline spaces are important for applications. In Remark 6.4 we noted that two of the C^1 macro-element spaces in that chapter can be used to create nested sequences of spline spaces. On the other hand, in Remark 7.6 we observed that none of the C^2 macro-element spaces in that chapter is suitable for creating nested sequences of spline spaces. The same is true for all of the macro-element spaces in this chapter with $r \geq 2$.

8.8. Historical Notes

We have already discussed the development of C^1 and C^2 macro-elements in Sections 6.8 and 7.9. C^r polynomial macro-elements were investigated in [Zen74], albeit without the use of Bernstein–Bézier methods. See also [SablL93].

Families of C^r macro-elements based on the Clough–Tocher and Powell–Sabin splits were studied in a number of papers, see [Sabl85b, Sabl87, LagS89a, LagS89b, LagS93, LagS94, Lag98]. Improved elements using lower degree polynomials were later constructed in [LaiS01, LaiS03], which we have followed here in Sections 8.2 and 8.4. The idea of natural degrees of freedom was introduced in [AlfS02a, AlfS02b], which we have followed here in Sections 8.3 and 8.5.

Families of C^r macro-elements based on triangulated quadrangulations were first studied in [LagS89a, LagS95]. Improved elements using lower degree splines were later constructed in [LaiS99, LaiS02]. The results in Section 8.6 are based on [LaiS02].

Dimension of Spline Spaces

In this chapter we discuss the problem of computing the dimension of spline spaces. This turns out to be a rather difficult problem, which in fact has only been partially solved. While we have very good lower bounds and relatively good upper bounds on the dimension of spline spaces of arbitrary degree and smoothness on general regular triangulations, there are many spaces for which we do not have an exact formula. This includes the space $\mathcal{S}_3^1(\triangle)$ for which we can establish a formula only for generic triangulations.

9.1. Dimension of Spline Spaces on Cells

In this section we give dimension formulae for the spline spaces

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d \text{ for all } T \in \triangle\} \tag{9.1}$$

in the special case when $\triangle$ is a cell.

Definition 9.1. *Suppose $\triangle_v$ is a triangulation consisting of a set of triangles which all share one common vertex v. Suppose every triangle in $\triangle_v$ has at least one neighbor with which it shares a common edge. Then we call $\triangle_v$ a* cell*. If v is an interior vertex of $\triangle_v$, then we call $\triangle_v$ an* interior cell*. Otherwise, we call it a* boundary cell*.*

Note that in the terminology of Chapter 4, a cell is a shellable triangulation which is the star of a vertex. Our first result deals with boundary cells.

Theorem 9.2. *Let $\triangle_v$ be a boundary cell with n interior edges attached to the vertex v, see Figure 9.1 (left). Then for any $0 \le r \le d$,*

$$dim\ \mathcal{S}_d^r(\triangle_v) = \binom{d+2}{2} + n\binom{d-r+1}{2}. \tag{9.2}$$

Proof: Since the space $\mathcal{S}_d^r(\triangle_v)$ contains the space of polynomials $\mathcal{P}_d$, it follows that the dimension of $\mathcal{S}_d^r(\triangle_v)$ is equal to $\dim \mathcal{P}_d = \binom{d+2}{2}$ plus the dimension of the space $\mathcal{S}_0 := \{s \in \mathcal{S}_d^r(\triangle_v) : s \equiv 0 \text{ on } T^{[0]}\}$, where $T^{[i]} := \langle v, v_i, v_{i+1}\rangle$ for $i = 0, \ldots, n$. Without loss of generality, we may assume that the cell is centered at $v = (0, 0)$ and is rotated so that all of the coordinates (x_i, y_i) of the points v_i are nonzero. Let $y + \alpha_i x = 0$ be

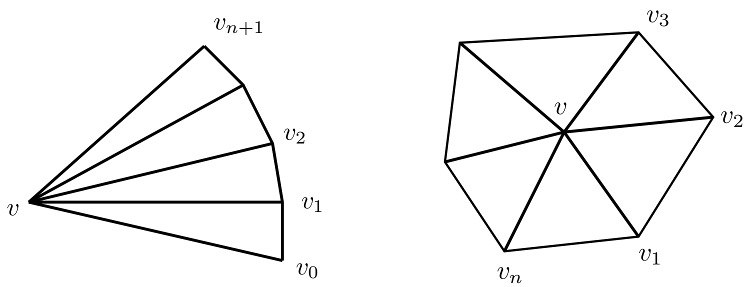


Fig. 9.1. Boundary and interior cells.

the equation of the i-th edge attached to v, where $\alpha_i := -y_i/x_i$. Suppose $s \in \mathcal{S}_0$. Then on the triangle $T^{[1]}$, s must have the form

$$s = \sum_{j=1}^{d-r} \sum_{k=1}^{j} a_{jk}^{[1]} (y + \alpha_1 x)^{r+k} x^{j-k}.$$

Since the polynomials in this sum are clearly linearly independent, we conclude that the dimension of $\mathcal{S}_d^r(\triangle_v)$ restricted to $T^{[0]} \cup\ T^{[1]}$ is $\binom{d+2}{2} + \binom{d-r+1}{2}$. Repeating this argument for each of the other interior edges leads immediately to (9.2). $\square$

We turn now to the case where $\triangle$ is an interior cell. This is a more complicated situation because as we go around the vertex, the smoothness conditions are connected. It turns out that for interior cells, the dimension of the spline space $\mathcal{S}_d^r(\triangle)$ depends on the slopes of the edges meeting at the vertex v. For any real number x, we write

$$(x)_+ = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{9.3}$$

Theorem 9.3. *Suppose $\triangle_v$ is an interior cell associated with a vertex v where n edges meet with m_v different slopes, see Figure 9.1 (right). Then for any $0 \le r \le d$,*

$$\dim \mathcal{S}_d^r(\triangle_v) = \binom{r+2}{2} + n\binom{d-r+1}{2} + \sigma_v, \tag{9.4}$$

where

$$\sigma_v := \sum_{j=1}^{d-r} (r + j + 1 - j m_v)_+ \,. \tag{9.5}$$

Proof: The dimension of $\mathcal{S}_d^r(\triangle_v)$ is equal to dim $\mathcal{P}_d = \binom{d+2}{2}$ plus the dimension of $\mathcal{S}_0 := \{s \in \mathcal{S}_d^r(\triangle_v) : s \equiv 0 \text{ on } T^{[n]}\}$, where $T^{[i]} := \langle v, v_i, v_{i+1} \rangle$ for $i = 1, \ldots, n$, and we identify $v_{n+1} := v_1$. Without loss of generality, we may assume that the cell is centered at $v = (0,0)$ and is rotated so that all of the coordinates (x_i, y_i) of the points v_i are nonzero. Let $y + \alpha_i x = 0$ be the equation of the i-th edge attached to v, where $\alpha_i := -y_i/x_i$. Suppose $s \in \mathcal{S}_0$. Then on the triangle $T^{[1]}$, s must have the form

$$s = \sum_{j=1}^{d-r} \sum_{k=1}^{j} a_{jk}^{[1]} (y + \alpha_1 x)^{r+k} x^{j-k}.$$

Adding similar terms as we cross each edge, we find that when we get back to $T^{[n]}$, s must have the form

$$s = \sum_{i=1}^{n} \sum_{j=1}^{d-r} \sum_{k=1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k}. \tag{9.6}$$

But on $T^{[n]}$ this expression must be identically zero. In particular, for each $j = 1, \ldots, d-r$, the coefficients of the powers $y^{r+j}, xy^{r+j-1}, \ldots, x^{r+j}$ must be zero. This translates into the system $A_j a_j = 0$, where

$$a_j := (a_{jj}^{[1]}, \ldots, a_{j1}^{[1]}, \ldots, a_{jj}^{[n]}, \ldots, a_{j1}^{[n]})^T$$

and

$$A_j := [A_j^{[1]}, \ldots, A_j^{[n]}],$$

with

$$A_j^{[\ell]} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \binom{r+j}{1}\alpha_\ell & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ \binom{r+j}{j-2}\alpha_\ell^{j-2} & \cdots & \cdots & 0 \\ \binom{r+j}{j-1}\alpha_\ell^{j-1} & \cdots & \cdots & 1 \\ \binom{r+j}{j}\alpha_\ell^{j} & \cdots & \cdots & \binom{r+1}{1}\alpha_\ell \\ \vdots & & & \vdots \\ \binom{r+j}{r+j}\alpha_\ell^{r+j} & \binom{r+j-1}{r+j-1}\alpha_\ell^{r+j-1} & \cdots & \binom{r+1}{r+1}\alpha_\ell^{r+1} \end{bmatrix}.$$

It follows that the dimension of $\mathcal{S}_0$ is equal to the number of linearly independent solutions of the system $Aa = 0$, where $a := (a_1, \ldots, a_{d-r})$ and

$$A := \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_{d-r} \end{bmatrix}. \tag{9.7}$$

This system consists of n_e equations in n_u unknowns, where

$$n_e := \sum_{j=1}^{d-r} (r+j+1) = \binom{d+2}{2} - \binom{r+2}{2}$$

and

$$n_u := n \sum_{j=1}^{d-r} j = n \binom{d-r+1}{2}.$$

The number of linear independent solutions is given by $n_u - n_r$, where n_r is the rank of A. By the block structure, it suffices to compute the rank of each A_j. We now show that

$$\text{rank}\,(A_j) = (r+j+1) - (r+j+1-jm_v)_+. \tag{9.8}$$

Clearly, the rank of A_j equals the rank of $B_j := [A_j^{[i_1]}, \ldots, A_j^{[i_{m_v}]}]$, where $i_1, \ldots, i_{m_v}$ is a set of indices such that the associated edges have different slopes. Examining the structure of B_j, we see that its transpose is a constant multiple of the matrix corresponding to Hermite interpolation up to the $(j-1)$-st derivative at each of the points $\alpha_{i_1}, \ldots, \alpha_{i_{m_v}}$ using the polynomials

$$1,\ \binom{r+j}{1} t,\ \binom{r+j}{2} t^2,\ \ldots,\ \binom{r+j}{r+j} t^{r+j}.$$

We now consider two cases.

Case 1: $r+j+1 \ge jm_v$. In this case the transpose of the $jm_v \times jm_v$ matrix obtained by taking the first jm_v rows of B_j corresponds to the interpolation problem of finding a polynomial p of degree $jm_v - 1$ satisfying

$$D^\nu p(\alpha_{i_\mu}) = 0, \qquad \nu = 0, \ldots, j-1, \quad \mu = 1, \ldots, m_v.$$

Since this problem has a unique solution, we conclude that B_j and thus also A_j has rank jm_v, and thus (9.8) holds.

Case 2: $r+j+1 < jm_v$. Let $q := \lfloor (r+j+1)/j \rfloor$, and let $u := r+j+1-qj$. Then the transpose of the $(r+j+1) \times (r+j+1)$ matrix formed from the first $r+j+1$ columns of B_j corresponds to the interpolation problem of finding a polynomial p of degree $r+j$ satisfying

$$\begin{aligned} D^\nu p(\alpha_{i_\mu}) &= 0, \qquad \nu = 0, \ldots, j-1, \quad \mu = 1, \ldots, q, \\ D^\nu p(\alpha_{i_{q+1}}) &= 0, \qquad \nu = 0, \ldots, u-1. \end{aligned}$$

Since this problem is uniquely solvable, we again conclude that B_j and thus A_j has rank $r+j+1$ as asserted in (9.8).

To complete the proof of the theorem, we note that by the block nature of A,

$$n_r = \text{rank } (A) = \sum_{j=1}^{d-r}(r+j+1) - \sum_{j=1}^{d-r}(r+j+1-jm_v)_+,$$

and the result follows. $\square$

Definition 9.4. *Suppose v is an interior vertex of a triangulation. Then we say that v is a* singular vertex *provided there are four edges attached to v where the edges lie on two lines that cross at v.*

Example 9.5. Let $\triangle_v$ be an interior cell with four interior edges. Then $\dim \mathcal{S}_2^1(\triangle_v) = 8$ if v is a singular vertex, and $\dim \mathcal{S}_2^1(\triangle_v) = 7$ if v is nonsingular.

Discussion: It is easy to see that for $r = 1$, the factor σ_v in (9.5) equals one whenever v is singular, and is zero otherwise. In Figure 9.2 we show the two cases and some associated minimal determining sets, where points in the determining sets are marked with black dots. $\square$

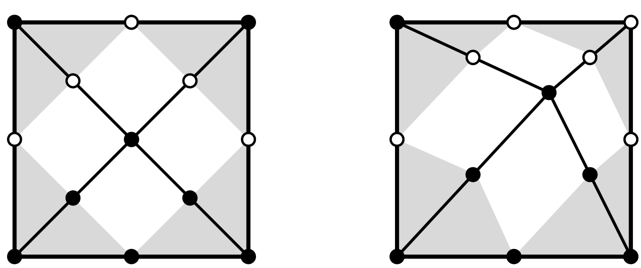


Fig. 9.2. A singular and a nonsingular vertex.

While it is possible to analyze the dimension of spline spaces on boundary cells directly in terms of minimal determining sets, it is not clear how to do this for interior cells. This is why we have used an algebraic approach in this section. We return to the problem of constructing minimal determining sets for spline spaces defined on cells in Chapter 11.

9.2. Dimension of Superspline Spaces on Cells

We now present dimension results for certain superspline spaces defined on cells. First we treat the case of a boundary cell.

Theorem 9.6. *Suppose $\triangle_v$ is a boundary cell as in Theorem 9.2, and let $0 \le r \le \rho \le d$. Then*

$$\dim \mathcal{S}_d^r(\triangle_v) \cap C^\rho(v) = \binom{d+2}{2} + n\left[\binom{d-r+1}{2} - \binom{\rho-r+1}{2}\right]. \tag{9.9}$$

Proof: The proof is very similar to the proof of Theorem 9.2. Following the notation introduced there, let $\mathcal{S}_0 := \{s \in \mathcal{S}_d^r(\triangle_v) \cap C^\rho(v) : s \equiv 0 \text{ on } T^{[0]}\}$. Let $s \in \mathcal{S}_0$. Then the fact that $s \in C^\rho(v)$ implies that on $T^{[1]}$, s must have the form

$$s = \sum_{j=\rho-r+1}^{d-r} \sum_{k=1}^{j} a_{jk}^{[1]} (y + \alpha_1 x)^{r+k} x^{j-k}.$$

Since the polynomials in this sum are clearly linearly independent, we conclude that the dimension of $\mathcal{S}_0$ restricted to $T^{[0]} \cup T^{[1]}$ is $\binom{d-r+1}{2} - \binom{\rho-r+1}{2}$. Repeating this argument for each of the other edges leads immediately to (9.9). $\square$

We have the following result for interior cells.

Theorem 9.7. *Suppose $\triangle_v$ is an interior cell associated with a vertex v where n edges meet with m_v different slopes. Then for all $0 \le r \le \rho \le d$,*

$$\dim \mathcal{S}_d^r(\triangle_v) \cap C^\rho(v) = \binom{\rho+2}{2} + n\left[\binom{d-r+1}{2} - \binom{\rho-r+1}{2}\right] + \sigma_v,$$

where

$$\sigma_v := \sigma_v(r,\rho,d) := \sum_{j=\rho-r+1}^{d-r} (r + j + 1 - j m_v)_+ .$$

Proof: The proof is similar to the proof of Theorem 9.3. Let $\mathcal{S}_0 := \{s \in \mathcal{S}_d^r(\triangle_v) \cap C^\rho(v) : s \equiv 0 \text{ on } T^{[n]}\}$. Let $s \in \mathcal{S}_0$. Then after passing over each of the n edges attached to v, when we get back to $T^{[n]}$, s must have the form (9.6). But the condition that $s \in C^\rho(v)$ implies that the terms with $1 \le j \le \rho - r$ must be zero. Thus, the dimension of $\mathcal{S}_0$ is equal to the number of nontrivial solutions of

$$\sum_{i=1}^{n} \sum_{j=\rho-r+1}^{d-r} \sum_{k=1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k} \equiv 0.$$

Setting powers of x and y to zero, we get the system of equations $Aa = 0$, where $a := (a_{\rho-r+1}, \ldots, a_{d-r})$, and

$$A := \begin{bmatrix} A_{\rho-r+1} & & \\ & \ddots & \\ & & A_{d-r} \end{bmatrix}.$$

Here a_j and A_j are as in the proof of Theorem 9.3. This system consists of n_e equations in n_u unknowns, where

$$n_e := \sum_{j=\rho-r+1}^{d-r} (r+j+1) = \binom{d+2}{2} - \binom{\rho+2}{2},$$

and

$$n_u := n \sum_{j=\rho-r+1}^{d-r} j = n\left[\binom{d-r+1}{2} - \binom{\rho-r+1}{2}\right].$$

The number of linear independent solutions is given by $n_u - n_r$, where n_r is the rank of A. Using (9.8), we see that

$$n_r = \operatorname{rank}(A) = \sum_{j=\rho-r+1}^{d-r} (r+j+1) - \sum_{j=\rho-r+1}^{d-r} (r+j+1-jm_v)_+,$$

and the result follows. $\square$

9.3. Bounds on the Dimension of $\mathcal{S}_d^r(\triangle)$

Throughout this section we assume that $\triangle$ is a shellable triangulation of a set Ω, see Definition 4.6. Our aim is to establish some fairly tight upper and lower bounds on the dimension of the spline spaces $\mathcal{S}_d^r(\triangle)$ for general r and d.

Definition 9.8. *A triangle T in a triangulation $\triangle$ of a domain Ω is called a* flap *provided two of its edges lie on the boundary of Ω. The triangle T is called a* fill *provided exactly one of its edges lies on the boundary of Ω.*

Let $\mathcal{V}_I$ be the set of interior vertices of $\triangle$. For each $v \in \mathcal{V}_I$, let m_v be the number of edges attached to v with different slopes. Suppose V_I and E_I are the numbers of interior vertices and edges of $\triangle$, respectively.

Theorem 9.9. *For all $0 \le r \le d$,*

$$D + \sigma \;\le\; dim\ \mathcal{S}_d^r(\triangle), \tag{9.10}$$

where

$$D := \binom{d+2}{2} + \binom{d-r+1}{2} E_I - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] V_I, \tag{9.11}$$

and

$$\sigma := \sum_{v \in \mathcal{V}_I} \sigma_v, \tag{9.12}$$

where σ_v is defined in (9.5).

Proof: The proof proceeds by induction on the number of triangles in $\triangle$. The result is trivial for a single triangle. Now suppose $\triangle$ is a triangulation of Ω with N triangles, and that the bound holds for all triangulations with $N-1$ triangles. Suppose some triangle T of $\triangle$ is a flap, and let $\widetilde{\triangle} = \triangle \setminus \{T\}$. Adding a flap T to $\widetilde{\triangle}$ increases the dimension by $\beta := \binom{d-r+1}{2}$, cf. the proof of Theorem 9.2. The desired bound follows for $\triangle$.

We now consider the case where $\triangle$ does not contain a flap. Then it must contain at least one fill T. Let v be the vertex of T which lies in the interior of $\triangle$. Let $\Omega_2 := \operatorname{star}(v)$ be the union of the triangles in $\triangle$ surrounding v, and let $\Omega_1 := \Omega \setminus \{T\}$ and $\Omega_3 := \Omega_1 \cap \Omega_2$. For $i = 1, 2, 3$, set $\mathcal{S}_i := \mathcal{S}_d^r(\triangle)|_{\Omega_i}$. Let $\mathcal{S}_1^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_1$ but not to $\mathcal{S}_2$. Similarly, let $\mathcal{S}_2^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_2$ but not to $\mathcal{S}_1$. Let $\mathcal{S}_{12}^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to both $\mathcal{S}_1$ and $\mathcal{S}_2$. Then the space of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_1$ is $\mathcal{S}_{12}^E \oplus \mathcal{S}_1^E$. Similarly, the space of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_2$ is $\mathcal{S}_{12}^E \oplus \mathcal{S}_2^E$. This implies

$$\mathcal{S}_3 = \mathcal{S}_{12}^E \oplus \mathcal{S}_1^E \oplus \mathcal{S}_2^E \oplus \mathcal{S}_{12}^{NE}, \tag{9.13}$$

where $\mathcal{S}_{12}^{NE}$ is the space of all splines in $\mathcal{S}_3$ that cannot be extended to either $\mathcal{S}_1$ or $\mathcal{S}_2$. Clearly, for $i = 1, 2$,

$$\dim \mathcal{S}_i = \dim \mathcal{S}_i^0 + \dim \mathcal{S}_{12}^E + \dim \mathcal{S}_i^E, \tag{9.14}$$

where

$$\mathcal{S}_i^0 := \{s \in \mathcal{S}_i : s \text{ vanishes on } \Omega_3\}.$$

Solving (9.14) for $\dim \mathcal{S}_i^0$ and inserting in

$$\dim \mathcal{S}_d^r(\triangle) = \dim \mathcal{S}_1^0 + \dim \mathcal{S}_2^0 + \dim \mathcal{S}_{12}^E,$$

it follows from (9.13) that

$$\begin{aligned}
\dim \mathcal{S}_d^r(\triangle) &= \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_1^E - \dim \mathcal{S}_2^E - \dim \mathcal{S}_{12}^E \\
&= \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_3 + \dim \mathcal{S}_{12}^{NE} \\
&\geq \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_3.
\end{aligned}$$

Now using the induction hypothesis, we have

$$\dim \mathcal{S}_d^r(\triangle) \geq \binom{d+2}{2} + \beta(E_I - 2) - \gamma(V_I - 1) + \sigma^{(1)} + 2\beta - \gamma + \sigma^{(2)},$$

where $\beta := \binom{d-r+1}{2}$, $\gamma := \binom{d+2}{2} - \binom{r+2}{2}$, and $\sigma^{(i)}$ is the number (9.12) associated with the triangulation $\triangle_i$ underlying $\mathcal{S}_i$. Combining terms leads to the lower bound (9.10). $\square$

Our aim now is to provide an upper bound on the dimension of $\mathcal{S}_d^r(\triangle)$ to complement the lower bound in Theorem 9.9. We first need a definition.

Definition 9.10. *Suppose $\triangle$ is a regular triangulation of a domain Ω without holes. Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be a grouping of the triangles of $\triangle$ into disjoint subsets, and for each $i = 1, \ldots, n$, let Ω_i be the union of the triangles in $\bigcup_{j=1}^{i} \mathcal{T}_j$. Let $\Omega_0 = \emptyset$. Suppose there exist vertices $v_1, \ldots, v_n$ such that:*

1) *For each $1 \le i \le n$, $\mathcal{T}_i$ is the union of all triangles in $\triangle \setminus \Omega_{i-1}$ that share the vertex v_i.*
2) *For each $2 \le i \le n$, v_i is on the boundary of Ω_{i-1}.*

Then we say that $\mathcal{T}_1, \ldots, \mathcal{T}_n$ is an admissible decomposition of $\triangle$ with centers *$v_1, \ldots, v_n$.*

We can construct an admissible decomposition of any regular triangulation $\triangle$ without holes by starting with $\mathcal{T}_1 := \mathrm{star}(v_1)$ for some arbitrary vertex of $\triangle$. Then we repeatedly choose a vertex on the boundary of Ω_{i-1} and take $\mathcal{T}_i$ to be all unchosen triangles attached to v_i. Since the starting vertex can be arbitrary, it is clear that any given $\triangle$ has several different admissible decompositions.

Example 9.11. *Let $\triangle_{MS}$ be the triangulation shown in Figure 9.3.*

Discussion: $\triangle_{MS}$ is called the Morgan–Scott triangulation. The sets $\mathcal{T}_1 := \{T_1, \ldots, T_4\}$, $\mathcal{T}_2 := \{T_5, T_6\}$, and $\mathcal{T}_3 := \{T_7\}$ provide an admissible decomposition with centers v_1, v_2, v_3. An alternative admissible decomposition is provided by $\mathcal{T}_1 := \{T_1, T_4, T_7\}$, $\mathcal{T}_2 := \{T_2, T_3\}$, and $\mathcal{T}_3 := \{T_5, T_6\}$. $\square$

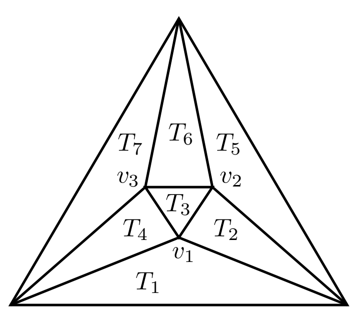


Fig. 9.3. The Morgan–Scott triangulation.

In the first decomposition given in Example 9.11, the three centers v_1, v_2, v_3 are all interior vertices. However, for the second decomposition, the first center is a boundary vertex. Figure 9.4 gives an example of a triangulation which is impossible to decompose without using at least one boundary vertex.

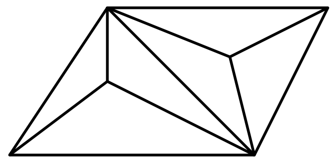


Fig. 9.4. A triangulation $\triangle$ such that every admissible decomposition must include one center which is a boundary vertex of $\triangle$.

We are now ready to establish an upper bound to go with the lower bound of Theorem 9.9.

Theorem 9.12. *Let $0 \le r \le d$, and suppose $\mathcal{T}_1, \ldots, \mathcal{T}_n$ is an admissible decomposition of $\triangle$ with centers $v_1, \ldots, v_n$. For each v_i, let n_i be the number of interior edges of $\triangle$ attached to v_i but not attached to any v_j with $j < i$. Let $\tilde{m}_i$ be the number of such edges, where we count only edges with different slopes. Finally, let*

$$\widetilde{\sigma}_i := \begin{cases} \displaystyle\sum_{j=1}^{d-r} (r + j + 1 - j\tilde{m}_i)_+ , & \text{if } v_i \in \mathcal{V}_I, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\dim \mathcal{S}_d^r(\triangle) \le D + \sum_{i=1}^n \widetilde{\sigma}_i, \tag{9.15}$$

where D is as in (9.11).

Proof: Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n = \Omega$ be the sets associated with an admissible decomposition of $\triangle$ as in Definition 9.10. If v_1 is a boundary vertex, then by Theorem 9.2 the dimension of $\mathcal{S}_d^r(\triangle)|_{\Omega_1}$ is equal to $\binom{d+2}{2} + n_1\beta$, where $\beta := \binom{d-r+1}{2}$. If v_1 is an interior vertex, then by Theorem 9.3, the dimension of $\mathcal{S}_d^r(\triangle)|_{\Omega_1}$ is equal to $\binom{d+2}{2} + n_1\beta - \gamma + \widetilde{\sigma}_1$ where $\gamma := \binom{d+2}{2} - \binom{r+2}{2}$. We now give a bound on how many ways a spline in $\mathcal{S}_d^r(\triangle)|_{\Omega_1}$ can be extended to be a spline in $\mathcal{S}_d^r(\triangle)|_{\Omega_2}$. If v_2 is a boundary vertex of $\triangle$, then by the proof of Theorem 9.2, it is clear that the number of linearly independent splines in $\mathcal{S}_d^r(\triangle)|_{\Omega_2 \setminus \Omega_1}$ that vanish on Ω_1 is at most $n_2\beta$. If v_2 is an interior vertex of $\triangle$, then as in the proof of Theorem 9.3, the number of linearly independent splines in $\mathcal{S}_d^r(\triangle)|_{\Omega_2 \setminus \Omega_1}$ that vanish on Ω_1 is equal to the number of linearly independent solutions of the system $Aa = 0$, where A is a matrix as in (9.7). This matrix has $n_2\beta$ columns and rank $\gamma - \widetilde{\sigma}_2$, and so there are at most $n_2\beta - \gamma + \widetilde{\sigma}_2 \ge 0$ linearly independent splines in $\mathcal{S}_d^r(\triangle)|_{\Omega_2 \setminus \Omega_1}$ that vanish on Ω_1. Repeating this argument for $v_3, \ldots, v_n$ gives (9.15). $\square$

Different admissible decompositions of a given triangulation $\triangle$ can lead to different upper bounds on the dimension of $\mathcal{S}_d^r(\triangle)$. Moreover, the lower and upper bounds of Theorems 9.9 and 9.12 will not agree in general, even if we find the best decomposition. Here is an explicit example.

Example 9.13. Let $\triangle_{MS}$ be the Morgan–Scott triangulation shown in Figure 9.3. Then $6 \le \dim \mathcal{S}_2^1(\triangle_{MS}) \le 7$.

Discussion: For this triangulation, $V_I = 3$ and $E_I = 9$. Now with $\beta = 1$ and $\gamma = 3$, Theorem 9.9 gives the lower bound $D = 6$ for the dimension of $\mathcal{S}_2^1(\triangle_{MS})$. Using the first decomposition of Example 9.11, we have $n_1 = \tilde{m}_1 = 4$, $n_2 = \tilde{m}_2 = 3$, and $n_3 = \tilde{m}_3 = 2$. Since $\widetilde{\sigma}_3 = 1$, this gives an upper bound of seven for the dimension of $\mathcal{S}_2^1(\triangle_{MS})$. The second decomposition in Example 9.11 also gives an upper bound of seven. It is easy to check that these upper bounds cannot be improved by choosing any other decomposition. □

This example has been intensively studied in the literature to determine exactly when the dimension is six and when it is seven, see Remark 9.1. It turns out that the dimension of $\mathcal{S}_2^1(\triangle)$ is only seven for very special choices of the interior vertices, including the symmetric configuration shown in Figure 9.3.

9.4. Dimension of $\mathcal{S}_d^r(\triangle)$ for $d \ge 3r+2$

In this section we construct a minimal determining set for $\mathcal{S}_d^r(\triangle)$ for all $d \ge 3r+2$, and use it to give a formula for the dimension of $\mathcal{S}_d^r(\triangle)$. First we need some further notation.

Definition 9.14. *Suppose that v is an interior vertex of a triangulation $\triangle$, and that w_1, w_2, w_3 are such that $\langle v, w_i\rangle$ are three consecutive edges connected to v in counterclockwise order. We say that the* edge $e := \langle v, w_2\rangle$ *is* degenerate at the vertex v *provided the two edges $\langle v, w_1\rangle$ and $\langle v, w_3\rangle$ are collinear. Otherwise we call e* nondegenerate.

Note that a vertex is singular provided there are exactly four edges attached to v, and they are all degenerate. It is impossible for an edge of a triangulation to be degenerate at both ends. Thus, a singular vertex cannot be a neighbor of another singular vertex. To state our next theorem we need some notation for various subsets of the domain points $\mathcal{D}_{d,T}$ lying in a triangle $T := \langle u, v, w\rangle$. Let

$$\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor. \tag{9.16}$$

Given a vertex u and an edge $e := \langle u, v \rangle$, we define the following sets:

$$\begin{aligned}
A^T(u) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi^T_{d-2r+i-1,r-j,r-i+j+1}\}, \\
C^T &:= \{\xi^T_{ijk} : i > r,\ j > r,\ k > r\}, \\
D^T_\mu(u) &:= \big\{\xi^T_{ijk} : i \geq d - \mu\big\}, \\
E^T(e) &:= F^T(e) \setminus \big[D^T_\mu(u) \cup D^T_\mu(v) \cup A^T(u) \cup A^T(v) \cup G^T_L(e) \cup G^T_R(e)\big], \\
F^T(e) &:= \big\{\xi^T_{ijk} : k \leq r\big\}, \\
G^T_L(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi^T_{d-2r+i-1,r+1+j,r-i-j}\}, \\
G^T_R(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi^T_{r+1+j,d-2r+i-1,r-i-j}\}.
\end{aligned} \tag{9.17}$$

The set $D^T_\mu(u)$ contains the points in a disk of radius μ around u. $A^T(u)$ is the set of points not in $D^T_\mu(u)$ but whose corresponding coefficients are involved in smoothness conditions of order up to r across both of the edges $\langle u, v \rangle$ and $\langle u, w \rangle$. The sets $E^T(e)$, $G^T_L(e)$ and $G^T_R(e)$ include only domain points whose corresponding coefficients are involved in C^r smoothness conditions across the edge e. Finally, C^T corresponds to coefficients in T which do not enter any C^r smoothness conditions across edges.

In Figure 9.5 we have marked the domain points for the case $d = 15$, $r = 4$, and $\mu = 6$ with different symbols to indicate which of the above sets they belong to. Points in the sets $A^T(u)$, C^T, $D^T_\mu(u)$, and $E^T(e)$ are marked with $\otimes$, squares, dots, and triangles, respectively. Points in $G^T_L(e) \cup G^T_R(e)$ are marked with $\oplus$. It is easy to check that

$$\#A^T(u) = \#G^T_L(e) = \#G^T_R(e) = \binom{2r-\mu+1}{2},$$
$$\#C^T = \binom{d-3r-1}{2}, \qquad \#D^T_\mu(u) = \binom{\mu+2}{2}.$$

Moreover, the cardinality of $E^T(e)$ is given by

$$\begin{aligned}
&\binom{d+2}{2} - \binom{d-r+1}{2} - 2\binom{\mu+2}{2} + 2\binom{\mu-r+1}{2} - 4\binom{2r-\mu+1}{2} \\
&\qquad = dr + d + 6\mu r - \frac{9}{2}r - 1 - \frac{15}{2}r^2 - 2\mu^2.
\end{aligned}$$

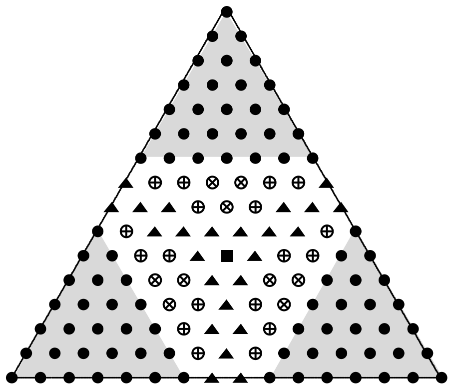


Fig. 9.5. The set $\mathcal{D}_{d,T}$ for $d = 15$, $r = 4$, $\mu = 6$.

Theorem 9.15. *Let $d \geq 3r + 2$ and let μ be as (9.16). Then the following set $\mathcal{M}$ of domain points is a minimal determining set for $\mathcal{S}_d^r(\triangle)$:*

1) *For each boundary vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_\mu^r(\triangle_v)$, where $\triangle_v = star(v)$.*
2) *For each interior vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_\mu^r(\triangle_v)$, where $\triangle_v = star(v)$.*
3) *For each triangle T in $\triangle$, choose the domain points in C^T.*
4) *For each edge e of $\triangle$, include the domain points in the set $E^T(e)$ for some triangle T with edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, T can be either of the two triangles sharing e. If e is a boundary edge, also include the domain points in the two sets $G_L^T(e)$ and $G_R^T(e)$.*
5) *For each triangle $T := \langle u, v, w\rangle$, include the sets $A^T(u)$, $A^T(v)$, and $A^T(w)$.*
6) *If $T_1 := \langle v, v_1, v_2\rangle$ and $T_2 := \langle v, v_2, v_3\rangle$ are two triangles sharing an edge $e := \langle v, v_2\rangle$ that is degenerate at v, then replace the set $A^{T_2}(v)$ by $G_L^{T_1}(e)$.*
7) *If v is a singular vertex, reinsert the set $A^T(v)$ for one triangle T attached to v.*

Proof: Let $s \in \mathcal{S}_d^r(\triangle)$ and suppose we fix its coefficients corresponding to points in $\mathcal{M}$. We now show that all remaining coefficients are uniquely and consistently determined by the smoothness conditions. First, by the choice of $\mathcal{M}$, all coefficients of s corresponding to domain points in the disks $D_\mu(v)$ around vertices v of $\triangle$ are uniquely determined. We now compute coefficients corresponding to domain points in the rings $R_{\mu+1}(v)$ for all v.

We do this by successively processing arcs of the form

$$a^r_{\mu+1,e}(v) := \{\xi \in R_{\mu+1}(v) : \operatorname{dist}(\xi, e) \leq r\},$$

where e is an edge attached to v, and where $\operatorname{dist}(\xi, e) \leq r$ describes the domain points that are either on the edge e, or in the first r rows parallel to e. Suppose $v_1, \ldots, v_n$ are the vertices attached to v in counterclockwise order. To get this process started, we note that $\mathcal{M}$ contains the set $A^{T_1}(v)$ for at least one of the triangles attached to v, which we may assume is $T_1 := \langle v, v_1, v_2 \rangle$. Now if the edge $e_2 := \langle v, v_2 \rangle$ is nondegenerate at v, $\mathcal{M}$ also contains the set $A^{T_2}(v)$ for a neighboring triangle $T_2 := \langle v, v_2, v_3 \rangle$. Then we can apply Lemma 2.30 to compute the coefficients associated with domain points on the arc $a^r_{\mu+1,e_2}(v)$. If the edge e_2 is degenerate at v, $\mathcal{M}$ contains the set $G_L^{T_2}(e_2)$. In this case, we simply use the C^r smoothness conditions across e_2 to compute coefficients on the arc. Now we continue this process in a *counterclockwise* direction around v to compute all coefficients associated with domain points on the ring $R_{\mu+1}(v)$. After computing all coefficients for the rings of radius $\mu+1$, we then do rings of radii $\mu+2$, etc., until we have computed all coefficients corresponding to domain points in the disks $D_{2r}(v)$.

Next using the C^r smoothness conditions across the edges, we can compute the coefficients corresponding to the remaining domain points in the sets $F^T(e)$ for all edges e. A careful examination of the way in which coefficients are computed shows that at this point we have uniquely determined all coefficients in a way that ensures that s satisfies all smoothness conditions for $\mathcal{S}_d^r(\triangle)$. It follows from Theorem 5.15 that $\mathcal{M}$ is a minimal determining set. $\square$

We now present the main result of this section. Since we are going to use Euler relations in the proof, we now have to restrict ourselves to regular triangulations, see Definition 4.7. To simplify matters even more, we treat only the case where $\triangle$ has no holes. For the more general case, see Remark 9.5. Let V_I, V_B be the numbers of interior and boundary vertices of $\triangle$, respectively. Similarly, let E_I, E_B be the numbers of interior and boundary edges, and let N be the number of triangles in $\triangle$.

Theorem 9.16. *Suppose $\triangle$ is a shellable triangulation, i.e., a regular triangulation with no holes. Then for all $d \geq 3r+2$,*

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) = {} & \frac{d^2+r^2-r+d-2rd}{2} V_B + (d-r)(d-2r)V_I \\ & + \frac{-2d^2+6rd-3r^2+3r+2}{2} + \sigma, \end{aligned} \tag{9.18}$$

where σ is as in (9.12).

Proof: By Theorem 5.13, the dimension of $\mathcal{S}_d^r(\triangle)$ is equal to the cardinality of the minimal determining set $\mathcal{M}$ constructed in Theorem 9.15. We now count the number of domain points in $\mathcal{M}$. Let $\mathcal{V}_B$ and $\mathcal{V}_I$ be the sets of boundary and interior vertices of $\triangle$, and let $\mathcal{E}_B$ be the set of boundary edges. Using Theorems 9.2 and 9.3 to count the number of points listed in 1) and 2), we get the following formula for the cardinality of $\mathcal{M}$:

$$\begin{aligned}
\#\mathcal{M} = \sum_{v\in\mathcal{V}_B} &\left[\binom{\mu+2}{2} + n_v\binom{\mu-r+1}{2}\right] \\
&+ \sum_{v\in\mathcal{V}_I}\left[\binom{r+2}{2} + n_v\binom{\mu-r+1}{2} + \sigma_v^\mu\right] \\
&+ 2\sum_{e\in\mathcal{E}_B}\binom{2r-\mu+1}{2} \\
&+ N\left[\binom{d-3r-1}{2} + 3\binom{2r-\mu+1}{2}\right] \\
&+ (E_I+E_B)\left[dr+d+6\mu r - \frac{9}{2}r - 1 - \frac{15}{2}r^2 - 2\mu^2\right] \\
&+ \sigma_{sing}\binom{2r-\mu+1}{2},
\end{aligned} \tag{9.19}$$

where n_v is the number of interior edges meeting at the vertex v, and m_v is the number of those edges with different slopes. Here σ_{sing} is the number of singular vertices, and

$$\sigma_v^\mu := \sum_{j=1}^{\mu-r}(r+j+1-jm_v)_+ .$$

If v is not a singular vertex, then σ_v^μ is equal to

$$\sigma_v := \sum_{j=1}^{d-r}(r+j+1-jm_v)_+ .$$

If v is a singular vertex, then σ_v^μ and the factor multiplying σ_{sing} combine to produce σ_v. Now using the Euler relations (4.5), we have

$$\sum_{v\in\mathcal{V}} n_v = 2E_I = 2(V_B + 3V_I - 3). \tag{9.20}$$

Substituting this in (9.19), and collecting terms, we get (9.18). $\square$

Using the Euler relations (4.5), it is easy to see that the formula (9.18) can be rewritten as

$$\dim \mathcal{S}_d^r(\triangle) = \binom{d+2}{2} + \binom{d-r+1}{2}E_I - \frac{d^2+3d-r^2-3r}{2}V_I + \sigma.$$

Comparing with (9.10), we see that this is exactly the lower bound given in Theorem 9.9.

The proof of Theorem 9.16 relies on being able to compute the cardinality of the minimal determining set $\mathcal{M}$ constructed in Theorem 9.15. This was possible, even though we did not give explicit choices for the minimal determining sets for splines on cells appearing in items 1) and 2) of that theorem. For explicit constructions of such determining sets, see Remark 9.3 and Chapter 11.

9.5. Dimension of Superspline Spaces

Given a regular triangulation $\triangle$, let $\mathcal{V}_I$ and $\mathcal{V}_B$ be the sets of interior and boundary vertices, respectively, and let $\mathcal{V} := \mathcal{V}_I \cup \mathcal{V}_B$. Let $\mathcal{E}_I$ and $\mathcal{E}_B$ be the sets of interior and boundary edges, respectively, and let $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_B$. Given $0 \le r \le d$, for each $v \in \mathcal{V}$, let ρ_v be an integer with $r \le \rho_v \le d$. We write $\rho := \{\rho_v\}_{v \in \mathcal{V}}$. In this section we compute the dimension of the superspline space

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \ s \in C^{\rho_v}(v), \ v \in \mathcal{V}\},$$

for $d \ge 3r+2$. To analyze these spaces, we need to assume that certain disks surrounding the vertices do not overlap. Let

$$k_v := \max\{\rho_v, \mu\}, \qquad v \in \mathcal{V}, \tag{9.21}$$

where μ is defined in (9.16). Throughout the remainder of this section we suppose that

$$k_u + k_v < d, \tag{9.22}$$

for all neighboring vertices u and v.

As in the previous section, to determine the dimension of $\mathcal{S}_d^{r,\rho}(\triangle)$ we shall construct a minimal determining set. To this end we need some additional notation. Given a vertex u and an edge $e := \langle u, v\rangle$ of a triangle $T := \langle u, v, w\rangle$, we define

$$\begin{aligned}
\tilde{A}^T(u) &:= A^T(u) \setminus D_{k_u}^T(u), \\
\tilde{C}^T &:= C^T \setminus \big(D_{k_u}^T(u) \cup D_{k_v}^T(v) \cup D_{k_w}^T(w)\big), \\
\widetilde{E}^T(e) &:= E^T(e) \setminus \big(D_{k_u}^T(u) \cup D_{k_v}^T(v)\big), \\
\tilde{G}_L^T(e) &:= G_L^T(e) \setminus D_{k_u}^T(u), \\
\tilde{G}_R^T(e) &:= G_R^T(e) \setminus D_{k_v}^T(v).
\end{aligned} \tag{9.23}$$

We define similar sets for the other vertices and edges.

Theorem 9.17. *Let $d \ge 3r+2$. Then the following set $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_d^{r,\rho}(\triangle)$:*

1) *For each boundary vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$, where $\triangle_v = star(v)$.*
2) *For each interior vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$.*
3) *For each triangle T in $\triangle$, choose the domain points in $\tilde{C}^T$.*
4) *For each edge e of $\triangle$, include the domain points in the set $\widetilde{E}^T(e)$ for some triangle T with edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, T can be either of the two triangles sharing e. If e is a boundary edge, also include the domain points in the two sets $\tilde{G}_L^T(e)$ and $\tilde{G}_R^T(e)$.*
5) *For each triangle $T := \langle u, v, w\rangle$, include the sets $\tilde{A}^T(u)$, $\tilde{A}^T(v)$, and $\tilde{A}^T(w)$.*
6) *If $T_1 := \langle v, v_1, v_2\rangle$ and $T_2 := \langle v, v_2, v_3\rangle$ are two triangles sharing an edge $e := \langle v, v_2\rangle$ that is degenerate at v, then replace the set $\tilde{A}^{T_2}(v)$ by $\tilde{G}_L^{T_1}(e)$.*
7) *If v is a singular vertex, include the set $\tilde{A}^T(v)$ for one triangle T attached to v.*

Proof: The proof is very similar to the proof of Theorem 9.15. First we use 1) and 2) to determine all coefficients in the disks $D_{k_v}(v)$ for all $v \in \mathcal{V}$. Then for each edge $e := \langle u, v\rangle$, we compute any undetermined coefficients on arcs across e until all coefficients in $D_{\ell_v}(v)$ and $D_{\ell_u}(u)$ whose distance to e is at most r have been determined, where $\ell_v := \max(\rho_v, 2r)$ for all $v \in \mathcal{V}$. Finally, any remaining undetermined coefficients corresponding to points in the sets $F^T(e)$ are determined for all edges e. $\square$

While Theorem 9.17 holds for arbitrary regular triangulations, for the following result we have to assume that $\triangle$ is a regular triangulation without holes.

Theorem 9.18. *Let $\triangle$ be a shellable triangulation, i.e., a regular triangulation with no holes, and suppose $d \ge 3r+2$. Let $\rho := (\rho_v)_{v\in\mathcal{V}}$ be a vector satisfying (9.22). For each interior vertex of $\triangle$, let*

$$\sigma_v^{k_v} := \sigma_v(r, \rho_v, k_v) := \sum_{j=\rho_v-r+1}^{k_v-r} (r+j+1-jm_v)_+ .$$

Let $\mathcal{V}_{sing}$ be the set of singular vertices of $\triangle$. Then the dimension of $\mathcal{S}_d^{r,\rho}(\triangle)$

is given by

$$
\begin{aligned}
&\sum_{v\in\mathcal{V}_B} \left[\binom{k_v+2}{2} + n_v\binom{k_v-r+1}{2} - n_v\binom{\rho_v-r+1}{2}\right] \\
&\quad + \sum_{v\in\mathcal{V}_I} \left[\binom{\rho_v+2}{2} + n_v\binom{k_v-r+1}{2} - n_v\binom{\rho_v-r+1}{2} + \sigma_v^{k_v}\right] \\
&\quad + \sum_{\langle u,v\rangle\in\mathcal{E}} \left[\binom{d+2}{2} - \binom{d-r+1}{2} - \binom{k_u+2}{2} + \binom{k_u-r+1}{2}\right. \\
&\qquad \left. - \binom{k_v+2}{2} + \binom{k_v-r+1}{2} - 2\binom{2r-k_u+1}{2} - 2\binom{2r-k_v+1}{2}\right] \\
&\quad + \sum_{\langle u,v\rangle\in\mathcal{E}_B} \left[\binom{2r-k_u+1}{2} + \binom{2r-k_v+1}{2}\right] \\
&\quad + \sum_{\langle u,v,w\rangle\in\triangle} \left[\binom{d-3r-1}{2} - \binom{k_u-2r}{2} - \binom{k_v-2r}{2} - \binom{k_w-2r}{2}\right. \\
&\qquad\qquad \left. + \binom{2r-k_u+1}{2} + \binom{2r-k_v+1}{2} + \binom{2r-k_w+1}{2}\right] \\
&\quad + \sum_{v\in\mathcal{V}_{sing}} \binom{2r-k_v+1}{2}.
\end{aligned}
$$

Proof: By Theorem 5.13, it suffices to compute the cardinality of the MDS $\mathcal{M}$ of Theorem 9.17. For each boundary vertex v, the set in item 1) of that theorem has cardinality

$$\binom{k_v+2}{2} + n_v\binom{k_v-r+1}{2} - n_v\binom{\rho_v-r+1}{2}$$

by Theorem 9.6. Similarly, by Theorem 9.7, for each interior vertex v, the set in 2) has cardinality

$$\binom{\rho_v+2}{2} + n_v\binom{k_v-r+1}{2} - n_v\binom{\rho_v-r+1}{2} + \sigma_v^{k_v}.$$

For each triangle $T := \langle u,v,w\rangle$ the set in 3) has cardinality

$$\#\tilde{C}^T = \binom{d-3r-1}{2} - \binom{k_u-2r}{2} - \binom{k_v-2r}{2} - \binom{k_w-2r}{2}.$$

For each edge $e = \langle u, v \rangle$,

$$\#\widetilde{E}(e) = \binom{d+2}{2} - \binom{d-r+1}{2} - \binom{k_u+2}{2} + \binom{k_u-r+1}{2} - \binom{k_v+2}{2} + \binom{k_v-r+1}{2} - 2\binom{2r-k_u+1}{2} - 2\binom{2r-k_v+1}{2}.$$

Moreover,

$$\#\tilde{A}^T(u) = \#\tilde{G}_L^T(e) = \binom{2r-k_u+1}{2}$$

and

$$\#\tilde{A}^T(v) = \#\tilde{G}_R^T(e) = \binom{2r-k_v+1}{2}.$$

Summing these quantities leads to the stated dimension formula. □

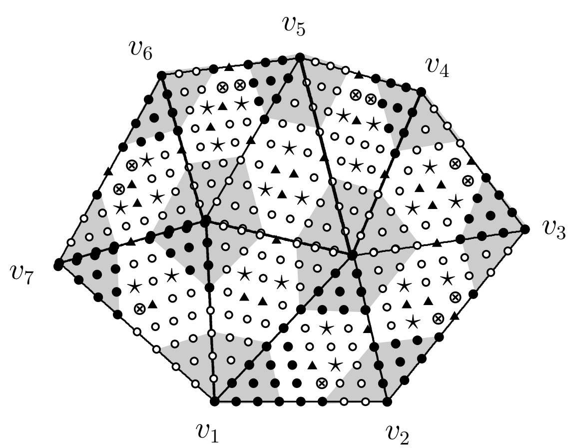


Fig. 9.6. A minimal determining set for Example 9.19.

Theorem 9.18 can be simplified in the case where all ρ_v are equal.

Example 9.19. Let $\triangle$ be the triangulation shown in Figure 9.6, and let $d = 8$, $r = 2$, and $\rho = (4, 2, 2, 3, 3, 2, 2, 3, 3)$.

Discussion: In this case $\mu = 3$, and by Theorem 9.18 the dimension of $\mathcal{S}_d^{r,\rho}(\triangle)$ is 165. In Figure 9.6 we mark the points in the minimal determining set of Theorem 9.17. For each vertex v, points in the minimal determining sets for $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$ are marked with black dots, while points in the sets $\tilde{A}^T(v)$ are marked with stars. We identify points in the sets $\widetilde{E}^T(e)$ with triangles, and points in the sets $\tilde{G}_L^T(e)$ and $\tilde{G}_R^T(e)$ with $\otimes$. In this example the sets $\tilde{C}^T$ are empty. □

Corollary 9.20. *Suppose $d \geq 3r+2$ and $\rho := (k, \ldots, k)$ with $r \leq k$ and $2k < d$. Then*

$$\begin{aligned}
\dim \mathcal{S}_d^{r,\rho}(\triangle) =& \frac{d^2 - 2rd - r^2 + d + r - 2k^2 + 4kr - 2k}{2} V_B \\
&+ \frac{2d^2 - 6rd - 3r^2 + 3r - 5k^2 + 12kr - 3k}{2} V_I \\
&+ \frac{-2d^2 + 6rd + 3r^2 - 3r + 6k^2 - 12rk + 6k + 2}{2} \\
&+ \sum_{v \in \mathcal{V}_I} \sum_{j=k-r+1}^{k_v - r} \Big(r + j + 1 - j m_v \Big)_+ .
\end{aligned} \tag{9.24}$$

Proof: We substitute $\rho_v = k$ for all v in the formula of Theorem 9.18 and use (9.20) to combine terms involving n_v. Then combining the $\sigma_v^{k_v}$ with the factor $\binom{2r-k_v+1}{2}$ leads to (9.24). $\square$

It is not hard to show that the MDS of Theorem 9.17 is local in the sense of Definition 5.16 with $\ell = 1$, but is not in general stable. The problem of constructing a stable local minimal determining set for $\mathcal{S}_d^{r,\rho}(\triangle)$ will be discussed in Chapter 11.

9.6. Splines on Type-I and Type-II Triangulations

In this section we present formulae for the dimensions of spline spaces $\mathcal{S}_d^r(\triangle)$ on uniform type-I and type-II triangulations. Given a rectangle $H := [a, b] \times [\tilde{a}, \tilde{b}]$, let

$$\begin{aligned}
a &= x_0 < x_1 < \cdots < x_k < x_{k+1} = b, \\
\tilde{a} &= y_0 < y_1 < \cdots < y_{\tilde{k}} < y_{\tilde{k}+1} = \tilde{b},
\end{aligned}$$

where we suppose that $x_{i+1} - x_i = h_x$ and $y_{j+1} - y_j = h_y$ for all i and j. The corresponding grid lines define a grid partition of H. Recall that a type-I partition is the triangulation that is obtained from this grid partition by drawing the diagonal connecting (x_i, y_j) to (x_{i+1}, y_{j+1}) in each rectangle, while the type-II partition is the triangulation that is obtained from this grid partition by drawing in both diagonals in each rectangle, see Section 4.13 and Figure 4.18.

Theorem 9.21. *Let $\triangle$ be a uniform type-I triangulation of a rectangle. Then for all $0 \leq r \leq d$,*

$$\begin{aligned}
\dim \mathcal{S}_d^r(\triangle) =& k\tilde{k}(d^2 - 3rd + 2r^2 + \sigma_g) \\
&+ (k + \tilde{k})(d^2 - 2rd + d - r + r^2) \\
&+ \frac{2d^2 + 4d - 2rd - r + r^2 + 2}{2},
\end{aligned} \tag{9.25}$$

where

$$\sigma_g := \begin{cases} r^2/4, & \text{if } r \text{ is even and } 3r+1 \le 2d, \\ (r^2-1)/4, & \text{if } r \text{ is odd and } 3r+1 \le 2d, \\ (d-r)(2r-d), & \text{otherwise.} \end{cases}$$

Proof: We make use of the upper and lower bounds given in Theorems 9.9 and 9.12. Following the notation of those theorems, it is easy to check that $V_I = k\tilde{k}$ and $E_I = 3k\tilde{k} + 2(k+\tilde{k}) + 1$. Now $m_v = 3$ for each interior vertex v, and thus

$$\sigma_v = \sum_{j=1}^{d-r} (r+1-2j)_+ = \sigma_g$$

in the lower bound (9.10). This shows that the dimension of $\mathcal{S}_d^r(\triangle)$ is bounded below by the formula in (9.25). To get an upper bound, we first need to construct an admissible decomposition of $\triangle$ (see Definition 9.10). Let $v_{ij} := (x_i, y_j)$, for all $0 \le i \le k+1$ and $0 \le j \le \tilde{k}+1$. Then we can choose the centers in the order $v_{0j}, v_{1j}, \ldots, v_{kj}$ for $j = 0, \ldots, \tilde{k}$. For each interior vertex, $\tilde{m}_v = 3$, and we get $\widetilde{\sigma}_v = \sigma_g$ in the upper bound. Since the upper and lower bounds coincide, we have established the stated dimension formula. □

Theorem 9.22. *Let $\triangle$ be a uniform type-II triangulation of a rectangle. Then for all $0 \le r \le d$,*

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) &= k\tilde{k}(2d^2 - 6rd + 4r^2 + \sigma_g + \sigma_c) \\ &\quad + (k+\tilde{k})(2d^2 - 5rd + d - r + 3r^2 + \sigma_c) \\ &\quad + (4d^2 + 4d - 8rd - r + 5r^2 + 2 + 2\sigma_c)/2, \end{aligned} \tag{9.26}$$

where

$$\sigma_g := \sum_{j=1}^{d-r} (r+1-3j)_+, \qquad \sigma_c := \sum_{j=1}^{d-r} (r+1-j)_+.$$

Proof: We again make use of Theorems 9.9 and 9.12. The number of interior edges is $E_I = 6k\tilde{k} + 5(k+\tilde{k}) + 4$. We call an interior vertex a grid vertex if it lies on the original rectangular grid. Otherwise, we call it a cross vertex. There are $k\tilde{k}$ grid vertices and $k\tilde{k} + (k+\tilde{k}) + 1$ cross vertices. Clearly, $m_v = 4$ for each grid vertex, and $m_v = 2$ for each cross vertex. It follows that

$$\sigma_v = \begin{cases} \sigma_g, & \text{at grid vertices}, \\ \sigma_c, & \text{at cross vertices}, \end{cases}$$

in the lower bound (9.10). This shows that the formula in (9.26) is a lower bound for $\dim \mathcal{S}_d^r(\triangle)$. To get an upper bound, we need to construct an admissible decomposition of $\triangle$. Let v_{ij} be as in the proof of Theorem 9.21, and let w_{ij} be the cross vertex in the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for all i, j. Then we can select the centers for the decomposition in the order $v_{0j}, v_{1j}, \ldots, v_{kj}$ followed by $w_{0j}, w_{1j}, \ldots, w_{kj}$ for $j = 0, \ldots, \tilde{k}$. This leads to an admissible decomposition with $\tilde{m}_v = 4$ for each grid vertex and $\tilde{m}_v = 2$ for each cross vertex. Thus, $\widetilde{\sigma}_v = \sigma_v$ for every interior vertex, and the upper bound is the same as the lower bound. □

9.7. Bounds on the Dimension of Superspline Spaces

Throughout this section we suppose that $\triangle$ is a regular triangulation. Our aim is to give upper and lower bounds on the dimension of the general superspline spaces $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ introduced in Definition 5.7. Recall that

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) := \{s \in \mathcal{S}_d^0(\triangle) : \tau s = 0, \ \text{all } \tau \in \mathcal{T}\}, \tag{9.27}$$

where $\mathcal{T}$ is a set of smoothness functionals of the form (5.7) associated with oriented edges of $\triangle$. To get useful results, we will have to restrict the nature of $\mathcal{T}$ somewhat.

Definition 9.23. *We say that the set $\mathcal{T}$ of smoothness functionals defined on a set $\mathcal{E}$ of oriented edges of $\triangle$ is* supported *provided that for every $e \in \mathcal{E}$,*

$$\tau_{j,e}^n \in \mathcal{T} \text{ implies } \tau_{j,e}^m \in \mathcal{T}, \quad \text{for } 0 \le m \le n.$$

We say that $\mathcal{T}$ is strongly supported *provided that for every $e \in \mathcal{E}$,*

$$\tau_{j,e}^n \in \mathcal{T} \text{ implies } \tau_{j-i,e}^m \in \mathcal{T}, \quad 0 \le i \le n-m \text{ and } 0 \le m \le n.$$

Figure 9.7 illustrates these concepts in the case where $d = 6$. The figure on the left shows the supports of the three smoothness functionals $\{\tau_{4,e}^n\}_{n=0}^2$, where the tip of each functional is indicated by a square showing the location of the first B-coefficient appearing in the definition (5.7) of $\tau_{j,e}^n$. This set of three functionals is supported but not strongly supported. The figure on the right shows a set of six smoothness functionals that is strongly supported. All of the classes of supersplines considered in previous sections correspond to strongly supported sets $\mathcal{T}$.

The following lemma shows that for strongly supported sets, we do not need to worry about the orientation of the underlying edges.

Lemma 9.24. *Let $T := \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2\rangle$ be neighboring triangles sharing the oriented edge $e := \langle v_2, v_3\rangle$, and suppose s is a piecewise polynomial defined on $T \cup \widetilde{T}$. Let $\{c_{ijk}\}$ and $\{\tilde{c}_{ijk}\}$ be the B-coefficients*

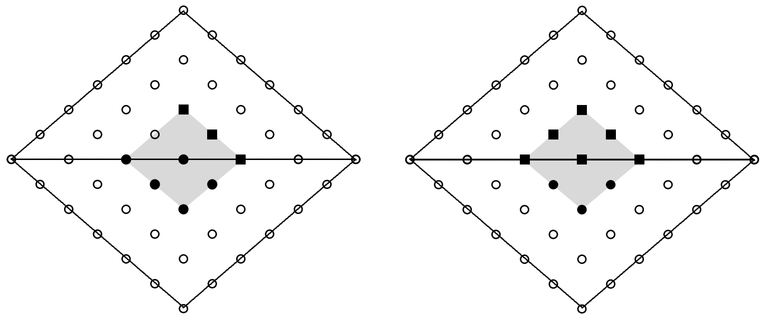


Fig. 9.7. Supported and strongly supported sets of smoothness conditions.

of $s|_T$ *and* $s|_{\widetilde{T}}$*, respectively. Fix* $0 \le n \le d$ *and* $d-n \le j \le d$*, and let* $\tilde{e} := \langle v_3, v_2 \rangle$*. Then* s *satisfies the smoothness conditions*

$$\tau^m_{j-i,e}\, s = 0, \qquad 0 \le i \le n-m, \quad 0 \le m \le n, \tag{9.28}$$

if and only if it satisfies the smoothness conditions

$$\tilde{\tau}^m_{d+j-n-i,\tilde{e}}\, s = 0, \qquad 0 \le i \le n-m, \quad 0 \le m \le n, \tag{9.29}$$

where

$$\tilde{\tau}^n_{j,\tilde{e}}\, s := \tilde{c}_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} c_{\nu,j-n+\mu,d-j+\kappa} B^n_{\nu\mu\kappa}(v_4).$$

Proof: We may regard the coefficients appearing in the conditions (9.28) and (9.29) to be the coefficients of polynomials q and $\tilde{q}$ of degree n. Then since both sets of smoothness conditions are equivalent to requiring that q and $\tilde{q}$ join with C^n smoothness across e, it follows that the sets of smoothness conditions are equivalent. □

We now establish a generalization of Theorem 9.7. Let $\triangle_v$ be an interior cell surrounding a vertex v. Without loss of generality, we may assume that v is located at the origin, and that the cell is rotated so that none of its interior edges is vertical. Let $v_1, \ldots, v_n$ be the boundary vertices of $\triangle_v$ in counterclockwise order, where for convenience we write $v_{n+1} := v_1$. Let $e_i := \langle v, v_i \rangle$ be the i-th interior edge with slope $-\alpha_i$, and let $T_i := \langle v, v_i, v_{i+1} \rangle$ be the i-th triangle, for $i = 1, \ldots, n$.

Given a set $\mathcal{T}$ of smoothness conditions associated with this cell, in view of Lemma 9.24, we may assume that all of the smoothness conditions are associated with edges oriented so that the first endpoint is at v. Given $1 \le i \le n$ and $0 \le j \le d$, let

$$r_{v,i,j} := \max\{k : \tau^k_{j,e_i} \in \mathcal{T}\}. \tag{9.30}$$

Let

$$\varepsilon_{v,j} := \sum_{i=1}^{n} m_{v,i,j},$$

where

$$m_{v,i,j} := \begin{cases} 0, & \text{if there exists } \ell \text{ with } \alpha_i = \alpha_\ell \text{ and } r_{v,\ell,j} < r_{v,i,j}, \\ 0, & \text{if there exists } \ell > i \text{ with } \alpha_i = \alpha_\ell \text{ and } r_{v,\ell,j} = r_{v,i,j}, \\ j - r_{v,i,j}, & \text{otherwise.} \end{cases} \tag{9.31}$$

Theorem 9.25. *Suppose $\mathcal{T}$ is a strongly supported set of smoothness conditions. Then*

$$\dim \mathcal{S}_d^{\mathcal{T}}(\triangle_v) = \sum_{i=1}^{n}\sum_{j=0}^{d}(j - r_{v,i,j}) + \sum_{j=0}^{d}(j + 1 - \varepsilon_{v,j})_+. \tag{9.32}$$

Proof: The proof is similar to the proof of Theorems 9.3 and 9.7. Clearly,

$$\mathcal{S}_d^{\mathcal{T}}(\triangle_v) = \mathcal{P}_d \oplus \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_d,$$

where $\mathcal{S}_j$ is the subspace of splines in $\mathcal{S}_d^{\mathcal{T}}(\triangle_v)$ which vanish on the triangle T_n, and whose restrictions to each triangle of $\triangle_v$ belong to the $(j+1)$-dimensional space $\mathcal{H}_j$ of bivariate homogeneous polynomials of degree j. Then

$$\dim \mathcal{S}_d^{\mathcal{T}}(\triangle_v) = \dim \mathcal{P}_d + \sum_{j=0}^{d} \dim \mathcal{S}_j. \tag{9.33}$$

For ease of notation, in the remainder of the proof we write $r_{i,j}$ in place of $r_{v,i,j}$. First note that (cf. the proof of Theorem 9.3), if $s \in \mathcal{S}_j$, then

$$s|_{T_\ell}(x,y) = \sum_{i=1}^{\ell} \sum_{k=r_{i,j}+1}^{j} a_{j,k}^{[i]} x^{j-k}(y + \alpha_i x)^k,$$

for $\ell = 1, \ldots, n$, where the coefficients satisfy

$$\sum_{i=1}^{n} \sum_{k=r_{i,j}+1}^{j} a_{j,k}^{[i]} x^{j-k}(y + \alpha_i x)^k \equiv 0. \tag{9.34}$$

Thus, the dimension of $\mathcal{S}_j$ is equal to the number of linearly independent coefficient vectors a satisfying (9.34). Clearly, (9.34) is satisfied if and only if the coefficient of each monomial is zero, and thus (9.34) is equivalent to the system of equations

$$A_j a = 0, \tag{9.35}$$

where $a := (a_{j,j}^{[1]}, \ldots, a_{j,r_{i,j}+1}^{[1]}, \ldots, a_{j,j}^{[n]}, \ldots, a_{j,r_{i,j}+1}^{[n]})^T$, $A_j := (A_j^{[1]}, \ldots, A_j^{[n]})$, and

$$A_j^{[i]} := \begin{bmatrix} 1 & 0 & & \\ \binom{j}{1}\alpha_i & 1 & & \\ \binom{j}{2}\alpha_i^2 & \binom{j-1}{1}\alpha_i & & \\ \vdots & \vdots & \ddots & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \binom{r_{i,j}+1}{1}\alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ \binom{j}{j}\alpha_i^j & \binom{j-1}{j-1}\alpha_i^{j-1} & \cdots & \binom{r_{i,j}+1}{r_{i,j}+1}\alpha_i^{r_{i,j}+1} \end{bmatrix}, \qquad i = 1, \ldots, n.$$

The number of linearly independent solutions of (9.35) is equal to the number of columns of A_j minus the rank of A_j. Clearly,

$$\#\,\text{columns}\,(A_j) = \sum_{i=1}^{n}(j - r_{i,j}), \qquad \#\,\text{rows}\,(A_j) = j + 1.$$

In computing the rank of A_j, if $\alpha_i = \alpha_\ell$ for some $i \neq \ell$, we can drop the smallest of the blocks $A_j^{[i]}$ and $A_j^{[\ell]}$, which accounts for the cases in (9.31) where $m_{v,i,j} = 0$. After dropping all such blocks arising from pairs of collinear edges, the number of remaining columns in A_j is $\varepsilon_{v,j}$. Using the same Hermite interpolation argument as in the proof of Theorem 9.3, we see that the rank of A_j is the smaller of $j + 1$ and $\varepsilon_{v,j}$, i.e.,

$$\text{rank}\,(A_j) = j + 1 - (j + 1 - \varepsilon_{v,j})_+.$$

We conclude that

$$\dim \mathcal{S}_j = nj - \sum_{i=1}^{n} r_{i,j} - (j + 1) + (j + 1 - \varepsilon_{v,j})_+,$$

for each $j = 1, \ldots, d$. Since $\dim \mathcal{P}_d = \binom{d+2}{2} = \sum_{j=0}^{d}(j + 1)$, (9.32) follows immediately from (9.33). $\square$

Examining the proof of this theorem, we see that the first $r_{v,i,j}$ coefficients on the ring $R_j(v)$ are determined by the smoothness conditions across the edge $e_i := \langle v, w_i \rangle$, where $r_{v,i,j}$ is the integer defined in (9.30). Thus, the total number of Bernstein–Bézier coefficients in the triangle T_i which are not determined by smoothness conditions across the edge e_i is

$$\beta_{e_i} := \sum_{j=0}^{d} (j - r_{v,i,j}). \tag{9.36}$$

With this notation, the formula (9.32) can be rewritten as

$$\dim \mathcal{S}_d^{\mathcal{T}}(\triangle_v) = \binom{d+2}{2} + \sum_{i=1}^{n} \beta_{e_i} - \gamma + \theta_v, \tag{9.37}$$

where $\gamma := \binom{d+2}{2}$ and

$$\theta_v := \sum_{j=0}^{d} (j+1-\varepsilon_{v,j})_+.$$

Formula (9.37) states that the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle_v)$ is equal to the dimension $\binom{d+2}{2}$ of the space of polynomials $\mathcal{P}_d$, plus a term which counts the number of free B-coefficients as we pass over each interior edge of $\triangle_v$, minus a term γ arising from the compatibility conditions at the vertex v, plus a correction factor θ_v relating to how the edges meet at v. This should be compared with the formulae in Theorems 9.3 and 9.7.

We are now ready to give a lower bound on the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ for an arbitrary triangulation $\triangle$. Let $\mathcal{E}_I$ and $\mathcal{V}_I$ be the sets of interior edges and vertices of $\triangle$, respectively. For each edge $e \in \mathcal{E}_I$, let β_e be defined as in (9.36), and let

$$D := \binom{d+2}{2} + \sum_{e\in\mathcal{E}_I} \beta_e - \binom{d+2}{2} V_I, \tag{9.38}$$

where $V_I := \#\mathcal{V}_I$. For each $v \in \mathcal{V}_I$, let $\mathcal{E}_v := \{e_i := \langle v, w_{v,i}\rangle\}_{i=1}^{n_v}$ be the set of edges attached to v. For each $1 \le i \le n_v$, let $m_{v,i,j}$ be defined as in (9.31), and let

$$\varepsilon_{v,j} := \sum_{i=1}^{n_v} m_{v,i,j}.$$

Set

$$\theta_v := \sum_{j=0}^{d} (j+1-\varepsilon_{v,j})_+, \qquad \theta := \sum_{v\in\mathcal{V}_I} \theta_v. \tag{9.39}$$

Theorem 9.26. *Suppose $\mathcal{T}$ is a strongly supported set of smoothness conditions associated with the edges of a regular triangulation $\triangle$. Then*

$$D + \theta \le \dim \mathcal{S}_d^{\mathcal{T}}(\triangle). \tag{9.40}$$

Proof: The proof proceeds by induction along the same lines as the proof of Theorem 9.9. The result is trivial for a single triangle. Now suppose $\triangle$ is a triangulation of Ω with N triangles, and that the bound holds for all triangulations with $N-1$ triangles. Suppose some triangle T of $\triangle$ is a flap, and let $\widetilde{\triangle} = \triangle \setminus \{T\}$. Clearly, adding a flap T to $\widetilde{\triangle}$ increases the dimension by β_e, where e is the edge where the flap joins $\widetilde{\triangle}$.

We now consider the case where $\triangle$ does not contain a flap. Then it must contain at least one fill T. Let v be the vertex of T which lies in the interior of $\triangle$. We denote the edges of T which are interior to Ω by e_1 and e_2. Let $\Omega_2 := \operatorname{star}(v)$ be the union of the triangles surrounding v, and let $\Omega_1 := \Omega \setminus \{T\}$ and $\Omega_3 := \Omega_1 \cap \Omega_2$. Set $\mathcal{S}_i := \mathcal{S}_d^{\mathcal{T}}(\triangle)|_{\Omega_i}$ for $i = 1, 2, 3$. Let $\mathcal{S}_1^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_1$ but not to $\mathcal{S}_2$. Similarly, let $\mathcal{S}_2^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_2$ but not to $\mathcal{S}_1$. Let $\mathcal{S}_{12}^E$ be the linear subspace of splines in $\mathcal{S}_3$ that can be extended to both $\mathcal{S}_1$ and $\mathcal{S}_2$. Then the space of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_1$ is $\mathcal{S}_{12}^E \oplus \mathcal{S}_1^E$. Similarly, the space of splines in $\mathcal{S}_3$ that can be extended to $\mathcal{S}_2$ is $\mathcal{S}_{12}^E \oplus \mathcal{S}_2^E$. This implies

$$\mathcal{S}_3 = \mathcal{S}_{12}^E \oplus \mathcal{S}_1^E \oplus \mathcal{S}_2^E \oplus \mathcal{S}_{12}^{NE}, \tag{9.41}$$

where $\mathcal{S}_{12}^{NE}$ is the space of all splines in $\mathcal{S}_3$ that cannot be extended to either $\mathcal{S}_1$ or $\mathcal{S}_2$. Clearly, for $i = 1, 2$,

$$\dim \mathcal{S}_i = \dim \mathcal{S}_i^0 + \dim \mathcal{S}_{12}^E + \dim \mathcal{S}_i^E, \tag{9.42}$$

where $\mathcal{S}_i^0 := \{s \in \mathcal{S}_i : s \text{ vanishes on } \Omega_3\}$. Solving (9.42) for $\dim \mathcal{S}_i^0$ and inserting in

$$\dim \mathcal{S} = \dim \mathcal{S}_1^0 + \dim \mathcal{S}_2^0 + \dim \mathcal{S}_{12}^E,$$

it follows from (9.41) that

$$\begin{aligned}
\dim \mathcal{S} &= \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_1^E - \dim \mathcal{S}_2^E - \dim \mathcal{S}_{12}^E \\
&= \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_3 + \dim \mathcal{S}_{12}^{NE} \\
&\ge \dim \mathcal{S}_1 + \dim \mathcal{S}_2 - \dim \mathcal{S}_3.
\end{aligned}$$

Let $\gamma := \binom{d+2}{2}$. Then using the induction hypothesis, we have

$$\dim \mathcal{S} \ge \binom{d+2}{2} + \sum_{e \in \mathcal{E}_1} \beta_e - \sum_{u \in \mathcal{V}_1} \gamma + \theta_1 + \beta_{e_1} + \beta_{e_2} - \gamma + \theta_2,$$

where $\mathcal{E}_1$ and $\mathcal{V}_1$ denote the number of interior edges and vertices of the triangulation $\triangle$ restricted to Ω_1, and where θ_i is the correction factor in the lower bound on the dimension of $\mathcal{S}_i$. Combining terms leads to the bound (9.40) for $\mathcal{S}_d^{\mathcal{T}}(\triangle)$. $\square$

It is clear from the proof of this theorem that strict inequality will hold in (9.40) whenever $\dim \mathcal{S}_{12}^{NE} \neq 0$ for every step of the induction process. We turn now to the problem of constructing an upper bound. To this end, we suppose that $\triangle$ has been decomposed as in Definition 9.10. Let

$v_1, \ldots, v_n$ be the corresponding centers. They have the property that for each $i > 1$, the vertex v_i is connected by an edge to some v_j with $j < i$.

For each k, let $\widetilde{\mathcal{E}}_{v_k}$ be the set of interior edges of $\triangle$ attached to v_k but not to any vertex v_j with $j < k$, and let $\tilde{n}_k := \#\widetilde{\mathcal{E}}_{v_k}$. Let $m_{v_k,i,j}$ be defined as in (9.31), and let

$$\tilde{\varepsilon}_{v_k,j} := \sum_{i=1}^{\tilde{n}_k} m_{v_k,i,j}.$$

Let

$$\tilde{\theta} := \sum_{k=1}^{n} \tilde{\theta}_{v_k}, \tag{9.43}$$

where

$$\tilde{\theta}_{v_k} := \begin{cases} \displaystyle\sum_{j=0}^{d} (j + 1 - \tilde{\varepsilon}_{v_k,j})_+, & \text{if } v_k \in \mathcal{V}_I, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 9.27. *Suppose $\mathcal{T}$ is a strongly supported set of smoothness conditions associated with the edges of a regular triangulation $\triangle$. Then*

$$\dim \mathcal{S}_d^{\mathcal{T}}(\triangle) \le D + \tilde{\theta}. \tag{9.44}$$

Proof: Suppose v_1 is an interior vertex. Then by (9.37), the dimension of $\mathcal{S}_d^r(\triangle)|_{\mathrm{star}(v_1)}$ is equal to

$$\binom{d+2}{2} + \sum_{e \in \widetilde{\mathcal{E}}_{v_1}} \beta_e - \gamma + \tilde{\theta}_{v_1},$$

where $\gamma := \binom{d+2}{2}$. Similarly, if v_1 is a boundary vertex, then the dimension of $\mathcal{S}_d^r(\triangle)|_{\mathrm{star}(v_1)}$ is equal to

$$\binom{d+2}{2} + \sum_{e \in \widetilde{\mathcal{E}}_{v_1}} \beta_e.$$

We now we examine s restricted to $\mathrm{star}(v_2) \cap \Omega_1$, where the Ω_i are the sets arising in the decomposition of $\triangle$. If v_2 is a boundary vertex of $\triangle$, then it is easy to see that the number of linearly independent splines in $\mathcal{S}_d^r(\triangle)|_{\mathrm{star}(v_1) \cup \mathrm{star}(v_2)}$ that vanish on $\mathrm{star}(v_1)$ is at most $\sum_{e \in \widetilde{\mathcal{E}}_{v_2}} \beta_e$. Similarly, if v_2 is an interior vertex of $\triangle$, then by the proof of Theorem 9.25, the number of linearly independent splines in $\mathcal{S}_d^r(\triangle)|_{\mathrm{star}(v_1) \cup \mathrm{star}(v_2)}$ that vanish on $\mathrm{star}(v_1)$ is at most $\sum_{e \in \widetilde{\mathcal{E}}_{v_2}} \beta_e - \gamma + \tilde{\theta}_{v_2}$. Repeating this argument for $v_3, \ldots, v_n$ gives (9.44). $\square$

It is clear that the bound (9.44) may depend on the decomposition of $\triangle$, as was the case for the spline spaces $\mathcal{S}_d^r(\triangle)$ treated in in Section 9.3. To get the best possible upper bound, we would have to examine all possible admissible decompositions.

Combining Theorems 9.26 and 9.27, we have shown that if $\mathcal{T}$ is a strongly supported set of smoothness conditions associated with the edges of a triangulation $\triangle$, then

$$D + \theta \le \dim \mathcal{S}_d^{\mathcal{T}}(\triangle) \le D + \tilde{\theta}, \tag{9.45}$$

where D is given in (9.38) and θ and $\tilde{\theta}$ are given in (9.39) and (9.43), respectively. Note that θ and $\tilde{\theta}$ are in general different with $\theta \le \tilde{\theta}$. This follows since for all $v \in \mathcal{V}_I$ and all $0 \le j \le d$, we have $\tilde{\varepsilon}_{v,j} \le \varepsilon_{v,j}$, since in computing $\varepsilon_{v,j}$, we sum over a subset of the vertices connected to v rather than over all of them.

The upper and lower bounds in (9.45) differ only by $\theta - \tilde{\theta}$, which depends on the exact geometry of the triangulation. Clearly, whenever $\tilde{\theta} = \theta$, (9.45) provides the actual dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ rather than just a bound. For some examples where this happens, see Section 9.6 where splines on type-I and type-II partitions are discussed.

To illustrate how the upper and lower bounds compare with true dimensions, in Tables 9.1 and 9.2 we give bounds and dimensions for several superspline spaces of $\mathcal{S}_d^1(\triangle_{MS})$ and $\mathcal{S}_d^2(\triangle_{MS})$ defined on the symmetric Morgan–Scott triangulation $\triangle_{MS}$ shown in Figure 9.3. Each space in the table has ρ_v supersmoothness at the three interior vertices of $\triangle_{MS}$, and also has C^{μ_e} smoothness across each of the three interior edges of the central triangle. The Java program described in Remark 5.6 was used to compute the true dimensions. The tables show that the actual dimension can be equal to the upper bound rather than the lower bound, and that in some cases, the lower bound turns out to be smaller than $\binom{d+2}{2}$, which of course is always a lower bound for the dimension since every spline space contains $\mathcal{P}_d$. Further experiments on a variety of spline spaces suggest that the lower bound is usually closer to the true dimension in general.

9.8. Generic Dimension

As we have seen above, the problem of finding the dimension of spline spaces is nontrivial since, in general, the dimension depends not only on the degree and smoothness of the space, but also on the precise geometry of the triangulation. In this section we show that for a given degree and set of smoothness conditions defining a spline space $\mathcal{S}$, the dimension of $\mathcal{S}$ is the same for almost all triangulations with the same connectivity.

Definition 9.28. *Let $\mathcal{L}$ be a list describing how to connect vertices in $\mathbb{R}^2$ to produce a triangulation. We define $\mathrm{Tri}(\mathcal{L})$ to be the set of all regular triangulations that can be constructed with this connectivity.*

If $\triangle$ is a triangulation in $\mathrm{Tri}(\mathcal{L})$, then $\mathcal{L}$ is a list of its edges. For given $\mathcal{L}$, $\mathrm{Tri}(\mathcal{L})$ contains infinitely many triangulations. In particular, if

d	μ_e	ρ_v	LB	dim	UB
2	1	1	6	7	7
3	1	1	16	16	17
3	2	1	13	13	13
3	1	2	13	13	13
4	1	1	33	33	33
4	2	1	27	27	27
4	1	2	30	30	30

Tab. 9.1. Dimensions of some spline spaces on the Morgan–Scott split for $r = 1$.

d	μ_e	ρ_v	LB	dim	UB
3	2	2	7	10	10
4	2	2	15	16	19
4	3	2	15	16	16
4	2	3	15	16	16
4	4	2	15	15	15
4	3	3	15	16	16
4	2	4	15	15	15
5	2	2	30	30	34
5	3	2	27	27	28
5	2	3	30	30	31

Tab. 9.2. Dimensions of some spline spaces on the Morgan–Scott split for $r = 2$.

$\triangle \in \mathrm{Tri}\,(\mathcal{L})$, then any triangulation obtained from $\triangle$ by perturbing its vertices by a sufficiently small amount will also belong to $\mathrm{Tri}\,(\mathcal{L})$. Now fix $d > 0$, and suppose $\mathcal{T}$ is an arbitrary set of smoothness conditions of the type (5.7) associated with the edges of any triangulation in $\mathrm{Tri}\,(\mathcal{L})$. For each $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, let $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ be the spline space defined in (9.27).

Definition 9.29. *Fix $d, \mathcal{T}$ and $\mathcal{L}$. Then a triangulation $\triangle^* \in \mathrm{Tri}\,(\mathcal{L})$ is said to be* generic with respect to $d, \mathcal{T}$ and $\mathcal{L}$ *provided that*

$$\dim S_d^{\mathcal{T}}(\triangle^*) = \min_{\triangle \in \mathrm{Tri}\,(\mathcal{L})} \dim S_d^{\mathcal{T}}(\triangle) =: n.$$

We call n the generic dimension *of* $\{\mathcal{S}_d^{\mathcal{T}}(\triangle)\}_{\triangle \in \mathrm{Tri}\,(\mathcal{L})}$.

We illustrate this idea with two examples.

Example 9.30. Let $\mathcal{L}$ describe the set of triangulations with connectivity as shown in Figure 9.2. Let $d = 2$, and suppose $\mathcal{T}$ defines the spline space $\mathcal{S}_2^1(\triangle)$ for any triangulation $\triangle \in \mathrm{Tri}\,(\mathcal{L})$.

Discussion: As discussed in Example 9.5, if $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, then the dimension of $\mathcal{S}_2^1(\triangle)$ is equal to $7+\sigma$, where $\sigma = 1$ if the interior vertex v is singular, and $\sigma = 0$ otherwise. It follows that all triangulations $\triangle \in \mathrm{Tri}\,(\mathcal{L})$ are generic relative to this $d, \mathcal{T}$ and $\mathcal{L}$, except those where the interior vertex v is singular. □

The following example shows that for fixed $\mathcal{L}$, the subset of triangulations in $\mathrm{Tri}\,(\mathcal{L})$ that are generic can be different if we change the degree and smoothness conditions.

Example 9.31. Let $\mathcal{L}$ describe the same set of triangulations as in Example 9.30. Let $d = 4$, and suppose $\mathcal{T}$ defines the superspline space $\mathcal{S}_4^1(\triangle) \cap C^3(v)$ for any triangulation $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, where v is the interior vertex of $\triangle$.

Discussion: By Theorem 9.7, if $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, then $\dim \mathcal{S}_4^1(\triangle) \cap C^3(v) = 22$, regardless of whether v is singular or not. Thus, in this example all triangulations in $\mathrm{Tri}\,(\mathcal{L})$ are generic relative to this choice of $d, \mathcal{T}$ and $\mathcal{L}$. □

We have the following interesting result concerning generic triangulations.

Theorem 9.32. *Suppose $\triangle^*$ is a generic triangulation relative to $d, \mathcal{T}$, and $\mathcal{L}$. Then all triangulations $\triangle \in \mathrm{Tri}\,(\mathcal{L})$ whose vertices are sufficiently small perturbations of the vertices of $\triangle$ are also generic relative to $d, \mathcal{T}$, and $\mathcal{L}$.*

Proof: As observed in Section 5.5.4, for each $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, there is a unique matrix $A_\triangle$ depending on $\triangle$ such that

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) = \{s \in \mathcal{S}_d^0(\triangle) : \; A_\triangle c = 0\}.$$

Recall that the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ is given by $n - r_\triangle$, where n is the dimension of $\mathcal{S}_d^0(\triangle)$ and $r_\triangle$ is the rank of $A_\triangle$. Since $\triangle^*$ is generic, we have

$$r_{\triangle^*} := \max_{\triangle \in \mathrm{Tri}\,(\mathcal{L})} r_\triangle.$$

Let D be the determinant of some square submatrix of $A_{\triangle^*}$ whose rank is equal to $r_{\triangle^*}$, and for any other triangulation $\triangle \in \mathrm{Tri}\,(\mathcal{L})$, let $D(\triangle)$ be the determinant of the same submatrix of the corresponding matrix $A_\triangle$. It is clear that $D(\triangle)$ is a continuous function of the $2m$ variables $(x_1, y_1, \ldots, x_m, y_m)$, where the (x_i, y_i) are the vertices of $\triangle$. Thus, if $D(\triangle)$ is nonzero at a given point in $\mathbb{R}^m$, then it must be nonzero for all points in an open neighborhood, and the claim follows. □

9.9. The Generic Dimension of $\mathcal{S}_3^1(\triangle)$

In this section we prove that if $\triangle$ is a generic triangulation, then dim $\mathcal{S}_3^1(\triangle) = 3V_B + 2V_I + 1$, where V_B and V_I are the number of boundary and interior vertices of $\triangle$, respectively. The proof will involve so-called smoothing cofactors.

Given a triangulation $\triangle$, let $v_1, \ldots, v_{V_I}$ be its interior vertices and $e_1, \ldots, e_{E_I}$ its interior edges, where each edge e_i has been assigned a specific orientation. Assuming that the triangulation $\triangle$ has been rotated so that no edge is vertical, we can find α_i, β_i such that e_i lies on the line $y+\alpha_i x+\beta_i = 0$. Now suppose T_1 and T_2 are two triangles sharing the edge e_i, where T_2 is the triangle on the left side of e_i. Then for any $s \in \mathcal{S}_3^1(\triangle)$, the C^1 smoothness across the edge e_i implies

$$s|_{T_2} - s|_{T_1} = (y + \alpha_i x + \beta_i)^2 f_i,$$

where

$$f_i := a_i + b_i y + c_i x,$$

for some constants a_i, b_i, c_i. The linear polynomial f_i is called a smoothing cofactor of s associated with the oriented edge e_i. It is clear that given any spline $s \in \mathcal{S}_3^1(\triangle)$, there is a unique set

$$\mathcal{C} := \{f_i : i = 1, \ldots, E_I\}$$

of smoothing cofactors associated with s. However, not every set $\mathcal{C}$ of linear polynomials can serve as the smoothing cofactors of a spline s in $\mathcal{S}_3^1(\triangle)$. For that to happen, the cofactors must be consistent. To describe this concept, we need to examine closed paths that start and end in one triangle of $\triangle$, and pass over a sequence of interior edges $e_{i_1}, \ldots, e_{i_k}$, but not through any vertex of $\triangle$.

Definition 9.33. *Let $\mathcal{C} := (f_1, \ldots, f_{E_I})$ be a vector of linear polynomials. Then we say that $\mathcal{C}$ is* consistent *provided for any closed path that crosses over a sequence of edges $e_{i_1}, \ldots, e_{i_k}$,*

$$\sum_{j=1}^{k} (y + \alpha_{i_j} x + \beta_{i_j})^2 (-1)^{\mu_{i_j}} f_{i_j} \equiv 0, \tag{9.46}$$

where

$$\mu_{i_j} = \begin{cases} 0, & \text{if the path crosses } e_{i_j} \text{ from right to left,} \\ 1, & \text{otherwise,} \end{cases} \tag{9.47}$$

for $j = 1, \ldots, k$. We say that $\mathcal{C}$ is locally consistent *provided (9.46) holds for any closed path that encloses just one interior vertex and crosses over each edge attached to that vertex exactly one time.*

Lemma 9.34. *A set $\mathcal{C}$ of cofactors is consistent if and only if it is locally consistent.*

Proof: Clearly, if $\mathcal{C}$ is consistent, then it must be locally consistent. We now show the converse. We first consider a closed path that encloses exactly two interior vertices v and w of $\triangle$. Suppose that $\mathcal{C}$ satisfies the local consistency conditions for both v and w, i.e.,

$$\begin{aligned} &\sum_{j=1}^{n_v} (y + \alpha_{i_j(v)} x + \beta_{i_j(v)})^2 (-1)^{\mu_{i_j(v)}} f_{i_j(v)} \equiv 0, \\ &\sum_{j=1}^{n_w} (y + \alpha_{i_j(w)} x + \beta_{i_j(w)})^2 (-1)^{\mu_{i_j(w)}} f_{i_j(w)} \equiv 0, \end{aligned} \tag{9.48}$$

where $i_1(v), \ldots, i_{n_v}(v)$ and $i_1(w), \ldots, i_{n_w}(w)$ are the indices of the edges attached to v and w, respectively. Now there are two cases. If v and w are connected by an interior edge $e := \langle v, w \rangle$, then adding the two expressions in (9.48) together and taking account of the fact that if we cross a given edge once in each direction, then the corresponding terms in (9.48) cancel, we see that the sum (9.46) vanishes for the closed path going around the pair v and w. On the other hand, if v and w are not connected by an interior edge, there must be some interior edge e of $\triangle$ that separate v and w, i.e., v and w lie on opposite sides of e. But then any closed path around both v and w must pass over e once in each direction, and we can again add the two expressions in (9.48) to see that (9.46) vanishes for the closed path going around the pair v and w. This argument can be repeated to show that (9.46) holds for closed paths around arbitrarily many interior vertices. $\square$

In view of this lemma, each spline $s \in \mathcal{S}_3^1(\triangle)$ is uniquely determined up to a polynomial by a set of coefficients $\{a_i, b_i, c_i\}_{i=1}^{E_I}$ such that for each $v \in \mathcal{V}_I$,

$$p_v(x, y) := \sum_{j=1}^{n_v} (y + \alpha_{i_j(v)} x + \beta_{i_j(v)})^2 (-1)^{\kappa_{ij}} (a_{i_j(v)} + b_{i_j(v)} y + c_{i_j(v)} x) \equiv 0, \tag{9.49}$$

where $i_1(v), \ldots, i_{n_v}(v)$ are the indices of the edges attached to v in counterclockwise order, and

$$\kappa_{ij} := \begin{cases} 0, & \text{if } e_{i_j} \text{ is oriented outwards from } v_i, \\ 1, & \text{otherwise.} \end{cases}$$

Since each of the polynomials p_v satisfies

$$p_v(v) = D_x p_v(v) = D_y p_v(v) = 0,$$

it follows that a set of coefficients $g := (a_1, b_1, c_1, \ldots, a_{E_I}, b_{E_I}, c_{E_I})^T$ satisfies (9.49) if and only if each of the p_v reduces to a linear polynomial. Clearly, the cubic polynomial p_v reduces to a linear polynomial if and only if the factors in (9.49) multiplying $y^2, xy, x^2, y^3, xy^2, x^2y, x^3$ are zero. Writing these seven conditions for each $v \in \mathcal{V}_I$ leads to the linear system

$$Mg = 0, \tag{9.50}$$

where M consists of $V_I \times E_I$ blocks M_{ij} of size 7×3, where

$$M_{ij} := M_{v_i,e_j} := (-1)^{\kappa_{ij}} \begin{bmatrix} 1 & 2\beta_j & 0 \\ 2\alpha_j & 2\alpha_j\beta_j & 2\beta_j \\ \alpha_j^2 & 0 & 2\alpha_j\beta_j \\ 0 & 1 & 0 \\ 0 & 2\alpha_j & 1 \\ 0 & \alpha_j^2 & 2\alpha_j \\ 0 & 0 & \alpha_j^2 \end{bmatrix} \tag{9.51}$$

if the j-th edge is attached to the i-th vertex, and M_{ij} is 0 otherwise. Note that the matrix M in (9.50) is not the same as the matrices $A_\triangle$ of smoothness conditions in Theorem 9.32.

Lemma 9.35. *For any triangulation* $\triangle$,

$$dim\ \mathcal{S}_3^1(\triangle) = 3V_B + 9V_I - rank\ (M) + 1, \tag{9.52}$$

where M *is the matrix in (9.50).*

Proof: The number of linearly independent splines in $\mathcal{S}_3^1(\triangle)$ that do not lie in $\mathcal{P}_3$ is equal to the number of linearly independent solutions of (9.50). M has $3E_I$ columns and $7V_I$ rows. By (4.5), $3E_I = 3V_B + 9V_I - 9 \geq 9V_I$, so the number of columns exceeds the number of rows. Thus, the number of linearly independent solutions of (9.50) is $3E_I - \text{rank}\ (M)$. Using $E_I = 3V_I + V_B - 3$, and the fact that $\mathcal{P}_3$ has dimension 10, we immediately get (9.52). □

We shall show below that for any generic triangulation, M has full rank, i.e., rank $(M) = 7V_I$. The proof will be based on an induction argument.

Definition 9.36. *An oriented interior edge* $e = \langle v, w \rangle$ *of a shellable triangulation* $\triangle$ *is said to be* contractible *provided that if we remove* e *and the two triangles containing* e*, and replace* v *by* w *in both the edge and triangle lists, we get a new shellable triangulation* $\widetilde{\triangle}$ *with one less interior vertex.*

In Figure 9.8 we show a triangulation that has both contractible and noncontractible edges. For example, the edges $\langle u, v \rangle$ and $\langle u, w \rangle$ are both contractible, but the edge $\langle v, w \rangle$ (marked with a dotted line) is not contractible.

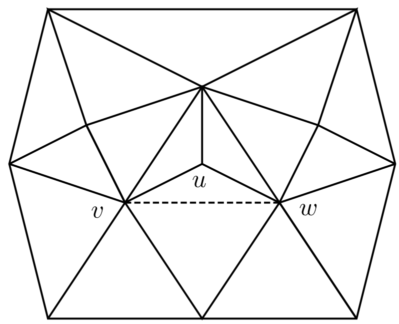


Fig. 9.8. A triangulation with a noncontractible edge.

If v is a vertex of a triangulation $\triangle$, then we say that v is of degree k provided there are k edges attached to v.

Lemma 9.37. *Let $\triangle$ be a shellable triangulation that contains an interior vertex v of degree 3, 4, or 5. Then there is at least one contractible edge attached to v.*

Proof: Let $\Omega_v := \operatorname{star}(v)$. If Ω_v is convex, then clearly any edge attached to v is contractible. This is the case when v is of degree 3. Now suppose v is of degree 4, and let $v_1, \ldots, v_4$ be the boundary vertices of Ω_v. If Ω_v is not convex, then the external angle at one of the vertices (say v_1) is less than π. But then we can contract the edge $\langle v, v_1 \rangle$. It remains to consider the case when v is of degree 5. Suppose $v_1, \ldots, v_5$ are the boundary vertices of Ω_v. If Ω_v has just one vertex (say v_1) whose exterior angle is less than π, then we can contract the edge $\langle v, v_1 \rangle$. Suppose now Ω_v has two vertices whose exterior angles are less than π. There are two cases. If these two vertices are v_1 and v_2, then the edge $\langle v, v_4 \rangle$ is contractible. If the two vertices are v_1 and v_3, then depending on the geometry, one of the edges $\langle v, v_1 \rangle$ or $\langle v, v_3 \rangle$ is contractible. $\square$

Theorem 9.38. *For any generic triangulation $\triangle$,*

$$dim\ \mathcal{S}_3^1(\triangle) = 3V_B + 2V_I + 1, \tag{9.53}$$

where V_B and V_I are the number of boundary and interior vertices of $\triangle$, respectively.

Proof: By Theorem 9.9 and the Euler relations, we know that the formula in (9.53) is a lower bound for the dimension of $\mathcal{S}_3^1(\triangle)$. We now show that for any generic triangulation $\triangle$, the dimension is actually equal to this number. We proceed by induction on the number N of triangles in $\triangle$.

Since the result is trivial if $N = 1$, we may suppose $N > 1$ and that (9.53) holds for generic triangulations with $N-1$ triangles. In view of Lemma 9.35, it suffices to show that the matrix M in (9.50) has full rank $7V_I$. There are three cases.

Case 1: There exists a boundary vertex v of $\triangle$ of degree 2. In this case we can write $\triangle = \widetilde{\triangle} \cup T$, where T is a flap (see Definition 9.8) attached to $\triangle$ along an edge e. Note that $V_I = \widetilde{V}_I$ and $V_B = \widetilde{V}_B + 1$. Let $\widetilde{M}$ be the matrix in (9.50) associated with $\widetilde{\triangle}$. M can be obtained from $\widetilde{M}$ by adding three columns corresponding to the coefficients of the smoothing cofactor associated with the new interior edge e. Thus, using the inductive hypothesis, we have rank $(M) =$ rank $(\widetilde{M}) = 7\widetilde{V}_I = 7V_I$.

Case 2: There exists a boundary vertex v of $\triangle$ of degree 3. In this case there exists a triangulation $\widetilde{\triangle}$ such that $\triangle$ can be obtained from $\widetilde{\triangle}$ by adding two neighboring triangles $T_1 := \langle v, v_1, v_2 \rangle$ and $T_2 := \langle v, v_2, v_3 \rangle$ with vertex at v. Note that $V_I = \widetilde{V}_I + 1$ and $V_B = \widetilde{V}_B$. In this case

$$M = \begin{bmatrix} \widetilde{M} & 0 \\ M_1 & M_2 \end{bmatrix},$$

where $M_2 = [M_{\langle v_2, v_3 \rangle}, M_{\langle v_2, v \rangle}, M_{\langle v_2, v_1 \rangle}]$. This is a 7×9 matrix with full rank 7 since it can be thought of as the matrix corresponding to a triangulation with one interior vertex v_2 and three interior edges. Thus, rank $(M) =$ rank $(\widetilde{M}) + 7 = 7\widetilde{V}_I + 7 = 7V_I$.

Case 3: All boundary vertices of $\triangle$ are of degree at least 4. In this case we claim that there is at least one interior vertex of $\triangle$ of degree 5 or less. To see this, suppose all interior vertices have degree 6 or more. Then counting edges, we have $E \geq (6V_I + 4V_B)/2 = 3V_I + 2V_B$, where V_I and V_B are the number of interior and boundary edges of $\triangle$, respectively. But this contradicts the Euler relation (4.5), which asserts that $E = 3V_I + 2V_B - 3$. Now let v be an interior vertex of $\triangle$ of degree at most 5. By Lemma 9.37, one of the edges attached to v can be contracted to get a triangulation $\widetilde{\triangle}$ with two fewer triangles, three fewer interior edges, and one less interior vertex. We now show that the rank of the matrix M of (9.50) associated with $\triangle$ is $7V_I$. After perturbing v slightly if necessary, we may assume $\widetilde{\triangle}$ is generic, see the proof of Theorem 9.32.

Suppose $e := \langle v, w \rangle$ is a contractible edge of $\triangle$. Let u and z be the other vertices of the two triangles sharing the edge e. In addition, let $z_1, \ldots, z_p$ be the neighbors of v other than u, w, z, and let $w_1, \ldots, w_q$ be the neighbors of w other than u, v, z. By the inductive hypothesis, the matrix $\widetilde{M}$ associated with $\widetilde{\triangle}$ has rank $\widetilde{V}_I = 7(V_I - 1)$.

We now analyze the rank of M. Let $M_{\langle w,u \rangle}$ be the matrix in (9.51) associated with the vertex w and edge $\langle w, u \rangle$, with similar definitions for $M_{\langle w,v \rangle}$ and $M_{\langle w,z \rangle}$. Consider the auxiliary matrix $(M_{\langle w,u \rangle}\, M_{\langle w,v \rangle}\, M_{\langle w,z \rangle})$.

This is a 7×9 matrix of full rank 7, since it can be thought of as the matrix corresponding to a triangulation with one interior vertex w and three interior edges. Let

$$M^{(1)} = \begin{bmatrix} \widetilde{M} & 0 & 0 & 0 \\ 0 & M_{\langle w,u\rangle} & M_{\langle w,v\rangle} & M_{\langle w,z\rangle} \end{bmatrix}.$$

Clearly, this is a full rank matrix whose rank is equal to the number of its rows, which is $7V_I$. Now, for each $i = 1, \ldots, p$, we add a linear combination of the last three block columns of $M^{(1)}$ to the block column of $M^{(1)}$ corresponding to the edge $\langle w, z_i \rangle$ to obtain the matrix $M_{\langle w,z_i\rangle}$ in the last block row. We denote the modified matrix $M^{(1)}$ by $M^{(2)}$.

Next, we subtract the last block row of $M^{(2)}$ from the block row corresponding to the vertex w, and denote the result by $M^{(3)}$. This step introduces zero blocks in the row corresponding to w and the columns corresponding to $\langle w, z_i \rangle$, $i = 1, \ldots, p$. Now, we add the block column corresponding to $\langle w, u \rangle$ in the first part (all but the last three columns) of $M^{(3)}$ to the block column corresponding to $\langle w, u \rangle$ in the second part (the last three columns). Similarly, we add the block column corresponding to $\langle w, z \rangle$ in the first part of $M^{(3)}$ to the block column corresponding to $\langle w, z \rangle$ in the second part. This results in a matrix $M^{(4)}$ with zero blocks in the row corresponding to w and the columns corresponding to $\langle w, u \rangle$ and $\langle w, z \rangle$ in the second part of the matrix. Clearly, the preceding three steps do not change the rank, and it follows that $M^{(4)}$ is a full rank matrix whose rank is equal to the number of its rows, which is $7V_I$.

The last step is more subtle. Given any point $\xi \in \mathbb{R}^2$, let $M^{(5)}(\xi)$ be the matrix obtained from $M^{(4)}$ by 1) replacing w by ξ in the last block column, and 2) replacing w by ξ in all block columns having a nonzero matrix in the last block row, except for the next to last block column (which contains $M_{\langle w,v\rangle}$), and 3) reversing the sign of the next to last block column. Note that $M^{(5)}(w) = M^{(4)}$, and up to a rearrangement of its columns, $M^{(5)}(v)$ is the matrix corresponding to the triangulation $\triangle$.

Since $M^{(4)}$ is of full rank, there exists a $7V_I \times 7V_I$ submatrix of it whose determinant is nonzero. Let $D(\xi)$ be the determinant of the corresponding submatrix of $M^{(5)}(\xi)$. Clearly, $D(\xi)$ is a continuous rational function of ξ. It is nontrivial since $D(w) \neq 0$. Let Z be the zero set of $D(\xi)$ as ξ runs over $\mathbb{R}^2$. Since $D(\xi)$ is rational, Z is a set of measure zero.

Now, since $M^{(5)}(v)$ corresponds to the *generic* triangulation $\triangle$, we conclude that v is not in Z, for if it were, we could move v to a nearby point ξ to obtain a perturbed triangulation $\triangle_\xi$ whose corresponding matrix $M^{(5)}(\xi)$ would be of full rank. This would imply that the spline space on $\triangle_\xi$ would have smaller dimension than the spline space on $\triangle$, contradicting our assumption that $\triangle$ is generic. It follows that $D(v) \neq 0$ and $M^{(5)}(v)$ is of full rank. $\square$

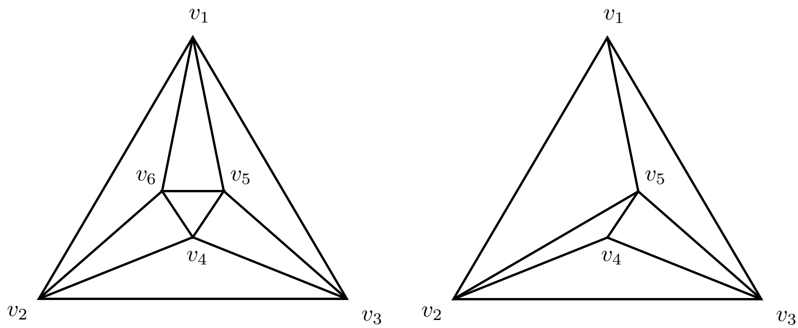


Fig. 9.9. The triangulations $\triangle$ and $\widetilde{\triangle}$ of Example 9.39.

To illustrate how edge contraction works in this proof, we consider the following example. For convenience, we abbreviate $M_{\langle v_i, v_j\rangle}$ to M_{ij}.

Example 9.39. Let $\triangle$ be the Morgan–Scott triangulation shown in Figure 9.9 (left), and let $\widetilde{\triangle}$ be the triangulation shown in Figure 9.9 (right) obtained from $\triangle$ by contracting the edge $\langle v_6, v_5\rangle$.

Discussion: In the notation of the proof of Theorem 9.38, $u = v_1$ and $z = v_4$, and we are contracting the edge $\langle v_6, v_5\rangle$. In this case, carrying out the matrix manipulations described in the proof, we get the following sequence of matrices:

$$\widetilde{M} = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 \\ 0 & 0 & M_{54} & M_{53} & M_{51} & M_{52} \end{bmatrix},$$

$$M^{(1)} = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{54} & M_{53} & M_{51} & M_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{51} & M_{56} & M_{54} \end{bmatrix},$$

$$M^{(2)} = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{54} & M_{53} & M_{51} & M_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{52} & M_{51} & M_{56} & M_{54} \end{bmatrix},$$

$$M^{(3)} = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{54} & M_{53} & M_{51} & 0 & -M_{51} & -M_{56} & -M_{54} \\ 0 & 0 & 0 & 0 & 0 & M_{52} & M_{51} & M_{56} & M_{54} \end{bmatrix},$$

$$M^{(4)} = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & M_{45} \\ 0 & 0 & M_{54} & M_{53} & M_{51} & 0 & 0 & -M_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{52} & M_{51} & M_{56} & M_{54} \end{bmatrix},$$

$$M^{(5)}(\xi) = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & M_{4\xi} \\ 0 & 0 & M_{54} & M_{53} & M_{51} & 0 & 0 & M_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{\xi 2} & M_{\xi 1} & M_{65} & M_{\xi 4} \end{bmatrix}.$$

$$M^{(5)}(v) = \begin{bmatrix} M_{42} & M_{43} & M_{45} & 0 & 0 & 0 & 0 & 0 & M_{46} \\ 0 & 0 & M_{54} & M_{53} & M_{51} & 0 & 0 & M_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{62} & M_{61} & M_{65} & M_{64} \end{bmatrix}.$$

After exchanging columns, this is the matrix corresponding to the Morgan–Scott triangulation $\triangle$. □

9.10. Remarks

Remark 9.1. The first indication that it might be very difficult to give closed formulae for dimensions of spline spaces was the discovery of Example 9.13 which appeared in an unpublished manuscript of Morgan and Scott. The dimension problem associated with this particular triangulation has been studied in several papers, see [Die90, Shi91, FenKZ96, DenFK00]. The following result is established in [Die90]. We denote the boundary vertices of $\triangle_{MS}$ by v_1, v_2, v_3 and the interior vertices by $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$, where $\tilde{v}_i$ is opposite to v_i. For each $i = 1, 2, 3$, let $(\alpha_i, \beta_i, \gamma_i)$ be the barycentric coordinates of v_i relative to the triangle $\langle \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \rangle$.

Theorem 9.40. *Let $\triangle$ be the triangulation in Figure 9.9 (left). Then*

$$\binom{2r+2}{2} + \sigma \le \dim \mathcal{S}_{2r}^r(\triangle) \le \binom{2r+2}{2} + \sigma + 1, \tag{9.54}$$

where

$$\sigma := 3 \sum_{j=1}^{r} (r + 1 - 3j)_+. \tag{9.55}$$

Moreover, the dimension is equal to the lower bound unless

$$\beta_1 \beta_2 \beta_3 = \begin{cases} \gamma_1 \gamma_2 \gamma_3, & \text{if } r \text{ is odd,} \\ \pm \gamma_1 \gamma_2 \gamma_3, & \text{if } r \text{ is even.} \end{cases} \tag{9.56}$$

The condition $\beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3$ holds if and only if the three lines $\langle v_1, \tilde{v}_1 \rangle$, $\langle v_2, \tilde{v}_2 \rangle$, and $\langle v_2, \tilde{v}_2 \rangle$ have a common intersection. For each $i = 1, 2, 3$, let u_i be the intersection of the two lines $\langle v_i, \tilde{v}_i \rangle$ and $\langle \tilde{v}_{i+1}, \tilde{v}_{i+2} \rangle$. Then the condition $\beta_1 \beta_2 \beta_3 = -\gamma_1 \gamma_2 \gamma_3$ holds if and only if the three points u_1, u_2, u_3 are collinear.

Remark 9.2. The following dimension result for $\mathcal{S}_4^1(\triangle)$ was established in [AlfPS87c] using graph-theoretic methods.

Theorem 9.41. *Let V be the number of vertices in a triangulation $\triangle$, and let σ_{sing} be the number of singular vertices. Then*

$$\dim \mathcal{S}_4^1(\triangle) = 6V - 3 + \sigma_{sing}.$$

This is exactly the lower bound of Theorem 9.9. This result is quite easy to prove for a nondegenerate triangulation, i.e., a regular triangulation which does not contain any degenerate edges, see [AlfPS87c]. In this case it is also easy to explicitly construct a basis of locally supported splines for $\mathcal{S}_4^1(\triangle)$. The question of whether or not such a basis exists for arbitrary regular triangulations remains open.

Remark 9.3. Theorem 9.15 describes a minimal determining set for $\mathcal{S}_d^r(\triangle)$ in the case $d \geq 3r+2$. However, this MDS is not prescribed in complete detail. In particular, the minimal determining sets for splines on cells needed in steps 1) and 2) have not been explicitly given. A construction of these sets was first given in [Sch88b], without regard for stability. In Section 11.5 we describe stable minimal determining sets for splines on cells based on [DavS02].

Remark 9.4. The problem of computing dimensions of spline spaces for $d < 3r+2$ on general triangulations seems to be very difficult. Besides the results for type-I and type-II partitions of Section 9.6, there are some results for other special triangulations. The following theorem was established in [AlfS90]. Given $\triangle$, let $\mathcal{V}_I$ is the set of interior vertices of $\triangle$, and for each $v \in \mathcal{V}_I$, let m_v be the number of edges with different slopes attached to v.

Theorem 9.42. *Suppose $\triangle$ is a nondegenerate triangulation. Then*

$$\dim \mathcal{S}_{3r+1}^r(\triangle) = \binom{2r+2}{2} V - 3\binom{r+1}{2} + \sigma,$$

where

$$\sigma := \sum_{v \in \mathcal{V}_I} \sum_{j=1}^{r} (r + j + 1 - j m_v)_+.$$

Remark 9.5. Some of the results in this chapter for shellable triangulations, i.e., regular triangulations without holes, can be extended to the case where $\triangle$ includes holes, see e.g. [AlfPS87b, Jia90].

Remark 9.6. It was proved in [Sch84b] that (9.25) also holds for nonuniform type-I partitions in the case $r = 1$, but not for $r > 1$. It was also shown in [Sch84b] that (9.26) holds for nonuniform type-I partitions in the case $r = 0, 1, 2$. For more on dimension of spline spaces on type-I and type-II partitions, see [ChuH89, ChuH90b].

Remark 9.7. The explicit formula for spline and superspline spaces given in this chapter all have terms that depend on the geometry of the underlying triangulation $\triangle$. In all cases these are nonnegative expressions which vanish for generic triangulations.

Remark 9.8. Splines on cross-cut partitions (see Remark 4.15) were introduced and studied in a series of papers by Chui and Wang [ChuW82a–ChuW84c] and the books [Chu88, Wan01]. The theory includes results on dimension, see [ChuH89, ChuH90b, Man92a, SablJ94]. For dimension results on general rectilinear partitions, see [Ibr89, Man91, Die97].

9.11. Historical Notes

The earliest paper we could find where the problem of finding the dimension of bivariate spline spaces is explictly formulated is due to Strang [Str73], where he made a conjecture about the dimension of the space $\mathcal{S}_d^r(\triangle)$ for type-I triangulations. In [Str74] it was conjectured that for general triangulations, the dimension of the spline space $\mathcal{S}_d^r(\triangle)$ should be equal to the quantity D appearing in the lower bound of Theorem 9.9. But as shown in that theorem, the actual lower bound is usually larger, depending on the geometry of the triangulation. In another early paper [MorS75], Morgan and Scott computed the dimension of $\mathcal{S}_d^1(\triangle)$ for $d \geq 5$ by constructing a nodal basis. Their result takes account of singular vertices, thus showing that Strang's conjecture is not valid for general triangulations.

The lower bounds presented in Theorem 9.9 were established in [Sch79]. This was the first paper to give formulae explaining in detail how the dimension depends on the geometry of the triangulations, and in particular on the slopes of the edges surrounding each interior vertex.

The upper bound results presented in Section 9.3 are taken from [Sch84b]. We have introduced the idea of admissible decompositions here to make the arguments in that paper more rigorous. For more on bounds, see [Wan85, Jia90, Man90, Man91, Man92a, Man92b, Rip95].

Example 9.13 was included in [Sch79] to show that taking account of slopes of edges as in Theorem 9.9 and 9.12 is not enough to describe the dimension of spline spaces in general. The example first appeared in unpublished work of Morgan and Scott. Diener [Die90] found a way to explain the dimension of this space (and more generally of the spaces $\mathcal{S}_{2r}^r(\triangle_{MS})$ in terms of the geometry, see Remark 9.1. Other authors who worked on this problem include [Shi91, FenKZ96, DenFK00].

It was observed in [Sch84b, ChuH89] that the upper and lower bounds of Section 9.3 can sometimes be combined to get exact dimension formulae. In particular, the results for type-I and type-II partitions given in Section 9.6 come from [Sch84b]. The upper and lower bounds also agree for cross-cut partitions, see [ChuH89].

The first paper to use Bernstein–Bézier methods to study the dimension of spline spaces is [AlfS87], where a formula for the dimension of $\mathcal{S}_d^r(\triangle)$ for $d \ge 4r+1$ was obtained by constructing a minimal determining set, a concept which was also introduced in that paper. Bases for these spaces were constructed in [AlfPS87a], and the result was extended to triangulations with holes in [AlfPS87b].

The results on the dimension of $\mathcal{S}_d^r(\triangle)$ for $d \ge 4r+1$ were extended to $d \ge 3r+2$ by Hong in his master's thesis, which was later published as [Hon91]. His techniques were adapted in [IbrS91] to compute the dimension of the superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ for $d \ge 3r+2$. We have followed this paper in Section 9.5.

There has not been much success in understanding the dimension of spline spaces $\mathcal{S}_d^r(\triangle)$ or superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ for degrees $d < 3r+2$ for general triangulations. The spaces $S_{3r+1}^r(\triangle)$ are treated in [AlfS90] under the assumption that $\triangle$ is nondegenerate, see Theorem 9.42. For general regular triangulations, the only case where the dimension of $S_{3r+1}^r(\triangle)$ is known is $r = 1$, see Remark 9.2. To get dimension results which hold for all $0 \le r \le d$ requires severe restrictions on the partition. Here we can point to the results in Section 9.6 on type-I and type-II partitions, and the results on cross-cut partitions, see Remark 9.8.

The upper and lower bounds on the dimension of the very general superspline spaces $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ presented in Section 9.7 are due to Alfeld and Schumaker [AlfS03].

Of the spaces which are not yet understood, the space $\mathcal{S}_3^1(\triangle)$ has certainly received the most attention. It has long been conjectured that the lower bound of Theorem 9.9 is the correct dimension, i.e., the dimension of $\mathcal{S}_3^1(\triangle)$ is $3V_B + 2V_I + 1 + \sigma$, where σ is the number of singular vertices. As shown in Section 9.9, this formula (without the σ term) is correct for generic triangulations.

The dimension question has attracted the attention of algebraists, who have tried to bring the power of homological algebra to bear on this problem, see [Bil88, Bil89, BilR89, BilR91, Whi91a–Whi91c, CoxLO98]. In fact, the book [CoxLO98] contains a chapter on splines. The first proof of the generic dimension of $\mathcal{S}_3^1(\triangle)$ was obtained by a combination of results in homological algebra by Billera [Bil88], and work of Whiteley [Whi91a–Whi91c] on so-called spline matrices. We have not followed this approach here – our results on $\mathcal{S}_3^1(\triangle)$ in Section 9.9 use only ordinary linear algebra.

Approximation Power of Spline Spaces

In this chapter we discuss how well smooth functions can be approximated by bivariate splines. The results are useful in deriving error bounds for various practical interpolation and approximation methods.

10.1. Approximation Power

Throughout this chapter we suppose Ω is a polygonal domain in $\mathbb{R}^2$ and that $\|\cdot\|_{q,\Omega}$ is the q-norm on Ω, for $1 \le q \le \infty$. Given $m \ge 1$, let $W_q^m(\Omega)$ be the Sobolev space with associated seminorm $|\cdot|_{m,q,\Omega}$ introduced in Section 1.6.

Definition 10.1. *Fix $0 \le r < d$ and $0 < \theta \le \pi/3$. Let m be the largest integer such that for every polygonal domain Ω and every regular triangulation $\triangle$ of Ω with smallest angle θ, for every $f \in W_q^m(\Omega)$, there exists a spline $s \in \mathcal{S}_d^r(\triangle)$ with*

$$\|f - s\|_{q,\Omega} \le K\,|\triangle|^m\ |f|_{m,q,\Omega}, \tag{10.1}$$

where the constant K depends only on r, d, θ, and the Lipschitz constant of the boundary of Ω. Then we say that $\mathcal{S}_d^r$ has approximation power m in the q-norm. *If this holds for $m = d+1$, we say that $\mathcal{S}_d^r$ has* full approximation power in the q-norm.

Our aim in this chapter is to explore the approximation power of $\mathcal{S}_d^r$ for various values of r and d. Our main results are as follows:

- If $d \ge 3r+2$, then the space $\mathcal{S}_d^r$ has full approximation power in all of the q-norms.
- If $(3r+2)/2 \le d \le 3r+1$ and $r > 0$, then in any q-norm, the space $\mathcal{S}_d^r$ has approximation power at most d.
- If $d < (3r+2)/2$ and $r > 0$, then in any q-norm, the space $\mathcal{S}_d^r$ has approximation power zero.

We prove the first statement in Section 10.3. The other two results will be established in Section 10.4.

10.2. C^0 Splines and Piecewise Polynomials

In this section we show that for any $d > 0$, $\mathcal{S}_d^0$ has full approximation power in all of the q-norms. For $d \geq 2$, this result is contained in more general results to be established in Section 10.3, but here we can give a much simpler proof.

Theorem 10.2. *Suppose $\triangle$ is a regular triangulation of a polygonal domain Ω, and let $1 \leq q \leq \infty$. Then for every $f \in W_q^{d+1}(\Omega)$, there exists a spline $s \in \mathcal{S}_d^0(\triangle)$ such that*

$$\|D_x^\alpha D_y^\beta (f - s)\|_{q,\Omega} \leq K \, |\triangle|^{d+1-\alpha-\beta} \, |f|_{d+1,q,\Omega},$$

for all $0 \leq \alpha + \beta \leq d$. The constant K depends only on d, the smallest angle in $\triangle$, and the Lipschitz constant of the boundary of Ω.

Proof: The set $\mathcal{M} := \mathcal{D}_{d,\triangle}$ of all domain points associated with $\triangle$ is a stable local minimal determining set for $\mathcal{S}_d^0(\triangle)$, and the result follows immediately from Theorem 5.19. □

Since for any triangulation $\triangle$, the space $\mathcal{PP}_d(\triangle)$ of piecewise polynomials of degree d defined on $\triangle$ contains the space $\mathcal{S}_d^0(\triangle)$, it follows that $\mathcal{PP}_d$ also has full approximation power in all of the q-norms.

10.3. Approximation Power of $\mathcal{S}_d^r(\triangle)$ for $d \geq 3r + 2$

In this section we show that for $d \geq 3r + 2$ the space $\mathcal{S}_d^r$ has full approximation power in all of the q-norms, see Theorem 10.10 below. To prove it, we need to develop some additional machinery.

10.3.1 Near-degenerate Edges and Near-Singular Vertices

Recall that an edge $e = \langle v_2, v_3 \rangle$ shared by two triangles $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ is said to be degenerate at v_2 provided that the points v_1, v_2, v_4 lie on a straight line. We call an edge which is nearly degenerate a near-degenerate edge. To make this concept more precise, we now introduce the following quantitative form.

Definition 10.3. *Suppose $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ are two triangles which share the edge $e = \langle v_2, v_3 \rangle$. Let $0 < \delta < \theta$, where θ is the smallest angle in the two triangles. We say that e is δ-*near-degenerate* at v_2 provided that the angle between the edges $\langle v_2, v_1 \rangle$ and $\langle v_2, v_4 \rangle$ is greater than $\pi - \delta$.*

Lemma 10.4. *Suppose $\triangle$ is a triangulation with smallest angle θ, and let $\delta < \theta$. Then no edge of $\triangle$ can be δ-near-degenerate at both ends.*

Proof: Suppose T and $\widetilde{T}$ are as in Definition 10.3 and that the edge $e := \langle v_2, v_3 \rangle$ is δ-near-degenerate at both v_2 and v_3. Then the sum of the angles in the quadrilateral $\langle v_1, v_2, v_3, v_4 \rangle$ is at least $2\pi - 2\delta + 2\theta > 2\pi$. This is impossible, and the lemma follows. $\square$

Lemma 10.5. *Suppose θ is the smallest angle in triangles $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$, and that $e_2 := \langle v_2, v_3 \rangle$ is not δ-near-degenerate at v_2 with $\delta < \theta$. Let $(\alpha_1, \alpha_2, \alpha_3)$ be the barycentric coordinates of v_4 in terms of the triangle T, i.e., $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. Then $|\alpha_3| \geq \sin\theta \sin\delta$.*

Proof: Let $e_1 := \langle v_2, v_1 \rangle$ and $e_3 := \langle v_2, v_4 \rangle$. Then since $|\alpha_3|$ is the ratio of the area of the triangle $\langle v_1, v_2, v_4 \rangle$ to the area of the triangle T, we have

$$|\alpha_3| = \frac{|e_1||e_3| \sin a}{|e_1||e_2| \sin b},$$

where a is the angle between e_1 and e_3, and b is the angle between the edges e_1 and e_2. By (4.3), $|e_3|/|e_2| \geq \sin\theta$, and the result follows from the fact that $|\sin a| \geq \sin\delta$. $\square$

This lemma is important for establishing the stabilty of the computation of B-coefficients using Lemma 2.30. Indeed, that computation involves solving a linear system of equations with a matrix whose determinant depends on the size of α_3. Cramer's rule then shows that the computation is stable as long as the edge appearing in the lemma is not near-degenerate.

We recall from Definition 9.4 that an interior vertex of a triangulation is called singular provided there are four edges attached to v, and the edges lie on two lines that cross at v.

Definition 10.6. *Suppose v is an interior vertex of a triangulation where four edges meet. If all four edges are δ-near-degenerate at v, then v is called a δ-near-singular vertex.*

10.3.2 Three Lemmas

In this section we present three lemmas that are useful in constructing a superspline subspace of $\mathcal{S}_d^r(\triangle)$ with a stable local basis. Throughout the remainder of this section we fix $0 < \theta \leq \pi/4$ and assume that $\triangle$ is a triangulation with smallest angle θ. Given $\delta < \theta$, we write $\mathcal{V}_{NS}^\delta$ for the set of all δ-near-singular vertices. Since no edge of a triangulation can be δ-near-degenerate at both ends, it impossible for two neighboring vertices of $\triangle$ to both belong to $\mathcal{V}_{NS}^\delta$. Given a vertex $v \in \triangle$, we write $\mathcal{E}_{ND}^\delta(v)$ for the collection of all δ-near-degenerate edges attached to v. The cardinality of $\mathcal{E}_{ND}^\delta(v)$ can only be one, two, or four. Fix $r > 0$, and set

$$\mu := r + \bar{r}, \qquad \bar{r} := \lfloor (r+1)/2 \rfloor. \tag{10.2}$$

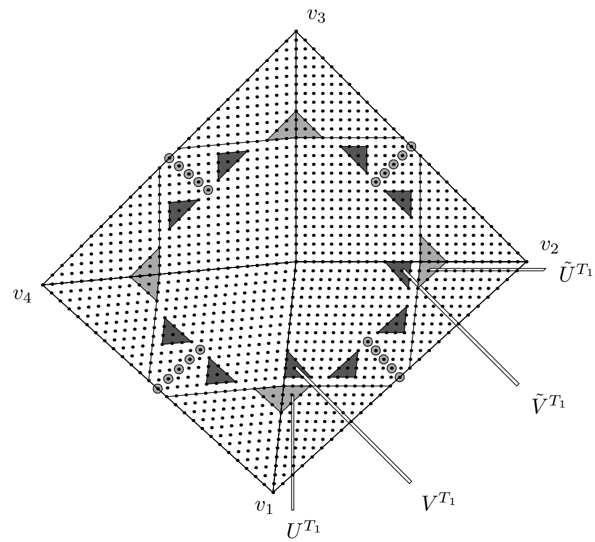


Fig. 10.1. Domain points in Lemma 10.7 with $r = 8$, $\bar{r} = 4$, $\mu = 12$, and $d = 26$.

For each triangle $T \in \triangle$, let

$$U^T := \bigcup_{k=0}^{\bar{r}-1} \{\xi^T_{i,d-i-k,k}\}_{i=r+1}^{\mu-k}, \qquad V^T := \bigcup_{k=0}^{\bar{r}-1} \{\xi^T_{i,d-i-k,k}\}_{i=\mu-k+1}^{\mu+\bar{r}-2k},$$

$$\widetilde{U}^T := \bigcup_{j=0}^{\bar{r}-1} \{\xi^T_{i,j,d-i-j}\}_{i=r+1}^{\mu-j}, \qquad \widetilde{V}^T := \bigcup_{j=0}^{\bar{r}-1} \{\xi^T_{i,j,d-i-j}\}_{i=\mu-j+1}^{\mu+\bar{r}-2j}.$$

These sets are illustrated in Figure 10.1. Our next lemma deals with splines defined on a cell consisting of four triangles surrounding an interior vertex.

Lemma 10.7. *Suppose $v \in \mathcal{V}^\delta_{NS}$. Let $v_1, \ldots, v_4$ be the vertices attached to v (in counterclockwise order), and let $\triangle_v$ be the triangulation consisting of the four triangles $T_i := \langle v, v_i, v_{i+1}\rangle$, $i = 1, \ldots, 4$, where v_5 is identified with v_1. Let*

$$\Lambda_v := \{\xi \in D_{d-r-1}(v) \cap T_1 : \xi \notin U^{T_1} \cup \widetilde{U}^{T_1} \cup V^{T_1} \cup \widetilde{V}^{T_1}\},$$

and let $s \in \mathcal{S}^r_d(\triangle_v) \cap C^{d-r-1}(v)$. Then if δ is sufficiently small, the coefficients of s associated with domain points in the disk $D_{d-r-1}(v)$ are uniquely determined by the coefficients associated with domain points in the set

$$\mathcal{M}_v := \Lambda_v \cup U^{T_1} \cup \widetilde{U}^{T_1} \cup \widetilde{U}^{T_2} \cup U^{T_4}.$$

Moreover, there exists a positive constant δ_0 depending only on d and the smallest angle θ in $\triangle_v$ such that if $\delta \le \delta_0$, then $|c_\xi| \le K \max_{\eta\in\mathcal{M}_v} |c_\eta|$ for all $\xi \in D_{d-r-1}(v)$, where K is a constant depending only on d and θ.

Proof: Let $\{c_{ijk}\}_{i+j+k=d}$ be the coefficients of $s|_{T_1}$, and let

$$\begin{aligned} v_3 =& \alpha_1 v + \alpha_2 v_1 + \alpha_3 v_2, \\ v_4 =& \beta_1 v + \beta_2 v_1 + \beta_3 v_2. \end{aligned}$$

Suppose that all of the coefficients of s corresponding to domain points in $\mathcal{M}_v$ have been fixed. Since s is in $C^{d-r-1}(v)$, it suffices to show that the unspecified coefficients in $T_1 \cap D_{d-r-1}(v)$ (namely those with subscripts lying in V^{T_1} and in $\widetilde{V}^{T_1}$) are uniquely determined by the smoothness conditions. We put these coefficients into a vector c in the order

$$c_{r+2,\tilde{d},\bar{r}-1}, c_{r+3,\tilde{d},\bar{r}-2}, c_{r+4,\tilde{d}-1,\bar{r}-2}, \ldots, c_{\mu+1,\tilde{d},0}, \ldots, c_{\mu+\bar{r},\tilde{d}-\bar{r}+1,0}, \tag{10.3}$$

followed by

$$c_{r+2,\bar{r}-1,\tilde{d}}, c_{r+3,\bar{r}-2,\tilde{d}}, c_{r+4,\bar{r}-2,\tilde{d}-1}, \ldots, c_{\mu+1,0,\tilde{d}}, \ldots, c_{\mu+\bar{r},0,\tilde{d}-\bar{r}+1}, \tag{10.4}$$

where $\tilde{d} = d - \mu - 1$. The vector c has length $2m$ with $m := 1 + 2 + \cdots + \bar{r} = \binom{\bar{r}+1}{2}$. Note that the coefficients in both (10.3) and (10.4) fall naturally into subsets of size $1, 2, \ldots, \bar{r}$.

Now we write down all smoothness conditions across the edge $e_2 := \langle v, v_2 \rangle$ which involve the coefficients in both U^{T_1} and $\widetilde{U}^{T_2}$. In addition, we write the conditions across $e_1 := \langle v, v_1 \rangle$ which involve the coefficients in both $\widetilde{U}^{T_1}$ and U^{T_4}. We need to exercise some care in the order in which we write down these conditions. We start with those associated with edge e_2. As the first equation, we write the $C^{d-\mu}$ condition which involves only the coefficient $c_{r+2,\tilde{d},\bar{r}-1}$ from V^{T_1}. Next we write two conditions, namely the $C^{d-\mu}$ and $C^{d-\mu+1}$ conditions which involve only the three coefficients from V^{T_1} with third subscript $k \ge \bar{r} - 2$. Continuing, we conclude by writing the $\bar{r}$ conditions for $C^{d-\mu}$ up to C^{d-r-1} which involve all the coefficients in V^{T_1}. So far this is a total of m conditions. We now repeat the process for the conditions across the edge e_1, and end up with a system of the form

$$\begin{bmatrix} A & B \\ \tilde{B} & \tilde{A} \end{bmatrix} c = R, \tag{10.5}$$

where all four blocks in the matrix are of size $m \times m$.

We now examine these blocks in detail. The matrix A is a lower triangular block matrix of the form

$$A = \begin{bmatrix} A_1 & & & \\ \times & A_2 & & \\ \times & \times & \ddots & \\ \times & \times & \ldots & A_{\bar{r}} \end{bmatrix},$$

where

$$A_i = g_i\, \alpha_1^{i^2}\, \alpha_2^{\kappa_i - i^2} \left[\frac{1}{(m+n+1)!}\right]_{m,n=0}^{i-1},$$

is an $i \times i$ matrix with $\kappa_i := \sum_{j=0}^{i-1}(d-\mu+j)$. Here g_i is a nonzero product of factorials. The matrix $\tilde{A}$ has a similar structure with

$$\tilde{A}_i = g_i\, \beta_1^{i^2}\, \beta_3^{\kappa_i - i^2} \left[\frac{1}{(m+n+1)!}\right]_{m,n=0}^{i-1}.$$

Now observe that every entry of B involves some positive power of α_3, while every entry of $\tilde{B}$ involves some positive power of β_2. Both α_3 and β_2 go to zero as $\delta \to 0$. The remaining α_i and β_i are bounded away from 0 by a constant depending on θ. Let $D(\delta)$ be the determinant of the matrix in (10.5). Then $D(0) = \det(A)\det(\tilde{A})$ is bounded below by a positive constant D_0 which depends only on d and θ. But then by continuity, there exists a δ_0 depending only on d and θ such that $D(\delta) \ge D_0/2$ for all $\delta \le \delta_0$. This shows that the computed coefficients satisfy the stated bound. □

The following lemma shows how to use smoothness conditions to compute certain coefficients of a spline in $\mathcal{S}_d^r(\triangle)$. For an explanation of the notation, see (9.17), (10.2), and Figure 9.5.

Lemma 10.8. *Suppose $T := \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2\rangle$ are two adjoining triangles sharing the edge $e := \langle v_2, v_3\rangle$, and that $e \notin \mathcal{E}_{ND}^\delta(v_2) \cup \mathcal{E}_{ND}^\delta(v_3)$. Suppose s is a spline in $\mathcal{S}_d^r(\triangle)$ whose coefficients are known for all domain points in $D_\mu^T(v_2)$, $D_\mu^T(v_3)$, and $E^T(e)$. Suppose the coefficients are also known for domain points in any two of $A^T(v_2)$, $A^{\widetilde{T}}(v_2)$, $G_L^T(e)$, and all points in any two of the sets $A^T(v_3)$, $A^{\widetilde{T}}(v_3)$, $G_R^{\widetilde{T}}(e)$. Then all unspecified coefficients of s in $\{\xi \in D_{2r}(v_2) \cup D_{2r}(v_3) : d(\xi, e) \le r\}$ are uniquely determined by the smoothness conditions.*

Proof: We alternately compute the coefficients in the arcs $a_{m,e}^r(v_2) := \{\xi \in R_m(v_2) : \operatorname{dist}(\xi, e) \le r\}$, and $a_{m,e}^r(v_3)$ for each $m = \mu+1, \ldots, 2r$, using Lemma 2.29 or Lemma 2.30, depending on which coefficients are given. □

Since the computation of coefficients in this lemma is based in part on Lemma 2.30, to ensure that the computation is stable, we should not use this lemma in situations where the edge e is near-degenerate at v_2 or at v_3. A careful examination of the computations involved in this lemma shows that if s has nonzero coefficients for some points in $D_{2r}(v_2)$, then the computed coefficients can be nonzero for some points in $D_{2r}(v_3)$. We refer to this as propagation. We are particularly concerned about getting nonzero coefficients in one of the sets $A^T(v_3)$ or $A^{\widetilde{T}}(v_3)$, since these can then propagate further. The following lemma shows how such propagation can be stopped.

Lemma 10.9. *Let T and $\widetilde{T}$ be as in Lemma 10.8 where $v_3 \notin \mathcal{V}_{NS}^{\delta}$. Suppose $s \in \mathcal{S}_d^r(\triangle)$ is a spline whose coefficients are zero for all domain points in a set $\mathcal{M}_0$ which contains the sets $D_\mu^T(v_2)$, $D_\mu^T(v_3)$, $A^{\widetilde{T}}(v_3)$, $G_R^{\widetilde{T}}(e)$, and $G_L^T(e)$, where e is the edge $\langle v_2, v_3\rangle$. Suppose $\mathcal{M}_0$ also contains one of the sets $E^T(e)$ or $E^{\widetilde{T}}(e)$. Then the coefficients of s associated with points in $A^T(v_3)$ must be zero.*

Proof: Suppose $\mathcal{M}_0$ contains $E^T(e)$. The other case is similar. Applying Lemma 2.29, it can be checked that the coefficients of s associated with domain points in $E^{\widetilde{T}}(e)$, $G_L^{\widetilde{T}}(e)$, and $G_R^T(e)$ must be zero. Then using the smoothness conditions of Lemma 2.29 to compute coefficients in $A^T(v_3)$ gives only zero values. □

10.3.3 Approximation with a Superspline Subspace of $\mathcal{S}_d^r(\triangle)$

Let Ω be a polygonal domain in $\mathbb{R}^2$, and suppose $d \ge 3r+2$ and $1 \le q \le \infty$. The following theorem implies that the space $\mathcal{S}_d^r$ has full approximation power in the q-norm.

Theorem 10.10. *Let $d \ge 3r+2$, and suppose $\triangle$ is a regular triangulation of Ω. Then for every $f \in W_q^{d+1}(\Omega)$, there exists a spline $s \in \mathcal{S}_d^r(\triangle)$ such that*

$$\|D_x^\alpha D_y^\beta (f-s)\|_{q,\Omega} \le K\, |\triangle|^{d+1-\alpha-\beta}\, |f|_{d+1,q,\Omega}, \tag{10.6}$$

for all $0 \le \alpha+\beta \le d$. If Ω is convex, then the constant K depends only on r, d, and the smallest angle in $\triangle$. If Ω is not convex, then K also depends on the Lipschitz constant of the boundary of Ω.

Proof: We show below that there exists a superspline subspace $\mathcal{S}$ of $\mathcal{S}_d^r(\triangle)$ with a stable local minimal determining set. Then the result follows by applying Theorem 5.19 to $\mathcal{S}$. □

We now define the superspline space $\mathcal{S}$ needed for the proof of Theorem 10.10. Let δ_0 be the minimum of $\delta_0(v)$ over all interior vertices $v \in \triangle$, where $\delta_0(v)$ is the constant defined in Lemma 10.7. Suppose $v_1, \ldots, v_n$ are the interior vertices of $\triangle$. Let $\rho := (\rho_1, \ldots, \rho_n)$ with

$$\rho_i = \begin{cases} d-r-1, & v_i \in \mathcal{V}_{NS}^{\delta_0}, \\ \mu, & \text{otherwise}, \end{cases}$$

where μ is defined in (10.2). Let

$$\mathcal{S} := \{s \in S_d^r(\triangle) : \ s \in C^{\rho_i}(v_i), \ i = 1, \ldots, n\}. \tag{10.7}$$

In the sequel we hold δ_0 fixed, and so for ease of notation we drop it from the notation, and write $\mathcal{V}_{NS} := \mathcal{V}_{NS}^{\delta_0}(\triangle)$ and $\mathcal{E}_{ND} := \mathcal{E}_{ND}^{\delta_0}(\triangle)$ for the corresponding sets of near-singular vertices and near-degenerate edges in $\triangle$, respectively.

Theorem 10.11. *Let $\mathcal{M}$ be the following set:*

1) *For each vertex $v \notin \mathcal{V}_{NS}$, pick a triangle T with vertex at v and choose all points in the set $D_\mu^T(v)$.*
2) *For each vertex $v \in \mathcal{V}_{NS}$, pick a triangle T with first vertex at v and choose all points in the set*

$$\mathcal{M}_v := \{\xi \in D_{d-r-1}^T(v) : \xi \notin U^T \cup \widetilde{U}^T \cup V^T \cup \widetilde{V}^T\}. \tag{10.8}$$

3) *For each edge $e := \langle v, u\rangle$ with $v, u \notin \mathcal{V}_{NS}$, include the set $E^T(e)$, where T is a triangle containing the edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, we can choose either of the two triangles containing e. If e is a boundary edge, also include the two sets $G_L^T(e)$ and $G_R^T(e)$.*
4) *Suppose $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in clockwise order. Let $T_i := \langle v, v_i, v_{i+1}\rangle$ for $i = 1, \ldots, n-1$, and set $T_0 := T_n := \langle v, v_n, v_1\rangle$ if v is an interior vertex. Suppose $1 \leq i_1 < \cdots < i_k < n$ are such that $e_{i_j} \in \mathcal{E}_{ND}(v_{i_j}) \cup \mathcal{E}_{ND}(v)$, where $e_i := \langle v, v_i\rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$.*
   a) *Include the sets $G_L^{T_{i_j-1}}(e_{i_j})$ for all $1 \leq j \leq k$ such that $v_{i_j} \notin \mathcal{V}_{NS}$.*
   b) *Include the sets $A^{T_i}(v)$ for all $1 \leq i \leq n-1$ such that $i \notin J_v$.*
   c) *Include $A^{T_n}(v)$ if v is an interior vertex.*
5) *For all triangles $T = \langle v, u, w\rangle$ with $u, v, w \notin \mathcal{V}_{NS}$, include the set C^T.*

Then $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}$.

Proof: We claim that $\mathcal{M}$ is well-defined. In particular, a simple geometric argument shows that for any interior vertex $v \notin \mathcal{V}_{NS}$, there is always at least one edge attached to v which is not near-degenerate at either end. In the numbering of the edges in item 4) above, we can choose this edge to be $\langle v, v_n\rangle$. The construction in 4) ensures that for each interior vertex $v \notin \mathcal{V}_{NS}$ and edge $e_i := \langle v, v_i\rangle$ attached to it, if $v_i \notin \mathcal{V}_{NS}$, then $\mathcal{M}$ includes exactly one of the two sets $A^{T_i}(v)$ or $G_L^{T_{i-1}}(e_i)$.

We now show that $\mathcal{M}$ is a determining set, i.e., if we prescribe the coefficients of a spline $s \in \mathcal{S}$ corresponding to all the points in $\mathcal{M}$, then all other coefficients of s can be uniquely computed. This can be done as follows:

Step 1: Compute coefficients for all domain points lying in disks of the form $D_\mu(v)$ for $v \notin \mathcal{V}_{NS}$. Note that for such vertices v, $s \in C^\mu(v)$ while $\mathcal{M}$ includes all points in one subtriangle intersected with $D_\mu(v)$. Then all coefficients in the disk $D_\mu(v)$ can be uniquely computed using Lemma 2.29.

Step 2: Use Lemma 10.7 to compute coefficients for points in the disks $D_{d-r-1}(v)$ for each near-singular vertex $v \in \mathcal{V}_{NS}$.

Step 3: Use Lemma 10.8 to compute coefficients corresponding to points in the disks $D_{2r}(v)$ for $v \notin \mathcal{V}_{NS}$. We proceed by first doing all rings of size $\mu+1$ around all such vertices, then all rings of size $\mu+2$, etc., until we have completed the rings of size $2r$. In computing coefficients in a ring $R_m(v)$, we process one arc $a^r_{m,e}(v)$ after another, always proceeding in a *clockwise* direction. To show that this process works, we have to show how to start it, and that once started we can continue all the way around the vertex. Consider the arc $a^r_{m,e_i}(v)$ associated with the edge $e_i := \langle v, v_i \rangle$, and suppose we already know the coefficients associated with $A^{T_{i-1}}(v)$. Then Lemma 10.8 can be applied to compute all coefficients on the arc. The set $\mathcal{M}$ includes the sets needed to apply the lemma since

a) if $v_i \in \mathcal{V}_{NS}$, then $G_L^{T_{i-1}}(e_i) \subset D_{d-r-1}(v_i) \subset \mathcal{M}$,

b) if $e_i \in \mathcal{E}_{ND}(v_i)$ but $v_i \notin \mathcal{V}_{NS}$, then $G_L^{T_{i-1}}(e_i) \subset \mathcal{M}$,

c) if $e_i \in \mathcal{E}_{ND}(v)$, then $G_L^{T_{i-1}}(e_i) \subset \mathcal{M}$,

d) otherwise $e_i \notin \mathcal{E}_{ND}(v) \cup \mathcal{E}_{ND}(v_i)$, and $A^{T_i}(v) \subset \mathcal{M}$.

It remains to show how to start the process. If v is a boundary vertex, we can start with the arc $a^r_{m,e_2}(v)$ since $A^{T_1}(v)$ is contained in $\mathcal{M}$. If v is an interior vertex, we can start with the arc $a^r_{m,e_1}(v)$ since $A^{T_n}(v)$ is contained in $\mathcal{M}$.

Step 4: Compute coefficients corresponding to domain points in sets of the form $E^{\widetilde{T}}(e) \setminus [D_{2r}(v) \cup D_{2r}(u)]$ which are not already known. In this case the points in $E^T(e)$ are in $\mathcal{M}$, where T and $\widetilde{T}$ are the two triangles sharing the edge $e = \langle v, u \rangle$ with $v, u \notin \mathcal{V}_{NS}$, and Lemma 2.29 can be applied.

We have shown that $\mathcal{M}$ is a minimal determining set. To complete the proof, it remains to show that it is stable and local in the sense of Definition 5.16. Suppose $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$ is a domain point of $\mathcal{S}$ that does not lie in $\mathcal{M}$, and suppose η lies in the triangle T_η. We shall show that there exists a set $\Gamma_\eta \subseteq \mathcal{M}$ with $\Gamma_\eta \subseteq \mathrm{star}^3(T_\eta)$ such that c_η depends only on the values of $\{c_\xi\}_{\xi \in \Gamma_\eta}$ and

$$|c_\eta| \le K \max_{\xi \in \Gamma_\eta} |c_\xi|. \tag{10.9}$$

Once we show the existence of the set Γ_η, (10.9) follows from the fact that all computed coefficients are obtained using Lemmas 2.29 and 2.30, where the latter is used only for edges that are not near-degenerate.

To find Γ_η explicitly, we would need to figure out which of the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ the coefficient c_η depends on. But we don't need an exact description of Γ_η. It is enough to show that $\Gamma_\eta \subseteq \mathrm{star}^3(T_\eta)$. To show this, we fix $\xi \in \mathcal{M}$ and let T_ξ be a triangle containing ξ. We now examine which coefficients depend on the value of c_ξ. Set $c_\xi = 1$ and $c_\zeta = 0$ for all $\zeta \in Z_\xi := \mathcal{M} \setminus \{\xi\}$, and suppose we use smoothness to compute the remaining coefficients of a spline $s_\xi \in \mathcal{S}_d^r(\triangle)$. We claim that the computed coefficients of s_ξ can be nonzero only if they correspond to a domain point in $\text{star}^3(T_\xi)$. The analysis divides into several cases depending on where ξ lies.

Case 1: Suppose $\xi \in C^T$ for some triangle T. Since the coefficients corresponding to points in C^T do not enter any smoothness conditions, the only nonzero coefficient of s_ξ is the one corresponding to ξ.

Case 2: Suppose $\xi \in E^T(e)$ where $e := \langle v, u \rangle$ is a boundary edge of a triangle T, and that $\xi \notin D_{2r}(v) \cup D_{2r}(u)$. Then the coefficient corresponding to ξ does not enter any smoothness conditions, and thus is the only nonzero coefficient of s_ξ.

Case 3: Suppose $\xi \in E^T(e) \setminus \big(D_{2r}^T(v_2) \cup D_{2r}^T(v_3)\big)$, where $T = \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} = \langle v_4, v_2, v_3 \rangle$ are two triangles sharing an interior edge $e = \langle v_2, v_3 \rangle$ with $v_2, v_3 \notin \mathcal{V}_{NS}$. Then the coefficients of s_ξ corresponding to points in $D_{2r}^T(v_2) \cup D_{2r}^T(v_3)$ will be zero, and carrying out Step 4), we can get nonzero coefficients for points in the set $E^{\widetilde{T}}(e)$. Since all other coefficients are zero, we conclude that the only nonzero coefficients of s_ξ correspond to domain points in $\text{star}^3(T_\xi)$.

Case 4: Suppose $u \notin \mathcal{V}_{NS}$ and that ξ lies in some set of the form $D_\mu(u)$, $A^T(u)$, $G_L^T(e)$, or $E^T(e) \cap D_{2r}(u)$, where T is a triangle attached to u and e is an edge attached to u. We assume u is an interior vertex (the case where it is a boundary vertex is similar). Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_m$ be the vertices in clockwise order which lie on the boundaries of $\text{star}(u)$ and of $\text{star}^2(u)$, respectively. Note that Z_ξ includes the disks $D_\mu(v)$ for all $v \neq u$. It also includes the set $\mathcal{M}_{u_i}$ for all $u_i \in \mathcal{V}_{NS}$. We now show that the nonzero coefficient c_ξ can propagate to points in the disks $D_{2r}(u_i)$, and even to some points in the disks $D_{2r}(w_j)$, but not to any points beyond $\text{star}^3(u)$. There are two subcases:

a) Suppose $u_i \notin \mathcal{V}_{NS}$. We show that propagation beyond $\text{star}(u_i)$ along the edge $e_{ij} := \langle u_i, w_j \rangle$ is blocked. This is clear if $w_j \in \mathcal{V}_{NS}$ since $D_\mu(u_i) \subset Z_\xi$. Now suppose $w_j \notin \mathcal{V}_{NS}$. Since we process the arcs around w_j in *clockwise* order, it suffices to show that the coefficients associated with points in $A^{T_{ij}}(w_j)$ are zero, where T_{ij} is the triangle with vertices u_i, w_j, v in counterclockwise order for some v. This is automatic if e_{ij} is not near-degenerate at either end, since then Z_ξ contains $A^{T_{ij}}(w_j)$ itself by the choice of $\mathcal{M}$, see item 4) in the statement of the theorem. Now suppose e_{ij} is near-degenerate at either u_i or w_j. Then by the choice of $\mathcal{M}$, Z_ξ contains both $G^{T_{ij}}(e_{ij})$ and $G^{\widetilde{T}_{ij}}(e_{ij})$, where $\widetilde{T}_{ij}$ is the other triangle sharing the edge e_{ij}. Lemma 10.9 then implies that the coefficients associated with points in $A^{T_{ij}}(w_j)$ are zero.

b) Suppose $u_i \in \mathcal{V}_{NS}$. Then applying Lemma 10.7, the nonzero coefficient c_ξ can propagate to the disk $D_{2r}(w_j)$ around the vertex w_j which lies on the opposite side from the near-singular vertex u_i. Note that $w_j \notin \mathcal{V}_{NS}$ and $D_\mu(w_j) \subset Z_\xi$. Now arguing as in case 4a) with u_i replaced by w_j, we see that there is no propagation beyond $\mathrm{star}(w_j)$, and thus not beyond $\mathrm{star}^2(u_i)$.

We conclude that the only nonzero coefficients of s_ξ lie in

$$\mathrm{star}(u) \;\cup \bigcup_{u_i \notin \mathcal{V}_{NS}} \mathrm{star}(u_i) \cup \bigcup_{u_i \in \mathcal{V}_{NS}} \mathrm{star}^2(u_i) \quad\subset\quad \mathrm{star}^3(u).$$

Case 5: Suppose $\xi \in \mathcal{M}_u$ where $u \in \mathcal{V}_{NS}$. All coefficients associated with points in the disks of the form $D_\mu^T(v)$ with $v \notin \mathcal{V}_{NS}$ are zero. Let $v_1, \ldots, v_4$ be the vertices attached to v. Since it is impossible for two near-singular vertices to be neighbors, $v_i \notin \mathcal{V}_{NS}$ for $i = 1, \ldots, 4$. Now nonzero coefficients associated with points in $\mathcal{M}_u$ may propagate to points in the disks of radius $2r$ around the vertices $v_1, \ldots, v_4$. However, since $D_\mu(v_i) \subset Z_\xi$, arguing as in Case 4a), we see that they cannot propagate any further, and thus all nonzero coefficients of s_ξ are contained in $\mathrm{star}^2(u)$.

We have now shown that in all cases a nonzero coefficient associated with ξ in a triangle T_ξ can propagate to at most $\mathrm{star}^3(T_\xi)$, and the proof is complete. $\square$

10.4. Approximation Power of $\mathcal{S}_d^r(\triangle)$ for $d < 3r + 2$

Let $H = [0,1] \times [0,1]$ be the unit square, and let $0 < r < d < 3r + 2$. In this section we show that for these values of r and d, the space $\mathcal{S}_d^r$ does not have full approximation power in any of the q-norms on H. We begin by showing that when $d < (3r+2)/2$, $\mathcal{S}_d^r$ has approximation power zero. Given a positive integer n, let

$$0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1,$$
$$0 = y_0 < y_1 < \cdots < y_n < y_{n+1} = 1,$$

with $x_i = y_i = ih$ for $i = 0, \ldots, n+1$, where $h := 1/(n+1)$. We write $\triangle_n$ for the associated uniform type-I triangulation of H obtained by drawing in the northeast diagonals.

10.4.1 The Case $d < (3r + 2)/2$

By Theorem 9.21,

$$\dim \mathcal{S}_d^r(\triangle_n) = 2n(d^2 - 2rd + d - r + r^2) + \frac{2d^2 + 4d - 2rd - r + r^2 + 2}{2}. \tag{10.10}$$

For any real x, let $(x)_+$ be defined in (9.3).

Theorem 10.12. *Suppose $d < (3r+2)/2$. Then the set*

$$\Phi_{r,d,n} := \bigcup_{\nu=0}^{d}\bigcup_{\mu=0}^{d-\nu}\{x^\nu y^\mu\} \cup \bigcup_{\nu=r+1}^{d}\bigcup_{\mu=0}^{d-\nu}\{(x-y)_+^\nu (x+y)^\mu\}$$
$$\cup \bigcup_{i=1}^{n}\bigcup_{\nu=r+1}^{d}\bigcup_{\mu=0}^{d-\nu}\{(x-x_i)_+^\nu y^\mu, (x-y-x_i)_+^\nu (x+y)^\mu\}$$
$$\cup \bigcup_{i=1}^{n}\bigcup_{\nu=r+1}^{d}\bigcup_{\mu=0}^{d-\nu}\{(y-y_i)_+^\nu x^\mu, (y-x-y_i)_+^\nu (x+y)^\mu\}$$

is a basis for $\mathcal{S}_d^r(\triangle_n)$.

Proof: Each of the functions in $\Phi_{r,d,n}$ clearly belongs to $\mathcal{S}_d^r(\triangle_n)$. A simple count shows that the cardinality of $\Phi_{r,d,n}$ is equal to the dimension of $\mathcal{S}_d^r(\triangle_n)$ as given in (10.10). Thus, to show that $\Phi_{r,d,n}$ is a basis, it suffices to show that the functions in $\Phi_{r,d,n}$ are linearly independent. Let $H_{ij} := [ih,(i+1)h] \times [jh,(j+1)h]$ for all $0 \le i,j \le n$. Let T_{ij}^+ and T_{ij}^- be the triangles above and below the diagonal of H_{ij}, respectively. Now suppose some linear combination g of the basis functions in $\Phi_{r,d,n}$ is identically zero on H. Then on T_{00}^+, g reduces to a linear combination of the functions $\{x^\nu y^\mu\}_{0\le\nu+\mu\le d}$. These functions are clearly linearly independent, and thus, the corresponding coefficients of g must be zero. Let g_1 be the remaining sum. It must still vanish on the rest of H, and in particular on T_{00}^-. On this triangle the only functions in g_1 that are nonzero are $\{(x-y)_+^\nu(x+y)^\mu\}_{\nu=r+1,\mu=0}^{d,d-\nu}$. Since these functions are linearly independent on this triangle, the corresponding coefficients of g_1 must be zero. The remainder g_2 of the original sum g must still vanish on the triangle T_{10}^+. On this triangle the only nonzero functions left in g_2 are $\{(x-x_1)_+^\nu y^\mu\}_{\nu=r+1,\mu=0}^{d,d-\nu}$. These are linearly independent, and the corresponding coefficients must vanish. Continuing through the triangles in the bottom row of the triangulation, we see that all coefficients of g must be zero except for those corresponding to the last collection in $\Phi_{r,d,n}$. We now repeat this argument moving upward from T_{00}^+ through the first column of triangles in $\triangle$ to show that all of these coefficients must also vanish. $\square$

Theorem 10.13. *Suppose $d < (3r+2)/2$ and $1 \le q \le \infty$. Then the approximation power of $\mathcal{S}_d^r$ in the q-norm is zero.*

Proof: We first deal with the case $q = \infty$. Let $F(x,y) := x^{d+1}y^{d+1}(x+y)^{d+1}$. We now show that assuming that for every $n > 0$ there exists a spline $s_n \in \mathcal{S}_d^r(\triangle_n)$ such that

$$\|F - s_n\|_H \le K\Big(\frac{1}{n+1}\Big)^m |F|_{m,H} \tag{10.11}$$

for some positive integer m leads to a contradiction.

Let $\delta = 1/(4(d+1))$, and let Δ_1 be the forward difference operator defined by $\Delta_1 f(x,y) = f(x+\delta,y) - f(x,y)$. Similarly, let $\Delta_2 f(x,y) = f(x,y+\delta) - f(x,y)$ and $\Delta_3 f(x,y) = f(x+\delta,y+\delta) - f(x,y)$. Let $\lambda f := \Delta_1^{d+1}\Delta_2^{d+1}\Delta_3^{d+1} f(0,0)$. We claim that $\lambda B = 0$ for every B in the basis $\Phi_{r,d,n}$ for $\mathcal{S}_d^r(\triangle_n)$ of Theorem 10.12. For example, if $B(x,y) = (x - x_i)_+^\nu y^\mu$ then for all $0 \le x \le 1$, $\Delta_2^{d+1} B = 0$ and thus $\lambda B = 0$. Similarly, for $B(x,y) = (x - y - x_i)_+^\nu (x+y)^\mu$, on every line where $x - y$ is constant, we have $\Delta_3^{d+1} B = 0$, and thus $\lambda B = 0$ throughout H. Note that for any $f \in C(H)$, λf is just a combination of values of f at points in H, and $|\lambda f| \le 2^{3(d+1)}\|f\|_H$. It is easy to check that $\lambda F = c\,\delta^{3(d+1)}$, with

$$c := \sum_{i=0}^{d+1} \binom{d+1}{i}^2 (d+i+1)!\,(2d-i+2)!\,.$$

Now if (10.11) holds, then $|\lambda F| \le 2^{3(d+1)}\|F - s_n\|_H \to 0$ as $n \to \infty$. This implies $\lambda F = 0$ which is a contradiction. This completes the proof for $q = \infty$.

The proof for the q-norm is similar. Now we suppose that for every $n > 0$ there exists a spline $s_n \in \mathcal{S}_d^r(\triangle_n)$ such that

$$\|F - s_n\|_{q,H} \le K\Big(\frac{1}{n+1}\Big)^m |F|_{m,q,H} \tag{10.12}$$

for some positive integer m, and show that this leads to a contradiction. Let $\tilde H = [0,1/2] \times [0,1/2]$, and let

$$\tilde\lambda f := \int_{\tilde H} \Delta_1^{d+1}\Delta_2^{d+1}\Delta_3^{d+1} f(x,y)\,dx\,dy.$$

Then $\tilde\lambda s = 0$ for all $s \in \mathcal{S}_d^r(\triangle_n)$. Using the fact that the area of $\tilde H$ is $1/4$, we have

$$\begin{aligned}
|\tilde\lambda F| &= |\tilde\lambda(F - s_n)| \\
&\le (1/4)^{1/q'} \bigg(\int_{\tilde H} |\Delta_1^{d+1}\Delta_2^{d+1}\Delta_3^{d+1}(F - s_n)(x,y)|^q dxdy\bigg)^{1/q} \\
&\le (1/4)^{1/q'} 2^{3(d+1)} \bigg(\int_H |(F - s_n)(x,y)|^q dxdy\bigg)^{1/q} \\
&= (1/4)^{1/q'} 2^{3(d+1)} \|F - s_n\|_{q,H},
\end{aligned}$$

where $1/q' + 1/q = 1$. Inserting (10.12) and letting $n \to \infty$, we see that $\tilde{\lambda} F = 0$, which is a contradiction since

$$\tilde{\lambda} F = \int_{\tilde{H}} \Delta_1^{d+1} \Delta_2^{d+1} \Delta_3^{d+1} F(x,y)\, dx\, dy = \frac{c\, \delta^{3(d+1)}}{4},$$

with c and δ as above. $\square$

10.4.2 The Case $2r+2 \le d \le 3r+1$

Throughout this subsection we suppose that $2r+2 \le d \le 3r+1$. By Theorem 9.21,

$$\dim \mathcal{S}_d^r(\triangle_n) = n^2(d^2 - 3rd + 2r^2 + \sigma_g) + 2n(d^2 - 2rd + d - r + r^2) + \frac{2d^2 + 4d - 2rd - r + r^2 + 2}{2}, \tag{10.13}$$

where

$$\sigma_g := \begin{cases} r^2/4, & \text{if } r \text{ is even}, \\ (r^2-1)/4, & \text{if } r \text{ is odd}. \end{cases}$$

We now construct a basis for $\mathcal{S}_d^r(\triangle_n)$. As part of the basis we take the set $\Phi_{r,d,n}$ of polynomials and one-sided splines defined in Theorem 10.12. To complete the basis, we make use of certain type-I box splines introduced in Section 12.1 below.

Type-I box splines are associated with the direction vectors $e_1 := (1,0)$, $e_2 := (0,1)$, and $e_3 := (1,1)$. Each such box spline B_{i_1,i_2,i_3} is defined by a triple of integers i_1, i_2, i_3 describing a direction set, where the direction e_j appears i_j times for $j = 1,2,3$. Type-I box splines are defined on the uniform type-I partition $\triangle_I$ corresponding to a rectangular mesh with mesh size 1. Here we are interested in the sets of box splines

$$\mathcal{B}_k := \{B_{k-r-i+1,r+i+1,0}\}_{i=1}^{k-2r-1} \cup \{B_{k-r-i+1,k-r,i+2r-k+1}\}_{i=k-2r}^{k-r} \cup \{B_{0,2k-2r-i+1,2r+i-k+1}\}_{i=k-r+1}^{2k-3r-1}, \tag{10.14}$$

for $(3r+2)/2 \le k \le d$.

Lemma 10.14. *For each $(3r+2)/2 \le k \le d$, the box splines in the set $\mathcal{B}_k$ belong to $\mathcal{S}_k^r(\triangle_I)$. Moreover, the restriction of these box splines to $H := [0,1] \times [0,1]$ are linearly independent.*

Proof: Let $m := \lceil (3r+1)/2 \rceil$, and fix $m \le k \le d$. Theorem 12.4 shows that each box spline in $\mathcal{B}_k$ is a C^r spline of degree k on $\triangle_I$. We now establish their linear independence. Suppose we split H into an upper triangle T^+

and a lower triangle T^- by drawing in the northeast diagonal. We can write $\mathcal{B}_k = \mathcal{B}_k^1 \cup \mathcal{B}_k^2 \cup \mathcal{B}_k^3$, where

$$\begin{aligned}
\mathcal{B}_k^1 &:= \{B_{k-r,r+2,0}, B_{k-r-1,r+3,0}, \ldots, B_{r+2,k-r,0}\}, \\
\mathcal{B}_k^2 &:= \{B_{r+1,k-r,1}, B_{r,k-r,2}, \ldots, B_{1,k-r,r+1}\}, \\
\mathcal{B}_k^3 &:= \{B_{0,k-r,r+2}, B_{0,k-r-1,r+3}, \ldots, B_{0,r+2,k-r}\}.
\end{aligned}$$

Here we define B_{ijk} to be identically zero whenever one of its subscripts is negative, or when two of its subscripts are zero.

Suppose g is a linear combination of box splines in $\mathcal{B}_k$ and that $g \equiv 0$ on H. We need to show that the coefficients of g must be zero. The proof divides into two cases.

Case 1: $m \le k \le 2r+1$. In this case we can write $g := g_2 + g_3$, where $g \in \mathcal{B}_k^2$ and $g_3 \in \mathcal{B}_k^3$. By the support properties of box splines, g_3 vanishes on T^-. We are left with $g_2|_{T^-} = g|_{T^-} \equiv 0$. We claim the splines in $\mathcal{B}_k^2$ are linearly independent on T^-. Indeed, for each $i = 1, \ldots, r+1$, $B_{r-i+1,k-r,i}$ can be obtained from $B_{r-i+1,0,i}$ by integrating $k-r$ times, see (12.4). The $B_{r+1,0,1}, B_{r,0,2}, \ldots, B_{1,0,r+1}$ can be considered as tensor-product splines in the variables x and $x+y$, and are easily seen to be linearly independent. We conclude that the coefficients of g_2 must be zero. We are left with $g_3 \equiv 0$ on T^+. But the splines in $\mathcal{B}_k^3$ are also tensor-product splines, each of a different degree in the y-variable, and thus are linearly independent. We conclude that the coefficients of g_3 must also be zero.

Case 2: $2r+2 \le 2k-3r-1$. In this case $g := g_1 + g_2 + g_3$, where $g_i \in \mathcal{B}_k^i$, for $i = 1,2,3$. Suppose $g \equiv 0$ on H. Then $D_y^{k-r} g$ also vanishes on H. Since $D_y^{k-r} g_1 \equiv 0$ while g_3 has no support on T^-, we conclude that $D_y^{k-r} g_2 \equiv 0$ on T^-. Now by Lemma 12.3, $D_y^{k-r} g_2$ is a linear combination of $\{\nabla_2^{k-r} B_{r+1,0,1}, \ldots, \nabla_2^{k-r} B_{1,0,r+1}\}$, where ∇_2 is the backward difference operator with respect to the variable y. This implies that the coefficients of g_2 must be zero. We are left with $g_1 \equiv 0$ on T^-. But as noted above g_1 is a linear combination of linearly independent tensor-product splines, and so the coefficients of g_1 must also be zero. We now have $g_3 \equiv 0$ on T^+. Since g_3 is a linear combination of linearly independent tensor-product splines, we find that its coefficients are also zero. $\square$

Let

$$\Psi_{k,n} := \Big\{\psi(x,y) := B\Big(\frac{x}{h} - i, \frac{y}{h} - j\Big) : B \in \mathcal{B}_k\Big\}_{i,j=1}^n,$$

where $h = 1/(n+1)$.

Theorem 10.15. *Fix $2r+2 \le d \le 3r+1$. Then the set*

$$\Phi_{r,d,n} \cup \bigcup_{k=m}^{d} \Psi_{k,n} \tag{10.15}$$

is a basis for $\mathcal{S}_d^r(\triangle_n)$, where $m := \lceil (3r+1)/2 \rceil$.

Proof: By Lemma 10.14, the box splines in $\Psi_{k,n}$ belong to $\mathcal{S}_k^r(\triangle_n) \subseteq \mathcal{S}_d^r(\triangle_n)$ for each $(3r+2)/2 \le k \le d$. By construction, the lower left corner of the support of each box spline in $\Psi_{k,n}$ is at lattice point (ih, jh) with $1 \le i, j \le n$. We now count the number of basis functions in (10.15). We have already seen in the proof of Theorem 10.12 that the number of functions in $\Phi_{r,d,n}$ is given by the formula in (10.10) which corresponds to the last two terms in (10.13). There are $n^2(2k-3r-1)$ splines in $\Psi_{k,n}$. Since

$$\sum_{k=m}^{d} (2k-3r-1) = \begin{cases} d^2 - 3rd + 9r^2/4, & \text{if } r \text{ is even}, \\ d^2 - 3rd + (9r^2-1)/4, & \text{if } r \text{ is odd}, \end{cases}$$

it follows that the cardinality of the set in (10.15) is equal to the dimension of $\mathcal{S}_d^r(\triangle_n)$. To complete the proof it suffices to show that the functions in (10.15) are linearly independent. Suppose g is a linear combination of these functions that vanishes identically on H. As noted above, none of the box splines have support in the L-shaped subset $[0,h] \times [0,1] \cup [0,1] \times [0,h]$ of H. But then the proof of Theorem 10.12 shows that all of the coefficients of g corresponding to functions in $\Phi_{r,d,n}$ must be zero. Let $\tilde{g}$ be the remaining sum, which involves only box splines.

We examine $\tilde{g}$ on the square H_{11}, where $H_{ij} := [ih, (i+1)h] \times [jh, (j+1)h]$. The only box splines that have values in this square are those whose lower left support corner is at the point (h,h). These are just scaled and translated versions of the box splines in the sets $\mathcal{B}_k$ of (10.14). Suppose $\tilde{g} := g_m + \cdots + g_d \equiv 0$, where $g_k \in \mathcal{B}_k$. Since the g_k are of different degrees, we conclude that $g_k \equiv 0$ for $k = m, \ldots, d$. But then Lemma 10.14 asserts that for each $m \le k \le d$, all of the coefficients of g_k must be zero. It follows that all coefficients of $\tilde{g}$ corresponding to box splines with corner at (h,h) must be zero. We can now repeat this argument for the squares $H_{2,1}, \ldots, H_{n,1}$, and then for the squares $H_{1,2}, \ldots, H_{1,n}$. We then repeat for $H_{i,i}$ and the corresponding row and column for $i = 2, \ldots, n$ to show that all coefficients of $\tilde{g}$ (and thus of g) must vanish. $\square$

Fix $d \ge 2r+2$. We now introduce some bivariate polynomials of exact degree d which play a key role in the remainder of this section. Let

$$q_m(x,y) := \begin{cases} x^{m-1} y^{r-m+1} (x+y)^{d-r}, & m = 1, \ldots, r+1, \\ x^{m-r-2} y^{d-r} (x+y)^{2r+2-m}, & m = r+2, \ldots, 2r+2. \end{cases} \tag{10.16}$$

Lemma 10.16. *The set of polynomials* $yq_1, \ldots, yq_{r+1}, (x+y)q_{r+2}, \ldots, (x+y)q_{2r+2}$ *are linearly independent.*

Proof: Suppose there exist real numbers $\{\alpha_i\}_{i=1}^{2r+2}$ such that $Q_1 + Q_2 \equiv 0$, where

$$Q_1(x,y) := \sum_{i=1}^{r+1} \alpha_i y q_i(x,y), \qquad Q_2(x,y) := \sum_{i=r+2}^{2r+2} \alpha_i (x+y) q_i(x,y).$$

Now setting $z := x + y$, we can write

$$Q_1(x,y) = \sum_{i=1}^{r+1} \alpha_i y (z-y)^{i-1} y^{r-i+1} z^{d-r} =: \sum_{j=d-r}^{d} \beta_j y^{d+1-j} z^j,$$

for some real numbers β_j. Similarly for $Q_2(x,y)$ we have

$$Q_2(x,y) = \sum_{i=r+2}^{2r+2} \alpha_i z (z-y)^{i-r-2} y^{d-r} z^{2r+2-i} =: \sum_{j=1}^{r+1} \gamma_j y^{d+1-j} z^j,$$

for some real numbers γ_j. Since $d - r > r + 1$, $Q_1 + Q_2 \equiv 0$ implies that $\gamma_j = 0$ for $j = 1, \ldots, r+1$ and $\beta_j = 0$ for $j = d-r, \ldots, d$. That is, $Q_1 \equiv 0$ and $Q_2 \equiv 0$. It follows that $\alpha_j = 0$ for $j = 1, \ldots, 2r+2$. $\square$

To state our next lemma we introduce the $(d+2)$-dimensional space

$$\mathcal{H}_{d+1} := \operatorname{span} \{x^i y^{d-i+1}\}_{i=0}^{d+1}$$

of bivariate homogeneous polynomials of degree $d+1$. We define an inner product on $\mathcal{H}_{d+1}$ by $\langle q, p \rangle := q(D_x, D_y)p$ for all $p, q \in \mathcal{H}_{d+1}$. It is clear that if $p(x,y) := \sum_{i+j=d+1} p_{ij} x^i y^j$ and $q(x,y) := \sum_{i+j=d+1} q_{ij} x^i y^j$, then

$$\langle q, p \rangle = \sum_{i+j=d+1} i\,!\, j\,!\, q_{ij}\, p_{ij}.$$

Now for each $q \in \mathcal{H}_{d+1}$, there exists a constant C dependent on q and d such that

$$|\langle q, p \rangle| \le C \, \|p\|_H, \qquad \text{for all } p \in \mathcal{H}_{d+1},$$

where $H = [0,1] \times [0,1]$. Let $D_z = D_{x+y}$ denote the directional derivative associated with the vector $x + y$, and let q_m be the polynomials in Lemma 10.16.

Lemma 10.17. *Given any nontrivial set of numbers $\beta_1, \ldots, \beta_{r+1}$, there exists a polynomial $p^* \in \mathcal{H}_{d+1}$ such that*

$$D_z q_m(D_x, D_y) p^* = 0, \qquad m = r+2, \ldots, 2r+2,$$

while

$$\sum_{i=1}^{r+1} \beta_i D_y q_i(D_x, D_y) p^* = 1.$$

Proof: Let $\mathcal{Q}$ be the space of homogeneous polynomials spanned by $zq_{r+2}, \ldots, zq_{2r+2}$, where $z = x + y$. Consider the nontrivial polynomial

$$p := \sum_{i=1}^{r+1} \beta_i y q_i(x, y).$$

By Lemma 10.16, p does not lie in $\mathcal{Q}$, and thus must lie in the orthogonal complement $\mathcal{Q}^\perp$. We can now take p^* to be the normalization of p such that $\langle p^*, p^* \rangle = 1$. $\square$

Our next lemma deals with derivatives of type-I box splines. Let $\nabla_1, \nabla_2, \nabla_3$ be the backward difference operators defined by $\nabla_1 f(x,y) := f(x,y) - f(x-1,y)$, $\nabla_2 f(x,y) := f(x,y) - f(x,y-1)$, and $\nabla_3 f(x,y) := f(x,y) - f(x-1,y-1)$.

Lemma 10.18. *Suppose we number the box splines in the set $\mathcal{B}_d$ of (10.14) as $\phi_1, \ldots, \phi_{2d-3r+1}$. Then for $m = 1, \ldots, r+1$,*

$$q_m(D_x, D_y)\phi_\ell = \begin{cases} \binom{d-r}{\ell+m-1} \nabla_1^{d-r-\ell} \nabla_2^{r+\ell} B_{110}, & 1 \le \ell \le d-2r-1, \\ \binom{2d-3r-1-\ell}{d-2r-2-m} \nabla_1^{d-r-\ell} \nabla_2^{d-r-1} \nabla_3^{\ell+2r+1-d} B_{110}, & d-2r \le \ell \le d-r, \end{cases}$$

and $q_m(D_x, D_y)\phi_\ell = 0$ for $d-r+1 \le \ell \le 2d-3r+1$.

Proof: Lemma 12.3 asserts that for i and j with $i + j = k + l - 2$,

$$D_x^i D_y^j B_{k,\ell,0}(x,y) = \begin{cases} \nabla_1^i \nabla_2^j B_{110}(x,y), & \text{for } i = k-1 \text{ and } j = \ell - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $1 \le \ell \le d-2r-1$. Then

$$D_x^{m+i-1} D_y^{d-m-i+1} \phi_\ell = \begin{cases} \nabla_1^{d-r-\ell} \nabla_2^{r+\ell} B_{110}(x,y), & \text{for } i + m = d - r - \ell + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since
$$\binom{d-r}{i} = \binom{d-r}{m+\ell-1}$$
when $i+m = d-r-\ell+1$, this proves the result for $\ell = 1,\ldots,d-2r-1$. The proof of the other cases is similar. □

The proof of the following lemma is similar to that of Lemma 10.18.

Lemma 10.19. *Let $\phi_1,\ldots,\phi_{2d-3r+1}$ be as in Lemma 10.18. Then for all $m = r+2,\ldots,2r+2$,*
$$q_m(D_x,D_y)\phi_\ell = \begin{cases} \nabla_1^{d-r-\ell}\nabla_2^{d-r}\nabla_3^{\ell-d+2r}B_{101}, & \ell = d-m+2, \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 10.20. *Suppose $2r+2 \le d \le 3r+1$ and let $1 \le q \le \infty$. Then the approximation power of the space $\mathcal{S}_d^r$ in the q-norm is at most d.*

Proof: We first deal with the case $q = \infty$. Choose nontrivial $\beta_1,\ldots,\beta_{r+1}$ such that
$$\sum_{m=1}^{r+1} \beta_m \binom{d-r}{\ell+m-1} = 0, \qquad \ell = 1,\cdots,d-2r-1,$$
and let p^* be the corresponding homogeneous polynomial of degree $d+1$ of Lemma 10.17. Let f be the polynomial in $\mathcal{H}_{4d+4}$ such that
$$(\nabla_1\nabla_2\nabla_3)^{d+1} f|_{\widehat{H}} = p^*|_{\widehat{H}},$$
where $\widehat{H} = [0,L]\times[0,L]$ with $L \ge 2(d+1)+1$. Let $\triangle$ be the uniform type-I triangulation of $\widehat{H}$ with vertices (i,j) for $i,j = 0,\ldots,L$. Given an integer $n > 0$, let $h = 1/(n+1)$. We abuse notation a little bit by letting $\triangle_n = h\triangle$, where $h\triangle$ is a now a triangulation with vertices at the points (ih,jh) for integer pairs (i,j), and each edge is an h multiple of an edge of $\triangle$. Note that restricted to the unit square H, $\triangle_n$ is just the triangulation defined at the beginning of Section 10.4.

Now suppose that for any $n > 0$, we can find $s_n \in S_d^r(\triangle_n|_{\widehat{H}})$ with
$$\|f - s_n\|_{\widehat{H}} \le \epsilon(n)h^d, \tag{10.17}$$
where $\epsilon(n) \to 0$ as $n \to \infty$. Then
$$\|(\nabla_1\nabla_2\nabla_3)^{d+1}(f-s_n)\|_H \le C\epsilon(n)h^d,$$
for a constant $C > 0$ and the subdomain $H = [0,1]\times[0,1] \subset \widehat{H}$. By

Theorem 10.15,

$$u_n := (\nabla_1\nabla_2\nabla_3)^{d+1} s_n \in \operatorname{span}\left(\bigcup_{k=\lceil(3r+1)/2\rceil}^{d} \Psi_{k,n}\right)$$

since $(\nabla_1\nabla_2\nabla_3)^{d+1} g = 0$ for any $g \in \Phi_{r,d,n}$. Indeed, for each $g \in \Phi_{r,d,n}$, one of $(\nabla_i)^{d+1}$, $i = 1,2,3$, will annihilate g. Thus, there exist coefficients $a(n,\ell,\nu)$ such that

$$u_n = \sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu)\phi_\ell(\cdot/h - \nu) + w_n,$$

where the sum on ν runs over an appropriate finite subset of $\mathbb{Z}^2$, and w_n is a sum of scaled box splines of degree strictly less than d. Thus, we have

$$|p^*(hx,hy) - \sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu)\phi_\ell((x,y)-\nu) - w_n(hx,hy)| \le C\epsilon(n)h^d, \tag{10.18}$$

for all $(x,y) \in [0,1/h] \times [0,1/h]$.

Now for $m = r+2,\ldots,2r+2$, $q_m(D_x,D_y)w_n(hx,hy) = 0$, where q_m are the polynomials defined in (10.16). By Lemma 10.19,

$$\begin{aligned}
q_m(D_x,D_y) &\sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu)\phi_\ell((x,y)-\nu) \\
&= \sum_{\nu} a(n,\ell,\nu)\nabla_1^{d-\ell-r}\nabla_2^{d-r}\nabla_3^{\ell-d+2r} B_{101}(\cdot-\nu) \\
&= \sum_{\nu} \nabla_1^{d-\ell-r}\big[\nabla_2^{d-r}\nabla_3^{\ell-d+2r} a(n,\ell,\nu)\big] B_{101}(\cdot-\nu),
\end{aligned}$$

where $\ell = d-m+2$ and the expression in the square brackets involves applying the backward difference operators to $a(n,\ell,\nu)$ as a function of ν. Since $(hx,hy) \in H$,

$$q_m(D_x,D_y)p^*(hx,hy) = h^d(q_m(D_x,D_y)p^*)(hx,hy).$$

For $T = \{(x,y) : \ 0 \le y \le x \le 1\}$, $B_{101}(x,y) \equiv 1$. By the Markov inequality over the triangle $T+\nu$ and (10.18),

$$\begin{aligned}
|h^d q_m(D_x,D_y)p^*(h((x,y)+\nu)) \\
- \nabla_1^{d-\ell-r}\nabla_2^{d-r}\nabla_3^{\ell-d+2r} a(n,\ell,\nu)| \le C\epsilon(n)h^d,
\end{aligned} \tag{10.19}$$

for another constant $C > 0$. In particular, for $(x,y) = (0,0)$ and all ν, we have

$$|h^d q_m(D_x,D_y)p^*(\nu h) - \nabla_1^{d-\ell-r}\nabla_2^{d-r}\nabla_3^{\ell-d+2r} a(n,\ell,\nu)| \le C\epsilon(n)h^d,$$

and

$$|h^d \nabla_3 q_m(D_x, D_y) p^*(\nu h) - \nabla_1^{d-\ell-r} \nabla_2^{d-r} \nabla_3^{\ell-d+2r+1} a(n,\ell,\nu)| \le C\epsilon(n) h^d.$$

By Lemma 10.17, the first term above is zero since $\nabla_3 q_m(D_x, D_y)p^* = D_z q_m(D_x, D_y)p^*$ and hence we have

$$|\nabla_1^{d-\ell-r} \nabla_2^{d-r} \nabla_3^{\ell-d+2r+1} a(n,\ell,\nu)| \le C\epsilon(n) h^d. \tag{10.20}$$

By Lemma 10.18, for $m = 1, \ldots, r+1$,

$$\begin{aligned} &q_m(D_x, D_y) u_n(\,\cdot\, h) \\ &\quad= \sum_{\ell=1}^{2d-3r-1} q_m(D_x, D_y) \sum_\nu a(n,\ell,\nu) \phi_\ell(\,\cdot - \nu) \\ &\quad= \sum_{\ell=1}^{d-2r-1} \binom{d-r}{\ell+m-1} \sum_\nu \big[\nabla_1^{d-r-\ell} \nabla_2^{r+\ell} a(n,\ell,\nu)\big] B_{110}(\,\cdot - \nu) \\ &\qquad + \sum_{\ell=d-2r}^{d-r} \binom{2d-3r-1-\ell}{d-2r-2+m} \sum_\nu b(n,\ell,\nu) B_{110}(\,\cdot - \nu), \end{aligned}$$

where $b(n,\ell,\nu) := \nabla_1^{d-r-\ell} \nabla_2^{d-r-1} \nabla_3^{\ell-d+2r+1} a(n,\ell,\nu)$. By (10.20),

$$\begin{aligned} &\left| \nabla_2 \binom{2d-3r-1-\ell}{d-2r-2+m} b(n,\ell,\nu) \right| \\ &\quad= \binom{2d-3r-1-\ell}{d-2r-2+m} |\nabla_1^{d-\ell-r} \nabla_2^{d-r} \nabla_3^{\ell-d+2r+1} a(n,\ell,\nu)| \le C\epsilon(n) h^d. \end{aligned}$$

It follows that for $m = 1, \ldots, r+1$,

$$\begin{aligned} &\big| h^d \nabla_2 q_m(D_x, D_y) p^*(\nu h) - \nabla_2 q_m(D_x, D_y) u_n(\nu h) \big| \\ &\quad\le \Big| h^d \nabla_2 q_m(D_x, D_y) p^*(\nu h) \\ &\qquad - \sum_{\ell=1}^{d-2r-1} \binom{d-r}{\ell+m-1} \nabla_1^{d-r-\ell} \nabla_2^{r+\ell+1} a(n,\ell,\nu) \Big| \le C\epsilon(n) h^d, \end{aligned}$$

for another constant $C > 0$. Then

$$\begin{aligned} &\Big| \sum_{m=1}^{r+1} \beta_m h^d \nabla_2 q_m(D_x, D_y) p^*(\nu h) \\ &\qquad - \sum_{\ell=1}^{d-2r-1} \sum_{m=1}^{r+1} \beta_m \binom{d-r}{\ell+m-1} \nabla_1^{d-r-\ell} \nabla_2^{r+\ell+1} a(n,\ell,\nu) \Big| \\ &\qquad \le C \sum_{m=1}^{r+1} |\beta_m| \epsilon(n) h^d \le C\epsilon(n) h^d. \end{aligned}$$

Hence, we have

$$\left|h^d \nabla_2 \sum_{m=1}^{r+1} \beta_m q_m(D_x, D_y) p^*(\,\cdot\, h)\right| \le C\epsilon(n) h^d.$$

Since p^* is a homogeneous polynomial of degree $d+1$, by Lemma 10.17,

$$\nabla_2 \sum_{m=1}^{r+1} \beta_m q_m(D_x, D_y) p^*(\,\cdot\, h) = D_y \sum_{m=1}^{r+1} \beta_m q_m(D_x, D_y) p^*(\,\cdot\, h) = 1.$$

But this gives

$$|h^d| \le C\epsilon(n) h^d,$$

and we conclude that (10.17) cannot hold.

Now suppose $1 \le q < \infty$. Instead of (10.17), we now assume that

$$\|f - s_n\|_{q,\widehat{H}} \le \epsilon(n) h^d.$$

Arguing as above, we have

$$h^2 \int_0^{n+1} \int_0^{n+1} \Big| p^*(hx, hy) - \sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu) \phi_\ell((x,y)-\nu) \\ - w_n(hx, hy)\Big|^q \, dx\, dy \le (C\epsilon(n) h^d)^q.$$

For at least one (i,j), we have

$$\int_i^{i+1} \int_j^{j+1} \Big| p^*(hx, hy) - \sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu) \phi_\ell((x,y)-\nu) \\ - w_n(hx, hy)\Big|^q \, dx\, dy \le (C\epsilon(n) h^d)^q.$$

Splitting the square $[i, i+1] \times [j, j+1]$ into two triangles and using the fact that for polynomials on triangles all norms are equivalent (see Theorem 1.1), we get

$$\left| p^*(hx, hy) - \sum_{\ell=1}^{2d-3r-1} \sum_{\nu} a(n,\ell,\nu) \phi_\ell((x,y)-\nu) - w_n(hx, hy)\right| \le C\epsilon(n) h^d,$$

for $(x,y) \in (i,j) + [0,1] \times [0,1]$. Using the Markov inequality (1.5) for each triangle, we get

$$|h^d q_m(D_x, D_y) p^*(((x,y)+\nu)h) - \nabla_1^{d-\ell-r} \nabla_2^{d-r} \nabla_3^{\ell-d+2r} a(n,\ell,\nu)| \le C\epsilon(n) h^d,$$

for $(x,y) \in [0,1] \times [0,1]$ and $\nu = (i,j)$. The rest of the proof is the same as before with $\nu = (i+1, j+1)$. $\square$

10.4.3 The Case $(3r+2)/2 \le d \le 2r+1$

Theorem 10.21. *Suppose* $(3r+2)/2 \le d \le 2r+1$ *and* $1 \le q \le \infty$. *Then the approximation power of* $\mathcal{S}_d^r$ *in the* q*-norm is at most* d.

Proof: We first consider the case $r \ge 3$. Then for each $(3r+2)/2 \le d \le 2r+1$, it is easy to see that there exists $1 \le \tilde{r} \le r$ with $2\tilde{r}+2 \le d \le 3\tilde{r}+1$. Theorem 10.20 asserts that the approximation power of $\mathcal{S}_d^{\tilde{r}}$ in any q-norm is at most d. But for any triangulation $\triangle$, we have $\mathcal{S}_d^r(\triangle) \subseteq \mathcal{S}_d^{\tilde{r}}(\triangle)$, and so the approximation power of $\mathcal{S}_d^r$ is also at most d.

It remains to consider the cases $r=1$ and $r=2$. For $r=1$, we have $5/2 \le d \le 3$, and so it suffices to consider $\mathcal{S}_3^1$. We show that this space has approximation power at most three in any q-norm in Section 10.4.4.

For $r=2$ we have to consider $4 \le d \le 5$. For any triangulation $\triangle$, we have $\mathcal{S}_4^2(\triangle) \subseteq \mathcal{S}_4^1(\triangle)$. We already know by Theorem 10.20 that $\mathcal{S}_4^1$ has approximation power at most four in any q-norm, and we conclude that $\mathcal{S}_4^2$ also has approximation power at most four. The fact that $\mathcal{S}_5^2$ has approximation power at most five follows from our next lemma. Let $\triangle_n$ be the uniform type-I triangulation of the unit square H defined at the beginning of Section 10.4.

Lemma 10.22. *Fix* $0 \le r < d$, $m > 0$, *and* $1 \le q \le \infty$. *Suppose that for all* $F \in C^\infty(H)$, *there exists a constant* K *depending only on* d *and* F *such that*

$$d(F, S_{d+1}^{r+1}(\triangle_n))_q \le K\,|\triangle_n|^{m+1}, \qquad \text{all } n > 0. \tag{10.21}$$

Then for all $f \in C^\infty(H)$, *there exists another constant* $\tilde{K}$ *depending only on* d *and* f *such that*

$$d(f, S_d^r(\triangle_n))_q \le \tilde{K}\,|\triangle_n|^m, \qquad \text{all } n > 0. \tag{10.22}$$

Proof: We give the proof for $1 \le q < \infty$. The proof of $q = \infty$ is similar and simpler. Given $f \in C^\infty(H)$, let $F(x,y) := \displaystyle\int_0^x f(u,y)du$. Let $s \in S_{d+1}^{r+1}(\triangle_n)$ be such that

$$\|F - s\|_{q,H} \le K_1\,|\triangle_n|^{m+1}. \tag{10.23}$$

By Theorem 1.7, for each triangle T in $\triangle_n$, there exists a polynomial $p := p_{F,T} \in \mathcal{P}_{m+1}$ such that

$$\|D_x^\alpha D_y^\beta(F-p)\|_{q,T} \le K_2\,|T|^{m+1-\alpha-\beta}|F|_{m+1,q,T},$$

for all $0 \le \alpha+\beta \le m$. Now

$$\|D_x(F-s)\|_{q,T} \le \|D_x(F-p)\|_{q,T} + \|D_x(p-s)\|_{q,T},$$

while by the Markov inequality (1.5),

$$\|D_x(p-s)\|_{q,T} \le K_3\,|T|^{-1}\|p-s\|_{q,T} \le K_3\,|T|^{-1}(\|p-F\|_{q,T} + \|F-s\|_{q,T}).$$

Combining the above and using the fact that $|T| = |\triangle_n|$, we have

$$\|D_x(F-s)\|_{q,T} \le K_2\,\big(1+K_3\big)|\triangle_n|^m|F|_{m+1,q,T} + K_3\,|\triangle_n|^{-1}\|F-s\|_{q,T}.$$

Taking the q-th power, using the discrete Hölder inequality, summing over all triangles T, and inserting (10.23), we get

$$\begin{aligned}\|D_x(F-s)\|_{q,H}^q &\le K_4^q\,|\triangle_n|^{mq}|F|_{m+1,q,H}^q + K_5\,|\triangle_n|^{-q}\|F-s\|_{q,H}^q \\ &\le K_6^q\,|\triangle_n|^{qm}.\end{aligned}$$

Taking the q-th root, we have

$$\|D_x(F-s)\|_{q,H} \le K_6\,|\triangle_n|^m.$$

Since $D_xF = f$, we conclude that

$$\|f-g\|_{q,H} \le K_6\,|\triangle_n|^m,$$

where $g = D_xs \in S_d^r(\triangle_n)$, and we have proved (10.22). □

This lemma shows that if the space $\mathcal{S}_d^r$ does not have full approximation power in a q-norm, then neither does the space $\mathcal{S}_{d+1}^{r+1}$.

10.4.4 The Space $\mathcal{S}_3^1$

In this section we show that in any q-norm, $1 < q \le \infty$, the approximation power of the space $\mathcal{S}_3^1$ is at most three. Our proof is based on the following lemma.

Lemma 10.23. *Suppose X is a normed linear space and that Y is a linear subspace of X. Let λ be a linear functional defined on X that annihilates Y, i.e., $\lambda g = 0$ for all $g \in Y$. Then for all $f \in X$,*

$$d(f,Y) := \inf_{g\in Y}\|f-g\|_X \ge \frac{|\lambda f|}{\|\lambda\|_X}, \tag{10.24}$$

where

$$\|\lambda\|_X := \sup_{f\in X,\, f\ne 0}\frac{|\lambda f|}{\|f\|_X}. \tag{10.25}$$

Proof: It immediately follows from (10.25) that for any $g \in Y$,

$$|\lambda f| = |\lambda(f-g)| \le \|\lambda\|_X\,\|f-g\|_X.$$

Since this holds for all $g \in Y$, (10.24) must hold. □

We will apply this lemma with $X := C(H)$ or $X := L_q(H)$, where H is the unit square, and $\triangle_n$ is the uniform type-I triangulation on H introduced at the beginning of Section 10.4. We take $Y := \mathcal{S}_3^1(\triangle_n) \cap L_q(H)$. We now need to construct a linear functional λ defined on X that anihilates Y. We first define λ for splines s in the space $\mathcal{S}_3^0(\triangle_n)$ in terms of the B-coefficients of s. To this end, it is convenient to label the domain points of $\triangle_n$ which fall in the square $H_{ij} := [ih, (i+1)h] \times [jh, (j+1)h]$ with fractional subscripts as follows:

$$\xi_{i+\frac{k-1}{3}, j+\frac{l-1}{3}} := \Big(i + \frac{k-1}{3}h, j + \frac{l-1}{3}h\Big),$$

for $k, l = 1, 2, 3, 4$.

Given $s \in \mathcal{S}_3^0(\triangle_n)$, we label its B-coefficients as $c_{i+\frac{k-1}{3}, j+\frac{l-1}{3}}$. Then for all $1 \le i \le n$ and $0 \le j \le n$, let

$$\begin{aligned}
\lambda_{11}^{ij} s &:= c_{i+\frac{1}{3}, j+\frac{1}{3}} + c_{i-\frac{1}{3}, j} - c_{i, j+\frac{1}{3}} - c_{i,j}, \\
\lambda_{21}^{ij} s &:= c_{i+\frac{1}{3}, j+\frac{2}{3}} + c_{i-\frac{1}{3}, j+\frac{1}{3}} - c_{i, j+\frac{2}{3}} - c_{i, j+\frac{1}{3}}, \\
\lambda_{31}^{ij} s &:= c_{i+\frac{1}{3}, j+1} + c_{i-\frac{1}{3}, j+\frac{2}{3}} - c_{i, j+1} - c_{i, j+\frac{2}{3}}.
\end{aligned}$$

Similarly, for all $0 \le i \le n$ and $1 \le j \le n$, let

$$\begin{aligned}
\lambda_{12}^{ij} s &:= c_{i+\frac{1}{3}, j+\frac{1}{3}} + c_{i, j-\frac{1}{3}} - c_{i,j} - c_{i+\frac{1}{3}, j}, \\
\lambda_{22}^{ij} s &:= c_{i+\frac{2}{3}, j+\frac{1}{3}} + c_{i+\frac{1}{3}, j-\frac{1}{3}} - c_{i+\frac{1}{3}, j} - c_{i+\frac{2}{3}, j}, \\
\lambda_{32}^{ij} s &:= c_{i+1, j+\frac{1}{3}} + c_{i+\frac{2}{3}, j-\frac{1}{3}} - c_{i+\frac{2}{3}, j} - c_{i+1, j}.
\end{aligned}$$

Finally, for all $0 \le i \le n$ and $0 \le j \le n$, let

$$\begin{aligned}
\lambda_{13}^{ij} s &:= c_{i, j+\frac{1}{3}} + c_{i+\frac{1}{3}, j} - c_{i,j} - c_{i+\frac{1}{3}, j+\frac{1}{3}}, \\
\lambda_{23}^{ij} s &:= c_{i+\frac{1}{3}, j+\frac{2}{3}} + c_{i+\frac{2}{3}, j+\frac{1}{3}} - c_{i+\frac{1}{3}, j+\frac{1}{3}} - c_{i+\frac{2}{3}, j+\frac{2}{3}}, \\
\lambda_{33}^{ij} s &:= c_{i+\frac{2}{3}, j+1} + c_{i+1, j+\frac{2}{3}} - c_{i+\frac{2}{3}, j+\frac{2}{3}} - c_{i+1, j+1}.
\end{aligned}$$

Let $\mathcal{I}$ be the set of all (i, j, k, l) such that λ_{kl}^{ij} is one of the linear functionals described above. Then $s \in \mathcal{S}_3^0(\triangle_n)$ belongs to $\mathcal{S}_3^1(\triangle_n)$ if and only if

$$\lambda_{kl}^{ij} s = 0, \qquad \text{all } (i, j, k, l) \in \mathcal{I}. \tag{10.26}$$

For every $f \in C(H)$ we now define

$$\lambda f := \sum_{(i,j,k,l) \in \mathcal{I}} a_{kl} \, \lambda_{kl}^{ij} \, Lf, \tag{10.27}$$

where Lf is the spline in $\mathcal{S}_3^0(\triangle_n)$ which interpolates f at the domain points, and

$$A := [a_{kl}] := \begin{bmatrix} -1 & -1 & 2 \\ 2 & 2 & -4 \\ -1 & -1 & 2 \end{bmatrix}.$$

Since $Ls = s$ for all $s \in \mathcal{S}_3^1(\triangle_n)$, it follows from (10.26) that $\lambda s = 0$ for all $s \in \mathcal{S}_3^1(\triangle_n)$. We now bound the norm of λ.

Lemma 10.24. *For any $f \in C(H)$,*

$$|\lambda f| \le 10240\, n \, \|f\|_\infty. \tag{10.28}$$

Proof: By the definition of λ, there exist coefficients α_{ijkl} such that for any $f \in C(H)$,

$$\lambda f = \sum_{i=0}^n \sum_{j=0}^n \sum_{k=1}^4 \sum_{l=1}^4 \alpha_{ijkl}\, c_{i+\frac{k-1}{3}, j+\frac{l-1}{3}}, \tag{10.29}$$

where the c's are the B-coefficients of the C^0 cubic spline Lf. Note that each α_{ijkl} is a sum of $\pm a_{kl}$, and thus

$$|\alpha_{ijkl}| \le 16 \qquad \text{all } i, j, k, l. \tag{10.30}$$

We now show that $\alpha_{ijkl} = 0$ for all $1 \le k, l \le 4$ whenever $1 \le i, j \le n - 1$. Consider for example $k = l = 1$. Then collecting all terms in (10.27) involving the coefficient c_{ij}, we have

$$\big(-\lambda_{11}^{ij} - \lambda_{12}^{ij} + 2\lambda_{13}^{ij} - \lambda_{31}^{i,j-1} - \lambda_{32}^{i-1,j-1} + 2\lambda_{33}^{i-1,j-1}\big) Lf = (1+1-2+1+1-2) c_{ij},$$

which shows that $\alpha_{ij11} = 0$. Consider $k = 2$ and $l = 1$. Collecting the terms in (10.27) involving the coefficient $c_{i+1/3,j}$, we get

$$\big(-\lambda_{12}^{ij} + 2\lambda_{22}^{ij} + 2\lambda_{13}^{ij} - \lambda_{31}^{i,j-1}\big) Lf = (1 - 2 + 2 - 1) c_{i+\frac{1}{3}j},$$

which shows that $\alpha_{ij21} = 0$. The other cases are similar.

It follows from the above that the only nonzero terms in (10.29) are those corresponding to coefficients that lie in one of the squares $H_{ij} := [ih, (i+1)h] \times [jh, (j+1)h]$ with $i, j \in \{0, n\}$. We write $\mathcal{I}_B$ for the index set of this set of $4n$ squares.

It remains to estimate the size of the coefficients of Lf appearing in (10.29) in terms of the size of f. For each triangle T of $\triangle_n$, Lf is the cubic polynomial that interpolates at the domain points of T. It follows that the coefficients of Lf are combinations of the values of $f_{i+\frac{k-1}{3}, j+\frac{l-1}{3}} :=$

$f(\xi_{i+\frac{k-1}{3},j+\frac{l-1}{3}})$. For example, in the lower triangle of the square H_{ij}, we have

$$\begin{aligned}
c_{ij} &= f_{ij},\\
c_{i+\frac{1}{3},j} &= 3f_{i+\frac{1}{3},j} - \frac{3}{2}f_{i+\frac{2}{3},j} - \frac{5}{6}f_{ij} + \frac{1}{3}f_{i+1,j},\\
c_{i+\frac{1}{3},j+\frac{1}{3}} &= 3f_{i+\frac{1}{3},j+\frac{1}{3}} - \frac{3}{2}f_{i+\frac{2}{3},j+\frac{2}{3}} - \frac{5}{6}f_{ij} + \frac{1}{3}f_{i+1,j+1},\\
c_{i+\frac{2}{3},j} &= 3f_{i+\frac{2}{3},j} - \frac{3}{2}f_{i+\frac{1}{3},j} - \frac{5}{6}f_{i+1,j} + \frac{1}{3}f_{i,j},\\
c_{i+\frac{2}{3},j+\frac{2}{3}} &= 3f_{i+\frac{2}{3},j+\frac{2}{3}} - \frac{3}{2}f_{i+\frac{1}{3},j+\frac{1}{3}} - \frac{5}{6}f_{i+1,j+1} + \frac{1}{3}f_{i,j},\\
c_{i+1,j} &= f_{i+1,j},\\
c_{i+1,j+\frac{1}{3}} &= 3f_{i+1,j+\frac{1}{3}} - \frac{3}{2}f_{i+1,j+\frac{2}{3}} - \frac{5}{6}f_{i+1,j} + \frac{1}{3}f_{i+1,j+1},\\
c_{i+1,j+\frac{2}{3}} &= 3f_{i+1,j+\frac{2}{3}} - \frac{3}{2}f_{i+1,j+\frac{1}{3}} - \frac{5}{6}f_{i+1,j+1} + \frac{1}{3}f_{i+1,j},\\
c_{i+1,j+1} &= f_{i+1,j+1},\\
c_{i+\frac{2}{3},j+\frac{1}{3}} &= \frac{9}{2}f_{i+\frac{2}{3},j+\frac{1}{3}} + \frac{1}{3}(f_{ij} + f_{i+1,j} + f_{i+1,j+1})\\
&\quad - \frac{3}{4}(f_{i+\frac{2}{3},j} + f_{i+\frac{1}{3},j} + f_{i+\frac{2}{3},j+\frac{2}{3}} + f_{i+\frac{1}{3},j+\frac{1}{3}} + f_{i+1,j+\frac{2}{3}} + f_{i+1,j+\frac{1}{3}}).
\end{aligned}$$

This immediately implies that

$$\left|c_{i+\frac{k-1}{3},j+\frac{l-1}{3}}\right| \le 10\,\|f\|_{\infty,H_{ij}}, \qquad \text{all } i,j = 0,\ldots,n \text{ and } k,l = 1,\ldots,4.$$

Combining this with (10.29), (10.30), and the fact that the cardinality of $\mathcal{I}_B$ is $4n$, we get

$$|\lambda f| \le 2560 \sum_{i,j\in\mathcal{I}_B} \|f\|_{\infty,H_{ij}} \le 10240\, n\, \|f\|_\infty. \quad \square$$

We are ready to deal with the case $q = \infty$.

Theorem 10.25. *The approximation power of $\mathcal{S}_3^1$ in the ∞-norm is at most three.*

Proof: Let $\triangle_1, \triangle_2, \ldots$ be the sequence of uniform type-I triangulations defined at the beginning of Section 10.4. Fix $n > 0$, and set $h = 1/(n+1)$. First we deal with the case $q = \infty$. To show that $\mathcal{S}_3^1(\triangle_n)$ does not satisfy $d(f, \mathcal{S}_3^1(\triangle_n) \le Kh^4$ for every function $f \in W_\infty^4(H)$, we apply Lemma 10.23 with the linear functional λ defined in (10.27). Let $F(x,y) = x^2y^2$. First we need to compute a lower bound for the value of $|\lambda F|$. Note that

$$(x - ih)^2(y - jh)^2 = x^2y^2 + q_{ij},$$

where q_{ij} is a cubic polynomial. Now since $\lambda_{kl}^{ij}\, q_{ij} = 0$, we conclude that for each $1 \le k, l \le 4$, $\lambda_{kl}^{ij}\, LF$ has a common value c for all $0 \le i, j \le n$. A simple computation shows that for all such i, j,

$$\begin{bmatrix} \lambda_{11}^{ij}\, LF & \lambda_{12}^{ij}\, LF & \lambda_{13}^{ij}\, LF \\ \lambda_{21}^{ij}\, LF & \lambda_{22}^{ij}\, LF & \lambda_{23}^{ij}\, LF \\ \lambda_{31}^{ij}\, LF & \lambda_{32}^{ij}\, LF & \lambda_{33}^{ij} LF \end{bmatrix} = \frac{h^4}{81} \begin{bmatrix} 6 & 6 & -6 \\ -3 & -3 & 12 \\ 6 & 6 & -6 \end{bmatrix}.$$

It follows that for $1 \le i, j \le n-1$,

$$\sum_{k=1}^{3} \sum_{l=1}^{3} a_{kl}\, \lambda_{kl}^{ij}\, LF = -\frac{4}{3} h^4,$$

while for all other i, j,

$$\Big| \sum_{k,l \in \mathcal{I}_{ij}} a_{kl}\, \lambda_{kl}^{ij}\, LF \Big| \le \frac{4}{3} h^4,$$

where $\mathcal{I}_{ij}$ is the set of (k,l) such that $(i,j,k,l) \in \mathcal{I}$. Thus,

$$|\lambda F| \ge \frac{4h^4}{3}[(n-1)^2 - 4n] = \frac{4h^4}{3}[n^2 - 6n + 1]. \tag{10.31}$$

Combining this with the bound on the norm of λ given in Lemma 10.24, we conclude from Lemma 10.23 that for $n \ge 13$,

$$d(f, \mathcal{S}_3^1(\triangle_n)) \ge \frac{4h^4(n^2 - 6n + 1)}{3 \times 10240 n} \ge \frac{h^3(n^2 - 6n + 1)}{7680\, n(n+1)} \ge \frac{h^3}{15360},$$

where in the last step we have used the fact that $(n^2 - 6n + 1)/(n^2 + n) \ge 1/2$ whenever $n \ge 13$. $\square$

To establish an analogous result for the q-norms, we need to extend the linear function λ defined above to L_q. First, we consider λ on $\mathcal{S}_4^1(\triangle_n) \cap L_q(H)$. As shown in the proof of Lemma 10.24, for any f in this space,

$$|\lambda f| \le 2560 \sum_{(i,j) \in \mathcal{I}_B} ||f||_{\infty, H_{ij}}.$$

Theorem 1.1 implies that $\|f\|_{\infty, H_{ij}} \le K \|f\|_{q, H_{ij}} / h^{2/q}$, where K is a fixed constant. Now by Hölder's inequality with $1/q + 1/\tilde{q} = 1$,

$$\begin{aligned} |\lambda f| &\le \frac{2560K}{h^{2/q}} \Big(\sum_{(i,j) \in \mathcal{I}_B} \Big)^{1/\tilde{q}} \|f\|_{q,H} \le \frac{2560K(4n)^{1/\tilde{q}}}{h^{2/q}} \|f\|_{q,H} \\ &\le 10240K(n+1)^{1+1/q} \|f\|_{q,H}. \end{aligned}$$

Using the Hahn–Banach theorem, we can define an extension of λ from $\mathcal{S}_4^1(\triangle_n) \cap L_q(H)$ to $L_q(H)$ such that

$$|\lambda(f)| \le 10240K\,(n+1)^{1+1/q}\,\|f\|_{q,H}, \tag{10.32}$$

for all $f \in L_q(H)$.

Theorem 10.26. *Let $1 < q \le \infty$. Then the approximation power of $\mathcal{S}_3^1$ in the q-norm is at most three.*

Proof: The proof is very similar to the proof of Theorem 10.25. Since λ was extended to $L_q(H)$ from $\mathcal{S}_4^1(\triangle_n)$, it follows that (10.31) still holds for $F := x^2y^2$. But then by Lemma 10.23 and (10.32),

$$d(f, \mathcal{S}_3^1(\triangle_n))_{q,H} \ge \frac{4h^4(n^2-6n+1)}{30720\,K(n+1)^{1+1/q}} \ge \frac{1}{15360\,K} h^{3+1/q},$$

for $n \ge 14$. Thus, the L_q approximation power of $\mathcal{S}_3^1(\triangle_n)$ cannot be four when $q > 1$. □

10.5. Remarks

Remark 10.1. In many of the papers in the literature, approximation power is defined in terms of a scale of spline spaces $\mathcal{S}_h$ which are obtained from a fixed space $\mathcal{S}$ defined over a triangulation $\triangle$ of the whole plane by replacing each function $s(x)$ in $\mathcal{S}$ by $s(x/h)$ for $h > 0$. Suppose Ω is a (finite) domain. Let h_n be an infinite sequence of real numbers with $h_n \to 0$ as $n \to \infty$ such that if we scale the triangulation $\triangle$ by $1/h_n$, it provides a triangulation $\triangle_n$ of Ω. Then the approximation power of the space $\mathcal{S}$ can be defined to be the largest integer k such that $d(f, \mathcal{S}_n)_\Omega = \mathcal{O}(h_n^k)$ as $n \to \infty$, where $\mathcal{S}_n := \{s(\,\cdot/h_n) : s \in \mathcal{S}\}$. As an example, suppose Ω is the unit square, and let $\triangle$ be the (infinite) uniform type-I triangulation of $\mathbb{R}^2$ with vertices on the unit spaced lattice. Then the sequence $h_n = 1/n$ satisfies the above conditions. However, the sequence $h_n = 1/\sqrt{n}$ does not. Moreover, if $\Omega = [0,1] \times [0,\pi]$, there is no such sequence h_n.

Remark 10.2. In Section 10.2 we have shown that the piecewise polynomials $\mathcal{PP}_d$ have approximation power $d+1$ in all of the q-norms. In view of Remarks 1.4 and 1.5, this is the highest power possible for $\mathcal{PP}_d$, and thus also for any smooth spline space $\mathcal{S}$ of degree d. This is why we say that a spline space has full approximation power if it has approximation power $d+1$.

Remark 10.3. Theorem 10.21 asserts that for $(3r+2)/2 \le d \le 3r+1$, the approximation power of $\mathcal{S}_d^r$ in any q-norm is at most d, i.e., is suboptimal. However, this result does not tell us the exact approximation power. It could be anything from zero to d. Finding the exact approximation power for this range of d remains an open question. For example, for $r = 1$, the exact approximation powers of $\mathcal{S}_3^1$ and $\mathcal{S}_4^1$ are not known.

Remark 10.4. The question of how well the spline spaces $\mathcal{S}_d^r(\triangle_n)$ defined on uniform type-I triangulations of the unit square approximate smooth functions has been studied by several authors. The problem has been solved for all values of $r \le 6$, but for higher values of r there are gaps in the theory. For example, it was shown in [Jia88] that these spaces have approximation power d for all choices of $2r+2 \le d \le 3r+1$. For $(3r+2)/2 \le d < 2r+2$, the situation is more complicated, and the exact approximation power has been determined only for some special values of r and d, see [Bam85].

Remark 10.5. The fact that for $d < 3r+2$ the space $\mathcal{S}_d^r$ does not have full approximation power in any of the q-norms on the unit square does not preclude the possibility that for special sequences of triangulations $\triangle_n$ with $|\triangle_n| \to 0$, the spaces $\mathcal{S}_d^r(\triangle_n)$ might have optimal approximation power in the sense of Remark 5.4, i.e., the bound (10.1) holds with $m = d+1$ for each $\triangle_n$. For example, consider the sequence $\diamondplus_1, \diamondplus_2, \ldots$ of induced triangulations obtained by applying uniform refinement to a given strictly convex quadrangulation of a polygonal domain Ω, see Method 4.56. Then for any $1 \le q \le \infty$, it follows from Theorem 6.18 that the spaces $\mathcal{S}_3^1(\diamondplus_n)$ satisfy (10.1) with $m = 4$, despite the fact that $\mathcal{S}_3^1$ does not have full approximation power as shown in Section 10.4.4.

Remark 10.6. The macro-element spaces discussed in Chapters 6–8 are superspline spaces defined on special refinements of arbitrary regular triangulations. As shown there, these spaces have optimal approximation power in the sense of Remark 5.4. Since (except for the polynomial macro-elements) they are defined on special refined triangulations, they do not fit into the framework of Definition 10.1. However, we can extend the definition by allowing the spaces to be defined on refinements of an arbitrary triangulation $\triangle$ with smallest angle θ. Then in this extended sense, all the macro-element spaces of Chapters 6–8 have full approximation power in all q-norms.

Remark 10.7. In this chapter we have focused on a study of the approximation power of the spaces $\mathcal{S}_d^r$. However, our definition of approximation power can also be applied to any other spline space $\mathcal{S}$ such that $\mathcal{S}(\triangle)$ is well defined for every regular triangulation $\triangle$. For example, the results of Section 10.3.3 show that for $d \ge 3r+2$, the special superspline space $\mathcal{S}$ defined in (10.7) has full approximation power in any q-norm.

Remark 10.8. The results of Section 11.6 below show that for $d \ge 3r+2$, the general superspline spaces $\mathcal{S}_d^{r,\rho}$ defined in (5.6) and (11.2) also have full approximation power in any q-norm.

Remark 10.9. We have shown that for $d < 3r+2$, the space $\mathcal{S}_d^r$ does not have full approximation power in any of the q-norms. This means that every superspline space $\mathcal{S}_d^{r,\rho}$ of $\mathcal{S}_d^r$ will also fail to have full approximation power.

Remark 10.10. It was conjectured by the first author that if the approximation power of $\mathcal{S}_d^r$ is $d+1$, then the approximation power of $\mathcal{S}_k^r$ will be $k+1$ for all $k \geq d$. This is referred to as Lai's conjecture in the survey [Boo93a], and remains open.

Remark 10.11. In Definition 10.1, we have defined the approximation power of $\mathcal{S}_d^r$ to be m provided that (10.1) holds for all polygonal domains Ω and all triangulations $\triangle$ of Ω with smallest angle θ, and for all $f \in W_q^m(\Omega)$. We could instead require that for all such f,

$$\|D_x^\alpha D_y^\beta (f-s)\|_{q,\Omega} \leq K\, |\triangle|^{m-\alpha-\beta}\, |f|_{m,q,\Omega},$$

for all $0 \leq \alpha+\beta \leq m-1$, where the constant K depends only on r, d, θ and the Lipschitz constant of the boundary of Ω. In this case we say that $\mathcal{S}_d^r$ has simultaneous approximation power m in the q-norm on Ω. Theorem 10.10 shows that for $d \geq 3r+2$, $\mathcal{S}_d^r$ has full approximation power in this stronger sense.

Remark 10.12. Suppose $\triangle_{II}$ is a uniform type-II partition of the unit square. It was shown in [DahM84a] that the space $\mathcal{S}_3^1(\triangle_{II})$ has approximation power four in the maximum norm, see also [Lai94] and Remark 12.7.

Remark 10.13. Theorem 10.26 shows that for all $1 < q \leq \infty$, the space $\mathcal{S}_3^1$ does not have full approximation power. The case $q=1$ is not covered by the theorem, but it was shown in [BooJ93] by a different argument that the result also holds in this case.

10.6. Historical Notes

The approximation power of univariate and tensor-product splines is well understood, see e.g. [Sch81]. Thus, it is somewhat surprising that even after many years of work by many spline researchers, the approximation properties of splines defined on triangulations are still far from being fully understood. The earliest results appear in the finite-element literature, where various special macro-element spaces were investigated. For a summary of what was known in the 1970's, see [Cia78a].

The strongest currently known positive result is Theorem 10.10, which asserts that $\mathcal{S}_d^r$ has full approximation power provided $d \geq 3r+2$. The first attempt to prove this theorem can be found in [BooH88]. Unfortunately, as pointed out in [Sch88b] and acknowledged in [Boo89, Boo90], there is a gap in the proof, since it does not deal with the possibility that, even if the smallest angle in a triangulation $\triangle$ is controlled, the constants in the error bound for approximating a smooth function with a spline in $\mathcal{S}_d^r(\triangle)$ can become arbitrarily large if the triangulation $\triangle$ contains near-singular vertices as defined in Section 10.3.1. A different proof based on super vertex splines can be found in [ChuL90b], but again the constants were not shown to be bounded in terms of the smallest angle in the triangulation. This deficiency was later removed in [ChuHoJ95], thus providing the first rigorous proof. The proof of Theorem 10.10 given here is based on a special superspline subspace, and follows [LaiS98]. The result can also be proved by combining Theorem 5.19 with the construction of stable local bases for $\mathcal{S}_d^r(\triangle)$ in Chapter 11. For the L_∞-norm, still another proof based on Lagrange interpolation can be found in [DavNZ01].

Much of the literature on approximation power of splines on triangulations deals with the case of type-I triangulations. The earliest result in this case seems to be the surprising Theorem 10.25, which shows that $\mathcal{S}_3^1$ does not have full approximation power on the unit square. Here we have followed the original proof in [BooH83a]. About the same time, it was shown in [BooD83] that with respect to the uniform norm, $\mathcal{S}_d^r$ has zero approximation power on the unit square when $d < (3r+2)/2$. Our proof of Theorem 10.13 is based on that paper. The proof for the q-norm is new.

The approximation power of spaces of splines defined on type-I partitions were studied in a series of papers which made use of the emerging theory of box splines, see [BooDH83, BooH88, BooJ93, DahM84a, DahM84b, DahM85a, Jia83, Jia86, Jia88]. A proof that for $d \geq 3r+2$, $\mathcal{S}_d^r(\triangle)$ has full approximation power on uniform type-I partitions (in the uniform norm) was given in [DavNZ98] by directly constructing an appropriate Hermite interpolation operator.

Our treatment of the case $2r+2 \leq d \leq 3r+1$, culminating in the proof of Theorem 10.20, is based on several sources, with heavy dependence on an unpublished manuscript of Jia, see also [BooJ93]. The basis result of Theorem 10.15 follows [BooH83b], while the lemmas in Section 10.4.2 are modifications of results in the unpublished manuscript. We have also followed Jia for the proof of Theorem 10.20.

Theorem 10.21 which deals with the case $(3r+2)/2 \leq d \leq 2r+1$ is due to [BooJ93], although here we give a different proof. It seems that Lemma 10.22 is new. The results of Section 10.4.4 on the approximation power of C^1 cubic splines are based on [BooH83a].

Stable Local Minimal Determining Sets

For practical applications, it is highly desirable that a spline space have an explicit stable local minimal determining set. In Chapters 6–8 we have shown that a variety of macro-element spaces have such determining sets. In this chapter we show how to construct stable local minimal determining sets for more general spline and superspline spaces defined on regular triangulations.

11.1. Introduction

Recall from Definition 5.16 that a minimal determining set (MDS) $\mathcal{M}$ for a spline space is said to be local provided that there exists an integer ℓ depending on the smallest angle in the triangulation $\triangle$ such that for every domain point $\eta \notin \mathcal{M}$,

$$\Gamma_\eta := \{\xi \in \mathcal{M} : \, c_\eta \text{ depends on } c_\xi\} \subseteq \operatorname{star}^\ell(T_\eta),$$

where T_η is a triangle containing η. Moreover, $\mathcal{M}$ is said to be stable provided there is a constant K depending only on ℓ and the smallest angle in $\triangle$ such that

$$|c_\eta| \le K \max_{\xi \in \Gamma_\eta} |c_\xi|, \qquad \text{all } \eta \notin \mathcal{M}.$$

Let $\triangle$ be a regular triangulation of a polygonal domain Ω and suppose throughout the remainder of this chapter that $d \ge 3r+2$. In Theorem 9.15 we described a minimal determining set for the spline space

$$\mathcal{S}_d^r(\triangle) := \mathcal{S}_d^0(\triangle) \cap C^r(\Omega). \tag{11.1}$$

while in Theorem 9.17 we gave an MDS for the superspline space

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \, s \in C^{\rho_v}(v), \ v \in \mathcal{V}\}, \tag{11.2}$$

where $\mathcal{V}$ is the set of vertices of $\triangle$, and $\rho := \{\rho_v\}_{v \in \mathcal{V}}$ with $r \le \rho_v \le d$. Neither of these minimal determining sets was completely specified in that we did not explicitly describe how to choose the needed domain points in disks around the vertices of $\triangle$. In this chapter we fill in this gap in the theory.

In addition, the minimal determining sets $\mathcal{M}$ constructed in Chapter 9 are local, but in general are not stable. In this chapter we give a construction of stable local minimal determining sets for $\mathcal{S}_d^r(\triangle)$ and the superspline subspace $\mathcal{S}_d^{r,\rho}(\triangle)$.

11.2. Supersplines on Four-cells

Suppose $\triangle_v$ is a triangulation consisting of exactly four triangles surrounding an interior vertex v. We call such a triangulation a four-cell. Let $T_i := \langle v, v_i, v_{i+1} \rangle$, $i = 1, \ldots, 4$, be the four triangles in $\triangle_v$, where $v_5 = v_1$ and $v_1, \ldots, v_4$ appear in counterclockwise order.

For our purposes, we need a slight generalization of the idea of a minimal determining set. Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle_v)$. Given $1 \le k \le 2r$, let $D_k(v)$ be the disk of radius k around v. Then we say that $\mathcal{M} \subseteq D_k(v)$ is a minimal determining set for $\mathcal{S}$ on $D_k(v)$ provided that for any spline $s \in \mathcal{S}$, setting the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ to arbitrary real numbers consistently determines all of the coefficients corresponding to domain points in $D_k(v)$. For $\mu + 1 \le \ell \le 2r$, we now introduce some special notation for certain domain points on the ring $R_\ell(v)$. Let

$$
\begin{aligned}
a_{\ell,j}^i &:= \xi_{d-\ell,\ell-r+j-1,r-j+1}^{T_i}, && 1 \le j \le n_\ell, \\
g_{\ell,j}^i &:= \xi_{d-\ell,\ell-r+n_\ell+j-1,r-n_\ell-j+1}^{T_i}, && 1 \le j \le n_\ell, \\
d_{\ell,j}^i &:= \xi_{d-\ell,\ell-r+2n_\ell+j-1,r-2n_\ell-j+1}^{T_i}, && 1 \le j \le r - 2n_\ell + 1,
\end{aligned}
\tag{11.3}
$$

where

$$
n_\ell := 2r + 1 - \ell. \tag{11.4}
$$

Note that $n_\ell \ge 1$ and $r - 2n_\ell + 1 \ge 1$. We illustrate this notation for $r = 4$, $\mu = 6$, and $d = 14$ in Figure 11.1. Let

$$
\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor. \tag{11.5}
$$

We want to describe stable minimal determining sets on disks for the superspline space in (11.2) in the special case where $\rho_v = \mu$ for all $v \in \mathcal{V}$. We abuse notation slightly by writing $\mathcal{S}_d^{r,\mu}(\triangle_v)$ for this space. There are two cases depending on whether v is singular or not.

11.2.1 The Vertex is Singular

Recall that an interior vertex v of a triangulation is called singular provided there are four edges attached to v, but they lie on only two different lines.

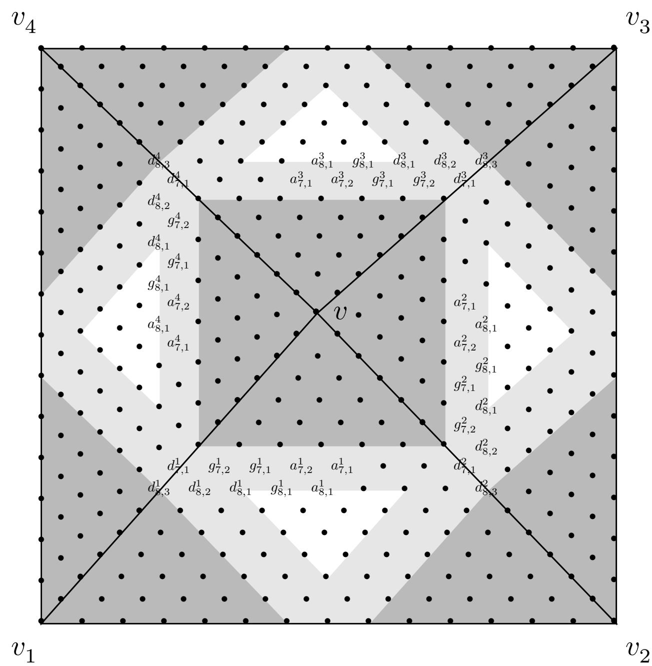


Fig. 11.1. The points in (11.3) for $r = 4$, $\mu = 6$, $d = 14$.

Theorem 11.1. *Suppose $\triangle_v$ is a four-cell associated with a singular vertex v. For each $\ell = \mu + 1, \ldots, 2r$, let*

$$\begin{aligned} \mathcal{M}_{v,\ell} &:= \{a^1_{\ell,1}, \ldots, a^1_{\ell,n_\ell}\} \ \cup \ \bigcup_{i=1}^{4} \{g^i_{\ell,1}, \ldots, g^i_{\ell,n_\ell}\}, \\ O_{v,\ell} &:= \bigcup_{i=1}^{4} \{d^i_{\ell,1}, \ldots, d^i_{\ell,r-2n_\ell+1}\}. \end{aligned} \tag{11.6}$$

Then for each $k = \mu, \ldots, 2r$,

$$\mathcal{M}^{[k]} := D^{T_1}_\mu(v) \ \cup \ \bigcup_{\ell=\mu+1}^{k} \left[\mathcal{M}_{v,\ell} \ \cup \ O_{v,\ell}\right]$$

is a stable MDS for the space $\mathcal{S}^{r,\mu}_d(\triangle_v)$ on $D_k(v)$.

Proof: Clearly, $\mathcal{M}^{[\mu]} = D^{T_1}_\mu(v)$ is a stable MDS for $\mathcal{S}^{r,\mu}_d(\triangle_v)$ on $D_\mu(v)$. We now proceed by induction on k. Fix $\mu < k \le 2r$. Suppose we set the coefficients c_ξ of $s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$ for $\xi \in \mathcal{M}^{[k]}$. Then by the inductive hypothesis, all coefficients c_ξ with $\xi \in D_{k-1}(v)$ are uniquely determined by those with $\xi \in \mathcal{M}^{[k-1]} \subseteq \mathcal{M}^{[k]}$. Now by Theorem 9.7,

$$\dim \mathcal{S}_k^{r,\mu}(\triangle_v) - \dim \mathcal{S}_{k-1}^{r,\mu}(\triangle_v) = 4(k-r) + n_k,$$

which is just the number of points in $\mathcal{M}^{[k]} \setminus \mathcal{M}^{[k-1]} = \mathcal{M}_{v,k} \cup O_{v,k}$. Thus, to complete the proof, we simply need to show that all of the coefficients associated with domain points on the ring R_k are determined from those corresponding to $\mathcal{M}^{[k]}$. But all of these coefficients can be computed from the standard smoothness conditions using Lemma 2.29. For each $i = 1,2,3,4$, we first use the coefficients corresponding to the domain points $g_{k,1}^{i+1}, \ldots, g_{k,n_k}^{i+1}, d_{k,1}^{i+1}, \ldots, d_{k,r-2n_k+1}^{i+1}$ along with those in $D_{k-1}(v)$ to compute the coefficients corresponding to $\{\xi_{d-k,0,k}^{T_i}, \ldots, \xi_{d-k,r-n_k,k-r+n_k}^{T_i}\}$. Then for domain points on R_k, the only undetermined coefficients are those associated with $a_{k,1}^i, \ldots, a_{k,n_k}^i$, $i = 2,3,4$. We now use the coefficients corresponding to $R_k^{T_1}(v)$ to compute the coefficients c_ξ with $\xi \in \{a_{k,1}^2, \ldots, a_{k,n_k}^2\}$, and, proceeding counterclockwise around v, successively compute the coefficients with $\xi \in \{a_{k,1}^3, \ldots, a_{k,n_k}^3\}$ and $\xi \in \{a_{k,1}^4, \ldots, a_{k,n_k}^4\}$. Note that although we have not used some of the smoothness conditions across the edge $e_1 := \langle v, v_1 \rangle$ which involve the coefficients c_ξ for $\xi \in \{a_{k,1}^1, \ldots, a_{k,n_k}^1\}$, these conditions will be automatically satisfied. Lemma 2.29 also asserts that the maximum of the computed coefficients with $\xi \in R_k(v)$ is bounded by a constant K times the maximum of $|c_\xi|$ over $\xi \in \mathcal{M}^{[k]}$, where K depends only on d and the smallest angle in $\triangle_v$, and we have stability. □

For later use in building stable local minimal determining sets for general spline spaces, it is critical that the stable MDS in Theorem 11.1 contains the sets $O_{v,\ell}$. In Section 11.5 we construct stable minimal determining sets for supersplines on cells with n edges. The construction there is simpler, but does not guarantee that the resulting MDS contains the needed sets $O_{v,\ell}$.

11.2.2 The Vertex is Nonsingular

In this section we prove a version of Theorem 11.1 for the case when v is not singular. Using the notation of the previous section, let $e_i := \langle v, v_i \rangle$ for $i = 1, \ldots, 4$, where for convenience we identify v_{i+4} with v_i for all i. Suppose the barycentric coordinates of v_{i-1} with respect to the triangle T_i are given by

$$v_{i-1} = r_i v_{i+1} + s_i v + t_i v_i,$$

for $i = 1, \ldots, 4$. Note that $t_i = 0$ if and only if the edge e_i is degenerate at v. Since v is assumed not to be a singular vertex, at least one t_i is nonzero. Moreover, the number of edges attached to v with different slopes is at least three. To describe minimal determining sets for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$ for $\mu \le k \le 2r$, we will proceed as in Theorem 11.1, but will replace the sets $\mathcal{M}_{v,\ell}$ in (11.6) by sets with $4n_\ell$ points insead of $5n_\ell$ points. Fix $\mu \le k \le 2r$. Theorem 9.7 implies

$$m_{k,r} := \dim \mathcal{S}_k^{r,\mu}(\triangle_v) - \dim \mathcal{S}_{k-1}^{r,\mu}(\triangle_v) = 4(k-r). \tag{11.7}$$

Thus, if we have a minimal determining set for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_{k-1}(v)$, to get a minimal determining set for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$, we need to add exactly $m_{k,r}$ points on the ring $R_k(v)$. Let $s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$, and suppose we know the coefficients of s corresponding to all domain points in $D_{k-1}(v)$. Let $z := (z_1, \ldots, z_{4r+4})$ be the vector of B-coefficients of s corresponding to the domain points

$$\bigcup_{i=1}^{4} \{a_{k,1}^i, \ldots, a_{k,n}^i, g_{k,1}^i, \ldots, g_{k,n}^i, d_{k,1}^i, \ldots, d_{k,r-2n+1}^i\},$$

where for ease of notation we write $n := n_k = 2r+1-k$. Then we can write the set of smoothness conditions across interior edges of $\triangle_v$ which connect the components of z to each other in matrix form as

$$Hz = h, \tag{11.8}$$

where h is a vector containing linear combinations of known coefficients, and

$$H := \begin{bmatrix} H_1^a & H_1^g & H_1^d & & & & & & & -I & & \\ -I & & & H_2^a & H_2^g & H_2^d & & & & & & \\ & & & -I & & & H_3^a & H_3^g & H_3^d & & & \\ & & & & & & -I & & & H_4^a & H_4^g & H_4^d \end{bmatrix},$$

$$H_i^a := \begin{bmatrix} & & & & r_i^{r-n+1} \\ & & & r_i^{r-n+2} & \binom{r-n+2}{r-n+1} r_i^{r-n+1} t_i \\ & & \iddots & & \vdots \\ & r_i^{r-1} & & \cdots & \binom{r-1}{r-n+1} r_i^{r-n+1} t_i^{n-2} \\ r_i^r & \binom{r}{r-1} r_i^{r-1} t_i & & \cdots & \binom{r}{r-n+1} r_i^{r-n+1} t_i^{n-1} \end{bmatrix},$$

$$H_i^g := \begin{bmatrix} \binom{r-n+1}{r-n} r_i^{r-n} t_i & \cdots & \binom{r-n+1}{r-2n+1} r_i^{r-2n+1} t_i^n \\ \vdots & & \vdots \\ \binom{r}{r-n} r_i^{r-n} t_i^n & \cdots & \binom{r}{r-2n+1} r_i^{r-2n+1} t_i^{2n-1} \end{bmatrix},$$

$$H_i^d := \begin{bmatrix} \binom{r-n+1}{r-2n} r_i^{r-2n} t_i^{n+1} & \cdots & \binom{r-n+1}{1} r_i t_i^{r-n} & t_i^{r-n+1} \\ \vdots & & \vdots & \vdots \\ \binom{r}{r-2n} r_i^{r-2n} t_i^{2n} & \cdots & \binom{r}{1} r_i t_i^{r-1} & t_i^r \end{bmatrix},$$

and I is the $n \times n$ identity matrix. The matrix H has $4n$ rows and $4(r+1)$ columns where $n < r+1$. We call a column of H a d-column when it passes through one of the matrices H_i^d. We define a-columns and g-columns similarly.

Lemma 11.2. *There is a choice of $4n$ indices $1 \le i_1 < \cdots < i_{4n} \le 4r+4$ such that the the submatrix $H(i_1, \ldots, i_{4n})$ obtained from H by selecting columns $i_1, \ldots, i_{4n}$ is nonsingular. The $i_1, \ldots, i_{4n}$ can be chosen so that no column of $H(i_1, \ldots, i_{4n})$ is a d-column.*

Proof: We claim the matrix H has full rank $4n$. To see this, we observe that the number of independent solutions of $Hz = 0$ is $4(r+1) - \operatorname{rank}(H)$. This must equal the number $m_{k,r}$ in (11.7), and it follows that $\operatorname{rank}(H) = 4n$. This means there is a choice of indices $1 \le i_1 < \cdots < i_{4n} \le 4r+4$ such that the the corresponding square submatrix $H(i_1, \ldots, i_{4n})$ of H is nonsingular.

We show now that $H(i_1, \ldots, i_{4n})$ can be chosen so that it does not contain a d-column. Suppose that $H(i_1, \ldots, i_{4n})$ contains a nontrivial d-column. Then we claim that there exists another submatrix $H(j_1, \ldots, j_{4n})$ with one less d-column such that

$$|\det H(i_1, \ldots, i_{4n})| \le C\ |\det H(j_1, \ldots, j_{4n})|, \tag{11.9}$$

where C depends only on d and the smallest angle $\theta_{\triangle_v}$ of $\triangle_v$. Suppose the nontrivial d-column of $H(i_1, \ldots, i_{4n})$ corresponds to $i_p = (r+1)(i-1)+2n+j$ with $1 \le i \le 4$ and $1 \le j \le r-2n+1$. Note that a column is nontrivial if and only if the corresponding t_i is nonzero. For any $1 \le j \le r-2n+1$, it is not difficult to see that

$$H_i^d(j) = \sum_{\kappa=1}^{n} x_\kappa^{[j]} \left(\frac{t_i}{r_i}\right)^{j+n-\kappa} H_i^g(\kappa),$$

where the numbers $x_\kappa^{[j]}$ are determined from the nonsingular linear system

$$\begin{bmatrix} \binom{r-n+1}{r-n} & \cdots & \binom{r-n+1}{r-2n+1} \\ \vdots & \ddots & \vdots \\ \binom{r}{r-n} & \cdots & \binom{r}{r-2n+1} \end{bmatrix} \begin{bmatrix} x_1^{[j]} \\ \vdots \\ x_n^{[j]} \end{bmatrix} = \begin{bmatrix} \binom{r-n+1}{r-2n+1-j} \\ \vdots \\ \binom{r}{r-2n+1-j} \end{bmatrix}. \tag{11.10}$$

A simple computation shows that the determinant of the matrix in (11.10) is equal to

$$C \det \begin{bmatrix} \frac{1}{1!} & \cdots & \frac{1}{n!} \\ \vdots & \cdots & \vdots \\ \frac{1}{n!} & \cdots & \frac{1}{(2n-1)!} \end{bmatrix},$$

where C is a positive constant depending only on r and n. It is well known that this determinant is nonzero for all choices of n.

Since the κ-th column of H_i^g corresponds to the $(r+1)(i-1)+n+\kappa$-th column of H, we have

$$\det H(i_1,\ldots,i_{4n}) = \sum_{\kappa=1}^{n} x_\kappa^{[j]} \left(\frac{t_i}{r_i}\right)^{j+n-\kappa} \det H_\kappa,$$

where

$$H_\kappa := H(i_1,\ldots,i_{p-1},(r+1)(i-1)+n+\kappa,i_{p+1},\ldots,i_{4n}).$$

Since $|r_i|$ is the quotient of the areas of two neighboring triangles T_{i-1} and T_i, by Lemma 4.14,

$$0 < K_1 \le |r_i| \le K_2, \tag{11.11}$$

where K_1, K_2 depend only on $\theta_\triangle$. Therefore,

$$|\det H(i_1,\ldots,i_{4n})| \le K_3 |t_i|^j \max_\kappa |\det H_\kappa|,$$

where K_3 depends only on d and $\theta_\triangle$. Now (11.9) follows since $|t_i| \le K_4$, where K_4 is a constant depending only on $\theta_{\triangle_v}$. In fact, $|t_i|$ is quite small if v is near-singular. $\square$

Let Σ be the set of indices of all a- and g-columns of H, and let $\{i_1^*,\ldots,i_{4n}^*\} \subseteq \Sigma$ be such that

$$|\det H(i_1^*,\ldots,i_{4n}^*)| = \max_{i_1,\ldots,i_{4n}\in\Sigma} |\det H(i_1,\ldots,i_{4n})|. \tag{11.12}$$

Let $\mathcal{M}_{v,k}$ be the set of domain points in $\mathcal{A}_{v,k}$ which correspond to the columns with indices in the set $\Sigma \setminus \{i_1^*,\ldots,i_{4n}^*\}$. Then $\mathcal{M}_{v,k} \cup \mathcal{O}_{v,k}$ is the set of domain points on $R_k(v)$ which correspond to the columns of H with indices in the set $J^* := \{1,\ldots,4r+4\} \setminus \{i_1^*,\ldots,i_{4n}^*\}$.

Theorem 11.3. *Suppose $\triangle_v$ is a four-cell associated with a nonsingular vertex v. For each $\mu+1 \le \ell \le 2r$, let $O_{v,\ell}$ be the sets in (11.6), and let n_ℓ be the integer defined in (11.4). Then there exists a set of $4n_\ell$ domain points*

$$\mathcal{M}_{v,\ell} \subseteq \mathcal{A}_{v,\ell} := \bigcup_{i=1}^{4} \left[\{a_{\ell,j}^i\}_{j=1}^{n_\ell} \cup \{g_{\ell,j}^i\}_{j=1}^{n_\ell}\right] \tag{11.13}$$

such that for each $k = \mu, \ldots, 2r$,

$$\mathcal{M}^{[k]} := D_\mu^{T_1}(v) \cup \bigcup_{\ell=\mu+1}^{k} \left[\mathcal{M}_{v,\ell} \cup O_{v,\ell}\right]$$

is a stable MDS for the space $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$.

Proof: We again proceed by induction on k as in the proof of Theorem 11.1. Clearly, the set $\mathcal{M}^{[\mu]} = D_\mu^{T_1}(v)$ is a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_\mu(v)$. Now fix $\mu + 1 \le k \le 2r$, and suppose that $\mathcal{M}^{[k-1]}$ is a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_{k-1}(v)$. To construct $\mathcal{M}^{[k]}$ which is a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$, we need to supplement $\mathcal{M}^{[k-1]}$ with an appropriate subset of the domain points on the ring $R_k(v)$.

To simplify the discussion of how to choose these m points, we first reduce the problem to one of considering splines whose coefficients are zero for all points in the disk $D_{k-1}(v)$. Given $s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$, let $\mathcal{I}_{k-1}s$ be the spline in $\mathcal{S}_d^r(\triangle_v)$ constructed in Lemma 11.4 below such that for each triangle attached to v, $g_T := \mathcal{I}_{k-1}s|_T$ interpolates the derivatives up to order $k-1$ of $s|_T$ at v. Note that since $s \in C^\mu(v)$, $\mathcal{I}_{k-1}s$ is also in $C^\mu(v)$. Then the spline $\hat{s} := s - \mathcal{I}_{k-1}s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$ has all zero coefficients in $D_{k-1}(v)$. Computing the coefficients of $\hat{s}$ associated with domain points on the ring $R_k(v)$ will stably and uniquely determine the corresponding coefficients of s, since by Lemma 11.4 the size of the coefficients of $\mathcal{I}_{k-1}s$ associated with domain points on this ring is bounded by the size of the coefficients of s associated with domain points in $D_{k-1}(v)$.

Now assuming that the coefficients $\{z_j\}_{j\in J^*}$ of $\hat{s}$ corresponding to points in $\mathcal{M}_{v,k} \cup \mathcal{O}_{v,k}$ have been fixed, we may compute the remaining coefficients corresponding to points in $\mathcal{A}_{v,k} \cup \mathcal{O}_{v,k}$ from the nonsingular system

$$H(i_1^*, \ldots, i_{4n}^*) \begin{bmatrix} z_{i_1^*} \\ \vdots \\ z_{i_{4n}^*} \end{bmatrix} = -\sum_{j\in J^*} z_j H(j), \tag{11.14}$$

where $H(j)$ is the j-th column of H. Using Cramer's rule and taking account of (11.12) and Lemma 11.2, we conclude that

$$|z_{i_\nu^*}| \le \frac{\sum_{j\in J^*} |z_j| |\det H(i_1^*, \ldots, i_{\nu-1}^*, j, i_{\nu+1}^*, \ldots, i_{4n}^*)|}{|\det H(i_1^*, \ldots, i_{4n}^*)|} \le K \max_{j\in J^*} |z_j|,$$

for $\nu = 1, \ldots, 4n$, where K is a constant depending only on d and the smallest angle in $\triangle_v$. This shows that the computation of $z_{i_1^*}, \ldots, z_{i_{4n}^*}$ is stable. $\square$

The following lemma was used in the proof of Theorem 11.3, and will also be useful in Section 11.7 below.

Lemma 11.4. *Let $\triangle_v$ be a cell, and let $0 \le r < k \le d$ be integers. Given a spline $s \in \mathcal{S}_d^r(\triangle_v)$, let $\mathcal{I}_{k-1}s$ be such that for each triangle T attached to v, $\mathcal{I}_{k-1}s|_T$ is the unique polynomial of degree $k-1$ which matches the derivatives of $s|_T$ at v up to order $k-1$. Then $\mathcal{I}_{k-1}s \in \mathcal{S}_{k-1}^r(\triangle_v) \subseteq \mathcal{S}_d^r(\triangle_v)$. Moreover, if*

$$s|_T := \sum c_\xi^T B_\xi^T, \qquad \mathcal{I}_{k-1}s|_T := \sum \hat{c}_\xi^T B_\xi^T,$$

where B_ξ^T are the Bernstein polynomials of degree d associated with a triangle T, then $\hat{c}_\xi^T = c_\xi^T$ for all $\xi \in D_{k-1}^T(v)$, and

$$\max_{\xi \in R_k^T(v)} |\hat{c}_\xi^T| \le K \max_{\xi \in D_{k-1}^T(v)} |c_\xi^T|, \tag{11.15}$$

where K is a constant depending only on d.

Proof: Comparing cross derivatives of neighboring pieces of $\mathcal{I}_{k-1}s$, it is easy to see that it satisfies C^r smoothness conditions across the interior edges of $\triangle_v$, and thus is a spline in $\mathcal{S}_{k-1}^r(\triangle_v) \subseteq \mathcal{S}_d^r(\triangle)$. Now fix a triangle $T := \langle v, v_i, v_{i+1}\rangle$ in $\triangle_v$. Then by the connection between derivatives and coefficients of a polynomial written in Bernstein–Bézier form, see Section 2.7, it follows that $\hat{c}_\xi^T = c_\xi^T$ for all $\xi \in D_{k-1}^T(v)$. Finally, to establish (11.15), we observe that since $\mathcal{I}_{k-1}s$ is a polynomial of degree $k-1$, its k-th derivatives are identically zero, and thus for all $\nu = 0, \dots, k$,

$$\begin{aligned} 0 &= D_{v_i-v}^{\nu} D_{v_{i+1}-v}^{k-\nu} \mathcal{I}_{k-1}s|_T(v) \\ &= \frac{d!}{(d-k)!} \sum_{j_1=0}^{\nu} \sum_{j_2=0}^{k-\nu} \binom{\nu}{j_1}\binom{k-\nu}{j_2} (-1)^{k-j_1-j_2} \hat{c}_{d-j_1-j_2,j_1,j_2}^T. \end{aligned}$$

It follows that

$$\begin{aligned} &\hat{c}_{d-k,\nu,k-\nu}^T \\ &\quad = -\frac{d!}{(d-k)!} \sum_{\substack{0 \le j_1 \le \nu,\ 0 \le j_2 \le k-\nu \\ j_1+j_2 \le k-1}} \binom{\nu}{j_1}\binom{k-\nu}{j_2} (-1)^{k-j_1-j_2} \hat{c}_{d-j_1-j_2,j_1,j_2}^T, \end{aligned}$$

which immediately implies (11.15). □

Example 11.5. Let $r = 5$, $\mu = 8$, $d = 17$ in Theorem 11.3.

Discussion: Figure 11.2 shows the set of domain points in this case, where some of the points are marked with symbols rather than dots. We have shaded the D_{2r} disks light gray and the D_μ disks dark gray. Consider the ring $R_9(v)$, where $n_9 = 2$. The points in $O_{v,9}$ are marked with the

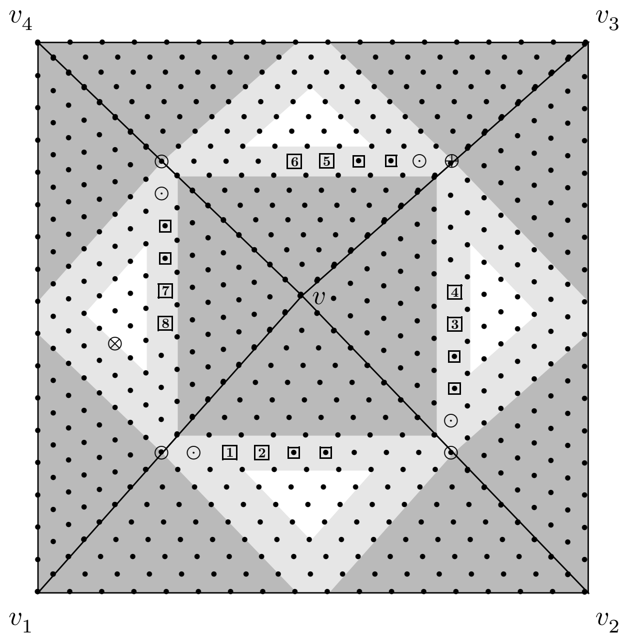


Fig. 11.2. The points in $\mathcal{M} \cap R_9(v)$ for $r = 5$, $\mu = 8$, $d = 17$.

symbol $\odot$ (except for the point $d^3_{9,2}$ which we have marked with a $\oplus$ for a later discussion). In this case the set $\mathcal{M}_{v,9}$ must contain eight of the sixteen points in the set $\mathcal{A}_{v,9}$ described in (11.13). These eight points are chosen by the method of maximization of the determinant in (11.12), and therefore depend on the exact geometry of the cell. In Figure 11.2 we show a possible constellation where we mark the eight points in $\mathcal{M}_{v,9} \cap \mathcal{A}_{v,9}$ with the symbol $\boxdot$. This leaves eight points which are computed by the linear system (11.8). They are marked with boxed numbers 1 through 8. $\square$

11.3. A Lemma on Near-Degenerate Edges

The concepts of δ-near-degenerate edges and δ-near-singular vertices were introduced and studied in Section 10.3.1. In this section we establish the following useful lemma.

Lemma 11.6. *Give a triangulation $\triangle$, let $\delta := 2\theta^2/\pi$, where θ is the smallest angle in $\triangle$. Then for any interior vertex v which is not δ-near-singular, there is at least one edge attached to v which is not δ-near-degenerate at either end.*

Proof: Let $v_1,\dots,v_n$ be the vertices attached to v in counter-clockwise order. For $i=1,\dots,n$, we set $T_i := \langle v, v_i, v_{i+1}\rangle$ and denote by $\theta_i, \phi_i, \omega_i$ the angles of T_i at v, v_i, v_{i+1}, respectively. We distinguish three cases.

Case 1: $n=3$. Consider the edge $e_1 := \langle v, v_1\rangle$, and let $\alpha := \phi_1 + \omega_3$ and $\beta := \theta_1 + \theta_3$. Then it is clear that $\beta \ge \pi + 2\theta$. Now $\alpha + \beta + \omega_1 + \phi_3 = 2\pi$. This implies $\alpha \le \pi - 4\theta$ since $\omega_1, \phi_3 \ge \theta$. This shows that the edge $\langle v, v_1\rangle$ is not δ-near-degenerate at either end.

Case 2: $n=4$. Since v is not δ-near-singular, there is at least one edge attached to v which is not δ-near-degenerate at v. Without loss of generality we can assume it is the edge $e_1 := \langle v, v_1\rangle$, and that the angle $\beta := \theta_1 + \theta_4$ is at least $\pi + 2\theta^2/\pi$. Then arguing as in Case 1, we see that $\alpha := \phi_1 + \omega_4 \le \pi - 2\theta - 2\theta^2/\pi \le \pi - 2\theta$.

Case 3: $n \ge 5$. Consider the edge $e_1 := \langle v, v_1\rangle$. Let $\alpha_i := \phi_i + \omega_{i-1}$ and $\beta_i := \theta_i + \theta_{i-1}$, for $i=1,\dots,n$, where we identify $\theta_{n+i} = \theta_i$, $\omega_{n+i} = \omega_i$. We claim that at least three of the α_i satisfy $\alpha_i \le \pi - 4\theta/(n-2)$. Indeed, if this were not the case, then

$$(n-2)\pi = \sum_{i=1}^{n} \alpha_i > (n-2)\Big(\pi - \frac{4\theta}{n-2}\Big) + 4\theta = (n-2)\pi.$$

On the other hand, we claim that at most two of the β_i satisfy $\beta_i \ge \pi - \theta/2$. Suppose to the contrary that there are three, say $\beta_k, \beta_l, \beta_m$. Then at least two of these do not overlap, say β_k, β_l. But then there are $n-4$ of the angles θ_i which are not covered by β_k or β_l, which would lead to the contradiction

$$2\pi \ge \beta_k + \beta_l + (n-4)\theta > 2\pi - \theta + (n-4)\theta \ge 2\pi.$$

Now $n\theta \le 2\pi$ implies $4/(n-2) \ge 4/n \ge 2\theta/\pi$. We conclude that for one of the edges, $\alpha_i \le \pi - 4\theta/(n-2) \le \pi - 2\theta^2/\pi$ and $\beta_i \le \pi - \theta/2 \le \pi - 2\theta^2/\pi$. It follows that this edge is not δ-near-degenerate at either end. $\square$

11.4. A Stable Local MDS for $\mathcal{S}_d^{r,\mu}(\triangle)$

Suppose $\triangle$ is a regular triangulation, and let $\mathcal{V}$ be its set of vertices. Let $\theta_\triangle$ be the smallest angle in $\triangle$, and let $\delta := 2\theta_\triangle^2/\pi$. Let $\mathcal{V}_S$ and $\mathcal{V}_{NS}$ be the sets of vertices of $\triangle$ which are singular and δ-near-singular, respectively. Given r and d with $d \ge 3r+2$, let $\mu = r + \lfloor (r+1)/2 \rfloor$. In this section we construct a stable local MDS for the superspline space

$$\mathcal{S}_d^{r,\mu}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) :\ s \in C^\mu(v),\ v \in \mathcal{V}\}.$$

This is a special case of the space in (11.2) with $\rho := (\mu,\dots,\mu)$. Given a triangle T in $\triangle$, let $A^T, C^T, E^T, F^T, G_L^T, G_R^T$ be the subsets of domain points in T defined in (9.17), see also Figure 9.5.

Theorem 11.7. *Let $\mathcal{M}$ be the following set of domain points:*

1) *For each triangle T, include C^T.*
2) *For each edge e, include $E^T(e)$, where T is some triangle sharing e.*
3) *For each edge of a triangle T such that e lies on the boundary of Ω, include $G_L^T(e)$ and $G_R^T(e)$.*
4) *For each $v \in \mathcal{V}$, include $D_\mu^T(v)$ for some triangle T with vertex v.*
5) *Suppose the vertex $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in counterclockwise order. Let $T_i := \langle v, v_i, v_{i+1}\rangle$ and set $T_0 := T_n = \langle v, v_n, v_1\rangle$ if v is an interior vertex. Let $1 \le i_1 < \cdots < i_k < n$ be such that e_{i_j} is δ-near-degenerate at either end, where $e_i := \langle v, v_i\rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$. Then*
   a) *include $G_L^{T_i}(e_i)$ for all $i \in J_v$,*
   b) *include $A^{T_i}(v)$ for all $1 \le i \le n-1$ such that $i \notin J_v$,*
   c) *include $A^{T_n}(v)$ if v is an interior vertex.*
6) *For each vertex $v \in \mathcal{V}_S$, include the sets $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 11.1.*
7) *For each $v \in \mathcal{V}_{NS} \setminus \mathcal{V}_S$ include the sets $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 11.3.*

Then $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}_d^{r,\mu}(\triangle)$.

Proof: We claim that $\mathcal{M}$ is well defined. In particular, if $v \notin \mathcal{V}_{NS}$, then by Lemma 11.6 there exists at least one edge attached to v which is not δ-near-degenerate at either end. In the numbering of the edges in item 5) above, we can choose this edge to be $\langle v, v_n\rangle$, and the construction ensures that for each interior vertex $v \notin \mathcal{V}_{NS}$ and edge $e_i := \langle v, v_i\rangle$ attached to it, if $v_i \notin \mathcal{V}_{NS}$, then $\mathcal{M}$ includes exactly one of the two sets $A^{T_i}(v)$ or $G_L^{T_i}(e_i)$. The construction also guarantees that for all vertices $v \notin \mathcal{V}_{NS}$, there is at least one triangle T with vertex at v such that $\mathcal{M}$ contains the set $A^T(v)$.

To see that $\mathcal{M}$ is a determining set for $\mathcal{S}_d^{r,\mu}(\triangle)$, we show that setting $c_\xi = 0$ for all $\xi \in \mathcal{M}$ implies s is identically zero. Since for every vertex v of $\triangle$ the set $\mathcal{M}$ contains $D_\mu^T(v)$ for some triangle attached to v, by Lemma 5.10 all coefficients of s associated with domain points in the disks $D_\mu(v)$ vanish.

Next we compute coefficients on the rings $R_{\mu+1}(v)$ for all v. First we do the vertices v which are not in $\mathcal{V}_{NS}$. We process arcs in a *counterclockwise* direction around v, starting with an edge e such that the preceding triangle T contains the set $A^T(v)$. These computations are based on Lemma 2.29, or (only if the corresponding edge is not δ-near-degenerate) Lemma 2.30. Next we use Theorem 11.1 for each vertex $v \in \mathcal{V}_S$, and Theorem 11.3 for each vertex in $\mathcal{V}_{NS} \setminus \mathcal{V}_S$. To do this, we need the coefficients corresponding to the sets $O_{v,\mu+1}$, but these will all have been set to zero or computed to be zero at this point. We now repeat this entire process one ring at a time until we have completed all of the rings up to $R_{2r}(v)$ for all v.

At this point we have shown that all coefficients of s corresponding to domain points in the disks $D_{2r}(v)$ are zero. Since $\mathcal{M}$ contains the sets C^T, the only remaining coefficients correspond to points in sets of the form

$$E^T(e) \setminus \Big[D_{2r}(u) \cup D_{2r}(v)\Big],$$

where $e = \langle u, v\rangle$ is an interior edge. These coefficients can be computed from the associated coefficients in the neighboring triangle (which will have been set to zero) using Lemma 2.29.

We have shown that $\mathcal{M}$ is a determining set for $\mathcal{S}_d^{r,\mu}(\triangle)$. To see that it is minimal, we simply check that its cardinality is equal to the dimension of $\mathcal{S}_d^{r,\mu}(\triangle)$ as given in Corollary 9.20. Let

$$\begin{aligned}
n_a &:= \#A^T(v) = \#G_L^T(e) = \#G_R^T(e) = \binom{2r-\mu+1}{2}, \\
n_c &:= \#C^T = \binom{d-3r-1}{2}, \\
n_d &:= \#D_\mu^T(v) = \binom{\mu+2}{2}, \\
n_e &:= \#E^T(e) = n_f - 4n_a, \\
n_f &:= \frac{(r+1)(2d-4\mu+r-2)}{2}.
\end{aligned}$$

It is easy to check that the number of points chosen in item 6) of Theorem 11.7 is $5n_a$, and in item 7) is $4n_a$. This is n_a points for each edge attached to v, and an additional n_a points when v is singular. Thus,

$$\#\mathcal{M} = n_d V + n_a(2E + S + E_B) + n_e E + n_c N, \tag{11.16}$$

where

$$\begin{aligned}
E &:= \text{number of edges of } \triangle, \\
E_B &:= \text{number of boundary edges of } \triangle, \\
N &:= \text{number of triangles of } \triangle, \\
S &:= \text{number of singular vertices of } \triangle, \\
V &:= \text{number of vertices of } \triangle.
\end{aligned}$$

Using the fact that $3N = 2E_I + E_B$, (11.16) reduces to

$$\#\mathcal{M} = n_d V + n_a(S - 3N) + n_f E + n_c N.$$

A simple computation shows that this is equal to formula (9.24) for the dimension of $\mathcal{S}_d^{r,\mu}(\triangle)$.

This completes the proof that $\mathcal{M}$ is an MDS for $\mathcal{S}_d^r(\triangle)$. We now claim that $\mathcal{M}$ is local in the sense of Definition 5.16. In particular, for every triangle T and every domain point η in T, we claim that the set Γ_η in the definition is a subset of $\mathrm{star}^3(T)$. To see this, we examine the supports of the basis splines in the $\mathcal{M}$-basis associated with the MDS $\mathcal{M}$. Recall that for each $\xi \in \mathcal{M}$, the basis function ψ_ξ is the spline whose coefficients satisfy

$$c_\eta = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}.$$

Our task is to determine which coefficients of ψ_ξ are nonzero, or in other words, how far does the coefficient $c_\xi = 1$ propagate?

Case 1: Suppose $\xi \in C^T$ for some triangle T. Then since no smoothness conditions involve c_ξ, all other coefficients of ψ_ξ must be zero, i.e., the support of ψ_ξ is contained in T.

Case 2: Suppose $\eta \in E^T(e_1) \setminus \big[D_{2r}(v_1) \cup D_{2r}(v_2)\big]$, where $T := \langle v_1, v_2, v_3\rangle$ and $e_1 := \langle v_1, v_2\rangle$ is a boundary edge of $\triangle$. In this case ψ_ξ has zero coefficients on the disks $D_{2r}(v_i)$ and on $E^T(e_j) \setminus [D_{2r}(v_j) \cup D_{2r}(v_{j+1})]$ for the other two edges $e_j := \langle v_j, v_{j+1}\rangle$ of T. It follows that the support of ψ_ξ is just the triangle T.

Case 3: Suppose $\xi \in E^T(e) \setminus \big[D_{2r}(v) \cup D_{2r}(u)\big]$, where $e = \langle v, u\rangle$ is an interior edge shared by T and a neighboring triangle $\widetilde{T}$. Then arguing as in Case 2, we see that the support of ψ_ξ is $T \cup \widetilde{T}$.

The situation is more complicated when ξ lies in some disk $D_{2r}(v)$. This is due to the fact that when $d < 4r + 1$, the $2r$-disks overlap, and nonzero coefficients in one such disk can propagate to a neighboring disk.

Case 4: Suppose $\xi \in \mathcal{M} \cap D_{2r}(v)$. Suppose $z_1, \ldots, z_n$ are the points on the boundary of $\mathrm{star}^3(v)$ in counterclockwise order. Then Lemma 11.8 below shows that the coefficients of ψ_ξ are zero on the disks $D_{2r}(z_i)$. Now for each $e_i := \langle z_i, z_{i+1}\rangle$, $E^{T_i}(e_i) \setminus [D_{2r}(z_i) \cup D_{2r}(z_{i+1})] \subseteq \mathcal{M}_0 := \mathcal{M} \setminus \{\xi\}$ for some triangle T_i sharing the edge e_i. It follows that the corresponding coefficients are also zero, and we conclude that the support of ψ_ξ is a subset of $\mathrm{star}^3(v)$.

Finally, we claim that $\mathcal{M}$ is stable. First, we note that the computations of coefficients corresponding to domain points in the rings $R_{\mu+1}(v)$, $\ldots, R_{2r}(v)$ for a singular or δ-near-singular vertex v are stable by Theorems 11.1 and 11.3. Now it is easy to see that the computation of all remaining coefficients is also stable. Indeed, each B-coefficient in $\mathcal{D}_{d,\triangle} \setminus \mathcal{M}$ is computed directly from smoothness conditions as in Lemma 2.29, or are computed indirectly from smoothness conditions as in Lemma 2.30. Since this latter lemma is only applied to edges which are not δ-near-degenerate, those computations are also stable. □

Since we have shown that $\mathcal{S}_d^{r,\mu}(\triangle)$ has a stable local MDS, it now follows from Theorem 5.19 that this space has full order approximation power. Since $\mathcal{S}_d^{r,\mu}(\triangle)$ is a subspace of $\mathcal{S}_d^r(\triangle)$, it follows that $\mathcal{S}_d^r(\triangle)$ also has full order approximation power. By Theorem 5.21 the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ provides a stable local basis for $\mathcal{S}_d^{r,\mu}(\triangle)$. This is also clear from the proof of Theorem 11.7, which makes use of the following lemma.

Lemma 11.8. *Let $\mathcal{M}$ be the MDS in Theorem 11.7, and let $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ be the corresponding $\mathcal{M}$-basis. Then for each $\xi \in \mathcal{M}$, ψ_ξ can have a nonzero coefficient associated with a domain point in a disk $D_{2r}(w)$ only if $\xi \in D_{2r}(v)$ for some v, and either $v = w$, or w is connected directly to v with an edge $\langle w, v\rangle$ or by a pair of edges $\langle w, u\rangle$, $\langle u, v\rangle$, where u is a δ-near-singular vertex.*

Proof: It is clear from the first part of the proof of Theorem 11.7 that the coefficients in a disk $D_{2r}(w)$ are computed from smoothness conditions which involve only coefficients in such disks. Hence, if ξ is not in any disk $D_{2r}(v)$, then ψ_ξ has zero coefficients on all disks $D_{2r}(w)$.

Suppose now that $\xi \in D_{2r}(v)$, and let $\mathcal{M}_0 := \mathcal{M} \setminus \{\xi\}$. Suppose $w \neq v$ is not connected directly to v with an edge $\langle w, v\rangle$ or by a pair of edges $\langle w, u\rangle$, $\langle u, v\rangle$, where u is a δ-near-singular vertex. Let $w_1, \ldots, w_n$ be the vertices attached to w and let $e_i := \langle w, w_i\rangle$ and $T_i := \langle w, w_i, w_{i+1}\rangle$. Clearly, $D_\mu(w) \subseteq \mathcal{M}_0$ and $E(e_i) \subseteq \mathcal{M}_0$ for each $1 \le i \le n$, where $E(e_i)$ is one of the sets $E^{T_i}(e_i)$ and $E^{T_{i-1}}(e_i)$. There are two cases.

Case 1: $w \in \mathcal{V}_{NS}$. In this case the four vertices $w_1, \ldots, w_4$ are all different from v, which ensures $D_\mu(w_i) \subseteq \mathcal{M}_0$ for $i = 1, \ldots, 4$. Moreover, none of the w_i is δ-near-singular since two δ-near-singular vertices cannot be neighbors. Since the edges $\langle w, w_i\rangle$ are all δ-near-degenerate at w, we conclude that $\mathcal{M}_0$ also contains the sets $G_R^{T_{i-1}}(e_i)$, $1 \le i \le 4$. This implies that all coefficients of ψ_ξ corresponding to domain points in $D_{2r}(w)$ must be zero. To see this, we first calculate the coefficients associated with domain points on the ring $R_{\mu+1}(w)$ from the nonsingular system (11.14) in the proof of Theorem 11.3. We will get zero coefficients if the right-hand side is zero, which happens as soon as the coefficients associated with domain points in $D_\mu^{T_1}(w)$ and the sets $\mathcal{M}_{w,\mu+1}$ and $O_{w,\mu+1}$ in the theorem are zero. Since $D_\mu^{T_1}(w) \cup \mathcal{M}_{w,\mu+1} \subseteq \mathcal{M}_0$, we only have to check $O_{w,\mu+1}$. It is easy to see that $O_{w,\mu+1} \subseteq \bigcup_{i=1}^4 E(e_i)$, and it follows that the coefficients corresponding to domain points in the disk $D_{\mu+1}(w)$ are zero. Repeating this process for each of the rings $R_{\mu+2}(w), \ldots, R_{2r}(w)$, we note that for each $k = \mu + 2, \ldots, 2r$,

$$O_{w,k} \subseteq \bigcup_{i=1}^{4} \big[D_\mu(w_i) \cup G_R^{T_{i-1}}(e_i) \cup G_R^{T_i}(e_i) \cup E^{T_{i-1}}(e_i) \cup E^{T_i}(e_i)\big].$$

Since the coefficients corresponding to $\mathcal{M}_0$ are zero, and the coefficients corresponding to the disk $D_{k-1}(w)$ are also zero by the induction hypothesis, it is easy to see that the coefficients associated with points in $O_{w,k}$ must be zero, and so we have only zero coefficients associated with $R_k(w)$.

Case 2: $w \notin \mathcal{V}_{NS}$. By Lemma 11.6, there is at least one edge attached to w which is not δ-near-degenerate at either end. Without loss of generality we may assume it is e_n. Then $A^{T_n}(w) \subseteq \mathcal{M}_0$, and the corresponding coefficients of ψ_ξ must be zero. We now compute coefficients associated with domain points on the ring $R_{\mu+1}(w)$ proceeding in counterclockwise order around w. For $i = 1, \ldots, n$, we show that coefficients corresponding to points on the arc $a^r_{\mu+1,e_i}(w)$ are all zero. Assuming this holds for for all $i \le k-1$, we now show it for $i = k$.

a) If e_k is not δ-near-degenerate at either end, then $\mathcal{M}_0$ contains $A^{T_k}(w)$ and either $E^{T_k}(e_k)$ or $E^{T_{k-1}}(e_k)$. Hence the coefficients for points in $a^r_{\mu+1,e_k}(w) \setminus (G_L^{T_{k-1}}(e_k) \cup G_L^{T_k}(e_k))$ are zero, and we get all zero coefficients for points on the arc $a^r_{\mu+1,e_k}(w)$ by Lemma 2.30.

b) If e_k is δ-near-degenerate at one end and $w_k \notin \mathcal{V}_{NS}$, then $\mathcal{M}_0$ contains all of the sets $D_\mu(w)$, $D_\mu(w_k)$, $G_R^{T_{k-1}}(e_k)$, $G_L^{T_k}(e_k)$ and either $E^{T_k}(e_i)$ or $E^{T_{k-1}}(e_i)$. Moreover, by the induction hypothesis, the coefficients for points in $a^r_{\mu+1,e_k}(w) \cap a^r_{\mu+1,e_{k-1}}(w)$ are also zero. Then using the smoothness conditions, it follows that all coefficients associated with points on the arc $a^r_{\mu+1,e_k}(w)$ must be zero.

c) If $w_k \in \mathcal{V}_{NS}$, then w_k is not connected directly to v with an edge $\langle w_k, v\rangle$ or by a pair of edges $\langle w_k, u\rangle$, $\langle u, v\rangle$ with u a δ-near-singular vertex. Then, by Case 1, all coefficients of ψ_ξ associated with points in $D_{2r}(w_k)$ must be zero, and the same argument as in b) shows that all coefficients associated with points on the arc $a^r_{\mu+1,e_k}(w)$ must be zero.

To complete the proof, we now repeat this process for each of the rings $R_{\mu+2}(w), \ldots, R_{2r}(w)$. □

Figure 11.3 illustrates Case 2b of Lemma 11.8 for $r = 4$, $\mu = 6$, and $d = 14$. Suppose e_2 is δ-near-degenerate at either w_2 or w. Then the coefficients corresponding to points in the sets $G_L^{T_2}(e_2)$ and $G_R^{T_1}(e_2)$ are zero. We have marked those points with the symbol $\otimes$. The coefficients associated with the points in $E^T(e_2)$ and $A^{T_1}(w)$ are also zero. They are marked with $\boxdot$ and $\odot$, respectively. Then using smoothness conditions, we see that all of the coefficients corresponding to points marked with $\odot$ along with those in $A^{T_2}(w)$ (marked with the number 4) must be zero. □

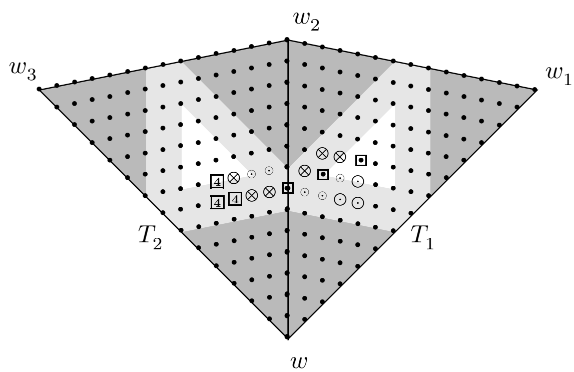


Fig. 11.3. Blocking propagation.

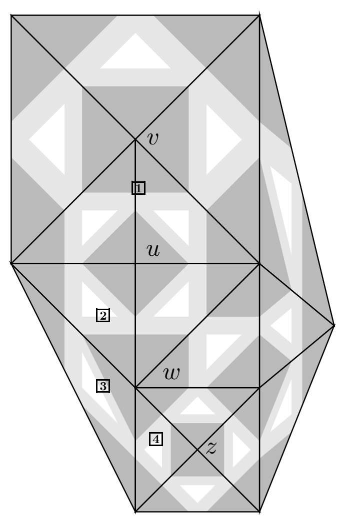


Fig. 11.4. Propagation to $\mathrm{star}^3(v)$.

We conclude this section with an example to illustrate that propagation to $\mathrm{star}^3(v)$ can actually happen.

Example 11.9. Let $\triangle$ be the triangulation shown in Figure 11.4, and let $r = 5$, $\mu = 8$, $d = 17$.

Discussion: For ease of understanding, we shade the disks D_μ and D_{2r} in dark and light gray, respectively. Suppose $\mathcal{M}$ contains the set $D_\mu^T(v)$ where T is the top triangle in the figure, and suppose ξ is the point at the vertex v. Then ψ_ξ has support on all of the triangles surrounding v, and in particular, it has a nonzero coefficient corresponding to the point $R_\mu(v) \cap \langle v, u\rangle$. This point is numbered 1 in the figure, and can be identified with the point marked with a $\oplus$ in Figure 11.2. As seen from that figure, the nonzero coefficient at point number 1 can propagate to a nonzero coefficient corresponding to the point in the set $A^{T_1}(w)$ which is marked with a $\otimes$ in Figure 11.2 and with the number 2 in Figure 11.4. We take $w_1 = u$ and $w_4 = z$. Assuming both $\langle w, w_2\rangle$ and $\langle w, w_3\rangle$ are δ-near-degenerate, we get further propagation to a point in the set $A^{T_3}(w)$ marked with the number 4. This set lies in $\mathrm{star}^3(v)$ but outside of $\mathrm{star}^2(v)$. $\square$

11.5. A Stable MDS for Splines on a Cell

In Section 11.2 we constructed a stable MDS for superspline spaces defined on four-cells. In this section we do the same for interior cells

$$\triangle_v := \{T_i := \langle v, v_i, v_{i+1}\rangle, \quad i = 1, \ldots, n\}$$

with an arbitrary number of edges. Fix $r \le \rho_v < \mu$. Our aim is to construct a stable MDS for the superspline space $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$.

Suppose the vertices $v_1, \ldots, v_n$ are in counterclockwise order, and let $v_{n+1} = v_1$. Let e be the number of edges attached to v with different slopes. Then by Theorem 9.7,

$$m := \dim \mathcal{S}_\mu^{r,\rho_v}(\triangle_v) = \binom{\rho_v + 2}{2} + n\left[\binom{\mu - r + 1}{2} - \binom{\rho_v - r + 1}{2}\right] + \sigma,$$

where

$$\sigma := \sum_{j=\rho_v - r+1}^{\mu - r} (r + j + 1 - je)_+.$$

Suppose $\{\xi_i\}_{i=1}^{n_c}$ are the domain points associated with the cell $\triangle_v$, where

$$n_c = n\left[\binom{\mu - 1}{2} + 2\mu - 1\right] + 1 = n\left[\frac{\mu^2 + \mu}{2}\right] + 1.$$

Given $s \in \mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$, we denote the B-coefficient associated with ξ_i by c_i for $i = 1, \ldots, n_c$. Associated with each interior edge of $\triangle_v$, there are $\mu - j + 1$ smoothness conditions to ensure C^j continuity across that edge, $j = 1, \ldots, r$, and $\rho_v - r - k + 1$ smoothness conditions to ensure C^{ρ_v} continuity at v, $k = 1, \ldots, \rho_v - r$. This gives a total of

$$\begin{aligned} n_s &:= n\left[\binom{\mu + 1}{2} - \binom{\mu - r + 1}{2} + \binom{\rho_v - r + 1}{2}\right] \\ &= nr\left[\frac{2\mu - r + 1}{2}\right] + n\binom{\rho_v - r + 1}{2} \end{aligned}$$

smoothness conditions to ensure that s lies in $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$. Note that $n_s < n_c$. These conditions can be written in matrix form

$$A\,c = 0, \tag{11.17}$$

where $c = (c_1, \ldots, c_{n_c})^T$, and A is an appropriate $n_s \times n_c$ matrix.

In general, the system (11.17) includes some redundant smoothness conditions, and so $n_r := \operatorname{rank}(A) < n_s$. Indeed, since $\dim \mathcal{S}_\mu^{r,\rho_v}(\triangle_v) = n_c - n_r$, it follows that

$$\begin{aligned} n_r &= n\Big[\frac{\mu^2+\mu}{2}\Big] + 1 - \binom{\rho_v+2}{2} - n\left[\binom{\mu-r+1}{2} - \binom{\rho_v-r+1}{2}\right] - \sigma \\ &= nr\Big[\frac{2\mu-r+1}{2}\Big] + 1 - \binom{\rho_v+2}{2} + n\binom{\rho_v-r+1}{2} - \sigma. \end{aligned}$$

This implies that the number of redundant equations in (11.17) is

$$n_{red} := \binom{\rho_v+2}{2} - 1 + \sigma.$$

Without loss of generality, we may assume that redundant equations have been dropped, and that (11.17) is written in the equivalent form

$$[A_1\, A_2]\, c = 0,$$

where A_1 is an $n_r \times m$ matrix and A_2 is an $n_r \times n_r$ matrix. We may also assume that the columns of A (and the corresponding components of c) have been numbered so that the determinant of A_2 has the maximal absolute value over all $n_r \times n_r$ subdeterminants of A.

Algorithm 11.10. *For each $i = 1, \ldots, m$, let s_i be the spline in $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ with B-coefficients $c := (c_1, \ldots, c_{n_c})^T$ chosen so that $c_i = 1$, $c_j = 0$ for $j = 1, \ldots, m$ with $j \neq i$, and $c_{m+1}, \ldots, c_{n_c}$ are determined from the linear system*

$$A_2 \begin{bmatrix} c_{m+1} \\ \vdots \\ c_{n_c} \end{bmatrix} = -A_1(i),$$

where $A_1(i)$ is the i-th column of the matrix A_1.

The splines $\{s_i\}_{i=1}^m$ are clearly linearly independent since

$$\lambda_j s_i = \delta_{i,j}, \qquad j = 1, \ldots, m,$$

where λ_j is a linear functional which picks off the j-th B-coefficient. It follows that they form a basis for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$. We now show that their construction is a stable process, i.e., for each i, all of the coefficients of s_i are uniformly bounded.

Theorem 11.11. *Suppose s_i is a spline constructed by Algorithm 11.10. Then its B-coefficients satisfy*

$$|c_j| \le 1, \qquad j = 1, \dots, n_c. \tag{11.18}$$

Proof: Fix $1 \le i \le m$, and let $c := (c_1, \dots, c_{n_c})$ be the vector of coefficients of s_i as computed from Algorithm 11.10. Then (11.18) clearly holds for $j = 1, \dots, m$. Let $m+1 \le j \le n_c$. Then by Cramer's rule,

$$c_j = \frac{\det(\tilde{A}_2)}{\det(A_2)},$$

where $\tilde{A}_2$ is the matrix obtained from A_2 by replacing the j-th column by $-A_1(i)$. But then $|c_j| \le 1$ follows by the choice of A_2. $\square$

A completely analogous algorithm can be used to create a stable MDS for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ in the case where $\triangle_v$ is a boundary cell.

11.6. A Stable Local MDS for $\mathcal{S}_d^{r,\rho}(\triangle)$

In this section we combine the constructions of the two previous sections to create a stable local MDS for the space of supersplines $\mathcal{S}_d^{r,\rho}(\triangle)$ defined in (11.2) for all $d \ge 3r+2$. As in Section 9.5, we assume that $k_v + k_u < d$ for each pair of neighboring vertices $v, u \in \mathcal{V}$, where $k_v := \max\{\rho_v, \mu\}$ for all $v \in \mathcal{V}$ with μ as in (11.5). Given a triangle $T = \langle u, v, w \rangle$, let $\tilde{A}^T, \tilde{C}^T, \widetilde{E}^T, \tilde{G}_L^T, \tilde{G}_R^T$ be as in (9.23).

Theorem 11.12. *Let $\mathcal{M}$ be the following set of domain points:*

1) *For each triangle T, include the set $\tilde{C}^T$.*
2) *For each edge e, include the set $\widetilde{E}^T(e)$, where T is some triangle sharing e.*
3) *For each edge of a triangle T with e on the boundary of Ω, include the sets $\tilde{G}_L^T(e)$ and $\tilde{G}_R^T(e)$.*
4) *For each vertex $v \in \mathcal{V}$,*
   a) *include the set $D_{\rho_v}^T(v)$ for some triangle attached to v if $\rho_v \ge \mu$,*
   b) *include the points in $D_\mu(v)$ corresponding to the stable minimal determining set $\mathcal{M}_v$ of Section 11.4 for $S_\mu^{r,\rho_v}(\triangle_v)$ if $\rho_v < \mu$.*
5) *Suppose the vertex $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \dots, v_n$, numbered in counterclockwise order. Let $T_i := \langle v, v_i, v_{i+1} \rangle$ and set $T_0 := T_n = \langle v, v_n, v_1 \rangle$ if v is an interior vertex. Let $1 \le i_1 < \cdots < i_k < n$ be such that e_{i_j} is δ-near-degenerate at either end, where $e_i := \langle v, v_i \rangle$ for $i = 1, \dots, n$. Set $J_v := \{i_1, \dots, i_k\}$, and*

   a) *include* $\tilde{G}_L^{T_i}(e_i)$ *for all* $i \in J_v$,
   b) *include* $\tilde{A}^{T_i}(v)$ *for all* $1 \le i \le n-1$ *such that* $i \notin J_v$,
   c) *include* $\tilde{A}^{T_n}(v)$ *if* v *is an interior vertex.*

6) *For each vertex* $v \in \mathcal{V}_S$, *include the sets* $\mathcal{M}_{v,k_v+1}, \ldots, \mathcal{M}_{v,2r}$ *constructed in Theorem 11.1.*
7) *For each* $v \in \mathcal{V}_{NS} \backslash \mathcal{V}_S$, *include the sets* $\mathcal{M}_{v,k_v+1}, \ldots, \mathcal{M}_{v,2r}$ *constructed in Theorem 11.3.*

Then $\mathcal{M}$ *is a stable local minimal determining set for* $\mathcal{S}_d^{r,\rho}(\triangle)$.

Proof: It is straightforward to check that $\mathcal{M}$ is a determining set for $\mathcal{S}_d^{r,\rho}(\triangle)$. To see that it is minimal, we check that its cardinality is equal to the dimension of $\mathcal{S}_d^{r,\rho}(\triangle)$ as given in Theorem 9.18. The localness and stability follow as in the proof of Theorem 11.7. □

Since we now know that $\mathcal{S}_d^{r,\rho}(\triangle)$ has a stable local MDS, it follows from Theorem 5.19 that it has full approximation power. Moreover, by Theorem 5.21, the $\mathcal{M}$ basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ associated with the stable local MDS of Theorem 11.12 provides a stable local basis for $\mathcal{S}_d^{r,\rho}(\triangle)$.

11.7. Stability and Local Linear Independence

A basis $\mathcal{B} = \{\psi_\nu\}_{\nu=1}^n$ for a spline space $\mathcal{S}$ is called locally linearly independent provided that for every $T \in \triangle$, the splines $\{\psi_\nu\}_{\nu\in\Sigma_T}$ are linearly independent on T, where

$$\Sigma_T := \{\nu : T \subseteq \operatorname{supp} \psi_\nu\}.$$

Since the classical univariate B-splines are both stable and locally linearly independent (see Theorems 4.18 and 4.41 in [Sch81]), it seems natural to expect that there also exist bases for bivariate spline spaces which possess both of these properties simultaneously. In this chapter we have constructed stable local bases for the spline spaces $\mathcal{S}_d^r(\triangle)$ and their superspline subspaces, while star-supported locally linearly independent bases for the same spaces were recently constructed in [DavS00a]. But these bases are different, and in fact we have the following surprising result.

Theorem 11.13. *Given* $r \ge 1$ *and* $d \ge 3r+2$, *there are triangulations such that no basis for* $\mathcal{S}_d^r(\triangle)$ *is simultaneously stable and locally linearly independent.*

Proof: Suppose $\mathcal{B} := \{\psi_\nu\}_{\nu=1}^n$ is a stable locally linearly independent basis for $\mathcal{S}_d^r(\triangle)$ on a triangulation which contains an interior near-singular vertex v. Following the notation introduced at the beginning of Section 11.2, suppose v is connected to v_1, v_2, v_3, v_4 in counterclockwise order. For each

$1 \le i \le 4$, let $e_i := \langle v, v_i \rangle$, and $T_i := \langle v, v_i, v_{i+1} \rangle$. Suppose $v_{i-1} = r_i v_{i+1} + s_i v + t_i v_i$, and suppose that none of the e_i is degenerate at v, i.e., $t_i \neq 0$. For convenience, we define $\alpha_i, \beta_i, \gamma_i, \mu_i$ to be the linear functionals that pick off the B-coefficients corresponding to the domain points $\xi^{T_i}_{d-2r,r,r}, \xi^{T_i}_{d-2r,r-1,r+1}, \xi^{T_i}_{d-2r,r+1,r-1}, \xi^{T_i}_{d-2r-1,r,r+1}$, respectively.

For each $1 \le j \le 4$, we claim that there is a unique spline $g_j \in \mathcal{S}^r_d(\triangle)$ whose only nonzero coefficients are

$$\alpha_j g_j = 1, \quad \gamma_j g_j = -r_j/(rt_j), \quad \gamma_{j+1} g_j = r^{1-r}_{j+1}/(rt_{j+1}),$$
$$\beta_{j-1} g_j = r^{r-1}_j \gamma_j g_j, \quad \beta_j g_j = r^{r-1}_{j+1} \gamma_{j+1} g_j,$$
$$\mu_{j-1} g_j = r r^{r-1}_j s_j \gamma_j g_j, \quad \mu_j g_j = r r^{r-1}_{j+1} s_{j+1} \gamma_{j+1} g_j.$$

It can be verified directly that g_j satisfies all C^r smoothness conditions, and thus belongs to $\mathcal{S}^r_d(\triangle)$. It is also easy to see that

$$\operatorname{supp} g_j = T_{j-1} \cup T_j \cup T_{j+1},$$

and by a property of locally linearly independent bases, see [CarP94] and [DavSoS99],

$$g_j = \sum_{\nu \in I_j} c^{[j]}_\nu \psi_\nu, \tag{11.19}$$

where $I_j := \{\nu : \operatorname{supp} \psi_\nu \subseteq T_{j-1} \cup T_j \cup T_{j+1}\}$ for $j = 1, 2, 3, 4$. We now define

$$g := r^r_2 g_1 + g_2 + r^{-r}_3 g_3 + (r_3 r_4)^{-r} g_4.$$

The definition of barycentric coordinates implies that $r_1\, r_2\, r_3\, r_4 = 1$. Using this fact, it is easy to check that all of the coefficients of g are zero except for

$$\alpha_1 g = r^r_2, \quad \alpha_2 g = 1, \quad \alpha_3 g = r^{-r}_3, \quad \alpha_4 g = (r_3 r_4)^{-r}.$$

For example, $\gamma_1 g = r^r_2 \gamma_1 g_1 + (r_3 r_4)^{-r} \gamma_1 g_4 = r_1(-r^r_2 + (r_1 r_3 r_4)^{-r})/(rt_1) = 0$. By (11.11) and Theorem 2.6, this immediately implies

$$\|g\|_\infty \le K_3,$$

where K_3 depends only on d and the smallest angle $\theta_\triangle$ in $\triangle$. In view of (11.19), we can write

$$g = \sum_{\nu \in I_1 \cup I_2 \cup I_3 \cup I_4} a_\nu \psi_\nu.$$

By the assumption that the basis $\mathcal{B}$ is stable, we have

$$\|a\|_\infty \le K^{-1}_1 \|g\|_\infty \le K_3/K_1.$$

For each ν, let $\tilde{B}_\nu = \psi_\nu - \mathcal{I}_{2r-1}\psi_\nu$, where $\mathcal{I}_{2r-1}\psi_\nu \in \mathcal{S}^r_{2r-1}(\triangle_v) \subseteq \mathcal{S}^r_d(\triangle_v)$ is the spline constructed in Lemma 11.4 which interpolates the derivatives of ψ_ν at v up to order $2r-1$. Then the B-coefficients of $\tilde{B}_\nu$ corresponding to domain points in the disk $D_{2r-1}(v)$ are zero. Moreover, since the basis $\mathcal{B}$ is stable, it follows from Lemma 11.4 that the B-coefficients of $\tilde{B}_\nu$ corresponding to domain points on the ring $R_{2r}(v)$ are bounded in absolute value by a constant K_4 depending only on d and $\theta_\triangle$.

Since all of the derivatives of g up to order $2r-1$ at v are zero, $\mathcal{I}_{2r-1}g = 0$, and on $\triangle_v$ we have

$$g = \sum_{\nu \in I_1 \cup I_2 \cup I_3 \cup I_4} a_\nu \tilde{B}_\nu.$$

Since the support of $\tilde{B}_\nu$ is a subset of the support of ψ_ν on $\triangle_v$, it follows that $\alpha_2 \tilde{B}_\nu \neq 0$ only if ν lies in the set

$$\tilde{I}_2 := \{\nu : \operatorname{supp} \psi_\nu = T_1 \cup T_2 \cup T_3\}.$$

This implies

$$1 = \alpha_2 g = \sum_{\nu \in \tilde{I}_2} a_\nu \, \alpha_2 \tilde{B}_\nu \leq \# \tilde{I}_2 \, \|a\|_\infty \max_{\nu \in \tilde{I}_2} |\alpha_2 \tilde{B}_\nu|.$$

Clearly, $\# \tilde{I}_2 \leq 3\binom{d+2}{2}$, and hence there exists $\nu_0 \in \tilde{I}_2$ such that

$$|\alpha_2 \tilde{B}_{\nu_0}| \geq K_5 > 0,$$

where K_5 depends only on d and $\theta_\triangle$.

Now consider the following C^r smoothness condition across the edge e_2:

$$\alpha_1 \tilde{B}_{\nu_0} = r_2^r \alpha_2 \tilde{B}_{\nu_0} + r r_2^{r-1} t_2 \gamma_2 \tilde{B}_{\nu_0} + \sum_{k=1}^{r-1} \binom{r}{r-k-1} r_2^{r-k-1} t_2^{k+1} \eta_{2,k} \tilde{B}_{\nu_0},$$

where $\eta_{2,k}$ is a linear functional which picks off the B-coefficient corresponding to $\xi^{T_i}_{d-2r,r+k+1,r-k-1}$ for $k = 1, \ldots, r-1$. Since $\alpha_1 \tilde{B}_{\nu_0} = 0$, this implies

$$|\gamma_2 \tilde{B}_{\nu_0}| + \frac{1}{r} \sum_{k=1}^{r-1} \binom{r}{r-k-1} \left| \frac{t_2}{r_2} \right|^k |\eta_{2,k} \tilde{B}_{\nu_0}| \geq \left| \frac{r_2}{t_2} \right| \frac{K_5}{r},$$

which is unbounded as $t_2 \to 0$. On the other hand, since the B-coefficients $\gamma_2 \tilde{B}_{\nu_0}$, $\eta_{2,k} \tilde{B}_{\nu_0}$, $k = 1, \ldots, r-1$, correspond to domain points on the ring

$R_{2r}(v)$, they cannot exceed K_4 in absolute value, which leads to a contradiction and completes the proof. □

The above proof also applies to the superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ in (11.2) whenever there exists a near-singular vertex v with $\rho_v < 2r$. On the other hand, if $d \geq 4r+1$ and $\rho_v \geq 2r$ for *all* vertices, then the basis corresponding to the MDS constructed in Section 11.5 is both stable and locally linearly independent.

11.8. Remarks

Remark 11.1. A collection $\{\phi_i\}_{i=1}^n$ of splines forming a basis for a spline space $\mathcal{S}$ on a triangulation $\triangle$ is said to be a star-supported basis for $\mathcal{S}$ provided that for each $1 \leq i \leq n$, there is a vertex v_i of $\triangle$ such that the support of ϕ_i is contained in $\mathrm{star}(v_i)$. We showed in Chapters 6–8 that for each of the macro-element spaces discussed there, the corresponding $\mathcal{M}$-basis and $\mathcal{N}$-basis are both stable star-supported bases.

Remark 11.2. For $d \geq 3r+2$, the space $\mathcal{S}_d^r(\triangle)$ and the superspline subspaces $\mathcal{S}_d^{r,\rho}(\triangle)$ have star-supported bases. This follows from the fact that the minimal determining sets for these spaces constructed in Theorems 9.15 and 9.17 are local in the sense of Definition 5.16 with a constant $\ell = 1$. However, these bases are not stable in general when $d < 4r+1$.

Remark 11.3. We constructed stable local bases for $\mathcal{S}_d^r(\triangle)$ and the superspline subspaces $\mathcal{S}_d^{r,\rho}(\triangle)$ in Theorem 11.12. However, these bases are not star-supported in general. In the worst case they are star^3-supported. It can shown that for $r \leq 2$, the basis is only star-supported, see [Dav02a]. In fact, the same holds for general $r > 2$ if $d \geq 3r + \lfloor (r+1)/2 \rfloor + 1$. Moreover, for $d = 3r + \lfloor (r+1)/2 \rfloor$ the basis is also star^2-supported. Thus, star^3-supported bases appear in the theorem only for $r \geq 5$.

Remark 11.4. It was shown in [AlfS00] that whenever $d < 3r+2$, the spaces $\mathcal{S}_d^r(\triangle)$ cannot have a star-supported basis. In particular, the space $\mathcal{S}_4^1(\triangle)$ does not have a star-supported basis. It is an open question whether it has a star^ℓ-supported basis for some larger value of ℓ.

Remark 11.5. We know by the results of Section 5.7 that if spline space defined on a triangulation $\triangle$ has a stable local basis, then it approximates smooth functions to order $\mathcal{O}(|\triangle|^{d+1})$, where $|\triangle|$ is the mesh size of $\triangle$. However, by the results of Chapter 10, we know that for $d \leq 3r+1$ the spaces $\mathcal{S}_d^r(\triangle)$ have approximation power at most d. It follows that these spaces cannot have a stable local basis. This includes both of the spaces $\mathcal{S}_3^1(\triangle)$ and $\mathcal{S}_4^1(\triangle)$.

Remark 11.6. Star-supported basis functions are also called vertex splines. This terminology was introduced in [ChuL85]. Vertex splines were studied further in [ChuH90a, ChuL90a, ChuL90b]. The idea is to construct enough vertex splines to build a basis and an associated quasi-interpolant. It was shown in [Boo89] that for arbitrary partitions and $d \geq 3r+2$, the space $\mathcal{S}_d^r(\triangle)$ has a basis of vertex splines.

Remark 11.7. A collection $\{\phi_i\}_{i=1}^n$ of splines forming a basis for a spline space $\mathcal{S}$ on a triangulation $\triangle$ is said to be a minimally supported provided that for each $1 \leq i \leq n$, there is no spline in $\mathcal{S}$ with support on a proper subset of the support of ϕ_i. Minimally supported bases were studied on type-I and type-II partitions in [ChuH88, ChuH90c]. It was shown in [CarP94] that any locally linearly independent basis is minimally supported.

Remark 11.8. The dimension of spline spaces on cells was determined in [Sch79] without explicitly constructing a minimal determining set. The first construction of minimal determining sets for such spline spaces was carried out in [Sch88a], but without regard for stability. The stable minimal determining sets for splines defined on cells described in Section 11.5 comes from [DavS02].

Remark 11.9. In this chapter we have focused on the construction of stable local bases for spline spaces defined on general triangulations. For some results on type-I, type-II, and certain cross-cut triangulations, see [ChuSW83a, ChuSW83b] and [ChuW82a–ChuW84c], respectively. Stability, linear independence, local linear independence, and the question of finding bases for spline spaces are also topics in the theory of box splines.

Remark 11.10. In this chapter we have discussed only bases for spline spaces that are constructed from minimal determining sets. As shown in Section 5.9, it is also possible to construct explicit stable local bases associated with nodal minimal determining sets. The basis constructed in [MorS75] for the spline spaces $\mathcal{S}_d^1(\triangle)$ with $d \geq 5$ are based on nodal functionals, but are not stable. For a construction of stable bases based on nodal functionals, see [DavS00b].

Remark 11.11. In Theorem 11.13 we showed that for $d \geq 3r+2$, there are triangulations such that no basis for $\mathcal{S}_d^r(\triangle)$ is simultaneously stable and locally linearly independent. For a construction of special superspline spaces with $d \geq 4r+1$ that have bases that are simultaneously stable and locally linearly independent, see [Dav02a]. There is an interesting connection between spaces with locally linearly independent bases and certain almost interpolation problems, see [DavSoS97a, DavSoS97b, DavSoS99].

Remark 11.12. In Theorem 10.10 we showed that the space $\mathcal{S}_d^r(\triangle)$ has full approximation power in all q-norms by showing that an especially constructed superspline subspace of $\mathcal{S}_d^r(\triangle)$ has a stable local minimal determining set. For an alternative proof, we can apply Theorem 5.19 to the subspace $\mathcal{S}_d^{r,\mu}(\triangle)$ of Section 11.4, since as shown in Theorem 11.7 it also has a stable local minimal determining set.

11.9. Historical Notes

Explicit local bases were constructed for the spaces $\mathcal{S}_d^1(\triangle)$ for all $d \geq 5$ in [MorS75]. Local bases for $\mathcal{S}_d^r(\triangle)$ with $d \geq 4r+1$ were described in [AlfPS87a, AlfPS87b]. The bases given there are essentially the $\mathcal{M}$-basis of Theorem 5.20, where $\mathcal{M}$ is the minimal determining set $\mathcal{M}$ described in [AlfS87]. These results were extended to $\mathcal{S}_d^r(\triangle)$ for $d \geq 3r+2$ in [Hon91], Local bases for the special superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ with common supersmoothness at all of the vertices, and with $2r \leq \rho$ and $2\rho+1 \leq d$, were constructed in [Sch89]. The more general superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ of (11.2) with variable smoothness at the vertices were treated in [IbrS91] for $d \geq 3r+2$.

The question of stability was not discussed in any of these papers, although for certain special cases, e.g. $r=0$ and for certain superspline spaces with $d \geq 4r+1$, the bases are both stable and locally linearly independent.

The first attempts to construct stable local bases in more general settings seems to have been motivated by a desire to use them as tools for establishing approximation results. In [ChuHoJ95] stable local bases were described for the superspline space $\mathcal{S}_d^{r,\mu}(\triangle)$ for $d \geq 3r+2$, where $\mu = r + \lfloor (r+1)/2 \rfloor$. Stable local bases for the superspline space $\mathcal{S} \subset \mathcal{S}_d^r(\triangle)$ of (10.7) were constructed in [LaiS98] for $d \geq 3r+2$.

The first construction of stable local bases for general superspline spaces and the full spline space $\mathcal{S}_d^r(\triangle)$ for $d \geq 3r+2$ was given in [DavS02], which we have closely followed here. The key idea is to base the construction on a stable local minimal determining set. This idea goes back to [DavS00b]. For more on the computation of stable local bases, including spline spaces on nested triangulations, see [Dav01, Dav02a, Dav02b].

The idea of local linear independence plays an important role in several parts of spline theory, including the theory of box splines and the theory of shift-invariant spaces, see [BooH83b, DahM85c, Jia85]. None of the bases in [AlfPS87a–AlfPS87c, Hon91, IbrS91] are locally linearly independent. Even the stable bases constructed in [ChuHoJ95] and [LaiS98] are not locally linearly independent, see [DavS00a].

Locally linearly independent bases for the spaces $\mathcal{S}_d^1(\triangle)$ for $d \geq 5$ were given in [Dav98]. For $d \geq 3r+2$, locally linearly independent bases for $\mathcal{S}_d^r(\triangle)$ and various superspline subspaces were constructed in [DavS00a]. The surprising results in Section 11.7 showing that no basis for $\mathcal{S}_d^r(\triangle)$ can be simultaneously stable and locally linearly independent is taken from [DavS00b].

Bivariate Box Splines

Our aim in this chapter is to show how to construct analogs of the classical univariate and tensor-product B-splines on type-I and type-II partitions, and to give a glimpse into the general theory of box splines. We also discuss certain classes of box-spline-like functions and their associated finite shift-invariant spaces.

12.1. Type-I Box Splines

Throughout this section we suppose that $\triangle_I$ is the uniform type-I triangulation associated with a bi-infinite grid with grid lines at the integers. This triangulation has vertices at all lattice points (i, j) in $\mathbb{Z}^2$. We denote points in the plane either by v or (x, y). We use the symbol μ for a multi-index, i.e., $\mu = (i, j)$. Our aim is to construct splines B in $\mathcal{S}_d^r(\triangle_I)$ with the following properties:

1) B has small support (the union of a few triangles).
2) B is positive in the interior of its support.
3) $\mathcal{S} :=$ span $\{B(v - \mu)\}_{\mu \in \mathbb{Z}^2}$ contains polynomials of degree d.
4) $\sum_{\mu \in \mathbb{Z}^2} B(v - \mu) \equiv 1$, all $v \in \mathbb{R}^2$.
5) The space $\mathcal{S}$ is capable of approximating smooth functions on compact subsets of $\mathbb{R}^2$.

To get started, let B_{111} be the classical Courant hat function with support on the star of the vertex $(1, 1)$. More precisely, let B_{111} be the spline in $\mathcal{S} := \mathcal{S}_1^0(\triangle_I)$ satisfying $B_{111}(1, 1) = 1$ and $B_{111}(i, j) = 0$ for all other vertices of $\triangle_I$, see Figure 12.1. It is easy to see that B_{111} has properties 1)–5).

To define higher degree box splines on $\triangle_I$, we need the notion of a direction set. Let

$$e^1 := (1, 0), \quad e^2 := (0, 1), \quad e^3 := (1, 1).$$

Given $n \geq 3$, we call $X_n := \{v_1, \cdots, v_n\}$ a type-I direction set provided $v_1, v_2, v_3 \in \{e^1, e^2, e^3\}$ and X_n contains each of e^1, e^2, e^3 at least once. Without loss of generality, we may assume that $v_i = e^i$ for $i = 1, 2, 3$.

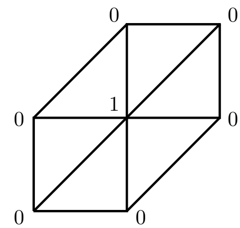


Fig. 12.1. The B-coefficients of B_{111}.

Definition 12.1. *Let $X_n := \{v_1, \ldots, v_n\}$ be a type-I direction set with $n > 3$, and let $X_i := \{v_1, \ldots, v_i\}$ for $i = 3, \ldots, n$. Then for $4 \le i \le n$, we define the associated* type-I box splines *$B(v|X_i)$ recursively as*

$$B(v|X_i) := \int_0^1 B(v - tv_i|X_{i-1})\, dt, \tag{12.1}$$

where $B(v|X_3)$ is the box spline B_{111} defined above.

It follows from Theorem 12.6 below that the splines $B(v|X_i)$ in Definition 12.1 do not depend on the order in which the directions appear in X_n. We use the simplified notation $B_{jkl}(v)$ for the type-I box spline $B(v|X_n)$ associated with a direction set X_n containing e^1 a total of j times, e^2 a total of k times, and e^3 a total of l times. Throughout this chapter we use the two notations interchangeably. The following result is an immediate consequence of (12.1).

Theorem 12.2. *Let X_n be type-I direction set. Then the associated box spline $B(v|X_n)$ has support on the closure of the set*

$$[X_n] := \Big\{ \sum_{j=1}^{n} t_j v_j \,:\, 0 \le t_j < 1, \quad j = 1, \cdots, n \Big\}.$$

Moreover, $B(v|X_n) > 0$ for all v in the interior of $[X_n]$.

The set $[X_n]$ appearing in Theorem 12.2 is called the affine cube associated with X_n. The supports of B_{jkl} for several choices of j, k, l are shown in Figure 12.2. As an aid to establishing additional properties of box splines, we now prove an elementary lemma on derivatives of box splines. Given a nontrivial vector $u = (u_1, u_2) \in \mathbb{R}^2$, let D_u be the corresponding directional derivative, and let Δ_u, ∇_u be the forward and backward difference operators defined by $\Delta_u f(\cdot) := f(\cdot + u) - f(\cdot)$ and $\nabla_u f(\cdot) = f(\cdot) - f(\cdot - u)$. In general, if $Y \subset \mathbb{R}^2$ is a finite set of nonzero vectors, we write

$$D_Y := \prod_{u \in Y} D_u, \quad \Delta_Y := \prod_{u \in Y} \Delta_u, \quad \nabla_Y := \prod_{u \in Y} \nabla_u.$$

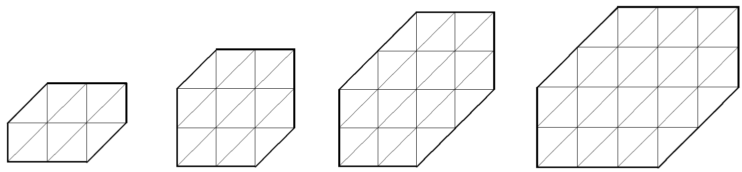


Fig. 12.2. The supports of B_{211}, B_{221}, B_{222}, and B_{322}.

Lemma 12.3. *Let X_n be a type-I direction set with $n \geq 4$. Then for any $4 \leq j \leq n$,*

$$D_{v_j} B(\cdot|X_n) = \nabla_{v_j} B(\cdot|X_n \backslash \{v_j\}) .$$

Proof: Let $X_{n-1} := X_n \backslash \{v_j\}$. Then by definition,

$$\begin{aligned} D_{v_j} B(v|X_n) &= D_{v_j} \int_0^1 B(v - tv_j|X_{n-1})\, dt \\ &= -\int_0^1 \frac{\partial}{\partial t} B(v - tv_j|X_{n-1})\, dt \\ &= -B(v - tv_j|X_{n-1})\Big|_0^1 = \nabla_{v_j} B(v|X_{n-1}). \quad \square \end{aligned}$$

We are ready to describe the structural and smoothness properties of type-I box splines.

Theorem 12.4. *Let X_n be a type-I direction set, where e^1, e^2, e^3 appear j, k, l times, respectively, with $j + k + l = n$. Then $B_{jkl} := B(\cdot|X_n) \in \mathcal{S}_{n-2}^r(\triangle_I)$, where $r := r(X_n) := \min\{j + l, j + k, k + l\} - 2$.*

Proof: By definition $B_{111} \in \mathcal{S}_1^0(\triangle_I)$, which proves the assertion for $n = 3$. Now let $Y := \{v_4, \cdots, v_n\}$. Then by Lemma 12.3,

$$D_Y B(v|X_n) = \nabla_Y B(v|X_n \backslash Y) = \nabla_Y B_{111}(v).$$

This is a piecewise linear function defined on the partition $\triangle_I$, and it follows that $B(v|X_n)$ is a piecewise polynomial function of degree $n-2$ on the same partition.

To check the smoothness of B_{jkl} on $\triangle_I$, let $r := \min\{j + l, j + k, k + l\} - 2$, and note that $D_x = D_{e^1} = D_{e^3} - D_{e^2}$ and $D_y = D_{e^2} = D_{e^3 - e^1}$. For any $\alpha \leq j - 1$, by Lemma 12.3,

$$D_x^\alpha B_{jkl}(v) = (\nabla_{e^1})^\alpha B_{j-\alpha,k,l}(v).$$

For $j \le \alpha \le r$, we have

$$D_x^\alpha B_{jkl}(v) = (D_{e^1})^{j-1}(D_{e^3} - D_{e^2})^{\alpha-j+1} B_{jkl}(v) = \sum_{\nu=0}^{\alpha-j+1} \binom{\alpha-j+1}{\nu} \times (\nabla_{e^1})^{j-1}(-\nabla_{e^2})^{\nu}(\nabla_{e^3})^{\alpha-j+1-\nu} B_{1,k-\nu,l-(\alpha-j+1-\nu)}(v).$$

Now $B_{1,k-\nu,l-\alpha+j-1+\nu}$ is a continuous function since $k-\nu \ge k-\alpha+j-1 \ge k+j-r-1 \ge 1$ and $l-(\alpha-j+1-\nu) \ge l+j-\alpha-1 \ge 1$, and hence $D_x^\alpha B_{jkl} \in C(\mathbb{R}^2)$ for $0 \le \alpha \le r$.

Similarly, $D_y^\beta B_{jkl} \in C(\mathbb{R}^2)$ for $0 \le \beta \le r$. For $\alpha+\beta \le r$, the above arguments lead to

$$D_x^\alpha D_y^\beta B_{jkl}(v) = \sum_{\nu=0}^{\alpha-j+1} \binom{\alpha-j+1}{\nu} (\nabla_{e^1})^{j-1}(-\nabla_{e^2})^{\nu+\beta} \times (\nabla_{e^3})^{\alpha-j+1-\nu} B_{1,k-\nu-\beta,l+j-\alpha-1+\nu}(v).$$

Since $k-\nu-\beta \ge k-(\alpha-j+1)-\beta = k+j-\alpha-\beta-1 \ge 1$, $D_x^\alpha D_y^\beta B_{jkl} \in C(\mathbb{R}^2)$ for $\alpha+\beta \le r$, and we conclude that $B_{jkl} \in C^r(\mathbb{R}^2)$. $\square$

This result shows that $B_{111} \in S_1^0(\triangle_I)$, $B_{221} \in S_3^1(\triangle_I)$, $B_{222} \in S_4^2(\triangle_I)$, $B_{322} \in S_5^2(\triangle_I)$, $B_{332} \in S_6^3(\triangle_I)$, and $B_{333} \in S_7^4(\triangle_I)$. We next establish a simple integral identity which will be used below to derive some additional properties of box splines.

Theorem 12.5. *For all $f \in C(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} B(v|X_n) f(v)\,dv = \int_{[0,1]^n} f\Big(\sum_{i=1}^n t_i v_i\Big)\,dt_1 \cdots dt_n. \tag{12.2}$$

Proof: We proceed by induction. For $n=3$, it is easy to see that the B-coefficients of $D_{e^3}B(\cdot|X_3)$ as a piecewise constant function are as shown in Figure 12.3. In fact, in terms of the characteristic function $\chi_{[0,1]^2}$ of $[0,1]^2 := [0,1]\times[0,1]$, we have $D_{e^3}B(v|X_3) = \chi_{[0,1]^2}(v) - \chi_{[0,1]^2}(v-e^3)$. Thus, for any $f \in C^1(\mathbb{R}^2)$, integration by parts gives

$$\begin{aligned}
\int_{\mathbb{R}^2} B(v|X_3) D_{e^3} f(v)\,dv &= -\int_{\mathbb{R}^2} D_{e^3} B(v|X_3) f(v)\,dv \\
&= -\int_{[0,1]^2} f(v)dv + \int_{[0,1]^2} f(v+e^3)\,dv \\
&= \int_{[0,1]^3} D_t f(v+te^3)\,dt\,dv \\
&= \int_{[0,1]^3} D_{e^3} f(t_1e^1 + t_2e^2 + t_3e^3)\,dt_1\,dt_2\,dt_3.
\end{aligned}$$

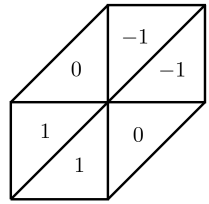


Fig. 12.3. The B-coefficients of $D_{e^3} B_{111}$.

Thus, (12.2) follows for $n = 3$. Now let $n \geq 4$. Then using the definition of $B(\cdot|X_n)$ and the induction hypothesis consecutively, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} B(v|X_n) f(v) dv &= \int_{\mathbb{R}^2} \int_0^1 B(v - tv_n | v_1, \ldots, v_{n-1}) f(v) \, dt \, dv \\ &= \int_0^1 \int_{\mathbb{R}^2} B(v|v_1, \cdots, v_{n-1}) f(v + tv_n) \, dv \, dt \\ &= \int_0^1 \int_{[0,1]^{n-1}} f\Big(\sum_{i=1}^{n-1} t_i v_i + tv_n\Big) dt_1 \cdots dt_{n-1} \, dt \\ &= \int_{[0,1]^n} f\Big(\sum_{i=1}^{n} t_i v_i\Big) dt_1 \cdots dt_n \, .\end{aligned}$$

This gives (12.2) for $f \in C^1(\mathbb{R}^2)$. It also holds for all $f \in C(\mathbb{R}^2)$ by the fact that $C^1(\mathbb{R}^2)$ is dense in $C(\mathbb{R}^2)$. $\square$

Inserting $f(v) = \exp(-iv \cdot \omega)$ with $\omega = (\omega_1, \omega_2)$ and $i := \sqrt{-1}$ in Theorem 12.5 leads immediately to the following theorem.

Theorem 12.6. *The* Fourier transform *of* $B(\cdot|X_n)$ *is*

$$\widehat{B}(\cdot|X_n)(\omega) = \prod_{j=1}^{n} \frac{1 - e^{-i\omega \cdot v_j}}{i\omega \cdot v_j}.$$

This formula for the Fourier transform of a type-I box spline is the direct analog of a similar formula for univariate B-splines. Theorem 12.6 immediately implies that $B(\cdot|X_n)$ is independent of the order of the vectors $v_1, \cdots, v_n$ in the direction set X_n. We can now prove the following Green's formula for box splines.

Theorem 12.7. *Suppose* $n \geq 3$. *Then, for any* $f \in C^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} B(v|X_n) D_{v_i} f(v) \, dv = - \int_{\mathbb{R}^2} D_{v_i} B(v|X_n) f(v) \, dv, \quad i = 1, \ldots, n.$$

Proof: Since $D_{v_i} f \in C(\mathbb{R}^2)$, Theorem 12.5 implies

$$\int_{\mathbb{R}^2} B(v|X_n) D_{v_i} f(v)\, dv = \int_{[0,1]^n} D_{v_i} f\Big(\sum_{j=1}^n t_j v_j\Big) dt_1 \cdots dt_n.$$

Observing that

$$D_{v_i} f\big(\sum_{j=1}^n t_j v_j\big) = \frac{\partial}{\partial t_i} f\big(\sum_{j=1}^n t_j v_j\big),$$

we obtain

$$\begin{aligned}
&\int_{[0,1]^n} D_{v_i} f\Big(\sum_{j=1}^n t_j v_j\Big) dt_1 \cdots dt_n \\
&\qquad = \int_{[0,1]^n} \frac{\partial}{\partial t_i} f\Big(\sum_{j=1}^n t_j v_j\Big) dt_1 \cdots dt_n \\
&\qquad = \int_{[0,1]^{n-1}} f\Big(\sum_{j=1}^n t_j v_j\Big)\Big|_{t_i=0}^{t_i=1} dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_n \\
&\qquad = \int_{\mathbb{R}^2} B(v|X_n \backslash \{v_i\}) \Delta_{v_i} f(v)\, dv \\
&\qquad = -\int_{\mathbb{R}^2} \nabla_{v_i} B(v|X_n \backslash \{v_i\}) f(v)\, dv \\
&\qquad = -\int_{\mathbb{R}^2} D_{v_i} B(v|X_n) f(v)\, dv
\end{aligned}$$

by Theorems 12.5 and 12.3. $\square$

We can now establish a Peano formula for box splines.

Theorem 12.8. *For any $f \in C^n(\mathbb{R}^2)$,*

$$\Delta_{X_n} f(0,0) = \int_{\mathbb{R}^2} B(v|X_n) D_{X_n} f(v)\, dv.$$

Proof: By Theorem 12.5,

$$\begin{aligned}
\int_{\mathbb{R}^2} B(v|X_n) D_{X_n} f(v)\, dv &= \int_{[0,1]^n} \prod_{i=1}^n D_{v_i} f\big(\sum_{j=1}^n t_j v_j\big)\, dt_1 \cdots dt_n \\
&= \int_{[0,1]^n} \prod_{i=1}^n \frac{\partial}{\partial t_i} f\big(\sum_{j=1}^n t_j v_j\big)\, dt_1 \cdots dt_n \\
&= \prod_{i=1}^n \Delta_{v_i} f(0,0) = \Delta_{X_n} f(0,0). \quad \square
\end{aligned}$$

We now describe the so-called refinement equation for box splines.

Theorem 12.9. *There exists a finite sequence $\{a_\nu\}_{\nu\in\mathbb{Z}^2}$ such that*

$$B(v|X_n) = \sum_{\nu\in\mathbb{Z}^2} a_\nu B(2v-\nu|X_n). \tag{12.3}$$

Proof: Using the Fourier transform of $B(\cdot|X_n)$ given in Theorem 12.6, we have

$$\begin{aligned}\widehat{B}(\cdot|X_n)(\omega) &= \prod_{j=1}^{n} \frac{1+e^{i\omega\cdot v_j/2}}{2}\;\frac{1-e^{i\omega\cdot v_j/2}}{i\omega\cdot v_j/2}\\ &= \prod_{j=1}^{n} \frac{1+e^{i\omega\cdot v_j/2}}{2}\widehat{B}(\cdot|X_n)(\omega/2),\end{aligned}$$

which is equivalent to (12.3). $\square$

We now show how to find the B-coefficients of an arbitrary type-I box spline. This will be useful as a means for evaluating box splines, since once we have the B-coefficients, we can apply the de Casteljau algorithm of Section 2.5. We already gave the B-coefficients of B_{111} in Figure 12.1. To compute the B-coefficients of B_{jkl} for higher degree type-I box splines, we note that any v_n in the direction set of B_{jkl} is one of the vectors e^1, e^2, and e^3. Thus, $D_{v_n}B(v|X_n)$ restricted to any triangle of $\triangle_I$ is a directional derivative along one side of the triangle. In this case, the derivative can be computed easily. Let

$$p_{n-2}(v) := \sum_{i+j+k=n-2} c_{ijk}B^{n-2}_{ijk}(v)$$

be the restriction of $B(\cdot|X_n)$ to the triangle T_0 with vertices (0,0), (1,0), and (1,1), where B^{n-2}_{ijk} are the Bernstein basis polynomials of degree $n-2$ with respect to T_0. Then

$$D_{e^1}p_{n-2}(v) = (n-2)\sum_{i+j+k=n-3}(c_{i,j+1,k}-c_{i+1,j,k})B^{n-3}_{ijk}(v),$$

$$D_{e^2}p_{n-2}(v) = (n-2)\sum_{i+j+k=n-3}(c_{i,j,k+1}-c_{i,j+1,k})B^{n-3}_{ijk}(v),$$

$$D_{e^3}p_{n-2}(v) = (n-2)\sum_{i+j+k=n-3}(c_{i,j,k+1}-c_{i+1,j,k})B^{n-3}_{ijk}(v).$$

On the other hand, by Theorem 12.3, $D_{v_n}B(v|X_n) = \Delta_{v_n}B(v|X_n\backslash\{v_n\})$. Suppose that all B-coefficients of $B(\cdot|X_n\backslash\{v_n\})$ are known. Then all B-coefficients of $\Delta_{v_n}B(\cdot|X_n\backslash\{v_n\})$ are also known. Hence, the B-coefficients

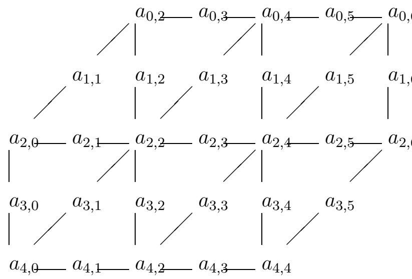


Fig. 12.4. The B-coefficients of B_{211}.

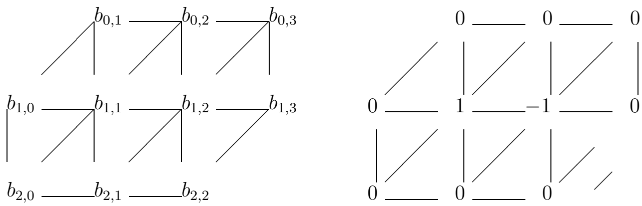


Fig. 12.5. The B-coefficients of $D_{e^1} B_{211}$.

of $B(\cdot|X_n)$ can be obtained by solving the first order difference equations with the initial condition that the B-coefficients of the box spline on the boundary of its support are zero. Thus, starting with the B-coefficients of B_{111}, we obtain the B-coefficients of B_{211}. It follows that for any $j, k, l \geq 1$, all of the B-coefficients of the box spline B_{jkl} can be computed recursively. We illustrate this computational procedure with an example.

Example 12.10. Let $X_4 := \{e^1, e^1, e^2, e^3\}$ and $X_3 := \{e^1, e^2, e^3\}$.

Discussion: In this case $B(\cdot|X_4) = B_{211}$. The B-coefficients of $D_{e^1} B_{211} = B(\cdot|X_3) - B(\cdot - e^1|X_3)$ can be easily found from the B-coefficients of $B_{111}(\cdot) = B(\cdot|X_3)$. Suppose we denote the B-coefficients of B_{211} by a_{ij} as in Figure 12.4, and those of $D_{e^1} B_{211}$ by b_{ij} as in Figure 12.5. The latter must be equal to the B-coefficients of $B(\cdot|X_3) - B(\cdot - e^1|X_3)$, i.e.,

$$
\begin{aligned}
b_{01} &= a_{12} - a_{11} = a_{03} - a_{02}, \\
b_{02} &= a_{04} - a_{03} = a_{14} - a_{13}, \\
b_{03} &= a_{06} - a_{05} = a_{16} - a_{15}, \\
b_{10} &= a_{21} - a_{20}, \\
b_{11} &= a_{22} - a_{21} = a_{32} - a_{31} = a_{23} - a_{22} = a_{13} - a_{12},
\end{aligned}
$$

$$
\begin{aligned}
b_{12} &= a_{24} - a_{23} = a_{15} - a_{14} = a_{25} - a_{24} = a_{34} - a_{33}, \\
b_{13} &= a_{26} - a_{25}, \\
b_{20} &= a_{31} - a_{30} = a_{41} - a_{40}, \\
b_{21} &= a_{42} - a_{41} = a_{33} - a_{32}, \\
b_{22} &= a_{44} - a_{43} = a_{35} - a_{34}.
\end{aligned}
$$

By equating these B-coefficients and recalling the fact that B_{211} has the support shown in Figure 12.4, we see that $a_{0i} = 0$ for $i = 1, \ldots, 6$, $a_{i0} = 0$ for $i = 2, 3, 4$, $a_{11} = a_{4i} = 0$, for $i = 1, 2, 3, 4$, and $a_{16} = a_{26} = a_{35} = 0$. But then we can immediately compute all of the remaining coefficients a_{ij} as shown in Figure 12.6. $\square$

The same procedure can be used to find the B-coefficients for the box splines B_{221} and B_{222}, see Figure 12.7.

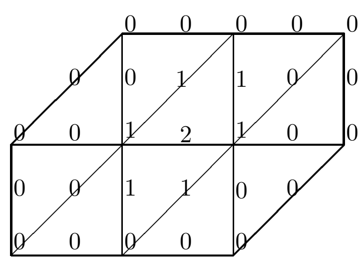


Fig. 12.6. The B-coefficients of $2B_{211}$.

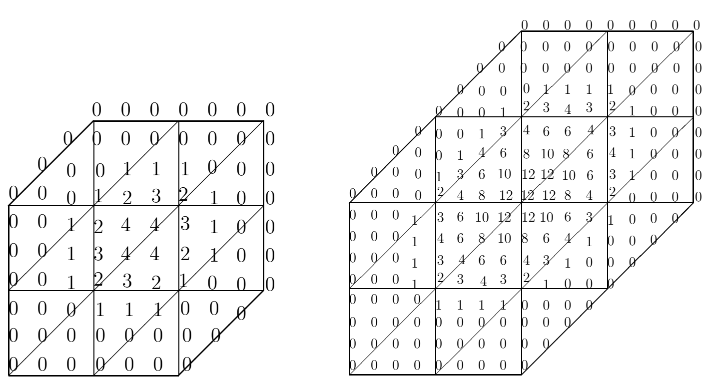


Fig. 12.7. Coefficients of the box splines $6B_{221}$ and $24B_{222}$.

We can now give a formula for the inner product of two box splines.

Theorem 12.11. *For any $j,k,l \geq 1$ and $\mu,\nu,\kappa \geq 1$,*

$$\int_{\mathbb{R}^2} B_{jkl}\, B_{\nu\mu\kappa}\, dv = B_{j+\nu,k+\mu,l+\kappa}\,(j+l,k+l).$$

Proof: By Parseval's identity, and the inverse Fourier transform,

$$\begin{aligned}\int_{\mathbb{R}^2} B_{jkl}\, B_{\nu\mu\kappa}\, dv &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{B}_{jkl}(\omega)\overline{\widehat{B}_{\nu\mu\kappa}(\omega)}\, d\omega \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{B}_{j+\nu,k+\mu,l+\kappa}(\omega)e^{-i\omega\cdot(j+l,k+l)}\, d\omega \\ &= B_{j+\nu,k+\mu,l+\kappa}(j+l,k+l). \quad \square\end{aligned}$$

As an example of Theorem 12.11, we note that

$$\int_{\mathbb{R}^2} B_{111}(x,y)B_{111}(x,y)\, dx\, dy = B_{222}(2,2) = \frac{1}{2}.$$

12.2. Type-II Box Splines

Throughout this section we suppose that $\triangle_{\mathrm{II}}$ is the uniform type-II triangulation associated with a bi-infinite grid with grid lines at the integers. This triangulation has vertices at all lattice points of the form $(i/2, j/2)$. On this triangulation, our basic building block will be the C^1 quadratic spline B_{1111} with support and B-coefficients as shown in Figure 12.8, where the vertex at the lower left corner is at (0,0). This spline belongs to $\mathcal{S}_2^1(\triangle_{\mathrm{II}})$.

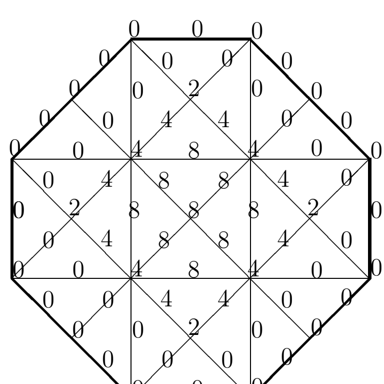


Fig. 12.8. Coefficients of $16B_{1111}$.

Now suppose $X_n := \{v_1, \ldots, v_n\}$, where $v_i \in \{e^1, e^2, e^3, e^4\}$ and $e^4 := (-1, 1)$ for $n \geq 4$. Without loss of generality we may assume $v_i = e^i$ for $i = 1, 2, 3, 4$. We call X_n a type-II direction set. We have the following analog of Definition 12.1, where $B(v|X_4) = B_{1111}$.

Definition 12.12. *Let $X_n := \{v_1, \ldots, v_n\}$ be a type-II direction set with $n > 4$, and let $X_i := \{v_1, \ldots, v_i\}$ for $i = 4, \ldots, n$. Then we define the associated* type-II box splines *$B(v|X_i)$ recursively as*

$$B(v|X_i) := \int_0^1 B(v - tv_i|X_{i-1})\, dt, \qquad i = 5, \ldots, n. \tag{12.4}$$

We write B_{jklm} for this spline, where j, k, l, m are the number of times that X_n contains e^1, e^2, e^3, e^4, respectively.

We now show that the type-II box splines have similar properties to the type-I box splines treated in the previous section. It follows directly from the definition that type-II box splines have support on the affine cube $[X_n]$ defined in Theorem 12.2, and are positive on the interior of $[X_n]$. Moreover, if X_n is a type-II direction set with $n \geq 5$, then the derivative formula of Lemma 12.3 holds. This leads immediately to the following analog of Theorem 12.4.

Theorem 12.13. *Let X_n be a type-II direction set where e^1, e^2, e^3, e^4 appear j, k, l, m times, respectively, with $j+k+l+m = n$. Then $B(\cdot|X_n) \in \mathcal{S}^r_{n-2}(\triangle_I)$, where $r := r(X_n) := \min\{j+l+m, k+l+m, j+k+l, j+k+m\} - 2$.*

Proof: By definition, $B(\cdot|X_4) \equiv B_{1111} \in \mathcal{S}^1_2(\triangle_{II})$. Let $Y := \{v_5, \ldots, v_n\}$. Then by Lemma 12.3,

$$D_Y B(v|X_n) = \nabla_Y B(v|X_n \backslash Y) = \nabla_Y B(v|v_1, v_2, v_3, v_4).$$

This is a piecewise linear function defined on the partition $\triangle_{II}$, and it follows that $B(v|X_n)$ is a piecewise polynomial function of degree $n-2$ on the same partition. The smoothness follows as in the proof of Theorem 12.4. $\square$

We now give the analog of Theorem 12.5 for type-II box splines.

Theorem 12.14. *For all $f \in C(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} B(v|X_n) f(v) dv = \int_{[0,1]^n} f\Big(\sum_{i=1}^n t_i v_i\Big) dt_1 \cdots dt_n. \tag{12.5}$$

Proof: We can write

$$D_{e^4} D_{e^3} B(v|X_4) = \chi_{[0,1]^2}(v) - \chi_{[0,1]^2}(v - e^3) - \chi_{[0,1]^2}(v - e^4) \\ + \chi_{[0,1]^2}(v - e^3 + e^4).$$

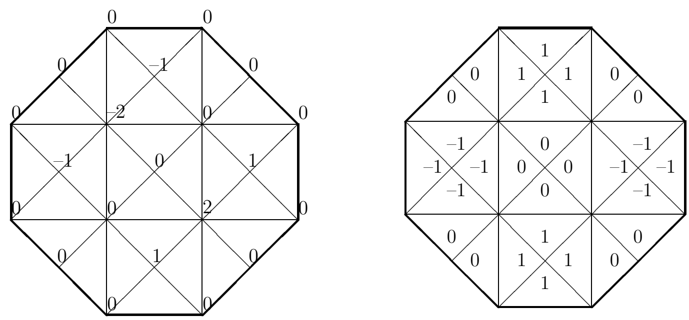


Fig. 12.9. The B-coefficients of $4D_{e^4}B_{1111}$ and $4D_{e^3}D_{e^4}B_{1111}$.

This gives the B-coefficients of $D_{e^4}B(v|X_4)$ and $D_{e^3}D_{e^4}B(v|X_4)$ shown in Figure 12.9. Now for any $f \in C^2(\mathbb{R}^2)$,

$$
\begin{aligned}
\int_{\mathbb{R}^2} & B(v|X_4) D_{e^4} D_{e^3} f(v)\, dv \\
&= \int_{\mathbb{R}^2} D_{e^4} D_{e^3} B(v|X_4) f(v)\, dv \\
&= \int_{[0,1]^2} f(v)\, dv - \int_{[0,1]^2} f(v+e^3)\, dv - \int_{[0,1]^2} f(v+e^4)\, dv \\
&\qquad + \int_{[0,1]^2} f(v+e^3+e^4)\, dv \\
&= \int_{[0,1]^3} D_{e^3} f(v+te^3)\, dt\, dv - \int_{[0,1]^3} D_{e^3} f(v+te^3+e^4)\, dt\, dv \\
&= \int_{[0,1]^4} D_{e^4} D_{e^3} f(v+te^3+se^4)\, ds\, dt\, dv \\
&= \int_{[0,1]^4} D_{e^4} D_{e^3} f(t_1e^1+t_2e^2+t_3e^3+t_4e^4)\, dt_1\, dt_2\, dt_3\, dt_4.
\end{aligned}
$$

This establishes (12.5) for $n = 4$ for all $f \in C^2(\mathbb{R}^2)$. The result then follows for all $f \in C(\mathbb{R}^2)$ by denseness. The proof for $n > 4$ follows by induction. □

Theorem 12.14 implies immediately that Theorem 12.6 on the Fourier transform also holds for type-II box splines. It then follows that the splines $B(v|X_i)$ in Definition 12.12 do not depend on the order in which the directions appear in X_n. This justifies our use of the notation B_{jklm} for the box spline corresponding to the direction set X_n containing e^1, e^2, e^3, e^4 a total of j, k, l, m times, respectively.

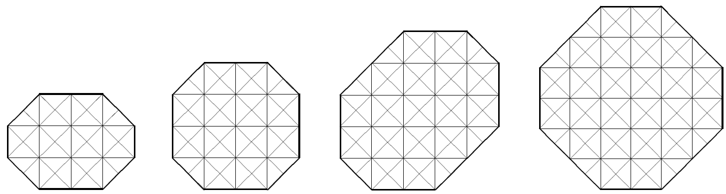


Fig. 12.10. The supports of B_{2111}, B_{2211}, B_{2221}, and B_{2222}.

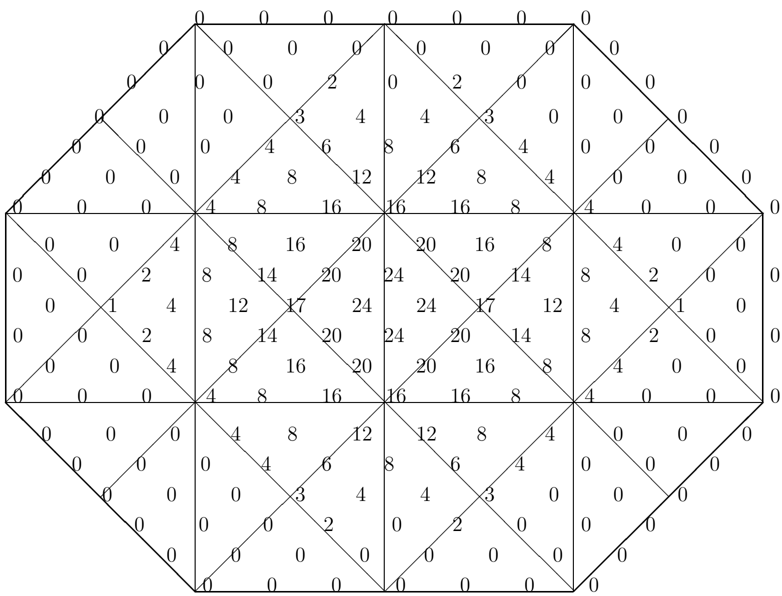


Fig. 12.11. Coefficients of the box spline $48B_{2111}$.

It follows from the above theorem that $B_{1111} \in S_2^1(\triangle_{II})$, $B_{2111} \in S_3^1(\triangle_{II})$, $B_{2211} \in S_4^2(\triangle_{II})$, $B_{2221} \in S_5^3(\triangle_{II})$, and $B_{2222} \in S_6^4(\triangle_{II})$. The supports of some of these type-II box splines are shown in Figure 12.10.

It is straightforward to establish the analogs of Theorems 12.7, 12.8, 12.9, and 12.11 giving a Green's formula, a Peano formula, a refinement equation, and a formula for inner products of type-II box splines. We conclude this section by pointing out that the B-coefficients of type-II box splines can also be computed recursively along the same lines as was done for type-I box splines. For an example, see Figure 12.11.

12.3. Box Spline Series

In this section we discuss box spline series, i.e., series of the form

$$s(\cdot) := \sum_{j\in\mathbb{Z}^2} c_j B(\cdot - j|X_n).$$

Let

$$\mathcal{S}(X_n) := \Big\{ \sum_{j\in\mathbb{Z}^2} c_j B(\cdot - j|X_n) : \, c_j \in \mathbb{R}, \text{ all } j \in \mathbb{Z}^2 \Big\}.$$

Clearly, $\mathcal{S}(X_n)$ is a linear space of splines, but is not finite dimensional. In this and the following sections, we explore various questions related to the space $\mathcal{S}(X_n)$, including whether $\{B(\cdot - j|X_n)\}_{j\in\mathbb{Z}^2}$ is a basis, which polynomials lie in $\mathcal{S}(X_n)$, and how well the scaled versions of $\mathcal{S}(X_n)$ can approximate smooth functions.

Theorem 12.15. *Suppose X_n is either a type-I or type-II direction set. Then the corresponding box splines form a partition of unity on $\mathbb{R}^2$, i.e., $\sum_{j\in\mathbb{Z}^2} B(\cdot - j|X_n) \equiv 1$.*

Proof: We first consider type-I box splines. For $n = 3$, the result follows from the definition. We now proceed by induction. Assume the result is true for $n \geq 3$. Then by the definition of the box splines,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^2} B(\cdot - j|X_{n+1}) &= \sum_{j\in\mathbb{Z}^2} \int_0^1 B(\cdot - tv^{(n+1)} - j|X_n)\, dt \\
&= \int_0^1 \sum_{j\in\mathbb{Z}^2} B(\cdot - tv^{(n+1)} - j|X_n)\, dt \\
&= \int_0^1 dt = 1,
\end{aligned}$$

where $X_n := X_{n+1} \setminus \{v_{n+1}\}$. The proof is similar for type-II box splines. $\square$

For type-II box splines, we have the following related result.

Theorem 12.16. *Let X_n be a type-II direction set. Then*

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2} B(\cdot - j_1 e^3 - j_2 e^4|X_n) = \frac{1}{2}. \tag{12.6}$$

Proof: Note that $\mathbb{R}^2$ is the essentially disjoint union of the sets

$$\{(t_1 - j_1)e^3 + (t_2 - j_2)e^4 : 0 < t_1, t_2 \leq 1\}, \qquad (j_1, j_2) \in \mathbb{Z}^2.$$

For any $f \in C(\mathbb{R}^2)$ with compact support,

$$\begin{aligned}
\int_{\mathbb{R}^2} \sum_{j\in\mathbb{Z}^2} B(v - j_1 e^3 - j_2 e^4 | X_n) f(v) dv \\
&= \sum_{j\in\mathbb{Z}^2} \int_{\mathbb{R}^2} B(v - j_1 e^3 - j_2 e^4 | X_n) f(v) dv \\
&= \sum_{j\in\mathbb{Z}^2} \int_{\mathbb{R}^2} B(v|X_n) f(v + j_1 e^3 + j_2 e^4) dv \\
&= \sum_{j\in\mathbb{Z}^2} \int_{[0,1]^n} f\Big(\sum_{i=1}^n t_i v_i + j_1 e^3 + j_2 e^4\Big) dt_1 \cdots dt_n \\
&= \int_{[0,1]^{n-2}} \int_{\mathbb{R}^2} f\Big(\sum_{\substack{i=1 \\ i\neq 3, i\neq 4}}^n t_i v_i + \lambda_1 e^3 + \lambda_2 e^4 \Big) d\lambda_1\, d\lambda_2\, \widehat{dt},
\end{aligned}$$

where $\widehat{dt}$ is obtained from $dt_1 \cdots dt_n$ by dropping dt_3 and dt_4. The change of variables $\lambda = (\lambda_1, \lambda_2) \mapsto \lambda_1 e^3 + \lambda_2 e^4$ leads to

$$\begin{aligned}
\int_{[0,1]^{n-2}} \int_{\mathbb{R}^2} \frac{1}{2} f\Big(\sum_{\substack{i=1 \\ i\neq 3, i\neq 4}}^n t_i v_i + v\Big)\, dx\, dy\, \widehat{dt} \\
&= \frac{1}{2} \int_{[0,1]^{n-2}} \int_{\mathbb{R}^2} f(v)\, dx\, dy\, \widehat{dt} \\
&= \frac{1}{2} \int_{\mathbb{R}^2} f(v)\, dv,
\end{aligned}$$

which implies (12.6). $\square$

We say that the shifted box splines $\{B(\cdot - \nu|X_n)\}_{\nu\in\mathbb{Z}^2}$ are linearly independent if

$$\sum_{\nu\in\mathbb{Z}^2} a_\nu B(v - \nu|X_n) \equiv 0, \qquad \text{all } v \in \mathbb{R}^2$$

implies that $a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$.

Theorem 12.17. *If $\sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \equiv 0$ with $\sum_{\nu\in\mathbb{Z}^2} |a_\nu| < \infty$, then $a_\nu = 0$, all $\nu \in \mathbb{Z}^2$.*

Proof: Suppose

$$\phi := \sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \equiv 0.$$

Then clearly the Fourier transform of ϕ is also zero. By assumption, $\sum_{\nu\in\mathbb{Z}^2} |a_\nu| < \infty$, and so we can interchange the integration and summation to get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^2} \sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \exp(-iv\cdot\omega)\, dv \\
&= \sum_{\nu\in\mathbb{Z}^2} a_\nu \int_{\mathbb{R}^2} B(v - \nu|X_n) \exp(-iv\cdot\omega)\, dv \\
&= \sum_{\nu\in\mathbb{Z}^2} a_\nu \exp(-i(\nu\cdot\omega)) \widehat{B}(\cdot|X_n)(\omega).
\end{aligned}$$

Hence,

$$\sum_{\nu\in\mathbb{Z}^2} a_\nu \exp(-i\nu\cdot\omega)) = 0, \qquad \text{all } \omega \in \mathbb{R}^2,$$

which implies that $a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$. $\square$

Theorem 12.18. *Suppose that $\sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \equiv 0$, where $a_\nu = p(\nu)$ for some bivariate polynomial p. Then $a_\nu = 0$, for all $\nu \in \mathbb{Z}^2$.*

Proof: Given p, there exists an integer M such that p is of one sign, say $p(v) > 0$, for all $\|v\|_2 \geq M$ and (x, y) in the first quadrant. But then for v large enough, say, $\|v\| \geq M + n/2$,

$$0 = \sum_{\nu\in\mathbb{Z}^2} p(\nu) B(v - \nu|X_n) > 0,$$

since $B(v - \nu|X_n) \geq 0$ and at least one of the terms is positive. This contradiction implies that $p \equiv 0$, and thus $a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$. $\square$

We now show that the integer translates of type-I box splines are linearly independent.

Theorem 12.19. *Let X_n be a type-I direction set. Then the corresponding box splines $\{B(\cdot - \nu|X_n)\}_{\nu\in\mathbb{Z}^2}$ are linearly independent.*

Proof: We use induction on the cardinality of X_n. If $n = 3$, it is clear that the box splines $B(\cdot - \nu|X_3)$ are linearly independent. Thus, we now assume that the result holds for all direction sets X_ℓ with $\ell < n$, and show that it holds for X_n. To show that $\{B(\cdot - \nu|X_n)\}_{\nu\in\mathbb{Z}^2}$ are linearly independent, we need to show that

$$\sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \equiv 0$$

implies $a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$. Let j, k, l be the number of times e^1, e^2, e^3 appear in X_n, respectively, where $j + k + l = n$. There are two cases.

Case 1: At least two indices are bigger than 1, say $j,k > 1$. Then by Lemma 12.3,

$$\begin{aligned}0 &= D_{e^i} \sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) \\ &= \sum_{\nu\in\mathbb{Z}^2} a_\nu \nabla_{e^i} B(\cdot - \nu|X_n\backslash\{e^i\}) \\ &= \sum_{\nu\in\mathbb{Z}^2} (a_{\nu+e^i} - a_\nu) B(\cdot - \nu + e^i|X_n\backslash\{e^i\}), \qquad i = 1,2.\end{aligned}$$

But then the inductive hypothesis implies that $a_{\nu+e^i} - a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$ and $i = 1,2$.

Case 2: Only one of the indices is bigger than 1, say $k > 1$. As above this implies $a_{\nu+e^2} - a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$. For any f in the space $C_0^1(\mathbb{R}^2)$ of compactly supported continuously differentiable functions,

$$\begin{aligned}0 &= \int_{\mathbb{R}^2} \frac{\partial}{\partial x} f(v) \sum_{\nu\in\mathbb{Z}^2} a_\nu B(v - \nu|X_n)\, dv \\ &= \int_{\mathbb{R}^2} \sum_{\nu\in\mathbb{Z}^2} a_\nu \frac{\partial}{\partial x} f(v) B(v - \nu|X_n)\, dv \\ &= \sum_{\nu\in\mathbb{Z}^2} a_\nu \int_{[0,1]^n} \frac{\partial}{\partial x} f\Big(\sum_{i=1}^n t_i v_i - \nu\Big)\, dt_1 \cdots dt_n \\ &= \int_{[0,1]^{n-1}} \sum_{\nu\in\mathbb{Z}^2} a_\nu \Big[f\Big(\sum_{i=2}^n t_i v_i + e^1 - \nu\Big) - f\Big(\sum_{i=2}^n t_i v_i - \nu\Big)\Big]\, dt_2 \cdots dt_n \\ &= \int_{[0,1]^{n-1}} \sum_{\nu\in\mathbb{Z}^2} (a_{\nu+e^1} - a_\nu) f\Big(\sum_{i=2}^n t_i v_i - \nu\Big)\, dt_2 \cdots dt_n \\ &= \sum_{\nu\in\mathbb{Z}^2} (a_{\nu+e^1} - a_\nu) \int_{[0,1]^{n-1}} f\Big(\sum_{i=2}^n t_i v_i - \nu\Big)\, dt_2 \cdots dt_n.\end{aligned}$$

It follows that $a_{e^1+\nu} - a_\nu = 0$, for all $\nu \in \mathbb{Z}^2$. We have shown that $a_{e^i+\nu} - a_\nu = 0$ for all $\nu \in \mathbb{Z}^2$ for $i = 1,2$ in either case. This implies that a_ν are equal to a constant c. But then

$$0 = \sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot - \nu|X_n) = c \sum_{\nu\in\mathbb{Z}^2} B(\cdot - \nu|X_n) = c,$$

by Theorem 12.15, and we conclude that $c = 0$. □

The following theorem shows that the integer translates of a type-I box spline $B(\cdot|X_n)$ form a stable basis for $\mathcal{S}(X_n)$.

Theorem 12.20. *Let $a := \{a_\nu\}_{\nu\in\mathbb{Z}^2}$ be a bounded sequence. Then*

$$A\|a\|_\infty \le \Big\| \sum_{\nu\in\mathbb{Z}^2} a_\nu B(\cdot|X_n)\Big\|_\infty \le \|a\|_\infty,$$

where $\|a\|_\infty = \max_{\nu\in\mathbb{Z}^2} |a_\nu|$.

Proof: The second inequality follows immediately from Theorem 12.15. To prove the first inequality, we assume the contrary. Let $a^m := \{a_\nu^m\}_{\nu\in\mathbb{Z}^2}$ be uniformly bounded sequences such that $\|a_\nu^m\|_\infty = 1$ and

$$\Big\| \sum_{\nu\in\mathbb{Z}^2} a_\nu^m B(\cdot|X_n)\Big\|_\infty \longrightarrow 0,$$

as $m \to \infty$. The boundedness of the sequence implies that there is a subsequence that converges to some $a^0 = \{a_\nu^0\}_{\nu\in\mathbb{Z}^2}$. It follows that $\|a^0\|_\infty = 1$ and

$$\Big\| \sum_{\nu\in\mathbb{Z}^2} a_\nu^0 B(\cdot|X_n)\Big\|_\infty = 0.$$

This contradicts the linear independence of $\{B(\cdot - \nu|X_n)\}_{\nu\in\mathbb{Z}^2}$. $\square$

The situation is different for type-II box splines. Coupling Theorems 12.15 and 12.16 with the definition of linear independence, we immediately get the following negative result.

Theorem 12.21. *Let X_n be a type-II direction set. Then the corresponding box splines $\{B(\cdot - \nu|X_n)\}_{\nu\in\mathbb{Z}^2}$ are linearly dependent.*

Proof: Since

$$\sum_{\nu\in\mathbb{Z}^2} B(\cdot - \nu|X_n) \equiv 1 \equiv \sum_{\nu\in\mathbb{Z}^2} 2B(\cdot - j_1 e^3 - j_2 e^4|X_n),$$

it follows that there exist coefficients a_ν which are either 1 or -2 such that

$$\sum_{\nu\in\mathbb{Z}^2} a_\nu B(v - \nu|X_n) \equiv 0. \quad \square$$

12.4. The Strang–Fix Conditions

Throughout this section we suppose that ϕ is a compactly supported continuous function defined on $\mathbb{R}^2$. For instance, ϕ could be one of the box splines discussed above, or it could be a finite linear combination of such box splines. Here we are interested in the space

$$\mathcal{S}(\phi) := \Big\{ s(\phi)(v) := \sum_{\nu\in\mathbb{Z}^2} a_\nu \phi(v - \nu) : \ a_\nu \in \mathbb{R} \Big\}.$$

In the literature this is a called a principal shift-invariant space. To explore its properties, we recall the following well-known Poisson summation formula.

Theorem 12.22. *Let ϕ be a compactly supported function in $C(\mathbb{R}^2)$, and suppose its Fourier transform $\widehat{\phi}$ is in $L_1(\mathbb{R}^2)$. Then*

$$\sum_{\nu\in\mathbb{Z}^2} \phi(\nu) = \sum_{\nu\in\mathbb{Z}^2} \widehat{\phi}(2\pi\nu). \tag{12.7}$$

Proof: Let

$$\psi(v) = \sum_{\nu\in\mathbb{Z}^2} \phi(v+\nu).$$

Then $\psi(v)$ is a well-defined continuous function which is 1-periodic in both variables. The Fourier coefficients of ψ are

$$\begin{aligned}
&\int_{[0,1]^2} \psi(v)\exp(-2i\pi v\cdot\nu)\,dv \\
&\qquad = \sum_{\mu\in\mathbb{Z}^2} \int_{[0,1]^2+\mu} \phi(v)\exp(-2i\pi v\cdot\nu)\,dv \\
&\qquad = \int_{\mathbb{R}^2} \phi(v)\exp(-2i\pi v\cdot\nu)\,dv = \widehat{\phi}(2\pi\nu), \qquad \text{all } \nu\in\mathbb{Z}^2.
\end{aligned}$$

Hence, by the hypothesis on ϕ, it belongs to $L_2[0,1]^2$ and

$$\psi(v) = \sum_{\nu\in\mathbb{Z}^2} \widehat{\phi}(2\pi\nu)\exp(-2i\pi v\cdot\nu).$$

Setting $v=(0,0)$ gives (12.7). $\square$

As an application, we now apply (12.7) to the functions $u^\alpha\phi(v-u)$ and $(v-u)^\alpha\phi(u)$, where $u,v\in\mathbb{R}^2$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2$.

Theorem 12.23. *Let ϕ be a continuous function of compact support on $\mathbb{R}^2$ whose Fourier transform $\widehat{\phi}$ belongs to $L_1(\mathbb{R}^2)$. Then for any $\alpha\in\mathbb{Z}_+^2$,*

$$\sum_{\nu\in\mathbb{Z}^2} \nu^\alpha\phi(v-\nu) = \sum_{\nu\in\mathbb{Z}^2} (-iD_u)^\alpha\big(\exp(iv\cdot u)\,\widehat{\phi}(u)\big)\Big|_{u=2\pi\nu} \tag{12.8}$$

and

$$\sum_{\nu\in\mathbb{Z}^2} (v-\nu)^\alpha\phi(\nu) = \sum_{\nu\in\mathbb{Z}^2} \exp(-iv\cdot\nu)(-iD_u)^\alpha\big(\exp(iv\cdot u)\,\widehat{\phi}(u)\big)\Big|_{u=2\pi\nu} \tag{12.9}$$

for all $u\in\mathbb{R}^2$. Here D_u is the directional derivative associated with u.

It is obvious that for all $v\in\mathbb{Z}^2$, the equations (12.8) and (12.9) agree, being simply the discrete convolution (or convolution over $\mathbb{Z}^2$) of $\{\phi(\nu)\}$

with $\{\nu^\alpha\}$. However, it is also clear that they are usually different if $v \notin \mathbb{Z}^2$. This leads us to the following definition.

Definition 12.24. *The* commutator *of a compactly supported continuous function ϕ on $\mathbb{R}^2$ is the operator on $C(\mathbb{R}^2)$ defined by*

$$[\phi|f](v) = \sum_{\nu\in\mathbb{Z}^2} \phi(v-\nu)f(\nu) - \sum_{\nu\in\mathbb{Z}^2} f(v-\nu)\phi(\nu), \qquad \textit{all } f \in C(\mathbb{R}^2).$$

Throughout the following we write $\mathcal{P}$ for the space of all bivariate polynomials, i.e., $\mathcal{P} := \bigcup_{k=0}^{\infty} \mathcal{P}_k$. Suppose that $f \in \mathcal{P}$. Then $[\phi|f] \equiv 0$ on $\mathbb{R}^2$ implies that

$$\sum_{\nu\in\mathbb{Z}^2} \phi(v-\nu)f(\nu) = \sum_{\nu\in\mathbb{Z}^2} f(v-\nu)\phi(\nu),$$

where the right-hand side is a polynomial, and the left-hand side is a linear combination of the translates of ϕ over $\mathbb{Z}^2$. The notion of commutator helps us understand which polynomials lie in $\mathcal{S}(\phi)$. We say that ϕ is normalized provided that $\displaystyle\sum_{\nu\in\mathbb{Z}^2} \phi(\nu) = 1$.

Theorem 12.25. *Let ϕ be a normalized continuous function of compact support. Then*

$$\mathcal{P} \cap \mathcal{S}(\phi) = \{f \in \mathcal{P} : [\phi|f] \equiv 0\} = \Big\{f \in \mathcal{P} : \sum_{\nu\in\mathbb{Z}^2} f(\nu)\phi(u-\nu) \in \mathcal{P}\Big\}.$$

Proof: $[\phi|f] \equiv 0$ and $f \in \mathcal{P}$ imply that $\sum_{\nu\in\mathbb{Z}^2} f(\nu)\phi(u-\nu) \in \mathcal{P}$. Moreover, $\int f \in \mathcal{P}$ and $\sum_{\nu\in\mathbb{Z}} f(\nu)\phi(u-\nu) \in \mathcal{P}$ imply that $[\phi|f] \equiv 0$, since $[\phi|f](\nu) = 0$ for all $\nu \in \mathbb{Z}^2$. Clearly, $\{f \in \mathcal{P} : [\phi|f] \equiv 0\} \subset \mathcal{P} \cap \mathcal{S}(\phi)$. On the other hand, for $f(\cdot) = \sum_{\nu\in\mathbb{Z}^2} c_\nu \phi(\cdot - \nu)$ in $\mathcal{S}(\phi)$,

$$\begin{aligned}
\sum_{\nu\in\mathbb{Z}^2} \phi(u-\nu)f(\nu) &= \sum_{\nu\in\mathbb{Z}^2}\sum_{\mu\in\mathbb{Z}^2} c_\mu \phi(\nu-\mu)\phi(u-\nu) \\
&= \sum_{\nu\in\mathbb{Z}^2} \phi(u-\nu) \sum_{\mu\in\mathbb{Z}^2} c_{\nu-\mu}\phi(\mu) \\
&= \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \phi(u-\nu-\mu)c_\nu \Big) \phi(\mu) \\
&= \sum_{\nu\in\mathbb{Z}^2} f(u-\nu)\phi(\nu).
\end{aligned}$$

It follows that $[\phi|f] \equiv 0$ for any $f \in \mathcal{S}(\phi)$, and thus $\mathcal{P} \cap \mathcal{S}(\phi) \subset \{f \in \mathcal{P} : [\phi|f] \equiv 0\}$. $\square$

We now explore some additional properties of the commutator of ϕ. In the sequel we use the following notation for monomials:

$$m_\alpha(v) := \frac{1}{\alpha_1 ! \, \alpha_2 !} x^{\alpha_1} y^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2. \tag{12.10}$$

Theorem 12.26. *Let ϕ be a continuous function of compact support with $\widehat{\phi} \in L_1(\mathbb{R}^2)$. Suppose $\alpha \in \mathbb{Z}_+^2$. Then*

$$[\phi|m_\alpha](v) = \sum_{\nu \in \mathbb{Z}^2} \Big[\sum_{0 \le \beta \le \alpha} (-i)^{|\beta|} m_{\alpha-\beta}(v) \frac{D^\beta}{\beta !} \widehat{\phi}(2\pi\nu) \Big] (\exp(i2\pi v \cdot \nu) - 1), \tag{12.11}$$

for any $v \in \mathbb{R}^2$. Furthermore,

$$[\phi|m_\gamma](v) = 0, \quad \textit{all } v \in \mathbb{R}^2, \quad \gamma \le \alpha, \tag{12.12}$$

if and only if

$$D^\gamma \widehat{\phi}(2\pi\nu) = 0 \quad \textit{all } \nu \in \mathbb{Z}^2 \backslash \{(0,0)\}, \quad \gamma \le \alpha. \tag{12.13}$$

Proof: The identity (12.11) follows immediately from Theorem 12.23. That (12.13) implies (12.12) is trivial. To show the converse, we use mathematical induction on α. Indeed, since

$$[\phi|m_0](v) = [\phi|1](v) = \sum_{\mu \in \mathbb{Z}^2 \backslash \{(0,0)\}} \widehat{\phi}(2\pi\mu)(\exp(i2\pi(\mu \cdot v) - 1),$$

we conclude that $[\phi|m_0] \equiv 0$ implies $\widehat{\phi}(2\pi j) = 0$ for all $j \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Suppose that (12.12) holds. Then, by the induction hypothesis, we have

$$D^\gamma \widehat{\phi}(2\pi\nu) = 0, \quad \nu \in \mathbb{Z}^2 \backslash \{(0,0)\},$$

for all $\gamma \le \alpha$ and $\gamma \ne \alpha$. Thus, (12.11) gives

$$\sum_{\nu \in \mathbb{Z}^2 \backslash \{(0,0)\}} D^\alpha \widehat{\phi}(2\pi\nu)(\exp(i2\pi\nu \cdot u) - 1),$$

or $D^\alpha \widehat{\phi}(2\pi\nu) = 0$, for all $\nu \in \mathbb{Z}^2 \backslash \{(0,0)\}$. $\square$

Definition 12.27. *A compactly supported function $\phi \in C(\mathbb{R}^2)$ with $\widehat{\phi}$ in $L_1(\mathbb{R}^2)$ is said to satisfy the* Strang–Fix conditions *with* SF index *$\alpha \in \mathbb{Z}_+^2$ if*

1) $\widehat{\phi}(0,0) = 1$,
2) $D^\gamma \widehat{\phi}(2\pi\nu) = 0$, *all* $\nu \in \mathbb{Z}^2 \backslash \{(0,0)\}$ *and* $\gamma \le \alpha$.

The collection Λ_ϕ of all SF indices of ϕ is called the SF indicator set *of ϕ. The largest integer n for which $\alpha \in \Lambda_\phi$ whenever $|\alpha| \le n$ is called the* SF degree *of ϕ.*

Using Theorem 12.6, we can directly compute the SF index set for type-I and type-II box splines.

Example 12.28. The SF indicator set of the box spline B_{jkl} contains the indices

$$\begin{aligned} &(j+l-i, i-1), \quad 1 \le i \le \min\{k, j+l\}, \\ &(i-1, k+l-i), \quad 1 \le i \le \min\{j, k+l\}. \end{aligned}$$

In particular, for B_{221}, the SF indicator set is $\{(2,0),(1,1),(0,2)\}$. Thus, the SF degree is two.

Example 12.29. The SF indicator set of the box spline B_{jklm} contains the indices

$$\begin{aligned} &(j+l+m-i, i-1), && 1 \le i \le \min\{k, j+l+m\}, \\ &(i-1, k+l+m-i), && 1 \le i \le \min\{k, k+l+m\}, \\ &(k+\alpha-i, l+i), && 0 \le i \le k+\alpha, \\ &(k+i, l+\alpha-i), && 0 \le i \le l+\alpha, \end{aligned}$$

where $\alpha := \min\{l, m, l+m-k, l+m-j\} - 1$.

Motivated by the above two examples, we now state the general result.

Theorem 12.30. *Let X_n be either a type-I or a type-II direction set. Then the SF degree of the corresponding box spline $B(\cdot|X_n)$ is $r(X_n)+1$, where $r(X_n)$ is the smoothness of $B(\cdot|X_n)$ as in Theorems 12.4 and 12.13, respectively.*

We conclude this section by giving the SF indicator sets for some linear combinations of type-I and type-II box splines.

Example 12.31. The SF indicator set of $\phi := (B_{l,l,l+1} + B_{l,l+1,l} + B_{l+1,l,l} - B_{l,l,l})/2$ contains the indices $\{(i,j) : i+j \le 2l\}$.

Example 12.32. The SF indicator set of $\phi := (B_{l+1,l,l,l} + B_{l,l+1,l,l} + B_{l,l,l+1,l} + B_{l,l,l,l+1} - B_{l,l,l,l})/3$ contains the indices $\{(i,j) : i+j \le 3l\}$.

12.5. Polynomial Reproducing Formulae

Suppose that μ is a bounded linear functional on $C(\mathbb{R}^2)$ with $\mu(m_0) = 1$, where, in general, m_α denotes the monomial of coordinate degree $\alpha \in \mathbb{Z}_+^2$ as defined in (12.10). Then we define a sequence $\{g_\alpha := \sum_{\gamma \le \alpha} a_\gamma m_\gamma(v)\}$ of polynomials by

$$\mu(D^\beta g_\alpha) = \delta_{\alpha,\beta}, \tag{12.14}$$

where $\delta_{\alpha,\beta}$ denotes the Kronecker delta for all $\alpha, \beta \in \mathbb{Z}_+^2$. A sequence satisfying (12.14) is called an Appell sequence. We note that

$$\mu(D^\beta g_\alpha) = \sum_{\gamma \le \alpha} a_\gamma \mu(m_{\gamma-\beta}), \tag{12.15}$$

with $m_{\gamma-\beta} = 0$ if $\gamma - \beta \notin \mathbb{Z}_+^2$. Thus, the matrix $[\mu(m_{\gamma-\beta})]_{\gamma\le\alpha;\beta\le\alpha}$ in the system (12.14) is upper triangular and has unit diagonal elements, so that g_α is uniquely determined by (12.14). The following identity can be easily verified by induction.

Lemma 12.33. *For $\alpha \in \mathbb{Z}_+^2$,*

$$g_\alpha(v) = m_\alpha(v) - \sum_{\substack{\beta \le \alpha \\ \beta \ne \alpha}} \mu(m_{\alpha-\beta}) g_\beta(v)$$

is an Appell sequence satisfying (12.14). Since $g_0(v) = m_0(v) = 1$, equation (12.15) provides an inductive scheme to compute $g_\alpha(v)$.

Fix a compactly supported continuous function ϕ which is normalized, e.g., a box spline $B(\cdot|X_n)$. Now we consider the bounded linear functional μ defined on $C(\mathbb{R}^2)$ by

$$\mu(f) := \sum_{\nu \in \mathbb{Z}^2} f(-\nu)\,\phi(\nu).$$

Theorem 12.34. *Set $L_0(v) = 1$, and define $L_\beta(v)$ inductively by*

$$L_\beta(v) := m_\beta(v) - \sum_{j \in \mathbb{Z}^2} \phi(j) \sum_{\substack{\gamma \le \beta \\ \gamma \ne \beta}} \frac{(-j)^{\beta-\gamma}}{(\beta-\gamma)\,!} L_\gamma(v), \tag{12.16}$$

for $\beta \in \mathbb{Z}_+^2$. Then

$$m_\alpha(v) = \sum_{\nu \in \mathbb{Z}^2} L_\alpha(\nu)\,\phi(v-\nu), \qquad \text{all } v \in \mathbb{R}^2 \text{ and } \alpha \in \Lambda_\phi.$$

Proof: With $L_0(v) = 1$ we may rewrite

$$L_\beta(v) = m_\beta(v) - \sum_{\substack{\gamma \le \beta \\ \gamma \ne \beta}} \mu(m_{\beta-\gamma}) L_\gamma(v)$$

in terms of the bounded linear functional μ defined above. Hence, $\{L_\beta\}$ is an Appell sequence by Lemma 12.33. Note that for $\alpha \in \Lambda_\phi$,

$$\sum_{\nu \in \mathbb{Z}^2} L_\alpha(\nu)\,\phi(v-\nu) = \sum_{\nu \in \mathbb{Z}^2} L_\alpha(v-\nu)\,\phi(\nu),$$

and

$$D^\beta \sum_{\nu\in\mathbb{Z}^2} L_\alpha(v-\nu)\,\phi(\nu)\big|_{v=0} = \sum_{\nu\in\mathbb{Z}^2} D^\beta L_\alpha(v-\nu)\big|_{v=0}\,\phi(\nu)$$
$$= \sum_{\nu\in\mathbb{Z}^2} D^\beta L_\alpha(-\nu)\,\phi(\nu)$$
$$= \mu(D^\beta L_\alpha) = \delta_{\alpha,\beta},$$

for $\beta \in \mathbb{Z}^2_+$. We conclude that

$$\sum_{\nu\in\mathbb{Z}^2} L_\alpha(v-\nu)\,\phi(\nu) = m_\alpha(v),$$

since the polynomial $\sum_{\nu\in\mathbb{Z}^2} L_\alpha(v-\nu)\,\phi(\nu)$ satisfies

$$D^\beta \sum_{\nu\in\mathbb{Z}^2} L_\alpha(v-\nu)\,\phi(\nu)\big|_{v=0} = \delta_{\alpha,\beta}. \quad \square$$

The function L_β defined above has the following properties.

Theorem 12.35. *Let $\beta \in \mathbb{Z}^2_+$. Then*

$$D^\gamma L_\beta(v) = L_{\beta-\gamma}(v), \qquad \textit{all } \gamma \le \beta, \tag{12.17}$$

and

$$L_\beta(v) = \sum_{\gamma\le\beta} L_{\beta-\gamma}(0,0) m_\gamma(v), \qquad \textit{all } v \in \mathbb{R}^2. \tag{12.18}$$

Proof: We first prove (12.17) by induction. For $\gamma = 0$ there is nothing to prove. For $\gamma \ge 0$, the induction hypothesis yields

$$D^\gamma L_\beta(v) = m_{\beta-\gamma}(v) - \sum_{\nu\in\mathbb{Z}^2} \phi(\nu) \sum_{\substack{\alpha\le\beta\\ \alpha\ne\beta}} \frac{(-\nu)^{\beta-\alpha}}{(\beta-\alpha)\,!} L_{\alpha-\gamma}(v)$$
$$= m_{\beta-\gamma}(v) - \sum_{\nu\in\mathbb{Z}^2} \phi(\nu) \sum_{\substack{\alpha\le\beta-\gamma\\ \alpha\ne\beta-\gamma}} \frac{(-\nu)^{\beta-\gamma-\alpha}}{(\beta-\gamma-\alpha)\,!} L_\alpha(v) = L_{\beta-\gamma}(v)$$

by a change of indices. This proves the first formula. Using (12.17), we see that the formula (12.18) is simply the Taylor expansion of L_β at $(0,0)$. $\square$

Theorem 12.36. *L_β can be written as*

$$L_\beta(v) = m_\beta(v) - \sum_{\substack{\gamma\le\beta\\ \beta\ne\gamma}} \frac{(-iD)^{\beta-\gamma}\,\widehat{\phi}(0,0)}{(\beta-\gamma)\,!} L_\gamma(v), \qquad \textit{all } \beta \in \Lambda_\phi. \tag{12.19}$$

Proof: Using the Poisson summation formula, we have

$$\sum_{\nu\in\mathbb{Z}^2} \phi(\nu)(-\nu)^{\beta-\gamma} = \sum_{\nu\in\mathbb{Z}^2} (-iD)^{\beta-\gamma}(\widehat{\phi}(u)\big|_{2\pi\nu}.$$

By the definition of the SF indicator set Λ_ϕ, we conclude that

$$\sum_{\nu\in\mathbb{Z}^2} \phi(\nu)(-\nu)^{\beta-\gamma} = (-iD)^{\beta-\gamma}\,\widehat{\phi}(0,0), \qquad \text{all } \beta\in\Lambda_\phi.$$

Thus, L_β can be written as in (12.19). □

We are now ready to prove a bivariate analog of the univariate Marsden identity, see [Sch81].

Theorem 12.37. *Let ϕ be a normalized function of compact support in $C(\mathbb{R}^2)$ with $\widehat{\phi}\in L_1(\mathbb{R}^2)$. Then for any $\alpha\in\Lambda_\phi$,*

$$m_\alpha(v-u) = \sum_{\nu\in\mathbb{Z}^2} L_\alpha(\nu-u)\,\phi(v-\nu), \qquad \text{all } v,u\in\mathbb{R}^2. \tag{12.20}$$

Proof: To prove this result, we appeal to Lemma 12.33 and consider the derivative of order β with respect to u. This gives

$$\begin{aligned} D_u^\beta\big[m_\alpha(v-u) &- \sum_{\nu\in\mathbb{Z}^2} L_\alpha(\nu-u)\,\phi(v-\nu)\big] \\ &= (-1)^\beta\big[m_{\alpha-\beta}(v-u) - \sum_{\nu\in\mathbb{Z}^2} L_{\alpha-\beta}(\nu-u)\,\phi(v-\nu)\big] \end{aligned}$$

by Theorem 12.35. Now the result follows by induction. Indeed, by the induction hypothesis, the above expression is zero, and hence

$$m_\alpha(v-u) - \sum_{j\in\mathbb{Z}^2} L_\alpha(j-u)\,\phi(v-j)$$

is independent of u. Since it is zero at $u=(0,0)$, (12.20) follows from Theorem 12.34. □

Example 12.38. Let B_{221} be a type-I box spline. Then

$$\begin{aligned}
1 &= \sum_{(i,j)\in\mathbb{Z}^2} B_{221}(x-i,y-j),\\
x &= \sum_{(i,j)\in\mathbb{Z}^2} \Big(i+\frac{3}{2}\Big)B_{221}(x-i,y-j),\\
y &= \sum_{(i,j)\in\mathbb{Z}^2} \Big(j+\frac{3}{2}\Big)B_{221}(x-i,y-j),\\
x^2 &= \sum_{(i,j)\in\mathbb{Z}^2} (i+1)(i+2)B_{221}(x-i,y-j),\\
xy &= \sum_{(i,j)\in\mathbb{Z}^2} \Big(\Big(i+\frac{3}{2}\Big)\Big(j+\frac{3}{2}\Big)-\frac{1}{12}\Big)B_{221}(x-i,y-j),\\
y^2 &= \sum_{(i,j)\in\mathbb{Z}^2} (j+1)(j+2)B_{221}(x-i,y-j).
\end{aligned}$$

Example 12.39. Let B_{1111} be a type-II box spline. Then

$$\begin{aligned}
1 &= \sum_{(i,j)\in\mathbb{Z}^2} B_{1111}(x-i,y-j),\\
x &= \sum_{(i,j)\in\mathbb{Z}^2} iB_{1111}(x-i,y-j),\\
y &= \sum_{(i,j)\in\mathbb{Z}^2} jB_{1111}(x-i,y-j),\\
x^2 &= \sum_{(i,j)\in\mathbb{Z}^2} \Big(i-\frac{1}{2}\Big)\Big(i+\frac{1}{2}\Big)B_{1111}(x-i,y-j),\\
xy &= \sum_{(i,j)\in\mathbb{Z}^2} ijB_{1111}(x-i,y-j),\\
y^2 &= \sum_{(i,j)\in\mathbb{Z}^2} \Big(j-\frac{1}{2}\Big)\Big(j+\frac{1}{2}\Big)B_{1111}(x-i,y-j).
\end{aligned}$$

12.6. Box Spline Quasi-interpolants

Suppose $\mathcal{S}(\phi)$ is as defined in a previous section. For any $h > 0$, the scaling operator σ_h is defined by

$$(\sigma_h f)(v) := f\Big(\frac{v}{h}\Big), \quad \text{all } f \text{ on } \mathbb{R}^2. \tag{12.21}$$

Set

$$\mathcal{S}_h(\phi) := \{\sigma_h f : f \in \mathcal{S}(\phi)\}.$$

Fix a domain $\Omega \subset \mathbb{R}^2$. We say that $\mathcal{S}(\phi)$ has approximation power m if m is the largest integer for which

$$d(f, \mathcal{S}(\phi)) := \inf\{\|f - s\|_{\infty,\Omega},\, s \in \mathcal{S}_h(\phi)\} = \mathcal{O}(h^m)$$

for all sufficiently smooth functions f. In this section we prove that the approximation power of $\mathcal{S}(\phi)$ is determined by the Strang–Fix degree of ϕ. In addition, we construct a spline quasi-interpolant to achieve this order.

We need some more notation and two lemmas. Let Γ be a finite lower subset of $\mathbb{Z}_+^2$ (see page 31), and let $\mathcal{P}_\Gamma$ be the space of polynomials of the form

$$p(v) = \sum_{\alpha \in \Gamma} b_\alpha v^\alpha.$$

Lemma 12.40. *The linear system*

$$\sum_{\nu \in \Gamma} a_\nu \nu^\beta = \beta!\, L_\beta(0,0), \quad \beta \in \Gamma, \tag{12.22}$$

has a unique solution $\{a_\nu\}_{\nu \in \Gamma}$, where L_β are the polynomials defined in Theorem 12.34.

Proof: We first consider the case that the set Γ has a unique largest index, say (m,n) in the lexicographical order on Γ. Then $\mathcal{P}_\Gamma$ is a polynomial space of coordinate degree less than or equal to (m,n). It is easy to see that in this case (12.22) has a unique solution over the grid of interpolation sites $\nu \in \Gamma$.

Next we consider the case where $\mathcal{P}_\Gamma$ is a polynomial space of total degree d. Then the interpolation sites Γ satisfy the conditions of Theorem 1.10, and it follows that the linear system (12.22) is invertible.

Finally we consider the general case. For simplicity, assume that Γ contains two largest indices, say (m_1,n_1) and (m_2,n_2) with $m_1 \ge n_1$ and $m_2 < n_2$. Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{(i,j), 0 \le i \le m_1, 0 \le j \le n_1\}$ and $\Gamma_2 = \{(i,j), 0 \le i \le m_2, 0 \le j \le n_2\}$. Let $\Gamma_3 = \Gamma_1 \cap \Gamma_2$. Then $\Gamma_3 = \{(i,j), 0 \le i \le m_2, 0 \le j \le n_1\}$. We divide the given interpolation values $\{\beta!\, L_\beta(0,0)\}_{\beta\in\Gamma}$ into three subsets $F_i = \{\beta!\, L_\beta(0,0) : \beta \in \Gamma_i\}$, for $i = 1,2,3$. Let $p_3 \in \mathcal{P}_{\Gamma_3}$ be such that $p_3(\beta) = \beta!\, L_\beta(0,0)$ for $\beta \in \Gamma_3$. Let $p_1 \in \mathcal{P}_{\Gamma_1}$, $p_2 \in \mathcal{P}_{\Gamma_2}$ be such that

$$p_1(\beta) = \begin{cases} \beta!\, L_\beta(0,0) - p_3(\beta), & \text{if } \beta \in \Gamma_1 \backslash \Gamma_3, \\ 0, & \text{if } \beta \in \Gamma_3, \end{cases}$$

and

$$p_2(\beta) = \begin{cases} \beta!\, L_\beta(0,0) - p_3(\beta), & \text{if } \beta \in \Gamma_2 \backslash \Gamma_3, \\ 0, & \text{if } \beta \in \Gamma_3. \end{cases}$$

By the above discussion, there exist unique such p_1, p_2, p_3. It is clear that $p = p_1 + p_2 + p_3 \in \mathcal{P}_\Gamma$. Then noting that $p_1(\beta) = 0$ for all $\beta \in \Gamma_2$ and $p_2(\beta) = 0$ for all $\beta \in \Gamma_1$, it is easy to check that p satisfies the interpolation conditions $p(\beta) = \beta!\, L_\beta(0,0)$ for all $\beta \in \Gamma$. That is, for any given interpolation values, we can find an interpolation polynomial p in this way. It follows that such a p is unique. $\square$

Lemma 12.41. *Let $\{a_\nu\}_{\nu\in\Gamma}$ be the solution of (12.22). Then for any $\mu \in \mathbb{Z}^2$,*

$$\sum_{\nu\in\Gamma} a_\nu (\mu+\nu)^\beta = \beta!\, L_\beta(\mu), \quad \beta \in \Gamma. \tag{12.23}$$

Proof: For any $\mu \in \mathbb{Z}^2_+$,

$$\sum_{\nu\in\Gamma} a_\nu(\mu+\nu)^\beta = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \mu^\gamma \sum_{\nu\in\Gamma} a_\nu \nu^{\beta-\gamma} = \sum_{\gamma\le\beta} \frac{\beta!}{\gamma!} \mu^\gamma L_{\beta-\gamma}(0,0) = \beta!\, L_\beta(\mu),$$

by Theorem 12.35. $\square$

We now use Lemma 12.41 to establish the following result.

Theorem 12.42. *Let ϕ be a compactly supported continuous function on $\mathbb{R}^2$ satisfying $\widehat{\phi} \in L_1(\mathbb{R}^2)$ and $\widehat{\phi}(0,0) = 1$. Let $\Gamma := \Lambda_\phi$ be its Strang–Fix indicator set. Then there exists a compactly supported function $\psi \in \mathcal{S}(\phi)$ such that*

$$m_\alpha(v) = \sum_{\nu\in\mathbb{Z}^2} m_\alpha(\nu)\psi(v-\nu), \qquad \textit{all } \alpha \in \Gamma.$$

Proof: With $\{a_\nu\}$ defined as in (12.22), let

$$\psi(v) := \sum_{\mu\in\Gamma} a_\mu \phi(v+\mu).$$

Then by Theorem 12.34 and (12.23), we have

$$\begin{aligned}
m_\alpha(v) &= \sum_{\nu\in\mathbb{Z}^2} L_\alpha(\nu)\phi(v-\nu) \\
&= \sum_{\nu\in\mathbb{Z}^2} \sum_{\mu\in\Gamma} a_\mu m_\alpha(\mu+\nu)\phi(v-\nu) \\
&= \sum_{\mu\in\Gamma} a_\mu \sum_{\nu\in\mathbb{Z}^2} m_\alpha(\mu+\nu)\phi(v-\nu) \\
&= \sum_{\nu\in\mathbb{Z}^2} m_\alpha(\nu) \sum_{\mu\in\Gamma} a_\mu \phi(v-\nu+\mu) \\
&= \sum_{\nu\in\mathbb{Z}^2} m_\alpha(\nu)\psi(v-\nu). \quad \square
\end{aligned}$$

With $\{a_\nu\}$ as in (12.22), for $g \in C(\mathbb{R}^2)$ we now define

$$\lambda_\mu(g) := \sum_{\nu\in\Gamma} a_\nu g(\mu+\nu),$$

for any integer $\mu \in \mathbb{Z}^2$. This defines a sequence of linear functionals on $C(\mathbb{R}^2)$. If $g = \sigma_{h^{-1}}(f)$, then

$$\lambda_\mu(\sigma_{h^{-1}} f) = \sum_{\nu\in\Gamma} a_\nu f(h(\mu+\nu)).$$

We now define the following quasi-interpolation operator:

$$Q^h_\Gamma f(v) := \sum_{\nu\in\mathbb{Z}^2} \lambda_\nu(\sigma_{h^{-1}} f)\, \phi(v/h-\nu).$$

Theorem 12.43. *Let ϕ be a compactly supported continuous function on $\mathbb{R}^2$ satisfying $\widehat{\phi} \in L_1(\mathbb{R}^2)$ and $\widehat{\phi}(0,0)=1$. Let $\Gamma := \Lambda_\phi$ be its Strang–Fix indicator set. Then $Q^h_\Gamma p = p$ for all $p \in \mathcal{P}_\Gamma$.*

Proof: Fix $\beta \in \Gamma$. Applying Theorem 12.34 and Lemma 12.41, we get

$$\begin{aligned} Q^h_\Gamma m_\beta(v) &= \sum_{\mu\in\mathbb{Z}^2} \frac{1}{\beta\,!} \sum_{\nu\in\Gamma} a_\nu (\mu+\nu)^\beta h^{|\beta|}\, \phi(v/h-\mu) \\ &= h^{|\beta|} \sum_{\mu\in\mathbb{Z}^2} L_\beta(\mu)\, \phi(v/h-\mu) \\ &= h^{|\beta|} m_\beta(v/h) = m_\beta(v). \quad \square \end{aligned}$$

Now let

$$\widehat{\Gamma} = \bigcup_{\alpha\in\Gamma} \{\beta :\ \beta \le \alpha + e^1\} \cup \{\beta :\ \beta \le \alpha + e^2\},$$

where e^1 and e^2 are the unit vectors defining the x and y-axes in $\mathbb{R}^2$.

Theorem 12.44. *Let Γ be the SF indicator set of ϕ. Then there exists a constant K such that for all sufficiently smooth f,*

$$\|Q^h_\Gamma f - f\|_{\infty,\Omega} \le K\, h^{|\beta|} \sum_{\beta\in\widehat{\Gamma}\setminus\Gamma} \|D^\beta f\|_{\infty,\Omega}. \tag{12.24}$$

In particular, $\|Q^h_\Gamma f - f\|_{\infty,\Omega} = \mathcal{O}(h^{m+1})$, where m is the Strang–Fix degree of ϕ.

Proof: Fix $v \in \Omega$. The Taylor expansion of f at v is

$$f(u) = \sum_{\beta\in\Gamma} D^\beta f(v) m_\beta(u-v) + \sum_{\beta\in\widehat{\Gamma}\setminus\Gamma} D^\beta f(v^*_\beta) m_\beta(u-v),$$

where v_β^* are points between v and u. Then by Theorem 12.43,

$$
\begin{aligned}
Q_\Gamma^h f(v) &= f(v) + \sum_{\beta\in\hat{\Gamma}\setminus\Gamma} Q_\Gamma^h\big[D^\beta f(v_\beta^*)\, m_\beta(\cdot - v)\big] \\
&= f(v) + \sum_{\beta\in\widehat{\Gamma}\setminus\Gamma}\sum_{\mu\in\mathbb{Z}^2} \lambda_\mu\big[\sigma_{h^{-1}} D^\beta f(v_\beta^*) m_\beta(\cdot - v)\big]\, \phi(v/h-\mu).
\end{aligned}
$$

From the definition of λ_μ, we know that these linear functionals are bounded by a constant independent of μ. Since ϕ is compactly supported, we conclude that there exists a positive constant K such that (12.24) holds. □

12.7. Half Box Splines

In this section we study a different class of box splines defined on type-II triangulations. Let L_{2200} and $\tilde{L}_{0022}$ be the splines in $S_1^0(\triangle_{II})$ whose B-coefficients are given in Figure 12.12. These splines are normalized so that $L_{2200}(0.5, 0.5) = 1$ and $\tilde{L}_{0022}(0,0) = 1$. It is easy to see that

$$
\begin{aligned}
\sum_{\nu\in\mathbb{Z}^2}\big[L_{2200}(v-\nu) + \tilde{L}_{0022}(v-\nu)\big] &\equiv 1, \\
\sum_{(i,j)\in\mathbb{Z}^2}\Big[\Big(i+\frac{1}{2}\Big)L_{2200}(v-(i,j)) + i\tilde{L}_{0022}(v-(i,j))\Big] &\equiv x, \\
\sum_{(i,j)\in\mathbb{Z}^2}\Big[\Big(j+\frac{1}{2}\Big)L_{2200}(v-(i,j)) + j\tilde{L}_{0022}(v-(i,j))\Big] &\equiv y.
\end{aligned}
$$

Thus, taken together, the shifts of L_{2200} and $\tilde{L}_{0022}$ form a partition of unity.

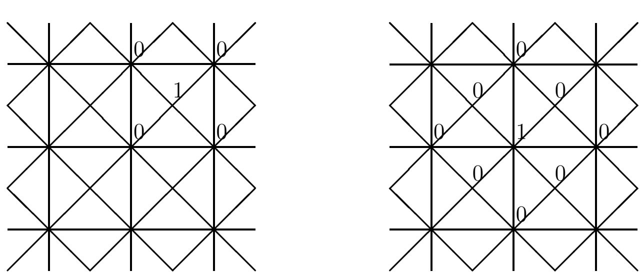


Fig. 12.12. The B-coefficients of L_{2200} and $\tilde{L}_{0022}$.

We identify the spline L_{2200} with the direction set $\{e^1, e^1, e^2, e^2\}$. Similarly, we identify $\tilde{L}_{0022}$ with the direction set $\{e^3, e^3, e^4, e^4\}$. Now given a type-II direction set X_n with $n > 4$ containing $\{e^1, e^1, e^2, e^2\}$, we define higher-degree half box splines recursively as

$$L(v|X_i) = \int_0^1 L(v - tv_i|X_{i-1})\,dt.$$

Given a type-II direction set X_n with $n > 4$ containing $\{e^3, e^3, e^4, e^4\}$, we define

$$\tilde{L}(v|X_i) := \int_0^1 \tilde{L}(v - tv_i|X_{i-1})\,dt.$$

Suppose X_n contains the directions e^1, e^2, e^3, e^4 a total of j, k, l, m times with $j, k \geq 2$. Then we write $L_{jklm}(v) := L(v|X_n)$. Similarly, if $l, m \geq 2$, we write $\tilde{L}_{jklm}(v) := \tilde{L}(v|X_n)$. It follows from the definitions that $L_{jklm} > 0$ for v in the interior of $[X_n \cup \{e^1, e^2\}]$, and vanishes outside of $[X_n \cup \{e^1, e^2\}]$. Similarly, $\tilde{L}_{jklm} > 0$ for v in the interior of $[X_n \cup \{e^3, e^4\}]$, and vanishes outside.

The following result is proved in the same way as Lemma 12.3. A similar result holds for $\tilde{L}(\cdot|X_n)$.

Lemma 12.45. *For any* $1 \leq j \leq n$, $D_{v_j} L(\cdot|X_n) = \nabla_{v_j} L(\cdot|X_n \backslash \{v_j\})$.

We now identify the structure of half box splines.

Theorem 12.46. *The splines* L_{jklm} *and* $\tilde{L}_{jklm}$ *are in* $\mathcal{S}^r_{n-3}(\triangle_{II})$*, where* $n = j + k + l + m$ *and* $r := r(X_n) = \min\{j + l + m, j + k + l, j + k + m, k + l + m\} - 2$.

Proof: Let $Y_n := X_n \setminus \{e^1, e^1, e^2, e^2\}$. Then Lemma 12.45 implies that

$$D_{Y_n} L(v|X_n) = \nabla_Y L(v|X_n \backslash Y_n) = \nabla_{Y_n} L_{2200},$$

which is a piecewise linear function. It follows that $L(v|X_n)$ is a piecewise polynomial function of degree $n-3$. The proof of smoothness follows along the same lines as the proof of Theorem 12.4. The proofs for $\tilde{L}(v|X_n)$ are similar. $\square$

The following result is the analog of Theorem 12.14.

Theorem 12.47. *For all* $f \in C(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} L(v|X_n) f(v) dv = \int_{[0,1]^n} F\Big(\sum_{i=1}^n t_i v_i\Big) dt_1 \cdots dt_n, \tag{12.25}$$

where $F(u) := \int_{\mathbb{R}^2} L_{2200}(v+u)f(v)\,dv$. *Similarly,*

$$\int_{\mathbb{R}^2} \tilde{L}(v|X_n)f(v)\,dv = \int_{[0,1]^n} G\Big(\sum_{i=1}^n t_i v_i\Big) dt_1 \cdots dt_n,$$

where $G(u) := \int_{\mathbb{R}^2} \tilde{L}_{0022}(v+u)f(v)\,dv$.

Proof: We proceed by induction. First consider $L(\cdot|X_n)$. Using the definition of $L(\cdot|X_n)$ and the induction hypothesis, we have

$$\begin{aligned}\int_{\mathbb{R}^2} L(v|X_n)f(v)dv &= \int_{\mathbb{R}^2}\int_0^1 L(v - tv_n|v_1,\ldots,v_{n-1})f(v)\,dt\,dv \\ &= \int_0^1 \int_{[0,1]^{n-1}} F\Big(\sum_{i=1}^{n-1} t_i v_i + tv_n\Big) dt_1 \cdots dt_{n-1}dt \\ &= \int_{[0,1]^n} F\Big(\sum_{i=1}^n t_i v_i\Big) dt_1 \cdots dt_n\,.\end{aligned}$$

The proof for $\tilde{L}(\cdot|X_n)$ is similar. $\square$

Inserting $f(v) = \exp(-iv\cdot\omega)$ with $\omega = (\omega_1, \omega_2)$ in Theorem 12.47, we get the following formulae for the Fourier transform.

Theorem 12.48. *The Fourier transform of* $L(\cdot|X_n)$ *is*

$$\widehat{L}(\cdot|X_n)(\omega) = \prod_{j=1}^n \frac{1 - e^{-i\omega\cdot v_j}}{i\omega\cdot v_j}\widehat{L}_{2200}(\omega),$$

where $\widehat{L}_{2200}(\omega)$ *denotes the Fourier transform of* L_{2200}. *A similar formula holds for the Fourier transform of* $\tilde{L}(v|X_n)$.

We now give a refinement equation for half box splines.

Theorem 12.49. *There exists a finite sequence of* 2×2 *matrices* A_ν *with*

$$\begin{bmatrix} L(v|X_n) \\ \tilde{L}(v|X_n) \end{bmatrix} = \sum_{\nu\in\mathbb{Z}^2} A_\nu \begin{bmatrix} L(2v-\nu|X_n) \\ \tilde{L}(2v-\nu|X_n) \end{bmatrix}. \tag{12.26}$$

Proof: Using the Fourier transform of $L(\cdot|X_n)$ given in Theorem 12.48, we have

$$\begin{aligned}\widehat{L}(\cdot|X_n)(\omega) &= \prod_{j=1}^n \frac{1+e^{i\omega\cdot v_j/2}}{2}\,\frac{1-e^{i\omega\cdot v_j/2}}{iv_j/2}\widehat{L}_{2200}(\omega) \\ &= \prod_{j=1}^n \frac{1+e^{i\omega\cdot v_j/2}}{2}\widehat{L}(\cdot|X_n)(\omega/2)\,\frac{\widehat{L}_{2200}(\omega)}{\widehat{L}_{2200}(\omega/2)},\end{aligned}$$

with a similar formula for the Fourier transform of $\tilde{L}$. It is easy to see that

$$\begin{bmatrix} L_{2200}(v) \\ \tilde{L}_{0022}(v) \end{bmatrix} = \sum_{\nu \in \mathbb{Z}^2} C_\nu \begin{bmatrix} L_{2200}(2v-\nu) \\ \tilde{L}_{0022}(2v-\nu) \end{bmatrix}$$

for some 2×2 coefficient matrices C_ν. Indeed,

$$\begin{aligned} L_{2200}(v) &= \frac{1}{2} L_{2200}(2v) + \frac{1}{2} L_{2200}(2v - e^1) + \frac{1}{2} L_{2200}(2v - e^2) \\ &\quad + \frac{1}{2} L_{2200}(2v - e^1 - e^2) + \tilde{L}_{0022}(2v - e^1 - e^2), \\ \tilde{L}_{0022}(v) &= \frac{1}{2} \tilde{L}_{0022}(2v - e^1) + \frac{1}{2} \tilde{L}_{0022}(2v - e^2) + \frac{1}{2} \tilde{L}_{0022}(2v + e^1) \\ &\quad + \frac{1}{2} \tilde{L}_{0022}(2v + e^2) + \frac{1}{2} L_{2200}(2v) + \frac{1}{2} L_{0022}(2v + e^1) \\ &\quad + \frac{1}{2} L_{0022}(2v + e^2) + \frac{1}{2} L_{0022}(2v + e^1 + e^2). \end{aligned}$$

Taking the Fourier transform of the above relations, we see that

$$\frac{\widehat{L}_{2200}(\omega)}{\widehat{L}_{2200}(\omega/2)}, \qquad \frac{\widehat{\tilde{L}}_{0022}(\omega)}{\widehat{\tilde{L}}_{0022}(\omega/2)}$$

are Laurent trigonometric polynomials, and (12.26) follows. □

The B-coefficients of L_{jklm} and $\tilde{L}_{jklm}$ can be computed in the same way as was done for the type-I and type-II box splines discussed in Sections 12.1 and 12.2. We have shown above that the integer translates of both L_{2200} and $\tilde{L}_{0022}$ taken together can generate constants and linear polynomials. This suggests that we should study finite shift-invariant spaces.

12.8. Finite Shift-invariant Spaces

Let $\Phi := \{\phi_1, \ldots, \phi_m\}$ be a finite set of functions in $C(\mathbb{R}^2)$. Suppose that $\phi_1, \ldots, \phi_m$ are normalized so that

$$\sum_{i=1}^{m} \sum_{\nu \in \mathbb{Z}^2} \phi_i(v + \nu) \equiv 1, \qquad \text{all } v \in \mathbb{R}^2.$$

Let

$$\mathcal{S}(\Phi) := \operatorname{span}_{L_2(\mathbb{R}^2)} \{\phi_1(\cdot - \nu), \ldots, \phi_m(\cdot - \nu)\}_{\nu \in \mathbb{Z}^2}$$

be the linear span of integer translates of the functions in Φ. Then $\mathcal{S}(\Phi)$ is called the finite shift-invariant space associated with Φ. We say that the set Φ is linearly independent if the periodic functions $\{\sum_{\nu \in \mathbb{Z}^2} \phi_i(\cdot + \nu)\}_{i=1}^m$ are linearly independent.

Theorem 12.50. *Suppose that $\Phi := \{\phi_1, \ldots, \phi_m\}$ is a collection of compactly supported functions. Then Φ is linearly independent if and only if the sequences $\{\widehat{\phi}_i(2\pi\nu)\}_{\nu\in\mathbb{Z}^2}$, for $i = 1, \ldots, m$, are linearly independent, where $\widehat{\phi}_i$ denotes the Fourier transform of ϕ_i.*

Proof: Suppose there are real numbers $c_1, \ldots, c_m$ such that

$$F(\cdot) := \sum_{i=1}^{m} c_i \sum_{\nu\in\mathbb{Z}^2} \phi_i(\cdot + \nu) \equiv 0.$$

Note that $F(\cdot)$ is 1-periodic in both variables. Thus, the Fourier coefficients of F are all zero, i.e.,

$$\sum_{i=1}^{m} c_i \widehat{\phi}_i(2\pi\nu) = 0, \qquad \text{all } \nu \in \mathbb{Z}^2.$$

It follows that the linear dependence of Φ implies the linear dependence of the sequences $\{\widehat{\phi}_i(2\pi\nu)\}_{\nu\in\mathbb{Z}^2}$. The converse is also true. $\square$

Theorem 12.51. *Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a collection of compactly supported functions in $L_1(\mathbb{R}^2)$. Then the following two statements are equivalent:*

1) *Φ is linearly independent,*
2) *$\sum_{\nu\in\mathbb{Z}^2} \sum_{k=1}^{m} \phi_k(v-\nu)\, q_k(\nu) \equiv 0$ for the polynomials $q_1, \ldots, q_m \in S(\Phi)$ implies that $q_1 = \cdots = q_m = 0$.*

Proof: Statement 2) implies 1) by the definition of the linear independence of Φ. To prove 1) implies 2), we use induction on the degree of the polynomials. Suppose that $\mathcal{S}(\Phi)$ contains all polynomials of degree $\le d$. When q_k are constants, the linear independence of Φ implies 2) immediately. Assume that 1) implies 2) when q_k are polynomials of degree $n < d$. For q_k of degree $n + 1$, we have

$$\sum_{\nu\in\mathbb{Z}^2} \sum_{k=1}^{m} \phi_k(v-\nu) q_k(\nu) \equiv \sum_{\nu\in\mathbb{Z}^2} \sum_{k=1}^{m} \phi_k(v-\nu+e^1) q_k(\nu) \equiv 0.$$

It follows that

$$\sum_{\nu\in\mathbb{Z}^2} \sum_{k=1}^{m} \phi_k(v-\nu) \nabla_{e^1} q_k(\nu) = 0.$$

Since $\nabla_{e^1} q_k$ are of degree n, we use the induction hypothesis to get $\nabla_{e^1} q_k = 0$ for all $k = 1, \cdots, m$. Similarly, $\nabla_{e^2} q_k = 0$ for all $k = 1, \ldots, m$, and it follows that q_k are constants. But then the linear independence of Φ implies that q_k must be zero, and we have 2). $\square$

Example 12.52. The set $\{L_{2200}, \tilde{L}_{0022}\}$ is linearly independent. However, for any $j \geq 1$ and $k \geq 1$, $\{L_{jklm}, \tilde{L}_{jklm}\}$ is linearly dependent. Indeed, by Theorem 12.48, $\widehat{L}_{jklm}(2\pi\nu) = 0$ and $\widehat{\tilde{L}}_{jkl,m}(2\pi\nu) = 0$ for all $\nu \in \mathbb{Z}^2\backslash\{0\}$. Clearly, the two sequences $\{\widehat{L}_{j,klm}(2\pi\nu)\}_{\nu\in\mathbb{Z}^2}$ and $\{\widehat{\tilde{L}}_{jkl,m}(2\pi\nu)\}_{\nu\in\mathbb{Z}^2}$ are linearly dependent. By Theorem 12.50, $\{L_{jklm}, \tilde{L}_{jklm}\}$ is linearly dependent. Similarly, $\{B_{221}, B_{212}, B_{122}\}$ is linearly dependent.

Next we discuss the characterization of polynomials lying in $\mathcal{S}(\Phi)$.

Theorem 12.53. *Suppose that $\Phi := \{\phi_1, \ldots, \phi_m\}$ where ϕ_i are compactly supported functions. If there exist a finite linear combination ψ of integer translates of $\phi_1, \ldots, \phi_m$ such that*

$$D^\nu \hat{\psi}(2\beta\pi) = \delta_{0,\nu}\, \delta_{0,\beta}, \qquad \beta \in \mathbb{Z}^2 \text{ and } |\nu| \leq d, \tag{12.27}$$

then $\mathcal{S}(\Phi)$ contains the monomials $m_\alpha(v)$ for all $|\alpha| \leq d$. On the other hand, if $\mathcal{S}(\Phi)$ contains all monomials $m_\alpha(v)$ for $|\alpha| \leq d$ and Φ is linearly independent, then there exists a ψ which is a finite linear combination of integer translates of $\phi_1, \cdots, \phi_m$ satisfying (12.27).

Proof: Suppose that there exists $\psi \in \mathcal{S}(\Phi)$ satisfying (12.27). Then $\hat{\psi}(\xi) = B(\xi)\hat{\Phi}(\xi)$ for a $1 \times m$ vector $B(\xi)$ of trigonometric functions. Let

$$\psi_\nu := \frac{1}{\nu\,!}(-iD)^\nu B(0)\Phi,$$

for $|\nu| \leq d$. Then by the Leibniz formula,

$$\begin{aligned}
\delta_{0,\nu}\, \delta_{0,\beta} &= \frac{1}{\nu\,!} D^\nu (B\,\hat{\Phi})(2\pi\beta) \\
&= \sum_{\mu\leq\nu} \frac{(-iD)^{\nu-\mu} B(0)}{(\nu-\mu)\,!} \frac{(-iD)^\mu\, \hat{\Phi}(2\pi\beta)}{\mu\,!} \\
&= \sum_{\mu\leq\nu} \frac{(-iD)^\mu\, \hat{\psi}_{\nu-\mu}(2\pi\beta)}{\mu\,!} \\
&= \sum_{\mu\leq\nu} \int_{\mathbb{R}^2} \frac{1}{\mu\,!}(-v)^\mu\, \psi_{\nu-\mu}(v)\, e^{-i2\pi\beta\cdot v} dv \\
&= \int_{[0,1]^2} \sum_{\mu\leq\nu} \sum_{\alpha\in\mathbb{Z}^2} \frac{1}{\mu\,!}(\alpha - v)^\mu\, \psi_{\nu-\mu}(v-\alpha)\, e^{-i2\pi\beta\cdot v} dv\,.
\end{aligned}$$

It follows that

$$\sum_{\mu\leq\nu} \sum_{\alpha\in\mathbb{Z}^2} \frac{1}{\mu\,!}(\alpha - v)^\mu \psi_{\nu-\mu}(v-\alpha) = \delta_{0,\nu},$$

which is equivalent to

$$\frac{v^\nu}{\nu\,!} = \sum_{\mu\le\nu}\sum_{\alpha\in\mathbb{Z}^2}\frac{\alpha^\mu}{\mu\,!}\psi_{\nu-\mu}(v-\alpha).$$

Thus, the span of Φ contains the monomials $m_\nu(v)$ for all $|\nu|\le d$.

Suppose that $\mathcal{S}(\Phi)$ contains polynomials $\mathcal{P}_d$ of degree d. For a monomial m_α, there exist polynomial coefficients $c_{k,\alpha}$ such that

$$m_\alpha(v) = \sum_{\nu\in\mathbb{Z}^2}\sum_{k=1}^{m}\phi_k(v-\nu)\,c_{k,\alpha}(\nu), \tag{12.28}$$

for $|\alpha|\le d$. For $v\in\mathbb{Z}^2$, we have

$$m_\alpha(v) = \sum_{\nu\in\mathbb{Z}^2}\sum_{k=1}^{m}\phi_k(\nu)\,c_{k,\alpha}(v-\nu).$$

Since both sides are polynomials and agree on all integers, the above equation is true for all $v\in\mathbb{R}^2$. For any $\beta\le\alpha$, there exist coefficients, which we denote by $D^\beta c_{k,\alpha}(v-\nu)$, such that

$$m_{\alpha-\beta}(v) = D^\beta m_\alpha(v) = \sum_{\nu\in\mathbb{Z}^2}\sum_{k=1}^{m}\phi_k(\nu)D^\beta c_{k,\alpha}(v-\nu).$$

By Theorem 12.51, the linear independence of Φ implies that $D^\beta c_{k,\alpha} = c_{k,\alpha-\beta}$. As in Section 12.6, we solve the linear systems

$$\sum_{|\beta|\le d} a_{k,\beta}\beta^\alpha = \beta\,!\;c_{k,\alpha}(0,0), \qquad \text{all } |\alpha|\le d.$$

As in the proof of Lemma 12.41, we have

$$c_{k,\alpha}(\nu) = \sum_{|\beta|\le d} a_{k,\beta}\frac{1}{\alpha\,!}(\nu+\beta)^\alpha,$$

and hence by (12.28),

$$\begin{aligned} m_\alpha(v) =& \sum_{k=1}^{m}\sum_{\nu\in\mathbb{Z}^2}\phi_k(v-\nu)\sum_{|\beta|\le d} a_{k,\beta}\frac{1}{\alpha\,!}(\nu+\beta)^\alpha \\ =& \sum_{\nu\in\mathbb{Z}^2}\frac{1}{\alpha}(\nu)^\alpha\sum_{k=1}^{m}\sum_{|\beta|\le d} a_{k,\beta}\,\phi_k(v-\nu+\beta). \end{aligned} \tag{12.29}$$

Letting

$$\psi(v) := \sum_{k=1}^{m} \sum_{|\beta|\le d} a_{k,\beta}\phi_k(v+\beta)$$

and noting that $m_\alpha(v) = v^\alpha/\alpha\,!$, (12.29) becomes

$$m_\alpha(v) = \sum_{\nu\in\mathbb{Z}^2} m_\alpha(\nu)\,\psi(v-\nu), \tag{12.30}$$

for all $|\alpha| \le d$. This leads to the desired conclusion

$$D^\beta \widehat{\psi}(2\pi\nu) = \delta_{0,\beta}\,\delta_{0,\nu}, \qquad \text{all } \nu \in \mathbb{Z}^2 \text{ and } |\beta| \le d.$$

Indeed, multiplying both sides of (12.30) by $\gamma\,!\; m_{\gamma-\alpha}(u)$ and summing over $\alpha \le \gamma$, we have

$$m_\gamma(v+u) = \sum_{\nu\in\mathbb{Z}^2} m_\gamma(\nu+u)\psi(v-\nu),$$

for $|\gamma| \le d$. Letting $u = -v$, we have

$$\delta_{0,\gamma} = \sum_{\nu\in\mathbb{Z}^2} m_\gamma(\nu - v)\psi(v-\nu).$$

Multiplying both sides by $e^{-2i\pi v\cdot\beta}$, and integrating over $[0,1]^2$, we get

$$\delta_{0,\gamma}\,\delta_{0,\beta} = \int_{\mathbb{R}^2} m_\gamma(-v)\psi(v)e^{-2i\pi v\cdot\beta}dv = (-iD)^\gamma\widehat{\psi}(2\pi\beta). \qquad \square$$

Equation (12.30) leads to the following result.

Corollary 12.54. *Suppose that there exists a $\psi \in \mathcal{S}(\Phi)$ satisfying (12.27). If Φ is linearly independent, then*

$$p(v) = \sum_{\nu\in\mathbb{Z}^2} p(\nu)\,\psi(v-\nu),$$

for any polynomial $p \in \mathcal{P}_d$.

We now study the approximation power of $\mathcal{S}(\Phi)$. Let G be a closed domain in $\mathbb{R}^2$, and let

$$d(f,\mathcal{S}(\Phi))_q := \inf_{g\in\mathcal{S}(\Phi)} ||f-g||_{q,G}.$$

For $h > 0$, let σ_h be the scaling operator defined in (12.21), and let $S^h(\Phi) := \{\sigma_h(f), f \in \mathcal{S}(\Phi)\}$. We say that $\mathcal{S}(\Phi)$ has approximation power k with respect to the q-norm provided that for any sufficiently smooth function $f \in L_q(\mathbb{R}^2)$,

$$d(f, S^h(\Phi))_q \leq K h^k, \qquad \text{all } h > 0,$$

where K is a constant independent of h.

For simplicity, suppose $\operatorname{supp}(\psi) \subset [0,1]^2$ and $\psi \geq 0$. Without loss of generality, we may assume that ψ is normalized so that

$$\int_{\mathbb{R}^2} \psi(v)\, dv = 1.$$

For $h > 0$, let $\psi_h(v) := \psi(v/h)/h^2$. Recall that ∇_u denotes the backward difference operator defined by

$$\nabla_u f(v) = f(v) - f(v-u).$$

We define $\nabla_u^k = \nabla_u \nabla_u^{k-1}$ for all positive integers k. For a given function $f \in L_p(\mathbb{R}^2)$, let

$$f_h(v) := \int_{\mathbb{R}^2} (f - \nabla_u^k f)(v) \psi_h(u)\, du.$$

It is easy to see that if $\psi \in C^k(\mathbb{R}^2)$, so is f_h. Indeed,

$$\nabla_u^k f(v) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(v - i\,u),$$

implies

$$\int_{\mathbb{R}^2} f(v - i\,u) \psi_h(u)\, du = \frac{1}{i^2} \int_{\mathbb{R}^2} f(u)\, \psi_h(v - u/i)\, du.$$

It follows that $f_h \in C^k(\mathbb{R}^2)$.

Lemma 12.55. *Fix $1 \leq q \leq \infty$. Then for any domain Ω,*

$$\|f_h\|_{\infty,\Omega} \leq K\, h^{-2/q} \|f\|_{q, B_{kh}(\Omega)}.$$

where $B_{kh}(\Omega) := \{x \in \mathbb{R}^2 : d(x,\Omega) \leq kh\}$, and K is a constant independent of f.

Proof: Let $\tilde{q}$ be such that $1/q + 1/\tilde{q} = 1$. Then by Hölder's inequality,

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^2} f(v - i\,u) \psi_h(u) du \Big| \\
&\qquad \leq \Big(\int_{[0,1]^2 h} |f(v - i\,u)|^q du \Big)^{1/q} \Big(\int_{[0,1]^2 h} |\psi_h(u)|^{\tilde{q}} du \Big)^{1/\tilde{q}} \\
&\qquad \leq \|f\|_{q, B_{kh}(v)} h^{-2/q} \Big(\int_{[0,1]^2} \psi(u)^{\tilde{q}} du \Big)^{1/\tilde{q}} \\
&\qquad \leq \|f\|_{q, B_{kh}(\Omega)} C h^{-2/q},
\end{aligned}$$

where $K_1 := \big(\int_{[0,1]^2} \psi(u)^{\tilde q} du \big)^{1/\tilde q}$. It follows that

$$\|f_h\|_{\infty,\Omega} \le K_1 \sum_{i=1}^{k} \binom{k}{i} \|f\|_{q,B_{kh}(\Omega)} h^{-2/q} \le K\, h^{-2/q} \|f\|_{q,B_{kh}(\Omega)}. \qquad \square$$

Lemma 12.56. *Let Ω be any domain in $\mathbb{R}^2$, and suppose $k \ge 1$. Then there exists a constant K depending only on k such that for any $f \in W_q^k(\Omega)$,*

$$|f - f_h|_{j,q,\Omega} \le K\, h^{k-j} |f|_{k,q,B_{kh}(\Omega)}, \tag{12.31}$$

for all $0 \le j \le k$.

Proof: First we assume that $f \in C_0^k(\mathbb{R}^2)$, and estimate $\|D^\alpha(f - f_h)\|_{q,\Omega}$ for $\alpha \in \mathbb{Z}_+^2$ and $|\alpha| \le k$. Since

$$f(v) - f_h(v) = \int_{\mathbb{R}^2} \nabla_u^k f(v) \psi_h(u)\, du,$$

we have

$$D^\alpha(f - f_h)(v) = \int_{\mathbb{R}^2} \nabla_u^k D^\alpha f(v) \psi_h(u)\, du.$$

For all $j \ge 1$, Peano's theorem for the backward different operator ∇ gives

$$\nabla^j g(v) = \int_0^j D^j g(v - t) B^j(t)\, dt, \qquad \text{all } g \in C^k(\mathbb{R}),$$

where $B^j(t)$ is the j-th order univariate B-spline, see [Boo78, Sch81]. Thus,

$$\nabla_u^k D^\alpha f(v) = \nabla_u^j \nabla_u^{k-j} D^\alpha f(v) = \int_0^{k-j} \nabla_u^j D_t^{k-j} D^\alpha f(v - tu) B^{k-j}(t)\, dt.$$

Hence, for $j = |\alpha|$,

$$D^\alpha(f - f_h)(v) = \int_{\mathbb{R}^2} \int_0^{k-j} \nabla_u^j D_t^{k-j} D^\alpha f(v - tu) B^{k-j}(t)\, du.$$

By the generalized Minkowski inequality,

$$\begin{aligned}
&\|D^\alpha(f - f_h)\|_{L_q(\Omega)} \\
&\qquad \le \int_{\mathbb{R}^2} \int_0^{k-j} \|\nabla_u^j D_t^{k-j} D^\alpha f(\cdot - tu)\|_{q,\Omega}\, B^{k-j}(t)\, \psi_h(u)\, dt\, du \\
&\qquad \le K_1 \int_{\mathbb{R}^2} \int_0^{k-j} |u|^{k-j} |f|_{k,q,B_{kh}(\Omega)}\, B^{k-j}(t)\, \psi_h(u)\, dt\, du \\
&\qquad \le K_2 h^{k-j} |f|_{k,q,B_{kh}(\Omega)},
\end{aligned}$$

where we have used the facts that $B^{k-j}(t) \geq 0$ and

$$\int_0^{k-j} B^{k-j}(t)dt = 1.$$

Here K_1 and K_2 are two positive constants. Since $C_0^k(\mathbb{R}^2) \cap W_q^k(\Omega)$ is dense in $W_q^k(\mathbb{R}^2)$ for all $1 \leq q < \infty$, we have established (12.31) for this range of q. For $q = \infty$, we note that with $g = D^\alpha(f - f_h)$, $\|g\|_{q,\Omega}$ converges to $\|g\|_{\infty,\Omega}$ as $q \to \infty$. $\square$

We are now ready to prove the main result in this section.

Theorem 12.57. *Suppose that Φ is linearly independent, and that Φ contains a finite linear combination ψ of integer translates of $\phi_1, \cdots, \phi_m$ satisfying (12.27) for some integer d. Then for any $f \in W_q^{d+1}(\mathbb{R}^2)$,*

$$\|f - S_h\|_{q,\mathbb{R}^2} \leq K\, h^{d+1} |f|_{d+1,q,\mathbb{R}^2},$$

where $S_h(v) := \sum_{\nu \in \mathbb{Z}^2} f_h(h\nu)\, \psi(v/h - \nu)$.

Proof: By Lemma 12.56, we it suffices to prove that

$$\|f_h - S_h\|_{q,\mathbb{R}^2} \leq K_1\, h^{d+1} |f|_{d+1,q,\mathbb{R}^2}.$$

For each integer $\nu \in \mathbb{Z}^2$, let $G_{\nu,h} := (\nu + [0,1]^2)h$. For $v \in G_{\nu,h}$, let T_f be the Taylor expansion of f_h of degree d at v. Since $\mathcal{S}(\Phi)$ reproduces polynomials of degree d by Corollary 12.54, we have

$$T_f(u) = \sum_{\alpha \in \mathbb{Z}^2} T_f(h\alpha)\, \psi(u/h - \alpha), \quad \text{all } u \in \mathbb{Z}^2,$$

and hence

$$\begin{aligned} f_h(v) - S_h(v) &= T_f(v) - S_h(v) \\ &= \sum_{\alpha \in \mathbb{Z}^2} (T_f(h\alpha) - f_h(h\alpha))\psi(v/h - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} (T_f(h\nu + h\alpha) - f_h(h\nu + h\alpha))\psi(v/h - \nu - \alpha). \end{aligned}$$

Since $v/h - \nu \in [0,1]^2$ and ψ is of compact support,

$$|\psi(v/h - \nu - \alpha)| \leq C(1 + |\alpha|)^{-d-6}, \quad \alpha \in \mathbb{Z}^2.$$

By the formula for the remainder of the Taylor expansion,

$$T_f(h\nu + h\alpha) - f_h(h\nu + h\alpha) = -\frac{1}{d!} \int_0^1 D^{d+1}_{h\nu + h\alpha - v} f_h(\xi_{\alpha,t}),$$

where $\xi_{\alpha,t} := (1-t)v + t(h\nu + h\alpha)$. It is easy to see that

$$|D^{k+1}_{h\nu + h\alpha - v} f_h(\xi_{\alpha,t})| \leq h^{d+1}(1 + |\alpha|)^{d+1} \int_0^1 F_h(\xi_{\alpha,t})dt,$$

where $F_h(\xi_{\alpha,t}) := \sum_{|\beta|=k+1} |D^\beta f_h(\xi_{\alpha,t})|$. Thus, for $v \in G_{\nu,h}$,

$$|f_h(v) - S_h(v)| \le K_1\, h^{d+1} \sum_{\alpha\in\mathbb{Z}^2} (1+|\alpha|)^{-5} \int_0^1 F_h(\xi_{\alpha,t})dt.$$

Let $\tilde{q}$ be such that $1/q + 1/\tilde{q} = 1$. By Hölder's inequality, the right-hand side of this inequality gives

$$\Big(\sum_{\alpha\in\mathbb{Z}^2}(1+|\alpha|)^{-5}\Big)^{1/\tilde{q}}\Big(\sum_{\alpha\in\mathbb{Z}^2}(1+|\alpha|)^{-5}\int_0^1 (F_h(\xi_{\alpha,t}))^q dt\Big)^{1/q}.$$

It follows that

$$\begin{aligned}\|f_h - S_h\|_q^q &= \sum_{\nu\in\mathbb{Z}^2}\int_{G_{\nu,h}} |f_h(v) - S_h(v)|^q dx \\ &\le K_2^q\, h^{(d+1)q} \sum_{\alpha\in\mathbb{Z}^2}(1+|\alpha|)^{-5} \sum_{\nu\in\mathbb{Z}^2}\int_{G_{\nu,h}}\int_0^1 \big(F_h(\xi_{\alpha,t})\big)^q\, dt\, dv.\end{aligned}$$

To estimate these integrals, we divide $[0,1]$ into $1+|\alpha|$ subintervals of equal length. Let

$$I_j := \Big[\frac{j}{1+|\alpha|}, \frac{j+1}{1+|\alpha|}\Big], \quad j = 0, \ldots, 1+|\alpha|.$$

Note that for $v \in G_{\nu,h}$ and $t \in I_j$,

$$\xi_{\alpha,t} = ((1-t)v + th\nu) + th\alpha \in G_{\nu,h} + \alpha h I_j,$$

and

$$\int_{I_j} |F_h(\xi_{\alpha,t})|^q dt \le \frac{1}{1+|\alpha|}\|F_h\|^q_{\infty, G_{\nu,h}+\alpha h I_j}.$$

The arguments in the proof of Lemma 12.55 can be applied to the right-hand side of the above inequality to give

$$\|F_h\|^q_{\infty,G_{\nu,h}+\alpha h I_j} \le K^q\, h^{-2}\|F\|^q_{B_{kh}(G_{\nu,h}+\alpha h I_j)}.$$

It follows that

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}^2}\int_{G_{\nu,h}}\int_0^1 |F_h(\xi_{\alpha,t})|^q\, dt\, dx &\le \frac{K^q}{1+|\alpha|}\sum_{j=0}^{1+|\alpha|}\sum_{\nu\in\mathbb{Z}^2}\|F\|^q_{q,B_{kh}(G_{\nu,h}+h\alpha I_j)} \\ &\le \frac{K^q}{1+|\alpha|}\sum_{j=0}^{1+|\alpha|}(2j+5)^2\|F\|^q_{q,\mathbb{R}^2}.\end{aligned}$$

Then $\|f_h - S_h\|_{q,\mathbb{R}^2} \le K_1K_2K\, h^{d+1}|f|_{d+1,q,\mathbb{R}^2}$ follows from

$$\|F\|^q_{q,\mathbb{R}^2} = \int_{\mathbb{R}^2}\sum_{|\beta|=d+1}|D^\beta f(v)|^q dv = |f|^2_{d+1,q,\mathbb{R}^2}. \quad \square$$

Example 12.58. The integer translates of L_{2200} and $\tilde{L}_{0022}$ reproduce linear polynomials and are linearly independent. By Theorem 12.57, the approximation power of the finite shift-invariant space generated by L_{2200} and $\tilde{L}_{0022}$ is two.

Example 12.59. The integer translates of $B_{221}, B_{122}, B_{212}$ contain all cubic polynomials, but are not linearly independent. The approximation power of $\mathcal{S}(B_{221}, B_{212}, B_{122})$ is three. Thus, the condition of linear independence is necessary in Theorem 12.57.

12.9. Remarks

Remark 12.1. For additional examples where the B-coefficients of box splines have been worked out, see [Lai92b].

Remark 12.2. Since box splines are bivariate splines on triangulations, we can associate a surface with any box spline series. Such surfaces have a nice convex preserving property: if the given data $\{c_j, j \in \mathbb{Z}^2\}$ is convex in the sense that the piecewise linear surface $\sum_{j\in\mathbb{Z}^2} c_j B_{111}(v - j)$ is convex, then $\sum_{j\in\mathbb{Z}^2} c_j B(v - j|X_n)$ is convex for any X_n which contains e^1, e^2, and e^3, see [DahM88].

Remark 12.3. In Theorem 12.19 we showed that shifted type-I box splines are (globally) linearly independent. In fact, it can be shown that they are also locally linearly independent in the following sense: for any open set A, the shifted box splines

$$\{B(\cdot - \nu|X) : \operatorname{supp}(B(\cdot - \nu|X) \cap A \neq \emptyset\}$$

are linearly independent. One proof of this fact uses arguments similar to those in the proof of Theorem 12.19, see [Jia85] for details. Another proof can be found in [DahM85c].

Remark 12.4. In this book we have focused on box splines associated with either three or four directions since these lead to splines on type-I and type-II partitions. Box splines can be constructed using an arbitrary number of directions. If we add $e^5 := 2e^1 + e^2$ and $e^6 := e^1 + 2e^2$ to the four directions e^1, e^2, e^3, e^4 used in Section 12.2, we get splines defined on a six-direction mesh. For example, if $X_6 := \{e^i\}_{i=1}^6$ then $B(\cdot|X_6)$ is a C^3 spline of degree four. Adding the directions $e^7 := -e^1 + 2e^2$ and $e^8 := -2e^1 + e^2$ leads to box splines on an eight-direction mesh. For example, with $X_8 := \{e^i\}_{i=1}^8$, the corresponding box spline $B(\cdot|X_8)$ is a C^5 spline of degree six.

Remark 12.5. Our discussion of box splines can be generalized to the multivariate setting. For example, we can replace the three direction vectors $e^1 := (1,0)$, $e^2 := (0,1)$, $e^3 := (1,1)$ in the plane by the four direction vectors $d^1 := (1,0,0)$, $d^2 := (0,1,0)$, $d^3 := (0,0,1)$, $d^4 := (1,1,1)$ in $\mathbb{R}^3$ to define box splines in the trivariate setting. For more on multivariate box splines, see the book [BooHR93].

Remark 12.6. We can also define a class of half-box splines for type-I partitions by starting with the functions

$$H(v) := \begin{cases} 1, & \text{if } v = (x,y) \text{ and } 0 \le x \le y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tilde{H}(v) := \begin{cases} 1, & \text{if } v = (x,y) \text{ and } 0 \le x < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

with an appropriate adjustment on the common boundary of the two triangles $\langle (0,0), (1,0), (1,1) \rangle$ and $\langle (0,0), (1,1), (0,1) \rangle$ such that

$$1 \equiv \sum_{\nu \in \mathbb{Z}^2} \big[H(v-\nu) + \tilde{H}(v-\nu) \big].$$

Thus, the integer translates of H and $\tilde{H}$ taken together form a partition of unity. Half box splines on type-I partitions first appeared in [Sab77], see also [Pra84].

Remark 12.7. When $l = 1$, the function ϕ in Example 12.32 is a spline in $S_3^1(\triangle_{II})$ with Strang–Fix degree three. Thus, we can use ϕ to construct a quasi-interpolant Q_Γ^h as in Theorem 12.44 which approximates smooth functions to order four. For another proof that this space has full approximation power, see [DahM84a].

Remark 12.8. As with univariate splines, it is possible to define a kind of discrete box spline. They were introduced in [CohLR84] as a tool in computer-aided geometric design, see also [DahM86, DahM87, BooHR93].

Remark 12.9. Box splines are a special case of a more general class of piecewise polynomials called simplex splines. They were heavily studied in the late 1970s and early 1980's. For a detailed survey of the subject and an extensive list of references, see [DahM83]. This class of splines also includes the so-called cone splines introduced in [Dah79].

Remark 12.10. For another construction of locally supported piecewise polynomials on triangulations that relates to both box splines and simplex splines, see [DahMS92]. These splines have been called triangular B-splines or DMS splines. There is also an analog defined on the sphere, see [PfeS95].

Remark 12.11. In Remark 5.9 we pointed out that very little work has been done on wavelet spaces associated with bivariate splines defined on general triangulations. However, much more has been done for box spline spaces. Some early papers are [RieS91, ChuStW92, BooDR93, DahlDL95, DahlGL97, LorRO97, HeWL98, RonS98, HeWL99]. For more recent work and further references, see [BuhDG01, BuhDG03, HeWL03, Lai06, LaiSt06].

12.10. Historical Notes

Box splines were first introduced by de Boor and DeVore in [BooD83], and were studied by many researchers in the 1980's, including de Boor, Dahmen, Höllig, Jia, and Micchelli. For a full-length book treatment and more detailed historical notes, see [BooHR93].

The computation of B-coefficients of box splines on type-I and type-II triangulations was discussed in [ChuL87c]. Fortran programs for such computations can be found in [Lai92b]. Graphs of various box spline surfaces can be found in [ChuL92]. The so-called line average and subdivision algorithms for (approximately) rendering box spline surfaces were described in [DahM85c] and [DahM84c], respectively. Although there exists a recurrence relation for box splines [BooHR93], it does not lead to a stable algorithm for the evaluation of box splines, see [Boo93b]. For stability and efficiency it is best to compute B-coefficients of the box splines.

Some of the material discussed in Section 12.3 can be found in [BooH82, BooH83a, BooH83b]. Additional properties of box splines can be found in [DahM83]. The proof of Theorem 12.19 follows ideas in [Jia84]. A different proof can be found in [DahM85a]. Local linearly independence of translates of a box spline is studied in [Jia85] and [DahM85b]. The concepts of principal shift-invariant spaces and finite shift invariant spaces were introduced in [BooDR94]. The Strang–Fix conditions were given in [StrF73], and the commutator was introduced in [ChuJW87]. The material for Sections 12.4, 12.5, and 12.6 is taken from [ChuL87b]. Examples 12.31 and 12.32 can be found in [DahM84b] and [DahM84a], respectively.

The definition of linear independence of finite shift-invariant spaces can be found in [Jia98]. The Strang–Fix conditions in the setting of finite shift-invariant spaces were studied by several researchers, see [CabHM98] and [JiaL93]. The proof of Theorem 12.57 follows [JiaL93].

Type-I box splines have been used to investigate the approximation order of the spline spaces $S_d^r(\triangle)$ for $d \le 3r+1$, see Chapter 10 and [Jia86] and [Jia88] for more details. Cardinal splines based on type-I box spline series, which are a generalization of Schoeberg's cardinal splines based on univariate B-splines with equally spaced knots, were studied by de Boor, Höllig and Riemenschneider in the 1980's, see [BooHR93].

The basic type-II box spline B_{1111} introduced in Section 12.2 can be found in [Zwa73], before box splines were formally introduced in [BooD83].

Spherical Splines

In this chapter and the next we discuss spline spaces defined on triangulations of the unit sphere S in $\mathbb{R}^3$. The spaces are natural analogs of the bivariate spline spaces discussed earlier in this book, and are made up of pieces of trivariate homogeneous polynomials restricted to S. Thus, they are piecewise spherical harmonics. As we shall see, virtually the entire theory of bivariate polynomial splines on planar triangulations carries over, although there are several significant differences. This chapter is devoted to the basic theory of spherical splines. Approximation properties of spherical splines are treated in the following chapter.

13.1. Spherical Polynomials

In this section we introduce the key building blocks for spherical splines. Throughout the chapter we write v for a point on the unit sphere S in $\mathbb{R}^3$. When there is no chance of confusion, at times we will also write v for the corresponding unit vector. Before introducing spherical polynomials, we need to discuss spherical triangles and spherical barycentric coordinates.

13.1.1 Spherical Triangles

Suppose v_1, v_2 are two points on the sphere which are not antipodal, i.e., they do not lie on a line through the origin. Then the points v_1, v_2 divide the great circle passing through v_1, v_2 into two circular arcs. We write $\langle v_1, v_2 \rangle$ for the shorter of the arcs. Its length is just the geodesic distance between v_1 and v_2.

Definition 13.1. *Suppose v_1, v_2, v_3 are three points on the unit sphere S which lie strictly in one hemisphere. Then we define the associated* spherical triangle *$T := \langle v_1, v_2, v_3 \rangle$ to be the set of points on S that lie in the region bounded by the three circular arcs $\langle v_i, v_{i+1} \rangle$, $i = 1, 2, 3$, where we identify $v_4 = v_1$. We say that T is* nondegenerate *provided that T has nonzero area.*

We call the points v_1, v_2, v_3 the vertices of T, and refer to the circular arcs $\langle v_i, v_{i+1} \rangle$ as the edges of T. Unless otherwise stated, we shall assume that the vertices of T are in clockwise order, viewed from the origin.

13.1.2 Spherical Barycentric Coordinates

In this subsection we introduce spherical barycentric coordinates and describe their basic properties as well as two important differences as compared to planar barycentric coordinates.

Definition 13.2. *Let $T := \langle v_1, v_2, v_3 \rangle$ be a nondegenerate spherical triangle. Given $v \in S$, let $b_i := b_i(v)$ be such that*

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3. \tag{13.1}$$

Then we call b_1, b_2, b_3 the spherical barycentric coordinates *of v* relative to *T.*

We claim that the spherical barycentric coordinates of a point $v \in S$ relative to a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$ are unique. To see this, suppose $v = (x, y, z) \in \mathbb{R}^3$, and suppose $v_i := (x_i, y_i, z_i)$, $i = 1, 2, 3$. Then

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{13.2}$$

The determinant of the matrix in (13.2) is equal to six times the volume of the tetrahedron $T := \langle 0, v_1, v_2, v_3 \rangle$. It follows that if T is nondegenerate, then the spherical barycentric coordinates of any point $v \in \mathbb{R}^3$ are uniquely defined. By Cramer's rule,

$$b_1(x, y, z) = \frac{\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ z & z_2 & z_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}}, \tag{13.3}$$

with similar formulae for b_2 and b_3.

Theorem 13.3. *Let T be a nondegenerate spherical triangle. Then*

1) $b_i(v_j) = \delta_{ij}$, *for all* $i, j = 1, 2, 3$,
2) b_1 *is the ratio* $vol(t_1)/vol(t)$ *of the signed volumes of the tetrahedra* $t_1 := \langle 0, v, v_2, v_3 \rangle$ *and* $t := \langle 0, v_1, v_2, v_3 \rangle$, *with a similar interpretation for* b_2 *and* b_3,
3) $b_1(v), b_2(v), b_3(v) \geq 0$ *for all* $v \in T$,
4) b_i *vanishes on the edge of* T *opposite to* v_i, *for all* $i = 1, 2, 3$.

Proof: Properties 1), 2), and 4) are obvious from (13.3) and the connection between determinants and volumes of tetrahedra. Property 3) follows from the formula $b_i = vol(t_i)/vol(t)$ and the observation that the volume of t_i is nonnegative whenever $v \in T$. $\square$

Since the volumes of tetrahedra do not change under rotation, it follows that the spherical barycentric coordinates of a point v relative to a spherical triangle T are rotation invariant, i.e., they do not change if we rotate v and the vertices of T by the same amount. This can also be seen directly. Recall that rotation on the sphere is described by a 3×3 orthogonal matrix with $\det M = 1$ and $M^T M = I$.

Theorem 13.4. *Suppose $T := \langle v_1, v_2, v_3 \rangle$ is a spherical triangle, and let $\widetilde{T} := \langle Mv_1, Mv_2, Mv_3 \rangle$, where M is an orthogonal matrix. Given $v \in S$, suppose b_1, b_2, b_3 and b_1^M, b_2^M, b_3^M are the spherical barycentric coordinates of v relative to T and $\widetilde{T}$, respectively. Then*

$$b_i^M(Mv) = b_i(v), \qquad i = 1, 2, 3.$$

Proof: Multiplying (13.1) by M, we have $Mv = b_1 M v_1 + b_2 M v_2 + b_3 M v_3$. $\square$

Theorem 13.3 shows that spherical barycentric coordinates have most of the properties of planar barycentric coordinates. There are some important differences though. Perhaps most significantly, spherical barycentric coordinates do not sum to 1, except at the vertices of T.

Lemma 13.5. *For any nondegenerate spherical triangle $T := \langle v_1, v_2, v_3 \rangle$,*

$$b_1(v) + b_2(v) + b_3(v) > 1, \qquad \text{all } v \in T \setminus \{v_1, v_2, v_3\}.$$

Proof: For each $i = 1, 2, 3$, $b_i = \mathit{vol}(t_i)/\mathit{vol}(t)$, where $t := \langle 0, v_1, v_2, v_3 \rangle$ and $t_i := \langle 0, v, v_{i+1}, v_{i+2} \rangle$. Here we identify $v_4 = v_1$ and $v_5 = v_2$. Now it is clear from the geometry that $\mathit{vol}(t_1 \cup t_2 \cup t_3) > \mathit{vol}(t)$ except when v is at one of the vertices of T. $\square$

There is another important difference between planar and spherical barycentric coordinates. If v is a point in a planar triangle T, then its planar barycentric coordinates relative to T are bounded by 1. However, if v is a point in a spherical triangle T, then its spherical barycentric coordinates relative to the spherical triangle T can be arbitrarily large unless we restrict the size of T. This is due to the fact that if the vertices of the spherical triangle $T := \langle v_1, v_2, v_3 \rangle$ are such that the planar triangle with the same vertices passes near the origin, then the volume of the tetrahedron $\langle 0, v_1, v_2, v_3 \rangle$ is very small. In this connection we have the following result.

Lemma 13.6. *Suppose $T := \langle v_1, v_2, v_3 \rangle$ is a spherical triangle which is small enough to be rotated into an octant of the unit sphere. Then for every $v \in T$,*

$$0 \le b_i(v) \le 1, \qquad i = 1, 2, 3.$$

Proof: It suffices to establish the result for b_1. Suppose we rotate T so that it lies in the octant $x, y, z \geq 0$, and so that the vertices v_2 and v_3 lie in the x–y plane. Let (x_1, y_1, z_1) be the Cartesian coordinates of v_1. Then the volume of the tetrahedron $t := \langle 0, v_1, v_2, v_3 \rangle$ is given by $vol\,(t) = Az_1/3$, where A is the area of the planar triangle with vertices $0, v_2, v_3$. Similarly, the volume of the tetrahedron $t_1 := \langle 0, v, v_2, v_3 \rangle$ is given by $vol\,(t_1) = Az/3$, where (x, y, z) are the Cartesian coordinates of v. Now the fact that v lies in the spherical triangle T implies that $z \leq z_1$, and thus $b_1 = vol\,(t_1)/vol\,(t) \leq 1$. $\square$

Combining property 1) of Theorem 13.3 with Lemma 13.5, it is easy to see that it is impossible to write the constant function 1 as a linear combination of b_1, b_2, b_3. However, we have the following useful expansion.

Lemma 13.7. *Let $T := \langle v_1, v_2, v_3 \rangle$, and for each $i = 1, 2, 3$, let a_i be the arc length of the edge of T opposite to vertex v_i. Then*

$$1 = b_1^2 + b_2^2 + b_3^2 + \beta_{011} b_2 b_3 + \beta_{101} b_1 b_3 + \beta_{110} b_1 b_2, \tag{13.4}$$

where

$$\beta_{011} = 2\cos a_1, \quad \beta_{101} = 2\cos a_2, \quad \beta_{110} = 2\cos a_3.$$

Proof: It follows from results in Section 13.1.8 on homogeneous polynomials that for (x, y, z) on the sphere, the function $1 = x^2 + y^2 + z^2$ can be written as a linear combination of the functions $\{b_1^2, b_2^2, b_3^2, b_1 b_2, b_1 b_3, b_2 b_3\}$. The associated coefficients can be found from the linear system corresponding to interpolation at the vertices of T and midpoints of the edges of T. $\square$

13.1.3 Spherical Bernstein Basis Polynomials

We now define spherical Bernstein basis polynomials as products of the barycentric coordinate functions b_1, b_2, b_3 of Section 13.1.2.

Definition 13.8. *Given a spherical triangle T and an integer d, let*

$$B_{ijk}^d := \frac{d!}{i!\, j!\, k!} b_1^i b_2^j b_3^k, \qquad i + j + k = d.$$

We call these spherical Bernstein basis polynomials of degree *d.*

Although they are direct analogs of the Bernstein basis polynomials of Section 2.2, spherical Bernstein basis polynomials B_{ijk}^d are not algebraic polynomials. As we shall see in Section 13.1.9, they are spherical harmonics.

It is clear from the definition of the spherical Bernstein basis polynomials B^d_{ijk} that they satisfy the same recurrence relation, see (2.20), as the classical bivariate Bernstein polynomials, namely

$$B^d_{ijk} = b_1 B^{d-1}_{i-1,j,k} + b_2 B^{d-1}_{i,j-1,k} + b_3 B^{d-1}_{i,j,k-1}, \tag{13.5}$$

for all $i+j+k = d$. Here we are using the convention that expressions with negative subscripts are defined as zero. For later use, we need a bound on the size of the spherical Bernstein basis polynomials $B^d_{ijk}(v)$ for $v \in T$. In the bivariate case, the Bernstein basis polynomials are bounded by 1.

Lemma 13.9. *Let $\{B^d_{ijk}\}_{i+j+k=d}$ be the spherical Bernstein basis polynomials associated with a spherical triangle T that can be rotated to lie in an octant of the unit sphere. Then for all $v \in T$,*

$$0 \le B^d_{ijk}(v) \le \frac{d\,!}{i\,!\,j\,!\,k\,!}, \qquad \text{all } i+j+k=d,$$

and

$$\sum_{i+j+k=d} B^d_{ijk}(v) \le 3^d. \tag{13.6}$$

Proof: The first statement follows immediately from Lemma 13.6. The second statement follows from the trinomial expansion

$$\sum_{i+j+k=d} B^d_{ijk} = (b_1+b_2+b_3)^d. \quad \square$$

13.1.4 The Spherical B-form

Let $\mathcal{B}_d :=$ span $\{B^d_{ijk}\}_{i+j+k=d}$. In Corollary 13.19 below we show that the $\{B^d_{ijk}\}_{i+j+k=d}$ are linearly independent. Thus, every $p \in \mathcal{B}_d$ has a unique expansion of the form

$$p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}. \tag{13.7}$$

Definition 13.10. *We call $\mathcal{B}_d$ the* space of spherical polynomials of degree *d. For every $p \in \mathcal{B}_d$, we call (13.7) the* spherical B-form *of p, and refer to the c_{ijk} as the* spherical B-coefficients of *p.*

It is convenient to associate the coefficients c_{ijk} in (13.7) with the points

$$v_{ijk} := \frac{iv_1 + jv_2 + kv_3}{\|iv_1 + jv_2 + kv_3\|}, \quad i+j+k=d, \tag{13.8}$$

in the spherical triangle $T := \langle v_1, v_2, v_3 \rangle$. These points are the radial projections upward onto the sphere S of the usual domain points associated with the planar triangle with vertices v_1, v_2, v_3.

Definition 13.11. *We call the set* $\mathcal{D}_{d,T} := \{v_{ijk}\}_{i+j+k=d}$ *the* set of spherical domain points associated with T and d.

Following the proof of Theorem 2.8, and using the recurrence relation (13.5), we can easily derive a de Casteljau algorithm for evaluating spherical polynomials written in spherical B-form.

Theorem 13.12. *Let p be a polynomial written in the spherical B-form (13.7) with coefficients*

$$c_{ijk}^{(0)} := c_{ijk}, \qquad i+j+k=d,$$

and suppose the point v on the sphere S has spherical barycentric coordinates $b := (b_1, b_2, b_3)$. For all $\ell = 1, \ldots, d$, let

$$c_{ijk}^{(\ell)} := b_1\, c_{i+1,j,k}^{(\ell-1)} + b_2\, c_{i,j+1,k}^{(\ell-1)} + b_3\, c_{i,j,k+1}^{(\ell-1)},$$

for $i+j+k = d-\ell$. Then

$$p(v) = \sum_{i+j+k=d-\ell} c_{ijk}^{(\ell)} B_{ijk}^{d-\ell}(v), \tag{13.9}$$

for all $0 \le \ell \le d$. In particular $p(v) = c_{000}^{(d)}(b)$.

Theorem 13.12 immediately leads to an algorithm for evaluating a spherical polynomial p in the B-form (13.7).

Algorithm 13.13. *(de Casteljau)*

For $\ell = 1, \ldots, d$
For all $i+j+k = d-\ell$

$$c_{ijk}^{(\ell)} := b_1 c_{i+1,j,k}^{(\ell-1)} + b_2 c_{i,j+1,k}^{(\ell-1)} + b_3 c_{i,j,k+1}^{(\ell-1)}$$

Following the proof of Theorem 2.10, it is easy to see that the coefficients in the de Casteljau algorithm and in (13.9) are given by

$$c_{ijk}^{(\ell)} = \sum_{\nu+\mu+\kappa=\ell} c_{i+\nu,j+\mu,k+\kappa} B_{\nu\mu\kappa}^{\ell}(v), \qquad i+j+k = d-\ell.$$

13.1.5 Subdividing Spherical Polynomials

Suppose that p is a spherical polynomial written in the spherical B-form (13.7) relative to a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$. Let w be a point in the interior of T, and suppose T is split into the three spherical subtriangles

$$T_1 := \langle w, v_2, v_3 \rangle, \quad T_2 := \langle w, v_3, v_1 \rangle, \quad T_3 := \langle w, v_1, v_2 \rangle,$$

see Figure 2.1 for the planar case. The following theorem shows how to write p in spherical B-form on each of the subtriangles. Its proof is analogous to that of Theorem 2.38.

Theorem 13.14. *Let $a = (a_1, a_2, a_3)$ be the spherical barycentric coordinates of the point w in the interior of T. For each $\ell = 1, 2, 3$, let $B^{T_\ell,d}_{ijk}$ be the spherical Bernstein basis polynomials associated with T_ℓ. Then*

$$p(v) = \begin{cases} \displaystyle\sum_{i+j+k=d} c^{(i)}_{0jk} B^{T_1,d}_{ijk}(v), & v \in T_1, \\ \displaystyle\sum_{i+j+k=d} c^{(j)}_{i0k} B^{T_2,d}_{ijk}(v), & v \in T_2, \\ \displaystyle\sum_{i+j+k=d} c^{(k)}_{ij0} B^{T_3,d}_{ijk}(v), & v \in T_3, \end{cases}$$

where $c^{(\nu)}_{ijk} := c^{(\nu)}_{ijk}(a)$ are the quantities obtained in the ν-th step of the de Casteljau algorithm based on the triple a, starting with $c^{(0)}_{ijk} = c_{ijk}$.

13.1.6 Degree Raising Spherical Polynomials

Let B^d_{ijk} be a spherical Bernstein basis polynomial of degree d, and suppose we multiply it by $1 = b_1^2 + b_2^2 + b_3^2 + \beta_{011} b_2 b_3 + \beta_{101} b_1 b_3 + \beta_{110} b_1 b_2$, where the β's are as in Lemma 13.7. This shows that B^d_{ijk} can be written as a linear combination of spherical Bernstein basis polynomials of degree $d+2$, and it follows that

$$\mathcal{B}_0 \subset \mathcal{B}_2 \subset \mathcal{B}_4 \subset \cdots \quad \text{and} \quad \mathcal{B}_1 \subset \mathcal{B}_3 \subset \mathcal{B}_5 \subset \cdots.$$

Thus, we can rewrite any spherical polynomial of degree d as a spherical polynomial of degree $d + 2$. More explicitly, we have the following degree-raising formula.

Theorem 13.15. *Suppose p is a spherical polynomial written in the B-form (13.7) relative to a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$. Then*

$$p = \sum_{i+j+k=d+2} \bar{c}_{ijk} B^{d+2}_{ijk},$$

with

$$\bar{c}_{ijk} = \frac{1}{(d+1)(d+2)} \big[i(i-1) c_{i-2,j,k} + ij\beta_{110} c_{i-1,j-1,k} + j(j-1) c_{i,j-2,k} \\ + ik\beta_{101} c_{i-1,j,k-1} + k(k-1) c_{i,j,k-2} + jk\beta_{011} c_{i,j-1,k-1} \big].$$

where $\beta_{011}, \beta_{101}, \beta_{110}$ are as in Lemma 13.7.

Proof: Suppose p is a linear combination of spherical Bernstein basis polynomials of degree d as in (13.7). Then multiplying p by (13.4) and collecting terms, we find that p is a linear combination of spherical Bernstein basis polynomials of degree $d + 2$ with the stated coefficients. $\square$

13.1.7 Restrictions of Spherical Polynomials to Edges

Suppose p is a spherical polynomial written in the spherical B-form (13.7) relative to a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$. Let $g := p|_e$ be the restriction of p to the edge $e := \langle v_1, v_2 \rangle$. Then since the barycentric coordinate function $b_3(v)$ vanishes at all points v on e, we have

$$g(v) = \sum_{i+j=d} c_{ij0} \frac{d\,!}{i\,!\,j\,!} b_1^i b_2^j = \sum_{i=0}^{d} c_{i,d-i,0} \frac{d\,!}{i\,!\,(d-i)\,!} b_1^i b_2^{d-i}. \tag{13.10}$$

Functions of the form (13.10) defined on circular arcs are called circular Bernstein–Bézier (CBB) polynomials. We now show that in terms of the usual angular coordinate system of the unit circle C, they are not algebraic polynomials, but instead are trigonometric polynomials. Let e be the circular arc connecting the two points

$$v_1 = (\cos\theta_1, \sin\theta_1)^T, \qquad v_2 = (\cos\theta_2, \sin\theta_2)^T,$$

on C. Then every point $v = (\cos\theta, \sin\theta)^T$ on C can be written uniquely as

$$v = b_1 v_1 + b_2 v_2,$$

where

$$\begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

The numbers b_1, b_2 are called the circular barycentric coordinates of v. They are given explicitly by

$$b_1(\theta) = \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)}, \qquad b_2(\theta) = \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}. \tag{13.11}$$

The associated functions

$$B_i^d(\theta) := \binom{d}{i} b_1(\theta)^{d-i} b_2(\theta)^i, \qquad i = 0, \ldots, d,$$

are called circular Bernstein basis polynomials.

Theorem 13.16. *The circular Bernstein basis polynomials $\{B_i^d(\theta)\}_{i=0}^d$ form a basis for the space*

$$\mathcal{T}_d := \text{span}\,\{\sin^{d-i}(\theta)\cos^i(\theta)\}_{i=0}^d$$

of trigonometric polynomials of degree *d.*

Proof: By (13.11), $b_1(\theta)$ and $b_2(\theta)$ are both linear combinations of $\sin\theta$ and $\cos\theta$. Thus, the products $b_1(\theta)^{d-i}b_2(\theta)^i$ lie in $\mathcal{T}_d$ for all $i = 0, 1, \ldots, d$, and it follows that the circular Bernstein basis polynomials also lie in $\mathcal{T}_d$. The linear independence of the B_i^d can be shown directly by induction on the degree d. $\square$

It is easy to verify that

$$\mathcal{T}_d = \begin{cases} \text{span}\,\{1, \cos(2\theta), \sin(2\theta), \ldots, \cos(d\theta), \sin(d\theta)\}, & d \text{ even}, \\ \text{span}\,\{\cos(\theta), \sin(\theta), \cos(3\theta), \sin(3\theta), \ldots, \cos(d\theta), \sin(d\theta)\}, & d \text{ odd}. \end{cases}$$

For more on circular Bernstein–Bézier polynomials, including evaluation with a de Casteljau algorithm, subdivision, and degree raising, see [AlfNS95].

13.1.8 Homogeneous Trivariate Polynomials

In this section we show that spherical polynomials are the restriction to S of certain homogeneous trivariate polynomials. We begin with a definition.

Definition 13.17. *Given an arbitrary integer k, we say that a function f defined on $\mathbb{R}^3$ is* homogeneous of degree k *provided that $f(rx, ry, rz) = r^k f(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$ and every real number r. Let*

$$\mathcal{H}_d := \{p \in \mathcal{P}_d : p \text{ is homogeneous of degree } d\},$$

where $\mathcal{P}_d$ is the space of trivariate polynomials of degree d. Then we refer to $\mathcal{H}_d$ as the space of homogeneous trivariate polynomials of degree d.

By definition, a trivariate polynomial p of degree d is homogeneous if and only if it can be written in the form

$$p = \sum_{i+j+k=d} c_{ijk} x^i y^j z^k.$$

Since the monomials are clearly linearly independent, it follows that the dimension of $\mathcal{H}_d$ is $\binom{d+2}{2}$. We now construct an alternate basis for $\mathcal{H}_d$.

Suppose T is a spherical triangle with vertices $v_i := (x_i, y_i, z_i)$, $i = 1, 2, 3$, and let $v = (x, y, z) \in \mathbb{R}^3$. Let $h_1(v), h_2(v), h_3(v)$ be the unique solution of the system

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} h_1(v) \\ h_2(v) \\ h_3(v) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{13.12}$$

If v is on the unit sphere S, then (h_1, h_2, h_3) are just the spherical barycentric coordinates (b_1, b_2, b_3) discussed in Section 13.1.2. Using Cramer's rule, we have

$$h_1(x,y,z) = \frac{\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ z & z_2 & z_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}}, \tag{13.13}$$

with similar formulae for h_2 and h_3. It follows that the functions $h_i(v)$ are homogeneous polynomials of degree 1.

Theorem 13.18. *The functions*

$$H^d_{ijk} := \frac{d!}{i!\,j!\,k!} h_1^i h_2^j h_3^k, \qquad i+j+k=d, \tag{13.14}$$

form a basis for $\mathcal{H}_d$.

Proof: The fact that the h_i are homogeneous polynomials of degree 1 implies that H^d_{ijk} belong to $\mathcal{H}_d$. To show that the H^d_{ijk} form a basis for $\mathcal{H}_d$, it suffices to show that they span $\mathcal{H}_d$, i.e., every monomial of the form $x^\nu y^\mu z^\kappa$ with $\nu+\mu+\kappa=d$ can be written as a linear combination of the $\{H^d_{ijk}\}_{i+j+k=d}$. This is obvious for $d=0$. For $d=1$ it follows from the fact that $x = h_1x_1 + h_2x_2 + h_3x_3$ and the analgous formulae for y and z. The general case can be established by the same kind of inductive argument as used in the proof of Theorem 2.4. □

The space $\mathcal{B}_d$ of spherical polynomials introduced in Section 13.1.4 is closely related to the space $\mathcal{H}_d$. Indeed, $\mathcal{B}_d = \mathcal{H}_d|_S$. Using this connection and Theorem 13.18, we can now prove the linear independence of the spherical Bernstein basis polynomials.

Corollary 13.19. *The space* $\mathcal{B}_d$ *has dimension* $\binom{d+2}{2}$, *and the spherical Bernstein basis polynomials* $\{B^d_{ijk}\}_{i+j+k=d}$ *form a basis for it.*

Proof: It suffices to show that the B^d_{ijk} are linearly independent. For each $i+j+k=d$, B^d_{ijk} is just the restriction to the sphere S of the homogeneous polynomial H^d_{ijk}. Now suppose $\sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v) = 0$ for all $v \in S$. This implies $\sum_{i+j+k=d} c_{ijk} H^d_{ijk}(v) = 0$ for all $v \in \mathbb{R}^3$. Theorem 13.18 implies that the c_{ijk} must be zero, and the proof is complete. □

13.1.9 Spherical Harmonics

In this section we show that the space $\mathcal{B}_d$ has a basis consisting of classical spherical harmonics defined on the sphere. The casual reader can skip this section if desired, since we will not make further use of spherical harmonics in this book. Given a point $v = (x, y, z)$ lying on S, we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos(\pi-\theta) \end{bmatrix}, \tag{13.15}$$

with $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. The θ and ϕ are called the spherical coordinates of v.

Definition 13.20. *A trivariate polynomial p is called* harmonic *provided $\Delta p \equiv 0$, where Δ is the* Laplace operator *defined by $\Delta f := f_{xx} + f_{yy} + f_{zz}$. The linear space $\mathcal{Y}_d := \{p|_S : p \in \mathcal{P}_d$ and p is harmonic and homogeneous of degree $d\}$ is called the* space of spherical harmonics of exact degree d.

Given d and ℓ, let

$$P_d^\ell(x) = (1-x^2)^{\ell/2} D_x^\ell P_d(x), \qquad -1 \le x \le 1,$$

be the Legendre function of degree d and order ℓ, where P_d are Legendre polynomials satisfying the recurrence formula

$$P_d(x) = -\frac{d-1}{d} P_{d-2}(x) + \frac{2d-1}{d} x P_{d-1}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$. The following result is well known, see e.g. [CouH53], page 314.

Theorem 13.21. *$\mathcal{Y}_d$ is a linear space of dimension $2d+1$, and the functions*

$$\begin{aligned} Y_{d,2\ell+1}(\theta,\phi) &:= \cos(\ell\phi)\ P_d^\ell(\cos\theta), \qquad \ell = 0,\dots,d, \\ Y_{d,2\ell}(\theta,\phi) &:= \sin(\ell\phi)\ P_d^\ell(\cos\theta), \qquad \ell = 1,\dots,d, \end{aligned}$$

form an orthogonal basis for $\mathcal{Y}_d$ with respect to the standard L_2 inner product on S. Moreover, the spaces $\mathcal{Y}_0,\dots,\mathcal{Y}_d$ are mutually orthogonal, and $\mathcal{P}_d|_S = \mathcal{Y}_d \oplus \mathcal{Y}_{d-1} \oplus \cdots \oplus \mathcal{Y}_0$.

Each of the $Y_{d,\ell}$ can be expanded in terms of sine and cosine functions. The formulae are simple for $d = 0, 1$. Indeed,

$$\begin{aligned} Y_{0,1}(\theta,\phi) &= 1, \\ Y_{1,1}(\theta,\phi) &= \cos\theta, \\ Y_{1,2}(\theta,\phi) &= \sin\phi\ \sin\theta, \\ Y_{1,3}(\theta,\phi) &= \cos\phi\ \sin\theta. \end{aligned}$$

The formulae become increasingly complicated for larger values of d. We now explore the connection between spherical polynomials and spherical harmonics.

Theorem 13.22. *For $d \geq 1$, dim $\mathcal{P}_d|_S = (d+1)^2$, and $\mathcal{P}_d|_S = \mathcal{B}_d \oplus \mathcal{B}_{d-1}$. Moreover,*

$$\mathcal{B}_d = \begin{cases} \mathcal{Y}_0 \oplus \mathcal{Y}_2 \oplus \cdots \oplus \mathcal{Y}_{2k}, & d = 2k, \\ \mathcal{Y}_1 \oplus \mathcal{Y}_3 \oplus \cdots \oplus \mathcal{Y}_{2k+1}, & d = 2k+1. \end{cases} \tag{13.16}$$

Proof: Suppose $d = 2k$. We claim that for each $i = 1, \ldots, k$, each of the spaces $\mathcal{Y}_{2i}$ is a subspace of $\mathcal{B}_d$. Indeed, suppose p is a polynomial in the $2i+1$ dimensional space $\mathcal{Y}_{2i}$. Then p is a polynomial of degree $2i$ which is homogeneous of degree $2i$. But on S, p is equivalent to the d-th degree polynomial $q = (x^2 + y^2 + z^2)^{d-2i} p$, which is homogeneous of degree d, and thus $p \in \mathcal{B}_d$. The orthogonality of the spherical harmonics stated in Theorem 13.21 implies the linear independence of the set of basis functions associated with $\bigcup_{i=0}^{k} \mathcal{Y}_{2i}$. For $d = 2k$, (13.16) follows from the fact that the dimension of $\mathcal{B}_d$ is $\binom{d+2}{2} = \sum_{i=0}^{k} (4i+1)$. The proof of (13.16) for d odd is similar. Now the fact that $\mathcal{P}_d|_S = \mathcal{B}_d \oplus \mathcal{B}_{d-1}$ follows from the fact (see Theorem 13.21) that $\mathcal{P}_d|_S = \mathcal{Y}_d \oplus \mathcal{Y}_{d-1} \oplus \cdots \oplus \mathcal{Y}_0$. $\square$

To illustrate these results, we consider the case $d = 2$. It is easy to check that

$$\begin{aligned} \mathcal{Y}_0 &= \operatorname{span}\{1\}, \\ \mathcal{Y}_1 &= \operatorname{span}\{x, y, z\}, \\ \mathcal{Y}_2 &= \operatorname{span}\{xy, xz, yz, x^2 - y^2, x^2 - z^2\}. \end{aligned}$$

Thus, $\mathcal{B}_2 = \mathcal{Y}_0 \oplus \mathcal{Y}_2$ has dimension six, while $\mathcal{B}_1 = \mathcal{Y}_1$ has dimension three. This gives dim $\mathcal{P}_2|_S = 9$. In contrast, the dimension of the space of trivariate polynomials $\mathcal{P}_2$ is ten, i.e., the dimension of $\mathcal{P}_2$ goes down by one when we restrict to S. Table 13.1 gives the dimensionality of the various spaces for $0 \leq d \leq 5$.

d	$\mathcal{Y}_d$	$\mathcal{B}_d$	$\mathcal{P}_d\|_S$	$\mathcal{P}_d$
0	1	1	1	1
1	3	3	4	4
2	5	6	9	10
3	7	10	16	20
4	9	15	25	35
5	11	21	36	56

Tab. 13.1. Dimensionality of various "polynomial" spaces.

13.1.10 Spherical Patches

Given any spherical polynomial p as in (13.7) defined on a spherical triangle $T := \langle v_1, v_2, v_3 \rangle$, we define the associated spherical patch to be the surface

$P := \{p(v)v : v \in T\}$ lying in $\mathbb{R}^3$. It can be shown [AlfNS96a] that the spherical patch associated with $p \in \text{span}\{b_1, b_2, b_3\}$ lies on a sphere passing through the origin.

To help understand the shape of general spherical patches, we can introduce a control structure similar to the one for bivariate Bernstein–Bézier polynomials discussed in Chapter 3. If P is a spherical patch associated with a spherical polynomial p as in (13.7), then we define the associated spherical control points to be

$$\bar{c}_{ijk} := c_{ijk} v_{ijk}, \qquad i + j + k = d,$$

where v_{ijk} are the domain points (13.8) associated with d and the spherical triangle T.

The points $\{iv_1 + jv_2 + kv_3\}_{i+j+k=d}$ are the vertices of a natural partition of the planar triangle $\bar{T}$ with vertices v_1, v_2, v_3 into d^2 congruent triangles. The associated points $\{v_{ijk}\}_{i+j+k=d}$ on the spherical triangle $T := \langle v_1, v_2, v_3 \rangle$ form the vertices of a corresponding collection of subtriangles of T, but these triangles are not congruent. In fact for $d > 1$, it is impossible to find any partition of T into d^2 congruent spherical triangles.

The function $p \equiv 1$ is contained in $\mathcal{B}_d$ for all even d. Thus, for even d and any constant a, we can construct a spherical patch that is at a constant height a above T. This patch is itself a spherical triangle, but lies on the sphere with radius $1 + a$ instead of on S. The following result shows that it is not possible to create such spherical patches when d is odd.

Theorem 13.23. *Let T be a spherical triangle, and suppose that d is odd. Then there is no spherical polynomial p of degree d defined on T with*

$$p(v) = 1, \qquad \text{all } v \in T. \tag{13.17}$$

Proof: Suppose there is a spherical polynomial p of degree $d = 2k+1$ which satisfies (13.17). Then the restriction of p to an edge e of T can be written in terms of circular Bernstein basis polynomials, which by Theorem 13.16 can be written as

$$p(\theta) = \sum_{j=0}^{k} [a_j \sin((2j + 1)\theta)) + b_j \cos((2j + 1)\theta)],$$

where $p(\theta) \equiv 1$ for some nontrivial interval $[a, b]$. Taking the derivative of p with respect to θ gives

$$0 = \sum_{j=0}^{k} [a_j (2j + 1) \cos((2j + 1)\theta) - b_j (2j + 1) \sin((2j + 1)\theta)]$$

on (a, b). The linear independence of the sin's and cos's implies that $a_j = b_j = 0$ for $j = 0 \ldots, k$, and thus $p(\theta) \equiv 0$ which is a contradiction. $\square$

It was observed earlier that when d is odd, $\mathcal{B}_d$ does not contain the function that is identically one on all of S. Theorem 13.23 is a stronger assertion since it says that $\mathcal{B}_d|_T$ does not contain the function that is identically one on T.

13.2. Derivatives of Spherical Polynomials

In this section we give formulae for derivatives of homogeneous and spherical polynomials, and describe conditions for smooth joins of spherical polynomials on neighboring spherical triangles.

13.2.1 Directional Derivatives of Functions on the Sphere

Suppose f is a function defined on the unit sphere S. Given a point v on S, we also write v for the vector with tip at the point v and tail at the origin. Let g be a unit vector that is perpendicular to the vector v at the point v, and let Π be the plane containing the vectors v and g. This plane defines a great circle arc a passing through the point v. Suppose we parametrize a by arc length so that $a(0) = v$. Note that g gives the direction of the tangent vector to the arc a at v.

Definition 13.24. *If f is a sufficiently smooth function on S, we define the* derivative of f at v in the direction g *by*

$$D_g f(v) := \left.\frac{df(a(\theta))}{d\theta}\right|_{\theta=0}.$$

The m-th order directional derivative is defined in a similar way for any $m > 0$.

We now show how to compute the directional derivative of f at the point v in terms of the usual directional derivative of an associated trivariate function F.

Lemma 13.25. *Given f, let F be any trivariate function such that $f = F|_S$. Then for any point v on S and any unit vector g perpendicular to the vector defined by v,*

$$D_g f(v) = D_g F(v).$$

Proof: Let $a(\theta) = (x(\theta), y(\theta), z(\theta))^T$. Then $g = (x'(0), y'(0), z'(0))^T$. By the chain rule,

$$\left.\frac{df(a(\theta))}{d\theta}\right|_{\theta=0} = \left.\frac{dF(a(\theta))}{d\theta}\right|_{\theta=0} = g^T \nabla F(v) = D_g F(v). \quad \square$$

Given f defined on S, we can construct the function F in Lemma 13.25 as any homogeneous extension of f to $\mathbb{R}^3 \setminus \{0\}$. The proof of the lemma shows that we get the same value for the derivative no matter what degree extension we take.

It is also possible to define higher order directional derivatives of functions f defined on the sphere. Suppose for each $v \in S$ that $g(v)$ is a vector field which is tangent to S at v, and that F is a trivariate function such that $f = F|_S$. Suppose also that $g(v)^T \nabla F(v)$ is continuously differentiable, and let $h(v)$ be a direction vector in the tangent plane of S at v. Then we can apply the above differentiation procedure to $g(v)^T \nabla F(v)$ to define a second order directional derivative. In particular, we first extend $g(v)^T \nabla F(v)$ homogeneously, take the directional derivative in the direction h, and restrict to the sphere S. This gives

$$D_h D_g f(v) := h(v)^T \nabla \big[G(v)^T \nabla F(v)\big], \qquad \text{all } v \in S,$$

where G is some homogeneous extension of g.

13.2.2 Directional Derivatives of Spherical B-polynomials

In view of the previous section, we can compute directional derivatives of spherical polynomials from directional derivatives of their homogeneous extensions. In particular, to compute derivatives of

$$p := \sum_{i+j+k=d} c_{ijk} B^d_{ijk},$$

we can work with the associated trivariate homogeneous polynomial

$$p := \sum_{i+j+k=d} c_{ijk} H^d_{ijk}. \tag{13.18}$$

We now give explicit formulae for directional derivatives of (13.18).

Lemma 13.26. *Let h_1, h_2, h_3 be the functions defined in (13.12), and let g be a vector in $\mathbb{R}^3$. Then*

$$D_g h_i = h_i(g), \qquad i = 1, 2, 3.$$

Proof: We establish the result for $i = 1$. Let v_1, v_2, v_3 be the vertices of T, and let $v \in \mathbb{R}^3$. Considering v and v_i as vectors and applying ∇ to the formula (13.13) for h_1, it is easy to see that $g^T \nabla h_1 = \det(g, v_2, v_3)/\det(v_1, v_2, v_3) = h_1(g)$, which is the desired result. $\square$

Applying the chain rule and Lemma 13.26, we immediately get the following formula for the directional derivative of an arbitrary homogeneous polynomial.

Lemma 13.27. *Suppose $p \in \mathcal{H}_d$ is a homogeneous polynomial of degree d. Then*

$$D_g p(v) = (h_1(g), h_2(g), h_3(g)) \nabla_h p, \tag{13.19}$$

where

$$\nabla_h := \left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \frac{\partial}{\partial h_3} \right)^T.$$

Using (13.19), we can now give explicit formulae for directional derivatives of arbitrary homogeneous polynomials in terms of their coefficients in the expansion (13.18). Let $c_{ijk}^{(0)} = c_{ijk}$ for $i + j + k = d$. Suppose $g_1, \ldots, g_m$ is a given set of unit vectors. For each $\ell = 1, \ldots, m$, let

$$c_{ijk}^{(\ell)} = h_1(g_\ell) c_{i+1,j,k}^{(\ell-1)} + h_2(g_\ell) c_{i,j+1,k}^{(\ell-1)} + h_3(g_\ell) c_{i,j,k+1}^{(\ell-1)}, \quad i + j + k = d - \ell.$$

It can be shown directly from the recurrence relation that the $c_{ijk}^{(\ell)}$ depend on the vectors $g_1, \ldots, g_\ell$, but not on their ordering. This fact also follows from the following theorem.

Theorem 13.28. *For any $0 \le m \le d$,*

$$D_{g_1,\ldots,g_m} p(v) := D_{g_1} \cdots D_{g_m} p(v) := \frac{d\,!}{(d-m)\,!} \sum_{i+j+k=d-m} c_{ijk}^{(m)} H_{ijk}^{d-m}(v). \tag{13.20}$$

Proof: By Lemma 13.26,

$$\begin{aligned} &D_{g_1} H_{ijk}^d(v) \\ &\quad = \frac{d!}{i\,!\, j\,!\, k\,!} \left[i h_1^{i-1} h_2^j h_3^k D_{g_1} h_1 + j h_1^i h_2^{j-1} h_3^k D_{g_1} h_2 + k h_1^i h_2^j h_3^{k-1} D_{g_1} h_3 \right] \\ &\quad = d \left[H_{i-1,j,k}^{d-1}(v) h_1(g_1) + H_{i,j-1,k}^{d-1}(v) h_2(g_1) + H_{i,j,k-1}^{d-1}(v) h_3(g_1) \right], \end{aligned}$$

for $i + j + k = d$. Substituting this in

$$D_{g_1} p(v) = \sum_{i+j+k=d} c_{ijk}^{(0)} D_{g_1} H_{ijk}^d(v)$$

and rearranging terms yields (13.20) for $m = 1$. The general result follows by induction. $\square$

It is clear from the properties of the coordinates h_1, h_2, h_3, that if p is a homogeneous polynomial written in the form (13.18), then $p(v_1) = c_{d00}$, $p(v_2) = c_{0d0}$, and $p(v_3) = c_{00d}$. The derivatives of p at the vertices of T also have a simple form. For example, if we evaluate the derivative in (13.20) at $v = v_1$ we get

$$D_{g_1,\ldots,g_m} p(v_1) = \frac{d\,!}{(d-m)\,!} c^{(m)}_{d-m,0,0}. \tag{13.21}$$

We can also give convenient formulae for derivatives of a homogeneous polynomial at the vertices of the spherical triangle $T := \langle v_1, v_2, v_3 \rangle$. Suppose that g is a unit vector lying in the plane passing through the origin and the two points v_1 and v_2. Then $h_3(g) = 0$, and (13.21) only involves the coefficients $c_{d,0,0}, \ldots, c_{d-m,m,0}$. For convenience, we write out the formulae for the first and second derivatives at v_1:

$$D_g p(v_1) = d[h_1(g) c_{d,0,0} + h_2(g) c_{d-1,1,0}],$$

while

$$D_g^2 p(v_1) = d(d-1)[h_1^2(g) c_{d,0,0} + 2h_1(g) h_2(g) c_{d-1,1,0} \; + h_2^2(g) c_{d-2,2,0}].$$

For the second order mixed derivatives, let g_1 be as above, and let g_2 be a vector in the plane passing through the origin and the two points v_1 and v_3. In this case $h_2(g_2) = 0$, and the formula (13.21) simplifies to

$$\begin{aligned} D_{g_1,g_2} p(v_1) = d(d-1)[&h_1(g_1) h_1(g_2) c_{d,0,0} + h_2(g_1) h_1(g_2) c_{d-1,1,0} \\ &+ h_1(g_1) h_3(g_2) c_{d-1,0,1} + h_2(g_1) h_3(g_2) c_{d-2,1,1}]. \end{aligned}$$

We conclude this section with a few remarks about cross derivatives associated with an edge $e := \langle v_1, v_2 \rangle$ of T. Consider the derivative D_g in a direction g which does not lie in the plane passing through the origin and the two points v_1 and v_2. Along e we have $h_3(v) \equiv 0$, and so by Theorem 13.28, for each $0 \le m \le d$, the m-fold cross-boundary derivative $D_g^m p$ reduces to a homogeneous polynomial of degree $d - m$ on e. For example, if p is cubic ($d = 3$) and $m = 1$, then $D_g p(v)$ is the quadratic polynomial

$$D_g p(v) = 3\big[c^{(1)}_{200} h_1(v)^2 + 2c^{(1)}_{110} h_1(v) h_2(v) + c^{(1)}_{020} h_2(v)^2\big], \quad v \in e.$$

13.2.3 Joining Two Spherical Polynomials Smoothly

In this section we establish necessary and sufficient conditions for two spherical polynomials on neighboring spherical triangles to join together smoothly across the common edge. First we discuss homogeneous polynomials associated with a pair of neighboring spherical triangles.

Theorem 13.29. *Let T and $\widetilde{T}$ be two spherical triangles with vertices $\{v_1, v_2, v_3\}$ and $\{v_4, v_3, v_2\}$. Let a_1, a_2, a_3 be the spherical barycentric coordinates of v_4 relative to T, i.e., $v_4 = \sum_{i=1}^3 a_i v_i$. Suppose*

$$p(v) := \sum_{i+j+k=d} c_{ijk} H_{ijk}^d(v),$$
$$\tilde{p}(v) := \sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{H}_{ijk}^d(v),$$

where $\{H_{ijk}^d\}$ and $\{\tilde{H}_{ijk}^d\}$ are the homogeneous Bernstein basis functions associated with the triangles T and $\widetilde{T}$. Then any derivative of p of order at most m agrees with the corresponding derivative of $\tilde{p}$ at every point on the plane passing through the origin and v_2, v_3 if and only if

$$\tilde{c}_{njk} = \sum_{\nu+\mu+\kappa=n} c_{\nu,k+\mu,j+\kappa} H_{\nu\mu\kappa}^n(v_4), \tag{13.22}$$

for all $j + k = d - n$ and $n = 0, \ldots, m$.

Proof: Consider the trivariate polynomials

$$P(v) := \sum_{i+j+k+l=d} C_{ijkl} B_{ijkl}^d(v),$$
$$\tilde{P}(v) := \sum_{i+j+k+l=d} \tilde{C}_{ijkl} \tilde{B}_{ijkl}^d(v),$$

where

$$C_{ijkl} := \begin{cases} c_{ijk}, & \text{if } l = 0, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \tilde{C}_{ijkl} := \begin{cases} \tilde{c}_{ijk}, & \text{if } l = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{13.23}$$

and $B_{ijkl}^d(v)$ are the trivariate B-polynomials of degree d associated with the tetrahedron $\{v_1, v_2, v_3, 0\}$ and $\tilde{B}_{ijkl}^d(v)$ are those associated with the tetrahedron $\{v_4, v_3, v_2, 0\}$, see Section 15.3 below. By Theorem 15.31, these trivariate polynomials join with C^m continuity if and only if

$$\tilde{C}_{ijkl} = \sum_{\nu+\mu+\kappa+\lambda=i} C_{\nu,k+\mu,j+\kappa,l+\lambda} B_{\nu\mu\kappa\lambda}^i(v_4), \tag{13.24}$$

for $i + j + k + l = d$ and $i = 0, \ldots, m$. In view of (13.23), we can choose $l = \lambda = 0$. In this case, (13.24) holds if and only if (13.22) holds. But $P = p$ and $\tilde{P} = \tilde{p}$, and the proof is complete. $\square$

We can now translate this to a result on spherical polynomials.

Theorem 13.30. *Let $T := \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2\rangle$ be two spherical triangles sharing an edge $e := \langle v_2, v_3\rangle$. Suppose*

$$p(v) := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v),$$
$$\tilde{p}(v) := \sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{B}^d_{ijk}(v),$$

where $\{B^d_{ijk}\}$ and $\{\tilde{B}^d_{ijk}\}$ are the spherical Bernstein basis polynomials associated with T and $\widetilde{T}$, respectively. Then any derivative of order at most m of p and the corresponding derivative of $\tilde{p}$ agree at all points on the edge e if and only if

$$\tilde{c}_{njk} = \sum_{\nu+\mu+\kappa=n} c_{\nu,k+\mu,j+\kappa} B^n_{\nu\mu\kappa}(v_4)$$

for all $j + k = d - n$ and $n = 0, \ldots, m$.

Proof: Each of the spherical Bernstein basis polynomials B^d_{ijk} can be extended to a homogeneous Bernstein basis polynomial H^d_{ijk}, and the result follows immediately from Theorem 13.29. □

13.3. Spherical Triangulations

In this section we discuss some basic properties of spherical triangulations.

Definition 13.31. *A set of spherical triangles $\triangle := \{T_i\}_1^N$ is called a* spherical triangulation *provided that the intersection of any two triangles in $\triangle$ is empty, or is a common vertex or common edge. We write $\Omega := \bigcup_{i=1}^N T_i$ for the associated domain. We are mostly interested in the case $\Omega = S$, in which case we say that $\triangle$* covers *S.*

To state results on the relationship between the number V of vertices, number E of edges, and number N of triangles in a spherical triangulation, we have to distinguish between the cases when $\triangle$ covers S and when it does not. First we consider the case when $\triangle$ does not cover S.

Definition 13.32. *Let $\triangle$ be a spherical triangulation of a domain $\Omega \subset S$. Then we say that $\triangle$ is* shellable *provided it consists of a single triangle, or if it can be obtained from a shellable triangulation $\widetilde{\triangle}$ by adding one triangle T such that T intersects $\widetilde{\triangle}$ precisely along one or two edges. We say that $\triangle$ is* regular *provided $\triangle$ is shellable, or it can be obtained from a shellable triangulation $\widetilde{\triangle}$ by removing one or more shellable subtriangulations, all of whose vertices are interior vertices of $\widetilde{\triangle}$.*

It is easy to show that for regular spherical triangulation that do not cover S, exactly the same Euler relations as in the planar case hold, see Theorem 4.10. The following result shows that the situation is different for the case where $\triangle$ covers S.

Theorem 13.33. *Let $\triangle$ be a spherical triangulation that covers S. Then*

1) $E = 3N/2$,
2) $N = 2V - 4$,
3) $E = 3V - 6$.

Here V, E, N denote the number of vertices, edges, and triangles in $\triangle$.

Proof: Equation 1) is obvious since every triangle has three edges, but if we count them all, each edge will be counted twice. To get 2), we look at the tetrahedral partition which is induced by $\triangle$, i.e., we replace each spherical triangle by a flat triangle which forms a face of a tetrahedron with its fourth vertex at the origin. By Theorem 16.13, $F_B = 2V_B - 4$, where F_B and V_B are the number of boundary faces and boundary vertices of this tetrahedral partition. Now 2) follows since $N = F_B$ and $V = V_B$. Finally, 3) follows by combining 1) and 2). $\square$

Spherical triangulation can be stored with the same kinds of data structures as discussed in Section 4.5 for planar triangulations. As for the planar case, there are a number of available algorithms for constructing spherical triangulations, including spherical Delaunay triangulations, see Remark 13.8. They can also be created as induced triangulations associated with quadrangulations as in Section 4.15. Spherical triangulations and quadrangulations can be refined in the same ways as in the planar case. In particular, we can apply any of the Clough–Tocher, Powell–Sabin, Powell–Sabin-12, Wang, Double-Clough–Tocher, or uniform refinement schemes described in Section 4.8.

Suppose $T := \langle v_1, v_2, v_3 \rangle$ is a spherical triangle. Then for each $i = 1, 2, 3$, the angle of T at the vertex v_i is defined to be the angle between the plane containing the vectors v_i and v_{i+1} and the plane containing the vectors v_i and v_{i+2}. It is well known that the sum of the angles of a spherical triangulation is larger than π. Given a spherical triangulation, we write $\theta_\triangle$ for the minimum angle among the triangles of $\triangle$.

13.4. Spaces of Spherical Splines

In this section we present a theory of spherical splines which is the direct analog of the theory of bivariate splines given in previous chapters. We begin with the space of continuous spherical splines.

13.4.1 C^0 Spherical Splines

Given a nonnegative integer d, let $\mathcal{B}_d$ be the space of spherical polynomials of degree d introduced in Definition 13.10. Then given any spherical triangulation $\triangle := \{T_i\}_{i=1}^N$ of a domain $\Omega \subseteq S$, we define

$$\mathcal{S}_d^0(\triangle) := \{s \in C^0(\Omega) : s|_{T_i} \in \mathcal{B}_d, \quad i = 1, \dots, N\}.$$

We now compute the dimension of this spline space. Given $s \in \mathcal{S}_d^0(\triangle)$, for each triangle $T \in \triangle$, we know by Corollary 13.19 that there exists a unique set of coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}}$ such that

$$s|_T = \sum_{\xi \in \mathcal{D}_{T,d}} c_\xi B_\xi^{T,d}, \tag{13.25}$$

where $B_\xi^{T,d}$ are the spherical Bernstein basis polynomials of degree d associated with the spherical triangle T, and $\mathcal{D}_{d,T}$ is the associated set of domain points in T, see Definition 13.11. Since s is continuous, if ξ is contained in two different triangles T and $\widetilde{T}$, then the coefficients c_ξ for $s|_T$ and $s|_{\widetilde{T}}$ are the same. Thus, for each $s \in \mathcal{S}_d^0(\triangle)$, there is a unique associated set of coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$, where $\mathcal{D}_{d,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{d,T}$ is the set of domain points for $\mathcal{S}_d^0(\triangle)$. Note that domain points on edges of $\triangle$ belong to two or more of the sets $\mathcal{D}_{d,T}$, but are included in $\mathcal{D}_{d,\triangle}$ just once. We call $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ the set of spherical B-coefficients of s. The converse also holds, i.e., given any $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ there is a unique spline $s \in \mathcal{S}_d^0(\triangle)$ defined by (13.25).

We have shown that the linear space $\mathcal{S}_d^0(\triangle)$ is in one-to-one correspondence with the set $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. It follows that the dimension of $\mathcal{S}_d^0(\triangle)$ is equal to the cardinality of $\mathcal{D}_{d,\triangle}$. A simple count leads to the following result.

Theorem 13.34. *Every spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely defined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. Moreover,*

$$\dim \mathcal{S}_d^0(\triangle) = \#\mathcal{D}_{d,\triangle} = V + (d-1)E + \binom{d-1}{2} N,$$

where V, E, and N are the number of vertices, edges, and triangles in $\triangle$.

This result shows that to store, evaluate, and render spherical splines, we can work with the B-form just as we did in the bivariate case discussed in Chapter 5. In particular, to store a spline, we can simply store the vector c of its B-coefficients. Moreover, it is clear that Algorithms 5.2 and 5.3 can be applied without change to evaluate a spherical spline or its mixed partial derivatives of any order.

We now construct locally supported basis functions in $\mathcal{S}_d^0(\triangle)$ which are analogs of the basis functions constructed in Section 5.4 for ordinary bivariate splines. For each $\xi \in \mathcal{D}_{d,\triangle}$, let ψ_ξ be the spline in $\mathcal{S}_d^0(\triangle)$ that satisfies

$$\gamma_\eta \psi_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle},$$

where γ_η is a linear functional which picks off the coefficient associated with the domain point η. Such functionals can be constructed explicitly in various ways, see Remark 13.7. By construction, ψ_ξ has all zero coefficients except for $c_\xi = 1$.

Since for each triangle T the associated Bernstein basis polynomials are nonnegative on T, it follows immediately that

$$\psi_\xi(v) \ge 0, \qquad \text{all } v \in \Omega.$$

Since ψ_ξ is identically zero on all triangles which do not contain ξ, it follows that the support of ψ_ξ is

1) a single triangle T, if ξ is in the interior of T,
2) the union of triangles T and $\widetilde{T}$, if ξ is in the interior of the edge between T and $\widetilde{T}$,
3) the union of all triangles sharing the vertex v, if $\xi = v$.

Theorem 13.35. *The set of splines $\mathcal{B} := \{\psi_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ forms a basis for $\mathcal{S}_d^0(\triangle)$.*

Proof: Since $\dim \mathcal{S}_d^0(\triangle) = \#\mathcal{D}_{d,\triangle}$, to show $\mathcal{B}$ is a basis, it suffices to show that the ψ_ξ are linearly independent. Suppose that

$$s := \sum_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi \psi_\xi \equiv 0, \qquad \text{on } \Omega.$$

Then on any spherical triangle $T \in \triangle$, the restriction $s|_T$ is a spherical polynomial of degree d which is identically 0, and so all the coefficients in its B-form must vanish. Corollary 13.19 implies that $c_\eta = 0$ for all $\eta \in \mathcal{D}_{d,T}$. Since this holds for every $T \in \triangle$, we have shown that all coefficients must be zero. □

13.4.2 Spaces of Smooth Spherical Splines

As in the bivariate case, we are also interested in spaces of spherical splines that have some additional smoothness beyond C^0 continuity. For example, we might want splines that have continuous derivatives up to order r everywhere on Ω, or we might want splines with supersmoothness at certain vertices or across certain edges.

Following our earlier treatment of bivariate splines, we now introduce some notation for describing smoothness of spherical splines. Suppose $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ are two spherical triangles sharing an interior edge $e := \langle v_2, v_3 \rangle$ of $\triangle$. Fix $0 \le n \le j \le d$. Then for any spherical spline $s \in \mathcal{S}_d^0(\triangle)$, let

$$\tau_{j,e}^n s := c_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} \tilde{c}_{\nu,j-n+\mu,d-j+\kappa} \tilde{B}_{\nu\mu\kappa}^n(v_1). \tag{13.26}$$

We call $\tau_{j,e}^n$ a smoothness functional of order m. Given a set $\mathcal{T}$ of linear functionals of the form (13.26) associated with oriented edges of $\triangle$, we define the corresponding space of smooth splines as

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) := \{s \in \mathcal{S}_d^0(\triangle) : \tau s = 0, \text{ all } \tau \in \mathcal{T}\}. \tag{13.27}$$

If e is an interior edge of the triangulation $\triangle$, then we say that $s \in \mathcal{S}_d^0(\triangle)$ is C^r smooth across the edge e provided that if T and $\widetilde{T}$ are the triangles sharing the edge, then the spherical polynomials $s|_T$ and $s|_{\widetilde{T}}$ join with C^r smoothness across e. Theorem 13.30 shows that all splines in $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ will be C^r smooth across an edge e if and only if $\mathcal{T}$ includes all of the linear functionals $\{\tau_{j,e}^n\}_{j=n}^d$, for $n = 1, \ldots, r$. Following the notation in the bivariate case, we write

$$\mathcal{S}_d^r(\triangle) := \mathcal{S}_d^0(\triangle) \cap C^r(\Omega) \tag{13.28}$$

for the space of splines of degree d which are C^r smooth across all interior edges of $\triangle$.

We can also define analogs of the superspline spaces described in Definition 5.6. In particular, we say that a spline $s \in \mathcal{S}_d^0(\triangle)$ is C^ρ smooth at v provided that all of the spherical polynomials $s|_T$ such that T is a triangle with vertex at v have common derivatives up to order ρ at the point v. In this case we write $s \in C^\rho(v)$. The following lemma shows that forcing a spline to belong to $C^\rho(v)$ can be achieved by enforcing an appropriate set of smoothness conditions across the interior edges attached to v. It can be proved in exactly the same way as Lemma 5.9.

Lemma 13.36. *Let $s \in \mathcal{S}_d^0(\triangle)$ and suppose v is a vertex of $\triangle$. Then $s \in C^\rho(v)$ if and only if for $i = 1, \ldots, m$,*

$$\tau_{j,e_i}^n s = 0, \qquad n \le j \le \rho \text{ and } 1 \le n \le \rho,$$

where $e_1, \ldots, e_m$ are the interior edges of $\triangle$ attached to v.

As in the bivariate case, smoothness conditions on a spline $s \in \mathcal{S}_d^0(\triangle)$ are just linear side conditions on the vector c of B-coefficients of s. Thus, for any given set $\mathcal{T}$ of smoothness conditions, there is a matrix A depending on $\mathcal{T}$ such that

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) = \{s \in \mathcal{S}_d^0(\triangle) : \, Ac = 0\}. \tag{13.29}$$

Clearly, the matrix A is of size $n_s \times n_d$, where n_s is the number of smoothness conditions in $\mathcal{T}$, and n_d is the dimension of $\mathcal{S}_d^0(\triangle)$. It is also clear that A is a relatively sparse matrix, since a typical C^r smoothness condition across an edge involves only $\binom{r+2}{2} + 1$ coefficients. Thus, for example, a C^1 condition involves only four coefficients, so the corresponding row in the matrix A has at most four nonzero entries. The proof of the following result is exactly the same as the proof of Theorem 5.11.

Theorem 13.37. *Let $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ be the space of smooth spherical splines defined in (13.29) corresponding to a matrix A. Then the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ is equal to $n_d - n_r$, where n_r is the rank of A.*

13.4.3 Minimal Determining Sets

Theorem 13.34 shows that every spherical spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. Suppose now that $\mathcal{S} := \mathcal{S}_d^{\mathcal{T}}(\triangle)$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$ defined by enforcing some set of smoothness conditions $\mathcal{T}$ across the interior edges of a spherical triangulation $\triangle$ as described in (13.27). Then for a spline $s \in \mathcal{S}_d^0(\triangle)$ to be in $\mathcal{S}$, its set of B-coefficients must satisfy the smoothness conditions in $\mathcal{T}$. This means that we cannot assign arbitrary values to every coefficient of s. Instead, we can only assign values to certain coefficients, and the values of the remaining coefficients will be determined by the smoothness conditions.

Suppose $\Gamma \subseteq \mathcal{D}_{d,\triangle}$ is such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \Gamma$, then $s \equiv 0$. Then we say that Γ is a **determining set for $\mathcal{S}$**. Clearly, for any $\mathcal{S} \subset \mathcal{S}_d^0(\triangle)$, the set of domain points $\mathcal{D}_{d,\triangle}$ is always a determining set for $\mathcal{S}$. But for any spline space $\mathcal{S}$ satisfying at least one smoothness condition, there will be determining sets with fewer points than the number of points in $\mathcal{D}_{d,\triangle}$. If $\mathcal{M}$ is a determining set for a spline space $\mathcal{S}$ and $\mathcal{M}$ has the smallest cardinality among all possible determining sets for $\mathcal{S}$, then we call $\mathcal{M}$ a **minimal determining set (MDS)** for $\mathcal{S}$. In general, there will be more than one minimal determining set corresponding to a given spline space $\mathcal{S}$.

Let $\mathcal{S}$ be a linear subspace of $\mathcal{S}_d^0(\triangle)$. Following the proof of Theorem 5.13, it is easy to see that a determining set $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$ if and only if $\#\mathcal{M} = \dim \mathcal{S}$. As in the bivariate case, it is a nontrivial task to construct minimal determining sets. If we know the dimension of $\mathcal{S}$, then we at least know how many domain points to put in a minimal determining set. On the other hand, it is not necessary to know the dimension of $\mathcal{S}$ to construct a minimal determining set. As in the bivariate case, we can try to determine which coefficients can be set independently and consistently, i.e., in such a way that all coefficients are uniquely determined and all smoothness conditions are satisfied.

We are especially interested in minimal determining sets $\mathcal{M}$ for spherical spline spaces $\mathcal{S}$ that have the following additional properties:

1) for all $\xi \notin \mathcal{M}$, c_ξ can be computed from a small set of coefficients c_η where $\eta \in \mathcal{M}$ is near ξ,
2) for all $\xi \notin \mathcal{M}$, the size of c_ξ is comparable to the size of the coefficients c_η used to compute it.

To make these properties more precise, we need some additional notation. Suppose $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$. Then if we fix the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ for a spline $s \in \mathcal{S}$, then all remaining B-coefficients of s can be computed using smoothness conditions. Given $\eta \notin \mathcal{M}$, we say that c_η *depends on* c_ξ, $\xi \in \mathcal{M}$, if changing the value of c_ξ also causes the value of c_η to change. Let

$$\Gamma_\eta := \{\xi \in \mathcal{M} : \, c_\eta \text{ depends on } c_\xi\}.$$

If v is a vertex of a spherical triangulation $\triangle$, then we define $\mathrm{star}(v) := \mathrm{star}^1(v)$ to be the set of all triangles sharing the vertex v, and define $\mathrm{star}^j(v)$ recursively to be the set of all triangles with vertices in common with the triangles of $\mathrm{star}^{j-1}(v)$. Similarly, we define $\mathrm{star}^0(T) := T$, and $\mathrm{star}^j(T) := \bigcup\{\mathrm{star}(v) : v \in \mathrm{star}^{j-1}(T)\}$.

Definition 13.38. *Suppose $\mathcal{M}$ is a minimal determining set for a linear space of sherical splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. We say that $\mathcal{M}$ is* local *provided there exists an integer ℓ not depending on $\triangle$ such that*

$$\Gamma_\eta \subseteq \mathit{star}^\ell(T_\eta), \qquad \mathit{all}\ \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}, \tag{13.30}$$

where T_η is a triangle containing η. We say that $\mathcal{M}$ is stable *provided there exists a constant K depending only on ℓ and the smallest angle in the triangulation $\triangle$ such that*

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \qquad \mathit{all}\ \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}.$$

13.4.4 Stable Local Bases

Suppose $\mathcal{M}$ is an MDS for a spherical spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ defined on a spherical triangulation $\triangle$. Then for each $\xi \in \mathcal{M}$, there is a unique spline $\psi_\xi \in \mathcal{S}$ such that

$$\gamma_\eta \psi_\xi = \delta_{\eta\xi}, \qquad \text{all } \eta \in \mathcal{M}.$$

Clearly, $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a basis for $\mathcal{S}$. We refer to it as the spherical $\mathcal{M}$-basis for $\mathcal{S}$. If $\mathcal{M}$ is a *stable local* minimal determining set, then the corresponding $\mathcal{M}$-basis is also stable and local. In particular, we have the following analog of Theorem 5.21.

Theorem 13.39. *Suppose $\mathcal{M}$ is a stable local minimal determining set for the linear space of spherical splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, and let $\Psi := \{\psi_\xi\}_{\xi\in\mathcal{M}}$ be the corresponding $\mathcal{M}$-basis. Then Ψ is a* stable local basis *for $\mathcal{S}$ in the sense that for all $\xi \in \mathcal{M}$,*

1) $\|\psi_\xi\|_\Omega \le K$,
2) $\mathit{supp}\, \psi_\xi \subseteq \mathit{star}^\ell(T_\xi)$, *where T_ξ is a triangle containing ξ,*

where ℓ is the integer constant in (13.30), and K is a constant depending only on ℓ and the smallest angle in $\triangle$.

13.4.5 Nodal Determining Sets

Suppose $\mathcal{S}$ is a space of spherical splines defined on a spherical triangulation $\triangle$. Let $\mathcal{N} = \{\lambda_i\}_{i=1}^n$ be a set of linear functionals of the form

$$\lambda_i := \varepsilon_{t_i} \sum_{\alpha+\beta=m_i} a^i_{\alpha,\beta} D^\alpha_{1,t_i} D^\beta_{2,t_i}, \qquad i = 1,\dots,n,$$

where ε_{t_i} denotes point evaluation at the point t_i, $a^i_{\alpha,\beta}$ are constants, and D_{1,t_i} and D_{2,t_i} are directional derivatives associated with two orthogonal vectors lying in the plane tangent to S at the point t_i. If $n = \dim \mathcal{S}$, then we call $\mathcal{N}$ a nodal minimal determining set for $\mathcal{S}$ provided that any $s \in \mathcal{S}$ is uniquely determined by the values $\lambda_1 s, \ldots, \lambda_n s$.

Suppose $\mathcal{N}$ is a nodal minimal determining set for a spherical spline space $\mathcal{S}$, and that $\bar{m}$ is the order of the highest derivative involved. Then for any function $f \in C^{\bar{m}}(S)$, there exists a unique spline $s_f \in \mathcal{S}$ satisfying

$$\lambda s_f = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}.$$

This defines a linear projector $\mathcal{I}_{\mathcal{S}}$ mapping $C^{\bar{m}}(S)$ onto $\mathcal{S}$.

13.4.6 Dimension of Spherical Spline Spaces

In this section we show how to extend results on the dimension of bivariate spline spaces to spherical spline spaces. As in the bivariate case, we define a cell $\triangle_v$ to be a collection of spherical triangles sharing a common vertex v such that eachy pair of triangles have at least one edge in common. As in the bivariate case, we distinguish between boundary cells, which correspond to the case where v is a boundary vertex of $\triangle_v$, and interior cells where where v is an interior vertex of $\triangle_v$. Let $|\triangle_v|$ be the maximum geodesic distance between any two points in $\triangle_v$.

Theorem 13.40. *Let $\triangle_v$ be a boundary cell with n interior edges attached to the vertex v, and suppose that $|\triangle_v| < 1$. Then for any $0 \le r \le d$,*

$$\dim \mathcal{S}^r_d(\triangle_v) = \binom{d+2}{2} + n\binom{d-r+1}{2}.$$

Proof: This result is the analog of Theorem 9.2. If we project $\triangle_v$ radially onto the tangent plane to S at v, we get a planar boundary cell $\widetilde{\triangle}_v$ attached to the vertex v. Let $\widetilde{\mathcal{S}}^r_d(\widetilde{\triangle}_v)$ be the space of bivariate splines obtained by taking the homogeneous extension of degree d of each spline $s \in \mathcal{S}^r_d(\triangle_v)$ to $\mathbb{R}^3$ and then restricting it to the tangent plane. Since the extension is unique, it follows that $\widetilde{\mathcal{S}}^r_d(\widetilde{\triangle}_v)$ is isomorphic to $\mathcal{S}^r_d(\triangle_v)$. Now the number of interior edges in $\widetilde{\triangle}_v$ is just n, and the result follows from Theorem 9.2. $\square$

Theorem 13.41. *Let $\triangle_v$ be an interior cell formed by n triangles sharing a vertex v, and suppose $|\triangle_v| < 1$. Suppose $v_1, \ldots, v_n$ are the boundary vertices of $\triangle_v$. For each $i = 1, \ldots, n$, let π_i be the plane spanned by the vectors v and v_i. Let m_v be the number of distinct planes in the set $\{\pi_1, \ldots, \pi_n\}$. Then for any $0 \le r \le d$,*

$$\dim \mathcal{S}^r_d(\triangle_v) = \binom{r+2}{2} + n\binom{d-r+1}{2} + \sigma_v,$$

where

$$\sigma_v := \sum_{j=1}^{d-r} (r+j+1-jm_v)_+ .$$

Proof: This result is the analog of Theorem 9.3, and follows from it by the same mapping argument as used in the proof of Theorem 13.40. $\square$

We call the π_i cutting planes associated with the edges $\langle v, v_i\rangle$. We now show how to use Theorem 13.41 to identify the dimension of the spherical spline space $\mathcal{S}_d^r(\triangle)$ defined in (13.28) for $d \geq 3r+2$, and to construct a minimal determining set for it. First we need to examine the concepts of degenerate and singular vertices, since they are slightly different on the sphere.

Definition 13.42. *Suppose that $\langle v_1, v_2, v_3\rangle$, $\langle v_4, v_3, v_2\rangle$ are two spherical triangles sharing an edge $e := \langle v_2, v_3\rangle$. Then we say that e is* degenerate at v_2 *provided the cutting planes π_2 and π_3 associated with the edges $\langle v_2, v_1\rangle$ and $\langle v_2, v_4\rangle$ coincide. We say that a vertex v is* singular *provided that there are exactly four edges attached to it, and they are all degenerate at v.*

In contrast to the planar case where an edge can be degenerate at only one endpoint, for spherical triangulations, it is possible for an edge to be degenerate at both ends. This can happen, however, only if the points v_2 and v_3 in Definition 13.42 are antipodal. Thus, this problem can be avoided by working only with spherical triangulations with no pairs of antipodal vertices. In order to apply Theorems 13.40 and 13.41, we assume more: namely, that $|\mathrm{star}(v)| < 1$ for every vertex v of $\triangle$. The following is an analog of Theorem 9.15, and can be proved in a similar way. Given $d \geq 3r+2$, let

$$\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor,$$

and let $A^T, C^T, D_\mu^T, E^T, F^T, G_L^T, G_R^T$ be the subsets of the set of domain points $\mathcal{D}_{d,\triangle}$ defined in (9.17).

Theorem 13.43. *Suppose $\triangle$ is a spherical triangulation with $|star(v)| < 1$ for every vertex v of $\triangle$, and suppose $d \geq 3r+2$. Then the following set $\mathcal{M}$ of domain points is a minimal determining set for the space $\mathcal{S}_d^r(\triangle)$ of spherical splines:*

1) *For each boundary vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_\mu^r(\triangle_v)$, where $\triangle_v = star(v)$.*
2) *For each interior vertex v of $\triangle$, choose a minimal determining set for $\mathcal{S}_\mu^r(\triangle_v)$.*
3) *For each triangle T in $\triangle$, include C^T.*

4) *For each edge e of $\triangle$, include $E^T(e)$ for some triangle T with edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, T can be either of the two triangles sharing e. If e is a boundary edge, also include $G_L^T(e)$ and $G_R^T(e)$.*
5) *For each triangle $T := \langle u, v, w \rangle$, include $A^T(u)$, $A^T(v)$, and $A^T(w)$.*
6) *If $T_1 := \langle v, w_1, w_2 \rangle$ and $T_2 := \langle v, w_2, w_3 \rangle$ are two triangles sharing a degenerate edge $e := \langle v, w_2 \rangle$, then replace $A^{T_2}(v)$ by $G_L^{T_1}(e)$.*
7) *If v is a singular vertex, reinsert $A^T(v)$ for one triangle T attached to v.*

If $\triangle$ covers S, there are no boundary vertices or boundary edges in $\triangle$, and the definition of $\mathcal{M}$ in Theorem 13.43 is somewhat simplified. We can now compute the dimension of the spherical spline space $\mathcal{S}_d^r(\triangle)$ by simply counting the number of points in $\mathcal{M}$, cf. Theorem 9.16. The result depends on whether $\triangle$ covers S or not.

We begin with the case where $\triangle$ does not cover S. Let $\mathcal{V}_I$ denote the set of interior vertices of $\triangle$. In addition, let

$$\sigma := \sum_{v \in \mathcal{V}_I} \sigma_v, \tag{13.31}$$

where for each $v \in \mathcal{V}_I$,

$$\sigma_v := \sum_{j=1}^{d-r} (r + j + 1 - j m_v)_+ ,$$

and m_v is the number of distinct cutting planes passing through v.

Theorem 13.44. *Let $\triangle$ be a regular triangulation of a spherical domain Ω with no holes. Suppose $\triangle$ does not cover S, and that $|star(v)| < 1$ for every vertex v of $\triangle$. Let V_B and V_I be the number of boundary and interior vertices of $\triangle$, respectively. Then for all $d \geq 3r + 2$,*

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) = {} & \frac{d^2 + r^2 - r + d - 2rd}{2} V_B + (d - r)(d - 2r) V_I \\ & + \frac{-2d^2 + 6rd - 3r^2 + 3r + 2}{2} + \sigma, \end{aligned}$$

where σ is as in (13.31).

When $\triangle$ covers S, we have the following somewhat simpler formula.

Theorem 13.45. *Let $\triangle$ be a spherical triangulation that covers S. Let V be the number of vertices of $\triangle$, and suppose $|star(v)| < 1$ for all vertices v of $\triangle$. Then*

$$\dim \mathcal{S}_d^r(\triangle) = (d - r)(d - 2r) V + \frac{-2d^2 + 6dr - 3r^2 + 3r + 2}{2} + \sigma,$$

where σ is as in (13.31).

To construct a basis for $\mathcal{S}_d^r(\triangle)$, we can take the set of $\mathcal{M}$-basis splines associated with the minimal determining set $\mathcal{M}$ of Theorem 13.43. However, just as in the bivariate case, while this basis is local, it not guaranteed to be stable as in Definition 13.38. To get a stable local basis for $\mathcal{S}_d^r(\triangle)$, we have to follow the more complicated construction described in Chapter 11 for bivariate splines.

Following Section 9.5, for $d \geq 3r+2$, we can also construct minimal determining sets and compute the dimension of the spherical superspline spaces

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : s \in C^\rho(v), \text{ all vertices } v\}$$

for both of the cases where $\triangle$ covers S and when it does not.

There are no analogs of the results of Section 9.6 on type-I and type-II triangulations, since on the sphere, type-I and type-II triangulations do not exist. Indeed, if $\triangle$ were a type-I triangulation on S, then there would be exactly six edges attached to each vertex, and it would follow that $E = 3V$, which contradicts Theorem 13.33. Similarly, if $\triangle$ were a type-II triangulation on S, then there would be exactly eight edges attached to each vertex, and it would follow that $E = 4V$, which also contradicts Theorem 13.33.

The upper and lower bounds established in Section 9.7 for general bivariate superspline spaces $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ associated with a set $\mathcal{T}$ of smoothness functionals can be carried over to the analogous spherical spline spaces $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ defined in (13.29).

13.5. Spherical Macro-element Spaces

We saw in Chapters 6–8 that one way to create spaces of bivariate splines with convenient stable local minimal determining sets as well as convenient stable local nodal minimal determining sets is to work with splines defined on a refinement $\triangle_R$ of a given general triangulation $\triangle$. The following is the analog of Definition 5.27.

Definition 13.46. *Suppose $\mathcal{N}$ is a nodal determining set for a space of spherical splines $\mathcal{S} \subset \mathcal{S}_d^0(\triangle_R)$. For each triangle $T \in \triangle$, suppose that the data $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$ uniquely determine $s|_T$, where $\mathcal{N}_T := \{\lambda \in \mathcal{N} :$ the carrier of λ is contained in $T\}$. Then we say that $\mathcal{S}$ is a* spherical macro-element space.

Suppose $\mathcal{S}$ is a spherical macro-element space with nodal determining set $\mathcal{N}$. Then for any sufficiently smooth function f, there is a unique Hermite interpolating spline $s \in \mathcal{S}$ such that $\lambda s = \lambda f$ for all $\lambda \in \mathcal{N}$. The fact that $\mathcal{S}$ is a macro-element space means that s can be computed one triangle at a time. As in the bivariate case, we are interested in macro-element spaces which have good approximation properties. Since spherical splines and bivariate splines share a common algebraic structure, it is clear that every bivariate macro-element space has a direct spherical analog. In particular, we have spherical analogs of all of the bivariate macro-element spaces discussed in Chapters 6–8. These include all of the macro-elements based on the Clough–Tocher, Powell–Sabin, Powell–Sabin-12, Wang, and Double-Clough–Tocher splits, as well as those based on triangulated quadrangulations.

13.6. Remarks

Remark 13.1. There is no universal agreement in the literature on the definition of spherical polynomials. Some authors refer to the $(d+1)^2$-dimensional space $\mathcal{P}_d|_S$ discussed in Section 13.1.9 as the space of spherical polynomials of degree d.

Remark 13.2. Derivatives of a spherical polynomial p can also be computed in terms of the spherical coordinates θ and ϕ using the chain rule. Let

$$\nabla_b p = \left(\frac{\partial p}{\partial b_1}, \frac{\partial p}{\partial b_2}, \frac{\partial p}{\partial b_3}\right),$$

where b_1, b_2, b_3 are the barycentric coordinate functions relative to some spherical triangle T. Then

$$\frac{\partial p}{\partial \theta}(v) = \nabla_b p \cdot \left(\nabla b_1 \cdot \frac{\partial v}{\partial \theta}, \nabla b_2 \cdot \frac{\partial v}{\partial \theta}, \nabla b_3 \cdot \frac{\partial v}{\partial \theta}\right).$$

A similar formula holds for the partial derivative with respect to ϕ. Higher derivatives can also be computed this way, but become more complicated.

Remark 13.3. The word spherical spline has been used in the literature for certain spaces of functions which do not have a piecewise structure, but instead are radial basis functions. These kinds of spherical splines arise out of certain natural variational problems, and have been intensively studied, see [FasS98, FreeGS98] and references therein.

Remark 13.4. In contrast to the bivariate case, there does not seem to be a simple explicit formula for integrals of spherical Bernstein basis polynomials. In fact, this difficulty arises already in the case of the circular Bernstein–Bézier polynomials discussed in Section 13.1.7. As shown there, these functions are essentially trigonometric polynomials. Although recurrence relations exist for computing integrals of products of trigonometric functions over an arbitrary interval, a convenient closed-form formula does not seem to be available.

Remark 13.5. It was shown in [BroW92] that it is impossible to construct barycentric coordinates on spherical triangles which have the usual properties of the barycentric coordinates associated with planar triangles, including the property that they sum to one. This observation may have held back the development of spherical splines, which began only after it was realized by Alfeld, Neamtu, and Schumaker [AlfNS96a–AlfNS96c] that the partition of unity property was not an essential property.

Remark 13.6. The definition of the functions h_1, h_2, h_3 in (13.12) does not require that the points v_1, v_2, v_3 lie on the unit sphere. The more general case was treated in [AlfNS96a] where the h_i were called trihedral coordinates and the associated functions H^d_{ijk} defined in (13.14) were called homogeneous Bernstein basis polynomials. Most of the properties of the H^d_{ijk} discussed here also hold in the more general case.

Remark 13.7. It is easy to construct explicit linear functionals $\{\gamma_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ that pick off the B-coefficients of a spherical spline in $\mathcal{S}^0_d(\triangle)$. For an explicit construction, see Remark 14.3. For a different construction, see [BarL05].

Remark 13.8. For some Fortran code for constructing Delaunay triangulations on the surface of the sphere, see [Ren97].

13.7. Historical Notes

The theory of spherical splines as presented in this chapter was initiated by the work of Alfeld, Neamtu, and Schumaker in the series of papers [AlfNS95, AlfNS96a–AlfNS96c]. The main breakthrough was the realization that one could define appropriate spherical barycentric coordinates by giving up the partition of unity property. It was pointed out later by Helmut Pottmann that the spherical barycentric coordinates introduced in [AlfNS96a] are precisely the same as those studied by Möbius over 120 years earlier, see [Moe86].

Bernstein basis methods for homogeneous polynomials on trihedral partitions were first studied in the paper [AlfNS96a], where the connection with spherical harmonics was also explained. They were later used in [AlfNS96c] to establish results on the dimension and existence of local bases for spherical spline spaces. Circular Bernstein–Bézier polynomials were introduced in [AlfNS95].

The fact that there are spherical analogs for all of the usual bivariate macro-element spaces was observed in [AlfNS96b], where several explicit methods including the C^1 polynomial, Clough–Tocher, and Powell–Sabin elements were explicitly discussed along with numerical experiments.

Approximation Power of Spherical Splines

Throughout this chapter we suppose that S is the unit sphere centered at the origin. Our aim is to discuss how well smooth functions defined on S can be approximated by spherical polynomials and spherical splines.

14.1. Radial Projection

In this section we introduce a natural radial projection operator which will be useful for establishing approximation results for spherical polynomials and splines. Given a point v_D on S and a real number $0 < r < \pi/2$, we define the associated spherical cap $D := D_{v_D,r}$ of radius r to be the set of all points v on S such that the geodesic distance from v to v_D is at most r.

Definition 14.1. *Suppose D is a spherical cap with center v_D and radius at most 1/2, and let π_D be the plane that is tangent to S at v_D. Then we define the* radial projection $\mathcal{R}_D$ *mapping D into π_D by*

$$\mathcal{R}_D\, v := \bar{v}, \quad v \in D,$$

where $\bar{v}$ is the unique point on π_D such that $v = \bar{v}/|\bar{v}|$.

The mapping $\mathcal{R}_D$ takes the spherical cap D one-to-one onto a disk $\bar{D}$ with center at v_D and lying in π_D. If r_D is the radius of D, then the radius of $\bar{D}$ is $\tan r_D$. An arbitrary spherical cap $B \subseteq D$ is mapped onto an ellipse $\bar{B}$ contained in $\bar{D}$. $\bar{B}$ is a disk if and only if the centers of B and D coincide. Clearly, $\mathcal{R}_D$ maps circular arcs in the cap D onto line segments in the disk $\bar{D}$. It thus maps spherical triangles into planar triangles, see Figure 14.1.

14.2. Projections of Triangulations

Throughout this section we suppose that D is a spherical cap of radius 1/2. Suppose $\triangle$ is a spherical triangulation contained in D. Then $\mathcal{R}_D$ maps $\triangle$ onto a planar triangulation $\bar{\triangle}$ contained in $\bar{D}$. The connectivity of $\triangle$ is preserved under this mapping, so $\bar{\triangle}$ has the same number of vertices, edges, and triangles as $\triangle$. The shapes of the triangles are of course different, see Figure 14.1. In this section we compare shape properties of triangles in $\triangle$ and $\bar{\triangle}$. First we compare the lengths of edges.

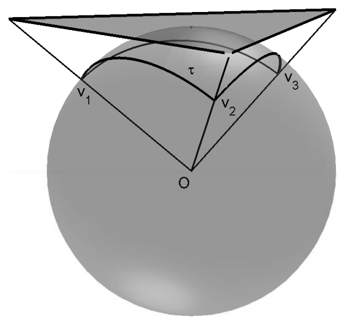


Fig. 14.1. Radial mapping of a spherical triangle to a planar triangle.

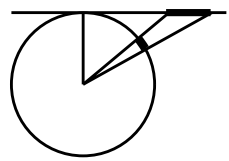


Fig. 14.2. Mapping of an arc.

Lemma 14.2. *Suppose D is a spherical cap with radius 1/2. Let a be a circular arc in D of geodesic length $|a|$, and let $\bar{a} := \mathcal{R}_D a$ be the corresponding line segment in $\bar{D}$. Then*

$$K|\bar{a}| \le |a| \le |\bar{a}|, \tag{14.1}$$

where $K := \cos^2(1/2)$.

Proof: We may suppose that the center of the cap D is the north pole of S, and that o is the origin of S. Let v_1 and v_2 be the endpoints of a, and let $\bar{v}_1$ and $\bar{v}_2$ be the endpoints of the line segment $\bar{a}$ in the tangent plane π_D. Let $A := \langle o, w_1, w_2 \rangle$ be a planar isosceles triangle that contains the arc a, and whose sides meeting at o have length $K_1 := 1/\cos(1/2)$. Then $\bar{a}$ is just the intersection of A with the tangent plane to S at the north pole.

To prove the first inequality, we now rotate A to make the length of $\bar{a}$ maximal, keeping the length of a constant. This happens when the arc a lies in a plane through the z-axis and v_2 is on the boundary of D. Figure 14.2 shows the arc and its image, both marked with thicker lines. With A in this position, we have $w_2 = \bar{v}_2$. Comparing A with a similar triangle with side lengths 1, we see that the edge $e := \langle w_1, w_2 \rangle$ has length at most $K_1|a|$. Now consider the triangle $B := \langle w_1, \bar{v}_1, \bar{v}_2 \rangle$. It is easy to see that the angle of B at $\bar{v}_2$ is at most 1/2, while the angle at w_1 is less than $\pi/2$. It follows that

$$|\bar{a}| \le K_1|e| \le K_1^2|a|.$$

To prove the second inequality, we observe that as we rotate A (which corresponds to moving a around within the cap), the length of $\bar{a}$ becomes minimal when v_1 and v_2 are equidistant from the north pole. It follows that

$$|\bar{a}| \geq 2\tan(|a|/2) \geq |a|. \quad \square$$

Given a spherical triangle T, let r_T be the radius of the smallest spherical cap containing T. Let $r_{\bar{T}}$ be the radius of the smallest disk containing the planar triangle $\bar{T} = \mathcal{R}_D T$.

Lemma 14.3. *Suppose $\triangle$ is a spherical triangulation contained in a spherical cap of radius 1/2. Then for any $T \in \triangle$,*

$$K\, r_{\bar{T}} \leq r_T \leq 2\, r_{\bar{T}}, \tag{14.2}$$

where $K := \cos^2(1/2)$.

Proof: The smallest spherical cap σ containing T is mapped onto an ellipse containing $\bar{T}$. Let o be the center of σ, and let $\bar{o}$ be its image in $\bar{T}$. Let $\partial\sigma$ be the boundary of σ. Then by Lemma 14.2,

$$r_{\bar{T}} \leq \max_{v \in \partial\sigma} |\bar{o} - \bar{v}| \leq \frac{r_T}{\cos^2(1/2)},$$

which establishes the first inequality in (14.2). On the other hand, again using Lemma 14.2,

$$r_{\bar{T}} \geq \frac{|\bar{T}|}{2} \geq \frac{|T|}{2} \geq \frac{r_T}{2},$$

where $|T|$ and $|\bar{T}|$ are the lengths of the longest edges in T and $\bar{T}$, respectively. $\square$

We turn now to the shape of triangles in $\triangle$ and $\bar{\triangle}$. Given a spherical triangle T, we define the inscribed cap to be the largest spherical cap contained in T. We call the center v_T of this inscribed cap the incenter of T, and refer to its radius ρ_T as the inradius of T. We now compare the inradius of T to the inradius of its image $\bar{T} = \mathcal{R}_D T$.

Lemma 14.4. *Suppose $\triangle$ is a spherical triangulation lying in a spherical cap D with radius 1/2. Then for any $T \in \triangle$,*

$$K\, \rho_{\bar{T}} \leq \rho_T \leq \rho_{\bar{T}}, \tag{14.3}$$

where $K := \frac{1}{2}\cos^2(1/2)$.

Proof: By the geometry, the largest spherical cap σ contained in T is mapped onto an ellipse ϵ contained in the triangle $\bar{T}$. Let o be the center of σ, and let $\bar{o}$ be its image in $\bar{T}$. Let r_ϵ be the radius of the largest circle with center at $\bar{o}$ that is contained in ϵ. Then

$$r_\epsilon := \min_{v \in \partial\sigma} |\bar{o} - \bar{v}|.$$

But then using Lemma 14.2, we have

$$\rho_{\bar{T}} \geq r_\epsilon \geq \rho_T,$$

which is the second inequality in (14.3). Now by Lemma 14.2, $\rho_T \geq K_1 \rho_\epsilon$, where

$$\rho_\epsilon := \max_{v \in \partial\sigma} |\bar{o} - \bar{v}|$$

and $K_1 := \cos^2(1/2)$. Since $\rho_{\bar{T}} \leq 2\rho_\epsilon$, the first inequality in (14.3) follows. □

We now compare the size of angles in the triangles of $\triangle$ with those in $\bar{\triangle}$. If $T := \langle v_1, v_2, v_3 \rangle$, is a spherical triangle, then we define the angle of T at v_1 to be the dihedral angle of T at v_1, i.e., the angle between the plane π_2 passing through the origin o and the points v_1, v_2 and the plane π_3 passing through o and the points v_1, v_3. Given a spherical triangle $T \in \triangle$, let θ_T be the smallest angle in T. Let $\theta_{\bar{T}}$ be the smallest angle in the planar triangle $\bar{T} = \mathcal{R}_D T$.

Lemma 14.5. *Suppose $\triangle$ is a spherical triangulation lying in a spherical cap D with radius 1/2. Then for any $T \in \triangle$,*

$$\theta_{\bar{T}} \leq \theta_T \leq K\,\theta_{\bar{T}}, \tag{14.4}$$

where $K := 4/(\sqrt{3}\cos^2(1/2))$.

Proof: Let $T := \langle v_1, v_2, v_3 \rangle$, and let π_2 be the plane passing through the origin o and the two points v_1, v_2. Similarly, let π_3 be the plane passing through o and the two points v_1, v_3. For $i = 2, 3$, let L_i be the line lying in π_i that is perpendicular to $e := \langle o, v_1 \rangle$ at v_1. Then the angle of T at v_1 is the angle between these two lines. Now the edge $\langle \bar{v}_1, \bar{v}_2 \rangle$ of $\bar{T}$ lies in the plane π_2, while $\langle \bar{v}_1, \bar{v}_3 \rangle$ lies in π_3. Since these edges are not perpedicular to e, the angle between them must be smaller than the angle between the planes, and we have established the first inequality in (14.4).

To establish the second inequality, let v_c be the center of the largest spherical cap contained in T. Without loss of generality, we may assume that the length ℓ of the arc from v_1 to v_c is larger than the lengths of the arcs from v_2 to v_c and from v_3 to v_c. By the spherical law of sines,

$$\sin(\theta_1/2) = \frac{\sin \rho_T}{\sin \ell},$$

where θ_1 denotes the angle of T at v_1. Using Lemmas 14.2 and 14.3, we have

$$\theta_T \le \frac{4\sin(\theta_1/2)}{\pi} \le \frac{2\rho_{_T}}{\ell} \le \frac{4\rho_{_T}}{|T|} \le \frac{4}{\cos^2(\frac{1}{2})}\frac{\rho_{_{\bar{T}}}}{|\bar{T}|}.$$

Since the smallest angle in a planar triangulation is at most $\pi/3$, we have

$$\frac{\rho_{_{\bar{T}}}}{|\bar{T}|} \le \tan(\theta_{\bar{T}}/2) \le \frac{\theta_{\bar{T}}}{\sqrt{3}},$$

and the second inequality in (14.4) follows. □

We conclude this section with a result on areas. Given $T \in \triangle$, let A_T and $A_{\bar{T}}$ be the areas of T and $\bar{T} = \mathcal{R}_{_D} T$, respectively.

Lemma 14.6. *Suppose $\triangle$ is a spherical triangulation lying in a spherical cap D with radius 1/2. Then for any $T \in \triangle$,*

$$K A_{\bar{T}} \le A_T \le A_{\bar{T}}, \tag{14.5}$$

where the constant K is positive and depends only on the smallest angle in T.

Proof: Since the area of a spherical cap of radius r is $2\pi(1-\cos r) = 4\pi\sin^2(r/2)$, it follows that

$$\frac{4}{\pi}\rho_{_T}^2 \le 4\pi\sin^2(\rho_{_T}/2) \le A_T \le 4\pi\sin^2(r_{_T}/2) \le \pi r_{_T}^2,$$

where $\rho_{_T}$ is the inradius of T and $r_{_T}$ is the radius of the smallest cap containing T. Simple trigonometry shows that

$$r_{_{\bar{T}}} \le |\bar{T}| \le \frac{2\rho_{_{\bar{T}}}}{\tan(\theta_{\bar{T}}/2)}.$$

We are ready to prove the first inequality in (14.5). Using Lemma 14.4, we have

$$A_{\bar{T}} \le \pi r_{_{\bar{T}}}^2 \le \frac{4\pi\,\rho_{_{\bar{T}}}^2}{\tan^2(\theta_{\bar{T}}/2)} \le \frac{16\pi\,\rho_{_T}^2}{\tan^2(\theta_{\bar{T}}/2)\cos^4(\frac{1}{2})} \le \frac{4\pi^2}{\tan^2(\theta_{\bar{T}}/2)\cos^4(\frac{1}{2})}\,A_T.$$

To prove the second inequality, we note that the volume of a tetrahedron with height 1 and base of area $A_{\bar{T}}$ is $A_{\bar{T}}/3$. The volume of the spherical wedge associated with T is $A_T/3$. Since the volume of the wedge is bounded above by the volume of the tetrahedron, it follows that $A_T \le A_{\bar{T}}$. □

14.3. Norms on the Sphere

In this section we introduce some notation and establish three useful results relating to norms of functions defined on the unit sphere S. Throughout the section we assume that Ω is the closure of a simply connected open subset of the unit sphere S. We call Ω a spherical domain. First we recall that the standard q-norms defined on a spherical domain Ω are defined by

$$\|f\|_{q,\Omega} := \begin{cases} \operatorname{ess\,sup}_{v\in\Omega}|f(v)|, & \text{if } q=\infty, \\ \left(\int_\Omega |f(v)|^q d\sigma\right)^{1/q}, & \text{if } 1\le q<\infty, \end{cases}$$

where σ is Lebesgue measure on S.

14.3.1 Norms of Homogeneous Extensions

Given a function f defined on S and an integer n, we define the homogeneous extension of f of degree n by

$$f_n(v) := |v|^n f(v/|v|), \qquad v \in \mathbb{R}^3 \setminus \{0\}.$$

Our first result compares the norm of a function defined on S with the norm of its homogeneous extension of degree n restricted to a tangent plane.

Lemma 14.7. *Suppose Ω is a spherical domain that is contained in a spherical cap D of radius 1/2, and let π_D be the plane that is tangent to S at the center of D. Let $f \in L_q(\Omega)$ for fixed $1 \le q \le \infty$. Given an integer n, let $\bar f_n$ be the restriction to $\bar\Omega$ of the homogeneous extension f_n of f of degree n. Then*

$$K_1 \|f\|_{q,\Omega} \le \|\bar f_n\|_{q,\bar\Omega} \le K_2 \|f\|_{q,\Omega}, \tag{14.6}$$

where

$$K_1 := \begin{cases} m_\Omega^{n+3/q}, & n+\frac{3}{q}\ge 0, \\ M_\Omega^{n+3/q}, & n+\frac{3}{q}<0, \end{cases} \qquad K_2 := \begin{cases} M_\Omega^{n+3/q}, & n+\frac{3}{q}\ge 0, \\ m_\Omega^{n+3/q}, & n+\frac{3}{q}<0, \end{cases} \tag{14.7}$$

with

$$m_\Omega := \inf\left\{|\bar v|,\ \bar v \in \bar\Omega\right\} \ge 1, \quad M_\Omega := \sup\left\{|\bar v|,\ \bar v\in\bar\Omega\right\} \le \frac{1}{\cos(1/2)}.$$

Here the exponents in (14.7) are understood to be equal to n for $q=\infty$.

Proof: We prove (14.6) for $q<\infty$. The case $q=\infty$ is similar and simpler. Let σ and $\bar\sigma$ denote the Lebesgue measures on S and π_D, respectively. Using the substitution $v \mapsto \bar v := \mathcal{R}_D v \in \bar\Omega$, it follows by a change of variable that

$$\int_\Omega |f(v)|^q d\sigma = \int_{\bar\Omega} \left|f(\mathcal{R}_D^{-1}\bar v)\right|^q |\bar v|^{-3} d\bar\sigma.$$

By the homogeneity of f_n and the identity $|\mathcal{R}_D^{-1}\bar{v}| = |v| = 1$ for $\bar{v} \in \bar{\Omega}$, we can write

$$\bar{f}_n(\bar{v}) = f_n(\bar{v}) = f_n\left(|\bar{v}|\mathcal{R}_D^{-1}\bar{v}\right) = |\bar{v}|^n f_n(\mathcal{R}_D^{-1}\bar{v}) = |\bar{v}|^n f(\mathcal{R}_D^{-1}\bar{v}),$$

and therefore

$$\int_\Omega |f(v)|^q \, d\sigma = \int_{\bar{\Omega}} |\bar{v}|^{-(nq+3)} |\bar{f}_n(\bar{v})|^q \, d\bar{\sigma}.$$

Now (14.6) follows immediately using $m_\Omega \le |\bar{v}| \le M_\Omega$. The bound $m_\Omega \ge 1$ is trivial, while $M_\Omega \le 1/\cos(\frac{1}{2})$ follows from the fact that Ω is contained in a cap of radius 1/2. □

14.3.2 Norms of Spherical Polynomials on Spherical Triangles

In this section we explore the connection between the q-norm and the ∞-norm of a spherical polynomial defined on a spherical triangle T.

Lemma 14.8. *Let T be a spherical triangle that is contained in a spherical cap of radius 1/2, and let $1 \le q \le \infty$. Then there exists a constant K depending only on d, q, and the smallest angle of T such that for every spherical polynomial p of degree d,*

$$A_T^{-1/q} \|p\|_{q,T} \le \|p\|_{\infty,T} \le K\, A_T^{-1/q} \|p\|_{q,T}, \tag{14.8}$$

where A_T is the area of T.

Proof: The first inequality in (14.8) is elementary. To prove the second inequality, let $\bar{p}_d := p_d|_{\bar{T}}$, where $\bar{T}$ is the image of T under the mapping $\mathcal{R}_D$ associated with the spherical cap of radius 1/2 centered at the incenter of T. Note that $\bar{p}_d$ is an ordinary bivariate polynomial defined on $\bar{T}$. Now using (1.4), we have

$$\|\bar{p}_d\|_{\infty,\bar{T}} \le K_1 A_{\bar{T}}^{-1/q} \|\bar{p}_d\|_{q,\bar{T}},$$

where K_1 depends only on d. Using this fact and Lemma 14.7 (with $n = d$), we obtain

$$\|p\|_{\infty,T} \le K_2\, \|\bar{p}_d\|_{\infty,\bar{T}} \le K_3 A_{\bar{T}}^{-1/q} \|\bar{p}_d\|_{q,\bar{T}} \le K_4 A_T^{-1/q} \|p\|_{q,T},$$

where in the last step we used (14.5) and Lemma 14.7. □

14.3.3 Stability of the Spherical B-form

In this section we establish the connection between the size of the B-coefficients of a spherical polynomial and the size of the polynomial itself. This is the analog of Theorem 2.7 for bivariate polynomials. Let T be a spherical triangle, and let $\{B_{ijk}^d\}_{i+j+k=d}$ be the associated spherical Bernstein basis polynomials of degree d.

Theorem 14.9. *Let T be a spherical triangle that is contained in a spherical cap of radius 1/2, and let $1 \le q \le \infty$. Then*

$$\frac{A_T^{1/q}}{K} \|c\|_q \le \|p\|_{q,T} \le 3^d A_T^{1/q} \|c\|_q \tag{14.9}$$

for every

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d.$$

The constant K depends only on d.

Proof: Using (13.6), we have

$$|p(v)| \le \|c\|_\infty \sum_{i+j+k=d} B_{ijk}^d(v) \le 3^d \|c\|_\infty \tag{14.10}$$

for all $v \in T$. This implies the second inequality in (14.9) for $q = \infty$. To prove it for $1 \le q < \infty$, we integrate the q-th power of (14.10) over T and use the fact that $\|c\|_\infty \le \|c\|_q$.

We now prove the first inequality in (14.9). The homogeneous extension $\bar{p}$ of p of degree d is given by $\bar{p} := \sum_{i+j+k=d} c_{ijk} \bar{B}_{ijk}^d$, where $\bar{B}_{ijk}^d$ are the bivariate Bernstein basis polynomials associated with $\bar{T}$. Let $A_{\bar{T}}$ be the area of $\bar{T} := \mathcal{R}_D T$. By (2.19), (14.6), and the fact that $A_T \le A_{\bar{T}}$ (see Lemma 14.6),

$$\|c\|_q \le K_1 A_{\bar{T}}^{-1/q} \|\bar{p}\|_{q,\bar{T}} = K_1 A_T^{-1/q} \left(\frac{A_T^{1/q}}{A_{\bar{T}}^{1/q}} \right) \|\bar{p}\|_{q,\bar{T}} \le K A_T^{-1/q} \|p\|_{q,T}. \quad \square$$

14.4. Spherical Sobolev Spaces

Let $1 \le q \le \infty$, and let Ω be a spherical domain. Our aim in this section is to define spherical analogs of the classical Sobolev spaces $W_q^k(B)$, and to construct corresponding seminorms which annihilate spherical polynomials. To get started, suppose that $\{(\Gamma_j, \phi_j)\}$ is an atlas for Ω, i.e., a finite collection of charts (Γ_j, ϕ_j), where Γ_j are open subsets of Ω whose union covers Ω, and where ϕ_j are infinitely differentiable mappings $\phi_j : \Gamma_j \to B_j$, B_j an open subset of $\mathbb{R}^2$, whose inverses ϕ_j^{-1} are also infinitely differentiable. Also, let $\{\alpha_j\}$ be a partition of unity subordinated to the atlas $\{(\Gamma_j, \phi_j)\}$, i.e., a set of infinitely differentiable functions α_j on Ω vanishing outside the sets Γ_j, such that $\sum_j \alpha_j = 1$ on Ω.

Definition 14.10. *Let $\Omega \subseteq S$. Then given $1 \le q \le \infty$ and $k > 0$, we define the associated* spherical Sobolev space *to be*

$$W_q^k(\Omega) := \{ f : (\alpha_j f) \circ \phi_j^{-1} \in W_q^k(B_j), \text{ for all } j \} \tag{14.11}$$

with norm $\|f\|_{k,q,\Omega} := \sum_j \|(\alpha_j f) \circ \phi_j^{-1}\|_{W_q^k(B_j)}$.

The Sobolev space $W_q^k(\Omega)$ is just the space of all functions f defined on Ω for which $\|f\|_{W_q^k(\Omega)}$ is finite. It is well known [Aub82, LioM72] that this definition does not depend on the choice of the atlas or the partition of unity, in the sense that other choices will give rise to the same space with a norm that is equivalent to the one introduced in Definition 14.10. We now relate functions in a spherical Sobolev space to bivariate functions in an ordinary Sobolev space.

Lemma 14.11. *Let Ω be a spherical domain contained in a spherical cap D of radius 1/2, and let $\bar\Omega$ be the image of Ω under the map $\mathcal{R}_D$ into the tangent plane π_D as in Section 14.1. Let k and n be positive integers. Suppose f is a function defined on Ω, and let $\bar f_n$ be the restriction to $\bar\Omega$ of the homogeneous extension f_n of degree n of f. Then for any $1 \le q \le \infty$, $f \in W_q^k(\Omega)$ if and only if $\bar f_n \in W_q^k(\bar\Omega)$.*

Proof: Let $\hat f : \bar\Omega \to \mathbb{R}$ be defined as $\hat f(\bar v) := f(\mathcal{R}_D^{-1}\bar v)$, $\bar v \in \bar\Omega$. It is well known that $f \in W_q^k(\Omega)$ if and only if $\hat f \in W_q^k(\bar\Omega)$. This is because in the definition of $W_q^k(\Omega)$, we can choose an atlas consisting of a single chart (Γ, ϕ), where $\Gamma = \Omega$ and $\phi : \Gamma \to B := \bar\Omega$, where $\phi(v) := \mathcal{R}_D v = \bar v \in \bar\Omega$, for $v \in \Omega$. Since Ω is contained in a spherical cap of radius 1/2, the mapping ϕ is a C^∞-diffeomorphism of Ω onto $\bar\Omega$. Thus, in this case (14.11) expresses the fact that $f \in W_q^k(\Omega)$ if and only if $f \circ \phi^{-1} = \hat f \in W_q^k(\bar\Omega)$.

Now note that $\bar f_n(\bar v) = |\bar v|^n \hat f(\bar v)$, $\bar v \in \bar\Omega$. Since the functions $|\bar v|^n$ and $|\bar v|^{-n}$ are bounded infinitely differentiable functions whose derivatives are also bounded, using the Leibnitz rule we see that multiplying any function in $W_q^k(\bar\Omega)$ by $|\bar v|^n$ and $|\bar v|^{-n}$ results in a new function in the same Sobolev space. We conclude that $\hat f \in W_q^k(\bar\Omega)$ if and only if $\bar f_n \in W_q^k(\bar\Omega)$, which combined with the above completes the proof. $\square$

The following lemma shows that the trivariate functions obtained as homogeneous extensions of functions belonging to a spherical Sobolev space are differentiable in some sense. Given a multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, we write $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$ and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$.

Lemma 14.12. *Let Ω be as in the previous lemma, and let $f \in W_q^k(\Omega)$ for some $k \ge 1$. Then $(D^\alpha f_{k-1})\,|_\Omega \in L_q(\Omega)$ for all multi-indices α with $|\alpha| = k$.*

Proof: Let $g := (D^\alpha f_{k-1})\,|_\Omega$. Note that whenever a trivariate homogeneous function is differentiated, the derivative is also homogeneous, and in particular, $g_{-1} := D^\alpha f_{k-1}$ is homogeneous of degree -1. It will be sufficient to show that $\bar g_{-1} := g_{-1}|_{\bar\Omega} \in L_q(\bar\Omega)$, since then by Lemma 14.7, $g = g_{-1}|_\Omega \in L_q(\Omega)$. We may assume without loss of generality that the center v_D of the smallest spherical cap D containing Ω is the north pole (i.e., π_D is the plane $z = 1$), and that the coordinates in π_D are the usual

(x, y)-coordinates. Let D_r denote differentiation in the radial direction, i.e., for $|r| = 1$ and a trivariate function h, we have

$$D_r h = x D_x h + y D_y h + z D_z h.$$

Since with $|\beta| = k - 1$ the function $D^\beta f_{k-1}$ is homogeneous of degree zero, it follows that

$$D_r D^\beta f_{k-1} = 0. \tag{14.12}$$

Using $D_z = z^{-1}(D_r - xD_x - yD_y)$, $z \neq 0$, and (14.12), we obtain

$$\begin{aligned} D_z D^\beta f_{k-1} &= z^{-1}(D_r - xD_x - yD_y) D^\beta f_{k-1} \\ &= -z^{-1}(x D_x D^\beta f_{k-1} + y D_y D^\beta f_{k-1}). \end{aligned}$$

Iterating this identity, we obtain the more general formula

$$D_z^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2} f_{k-1} = (-z)^{-\alpha_3} \sum_{\ell=0}^{\alpha_3} \binom{\alpha_3}{\ell} x^\ell y^{\alpha_3-\ell} D_x^{\alpha_1+\ell} D_y^{\alpha_2+\alpha_3-\ell} f_{k-1}, \tag{14.13}$$

which holds whenever $\alpha_1 + \alpha_2 + \alpha_3 = k$.

Let $(x, y, z) \in \bar{\Omega}$. By our assumption on π_D, we have $z = 1$. Moreover,

$$|x| \leq M_\Omega, \qquad |y| \leq M_\Omega,$$

where M_Ω is the constant in Lemma 14.7. It follows that $|\binom{\alpha_3}{\ell} x^\ell y^{\alpha_3-\ell}| \leq (|x| + |y|)^{\alpha_3} \leq (2M_\Omega)^{\alpha_3}$. We can now bound (14.13) as

$$\begin{aligned} \|\bar{g}_{-1}\|_{q,\bar{\Omega}} &= \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{q,\bar{\Omega}} \\ &= \Big\| \sum_{\ell=0}^{\alpha_3} \binom{\alpha_3}{\ell} x^\ell y^{\alpha_3-\ell} D_x^{\alpha_1+\ell} D_y^{\alpha_2+\alpha_3-\ell} f_{k-1} \Big\|_{q,\bar{\Omega}} \\ &\leq (2M_\Omega)^{\alpha_3} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{q,\bar{\Omega}} \\ &= (2M_\Omega)^{\alpha_3} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar{f}_{k-1}\|_{q,\bar{\Omega}} \\ &\leq (2M_\Omega)^{\alpha_3} \|\bar{f}_{k-1}\|_{W_q^k(\bar{\Omega})} < \infty, \end{aligned} \tag{14.14}$$

where it is understood that the trivariate homogeneous functions involved in the above inequalities are first restricted to $\bar{\Omega}$ before we take their L_q norms. The last inequality follows from Lemma 14.11 with $n = k - 1$ since $f \in W_q^k(\Omega)$. □

Our next result shows that the Sobolev norm of $\bar{f}_n = f_n|_{\bar{\Omega}}$ does not depend in an essential way on the degree n of the homogeneous extension of f that is used to define f_n.

Lemma 14.13. *Let $\Omega \subset S$ be as in Lemma 14.11, and suppose $f \in W_q^k(\Omega)$. Let $\bar{f}_m$ and $\bar{f}_n$ be two homogeneous extensions of f restricted to $\bar{\Omega}$. Then*

$$\|\bar{f}_m\|_{k,q,\bar{\Omega}} \le K \|\bar{f}_n\|_{k,q,\bar{\Omega}},$$

for some constant K depending only on k, m, and n.

Proof: Note that $\bar{f}_m = g\bar{f}_n$, where $g(u) := \|u\|^{m-n}$, $u \in \pi_D$, i.e., g is the restriction of the trivariate function $\|\cdot\|^{m-n}$ to π_D. It is not difficult to see that g is infinitely differentiable, and hence all of its partial derivatives are bounded on $\bar{\Omega}$, since $\bar{\Omega}$ is bounded. Let (ξ, η) be a Cartesian coordinate system in π_D, and let

$$K_1 := \sup\{\|D^\gamma g\|_{\infty,\bar{\Omega}} : |\gamma| \le k\} < \infty,$$

where $D^\gamma = D_\xi^{\gamma_1} D_\eta^{\gamma_2}$. Since Ω is contained in a cap of radius $1/2$, $\bar{\Omega}$ is contained in a disk of radius $\tan(1/2)$, and it follows that K_1 is bounded by

$$\sup\{\|D^\gamma g\|_{\infty,B} : |\gamma| \le k\} < \infty,$$

which depends only on $m - n$ and k. Now by the Leibnitz rule,

$$\begin{aligned}
\|\bar{f}_m\|_{k,q,\bar{\Omega}} &= \sum_{|\alpha|\le k} \|D^\alpha(g\bar{f}_n)\|_{q,\bar{\Omega}} \le K_1 \sum_{|\alpha|\le k} \sum_{\beta\le\alpha} \|D^\beta \bar{f}_n\|_{q,\bar{\Omega}} \\
&= K_1 \sum_{|\beta|\le k} \#\{\alpha : |\alpha| \le k,\, \alpha \ge \beta\}\, \|D^\beta \bar{f}_n\|_{q,\bar{\Omega}} \\
&\le K_1 \binom{k+2}{2} \|\bar{f}_n\|_{k,q,\bar{\Omega}}. \quad \square
\end{aligned}$$

14.5. Sobolev Seminorms

We now turn to the problem of defining seminorms for the spaces $W_q^k(\Omega)$. In analogy with the bivariate case, where the Sobolev seminorms annihilate ordinary polynomials, we want to construct seminorms that annihilate spherical polynomials.

Definition 14.14. *Let $\Omega \subseteq S$, and let $f \in W_q^k(\Omega)$ for some $k \ge 0$ and $1 \le q \le \infty$. Then we define the* Sobolev-type seminorm *of f to be*

$$|f|_{k,q,\Omega} := \sum_{|\alpha|=k} \|D^\alpha f_{k-1}\|_{q,\Omega}, \tag{14.15}$$

where $\|D^\alpha f_{k-1}\|_{q,\Omega}$ should be understood as the L_q-norm of the restriction of the trivariate function $D^\alpha f_{k-1}$ to Ω.

For $k = 0$, the above seminorm reduces to the usual L_q-norm, i.e.,

$$|f|_{0,q,\Omega} = \|f\|_{q,\Omega}.$$

One reason why the seminorms (14.15) make sense is that they are locally equivalent to the usual Sobolev seminorms of functions defined in a plane. The precise statement is as follows. Let Ω be a spherical domain contained in a cap D of radius 1/2, i.e.,

$$|\Omega| := \sup\{\arccos(u \cdot v) : u, v \in \Omega\} < 1.$$

Let (ξ, η) be a local Cartesian system in the tangent plane π_D associated with Ω, i.e., $|\xi| = |\eta| = 1$, $\xi \cdot \eta = 0$, and $\xi \cdot v_D = \eta \cdot v_D = 0$. Let $|\cdot|_{k,q,\bar{\Omega}}$ be the usual Sobolev seminorm on $\bar{\Omega}$, i.e.,

$$|g|_{k,q,\bar{\Omega}} := \sum_{\gamma_1+\gamma_2=k} \|D_\xi^{\gamma_1} D_\eta^{\gamma_2} g\|_{q,\bar{\Omega}}, \quad g \in W_q^k(\bar{\Omega}).$$

Combining Lemmas 14.7 and 14.13 leads to the following result.

Lemma 14.15. *Let $\Omega \subseteq S$ with $|\Omega| \le 1$. Then the seminorms $|\cdot|_{k,q,\Omega}$ and $|\cdot|_{k,q,\bar{\Omega}}$ are equivalent in the sense that for every $f \in W_q^k(\Omega)$,*

$$K_3\,|f|_{k,q,\Omega} \le |\bar{f}_{k-1}|_{k,q,\bar{\Omega}} \le K_4\,|f|_{k,q,\Omega}. \tag{14.16}$$

Here K_3 and K_4 are positive constants depending only on k and p.

Proof: Suppose $f \in W_q^k(\Omega)$. Then

$$\begin{aligned}
|f|_{k,p,\Omega} &= \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\Omega)} \\
&\le K_1^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\bar{\Omega})} \\
&\le K_1^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} (2M_\Omega)^{\alpha_3} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar{f}_{k-1}\|_{L^p(\bar{\Omega})} \\
&\le K_1^{-1} (2M_\Omega)^k \binom{k+2}{2} |\bar{f}_{k-1}|_{k,p,\bar{\Omega}},
\end{aligned}$$

where above, in the first inequality, we used Lemma 14.7 with $n = -1$, and in the second inequality, we employed (14.14). This proves the left-hand inequality in (14.16) with $K_3 = K_1 (2M_\Omega)^{-k} \binom{k+2}{2}^{-1} > 0$. On the other hand,

$$\begin{aligned}
|f|_{k,p,\Omega} &= \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\Omega)} \\
&\ge K_2^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\bar{\Omega})} \\
&\ge K_2^{-1} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{L^p(\bar{\Omega})} \\
&= K_2^{-1} |\bar{f}_{k-1}|_{k,p,\bar{\Omega}},
\end{aligned}$$

where in the first inequality, we used (14.6). This gives the right-hand inequality in (14.16) with $K_4 = K_2$. $\square$

The main motivation behind the definition (14.15) of Sobolev seminorms for spherical functions is the requirement that they annihilate appropriate spaces of spherical polynomials.

Lemma 14.16. *Suppose Ω is a spherical domain with $|\Omega| \le 1$. For all $f \in W_q^k(\Omega)$ with $k \ge 1$, $|f|_{k,q,\Omega} = 0$ if and only if f is a spherical polynomial of degree $k-1$.*

Proof: Clearly, $|f|_{k,p,\Omega} = 0$ if and only if $|f|_{k,p,\Omega'} = 0$, for all $\Omega' \subseteq \Omega$ such that $|\Omega'| \le 1$. By Lemma 14.15 applied to Ω', $|f|_{k,p,\Omega'} = 0$ if and only if $|\bar{f}_{k-1}|_{k,p,\bar{\Omega}'} = 0$. Since $\bar{\Omega}'$ is a planar region, $|\bar{f}_{k-1}|_{k,p,\bar{\Omega}'} = 0$ if and only if $\bar{f}_{k-1}$ is a bivariate polynomial of degree at most $k-1$ on every open subset of $\bar{\Omega}$. Since $\bar{\Omega}$ is connected, this is equivalent to f_{k-1} being a trivariate homogeneous polynomial of degree $k-1$ on $\bar{\Omega}$. This in turn is equivalent to f being a spherical polynomial of degree $k-1$, since the space of such polynomials is just the space of trivariate homogeneous polynomials of degree $k-1$ restricted to Ω, see Section 13.1.8. $\square$

In view of the results in Section 13.1.9, the space $\mathcal{B}_{k-1}$ of spherical polynomials contains the spaces $\mathcal{B}_j$ of spherical polynomials of degree j with $0 \le j \le k-1$ only for $j = k-1 \,(\mathrm{mod}\ 2)$. Thus, $|p|_{k,q,\Omega} = 0$ for all spherical polynomials p in $\mathcal{B}_j$ with $0 \le j \le k-1$ and $j = k-1 \,(\mathrm{mod}\ 2)$, but not for any other spherical polynomials.

14.6. Clusters of Spherical Triangles

Suppose $\triangle$ is a spherical triangulation of a spherical domain Ω lying in a spherical cap D of radius $1/2$. We say that a collection $\mathcal{T}$ of triangles in $\triangle$ is an ℓ-cluster of spherical triangles provided there is a vertex v in $\triangle$ such that all of the triangles in $\mathcal{T}$ are contained in $\mathrm{star}^\ell(v)$. In this section we establish several useful properties of clusters. While the results of this section could be proved directly, here we will instead make use of radial projection coupled with the results of Section 4.7 for planar clusters. We write $\bar{\mathcal{T}}$ for the planar triangulation corresponding to $\mathcal{T}$ under the mapping $\mathcal{R}_D$ associated with the spherical cap D.

Recall that if $T := \langle v_1, v_2, v_3 \rangle$ is a spherical triangle, then the angle of T at the vertex v_1 is defined to be the angle between the planes π_1 and π_2 passing through the origin and containing the edges $\langle v_1, v_2 \rangle$ and $\langle v_1, v_3 \rangle$, respectively. We write $\theta_{\mathcal{T}}$ for the smallest angle appearing in the triangulation $\mathcal{T}$. Similarly, we write $\theta_{\bar{\mathcal{T}}}$ for the smallest angle in $\bar{\mathcal{T}}$. In view of Lemma 14.5,

$$\theta_{\bar{\mathcal{T}}} \le \theta_{\mathcal{T}} \le \frac{4}{\sqrt{3}\cos^2(1/2)} \theta_{\bar{\mathcal{T}}}. \tag{14.17}$$

Lemma 14.17. *Suppose $\mathcal{T}$ is an ℓ-cluster of spherical triangles whose union is a spherical domain Ω. Suppose Ω lies in a spherical cap of radius 1/2. Then the number of triangles in $\mathcal{T}$ is bounded by a constant depending on ℓ and $\theta_{\mathcal{T}}$.*

Proof: Since the planar triangulation $\bar{\mathcal{T}} = \mathcal{R}_D \mathcal{T}$ has exactly the same combinatorial structure as $\mathcal{T}$, the result follows from Lemma 4.13 coupled with Lemma 14.5 and (14.17). □

If T is a spherical triangle, we write $|T|$ for the (geodesic) length of the longest edge of T. The following is the spherical analog of Lemma 4.14.

Lemma 14.18. *Suppose $\mathcal{T}$ is an ℓ-cluster of spherical triangles as in the previous lemma. Then for any two triangles T and $\widetilde{T}$ in $\mathcal{T}$,*

$$\frac{|T|}{|\widetilde{T}|} \le K_1, \tag{14.18}$$

and

$$\frac{A_T}{A_{\widetilde{T}}} \le K_2, \tag{14.19}$$

where K_1 and K_2 are constants depending only on ℓ and $\theta_{\mathcal{T}}$.

Proof: The bound (14.18) follows immediately from (4.9) coupled with (14.1) and (14.17). Similarly, the bound (14.19) follows from (4.10). □

Suppose T is a spherical triangle and ρ_T is the radius of the largest spherical cap contained in T. Then combining Lemmas 14.2, 14.4, and 14.5, it follows that the ratio

$$\kappa_T := |T|/\rho_T$$

is bounded by a constant depending only on the smallest angle in T. As in the planar case, we can regard κ_T as a shape parameter for the spherical triangle T.

Given an ℓ-cluster $\mathcal{T}$ of spherical triangles, let Ω_T be the union of the triangles in $\mathcal{T}$. Then

$$|\Omega_T| \le 2\ell\, |\mathcal{T}|, \tag{14.20}$$

where $|\mathcal{T}|$ denotes the length of the longest edge in the triangles of $\mathcal{T}$. Moreover, if $\rho_{\mathcal{T}}$ is the smallest inradius of the triangles in $\mathcal{T}$, then

$$\frac{|\Omega_{\mathcal{T}}|}{\rho_{\mathcal{T}}} \le K, \tag{14.21}$$

for some constant K depending only on ℓ and the smallest angle in $\mathcal{T}$. This is the spherical version of (4.12).

14.7. Local Approximation by Spherical Polynomials

In this section we establish a spherical analog of Theorem 1.9 which will describe how well functions in a Sobolev space on the sphere can be approximated locally by spherical polynomials. Suppose that Ω is the closure of a spherical domain lying in a spherical cap D of radius 1/2. Then we define the convex hull $\mathrm{co}(\Omega)$ of Ω as the set of all points lying on great circle arcs $\langle u, v\rangle$ where both u and v lie in Ω. We say that Ω is convex whenever it equals its convex hull. Now let π_D be the tangent plane associated with D, i.e., the plane tangent to S at the center of D. Let $\mathcal{R}_D$ be the associated mapping described in Section 14.1, and let $\bar{\Omega} = \mathcal{R}_D \Omega$. Then it is easy to see that Ω is convex if and only if $\bar{\Omega}$ is convex. If Ω is not convex, we define the Lipschitz constant of the boundary of Ω to be equal to the Lipschitz constant of the boundary of $\bar{\Omega}$.

Suppose that Ω is convex. Following Section 1.6, we now introduce a linear mapping $F_{d,\Omega}$ which takes functions in $L_1(\Omega)$ to spherical polynomials of degree d. Given $f \in L_1(\Omega)$, let $\bar{f}_d$ be the bivariate function defined on $\bar{\Omega}$ obtained by taking the homogeneous extension of f of degree d and then restricting it to $\bar{\Omega}$. Let $\bar{F}_{d,\bar{\Omega}}\bar{f}_d$ be the averaged Taylor polynomial defined in Section 1.5. Then we define $F_{d,\Omega}f$ to be the unique spherical polynomial of degree d whose homogeneous extension of degree d restricted to $\bar{\Omega}$ is $\bar{F}_{d,\bar{\Omega}}\bar{f}_d$. Thus, $\overline{(F_{d,\Omega}f)_d} = \bar{F}_{d,\bar{\Omega}}\bar{f}_d$. This defines the operator $F_{d,\Omega}$. We now give a bound on its norm.

Theorem 14.19. *Suppose Ω is the closure of a convex domain lying in a spherical cap of radius 1/2, and let $d \ge 0$. Let B_Ω be the largest spherical cap contained in Ω. Then*

$$||F_{d,B_\Omega} f||_{q,\Omega} \le K ||f||_{q,B_\Omega},$$

for all $f \in L_q(B_\Omega)$ with $1 \le q \le \infty$. The constant K depends only on d and $\kappa_\Omega := |\Omega|/\rho_\Omega$, where ρ_Ω is the radius of B_Ω.

Proof: Using Lemma 1.6 and Lemma 14.7, it immediately follows that

$$\|F_{d,B_\Omega} f\|_{q,\Omega} \le K_1 \|\bar{F}_{d,B_{\bar{\Omega}}} \bar{f}_d\|_{q,\bar{\Omega}} \le K_2 \|\bar{f}_d\|_{q,B_{\bar{\Omega}}} \le K_3 \|f\|_{q,B_\Omega}. \quad \square$$

The following is the analog of Theorem 1.9.

Theorem 14.20. *Let Ω be the the closure of a convex spherical domain lying in a spherical cap of radius 1/2, and let $d \ge 0$. Then for every $f \in W_q^{d+1}(\Omega)$ with $1 \le q \le \infty$,*

$$|f - F_{d,B_\Omega} f|_{k,q,\Omega} \le K\, |\Omega|^{d+1-k}\, |f|_{d+1,q,\Omega},$$

for all $0 \le k \le d+1$. The constant K depends only on d and the shape parameter κ_Ω defined in Theorem 14.19.

Proof: Let $p = F_{d,\Omega} f$. Using Theorem 1.9 and Lemmas 14.13 and 14.15, we have

$$\begin{aligned}|f-p|_{k,q,\Omega} &\le K_1\,|\bar f_{k-1}-\bar p_{k-1}|_{k,q,\bar\Omega} \le K_1\|\bar f_{k-1}-\bar p_{k-1}\|_{k,q,\bar\Omega}\\ &\le K_2\,\|\bar f_d-\bar p_d\|_{k,q,\bar\Omega} = K_2\sum_{\ell=0}^{k}|\bar f_d-\bar p_d|_{\ell,q,\bar\Omega}\\ &\le K_3\sum_{\ell=0}^{k}|\bar\Omega|^{d+1-\ell}|\bar f_d|_{d+1,q,\bar\Omega}\\ &\le K_4\Big(\sum_{\ell=0}^{k}|\Omega|^{\ell}\Big)|\Omega|^{d+1-k}|f|_{d+1,q,\Omega}\\ &\le K_5\,|\Omega|^{d+1-k}|f|_{d+1,q,\Omega}.\end{aligned}$$

Here we have used the fact that $|\Omega| \le 1$, which in turn implies $\sum_{\ell=0}^{k}|\Omega|^{\ell} \le k+1 \le d+1$. $\square$

Using Theorems 1.8 and 1.9, this result can be extended to nonconvex Ω. In this case the constant K also depends on the Lipschitz constant of $\partial\Omega$. In Section 14.9 we use the approximation theorems of this section to determine the approximation power of spherical spline spaces. There Ω will be an ℓ-cluster of spherical triangles, and the shape parameter B_{Ω} associated with Ω will be bounded by a constant depending on the smallest angle in the triangles of the cluster.

14.8. The Markov Inequality for Spherical Polynomials

In this section we establish a bound on the norm of the derivatives of a spherical polynomial in terms of the norm of the polynomial itself. Given a spherical triangle T, let $\mathcal{R}_{D}$ be the radial mapping associated with the smallest cap containing T with center at the incenter of T. Let $\bar T := \mathcal{R}_{D} T$ be the corresponding planar triangle, and let ρ_{T} and $\rho_{\bar T}$ be the inradii of T and $\bar T$, respectively. Recall from (14.3) that

$$\rho_{T} \le \rho_{\bar T} \le \frac{2\rho_{T}}{\cos^2(1/2)}. \tag{14.22}$$

Theorem 14.21. *Suppose T is a spherical triangle with $|T| \le 1$. Then there exists a constant K depending only on d such that for any spherical polynomial p of degree d and any $1 \le q \le \infty$,*

$$|p|_{k,q,T} \le K\rho_{T}^{-k}\|p\|_{q,T}, \qquad \textit{all } 0 \le k \le d. \tag{14.23}$$

Proof: Let π_D be the plane tangent to S at the incenter of T. Let $\bar{p}_d$ be the restriction to π_D of the homogeneous extension p_d of p. Then by the Markov inequality for bivariate polynomials (1.5),

$$|\bar{p}_d|_{\ell,q,\bar{T}} \le K_1\, \rho_{\bar{T}}^{-\ell}\, \|\bar{p}_d\|_{q,\bar{T}}, \qquad \ell = 0,\ldots,d,$$

for some constant K_1 depending on d. Thus,

$$\begin{aligned} |p|_{k,q,T} \le{}& K_2\, |\bar{p}_{k-1}|_{k,q,\bar{T}} \le K_2\, \|\bar{p}_{k-1}\|_{k,q,\bar{T}} \le K_3\, \|\bar{p}_d\|_{k,q,\bar{T}} \\ &\le K_4 \sum_{\ell=0}^{k} \rho_{\bar{T}}^{-\ell}\, \|\bar{p}_d\|_{q,\bar{T}} = K_4 \Big(\sum_{\ell=0}^{k} \rho_{\bar{T}}^{\ell} \Big) \rho_{\bar{T}}^{-k}\, \|\bar{p}_d\|_{q,\bar{T}}. \end{aligned}$$

Now (14.22) implies $\rho_{\bar{T}} \le K_5 \rho_T \le K_5 |T| \le K_5$ which in turn implies $\sum_{\ell=0}^{k} \rho_{\bar{T}}^{\ell} \le K_5^d (d+1)$. Since (14.22) also gives $\rho_T \le \rho_{\bar{T}}$, we get (14.23). $\square$

14.9. Spaces with Full Approximation Power

In this section we discuss three cases where spherical spline spaces have full approximation power in the sense of Definition 10.1.

14.9.1 Spherical Spline Spaces with a Stable Local MDS

Let $\triangle$ be a spherical triangulation of a set $\Omega \subseteq S$, where $\triangle$ covers all or part of S. Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spherical spline space associated with $\triangle$, and suppose $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}$. We now show how to use $\mathcal{M}$ to construct an explicit quasi-interpolation operator Q mapping $L_1(S)$ onto $\mathcal{S}$ which provides full approximation power. Let $f \in L_1(S)$. We define a spline Qf in $\mathcal{S}$ by explicitly constructing its coefficients. Given $\xi \in \mathcal{M}$, let γ_ξ be a linear functional that for any spline $s \in \mathcal{S}_d^0(\triangle)$ picks off the B-coefficient c_ξ corresponding to ξ, see Remark 14.3. Let T_ξ be a triangle containing ξ, and let

$$c_\xi = \gamma_\xi(F_{d,T_\xi} f), \tag{14.24}$$

where $F_{d,T_\xi} f$ is the averaged Taylor polynomial of degree d associated with T_ξ, see Theorem 14.19.

We have now defined coefficients c_ξ of a spline Qf for all $\xi \in \mathcal{M}$. Since $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}$, for each $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$ we can use smoothness conditions to compute the coefficient c_η of Qf as a linear combination of $\{c_\xi\}_{\xi \in \mathcal{M}_\eta}$ for some set of domain points $\mathcal{M}_\eta \subseteq \mathrm{star}^\ell(T_\eta)$. Here T_η is a triangle containing η, and ℓ is the integer constant in (13.30).

By stability, there is a constant K depending only on ℓ and the smallest angle $\theta_\triangle$ in $\triangle$ such that

$$|c_\eta| \le K \max_{\xi \in \mathcal{M}_\eta} |c_\xi|. \qquad \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}. \tag{14.25}$$

Theorem 14.22. *Q is a linear projector mapping $L_1(S)$ onto $\mathcal{S}$ such that for any triangle $T \in \triangle$ and any $1 \le q \le \infty$,*

$$\|Qf\|_{q,T} \le K\|f\|_{q,\Omega_T}, \qquad \text{all } f \in L_1(\Omega_T), \tag{14.26}$$

where $\Omega_T := \mathrm{star}^\ell(T)$. The constant K depends only on d, ℓ, and the smallest angle in Ω_T.

Proof: By construction, Q is defined for all functions $f \in L_1(S)$, and is a linear operator. For each $\xi \in \mathcal{M}$, $F_{d,T_\xi} p = p$ for any polynomial of degree d. It follows that $Qs = s$ for all splines $s \in \mathcal{S}$. We now establish (14.26) in the case $1 \le q < \infty$. The case $q = \infty$ is similar and simpler.

We first bound the coefficients of Qf. Suppose $\xi \in \mathcal{M}$, and let T_ξ be the triangle containing ξ. Then applying Theorem 14.9 to the polynomial $F_{d,T_\xi} f := \sum c_\eta B_\eta^{T_\xi}$ and using Theorem 14.19 on the triangle T_ξ, we get

$$|c_\xi| = |\gamma_\xi(F_{d,T_\xi} f)| \le \frac{K_1}{A_{T_\xi}^{1/q}} \|F_{d,T_\xi} f\|_{q,T} \le \frac{K_2}{A_{T_\xi}^{1/q}} \|f\|_{q,T_\xi}.$$

Here the constant K_2 depends on the shape parameter κ_T of T_ξ, which as shown in Section 14.6 depends only on the smallest angle in T_ξ. Now fix a triangle $T \in \triangle$. Using (14.25), we see that for all $\eta \in \mathcal{D}_{d,T}$,

$$|c_\eta| \le \frac{K_3}{A_{min}^{1/q}} \|f\|_{q,\Omega_T},$$

where A_{min} is the minimum of the areas of the triangles in Ω_T. Using (14.19) (which says that all triangles in Ω_T have comparable areas), and (13.6), this immediately implies

$$\|Qf\|_{q,T}^q = \int_T \Big| \sum_{\eta \in \mathcal{D}_{d,T}} c_\eta B_\eta^T \Big|^q \le K_4 \|f\|_{q,\Omega_T}^q. \quad \square$$

In working with spherical polynomials, it is important to keep in mind the fact that the space $\mathcal{B}_d$ of spherical polynomials of degree d contains the spaces $\mathcal{B}_j$ of spherical polynomials of degree j with $0 \le j \le d-1$ only for $j = d \,(\mathrm{mod}\ 2)$, see (13.16). This means that Q reproduces spherical polynomials of degree j with $0 \le j \le d-1$ only for $j = d \,(\mathrm{mod}\ 2)$. We can now give a local approximation result for Q.

Theorem 14.23. *Let $0 \le m \le d$ with $m = d \,(\text{mod } 2)$. Given a triangle T in $\triangle$, suppose the set $\Omega_T := \text{star}^\ell(T)$ of Theorem 14.22 is such that $|\Omega_T| \le 1$. Then for all $f \in W_q^{m+1}(\Omega_T)$ with $1 \le q \le \infty$,*

$$|f - Qf|_{k,q,T} \le K\,|T|^{m+1-k}\,|f|_{m+1,q,\Omega_T}, \tag{14.27}$$

for all $0 \le k \le m$. If Ω_T is convex, the constant K depends only on d, ℓ, and the smallest angle in the triangles of Ω_T. If Ω_T is not convex, K also depends on the Lipschitz constant of the boundary of the convex hull of Ω_T.

Proof: Fix $0 \le m \le d$ with $m = d\,(\text{mod } 2)$. By Theorem 14.20, there exists a spherical polynomial p of degree m depending on f so that

$$|f - p|_{j,q,\Omega_T} \le K_1\,|\Omega_T|^{m+1-j}\,|f|_{m+1,q,\Omega_T}, \tag{14.28}$$

for all $0 \le j \le m$, where K_1 is a constant depending on m and the smallest angle in $\triangle$. If Ω_T is convex, then K_1 is a constant depending on m and the shape parameter $\kappa_{\Omega_T} := |\Omega_T|/\rho_{\Omega_T}$ of Ω_T, where $|\Omega_T|$ is the diameter of Ω_T and ρ_{Ω_T} is the radius of the largest disk contained in Ω_T. If Ω_T is nonconvex, then K_1 also depends on the Lipschitz constant of the boundary of the convex hull of Ω_T. Now by (14.20) and Lemma 14.18, $|\Omega_T| \le K_2|T|$, where K_2 is a constant depending on the smallest angle in Ω_T. It follows that $|\Omega_T|/\rho_{\Omega_T} \le K_2|T|/\rho_T$, where ρ_T is the radius of the largest disk contained in T. Thus, the shape parameter κ_{Ω_T} of Ω_T is bounded by K_2 times the shape parameter κ_T of T, which as observed on page 422 is bounded by a constant depending on the smallest angle in Ω_T.

Now fix $0 \le k \le m$. Since Q reproduces spherical polynomials of degree d, and thus also spherical polynomials of degree m with $m = d\,(\text{mod } 2)$, we have

$$|f - Qf|_{k,q,T} \le |f - p|_{k,q,T} + |Q(f - p)|_{k,q,T}.$$

In view of (14.28), it suffices to estimate the second term. For any function g, the restriction of Qg to T is a polynomial of degree d. Thus, using (14.26) and the Markov inequality (14.23), it follows that

$$|Q(f-p)|_{k,q,T} \le K_2\,\rho_T^{-k}\|Q(f-p)\|_{q,T} \le K_3\,\rho_T^{-k}\|f-p\|_{q,\Omega_T},$$

where ρ_T is the radius of the largest spherical cap contained in T. Using (14.18) and (14.28), we immediately get (14.27). $\square$

We now give a global version of the above approximation result. Let $\triangle$ be a spherical triangulation of a spherical domain Ω on S, and let $|\triangle|$ be the mesh size of $\triangle$, i.e., the length of the longest edge in $\triangle$. We assume that the mesh is sufficiently fine so that $\ell|\triangle| \le 1$, where ℓ is the constant appearing in the stability of the MDS $\mathcal{M}$ for $\mathcal{S}$, see (14.25).

Theorem 14.24. *Let $0 \le m \le d$ with $m = d \,(\mathrm{mod}\ 2)$. Then for all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$,*

$$|f - Qf|_{k,q,\Omega} \le K\, |\triangle|^{m+1-k}\, |f|_{m+1,q,\Omega}, \tag{14.29}$$

for all $0 \le k \le m$. If Ω covers S, the constant K depends only on d, ℓ, and the smallest angle $\theta_\triangle$ in the triangles of $\triangle$. Otherwise, K may also depend on the Lipschitz constant of $\partial\Omega$.

Proof: For $q = \infty$, (14.29) follows immediately from (14.27) by taking the maximum over all triangles T in $\triangle$ and using the fact that $|\Omega_T| \le 2\ell|\triangle|$. To get the result for $1 \le q < \infty$, we take the q-th power of both sides of (14.27) and sum over all triangles in $\triangle$. Since Ω_T contains other triangles besides T, some triangles appear more than once in the sum on the right. However, a given triangle T_R appears on the right only if it is associated with a triangle T_L on the left which lies in $\mathrm{star}^\ell(T_R)$. But Lemma 14.17 implies that there is a constant K_1 depending only on ℓ and $\theta_\triangle$ such that T_R enters at most K_1 times on the right, and (14.29) follows . □

14.9.2 The Space $\mathcal{S}_d^r(\triangle)$ for $d \ge 3r + 2$

In this section we describe the approximation power of the spherical spline space $\mathcal{S}_d^r(\triangle)$ for $d \ge 3r+2$. As in the bivariate case, it suffices to establish the approximation power of any subspace of $\mathcal{S}_d^r(\triangle)$, which for convenience we take to be the space of spherical supersplines

$$\mathcal{S}_d^{r,\mu}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) :\ s \in C^\mu(v),\ \text{all } v \in \mathcal{V}\},$$

where

$$\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor, \tag{14.30}$$

and $\mathcal{V}$ is the set of vertices of $\triangle$.

To get results on the approximation power of $\mathcal{S}_d^{r,\mu}(\triangle)$, we construct a stable local minimal determining set and apply the results of the previous section. We can follow the treatment of the bivariate case in Section 11.4. First, we note that the concepts of δ-near-degenerate edges and δ-near-singular vertices introduced in Definitions 10.3 and 10.6 also make sense for spherical triangulations provided we measure the angle between two edges by the dihedral angle at the common vertex.

Suppose $\triangle$ is a spherical triangulation of a domain $\Omega \subseteq S$ with $|\triangle| \le 1/6$. Let $\mathcal{V}_S$ and $\mathcal{V}_{NS}$ be the sets of vertices of $\triangle$ which are singular and $\theta_\triangle$-near-singular, respectively, where $\theta_\triangle$ is the smallest angle in $\triangle$. Given a spherical triangle T in $\triangle$, let $A^T, C^T, E^T, F^T, G_L^T, G_R^T$ be the subsets of domain points in T defined in (9.17), see also Figure 9.5.

The proof of the following theorem follows along the same lines as the proof of Theorem 11.7, and rests on spherical analogs of the theorems and lemmas of Sections 11.2–11.4. The proofs of the spherical versions of these results are essentially the same as in the bivariate case, although in some cases minor adjustments are needed. For example, Lemma 11.6 is not true for arbitrary spherical triangulations, but does hold if we require that $|\triangle| \le 1/6$.

Theorem 14.25. *Let $\mathcal{M}$ be the following set of domain points:*

1) *For each triangle T, include C^T.*
2) *For each edge e of $\triangle$, include $E^T(e)$, where T is a triangle sharing e.*
3) *For each edge of a triangle T such that e lies on the boundary of Ω, include $G_L^T(e)$ and $G_R^T(e)$.*
4) *For each vertex v of $\triangle$, include $D_\mu^T(v)$ for some triangle T attached to v.*
5) *Suppose the vertex $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in counterclockwise order. Let $T_i := \langle v, v_i, v_{i+1} \rangle$ and set $T_0 := T_n = \langle v, v_n, v_1 \rangle$. if v is an interior vertex. Let $1 \le i_1 < \cdots < i_k < n$ be such that e_{i_j} is $\theta_\triangle$-near-degenerate at either end, where $e_i := \langle v, v_i \rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$ and*
   a) *include $G_L^{T_i}(e_i)$ for all $i \in J_v$,*
   b) *include $A^{T_i}(v)$ for all $1 \le i \le n-1$ such that $i \notin J_v$,*
   c) *include $A^{T_n}(v)$, if v is an interior vertex,*
6) *For each vertex $v \in \mathcal{V}_S$, include the $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 11.1.*
7) *For each $v \in \mathcal{V}_{NS} \setminus \mathcal{V}_S$, include the $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 11.3.*

Then $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}_d^{r,\mu}(\triangle)$.

If $\triangle$ covers all of S, then all vertices and edges are interior, and there is no need to include the sets in item 3) of this theorem. We can now establish the approximation power of $\mathcal{S}_d^{r,\mu}(\triangle)$, and thus also of $\mathcal{S}_d^r(\triangle)$.

Theorem 14.26. *Let $0 \le m \le d$ with $m = d \,(\mathrm{mod}\ 2)$. Suppose $\triangle$ is a spherical triangulation of a set Ω on S, and that $|\triangle| \le 1/6$. Let $d \ge 3r+2$, and let μ be as in (14.30). Suppose $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$. Then there exists a spline $s \in \mathcal{S}_d^{r,\mu}(\triangle)$ such that*

$$|f - s|_{k,q,\Omega} \le K \, |\triangle|^{m+1-k} \, |f|_{m+1,q,\Omega}, \tag{14.31}$$

for all $0 \le k \le m$. If Ω covers S, the constant K depends only on d and the smallest angle in the triangles of $\triangle$. Otherwise, K may also depend on the Lipschitz constant of $\partial\Omega$.

Proof: We apply Theorems 14.24 and 14.25. Note that the construction of the MDS in Theorem 14.25 ensures that the constant in the localness of $\mathcal{M}$ is $\ell = 3$. The assumption $|\triangle| \le 1/6$ ensures that $|\Omega_T| := |\text{star}^3(T)| \le 1$ for any $T \in \triangle$. $\square$

14.9.3 Spaces with a Stable Local NMDS

Suppose $\triangle$ is a spherical triangulation of a set $\Omega \subseteq S$. As in the previous subsection, $\triangle$ can cover all of S or just a part of it. In this section we investigate the approximation power of spaces of spherical splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ which have stable local nodal minimal determining sets.

We recall from Section 13.4.5 that a nodal minimal determining set for $\mathcal{S}$ is a set of $n := \dim \mathcal{S}$ linear functionals $\mathcal{N} = \{\lambda_i\}_{i=1}^n$ based on point evaluation of derivatives such that every $s \in \mathcal{S}$ is uniquely determined by the values $\{\lambda_i s\}_{i=1}^n$. We focus on functionals of the form

$$\lambda_i := \varepsilon_{v_i} \sum_{\alpha+\beta=m_i} a^i_{\alpha,\beta} D^\alpha_{1,v_i} D^\beta_{2,v_i},$$

where ε_{v_i} denotes point evaluation at a point v_i on the sphere S, and where D_{1,v_i} and D_{2,v_i} stand for differentiation in the directions of the axes of some Cartesian coordinate system with origin at v_i and lying in the tangent plane to S at v_i.

Definition 14.27. *Suppose $\mathcal{N}$ is a nodal minimal determining set for a spherical spline space $\mathcal{S}$. Let $\bar m$ be the order of the highest derivative involved in the linear functionals defining $\mathcal{N}$. We say that $\mathcal{N}$ is* local *provided there exists an integer constant $\ell > 0$ such that for every $s \in \mathcal{S}$ and $\xi \in \mathcal{D}_{d,\triangle}$, the B-coefficient c_ξ of s depends on λs only if λ involves point evaluation at a point contained in $\Omega_\xi := \text{star}^\ell(T_\xi)$, where T_ξ is a triangle containing ξ, We say that $\mathcal{N}$ is* stable *provided there exists a constant K depending only on ℓ and the smallest angle in $\triangle$ such that for every $s \in \mathcal{S}$,*

$$|c_\xi| \le K \sum_{\nu=0}^{\bar m} |\Omega_\xi|^\nu |s|_{\nu,\Omega_\xi}. \tag{14.32}$$

All of the spherical macro-elements spaces mentioned in Section 13.5 have stable local nodal minimal determining sets. Suppose $\mathcal{S}$ is a spherical spline space with a stable local nodal minimal determining set $\mathcal{N}$. Then for any $f \in C^{\bar m}(S)$, there is a unique spline $s \in \mathcal{S}$ satisfying the Hermite interpolation conditions

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}. \tag{14.33}$$

This defines a linear operator $\mathcal{I}$ mapping $C^{\bar m}(S)$ onto $\mathcal{S}$. We now give a bound for how well $\mathcal{I} f$ approximates f. We restrict our attention to the uniform norm.

Theorem 14.28. *Let $\triangle$ be a spherical triangulation of a set Ω on S. Let $\mathcal{S}$ be a spherical spline space with a stable local nodal minimal determining set, and suppose $|\triangle| \le 1/\ell$, where ℓ is the constant in Definition 14.27. Let $\mathcal{I}$ be the Hermite interpolation operator defined by (14.33), and let $\bar{m} \le m \le d$ with $m = d\,(\mathrm{mod}\ 2)$. Then for all $f \in W_\infty^{m+1}(S)$,*

$$|f - \mathcal{I}f|_{k,\Omega} \le K\,|\triangle|^{m+1-k}\,|f|_{m+1,\Omega}, \tag{14.34}$$

for all $0 \le k \le m$. If Ω covers S, the constant K depends only on d, ℓ, and the smallest angle in the triangles of $\triangle$. Otherwise, K may also depend on the Lipschitz constant of $\partial\Omega$.

Proof: Let $\bar{m} \le m \le d$ with $m = d\,(\mathrm{mod}\ 2)$. Fix $T \in \triangle$, and let $\Omega_T := \mathrm{star}^\ell(T)$. It suffices to show that

$$|f - \mathcal{I}f|_{k,T} \le K\,|\triangle|^{m+1-k}\,|f|_{m+1,\Omega_T}, \qquad 0 \le k \le m,$$

for all $0 \le k \le m$, since then (14.34) follows by taking the maximum over all triangles in $\triangle$. By Theorem 14.20, there exists a spherical polynomial p of degree m with

$$|f - p|_{j,\Omega_T} \le K_1\,|\Omega_T|^{m+1-j}\,|f|_{m+1,\Omega_T}, \tag{14.35}$$

for all $0 \le j \le m$. Now fix $0 \le k \le m$. Then by the linearity of $\mathcal{I}$ and the fact that it reproduces spherical polynomials of degree d,

$$|f - \mathcal{I}f|_{k,T} \le |f - p|_{k,T} + |\mathcal{I}(f - p)|_{k,T}.$$

In view of (14.35), it suffices to estimate the second term. Combining (13.6) with (14.32), we see that

$$\|\mathcal{I}(f - p)\|_T \le K_2 \sum_{\nu=0}^{\bar{m}} |\Omega_T|^\nu |f - p|_{\nu,\Omega_T}.$$

Since $\mathcal{I}(f - p)$ is a spherical polynomial, we can use the Markov inequality (14.23), and it follows that

$$|\mathcal{I}(f - p)|_{k,T} \le \frac{K_3}{\rho_T^k}\,\|\mathcal{I}(f - p)\|_T \le \frac{K_4}{\rho_T^k} \sum_{\nu=0}^{\bar{m}} |\Omega_T|^\nu |f - p|_{\nu,\Omega_T}, \tag{14.36}$$

where ρ_T is the inradius of T. By (14.21), $|\Omega_T|/\rho_T \le K_5$ for some constant K_5 depending only on ℓ and the smallest angle in Ω_T. Now in view of (14.20), $|\Omega_T| \le 2\ell|\triangle|$, and inserting (14.35) into (14.36), we immediately get (14.34). $\square$

14.10. Remarks

Remark 14.1. For a different proof of the Markov inequality (14.23) with an explicit constant, see [BarL05].

Remark 14.2. The projection techniques developed in this chapter can be used to establish several interesting properties of the spherical Bernstein basis polynomials introduced in Section 13.1.3. The following result concerns interpolation. Given $T := \langle v_1, v_2, v_3 \rangle$, let D be the spherical cap D with center v_c whose boundary passes through the vertices v_1, v_2, v_3. We call v_c the spherical circumcenter of T and D the spherical circumcap of T.

Theorem 14.29. *Suppose T is a spherical triangle whose spherical circumcap has radius at most 1/2. Let $\{B^d_{ijk}\}_{i+j+k=d}$ be the associated spherical Bernstein basis polynomials of degree d, and let $\mathcal{D}_{d,T} := \{v_{ijk}\}_{i+j+k=d}$ be the associated domain points defined in (13.8). Suppose Γ is the associated index set in lexicographical order. Then the matrix*

$$M := \left[B^d_\alpha(v_\beta)\right]_{\alpha,\beta\in\Gamma}$$

is nonsingular.

Proof: Let $\bar{T}$ be the image of under the map $\mathcal{R}_D$ associated with the spherical cap whose center is at the circumcenter v_c of T and whose radius is 1/2. Then the v_i are all equidistant from v_c, and thus

$$\|\bar{v}_1\| = \|\bar{v}_2\| = \|\bar{v}_3\|.$$

It follows that

$$\begin{aligned} v_{ijk} &= \frac{iv_1 + jv_2 + kv_3}{\|iv_1 + jv_2 + kv_3\|} = \frac{d\|\bar{v}_1\|}{\|iv_1 + jv_2 + kv_3\|}\left(\frac{i}{d}\bar{v}_1 + \frac{j}{d}\bar{v}_2 + \frac{k}{d}\bar{v}_3\right) \\ &= \frac{d\|\bar{v}_1\|}{\|iv_1 + jv_2 + kv_3\|}\bar{\xi}_{ijk}, \end{aligned}$$

where $\bar{\xi}_{ijk}$ are the domain points on $\bar{T}$. Let $\bar{B}^d_{ijk}$ be the Bernstein basis polynomials associated with the planar triangle $\bar{T}$. Then for all $\nu+\mu+\kappa = d$ and all $i + j + k = d$,

$$B^d_{\nu\mu\kappa}(\xi_{ijk}) = \left(\frac{d\|\bar{v}_1\|}{\|\nu v_1 + \mu v_2 + \kappa v_3\|}\right)^d \bar{B}^d_{\nu\mu\kappa}(\bar{\xi}_{ijk}).$$

This implies

$$[B_\alpha(v_\beta)]_{\alpha,\beta\in\Gamma} = \prod_{\nu+\mu+\kappa=d}\left(\frac{d\|\bar{v}_1\|}{\|\nu v_1 + \mu v_2 + \kappa v_3\|}\right)^d \left[\bar{B}^d_\alpha(\xi_\beta)\right]_{\alpha,\beta\in\Gamma}.$$

Theorem 1.11 ensures that the matrix on the right is nonsingular, and the proof is complete. □

Remark 14.3. Using the theorem in the previous remark, we can now give an explicit construction of dual linear functionals for the spherical Bernstein basis polynomials $\{B_{ijk}\}_{i+j+k=d}$ associated with a spherical triangle T. Suppose that T is contained in a spherical cap of radius less than $1/2$. Let Γ and M be as in the proof of Theorem 14.29, and let $n := \binom{d+2}{2}$. For each $\alpha \in \Gamma$, let $w^\alpha := \{w^\alpha_\beta\}_{\beta \in \Gamma}$ be the solution of the linear system $Mw^\alpha = g^\alpha$, where g^α is a vector with all zero entries except for the entry corresponding to α, which we set to 1. Theorem 14.29 ensures that M is nonsingular. For all $\alpha \in \Gamma$ and any function $f \in C(T)$, let $\gamma_\alpha := \sum_{\beta \in \Gamma} w^\alpha_\beta f(v_\beta)$. Then $\gamma_\alpha B_\beta = \delta_{\alpha,\beta}$, for all $\alpha, \beta \in \Gamma$. Thus, given $p := \sum_{\beta \in \Gamma} B_\beta$, then for each $\alpha \in \Gamma$, $\gamma_\alpha p$ picks off the B-coefficient c_α.

14.11. Historical Notes

Much of the classical theory of approximation on the sphere deals with the use of spherical harmonics as approximants, see [Mue66, FreeGS98]. For an extensive list of results (including both direct and inverse theorems), see the survey [FasS98]. The results concentrate on the global case, whereas for our study of spherical splines we need local results. Local approximation on manifolds using polynomials was investigated in [LevR00]. But these results are not exactly what we need either since they are not formulated in terms of an appropriate seminorm associated with spherical Sobolev spaces.

The key to the development of the results in this chapter was the use in [NeaS04] of projection methods to define Sobolev spaces on the sphere with associated seminorms. The results in this chapter are based largely on [NeaS04], although we have changed some of the notation, and there are a number of new and different results here. In some cases we have given different proofs for some of the results drawn from that paper. In particular, Neamtu and Schumaker construct a different quasi-interpolation operator (based on the Hahn–Banach theorem) in order to derive the approximation power of spherical spline spaces.

Trivariate Polynomials

In this chapter we discuss basic properties of trivariate polynomials. Our treatment parallels the developments in Chapters 1 and 2 for bivariate polynomials, with a special emphasis on the Bernstein–Bézier representation of polynomials relative to a tetrahedron.

15.1. The space $\mathcal{P}_d$

Throughout the remainder of the book we write $\mathcal{P}_d$ for the space of trivariate polynomials of degree d, i.e., the finite dimensional linear space of all functions of the form

$$p(x,y,z) := \sum_{0 \le i+j+k \le d} a_{ijk} x^i y^j z^k, \tag{15.1}$$

where a_{ijk} are real numbers. Although we are using the same notation as used in previous chapters for the space of bivariate polynomials, in the rest of the book we will not use the symbol $\mathcal{P}_d$ for bivariate polynomials.

Lemma 15.1. *The space $\mathcal{P}_d$ has dimension $\binom{d+3}{3}$. Moreover, the monomials $\{x^i y^j z^k\}_{0 \le i+j+k \le d}$ form a basis.*

Proof: The monomials $\{x^i y^j z^k\}_{0 \le i+j+k \le d}$ span $\mathcal{P}_d$ by definition. To show that they are linearly independent, suppose the sum in (15.1) is identically zero. Then $D_x^i D_y^j D_z^k p(0,0,0) = a_{ijk} = 0$ for all $0 \le i+j+k \le d$. Arranging the monomials in lexicographical order, it is easy to see that the cardinality of this set is $\binom{d+3}{3}$, which is therefore the dimension of $\mathcal{P}_d$. $\square$

Most of the classical calculus of trivariate polynomials is based on the monomial basis. Since we are going to introduce an alternative basis in the following section which is far more useful for our purposes, we do not go any further into properties of polynomials written in monomial form. Instead, we now introduce some notation which will be useful throughout the remainder of the book, and state an important result connecting the size of the derivative of a polynomial to the size of the polynomial itself.

In dealing with derivatives of trivariate functions, we write D_x, D_y, D_z for the usual partial derivatives with respect to x, y, z, respectively. We shall also make use of the multi-index notation $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$ where $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ is a vector of nonnegative integers. The order of the derivative D^α is then given by $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$.

To measure the size of functions defined on a domain Ω, we use the standard L_q-norms defined by

$$\|f\|_{L_q(\Omega)} := \begin{cases} \Big[\int_\Omega |f(x,y,z)|^q dx\,dy\,dz \Big]^{1/q}, & \text{if } 1 \le q < \infty, \\ \text{ess sup}_{(x,y,z)\in\Omega} |f(x,y,z)|, & \text{if } q = \infty. \end{cases}$$

For convenience, we usually write $\|f\|_{q,\Omega}$ instead of the more cumbersome $\|f\|_{L_q(\Omega)}$. For the case $q = \infty$, we usually write $\|f\|_\Omega$ in place of $\|f\|_{\infty,\Omega}$. The following result compares the size of the various norms when f is a polynomial and Ω is a nondegenerate tetrahedron.

Lemma 15.2. *There exists a constant K depending only on d such that for any $p \in \mathcal{P}_d$ and any $1 \le q \le \infty$,*

$$V_T{}^{-1/q}\|p\|_{q,T} \le \|p\|_{\infty,T} \le K\, V_T{}^{-1/q}\, \|p\|_{q,T}, \tag{15.2}$$

for every tetrahedron T with volume V_T.

Proof: The first inequality is elementary. The second inequality follows by mapping T to the standard tetrahedron $\widetilde{T}$ with vertices at $v_1 := (0,0,0)$, $v_2 := (1,0,0)$, $v_3 := (0,1,0)$, and $v_4 := (0,0,1)$, and then using the fact that all norms on a finite dimensional space are equivalent. $\square$

In stating results involving the size of derivatives, we also make use of the standard seminorms defined for arbitrary domains Ω and sufficiently smooth functions f by

$$|f|_{k,q,\Omega} := \begin{cases} \Big[\sum_{|\alpha|=k} \|D^\alpha f\|_{q,\Omega}^q \Big]^{1/q}, & \text{if } 1 \le q < \infty, \\ \max_{|\alpha|=k} \|D^\alpha f\|_\Omega, & \text{if } q = \infty. \end{cases}$$

We usually write $|f|_{k,\Omega}$ instead of $|f|_{k,\infty,\Omega}$.

Given a tetrahedron T, let ρ_T be the radius of the largest ball which can be inscribed in T. The following result is the Markov inequality for trivariate polynomials. We give a proof in Section 15.12 below.

Theorem 15.3. *There exists a constant K depending only on d such that for every polynomial $p \in \mathcal{P}_d$,*

$$|p|_{k,q,T} \le \frac{K}{\rho_T^k}\, \|p\|_{q,T}, \tag{15.3}$$

for all $0 \le k \le d$ and all $1 \le q \le \infty$.

15.2. Barycentric Coordinates

Our aim in this section is to describe an extremely useful basis for $\mathcal{P}_d$ which is based on the Bernstein-basis polynomials relative to a tetrahedron. First we need to introduce barycentric coordinates with respect to a tetrahedron.

Definition 15.4. *We say that a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ is* nondegenerate *provided that it has nonzero volume. We say that the vertices of T are in* canonical order *provided that if we rotate and translate T so that the triangular face $\langle v_1, v_2, v_3\rangle$ lies in the x–y-plane with v_1, v_2, v_3 in counterclockwise order, then $z_4 > 0$.*

Given a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ with $\{v_i := (x_i, y_i, z_i)\}_{i=1}^4$, let

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}.$$

It is an elementary fact that

$$\det(M) = 6\, V_T, \tag{15.4}$$

where V_T is the volume of T. Throughout the remainder of the book, whenever we deal with tetrahedra we suppose they have nonzero volume, It is easy to see that the volume of a nondegenerate tetrahedron will be positive whenever its vertices are in canonical order and do not all lie in one plane.

Lemma 15.5. *Let $T := \langle v_1, v_2, v_3, v_4\rangle$ be a nondegenerate tetrahedron. Then every point $v := (x, y, z) \in \mathbb{R}^3$ can be written uniquely in the form*

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4, \tag{15.5}$$

with

$$b_1 + b_2 + b_3 + b_4 = 1. \tag{15.6}$$

The numbers b_1, b_2, b_3, b_4 are called the barycentric coordinates of v relative to the tetrahedron T.

Proof: Equations (15.5) and (15.6) are equivalent to the nonsingular system

$$M \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}. \quad \square \tag{15.7}$$

By Cramer's rule

$$b_1 = \frac{1}{\det(M)} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x_2 & x_3 & x_4 \\ y & y_2 & y_3 & y_4 \\ z & z_2 & z_3 & z_4 \end{vmatrix}, \tag{15.8}$$

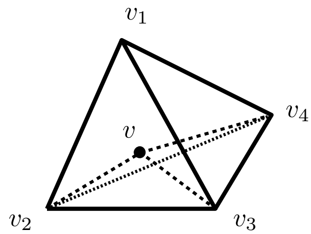


Fig. 15.1. Barycentric coordinates as a ratio of volumes.

with similar expressions for b_2, b_3, and b_4. Barycentric coordinates relative to a tetrahedron have very similar properties to those of the barycentric coordinates associated with triangles in the bivariate case. For example, it is clear from (15.8) that b_1 can be interpreted as the ratio of the volume of the tetrahedron $\langle v, v_2, v_3, v_4 \rangle$ to the volume of T, with similar interpretations for b_2, b_3, b_4, see Figure 15.1.

Lemma 15.6. *For each $i = 1, 2, 3, 4$, the function b_i is a linear polynomial in x, y, z which assumes the value 1 at the vertex v_i and vanishes at all points on the face of T opposite to v_i. Moreover, $0 \le b_i \le 1$ whenever (x, y, z) lies in T.*

15.3. Bernstein Basis Polynomials

Following the bivariate case, see Section 2.2, we make the following definition.

Definition 15.7. *Given a tetrahedron T, let b_1, b_2, b_3, b_4 be the associated barycentric coordinate functions. Then we define the* trivariate Bernstein basis polynomials of degree d relative to T *as*

$$B^d_{ijkl} := \frac{d\,!}{i\,!\,j\,!\,k\,!\,l!}\, b_1^i b_2^j b_3^k b_4^l, \qquad i + j + k + l = d.$$

As in the bivariate case, we define B^d_{ijkl} to be identically zero whenever any of its subscripts is negative. Since each b_1, b_2, b_3, b_4 is a linear polynomial, it is clear that each of the B^d_{ijkl} is a polynomial of degree d. These basis polynomials have many other properties which are similar to those of the bivariate Bernstein basis polynomials.

Theorem 15.8. *The set $\mathcal{B}^d := \{B^d_{ijkl}\}_{i+j+k+l=d}$ of Bernstein basis polynomials is a basis for the space of trivariate polynomials $\mathcal{P}_d$. Moreover,*

$$\sum_{i+j+k+l=d} B^d_{ijkl}(v) = 1, \qquad \text{all } v \in \mathbb{R}^3, \tag{15.9}$$

and

$$0 \le B^d_{ijkl}(v) \le 1, \qquad \text{all } v \text{ in the tetrahedron } T. \tag{15.10}$$

Proof: The following is an analog of the trinomial expansion (2.8):

$$1 = (b_1 + b_2 + b_3 + b_4)^d = \sum_{i+j+k+l=d} \frac{d\,!}{i\,!\,j\,!\,k\,!\,l\,!}\, b_1^i b_2^j b_3^k b_4^l. \tag{15.11}$$

It immediately implies the partition of unity property (15.9) which shows that 1 is in the span of $\mathcal{B}^d$. Now multiplying $x = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$ by $1 = \sum_{i+j+k=d-1} B_{ijk}^{d-1}$ and collecting terms, we see that x is also in the span of $\mathcal{B}^d$. Using the analogous expansions of y and z, we get the following analog of (2.12):

$$\begin{aligned} x &= \sum_{i+j+k+l=d} \frac{(ix_1 + jx_2 + kx_3 + lx_4)}{d}\, B_{ijkl}^d(x,y,z), \\ y &= \sum_{i+j+k+l=d} \frac{(iy_1 + jy_2 + ky_3 + ly_4)}{d}\, B_{ijkl}^d(x,y,z), \\ z &= \sum_{i+j+k+l=d} \frac{(iz_1 + jz_2 + kz_3 + lz_4)}{d}\, B_{ijkl}^d(x,y,z). \end{aligned} \tag{15.12}$$

An inductive proof (see the proof of Theorem 2.4 in the bivariate case) shows that all of the monomials $\{x^\nu y^\mu z^\kappa\}_{0\le \nu+\mu+\kappa\le d}$ are in the span of $\mathcal{B}^d$. Since the number of basis functions in $\mathcal{B}^d$ is equal to the dimension $\binom{d+3}{2}$ of $\mathcal{P}_d$, it follows that $\mathcal{B}^d$ is a basis for $\mathcal{P}_d$. The fact that the B_{ijkl}^d are nonnegative for $v \in T$ follows from the fact that the b_1, b_2, b_3, b_4 have this property. The upper bound in (15.10) then follows from (15.9). $\square$

15.4. The B-form of a Trivariate Polynomial

Fix a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$. Then by Theorem 15.8 below, any trivariate polynomial p of degree d can be written uniquely in the form

$$p := \sum_{i+j+k+l=d} c_{ijkl} B_{ijkl}^d, \tag{15.13}$$

where the B_{ijkl}^d are the Bernstein basis polynomials relative to T. We refer to (15.13) as the B-form of p. We call the c_{ijkl} the B-coefficients of p, and define the associated set of domain points to be

$$\mathcal{D}_{d,T} := \left\{ \xi_{ijkl}^T := \frac{iv_1 + jv_2 + kv_3 + lv_4}{d} \right\}_{i+j+k+l=d}. \tag{15.14}$$

Here again we have committed a slight abuse of notation by writing $\mathcal{D}_{d,T}$, which is the same symbol as used in the bivariate case with triangles T. But the meaning will be clear from the context.

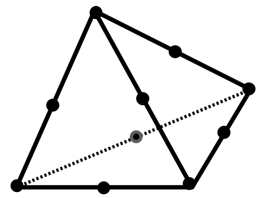


Fig. 15.2. Domain points for $d = 2$.

Figure 15.2 shows $\mathcal{D}_{2,T}$ for a typical tetrahedron. To help identify where domain points are located in a tetrahedron, we say that

ξ^T_{ijkl} is at a distance $d - i$ from the vertex v_1,

ξ^T_{ijkl} is at a distance i from the face $\langle v_2, v_3, v_4 \rangle$ opposite v_1,

ξ^T_{ijkl} is at a distance $i + j$ from the edge $\langle v_3, v_4 \rangle$,

with similar definitions for the other vertices, edges, and faces of T. We write $\operatorname{dist}(\xi^T_{ijkl}, v_1) = d - i$, with similar notation for the other cases. We identify the B-coefficient c_{ijkl} of a polynomial p with the domain point ξ^T_{ijkl} for each $i + j + k + l = d$. We can also index the B-coefficients of p directly in terms of $\mathcal{D}_{d,T}$ as $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}}$ so that if $\xi = \xi^T_{ijkl}$, then $c_\xi = c_{ijkl}$. We also make use of the following subsets of domain points, with similar definitions for other vertices and edges:

The shell of radius m around the vertex v_1: $R^T_m(v_1) := \{\xi^T_{ijkl} : i = d - m\}$.

The ball of radius m around the vertex v_1: $D^T_m(v_1) := \{\xi^T_{ijkl} : i \geq d - m\}$.

The tube of radius ρ around an edge e: $t_\rho(e) := \{\xi : d(\xi, e) \leq \rho\}$.

We conclude this section by observing that the restriction of a trivariate polynomial p of degree d to any plane in $\mathbb{R}^3$ is a bivariate polynomial. Thus, in particular, the restriction of p to a face of a tetrahedron T is a bivariate polynomial. Assuming p is written in the form (15.13), and using the fact that b_1 vanishes on the face $F_1 := \langle v_2, v_3, v_4 \rangle$ of T, it is clear that

$$p|_{F_1} = \sum_{j+k+l=d} c_{0jkl} B^d_{0jkl} = \sum_{j+k+l=d} c^{F_1}_{jkl} B^{F_1,d}_{jkl},$$

where $c^{F_1}_{jkl} := c_{0jkl}$, and $B^{F_1,d}_{jkl}$ are the Bernstein basis polynomials of degree d relative to the triangle F_1. Similar formulae hold for the restrictions of p to the other faces of T.

15.5. Stability of the B-form

In this section we show that the B-form for trivariate polynomials is stable, i.e., the size of a polynomial and the size of its B-coefficients are closely connected. To state our result, we need to agree on an ordering for the B-coefficients and their associated domain points. Suppose T is a tetrahedron whose vertices are numbered in canonical order as in Definition 15.4. To assign an ordering, let $\mathcal{C}_i$ be the set of coefficients whose first index is i. We consider these sets in the order $\mathcal{C}_d, \ldots, \mathcal{C}_0$, and order the coefficients in each $\mathcal{C}_i$ according to the lexicographical ordering defined in Section 2.3 associated with the triangle $\langle v_2, v_3, v_4 \rangle$. For example, for $d = 2$, this gives the lexicographical order

$$c_{2000},\ \underbrace{c_{1100}, c_{1010}, c_{1001}},\ \underbrace{c_{0200}, c_{0110}, c_{0101}, c_{0020}, c_{0011}, c_{0002}}.$$

Given a trivariate polynomial $p \in \mathcal{P}_d$, we write c for the vector of its B-coefficients in the above lexicographical order. We now have the following result on the stability of the B-form, where we measure the size of c by

$$\|c\|_q := \begin{cases} \Big(\displaystyle\sum_{i+j+k+l=d} |c_{ijkl}|^q \Big)^{1/q}, & 1 \le q < \infty, \\ \displaystyle\max_{i+j+k+l=d} |c_{ijkl}|, & q = \infty. \end{cases}$$

Theorem 15.9. *Suppose T is a tetrahedron whose volume is V_T. Then for any polynomial p written in the B-form (15.13),*

$$\frac{V_T^{1/q}}{K} \|c\|_q \le \|p\|_{q,T} \le V_T^{1/q} \|c\|_q. \tag{15.15}$$

The constant K depends only on d.

Proof: We deal first with the case $q = \infty$, and follow the proofs of Theorems 2.6 and 2.7. The inequality on the right follows immediately from the fact that the B^d_{ijkl} form a partition of unity. To prove the inequality on the left, let $\{g_1, \ldots, g_n\}$ be the Bernstein basis polynomials in lexicographical order with $n := \binom{d+3}{3}$, and let $\{t_1, \ldots, t_n\}$ be the domain points $\mathcal{D}_{d,T}$ in the same order. Then organizing the B-coefficients in the same order, we have $Mc = r$, where $r := (p(t_1), \ldots, p(t_n))^T$ and $M = [g_j(t_i)]_{i,j=1}^n$. M is nonsingular by Theorem 15.38 below, and thus

$$\|c\|_\infty \le \|M^{-1}\|_\infty \, \|r\|_\infty \le K \, \|p\|_\infty,$$

where $K := \|M^{-1}\|_\infty$. This proves the inequality on the left in (15.15) for $q = \infty$ since M depends only on d. To prove (15.15) for $q < \infty$, we use Lemma 15.2. $\square$

15.6. The de Casteljau Algorithm

It is clear from Theorem 15.8 that to *store* a polynomial p written in B-form, we need only store its coefficient vector c. We now present an efficient and stable algorithm for *evaluating* p at a given point $v := (x, y, z)$. The algorithm is based on the simple recurrence relation

$$B^d_{ijkl} = b_1 B^{d-1}_{i-1,j,k,l} + b_2 B^{d-1}_{i,j-1,k,l} + b_3 B^{d-1}_{i,j,k-1,l} + b_4 B^{d-1}_{i,j,k,l-1},$$

which is an immediate consequence of the definition of B^d_{ijkl}. Here we are using the convention that expressions with negative subscripts are zero. The proof of the following result is based on this recurrence relation, and is very similar to the proof of Theorem 2.8.

Theorem 15.10. *Suppose p is a trivariate polynomial written in the B-form (15.13). Let $c^{(0)}_{ijkl} := c_{ijkl}$, $i+j+k+l = d$, be its coefficients. Suppose v has barycentric coordinates $b := (b_1, b_2, b_3, b_4)$. Then*

$$p(v) = \sum_{i+j+k+l=d-m} c^{(m)}_{ijkl} B^{d-m}_{ijkl}(v), \tag{15.16}$$

where for $m = 1, \ldots, d$, $c^{(m)}_{ijkl} := c^{(m)}_{ijkl}(b)$ are computed by the recursion

$$c^{(m)}_{ijkl} := b_1 c^{(m-1)}_{i+1,j,k,l} + b_2 c^{(m-1)}_{i,j+1,k,l} + b_3 c^{(m-1)}_{i,j,k+1,l} + b_4 c^{(m-1)}_{i,j,k,l+1},$$

for $i + j + k + l = d - m$.

It is easy to see by induction that

$$c^{(m)}_{ijkl} = \sum_{\alpha+\beta+\gamma+\delta=m} c_{i+\alpha,j+\beta,k+\gamma,l+\delta} B^m_{\alpha,\beta,\gamma,\delta}(v), \quad i + j + k + l = d - m.$$

Theorem 15.10 immediately leads to an algorithm for evaluating a polynomial p in B-form.

Algorithm 15.11. *(de Casteljau)*

For $m = 1, \ldots, d$

For all $i + j + k + l = d - m$

$$c^{(m)}_{ijkl} := b_1 c^{(m-1)}_{i+1,j,k,l} + b_2 c^{(m-1)}_{i,j+1,k,l} + b_3 c^{(m-1)}_{i,j,k+1,l} + b_4 c^{(m-1)}_{i,j,k,l+1}$$

Discussion: By Theorem 15.10, the value of $p(v)$ is given by $c^{(d)}_{0000}$. We illustrate this algorithm for the case $d = 2$ in Figure 15.3. □

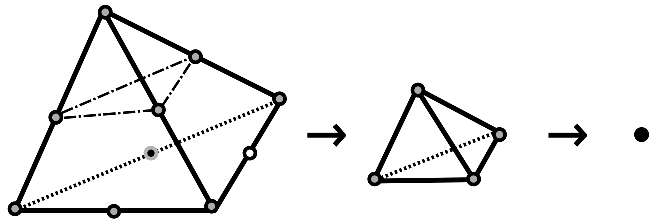


Fig. 15.3. Steps of the de Casteljau Algorithm.

15.7. Directional Derivatives

Given a nontrivial vector $u := (u_x, u_y, u_z) \in \mathbb{R}^3$, we define the associated directional derivative of a trivariate function p by

$$D_u p(x,y,z) := \frac{d}{dt} p(x + tu_x, y + tu_y, z + tu_z)\Big|_{t=0}. \qquad (15.17)$$

It is well known from calculus that

$$D_u p(x,y,z) = u_x D_x p(x,y,z) + u_y D_y p(x,y,z) + u_z D_z p(x,y,z). \qquad (15.18)$$

This shows that the directional derivative of a polynomial of degree d is a polynomial of degree $d-1$.

Suppose now that T is a tetrahedron. Then as in the bivariate case, u is described by the directional coordinates $a := (a_1, a_2, a_3, a_4)$ relative to T, where $a_i := \beta_i - \alpha_i$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are the barycentric coordinates of u and 0 relative to T, respectively. The proof of the following result is almost identical to the proof of Lemma 2.11.

Lemma 15.12. *Suppose u is a vector whose associated directional coordinates are (a_1, a_2, a_3, a_4). Then for any $i + j + k + l = d$,*

$$\begin{aligned} D_u B^d_{ijkl}(v) = d\big[&a_1 B^{d-1}_{i-1,j,k,l}(v) + a_2 B^{d-1}_{i,j-1,k,l}(v) \\ &+ a_3 B^{d-1}_{i,j,k-1,l}(v) + a_4 B^{d-1}_{i,j,k,l-1}(v)\big]. \end{aligned} \qquad (15.19)$$

We now give a formula for the directional derivative of an arbitrary trivariate polynomial p in B-form. Lemma 15.12 immediately implies the following result.

Theorem 15.13. *Let p be a polynomial written in the B-form (15.13) relative to a tetrahedron T, and let u have directional coordinates $a := (a_1, a_2, a_3, a_4)$. Then the directional derivative at v of p in the direction u*

is given by

$$D_u p(v) = d \sum_{i+j+k+l=d-1} c^{(1)}_{ijkl}(a) B^{d-1}_{ijkl}(v), \tag{15.20}$$

where $c^{(1)}_{ijkl}(a)$ *are the numbers arising in the first step of the de Casteljau algorithm based on the 4-tuple* a.

Repeatedly applying Theorem 15.13, we get the following formula for an arbitrary higher-order directional derivative.

Theorem 15.14. *Suppose we are given a set* $u_1, \ldots, u_m$ *of* m *directions with associated directional coordinates* $a^{(i)} := (a^{(i)}_1, a^{(i)}_2, a^{(i)}_3, a^{(i)}_4)$, $i = 1, \ldots, m$. *Then*

$$D_{u_m} \cdots D_{u_1} p(v) = \frac{d!}{(d-m)!} \sum_{i+j+k+l=d-m} c^{(m)}_{ijkl}(a^{(1)}, \ldots, a^{(m)}) B^{d-m}_{ijkl}(v), \tag{15.21}$$

where $c^{(m)}_{ijkl}(a^{(1)}, \ldots, a^{(m)})$ *are the numbers obtained after carrying out* m *steps of the de Casteljau algorithm, using* $a^{(1)}, \ldots, a^{(m)}$ *in order.*

Formula (15.21) reaffirms that the m-th mixed directional derivative of a polynomial p is a polynomial of degree $d-m$. To evaluate $D_{u_m} \cdots D_{u_1} p(v)$ at a point v with barycentric coordinates $b := (b_1, b_2, b_3, b_4)$, we simply apply the de Casteljau algorithm to the coefficient vector of p using the 4-tuples $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ in order, and then follow these m steps with an additional $d - m$ steps of the algorithm using the 4-tuple b of barycentric coordinates of v.

Since polynomials are infinitely differentiable functions, the order in which the derivatives are taken in (15.21) does not matter. This also follows directly from the fact that if we apply the de Casteljau algorithm with two different 4-tuples, we get the same result no matter which one we use first.

15.8. B-coefficients and Derivatives at a Vertex

Suppose that p is a polynomial of degree d and that $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}}$ are its B-coefficients with respect to a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$. In this section we give explicit formulae for computing the coefficients of p corresponding to domain points in a ball $D^T_m(v)$ around a vertex v of T from derivatives up to order m at v, and vice versa. These formulae will be particularly useful in Chapter 18 where we construct certain trivariate macro-element spaces.

We begin with a technical lemma. A set L of triples of nonnegative integers is called a lower set provided that for any $(l, m, n) \in L$, all triples of the form (i, j, k) with $0 \le i \le l$, $0 \le j \le m$, and $0 \le k \le n$ also belong to L. The following lemma is easily proved by induction.

Lemma 15.15. *Suppose that L is a lower set, and that*

$$f(l,m,n) := \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{l}{i}\binom{m}{j}\binom{n}{k}(-1)^{i+j+k}g(i,j,k), \quad (l,m,n)\in L.$$

Then

$$g(i,j,k) = \sum_{l=0}^{i}\sum_{m=0}^{j}\sum_{n=0}^{k}\binom{i}{l}\binom{j}{m}\binom{k}{n}(-1)^{l+m+n}f(l,m,n), \quad (i,j,k)\in L.$$

The following theorem is stated for the vertex v_1 of T. Analogous formulae hold for the other vertices of T.

Theorem 15.16. *For all $0 \le l+m+n \le d$,*

$$D^l_{v_2-v_1}D^m_{v_3-v_1}D^n_{v_4-v_1}p(v_1) = \frac{(-1)^{l+m+n}d\,!}{(d-l-m-n)\,!}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{i+j+k} \times \binom{l}{i}\binom{m}{j}\binom{n}{k}c_{d-i-j-k,i,j,k}. \tag{15.22}$$

Conversely,

$$c_{d-i-j-k,i,j,k} = \sum_{l=0}^{i}\sum_{m=0}^{j}\sum_{n=0}^{k}\binom{i}{l}\binom{j}{m}\binom{k}{n}\frac{(d-l-m-n)\,!}{d\,!} \times D^l_{v_2-v_1}D^m_{v_3-v_1}D^n_{v_4-v_1}p(v_1), \tag{15.23}$$

for all $0 \le i+j+k \le d$.

Proof: To establish (15.22), we evaluate (15.21) at v_1. The converse follows from Lemma 15.15. $\square$

It follows easily from (15.22) that for all $0 \le l+m+n \le d$,

$$\left|D^l_{v_2-v_1}D^m_{v_3-v_1}D^n_{v_4-v_1}p(v_1)\right| \le \frac{2^{l+m+n}d\,!}{(d-l-m-n)\,!}\max_{\xi\in D_{l+m+n}(v_1)}|c_\xi|. \tag{15.24}$$

Similarly, (15.23) implies that for all $0 \le \rho \le d$ and all $\xi \in D_\rho(v_1)$,

$$|c_\xi| \le 2^\rho \max_{l+m+n\le\rho}\left|D^l_{v_2-v_1}D^m_{v_3-v_1}D^n_{v_4-v_1}p(v_1)\right|. \tag{15.25}$$

Theorem 15.16 deals with directional derivatives. For many applications we would like an analogous result for the derivatives relative to the standard Cartesian coordinate system. Let ρ_T be the radius of the largest ball which can be inscribed in the tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$.

Theorem 15.17. *For each $0 \le i \le 4$, $D^\alpha p(v_i)$ can be computed from the set of coefficients of p corresponding to domain points lying in the ball $D^T_{|\alpha|}(v_i)$, and*

$$\bigl|D^\alpha p(v_i)\bigr| \le \frac{d\,!}{(d-|\alpha|)\,!}\rho_T^{-|\alpha|} \max_{\xi \in D^T_{|\alpha|}(v_i)} |c_\xi|. \tag{15.26}$$

Conversely, for all $0 \le m \le d$, the coefficients corresponding to $\xi \in D^T_m(v_i)$ can be computed from the derivatives of p up to order m at v_i, and

$$|c_\xi| \le 2^m \sum_{j=0}^m 3^j |T|^j \max_{|\alpha|=j} \bigl|D^\alpha p(v_i)\bigr|. \tag{15.27}$$

Proof: It suffices to consider the case of v_1. Let $v_i = (x_i, y_i, z_i)$ for $i = 1, \ldots, 4$, and let V_T be the volume of T. Then the unit direction vector pointing in the direction of the x-axis has directional coordinates (a_1, a_2, a_3, a_4) where

$$a_1 := \frac{-1}{6V_T}\begin{vmatrix} y_3 - y_2 & y_4 - y_2 \\ z_3 - z_2 & z_4 - z_2 \end{vmatrix}, \qquad a_2 := \frac{1}{6V_T}\begin{vmatrix} y_3 - y_1 & y_4 - y_1 \\ z_3 - z_1 & z_4 - z_1 \end{vmatrix},$$

$$a_3 := \frac{-1}{6V_T}\begin{vmatrix} y_2 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_4 - z_1 \end{vmatrix}, \qquad a_4 := \frac{1}{6V_T}\begin{vmatrix} y_2 - y_1 & y_3 - y_1 \\ z_2 - z_1 & z_3 - z_1 \end{vmatrix}.$$

Note that the absolute value of the determinant in the formula for a_1 is twice the area of the triangle $\langle v_2, v_3, v_4 \rangle$ projected onto the (y, z)-plane. The other determinants have similar interpretations as the areas of the triangular faces of T projected onto the (y, z)-plane. Using the fact that V_T is equal to the surface area of T times $\rho_T/3$, it follows that

$$|a_1| + |a_2| + |a_3| + |a_4| \le \frac{1}{\rho_T}. \tag{15.28}$$

The same bound holds for the direction coordinates associated with the unit vectors in the direction of the y- and z–axes. Now by Theorem 15.14,

$$D^\alpha p(v_1) = \frac{d\,!}{(d-|\alpha|)\,!} c^\alpha_{d-|\alpha|,0,0,0},$$

where $c^\alpha_{d-|\alpha|,0,0,0}$ is obtained from the coefficients of p associated with domain points in $D^T_{|\alpha|}(v_1)$ by applying $|\alpha|$ steps of the de Casteljau algorithm using the direction coordinates defining D^α. In each step of the algorithm the coefficients can grow by a factor of at most $1/\rho_T$, and (15.26) follows. The bound (15.27) follows directly from (15.25) using (15.18). $\square$

15.9. B-coefficients and Derivatives on Edges

In this section we explore the connection between B-coefficients of a polynomial p relative to a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ and the values of derivatives of p at points on the edges of T. As a first observation, we note that by the formulae for derivatives in Section 15.7, it is clear that for any direction vector u not parallel to an edge e and any point η on e, the value of $D_u^\alpha p(\eta)$ depends only on the B-coefficients of p lying in the tube $t_m(e)$ around e, where $m = |\alpha|$. We now show a kind of converse, namely, how to compute the B-coefficients of p corresponding to domain points in the tube $t_m(e)$ from values of derivatives of p at points on e.

We focus on the case $e := \langle v_3, v_4\rangle$. Given $0 \le m \le \rho$ and $d > 2\rho$, suppose that we already know the coefficients of p corresponding to all domain points in

$$\Gamma_{m-1}(e) := t_{m-1}(e) \cup D_\rho^T(v_3) \cup D_\rho^T(v_4).$$

Our aim is to compute the coefficients of p corresponding to domain points in $t_m(e) \setminus \Gamma_{m-1}(e)$. These domain points lie on $m+1$ lines parallel to e, where there are exactly $n := d - 2\rho + m - 1$ points on each line. In particular, for each $0 \le i \le m$, the associated line of domain points has the form

$$L_m^i(e) := \bigcup_{k=\rho+1-m}^{d-\rho-1} \{\xi^T_{i,m-i,k,d-m-k}\}.$$

For $i = 0$ these points lie in the face $\langle v_2, v_3, v_4\rangle$ of T, while for $i = m$, they lie in the face $\langle v_1, v_3, v_4\rangle$. Let D_1 and D_2 be the directional derivatives associated with the vectors $v_1 - v_3$ and $v_2 - v_3$.

Theorem 15.18. *Fix $0 \le i \le m \le \rho$ with $d > 2\rho$. Suppose that we know the coefficients of the polynomial p corresponding to all domain points in $\Gamma_{m-1}(e)$ for some edge e. Let $n := d - 2\rho + m - 1$, and let $\eta_i := iv_2 + (n - i + 1)(v_3 - v_2))/(n + 1)$ for $i = 1, \dots, n$. Then the coefficients of p associated with the domain points in $L_m^i(e)$ can be computed from the known coefficients together with the values $\{D_1^i D_2^{m-i} p(\eta_j)\}_{j=1}^n$.*

Proof: By Theorem 15.14,

$$D_1^i D_2^{m-i} p|_e = \frac{d!}{(d-m)!} \sum_{k+l=d-m} a_{00kl} B_{00kl}^{d-m},$$

where a_{00kl} are obtained from the coefficients of p by performing i steps of the de Casteljau algorithm using the directional coordinates $(1, 0, -1, 0)$ corresponding to the derivative D_1, followed by $m - i$ steps of the de Casteljau algorithm using the directional coordinates $(0, 1, -1, 0)$ corresponding to the derivative D_2. In particular, each a_{00kl} has the form

$$a_{0,0,k,l} = c_{i,m-i,k,l} + d_{0,0,k,l}, \tag{15.29}$$

where $d_{0,0,k,l}$ is a linear combination of known coefficients. Now evaluating at the points $\eta_1, \ldots, \eta_n$ leads to an $n \times n$ linear system of the form

$$Ma = r, \tag{15.30}$$

where a is the vector of n unknown coefficients $\{a_{00k\ell}\}_{k+\ell=d-m}$, and where the j-th component of r is $\frac{(d-m)!}{d!} D_1^i D_2^{m-i} p(\eta_j)$ plus a linear combination of the known coefficients. The matrix of this system is $M = [\phi_j(\eta_i)]_{i,j=1}^n$, where

$$\{\phi_1, \ldots, \phi_n\} := \{B^{d-m}_{0,0,\rho+1-m,d-\rho-1}, \ldots, B^{d-m}_{0,0,d-\rho-1,\rho+1-m}\}.$$

Since the matrix M reduces to a matrix which arises in interpolation with univariate Bernstein basis polynomials, it is clearly nonsingular, and the proof is complete. $\square$

It is clear from the proof of Theorem 15.18 that the matrix M entering there does not depend on i, i.e., we have to solve the same linear system for each line of domain points $L_m^i(e)$. Moreover, the inverse of this matrix is bounded by a constant which is independent of the size and shape of the tetrahedron T. We illustrate Theorem 15.18 with two examples.

Example 15.19. Let $d = 9$, $\rho = 4$, and $m = 1$.

Discussion: In this case there is exactly one domain point in each of the sets $L_1^0(e)$ and $L_1^1(e)$. The coefficient corresponding to the domain point on $L_1^0(e)$ is c_{0144}. It is computed from the equation

$$B^8_{0044}(\eta_1)(c_{0144} - c_{0054}) = \frac{1}{9} D_2 p(\eta_1) - \sum_{\substack{j=0 \\ j \neq 4}}^{8} (c_{0,1,8-j,j} - c_{0,0,9-j,j}) B^8_{0,0,8-j,j}(\eta_1).$$

The coefficient corresponding to the domain point on $L_1^1(e)$ is c_{1044}. It is computed from the equation

$$B^8_{0044}(\eta_1)(c_{1044} - c_{0054}) = \frac{1}{9} D_1 p(\eta_1) - \sum_{\substack{j=0 \\ j \neq 4}}^{8} (c_{1,0,8-j,j} - c_{0,0,9-j,j}) B^8_{0,0,8-j,j}(\eta_1).$$

Example 15.20. Let $d = 9$ and $\rho = 4$, and let $m = 2$.

Discussion: In this case there are two domain points in each of the sets $L_2^i(e)$ for $i = 0, 1, 2$. The two in $L_2^0(e)$ lie on the face $\langle v_2, v_3, v_4 \rangle$, and the corresponding coefficients are computed from the following 2×2 system:

$$\begin{bmatrix} B^7_{0034}(\eta_1) & B^7_{0043}(\eta_1) \\ B^7_{0034}(\eta_2) & B^7_{0043}(\eta_2) \end{bmatrix} \begin{bmatrix} c_{0234} \\ c_{0243} \end{bmatrix} = \begin{bmatrix} \frac{1}{72} D_1^2 p(\eta_1) + g_1 \\ \frac{1}{72} D_1^2 p(\eta_2) + g_2 \end{bmatrix},$$

where g_1 and g_2 are combinations of known coefficients. The two coefficients corresponding to $L_2^2(e)$ are computed similarly. The two domain points corresponding to $L_2^1(e)$ are inside T, and the associated coefficients are computed from

$$\begin{bmatrix} B^7_{0034}(\eta_1) & B^7_{0043}(\eta_1) \\ B^7_{0034}(\eta_2) & B^7_{0043}(\eta_2) \end{bmatrix} \begin{bmatrix} c_{1134} \\ c_{1143} \end{bmatrix} = \begin{bmatrix} \frac{1}{72} D_1 D_2 p(\eta_1) + h_1 \\ \frac{1}{72} D_1 D_2 p(\eta_2) + h_2 \end{bmatrix},$$

where h_1 and h_2 are combinations of known coefficients. $\square$

We now show that the computation of the coefficients corresponding to domain points in $t_m(e) \setminus \Gamma_{m-1}(e)$ described in Theorem 15.18 is stable. Let $|T|$ be the diameter of T, and let $|p|_{\nu,T}$ be the usual Sobolev seminorm defined in Section 15.1.

Theorem 15.21. *For all $\xi \in t_m(e) \setminus \Gamma_{m-1}(e)$,*

$$|c_\xi| \le K \left[|T|^m \, |p|_{m,T} + \max_{\eta \in \Gamma_{m-1}(e)} |c_\eta| \right], \tag{15.31}$$

where K is a constant depending only d.

Proof: By the proof of Theorem 15.18, the computed coefficients associated with domain points in the set $L_m^i(e)$ are given by (15.29) with $a = M^{-1} r$, where M and a are as in the proof of Theorem 15.18. This matrix is independent of i, and corresponds to interpolation by univariate Bernstein basis polynomials at equally spaced points $\eta_1, \ldots, \eta_n$. We conclude that $\|M^{-1}\|_\infty$ is bounded by a constant depending only on d, and it follows that $\|a\|_\infty \le \|M^{-1}\|_\infty \|r\|_\infty$, where the right-hand side r is a linear combination of known coefficients and derivatives of the form $D_1^i D_2^{m-i} p(\eta_j)$. Using (15.18) to write these derivatives in terms of D_x, D_y, D_z, we get (15.31). $\square$

When e is an edge shared by two or more tetrahedra, it is often more convenient to replace the derivatives D_1 and D_2 appearing in Theorem 15.18 and 15.21 by derivatives which are independent of the shape of the attached tetrahedra.

Definition 15.22. *Given an edge $e := \langle u, v \rangle$ of a tetrahedron T, let X_e be the plane perpendicular to e at u. We endow X_e with Cartesian coordinate axes whose origin lies at the point u. Then for any multi-index $\beta = (\beta_1, \beta_2)$, we define D_e^β to be the corresponding directional derivative of order $|\beta| := \beta_1 + \beta_2$ in a direction lying in X_e.*

Since the derivatives D_e^β lie in a plane perpendicular to the edge e, we can always compute the derivatives $\{D_1^i D_2^{m-i} p\}_{i=0}^m$ needed in Theorem 15.18 from the derivatives $\{D_e^\beta p\}_{|\beta|=m}$.

Theorem 15.23. *Suppose $\triangle := \{T_i\}_{i=1}^k$ is a set of tetrahedra which share an edge $e := \langle u, v\rangle$, and suppose s is a piecewise polynomial function of degree d with $s \in C^m(\Omega)$, where $\Omega := \bigcup_{i=1}^k T_i$. Then we can set the coefficients of $s_1 := s|_{T_1}$ corresponding to the domain points in $t_m(e) \cap T_1$ to arbitrary values, and all other coefficients of s corresponding to domain points in the tube $t_m(e)$ are uniquely and stably determined.*

Proof: By the above results, setting the coefficients of s corresponding to the domain points in $t_m(e) \cap T_1$ uniquely defines the derivatives $\{D^\beta s(v)\}_{|\beta|\le m}$ for all points v on e, where D^β are the derivatives in a plane perpendicular to e. Now for each T_i, these derivatives uniquely define the B-coefficients of the polynomial $s|_{T_i}$ corresponding to domain points in $t_m(e) \cap T_i$. $\square$

Theorem 15.23 holds even if $\triangle$ consists of a chain of tetrahedra for which T_1 and T_k share a common face, i.e., the edge e is an interior edge of $\triangle$. This configuration is called an orange in [AlfSS92].

15.10. B-coefficients and Derivatives on Faces

In this section we explore the connection between certain B-coefficients of a polynomial p relative to a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$, and the values of its derivatives at points on a face F of T. As a first observation, we note that by the formulae for derivatives in Section 15.7, it is clear that for any direction vector u not parallel to F and any point η on F, the value of $D_u^m p(\eta)$ depends only on the B-coefficients of p corresponding to domain points lying in the set $G_m(F) := \{\xi \in \mathcal{D}_{d,T} : \operatorname{dist}(\xi, F) \le m\}$ of domain points which lie within a distance m of F. We now show a kind of converse, namely, how to compute the B-coefficients of p corresponding to domain points in $G_m(F)$ from values of derivatives at points on F.

We focus on the case $F := \langle v_2, v_3, v_4\rangle$. The other faces can be dealt with in a similar way. We write D_F for the derivative corresponding to the unit vector perpendicular to F and pointing into T. For any n, we write $\xi_{jkl}^{F,n} := (jv_2 + kv_3 + lv_4)/n$, for $j + k + l = n$. Note that when $n = d$, these points coincide with some of the domain points associated with the B-representation of p.

Given $0 \le m \le d$, suppose that we already know the coefficients of p corresponding to all domain points in $G_{m-1}(F)$. In addition, suppose we know the coefficients of p for all domain points in the set

$$\Lambda := \{\xi_{mjkl}^T : (j, k, l) \in J\},$$

where J is a subset of $I_m := \{(j,k,l) : j+k+l = d-m\}$. Our aim is to compute the coefficients corresponding to the set

$$\Lambda' := \{\xi^T_{mjkl} : (j,k,l) \in J'\},$$

where $J' := I_m \setminus J$. Let

$$\Gamma := \{\xi^{F,d-m}_{jkl} : (j,k,l) \in J'\}. \tag{15.32}$$

Theorem 15.24. *Suppose we know the coefficients of p corresponding to all domain points in $G_{m-1}(F) \cup \Lambda$ as described above, and suppose that the set Γ in (15.32) is such that the matrix*

$$M := [B^{F,d-m}_\xi(\eta)]_{\xi,\eta\in\Gamma}$$

is nonsingular. Then the coefficients of p corresponding to domain points in the set Λ' can be computed from the values $\{D^m_F p(\eta)\}_{\eta\in\Gamma}$.

Proof: By Theorem 15.14,

$$D^m_F p|_F = \frac{d!}{(d-m)!} \sum_{j+k+l=d-m} a_{0jkl} B^{d-m}_{0jkl}, \tag{15.33}$$

where a_{0jkl} are obtained from the coefficients of p by performing m steps of the de Casteljau algorithm using the directional coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ associated with the unit vector defining D_F. Note that for each $j+k+l = d-m$, a_{0jkl} is equal to $\alpha_1^m c_{mjkl}$ plus a sum of known coefficients. The fact that $\alpha_1 > 0$, implies that we can solve for c_{mjkl} once we have a_{0jkl}. To get these, we write the equations for $\{D^m_F p(\eta)\}_{\eta\in\Gamma}$. This leads immediately to a system of the form

$$Ma = r$$

for the vector a of coefficients in (15.33) with $(j,k,l) \in J'$, where the components of r are a combination of known coefficients and the values $\{D^m_F p(\eta)\}_{\eta\in\Gamma}$. The result follows since we have assumed that M is nonsingular. $\square$

For a discussion of how to choose sets Γ such that the matrix M in this theorem is nonsingular, see Section 2.9, and in particular Conjecture 2.22. We apply this result in Chapter 18 to help analyze various trivariate macro-element spaces. We have the following stability result.

Theorem 15.25. *The coefficients computed in Theorem 15.24 satisfy*

$$|c_\xi| \le K \left[|T|^m |p|_{m,T} + \max_{\eta\in G_{m-1}(F)\cup\Lambda} |c_\eta|\right], \qquad \textit{all } \xi \in \Lambda'.$$

Here K is a constant depending only on d and the smallest solid and face angles in T (see Definition 16.2).

Proof: The equation $a = M^{-1}r$ leads immediately to bounds on the a_{0jkl} which translate immediately into analogous bounds on the c_{mjkl} in terms of $\|M^{-1}\|_\infty$. This norm can be bounded by the smallest angles in T. □

15.11. B-coefficients and Hermite Interpolation

In this section we show how to compute certain B-coefficients of a trivariate polynomial from Hermite interpolation conditions at a point v inside of T, assuming that all other coefficients of p are already known. This result will be useful later in our study of trivariate macro-elements.

Theorem 15.26. *Let v_c be an arbitrary point in a tetrahedron T, and let $d \geq 4r+4$. Suppose that the B-coefficients of a polynomial p of degree d are known except for those corresponding to the domain points*

$$\Gamma := \{\xi^T_{ijkl} : i,j,k,l > r\}.$$

Then the coefficients $\{c_\xi\}_{\xi \in \Gamma}$ of p are uniquely determined from the values $\{D^\alpha p(v_c)\}_{|\alpha| \leq d-4r-4}$.

Proof: The known coefficients are associated with domain points that lie on the outer faces of T and on the r layers next to those outer faces. We are left with $N := \binom{d-4r-1}{3}$ coefficients which are to be determined from the same number of Hermite interpolation conditions. Enforcing these conditions leads to a $N \times N$ linear system of equations. We need to show that the associated matrix M is nonsingular. It suffices to show that if the coefficients corresponding to $\mathcal{D}_{d,T} \setminus \Gamma$ are all zero and we set $D^\alpha p(v_c) = 0$ for $|\alpha| \leq d-4r-4$, then $p \equiv 0$. Now by Bezout's theorem, we can write $p = \ell_1^{r+1}\,\ell_2^{r+1}\,\ell_3^{r+1}\,\ell_4^{r+1}\,q$, where ℓ_i is a linear polynomial which vanishes on the i-th face of T, and q is a polynomial of degree $d-4r-4$. Since v_c is inside of T, setting $D^\alpha p(v_c) = 0$ for $|\alpha| \leq d-4r-4$ is equivalent to setting $D^\alpha q(v_c) = 0$ for $|\alpha| \leq d-4r-4$. But this implies $q \equiv 0$, and it follows that $p \equiv 0$. We conclude that M is nonsingular and the proof is complete. □

In applications we would typically take v_c to be the barycenter of T. Using the formulae for derivatives of trivariate polynomials written in B-form, it is easy to see that the matrix M of Theorem 15.26 depends only on the barycentric coordinates of v_c, and is independent of the size and shape of the tetrahedron T. Thus, the computation of the unknown coefficients in this theorem is a stable process.

Theorem 15.27. *There exists a constant K depending only on d such that the computation of coefficients in Theorem 15.26 is stable in the sense that for all $\xi \in \mathcal{D}_{d,T} \setminus \Gamma$,*

$$|c_\xi| \le K \Big[\sum_{\nu=0}^{d-4r-4} |T|^\nu |p|_{\nu,T} + \max_{\eta\in\Gamma} |c_\eta| \Big]. \tag{15.34}$$

15.12. The Markov Inequality on Tetrahedra

In this section we prove the Markov inequality (15.3) for trivariate polynomials by establishing the following result which bounds the size of particular derivatives rather than a seminorm.

Theorem 15.28. *Let $T := \langle v_1, v_2, v_3, v_4 \rangle$ be a tetrahedron, and fix $1 \le q \le \infty$. Let ρ_T be the radius of the largest sphere that can be inscribed in T. Then for any trivariate polynomial $p \in \mathcal{P}_d$,*

$$\|D^\alpha p\|_{q,T} \le \frac{K}{\rho_T^{|\alpha|}} \|p\|_{q,T}, \tag{15.35}$$

for all α with $0 \le |\alpha| \le d$. The constant K depends only on d.

Proof: By Theorem 15.14, for any $v \in T$,

$$D^\alpha p(v) = \frac{d\,!}{(d-|\alpha|)\,!} \sum_{i+j+k+l=d-|\alpha|} c^\alpha_{ijkl} B^{d-|\alpha|}_{ijkl}(v),$$

where c^α_{ijkl} are obtained from the coefficients of p by applying $|\alpha|$ steps of the de Casteljau algorithm using the direction coordinates defining D^α. Now as shown in (15.28), the coefficients arising in the de Casteljau algorithm can grow by a factor of at most $1/\rho_T$ in each step, and it follows that $|c^\alpha_{ijkl}| \le \rho_T^{-|\alpha|} \|c\|_\infty$ for all $i+j+k+l = d-|\alpha|$. Since the Bernstein basis polynomials form a partition of unity, using (15.15), we get

$$\|D^\alpha p\|_T \le \frac{d\,!}{(d-|\alpha|)\,!} \frac{1}{\rho_T^{|\alpha|}} \|c\|_\infty \le \frac{K}{\rho_T^{|\alpha|}} \|p\|_T,$$

which is (15.35) for $q = \infty$. We now use Lemma 15.2 to get the result for arbitrary $1 \le q \le \infty$. $\square$

15.13. Integrals and Inner Products

As in the bivariate case, there are simple formulae for integrals and inner products of trivariate polynomials in B-form. The proof of the following result is similar to the proof of Theorem 2.33.

Lemma 15.29. *For any tetrahedron T,*

$$\int_T B^d_{ijkl}(x,y,z)\,dx\,dy\,dz = \frac{V_T}{\binom{d+3}{3}}, \qquad i+j+k+l=d. \tag{15.36}$$

Moreover, for any trivariate polynomial p of degree d with B-coefficients $\{c_{ijkl}\}_{i+j+k+l=d}$ relative to T,

$$\int_T p(x,y,z)\,dx\,dy\,dz = \frac{V_T}{\binom{d+3}{3}} \sum_{i+j+k+l=d} c_{ijkl}. \tag{15.37}$$

Using

$$B^d_{ijkl}\, B^d_{\nu\mu\kappa\delta} = \frac{\binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}\binom{l+\delta}{l}}{\binom{2d}{d}} B^{2d}_{i+\nu,j+\mu,k+\kappa,l+\delta}$$

leads immediately to the following formula for the inner product of two polynomials in B-form.

Lemma 15.30. *Let p and q be trivariate polynomials with B-coefficients $\{c_{ijkl}\}_{i+j+k+l=d}$ and $\{a_{ijkl}\}_{i+j+k+l=d}$, respectively. Then*

$$\begin{aligned}\langle p,q\rangle &:= \int_T p(x,y,z)q(x,y,z)\,dx\,dy\,dz \\ &= \frac{V_T}{\binom{2d}{d}\binom{2d+3}{3}} \sum_{\substack{i+j+k+l=d\\ \nu+\mu+\kappa+\delta=d}} c_{ijkl}\, a_{\nu\mu\kappa\delta} \binom{i+\nu}{i}\binom{j+\mu}{j}\binom{k+\kappa}{k}\binom{l+\delta}{l}.\end{aligned} \tag{15.38}$$

The inner product formula (15.38) can also be written in the form

$$\langle p,q\rangle = \frac{V_T}{\binom{2d}{d}\binom{2d+3}{3}} c^T G\, a,$$

where c and a are the vectors of B-coefficients of p and q respectively, and where G is a symmetric matrix with binomial coefficients as in (15.38).

15.14. Conditions for Smooth Joins

As in the bivariate case, smoothness between two polynomials defined on adjoining tetrahedra can be easily described in terms of B-coefficients. The proof of the following result is similar to the proof of Theorem 2.28.

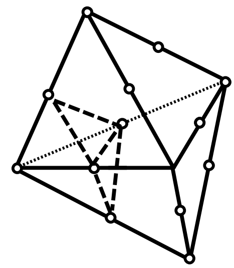


Fig. 15.4. A typical C^1 smoothness condition.

Theorem 15.31. *Suppose $T := \langle v_1, v_2, v_3, v_4\rangle$ and $\widetilde{T} := \langle v_5, v_2, v_4, v_3\rangle$ are two tetrahedra sharing the face $F := \langle v_2, v_3, v_4\rangle$ of $\triangle$. Suppose p and $\tilde{p}$ are two polynomials of degree d, and suppose the B-coefficients of p relative to T are $\{c_{ijkl}\}$ while the B-coefficients of $\tilde{p}$ relative to $\widetilde{T}$ are $\{\tilde{c}_{ijkl}\}$. Then p and $\tilde{p}$ join together with C^r continuity across the face F if and only if for $m = 0, \ldots, r$,*

$$\tilde{c}_{mijk} = \sum_{\nu+\mu+\kappa+\delta=m} c_{\nu,i+\mu,k+\kappa,j+\delta} B^m_{\nu\mu\kappa\delta}(v_5), \qquad \text{all } i+j+k = d-m.$$

Here $B^m_{\nu\mu\kappa\delta}$ are the Bernstein basis polynomials of degree m associated with the tetrahedron T.

Figure 15.4 illustrates a typical C^1 smoothness condition between two trivariate polynomials of degree two. The condition involves the five coefficients corresponding to the domain points in the pair of small tetrahedra with some dotted edges.

15.15. Approximation Power in the Maximum Norm

In this section we establish a bound on how well trivariate polynomials of degree d can approximate functions in $C^{d+1}(\Omega)$, where Ω is the closure of a convex domain in $\mathbb{R}^3$. The bound in this section is implied by the more general results for q-norms presented in the following section, but the proof here is much simpler. We define the diameter of Ω to be

$$|\Omega| := \max_{u,v\in\Omega} \|v-u\|. \tag{15.39}$$

Theorem 15.32. *Fix $d \ge 0$. Then there exists a constant K depending only on d such that for every $f \in C^{d+1}(\Omega)$, there exists a polynomial $p_f \in \mathcal{P}_d$ with*

$$\|D^\alpha(f-p_f)\|_\Omega \le K\ |\Omega|^{d+1-|\alpha|}\ |f|_{d+1,\Omega}, \tag{15.40}$$

for all $0 \le |\alpha| \le d$.

Proof: Let (u, v, w) be the center of the largest ball contained in Ω, and let

$$T_d f(x,y,z) := \sum_{i+j+k\le d} \frac{D_x^i D_y^j D_z^k f(u,v,w)}{i\,!\,j\,!\,k\,!}(x-u)^i(y-v)^j(z-w)^k \quad (15.41)$$

be the trivariate Taylor polynomial of degree d centered at (u, v, w). Then

$$f(x,y,z) - T_d f(x,y,z) = \sum_{i+j+k=d+1} \frac{(x-u)^i(y-v)^j(z-w)^k}{i\,!\,j\,!\,k\,!} D_x^i D_y^j D_z^k f(\tilde{x},\tilde{y},\tilde{z}),$$

where $(\tilde{x},\tilde{y},\tilde{z})$ is some point on the line between (x, y, z) and (u, v, w). This immediately gives (15.40) with $p_f := T_d f$ for the case $\alpha := (0, 0, 0)$. To get (15.40) for general α, we note that $D^\alpha T_d f = T_{d-|\alpha|} D^\alpha f$. $\square$

15.16. Averaged Taylor Polynomials

Let $B := B(x_0, y_0, z_0, \rho) := \{(x, y, z) \in \mathbb{R}^3 : ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{1/2} \le \rho\}$ be the largest ball contained in T. Let

$$g_B(x,y,z) := \begin{cases} ce^{-\rho^2/(\rho^2-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2)}, & (x,y,z) \in B, \\ 0, & \text{otherwise,} \end{cases}$$

where we choose c so that $\int_{\mathbb{R}^3} g_B(x,y,z)\,dx\,dy\,dz = 1$. We call g a mollifier. We now define the averaged Taylor polynomial of degree d with respect to $B := B(x_0, y_0, z_0, \rho)$ as

$$F_{d,B} f(x,y,z) := \int_B T_{d,(u,v,w)} f(x,y,z)\, g_B(u,v,w)\,du\,dv\,dw,$$

where $T_{d,(u,v,w)} f$ is the trivariate Taylor polynomial of degree d of f centered at (u, v, w), see (15.41). Integrating by parts, we have the equivalent formula

$$\begin{aligned} F_{d,B} f(x,y,z) &= \sum_{i+j+k\le d} \frac{1}{i\,!\,j\,!\,k\,!} \int_B D_u^i D_v^j D_w^k f(u,v,w) G_{ijk}(x,y,z,u,v,w)\,du\,dv\,dw \\ &= \sum_{i+j+k\le d} \frac{(-1)^{i+j+k}}{i\,!\,j\,!\,k\,!} \int_B f(u,v,w) H_{ijk}(x,y,z,u,v,w) du\,dv\,dw, \end{aligned}$$

where

$$G_{ijk}(x,y,z,u,v,w) := (x-u)^i(y-v)^j(z-w)^k g_B(u,v,w),$$
$$H_{ijk}(x,y,z,u,v,w) := D_u^i D_v^j D_w^k \left[(x-u)^i(y-v)^j(z-w)^k g_B(u,v,w)\right].$$

This shows that the averaged Taylor polynomial is well defined for any integrable function $f \in L_1(B(x_0,y_0,z_0,\rho))$, and $F_{d,B}f$ is a trivariate polynomial of degree at most d. The proof of the following lemma is similar to the proof of Lemma 1.5. The statement involves trivariate Sobolev spaces which are defined in the following section.

Lemma 15.33. *Given a multi-index α with $|\alpha| \le d$, suppose that f belongs to the Sobolev space $W_1^{|\alpha|}(B)$. Then $D^\alpha F_{d,B}f = F_{d-|\alpha|,B}(D^\alpha f)$. Moreover, $f = F_{d,B}f$ for any polynomial $f \in \mathcal{P}_d$.*

Now suppose Ω is the closure of a convex domain in $\mathbb{R}^3$, and let B_Ω be the largest ball that can be inscribed in Ω. Then the associated averaged Taylor expansion F_{d,B_Ω} maps functions defined on B_Ω into trivariate polynomials of degree d. We now give a bound on the size of $F_{d,B_\Omega}f$ in terms of the size of f. The bound will depend on the shape of Ω, as measured by

$$\kappa_\Omega := \frac{|\Omega|}{\rho_\Omega}, \tag{15.42}$$

where ρ_Ω is the radius of B_Ω and $|\Omega|$ is the diameter of Ω, see (15.39). The shape parameter κ_Ω can be large if the set Ω is thin. The proof of the following theorem is similar to the proof of Lemma 1.6 in the bivariate case.

Theorem 15.34. *There exists a constant K depending only on d and κ_Ω such that for any $f \in L_q(\Omega)$ with $1 \le q \le \infty$,*

$$\|F_{d,B_\Omega}f\|_{q,\Omega} \le K\,\|f\|_{q,\Omega}. \tag{15.43}$$

15.17. Approximation Power in the q-Norms

In this section we examine the approximation of functions in the Sobolev spaces

$$W_q^{d+1}(\Omega) := \{f : \|f\|_{d+1,q,\Omega} < \infty\},$$

where

$$\|f\|_{d+1,q,\Omega} := \begin{cases} \left(\displaystyle\sum_{k=0}^{d+1} |f|_{k,q,\Omega}^q\right)^{1/q}, & 1 \le q < \infty \\ \displaystyle\sum_{k=0}^{d+1} |f|_{k,\infty,\Omega}, & q = \infty, \end{cases}$$

and the corresponding Sobolev seminorms are

$$|f|_{k,q,\Omega} := \begin{cases} \Big(\sum_{|\alpha|=k} \|D^\alpha f\|_{q,\Omega}^q \Big)^{1/q}, & \text{if } 1 \le q < \infty, \\ \max_{|\alpha|=k} \|D^\alpha f\|_\Omega, & \text{if } q = \infty. \end{cases}$$

Suppose Ω is the closure of an arbitrary (not necessarily convex) domain in $\mathbb{R}^3$ with a Lipschitz smooth boundary. Let B_Ω be the largest disk that can be inscribed in Ω. Given a function f defined on Ω, let $F_{d,B_\Omega} f$ be its associated averaged Taylor polynomial. Then using the Stein extension theorem if necessary, and following the proof of Theorem 1.7, we can establish the following result.

Theorem 15.35. *For all $f \in W_q^{d+1}(\Omega)$,*

$$\|D^\alpha (f - F_{d,B_\Omega} f)\|_{q,\Omega} \le K \, |\Omega|^{d+1-|\alpha|} \, |f|_{d+1,q,\Omega},$$

for all $|\alpha| \le d$. If Ω is convex, the constant K depends only on d and the shape parameter κ_Ω of Ω. If Ω is nonconvex, it also depends on the Lipschitz constant of the boundary of Ω.

15.18. Subdivision

Suppose $T := \langle v_1, v_2, v_3, v_4 \rangle$ is a given tetrahedron, and let w be a point in the interior of T. Let $T_i := \langle w, v_{i+1}, v_{i+2}, v_{i+3} \rangle$ for $i = 1, 2, 3, 4$, where we identify $v_{4+j} = v_j$ for all j. Then we have the following result which can be proved along the same lines as Theorem 2.38 using the de Casteljau algorithm.

Theorem 15.36. *Suppose p is a trivariate polynomial whose B-coefficients relative to T are $\{c_{ijkl}^{(0)}\}_{i+j+k+l=d}$. Then for all $v \in T$,*

$$p(v) = \begin{cases} \sum_{i+j+k+l=d} c_{0jkl}^{(i)} B_{ijkl}^{T_1,d}(v), & v \in T_1, \\ \sum_{i+j+k+l=d} c_{i0kl}^{(j)} B_{ijkl}^{T_2,d}(v), & v \in T_2, \\ \sum_{i+j+k+l=d} c_{ij0l}^{(k)} B_{ijkl}^{T_3,d}(v), & v \in T_3, \\ \sum_{i+j+k+l=d} c_{ijk0}^{(l)} B_{ijkl}^{T_4,d}(v), & v \in T_4, \end{cases}$$

where $B_{ijkl}^{T_\nu,d}$ are the Bernstein basis polynomials of degree d relative to T_ν, and the coefficients $c_{ijkl}^{(m)}$ are the intermediate quantities generated by applying m steps of the de Casteljau Algorithm 15.11 based on the 4-tuple a of barycentric coordinates of w relative to T.

15.19. Degree Raising

It is clear that a trivariate polynomial of degree d can also be regarded as a polynomial of degree $\tilde{d}$ for any $\tilde{d} > d$. In this section we show how to find the B-coefficients of p after degree raising it.

Theorem 15.37. *Let p be a polynomial of degree d defined on a tetrahedron T written in the B-form (15.13). Let $c_{ijkl}^{[d]} = c_{ijkl}$ be its coefficients. Then*

$$p = \sum_{i+j+k+l=d+1} c_{ijkl}^{[d+1]} B_{ijkl}^{d+1}, \tag{15.44}$$

where B_{ijkl}^{d+1} are the Bernstein basis polynomials of degree $d+1$ associated with T, and where

$$c_{ijkl}^{[d+1]} := \frac{ic_{i-1,j,k,l}^{[d]} + jc_{i,j-1,k,l}^{[d]} + kc_{i,j,k-1,l}^{[d]} + lc_{i,j,k,l-1}^{[d]}}{d+1}, \tag{15.45}$$

for $i+j+k+l = d+1$. Here coefficients with negative subscripts are taken to be zero.

Proof: Multiplying both sides of (15.13) by $1 = b_1 + b_2 + b_3 + b_4$, we get

$$p = \sum_{i+j+k+l=d} c_{ijkl}^{[d]} \frac{d!}{i!\,j!\,k!\,l!} b_1^i b_2^j b_3^k b_4^l (b_1 + b_2 + b_3 + b_4).$$

Then multiplying out and collecting terms, we get (15.45). $\square$

We can repeat this process to raise the degree of a polynomial by more than one.

15.20. Remarks

Remark 15.1. As in the bivariate case, it is a nontrivial task to find sets of points A which are poised with respect to $\mathcal{P}_d$, i.e., such that for every set of real numbers $\{z_\nu\}_{\nu=1}^n$ with $n := \binom{d+3}{3}$ there exists a unique trivariate polynomial $p \in \mathcal{P}_d$ such that $p(t_\nu) = z_\nu$, for $\nu = 1, \ldots, n$. The following result is the analog of Theorem 1.10.

Theorem 15.38. *Fix $0 \le d$, and let $n := \binom{d+3}{3}$. Suppose*

$$A := \{t_\nu\}_{\nu=1}^n := \bigcup_{i=1}^{d+1} \bigcup_{j=1}^{i} \{t_{ijk}\}_{k=1}^j$$

is a set of distinct points in $\mathbb{R}^3$ such that for some collection $\{F_i\}_{i=1}^{d+1}$ of distinct planes,

1) *for $1 \le i \le d+1$, the point set $A_i := \bigcup_{j=1}^i \{t_{ijk}\}_{k=1}^j$ lies on F_i but not on $F_{i+1} \cup \cdots \cup F_{d+1}$,*
2) *for $1 \le i \le d+1$, A_i satisfies the conditions of Theorem 1.10.*

Then A is poised with respect to $\mathcal{P}_d$.

This theorem is a generalization of a similar result in [ChunY77]. It is easy to see that for any tetrahedron T, the set $A := \mathcal{D}_{d,T}$ of domain points associated with T satisfies the hypotheses of this theorem. For some error bounds for this type of polynomial interpolation, see [Boo97].

Remark 15.2. Given a tetrahedron T and an integer d, let $\{\xi_i\}_{i=1}^n$ be the set of domain points associated with T, arranged in lexicographical order, where $n := \binom{d+3}{3}$. Let $\{B_i\}_{i=1}^n$ be the corresponding Bernstein basis polynomials, arranged in the same order. We can now use the previous remark to construct linear functionals $\{\lambda_i\}_{i=1}^n$ with the property

$$\lambda_i B_j = \delta_{i,j}, \qquad \text{all } i, j = 1, \ldots, n. \tag{15.46}$$

Let

$$M := [B_i(\xi_j)]_{i,j=1}^n.$$

Lemma 15.39. *For each $1 \le i \le n$, let $a_i := (a_{i1}, \ldots, a_{in})^T$ be the solution of the linear system*

$$M a_i = r_i,$$

where r_i is the n-vector with all zero entries except for the i-th, which is one. Let λ_i be the linear functional defined on $C(T)$ by

$$\lambda_i f := \sum_{j=1}^n a_{ij} f(\xi_j).$$

Then $\{\lambda_i\}_{i=1}^n$ are dual functionals to $\{B_i\}_{i=1}^n$, i.e., (15.46) holds.

Proof: We know from Remark 15.1 that M is nonsingular, and so the λ_i are well-defined. The fact that they satisfy (15.46) follows immediately from the choice of coefficients a_{ij}. $\square$

Remark 15.3. Theorem 15.23 ensures that if we set the coefficients of a piecewise polynomial s that belongs to $C^m(\Omega)$, where Ω is the union of a set of tetrahedra forming an orange with respect to an edge e, then we can set the B-coefficients of s for domain points in $t_m(e) \cap T_1$ and all other B-coefficients will be consistently determined by the smoothness conditions across common faces.

Remark 15.4. As shown in [SchV86], dropping the normalization factors in the Bernstein basis polynomials of Definition 15.7 gives an alternative basis for trivariate polynomials of degree d that can be evaluated with fewer operations than are required for the de Casteljau algorithm applied to a polynomial in B-form. As discussed in Remark 2.12, in the bivariate case the analogous unnormalized basis does not seem to be quite as stable as the Bernstein polynomial basis. However, a detailed round-off analysis and extensive numerical experiments carried out in [MaiP06b] show that in the trivariate case, this alternative basis is at least as stable as using the Bernstein basis polynomials.

15.21. Historical Notes

Barycentric coordinates relative to a tetrahedron go back to at least Möbius, see [Moe86]. Bernstein basis polynomials on tetrahedra probably were treated somewhere in the approximation literature, but we have not found a reference. Some early paper where the B-form of trivariate polynomials was used in the CAGD/spline setting include [Gol82, Gol83, Alf84b, Pra84, Far86, Las87]. The use of the B-form seems to have been well-established in the CAGD community by the time the first CAGD books [Far88, HosL93] were written.

The de Casteljau algorithm, degree raising, and subdivision were studied in [Gol82, Gol83], see also the thesis [Las87]. This thesis also contains derivative formulae for polynomials in B-form, along with various other results on the trivariate B-form. The earliest source we could find for the formulae for integrals and inner products given in Section 15.13 is [ChuL90a].

For the ∞-norm, the Markov inequality for trivariate polynomials was established in [Wil74] by reducing the problem to a univariate problem where the classical Markov inequality can be applied. Our proof for general $1 \le q \le \infty$ in Section 15.12 is based on the B-form, and follows the proof for the bivariate case given in Section 2.12, which in turn was based on ideas from our paper [LaiS98].

Conditions for two trivariate polynomials in B-form to join with C^1 smoothness were introduced in the construction of a macro-element in [Alf84b]. It is not clear who first discovered the analogous C^r smoothness conditions, but they are given explicitly in [Boo87, Las87, Far88].

Tetrahedral Partitions

In this chapter we explore various properties of tetrahedral partitions. The discussion includes shape properties, regular and shellable partitions, Euler relations, clusters, various refinement schemes, and Delaunay partitions.

16.1. Properties of a Tetrahedron

Before defining tetrahedral partitions in the next section, we present some facts about a single tetrahedron. Suppose we are given four noncoplanar points v_1, v_2, v_3, v_4 in $\mathbb{R}^3$. Then the convex hull of these points form a tetrahedron which we write as $T := \langle v_1, v_2, v_3, v_4 \rangle$.

We call the points $v_i := (x_i, y_i, z_i)$ the vertices of T. Throughout the remainder of the book, whenever we deal with a tetrahedron we assume its vertices are noncoplanar, and are arranged in canonical order as described in Definition 15.4. This ensures that T has a positive volume

$$V_T = \frac{1}{6} \det(M),$$

where

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}.$$

A tetrahedron has six edges and four triangular faces. We write $\langle v_1, v_2 \rangle$ for the edge connecting vertices v_1 and v_2, with similar notation for the other edges. We write $\langle v_1, v_2, v_3 \rangle$ for the triangular face with vertices v_1, v_2, v_3, with similar notation for the other faces. We now describe how to measure the size and shape of a tetrahedron.

Definition 16.1. *We define $|T|$ to be the length of the longest edge of T. We write ρ_T for the radius of the largest ball which can be inscribed in T, and call it the* inradius *of T. We call the center of this ball the* incenter *of T. The ratio $\kappa_T := |T|/\rho_T$ is called the* shape parameter *of T.*

The shape parameter κ_T describes the shape of T. If T is a tetrahedron whose six edges are all of the same length, then $\kappa_T = 12/\sqrt{6}$. Such a tetrahedron is called a regular tetrahedron. For more on regular tetrahedra, see Remark 16.1. For any other tetrahedron, κ_T is larger, and the larger κ_T becomes, the flatter the tetrahedron T becomes. Another way to describe the shape of a tetrahedron is in terms of certain angles at the vertices of T.

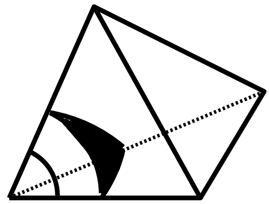


Fig. 16.1. A face angle and the solid angle at a vertex.

Definition 16.2. *Let $T := \langle v_1, v_2, v_3, v_4 \rangle$ be a tetrahedron. We call the angles of the triangles forming the faces of T the* face angles *of T. If F_1 and F_2 are two faces of T, then the angle between these faces is called the* dihedral angle *between F_1 and F_2. If v is one of the vertices of T, then we define the* solid angle of T at v *to be the spherical area of the intersection of the unit ball centered at v with the* trihedron $\widetilde{T}$ *which is obtained by extending the three faces of T that join at v. We write θ_T for the smallest of the four solid angles of T, and ϕ_T for the smallest of the twelve face angles of T.*

Figure 16.1 illustrates one of the three face angles (shown as an arc) and the solid angle (shown as a spherical surface) at one vertex of a tetrahedron. For more on solid angles, see Remark 16.2.

Example 16.3. The angle θ_T can be arbitrarily small even when ϕ_T is large.

Discussion: To see this, suppose $T := \langle v_1, v_2, v_3, v_4 \rangle$ is a tetrahedron where v_1 lies on the line perpendicular to the face $F := \langle v_2, v_3, v_4 \rangle$ at the barycenter v_T of F. Suppose F is an equilateral triangle. Now gradually adjust the shape of the tetrahedron by moving v_1 towards v_T. Then θ_T goes to zero, but all face angles of T remain bounded below by 30 degrees. □

It is clear that the size of the shape parameter $\kappa_T := |T|/\rho_T$ of a tetrahedron is connected to the size of its angles. The following lemma makes this connection more precise.

Lemma 16.4. *For any tetrahedron,*

$$\frac{3^{3/4}}{2\theta_T^{1/2}} \le \kappa_T \le \frac{8\sqrt{3}\pi^2}{\theta_T^2}. \tag{16.1}$$

Proof: We make use of Theorem 6.1 of [JoeL94], which states that

$$\frac{\sqrt{3}}{24} \left(\frac{3\rho_T}{r_T}\right)^2 \le \sin\left(\frac{\theta_T}{2}\right) \le \frac{2}{\sqrt[4]{3}} \left(\frac{3\rho_T}{r_T}\right)^{1/2},$$

where r_T is the circumradius of T, i.e., the radius of the smallest ball that encloses T. The result then follows from the inequalities $r_T \leq |T| \leq 2r_T$ and $\theta_T/\pi \leq \sin(\theta_T/2) \leq \theta_T/2$. $\square$

16.2. General Tetrahedral Partitions

Definition 16.5. *A collection $\triangle := \{T_i\}_{i=1}^N$ of tetrahedra in $\mathbb{R}^3$ is called a* tetrahedral partition *of a polygonal set $\Omega := \bigcup_{i=1}^N T_i$ provided that any pair of tetrahedra in $\triangle$ intersect at most at a vertex, along a common edge, or along a common triangular face.*

This definition allows quite general tetrahedral partitions such as

– two tetrahedra which are completely separated,

– two tetrahedra which touch only at a vertex,

– two tetrahedra which share only an edge.

The definition also allows the possibility that

– Ω has holes cutting through it, e.g. when Ω has the shape of a torus,

– Ω has cavities.

Such partitions arise frequently in the finite-element method for solving partial differential equations.

Definition 16.6. *The vertices of the tetrahedra in $\triangle$ are called the vertices of the partition $\triangle$. If a vertex v is a boundary point of Ω, we say that it is a* boundary vertex. *Otherwise, we call it an* interior vertex. *Similarly, the edges and faces of the tetrahedra of $\triangle$ are called the edges and faces of the partition $\triangle$. If an edge or face lies on the boundary of Ω, we say that it is a* boundary edge *or* boundary face, *respectively. All other edges and faces are* interior edges *or* interior faces.

Given a tetrahedral partition $\triangle$, we shall often need to work with certain subpartitions called stars.

Definition 16.7. *If v is a vertex of a tetrahedral partition $\triangle$, we define $star(v)$ to be the set of all tetrahedra in $\triangle$ which share the vertex v. Setting $star^1(v) := star(v)$, we define $star^i(v)$ inductively for $i > 1$ to be the set of all tetrahedra in $\triangle$ which have a nonempty intersection with $star^{i-1}(v)$. Similarly, we define $star^0(T) := T$, and $star^j(T) := \bigcup\{star(v) : v \in star^{j-1}(T)\}$.*

16.3. Regular Tetrahedral Partitions

We emphasize that most of the results in the remainder of this book hold for general tetrahedral partitions as described in Definition 16.5. However, there are a few results which require partitions with more structure.

Definition 16.8. *A tetrahedral partition $\triangle$ is called* shellable *provided it consists of a single tetrahedron, or it can be obtained from a shellable tetrahedral partition $\widetilde{\triangle}$ by adding one tetrahedron T that intersects $\widetilde{\triangle}$ precisely along one, two, or three triangular faces.*

Not all tetrahedral partitions are shellable. For example, two tetrahedra touching only at a vertex, or two tetrahedra touching only along an edge are not shellable. By definition, if $\triangle$ is a shellable tetrahedral partition of a set Ω, then Ω is homeomorphic to the unit ball. The converse does not hold, i.e., there are nonshellable tetrahedral partitions of sets Ω which are homeomorphic to a ball, see Remark 16.3.

For many applications, the class of shellable tetrahedral partitions is too restrictive. In particular, we would like to allow partitions with holes and cavities. Partitions with holes or cavities cannot be shellable since they are not homeomorphic to the ball. We now show how to construct well-behaved tetrahedral partitions by starting with a shellable partition and successively removing shellable subpartitions.

Definition 16.9. *We say that a tetrahedral partition $\triangle$ has a* regular cavity *provided that it can be obtained from a larger tetrahedral partition by removing a shellable subpartition $\mathcal{T}$, all of whose vertices are interior vertices of $\triangle$.*

Definition 16.10. *We say that a tetrahedral partition $\triangle$ of a set Ω has a* regular hole *provided that it can be obtained from a larger tetrahedral partition by removing a shellable subpartition $\mathcal{T}$ with the property that if F is a triangular face of a tetrahedron in $\mathcal{T}$ and F lies on the boundary of Ω, then F lies in one of two triangulations $\mathcal{F}_1$ or $\mathcal{F}_2$ on the boundary of Ω, where*

1) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are nonempty shellable triangulations,*
2) *the triangles of $\mathcal{F}_1$ do not touch those in $\mathcal{F}_2$.*

The idea behind this definition is as follows: to create a regular hole in a tetrahedral partition $\triangle$, we remove a tetrahedron with a face on the boundary of $\triangle$, and then continue removing tetrahedra until finally we have removed one or more tetrahedra with faces somewhere else on the boundary of $\triangle$.

Definition 16.11. *We say that a tetrahedral partition $\triangle$ is* regular *provided that one of the following holds:*

1) *$\triangle$ is shellable, or*
2) *$\triangle$ can be obtained from a regular tetrahedral partition by creating a regular hole or a regular cavity.*

This definition allows multiple cavities and multiple holes which may have branches. It is general enough to include all of the tetrahedral partitions typically used in practice. Many tetrahedral partitions are not regular. A simple example is provided by a pair of tetrahedra that touch only at a vertex.

16.4. Euler Relations

In this section we explore the relationship between the number of vertices, edges, and faces of a tetrahedral partition $\triangle$. Let V_I, V_B be the number of interior and boundary vertices of $\triangle$, respectively. Let E_I and E_B be the number of interior and boundary edges, and let F_I and F_B be the number of interior and boundary faces of $\triangle$. Suppose N is the number of tetrahedra in $\triangle$. We begin with a result for shellable partitions.

Theorem 16.12. *Suppose $\triangle$ is a shellable tetrahedral partition. Suppose we can build $\triangle$ by starting with one tetrahedron and adding tetrahedra one at a time so that the number of times a tetrahedron touches the preceding partition on exactly i faces is α_i for $i = 1, 2, 3$. Then*

1) $N = 1 + \alpha_1 + \alpha_2 + \alpha_3$,
2) $F_I = \alpha_1 + 2\alpha_2 + 3\alpha_3$,
3) $F_B = 2\alpha_1 - 2\alpha_3 + 4$,
4) $E_I = \alpha_2 + 3\alpha_3$,
5) $E_B = 3\alpha_1 - 3\alpha_3 + 6$,
6) $V_B = \alpha_1 - \alpha_3 + 4$,
7) $V_I = \alpha_3$.

Proof: The proof is just a simple matter of counting. To get formula 1), we start with one tetrahedron, and note that α_i is the number of times that we add a tetrahedron touching on exactly i faces, so the total number of tetrahedra added is $N = 1 + \alpha_1 + \alpha_2 + \alpha_3$. To establish formula 2), we note that each time we add a tetrahedron to an existing shellable tetrahedral partition that touches on exactly i faces, the number of interior faces is increased by i. We conclude that the total number of interior faces is given by $F_I = \alpha_1 + 2\alpha_2 + 3\alpha_3$. The proofs of the other formulae are similar. $\square$

The formulae in Theorem 16.12 can be combined in various ways to yield relationships between the number of vertices, edges, and faces. We give four typical such relationships in the following theorem.

Theorem 16.13. *Suppose $\triangle$ is a shellable tetrahedral partition. Then*

1) $N = E_I + V_B - V_I - 3$,
2) $N = F_I/2 + F_B/4$,
3) $E_B = 3V_B - 6$,
4) $F_B = 2E_B/3$.

Proof: The equations in Theorem 16.12 imply $\alpha_1 = V_B + V_I - 4$, $\alpha_2 = E_I - 3V_I$, and $\alpha_3 = V_I$. They also imply $\alpha_1 = (F_B + 2V_I - 4)/2$ and $\alpha_2 = F_I/2 - F_B/4 - 2V_I + 1$. The results follow after substituting these expressions in the formulae in Theorem 16.12. $\square$

The formulae in the proof of Theorem 16.13 show that the shelling parameters $\alpha_1, \alpha_2, \alpha_3$ are uniquely determined by the partition, i.e., they don't depend on the order in which the partition is decomposed. We now extend Theorem 16.13 to the class of regular tetrahedral partitions.

Theorem 16.14. *Suppose $\triangle$ is a regular tetrahedral partition with H holes and C cavities. Then*

1) $N = E_I + V_B - V_I + 3(H - C - 1)$,
2) $N = F_I/2 + F_B/4$,
3) $E_B = 3V_B + 6(H - C - 1)$,
4) $F_B = 2E_B/3$.

Proof: We proceed by induction on the number of holes and cavities. If there are no holes or cavities, the formulae follow from Theorem 16.13. We consider two cases.

Case 1: Suppose $\triangle$ is a partition with $C > 0$ cavities. Then we can get $\triangle$ by removing a shellable cluster $\mathcal{T}$ of tetrahedra from a partition $\widetilde{\triangle}$ with $C - 1$ cavities. By the induction hypothesis,

$$
\begin{aligned}
\widetilde{N} &= \widetilde{E}_I + \widetilde{V}_B - \widetilde{V}_I + 3(H - C), \\
\widetilde{N} &= \frac{1}{2}\widetilde{F}_I + \frac{1}{4}\widetilde{F}_B, \\
\widetilde{E}_B &= 3\widetilde{V}_B + 6(H - C), \\
\widetilde{F}_B &= \frac{2}{3}\widetilde{E}_B,
\end{aligned}
$$

where $\widetilde{V}, \widetilde{E}, \widetilde{F}, \widetilde{N}$ are the number of vertices, edges, faces, and tetrahedra in $\widetilde{\triangle}$, respectively, and the subscripts B and I denote boundary and interior.

For the cluster $\mathcal{T}$, Theorem 16.13 implies

$$\begin{aligned}\widehat{N} &= \widehat{E}_I + \widehat{V}_B - \widehat{V}_I - 3,\\ \widehat{N} &= \frac{1}{2}\widehat{F}_I + \frac{1}{4}\widehat{F}_B,\\ \widehat{E}_B &= 3\widehat{V}_B - 6,\\ \widehat{F}_B &= \frac{2}{3}\widehat{E}_B,\end{aligned} \tag{16.2}$$

where $\widehat{V}, \widehat{E}, \widehat{F}, \widehat{N}$ are the number of vertices, edges, faces, and tetrahedra in $\mathcal{T}$, respectively. It is easy to check that

$$\begin{aligned}V_B &= \widetilde{V}_B + \widehat{V}_B,\\ V_I &= \widetilde{V}_I - \widehat{V}_I - \widehat{V}_B,\\ E_B &= \widetilde{E}_B + \widehat{E}_B,\\ E_I &= \widetilde{E}_I - \widehat{E}_I - \widehat{E}_B,\\ F_B &= \widetilde{F}_B + \widehat{F}_B,\\ F_I &= \widetilde{F}_I - \widehat{F}_I - \widehat{F}_B.\end{aligned}$$

We are now ready to prove 1). Using $\widehat{E}_B = 3\widehat{V}_B - 6$, we have

$$\begin{aligned}N &= \widetilde{N} - \widehat{N}\\ &= (\widetilde{E}_I - \widehat{E}_I - \widehat{E}_B) + (\widetilde{V}_B + \widehat{V}_B) - (\widetilde{V}_I - \widehat{V}_I - \widehat{V}_B) + 3(H - C - 1)\\ &= E_I + V_B - V_I + 3(H - C - 1).\end{aligned}$$

Similarly, using the fact that the boundary faces of $\mathcal{T}$ are interior faces of $\triangle$, we get

$$\widetilde{N} - \widehat{N} = \frac{1}{2}(\widetilde{F}_I - \widehat{F}_I - \widehat{F}_B) + \frac{1}{4}(\widetilde{F}_B - \widehat{F}_B + 2\widehat{F}_B),$$

which reduces to formula 2). To prove 3), we add the expressions for $\widetilde{E}_B$ and $\widehat{E}_B$ to get

$$E_B = \widetilde{E}_B + \widehat{E}_B = 3(\widetilde{V}_B + \widehat{V}_B) + 6(H - C - 1) = 3V_B + 6(H - C - 1).$$

To prove 4), we have

$$F_B = \widetilde{F}_B + \widehat{F}_B = \frac{2}{3}(\widetilde{E}_B + \widehat{E}_B) = \frac{2}{3}E_B.$$

Case 2: Suppose $\triangle$ is a partition with H holes. Then $\triangle$ can be obtained from a partition $\widetilde{\triangle}$ with $H - 1$ holes by removing a shellable cluster $\mathcal{T}$. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are the triangulations on the boundary of Ω in Definition 16.10. For combinatorial purposes, we may think of $\mathcal{F}_i$ as a planar triangulation. For $i = 1, 2$, let V_B^i, V_I^i, E_I^i be the number of boundary vertices, interior vertices, and interior edges of $\mathcal{F}_i$. Using Euler's relations for triangulations (4.5), we have $E_I^i = V_B^i + 3V_I^i - 3$ and $E_B^i = V_B^i$ for $i = 1, 2$. It is straightforward to see that

$$\begin{aligned}
V_B &= \widetilde{V}_B + \widehat{V}_B - V_B^1 - 2V_I^1 - V_B^2 - 2V_I^2, \\
V_I &= \widetilde{V}_I - \widehat{V}_I - \widehat{V}_B + V_B^1 + V_I^1 + V_B^2 + V_I^2, \\
E_B &= \widetilde{E}_B + \widehat{E}_B - E_B^1 - 2E_I^1 - E_B^2 - 2E_I^2, \\
E_I &= \widetilde{E}_I - \widehat{E}_I - \widehat{E}_B + E_B^1 + E_I^1 + E_B^2 + E_I^2, \\
F_B &= \widetilde{F}_B + \widehat{F}_B - 2F^1 - 2F^2, \\
F_I &= \widetilde{F}_I - \widehat{F}_I - \widehat{F}_B + F^1 + F^2,
\end{aligned} \tag{16.3}$$

where F^1, F^2 are the number of faces in $\mathcal{F}_1, \mathcal{F}_2$, respectively. By the induction hypothesis,

$$\begin{aligned}
\widetilde{N} &= \widetilde{E}_I + \widetilde{V}_B - \widetilde{V}_I + 3(H - C - 2), \\
\widetilde{N} &= \frac{1}{2}\widetilde{F}_I + \frac{1}{4}\widetilde{F}_B, \\
\widetilde{E}_B &= 3\widetilde{V}_B + 6(H - C - 2), \\
\widetilde{F}_B &= \frac{2}{3}\widetilde{E}_B.
\end{aligned} \tag{16.4}$$

Combining the above formulae for $\widetilde{N}$ and $\widehat{N}$ with the fact that $\widehat{E}_B = 3\widehat{V}_B - 6$, we have

$$\begin{aligned}
N = \widetilde{N} - \widehat{N} &= (\widetilde{E}_I - \widehat{E}_I) + (\widetilde{V}_B - \widehat{V}_B) - (\widetilde{V}_I - \widehat{V}_I) + 3(H - C - 1) \\
&= (\widetilde{E}_I - \widehat{E}_I - \widehat{E}_B) + (\widetilde{V}_B + \widehat{V}_B) - (\widetilde{V}_I - \widehat{V}_I - \widehat{V}_B) + 3(H - C - 3).
\end{aligned}$$

Using $E_I^i = V_B^i + 3V_I^i - 3$ and $E_B^i = V_B^i$ for $i = 1, 2$, we get

$$\begin{aligned}
N &= (\widetilde{E}_I - \widehat{E}_I - \widehat{E}_B + E_B^1 + E_I^1 + E_B^2 + E_I^2) \\
&\quad + (\widetilde{V}_B + \widehat{V}_B - V_B^1 - 2V_I^1 - V_B^2 - 2V_I^2) \\
&\quad - (\widetilde{V}_I - \widehat{V}_I - \widehat{V}_B + V_B^1 + V_I^1 + V_B^2 + V_I^2) + 3(H - C - 1) \\
&= E_I + V_B - V_I + 3(H - C - 1),
\end{aligned}$$

which gives

$$\begin{aligned}
N = \widetilde{N} - \widehat{N} &= \frac{1}{2}(\widetilde{F}_I - \widehat{F}_I) + \frac{1}{4}(\widetilde{F}_B - \widehat{F}_B) \\
&= \frac{1}{2}(\widetilde{F}_I - \widehat{F}_I - \widehat{F}_B + F^1 + F^2) + \frac{1}{4}(\widetilde{F}_B + \widehat{F}_B - 2F^1 - 2F^2) \\
&= \frac{1}{2}F_I + \frac{1}{4}F_B.
\end{aligned}$$

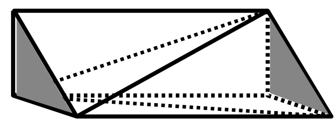


Fig. 16.2. A prism cut into three tetrahedra.

The proof of 3) follows by substituting (16.2) and (16.4) in the formula for E_B in (16.3). Similarly, 4) follows by substituting (16.2) and (16.4) in the formula for F_B in (16.3), and observing that $F^i = (2E_I^i + E_B^i)/3$ for $i = 1, 2$. $\square$

The formulae in Theorem 16.14 do not hold for general nonregular tetrahedral partitions. For example, it is easy to check that they fail for the simple case where $\triangle$ consists of two tetrahedra which touch only at a vertex. We now give a simple example to illustrate Theorem 16.14 for a partition with a hole.

Example 16.15. Let $\triangle$ be the tetrahedral partition obtained by connecting $m \geq 3$ copies of a prism-shaped complex $\mathcal{T}$ as shown in Figure 16.2 end to end to create a torus-shaped object Ω.

Discussion: In order to create a torus-shaped object from prisms as in Figure 16.2, we have to choose the end triangles (shaded in the figure) to be nonparallel. Each copy of the complex $\mathcal{T}$ consists of three tetrahedra. A simple count shows that $V_B = 3m$, $V_I = 0$, $E_B = 9m$, $E_I = 0$, $F_B = 6m$, $F_I = 3m$, $N = 3m$, $C = 0$, and $H = 1$. Then it is easily checked that each of the Euler relations of Theorem 16.14 holds for $\triangle$. $\square$

16.5. Constructing and Storing Tetrahedral Partitions

Tetrahedral partitions can be stored in a computer using data structures similar to those described in Section 4.5 for storing triangulations. Special care must be exercised when Ω has either holes or cavities. It is a nontrivial problem in computational geometry to design efficient algorithms for constructing tetrahedral partitions with given vertices (and possibly given edges and faces), especially if the desired set Ω is nonconvex and has holes or cavities. The task is further complicated by the fact that usually we want our tetrahedral partition to have good shape properties.

On the other hand, if we want to use splines on tetrahedral partitions as a tool for solving partial differential equations, usually we do not require the partition to have a specified set of vertices, and the locations of the vertices can be used as free parameters to help get a good partition. This is the subject of grid generation, see Remark 16.4.

16.6. Clusters of Tetrahedra

In dealing with spline spaces on tetrahedral partitions $\triangle$, we often have to work with small clusters of tetrahedra contained in a given tetrahedral partition $\triangle$.

Definition 16.16. *Let v be a vertex of a tetrahedral partition $\triangle$, and let ℓ be a positive integer. Suppose $\mathcal{T}$ is a shellable subpartition of $\triangle$ such that all tetrahedra in $\mathcal{T}$ lie in $\mathrm{star}^\ell(v)$ for some vertex v of $\triangle$. Then we say that $\mathcal{T}$ is an ℓ-*cluster with center at* v.*

In this section we establish several useful properties of clusters. Our first result gives a bound on how many tetrahedra there can be in an ℓ-cluster.

Lemma 16.17. *Suppose that $\mathcal{T}$ is an ℓ-cluster of tetrahedra. Then*

$$N := \#\mathcal{T} \le \begin{cases} a\displaystyle\sum_{\nu=0}^{k}(a+4)^{2\nu}, & \ell = 2k+1, \\ a\displaystyle\sum_{\nu=1}^{k}(a+4)^{2\nu-1}, & \ell = 2k, \end{cases} \tag{16.5}$$

where $a := 4\pi/\theta$, and θ is the smallest solid angle in the tetrahedra of $\mathcal{T}$.

Proof: We first consider the case where $\mathcal{T} = \mathrm{star}(v)$ with N tetrahedra attached to v. Taking the sum of the solid angles at v over all N tetrahedra, we see that $N\theta \le 4\pi$ which implies $N \le 4\pi/\theta = a$. This establishes (16.5) for $\ell = 1$. Formula 1) of Theorem 16.13 implies that the number of boundary vertices of $\mathrm{star}(v)$ is at most $N + 4 \le a + 4$.

Now consider $\mathcal{T} = \mathrm{star}^\ell(v)$. We say that a vertex w is at level j with respect to v if we have to follow at most j edges to get from w to v. $\mathcal{T}$ has vertices at all levels from 0 to ℓ. Since there are at most a tetrahedra surrounding any vertex, it follows that there are at most $(a+4)^j$ vertices at level j. To count the total number of tetrahedra, it suffices to consider vertices at levels $0, 2, \ldots, 2k$ if $\ell = 2k+1$, and at levels $1, 3, \ldots, 2k-1$ if $\ell = 2k$. $\square$

Our next result gives a bound on the relative sizes of the edge lengths and volumes of tetrahedra in a cluster.

Lemma 16.18. *Suppose $\mathcal{T}$ is an ℓ-cluster of tetrahedra, and let θ and ϕ be the smallest solid and face angles in $\mathcal{T}$, respectively. Then for any two tetrahedra T and $\widetilde{T}$ in $\mathcal{T}$,*

$$\frac{|T|}{|\widetilde{T}|} \le b^n, \tag{16.6}$$

where $b := \frac{1}{\sin\phi}$ *and* $n := \lceil 4(2\ell-1)\pi/\theta + 2\rceil$. *Moreover, the ratio of the volumes of* T *and* $\widetilde{T}$ *satisfies*

$$\frac{V_T}{V_{\widetilde{T}}} \le \frac{b^{3n}\kappa_{\mathcal{T}}^3}{8\pi}, \tag{16.7}$$

where $\kappa_{\mathcal{T}} := \max_{T\in\mathcal{T}} \kappa_T$.

Proof: Suppose e and $\tilde{e}$ are any two edges which lie in a common face of a tetrahedron T in $\mathcal{T}$. Then by (4.3),

$$\frac{|e|}{|\tilde{e}|} \le b. \tag{16.8}$$

Now any two vertices in $\mathcal{T}$ are connected by a path of edges which passes through at most $2\ell-1$ vertices. Since at most $4\pi/\theta$ tetrahedra can touch any given vertex, this means that we can get from any edge in $\mathcal{T}$ to any other edge by making at most n comparisons, and (16.6) follows.

To prove (16.7), let $T, \widetilde{T} \in \mathcal{T}$. Then $V_T \le \frac{1}{6}|e_1||e_2||e_3|$, where e_1, e_2, e_3 are any three edges of T. On the other hand, the volume of $\widetilde{T}$ is at least as great as the volume of the inscribed sphere associated with $\widetilde{T}$. Its volume is $\frac{4\pi}{3}\rho_{\widetilde{T}}^3$. Since $\kappa_{\widetilde{T}} = |\widetilde{T}|/\rho_{\widetilde{T}}$, using (16.8), we get

$$\frac{V_T}{V_{\widetilde{T}}} \le \frac{b^{3n}\kappa_{\widetilde{T}}^3}{8\pi}.$$

Since $\kappa_{\widetilde{T}} \le \kappa_{\mathcal{T}}$, (16.7) follows. $\square$

Lemma 16.4 shows that the shape parameter $\kappa_{\mathcal{T}}$ in (16.7) is bounded above in terms of the smallest solid angle in $\mathcal{T}$. It follows that the ratio of volumes in (16.7) is bounded by a constant depending only on the smallest solid and face angles in $\mathcal{T}$. We conclude this section with a useful bound on the diameter of a cluster of tetrahedra, where in general for any subset A of $\mathbb{R}^3$, we define its diameter by

$$|A| := \max_{v,w\in A} \|v-w\|.$$

Lemma 16.19. *Let* $\mathcal{T}$ *be an* ℓ*-cluster of tetrahedra in an arbitrary tetrahedral partition* $\triangle$, *and let* $\Omega_{\mathcal{T}} := \bigcup_{T\in\mathcal{T}} T$. *Let* $|\mathcal{T}|$ *be the length of the longest edge in* $\mathcal{T}$. *Then*

$$|\Omega_{\mathcal{T}}| \le 2\,\ell\,|\mathcal{T}|. \tag{16.9}$$

Proof: Since $\mathcal{T}$ is an ℓ-cluster, there is some vertex v such that $\Omega_{\mathcal{T}} \subseteq \mathrm{star}^{\ell}(v)$. Since $\Omega_{\mathcal{T}}$ has a polygonal boundary, its diameter must be equal to $\|u - w\|$ for some pair of boundary vertices. But for any pair of vertices u, w of $\Omega_{\mathcal{T}}$, we can find a polygonal path from u to w involving at most 2ℓ edges of $\mathcal{T}$. □

16.7. Refinements of Tetrahedral Partitions

Suppose $\triangle$ and $\triangle_R$ are two tetrahedral partitions of Ω.

Definition 16.20. *We say that $\triangle_R$ is a* refinement *of $\triangle$ provided*

1) *every vertex of $\triangle$ is a vertex of $\triangle_R$,*
2) *every tetrahedron $t \in \triangle_R$ is a subtetrahedron of some tetrahedron $T \in \triangle$.*

When $\triangle_R$ is a refinement of $\triangle$, we call $\triangle$ the coarser tetrahedral partition and $\triangle_R$ the finer tetrahedral partition. In this case we also say that the two tetrahedral partitions are nested.

There are many ways to refine a given tetrahedral partition. In practice we are interested in systematic refinement algorithms which produce refinements with good shape properties. We now describe several refinement algorithms which will be especially useful for constructing trivariate macro-element spaces in Chapter 18.

16.7.1 The Alfeld Refinement

Definition 16.21. *Given a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$, let $v_T := (v_1 + v_2 + v_3 + v_4)/4$ be the* barycenter *of T. Then we define the* Alfeld split *T_A of T to consist of the four subtetrahedra obtained by connecting v_T to each of the vertices of T, see Figure 16.3. If $\triangle$ is a general tetrahedral partition, we write $\triangle_A$ for the partition which results from applying the Alfeld split to each tetrahedron in $\triangle$.*

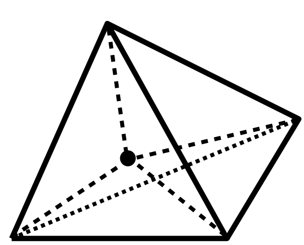


Fig. 16.3. The Alfeld split of a tetrahedron.

We now investigate the shape properties of the subtetrahedra in the Alfeld split.

Theorem 16.22. *Given a tetrahedron T, let T_A be the associated Alfeld refinement. Suppose θ_T and ϕ_T are the smallest solid angle and smallest face angle in T, respectively. Let θ_A and ϕ_A be the corresponding angles for T_A. Then*

$$\theta_A \ge \pi \left(\frac{\sin(\phi_T)}{8\sqrt{3}\pi^2} \right)^3 \theta_T^6, \qquad \phi_A \ge \frac{2}{3\pi} \phi_T. \tag{16.10}$$

Proof: Let $\widetilde{T}$ be one of the subtetrahedra of T_A. Suppose e_A is a longest edge of $\widetilde{T}$. Then since the volume of $\widetilde{T}$ is one-quarter the volume of T, comparing with the volumes of spherical wedges, we have

$$\frac{\theta_A \, |e_A|^3}{3} \ge V_{\widetilde{T}} = \frac{1}{4} V_T \ge \frac{\pi}{3} \rho_{_T}^3 = \frac{\pi |T|^3}{3\kappa_T^3},$$

where $\kappa_{_T}$ is the shape parameter associated with T. Using (16.1) and (16.8), this gives

$$\theta_A \ge \pi \left(\frac{|T|}{|e_A| \kappa_T} \right)^3 \ge \pi \left(\frac{\sin(\phi_T)\theta_T^2}{8\sqrt{3}\pi^2} \right)^3,$$

which reduces to the first inequality in (16.10).

To prove the second inequality in (16.10), we first consider the face angle ϕ_1 formed by the vertices $v_1, v_{_T}, v_2$. Let e be the line passing through $v_{_T}$ and v_3, and let w be the point where e intersects the face opposite v_3. Now there exists a point v^* on e such that the angle formed by v_1, v^*, v_2 is maximal. Moreover, for every point v on e, the angle formed by v_1, v, v_2 is decreasing as v moves away from v^* in either direction. We conclude that ϕ_1 is at least as large as the angle formed by v_1, v_3, v_2 or the one formed by v_1, w, v_2. The first of these angles is bounded below by ϕ, while by Lemma 4.17, the second is bounded below by $2\phi/3\pi$. This establishes the second inequality in (16.10) for all face angles at $v_{_T}$.

We now consider a typical face angle of a subtetrahedron at a vertex of T. Let α be the angle at v_1 of the triangle $\langle v_{_T}, v_1, v_2 \rangle$, β be the angle at v_1 of $\langle v_m, v_1, v_2 \rangle$, and γ be the angle at v_2 of $\langle v_1, v_2, v_m \rangle$, where $v_m = (v_3 + v_4)/2$ is the midpoint of the line segment $\langle v_3, v_4 \rangle$. It is easy to see that the plane passing through $v_1, v_2, v_{_T}$ contains both v_m and the barycenter v_F of the triangle $F := \langle v_2, v_3, v_4 \rangle$. Moreover $v_F = (2v_m + v_2)/3$. This implies that the area A_1 of the triangle $\langle v_1, v_2, v_F \rangle$ is two-thirds of the area A_2 of the triangle $\langle v_1, v_2, v_m \rangle$. Let $d_1 := |\langle v_1, v_m \rangle|$, $d_2 := |\langle v_2, v_m \rangle|$, $d := |\langle v_1, v_F \rangle|$, and $L := |\langle v_1, v_2 \rangle|$. Suppose now that $d_1 > d_2$. Then using $A_1 = dL \sin(\alpha)/2$, $A_2 = d_1 L \sin(\beta)/2$, and $A_1 = 2A_2/3$, we get

$$\alpha \ge \sin \alpha = \frac{2}{3} \frac{d_1}{d} \sin \beta \ge \frac{2}{3} \frac{d_1}{d_1 + d_2/3} \sin \beta = \frac{1}{\pi} \beta \ge \frac{1}{\pi} \phi.$$

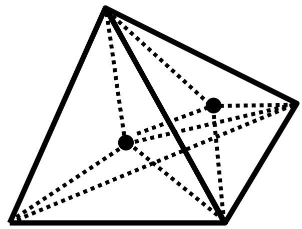


Fig. 16.4. A partial Worsey–Farin split of a tetrahedron.

If $d_1 < d_2$, we apply the law of sines to get

$$\alpha \geq \sin\alpha = \frac{2}{3}\frac{d_2}{d}\sin\gamma \geq \frac{2}{3}\frac{d_2}{d_1 + d_2/3}\sin\gamma = \frac{1}{\pi}\gamma \geq \frac{1}{\pi}\phi. \quad \square$$

16.7.2 The Worsey–Farin Refinement

Definition 16.23. *Suppose $\triangle$ is tetrahedral partition. For each tetrahedron T in $\triangle$, let v_T be the incenter of T, and let T_A be the corresponding Alfeld split of T. For each interior face F of $\triangle$, let v_F be the point where the line segment joining the incenters of the two tetrahedra sharing F intersect F. For each boundary face F, let v_F be the barycenter of F. Now for each face F, connect v_F to the vertices of F and to the centers v_T of each tetrahedron sharing the face F. We call the resulting refined partition $\triangle_{WF}$ the* Worsey–Farin refinement *of $\triangle$.*

The following lemma ensures that the Worsey–Farin refinement is well defined.

Lemma 16.24. *Suppose T and $\widetilde{T}$ are two tetrahedra sharing a face F. Then the line joining the incenters v_T and $v_{\tilde{T}}$ intersects F at a point v_F in the interior of F.*

Proof: Let $\ell := \langle v_T, w_T\rangle$ be a line segment perpendicular to F at the point w_T. Let $\widetilde{\ell} := \langle v_{\tilde{T}}, w_{\tilde{T}}\rangle$ be an analogous line segment through $v_{\tilde{T}}$. Note that both w_T and $w_{\tilde{T}}$ are in the interior of F. Since these two line segments are parallel, there exists a plane π containing both of them. Then the line from v_T to $v_{\tilde{T}}$ lies in π, and the point v_F where it intersects F must lie on the line segment $\langle w_T, w_{\tilde{T}}\rangle$. $\square$

Figure 16.4 illustrates a partial Worsey–Farin split of a tetrahedron T, where only one of the four tetrahedra in the Alfeld split T_A of T has been further split into three subtetrahedra. Clearly, if we apply the Worsey–Farin refinement to $\triangle$, then each tetrahedron T in $\triangle$ is split into twelve subtetrahedra. Moreover, each triangular face of $\triangle$ is split into three subtriangles similar to the bivariate Clough–Tocher split discussed in Section 4.8.1.

16.7.3 The Worsey–Piper Refinement

Definition 16.25. *Suppose $\triangle$ is tetrahedral partition. For each tetrahedron T in $\triangle$, choose a point v_T in the interior of T. For each edge of $\triangle$, choose a point v_e in the interior of e, and for each face F of $\triangle$, choose a point v_F in the interior of F. Split each tetrahedron T of $\triangle$ into twenty-four subtetrahedra as follows. First, connect v_T to each of the vertices of T, to each point v_e on an edge of T, and to each point v_F on a face of T. Then for each face F of T, connect v_F to each of the vertices of F as well as to each of the points v_e associated with edges e of F. We call the resulting collection $\triangle_{WP}$ of tetrahedra the* Worsey–Piper refinement $\triangle$.

The Worsey–Piper refinement splits each tetrahedron in $\triangle$ into 24 subtetrahedra. Each face of $\triangle$ is split into six subtriangles. For later use, we define a special restricted class of Worsey–Piper refinements which plays a role in building trivariate macro-element spaces, see Section 18.5.

Definition 16.26. *We say that the Worsey–Piper refinement $\triangle_{WP}$ of $\triangle$ is* proper *provided that*

1) *for every pair of tetrahedra T and $\widetilde{T}$ sharing a face F, the point v_F lies on the line segment $\langle v_T, v_{\tilde{T}} \rangle$,*
2) *for every collection $T_1, \ldots, T_m$ of tetrahedra sharing an edge e of $\triangle$, the points v_e and $v_{T_1}, \ldots, v_{T_m}$ are coplanar.*

In general Worsey–Piper refinements will not be proper. The following theorem gives some conditions on a given triangulation $\triangle$ which ensure that $\triangle_{WP}$ is proper.

Theorem 16.27. *Suppose $\triangle$ is such that for every tetrahedron in $\triangle$, the center of the sphere passing through its four vertices lies in the interior of T. In addition, suppose that*

1) *for each tetrahedron of $\triangle$, v_T is the center of the circumscribed sphere,*
2) *for each face F of $\triangle$, v_F is the center of the circumscribed circle,*
3) *for each edge e of $\triangle$, v_e is the midpoint of e.*

Then the associated Worsey–Piper refinement is proper.

Proof: Fix a tetrahedron T. Then since v_T is the circumcenter of T, for each face F of T, the line passing through v_T and perpendicular to F intersects F at the center v_F of the circumcircle of F. If F is shared by $\widetilde{T}$, the same holds for $\widetilde{T}$. It follows that the line segment connecting v_T and $v_{\tilde{T}}$ intersects the face F at v_F. For each edge e, the line segment $\langle v_T, v_e \rangle$ is perpendicular to e for each tetrahedron T containing the edge e. Thus, for all tetrahedra T sharing e, v_T are in the same plane. $\square$

The conditions of Theorem 16.27 are satisfied whenever $\triangle$ contains only acute tetrahedra, i.e., tetrahedra where the dihedral angle between any two adjoining faces is at most $\pi/2$.

16.7.4 The Alfeld-16 Refinement

Definition 16.28. *Given a tetrahedron T, let T_A be the Alfeld split of T into four subtetrahedra $T_1, \ldots, T_4$ based on the barycenter v_T of T. For each $i = 1, \ldots, 4$, let v_T^i be the barycenter of T_i. For each $i = 1, \ldots, 4$, connect v_T^i to the vertices of T_i to split T_i into four subtetrahedra. We call the resulting partition of T into 16 subtetrahedra the* Alfeld-16 split *of T and denote it by T_{A16}. If this split is applied to every tetrahedron in a partition $\triangle$, we call the resulting refined partition $\triangle_{A16}$ the* Alfeld-16 refinement *of $\triangle$.*

We can get bounds on the size of the angles of the subtetrahedra appearing in an Alfeld-16 split of a tetrahedron T by applying Theorem 16.22 twice.

16.7.5 Quasi-uniform Refinement

The refinement methods discussed above produce partitions whose angles are smaller than the angles of the original partition. Thus, if a given partition is repeatedly refined by one of these methods, the smallest angle of the resulting partitions will approach zero, and the shape parameters will become unbounded. On the other hand, there are important applications, including, for example, the finite-element method and various multiresolution methods, where it is important to refine a number of times while keeping the shape of the tetrahedra under control.

Ideally, we would like a uniform refinement method where the shape parameter associated with a refined tetrahedral partition is the same as the shape parameter associated with the original partition. There is a uniform refinement algorithm for triangulations, see Algorithm 4.23, but it is impossible to design such an algorithm for tetrahedral partitions. In this section we discuss a so-called quasi-uniform refinement algorithm.

Let $u_1 := (-\sqrt{2/3}, 0, 0)$, $u_2 := (0, -\sqrt{1/3}, 0)$, $u_3 := (0, \sqrt{1/3}, 0)$, and $u_4 := (0, 0, \sqrt{2/3})$. Then $T^* := \langle u_1, u_2, u_3, u_4 \rangle$ is the rhombic tetrahedron. This tetrahedron has four edges of length 1, and two edges of length $2/\sqrt{3}$. Let $u_{ij} := (u_i + u_j)/2$ be the midpoints of the edges of T^*. Then it is easy to see that the eight tetrahedra

$$
\begin{aligned}
T_1 &:= \langle u_1, u_{12}, u_{13}, u_{14} \rangle, & T_2 &:= \langle u_2, u_{12}, u_{23}, u_{24} \rangle, \\
T_3 &:= \langle u_3, u_{13}, u_{23}, u_{34} \rangle, & T_4 &:= \langle u_4, u_{14}, u_{24}, u_{34} \rangle, \\
T_5 &:= \langle u_{14}, u_{23}, u_{12}, u_{13} \rangle, & T_6 &:= \langle u_{14}, u_{23}, u_{12}, u_{24} \rangle, \\
T_7 &:= \langle u_{14}, u_{23}, u_{13}, u_{34} \rangle, & T_8 &:= \langle u_{14}, u_{23}, u_{24}, u_{34} \rangle,
\end{aligned}
\tag{16.11}
$$

form a uniform partition of T^*, where each of these eight tetrahedra is just a copy of T^* scaled by a factor of 1/2. The tetrahedra T_1, T_2, T_3, T_4 are obtained by cutting off corners of T^*. The other four tetrahedra share the edge $e := \langle u_{14}, u_{23} \rangle$. Given a positive integer n, let $t_1^*, \ldots, t_N^*$ be the $N := 8^n$ subtetrahedra of T obtained by applying the above uniform refinement process n times to the rhombic tetrahedron T^*.

We now describe an algorithm for refining an arbitrary tetrahedron T. Let $M(T)$ be the 3×3 matrix that describes the affine map taking T into T^*. We may assume that $M(T)$ is chosen so that a longest edge of T is mapped onto the edge $\langle u_2, u_3 \rangle$ of the rhombic tetrahedron, or equivalently, the midpoint of a longest edge of T is mapped to the origin.

Algorithm 16.29. *(Rhombic Refinement of a tetrahedron T)*

For $j = 1, \ldots, N$, set $t_j := M(T)^{-1} t_j^*$

We now show that for arbitrary T, the subtetrahedra in the rhombic refinement of T have at most three different shapes.

Lemma 16.30. *For any tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$, there exist κ_1, κ_2, κ_3 such that for every positive integer n, the shape parameter of every subtetrahedron in the n-th level refinement of T produced by Algorithm 16.29 is one of the three numbers κ_1, κ_2, or κ_3.*

Proof: Let κ_1 be the shape parameter of T, and let $t_1, \ldots, t_8$ be the eight subtetrahedra of T. For a list of their vertices, simply replace the u_i in (16.11) with v_i and the u_{ij} with $v_{ij} = (v_i + v_j)/2$. The four tetrahedra $t_1, \ldots, t_4$ are just scaled versions of T, and thus have the same shape parameter κ_1. We claim that t_5 and t_8 have a common shape parameter which we call κ_2. To see this, we observe that

$$t_8 = -t_5 + \frac{(v_1 + v_2 + v_3 + v_4)}{2},$$

i.e., t_8 can be obtained from t_5 by scaling it by -1 and translating by $(v_1 + v_2 + v_3 + v_4)/2$. To check this, it suffices to examine vertices:

$$\begin{aligned}
v_{23} &= -\frac{(v_1 + v_4)}{2} + \frac{(v_1 + v_2 + v_3 + v_4)}{2}, \\
v_{14} &= -\frac{(v_2 + v_3)}{2} + \frac{(v_1 + v_2 + v_3 + v_4)}{2}, \\
v_{12} &= -\frac{(v_3 + v_4)}{2} + \frac{(v_1 + v_2 + v_3 + v_4)}{2}, \\
v_{13} &= -\frac{(v_2 + v_4)}{2} + \frac{(v_1 + v_2 + v_3 + v_4)}{2}.
\end{aligned}$$

Similarly,

$$t_7 = -t_6 + \frac{(v_1 + v_2 + v_3 + v_4)}{2},$$

and thus t_6 and t_7 have a common shape parameter which we call κ_3. The next cycle of the refinement process divides each of the t_i into eight subtetrahedra. A similar calculation shows that each of these is a scaled and translated copy of one of the three types of tetrahedra $\{t_1, t_2, t_3, t_4\}$, $\{t_5, t_8\}$, and $\{t_6, t_7\}$. $\square$

Clearly, if we start with a rhombic tetrahedron T, then the algorithm produces a uniform refinement of T, i.e., all subtetrahedra are similar to T. Suppose now that $\triangle$ is an arbitrary tetrahedral partition. To refine $\triangle$ we can apply Algorithm 16.29 to each tetrahedron in $\triangle$. This process is well defined since the refinement process of Algorithm 16.29 uses the midpoints of the edges to split each face F of a tetrahedron T into four subtriangles which are all similar to F. Thus, if T and $\widetilde{T}$ share the face F, the refinements of T and $\widetilde{T}$ will split F in the same way. Now it is clear from the above discussion that this refinement process is quasi-uniform in the sense that the maximum of the shape parameters of tetrahedra in the n-th level refinement $\triangle_n$ of $\triangle$ is given by

$$\kappa_{\triangle_n} := \max_{T \in \triangle} \max_{1 \le i \le 3} \kappa_i(T), \tag{16.12}$$

where the $\kappa_1(T), \kappa_2(T), \kappa_3(T)$ are the shape parameters in Lemma 16.30 associated with T. The quantity on the right in (16.12) is independent of n, and in fact depends only on the original tetrahedral partition $\triangle$.

For some applications it is useful to have local refinement algorithms where some (but not all) of the tetrahedra in a partition $\triangle$ are split. A full treatment of such algorithms is beyond the scope of this book, but we list here three simple strategies:

1) (Edge splitting). Suppose $e := \langle v_1, v_2 \rangle$ is an edge of $\triangle$, and let u be its midpoint. Then for each tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ of $\triangle$ sharing the edge e, we connect u to the vertices v_3 and v_4 of T.

2) (Face splitting). Suppose $F := \langle v_1, v_2, v_3 \rangle$ is a triangular face of a tetrahedron in $\triangle$. Split F into four triangles using the midpoints of its edges. For each tetrahedron T sharing F, connect these points to the vertex of T not lying on F. Finally, for each remaining tetrahedron T sharing an edge e of F, split T by connecting the midpoint of e to the two vertices of T not lying on e.

3) (Tetrahedral splitting). Pick a tetrahedron T and split it into eight subtetrahedra using Algorithm 16.29. For each tetrahedron $\widetilde{T}$ sharing

a face F with T, connect the vertex of $\widetilde{T}$ not lying on F to each of the midpoints of the edges of F. Finally, for each remaining tetrahedron $\widetilde{T}$ sharing an edge e with T, connect the midpoint of e to the two vertices of $\widetilde{T}$ not on e.

16.8. Delaunay Tetrahedral Partitions

Given a set $\mathcal{V}$ of points in $\mathbb{R}^3$, let Ω be the associated convex hull. Suppose $\triangle$ is a tetrahedral partition of Ω with vertices at the points of $\mathcal{V}$, and let $\theta_\triangle$ be the minimal solid angle appearing in the tetrahedra in $\triangle$. Then we say that $\triangle$ is a maxmin-angle partition provided that there is no other tetrahedral partition $\widetilde{\triangle}$ of Ω with vertices $\mathcal{V}$ such that the minimal solid angle of the tetrahedra in $\widetilde{\triangle}$ is larger than $\theta_\triangle$. As in the bivariate case, it turns out that a max-min angle partition need not be unique, and finding one is equivalent to constructing a so-called Delaunay tetrahedral partition.

Definition 16.31. *Suppose $\triangle$ is a tetrahedral partition corresponding to a set of vertices $\mathcal{V}$. Then $\triangle$ is said to be a* Delaunay tetrahedral partition *provided that for every tetrahedron T in $\triangle$, there is no vertex $v \in \mathcal{V}$ lying in the interior of the ball passing through the vertices of T.*

We do not have space here to go further into the theory and practice of constructing Delaunay tetrahedral partitions, see e.g. [PreS85, Ede87, Joe91] for more details and references.

16.9. Remarks

Remark 16.1. Suppose T is a regular tetrahedron with edges of length a. Then the surface area of T is $\sqrt{3}a^2$ while the volume is $\sqrt{2}a^3/12$. All dihedral angles of T are equal to $\arccos(1/3)$, while the face angles are equal to $\pi/3$. The solid angles of T are $3\arccos(1/3) - \pi$, and the inradius of T is $\sqrt{6}a/12$.

Remark 16.2. Solid angles are measured in steradians. Note that the sum of the solid angles in a tetrahedron depends on the shape of the tetrahedron, but is bounded by 2π. For a regular tetrahedron, the sum is $4(3\arccos(1/3) - \pi)$ which is somewhat smaller than 2π. There are tetrahedra where the sum is arbitrary close to 2π. Indeed, as v_1 approaches the face F in Example 16.3, the solid angle at v_1 approaches 2π which is the area of a hemisphere. The solid angle at a vertex of a tetrahedron is equal to the sum of the dihedral angles at that vertex minus π.

Remark 16.3. There exist tetrahedral partitions which are homeomorphic to a ball, but are not shellable. The first such example is due to Rudin [Rud58], and involves 41 tetrahedra. The smallest known example contains only five tetrahedra.

Remark 16.4. Let Ω be a given domain, possibly with holes and cavities. Then an important problem in using splines to solve partial differential equations is how to construct a tetrahedral partition $\triangle$ with good shape properties. This is called the grid generation problem, and has been heavily studied in the finite-element literature, see e.g. [Ede01].

Remark 16.5. It is not clear how small the angles in the Worsey–Farin refinement $\triangle_{WF}$ of a tetrahedral partition $\triangle$ can be as compared to those in $\triangle$. We conjecture that there exist constants K_1 and K_2 depending on $\theta_\triangle$ and $\phi_\triangle$ such that

$$\theta_{WF} \geq K_1 \theta_\triangle, \qquad \phi_{WF} \geq K_2 \phi_\triangle, \tag{16.13}$$

where $\theta_\triangle$ and $\phi_\triangle$ are the smallest solid and face angles in $\triangle$, respectively, and θ_{WF} and ϕ_{WF} are the corresponding angles for $\triangle_{WF}$.

16.10. Historical Notes

As with triangulations, there are a number of different definitions of a tetrahedral partition in the literature. The concept of shellable tetrahedral partition arose in topology, see [Rud58] and references therein. The terminology *regular tetrahedral partition* also seems to have different meanings to different people. Here we have adopted a definition which is general enough to include all of the partitions that one is likely to use in practice, including those with holes and cavities.

The combinatorial formulae in Theorems 16.13 and 16.14 relating numbers of vertices, edges, faces, and tetrahedra in a tetrahedral partition are well known in the finite-element community, but it is perhaps less well known that they do not hold for arbitrary tetrahedral partitions. The more general formulae in Theorem 16.14 for domains with holes and cavities were claimed in [EwiFG70], but the inductive argument given there does not work without shellability. We have tailored our definition of regularity to allow us to give a rigorous proof of this theorem.

The Alfeld, Worsey–Farin, Worsey–Piper, and Alfeld-16 refinements described in Section 16.7 were introduced in [Alf84b], [WorF87], [WorP88], and [AlfS05b], respectively, where they were used to construct certain trivariate macro-element spaces.

It is a nontrivial problem to construct quasi-uniform refinement algorithms in the tetrahedral case. The algorithm described in Section 16.7.5 is based on [LiuJ96]. For further references, see [Ong94, LiuJ95, LiuJ96].

Trivariate Splines

In this chapter we develop the basic properties of trivariate splines, i.e., smooth piecewise polynomial functions defined over tetrahedral partitions in $\mathbb{R}^3$.

17.1. C^0 Trivariate Spline Spaces

Suppose $\triangle$ is a tetrahedral partition of a bounded domain $\Omega \in \mathbb{R}^3$. Given an integer $d \geq 0$, let $\mathcal{P}_d$ be the space of trivariate polynomials of degree d. Then we define the associated space of C^0 polynomial splines of degree d over $\triangle$ as

$$\mathcal{S}_d^0(\triangle) := \{s \in C^0(\Omega),\ s|_T \in \mathcal{P}_d,\ \text{for all } T \in \triangle\}.$$

For each tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ in $\triangle$, let

$$\mathcal{D}_{d,T} := \Big\{ \xi_{ijkl}^T := \frac{iv_1 + jv_2 + kv_3 + lv_4}{d} \Big\}_{i+j+k+l=d}$$

be the set of domain points associated with T as introduced in (15.14). We define the set of domain points associated with $\triangle$ as

$$\mathcal{D}_{d,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{d,T},$$

where if a domain point lies in more than one tetrahedron, it is included just once in $\mathcal{D}_{d,\triangle}$.

We now show how to use the set $\mathcal{D}_{d,\triangle}$ to parametrize the space $\mathcal{S}_d^0(\triangle)$. For each tetrahedron $T \in \triangle$ and each $\xi := \xi_{ijkl}^T \in \mathcal{D}_{d,T}$, let $B_\xi^d := B_{ijkl}^d$ be the associated Bernstein basis polynomial, and let c_ξ be a real number. Then

$$s|_T := \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^d$$

defines a unique polynomial in $\mathcal{P}_d$. Moreover, if T and $\widetilde{T}$ are two tetrahedra sharing a face F, then $s|_T$ and $s|_{\widetilde{T}}$ restricted to F are both bivariate polynomials and have the same set of coefficients. It follows that $s|_T$ and $s|_{\widetilde{T}}$ join together with C^0 smoothness across F. Thus, assigning values to $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ uniquely defines a spline $s \in \mathcal{S}_d^0(\triangle)$. Conversely, for each spline in this space there is a unique set of coefficients. We call these the B-coefficients of s. Counting the number of points in $\mathcal{D}_{d,\triangle}$, we immediately get the following result.

Theorem 17.1. *Every spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely defined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$, and the dimension of $\mathcal{S}_d^0(\triangle)$ is*

$$n := n_{\scriptscriptstyle V} + (d-1)n_{\scriptscriptstyle E} + \binom{d-1}{2} n_{\scriptscriptstyle F} + \binom{d-1}{3} n_{\scriptscriptstyle T},$$

where $n_{\scriptscriptstyle V}, n_{\scriptscriptstyle E}, n_{\scriptscriptstyle F}, n_{\scriptscriptstyle T}$ are the number of vertices, edges, faces, and tetrahedra in $\triangle$, respectively.

To *store* a spline in $\mathcal{S}_d^0(\triangle)$, it suffices to store its set of B-coefficients. To *evaluate* a spline s at a point v, it suffices to find the tetrahedron in which v lies, and then apply the de Casteljau Algorithm 15.11 to the coefficients of the polynomial $s|_T$. As in the bivariate case, it is easy to construct locally supported basis functions for $\mathcal{S}_d^0(\triangle)$.

Definition 17.2. *For each $\xi \in \mathcal{D}_{d,\triangle}$, let ψ_ξ be the spline in $\mathcal{S}_d^0(\triangle)$ that satisfies*

$$\gamma_\eta \psi_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle},$$

where γ_η is a linear functional which picks off the coefficient associated with the domain point η.

Here we have not explicitly constructed the linear functionals γ_η appearing in Definition 17.2, but it is a straightforward process based on dual linear functionals for the Bernstein basis polynomials, see Remark 15.2. Note that for each $\xi \in \mathcal{D}_{d,\triangle}$, the basis spline ψ_ξ in Definition 17.2 has all zero coefficients except for c_ξ which is 1. Since for each tetrahedron the associated Bernstein basis polynomials are nonnegative, it follows immediately that

$$\psi_\xi(v) \ge 0, \qquad \text{all } v \in \Omega.$$

We claim that the ψ_ξ also have small support. In particular, since ψ_ξ is identically zero on all tetrahedra which do not contain ξ, it follows that the support of ψ_ξ is as follows:

1) a single tetrahedron T, if ξ is in the interior of T,
2) the union of the two tetrahedra sharing F, if ξ is in the interior of a face F of $\triangle$,
3) the union of all tetrahedra containing e, if ξ is in the interior of an edge e of $\triangle$,
4) the union of all tetrahedra sharing the vertex v, if $\xi = v$.

The following result can be proved in the same way as Theorem 5.5.

Theorem 17.3. *The set of splines* $\{\psi_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ *forms a basis for* $\mathcal{S}_d^0(\triangle)$. *Moreover, these basis functions form a partition of unity on* Ω, *i.e.,*

$$\sum_{\xi \in \mathcal{D}_{d,\triangle}} \psi_\xi \equiv 1.$$

17.2. Spaces of Smooth Splines

Suppose $\triangle$ is a tetrahedral partition of a bounded domain $\Omega \in \mathbb{R}^3$. Then given $0 \le r \le d$, we define the associated space of C^r polynomial splines of degree d and smoothness r over $\triangle$ as

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega),\ s|_T \in \mathcal{P}_d,\ \text{for all } T \in \triangle\}. \tag{17.1}$$

For later use, we also introduce certain supерspline subspaces of $\mathcal{S}_d^r(\triangle)$. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$, respectively. Fix $0 \le r \le \mu \le \rho$. Then we define

$$\begin{aligned}\mathcal{S}_d^{r,\rho,\mu}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) :\ & s \in C^\rho(v),\ \text{all } v \in \mathcal{V}, \\ & s \in C^\mu(e),\ \text{all } e \in \mathcal{E}\}.\end{aligned} \tag{17.2}$$

Here $s \in C^\rho(v)$ means that all polynomial pieces $s|_T$ associated with tetrahedra T sharing the vertex v have common derivatives up to order ρ at v. Similarly, $s \in C^\mu(e)$ means that all polynomial pieces $s|_T$ associated with tetrahedra T sharing the edge e have common derivatives up to order μ at all points along the edge e.

We now introduce a much larger class of splines which includes the spaces $\mathcal{S}_d^r(\triangle)$ and $\mathcal{S}_d^{r,\rho,\mu}(\triangle)$. First we need some additional notation. Suppose T and $\widetilde{T}$ are two tetrahedra in $\triangle$ sharing a face F. In particular, suppose the vertices of T are v_1, v_2, v_3, v_4 in canonical order, while those of $\widetilde{T}$ are v_5, v_2, v_4, v_3 in canonical order. Then the common face is $F := \langle v_2, v_3, v_4\rangle$. Fix $0 \le m \le d$, and suppose $s \in \mathcal{S}_d^0(\triangle)$. Let c_{ijkl} and $\tilde{c}_{ijkl}$ be the B-coefficients of $s|_T$ and $s|_{\widetilde{T}}$, respectively. Then for all i, j, k with $i + j + k = d - m$, we define

$$\tau_{ijk}^{F,m} s := c_{mijk} - \sum_{\nu+\mu+\kappa+\delta=m} \tilde{c}_{\nu,i+\mu,k+\kappa,j+\delta}\, \tilde{B}_{\nu\mu\kappa\delta}^m(v_1), \tag{17.3}$$

where the $\tilde{B}_{\nu\mu\kappa\delta}^m$ are the Bernstein basis polynomials of degree m associated with the tetrahedron $\widetilde{T}$. We call $\tau_{ijk}^{F,m}$ a smoothness functional of order m, and refer to ξ_{mijk}^T as the tip of the smoothness functional. It is important to note that we are associating $\tau_{ijk}^{F,m}$ with the oriented face F of T, and not with the oriented face $\widetilde{F} := \langle v_2, v_4, v_3\rangle$ of $\widetilde{T}$, even though these two faces represent a common triangular face of $\triangle$.

Definition 17.4. *Given a set $\mathcal{T}$ of linear functionals of the form (17.3) associated with oriented faces of $\triangle$, we define the corresponding space of smooth splines as*

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) := \{s \in \mathcal{S}_d^0(\triangle) : \tau s = 0, \ \text{all } \tau \in \mathcal{T}\}.$$

Following the proof of Theorem 2.28, we get the following result giving conditions for two pieces of a spline defined on adjoining tetrahedra to join with C^r smoothness across the common face.

Theorem 17.5. *Let $T := \langle v_1, v_2, v_3, v_4\rangle$ and $\widetilde{T} := \langle v_5, v_2, v_4, v_3\rangle$ be two tetrahedra with common face $F := \langle v_2, v_3, v_4\rangle$. Then for any $s \in \mathcal{S}_d^0(\triangle)$, the two polynomial pieces $s|_T$ and $s|_{\widetilde{T}}$ join with C^r smoothness across the face F if and only if*

$$\tau_{ijk}^{F,m} s = 0, \qquad \text{all } i + j + k = d - m \ \text{ and } \ 0 \le m \le r.$$

Since smoothness conditions on a spline $s \in \mathcal{S}_d^0(\triangle)$ are just linear side conditions on the vector c of B-coefficients of s, it is clear that for any given set $\mathcal{T}$ of smoothness conditions, there is a matrix $A := A_{\mathcal{T}}$ such that

$$\mathcal{S}_d^{\mathcal{T}}(\triangle) = \{s \in \mathcal{S}_d^0(\triangle) : Ac = 0\}. \tag{17.4}$$

Clearly, the matrix A is of size $m \times n$, where m is the number of smoothness conditions in $\mathcal{T}$, and n is the dimension of $\mathcal{S}_d^0(\triangle)$. It is also clear that A is a relatively sparse matrix, since a typical C^r smoothness condition across a face involves only $\binom{r+3}{3} + 1$ coefficients. Thus, for example, a C^1 condition involves at most five nonzero coefficients, so the corresponding row in the matrix A has only five nonzero entries. The proof of the following analog of Theorem 5.11 is straightforward.

Theorem 17.6. *Let $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ be the space of smooth splines defined in (17.4) corresponding to a matrix A, and let $n = \dim \mathcal{S}_d^0(\triangle)$. Then the dimension of $\mathcal{S}_d^{\mathcal{T}}(\triangle)$ is equal to $n - k$, where k is the rank of A.*

17.3. Minimal Determining Sets

Given a tetrahedral partition $\triangle$, suppose $\mathcal{S} := \mathcal{S}_d^{\mathcal{T}}(\triangle)$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$ that is defined by enforcing some set of smoothness conditions $\mathcal{T}$ across the faces of $\triangle$ as described in the previous section. As an aid to analyzing the dimension and structure of such spaces, we now introduce the analog of the minimal determining sets used earlier in our study of bivariate spline spaces.

Definition 17.7. *Suppose $\Gamma \subseteq \mathcal{D}_{d,\triangle}$ is such that if we set the coefficients $\{c_\xi\}_{\xi\in\Gamma}$ of a spline $s \in \mathcal{S}$ to zero, then all other coefficients must also be zero, i.e., $s \equiv 0$. Then we say that Γ is a* determining set *for $\mathcal{S}$. If $\mathcal{M}$ is a determining set for a spline space $\mathcal{S}$, and if $\mathcal{M}$ has the smallest cardinality among all possible determining sets for $\mathcal{S}$, then we call $\mathcal{M}$ a* minimal determining set (MDS) *for $\mathcal{S}$.*

Clearly, for any $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, the set of domain points $\mathcal{D}_{d,\triangle}$ is always a determining set for $\mathcal{S}$. But for any spline space $\mathcal{S}$ satisfying at least one additional smoothness condition, there will be determining sets with fewer points than the number of points in $\mathcal{D}_{d,\triangle}$. In general, there will be more than one minimal determining set corresponding to a given spline space $\mathcal{S}$. The proof of the following result is very similar to the proof of Theorem 5.13 in the bivariate case.

Theorem 17.8. *Suppose $\mathcal{S}$ is an m-dimensional linear subspace of $\mathcal{S}_d^0(\triangle)$, and suppose that Γ is a determining set for $\mathcal{S}$. Then $\#\,\Gamma \ge m$. Moreover, if $\mathcal{M}$ is a determining set for $\mathcal{S}$ with $\#\mathcal{M} = m$, then $\mathcal{M}$ is minimal.*

In general, it is a nontrivial task to construct minimal determining sets for spline spaces. Indeed, in practice we often don't know the dimension of $\mathcal{S}$, and so don't even know how many domain points to put in a minimal determining set.

Definition 17.9. *If $\mathcal{M}$ is a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, we say that it is* consistent *provided that if we fix the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \mathcal{S}$, then all of the coefficients of s are determined, and all smoothness conditions defining $\mathcal{S}$ are satisfied with these coefficients.*

The following theorem provides an important tool for constructing minimal determining sets. It is a direct analog of Theorem 5.15, and can be proved in the same way.

Theorem 17.10. *Suppose $\mathcal{M}$ is a consistent determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then $\mathcal{M}$ is minimal.*

Given a minimal determining set $\mathcal{M}$ for $\mathcal{S}$, suppose we assign values to the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$. Then for every $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$, the coefficient c_η can be computed from the known coefficients by using the smoothness conditions. We say that c_η *depends on* c_ξ, $\xi \in \mathcal{M}$, if changing the value of c_ξ also causes the value of c_η to change. We write

$$\Gamma_\eta := \{\xi \in \mathcal{M} : \, c_\eta \text{ depends on } c_\xi\}.$$

Minimal determining sets $\mathcal{M}$ are especially useful if they are local and stable in the sense that

1) for each $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$, c_η can be computed from a small set of coefficients $\{c_\xi\}_{\xi\in\Gamma_\eta}$, where the domain points in Γ_η are near η,
2) for each $\eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}$, the size of c_η is comparable to the size of the coefficients in the set Γ_η.

We now make these properties more precise. Let $\theta_\triangle$ be the smallest solid angle in the tetrahedra of $\triangle$, and $\phi_\triangle$ the smallest of the face angles in the tetrahedra of $\triangle$, see Definition 16.2.

Definition 17.11. *Suppose $\mathcal{M}$ is a minimal determining set for a linear space of trivariate splines $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then we say that $\mathcal{M}$ is* local *provided that there exists an integer ℓ not depending on $\triangle$ such that*

$$\Gamma_\eta \subseteq star^\ell(T_\eta), \qquad all\ \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}, \tag{17.5}$$

where T_η is a tetrahedron containing η. We say that $\mathcal{M}$ is stable *provided that there exists a constant K depending only on ℓ, $\theta_\triangle$, and $\phi_\triangle$ such that*

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \qquad all\ \eta \in \mathcal{D}_{d,\triangle} \setminus \mathcal{M}. \tag{17.6}$$

For the meaning of $\mathrm{star}^\ell(v)$, see Definition 16.7. We shall see in Chapter 18 that several trivariate macro-element spaces have stable local bases.

17.4. Approximation Power of Trivariate Spline Spaces

In this section we give a general result on the approximation power of trivariate spline spaces which have stable local minimal determining sets. For approximation results on some specific trivariate spline spaces with this property, see Chapter 18.

Suppose $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a spline space associated with a tetrahedral partition $\triangle$ of a domain Ω, and suppose $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}$ as defined in the previous section. We now construct an explicit linear operator Q mapping $L_1(\Omega)$ into $\mathcal{S}$ which provides full approximation power, see Remark 17.4.

Definition 17.12. *Suppose $f \in L_1(\Omega)$. For each domain point ξ in $\mathcal{M}$, let T_ξ be a tetrahedron containing ξ, and let $F_\xi f$ be the averaged Taylor polynomial of degree d associated with the largest ball contained in T_ξ. Let γ_ξ be a linear functional which for any spline $s \in \mathcal{S}_d^0(\triangle)$ picks off the B-coefficient corresponding to ξ, and set $c_\xi := \gamma_\xi(F_\xi f)$. Let $\{c_\eta\}_{\eta\in\mathcal{D}_{d,\triangle}\setminus\mathcal{M}}$ be such that the piecewise polynomial Qf defined by*

$$Qf|_T := \sum_{\eta\in\mathcal{D}_{d,T}} c_\eta B_\eta^T, \qquad all\ T \in \triangle,$$

satisfies all smoothness conditions required to ensure $Qf \in \mathcal{S}$. We call Q the quasi-interpolation operator *associated with the spline space $\mathcal{S}$ and minimal determining set $\mathcal{M}$.*

The quasi-interpolation operator Q is well defined since $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$, which guarantees that once we set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$, all remaining coefficients of Qf are uniquely determined from the smoothness conditions. Moreover, by the assumption that $\mathcal{M}$ is local and stable, we know that for each $\eta \in \mathcal{D}_{d,\triangle}\backslash\mathcal{M}$, there exists a set $\Gamma_\eta \subseteq \mathrm{star}^\ell(T_\eta)$ such that c_η can be computed from the coefficients $\{c_\xi\}_{\xi\in\Gamma_\eta}$ by smoothness conditions. The stability of $\mathcal{M}$ guarantees that the size of $|c_\eta|$ is controlled in the sense that (17.6) holds, where the constant K depends only on ℓ and the smallest solid and face angles $\theta_\triangle$ and $\phi_\triangle$ associated with $\triangle$.

Theorem 17.13. *The above process defines a linear operator Q mapping $L_1(\Omega)$ onto $\mathcal{S}$ with $Qs = s$ for all splines $s \in \mathcal{S}$. Moreover, for every tetrahedron $T \in \triangle$ and any $1 \le q \le \infty$,*

$$\|Qf\|_{q,T} \le K\,\|f\|_{q,\Omega_T}, \qquad \text{all } f \in L_1(\Omega_T), \tag{17.7}$$

where $\Omega_T := \mathrm{star}^\ell(T)$ and ℓ is the integer constant in (17.5). Here K depends only on d, ℓ, and the angles $\theta_\triangle$ and $\phi_\triangle$.

Proof: The linearity of Q follows from its definition. By Lemma 15.33, the averaged Taylor polynomial of degree d associated with a polynomial $p \in \mathcal{P}_d$ is p itself, and it follows that $Qs = s$ for all $s \in \mathcal{S}$. We now establish (17.7) in the case $1 \le q < \infty$. The case $q = \infty$ is similar and simpler. Let $\mathcal{M}$ be a stable local minimal determining set for $\mathcal{S}$. Then for every $\xi \in \mathcal{M}$,

$$\begin{aligned}
|c_\xi| = |\gamma_\xi(F_\xi f)| &\le K_1\,\|F_\xi f\|_{\infty,T_\xi}\\
&\le K_1K_2\,V_{T_\xi}^{-1/q}\|F_\xi f\|_{q,T_\xi}\\
&\le K_1K_2K_3\,V_{T_\xi}^{-1/q}\|f\|_{q,T_\xi},
\end{aligned}$$

where T_ξ is a tetrahedron containing ξ and V_{T_ξ} is its volume. Here K_1 is the constant appearing in (15.15), K_2 is the constant appearing in (15.2), and K_3 is the constant appearing in (15.43). Note that K_1, K_2 depend only on d, while K_3 depends on d and the shape parameter of T_ξ, which by Lemma 16.4 can be bounded in terms of $\theta_\triangle$ and $\phi_\triangle$. Now fix $T \in \triangle$. Then using (17.6), we conclude that

$$|c_\eta| \le K_4\,V_{T_{min}}^{-1/q}\|f\|_{q,\Omega_T}, \qquad \eta \in \mathcal{D}_{d,T}, \tag{17.8}$$

where T_{min} is a tetrahedron in Ω_T with minimal volume. The fact that the Bernstein basis polynomials form a partition of unity implies

$$\|Qf\|_{q,T} = \left[\int_T \Big|\sum_{\eta\in\mathcal{D}_{d,T}} c_\eta B_\eta^T\Big|^q\right]^{1/q} \le {V_T}^{1/q} \max_{\eta\in\mathcal{D}_{d,T}} |c_\eta|. \tag{17.9}$$

To complete the proof, we insert (17.8) in (17.9) and use Lemma 16.18 to bound the ratio $V_T/V_{T_{min}}$ by a constant depending on the angles. □

We now give a local approximation result for Q.

Theorem 17.14. *Given a tetrahedron $T \in \triangle$, let $\Omega_T := star^\ell(T)$, where ℓ is the integer constant in (17.5). Suppose $f \in W_q^{m+1}(\Omega_T)$ for some $0 \le m \le d$ and $1 \le q \le \infty$. Then*

$$\|D^\alpha(f - Qf)\|_{q,T} \le K\,|T|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega_T}, \tag{17.10}$$

for all $0 \le |\alpha| \le m$. If Ω_T is convex, the constant K depends only on d, ℓ, and the smallest angles θ_Ω and ϕ_Ω associated with $\triangle$. If Ω_T is not convex, K also depends on the Lipschitz constant of the boundary of Ω_T.

Proof: By Theorem 15.35, there exists a polynomial $p \in \mathcal{P}_m$ depending on f so that

$$\|D^\beta(f - p)\|_{q,\Omega_T} \le K_1\,|\Omega_T|^{m+1-|\beta|}\,|f|_{m+1,q,\Omega_T}, \tag{17.11}$$

for all $0 \le |\beta| \le m$. The constant K_1 depends on d and the shape parameter κ_Ω of (15.42), and also on the Lipschitz constant of the boundary of Ω_T when Ω_T is nonconvex. The results of Section 16.6 coupled with Lemma 16.4 show that κ_Ω can be bounded in terms of the angles only. Now since Q reproduces polynomials of degree d,

$$\|D^\alpha(f - Qf)\|_{q,T} \le \|D^\alpha(f-p)\|_{q,T} + \|D^\alpha Q(f-p)\|_{q,T}.$$

By (17.11) with $\beta = \alpha$ and the fact that $|\Omega_T| \le K_2|T|$ by Lemmas 16.18 and 16.19, it suffices to consider the second term. Since the restriction of $Q(f-p)$ to T is a polynomial of degree d, we can use the Markov inequality (15.3) to estimate its derivatives. Then by (17.7),

$$\|D^\alpha Q(f-p)\|_{q,T} \le \frac{K_3}{\rho_T^{|\alpha|}}\,\|Q(f-p)\|_{q,T} \le \frac{K_3K_4}{\rho_T^{|\alpha|}}\,\|f-p\|_{q,\Omega_T},$$

where ρ_T is the radius of the largest ball contained in T. Now Lemma 16.4 implies $|T| \le K_5\rho_T$, and combining the above we immediately get (17.10). When Ω_T is not convex the dependence of the constant in (17.10) on the Lipschitz constant of the boundary of Ω_T enters in the use of the Stein extension theorem, see Section 15.17. □

We can now give a global version of this approximation result. Let $\triangle$ be a tetrahedral partition of a set Ω. Suppose $\mathcal{M}$ is a stable local minimal determining set for a spline space $\mathcal{S}$, and suppose Q is the quasi-interpolation operator of Definition 17.12. Let $\theta_\triangle$ and $\phi_\triangle$ be the smallest solid and face angles in the tetrahedra of $\triangle$, respectively.

Theorem 17.15. *There exists a constant K such that if $f \in W_q^{m+1}(\Omega)$ for some $0 \le m \le d$ and $1 \le q \le \infty$, then*

$$\|D^\alpha(f - Qf)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega}, \tag{17.12}$$

for all $0 \le |\alpha| \le m$. If Ω is convex, the constant K depends only on d, ℓ, $\theta_\triangle$, and $\phi_\triangle$. If Ω is not convex, K also depends on the Lipschitz constant of the boundary of Ω.

Proof: For $q = \infty$, (17.12) follows immediately from (17.10) by taking the maximum over all tetrahedra T in $\triangle$. To get the result for $q < \infty$, we take the q-th power of both sides of (17.10) and sum over all tetrahedra in $\triangle$. Since Ω_T contains other tetrahedra besides T, some tetrahedra appear more than once in the sum on the right. However, a given tetrahedron T_R appears on the right only if it is associated with a tetrahedron T_L on the left which lies in $\mathrm{star}^\ell(T_R)$, where ℓ is the constant in the localness of $\mathcal{M}$. But then Lemma 16.17 implies that there is a constant K depending only on the angles such that T_R enters at most K times on the right, and (17.12) follows. $\square$

17.5. Stable Local Bases

In this section we show how minimal determining sets can be used to construct bases for spline spaces. We emphasize that this construction is mostly for theoretical completeness. We do not need these bases to establish results on the approximation power of splines. Moreover, we do not advocate their use in computation. It is almost always more convenient to work directly with the B-representation.

Theorem 17.16. *Suppose $\mathcal{M}$ is a minimal determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then for each $\xi \in \mathcal{M}$, there is a unique spline $\psi_\xi \in \mathcal{S}$ such that*

$$\gamma_\eta \psi_\xi = \delta_{\eta\xi}, \qquad \text{all } \eta \in \mathcal{M}, \tag{17.13}$$

where γ_η is a functional that picks off the B-coefficient corresponding to η. Moreover, $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ is a basis for $\mathcal{S}$ which we call the $\mathcal{M}$-basis of $\mathcal{S}$.

Proof: By Theorem 17.8, the cardinality of $\mathcal{M}$ is equal to the dimension of $\mathcal{S}$, and so it suffices to show that the splines ψ_ξ are linearly independent. But this follows immediately from (17.13). $\square$

Following the proof of Theorem 5.21, it is relatively easy to show that if the space $\mathcal{S}$ has a *stable local* minimal determining set $\mathcal{M}$, then the corresponding $\mathcal{M}$-basis is local and stable. Let $\theta_\triangle$ and $\phi_\triangle$ be the smallest angles associated with $\triangle$ as in Theorem 17.15.

Theorem 17.17. *Suppose $\mathcal{M}$ is a stable local minimal determining set for the space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ of trivariate splines, and let $\Psi := \{\psi_\xi\}_{\xi \in \mathcal{M}}$ be the $\mathcal{M}$-basis for $\mathcal{S}$ described in Theorem 17.16. Then Ψ is a* stable local basis *for $\mathcal{S}$ in the sense that for all $\xi \in \mathcal{M}$,*

1) $\|\psi_\xi\|_\Omega \le K$,
2) $\sigma(\psi_\xi) := \operatorname{supp} \psi_\xi \subseteq \operatorname{star}^\ell(T_\xi)$, *where T_ξ is a tetrahedron containing ξ.*

Here ℓ is the integer constant in (17.5), and K is a constant depending only on ℓ, $\theta_\triangle$, and $\phi_\triangle$.

As in the bivariate case (cf. Theorem 5.22), the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi \in \mathcal{M}}$ for $\mathcal{S}$ can always be renormed to form a stable basis for $\mathcal{S}$ in the q-norm for any $1 \le q \le \infty$.

Theorem 17.18. *Let $\Psi_q := \{\psi_{\xi,q} := V_{T_\xi}^{-1/q} \psi_\xi\}_{\xi \in \mathcal{M}}$, where for each ξ, T_ξ is a tetrahedron containing ξ and V_{T_ξ} is its volume. Then*

$$K_1 \, \|a\|_q \;\le\; \Big\| \sum_{\xi \in \mathcal{M}} a_\xi \psi_{\xi,q} \Big\|_q \;\le\; K_2 \, \|a\|_q, \tag{17.14}$$

for all choices of the coefficient vector $a = (a_\xi)_{\xi \in \mathcal{M}}$. The constants K_1 and K_2 depend only on d, ℓ, $\theta_\triangle$, and $\phi_\triangle$.

17.6. Nodal Minimal Determining Sets

So far we have concentrated on parametrizing trivariate spline spaces $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ using the Bernstein–Bézier form, i.e., in terms of B-coefficients. However, at times it is more useful to parametrize spline spaces in terms of so-called nodal parameters, also called degrees of freedom.

For any multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, we write $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$. Suppose $\mathcal{N} = \{\lambda_i\}_{i=1}^n$ is a set of linear functionals of the form

$$\lambda_i := \varepsilon_{u_i} \sum_{|\alpha| \le m_i} a_i^\alpha D^\alpha, \qquad i = 1, \ldots, n,$$

where ε_{u_i} is point evaluation at the point u_i. We say that λ_i is a nodal functional, and call u_i its carrier.

Definition 17.19. *Suppose $\mathcal{N}$ is a set of nodal functionals defined on a trivariate spline space $\mathcal{S}$, and suppose that if $s \in \mathcal{S}$ with $\lambda s = 0$ for all $\lambda \in \mathcal{N}$, then $s \equiv 0$. Then we call $\mathcal{N}$ a* nodal determining set (NDS) *for $\mathcal{S}$. If there is no smaller nodal determining set for $\mathcal{S}$, then we say that $\mathcal{N}$ is a* nodal minimal determining set (NMDS) *for $\mathcal{S}$.*

The analog of Theorem 17.8 holds, i.e., a nodal determining set $\mathcal{N}$ is minimal if and only if $\dim \mathcal{S} = \#\mathcal{N}$. To check that a set $\mathcal{N}$ is a nodal minimal determining set, it is enough to show that if we set $\{\lambda s\}_{\lambda \in \mathcal{N}}$, then all B-coefficients of s are determined in such a way that all smoothness conditions associated with $\mathcal{S}$ are satisfied.

Example 17.20. Suppose $\triangle$ is a tetrahedral partition, and let $\mathcal{V}$ and $\mathcal{F}$ be the sets of vertices and faces of $\triangle$, respectively. Let

$$\mathcal{S} := \{s \in \mathcal{S}_3^0(\triangle) : \; s \in C^1(v) \text{ all } v \in \mathcal{V}\}.$$

For each face $F \in \mathcal{F}$, let u_F be the barycenter of F. Then

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{F \in \mathcal{F}} \mathcal{N}_F$$

is a nodal minimal determining set for $\mathcal{S}$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha| \le 1}$,
2) $\mathcal{N}_F := \{\varepsilon_{u_F}\}$.

Discussion: Let $s \in \mathcal{S}$. For each $v \in \mathcal{V}$, the values $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$ determine all B-coefficients of $s \in \mathcal{S}_3^0(\triangle)$ corresponding to domain points in the ball $D_1(v)$ in such a way that $s \in C^1(v)$, see Theorem 15.17. Then for each face F of $\triangle$, all B-coefficients of $s|_F$ are already determined except for the coefficient corresponding to the domain point ξ_{111}^F. But this is given by the equation

$$c_{111}^F = \frac{1}{B_{111}^F(u_F)} \Big[s(u_F) - \sum_{\substack{i+j+k=3 \\ (i,j,k) \ne (1,1,1)}} c_{ijk}^F B_{ijk}^F(u_F) \Big],$$

see also Theorem 15.24. This automatically ensures that s is continuous across the faces of $\triangle$. At this point, all B-coefficients of s have been determined, and s satisfies all smoothness conditions for s to belong to $\mathcal{S}$, and thus $\mathcal{N}$ is a nodal minimal determining set. It follows that $\dim \mathcal{S} = 4n_V + n_F$, where n_V is the number of vertices of $\triangle$ and n_F is the number of faces. $\square$

If $\mathcal{N} := \{\lambda_i\}_{i=1}^n$ is a nodal minimal determining set for a spline space $\mathcal{S}$, then for each $1 \le i \le n$, there is a unique spline $\phi_i \in \mathcal{S}$ such that

$$\lambda_j \phi_i = \delta_{ij}, \qquad j = 1, \ldots, n. \tag{17.15}$$

Since the splines $\Phi := \{\phi_i\}_{i=1}^n$ are clearly linearly independent, it follows that Φ is a basis for $\mathcal{S}$. We refer to it as the $\mathcal{N}$-basis. For approximation purposes, we need nodal minimal determining sets which are local and stable.

Definition 17.21. *Suppose $\mathcal{N}$ is a nodal minimal determining set for the spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$.*

1) *We say that $\mathcal{N}$ is* local *provided that there exists an integer ℓ not depending on $\triangle$ such that for every $s \in \mathcal{S}$, every tetrahedron $T \in \triangle$, and every $\xi \in \mathcal{D}_{d,T}$, the coefficient c_ξ depends only on derivatives of s at points lying in $\mathcal{M}_\xi := \text{star}^\ell(T_\xi)$, where T_ξ is a tetrahedron containing ξ.*

2) *We say that $\mathcal{N}$ is* stable *provided that there exists a constant K depending only on d, ℓ, $\theta_\triangle$, and $\phi_\triangle$, such that for every $s \in \mathcal{S}$, every tetrahedron $T \in \triangle$, and every $\xi \in \mathcal{D}_{d,T}$, the coefficient c_ξ satisfies*

$$|c_\xi| \le K \sum_{\nu=0}^{m_\xi} |T_\xi|^\nu |s|_{\nu,\mathcal{M}_\xi}, \tag{17.16}$$

for some integer $m_\xi \le d$.

Finding trivariate spline spaces with stable local nodal minimal determining sets is nontrivial, but we shall give several important examples in the next chapter.

17.7. Hermite Interpolation

Suppose that $\mathcal{N}$ is a nodal minimal determining set for a trivariate spline space $\mathcal{S}$ defined on a tetrahedral partition $\triangle$ of a set Ω. Let $\bar{m}$ be the order of the highest derivative involved in the nodal functionals in $\mathcal{N}$. Then $\mathcal{N}$ defines a natural Hermite interpolation operator $\mathcal{I}_\mathcal{S}$ mapping $C^{\bar{m}}(\Omega)$ onto $\mathcal{S}$. Indeed, for each $f \in C^{\bar{m}}(\Omega)$, there is a unique spline $s \in \mathcal{S}$ such that

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}. \tag{17.17}$$

In terms of the $\mathcal{N}$-basis $\Phi := \{\phi_i\}_{i=1}^n$ associated with $\mathcal{N} := \{\lambda_i\}_{i=1}^n$, we can write

$$\mathcal{I}_\mathcal{S} f = \sum_{i=1}^n (\lambda_i f)\, \phi_i.$$

It is clear that $\mathcal{I}_\mathcal{S}$ is a linear projector onto $\mathcal{S}$, and thus reproduces polynomials of degree d, i.e.,

$$\mathcal{I}_\mathcal{S}\, p = p, \qquad \text{all } p \in \mathcal{P}_d.$$

We emphasize that if $\mathcal{N}$ is chosen properly, we can solve the Hermite interpolation problem (17.17) without actually constructing any basis. Indeed, we can compute certain of the coefficients of the interpolating spline directly from the nodal data, and the rest using smoothing conditions.

We now give an error bound in the maximum norm for this Hermite interpolation operator, under the assumption that the nodal minimal determining set $\mathcal{N}$ for $\mathcal{S}$ is stable and local.

Theorem 17.22. *Suppose $\mathcal{N}$ is a stable local nodal minimal determining set for a spline space $\mathcal{S}$, and let ℓ be the constant in Definition 17.21. Let $\mathcal{I}_{\mathcal{S}}$ be the associated Hermite interpolation operator. Given a tetrahedron T in $\triangle$, let $\Omega_T := \mathrm{star}^\ell(T)$. Then for every $f \in C^{m+1}(\Omega_T)$ with $\bar{m} \le m \le d$,*

$$\|D^\alpha(f - \mathcal{I}_{\mathcal{S}} f)\|_T \le K\, |T|^{m+1-|\alpha|}\; |f|_{m+1,\Omega_T}, \qquad |\alpha| \le m. \tag{17.18}$$

If Ω_T is convex, the constant K depends only on ℓ, d, and the smallest solid and face angles θ_Ω and ϕ_Ω associated with $\triangle$. If Ω_T is not convex, K also depends on the Lipschitz constant of the boundary of Ω_T.

Proof: The proof is very much like the proof of Theorem 5.26. Fix T, and let ρ_T be the radius of the largest ball contained in T. By Theorem 15.35, there exists a polynomial $p := p_f \in \mathcal{P}_m$ so that

$$\|D^\beta(f - p)\|_{\Omega_T} \le K_1\, |\Omega_T|^{m+1-|\beta|}\, |f|_{m+1,\Omega_T}, \tag{17.19}$$

for all $0 \le |\beta| \le m$. The constant K_1 depends on d and the shape parameter κ_Ω of (15.42) associated with Ω_T. It also depends on the Lipschitz constant of the boundary of Ω_T when Ω_T is nonconvex. The results of Section 16.6 coupled with Lemma 16.4 show that κ_Ω can be bounded in terms of the angles θ_Ω and ϕ_Ω.

Now fix a multi-index α with $0 \le |\alpha| \le m$. By the linearity of $\mathcal{I}_{\mathcal{S}}$ and the fact that it reproduces polynomials of degree d,

$$\|D^\alpha(f - \mathcal{I}_{\mathcal{S}} f)\|_T \le \|D^\alpha(f - p)\|_T + \|D^\alpha \mathcal{I}_{\mathcal{S}}(f - p)\|_T.$$

Using (17.19) with $\beta = \alpha$ and the fact that $|\Omega_T| \le K_2|T|$ by Lemmas 16.18 and 16.19, we see that it suffices to estimate the second term.

Since the Bernstein basis polynomials form a partition of unity, (17.16) implies

$$\|\mathcal{I}_{\mathcal{S}}(f - p)\|_T \le K_3 \sum_{\nu=0}^{\bar{m}} |T|^\nu |f - p|_{\nu,\Omega_T}.$$

Using the Markov inequality (15.3), it follows that

$$\|D^\alpha \mathcal{I}_{\mathcal{S}}(f - p)\|_T \le \frac{K_4}{\rho_T^{|\alpha|}} \|\mathcal{I}_{\mathcal{S}}(f - p)\|_T \le \frac{K_3 K_4}{\rho_T^{|\alpha|}} \sum_{\nu=0}^{\bar{m}} |T|^\nu |f - p|_{\nu,\Omega_T}.$$

Since $|T| \le \kappa_T \rho_T$, we can combine the above to get (17.18). $\square$

The global version of (17.18) also holds with T and Ω_T replaced by Ω, and with $|T|$ replaced by $|\triangle|$, as is easily seen by taking the maximum over all tetrahedra T in $\triangle$.

17.8. Dimension of Trivariate Spline Spaces

In Chapter 9 we gave a detailed treatment of the dimension problem for bivariate spline spaces on triangulations. In this section we discuss dimension of trivariate spline spaces on tetrahedral partitions. However, the problem is much harder in the trivariate case, and we have much less to report than in the bivariate case.

17.8.1 Connection with the Bivariate Problem

We first give an example to show that before we can hope to find minimal determining sets for the spline spaces $\mathcal{S}_d^r(\triangle)$ on general tetrahedral partitions, we have to be able to find minimal determining sets for C^r bivariate spline spaces of all degrees i with $r \le i \le d$.

Example 17.23. Suppose $F := \{F_j\}_{j=1}^m$ is a planar triangulation with vertices $\{v_i := (x_i, y_i, 0)\}_{i=1}^n$. Let $v_{n+1} := (x_1, y_1, 1) \in \mathbb{R}^3$, and let $\triangle$ be the tetrahedral partition which is obtained by connecting each of the v_i to v_{n+1}.

Discussion: Let $\{T_\nu\}_{\nu=1}^m$ be the tetrahedra of $\triangle$, where for each T_ν we choose its first vertex to be at v_{n+1}. Let $\{\xi_{ijkl}^{T_\nu}\}_{i+j+k+l=d}$ be the set of domain points associated with T_ν. Then for each $0 \le i \le d$, all of the domain points in $\Gamma_i := \bigcup_{\nu=1}^m \{\xi_{ijkl}^{T_\nu}\}_{j+k+l=d-i}$ lie on a plane π_i parallel to the (x,y)-plane. Thus, any smoothness condition which involves a coefficient associated with a domain point on π_i can involve only coefficients associated with points on π_i. In other words, all smoothness conditions reduce to bivariate smoothness conditions. Now suppose that we have a minimal determining set $\mathcal{M}$ for the trivariate spline space $\mathcal{S}_d^r(\triangle)$. Then for each $i = r+1, \ldots, d$, the set $\mathcal{M}_i := \mathcal{M} \cap \Gamma_i$ is a consistent determining set for the bivariate spline space $\mathcal{S}_i^r(F)$, and thus is a minimal determining set. □

In analogy with the bivariate case, we say that a shellable tetrahedral partition $\triangle$ of a polyhedral set Ω is a cell provided there is a vertex v such that all of the tetrahedra in $\triangle$ share the vertex v. If v is on the boundary of Ω, then we call $\triangle$ a boundary cell. Example 17.23 shows that even for boundary cells, the trivariate dimension problem cannot be solved until we have a full understanding of the dimension of bivariate spline spaces.

17.8.2 Upper and Lower Bounds on Dimension

Fix $0 \le r < d$. In this subsection we give upper and lower bounds on the dimension of the trivariate spline space $\mathcal{S}_d^r(\triangle)$ for a large class of tetrahedral partitions which contains all shellable partitions as well as many partitions with holes and cavities. Our results here are the analogs of the upper and lower bounds of Section 9.7, but are not as precise since here we are not able to take account of the geometry of cells.

Our discussion is restricted to tetrahedral partitions $\triangle$ that can be built up by starting with a single tetrahedron and successively adding one tetrahedron T at a time so that one of the following holds:

1) T touches the existing partition on exactly one, two, or three faces.
2) T touches the existing partition on exactly one face and at the opposite vertex.

Every shellable tetrahedral partition can be built in this way using only the first type of building block. The second type of building block is needed to get partitions with holes and cavities. To get upper and lower bounds on the dimension of spline spaces on these types of partitions, we establish four lemmas which give bounds on

$$\delta_T := \dim \mathcal{S}_d^r(\triangle \cup T) - \dim \mathcal{S}_d^r(\triangle),$$

i.e., on the change in dimension when we add a new tetrahedron T to an existing partition $\triangle$.

Lemma 17.24. *Suppose we add a tetrahedron T to an existing partition such that T touches on exactly one face. Then $\ell_1 \le \delta_T \le u_1$, where*

$$\ell_1 = u_1 := \binom{d-r+2}{3}.$$

Proof: Let F be the face where T joins $\triangle$. Then the coefficients of s associated with domain points within a distance r of F are determined from the C^r smoothness conditions across F. The coefficients associated with the remaining $\binom{d-r+2}{3}$ points do not enter any smoothness conditions, and thus can be assigned arbitrary values. $\square$

Lemma 17.25. *Suppose we add a tetrahedron T to an existing partition such that T touches on exactly two faces. Then $\ell_2 \le \delta_T \le u_2$, where*

$$\ell_2 := 2\binom{d+2-r}{3} - \binom{d+3}{3} + \binom{r+3}{3} + (d-r)\binom{r+2}{2},$$
$$u_2 := \binom{d-2r+1}{3}.$$

Proof: The C^r smoothness conditions across the two faces where T touches the existing partition determine all coefficients associated with domain points of T within a distance r of these faces. The number of remaining coefficients is $\binom{d-2r+1}{3}$, which gives the upper bound. To get the lower bound, let m_s be the number of points within a distance r of a face, and let m_t be the number of points in the tube of radius r around an edge. It is easy to see that $m_s = \binom{d+3}{3} - \binom{d+2-r}{3}$ while $m_t = \binom{r+3}{3} + (d-r)\binom{r+2}{2}$. Now $\ell_2 = \binom{d+3}{3} - 2m_s + m_t$, which reduces to the stated lower bound. $\square$

Lemma 17.26. *Suppose we add a tetrahedron T to an existing partition such that T touches on exactly three faces. Then $\ell_3 \le \delta_T \le u_3$, where*

$$\begin{aligned}\ell_3 &:= 3\binom{d+2-r}{3} - 2\binom{d+3}{3} + 2\binom{r+3}{3} \\ &\qquad + 3(d-r)\binom{r+2}{2} - \sum_{k=r+1}^{d} \binom{3r+2-2k}{2}, \\ u_3 &:= \binom{d-3r}{3}.\end{aligned}$$

Proof: The upper bound is simply the number of domain points in T that are at a distance at least $r+1$ from the three faces where T touches the existing partition. To get the lower bound, let m_s and m_t be as in the proof of Lemma 17.25, and let m_3 be the number of points in the intersection of three tubes of radius r all sharing one vertex. It is easy to see that $m_3 = \binom{r+3}{3} + \sum_{k=r+1}^{d} \binom{3r+2-2k}{2}$. Now $\ell_2 = \binom{d+3}{3} - 3m_s + 3m_t - m_3$, which reduces to the stated lower bound. $\square$

Lemma 17.27. *Suppose we add a tetrahedron T to an existing partition such that T touches on exactly one face and at the opposite vertex. Then $\ell_o \le \delta_T \le u_o$, where*

$$\ell_o := \binom{d-r+2}{3} - \binom{r+3}{3},$$

$$u_o := \begin{cases} \ell_o, & d > 2r+1, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof: If $d > 2r+1$, then the ball $D_r(v)$ does not intersect the set of points $G_r(F)$ that lie within a distance r of the face F. In this case both the upper and lower bounds are equal to the number of points not in either set. If $d \le 2r+1$, the sets overlap, and the upper bound is zero. In this case the dimension can go down by the number of points in the overlap, which is just ℓ_o. $\square$

Theorem 17.28. *Let $\triangle$ be a tetrahedral partition that can be built starting with one tetrahedron and adding tetrahedra that touch on exactly i faces a total of n_i times, $1 \le i \le 3$, and by adding tetrahedra that touch on one face and the opposite vertex a total of n_o times. Then for any $d \ge r \ge 0$,*

$$L(r,d) \le \dim \mathcal{S}_d^r(\triangle) \le U(r,d),$$

where

$$\begin{aligned} L(r,d) &:= \binom{d+3}{3} + n_1\ell_1 + n_2\ell_2 + n_3\ell_3 + n_o\ell_o, \\ U(r,d) &:= \binom{d+3}{3} + n_1u_1 + n_2u_2 + n_3u_3 + n_ou_o. \end{aligned}$$

Proof: The spline space $\mathcal{S}_d^r(\triangle)$ restricted to the single tetrahedron T_1 has dimension $\binom{d+3}{3}$. Now each time we add a new tetrahedron, the dimension can change by an amount that is bounded above and below by the formulae in Lemmas 17.24–17.27. $\square$

In general the upper and lower bounds in Theorem 17.28 are both cubic polynomials in d and r. If $\triangle$ is shellable, then the terms involving n_o can be dropped.

17.8.3 The Case $d \geq 8r+1$

In this section we describe minimal determining sets $\mathcal{M}$ for the spline spaces $\mathcal{S}_d^r(\triangle)$ with $d \geq 8r+1$ in enough detail to show that these spaces have star-supported bases. The key idea is to localize the construction of $\mathcal{M}$ as was done in Theorem 9.15 in the bivariate case. Fix $d \geq 8r+1$.

1) For each vertex $v \in \triangle$, let $\mathcal{M}_v$ be a smallest subset of $D_{4r}(v)$ such that if we fix the coefficients of $s \in \mathcal{S}_d^r(\triangle)$ corresponding to the domain points in $\mathcal{M}_v$, then all coefficients of s corresponding to domain points in $D_{4r}(v)$ are consistently determined.
2) For each edge $e := \langle u, v\rangle$ of $\triangle$, let $\mathcal{M}_e$ be a smallest subset of $E_{2r}(e) := \{\xi \in \mathcal{D}_{d,\triangle} : \operatorname{dist}(\xi, e) \leq 2r\} \setminus \big(D_{4r}(u) \cup D_{4r}(v)\big)$ such that if we now set the coefficients of s corresponding to domain points in $\mathcal{M}_e$, then all coefficients of s corresponding to domain points in the tube $t_{2r}(e)$ are consistently determined.
3) For each face F of $\triangle$ with edges e_1, e_2, e_3, let $\mathcal{M}_F$ be a smallest subset of $G_r(F) := \{\xi \in \mathcal{D}_{d,\triangle} : \operatorname{dist}(\xi, F) \leq r\} \setminus \big(t_{2r}(e_1) \cup t_{2r}(e_2) \cup t_{2r}(e_3)\big)$ such that if we now set the coefficients of s corresponding to domain points in $\mathcal{M}_F$, then all coefficients of s corresponding to domain points in the set $H_r(F) := \{\xi \in \mathcal{D}_{d,\triangle} : \operatorname{dist}(\xi, F) \leq r\}$ are consistently determined.
4) For each tetrahedron T of $\triangle$ with faces F_1, F_2, F_3, F_4, let $\mathcal{M}_T$ be a smallest subset of $\mathcal{D}_T \setminus \big(H_r(F_1) \cup H_r(F_2) \cup H_r(F_3) \cup H_r(F_4)\big)$ such that if we now set the coefficients of s corresponding to domain points in $\mathcal{M}_T$, then all coefficients of s corresponding to domain points in $\mathcal{D}_T$ are consistently determined.

Theorem 17.29. *Let $\mathcal{V}, \mathcal{E}, \mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. Then the set*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e \cup \bigcup_{F \in \mathcal{F}} \mathcal{M}_F \cup \bigcup_{T \in \triangle} \mathcal{M}_T$$

is a minimal determining set for $\mathcal{S}_d^r(\triangle)$ and the corresponding $\mathcal{M}$-basis of Theorem 17.16 contains only star-supported splines.

Proof: The construction of $\mathcal{M}$ ensures that it is a consistent determining set of $\mathcal{S}_d^r(\triangle)$. It follows from Theorem 17.10 that it is a minimal determining set for $\mathcal{S}_d^r(\triangle)$. The support properties of the corresponding $\mathcal{M}$-basis are evident. □

This process of localizing the construction of a minimal determining set can also be used to show that various superspline spaces of $\mathcal{S}_d^r(\triangle)$ with $d \geq 8r+1$ also have star-supported bases. In Section 18.11 we give an explicit construction for a special superspline subspace of $\mathcal{S}_{8r+1}^r(\triangle)$.

17.8.4 Oranges

In this section we discuss a special class of tetrahedral partitions $\triangle$ for which the dimension of $\mathcal{S}_d^r(\triangle)$ can be identified for all d and r.

Definition 17.30. *We say that a set $\triangle$ of tetrahedra $T_1,\ldots,T_n$ is an* orange *provided*

1) *$T_1,\ldots,T_n$ share a common edge e,*
2) *for $i=1,\ldots,n$, T_i and T_{i+1} share a common face, where we identify T_{n+1} with T_1.*

Given an orange $\triangle$, we may assume that the common edge $e := \langle v_T, v_B\rangle$ lies on the z-axis. We number the remaining vertices of the tetrahedra in $\triangle$ as $v_1,\ldots,v_n$, so that $T_i = \langle v_T, v_B, v_i, v_{i+1}\rangle$, $i=1,\ldots,n$, where $v_{n+1}=v_1$. Let $v_i := \langle x_i, y_i, z_i\rangle$. We may assume that the orange is rotated so that none of the points (x_i,y_i) lie on the y axis. Let $\alpha_i := y_i/x_i$ for $i=1,\ldots,n$, and let m be the number of distinct numbers in the sequence $\alpha_1,\ldots,\alpha_n$.

Theorem 17.31. *For all $0 \leq r < d$,*

$$\begin{aligned} \dim \mathcal{S}_d^r(\triangle) = &\binom{d+3}{3} + n\binom{d-r+2}{3} - (d+3)\binom{d-r+1}{2} \\ &+ 2\binom{d-r+2}{3} + \sigma, \end{aligned} \tag{17.20}$$

where

$$\sigma := \sum_{j=1}^{d-r} (d-r-j+1)(r+j+1-jm)_+.$$

Proof: The proof is very similar to the proof of Theorem 9.3 for bivariate splines on interior cells. For complete details, see [AlfSS92], where a minimal determining set is also constructed. □

17.9. Remarks

Remark 17.1. Let H be the unit cube in $\mathbb{R}^3$. Given a positive integer n, let $\diamondsuit$ be the partition into n^3 subboxes

$$H_{ijk} := [(i-1)h, ih] \times [(j-1)h, jh] \times [(k-1)h, kh], \quad 1 \le i, j, k \le n.$$

where $h = 1/n$. We call $\diamondsuit$ a cube partition. We now partition each subcube H_{ijk} into 24 congruent tetrahedra by applying six cutting planes, where each plane passes through two diagonally opposite edges of H_{ijk}. Let $\triangle$ be the resulting tetrahedral partition. The total number of tetrahedra in $\triangle$ is $24n^3$. The following dimension result was established in [HanNRSZ04]. Because of the special nature of $\triangle$, bivariate arguments could be used.

Theorem 17.32. *For any $n > 1$, dim $\mathcal{S}_2^1(\triangle) = 3n^2 + 9n + 4$. Moreover, dim $\mathcal{S}_d^1(\triangle) = (4d^3 - 24d^2 + 53d - 45)n^3 + (2d^2 - 7d + 7)n^2 + 9(d-1)n + 4$, for all $d > 2$.*

Remark 17.2. Trivariate splines have also been studied on another special tetrahedral partition that can be created from a cube partition by splitting each subbox into five tetrahedra. In particular it is shown in [SchS04] that the space $\mathcal{S}_5^1(\triangle)$ on this partition has a stable local minimal determining set and full approximation power.

Remark 17.3. In Section 9.8 we presented dimension results for bivariate spline spaces defined on generic triangulations. There is an analogous concept for tetrahedral partitions. Suppose $\mathcal{L}$ describes the connectivity of a given tetrahedral partition. For example $\mathcal{L}$ could describe how the vertices are connected together. Let $\mathrm{Tri}\,(\mathcal{L})$ be the set of all tetrahedral partitions with this connectivity. We now fix r and d, and consider all spaces of the form $\mathcal{S}_d^r(\triangle)$, where $\triangle \in \mathrm{Tri}\,(\mathcal{L})$. Then a partition $\triangle^* \in \mathrm{Tri}\,(\mathcal{L})$ is called generic with respect to $r, d, \mathcal{L}$ provided that

$$\dim S_d^r(\triangle^*) = \min_{\triangle \in \mathrm{Tri}\,(\mathcal{L})} \dim S_d^r(\triangle).$$

Following the proof of Theorem 9.32, it can be shown that all tetrahedral partitions whose vertices are sufficiently close to a generic partition $\triangle^*$ are also generic. The following result was established in [AlfSW93] with a technique involving projecting a tetrahedral partition onto a so-called generalized triangulation. Let N be the number of tetrahedra in $\triangle$, and let V_I and V_B be the number of interior and boundary vertices, respectively.

Theorem 17.33. *Suppose $\triangle$ is a generic tetrahedral partition and that $d \ge 8$. Then*

$$\dim \mathcal{S}_d^1(\triangle) = \frac{d(d-1)(d-5)}{6} N + 3(d-1)V_I + d(d-1)V_B - 2d^2 + 5d + 1.$$

Remark 17.4. In Definition 10.1 we defined the concepts of approximation power and full approximation power for bivariate spline spaces. It is straightforward to extend these concepts to the trivariate case.

Definition 17.34. *Fix $0 \le r < d$ and $0 < \theta \le \pi/3$. Let m be the largest integer such that for every polyhedral domain Ω and every regular tetrahedral partition $\triangle$ of Ω with smallest solid and face angles $\theta_\triangle$ and $\phi_\triangle$, for every $f \in W_q^m(\Omega)$, there exists a spline $s \in \mathcal{S}_d^r(\triangle)$ with*

$$\|f - s\|_{q,\Omega} \le K\,|\triangle|^m\,|f|_{m,q,\Omega}, \tag{17.21}$$

where the constant K depends only on $r, d, \theta_\triangle, \phi_\triangle$, and the Lipschitz constant of the boundary of Ω. Then we say that $\mathcal{S}_d^r$ has approximation power m in the q-norm. *If this holds for $m = d + 1$, we say that $\mathcal{S}_d^r$ has* full approximation power in the q-norm.

The results of Section 17.4 show that any spline space that has a stable local minimal determining set will provide full approximation power. Thus, in particular, the spaces $\mathcal{S}_d^0(\triangle)$ and $\mathcal{PP}(\triangle)$ have full approximation power, cf. Section 10.2 for the bivariate case.

Remark 17.5. The bounds in Theorem 17.28 depend on the order in which the partition $\triangle$ is built. Indeed, partitions can be built in many different ways leading to different values of n_i and n_o. To get the best bound, we could try all possibilities, but except for very small partitions, this would be computationally expensive. For examples, see [AlfS06].

Remark 17.6. Definition 16.5 allows very general tetrahedral partitions which cannot be built using the building blocks of Section 17.8.2. For example, the partition $\triangle$ consisting of two tetrahedra touching only at one vertex would require a different kind of building block. To get upper and lower bounds on the dimension of $\mathcal{S}_d^r(\triangle)$ for general partitions, one has to consider 27 different building blocks and find bounds on the associated change in dimension when they are added to an existing partition. For details, see [AlfS06].

17.10. Historical Notes

Spaces of piecewise polynomials defined on tetrahedral partitions have been used by engineers for quite some time to solve boundary value problems by the finite-element method, see [Zla68, Zla70, Zen73a, Zen73b] and the books [Cia78a, BreS94, Bra97]. Bernstein–Bézier methods are not used in this literature. Bernstein–Bézier methods were first used for investigating trivariate macro-element spaces in [Alf84a, Alf84c, WorF87, WorP88].

The dimension problem for trivariate spline spaces $\mathcal{S}_d^r(\triangle)$ on general tetrahedral partitions was studied in [AlfSir89, AlfSir91, AlfSS92, AlfSW93] using Bernstein–Bézier methods. The concept of minimal determining sets played a key role. Example 17.23 showing that the dimension problem for the trivariate case cannot be settled until we fully understand the dimension problem for bivariate splines is taken from [AlfSW93].

The problem of finding upper and lower bounds on the dimension of trivariate spline spaces was first studied in [Alf86, Alf87, Alf96]. The results in Section 17.8.2 follow [AlfS06], where bounds are established for considerably more general tetrahedral partitions than considered here. The approach described in Section 17.8.3 for localizing the study of trivariate spline spaces comes from [AlfSS92], see also [AlfSir89, AlfSir91]. The concept of oranges was introduced in [AlfSS92] where Theorem 17.31 was first established. A variant of the dimension formula (17.20) was given in [Lau06] where it was used to get a lower bound on the trivariate spline space $\mathcal{S}_d^r(\triangle)$ for simply connected tetrahedral partitions. These lower bounds still do not take account of the geometry of cells. Finding a way to do so remains a major open problem in trivariate dimension theory. In Section 17.5 we showed how to construct stable local bases for spline spaces from stable local minimal determining sets or nodal minimal determining sets. For an algorithm for constructing such bases without knowing a determining set, see [Dav02b].

Trivariate Macro-element Spaces

Our aim in this chapter is to describe several useful C^1 and C^2 trivariate macro-element spaces with the following properties:

1) their dimension can be given explicitly,
2) they have stable local bases,
3) they have optimal order approximation power,
4) they can be used to construct convenient local Hermite interpolation operators which approximate smooth functions well,
5) they can be used in the finite-element method to solve boundary value problems.

18.1. Introduction

Let $\triangle$ be a tetrahedral partition of a polyhedral set Ω in $\mathbb{R}^3$, and let $\triangle_R$ be a tetrahedral partition of Ω which is obtained from $\triangle$ by applying a fixed refinement process to each tetrahedron in $\triangle$.

Definition 18.1. *Let $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle_R)$ be a space of trivariate splines, and let $\mathcal{N} := \{\lambda_i\}_{i=1}^n$ be a corresponding nodal minimal determining set, see Section 17.6. Suppose $\mathcal{N}$ is such that for each tetrahedron $T \in \triangle$, the data $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$ uniquely determine $s|_T$, where*

$$\mathcal{N}_T := \{\lambda \in \mathcal{N} : \text{ the carrier of } \lambda \text{ is contained in } T\}.$$

Then we say that $\mathcal{S}$ is a macro-element space*. We refer to the linear functionals in $\mathcal{N}$ as the* nodal degrees of freedom *of $\mathcal{S}$.*

According to this definition, if $\mathcal{S}$ is a macro-element space with nodal determining set $\mathcal{N}$, then the Hermite-interpolating spline s defined in Section 17.7 can be computed one tetrahedron at a time, and in particular, $s|_T$ can be computed from data at points in T. Trivariate macro-element spaces are useful for a variety of approximation purposes, including Hermite interpolation of scattered data, and the numerical solution of partial differential equations.

Our analysis of the trivariate macro-element spaces presented in this chapter will be based on the Bernstein–Bézier theory described in Chapter 17. For each of the macro-element spaces $\mathcal{S}$ discussed in this chapter, we first find a stable local minimal determining set $\mathcal{M}$ for $\mathcal{S}$, see Definition 17.11. This approach gives us the dimension of $\mathcal{S}$, and also shows that it has full approximation power, cf. Theorem 17.15. For each macro-element space presented here, we also give a stable local nodal minimal determining set, and use it to construct a Hermite interpolation operator with full approximation power.

As an aid to describing nodal minimal determining sets, we now introduce some additional notation. Throughout this chapter, we will write T for a tetrahedron in $\triangle$, and t for a tetrahedron in the refined partition $\triangle_R$. Given an edge $e := \langle u, v\rangle$ of a tetrahedron, let X_e be the plane perpendicular to e at u. We endow X_e with Cartesian coordinate axes whose origin lies at the point u. Then for any multi-index $\beta = (\beta_1, \beta_2)$, we define D_e^β to be the corresponding directional derivative of order $|\beta| := \beta_1 + \beta_2$. Given $i > 0$, we introduce the following notation for equally spaced points in the interior of e:

$$\eta^i_{e,j} := \frac{(i-j+1)u + jv}{i+1}, \qquad j = 1, \ldots, i.$$

If $F := \langle u, v, w\rangle$ is an oriented triangular face of T, then we write D_F for a unit normal derivative associated with F and define $\xi^{F,m}_{ijk} := (iu + jv + kw)/m$ for all $i + j + k = m$. For any point $u \in \mathbb{R}^3$, we write ε_u for the point evaluation functional defined by $\varepsilon_u f := f(u)$.

Throughout this chapter we shall give a number of approximation results. In most cases the constants in these results will depend on d and the smallest solid and face angles of $\triangle_R$ or $\triangle$, see Definition 16.2. When Ω is nonconvex, the constants will also depend on the Lipschitz constant of the boundary of Ω, which in turn depends on the smallest external solid angles of $\triangle$.

18.2. A C^1 Polynomial Macro-element

Suppose $\triangle$ is a tetrahedral partition of a polyhedral set Ω in $\mathbb{R}^3$, and let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. In this section we discuss the C^1 polynomial macro-element space

$$\mathcal{S}_1(\triangle) := \{s \in \mathcal{S}^1_9(\triangle) : s \in C^2(e), \text{ all } e \in \mathcal{E}, \text{ and } s \in C^4(v), \text{ all } v \in \mathcal{V}\}.$$

For each vertex v of $\triangle$, let T_v be some tetrahedron with vertex at v. For each edge e of $\triangle$, let T_e be some tetrahedron in $\triangle$ containing e, and let $E_2(e)$ be the set of domain points which lie in the tube of radius 2 around $e := \langle u, v\rangle$, but which are not contained in the balls $D_4(u)$ or $D_4(v)$. For each face F of $\triangle$, let T_F be a tetrahedron in $\triangle$ containing F, and let $G_1(F)$ be the set of domain points which lie within a distance 1 of F, but which are not in any of the balls $D_4(v)$ or sets $E_2(e)$ associated with vertices and edges of F. Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.2. *dim* $\mathcal{S}_1(\triangle) = 35n_{_V} + 8n_{_E} + 7n_{_F} + 4n_{_T}$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{M}_F \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_4(v) \cap T_v$,
2) $\mathcal{M}_e := E_2(e) \cap T_e$,
3) $\mathcal{M}_F := G_1(F) \cap T_F$,
4) $\mathcal{M}_T := \{\xi^T_{ijkl} : i,j,k,l \ge 2\}$.

Proof: We make use of Theorem 17.10 to show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_1(\triangle)$. In particular, we show that the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_1(\triangle)$ can be set to arbitrary values, and that all other coefficients of s are then consistently determined in such a way that all smoothness conditions are satisfied. First, for each vertex v, we set the coefficients corresponding to domain points in the set $\mathcal{M}_v$ to arbitrary values. By the results of Section 15.8, this determines $\{D^\alpha s(v)\}_{|\alpha|\le 4}$, which in turn uniquely determine all coefficients corresponding to the remaining domain points in the ball $D_4(v)$. By the formulae in Section 15.8, this is a stable local process. In particular, for all $\eta \in D_4(v) \setminus \mathcal{M}_v$, c_η can be computed from coefficients in the set $\Gamma_\eta := \mathcal{M}_v \subseteq D_4(v)$, and

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \tag{18.1}$$

where K is a constant depending on the smallest solid angle $\theta_\triangle$ and smallest face angle $\phi_\triangle$ in $\triangle$. No smoothness conditions have been violated since the balls $D_4(v)$ do not overlap.

For each edge $e := \langle u, v\rangle$, we now fix the coefficients corresponding to $\mathcal{M}_e$. We then use the C^2 supersmoothness around e to stably determine all coefficients c_η corresponding to the remaining domain points η in the tube of radius 2 around e, see Theorem 15.23. For each such η, c_η has been computed from coefficients corresponding to domain points in the set $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$, and (18.1) holds for all $\eta \in E_2(e)$ with $\eta \notin \mathcal{M}$. The sets $E_2(e)$ are disjoint from each other and from all balls $D_4(v)$, and so we can be sure that no smoothness conditions have been violated.

For each face $F := \langle v_2, v_3, v_4\rangle$, let $T_F := \langle v_1, v_2, v_3, v_4\rangle$ be the tetrahedron associated with F. Note that $\mathcal{M}_F = \mathcal{M}^0_F \cap \mathcal{M}^1_F$, where $\mathcal{M}^0_F := \{\xi^{T_F,9}_{0jkl} : j,k,l \ge 2\}$ and $\mathcal{M}^1_F := \{\xi^{T_F,9}_{1jkl} : j,k,l \ge 1\}$. We mark the points in these sets with $\oplus$ in Figure 18.1. Coefficients of s corresponding to points marked with circles or with triangles are already determined. We now fix the coefficients of s corresponding to $\mathcal{M}_F$. There cannot be any incompatibilities since the sets $G_1(F)$ are disjoint from each other, and

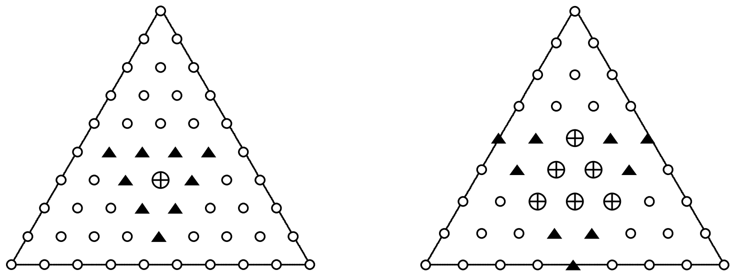


Fig. 18.1. Points $\oplus$ in the sets $\mathcal{M}_F^0$ and $\mathcal{M}_F^1$ in the proof of Theorem 18.2.

there are no smoothness conditions connecting coefficients associated with domain points in two different such sets.

If F is a boundary face of $\triangle$, this uniquely determines all coefficients corresponding to domain points in $G_1(F)$. If F is an interior face and $\widetilde{T}_F$ is the other tetrahedron containing the face F, then the coefficients corresponding to domain points in $G_1(F) \cap \widetilde{T}_F$, are uniquely determined by the C^1 smoothness conditions across F. This is a stable local process, i.e., (18.1) holds where Γ_η is the union of the set $\mathcal{M}_F$ with all sets of the form $\mathcal{M}_v$ and $\mathcal{M}_e$, where v and e are vertices or edges of F. At this point, we are left only with domain points in the sets $\mathcal{M}_T$. These sets are clearly disjoint from each other, and there are no smoothness conditions connecting coefficients associated with domain points in two different such sets. Thus, the corresponding coefficients can be set to arbitrary values.

To complete the proof, we note that since $\mathcal{M}$ is a minimal determining set, by Theorem 17.8, the dimension of $\mathcal{S}_1(\triangle)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the stated formula. $\square$

Since $\mathcal{S}_1(\triangle)$ has a stable local MDS, Theorem 17.15 immediately implies the following result showing that $\mathcal{S}_1(\triangle)$ has full approximation power. Let $\theta_\triangle$ and $\phi_\triangle$ be the smallest solid angle and smallest face angle in $\triangle$, respectively.

Theorem 18.3. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 9$, there exists a spline $s_f \in \mathcal{S}_1(\triangle)$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d, $\theta_\triangle$ and $\phi_\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set for $\mathcal{S}_1(\triangle)$. For each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$, let v_F be the barycenter of F, and let

$$A_F^1 := \{\xi_{ijk}^{F,8} : i, j, k \geq 2\},$$

where $\xi_{ijk}^{F,8} := (iv_1 + jv_2 + kv_3)/8$ for all $i + j + k = 8$. Note that the six points in A_F^1 lie on F, but are not in the set $\mathcal{D}_{9,\triangle}$ of domain points of $\mathcal{S}_1(\triangle)$. Their locations are marked with $\oplus$ in Figure 18.1 (right) (which was earlier used to depict domain points in $\mathcal{D}_{9,\triangle}$ lying at a distance 1 from a face). For each tetrahedron $T \in \triangle$, let v_T be the barycenter of T.

Theorem 18.4. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{F \in \mathcal{F}} (\mathcal{N}_F^0 \cup \mathcal{N}_F^1) \cup \bigcup_{T \in \triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}_1(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha| \leq 4}$,
2) $\mathcal{N}_e := \bigcup_{\ell=1}^2 \bigcup_{m=1}^\ell \{\varepsilon_{\eta_{e,m}^\ell} D_e^\beta\}_{|\beta|=\ell}$,
3) $\mathcal{N}_F^0 := \{\varepsilon_{v_F}\}$,
4) $\mathcal{N}_F^1 := \{\varepsilon_\xi D_F\}_{\xi \in A_F^1}$,
5) $\mathcal{N}_T := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha| \leq 1}$.

Proof: It is easy to check that $\#\mathcal{N}$ is equal to the dimension of $\mathcal{S}_1(\triangle)$ as given in Theorem 18.2. Thus, to show that $\mathcal{N}$ is a stable local NMDS for $\mathcal{S}_1(\triangle)$, it suffices to show that given the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$ for a spline $s \in \mathcal{S}_1(\triangle)$, all of its B-coefficients can be stably and locally computed. First, we examine the coefficients in the balls $D_4(v)$. For each $v \in \mathcal{V}$, we can use the formulae in Theorem 15.16 to compute the coefficients $\{c_\xi\}_{\xi \in D_4(v)}$ from the values $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$. By Theorem 15.17, this is a stable process. Indeed, if T_ξ is a tetrahedron containing ξ, then for all $\xi \in D_4(v)$,

$$|c_\xi| \leq K \sum_{\nu=0}^{4} |T_\xi|^\nu |s|_{\nu, T_\xi}, \tag{18.2}$$

for some constant K depending only on the angles $\theta_\triangle$ and $\phi_\triangle$.

For each edge $e \in \mathcal{E}$, we now use Theorem 15.18 to compute the coefficients of s corresponding to $\xi \in E_2(e)$ directly from the values $\{\lambda s\}_{\lambda \in \mathcal{N}_e}$. For each edge, this involves solving a system of equations with a nonsingular matrix that does not depend on the size and shape of T. Thus, (18.2) also holds for all domain points ξ in $E_2(e)$, see Theorem 15.23.

For each face F of $\triangle$, we now use Theorem 15.24 and Lemma 2.25 to compute the coefficient corresponding to the domain point $\xi_{333}^{F,9}$ from the value $s(v_F)$. Using the lemma again, we can compute the coefficients corresponding to the six domain points in $\mathcal{M}_F$ which are at a distance of 1 from F from the derivatives $\{D_F s(\xi)\}_{\xi \in A_F^1}$. These computations involve solving nonsingular linear systems whose matrices do not depend on the size or shape of T. It follows (see Theorem 15.25) that (18.2) holds for all domain points ξ lying in $G_1(F) \cap T_F$. If F is an interior face, then the coefficients associated with the points $G_1(F) \cap \widetilde{T}_F$, where $\widetilde{T}_F$ is the other tetrahedron containing F can be computed using the C^1 smoothness conditions across F. This is a stable local process, so (18.2) holds for these coefficients too.

For each tetrahedron $T \in \triangle$, we have already uniquely determined the coefficients of $s|_T$ corresponding to the 216 domain points which are either on outer faces of T or within a distance of 1 of an outer face. This leaves the four coefficients corresponding to the domain points in $\mathcal{M}_T$. By Theorem 15.26 these can be stably computed from the values $\{D^\alpha s(v_T)\}_{|\alpha| \le 1}$. This involves solving a nonsingular 4×4 system whose corresponding matrix is the same for every tetrahedron. We conclude that (18.2) holds for all $\xi \in \mathcal{D}_{9,\triangle}$. $\square$

Theorem 18.4 shows that for any function $f \in C^4(\Omega)$, there is a unique spline $s \in \mathcal{S}_1(\triangle)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 4$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,j}^i) = D_e^\beta f(\eta_{e,j}^i)$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 2$, and for all edges e of $\triangle$,
3) $s(v_F) = f(v_F)$, all faces F of $\triangle$,
4) $D_F s(\xi) = D_F f(\xi)$, all $\xi \in A_F^1$ and all faces F of $\triangle$,
5) $D^\alpha s(v_T) = D^\alpha f(v_T)$, all $|\alpha| \le 1$.

This defines a linear projector $\mathcal{I}_P^1$ mapping $C^4(\Omega)$ onto the macro-element space $\mathcal{S}_1(\triangle)$. Thus, $\mathcal{I}_P^1$ reproduces polynomials of degree nine. Since the NMDS of Theorem 18.4 is local and stable, Theorem 17.22 implies the following error bound.

Theorem 18.5. *For every $f \in C^{m+1}(\Omega)$ with $3 \le m \le 9$,*

$$\|D^\alpha(f - \mathcal{I}_P^1 f)\|_\Omega \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We conclude this section by noting that $\mathcal{S}_1(\triangle)$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$ of Theorem 18.2, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.4 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_1(\triangle)$, where each basis function is star-supported.

18.3. A C^1 Macro-element on the Alfeld Split

Let $\triangle$ be an arbitrary tetrahedral partition of a polyhedral set $\Omega \in \mathbb{R}^3$, and let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. Let $\triangle_A$ be be the Alfeld refinement of $\triangle$ obtained by splitting each tetrahedron $T \in \triangle$ into four subtetrahedra using its barycenter v_T, see Definition 16.21. In this section we discuss the following C^1 macro-element space:

$$\begin{aligned}\mathcal{S}_1(\triangle_A) := \{s \in \mathcal{S}_5^1(\triangle_A) :\ & s \in C^2(v),\ \text{ all } v \in \mathcal{V}, \\ & s \in C^4(v_T),\ \text{ all } T \in \triangle\}.\end{aligned} \tag{18.3}$$

Before proceeding, we note that the splitting process used to create the Alfeld refinement is stable in the sense that the smallest solid and face angles associated with the the refined partition $\triangle_A$ can be bounded below in terms of the smallest solid and face angles $\theta_\triangle$ and $\phi_\triangle$ associated with the initial partition $\triangle$, see Theorem 16.22. This is critical for the stability of the MDS and NMDS constructed in this section, which in turn control the constants in our approximation results.

For each $v \in \mathcal{V}$, let t_v be some tetrahedron in $\triangle_A$ with vertex at v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let t_e be some tetrahedron in $\triangle_A$ containing e, and let $E_1(e)$ be the set of domain points which lie in the tube of radius 1 around e, but which are not contained in either $D_2(u)$ or $D_2(v)$. For each face F of $\triangle$, let t_F be a tetrahedron in $\triangle_A$ containing F, and let $G_1(F)$ be the set of domain points which lie within a distance 1 of F, but which are not in any of the balls $D_2(v)$ or sets $E_1(e)$. Finally, let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.6. *dim* $\mathcal{S}_1(\triangle_A) = 10n_{_V} + 2n_{_E} + 3n_{_F} + n_{_T}$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{M}_F \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_2(v) \cap t_v$,
2) $\mathcal{M}_e := E_1(e) \cap t_e$,
3) $\mathcal{M}_F := G_1(F) \cap t_F$,
4) $\mathcal{M}_T := \{v_{_T}\}$.

Proof: We use Theorem 17.10 to show that $\mathcal{M}$ is a MDS for $\mathcal{S}_1(\triangle_A)$. In particular, we show that if $s \in \mathcal{S}_1(\triangle_A)$, then we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ to arbitrary values, and that the remaining coefficients are determined in such a way that all smoothness conditions are satisfied. The proof is quite similar to that of Theorem 18.2. First we set all of the coefficients corresponding to the sets $\mathcal{M}_v$ to arbitrary values. Then for each $v \in \mathcal{V}$, by the results of Section 15.8 this uniquely determines $\{D^\alpha s(v)\}_{|\alpha|\le 2}$, which in turn uniquely determines all other coefficients corresponding to domain points in the ball $D_2(v)$. By the formulae in Section 15.8, this is a stable local process. In particular, for each $\eta \in D_2(v) \setminus \mathcal{M}_v$,

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \tag{18.4}$$

with $\Gamma_\eta = \mathcal{M}_v$, where K is a constant depending on the smallest solid and face angles in $\triangle_A$. Since the balls $D_2(v)$ do not overlap, none of the smoothness conditions is violated.

Next we fix all of the coefficients corresponding to the sets $\mathcal{M}_e$. Then for each edge $e := \langle u, v\rangle$, all coefficients corresponding to domain points in the set $E_1(e)$ can be uniquely computed from those in $\mathcal{M}_e$ using the C^1 smoothness conditions, see Theorem 15.18. By Theorem 15.23 this is a stable process, and (18.4) holds with $\Gamma_\eta = \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. No smoothness conditions have been violated since the sets $E_1(e)$ are disjoint from each other and from all balls $D_2(v)$.

For each face F of $\triangle$, we have already determined the B-coefficients of s corresponding to all domain points on F. Now set the coefficients corresponding to the domain points in $\mathcal{M}_F := G_1(F) \cap t_F = \{\xi^{t_F}_{1211}, \xi^{t_F}_{1121}, \xi^{t_F}_{1112}\}$. If F is an interior face of $\triangle$ and $\tilde{t}$ is the other tetrahedron in $\triangle_A$ which contains the face F, then we can stably compute the coefficients corresponding to $G_1(F) \cap \tilde{t}$ by the C^1 smoothness of s across the face F. It follows that (18.4) holds for all $\eta \in G_1(F)$ with Γ_η equal to the union of the sets $\mathcal{M}_v$, $\mathcal{M}_e$ and $\mathcal{M}_F$, where v and e are vertices or edges of F.

For each tetrahedron $T \in \triangle$, we have now determined all B-coefficients of s on the shells $R_4(v_{_T})$ and $R_5(v_{_T})$. By the C^4 smoothness at $v_{_T}$, we can regard the coefficients of s in the ball $D_4(v_T)$ as those of a trivariate polynomial g of degree 4 on a tetrahedron $\hat{T}$ which is congruent to T and which has been subjected to the Alfeld split. By the above, we have already uniquely determined the coefficients of g corresponding to the 34 domain points on the outer faces of $\hat{T}$. Now fixing the coefficient of s corresponding to $\mathcal{M}_T$ is equivalent to setting $g(v_T)$, which in turn uniquely determines all coefficients of g by Theorem 15.26. Once we have the coefficients of g, the remaining coefficients of s corresponding to domain points in T can be stably computed by subdivision using the de Casteljau algorithm.

To complete the proof, we note that by Theorem 17.8, the dimension of $\mathcal{S}_1(\triangle_A)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the given formula. $\square$

The proof of this theorem shows that the constant of stability for the minimal determining set $\mathcal{M}$ depends on the smallest solid and face angles in $\triangle_A$. However, in view of Theorem 16.22, these are bounded below in terms of the corresponding angles in $\triangle$. We now illustrate Theorem 18.6 in the case where $\triangle$ consists of a single tetrahedron.

Example 18.7. Let $\mathcal{S}_1(T_A)$ be the C^1 macro-element space defined in (18.3) associated with the Alfeld split T_A of a single tetrahedron T. Then $\dim \mathcal{S}_1(T_A) = 65$.

Discussion: The dimension statement follows immediately from Theorem 18.6 since $n_V = 4$, $n_E = 6$, $n_F = 4$, and $n_T = 1$. $\square$

Since $\mathcal{S}_1(\triangle_A)$ has a stable local MDS, Theorem 17.15 immediately implies the following result which shows that $\mathcal{S}_1(\triangle_A)$ has full approximation power. Let $\theta_\triangle$ and $\phi_\triangle$ be the smallest solid and face angles in $\triangle$.

Theorem 18.8. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 5$, there exists a spline $s_f \in \mathcal{S}_1(\triangle_A)$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d, $\theta_\triangle$ and $\phi_\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of $\partial\Omega$.

We now construct a nodal minimal determining set for $\mathcal{S}_1(\triangle_A)$ and then use it to construct a Hermite interpolating spline. For each face $F := \langle v_1, v_2, v_3\rangle \in \mathcal{F}$, let

$$A_F^0 := \{\xi_{211}^{F,4}, \xi_{121}^{F,4}, \xi_{112}^{F,4}\}, \tag{18.5}$$

where $\xi_{ijk}^F = (iv_1 + jv_2 + kv_3)/4$ for all $i+j+k = 4$. Note that these points lie on F, but are not in the set of domain points $\mathcal{D}_{5,\triangle_A}$ corresponding to $\mathcal{S}_1(\triangle_A)$. They are marked with the symbol $\oplus$ in Figure 18.2.

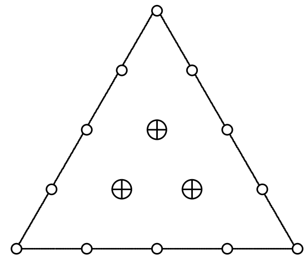


Fig. 18.2. Points $\oplus$ in the set A_F^0 of (18.5).

Theorem 18.9. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{N}_F \cup \bigcup_{T\in\triangle} \mathcal{N}_T \tag{18.6}$$

is a stable local nodal minimal determining set for $\mathcal{S}_1(\triangle_A)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 2}$,
2) $\mathcal{N}_e := \{\varepsilon_{\eta^1_{e,1}} D_e^\alpha\}_{|\alpha|=1}$,
3) $\mathcal{N}_F := \{\varepsilon_\xi D_{F}\}_{\xi\in A_F^0}$,
4) $\mathcal{N}_T := \{\varepsilon_{v_T}\}$.

Proof: It is easy to check that $\#\mathcal{N}$ is equal to the dimension of $\mathcal{S}_1(\triangle_A)$ as given in Theorem 18.6. Thus, to show that $\mathcal{N}$ is a nodal minimal determining set for $\mathcal{S}_1(\triangle_A)$, it suffices to show that setting $\{\lambda s\}_{\lambda\in\mathcal{N}}$ for a spline $s \in \mathcal{S}_1(\triangle_A)$ determines all B-coefficients of s corresponding to domain points in the set $\mathcal{M}$ of Theorem 18.6. For each $v \in \mathcal{V}$, we use the formulae in Theorem 15.16 to determine all B-coefficients of s corresponding to domain points ξ in the ball $D_2(v)$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_v}$. By Theorem 15.17, this is a stable local process. In particular,

$$|c_\xi| \le K \sum_{\nu=0}^{2} |T_\xi|^\nu |s|_{\nu,T_\xi}, \tag{18.7}$$

where T_ξ is a tetrahedron containing ξ, and K is a constant depending on $\theta_\triangle$ and $\phi_\triangle$. For each edge e of $\triangle$, we now use Theorem 15.18 to stably compute the coefficients of s corresponding to the domain points in $E_1(e)$ from $\{\lambda s\}_{\lambda\in\mathcal{N}_e}$.

Now for each face F, we can use Theorem 15.24 and Lemma 2.25 to compute the three coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_F} = \{c^{t_F}_{1211}, c^{t_F}_{1121}, c^{t_F}_{1112}\}$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_F}$. This involves solving a nonsingular 3×3 system whose matrix is independent of the size and shape of t_F. Thus, (18.7) also holds for these coefficients.

For each $T \in \triangle$, we note that the coefficient c_ξ corresponding to the domain point $\xi \in \mathcal{M}_T$ is equal to the value of $s(v_T)$. We have now computed all coefficients of s corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 18.6, and by that theorem, all coefficients of s are stably determined. $\square$

Theorem 18.9 shows that for any function $f \in C^2(\Omega)$, there is a unique spline $s \in \mathcal{S}_1(\triangle_A)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \leq 2$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,1}^1) = D_e^\beta f(\eta_{e,1}^1)$, all $|\beta| = 1$,
3) $D_F s(\xi) = D_F f(\xi)$, all $\xi \in A_F^0$ and all faces F of $\triangle$,
4) $s(v_T) = f(v_T)$, all tetrahedra $T \in \triangle$.

This defines a linear projector $\mathcal{I}_A^1$ mapping $C^2(\Omega)$ onto the superspline space $\mathcal{S}_1(\triangle_A)$. In particular, $\mathcal{I}_A^1$ reproduces polynomials of degree five. We can now give an error bound for this interpolation operator. We note that for any tetrahedron T in $\triangle$ and any subtetrahedron $t \in T \cap \triangle_A$, $|T|/|t| \leq 4$. Since the NMDS of Theorem 18.9 is both local and stable, Theorem 17.22 implies the following error bound.

Theorem 18.10. *For every $f \in C^{m+1}(\Omega)$ with $1 \leq m \leq 5$,*

$$\|D^\alpha(f - \mathcal{I}_A^1 f)\|_\Omega \leq K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega},$$

for all $0 \leq |\alpha| \leq m$. If Ω is convex, K depends on d and the the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We conclude this section by noting that $\mathcal{S}_1(\triangle_A)$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$ of Theorem 18.6, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.9 is a stable local NMDS for $\mathcal{S}_1(\triangle_A)$, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_1(\triangle_A)$, where each basis function is star–supported.

18.4. A C^1 Macro-element on the Worsey–Farin Split

Let $\triangle$ be an arbitrary tetrahedral partition of a polyhedral set $\Omega \in \mathbb{R}^3$, and let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of $\triangle$, respectively. Let $\triangle_{WF}$ be the Worsey–Farin refinement of $\triangle$ based on the incenters v_T of the tetrahedra T of $\triangle$, see Definition 16.23. In this section we discuss the following C^1 Worsey–Farin macro-element space:

$$\mathcal{S}_1(\triangle_{WF}) := \{s \in \mathcal{S}_3^1(\triangle_{WF}) : \, s \in C^2(v_T), \text{ all } T \in \triangle\}. \tag{18.8}$$

To define an MDS for $\mathcal{S}_1(\triangle_{WF})$, we need some more notation. For each vertex v of $\triangle$, let t_v be one of the tetrahedra in $\triangle_{WF}$ attached to v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let t_e be one of the tetrahedra in $\triangle_{WF}$ containing e, and let $E_1(e)$ denote the set of domain points in the tube of radius 1 around e which do not lie in the balls $D_1(u)$ or $D_1(v)$. Finally, let n_V and n_E be the number of vertices and edges in $\triangle$.

Theorem 18.11. *The space $\mathcal{S}_1(\triangle_{WF})$ has dimension $4n_V + 2n_E$. Moreover, the set*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_1(v) \cap t_v$,
2) $\mathcal{M}_e := E_1(e) \cap t_e$.

Proof: We use Theorem 17.10 to show that $\mathcal{M}$ is an MDS for $\mathcal{S}_1(\triangle_{WF})$. It then follows from Theorem 17.8 that the dimension of $\mathcal{S}_1(\triangle_{WF})$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the given formula. We need to show that if $s \in \mathcal{S}_1(\triangle_{WF})$, then we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ to arbitrary values, and all other coefficients will be determined such that no smoothness conditions are violated.

First, we fix all of the coefficients corresponding to the sets $\mathcal{M}_v$ to arbitrary values. Then by Theorem 15.17, all other coefficients corresponding to domain points in the balls $D_1(v)$ are uniquely and stably determined. In particular, for all $\eta \in D_1(v) \setminus \mathcal{M}_v$, c_η can be computed from coefficients in the set $\Gamma_\eta := \mathcal{M}_v \subseteq D_1(v)$, and

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \tag{18.9}$$

where K is a constant depending only on the smallest solid and face angles in $\triangle_{WF}$. Since the balls $D_1(v)$ do not overlap, none of the smoothness conditions are violated by the coefficients we have set so far.

Next, for each $e := \langle u, v \rangle$ of $\triangle$, we fix all of the coefficients corresponding to the set $\mathcal{M}_e$. Then by the C^1 smoothness around edges, all other coefficients corresponding to domain points in the set $E_1(e)$ will be uniquely determined, see Theorem 15.18. By Theorem 15.23, this is a stable local process, and (18.9) holds for these coefficients with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. None of the smoothness conditions are violated since the sets $E_1(e)$ do not overlap each other or any of the balls $D_1(v)$.

For each face F of $\triangle$, we now use the C^1 smoothness conditions in the face to compute the coefficients corresponding to the remaining domain points on the face. This computation is exactly the same as for the bivariate C^1 cubic spline space on the Clough–Tocher split of F, see Figure 6.2, and (18.9) holds for these coefficients with Γ_η equal to the union of the sets $\mathcal{M}_v$ and $\mathcal{M}_e$ over the vertices and edges of F.

Now let T be a tetrahedron in $\triangle$. We have already determined the coefficients of s corresponding to domain points on the shell $R_3(v_T)$. Using the C^1 smoothness conditions, we can compute all of the coefficients of s corresponding to the ten domain points on the edges of the shell $R_2(v_T)$. By the C^2 smoothness at v_T, we can consider the coefficients of s corresponding to the domain points in $D_2(v_T)$ to be those of a quadratic polynomial g. It is easy to see that the ten coefficients corresponding to domain points on the faces of $D_2(v_T)$ determine all other coefficients, and since the dimension of $\mathcal{P}_2$ is 10, we conclude that the coefficients of g corresponding to domain points in $D_2(v_T)$ are uniquely determined. Then all coefficients of s corresponding to domain points in $D_2(v_T)$ are uniquely and stably determined by subdivision.

We are not quite done with the proof. We still have to check that the coefficients satisfy the C^1 smoothness conditions across faces of $\triangle$. Suppose $t := \langle v_T, v_1, v_2, v_3 \rangle$ and $\tilde{t} := \langle v_{\tilde{T}}, v_1, v_2, v_3 \rangle$ are two tetrahedra in $\triangle_{WF}$ sharing a face $F := \langle v_1, v_2, v_3 \rangle$. Note that the split point v_F in the Clough–Tocher split F_{CT} of F lies on the line from v_T to $v_{\tilde{T}}$, and thus each of these C^1 conditions reduces to a univariate smoothness condition. Set $g := s|_t$ and $\tilde{g} := s|_{\tilde{t}}$, and let $f_2 := R_2(v_T) \cap t$ and $\tilde{f}_2 := R_2(v_{\tilde{T}}) \cap \tilde{t}$. Let $b_1, \ldots, b_6$ be the coefficients of g corresponding to the domain points $\xi_1, \ldots, \xi_6$ which lie on the edges of f_2. These six domain points are marked with black dots in Figure 18.3. Let $\tilde{b}_1, \ldots, \tilde{b}_6$ be the coefficients of $\tilde{g}$ associated with the corresponding domain points $\tilde{\xi}_1, \ldots, \tilde{\xi}_6$ on the edges of $\tilde{f}_2$. Let r, s be such that $v_{\tilde{T}} = s v_F + r v_T$. Then each of the C^1 smoothness conditions across F with a tip at one of the points ξ_i is already satisfied. Explicitly,

$$b_i = s c_i + r \tilde{b}_i. \tag{18.10}$$

Now let b be any other coefficient of g at a distance 1 from F. We have seen that b can be written as a linear combination of the $b_1, \ldots, b_6$, i.e, there

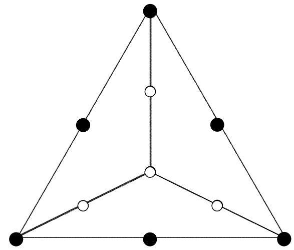


Fig. 18.3. The set f_2 in the proof of Theorem 18.11.

exist $\{\alpha_i\}_{i=1}^6$ such that $b = \sum_{i=1}^6 \alpha_i b_i$. The same equation also holds with b's replaced by either c's or $\tilde{b}$'s. But then using (18.10), we have

$$[1,\, -s,\, -r] \begin{bmatrix} b \\ c \\ \tilde{b} \end{bmatrix} = [1,\, -s,\, -r] \begin{bmatrix} b_1 \cdots b_6 \\ c_1 \cdots c_6 \\ \tilde{b}_1 \cdots \tilde{b}_6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_6 \end{bmatrix} = 0,$$

which shows that the C^1 smoothness condition involving $b, c, \tilde{b}$ is also satisfied. □

The constant in the stability of the MDS in Theorem 18.11 depends on the smallest solid and face angles in $\triangle_{WF}$. Although we believe that these angles are bounded below in terms of the smallest solid and face angles in $\triangle$, this remains an open conjecture. We now illustrate Theorem 18.11 for a single tetrahedron.

Example 18.12. Let T_{WF} be the Worsey–Farin split of a single tetrahedron T, and let $\mathcal{S}_1(T_{WF})$ be the associated macro-element space as defined in (18.8). Then $\dim \mathcal{S}_1(T_{WF}) = 28$.

Since $\mathcal{S}_1(\triangle_{WF})$ has a stable local MDS, Theorem 17.15 immediately implies that it has full approximation power.

Theorem 18.13. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 3$, there exists a spline $s_f \in \mathcal{S}_1(\triangle_{WF})$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,q,\Omega}, \tag{18.11}$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WF}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a stable local nodal determining set for $\mathcal{S}_1(\triangle_{WF})$, using the notation introduced at the end of Section 18.1.

Theorem 18.14. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e$$

is a stable local nodal minimal determining set for $\mathcal{S}_1(\triangle_{WF})$, *where*

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha| \le 1}$,
2) $\mathcal{N}_e := \{\varepsilon_{\eta^1_{e,1}} D_e^\beta\}_{|\beta|=1}$.

Proof: It is easy to see that the cardinality of the set $\mathcal{N}$ matches the dimension of $\mathcal{S}_1(\triangle_{WF})$ as given in Theorem 18.11. We already know that the set $\mathcal{M}$ defined in that theorem is an MDS for $\mathcal{S}_1(\triangle_{WF})$. Thus, to show that $\mathcal{N}$ is an NMDS, it suffices to show that if $s \in \mathcal{S}_1(\triangle_{WF})$, then fixing the values $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determines all coefficients in the set $\{c_\xi\}_{\xi \in \mathcal{M}}$.

For each $v \in \mathcal{V}$, we can use the formulae in Theorem 15.16 to determine all B-coefficients of s corresponding to domain points ξ in the ball $D_1(v)$ from the values $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$. By Theorem 15.17, this is a stable local process, and in fact we have

$$|c_\xi| \le K \sum_{\nu=0}^{1} |T_\xi|^\nu |s|_{\nu, T_\xi},$$

where T_ξ is a tetrahedron containing ξ, and K is a constant depending on the smallest angle in $\triangle_{WF}$. Now for each edge of $\triangle$, we can use Theorem 15.18 to compute the coefficients of s corresponding to domain points in $E_1(e)$, where $E_1(e)$ is as in Theorem 18.11. By Theorem 15.23 this is a stable process.

At this point we have stably computed all of the coefficients of s corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 18.11. But then all other coefficients are also locally and stably determined. $\square$

Theorem 18.14 shows that for any function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}_1(\triangle_{WF})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 1$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta^1_{e,1}) = D_e^\beta f(\eta^1_{e,1})$, all $|\beta| = 1$ and all edges e of $\triangle$.

The mapping which takes functions $f \in C^1(\Omega)$ to this Hermite interpolating spline defines a linear projector $\mathcal{I}^1_{WF}$ mapping $C^1(\Omega)$ onto $\mathcal{S}_1(\triangle_{WF})$. In particular, $\mathcal{I}^1_{WF} p = p$ for all trivariate polynomials of degree three. Now Theorem 17.22 implies the following error bound.

Theorem 18.15. *For every $f \in C^{m+1}(\Omega)$ with $0 \le m \le 3$,*

$$\|D^\alpha(f - \mathcal{I}^1_{WF} f)\|_\Omega \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega},$$

for all $|\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WF}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

As for the other macro-element spaces in this chapter, $\mathcal{S}_1(\triangle_{WF})$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$ of Theorem 18.11, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e.

Since the set $\mathcal{N}$ given in Theorem 18.14 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_1(\triangle_{WF})$, where each basis function is star-supported.

18.5. A C^1 Macro-element on the Worsey–Piper Split

Let $\triangle$ be an arbitrary tetrahedral partition of a polyhedral set $\Omega \in \mathbb{R}^3$, and let $\mathcal{V}$ be the set of vertices of $\triangle$. In this section we discuss the following C^1 Worsey–Piper macro-element space:

$$\mathcal{S}_1(\triangle_{WP}) := \mathcal{S}^1_2(\triangle_{WP}),$$

where $\triangle_{WP}$ is a *proper* Worsey–Piper refinement of $\triangle$, see Definitions 16.25 and 16.26. For each $v \in \mathcal{V}$, let t_v be some tetrahedron in $\triangle_{WP}$ with vertex at v. Let n_V be the number of vertices of $\triangle$.

Theorem 18.16. *dim $\mathcal{S}_1(\triangle_{WP}) = 4n_V$, and $\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v$ is a stable local minimal determining set, where $\mathcal{M}_v := D_1(v) \cap t_v$.*

Proof: By the results of Section 15.8, fixing the coefficients of s corresponding to the set $\mathcal{M}$ determines all the coefficients of s corresponding to domain points in the balls $D_1(v)$ for $v \in \mathcal{V}$. We now show that all other coefficients of s are determined in a consistent way, i.e., in such a way that all C^1 smoothness conditions are satisfied. First, for each face F of $\triangle$, since F has been subjected to a Powell–Sabin split, by Theorem 6.9, the coefficients corresponding to domain points in the disks $D_1(v)$ around the vertices of F stably determine the coefficients associated with the remaining domain points on F. Now let $T \in \triangle$, and consider the ball $D_1(v_T)$. The C^1

smoothness at v_T implies that the coefficients of s restricted to this ball can be considered as the coefficients of a linear polynomial g on a tetrahedron $\hat{T}$ which is congruent to T and which has been subjected to the Worsey–Piper split. We already have the coefficients of g at the four vertices of $\widetilde{T}$. Since the space $\mathcal{P}_1$ is of dimension four, this stably and uniquely determines all coefficients of g in $D_1(v_T)$, see Theorem 15.38. The coefficients of s corresponding to domain points in $D_1(v_T)$ can then be computed by subdivision using the de Casteljau algorithm.

To complete the proof, we have to check that the coefficients of s satisfy all C^1 smoothness conditions which have not already been explicitly used. First, for each edge e, we note that by the assumption that $\triangle_{WP}$ is proper, all domain points that lie within a distance of $e := \langle u, v \rangle$ and outside the disks $D_1(u)$ and $D_1(v)$ lie in a plane passing through the point v_e. Then arguing as in the proof of Theorem 18.11, it follows that the C^1 smoothness conditions involving these coefficients are satisfied. We also have to check the C^1 smoothness conditions across interior faces of $\triangle$. Let F be such a face. Then by the assumption that $\triangle_{WP}$ is proper, we know that v_F lies on the line connecting v_T and $v_{\tilde{T}}$, where T and $\widetilde{T}$ are the two tetrahedra in $\triangle$ which share the face F. Then the fact that the C^1 conditions involving coefficients within a distance of F are satisfied can be established using the same argument as was employed in Theorem 18.11. $\square$

The constant in the stability of the MDS in Theorem 18.16 depends on the smallest solid and face angles in $\triangle_{WP}$. Although we believe that these angles are bounded below in terms of the smallest solid and face angles in $\triangle$, this remains an open conjecture. We now illustrate Theorem 18.16 in the case when $\triangle$ consists of a single tetrahedron.

Example 18.17. Let T_{WP} be the Worsey–Piper split of a single tetrahedron T. Then $\dim \mathcal{S}_1(T_{WP}) = 16$.

Since $\mathcal{S}_1(\triangle_{WP})$ has a stable local MDS, Theorem 17.15 immediately implies that it has full approximation power.

Theorem 18.18. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 2$, there exists a spline $s_f \in \mathcal{S}_1(\triangle_{WP})$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WP}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set for $\mathcal{S}_1(\triangle_{WP})$ and then use it to construct a Hermite interpolating spline.

Theorem 18.19. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \{\varepsilon_v D^\alpha\}_{|\alpha| \le 1}$$

is a stable local nodal minimal determining set for $\mathcal{S}_1(\triangle_{WP})$.

Proof: Since $\#\mathcal{N}$ is equal to the dimension of $\mathcal{S}_1(\triangle_{WP})$, it suffices to show that setting $\{\lambda s\}_{\lambda \in \mathcal{N}}$ for a spline $s \in \mathcal{S}_1(\triangle_A)$ determines all B-coefficients of s corresponding to the MDS $\mathcal{M}$ of Theorem 18.16. For each $v \in \mathcal{V}$, we can compute all coefficients of s in the ball $D_1(v)$ directly from $\{D^\alpha s(v)\}_{|\alpha| \le 1}$ using the results of Section 15.8. This gives all coefficients corresponding to $\mathcal{M}$. The computation of coefficients is local and stable, and in particular if T_ξ is a tetrahedron containing ξ, then

$$|c_\xi| \le K \sum_{\nu=0}^{1} |T_\xi|^\nu |s|_{\nu, T_\xi},$$

where K is a constant depending on the smallest angle in $\triangle_{WP}$. $\square$

Theorem 18.19 shows that for any function $f \in C^1(\Omega)$, there is a unique spline $s \in \mathcal{S}_1(\triangle_{WP})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently

$$D^\alpha s(v) = D^\alpha f(v), \qquad \text{all } |\alpha| \le 1 \text{ and all } v \in \mathcal{V}.$$

This defines a linear projector $\mathcal{I}^1_{WP}$ mapping $C^1(\Omega)$ onto the superspline space $\mathcal{S}_1(\triangle_{WP})$. Theorem 17.22 implies the following error bound.

Theorem 18.20. *For every* $f \in C^{m+1}(\Omega)$ *with* $0 \le m \le 2$,

$$\|D^\alpha (f - \mathcal{I}^1_{WP} f)\|_\Omega \le K \, |\triangle|^{m+1-|\alpha|} |f|_{m+1,\Omega},$$

for all $0 \le |\alpha| \le m$. *If* Ω *is convex,* K *depends on* d *and the smallest solid and face angles in* $\triangle_{WP}$. *If* Ω *is nonconvex, then* K *also depends on the Lipschitz constant of the boundary of* Ω.

To get a stable local basis for $\mathcal{S}_1(\triangle_{WP})$, we can use the $\mathcal{M}$-basis of Theorem 17.16 corresponding to the MDS $\mathcal{M}$ of Theorem 18.16. The $\mathcal{N}$-basis corresponding to the NMDS $\mathcal{N}$ of Theorem 18.19 provides another stable local basis.

18.6. A C^2 Polynomial Macro-element

In this section we discuss the C^2 analog of the C^1 element of Section 18.2. Suppose $\triangle$ is a tetrahedral partition of a polyhedral set Ω in $\mathbb{R}^3$, and let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. In this section we discuss the following C^2 polynomial macro-element space

$$\mathcal{S}_2(\triangle) := \{s \in \mathcal{S}^2_{17}(\triangle) : s \in C^4(e), \text{ all } e \in \mathcal{E}, \text{ and } s \in C^8(v), \text{ all } v \in \mathcal{V}\}.$$

For each $v \in \mathcal{V}$, let T_v be some tetrahedron with vertex at v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let $E_4(e)$ be the set of domain points which lie in the tube of radius 4 around $e := \langle u, v\rangle$, but which are not contained in the balls $D_8(u)$ or $D_8(v)$, and let T_e be some tetrahedron in $\triangle$ containing the edge e. Finally, for each face F of $\triangle$, let $G_2(F)$ be the set of domain points which lie within a distance 2 of F, but which are not in any of the balls $D_8(v)$ or sets $E_4(e)$. Let T_F be some tetrahedron containing the face F. Let $n_{_V}, n_{_E}, n_{_F}, n_{_T}$ be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.21. *dim* $\mathcal{S}_2(\triangle) = 165n_{_V} + 40n_{_E} + 46n_{_F} + 56n_{_T}$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{M}_F \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_8(v) \cap T_v$,
2) $\mathcal{M}_e := E_4(e) \cap T_e$,
3) $\mathcal{M}_F := G_2(F) \cap T_F$,
4) $\mathcal{M}_T := \{\xi^T_{ijkl} : i, j, k, l \ge 3\}$.

Proof: To show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2(\triangle)$ we make use of Theorem 17.10. We need to show that the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline $s \in \mathcal{S}_2(\triangle)$ can be set to arbitrary values, and that the remaining coefficients of s are determined in such a way that all smoothness conditions are satisfied. For each vertex v, we begin by fixing the coefficients corresponding to domain points in the set $\mathcal{M}_v$. By the C^8 smoothness at v and the results of Section 15.8, all coefficients corresponding to domain points in the ball $D_8(v)$ are stably determined. In particular, for all $\eta \in D_8(v) \setminus \mathcal{M}_v$, c_η depends only on coefficients corresponding to domain points in $\Gamma_\eta = \mathcal{M}_v$, and by Theorem 15.17,

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \tag{18.12}$$

where K is a constant depending on the smallest solid and face angles in $\triangle$. It is also local since each such c_η depends only on other coefficients in the same ball. None of the smoothness conditions is violated since the balls $D_8(v)$ do not overlap.

Next we fix the coefficients corresponding to $\mathcal{M}_e$. Then for each edge $e := \langle u, v\rangle$, using the C^4 smoothness around e, Theorem 15.23 can be used to determine all coefficients c_η corresponding to the remaining domain points in the set $E_4(e)$. The computation of these coefficients is a stable local process, and (18.12) holds for all $\eta \in E_4(e)$ with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. None of the smoothness conditions has been violated since the sets $E_4(e)$ are disjoint from each other and from all balls $D_8(v)$.

For each face F of $\triangle$, we now fix the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_F}$. This determines all coefficients corresponding to domain points in $G_2(F) \cap T_F$. If F is an interior face, then the coefficients corresponding to $G_2(F) \cap \widetilde{T}_F$ are uniquely determined from the C^2 smoothness across F, where $\widetilde{T}_F$ is the other tetrahedron in $\triangle$ sharing the face F. This is a stable local process, and for each of these ξ, (18.12) holds with Γ_η equal to the union of $\mathcal{M}_F$ with all $\mathcal{M}_v$ and $\mathcal{M}_e$, where v and e are vertices and edges of F.

We have now determined all coefficients of s except for those corresponding to domain points in the sets $\mathcal{M}_T$, which can now be given arbitrary values. These sets are disjoint from each other, and so there are no smoothness conditions connecting coefficients associated with domain points in two such sets. We have shown that $\mathcal{M}$ is a stable local MDS. To finish the proof, we apply Theorem 17.8 which says that the dimension of $\mathcal{S}_2(\triangle)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the given formula. $\square$

Since $\mathcal{S}_2(\triangle)$ has a stable local MDS, Theorem 17.15 immediately implies that it has full approximation power.

Theorem 18.22. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 17$, there exists a spline $s_f \in \mathcal{S}_2(\triangle)$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a nodal minimal determining set for $\mathcal{S}_2(\triangle)$. For each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let

$$\begin{aligned} A_F^0 &:= \{\xi_{ijk}^{F,17} : i,j,k \ge 5\}, \\ A_F^1 &:= \{\xi_{ijk}^{F,16} : i,j,k \ge 4\}, \\ A_F^2 &:= \{\xi_{ijk}^{F,15} : i,j,k \ge 3\} \setminus \{\xi_{933}^{F,15}, \xi_{393}^{F,15}, \xi_{339}^{F,15}\}, \end{aligned} \tag{18.13}$$

where $\xi_{ijk}^{F,n} := (iv_1 + jv_2 + kv_3)/n$, for all $i+j+k=n$. Points in these sets are marked with $\oplus$ in Figure 18.4. Note that the points in A_F^1 and A_F^2 are

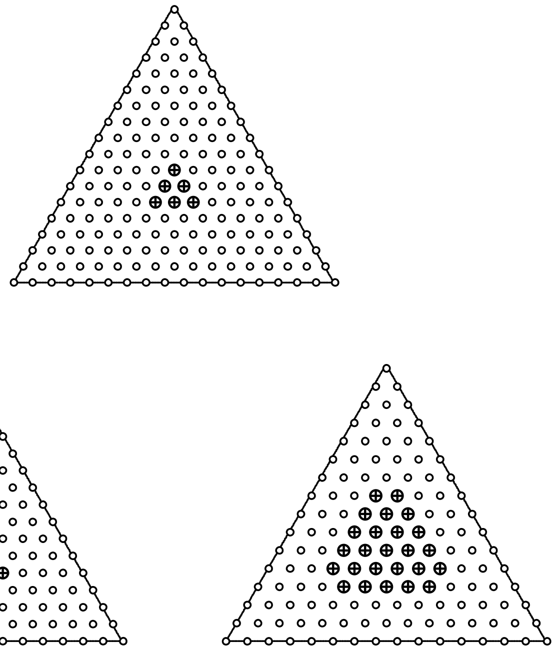


Fig. 18.4. Points $\oplus$ in the sets A_F^0, A_F^1, and A_F^2 of (18.13).

not in the set of domain points $\mathcal{D}_{17,\triangle}$ for $\mathcal{S}_2(\triangle)$. For each tetrahedron T, let v_T be its barycenter.

Theorem 18.23. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{F \in \mathcal{F}} \left(\mathcal{N}_F^0 \cup \mathcal{N}_F^1 \cup \mathcal{N}_F^2 \right) \cup \bigcup_{T \in \triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha| \le 8}$,
2) $\mathcal{N}_e := \bigcup_{\ell=1}^4 \bigcup_{m=1}^{\ell} \{\varepsilon_{\eta_{e,m}^\ell} D_e^\beta\}_{|\beta|=\ell}$,
3) $\mathcal{N}_F^0 := \{\varepsilon_\xi\}_{\xi \in A_F^0}$,
4) $\mathcal{N}_F^1 := \{\varepsilon_\xi D_F\}_{\xi \in A_F^1}$,
5) $\mathcal{N}_F^2 := \{\varepsilon_\xi D_F^2\}_{\xi \in A_F^2}$,
6) $\mathcal{N}_T := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha| \le 5}$.

Proof: It is easy to check that $\#\mathcal{N}$ is equal to the dimension of $\mathcal{S}_2(\triangle)$ as given in Theorem 18.21. Thus, to show that $\mathcal{N}$ is a stable local NMDS for $\mathcal{S}_2(\triangle)$, it suffices to show that given the values $\{\lambda s\}_{\lambda\in\mathcal{N}}$ for a spline $s \in \mathcal{S}_2(\triangle)$, all of its B-coefficients can be stably and locally computed. For each $v \in \mathcal{V}$, we can use the results of Section 15.8 to compute the coefficients $\{c_\xi\}_{\xi\in D_8(v)}$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_v}$. By Theorem 15.17, this is a local and stable process. Indeed, if T_ξ is a tetrahedron containing ξ, then for all $\xi \in D_4(v)$,

$$|c_\xi| \le K \sum_{\nu=0}^{8} |T_\xi|^\nu |s|_{\nu,T_\xi}, \tag{18.14}$$

for some constant K depending only on the smallest solid and face angles in $\triangle$.

Next, for each edge $e \in \mathcal{E}$, we compute the coefficients corresponding to $\xi \in E_4(e)$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_e}$. By Theorem 15.23 these computations are local and stable in the sense that (18.14) holds.

Now fix a face F of $\triangle$. All coefficients of s corresponding to domain points on F have been computed except for the six corresponding to the points in A_F^0. By Theorem 15.24 and Lemma 2.25 they can be computed from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_F^0}$ by solving an appropriate 6×6 linear system. Next we consider coefficients corresponding to the remaining domain points in T_F at a distance of 1 to F. Using Theorem 15.24 and Lemma 2.25 we can compute these coefficients from the values $\{D_{{}_F} s(\eta)\}_{\eta\in A_F^1}$ by solving a nonsingular 15×15 system whose matrix is the same for every face F. By Theorem 15.25, this computation is also local and stable. If F is an interior face, then the coefficients associated with the analogous points in the other tetrahedron containing F can be computed using the C^1 smoothness conditions across F. Using Theorem 15.24 and Lemma 2.23 we can now compute the coefficients of s that lie at a distance of 2 from a face by solving a nonsingular 25×25 linear system which is the same for every face.

Finally, for each T in $\triangle$, we compute the coefficients corresponding to the remaining domain points in T from the data $\{D^\alpha s(v_{{}_T})\}_{|\alpha|\le 5}$. This involves solving a nonsingular 56×56 system which is the same for every tetrahedron, see Theorem 15.26. $\square$

Theorem 18.23 shows that for any function $f \in C^8(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 8$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,j}^i) = D_e^\beta f(\eta_{e,j}^i)$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 4$, and all edges e of $\triangle$,
3) $s(\xi) = f(\xi)$, all $\xi \in A_F^0$ and all faces F of $\triangle$,

4) $D_F s(\xi) = D_F f(\xi)$, all $\xi \in A_F^1$ and all faces F of $\triangle$,

5) $D_F^2 s(\xi) = D_F^2 f(\xi)$, all $\xi \in A_F^2$ and all faces F of $\triangle$,

6) $D^\alpha s(v_T) = D^\alpha f(v_T)$, all $|\alpha| \le 5$.

This defines a linear projector $\mathcal{I}_P^2$ mapping $C^8(\Omega)$ onto the superspline space $\mathcal{S}_2(\triangle)$. In particular, $\mathcal{I}_P^2$ reproduces polynomials of degree 17. Since the NMDS of Theorem 18.23 is local and stable, Theorem 17.22 implies the following error bound.

Theorem 18.24. *For every $f \in C^{m+1}(\Omega)$ with $7 \le m \le 17$,*

$$\|D^\alpha(f - \mathcal{I}_P^2 f)\|_\Omega \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega}, \tag{18.15}$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We conclude this section by noting that $\mathcal{S}_2(\triangle)$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$ in Theorem 18.21, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),

2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,

3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,

4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.23 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_2(\triangle)$, where each basis function is star-supported.

18.7. A C^2 Macro-element on the Alfeld Split

In this section we give a C^2 analog of the C^1 macro-element discussed in Section 18.3. Given an arbitrary tetrahedral partition $\triangle$ of a polyhedral set $\Omega \in \mathbb{R}^3$, let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. Let $\triangle_A$ be the tetrahedral partition which is obtained from $\triangle$ by applying the Alfeld split to each tetrahedron T in $\triangle$ using the barycenter v_T, see Definition 16.21. In this section we discuss the following C^2 macro-element space:

$$\begin{aligned}\mathcal{S}_2(\triangle_A) := \{s \in \mathcal{S}_{13}^2(\triangle_A) :\ & s \in C^3(e), \ \text{all } e \in \mathcal{E}, \\ & s \in C^6(v), \ \text{all } v \in \mathcal{V}, \\ & s \in C^{12}(v_T), \ \ \text{all } T \in \triangle\}.\end{aligned} \tag{18.16}$$

For each $v \in \mathcal{V}$, let t_v be some tetrahedron in $\triangle_A$ with vertex at v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let $E_3(e)$ be the set of domain points which lie on the tube of radius 3 around e, but which are not contained in either $D_6(u)$ or $D_6(v)$. Let t_e be some tetrahedron in $\triangle_A$ containing e. For each face F of $\triangle$, let t_F be some tetrahedron in $\triangle_A$ containing the face F, and let $G_2(F)$ be the set of all domain points which lie within a distance 2 of F, but which are not in any of the balls $D_6(v)$ or tubes $E_2(e)$. Finally, for each tetrahedron T of $\triangle$, let t_T be some tetrahedron in $\triangle_A$ containing v_T. Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.25. *dim* $\mathcal{S}_2(\triangle_A) = 84n_V + 20n_E + 31n_F + 35n_T$, *and*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} (\mathcal{M}_F^0 \cup \mathcal{M}_F^1 \cup \mathcal{M}_F^2) \cup \bigcup_{T\in\triangle} \mathcal{M}_T \tag{18.17}$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_6(v) \cap t_v$,
2) $\mathcal{M}_e := E_3(e) \cap t_e$,
3) $\mathcal{M}_F^0 := \{\xi_{0ijk}^{t_F} : i,j,k \ge 4, \ i+j+k = 13\}$,
4) $\mathcal{M}_F^1 := \{\xi_{1ijk}^{t_F} : i,j,k \ge 3, \ i+j+k = 12\}$,
5) $\mathcal{M}_F^2 := \{\xi_{2ijk}^{t_F} : i,j,k \ge 2, \ i+j+k = 11\} \setminus \{\xi_{2272}^{t_F}, \xi_{2227}^{t_F}, \xi_{2722}^{t_F}\}$,
6) $\mathcal{M}_T := D_4(v_T) \cap t_T$.

Proof: The proof is similar to the proof of Theorem 18.6. To show that $\mathcal{M}$ is a stable local MDS for $\mathcal{S}_2(\triangle_A)$, using Theorem 17.10, we need to show that if $s \in \mathcal{S}_2(\triangle_A)$, then we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ to arbitrary values, and then all other coefficients are determined in a stable and local way while not violating any smoothness conditions. First, we fix all of the coefficients corresponding to domain points in the sets $\mathcal{M}_v$. Then for each $v \in \mathcal{V}$, by the C^6 smoothness at v, the results of Section 15.8 show that all other coefficients corresponding to domain points in the ball $D_6(v)$ can be stably computed. By Theorem 15.17, for all $\eta \in D_6(v) \setminus \mathcal{M}_v$,

$$|c_\eta| \le K \max_{\xi\in\Gamma_\eta} |c_\xi|, \tag{18.18}$$

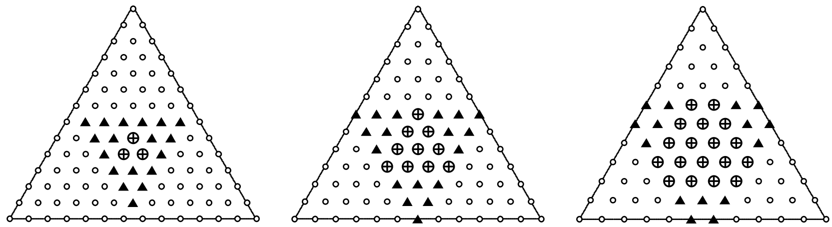


Fig. 18.5. Points $\oplus$ in the sets $\mathcal{M}_F^0$, $\mathcal{M}_F^1$, and $\mathcal{M}_F^2$ of Theorem 18.25.

where $\Gamma_\eta := \mathcal{M}_v$ and K is a constant depending on the smallest solid and face angles in $\triangle_A$. None of the smoothness conditions defining $\mathcal{S}_2(\triangle_A)$ has been violated since the balls $D_6(v)$ do not overlap.

Next we fix all of the coefficients corresponding to the sets $\mathcal{M}_e$. Then for each edge $e := \langle u, v \rangle$, using the C^3 smoothness conditions, we can stably compute all coefficients corresponding to the domain points in $E_3(e)$ from those in $\mathcal{M}_e$. By Theorem 15.23, (18.18) holds for these coefficients with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. No smoothness conditions have been violated since the sets $\mathcal{M}_e$ do not overlap each other or any of the balls $D_6(v)$.

Given a face $F := \langle v_1, v_2, v_3 \rangle \in \mathcal{F}$, suppose $t_F := \langle v_1, v_2, v_3, v_T \rangle$ is the corresponding tetrahedron used in defining $\mathcal{M}$. We now focus on points in the set $G_2(F) \cap t_F = \mathcal{M}_F^0 \cup \mathcal{M}_F^1 \cup \mathcal{M}_F^2$. The points in $\mathcal{M}_F^0 = \{\xi_{5440}^{t_F}, \xi_{4540}^{t_F}, \xi_{4450}^{t_F}\}$ are marked with $\oplus$ in Figure 18.5 (left). Similarly, the points in $\mathcal{M}_F^1$ and $\mathcal{M}_F^2$ are marked with $\oplus$ in Figure 18.5 (mid) and Figure 18.5 (right), respectively. Observe that on F, we have already determined all coefficients except those corresponding to $\mathcal{M}_F^0$ which we now fix. On the subface of t_F immediately behind F, we have already determined all coefficients except for those corresponding to $\mathcal{M}_F^1$, which we now fix. Similarly, on the subface of t_F at a distance 2 behind F we have already determined all coefficients except for those corresponding to $\mathcal{M}_F^2$, which we now fix. If F is a boundary face, this determines all coefficients corresponding to domain points in the set $G_2(F)$. If F is an interior face, then using the C^2 smoothness across F, we can now uniquely compute the coefficients of s corresponding to the remaining points in $G_2(F)$. Since only smoothness conditions are used, this is a stable process, and (18.18) holds for all $\xi \in G_2(F)$. No inconsistencies in the smoothness conditions can arise since the sets $G_2(F)$ are disjoint from each other.

Now fix T in $\triangle$. Then by the C^{12} smoothness at v_T, we may consider the B-coefficients of s corresponding to domain points in the ball $D_{12}(v_T)$ to be those of a polynomial g of degree 12 on a tetrahedron $\hat{T}$ which is congruent to T, and which has been subjected to the Alfeld split. The space $\mathcal{P}_{12}$ has dimension 455. We have already uniquely determined the coefficients of g corresponding to domain points of g in balls of radius 5 around each vertex of $\hat{T}$, in tubes of radius 2 around each edge, and on the faces of the shells $R_{12}(v_T)$ and $R_{11}(v_T)$. Thus, a total of $4 \times 56 + 6 \times 14 + 4 \times 10 + 4 \times 18 = 420$ coefficients are already determined. Now setting the coefficients of s corresponding to $\mathcal{M}_T := D_4(v_T) \cap t_v$ determines the values $\{D^\alpha g(v_T)\}_{|\alpha| \le 4}$. By Theorem 15.26 this stably determines all of the remaining coefficients of g. Applying subdivision, we see that all remaining coefficients of s are uniquely and stably determined.

To complete the proof, we note that the dimension of $\mathcal{S}_2(\triangle_A)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the given formula. □

The proof of this theorem shows that the constant of stability for the minimal determining set $\mathcal{M}$ depends on the smallest solid and face angles in $\triangle_A$. However, in view of Theorem 16.22, these are bounded below in terms of the corresponding angles in $\triangle$. We now illustrate Theorem 18.25 in the case of a single tetrahedron.

Example 18.26. Let $\mathcal{S}_2(T_A)$ be the macro-element space (18.16) associated with the Alfeld split T_A of a single tetrahedron T. Then $\dim \mathcal{S}_2(T_A) = 615$.

Since $\mathcal{S}_2(\triangle_A)$ has a stable local MDS, Theorem 17.15 immediately implies that it has full approximation power.

Theorem 18.27. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 13$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_A)$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\, |\triangle|^{m+1-|\alpha|}\; |f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a stable local nodal determining set for $\mathcal{S}_2(\triangle_A)$. For each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$, let

$$\begin{aligned}
A_F^0 &:= \{\xi_{ijk}^{F,13} : i,j,k \ge 4\} = \{\xi_{544}^{F,13}, \xi_{454}^{F,13}, \xi_{445}^{F,13}\}, \\
A_F^1 &:= \{\xi_{ijk}^{F,12} : i,j,k \ge 3\}, \\
A_F^2 &:= \{\xi_{ijk}^{F,11} : i,j,k \ge 2\} \setminus \{\xi_{272}^{F,11}, \xi_{227}^{F,11}, \xi_{722}^{F,11}\},
\end{aligned}$$

where $\xi_{ijk}^{F,n} := (iv_1 + jv_2 + kv_3)/n$ for $i+j+k = n$. We mark the position of these points with $\oplus$ in Figure 18.5. These are the same figures used to illustrate $\mathcal{M}_F^0$, $\mathcal{M}_F^1$, and $\mathcal{M}_F^2$ earlier, but they are not the same points. The points in A_F^1 and A_F^2 are not in the set of domain points of $\mathcal{S}_2(\triangle_A)$. The cardinalities of these three sets are 3, 10, and 18, respectively.

Theorem 18.28. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} (\mathcal{N}_F^0 \cup \mathcal{N}_F^1 \cup \mathcal{N}_F^2) \cup \bigcup_{T\in\triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_A)$, *where*

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 6}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^3 \bigcup_{j=1}^i \{\varepsilon_{\eta_{e,j}^i} D_e^\alpha\}_{|\alpha|=i}$,
3) $\mathcal{N}_F^0 := \{\varepsilon_\xi\}_{\xi\in A_F^0}$,
4) $\mathcal{N}_F^1 := \{\varepsilon_\xi D_F\}_{\xi\in A_F^1}$,
5) $\mathcal{N}_F^2 := \{\varepsilon_\xi D_F^2\}_{\xi\in A_F^2}$,
6) $\mathcal{N}_T := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha|\le 4}$.

Proof: It is easy to check that $\#\mathcal{N}$ is equal to the dimension of $\mathcal{S}_2(\triangle_A)$ as given in Theorem 18.25. Thus, to show that $\mathcal{N}$ is a stable local NMDS for $\mathcal{S}_2(\triangle_A)$, it suffices to show that given the values $\{\lambda s\}_{\lambda\in\mathcal{N}}$ for a spline $s \in \mathcal{S}_2(\triangle)$, all of its B-coefficients can be stably and locally computed. First, we examine the coefficients in the balls $D_6(v)$. For each $v \in \mathcal{V}$, it is clear that we can use the formulae in Theorem 15.16 to compute the coefficients $\{c_\xi\}_{\xi\in D_6(v)}$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_v}$. This is a stable process, and indeed if T_ξ is a tetrahedron containing ξ, then for all $\xi \in D_6(v)$,

$$|c_\xi| \le K \sum_{\nu=0}^{6} |T_\xi|^\nu |s|_{\nu,T_\xi}, \tag{18.19}$$

where K depends only on the smallest solid and face angles in $\triangle_A$.

Now for each edge $e \in \mathcal{E}$, we use the results of Section 15.9 to stably compute the coefficients corresponding to $\xi \in E_3(e)$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_e}$. Next, for each face F of $\triangle$, we use Theorem 15.24 to compute the coefficients corresponding to $\xi \in G_2(F) \cap t_F$, where $G_2(F)$ and t_F are as in Theorem 18.25. This involves solving 3×3, 10×10, and 18×18 linear systems corresponding to the sets A_F^0, A_F^1, and A_F^2, respectively. The fact that these systems are nonsingular follows from Lemmas 2.23 and 2.25.

If F is an interior face of $\triangle$, we use C^2 smoothness across F to compute the coefficients in $G_2(F) \cap \tilde{t}_F$, where $\tilde{t}_F$ is the other tetrahedron sharing the face F. These computations are also stable.

At this point, we have determined all the coefficients of s corresponding to domain points in the MDS $\mathcal{M}$ of Theorem 18.25, except for those in the sets $\mathcal{M}_T$. But these can be computed directly from the data $\{\lambda s\}_{\lambda\in\mathcal{N}_T}$ using Theorem 15.26. Since $\mathcal{M}$ is a stable local MDS, it follows that all coefficients of s can be computed from the nodal data and (18.19) holds. $\square$

Theorem 18.28 shows that for any function $f \in C^6(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_A)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 6$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,j}^i) = D_e^\beta f(\eta_{e,j}^i)$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 3$, for all edges e of $\triangle$,
3) $s(\xi) = f(\xi)$, all $\xi \in A_F^0$, all faces F of $\triangle$,
4) $D_F s(\xi) = D_F f(\xi)$, all $\xi \in A_F^1$ and all faces F of $\triangle$,
5) $D_F^2 s(\xi) = D_F^2 f(\xi)$, all $\xi \in A_F^2$ and all faces F of $\triangle$,
6) $D^\alpha s(v_T) = D^\alpha f(v_T)$, all $|\alpha| \le 4$.

This defines a linear projector $\mathcal{I}_A^2$ mapping $C^6(\Omega)$ onto the superspline space $\mathcal{S}_2(\triangle_A)$. In particular, $\mathcal{I}_A^2$ reproduces polynomials of degree thirteen. Since the the NMDS of Theorem 18.28 is local and stable, Theorem 17.22 implies the following error bound.

Theorem 18.29. *For every $f \in C^{m+1}(\Omega)$ with $5 \le m \le 13$,*

$$\|D^\alpha(f - \mathcal{I}_A^2 f)\|_\Omega \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega}, \tag{18.20}$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We conclude this section by noting that $\mathcal{S}_2(\triangle_A)$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$ in Theorem 18.25, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.28 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_2(\triangle_A)$, where each basis function is star-supported.

18.8. A C^2 Macro-element on the Worsey–Farin Split

Given an arbitrary tetrahedral partition $\triangle$ of a polyhedral set $\Omega \in \mathbb{R}^3$, let $\mathcal{V}, \mathcal{F}$ and $\mathcal{E}$ be the sets of vertices, faces and edges of $\triangle$, respectively. In this section we discuss a C^2 macro-element space which is defined over the tetrahedral partition $\triangle_{WF}$ defined in Definition 16.23. Let $\mathcal{E}^c$ be the set all edges in $\triangle_{WF}$ that connect the incenters v_T to the face split points v_F. In this section we discuss the following C^2 Worsey–Farin macro-element space:

$$\begin{aligned} \mathcal{S}_2(\triangle_{WF}) := \{s \in \mathcal{S}_9^2(\triangle_{WF}) :\ & s \in C^3(e), \ \text{all } e \in \mathcal{E}, \\ & s \in C^7(e), \ \text{all } e \in \mathcal{E}^c, \\ & s \in C^4(v), \ \text{all } v \in \mathcal{V}, \\ & s \in C^7(v_T), \ \text{all } T \in \triangle\}. \end{aligned} \tag{18.21}$$

To define an MDS for $\mathcal{S}_2(\triangle_{WF})$ we need some more notation. For each vertex v of $\triangle$, let t_v be one of the tetrahedra in $\triangle_{WF}$ attached to v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let t_e be one of the tetrahedra in $\triangle_{WF}$ containing e, and let $E_3(e)$ denote the set of domain points in the tube of radius 3 around e which do not lie in the balls $D_4(u)$ or $D_4(v)$. Finally, for each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let v_T be the barycenter of some tetrahedron T containing F, and let $t_F := \langle v_T, v_F, v_2, v_3\rangle$, where v_F is the split point in the face F. Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.30. *The space $\mathcal{S}_2(\triangle_{WF})$ has dimension $35n_V + 20n_E + 9n_F + 20n_T$. Moreover, the set*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} (\mathcal{M}_F^0 \cup \mathcal{M}_F^1 \cup \mathcal{M}_F^2) \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_4(v) \cap t_v$,
2) $\mathcal{M}_e := E_3(e) \cap t_e$,
3) $\mathcal{M}_F^0 := \{\xi_{0900}^{t_F}, \xi_{0810}^{t_F}, \xi_{0180}^{t_F}\}$,
4) $\mathcal{M}_F^1 := \{\xi_{1800}^{t_F}, \xi_{1710}^{t_F}, \xi_{1170}^{t_F}\}$,
5) $\mathcal{M}_F^2 := \{\xi_{2700}^{t_F}, \xi_{2610}^{t_F}, \xi_{2160}^{t_F}\}$,
6) $\mathcal{M}_T := D_3(v_T) \cap t_{v_T}$.

Proof: We apply Theorem 17.10 to show that $\mathcal{M}$ is an MDS for $\mathcal{S}_2(\triangle_{WF})$. Theorem 17.8 shows that the dimension of $\mathcal{S}_2(\triangle_{WF})$ is just the cardinality of $\mathcal{M}$, which is easily seen to be equal to the given formula. To show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2(\triangle_{WF})$, we need to show that if $s \in \mathcal{S}_2(\triangle_{WF})$, then we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ to arbitrary values, and all remaining coefficients will be determined in such a way that all smoothness conditions are satisfied.

First, we fix all coefficients corresponding to the sets $\mathcal{M}_v$. Then by the C^4 smoothness at vertices and the results of Section 15.8, all coefficients corresponding to domain points in $D_4(v) \setminus \mathcal{M}_v$ will be uniquely determined by the formulae in Theorem 15.16. By Theorem 15.17 this is a stable local process. In particular, for all $\eta \in D_4(v) \setminus \mathcal{M}_v$, c_η can be computed from coefficients in the set $\Gamma_\eta := \mathcal{M}_v \subseteq D_4(v)$, and

$$|c_\eta| \leq K \max_{\xi \in \Gamma_\eta} |c_\xi|, \tag{18.22}$$

where K is a constant depending only on the smallest solid and face angles in $\triangle_{WF}$. Since the balls $D_4(v)$ do not overlap, none of the smoothness conditions involving these coefficients can be violated.

Next we fix the coefficients corresponding to the sets $\mathcal{M}_e$. Then by the C^3 smoothness around edges, all other coefficients corresponding to domain points in the sets $E_3(e)$ will be uniquely determined, see Theorem 15.18. By Theorem 15.23, this is a stable local process, and (18.22) holds for these coefficients with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. None of the smoothness conditions has been violated since the sets $E_3(e)$ do not overlap.

We now fix the coefficients of s for all remaining domain points in the set $\mathcal{M}$, and show that s is consistently determined by smoothness conditions. Let T be a tetrahedron T in $\triangle$. First we examine the shell $R_9(v_T)$. Let $F := \langle v_1, v_2, v_3 \rangle$ be a typical face of this shell, and let v_F be the associated barycenter. We may regard the coefficients of s corresponding to domain points on this face as those of a bivariate spline $g \in \mathcal{S}_9^2(F_{CT})$ defined on the Clough–Tocher split F_{CT} of F, see Figure 18.6 (left). The fact that $s \in C^7(e)$ for the edge e connecting the barycenter v_T of T with v_F implies that g is also in $C^7(v_F)$. We have already consistently determined the coefficients of g corresponding to the domain points marked with black dots and triangles in Figure 18.6 (left), and have set the three coefficients corresponding to the domain points marked with $\oplus$. We now claim that the coefficients corresponding to the remaining domain points on F are consistently determined by smoothness conditions.

To prove this claim, it suffices to focus on the domain points in the disk $D_7(v_F)$. Due to the C^7 smoothness at v_F, we can regard these coefficients as those of a (subdivided) polynomial p of degree seven on a triangle $\widetilde{F} := \langle u_1, u_2, u_3 \rangle$ as in Figure 18.7. The coefficients of p corresponding to domain points marked with black dots, triangles, and $\oplus$ are all determined. We claim that this determines all coefficients of p. Suppose that we assign the value zero to all coefficients of p corresponding to these points. This implies that p and its first cross derivatives at all points along the three edges of the must be zero. For each $i = 1, 2, 3$, let ℓ_i be the linear polynomial that vanishes on e_i. Then by Bezout's theorem, we can write $p = \ell_1^2 \, \ell_2^2 \, \ell_3^2 \, q$, where

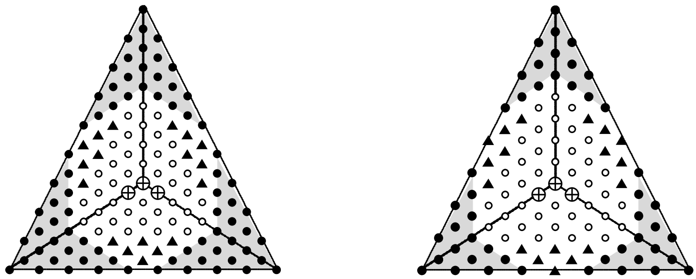


Fig. 18.6. Domain points of $\mathcal{S}_2(\triangle_{WF})$ on $R_9(v_T)$ and $R_8(v_T)$, respectively.

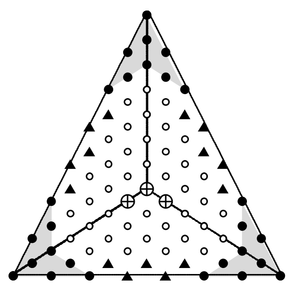


Fig. 18.7. Domain points of $\mathcal{S}_2(\triangle_{WF})$ on $R_7(v_T)$.

q is a linear polynomial. Since the three coefficients of p corresponding to the domain points marked with $\oplus$ are zero, we know that $D^\alpha p(v_F) = 0$ for $|\alpha| \leq 1$. This implies $D^\alpha q(v_F) = 0$ for $|\alpha| \leq 1$, and we conclude that $q \equiv 0$ and thus $p \equiv 0$. Now exactly 36 of the marked points in Figure 18.7 can be set independently, and since the dimension of the space of bivariate polynomials of degree seven is $36 = \#\Gamma$, it follows that these points consistently determine all coefficients of p.

Our next step is to show that all coefficients of s corresponding to domain points on the shell $R_8(v_T)$ are consistently determined. Let $F_8 := R_8(v_T) \cap T$. We can regard the domain points on F_8 as those of a bivariate spline $g \in \mathcal{S}_8^2(F_{CT})$ defined on the Clough–Tocher split of F_8, see Figure 18.6 (right). The fact that $s \in C^7(e)$ for the edge e connecting the barycenter v_T of T with v_F implies that g is also in C^7 at the split point v_F. The coefficients of g corresponding to the domain points marked with black dots and triangles are already determined. If the tetrahedron t_F used in defining $\mathcal{M}_F^1$ is a subtetrahedron of T, then we have already set the coefficients of s corresponding to the three points marked with $\oplus$ in

Figure 18.6 (right). If not, using C^1 smoothness across F, we can compute these coefficients from the analogous ones in the tetrahedron on the other side of F. Now the same argument as in the previous paragraph shows that the coefficients of g and thus of s are consistently determined on F_8.

Before proceeding, we have to make sure that the coefficients computed so far satisfy all of the smoothness conditions across the faces of $\triangle$. Suppose $T := \langle v_T, v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_{\tilde{T}}, v_1, v_2, v_3\rangle$ are two tetrahedra of $\triangle$ sharing a face $F := \langle v_1, v_2, v_3\rangle$. Note that the split point v_F lies on the line from v_T to $v_{\tilde{T}}$. Let $g := s|_T$ and $\tilde{g} := s|_{\widetilde{T}}$. We now check that g and $\tilde{g}$ satisfy all C^1 smoothness conditions across F. By the geometry, there exist r, s such that for each subtetrahedron $\tilde{t} := \langle v_{\tilde{T}}, v_F, v_1, v_2\rangle$ of T, the barycentric coordinates of v_T with respect to $\tilde{t}$ are $(r, s, 0, 0)$ Thus, each of the C^1 smoothness conditions involving coefficients associated with domain points on F reduces to a relationship of the form

$$b = sc + r\tilde{b}.$$

Here b is a coefficient of g corresponding to a domain point ξ_b in T which lies at a distance 1 from F, i.e., in F_8. Similarly, $\tilde{b}$ is a coefficient of $\tilde{g}$ corresponding to a domain point $\xi_{\tilde{b}}$ in $\widetilde{T}$ which lies at a distance 1 from F, i.e., in $\widetilde{F}_8 := R_8(v_{\tilde{T}}) \cap \widetilde{T}$. The coefficient c corresponds to the domain point on F which lies on the straight line between ξ_b and $\xi_{\tilde{b}}$. Let Γ_8 be the set of $n := 69$ domain points on F_8 marked with either a black dot, a triangle, or $\oplus$ in Figure 18.6 (right). Let $\{b_i\}_{i=1}^n$ be the corresponding coefficients of g, and let $\{\tilde{b}_i\}_{i=1}^n$ the analogous coefficients of $\tilde{g}$. Let $\{c_i\}_{i=1}^n$ be the coefficients of g which are involved in the C^1 smoothness conditions with the b_i and $\tilde{b}_i$. Then by the smoothness of s at vertices and around edges, it is clear that all C^1 continuity conditions with tips at points in Γ_8 are satisfied, i.e.,

$$b_i = sc_i + r\tilde{b}_i, \qquad i = 1, \ldots, n. \tag{18.23}$$

Now let ξ be any other domain point on F_8, and let $b, c, \tilde{b}$ be the coefficients entering into the C^1 smoothness condition with tip at ξ. Then as we saw above, b can be computed as a linear combination of the $b_1, \ldots, b_n$, i.e., there exist $\{\alpha_i\}_{i=1}^n$ such that

$$b = \sum_{i=1}^n \alpha_i b_i. \tag{18.24}$$

Since F_8 and $\widetilde{F}_8$ are just scaled versions of $F_9 := F$, it follows that (18.24) also holds with b's replaced by either c's or $\tilde{b}$'s. But then using (18.23), we have

$$[1,\ -s,\ -r] \begin{bmatrix} b \\ c \\ \tilde{b} \end{bmatrix} = [1,\ -s,\ -r] \begin{bmatrix} b_1 \cdots b_n \\ c_1 \cdots c_n \\ \tilde{b}_1 \cdots \tilde{b}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0,$$

which shows that the C^1 smoothness condition with tip at ξ is also satisfied.

For each tetrahedron T, we now compute coefficients corresponding to domain points on the shells $R_7(v_T)$. Let $F_7 := R_7(v_T) \cap T$. Figure 18.7 shows the domain points on F_7. Coefficients corresponding to the black dots and triangles are already known. Coefficients corresponding to the domain points marked with $\oplus$ in the figure have either been set, or can be computed from the corresponding coefficients in a neighboring tetrahedron $\widetilde{T}$ by C^2 smoothness conditions across the common face F. As argued above, it follows that the coefficients of s corresponding to all remaining domain points on F are consistently determined. We must now check that these computed coefficients satisfy all of the C^2 smoothness conditions across F. Each domain point in F_7 is the tip of a C^2 smoothness condition. Assuming a, c, d, e are the coefficients corresponding to domain points on $F_7, F_9, \widetilde{F}_8, \widetilde{F}_7$, the typical condition has the form

$$a = s^2 c + 2rsd + r^2 e,$$

where r, s are as before. By construction, these smoothness conditions are satisfied for all points ξ marked with black dots, triangles, or $\oplus$ in Figure 18.7. There are $n := 45$ such points. Writing $\{a_i, c_i, d_i, e_i\}_{i=1}^n$ for the associated coefficients, we have

$$a_i = s^2 c_i + 2rsd_i + r^2 e_i, \qquad i = 1, \ldots, n.$$

Now if ξ is any other domain point on F_7, then arguing as before, we see that there exist α_i such that

$$[1,\ -s^2,\ -2rs,\ -r^2] \begin{bmatrix} a \\ c \\ d \\ e \end{bmatrix} = [1,\ -s^2,\ -2rs,\ -r^2] \begin{bmatrix} a_1 \cdots a_n \\ c_1 \cdots c_n \\ d_1 \cdots d_n \\ e_1 \cdots e_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0,$$

which shows that the C^2 smoothness condition with tip at ξ is also satisfied.

To complete the proof, for each T in $\triangle$, we now examine the ball $D_7(v_T)$. By the C^7 smoothness at v_T, on this ball s can be considered to be a (subdivided) polynomial g of degree seven on a tetrahedron $\widetilde{T}$ similar to T. We already know all B-coefficients of g on the faces of $\widetilde{T}$. This leaves the 20 coefficients $\{\xi^{\widetilde{T}}_{ijkl} : i, j, k, l \geq 1\}$. Setting the coefficients of s corresponding to the domain points in $\mathcal{M}_T$ is equivalent to setting the derivatives of s up to order 3 at v_T. But then by Theorem 15.26, g is uniquely and stably determined. $\square$

The constant in the stability of the MDS $\mathcal{M}$ in Theorem 18.30 depends on the smallest solid and face angles in $\triangle_{WF}$. As mentioned on page 518, we believe that these angles are bounded below in terms of the smallest solid and face angles in $\triangle$, but this remains an open conjecture. We now illustrate the theorem for a single tetrahedron.

Example 18.31. Let T_{WF} be the Worsey–Farin split of a single tetrahedron T, and let $\mathcal{S}_2(T_{WF})$ be the associated macro-element space as defined in (18.21). Then $\dim \mathcal{S}_2(T_{WF}) = 316$.

Since $\mathcal{S}_2(\triangle_{WF})$ has a stable local MDS, Theorem 17.15 immediately implies that $\mathcal{S}_2(\triangle_{WF})$ has full approximation power.

Theorem 18.32. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 9$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_{WF})$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\;|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WF}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a stable local nodal determining set for $\mathcal{S}_2(\triangle_{WF})$. For each face F of a tetrahedron T in $\triangle$, let D_F be the derivative associated with a unit vector perpendicular to F, and let $D_{F,i}$ be the directional derivatives associated with the vectors $\langle v_i, v_F\rangle$ for $i = 1, 2, 3$, where as before v_F is the split point in the face F.

Theorem 18.33. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} \left(\mathcal{N}_F^0 \cup \mathcal{N}_F^1 \cup \mathcal{N}_F^2\right) \cup \bigcup_{T\in\triangle} \mathcal{N}_T$$

is a stable nodal minimal determining set for $\mathcal{S}_2(\triangle_{WF})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 4}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^3 \bigcup_{j=1}^i \{\varepsilon_{\eta^i_{e,j}} D_e^\beta\}_{|\beta|=i}$,
3) $\mathcal{N}_F^0 := \{\varepsilon_{v_F} D_{F,1}^i D_{F,2}^j\}_{0\le i+j\le 1}$,
4) $\mathcal{N}_F^1 := \{\varepsilon_{v_F} D_F D_{F,1}^i D_{F,2}^j\}_{0\le i+j\le 1}$,
5) $\mathcal{N}_F^2 := \{\varepsilon_{v_F} D_F^2 D_{F,1}^i D_{F,2}^j\}_{0\le i+j\le 1}$,
6) $\mathcal{N}_T := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha|\le 3}$.

Proof: It is easy to see that the cardinality of the set $\mathcal{N}$ matches the dimension of $\mathcal{S}_2(\triangle_{WF})$ as given in Theorem 18.30. Thus, to show that $\mathcal{N}$ is a stable NMDS, it suffices to show that all coefficients of $s \in \mathcal{S}_2(\triangle_{WF})$ can be stably computed from $\{\lambda s\}_{\lambda\in\mathcal{N}}$. For each $v \in \mathcal{V}$, we can use the formulae in Theorem 15.16 to compute the coefficients in $D_4(v)$ from the values of the derivatives $D^\alpha s(v)$ corresponding to $\mathcal{N}_v$. This is a stable process, and indeed if T_ξ is a tetrahedron containing ξ, then for all $\xi \in D_6(v)$,

$$|c_\xi| \le K \sum_{\nu=0}^{4} |T_\xi|^\nu |s|_{\nu,T_\xi}, \tag{18.25}$$

where K is a constant depending only on the smallest solid and face angles in $\triangle_A$.

For each edge $e \in \mathcal{E}$, we can use the results of Section 15.9 to compute the coefficients corresponding to $E_3(e)$ from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_e}$. Now for each face $F \in \mathcal{F}$, we can use the data corresponding to $\mathcal{N}_F^0$ to compute the coefficients corresponding to the domain points $\mathcal{M}_F^0$. Then arguing as in the proof of Theorem 18.30, we can compute all coefficients corresponding to domain points on F. Next we use the data corresponding to $\mathcal{N}_F^1$ to compute the coefficients corresponding to the domain points $\mathcal{M}_F^1$. Then arguing as in the proof of Theorem 18.30, we can compute all coefficients corresponding to domain points on the shells $R_8(v_T)$. Next we use the data corresponding to $\mathcal{N}_F^2$ to compute the coefficients corresponding to the domain points $\mathcal{M}_F^2$. Then arguing as in the proof of Theorem 18.30, we can compute all coefficients corresponding to domain points on the shells $R_7(v_T)$. Finally, for each tetrahedron T in $\triangle$, we can use the values $\{\lambda s\}_{\lambda \in \mathcal{N}_T}$ to compute the coefficients c_ξ of s for $\xi \in \mathcal{M}_T$, see the formulae in Theorem 15.16. $\square$

Theorem 18.33 shows that for any function $f \in C^4(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_{WF})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $0 \le |\alpha| \le 4$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,j}^i) = D_e^\beta f(\eta_{e,j}^i)$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 3$, and for all edges e of $\triangle$,
3) $D_{F,1}^i D_{F,2}^j s(v_F) = D_{F,1}^i D_{F,2}^j f(v_F)$, $0 \le i+j \le 1$, for each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$,
4) $D_F D_{F,1}^i D_{F,2}^j s(v_F) = D_F D_{F,1}^i D_{F,2}^j f(v_F)$, $0 \le i+j \le 1$, for each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$,
5) $D_F^2 D_{F,1}^i D_{F,2}^j s(v_F) = D_F^2 D_{F,1}^i D_{F,2}^j f(v_F)$, $0 \le i+j \le 1$, for each face $F := \langle v_1, v_2, v_3 \rangle$ of $\triangle$,
6) $D^\alpha s(v_T) = D^\alpha f(v_T)$, all $0 \le |\alpha| \le 3$ and all tetrahedra $T \in \triangle$.

The nodal functionals described in Theorem 18.33 involve some derivatives of order higher than two, even though s is only C^2 globally. However, s is in $C^4(v)$ at vertices and in $C^3(e)$ around edges, and so the third and fourth derivatives appearing in $\mathcal{N}_e$ and $\mathcal{N}_v$ are well defined.

The mapping which takes functions $f \in C^4(\Omega)$ to this Hermite interpolating spline defines a linear projector $\mathcal{I}_{WF}^2$ mapping $C^4(\Omega)$ onto $\mathcal{S}_2(\triangle_{WF})$. In particular, $\mathcal{I}_{WF}^2 p = p$ for all trivariate polynomials of degree nine. Since the nodal basis in Theorem 18.33 is stable and local, Theorem 17.22 implies the following error bounds.

Theorem 18.34. *For every $f \in C^{m+1}(\Omega)$ with $3 \le m \le 9$,*

$$\|D^\alpha(f - \mathcal{I}^2_{WF} f)\|_\Omega \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WF}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

As for the other macro-element spaces in this chapter, $\mathcal{S}_2(\triangle_{WF})$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in star(v),
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.33 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_2(\triangle_{WF})$, where each basis function is star-supported.

18.9. Another C^2 Worsey–Farin Macro-element

In this section we show how to remove some unnecessary degrees of freedom from the macro-element constructed in the previous section by working with an appropriate subspace. Suppose $\triangle_{WF}$ is the Worsey–Farin refinement of an arbitrary tetrahedral partition $\triangle$. Let $\mathcal{V}$, $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{E}^c$ be as in the previous section. We write $\mathcal{F}^0$ for the set of all faces of $\triangle_{WF}$ of the form $\langle v_T, v_F, v\rangle$, where v_T is a split point in the interior of a tetrahedron T of $\triangle$, v_F is a split point on a face of T, and $v \in \mathcal{V}$.

Suppose $t := \langle v_T, v_F, v_1, v_2\rangle$ and $\tilde t := \langle v_T, v_F, v_2, v_3\rangle$ are two tetrahedra in $\triangle_{WF}$ which share the face $F := \langle v_T, v_F, v_2\rangle \in \mathcal{F}^0$. Let c_{ijkl} and $\tilde c_{ijkl}$ be the coefficients of the B-representations of $s|_t$ and $s|_{\tilde t}$, respectively. Given $s \in \mathcal{S}_2(\triangle_{WF})$, we define the linear functionals ν_F and μ_F by

$$\begin{aligned} \nu_F s &:= \tilde c_{0,1,3,5} - \sum_{i+j+k=5} c_{0,i+1,j,k+3} B^{f,5}_{ijk}(v_3), \\ \mu_F s &:= \tilde c_{1,0,3,5} - \sum_{i+j+k=5} c_{1,i,j,k+3} B^{f,5}_{ijk}(v_3), \end{aligned} \tag{18.26}$$

where $B^{f,5}_{ijk}$ are the Bernstein polynomials of degree five with respect to the triangle $f := \langle v_F, v_1, v_2\rangle$. Note that $\nu_F s$ involves coefficients of s on the shell $R_9(v_T)$, while $\mu_F s$ involves coefficients of s on the shell $R_8(v_T)$. In this section we discuss the subspace

$$\widetilde{\mathcal{S}}_2(\triangle_{WF}) := \{s \in \mathcal{S}_2(\triangle_{WF}) : \nu_F s = \mu_F s = 0, \text{ all } F \in \mathcal{F}^0\} \qquad (18.27)$$

of the C^2 Worsey–Farin macro-element space $\mathcal{S}_2(\triangle_{WF})$. To define an MDS for $\widetilde{\mathcal{S}}_2(\triangle_{WF})$ we need some more notation. For each vertex v of $\triangle$, let t_v be one of the tetrahedra in $\triangle_{WF}$ attached to v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let t_e be one of the tetrahedra in $\triangle_{WF}$ containing e, and let $E_3(e)$ denote the set of domain points in the tube of radius 3 around e which do not lie in the balls $D_4(u)$ or $D_4(v)$. Finally, for each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let $t_{F,i} := \langle v_T, v_F, v_i, v_{i+1}\rangle$, $i = 1, 2, 3$, where v_T is the split point of some tetrahedron in $\triangle$ containing F (if F is a boundary face, there is just one such tetrahedron – otherwise, there are two). Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.35. *The space $\widetilde{\mathcal{S}}_2(\triangle_{WF})$ has dimension $35n_V + 20n_E + 3n_F + 20n_T$. Moreover, the set*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{M}_F \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable minimal determining set, where

1) $\mathcal{M}_v := D_4(v) \cap t_v$,
2) $\mathcal{M}_e := E_3(e) \cap t_e$,
3) $\mathcal{M}_F := \{\xi^{t_{F,1}}_{2430}, \xi^{t_{F,2}}_{2430}, \xi^{t_{F,3}}_{2430}\}$,
4) $\mathcal{M}_T := D_3(v_T) \cap t_{v_T}$.

Proof: The proof is very similar to the proof of Theorem 18.30. The key difference is that now we use Lemmas 18.37–18.39 below to compute coefficients on faces of the shells $R_9(v_T)$, $R_8(v_T)$, and $R_7(v_T)$, respectively. Figures 18.8 and 18.9 show domain points on the faces of these shells. $\square$

To give an example of Theorem 18.35, we consider a single tetrahedron.

Example 18.36. Let T_{WF} be the Worsey–Farin split of a single tetrahedron T, and let $\widetilde{\mathcal{S}}_2(T_{WF})$ be the associated macro-element space as defined in (18.27). Then $\dim \mathcal{S}_2(T_{WF}) = 292$.

Given $F := \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^2$, let F_{CT} be the split of F into three triangles based on the split point v_F. Let $F_l := \langle v_F, v_l, v_{l+1}\rangle$, $e_i := \langle v_i, v_{l+1}\rangle$, and $\tilde{e}_l := \langle v_l, v_F\rangle$, $l = 1, 2, 3$, where $v_4 := v_1$. For each $1 \le l \le 3$, suppose $\{c^l_{ijk}\}$ and $\{\tilde{c}^l_{ijk}\}$ are the coefficients of $s \in \mathcal{S}^2_d(F_{CT})$ relative to F_{l-1}

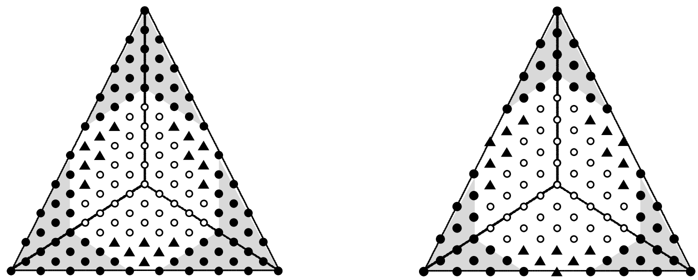


Fig. 18.8. Domain points of $\mathcal{S}_2(T_{WF})$ on $R_9(v_T)$ and $R_8(v_T)$.

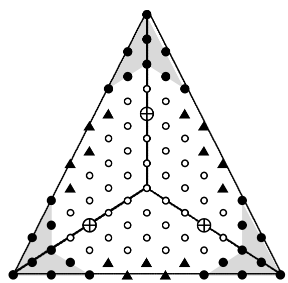


Fig. 18.9. Domain points of $\mathcal{S}_2(T_{WF})$ on $R_7(v_T)$.

and F_l, respectively, where we identify $v_4 = v_1$. Then we define the linear functional $\tau^n_{l,m,d}$ by

$$\tau^n_{l,m,d} s := \tilde{c}^l_{m-n,d-m,n} - \sum_{i+j+k=n} c^l_{i+m-n,j,k+d-m} B^{l-1,n}_{ijk}(v_{l+1}),$$

where $B^{l-1,n}_{ijk}$ are the Bernstein polynomials of degree n relative to the triangle F_{l-1}. Note that $\tau^n_{l,m,d}$ describes an individual C^n smoothness condition involving the coefficients on ring $R_m(v_l)$.

Lemma 18.37. *Let*

$$\widetilde{\mathcal{S}}^2_9(F_{CT}) := \{s \in \mathcal{S}^2_9(F_{CT}) \cap C^7(v_F) : \ s \in C^4(v_l) \text{ and } \tau^5_{l,6,9} s = 0,\ l = 1,2,3\}.$$

Then dim $\widetilde{\mathcal{S}}^2_9(F_{CT}) = 63$, *and the set*

$$\mathcal{M}_9 := \bigcup_{i=1}^{3} (\mathcal{M}_{v_i} \cup \mathcal{M}_{e_i})$$

is a stable minimal determining set, where

1) $\mathcal{M}_v := D_4(v) \cap t_v$, *where* t_v *is some triangle of* F_{CT} *attached to* v,
2) $\mathcal{M}_e$ *is the set of domain points whose distance to* $e := \langle u, v\rangle$ *is at most three, and which do not lie in the disks* $D_4(u)$ *or* $D_4(v)$.

Proof: Figure 18.8 (left) shows the domain points for this space. Points in the sets $\mathcal{M}_e$ are marked with small triangles. Theorem 9.7 implies that $\dim \mathcal{S}_9^2(F_{CT}) \cap C^7(v_F) = 75$. To get the subspace $\widetilde{\mathcal{S}}_9^2(F_{CT})$, for each $l = 1, 2, 3$ we have to enforce three extra smoothness conditions at the vertex v_l to get $C^4(v_l)$ as well as the special smoothness condition corresponding to $\tau_{l,6,9}^5$. It follows that $\dim \widetilde{\mathcal{S}}_9^2(F_{CT}) \geq 63$. Since the cardinality of $\mathcal{M}_9$ is 63, to show that $\mathcal{M}_9$ is a minimal determining set for $\widetilde{\mathcal{S}}_9^2(F_{CT})$ and $\dim \widetilde{\mathcal{S}}_9^2(F_{CT}) = 63$, it suffices to show that if s is a spline in $\widetilde{\mathcal{S}}_9^2(F_{CT})$ whose coefficients satisfy $c_\xi = 0$ for all $\xi \in \mathcal{M}_9$, then $s \equiv 0$. By the definition of $\mathcal{M}_9$, it is clear that all coefficients of s marked with circles or triangles in Figure 18.8 (left) are zero. We now examine the coefficients corresponding to the remaining domain points.

First consider the ring $R_5(v_1)$. All coefficients corresponding to domain points on this ring are already zero except for the three corresponding to domain points within a distance 1 of the edge $\tilde{e}_1$. To compute these three coefficients, we use Lemma 2.30. The C^7 smoothness at v_F implies that s satisfies individual C^1, C^2, and C^3 continuity conditions on ring $R_5(v_1)$, i.e., $\tau_{1,5,9}^n s = 0$ for $n = 1, 2, 3$. This leads to a linear system of equations with matrix

$$M_3 := \begin{bmatrix} a_2 & a_1 & -1 \\ 2a_2a_1 & a_1^2 & 0 \\ 3a_2a_1^2 & a_1^3 & 0 \end{bmatrix}, \tag{18.28}$$

where (a_1, a_2, a_3) are the barycentric coordinates of v_3 relative to the triangle F_1. This matrix is nonsingular since its determinant is $-a_2a_1^4$ and a_1, a_2 are both nonzero. Coefficients on the rings $R_5(v_2)$ and $R_5(v_3)$ can be computed in a similar way.

Now consider the ring $R_6(v_1)$. So far, all coefficients corresponding to domain points on the ring $R_6(v_1)$ are determined to be zero except for the five corresponding to domain points within a distance 2 of $\tilde{e}_1$. Now the C^7 smoothness at v_F implies that s satisfies individual C^1 through C^4 smoothness conditions on ring $R_6(v_1)$. Coupling this with the special smoothness condition $\tau_{1,6,9}^5 s = 0$, we are led to the system of equations $\tau_{1,6,9}^n s = 0$ for $n = 1, \ldots, 5$. The matrix of this system is

$$M_5 := \begin{bmatrix} 0 & a_2 & a_1 & -1 & 0 \\ a_2^2 & 2a_2a_1 & a_1^2 & 0 & -1 \\ 3a_2^2a_1 & 3a_2a_1^2 & a_1^3 & 0 & 0 \\ 6a_2^2a_1^2 & 4a_2a_1^3 & a_1^4 & 0 & 0 \\ 10a_2^2a_1^3 & 5a_2a_1^4 & a_1^5 & 0 & 0 \end{bmatrix}.$$

This is a nonsingular matrix since its determinant is equal to $-a_2^3 a_1^9$. Coefficients on the rings $R_6(v_2)$ and $R_6(v_3)$ can be computed in a similar way. Now all remaining coefficients of s can be computed from the smoothness conditions. We conclude that all coefficients of s must be zero, which completes the proof of the lemma. □

Lemma 18.38. *Let*

$$\widetilde{\mathcal{S}}_8^2(F_{CT}) := \{s \in \mathcal{S}_8^2(F_{CT}) \cap C^7(v_F) : s \in C^3(v_l) \\ \text{and } \tau_{l,5,8}^5 s = 0,\ l = 1,2,3\}. \tag{18.29}$$

Then $\dim \widetilde{\mathcal{S}}_8^2(F_{CT}) = 48$, *and the set*

$$\mathcal{M}_8 := \bigcup_{i=1}^{3} (\mathcal{M}_{v_i} \cup \mathcal{M}_{e_i})$$

is a stable minimal determining set, where

1) $\mathcal{M}_v := D_3(v) \cap t_v$, *where t_v is some triangle of F_{CT} attached to v,*
2) $\mathcal{M}_e$ *is the set of domain points whose distance to $e := \langle u, v\rangle$ is at most two, and which do not lie in the disks $D_3(u)$ or $D_3(v)$.*

Proof: The proof is very similar to proof of Lemma 18.37, so we can be brief. By Theorem 9.7, $\dim \mathcal{S}_8^2(F_{CT}) \cap C^7(v_F) = 54$. To get the subspace $\widetilde{\mathcal{S}}_8^2(F_{CT})$, for each $l = 1,2,3$, we have to enforce one extra smoothness condition at the vertex v_l to get $C^3(v_l)$ along with the special smoothness condition corresponding to $\tau_{l,5,8}^5$. It follows that $\dim \widetilde{\mathcal{S}}_8^2(F_{CT}) \ge 48$. Since the cardinality of $\mathcal{M}_8$ is 48, to show that it is a minimal determining set for $\widetilde{\mathcal{S}}_8^2(F_{CT})$ and $\dim \widetilde{\mathcal{S}}_8^2(F_{CT}) = 48$, it suffices to show that if $s \in \widetilde{\mathcal{S}}_8^2(F_{CT})$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}_8$, then $s \equiv 0$. We already know that all coefficients of s corresponding to domain points marked with circles or triangles in Figure 18.8 (right) are zero. The remaining coefficients can be shown to be zero in the same way as in Lemma 18.37. □

Lemma 18.39. *The set*

$$\mathcal{M}_7 := \mathcal{M}_F \cap \bigcup_{i=1}^{3} (\mathcal{M}_{v_i} \cup \mathcal{M}_{e_i})$$

is a stable minimal determining set for $\mathcal{S}_7^2(F_{CT}) \cap C^7(v_F)$, where

1) $\mathcal{M}_v := D_2(v) \cap t_v$, *where t_v is some triangle of F_{CT} attached to v,*
2) $\mathcal{M}_e$ *is the set of domain points whose distance to $e := \langle u, v\rangle$ is at most one, and which do not lie in the disks $D_2(u)$ or $D_2(v)$,*

3) $\mathcal{M}_F := \{\xi_{430}^{F_1}, \xi_{430}^{F_2}, \xi_{430}^{F_3}\}$.

Proof: This space reduces to the space of bivariate polynomials of degree seven. It has dimension 36 which is also the cardinality of $\mathcal{M}$, and thus it suffices to show that $\mathcal{M}$ is a determining set. Figure 18.9 shows the domain points of $\mathcal{S}_7^2(T_{W_F})$. Points in $\mathcal{M}_F$ are marked with $\oplus$. Suppose $s \in \mathcal{P}_7^2$, and $c_\xi = 0$ for all $\xi \in \mathcal{M}$. Then the B-coefficients of s corresponding to all domain points marked with either $\oplus$ or black dots in Figure 18.9 are zero. The coefficients corresponding to the three remaining domain points on $R_3(v_1)$ can be computed from a nonsingular 3×3 linear system with the matrix M_3 given in (18.28), and we conclude they must also be zero. The same holds for the rings $R_3(v_2)$ and $R_3(v_3)$. Now consider $R_4(v_1)$. There are four unknown coefficients corresponding to the unmarked points on this ring, and they can be computed from a system of four equations with matrix

$$M_4 := \begin{bmatrix} 0 & a_2 & -1 & 0 \\ a_2^2 & 2a_2a_1 & 0 & -1 \\ 3a_2^2a_1 & 3a_2a_1^2 & 0 & 0 \\ 6a_2^2a_1^2 & 4a_2a_1^3 & 0 & 0 \end{bmatrix}.$$

The determinant of this matrix is $-6a_1^4a_2^3 \neq 0$. We can repeat this for the other two vertices v_2, v_3. Then using smoothness conditions we see that the remaining coefficients of s are zero. $\square$

Since $\widetilde{\mathcal{S}}_2(\triangle_{W_F})$ has a stable local MDS, Theorem 17.15 immediately implies that $\widetilde{\mathcal{S}}_2(\triangle_{W_F})$ has full approximation power.

Theorem 18.40. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 9$, there exists a spline $s_f \in \widetilde{\mathcal{S}}_2(\triangle_{W_F})$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{W_F}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now construct a stable local nodal determining set for $\widetilde{\mathcal{S}}_2(\triangle_{W_F})$. For each face F of T, let D_F be the directional derivative associated with a vector perpendicular to F. For each $i = 1, 2, 3$, let $D_{F,i}$ be the directional derivatives appearing in Theorem 18.33.

Theorem 18.41. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{N}_e \cup \bigcup_{F \in \mathcal{F}} \mathcal{N}_F \cup \bigcup_{T \in \triangle} \mathcal{N}_T$$

is a stable nodal minimal determining set for $\widetilde{\mathcal{S}}_2(\triangle_{WF})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 4}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^3 \bigcup_{j=1}^i \{\varepsilon_{\eta^i_{e,j}} D_e^\beta\}_{|\beta|=i}$,
3) $\mathcal{N}_F := \{\varepsilon_{v_i} D_F^2 D_{F,i}^4\}_{i=1}^3$,
4) $\mathcal{N}_T := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha|\le 3}$.

Proof: It is easy to see that the cardinality of the set $\mathcal{N}$ matches the dimension of $\mathcal{S}_2(\triangle_{WF})$ as given in Theorem 18.35. Thus, to show that $\mathcal{N}$ is a stable NMDS, it suffices to show that all coefficients of $s \in \mathcal{S}_2(\triangle_{WF})$ can be stably computed from $\{\lambda s\}_{\lambda\in\mathcal{N}}$. The proof is essentially the same as the proof of Theorem 18.33. □

Theorem 18.41 shows that for any function $f \in C^6(\Omega)$, there is a unique spline $s \in \widetilde{\mathcal{S}}_2(\triangle_{WF})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{for all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 4$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta^i_{e,j}) = D_e^\beta f(\eta^i_{e,j})$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 3$, and for all edges e of $\triangle$,
3) $D_F^2 D_{F,i}^4 s(v_i) = D_F^2 D_{F,i}^4 f(v_i)$, $i = 1, 2, 3$, for each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$,
4) $D^\alpha s(v_T) = D^\alpha f(v_T)$, all $|\alpha| \le 3$ and all tetrahedra $T \in \triangle$.

The nodal functionals described in Theorem 18.41 involve some derivatives of order higher than two, even though s is only C^2 globally. However, s is in $C^4(v)$ at vertices and in $C^3(e)$ around edges, and so the third and fourth derivatives appearing in $\mathcal{N}_e$ and $\mathcal{N}_v$ are well defined. But it is not in $C^6(v)$ at a vertex v, and so if F is an interior face, then the derivatives in $\mathcal{N}_F$ are applied to just one of the polynomial pieces of s which share F.

The mapping which takes functions $f \in C^6(\Omega)$ to this Hermite interpolating spline defines a linear projector $\tilde{\mathcal{I}}^2_{WF}$ mapping $C^6(\Omega)$ onto $\widetilde{\mathcal{S}}_2(\triangle_{WF})$. In particular, $\tilde{\mathcal{I}}^2_{WF} p = p$ for all trivariate polynomials of degree nine. Theorem 17.22 now gives the following error bound for this interpolation operator.

Theorem 18.42. *For every $f \in C^{m+1}(\Omega)$ with $5 \le m \le 9$,*

$$\|D^\alpha(f - \tilde{\mathcal{I}}^2_{WF} f)\|_\Omega \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,\Omega},$$

for all $|\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle_{WF}$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We can construct two natural stable local bases for $\widetilde{\mathcal{S}}_2(\triangle_{WF})$ in the same way as was done above for $\mathcal{S}_2(\triangle_{WF})$.

18.10. A C^2 Macro-element on the Alfeld-16 Split

Let $\triangle$ be an arbitrary tetrahedral partition of a polyhedral set $\Omega \in \mathbb{R}^3$, and let $\mathcal{V}, \mathcal{F}$ and $\mathcal{E}$ be the sets of vertices, faces, and edges of $\triangle$, respectively. Suppose $\triangle_{A16}$ is the Alfeld-16 refinement of $\triangle$, see Definition 16.28. As usual we write v_T for the barycenters of the tetrahedra of $\triangle$. Let $\mathcal{V}^c$ be the set of all subcenters v_T^i introduced to form the refined partition $\triangle_{A16}$. We write $\mathcal{F}^1$ for the set of triangular faces of the form $\langle v_T, u, v\rangle$, where v_T is the center of a tetrahedron T and u, v are two vertices of T.

In this section we consider the following space of supersplines defined on $\triangle_{A16}$:

$$\begin{aligned} \mathcal{S}_2(\triangle_{A16}) := \{s \in \mathcal{S}_9^2(\triangle_{A16}) : \ & s \in C^4(v), \ \text{all } v \in \mathcal{V}, \\ & s \in C^7(v_T), \\ & s \in C^8(v), \ \text{all } v \in \mathcal{V}^c, \\ & s \in C^3(F), \ \text{all } F \in \mathcal{F}^1\}. \end{aligned} \tag{18.30}$$

We call this the C^2 Alfeld-16 macro-element space.

For each vertex v of $\triangle_{A16}$, let t_v be one of the tetrahedra in $\triangle_{A16}$ attached to v. For each edge $e := \langle u, v\rangle$ of $\triangle$, let t_e be one of the four tetrahedra in $\triangle_{A16}$ containing e, and let $E_2(e)$ denote the set of domain points in the tube of radius 2 around e which do not lie in the disks $D_4(u)$ or $D_4(v)$. Finally, for each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let $t_F := \langle w, v_1, v_2, v_3\rangle$ with $w \in \mathcal{V}^c$ be the tetrahedron in $\triangle_{A16}$ containing F. Let n_V, n_E, n_F, n_T be the number of vertices, edges, faces, and tetrahedra in $\triangle$.

Theorem 18.43. *The space $\mathcal{S}_2(\triangle_{A16})$ has dimension $35n_V + 8n_E + 19n_F + 16n_T$. Moreover, the set*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} (\mathcal{M}_F^0 \cup \mathcal{M}_F^1 \cup \mathcal{M}_F^2) \cup \bigcup_{T\in\triangle} \mathcal{M}_T$$

is a stable local minimal determining set, where

1) $\mathcal{M}_v := D_4(v) \cap t_v$,
2) $\mathcal{M}_e := E_2(e) \cap t_e$,
3) $\mathcal{M}_F^0 := \{\xi_{0333}^{t_F}\}$,
4) $\mathcal{M}_F^1 := \{\xi_{1ijk}^{t_F} : i,j,k \ge 2\}$,
5) $\mathcal{M}_F^2 := \{\xi_{2ijk}^{t_F} : i,j,k \ge 1\} \setminus \{\xi_{2115}^{t_F}, \xi_{2151}^{t_F}, \xi_{2511}^{t_F}\}$,

6) $\mathcal{M}_T := \bigcup_{i=1}^4 \big[D_1(v_T^i) \cap T_{v_T^i}\big]$.

Proof: We first consider the case where $\triangle$ consists of a single tetrahedron T. Let T_{A16} be the associated Alfeld-16 split. Then using the Java program described in Remark 18.1, Alfeld and Schumaker [AlfS05c] have verified that $\mathcal{M}$ is an MDS and that $\dim \mathcal{S}_2(T_{A16}) = 280$.

To establish the result for a general partition $\triangle$, we appeal to Theorem 17.10. Thus, we need to show that if $s \in \mathcal{S}_2(\triangle_{A16})$, then we can set the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ to arbitrary values, and all other coefficients will be determined in such a way that all smoothness conditions are satisfied. First, we set all of the coefficients corresponding to the sets $\mathcal{M}_v$ to arbitrary values. Then by the C^4 smoothness at vertices, all other coefficients corresponding to domain points in the balls $D_4(v)$ are stably determined. In particular, for each $\eta \in D_4(v) \setminus \mathcal{M}_v$,

$$|c_\eta| \le K \max_{\xi \in \Gamma_\eta} |c_\xi|, \tag{18.31}$$

where $\Gamma_\eta := \mathcal{D}_4(v)$ and K is a constant depending only on the smallest solid angle and smallest face angle in $\triangle_{A16}$. No smoothness conditions have been violated since the balls $D_4(v)$ do not overlap.

Next for each edge $e := \langle u, v\rangle$ of $\triangle$, we fix all of the coefficients corresponding to $\mathcal{M}_e$. By the C^2 smoothness of s, all other coefficients corresponding to domain points in the set $E_2(e)$ are uniquely determined, see Theorem 15.18. By Theorem 15.23, this is a stable process, and (18.31) holds with with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. No smoothness conditions have been violated since the sets $E_2(e)$ do not overlap each other or any of the balls $D_4(v)$.

Now let F be a face of a tetrahedron $T \in \triangle$, and let $t_F \in \triangle_{A16}$ be the tetrahedron associated with F in the definition of $\mathcal{M}_F^0$, $\mathcal{M}_F^1$, and $\mathcal{M}_F^2$. Suppose we set the coefficients of $s|_{T_F}$ corresponding to $\mathcal{M}_F^0 \cup \mathcal{M}_F^1 \cup \mathcal{M}_F^2$. If F is an interior face of $\triangle$, we can use the C^2 smoothness across F to determine the coefficients of s corresponding to domain points that are within a distance two of F and lie in the other tetrahedron in $\triangle_{A16}$ that shares the face F.

For each tetrahedron t in $\triangle_{A16}$, we have now uniquely determined all of the coefficients of s corresponding to domain points in the minimal determining set for $s|_t$. It follows that all coefficients of s are consistently determined and so $\mathcal{M}$ is an MDS. The dimension statement follows from Theorem 17.8. $\square$

The constant in the stability of the MDS in Theorem 18.43 depends on the smallest solid and face angles in $\triangle_{A16}$. These angles are bounded below in terms of the smallest solid and face angles in $\triangle$, see Section 16.7.4. We now illustrate Theorem 18.43 for a single tetrahedron.

Example 18.44. Let T_{A16} be the Alfeld-16 split of a single tetrahedron T, and let $\mathcal{S}_2(T_{A16})$ be the associated macro-element space as defined in (18.30). Then dim $\mathcal{S}_2(T_{A16}) = 280$.

Discussion: The cardinalities of the sets $\mathcal{M}_v$, $\mathcal{M}_e$, $\mathcal{M}_F^0$, $\mathcal{M}_F^1$, $\mathcal{M}_F^2$, and $\mathcal{M}_T$ are 35, 8, 1, 6, 12, and 16, respectively. Since T has four vertices, six edges, and four faces, we find that $\#\mathcal{M} = 4\times 35+6\times 8+4\times(1+6+12)+16 = 280$. □

Since $\mathcal{S}_2(\triangle_{A16})$ has a stable local MDS, Theorem 17.15 immediately implies that it has full approximation power.

Theorem 18.45. *For all $f \in W_q^{m+1}(\Omega)$ with $1 \le q \le \infty$ and $0 \le m \le 9$, there exists a spline $s_f \in \mathcal{S}_2(\triangle_{A16})$ such that*

$$\|D^\alpha(f - s_f)\|_{q,\Omega} \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

We now give a nodal basis for $\mathcal{S}_2(\triangle_{A16})$. For each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let

$$A_F^1 := \{\xi_{ijk}^{F,8} : i,j,k \ge 2\},$$
$$A_F^2 := \{\xi_{ijk}^{F,7} : i,j,k \ge 1\} \setminus \{\xi_{115}^{F,7}, \xi_{151}^{F,7}, \xi_{511}^{F,7}\},$$

where $\xi_{ijk}^{F,d} := (iv_1 + jv_2 + kv_3)/d$. We emphasize that all of these points are on the face F, and are not inside any tetrahedron. Note that there are six points in A_F^1 and twelve points in A_F^2.

Theorem 18.46. *The set*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} (\mathcal{N}_F^0 \cup \mathcal{N}_F^1 \cup \mathcal{N}_F^2) \cup \bigcup_{T\in\triangle} \mathcal{N}_T$$

is a stable local nodal minimal determining set for $\mathcal{S}_2(\triangle_{A16})$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 4}$,
2) $\mathcal{N}_e := \bigcup_{i=1}^2 \bigcup_{j=1}^i \{\varepsilon_{\eta_j^i} D_e^\alpha\}_{|\alpha|=j}$,
3) $\mathcal{N}_F^0 := \{\varepsilon_{\eta_F}\}$, *where* $\eta_F = \xi_{333}^{F,9}$ *is the barycenter of* F,
4) $\mathcal{N}_F^1 := \{\varepsilon_\eta D_F\}_{\eta\in A_F^1}$,
5) $\mathcal{N}_F^2 := \{\varepsilon_\eta D_F^2\}_{\eta\in A_F^2}$,
6) $\mathcal{N}_T := \bigcup_{i=1}^4 \{\varepsilon_{v_T^i} D^\alpha\}_{|\alpha|\le 1}$.

Proof: It is easy to see that the cardinality of the set $\mathcal{N}$ matches the dimension of $\mathcal{S}_2(\triangle_{A16})$ as given in Theorem 18.43. We already know that the set $\mathcal{M}$ defined in that theorem is an MDS for $\mathcal{S}_2(\triangle_{A16})$. Thus, to show that $\mathcal{N}$ is an MNDS, it suffices to show that if $s \in \mathcal{S}_2(\triangle_{A16})$, then setting the values $\{\lambda s\}_{\lambda\in\mathcal{N}}$ determines all coefficients in the set $\{c_\xi\}_{\xi\in\mathcal{M}}$.

For each $v \in \mathcal{V}$, using the the C^4 smoothness at v and the results of Section 15.8, we can stably compute the B-coefficients corresponding to domain points in $D_4(v)$ from the values $\{\lambda s\}_{\lambda\in\mathcal{N}_v}$. For each edge $e \in \mathcal{E}$, using the C^2 smoothness around e and the results of Section 15.9, we can stably compute the B-coefficients of s corresponding to all domain points in the tube $E_2(e)$ from the data $\{\lambda s\}_{\lambda\in\mathcal{N}_e}$.

Now suppose $F := \langle v_1, v_2, v_3\rangle$ is a face of $\triangle$. Using Theorem 15.24, we can compute all coefficients corresponding to domain points in $G_2(F) \cap t_F$. If F is an interior face of $\triangle$, then we can use the C^2 smoothness conditions across F to compute the coefficients corresponding to the domain points in $G_2(F) \cap \tilde{t}_F$, where $\tilde{t}_F$ is the tetrahedron on the other side of F.

Finally, it is clear that for each tetrahedron T in $\triangle$, setting $\{\lambda s\}_{\lambda\in\mathcal{N}_T}$ determines the B-coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_T}$ via the formulae in Theorem 15.16. $\square$

Theorem 18.46 shows that for any function $f \in C^4(\Omega)$, there is a unique spline $s \in \mathcal{S}_2(\triangle_{A16})$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N},$$

or equivalently:

1) $D^\alpha s(v) = D^\alpha f(v)$, all $|\alpha| \le 4$ and all $v \in \mathcal{V}$,
2) $D_e^\beta s(\eta_{e,j}^i) = D_e^\beta f(\eta_{e,j}^i)$, all $|\beta| = i$ with $1 \le j \le i$ and $1 \le i \le 2$, and for all edges e of $\triangle$,
3) $s(\xi_{333}^{F,9}) = f(\xi_{333}^{F,9})$, for each face F of $\triangle$,
4) $D_F s(\xi) = D_F f(\xi)$, all $\xi \in A_F^1$, for each face F of $\triangle$,
5) $D_F^2 s(\xi) = D_F^2 f(\xi)$, all $\xi \in A_F^2$, for each face F of $\triangle$,
6) $D^\alpha s(v) = D^\alpha f(v)$, $|\alpha| \le 1$, for all $v \in \mathcal{V}^c$.

The mapping which takes functions $f \in C^4(\Omega)$ to this Hermite interpolating spline defines a linear projector $\mathcal{I}_{A16}^2$ mapping $C^4(\Omega)$ onto $\mathcal{S}_2(\triangle_{A16})$. In particular, $\mathcal{I}_{A16}^2\, p = p$ for all trivariate polynomials of degree nine. Since the nodal determining set in Theorem 18.46 is stable and local, Theorem 17.22 immediately implies the following error bound for this interpolation process.

Theorem 18.47. *For every $f \in C^{m+1}(\Omega)$ with $3 \le m \le 9$,*

$$\|D^\alpha(f - \mathcal{I}_{A16}^2 f)\|_\Omega \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,\Omega},$$

for all $|\alpha| \le m$. If Ω is convex, K depends on d and the smallest solid and face angles in $\triangle$. If Ω is nonconvex, then K also depends on the Lipschitz constant of the boundary of Ω.

As for the other macro-element spaces in this chapter, $\mathcal{S}_2(\triangle_{A16})$ has two natural stable local bases. Starting with the stable local minimal determining set $\mathcal{M}$, it follows from Theorem 17.17 that the $\mathcal{M}$-basis $\{\psi_\xi\}_{\xi\in\mathcal{M}}$ defined in Theorem 17.16 is a stable local basis. In particular,

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

Since the set $\mathcal{N}$ given in Theorem 18.46 is a stable local NMDS, the associated $\mathcal{N}$-basis $\{\varphi_\lambda\}_{\lambda\in\mathcal{N}}$ is also a stable local basis for $\mathcal{S}_2(\triangle_{A16})$, where each basis function is star-supported.

18.11. A C^r Polynomial Macro-element

In this section we describe a family of C^r polynomial macro-elements that includes the C^1 and C^2 elements discussed in Sections 18.2 and 18.6. Suppose $\triangle$ is a tetrahedral partition of a polyhedral set Ω in $\mathbb{R}^3$, and let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of $\triangle$, respectively. Let

$$\begin{aligned} \mathcal{S}_r(\triangle) := \{s \in \mathcal{S}^r_{8r+1}(\triangle) :\ & s \in C^{4r}(v),\ \text{all } v \in \mathcal{V}, \\ & s \in C^{2r}(e),\ \text{all } e \in \mathcal{E}\}. \end{aligned} \tag{18.32}$$

For ease of notation, throughout this section we write $d := 8r+1$, $\rho := 4r$, and $\mu := 2r$. The space $\mathcal{S}_r(\triangle)$ is an example of the general trivariate superspline spaces described in Section 17.2.

Our first goal in this section is to describe a stable local minimal determining set for $\mathcal{S}_r(\triangle)$. To this end, we introduce some notation for subsets of the set of domain points $\mathcal{D}_{d,\triangle}$ associated with $\triangle$:

1) For each vertex v of $\triangle$, let T_v be some tetrahedron containing v, and let $\mathcal{M}_v := D_\rho(v) \cap T_v$. This set has cardinality $\binom{\rho+3}{3} = \binom{4r+3}{3} = (32r^3 + 48r^2 + 22r + 3)/3$.
2) For each edge $e := \langle u, v\rangle$ of $\triangle$, we write $E_\mu(e)$ for the set of all domain points which lie in the tube of radius μ around e, but which do not lie in either of the balls $D_\rho(u)$ or $D_\rho(v)$. Since $\rho \ge 2\mu$, the sets $E_\mu(e)$ are disjoint. For each edge e of $\triangle$, let T_e be some tetrahedron containing e, and let $\mathcal{M}_e := E_\mu(e) \cap T_e$. This set has cardinality $(r+1)(2r+1)(4r)/3 = (8r^3 + 12r^2 + 4r)/3$.
3) For each face $F := \langle v_1, v_2, v_3\rangle$ of $\triangle$, let $G_r(F)$ be the set of domain points which are at a distance at most r from F, but which do not lie

in any of the sets $D_\rho(v)$ or $E_\mu(e)$. Suppose T_F is a tetrahedron in $\triangle$ which contains F, and let $\mathcal{M}_F := G_r(F) \cap T_F$. The cardinality of this set is $(25r^3 + 21r^2 - 4r)/6$.

4) For each tetrahedron T, let $\mathcal{M}_T$ be the set of domain points in $\mathcal{D}_{d,T}$ which do not lie in any of the previous sets. The cardinality of this set is $\binom{4r}{3} - 4\binom{r}{3} = 10r^3 - 6r^2$.

Let $n_{_V}, n_{_E}, n_{_F}, n_{_T}$ be the number of vertices, edges, faces, and tetrahedra in $\triangle$, respectively.

Theorem 18.48. *The set*

$$\mathcal{M} := \bigcup_{v \in \mathcal{V}} \mathcal{M}_v \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}_e \cup \bigcup_{F \in \mathcal{F}} \mathcal{M}_F \cup \bigcup_{T \in \triangle} \mathcal{M}_T$$

is a stable local minimal determining set for $\mathcal{S}_r(\triangle)$, and

$$\begin{aligned} dim\ \mathcal{S}_r(\triangle) = \frac{(32r^3 + 48r^2 + 22r + 3)}{3} n_{_V} + \frac{(8r^3 + 12r^2 + 4r)}{3} n_{_E} \\ + \frac{(25r^3 + 21r^2 - 4r)}{6} n_{_F} + (10r^3 - 6r^2) n_{_T}. \end{aligned} \tag{18.33}$$

Proof: We use Theorem 17.10 to show that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$. To apply it, we need to show that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ of a spline $s \in \mathcal{S}_d^0(\triangle)$ can be set to arbitrary values, and that all other coefficients of s are determined in such a way that s satisfies all smoothness conditions that are required for s to belong to $\mathcal{S}_r(\triangle)$.

First for each vertex $v \in \mathcal{V}$, we set the coefficients of s corresponding to domain points in the set $\mathcal{M}_v$ to arbitrary values. Then using the C^ρ smoothness at v and the results of Section 15.8, we can compute all remaining coefficients corresponding to domain points in the ball $D_\rho(v)$ from smoothness conditions. This is a stable local process, and in particular for each $\eta \in D_\rho(v) \setminus \mathcal{M}_v$,

$$|c_\eta| \le K \max_{\xi \in \Gamma_\eta} |c_\xi|, \tag{18.34}$$

with $\Gamma_\eta = \mathcal{M}_v$. None of the smoothness conditions has been violated so far since the balls $D_\rho(v)$ do not overlap.

For each edge $e := \langle u, v \rangle$ of $\triangle$, we now fix the coefficients of s corresponding to domain points in $\mathcal{M}_e$. Then using the C^μ smoothness around e, we can use the results of Section 15.9 to determine all of the coefficients of s corresponding to the domain points in $E_\mu(e)$. By Theorem 15.23, the computation of these coefficients is a stable local process, and (18.34) holds with $\Gamma_\eta := \mathcal{M}_e \cup \mathcal{M}_u \cup \mathcal{M}_v$. The sets $E_\mu(e)$ are disjoint from each other and from all balls $D_\rho(v)$, and thus we can be sure that none of the smoothness conditions defining $\mathcal{S}_r(\triangle)$ have been violated.

For each face F of $\triangle$, we now fix the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_F}$, where $\mathcal{M}_F := G_r(F) \cap T_F$. Since the sets $G_r(F)$ are disjoint from each other, there are no smoothness conditions connecting coefficients associated with domain points in two different such sets. If F is an interior face, then the coefficients corresponding to $G_r(F) \cap \widetilde{T}_F$ are uniquely determined from the C^r smoothness across F, where $\widetilde{T}_F$ is the other tetrahedron in $\triangle$ sharing the face F. This is a stable local process, and (18.34) holds with Γ_η equal to the union of $\mathcal{M}_F$ with all $\mathcal{M}_v$ and $\mathcal{M}_e$ such that v and e are vertices and edges of F.

We have now determined all coefficients of s except for those corresponding to domain points in the sets $\mathcal{M}_T$. These sets are disjoint from each other, and there are no smoothness conditions connecting coefficients associated with domain points in two such sets. Thus, for each T, the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_T}$ can be set to arbitrary values. Since all coefficients of s have been fixed or have been stably and locally computed using smoothness conditions, we have shown that $\mathcal{M}$ is a stable local MDS.

To finish the proof, we appeal to Theorem 17.8 which says that the dimension of $\mathcal{S}_r(\triangle)$ is just the cardinality of $\mathcal{M}$. It is easily seen to be given by the formula (18.33). $\square$

Example 18.49. Let $\triangle$ consist of a single tetrahedron.

Discussion: In this case $n_V = n_F = 4$, $n_E = 6$, and $n_T = 1$. Thus, (18.33) reduces to dim $\mathcal{S}_r(\triangle) = (256r^3 + 288r^2 + 104r + 12)/3$. This is equal to dim $\mathcal{P}_{8r+1} = \binom{8r+4}{3}$. $\square$

Since $\mathcal{S}_r(\triangle)$ has a stable local MDS $\mathcal{M}$, we can apply Theorem 17.15 to conclude that $\mathcal{S}_r(\triangle)$ provides optimal order approximation of smooth functions.

Theorem 18.50. *For any $f \in W_q^{m+1}(\Omega)$ with $0 \le m \le 8r+1$ and all $1 \le q \le \infty$,*

$$\|D^\alpha(f - Qf)\|_{q,\Omega} \le K\, |\triangle|^{m+1-|\alpha|}\, |f|_{m+1,q,\Omega},$$

for all $0 \le |\alpha| \le m$. If Ω is convex, the constant K depends only on r and the smallest solid and face angles in $\triangle$. If Ω is not convex, K also depends on the Lipschitz constant of the boundary of Ω.

For $r = 1, 2$, the space $\mathcal{S}_r(\triangle)$ coincides with the classical trivariate finite-element spaces treated in Sections 18.2 and 18.6. In the literature, finite-element spaces have traditionally been parametrized in terms of nodal functionals. We now construct a stable nodal minimal determining set for $\mathcal{S}_r(\triangle)$.

Suppose $T := \langle v_1, v_2, v_3, v_4\rangle$ is a tetrahedron in $\triangle$. Then corresponding to the edge $e := \langle v_1, v_2\rangle$ we define $D_{T,e,1}$ to be the directional derivative associated with a unit vector perpendicular to e and lying in the face $\langle v_1, v_2, v_3\rangle$. Similarly, we define $D_{T,e,2}$ to be the directional derivative associated with a unit vector perpendicular to e and lying in the face $\langle v_1, v_2, v_4\rangle$. For each triangular face F of $\triangle$, we write D_F for the directional derivative associated with a unit vector that is perpendicular to F. For each e of F, let $D_{F,e}$ be the directional derivative associated with a unit vector that lies in F and is perpendicular to e.

We also need some notation for certain sets of points lying on faces and edges of tetrahedra in $\triangle$. Given any face $F := \langle v_1, v_2, v_3\rangle$ and integer $m > 0$, let

$$\mathcal{D}_{F,m} := \Big\{\xi^{F,m}_{ijk} := \frac{iv_1 + jv_2 + kv_3}{m}\Big\}_{i+j+k=m}$$

be the set of domain points associated with bivariate polynomials of degree m on F. For any $\ell \ge 0$, let

$$A_{F,\ell} := \{\xi^{F,8r+1-\ell}_{ijk} : i, j, k \ge 2r + 1 - \ell + \lfloor \ell/2\rfloor\}.$$

These sets depend on r, but for ease of notation we do not write this dependence explicitly. For $r = 3$ and $\ell = 0, \dots, 4$, we have marked the points in the sets $A_{F,\ell}$ with $\odot$ in Figures 18.10–18.12. For any $i > 0$, we define equally spaced points in the interior of $e := \langle v_1, v_2\rangle$ as follows:

$$\eta^i_{e,j} := \frac{(i - j + 1)v_1 + jv_2}{i + 1}, \qquad j = 1, \dots, i.$$

For each tetrahedron T of $\triangle$, let v_T be its barycenter, $\mathcal{E}_T$ its set of edges, and $\mathcal{F}_T$ its set of faces. For each face F of $\triangle$, let $\mathcal{E}_F$ be its set of edges. For each edge e of $\triangle$, pick some tetrahedron T_e containing e. For any point $t \in \mathbb{R}^3$, let ε_t be the point evaluation functional at t.

Theorem 18.51. *Given $r > 0$, let $n := \lfloor r/3\rfloor$. Then*

$$\mathcal{N} := \bigcup_{v\in\mathcal{V}} \mathcal{N}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{N}_e \cup \bigcup_{F\in\mathcal{F}} \big(\mathcal{N}^1_F \cup \mathcal{N}^2_F\big) \cup \bigcup_{T\in\triangle} \big(\mathcal{N}^1_T \cup \mathcal{N}^2_T \cup \mathcal{N}^3_T \cup \mathcal{N}^4_T\big)$$

is a stable nodal minimal determining set for $\mathcal{S}_r(\triangle)$, where

1) $\mathcal{N}_v := \{\varepsilon_v D^\alpha\}_{|\alpha|\le 4r}$,

2) $\mathcal{N}_e := \displaystyle\bigcup_{\ell=1}^{2r} \bigcup_{m=0}^{\ell} \{\varepsilon_{\eta^\ell_{e,k}} D^m_{T_e,e,1} D^{\ell-m}_{T_e,e,2}\}^{\ell}_{k=1}$,

3) $\mathcal{N}^1_F := \displaystyle\bigcup_{\ell=0}^{r} \{\varepsilon_\xi D^\ell_F\}_{\xi\in A_{F,\ell}}$,

4) $\mathcal{N}_F^2 := \bigcup_{e\in\mathcal{E}_F} \bigcup_{\ell=2}^{r} \bigcup_{m=1}^{\lfloor \ell/2\rfloor} \{\epsilon_{\eta^{2r+m}_{e,k}} D_F^\ell D_{F,e}^{2r-\ell+m}\}_{k=1}^{2r+m}$,

5) $\mathcal{N}_T^1 := \bigcup_{e\in\mathcal{E}_T} \bigcup_{\ell=r+1}^{r+n} \{\epsilon_{\eta^{2\ell}_{e,k}} D_{T,e,1}^\ell D_{T,e,2}^\ell\}_{k=1}^{2\ell}$,

6) $\mathcal{N}_T^2 := \bigcup_{F\in\mathcal{F}_T} \bigcup_{e\in\mathcal{E}_F} \bigcup_{\ell=r+1}^{r+n} \bigcup_{m=1}^{2r-2\ell+\lfloor \ell/2\rfloor} \{\epsilon_{\eta^{2\ell+m}_{e,k}} D_F^\ell D_{F,e}^{\ell+m}\}_{k=1}^{2\ell+m}$,

7) $\mathcal{N}_T^3 := \bigcup_{F\in\mathcal{F}_T} \bigcup_{\ell=r+1}^{r+n} \{\varepsilon_\xi D_F^\ell\}_{\xi\in A_{F,\ell}}$,

8) $\mathcal{N}_T^4 := \{\varepsilon_{v_T} D^\alpha\}_{|\alpha|\le 4r-4n-3}$.

Proof: First we show that $\mathcal{N}$ is a nodal determining set for $\mathcal{S}_r(\triangle)$. We show later that it is minimal and stable. Suppose $s \in \mathcal{S}_r(\triangle)$ and that we have assigned values for $\{\lambda s\}_{\lambda\in\mathcal{N}}$. We show how to compute all B-coefficients of s from this derivative data. For each $v \in \mathcal{V}$, we can compute all of the coefficients $\{c_\xi\}_{\xi\in D_{4r}(v)}$ from $\{D^\alpha s(v)\}_{|\alpha|\le 4r}$, which corresponds to $\mathcal{N}_v$. Similarly, for each edge e of $\triangle$, using the derivative information associated with $\mathcal{N}_e$, we can compute all coefficients c_ξ associated with points $\xi \in E_{2r}(e)$.

Fix a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$. We now show how to compute the remaining coefficients of s associated with domain points in $\mathcal{D}_{d,T}$. We start with domain points on the outer faces of T and work our way inward. Consider the face $F := \langle v_1, v_2, v_3\rangle$. We already know the coefficients of $s|_F$ corresponding to domain points in the disks $D_{4r}(v_i)$ for $i = 1, 2, 3$. We also know the coefficients of $s|_F$ corresponding to domain points within a distance of $2r$ of any edge of F. This leaves only the coefficients associated with domain points in the set $\{\xi^{T,8r+1}_{ijk0} : i, j, k \ge 2r + 1\}$. The coefficients corresponding to these domain points can be computed from the values of s at the points of $A_{F,0}$, which is part of the data corresponding to $\mathcal{N}_F^1$. This leads to a linear system with matrix $M_0 := [B^{F,8r+1}_\eta(\xi)]_{\xi,\eta\in A_{F,0}}$. This matrix is independent of the size and shape of F since the entries depend only on barycentric coordinates. It is nonsingular by Lemma 2.25. Figure 18.10 (left) shows the domain points of s on F for the case $r = 3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ (i.e., those in the disks $D_{12}(v_i)$) are marked with circles, while those corresponding to coefficients determined by the sets $\mathcal{N}_e$ (i.e. those within a distance 6 of edges) are marked with triangles. Points corresponding to coefficients determined by the set $\mathcal{N}_F^1$ are marked with $\odot$.

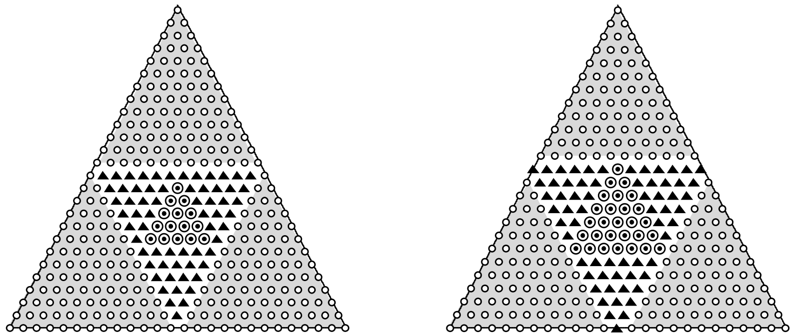


Fig. 18.10. Domain points on layers $\ell = 0$ and $\ell = 1$ for $r = 3$.

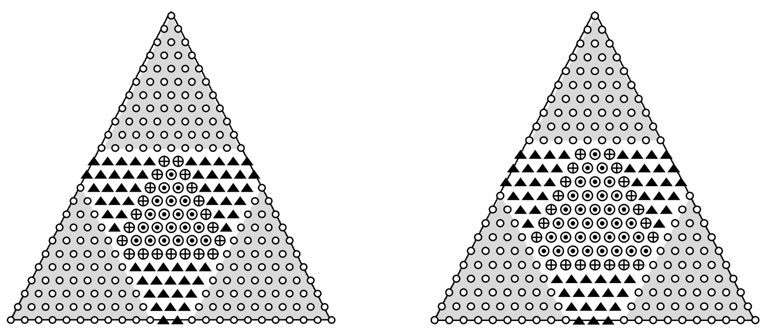


Fig. 18.11. Domain points on layers $\ell = 2$ and $\ell = 3$ for $r = 3$.

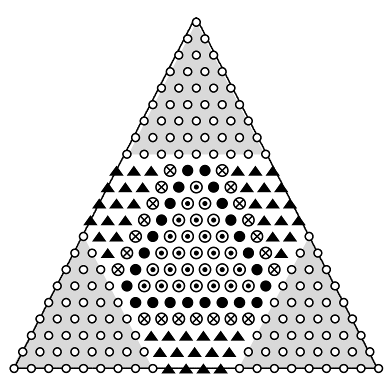


Fig. 18.12. Domain points on layer $\ell = 4$ for $r = 3$.

We now compute the coefficients associated with domain points on the first layer F_1 of domain points inward from an outer face F of T. We have determined the coefficients of s corresponding to the balls of radius $4r$ around the vertices of T which correspond to disks of radius $4r-1$ around the vertices of F_1. In addition, we know the coefficients of s corresponding to tubes of radius $2r$ around the edges of T which gives us the points within a distance $2r-1$ of the edges of F_1. It remains to compute the coefficients corresponding to domain points on F_1 with indices $i,j,k \geq 2r$. We use Theorem 15.24 to compute these coefficients from the values $\{D_F s(\xi)\}_{\xi \in A_{F,1}}$, which are part of the data associated with $\mathcal{N}_F^1$. This involves solving a linear system with matrix $M_1 := [B_\eta^{F,8r}(\xi)]_{\xi,\eta \in A_{F,1}}$. This matrix is independent of the size and shape of F, and is nonsingular by Lemma 2.25. Figure 18.10 (right) shows the domain points on this layer for the case $r=3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ (i.e., those in the disks of radius 11 around the vertices of F_1) are marked with circles, while those corresponding to coefficients that are determined from the sets $\mathcal{N}_e$ (i.e., those within a distance 5 of edges of F_1) are marked with triangles. Points corresponding to coefficients that are determined from $\mathcal{N}_F^1$ are marked with $\odot$.

We continue with layers that are a distance $\ell = 2,\ldots,r$ from the faces of T. The analysis of these layers is a little different from layers 0 and 1 since now we have to make use of the data associated with the functionals in the sets $\mathcal{N}_F^2$. Let F_ℓ be a triangular face of layer ℓ. We have determined the coefficients of s corresponding to the balls of radius $4r$ around the vertices of T which correspond to disks of radius $4r-\ell$ around the vertices of F_ℓ. In addition, we know the coefficients of s corresponding to tubes of radius $2r$ around the edges of T which gives us the points within a distance $2r-\ell$ of the edges of F_ℓ. To compute the remaining coefficients on F_ℓ, we first use the data associated with the sets $\mathcal{N}_F^2$ to compute the remaining unknown coefficients of s corresponding to domain points at a distance $2r-\ell+m$ from the edges of F_ℓ for $m=1,\ldots,\lfloor \ell/2 \rfloor$. Then we use the values $\{D_F^\ell s(\xi)\}_{\xi \in A_{F,\ell}}$ (which come from $\mathcal{N}_F^1$) to solve for the coefficients of s corresponding to the domain points $\{\xi_{ijk\ell}^{T,8r+1} : i,j,k \geq 2r+1-\ell+\lfloor \ell/2 \rfloor\}$. This involves solving a linear system with matrix $M_\ell := [B_\eta^{F,8r-\ell+1}(\xi)]_{\xi,\eta \in A_{F,\ell}}$. This matrix is independent of the size and shape of F, and is nonsingular by Lemma 2.25. Figure 18.11 shows the domain points on layers $\ell = 2,3$ for the case $r=3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ and $\mathcal{N}_e$ (i.e., those that lie in the disks of radius $12-\ell$ around vertices of F_ℓ or in tubes of radius $6-\ell$ around edges of F_ℓ are marked with circles and triangles, respectively. Points marked with $\oplus$ indicate coefficients that are computed from the sets $\mathcal{N}_F^2$. Points corresponding to coefficients that are determined from $\mathcal{N}_F^1$ are marked with $\odot$.

We now proceed to compute unknown coefficients on layers $\ell = r+1,\ldots,r+n$. Let F_ℓ be a triangular face on layer ℓ. We already know the coefficients corresponding to domain points in disks of radius $4r - \ell$ around the vertices of F_ℓ. We also know the coefficient associated with all domain points within a distance $\ell - 1$ of the edges of F. We now use the data associated with $\mathcal{N}_T^1$ to compute the remaining unknown coefficients of s corresponding to domain points at a distance ℓ from the edges of F_ℓ. Similarly, we use the data associated with $\mathcal{N}_T^2$ to compute the coefficients of s corresponding to domain points at a distance $\ell + 1, \ldots, 2r - \ell + \lfloor \ell/2 \rfloor$ from the edges of F_ℓ. Finally, we use the values $\{D_F^\ell s(\xi)\}_{\xi \in A_{F,\ell}}$ (which come from $\mathcal{N}_T^3$) to solve for the coefficients of s corresponding to the domain points $\{\xi_{ijk\ell}^{T,8r+1} : i, j, k \geq 2r + 1 - \ell + \lfloor \ell/2 \rfloor\}$. This involves solving a linear system with matrix $M_\ell := [B_\eta^{F,8r-\ell+1}(\xi)]_{\xi,\eta \in A_{F,\ell}}$. This matrix is independent of the size or shape of F, and is nonsingular by Lemma 2.25. Figure 18.12 shows the domain points on layer $\ell = 4$ for the case $r = 3$. Points corresponding to coefficients that are determined from the sets $\mathcal{N}_v$ or $\mathcal{N}_e$ (i.e., those that lie in the disks of radius 8 around the vertices of F_4 or within a distance 2 of the edges of F_4) are marked with circles and triangles, respectively. Points marked with $\otimes$ indicate coefficients that were computed in previous steps, while those marked with black dots correspond to coefficients that are determined from the data of $\mathcal{N}_T^1$. $\mathcal{N}_T^2$ is empty for this case. Points on F_4 corresponding to coefficients that are determined from $\mathcal{N}_T^3$ are marked with $\odot$.

After completing the above steps for layers $0, \ldots, r + n$, it remains to compute the coefficients of s corresponding to the domain points whose distance to the boundary of T are greater than or equal to $r + n + 1$, i.e., coefficients of the form c_{ijkl}^T with $i, j, k, l \geq r+n+1$. Theorem 15.26 shows how to compute these coefficients from the data corresponding to $\mathcal{N}_T^4$.

We have shown that $\mathcal{N}$ is a nodal determining set. We claim it is stable in the sense of (17.16) with $m_\xi = 4r$. This follows from the fact that all coefficients are computed directly from derivatives using the results of Sections 15.8–15.10, or by solving linear systems of equations whose matrices are nonsingular and whose determinants depend only on the smallest angles in $\triangle$. The highest derivative involved is of order $4r$.

To show that $\mathcal{N}$ is minimal, we have to show that its cardinality is equal to the dimension of $\mathcal{S}_r(\triangle)$ as given in (18.33). It is clear that for every $v \in \mathcal{V}$,

$$\#\mathcal{N}_v = \#\mathcal{M}_v = \binom{4r+3}{3} = \frac{32r^3 + 48r^2 + 22r + 3}{3},$$

and for every $e \in \mathcal{E}$,

$$\#\mathcal{N}_e = \#\mathcal{M}_e = \frac{(r+1)(2r+1)(4r)}{3} = \frac{8r^3 + 12r^2 + 4r}{3}.$$

This gives the first two terms in (18.33). To get the term involving n_F, we note that there is a one-to-one correspondence between the functionals in the sets $\mathcal{N}_F^1 \cup \mathcal{N}_F^2$ and the point in $\mathcal{M}_F$, and so the the two sets have the same cardinality. To see this directly, note that the cardinality of $A_{F,\ell}$ is $\binom{2r+2\ell-3\lfloor \ell/2 \rfloor}{2}$. Thus,

$$\#(\mathcal{N}_F^1 \cup \mathcal{N}_F^2) = \sum_{\ell=0}^{r} \binom{2r+2\ell-3\lfloor \ell/2 \rfloor}{2} + 3 \sum_{\ell=2}^{r} \sum_{m=1}^{\lfloor \ell/2 \rfloor} (2r+m),$$

which reduces to $(25r^3 + 21r^2 - 4r)/6 = \#\mathcal{M}_F$. Finally, we deal with the term in (18.33) involving N. We have

$$\begin{aligned} &\#\big[\mathcal{N}_T^1 \cup \mathcal{N}_T^2 \cup \mathcal{N}_T^3 \cup \mathcal{N}_T^4\big] \\ &= \sum_{\ell=r+1}^{r+n} \Big[12\ell + \sum_{m=1}^{2r-2\ell+\lfloor \ell/2 \rfloor} 12(2\ell+m) + 4\binom{2r+2\ell-3\lfloor \ell/2 \rfloor}{2}\Big] + \binom{4r-4n}{3}, \end{aligned}$$

which reduces to $10r^3 - 6r^2 = \#\mathcal{M}_T$. $\square$

Theorem 18.51 asserts that if we assign values to all of the derivatives of s listed there, then s is uniquely determined. We emphasize that some of this data applies to the polynomial pieces of s rather than s itself. For example (cf. $\mathcal{N}_T^3$), for every interior face F of T, every $r+1 \le \ell \le r+n$, and every point $t \in A_{F,\ell}$, we have to assign values to both $D_F^\ell s|_T(t)$ and $D_F^\ell s|_{\widetilde{T}}(t)$, where T and $\widetilde{T}$ are the tetrahedra sharing the face F. We are allowed to assign different values to these derivatives since s is not not required to be C^ℓ across the face F. The situation is similar for the data associated with $\mathcal{N}_T^1$ and $\mathcal{N}_T^2$, since they also involve derivatives of order greater than r.

Theorem 18.51 shows that for any function $f \in C^{4r}(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\triangle)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \qquad \text{all } \lambda \in \mathcal{N}.$$

This defines a linear projector $\mathcal{I}_P^r$ mapping $C^{4r}(\Omega)$ onto the superspline space $\mathcal{S}_r(\triangle)$. In particular, $\mathcal{I}_P^r$ reproduces all polynomials of degree $d := 8r+1$. Since the NMDS of Theorem 18.51 is local and stable, Theorem 17.22 implies the following error bound.

Theorem 18.52. *For every $f \in C^{m+1}(\Omega)$ with $4r-1 \le m \le 8r+1$,*

$$\|D^\alpha(f - \mathcal{I}_P^r f)\|_\Omega \le K\,|\triangle|^{m+1-|\alpha|}\,|f|_{m+1,\Omega},$$

for all $0 \le |\alpha| \le m$. Here K depends only on r and the smallest solid and face angles of $\triangle$.

As with the other macro-element spaces in this chapter, it is easy to construct two different stable local bases for $\mathcal{S}_r(\triangle)$. Clearly, the $\mathcal{M}$-basis

$\{\psi_\xi\}_{\xi\in\mathcal{M}}$ of Theorem 17.16 provides a stable local basis for $\mathcal{S}_r(\triangle)$. These basis functions have the following support sets:

1) if $\xi \in \mathcal{M}_v$ for some vertex v of $\triangle$, then the support of ψ_ξ lies in $\mathrm{star}(v)$,
2) if $\xi \in \mathcal{M}_e$ for some edge e of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing e,
3) if $\xi \in \mathcal{M}_F$ for some face F of $\triangle$, then the support of ψ_ξ is contained in the union of the tetrahedra containing F,
4) if $\xi \in \mathcal{M}_T$ for some tetrahedron T of $\triangle$, then the support of ψ_ξ is contained in T.

The $\mathcal{N}$-basis associated with the nodal determining set of Theorem 18.51 provides another stable local basis for $\mathcal{S}_r(\triangle)$, where each basis function is star-supported.

18.12. Remarks

Remark 18.1. Peter Alfeld has written a Java program which is an extremely useful tool for experimenting with trivariate spline spaces. Using exact arithmetic, it can compute the dimension of trivariate spline spaces and help find minimal determining sets. The code can be downloaded from www.math.utah/∼pa.

Remark 18.2. For a given split and a given smoothness, there generally is more than one way to define a corresponding macro-element space, especially when individual smoothness conditions are used to remove unnecessary degrees of freedom.

Remark 18.3. As in the bivariate case, we say that a macro-element space has natural degrees of freedom provided that it has a nodal minimal determining set that involves data only at the vertices or at points on the edges and faces of the initial tetrahedral partition $\triangle$. Many of the macro-elements discussed in this chapter include some nonnatural degrees of freedom. Such extraneous degrees of freedom can always be removed by working with subspaces defined by adding appropriate individual smoothness conditions, although finding exactly which conditions to enforce is not always easy.

Remark 18.4. Another way to remove degrees of freedom from macro-element spaces is via the method of condensation. The idea is to require the restrictions of the spline to edges or faces to be of lower degree than the overall element. The disadvantage of using condensed elements is that they do not have full approximation power.

Remark 18.5. There is a trivariate analog of the quadrilateral macro-elements discussed in Chapters 6–8. In the trivariate setting they are defined on octahedra. For details, see [LaiL04] for the C^1 case, and [LaiLS06] for the C^2 case.

Remark 18.6. In [SchS05] a trivariate macro-element was constructed based on splitting a cube into twenty-four congruent subtetrahedra as in Remark 17.1. C^1 splines of degree six are used.

18.13. Historical Notes

Although C^0 trivariate macro-elements were used by the finite-element community much earlier, the first C^1 macro-element was introduced in 1973 in [Zen73a]. This paper deals with the ninth-degree polynomial element discussed here in Section 18.2. Bernstein–Bézier methods were not used.

The first C^1 trivariate macro-element based on a split tetrahedron is contained in [Alf84b]. In this paper Alfeld discusses a condensed version of the macro-element of Section 18.3. What we call the Alfeld split here was called the Clough–Tocher split in that paper. A noncondensed version of this element was suggested by Lai and Le Méhaute, see [AwaL02]. The C^1 elements on the Worsey–Farin and Worsey–Piper splits were described first in [WorF87] and [WorP88], respectively, again without using Bernstein–Bézier methods. The approximation power of the macro-element spaces was not addressed in these papers.

The C^2 trivariate macro-element based on polynomials of degree 17 in Section 18.6 was described in nodal form in [Zen73b]. It was also studied in Le Méhauté's dissertation [Lem84]. Neither reference uses Bernstein–Bézier methods, and the approximation power of the macro-element space was not addressed.

The C^2 macro-element spaces of Sections 18.7–18.10 were developed much more recently, see [AlfS05a–AlfS05c]. These papers make use of Bernstein-Bézier methods, and take special care to deal with the question of stability and its impact on approximation power. Although it was recognized in [Zen73b, Lem84] that to get C^r polynomial macro-elements we need to work with polynomials of degree $8r + 1$, a description and analysis of such elements appeared for the first time in our recent paper [LaiS06], which we follow in Section 18.11.

References

ADAMS, R. A.

[Ada75] *Sobolev Spaces*, New York, Academic Press, 1975.

ALFELD, P.

[Alf84a] A bivariate C^2 Clough–Tocher scheme, *Comput. Aided Geom. Design* **1** (1984), 257–267.

[Alf84b] A trivariate Clough–Tocher scheme for tetrahedral data, *Comput. Aided Geom. Design* **1** (1984), 169–181.

[Alf84c] A discrete C^1 interpolant for tetrahedral data, *Rocky Mountain J. Math.* **14** (1984), 5–16.

[Alf86] On the dimension of multivariate piecewise polynomials, in *Numerical Analysis 1985*, D. F. Griffiths and G. A. Watson (eds.), Essex, Longman Scientific and Technical, 1986, 1–23.

[Alf87] A case study of multivariate piecewise polynomials, in *Geometric Modeling: Algorithms and New Trends*, G. E. Farin (ed.), Philadelphia, SIAM, 1987, 149–159.

[Alf96] Upper and lower bounds on the dimension of multivariate spline spaces, *SIAM J. Numer. Anal.* **33** (1996), 571–588.

[Alf00] Bivariate splines and minimal determining sets, *J. Comput. Appl. Math.* **119** (2000), 13–27.

ALFELD, P., M. NEAMTU, AND L. L. SCHUMAKER

[AlfNS95] Circular Bernstein–Bézier polynomials, in *Mathematical Methods for Curves and Surfaces*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1995, 11–20.

[AlfNS96a] Bernstein–Bézier polynomials on spheres and sphere–like surfaces, *Comput. Aided Geom. Design* **13** (1996), 333–349.

[AlfNS96b] Fitting scattered data on sphere-like surfaces using spherical splines, *J. Comput. Appl. Math.* **73** (1996), 5–43.

[AlfNS96c] Dimension and local bases of homogeneous spline spaces, *SIAM J. Math. Anal.* **27** (1996), 1482–1501.

ALFELD, P., B. PIPER, AND L. L. SCHUMAKER

[AlfPS87a] Minimally supported bases for spaces of bivariate piecewise polynomials of smoothness r and degree $d \geq 4r+1$, *Comput. Aided Geom. Design* **4** (1987), 105–123.

[AlfPS87b] Spaces of bivariate splines on triangulations with holes, *Approx. Theory Appl.* **3** (1987), 1–10.

[AlfPS87c] An explicit basis for C^1 quartic bivariate splines, *SIAM J. Numer. Anal.* **24** (1987), 891–911.

ALFELD, P. AND L. L. SCHUMAKER

[AlfS87] The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r+1$, *Constr. Approx.* **3** (1987), 189–197.

[AlfS90] On the dimension of bivariate splines spaces of smoothness r and degree $d = 3r+1$, *Numer. Math.* **57** (1990), 651–661.

[AlfS00] Non-existence of star-supported spline bases, *SIAM J. Math. Anal.* **31** (2000), 455–465.

[AlfS02a] Smooth macro-elements based on Clough–Tocher triangle splits, *Numer. Math.* **90** (2002), 597–616.

[AlfS02b] Smooth macro-elements based on Powell–Sabin triangle splits, *Adv. Comp. Math.* **16** (2002), 29–46.

[AlfS03] Upper and lower bounds on the dimension of superspline spaces, *Constr. Approx.* **19** (2003), 145–161.

[AlfS05a] A C^2 trivariate macro-element based on the Clough–Tocher split of a tetrahedron, *Comput. Aided Geom. Design* **22** (2005), 710–721.

[AlfS05b] A C^2 trivariate double-Clough–Tocher macro-element, in *Approximation Theory XI: Gatlinburg 2004*, C. K. Chui, M. Neamtu, and L. L. Schumaker (eds.), Brentwood, Nashboro Press, 2005, 1–14.

[AlfS05c] A C^2 trivariate macro-element based on the Worsey-Farin split of a tetrahedron, *SIAM J. Numer. Anal.* **43** (2005), 1750–1765.

[AlfS06] Bounds on the dimension of trivariate spline spaces, manuscript, 2006.

ALFELD, P., L. L. SCHUMAKER, AND M. SIRVENT

[AlfSS92] On dimension and existence of local bases for multivariate spline spaces, *J. Approx. Theory* **70** (1992), 243–264.

ALFELD, P., L. L. SCHUMAKER, AND W. WHITELEY

[AlfSW93] The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions, *SIAM J. Numer. Anal.* **30** (1993), 889–920.

ALFELD, P. AND M. SIRVENT

[AlfSir89] A recursion formula for the dimension of super spline spaces of smoothness r and degree $d > r2^k$, in *Multivariate Approximation Theory IV, ISNM 90*, C. Chui, W. Schempp, and K. Zeller (eds.), Basel, Birkhäuser Verlag, 1989, 1–8.

[AlfSir91] The structure of multivariate superspline spaces of high degree, *Math. Comp.* **57** (1991), 299–308.

Arcangéli, R., M. Cruz López de Silanes, and J. J. Torrens

[ArcCT04] *Multidimensional Minimizing Splines*, Dordrecht, Kluwer, 2004.

Aubin, T.

[Aub82] *Nonlinear Analysis on Manifolds. Monge-Ampére Equations*, Berlin, Springer, 1982.

Awanou, G. and M. J. Lai

[AwaL02] C^1 quintic spline interpolation over tetrahedral partitions, in *Approximation Theory X: Wavelets, Splines, and Applications*, C. K. Chui, L. L. Schumaker, and J. Stöckler (eds.), Nashville, Vanderbilt University Press, 2002, 1–16.

Bamberger, L.

[Bam85] Zweidimensionale Splines auf regulären Triangulationen, dissertation, University of Munich, 1985.

Baramidze, V. and M. J. Lai

[BarL05] Error bounds for minimal energy interpolatory spherical splines, in *Approximation Theory XI: Gatlinburg 2004*, C. K. Chui, M. Neamtu, and L. L. Schumaker (eds.), Brentwood, Nashboro Press, 2005, 25–50.

Barnhill, R. E. and G. Farin

[BarnF81] C^1 quintic interpolation over triangles: two explicit representations, *Int. J. Numer. Meth. Engr.* **17** (1981), 1763–1778.

Barnhill, R. E. and T. Whelan

[BarnW84] A geometric interpretation of convexity conditions for surfaces, *Comput. Aided Geom. Design* **1** (1984), 285–287.

Bezhaev, A. Y. and V. A. Vasilenko

[BezV01] *Variational Theory of Splines*, Dordrecht, Kluwer, 2001.

Billera, L. J.

[Bil88] Homology of smooth splines: Generic triangulations and a conjecture of Strang, *Trans. Amer. Math. Soc.* **310** (1988), 325–340.

[Bil89] The algebra of continuous piecewise polynomials, *Advances of Math.* **76** (1989), 170–183.

Billera, L. J. and L. L. Rose

[BilR89] Gröbner basis methods for multivariate splines, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker (eds.), New York, Academic Press, 1989, 93–104.

[BilR91] A dimension series for multivariate splines, *Discrete Computat. Geom.* **6** (1991), 107–128.

Boehm, W., G. Farin, and J. Kahmann

[BoeFK84] A survey of curve and surface methods in CAGD, *Comput. Aided Geom. Design* **1** (1984), 1–60.

Boehm, W. and A. Müller

[BoeM99] On de Casteljau's algorithm, *Comput. Aided Geom. Design* **16** (1999), 587–605.

Boor, C. de

[Boo68] On uniform approximation by splines, *J. Approx. Theory* **1** (1968), 219–235.

[Boo73] The quasi-interpolant as a tool in elementary polynomial spline theory, in *Approximation Theory*, G. G. Lorentz *et al.* (eds.), New York, Academic Press, 1973, 269–276.

[Boo78] *A Practical Guide to Splines*, New York, Springer, 1978.

[Boo87] *B*–form basics, in *Geometric Modeling: Algorithms and New Trends*, G. E. Farin (ed.), Philadelphia, SIAM, 1987, 131–148.

[Boo88] What is a multivariate spline?, in *Proc. First Intern. Conf. Industr. Applied Math., Paris 1987*, J. McKenna and R. Temam (eds.), Philadelphia, SIAM, 1988, 90–101.

[Boo89] A local basis for certain smooth bivariate pp spaces, in *Multivariate Approximation Theory IV, ISNM 90*, C. Chui, W. Schempp, and K. Zeller (eds.), Basel, Birkhäuser Verlag, 1989, 25–30.

[Boo90] Quasiinterpolants and approximation power of multivariate splines, in *Computation of Curves and Surfaces*, W. Dahmen, M. Gasca and C. Micchelli (eds.), Dordrecht, Kluwer, 1990, 313–345.

[Boo92] Approximation order without quasi-interpolants, in *Approximation Theory VII*, E. W. Cheney, C. Chui, and L. Schumaker (eds.), New York, Academic Press, 1992, 1–18.

[Boo93a] Multivariate piecewise polynomials, *Acta Numerica* (1993), 65–109.

[Boo93b] On the evaluation of box splines, *Numer. Algorithms* **5** (1993), 5–23.

[Boo97] The error in polynomial tensor-product, and Chung-Yao interpolation, in *Surface Fitting and Multiresolution Methods*, A. LeMéhauté, C. Rabut, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1997, 35–50.

Boor, C. de and R. DeVore

[BooD83] Approximation by smooth multivariate splines, *Trans. Amer. Math. Soc.* **276** (1983), 775–788.

Boor, C. de, R. DeVore, and K. Höllig

[BooDH83] Approximation order from smooth bivariate pp functions, in *Approximation Theory IV*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1983, 353–357.

Boor, C. de, R. DeVore, and A. Ron

[BooDR93] On the construction of multivariate (pre)wavelets, *Constr. Approx.* **9** (1993), 123–166.

[BooDR94] The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, *J. Funct. Anal.* **119(1)** (1994), 37–78.

Boor, C. de and G. J. Fix

[BooF73] Spline approximation by quasi-interpolants, *J. Approx. Theory* **8** (1973), 19–45.

Boor, C. de and K. Höllig

[BooH82] B-splines from parallelepipeds, *J. Analyse Math.* **42** (1982), 99–115.

[BooH83a] Approximation order from bivariate C^1-cubics: a counterexample, *Proc. Amer. Math. Soc.* **87** (1983), 649–655.

[BooH83b] Bivariate box splines and smooth pp functions on a three direction mesh, *J. Comput. Appl. Math.* **9** (1983), 13–28.

[BooH88] Approximation power of smooth bivariate pp functions, *Math. Z.* **197** (1988), 343–363.

Boor, C. de, K. Höllig, and S. D. Riemenschneider

[BooHR93] *Box Splines*, New York, Springer, 1993.

Boor, C. de and R. Q. Jia

[BooJ93] A sharp upper bound on the approximation order of smooth bivariate pp function, *J. Approx. Theory* **72** (1993), 24–33.

Bos, L.

[Bos91] On certain configurations of points in $\mathbb{R}^n$ which are unisolvent for polynomial interpolation, *J. Approx. Theory* **64** (1991), 271–280.

Bose, P., S. Ramaswami, G. Toussaint, and A. Turki

[BoseRTT02] Experimental results on quadrangulations of sets of fixed points, *Comput. Aided Geom. Design* **19** (2002), 533-552.

Bose, P. and G. Toussaint

[BoseT97] Characterizing and efficiently computing quadrangulations of planar point sets, *Comput. Aided Geom. Design* **14** (1997), 763–785.

Braess, D.

[Bra97] *Finite Elements*, Cambridge, Cambridge University Press, 1997.

Bramble, J. H. and S. R. Hilbert

[BraH71] Bounds for a class of linear functionals with applications to Hermite interpolation, *Numer. Math.* **16** (1971), 362–369.

Brenner, S. C. and L. R. Scott

[BreS94] *The Mathematical Theory of Finite Element Methods*, New York, Springer, 1994.

Brown, J. L. and A. J. Worsey

[BroW92] Problems with defining barycentric coordinates for the sphere, *Math. Modelling and Numer. Anal.* **26** (1992), 37–49.

Buhmann, M. D., O. Davydov, and T. N. T. Goodman

[BuhDG01] Prewavelets of small support, *J. Approx. Theory* **112** (2001), 16–27.

[BuhDG03] Cubic spline prewavelets on the four-directional mesh, Found. Comput. Math. **3** (2003), 113–133.

Cabrelli, C., C. Heil, and U. Molter

[CabHM98] Accuracy of lattice translates of several multidimensional refinable functions, *J. Approx. Theory* **95** (1998), 5–52.

Carnicer, J. M., M. S. Floater, and J. M. Peña

[CarFP97] Linear convexity conditions for rectangular and triangular Bernstein–Bézier surfaces, *Comput. Aided Geom. Design* **15** (1997), 27–38.

Carnicer, J. M. and J. M. Peña

[CarP94] Least supported bases and local linear independence, *Numer. Math.* **67** (1994), 289–301.

Casteljau, P. de

[Cas59] Outillage méthodes calcul, Paris, André Citroën Automobiles, 1959.

[Cas63] Courbes et Surfaces à Pôles, Paris, André Citroën Automobiles SA, 1963.

Chang, G. Z. and P. J. Davis

[ChaD84] The convexity of Bernstein polynomials over triangles, *J. Approx. Theory* **40** (1984), 11–28.

Chang, G. Z. and Y. Y. Feng

[ChaF84] An improved condition for the convexity of Bernstein–Bézier surfaces over triangles, *Comput. Aided Geom. Design* **1** (1984), 279–283.

Chang, G. Z. and J. Hoschek

[ChaH85] Convexity and variation diminishing property of Bernstein polynomials over triangles, in *Multivariate Approximation Theory III, ISNM 75*, W. Schempp and K. Zeller (eds.), Basel, Birkhäuser, 1985, 61–71.

Chang, G. Z. and T. Sederberg

[ChaS94] Nonnegative quadratic Bézier triangular patches, *Comput. Aided Geom. Design* **11** (1994), 113–116.

Chang, G. Z. and B. Su

[ChaSu85] Families of adjoint patches for a Bézier triangular surface, *Comput. Aided Geom. Design* **2** (1985), 37–42.

Chang, G. Z. and J. H. Wu

[ChaW81] Mathematical foundations of Bézier's technique, *Computer-Aided Design* **13** (1981), 134–136.

Chui, C. K.

[Chu88] *Multivariate Splines*, Philadelphia, CBMS-NSF Reg. Conf. Series in Appl. Math., vol. 54, SIAM, 1988.

Chui, C. K. and T. X. He

[ChuH88] On minimal and quasi-minimal supported bivariate splines, *J. Approx. Theory* **52** (1988), 217–238.

[ChuH89] On the dimension of bivariate super-spline spaces, *Math. Comp.* **53** (1989), 219–234.

[ChuH90a] Bivariate C^1 quadratic finite elements and vertex splines, *Math. Comp.* **54** (1990), 169–187.

[ChuH90b] Corrigenda: On the dimension of bivariate superspline spaces, *Math. Comp.* **55** (1990), 407–409.

[ChuH90c] Computation of minimal and quasi-minimal supported bivariate splines and quasi-minimal supported bivariate splines, *J. Comput. Math.* **8** (1990), 108–117.

Chui, C. K., D. Hong, and R. Q. Jia

[ChuHoJ95] Stability of optimal-order approximation by bivariate splines over arbitrary triangulations, *Trans. Amer. Math. Soc.* **347** (1995), 3301–3318.

Chui, C. K., K. Jetter, and J. D. Ward

[ChuJW87] Cardinal interpolation by multivariate splines, *Math. Comp.* **48(178)** (1987), 711–724.

Chui, C. K. and M. J. Lai

[ChuL85] On bivariate vertex splines, in *Multivariate Approximation Theory III, ISNM 75*, W. Schempp and K. Zeller (eds.), Basel, Birkhäuser, 1985, 84–115.

[ChuL87a] Vandermonde determinants and Lagrange interpolation in $\mathbb{R}^s$, in *Nonlinear and Convex Analysis*, B. L. Lin and S. Simons (eds.), New York, Marcel Dekker, 1987, 23–32.

[ChuL87b] A multivariate analog of Marsden's identity and a quasi-interpolation scheme, *Constr. Approx.* **3** (1987), 111–122.

[ChuL87c] Computation of box splines and B-splines on triangulations of nonuniform rectangular partitions, *Approx. Theory Appl.* **3** (1987), 37–62.

[ChuL90a] Multivariate vertex splines and finite elements, *J. Approx. Theory* **60** (1990), 245–343.

[ChuL90b] On bivariate super vertex splines, *Constr. Approx.* **6** (1990), 399–419.

[ChuL92] Algorithms for generating B-nets and graphically displaying box spline surfaces, *Comput. Aided Geom. Design* **8** (1992), 479–493.

Chui, C. K., L. L. Schumaker, and R. H. Wang

[ChuSW83a] On spaces of piecewise polynomials with boundary conditions, II, Type-1 triangulations, *Canad. Math. Soc. Conf. Proceedings* **3** (1983), 51–66.

[ChuSW83b] On spaces of piecewise polynomials with boundary conditions, III, Type-2 triangulations, *Canad. Math. Soc. Conf. Proceedings* **3** (1983), 67–80.

Chui, C. K., J. Stöckler, and J. D. Ward

[ChuStW92] Compactly supported box spline wavelets, *Approx. Theory Appl.* **8** (1992), 77–100.

Chui, C. K. and R. H. Wang

[ChuW82a] A generalization of univariate splines with equally spaced knots to multivariate splines, *J. Math. Res. Exposit.* **2** (1982), 99–104.

[ChuW82b] Bases of bivariate spline spaces with cross-cut grid partitions, *J. Math. Res. Exposit.* **2** (1982), 1–4.

[ChuW83a] Multivariate spline spaces, *J. Math. Anal. Appl.* **94** (1983), 197–221.

[ChuW83b] Multivariate B-splines on triangulated rectangles, *J. Math. Anal. Appl.* **92** (1983), 533–551.

[ChuW83c] On smooth multivariate spline functions, *Math. Comp.* **41** (1983), 131–142.

[ChuW83d] Bivariate cubic B-splines relative to cross-cut triangulations, *Chinese Ann. Math.* **4** (1983), 509–523.

[ChuW83e] Bivariate B-splines on triangulated rectangles, in *Approximation Theory IV*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1983, 413–418.

[ChuW84a] Spaces of bivariate cubic and quartic splines on type-1 triangulations, *J. Math. Anal. Appl.* **101** (1984), 540–554.

[ChuW84b] On a bivariate B-spline basis, *Sci. Sinica* **27** (1984), 1129–1142.

[ChuW84c] Concerning C^1 B-splines on triangulations of non-uniform rectangular partitions, *Approx. Theory Appl.* **1** (1984), 11–18.

Chung, K. C. and T. H. Yao

[ChunY77] On lattices admitting unique Lagrange interpolations, *SIAM J. Numer. Anal.* **14** (1977), 735–743.

Ciarlet, Philippe G.

[Cia78a] *The Finite Element Method for Elliptic Problems*, Amsterdam, North Holland, 1978.

[Cia78b] Interpolation error estimates for the reduced Hsieh–Clough-Tocher triangle, *Math. Comp.* **32** (1978), 335–344.

Ciarlet, P. G. and P. A. Raviart

[CiaR72] General Lagrange and Hermite interpolation in $\mathbb{R}^N$ with applications to finite element methods, *Arch. Rational Mech. Anal.* **46** (1972), 177–199.

Ciavaldini, J. F. and J. C. Nedelec

[CiavN74] Sur l'élément de Fraeijs de Veubeke et Sander, *Rev. Française Automat. Informat. Rech. Opér., Anal. Numer.* **2** (1974), 29–45.

Clough, R. and J. Tocher

[CloT65] Finite element stiffness matrices for analysis of plates in bending, in *Proc. of Conference on Matrix Methods in Structural Analysis*, Wright–Patterson Air Force Base, 1965.

Coatmelec, C.

[Coa66] Approximation et interpolation des fonctions differentiables de plusieurs variables, *Ann. Sci. Ecole Norm. Sup.* **83** (1966), 271–341.

Cohen, E., T. Lyche, and R. Riesenfeld

[CohLR84] Discrete box splines and refinement algorithms, *Comput. Aided Geom. Design* **1** (1984), 131–148.

COHEN, E., R. RIESENFELD, AND G. ELBER

[CohRE01] *Geometric Modelling with Splines*, Natik, MA, AK Peters, 2001.

COHEN, E. AND L. L. SCHUMAKER

[CohS85] Rates of convergence of control polygons, *Comput. Aided Geom. Design* **2** (1985), 229–235.

COURANT, D. AND D. HILBERT

[CouH53] *Methods of Mathematical Physics, Vol. 1*, New York, Interscience, 1953.

COX, D., L. LITTLE, AND D. OSHEA

[CoxLO98] *Using Algebraic Geometry*, New York, Springer, 1998.

CURRY, H. B. AND I. J. SCHOENBERG

[CurS66] On Pólya frequency functions IV: the fundamental spline functions and their limits, *J. Analyse Math.* **17** (1966), 71–107.

DÆHLEN, M. AND T. LYCHE

[DaeL91] Box splines and applications, in *Geometric Modeling, Methods and Applications*, H. Hagen and D. Roller (eds.), Berlin, Springer Verlag, 1991, 35–93.

DAHLKE, S., W. DAHMEN, AND V. LATOUR

[DahlDL95] Smooth refinable functions and wavelets obtained by convolution products, *Appl. Comput. Harmonic Anal.* **2** (1995), 68–84.

DAHLKE, S., K. GRÖCHENIG, AND V. LATOUR

[DahlGL97] Biorthogonal box spline wavelet bases, in *Surface Fitting and Multiresolution Methods*, A. LeMéhauté, C. Rabut, and L.L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1997, 83–92.

DAHMEN, W.

[Dah79] Multivariate B-splines — Recurrence relations and linear combinations of truncated powers, in *Multivariate Approximation Theory*, W. Schempp and K. Zeller (eds.), Basel, Birkhäuser, 1979, 64–82.

[Dah86] Subdivision algorithms converge quadratically, *J. Comput. Appl. Math.* **16** (1986), 145–158.

[Dah90] Subdivision algorithms – recent results, some extensions and further developments, in *Algorithms for Approximation II*, J. Mason and M. Cox (eds.), London, Chapman & Hall, 1990, 21–49.

[Dah91] Convexity and Bernstein–Bèzier polynomials, in *Curves and Surfaces*, P.-J. Laurent, A. LeMéhauté, and L. L. Schumaker (eds.), New York, Academic Press, 1991, 107–134.

Dahmen, W. and C. A. Micchelli

[DahM83] Recent progress in multivariate splines, in *Approximation Theory IV*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1983, 27–121.

[DahM84a] On the optimal approximation rates for criss-cross finite element spaces, *J. Comput. Appl. Math.* **10** (1984), 255–273.

[DahM84b] On the approximation order from certain multivariate spline spaces, *J. Austral. Math. Society, Ser. B* **26** (1984), 233–246.

[DahM84c] Subdivision algorithms for the generation of box-spline surfaces, *Comput. Aided Geom. Design* **1** (1984), 115–129.

[DahM85a] On the solution of certain systems of partial difference equations and linear independence of translates of box splines, *Trans. Amer. Math. Soc.* **292** (1985), 305–320.

[DahM85b] On the local linear independence of translates of a box spline, *Studia Math.* **82** (1985), 243–263.

[DahM85c] Line average algorithm: a method for the computer generation of smooth surfaces, *Comput. Aided Geom. Design* **2** (1985), 77–85.

[DahM86] On the piecewise structure of discrete box splines, *Comput. Aided Geom. Design* **3** (1986), 185–191.

[DahM87] Algebraic properties of discrete box splines, *Constr. Approx.* **3** (1987), 209–221.

[DahM88] Convexity of multivariate Bernstein polynomials and box spline surfaces, *Studia Scientiarum Math. Hungarica* **23** (1988), 265–287.

Dahmen, W., C. A. Micchelli, and H.-P. Seidel

[DahMS92] Blossoming begets B-spline bases built better by B-patches, *Math. Comp.* **59** (1992), 97–115.

Dahmen, W., P. Oswald, and X. Q. Shi

[DahOS94] C^1-hierarchical bases, *J. Comput. Appl. Math.* **51** (1994), 37–56.

Davydov, O.

[Dav98] Locally linearly independent basis for C^1 bivariate splines of degree $q \geq 5$, in *Mathematical Methods for Curves and Surfaces II*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1998, 71–78.

[Dav01] On the computation of stable local bases for bivariate polynomial splines, in *Trends in Approximation Theory*, K. Kopotun, T. Lyche, and M. Neamtu (eds.), Nashville, Vanderbilt University Press, 2001, 83–92.

[Dav02a] Locally stable spline bases on nested triangulations, in *Approximation Theory X: Wavelets, Splines, and Applications*, C. K. Chui, L. L. Schumaker, and J. Stöckler (eds.), Nashville, Vanderbilt University Press, 2002, 231–240.

[Dav02b] Stable local bases for multivariate spline spaces, *J. Approx. Theory* **111** (2002), 267–297.

DAVYDOV, O., G. NÜRNBERGER, AND F. ZEILFELDER

[DavNZ98] Approximation order of bivariate spline interpolation for arbitrary smoothness, *J. Comput. Appl. Math.* **90** (1998), 117–134.

[DavNZ01] Bivariate spline interpolation with optimal approximation order, *Constr. Approx.* **17** (2001), 181–208.

DAVYDOV, O. AND L. L. SCHUMAKER

[DavS00a] Locally linearly independent bases for bivariate polynomial spline spaces, *Adv. Comp. Math.* **13** (2000), 355–373.

[DavS00b] Stable local nodal bases for C^1 bivariate polynomial splines, in *Curve and Surface Fitting: Saint-Malo 1999*, A. Cohen, C. Rabut, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 2000, 171–180.

[DavS02] On stable local bases for bivariate polynomial spline spaces, *Constr. Approx.* **18** (2002), 87–116.

DAVYDOV, O., M. SOMMER, AND H. STRAUSS

[DavSoS97a] On almost interpolation by multivariate splines, in *Multivariate Approximation and Splines, ISNM 125*, G. Nürnberger, J. W. Schmidt, and G. Walz (eds.), Basel, Birkhäuser, 1997, 45–58.

[DavSoS97b] Locally linearly independent systems and almost interpolation, in *Multivariate Approximation and Splines, ISNM 125*, G. Nürnberger, J. W. Schmidt, and G. Walz (eds.), Basel, Birkhäuser, 1997, 59–72.

[DavSoS99] On almost interpolation and locally linearly independent bases, *East J. Approx.* **5** (1999), 67–88.

DENG, J. S., Y. Y. FENG, AND J. KOZAK

[DenFK00] A note on the dimension of the bivariate spline space over the Morgan–Scott triangulation, *SIAM J. Numer. Anal.* **37** (2000), 1021–1028.

DIENER, D.

[Die90] Instability in the spaces of bivariate piecewise polynomials of degree $2r$ and smoothness order r, *SIAM J. Numer. Anal.* **27** (1990), 543–551.

[Die97] Geometry dependence of the dimension of spaces of piecewise polynomials on rectilinear partitions, *Comput. Aided Geom. Design* **14** (1997), 43–50.

DYN, N., I. GOREN, AND S. RIPPA

[DynGR93] Transforming triangulations in polygonal domains, *Comput. Aided Geom. Design* **10** (1993), 31–536.

EDELSBRUNNER, H.

[Ede87] *Algorithms for Combinatorial Geometry*, Heidelberg, Springer-Verlag, 1987.

[Ede01] *Geometry and Topology for Mesh Generation*, Cambridge, Cambridge University Press, 2001.

EWING, D. J. E., A. J. FAWKES, AND J. R. GRIFFITHS

[EwiFG70] Rules governing the numbers of nodes and elements in a finite element mesh, *Internat. J. Numer. Meth. Engr.* **2** (1970), 597–600.

FARIN, G.

[Far77] Konstruktion und Eigenschaften von Bézier-Kurven und Bézier Flächen, Diplom-Arbeit, University of Braunschweig, 1977.

[Far79] Subsplines über Dreiecken, dissertation, Braunschweig, 1979.

[Far80] Bézier polynomials over triangles and the construction of piecewise C^r polynomials, Technical Report TR/91, Brunel University, 1980.

[Far82] Designing C^1 surfaces consisting of triangular cubic patches, *Computer-Aided Design* **14** (1982), 253–256.

[Far86] Triangular Bernstein–Bézier patches, *Comput. Aided Geom. Design* **3** (1986), 83–127.

[Far88] *Curves and Surfaces for Computer Aided Geometric Design*, NY, Academic Press, 1988.

FARIN, G., J. HOSCHEK, AND M. S. KIM (EDS).

[FarHK02] *Handbook of CAGD*, Amsterdam, North Holland, 2002.

FAROUKI, R. T. AND T. N. T. GOODMAN

[FaroG96] On the optimal stability of the Bernstein basis, *Math. Comp.* **65** (1996), 1553–1566.

FAROUKI, R. T., T. N. T. GOODMAN, AND T. SAUER

[FaroGS03] Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains, *Comput. Aided Geom. Design* **20** (2003), 209–230.

Fasshauer, G. and L. L. Schumaker

[FasS98] Scattered data fitting on the sphere, in *Mathematical Methods for Curves and Surfaces II*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1998, 117–166.

Feng, Y. Y.

[Fen87] Rates of convergence of Bézier net over triangles, *Comput. Aided Geom. Design* **4** (1987), 245–249.

Feng, Y. Y., F. L. Chen, and H. L. Zhou

[FenCZ94] The invariance of weak convexity conditions of B-nets with respect to subdivision, *Comput. Aided Geom. Design* **11** (1994), 97–107.

Feng, Y. Y. and J. Kozak

[FenK91] The convexity of families of adjoint patches for a Bézier triangular surface, *Contemp. Math.* **9** (1991), 301–304.

[FenK92] Asymptotic expansion formula for Bernstein polynomials defined on a simplex, *Constr. Approx.* **8** (1992), 49–58.

Feng, Y. Y., J. Kozak, and M. Zhang

[FenKZ96] On the dimension of the C^1 spline space for the Morgan–Scott triangulation from the blossoming approach, in *Advanced Topics in Multivariate Approximation*, F. Fontanella, K. Jetter, and P.-J. Laurent (eds.), Singapore, World Scientific Publishing Co., 1996, 71–86.

Fraeijs de Veubeke, B.

[Fra68] A conforming finite element for plate bending, *J. Solids Structures* **4** (1968), 95–108.

Floater, M. S. and E. G. Quak

[FloQ98] A semi-prewavelet approach to piecewise linear prewavelets on triangulations, in *Approximation Theory IX, Vol. 2: Computational Aspects*, C. K. Chui and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1998, 63–70.

[FloQ99] Piecewise linear prewavelets on arbitrary triangulations, *Numer. Math.* **82** (1999), 221–252.

Freeden, W., T. Gervens, and M. Schreiner

[FreeGS98] *Constructive Approximation on the Sphere*, Oxford, Oxford University Press, 1998.

Gallier, J.

[Gal00] *Curves and Surfaces in Geometric Modeling*, San Francisco, Morgan Kaufmann, 2000.

Gao, J.

[Gao92] A new finite element of C^1 cubic splines, *J. Comput. Appl. Math.* **40** (1992), 305–312.

[Gao93] A C^2 finite element and interpolation, *Computing* **50** (1993), 69–76.

Goldman, R. N.

[Gol82] Using degenerate Bézier triangles and tetrahedra to subdivide Bézier curves, *Computer-Aided Design* **14(6)** (1982), 307–311.

[Gol83] Subdivision algorithms for Bézier triangles, *Computer-Aided Design* **15(3)** (1983), 159–166.

Gonska, H. H. and J. Meir

[GonM83] A bibliography on approximation of functions by Bernstein-type operators (1955–1982), in *Approximation Theory IV*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1983, 739–785.

[GonM86] A bibliography on approximation of functions by Bernstein-type operators: Supplement 1986, in *Approximation Theory V*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1986, 621–654.

Goodman, T. N. T.

[Goo87] Variation diminishing properties of Bernstein polynomials on triangles, *J. Approx. Theory* **50** (1987), 111–126.

[Goo91] Convexity of Bézier nets on triangulations, *Comput. Aided Geom. Design* **2** (1991), 175–180.

Goodman, T. N. T. and J. Peters

[GooP95] Bézier nets, convexity and subdivision on higher-dimensional simplices, *Comput. Aided Geom. Design* **12** (1995), 53–65.

Grandine, T.

[Gra89] On convexity of piecewise polynomial functions on triangulations, *Comput. Aided Geom. Design* **6** (1989), 181–187.

Gregory, J. A. and J. W. Zhou

[GreZ91] Convexity of Bézier nets on sub-triangles, *Comput. Aided Geom. Design* **8** (1991), 207–211.

Hangelbroek, T., G. Nürnberger, C. Rössl, H.-P. Seidel, and F. Zeilfelder

[HanNRSZ04] Dimension of C^1 splines on type-6 tetrahedral partitions, *J. Approx. Theory* **131** (2004), 157–184.

He, T. X.

[He95] Shape criteria of Bernstein–Bézier polynomials over simplexes, *Computers Math. Appl.* **30** (1995), 317–333.

[He97] Positivity and convexity criteria for Bernstein–Bézier polynomials over simplices, in *Curves and Surfaces with Applications in CAGD*, A. LeMéhauté, C. Rabut, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1997, 169–176.

He, W. J. and M. J. Lai

[HeWL98] Bivariate box spline wavelets in Sobolev spaces, in *Proceedings of SPIE, Vol. 3458*, 1998, 56–66.

[HeWL99] Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities, *J. Appl. Comput. Harm. Anal.* **6** (1999), 53–74.

[HeWL03] Construction of trivariate compactly supported biorthogonal box wavelets, *J. Approx. Theory* **120** (2003), 1–19.

Heighway, E.

[Hei83] A mesh generator for automatically subdividing irregular polygons into quadrilaterals, *IEEE Trans. Magnetics* **19** (1983), 2535–2538.

Hong, D.

[Hon91] Spaces of bivariate spline functions over triangulation, *Approx. Theory Appl.* **7** (1991), 56–75.

[Hon95] A new formulation of Bernstein–Bézier based smoothness conditions for pp functions, *Approx. Theory Appl.* **11** (1995), 67–75.

Hong, D. and L. L. Schumaker

[HonS04] Surface compression using a space of C^1 cubic splines with a hierarchical basis, *Computing* **72** (2004), 79–92.

Hoschek, J. and D. Lasser

[HosL93] *Fundamentals of Computer Aided Geometric Design*, Wellesley, MA, AK Peters, 1993.

Hu, S. M.

[Hu96] Conversion of a triangulated Bézier patch into three rectangular Bézier patches, *Comput. Aided Geom. Design* **13** (1996), 219–226.

Ibrahim, A. Kh.

[Ibr89] Dimension of superspline spaces defined over rectilinear partitions, in *Approximation Theory VI*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1989, 337–340.

IBRAHIM, A. AND L. L. SCHUMAKER

[IbrS91] Super spline spaces of smoothness r and degree $d \geq 3r + 2$, *Constr. Approx.* **7** (1991), 401–423.

JIA, R. Q.

[Jia83] Approximation by smooth bivariate splines on a three-direction mesh, in *Approximation Theory IV*, C. Chui, L. Schumaker, and J. Ward (eds.), New York, Academic Press, 1983, 539–545.

[Jia84] Linear independence of translates of a box spline, *J. Approx. Theory* **40** (1984), 158–160.

[Jia85] Local linear independence of the translates of a box spline, *Constr. Approx.* **1** (1985), 175–182.

[Jia86] Approximation order from certain spaces of smooth bivariate splines on a three-direction mesh, *Trans. Amer. Math. Soc.* **295** (1986), 199–212.

[Jia88] Local approximation order of box splines, *Scientia Sinica* **31** (1988), 274–285.

[Jia90] Lower bounds on the dimension of spaces of bivariate splines, in *Multivariate Approximation and Interpolation, ISNM 94*, W. Haussmann and K. Jetter (eds.), Basel, Birkhäuser, 1990, 155–165.

[Jia98] Stability of the shifts of a finite number of functions, *J. Approx. Theory* **95** (1998), 194–202.

JIA, R. Q. AND J. J. LEI

[JiaL93] A new version of the Strang–Fix conditions, *J. Approx. Theory* **74(2)** (1993), 221–225.

JIA, R. Q. AND S.-T. LIU

[JiaLiu06] C^1 spline wavelets on triangulations, manuscript, 2006.

JOE, B.

[Joe91] Construction of three-dimensional Delaunay triangulations using local transformations, *Comput. Aided Geom. Design* **8** (1991), 123–142.

JOE, B. AND A. LIU

[JoeL94] Relationship between tetrahedron shape measures, *BIT* **34** (1994), 268–287.

JOHNSTON, B. P., J. M. SULLIVAN, AND A. KWASNIK

[JohSK91] Automatic conversion of a triangular finite meshes to quadrilateral elements, *Intern. J. Numer. Methods Eng.* **31** (1991), 67–84.

KASHYAP, P.

[Kas98] Geometric interpretation of continuity over triangular domains, *Comput. Aided Geom. Design* **15** (1998), 773–786.

Kotyczka, U. and P. Oswald

[KotO95] Piecewise linear prewavelets of small support, in *Approximation Theory III*, E. W. Cheney (ed.), New York, Academic Press, 1980, 235–242.

Laghchim-Lahlou, M.

[Lag98] The C^r-fundamental splines of Clough–Tocher and Powell–Sabin types of Lagrange interpolation on a three direction mesh, *Adv. Comp. Math.* **8** (1998), 353–366.

Laghchim-Lahlou, M. and P. Sablonnière

[LagS89a] Composite quadrilateral finite elements of class C^r, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker (eds.), New York, Academic Press, 1989, 413–418.

[LagS89b] C^r finite elements of HCT, PS and FVS types, in *Proc. Fifth International Symposium on Numerical Methods in Engineering, Vol. 2*, R. Gruber, J. Periaux and R. P. Shaw (eds.), Berlin, Springer, 1989, 163–168.

[LagS93] Eléments finis polynomiaux composés de classe C^r, *C. R. Acad. Sci. Paris, série I* **136** (1993), 503–508.

[LagS94] Triangular finite elements of HCT type and class C^ρ, *Adv. Comp. Math.* **2** (1994), 101–122.

[LagS95] Quadrilateral finite elements of FVS type and class C^ρ, *Numer. Math.* **70** (1995), 229–243.

Lai, M. J.

[Lai91] On dual functionals of polynomials in B-form, *J. Approx. Theory* **67** (1991), 19–37.

[Lai92a] Asymptotic formulae of multivariate Bernstein approximation, *J. Approx. Theory* **70** (1992), 229–242.

[Lai92b] Fortran subroutines for B-nets of box splines on three- and four-directional meshes, *Numer. Algorithms* **2** (1992), 33–38.

[Lai93] Some sufficient conditions for convexity of multivariate Bernstein-Bézier polynomials and box spline surfaces, *Studia Scientiarum Math. Hungarica* **28** (1993), 363–374.

[Lai94] Approximation order from bivariate C^1 cubics on a four directional mesh is full, *Comput. Aided Geom. Design* **11** (1994), 215–223.

[Lai95] Bivariate spline spaces on FVS-triangulations, in *Approximation Theory VIII, Vol. 1: Approximation and Interpolation*, C. K. Chui and L. L. Schumaker (eds.), Singapore, World Scientific Publishing Co., Inc., 1995, 309–316.

[Lai96a] Scattered data interpolation and approximation using bivariate C^1 piecewise cubic polynomials, *Comput. Aided Geom. Design* **13** (1996), 81–88.

[Lai96b] On C^2 quintic spline functions over triangulations of Powell–Sabin's type, *J. Comput. Appl. Math.* **73** (1996), 135–155.

[Lai97] Geometric interpretation of smoothness conditions of triangular polynomial patches, *Comput. Aided Geom. Design* **14** (1997), 191–199.

[Lai06] Construction of multivariate compactly supported prewavelets in L_2 spaces and pre-Riesz basis in Sobolev spaces, *J. Approx. Theory* **142** (2006), 83–115.

Lai, M. J. and A. LeMéhauté

[LaiL04] A new kind of trivariate C^1 spline, *Adv. Comp. Math.* **21** (2004), 273–292.

Lai, M. J., A. LeMéhauté, and T. Sorokina

[LaiLS06] An octahedral C^2 macro-element, *Comput. Aided Geom. Design* **23** (2006), 640–654.

Lai, M. J. and L. L. Schumaker

[LaiS97] Scattered data interpolation using C^2 supersplines of degree six, *SIAM J. Numer. Anal.* **34** (1997), 905–921.

[LaiS98] On the approximation power of bivariate splines, *Adv. Comp. Math.* **9** (1998), 251–279.

[LaiS99] On the approximation power of splines on triangulated quadrangulations, *SIAM J. Numer. Anal.* **36** (1999), 143–159.

[LaiS01] Macro-elements and stable local bases for splines on Clough–Tocher triangulations, *Numer. Math.* **88** (2001), 105–119.

[LaiS02] Quadrilateral macro-elements, *SIAM J. Math. Anal.* **33** (2002), 1107–1116.

[LaiS03] Macro-elements and stable local bases for splines on Powell–Sabin triangulations, *Math. Comp.* **72** (2003), 335–354.

[LaiS06] Trivariate C^r polynomial macro-elements, Constr. Approx., to appear.

Lai, M. J. and J. Stöckler

[LaiSt06] Construction of multivariate compactly supported tight wavelet frames, *Appl. Comput. Harmonic Anal.* **21** (2006), 324–348.

Lai, M. J. and P. Wenston

[LaiW96] On multilevel bases for elliptic boundary value problems, *J. Comput. Appl. Math.* **71** (1996), 95–113.

Lasser, D.

[Las87] Bernstein–Bézier-Darstellung trivariater Splines, dissertation, University of Darmstadt, 1987.

[Las02] Tensor product Bézier surfaces on triangle Bézier surfaces, *Comput. Aided Geom. Design* **19** (2002), 625–643.

Lau, W. W.

[Lau06] A lower bound for the dimension of trivariate spline spaces, *Constr. Approx.* **23** (2006), 23–31.

Laurent, P. J. and P. Sablonnière

[LauS01] Pierre Bézier: An engineer and a mathematician, *Comput. Aided Geom. Design* **18** (2001), 609–617.

Lawson, C. L.

[Law72] Transforming triangulations, *Discrete Math.* **3** (1972), 365–372.

[Law86] Properties of n-dimensional triangulations, *Comput. Aided Geom. Design* **3** (1986), 231–246.

Le Méhauté, A.

[Lem84] Interpolation et approximation par des fonctions polynomiales par morceaux dans $\mathbb{R}^n$, dissertation, University of Rennes, 1984.

Levesley, J. and D. L. Ragozin

[LevR00] Local approximation on manifolds using radial functions and polynomials, in *Curve and Surface Fitting: Saint-Malo 1999*, A. Cohen, C. Rabut, and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville TN, 2000, 291–300.

Lino, L and J. Wilde

[LinW91] Subdivision of triangular Bézier patches into rectangular Bézier patches, *Adv. Design Automation* **32** (1991), 1–6.

Lions, J. L. and E. Magenes

[LioM72] *Non-Homogeneous Boundary Value Problems and Applications I*, Berlin, Springer-Verlag, 1972.

Liu, A. and B. Joe

[LiuJ95] Quality local refinement of tetrahedral meshes based on bisection, *SIAM J. Sci. Comput.* **16** (1995), 1269–1291.

[LiuJ96] Quality local refinement of tetrahedral meshes based on 8-subtetrahedron subdivision, *Math. Comp.* **65** (1996), 1183–1200.

Lorentz, G. G.

[Lor53] *Bernstein Polynomials*, Toronto, University of Toronto Press, 1953.

Lorentz, R. A. and P. Oswald

[LorRO97] Nonexistence of compactly supported box spline prewavelets in Sobolev spaces, in *Surface Fitting and Multiresolution Methods*, A. LeMéhauté, C. Rabut, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1997, 235–244.

Lyche, T. and L. L. Schumaker

[LycS75] Local spline approximation methods, *J. Approx. Theory* **15** (1975), 294–325.

Mainar, E. and J. Peña

[MaiP06a] Running error analysis of evaluation algorithms for bivariate polynomials in barycentric Bernstein form, *Computing* **77** (2006), 97–111.

[MaiP06b] Evaluation algorithms for multivariate polynomials in Bernstein Bézier form, J. Approx. Theory, to appear.

Manni, C.

[Man90] The dimension of bivariate spline spaces over special partitions, in *Contributions to the Computation of Curves and Surfaces, Puerto de la Cruz, 1989*, Zaragoza, Monogr. Acad. Ci. Exact. Nat. Zaragoza, 1990, 35–44.

[Man91] On the dimension of bivariate spline spaces over rectilinear partitions, *Approx. Theory Appl.* **7** (1991), 23–34.

[Man92a] On the dimension of bivariate spline spaces on generalized quasi-cross-cut partitions, *J. Approx. Theory* **69** (1992), 141–155.

[Man92b] Lower bounds on the dimension of bivariate spline spaces and generic triangulations, in *Mathematical Methods in Computer Aided Geometric Design II*, T. Lyche and L. L. Schumaker (eds.), New York, Academic Press, 1992, 401–412.

Micchelli, C. A.

[Mic95] *Mathematical Aspects of Geometric Modeling*, CMBS-NSF Regional Conference Series in Applied Mathematics v. 65, Philadelphia, SIAM, 1995.

Micchelli, C. A. and A. Pinkus

[MicP89] Some remarks on nonnegative polynomials on polyhedra, in *Probability, Statistics, and Mathematics*, T. Anderson, K. Athreya and D. Iglehart (eds.), Boston, Academic Press, 1989, 163–186.

Möbius, A. F.

[Moe86] Ueber eine neue Behandlungsweise der analytischen Sphärik, in *Abhandlungen bei Begründung der Königl. Sächs. Gesellschaft der Wissenschaften*, Jablonowski Gesellschaft, Leipzig, 45-86. (See also

A. F. Möbius, Gesammelte Werke, F. Klein (ed.), vol. 2, Leipzig, 1886, 1–54.

Morgan, J. and R. Scott

[MorS75] A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$, *Math. Comp.* **29** (1975), 736–740.

Müller, C.

[Mue66] *Spherical Harmonics*, Berlin, Lecture Notes in Mathematics Springer-Verlag, 1966.

Nadler, E. J.

[Nad93] Non-negativity of a bivariate quadratic function on a triangle, *Comput. Aided Geom. Design* **9** (1993), 195–205.

Nairn, D., J. Peters, and D. Lutterkort

[NaiPL99] Sharp quantitative bounds on the distance between a polynomial piece and its Bézier control polygon, *Comput. Aided Geom. Design* **16** (1999), 613–631.

Neamtu, M. and L. L. Schumaker

[NeaS04] On the approximation order of splines on spherical triangulations, *Adv. Comp. Math.* **21** (2004), 3–20.

Nielson, G. M.

[Nie93] Characterization of an affine invariant triangulation, in *Geometric Modelling, Computing Supplementum 8*, G. Farin, H. Hagen, H. Noltemeir, and W. Knoedel (eds), Berlin, Springer-Verlag, 1993, 191–210.

Nürnberger, G., V. Rayevskaya, L. L. Schumaker, and F. Zeilfelder

[NurRSZ04] Local Lagrange interpolation with C^2 splines of degree seven on triangulations, in *Advances in Constructive Approximation: Vanderbilt 2003*, M. Neamtu and E. W. Saff (eds.), Brentwood, Nashboro Press, 2004, 345–370.

Ong, M. E.

[Ong94] Uniform refinement of a tetrahedron, *SIAM J. Sci. Comput.* **15** (1994), 1134–1144.

Oswald, P.

[Osw88] *Multilevel Finite Element Approximation: Theory and Applications*, Stuttgart, Teubner, 1988.

Pfeifle, R. and H.-P. Seidel

[PfeS95] Spherical triangular B-splines with application to data fitting, in *Computer Graphics Forum, Vol. 14*, F. Post and M. Göbel (eds.), Blackwell, 1995, 89–96.

Powell, M. J. D. and M. A. Sabin

[PowS77] Piecewise quadratic approximations on triangles, *ACM Trans. Math. Software* **3** (1977), 316–325.

Prautzsch, H.

[Pra84] Unterteilungsalgorithmen für multivariate Splines – ein geometrischer Zugang, dissertation, University of Braunschweig, 1984.

[Pra92] On convex Bézier triangles., *Rev. Française Automat. Informat. Rech. Opér., Anal. Numer.* **26** (1992), 23–36.

Prautzsch, H., W. Boehm, and M. Paluszny

[PraBP02] *Bézier and B-spline Techniques*, Berlin, Springer, 2002.

Prautzsch, H. and L. Kobbelt

[PraK94] Convergence of subdivision and degree elevation, *Adv. Comput. Math.* **2** (1994), 143–154.

Preparata, F. P. and M. I. Shamos

[PreS85] *Computational Geometry: An Introduction*, New York, Springer-Verlag, 1985.

Ramaswami, S., P. Ramos, and G. T. Toussain

[RamRT98] Converting triangulations to quadrangulations, *Computat. Geom.* **9** (1998), 257–276.

Rayevskaya, V. and L. L. Schumaker

[RayS05] Multi-sided macro-element spaces based on Clough–Tocher triangle splits, *Comput. Aided Geom. Design* **22** (2005), 57–79.

Reif, U.

[Rei00] Best bounds on the approximation of polynomials and splines by their control structure, *Comput. Aided Geom. Design* **17** (2000), 579–589.

Renka, R. J.

[Ren84] Algorithm 624: Triangulation and interpolation of arbitrarily distributed points in the plane, *ACM Trans. Math. Software* **10** (1984), 440–442.

[Ren97] Algorithm 772: STRIPACK: Delaunay triangulation and Voronoi diagram on the surface of a sphere, *ACM Trans. Math. Software* **23** (1997), 416–434.

Riemenschneider, S. and Z. W. Shen

[RieS91] Box splines, cardinal series, and wavelets, in *Approximation Theory and Functional Analysis*, C.K. Chui (ed.), New York, Academic Press, 1991, 133–149.

RIPMEESTER, D. J.

[Rip95] Upper bounds on the dimension of bivariate spline spaces and duality in the plane, in *Mathematical Methods for Curves and Surfaces*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Nashville, Vanderbilt University Press, 1995, 455–466.

RON, A. AND Z. SHEN

[RonS98] Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$, *Math. Comp.* **67** (1998), 191–207.

RUDIN, M. E.

[Rud58] An unshellable triangulation of a tetrahedron, *Bull. Amer. Math. Soc.* **64** (1958), 90–91.

RUSHING, B.

[Rus73] *Topological Embeddings*, New York, Academic Press, 1973.

SABIN, M. A.

[Sab77] The use of piecewise forms for the numerical representation of shape, dissertation, MTA Budapest, 1977.

SABLONNIÈRE, P.

[Sabl85a] Bernstein—Bézier methods for the construction of bivariate spline approximants, *Comput. Aided Geom. Design* **2** (1985), 29–36.

[Sabl85b] Composite finite elements of class C^k, *J. Comp. Appl. Math.* **12** (1985), 541–550.

[Sabl87] Composite finite elements of class C^2, in *Topics in Multivariate Approximation*, C. K. Chui, L. L. Schumaker, and F. Utreras (eds.), New York, Academic Press, 1987, 207–217.

SABLONNIÈRE, P. AND F. JEEAWOCK-ZEDEK

[SablJ94] Hermite and Lagrange interpolation by quadratic splines on non-uniform criss-cross triangulations, in *Wavelets, Images, and Surface Fitting*, P.-J. Laurent, A. LeMéhauté, and L. L. Schumaker (eds.), Wellesley, MA, A. K. Peters, 1994, 445–452.

SABLONNIÈRE, P. AND M. LAGHCHIM-LAHLOU

[SablL93] Eléments finis polynomiaux composés de classe C^r, *C. R. Acad. Sci. Paris* **316** (1993), 503–508.

SANDER, G.

[San64] Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion, *Bull. Soc. Royale Sciences Liège* **33** (1964), 456–494.

Sauer, T.

[Sau91] Multivariate Bernstein polynomials and convexity, *Comput. Aided Geom. Design* **8** (1991), 465–478.

Schumaker, L. L.

[Sch79] On the dimension of spaces of piecewise polynomials in two variables, in *Multivariate Approximation Theory*, W. Schempp and K. Zeller (eds.), Basel, Birkhäuser, 1979, 396–412.

[Sch81] *Spline Functions: Basic Theory*, New York, Wiley, 1981.

[Sch84a] On spaces of piecewise polynomials in two variables, in *Approximation Theory and Spline Functions*, S. P. Singh, J. H. W. Burry, and B. Watson (eds.), Dordrecht, Reidel, 1984, 151–197.

[Sch84b] Bounds on the dimension of spaces of multivariate piecewise polynomials, *Rocky Mountain J. Math.* **14** (1984), 251–264.

[Sch87] Triangulation methods, in *Topics in Multivariate Approximation*, C. K. Chui, L. L. Schumaker, and F. Utreras (eds.), New York, Academic Press, 1987, 219–232.

[Sch88a] Dual bases for spline spaces on cells, *Comput. Aided Geom. Design* **5** (1988), 277–284.

[Sch88b] Constructive aspects of bivariate piecewise polynomials, in *Mathematics of Finite Elements and Applications VI*, J. Whiteman (ed.), London, Academic Press, 1988, 513–520.

[Sch89] On super splines and finite elements, *SIAM J. Numer. Anal.* **26** (1989), 997–1005.

[Sch93a] Computing optimal triangulations using simulated annealing, *Comput. Aided Geom. Design* **10** (1993), 329–345.

[Sch93b] Triangulation methods in CAGD, *IEEE Comp. Graph. Appl.* **13** (1993), 47–52.

Schumaker, L. L. and T. Sorokina

[SchS04] C^1 quintic splines on type-4 tetrahedral partitions, *Adv. Comp. Math.* **21** (2004), 421–444.

[SchS05] A trivariate box macro-element, *Constr. Approx.* **21** (2005), 413–431.

[SchS06] A family of C^r macro-elements on Powell–Sabin-12 splits, *Math. Comp.* **75** (2006), 711–726.

Schumaker, L. L. and W. Volk

[SchV86] Efficient evaluation of multivariate polynomials, *Comput. Aided Geom. Design* **3** (1986), 149–154.

Seidel, H. P.

[Sei89] A general subdivision theorem for Bézier triangles, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker (eds.), New York, Academic Press, 1989, 573–581.

Shewchuk, J. R.

[She02] Delaunay refinement algorithms for triangular mesh generation, *Comp. Geom.* **22** (2002), 21–74.

Shi, X. Q.

[Shi91] The singularity of Morgan–Scott's triagulation, *Comput. Aided Geom. Design* **8** (1991), 201–206.

Stancu, D. D.

[Sta59] De l'approximation par des polynomes du type Bernstein des fonctions des deux variables, *Comm. Akad. R. P. Romaine* **9** (1959), 773–777.

[Sta80] Some Bernstein polynomials in two variables and their applications, *Soviet Math.* **1** (1980), 1025–1028.

Stein, E. M.

[Ste70] *Singular Integrals and Differentiability Properties of Functions*, New Jersey, Princeton University Press, 1970.

Strang, G.

[Str73] Piecewise polynomials and the finite element method, *Bull. Amer. Math. Soc.* **79** (1973), 1128–1137.

[Str74] The dimension of piecewise polynomials, and one-sided approximation, in *Numerical Solution of Differential Equations*, G. A. Watson (ed.), Berlin, Springer, 1974, 144–152.

Strang, G. and G. Fix

[StrF73] A Fourier analysis of the finite element variational method, in *Constructive Aspects of Functional Analysis*, G. Geymonat (ed.), C.I.M.E. II Ciclo 1971, 1973, 793–840.

Vigo, M., P. Núria, and J. Cotria

[VigNC02] Regular triangulations of dynamic sets of points, *Comput. Aided Geom. Design* **19** (2002), 127–149.

Wahba, G.

[Wah90] *Spline Models for Observational Data*, Philadelphia, CBMS NSF Regional Conference Series in Applied Mathematics 59, SIAM, 1990.

Walz, G.

[Wal97] Trigonometric Bézier and Stancu polynomials over intervals and triangles, *Comput. Aided Geom. Design* **14** (1997), 393–397.

Wang, R. H.

[Wan85] The dimension and basis of spaces of multivariate splines, *J. Comput. Appl. Math.* **12** (1985), 163–177.

[Wan01] *Multivariate Spline Functions and their Applications,* Translated from the 1994 Chinese original by Shao-Ming Wang, Dordrecht, Kluwer, 2001.

Wang, T.

[Wang92] A C^2-quintic spline interpolation scheme on a triangulation, *Comput. Aided Geom. Design* **9** (1992), 379–386.

Wang, Z. B. and Q. M. Liu

[WangL88] An improved condition for the convexity and positivity of Bernstein–Bézier surfaces over triangles, *Comput. Aided Geom. Design* **5** (1988), 269–275.

Whelan, T.

[Whe86] A representation of a C^2 interpolant over triangles, *Comput. Aided Geom. Design* **3** (1986), 53–66.

Whiteley, W.

[Whi91a] The combinatorics of bivariate splines, in *Applied Geometry and Discrete Mathematics, Victor Klee Festschrift*, P. Gritzmann and B. Sturmfels (eds.), Providence, DIMACS series, AMS, 1991, 567–608.

[Whi91b] A matrix for splines, in *Progress in Approximation Theory*, P. Nevai and A. Pinkus (eds.), New York, Academic Press, 1991, 821–828.

[Whi91c] Vertex splitting in isostatic frameworks, *Structural Topology* **16** (1991), 23–30.

Wilhelmsen, D. R.

[Wil74] A Markov inequality in several dimensions, *J. Approx. Theory* **11** (1974), 216–220.

Worsey, A. J. and G. Farin

[WorF87] An n-dimensional Clough–Tocher interpolant, *Constr. Approx.* **3** (1987), 99–110.

Worsey, A. J. and B. Piper

[WorP88] A trivariate Powell–Sabin interpolant, *Comput. Aided Geom. Design* **5** (1988), 177–186.

Wu, D. B.

[Wu93] Dual bases of a Bernstein polynomial basis on simplices, *Comput. Aided Geom. Design* **10** (1993), 483–489.

Ženíšek, A.

[Zen70] Interpolation polynomials on the triangle, *Numer. Math.* **15** (1970), 283–296.

[Zen73a] Polynomial approximation on tetrahedrons in the finite element method, *J. Approx. Theory* **7** (1973), 334–351.

[Zen73b] Hermite interpolation on simplexes and the finite element method, in *Proc. Equadiff III, Brno*, 1973, 271–277.

[Zen74] A general theorem on triangular finite C^m elements, *Rev. Française Automat. Informat. Rech. Opér., Ser. Rouge* **8** (1974), 119–127.

Zhao, K. and J. C. Sun

[ZhaS88] Dual bases of multivariate Bernstein–Bézier polynomials, *Comput. Aided Geom. Design* **5** (1988), 119–125.

Ziegler, G. M.

[Zie95] *Lectures on Polytopes*, Berlin, Springer-Verlag, 1995.

Zlamal, M

[Zla68] On the finite element method, *Numer. Math.* **12** (1968), 394–409.

[Zla70] A finite element procedure of the second order of accuracy, *Numer. Math.* **14** (1970), 394–402.

Zwart, P. B.

[Zwa73] Multivariate splines with non-degenerate partitions, *SIAM J. Numer. Anal.* **10** (1973), 665–673.

INDEX