NEW AGE

A TEXTBOOK OF COMPUTER BASED NUMERICAL AND STATISTICAL TECHNIQUES

A.K. Jaiswal
Anju Khandelwal

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Preface

The present book 'Computer Based Numerical and Statistical Techniques' is primarily written according to the unified syllabus of Mathematics for B. Tech. II year and M.C.A. I year students of all Engineering colleges affiliated to U.P. Technical University, Lucknow.

The subject matter is presented in a very systematic and logical manner. In each chapter, all concepts, definitions and large number of examples in the best possible way have been discussed in detail and lucid manner so that the students should feel no difficulty to understand the subject. A unique feature of this book is to provide with an algorithm and computer program in C-language to understand the steps and methodology used in writing the program.

Thorough care has been taken to eradicate errors but perfection cannot be claimed. The reader are requested for constructive suggestions to improve the book and it will be gratefully accepted.

The authors wish to express their thanks to New Age International (P) Limited, Publishers and the editorial department for the inspiration given by them to undertake and complete this project.

AUTHORS

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CHAPTER **1**

Errors and Floating Point

1.1 INTRODUCTION

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique.

1.2 ACCURACY OF NUMBERS

(*i*) **Exact Number:** Number with which no uncertainly is associated to no approximation is taken, are known as exact numbers *e.g.,* 5, 21/6, 12/3, ... etc. are exact numbers.

(*ii*) **Approximate Number:** There are numbers, which are not exact, *e.g.,* $\sqrt{2}$ = 1.41421 ..., e = 2.7183 ...., etc. are not exact numbers since they contain infinitely many non-recurring digits. Therefore the numbers obtained by retaining a few digits, are called approximates numbers, *e.g.*, 3.142, 2.718 are the approximate values of π and e.

(*iii*) **Significant Figures:** The significant figures are the number of digits used to express a number. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant digits. '0' is also a significant figure except when it is used to fix the decimal point or to fill the places of unknown or discarded digits.

For example, each number 5879, 3.487, 0.4762 contains four significant figures while the numbers 0.00486, 0.000382, 0.0000376 contains only three significant figures since zeros only help to fix the position of the decimal point.

Similarly, in the number 0.0002070, the first four '0's are not significant figure since they serve only to fix the position of decimal point and indicate the place values of the other digits. The other two '0's are significant.

Some example to be more clear, the number 2.0683 contain five significant figure.

3900	two
39.0	two
3.9×10^6	two

(*iv*) **Round off Number:** If we divide 2 by 7, we get 0.285714... a quotient which is a non-terminating decimal fraction. For using such a number in practical computation, it is to be cut-off to a manageable size such as 0.29, 0.286, 0.2857,.... etc. The process of cutting off super-flouts digits and retaining as many digits as desired is known as rounding off a number or we can say that process of dropping unwanted digits is called rounding-off. Number are rounded-off according to the following rules:

To round-off the number to *n* significant figures, discard all digits to the right of nth digit and if this discarded number is

(1) Less than 5 in $(n + 1)$th place, leave the nth digit unaltered *e.g.*, 8.893 to 8.89

(2) Greater than 5 in $(n + 1)$th place, increase the nth digit by unity *e.g.*, 5.3456 to 5.346

(3) Exactly 5 in $(n + 1)$th place, increase the *n*th digit by unity if it is odd otherwise leave it unchanged. *e.g.*, 11.675 to 11.68, 11.685 to 11.68.

Example 1. *Round-off the following numbers correct to four significant figures: 58.3643, 979.267, 7.7265, 56.395, 0.065738 and 7326853000.*

Sol. After retaining first four significant figures we have:

(*i*) 58.3643 becomes 58.36

(*ii*) 979.267 becomes 979.3

(*iii*) 7.7265 becomes 7.726 (digit in the fourth place is even)

(*iv*) 56.395 becomes 56.40 (digit in the fourth place is odd)

(*v*) 0.065738 becomes 0.06574 (because zero in the left is not significant)

(*vi*) 7326853000 becomes 7327×10^6.

1.3 ERRORS

Error = True value – Approximate value

A computer has a finite word length and so only a fixed number of digits are stored and used during computation. This would mean that even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependent and is called machine epsilon. After the computation is over, the result in the machine form (with base *b*) is again converted to decimal form understandable to the users and some more error may be introduced at this stage. In general, we can say that *Error = True value – Approximate value*. The errors may be divided into the following different types:

1. **Inherent Error:** The inherent error is that quantity which is already present in the statement of the problem before its solution. The inherent error arises either due to the simplified assumptions in the mathematical formulation of the problem or due to the errors in the physical measurements of the parameters of the problem.

   Inherent error can be minimized by obtaining better data, by using high precision computing aids and by correcting obvious errors in the data.

2. **Round-off Error:** The round-off error is the quantity, which arises from the process of rounding off numbers. It sometimes also called numerical error. Also round off denote a quantity, which must be added to the finite representation of a compound number in order to make it the true representation of that number. The round-off error can be reduced by carrying the computation to more significant figures at each step of computation. At each step of computations, retain at least one more significant figure than that given in the data, perform the last operation, and then round off.

3. **Truncation Error:** Three types of errors caused by using appropximate formulae in computation or on replace an infinite process by a finite one that is when a function $f(x)$

is evaluated from an infinite series for x after 'truncating' it at a certain stage, we have this type of error. The study of this type of error is usually associated with the problem of convergence.

4. **Absolute Error:** Absolute error is the numerical difference between the true value of a quantity and its approximate value. Thus if x' is the approximate value of quantity x then $|x - x'|$ is called the absolute error and denoted by E_a. Therefore $E_a = |x - x'|$. The unit of exact or unit of approximate values expresses the absolute error.

5. **Relative Error:** The relative error E_r defined by $E_r = \left|\frac{x - x'}{x}\right| = \frac{E_a}{True\ Value}$. Where x' is the approximate value of quantity x. The relative error is independent of units.

6. **Percentage Error:** The percentage error in x' which is the approximate value of x is given by $E_p = 100 \times E_r = 100 \times \left|\frac{x - x'}{x}\right|$. The percentage error is also independent of units.

1.4 A GENERAL FORMULA FOR ERROR

Let $X = f(x_1, x_2, \ldots, x_n)$ be the function having n variables. To determined the error δX in X due to the errors $\delta x_1, \delta x_2, \ldots, \delta x_n$ in $x_1, x_2, \ldots, x_n$ respectively.

$$X + \delta X = f(x_1 + \delta x_1, x_2 + \delta x_2, \ldots\ldots, x_n + \delta x_n)$$

Using Taylor's series for more than two variables, to expand the R.H.S. of above, we get

$$X + \delta X = f(x_1, x_2 \ldots\ldots, x_n) + \left(\delta x_1 \frac{\partial X}{\partial x_1} + \delta x_2 \frac{\partial X}{\partial x_2} + \ldots\ldots + \delta x_n \frac{\partial X}{\partial x_n}\right)$$

$$+ \frac{1}{2}\left[(\delta x_1)^2 \frac{\partial^2 X}{\partial x_1^2} + (\delta x_2)^2 \frac{\partial^2 X}{\partial x_2^2} + \ldots + (\delta x_n)^2 \frac{\delta^2 x}{\delta x_n^2} + 2\delta x_1 \delta x_2 \frac{\partial^2 X}{\partial x_1 \partial x_2} + \ldots\right] + \ldots$$

Errors $\delta x_1, \delta x_2, \ldots, \delta x_n$ all are small so that the terms containing $(\delta x_1)^2, (\delta x_2)^2, \ldots (\delta x_n)^2$ and higher powers of $\delta x_1, \delta x_2, \ldots, \delta x_n$ are being neglected.

Therefore $X + \delta X = f(x_1, x_2, \ldots, (x_n) + \left(\delta x_1 \frac{\partial X}{\partial x_1} + \delta x_2 \frac{\partial X}{\partial x_2} + \ldots\ldots + \delta x_n \frac{\partial X}{\delta x_n}\right)$ ... (1)

$$\delta X = \delta x_1 \frac{\partial X}{\partial x_1} + \delta x_2 \frac{\partial X}{\partial x_2} + \ldots\ldots + \delta x_n \frac{\partial X}{\partial x_n} \qquad \text{.... (2)}$$

Because $X = f(x_1, x_2, \ldots\ldots, x_n)$.

Equation (2) represents the general formula for Errors. If equation (2) divided by X we get relative error

$$E_r = \frac{\delta X}{X} = \frac{\delta x_1}{X}\frac{\partial X}{\partial x_1} + \frac{\delta x_2}{X}\frac{\delta X}{\delta x_2} + \ldots\ldots + \frac{\delta x_n}{X}\frac{\partial X}{\partial x_n}$$

On taking modulus both of the sides, we get maximum relative error.

$$\left|\frac{\delta X}{X}\right| \le \left|\frac{\delta x_1}{X}\frac{\partial X}{\partial x_1}\right| + \left|\frac{\delta x_2}{X}\frac{\partial X}{\partial x_2}\right| + \ldots\ldots\ldots\ldots + \left|\frac{\delta x_n}{X}\frac{\partial X}{\partial x_n}\right|$$

Also from equation (2), by taking modulus we get maximum absolute error.

$$|\delta X| \le \left|\delta x_1 \frac{\partial X}{\partial x_1}\right| + \left|\delta x_2 \frac{\partial X}{\partial x_2}\right| + \ldots\ldots\ldots\ldots + \left|\delta x_n \frac{\partial X}{\partial x_n}\right|$$

1.4.1 Error in Addition of Numbers

Let $X = f(x_1 + x_2 + \ldots\ldots + x_n)$

$\therefore \quad X + \delta X = (x_1 + \delta x_1) + (x_2 + \delta x_2) + \ldots\ldots\ldots\ldots + (x_n + \delta x_n)$

$\qquad\qquad = (x_1 + x_2 + \ldots\ldots\ldots + x_n) + (\delta x_1 + \delta x_2 + \ldots\ldots\ldots + \delta x_n)$

Therefore, $\delta X = \delta x_1 + \delta x_2 + \ldots\ldots\ldots\ldots\ldots + \delta x_n$; this is an absolute error.

Dividing by X we get, $\frac{\delta X}{X} = \frac{\delta x_1}{X} + \frac{\delta x_2}{X} + \ldots\ldots\ldots \frac{\delta x_n}{X}$; which is a relative error. A gain, $\left|\frac{\delta X}{X}\right| \le \left|\frac{\delta x_1}{X}\right| + \left|\frac{\delta x_2}{X}\right| + \ldots\ldots\ldots\ldots + \left|\frac{\delta x_n}{X}\right|$; which is a maximum relative error. Therefore it shows that when the given numbers are added then the magnitude of absolute error in the result is the sum of the magnitudes of the absolute errors in that numbers.

1.4.2 Error in Subtraction of Numbers

Let $X = x_1 - x_2$ then we have

$X + \delta X = (x_1 + \delta x_1) - (x_2 + \delta x_2)$ Or $X + \delta X = (x_1 - x_2) + (\delta x_1 + \delta x_2)$

$\therefore$ Absolute error is given by $\delta X = \delta x_1 - \delta x_2$

and Relative error is $\frac{\delta X}{X} = \frac{\delta x_1}{X} - \frac{\delta x_2}{X}$.

But we know that $|\delta X| \le |\delta x_1| + |\delta x_2|$ and $\left|\frac{\delta X}{X}\right| \le \left|\frac{\delta x_1}{X}\right| + \left|\frac{\delta x_2}{X}\right|$ therefore on taking modulus of relative errors and absolute errors to get its maximum value, we have $|\delta X| \le |\delta x_1| + |\delta x_2|$ which is the maximum absolute error and $\left|\frac{\delta X}{X}\right| \le \left|\frac{\delta x_1}{X}\right| + \left|\frac{\delta x_2}{X}\right|$ which gives the maximum relative error in subtraction of numbers.

1.4.3 Error in Product of Numbers

Let $X = x_1 x_2 x_3 \ldots\ldots, x_n$ then using general formula for error

$$\delta X = \delta x_1 \frac{\partial X}{\partial x_1} + \delta x_2 \frac{\partial X}{\partial x_2} + \ldots\ldots\ldots + \delta x_n \frac{\partial X}{\partial x_n}$$

We have
$$\frac{\delta X}{X} = \frac{\delta x_1}{X}\frac{\partial X}{\partial x_1} + \frac{\delta x_2}{X}\frac{\partial X}{\partial x_2} + \ldots\ldots\ldots + \frac{\delta x_n}{X}\frac{\partial X}{\partial x_n}$$

Now
$$\frac{1}{X}\cdot\frac{\partial X}{\partial x_1} = \frac{x_2 x_3 \ldots\ldots\ldots x_n}{x_1 x_2 x_3 \ldots\ldots\ldots x_n} = \frac{1}{x_1}$$
$$\frac{1}{X}\cdot\frac{\partial X}{\partial x_2} = \frac{x_1 x_3 \ldots\ldots\ldots x_n}{x_1 x_2 x_3 \ldots\ldots\ldots x_n} = \frac{1}{x_2}$$
$$\ldots \quad \ldots\ldots \qquad \ldots\ldots \quad \ldots\ldots$$
$$\frac{1}{X}\cdot\frac{\partial X}{\partial x_n} = \frac{x_1 x_2 \ldots\ldots\ldots x_{n-1}}{x_1 x_2 x_3 \ldots\ldots\ldots x_{n-1} x_n} = \frac{1}{x_n}$$

Therefore
$$\frac{\delta X}{X} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \ldots\ldots\ldots + \frac{\delta x_n}{x_n}$$

Therefore maximum Relative and Absolute errors are given by

$$\text{Relative Error} = \left|\frac{\delta X}{X}\right| \le \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right| + \ldots\ldots\ldots + \left|\frac{\delta x_n}{x_n}\right|$$

$$\text{Absolute Error} = \left|\frac{\delta X}{X}\right| X = \left|\frac{\delta X}{X}\right| \times (x_1 x_2 \ldots\ldots\ldots x_n)$$

1.4.4 Error in Division of Numbers

Let $X = \dfrac{x_1}{x_2}$ then again using general formula for error

$$\delta X = \delta x_1 \frac{\partial X}{\partial x_1} + \delta x_2 \frac{\partial X}{\partial x_2} + \ldots\ldots\ldots + \delta x_n \frac{\partial X}{\partial x_n}$$

We have
$$\frac{\delta X}{X} = \frac{\delta x_1}{X}\frac{\partial X}{\partial x_1} + \frac{\delta x_2}{X}\frac{\partial X}{\partial x_2} = \frac{\delta x_1}{\frac{x_1}{x_2}} \times \frac{1}{x_2} + \frac{\delta x_2}{\frac{x_1}{x_2}} \times \left(-\frac{x_1}{x_2^2}\right) = \frac{\delta x_1}{x_1} - \frac{\delta x_2}{x_2}$$

Therefore $\left|\dfrac{\delta X}{X}\right| \le \left|\dfrac{\delta x_1}{x_1}\right| + \left|\dfrac{\delta x_2}{x_2}\right|$ or Relative Error $\le \left|\dfrac{\delta x_1}{x_1}\right| + \left|\dfrac{\delta x_2}{x_2}\right|$ and

Absolute Error = $|\delta X| \le \left|\dfrac{\delta X}{X}\right| X$.

1.4.5 Inverse Problem

To find the error in the function $X = f(x_1, x_2, \ldots\ldots x_n)$ is to have a desired accuracy and to evaluate errors δx_1, δx_2, ... δx_n in x_1, x_2, ...; x_n we have $\delta X = \delta x_1 \dfrac{\partial X}{\partial x_1} + \delta x_2 \dfrac{\partial X}{\partial x_2} + \ldots\ldots\ldots + \delta x_n \dfrac{\partial X}{\partial x_n}$.

Using the principle of equal effects, which states $\delta x_1 \frac{\partial X}{\partial x_1} = \delta x_2 \frac{\partial X}{\partial x_2}$ = ... = $\delta x_n \frac{\partial X}{\partial x_n}$ this implies that $\delta X = n\delta x_1 \frac{\partial X}{\partial x_1}$ or $\delta x_1 = \frac{\delta X}{n\frac{\partial X}{\partial x_1}}$. Similarly we get $\delta x_2 = \frac{\delta X}{n\frac{\partial X}{\partial x_2}}, \delta x_3 = \frac{\delta X}{n\frac{\partial X}{\partial x_3}}$, ..., $\delta x_n = \frac{\delta X}{n\frac{\partial X}{\partial x_n}}$ and so on.

This form is useful where error in dependent variable is given and also we are to find errors in both independent variables.

> **Remark:** The Error $= \frac{1}{2} \times 10^{-n}$, if a number is correct to n decimal places. Also Relative error is less than $\frac{1}{l \times 10^{n-1}}$, if number is correct to n significant digits and l is the first significant digit of a number.

1.4.6 Error in Evaluating x^k

Let x^k be the function having k is an integer or fraction then Relative Error for this function is given

$$\text{Relative Error} = k\left|\frac{\delta x}{x}\right| \text{ or } \left|\frac{\delta X}{X}\right| \le k\left|\frac{\delta x}{x}\right|$$

Example 2. *Find the absolute, percentage and relative errors if x is rounded-off to three decimal digits. Given x = 0.005998.*

Sol. If x is rounded-off to three decimal places we get $x = 0.006$. Therefore

$$Error = True\ value - Approximate\ value$$

$$Error = .005998 - .006 = -.000002$$

$$\text{Absolute Error} = E_a = |Error| = 0.000002$$

$$\text{Relative Error} = E_r = \frac{E_a}{True\ value} = \frac{E_a}{0.005998} = \frac{0.000002}{0.005998} = 0.0033344 \text{ and}$$

$$\text{Percentage Error} = E_p = E_r \times 100 = 0.33344.$$

Example 3. *Find the number of trustworthy figure in $(0.491)^3$ assuming that the number 0.491 is correct to last figure.*

Sol. We know that Relative Error, $E_r = \left|\frac{\delta X}{X}\right| \le k\left|\frac{\delta x}{x}\right|$

Here $\delta x = 0.0005$ because $\frac{1}{2} \times 10^{-3} = 0.0005$

or $$k\frac{\delta x}{x} = 3\times\frac{0.0005}{(0.491)^3} = \frac{3\times 0.0005}{0.118371} = 0.01267$$

Therefore, Absolute Error $= E_r.X$

or Absolute Error $< 0.01267 \times (0.491)^3$

$= 0.01267 \times 0.118371$

$= 0.0015$

The error affects the third decimal place, therefore, $(0.491)^3 = 0.1183$ is correct to second decimal places.

Example 4. *If 0.333 is the approximate value of* $\frac{1}{3}$*, then find its absolute, relative and percentage errors.*

Sol. Given that True value $(x) = \frac{1}{3}$, and its Approximate value $(x') = 0.333$

Therefore, Absolute Error, $E_a = |x - x'| = \left|\frac{1}{3} - 0.333\right| = |0.333333 - 0.333| = 0.000333$

Relative Error, $E_r = \frac{E_a}{x} = \frac{0.000333}{0.333333} = 0.000999$ and

Percentage Error, $E_p = E_r \times 100 = 0.000999 \times 100 = 0.099\%$.

Example 5. *Round-off the number 75462 to four significant digits and then calculate its absolute error, relative error and percentage error.*

Sol. After rounded-off the number to four significant digits we get 75460.

Therefore Absolute Error $E_a = |75462 - 75460| = 2$

Relative Error $E_r = \frac{E_a}{true\ value} = \frac{E_a}{75462} = \frac{2}{75462} = 0.0000265$

Percentage Error $E_p = E_r \times 100 = 0.00265$.

Example 6. *Find the relative error of the number 8.6 if both of its digits are correct.*

Sol. Since $E_a = \frac{1}{2}\times 10^{-1} = 0.05$ therefore, Relative Error $= E_r = \frac{0.5}{8.6} = 0.0058$.

Example 7. *Three approximate values of number* $\frac{1}{3}$ *are given as 0.30, 0.33 and 0.34. Which of these three is the best approximation?*

Sol. The number, which has least absolute error, gives the best approximation.

True value $x = \frac{1}{3} = 0.33333$

When approximate value x' is 0.30 the Absolute Error is given by:

$$E_a = |x - x'| = |0.33333 - 0.30| = 0.03333$$

When approximate value x' is 0.33 the Absolute Error is given by:

$$E_a = |x - x'| = |0.33333 - 0.33| = 0.00333$$

When approximate value x' is 0.34 the Absolute Error is given by:

$$E_a = |x - x'| = |0.33333 - 0.34| = 0.00667$$

Here absolute error is least when approximate value is 0.33. Hence 0.33 is the best approximation.

Example 8. *Calculate the sum of $\sqrt{3}, \sqrt{5}$ and $\sqrt{7}$ to four significant digits and find its absolute and relative errors.*

Sol. Here $\sqrt{3} = 1.732, \sqrt{5} = 2.236, \sqrt{7} = 2.646$

Hence Sum = 6.614 and

Absolute Error = E_a = 0.0005 + 0.0005 + 0.0005 = 0.0015

(Because $\frac{1}{2} \times 10^{-3} = 0.0005$). Also the total absolute error shows that the sum is correct up to 3 significant figures. Therefore S = 6.61 and

Relative Error, $E_r = \dfrac{0.0015}{6.61} = 0.0002.$

Example 9. *Approximate values of $\frac{1}{7}$ and $\frac{1}{11}$, correct to 4 decimal places are 0.1429 and 0.0909 respectively. Find the possible relative error and absolute error in the sum of 0.1429 and 0.0909.*

Sol. The maximum error in each case $= \frac{1}{2} \times 10^{-4} = \frac{1}{2} \times 0.0001 = 0.00005$

1. Relative Error, $E_r = \left|\dfrac{\delta X}{X}\right| < \left|\dfrac{0.00005}{0.2338}\right| + \left|\dfrac{0.00005}{0.2338}\right|$ (Because X = 0.1429 + 0.0909)

Therefore, $\left|\dfrac{\delta X}{X}\right| < \dfrac{0.0001}{0.2338} = 0.00043$

2. Absolute Error, $E_a = \left|\dfrac{\delta X}{X}\right| \times |X| = \left|\dfrac{0.0001}{0.2338}\right| \times |0.2338| = 0.0001.$

Example 10. *Find the number of trustworthy figures in $(367)^{1/5}$ where 367 is correct to three significant figures.*

Sol. Relative Error $E_r < \dfrac{1}{5}\dfrac{\delta x}{x}$;

Therefore, $\dfrac{1}{5}\dfrac{\delta x}{x} = \dfrac{1}{5} \times \dfrac{0.5}{367} = 0.0003$

Similarly, Absolute Error $E_a < (367)^{1/5} \times 0.0003 = 3.258 \times 0.0003 = 0.001$

Hence Absolute Error <0.001.

Thus error effects fourth significant figure and hence $(367)^{1/5} \approx 3.26$ correct to the three figures.

Example 11. *Find the relative error in calculation of* $\frac{7.342}{0.241}$. *Where numbers 7.342 and 0.241 are correct to three decimal places. Determine the smallest interval in which true result lies.*

Sol. Relative Error $\leq \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right|$

Here $\delta x_1 = \delta x_2 = 0.0005$, $x_1 = 7.342$, $x_2 = 0.241$

Therefore, Relative Error $\leq \left|\frac{0.0005}{7.342}\right| + \left|\frac{0.0005}{0.241}\right|$

$$\leq 0.0005\left(\frac{1}{7.342} + \frac{1}{0.241}\right) = \frac{0.0005 \times 7.583}{7.342 \times 0.241} = 0.0021$$

Similarly, Absolute Error $\leq 0.0021 \times \left(\frac{x_1}{x_2}\right) = \frac{0.0021 \times 7.342}{0.241} = 0.0639$

Here $\frac{x_1}{x_2} = \frac{7.342}{0.241} = 30.4647$

Hence true value of $\frac{7.342}{0.241}$ lies between 30.4647 – 0.0639 = 30. 4008 and 30.4647 + 0.0639 = 30.5286.

Example 12. *Find the product of 346.1 and 865.2 and state how many figures of the result are trustworthy, given that the numbers are correct to four significant figures.*

Sol. For given numbers 346.1 and 865.2,

$$\delta x_1 = 0.05 = \delta x_2 \text{ Because } Error = \frac{1}{2} \times 10^{-n}$$

Also, $X = 346.1 \times 865.2 = 299446$ (correct to six significant figures)

Therefore Relative Error $E_r \leq \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right| = \left|\frac{0.05}{346.1}\right| + \left|\frac{0.05}{865.2}\right|$

$$= 0.000144 + 0.000058 = 0.000202$$

Similarly, Absolute Error $E_a = E_r X \leq 0.000202 \times 299446 \approx 60$

So, true value of the product of the given numbers lies between

$$299446 - 60 = 299386 \text{ And } 299446 + 60 = 299506.$$

Hence the mean of these values is $\frac{299386 + 299506}{2} = 299446$ which is written as 299.4×10^3. This is correct to four significant figures.

Example 13. *Find the relative error in the calculation of 3.724 × 4.312 and determine the interval in which true result lies. Given that the numbers 3.724 and 4.312 are correct to last digit?*

Sol. For product of numbers, Relative Error = $\frac{\delta X}{X} \leq \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right| + \ldots\ldots\ldots\ldots + \left|\frac{\delta x_n}{x_n}\right|$

Therefore, Relative Error, $E_r = \left|\frac{0.0005}{3.724}\right| + \left|\frac{0.0005}{4.312}\right| = 0.0002501$

(Because Error $= \frac{1}{2} \times 10^{-n} = \frac{1}{2} \times 10^{-3} = 0.0005$)

Absolute Error, $E_a = E_r X = 0.002501 \times 3.724 \times 4.312$
$= 0.0040157$

Product $x_1 x_2 = 3.724 \times 4.312 = 16.057888$

Lower limit is given by 16.057888 – 0.004016 = 16.053872

Upper limit is given by 16.057888 + 0.004016=16.061904

Hence true value lies between 16.0539 and 16.0619.

Example 14. *Find the absolute error in calculating* $(768)^{1/5}$ *and determine the interval in which true value lies 768 is correct its last digit.*

Sol. Relative Error, $E_r = k\frac{\delta x}{x} = \frac{1}{5} \times \frac{0.5}{768}$

$= \frac{0.1}{768} = 0.0001302$

Absolute Error, $E_a = E_r \times (768)^{1/5}$
$= 0.0001302 \times 3.77636$
$= 0.0004916$

Therefore, lower limit $= 3.77636 - 0.00049 = 3.77587$ and

Upper limit $= 3.77636 + 0.00049 = 3.77685$

Hence value of $(768)^{1/5}$ lies between 3.77587 and 3.77685.

Example 15. *Find the number of correct figure in the quotient* $\frac{65.3}{\sqrt{7}}$*, assuming that the numerator is correct to last figure.*

Sol. Since Relative Error, $E_r \le \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right|$

Here $\delta x_1 = 0.05 = \delta x_2$, $x_1 = 65.7$ and $x_2 = 2.6$

Therefore, Relative Error, $E_r \le \left|\frac{0.05}{65.7}\right| + \left|\frac{0.05}{2.6}\right| \le \frac{0.05 \times 68.3}{65.7 \times 2.6} = 0.01999$

Also, Absolute Error, $E_a \le 0.01999 \times \frac{65.3}{2.6} = 0.502$ (since the error affects the first decimal place).

Example 16. *Find the percentage error if 625.483 is approximated to three significant figures.*

Sol. Here $x = 625.483$ and $x' = 625.0$ therefore,

Absolute Error, $E_a = |625.483 - 625| = 0.483$, Relative Error, $E_r = \frac{E_a}{625.483} = \frac{0.483}{625.483} = 0.000772$ and

Percentage Error, $E_p = E_r \times 100 = 0.077\%$.

Example 17. *Find the relative error in taking the difference of numbers* $\sqrt{5.5} = 2.345$ *and* $\sqrt{6.1} = 2.470$. *Numbers should be correct to four significant figures.*

Sol. Relative Error $E_r \le \left|\frac{\delta x_1}{X}\right| + \left|\frac{\delta x_2}{X}\right|$

Here $\delta x_1 = 0.0005 = \delta x_2$

Therefore, Relative Error $= 2\left|\frac{\delta x_1}{X}\right| = 2\left|\frac{0.0005}{2.470 - 2.345}\right|$

$$= 2\left|\frac{0.0005}{0.125}\right| = \frac{0.001}{0.125} = 0.008.$$

Example 18. *If* $X = x + e$ *prove that* $\sqrt{X} - \sqrt{x} \approx \frac{e}{2\sqrt{X}}$.

Sol. L.H.S. $\sqrt{X} - \sqrt{x} = \sqrt{X} - \sqrt{X - e} = \sqrt{X} - \sqrt{X}\left(1 - \frac{e}{x}\right)^{\frac{1}{2}}$

$$= \sqrt{X} - \sqrt{X}\left(1 - \frac{e}{2X}\right)$$

$$= \sqrt{X} - \sqrt{X} + \frac{e}{2\sqrt{X}} \approx \frac{e}{2\sqrt{X}}.$$ **R.H.S. proved.**

Example 19. *If* $u = \frac{4x^2y^3}{z^4}$ *and errors in x, y, z be 0.001, compute the relative maximum error in u when* $x = y = z = 1$.

Sol. We know $\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$

Since $\frac{\partial u}{\partial x} = \frac{8xy^3}{z^4}, \frac{\partial u}{\partial y} = \frac{12x^2y^2}{z^4}, \frac{\partial u}{\partial z} = \frac{-16x^2y^3}{z^5}$

Also the errors δx, δy, δz may be positive or negative, therefore absolute values of terms on R.H.S. is,

$$(\delta u)_{\max} = \left|\frac{8xy^3}{z^4}\delta x\right| + \left|\frac{12x^2y^2}{z^4}\delta y\right| + \left|\frac{12x^2y^3}{z^5}\delta z\right|$$

$$= 8(0.001) + 12(0.001) + 16(0.001) = 0.036$$

Also, Max. Relative Error $= \frac{0.036}{4} = 0.009$ (Because $E_{r(\max)} = \frac{\delta u}{u}$; $u = 4$ at $x = y = z = 1$).

Example 20. *It is required to obtain the roots of $X^2 - 2X + \log_{10} 2 = 0$ to four decimal places. To what accuracy should $\log_{10} 2$ be given?*

Sol. Roots of the equation $X^2 - 2X + \log_{10} 2 = 0$ are given by,

$$X = \frac{2 \pm \sqrt{4 - 4\log_{10} 2}}{2} = 1 \pm \sqrt{1 - \log_{10} 2}$$

Therefore,
$$|\delta X| = \frac{1}{2}\frac{\delta(\log 2)}{\sqrt{1 - \log 2}} < 0.5 \times 10^{-4}$$

or
$$\delta(\log 2) < 2 \times 0.5 \times 10^{-4}(1 - \log 2)^{1/2} < 0.83604 \times 10^{-4}$$
$$\approx 8.3604 \times 10^{-5}.$$

Example 21. *If $r = 3h(h^6 - 2)$, find the percentage error in r at $h = 1$, if the percentage error in h is 5.*

Sol. We know
$$\delta x_n = \frac{\delta X}{n \dfrac{\partial X}{\partial x_n}} \quad \text{where } X = f(x_1, x_2, \ldots\ldots, x_n)$$

Therefore,
$$\delta r = \frac{\delta r}{\partial h}\delta h = (21h^6 - 6)\delta h$$

$$\frac{\delta r}{r} \times 100 = \left(\frac{21h^6 - 6}{3h^7 - 6h}\right)\delta h \times 100$$

$$= \left(\frac{21 - 6}{3 - 6}\right)\left(\frac{\delta h}{h} \times 100\right) = \frac{15}{-3} \times 5 = -25\%$$

$$\text{Percentage Error} = E_p = \left|\frac{\delta r}{r} \times 100\right| = 25\%.$$

Example 22. *Find the relative error in the function $y = ax_1^{m_1} x_2^{m_2} \ldots\ldots\ldots\ldots\ldots\ldots x_n^{m_n}$.*

Sol. Given function $y = ax_1^{m_1} x_2^{m_2} \ldots\ldots\ldots\ldots\ldots\ldots x_n^{m_n}$.

On taking log both the sides, we get

$$\log y = \log a + m_1 \log x_1 + m_2 \log x_2 + \ldots\ldots + m_n \log x_n$$

Therefore,
$$\frac{1}{y}\left(\frac{\partial y}{\partial x_1}\right) = \frac{m_1}{x_1}, \frac{1}{y}\left(\frac{\partial y}{\partial x_2}\right) = \frac{m_2}{x_2}, \frac{1}{y}\left(\frac{\partial y}{\partial x_3}\right) = \frac{m_3}{x_3} \ldots\ldots\text{etc.}$$

Hence Relative Error,
$$E_r = \frac{\partial y}{\partial x_1}\frac{\delta x_1}{y} + \frac{\partial y}{\partial x_2}\frac{\delta x_2}{y} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial y}{\partial x_n}\frac{\delta x_n}{y}$$

$$= m_1\frac{\delta x_1}{x_1} + m_2\frac{\delta x_2}{x_2} + \ldots\ldots\ldots\ldots + m_n\frac{\delta x_n}{x_n}$$

Since errors δx_1, δx_2 may be positive or negative, therefore absolute values of terms on R.H.S. give,

$$(E_r)_{\max} \le m_1 \left|\frac{\delta x_1}{x_1}\right| + m_2 \left|\frac{\delta x_2}{x_2}\right| + \ldots\ldots\ldots\ldots + m_n \left|\frac{\delta x_n}{x_n}\right|.$$

Remark: If $y = x_1 x_2 x_3 \ldots\ldots\ldots\ldots\ldots\ldots\ x_n$, then relative error is given by

$$E_r \approx \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \ldots\ldots\ldots\ldots\ldots + \frac{\delta x_n}{x_n}$$

Therefore relative error of n product of n numbers is approximately equal to the algebraic sum of their relative errors.

Example 23. *The discharge Q over a notch for head H is calculated by the formula $Q = kH^{5/2}$, where k is a given constant. If the head is 75 cm and an error of 0.15 cm is possible in its measurement, estimate the percentage error in computing the discharge.*

Sol. Given that $Q = kH^{5/2}$

On taking Log both the sides of the equation, we have

$$\log Q = \log k + \frac{5}{2} \log H$$

On differentiating, we get

$$\frac{\delta Q}{Q} = \frac{5}{2} \frac{\delta H}{H}$$

$$\frac{\delta Q}{Q} \times 100 = \frac{5}{2} \times \frac{0.15}{75} \times 100 = \frac{1}{2} = 0.5.$$

Example 24. *Compute the percentage error in the time period $T = 2\pi\sqrt{\frac{1}{g}}$ for l = 1m if the error in the measurement of l is 0.01.*

Sol. Given $T = 2\pi\sqrt{\frac{l}{g}}$

On taking log both the sides, we get

$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

$$\Rightarrow \quad \frac{1}{T}\delta T = \frac{1}{2}\frac{\delta l}{l}$$

$$\Rightarrow \quad \frac{\delta T}{T} \times 100 = \frac{\delta l}{2l} \times 100 = \frac{0.01}{2 \times 1} \times 100 = 0.5\%.$$

Example 25. *If $u = 2V^6 - 5V$, find the percentage error in u at V = 1, if error in V is 0.05.*

Sol. Given $u = 2V^6 - 5V$

$$\delta u = \frac{\partial u}{\partial V}\delta V = (12V^5 - 5)\delta V$$

$$\frac{\partial u}{u}\times 100 = \left(\frac{12V^5-5}{2V^6-5V}\right)\delta V\times 100$$

$$= \frac{(12-5)}{(2-5)}\times 0.05\times 100 = -\frac{7}{3}\times 5 = -11.667\%$$

Hence maximum percentage error $(E_p)_{\max}$ = 11.667%.

Example 26. *How accurately should the length and time of vibration of a pendulum should be measured in order that the computed value of g is correct to 0.01%.*

Sol. Period of vibration T is given by $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of pendulum.

Therefore, $$g = \frac{4\pi^2 l}{T^2} \Rightarrow \frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2} \text{ and } \frac{\partial g}{\partial T} = (-2)\frac{4\pi^2 l}{T^3}$$

$$\delta l = \frac{\delta g}{2\left(\frac{\partial g}{\partial l}\right)} \text{ and } \delta T = \frac{\delta g}{2\left(\frac{\partial g}{\partial T}\right)} \qquad ....(1)$$

But the percentage error in g is

$$\frac{\delta g}{g}\times 100 = 0.01 \Rightarrow \frac{\delta g}{\frac{4\pi^2 l}{T^2}}\times 100 = 0.01 \qquad ...(2)$$

(*a*) Percentage Error in l $$= \frac{\delta l}{l}\times 100$$

$$= \frac{1}{l}\left(\frac{\delta g}{2\frac{\partial g}{\partial l}}\right)\times 100 \quad \left[\text{Because } \delta l = \frac{\delta g}{2\left(\frac{\partial g}{\partial l}\right)}\right]$$

$$= \frac{1}{l}\frac{\delta g}{2\left(4\pi^2/T^2\right)}\times 100 = \frac{1}{2}\left(\frac{\delta g}{\left(4\pi^2 l/T^2\right)}\times 100\right)$$

$$= \frac{1}{2}\times 0.01 = 0.005 \qquad \text{(From 2)}$$

(*b*) Percentage Error in T $$= \left|\frac{\delta T}{T}\times 100\right|$$

$$= \left|\frac{1}{T}\left(\frac{\delta g}{2\partial g/\partial T}\right)\times 100\right|$$

$$= \frac{\delta g}{4\left(\frac{4\pi^2 l}{T^3}\right)}\times 100$$

$$= \frac{1}{4}\times 100 = 0.0025. \qquad \text{(From 2)}$$

Example 27. *Calculate the value of x − x cos θ correct to three significant figures if x = 10.2 cm, and θ = 5°. Find permissible errors also in x and θ.*

Sol. Given that $\theta = 5° = \frac{5\pi}{180} = \frac{11}{126}$ radian

$$1 - \cos\theta = 1 - \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots\ldots\ldots\right]$$

$$= \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \ldots\ldots\ldots = \frac{1}{2}\left(\frac{11}{126}\right)^2 - \frac{1}{24}\left(\frac{11}{126}\right)^4 + \ldots\ldots\ldots$$

$$= 0.0038107 - 0.0000024 \approx 0.0038083$$

Therefore $X = x(1 - \cos\theta) = 10.2(0.0038083) = 0.0388446 \approx 0.0388$

Further, $$\delta x = \frac{\delta X}{2\left(\frac{\partial X}{\partial x}\right)} = \frac{0.0005}{2 \times 0.0038083} \approx 0.0656$$

$$\delta\theta = \frac{\delta X}{2\left(\frac{\partial X}{\partial \theta}\right)} = \frac{0.0005}{2x\sin\theta} = \frac{0.0005}{2 \times 10.2 \times 0.0871907}$$

where $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots\ldots = \frac{11}{126} - \frac{1}{6}\left(\frac{11}{126}\right)^3 + \ldots\ldots\ldots\ldots = 0.0871907$$

Therefore $$\delta\theta = \frac{0.0005}{20.4 \times 0.0871907} \approx 0.0002809 \approx 0.00028.$$

Example 28. *The percentage error in R which is given by* $R = \frac{r^2}{2h} + \frac{h}{2}$*, is not allowed to exceed 0.2%, find allowable error in r and h when r = 4.5 cm and h = 5.5 cm.*

Sol. The percentage error in $R = \frac{\delta R}{R} \times 100 = 0.2$

Therefore $$\delta R = \frac{0.2}{100} \times R = \frac{0.2}{100}\left[\frac{(4.5^2)}{2 \times 5.5} + \frac{5.5}{2}\right]$$

Because $$R = \frac{r^2}{2h} + \frac{h}{2}$$

$$= \frac{0.2}{100} \times \frac{50.50}{11} = \frac{0.002 \times 50.50}{11} \qquad \ldots(1)$$

Percentage Error in $r = \frac{\delta r}{r}\times 100 = \frac{100}{r}\times\left(\frac{\delta R}{2\frac{\partial R}{\partial r}}\right)$

Because $\delta r = \frac{\delta R}{2\frac{\partial R}{\partial r}} = \frac{100}{r}\times\frac{\delta R}{2\left(\frac{r}{h}\right)} = \frac{100(\delta R)h}{2r^2}$

On substituting $r = 4.5$ and value of δR from (1)

$$= \frac{100}{2\times(4.5)^2}\times\frac{0.002\times 50.50}{11}\times h = \frac{0.1\times 50.50\times 5.5}{11\times 20.25}$$

$$= 0.12$$

Percentage Error in $h = \frac{\delta h}{h}\times 100 = \frac{100}{h}\times\frac{\delta R}{2\frac{\partial R}{\partial h}} = \frac{100}{h}\frac{\delta R}{2\left(-\frac{r^2}{2h^2}+\frac{1}{2}\right)}$

$$= \frac{100\delta R}{\left(-\frac{r^2}{2h^2}+\frac{1}{2}\right)} = \frac{100}{20/11}\times\frac{50.5\times 0.002}{11} = 0.505.$$

Example 29. *Two sides and included angle of a triangle are 9.6 cm, 7.8 cm and 45° respectively. Find the possible error in the area of a triangle if the error in sides is correct to a millimeter and the angle is measured correct to one degree.*

Sol. Assume that the area of the triangle $ABC \Rightarrow X = \frac{1}{2}\ bc \sin A$

Error in the measurement of sides and angles are

$\angle b = 0.05$ cm, $\angle c = 0.05$ cm, and $\angle A = \frac{1}{2}\times 0.01745 = 0.008725$ radians

$$\frac{\partial X}{\partial b} = \frac{1}{2}c\sin A,\ \frac{\partial X}{\partial c} = \frac{1}{2}b\ \sin A \text{ and } \frac{\partial X}{\partial A} = \frac{1}{2}bc\ \cos A$$

$$\delta X < \left|\delta b\frac{\partial X}{\partial b}\right| + \left|\delta c\frac{\partial X}{\partial c}\right| + \left|\delta A\frac{\partial X}{\partial c}\right|$$

$$< 0.05\times\frac{1}{2}\times 9.6\times\frac{1}{\sqrt{2}} + 0.05\times\frac{1}{2}\times 7.8\times\frac{1}{\sqrt{2}} + 0.008725\times\frac{1}{2}\times 9.6\times 7.8\times\frac{1}{\sqrt{2}}$$

$$< \frac{1}{\sqrt{2}}[0.05\times 4.8 + 0.05\times 3.9 + 0.008725\times 4.8\times 7.8]$$

$$< \frac{0.761664}{1.4142135} = 0.5385778 \approx 0.539 \text{ sq. cm.}$$

Example 30. *The error in the measurement of area of a circle is not allowed to exceed 0.5%. How accurately the radius should be measured.*

Sol. Area of the circle $=\pi r^2 = A$ (say)

$$\frac{\partial A}{\partial r} = 2\pi r$$

Percentage Error in $A = \dfrac{\delta A}{A}\times 100 = 0.5$

Therefore $\delta A = \dfrac{0.5}{100}\times A = \dfrac{1}{200}\pi r^2$

Percentage Error in $r = \dfrac{\delta r}{r}\times 100$

$$= \frac{100}{r}\frac{\delta A}{\frac{\partial A}{\partial r}} = \frac{100}{r}\frac{\frac{1}{200}\pi r^2}{2\pi^2}$$

$$= \frac{1}{4} = 0.25.$$

Example 31. *The error in the measurement of the area of a circle is not allowed to exceed 0.1%. How accurately should the diameter be measured?*

Sol. Let d is the diameter of a circle, and then its area is given by $A = \dfrac{\pi d^2}{4}$. Therefore,

$$\frac{\partial A}{\partial d} = \frac{\pi d}{2}$$

Since $\delta A = \delta d \dfrac{\partial A}{\partial d}$, therefore $\delta d = \dfrac{\delta A}{\partial A/\partial d}$

Now Percentage Error in $A = \dfrac{\delta A}{A}\times 100 = 0.1$

Therefore, $\delta A = \dfrac{0.1\times A}{100} = 0.001\times A = \dfrac{0.001\times \pi d^2}{4}$

Similarly, Percentage Error in $d = \dfrac{\delta d}{d}\times 100$

$$= \frac{100}{d}\times\frac{\delta A}{\frac{\partial A}{\partial d}} = \frac{100}{d}\left(\frac{0.001\times\pi d^2}{4}\right)\times\frac{2}{\pi d}$$

$$= \frac{0.1\times\pi d^2}{4d}\times\frac{2}{\pi d} = \frac{0.1}{2} = 0.05\cdot$$

Example 32. *In a ΔABC, b = 9.5 cm, c = 8.5 cm and A = 45°, find allowable errors in b, c, and A such that the area of ΔABC may be determined nearest to a square centimeter.*

Sol. Let area of the ΔABC be given by,

$$X = \frac{1}{2}bc\sin A$$

(1) $$\delta b = \frac{\delta X}{3\dfrac{\partial X}{\partial b}} = \frac{0.5}{3\times\frac{1}{2}c\sin A} = \frac{0.5}{\frac{3}{2}\times 8.5\times\frac{1}{\sqrt{2}}} = 0.055 \text{ cm.}$$

(2) $$\delta c = \frac{\delta X}{3\dfrac{\partial X}{\partial c}} = \frac{0.5}{3\times\frac{1}{2}b\sin A} = \frac{0.5}{\frac{3}{2}\times 9.5\times\frac{1}{\sqrt{2}}} = 0.049 \text{ cm.}$$

(3) $$\delta A = \frac{\delta X}{3\dfrac{\partial X}{\partial A}} = \frac{0.5}{3\times\frac{1}{2}bc\cos A} = \frac{0.5}{\frac{3}{2}\times 9.5\times 8.5\times\frac{1}{\sqrt{2}}} = 0.006 \text{ radians.}$$

Example 33. *In a triangle ΔABC, a = 2.3 cm, b = 5.7 cm and $\angle B = 90^o$. If possible errors in the computed value of b and a are 2 mm and 1 mm respectively, find the possible error in the measurement of angle A.*

Sol. Given $\delta b = 2 \text{ mm} = 0.2 \text{ cm}$

$\delta a = 1 \text{ mm} = 0.1 \text{ cm}$

$$\sin A = \frac{a}{b} \Rightarrow A = \sin^{-1}\frac{a}{b}$$

$$\frac{\partial A}{\partial a} = \frac{1}{\sqrt{1-\frac{a^2}{b^2}}}\cdot\frac{1}{b} = \frac{1}{\sqrt{b^2-a^2}}$$

$$\frac{\partial A}{\partial b} = \frac{1}{\sqrt{1-\frac{a^2}{b^2}}}\cdot -\frac{a}{b^2} = -\frac{a}{b\sqrt{b^2-a^2}}$$

$$\delta A < \left|\delta a\frac{\partial A}{\partial a}\right| + \left|\delta b\frac{\partial A}{\partial a}\right|$$

$$< 0.1\times\frac{1}{\sqrt{(5.7)^2-(2.3)^2}} + 0.2\times\frac{2.3}{5.7\sqrt{(5.7)^2-(2.3)^2}}$$

$$< \frac{0.1}{5.2154} + \frac{0.46}{29.7276} = 0.0346 \text{ radians}$$

Example 34. *In a triangle ΔABC, a = 30 cm, b = 80 cm and $\angle B = 90^o$. Find the maximum error in the computed value of A, if possible errors in a and b are $\frac{1}{3}\%$ and $\frac{1}{4}\%$ respectively.*

Sol. Since $\sin A = \frac{a}{b} \Rightarrow A = \sin^{-1}\frac{a}{b}$

$$\delta A < \left|\delta a\frac{\partial A}{\partial a}\right| + \left|\delta b\frac{\partial A}{\partial b}\right| \quad ....(1)$$

Given that $\dfrac{\delta a}{a\times 100} = \dfrac{1}{3} \Rightarrow \delta a = 0.1$

Also $\dfrac{\delta b}{b\times 100} = \dfrac{1}{4} \Rightarrow \delta b = 0.2$

We have $\frac{\partial A}{\partial a} = \frac{1}{\sqrt{b^2 - a^2}}$ and $\frac{\partial A}{\partial b} = \frac{-a}{b\sqrt{b^2 - a^2}}$

Substituting these values in equation (1), we have

$$\delta A < 0.00135 + 0.00100 < 0.00235 \text{ radians}$$

or $$\delta A < 8°5'.$$

Example 35. *Find the smaller root of the equation $x^2 - 30x + 1 = 0$ correct to three places of decimal. State different algorithm, which algorithm is better and why?*

Sol. Roots of the given equation $x^2 - 30x + 1 = 0$ are

$$x = \frac{30 \pm \sqrt{900-4}}{2} = \frac{30 \pm \sqrt{896}}{2}$$

and smaller root is $\frac{30 - \sqrt{896}}{2}$

(1) First method: $15 - \sqrt{224} = 0.0333704$

(2) Second method: $\frac{\left(15 - \sqrt{224}\right)\left(15 + \sqrt{224}\right)}{\left(15 + \sqrt{224}\right)}$

$$= \frac{225 - 224}{15 + \sqrt{224}} = \frac{1}{15 + \sqrt{224}}$$

$$= \frac{1}{15 + 14.966629} = \frac{1}{29.966629} = 0.0333704.$$

Therefore second algorithm is comparatively a better one as this gives the result correct to four figures.

Example 36. *Find the smaller root of the equation $x^2 - 400x + 1 = 0$ using four-digit arithmetic.*

Sol. We know that the smaller root of the equation $ax^2 + bx + c = 0$, $b > 0$ is given by,

$$x = \frac{b - \sqrt{b^2 - 4ac}}{2a}$$

Here
$$a = 1 = 0.1000 \times 10^1$$
$$b = 400 = 0.4000 \times 10^3$$
$$c = 1 = 0.1000 \times 10^1$$
$$b^2 - 4ac = 0.1600 \times 10^6 - 0.4000 \times 10^1$$
$$= 0.1600 \times 10^6 \text{ (To four-digit accuracy)}$$
$$\sqrt{b^2 - 4ac} = 0.4000 \times 10^3$$

On substituting these values in the above formula we obtain $x = 0.0000$.

However, this formula can also be written as

$$x = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

or $$x = \frac{0.2000 \times 10^1}{0.4000 \times 10^3 + 0.4000 \times 10^3}$$

$$x = \frac{0.2000 \times 10^1}{0.8000 \times 10^3} = 0.0025.$$

This is the exact root of the given equation.

Remark: When two nearly equal numbers are subtracted then there is a loss of significant figures.

e.g., $43.206 - 42.995 = 0.211$

Here given numbers are correct to five figures while the result 0.211 is correct to three figures only. Similarly numbers 12450 and 12360 are correct to four figures and their difference 90 is correct to one figure only. The error due to loss of significant figures sometimes renders the result of computation worthless. Using techniques below can minimize such error:

(1) $\sqrt{a} - \sqrt{b}$ by $\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

(2) $\sin a - \sin b$ by $2 \cos \dfrac{a+b}{2} \sin \dfrac{a-b}{2}$

(3) $1 - \cos a$ by $2 \sin^2 \dfrac{a}{2}$ or $\dfrac{a^2}{2!} - \dfrac{a^4}{4!} + ...$

(4) $\log a - \log b$ by $\log \dfrac{a}{b}$ etc.

1.5 ERROR IN SERIES APPROXIMATION

The error committed in a series approximation can be evaluated by using the remainder after n terms. Taylor's series for $f(x)$ at $x = a$ is given by,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + ... + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n(x)$$

where $R_n(x) = \dfrac{(x-a)^n}{n!} f^n(\zeta); a < \zeta < x$.

This term $R_n(x)$ is called remainder term and for a convergent series it tends to zero as $n \to \infty$. Thus if we approximate $f(x)$ by the first n terms of a series then maximum error committed in this approximation is given by the $R_n(x)$ and if accuracy required is already given then it is possible to find the number of terms n such that the finite series yields the required accuracy.

Example 37. *The Maclaurin's expansion for e^x is given by,*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\xi}, 0 < \xi < x$$

Find the number of terms, such that their sum yields the value of e^x correct to 8 decimal places at $x = 1$.

Sol. Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\xi}, 0 < \xi < x$

Then the remainder term is,

$$R_n(x) = \frac{x^n}{n!}e^{\xi}$$

So that $\xi = x$ gives maximum absolute error

$$E_{a(\max)} = \frac{x^n}{n!}e^x$$

and

$$E_{r(\max)} = \frac{x^n}{n!}$$

For an 8 decimal accuracy at $x = 1$, $\frac{1}{n!} < \frac{1}{2}10^{-8} \Rightarrow n = 12$

Hence we have 12 terms of the expansion in order that its sum is correct to 8 decimal places.

Example 38. *Find the number of terms of the exponential series such that their sum gives the value of e^x correct to six decimal places at $x = 1$.*

Sol. The exponential series is given by,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n-1}}{(n-1)!} + R_n(x) \quad ...(1)$$

where $R_n(x) = \frac{x^n}{n!}e^{\theta}, 0 < \theta < x$.

Maximum absolute error at $\theta = x$ is $E_{a(\max)} = \frac{x^n}{n!}e^x$

and Maximum Relative Error is $E_{r(\max)} = \frac{x^n}{n!}$

Hence $E_{r(\max)}$ at $x = 1$ is $\frac{1}{n!}$

For a six decimal accuracy at $x = 1$, we get

$$\frac{1}{n!} < \frac{1}{2}10^{-6} \Rightarrow n! > 2\times10^6 \Rightarrow n = 10$$

Therefore, we get $n = 10$.

Hence we have 10 terms of series (1) to obtain the sum correct to 6 decimal places.

Example 39. *Obtain a second-degree polynomial approximation to $f(x) = (l + x)^{1/2}$, $x \in [0,0.1]$ using the Taylor series expansion about $x = 0$. Use the expansion to approximate $f(0.05)$ and find a bound of the truncation error.*

Sol. Given that

$$f(x) = (1 + x)^{1/2}, \quad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \quad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}, \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}, \quad f'''(0) = \frac{3}{8}$$

Thus, the Taylor series expansion with remainder term is given by

$$(1 + x)^{1/2} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{1}{16}\frac{x^3}{\left[(1+\xi)^{1/2}\right]^5}, 0<\xi<0.1$$

The Truncation term is as,

$$T = (1 + x)^{1/2} - \left(1+\frac{x}{2}-\frac{x^2}{8}\right) = \frac{1}{16}\frac{x^3}{[(1+\xi)^{1/2}]^5}$$

Also $f(0.05) \approx 1+\frac{0.05}{2}-\frac{(0.05)^2}{8} = 0.10246875\times 10^1$

The bound of the truncation error, for $x \in [0, 0.1]$ is

$$|T| \le \max_{0\le x\le 0.1} \frac{(0.1)^3}{16(1+x)^{1/2}]^5}$$

$$\le \frac{(0.1)^3}{16} = 0.625\times 10^{-4}.$$

Example 40. *The function f(x) = tan^{-1} x can be expanded as*

$$\tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-.......+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+........$$

Find n such that series determines tan^{-1} (1) correct to eight significant digits.

Sol. If we retain n terms then $(n + 1)$th term $= (-1)^n \dfrac{x^{2n+1}}{2n+1}$

For $x = 1$, $(n + 1)$th term $= \dfrac{(-1)^n}{2n+1}$

To determine a tan^{-1} (1) correct up to eight significant digits,

$$\left|\frac{(-1)^n}{2n+1}\right| < \frac{1}{2}\times 10^{-8} \Rightarrow 2n+1 > 2\times 10^8$$

$$\Rightarrow n = 10^8+1. \quad \text{Satisfies}$$

Example 41. *Use the series* $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+.......\right)$ *to compute the value of log (1.2) correct to seven decimal places and find the number of terms retained.*

Sol. Assume $\dfrac{1+x}{1-x} = 1.2 \Rightarrow x = \dfrac{1}{11}$

If we retain n terms then, $(n + 1)$th terms $= \dfrac{x^{2n+1}}{2n+1} = \dfrac{2(1/11)^{2n+1}}{2n+1}$

For seven decimal accuracy, $\frac{2}{2n+1}\left(\frac{1}{11}\right)^{2n+1} < \frac{1}{2}\times 10^{-7}$

$$(2n+1)(11)^{2n+1} > 4\times 10^{7}$$

This gives $n \geq 3$.

After retaining the first three terms of the series, we get

$$\log_e (1.2) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}\right)$$

At $\left(x=\frac{1}{11}\right)$ $= 0.1823215.$

PROBLEM SET 1.1

1. Prove that the relative error of a product of three non-zero numbers does not exceed the sum of the relative errors of the given numbers.

2. If $y = (0.31x + 2.73)/(x + 0.35)$ where the coefficients are round off, find the absolute and relative errors in y when $x = 0.5 \pm 0.1$. [**Ans.** e_a = 2.9047, 4.6604; e_r = 0.9464, 1.225]

3. Find the quotient $q = x/y$, where $x = 4.536$ and $y = 1.32$, both x and y being correct to the digits given. Find also the relative error in the result. [**Ans.** q = 3.44, e_r = 0.0039]

4. If $S = 4x^2y^3z^{-4}$, find the maximum absolute error and maximum relative errors in S. When errors in $x = 1$, $y = 2$, $z = 3$ respectively are equal to 0.001, 0.002, 0.003.

[**Ans.** 0.0035, 0.0089]

5. Obtain the range of values within which the exact value of $\frac{1.265(10.21-7.54)}{47}$ lies, if all the numerical quantities are rounded-off. [**Hint:** on taking e_a<1%] [**Ans.** $0.06186<x<0.08186$]

6. Find the number of terms of the exponential series such that their sum yields the value of e^x correct to 8 decimal places at $x = 1$. [**Ans.** n = 12]

7. Estimate the error in evaluating $f(x) = \cos xe^{\log^2}$ around $x = 2$ if the absolute error in x is 10^{-6}. [**Ans.** 0.0053×10^{-3}]

8. (*a*) $S=\sum_{0}^{\infty} 6^{-k}$, calculate the actual sum by using the infinite series. [**Ans.** 12]

(*b*) Assume three-digit arithmetic. Find the sum (up to 11 terms) by adding largest to smallest. Also find the absolute, relative and percentage errors.

(*c*) Find the sum up to 11 terms by adding smallest to largest and also find the absolute, relative and percentage errors.

9. Find the absolute, relative and percentage errors of the approximations as

(*a*) $\frac{1}{11} \approx 0.1$ (*b*) $\frac{1}{11} \approx 0.00$ (*c*) $\frac{5}{9} \approx 0.56$ (*d*) $\frac{4}{9} \approx 0.44$

[**Ans.** (*a*) e_a = 0.009, e_r = 0.0999, e_p = 9.9]

[**Ans.** (*b*) e_a = 0.009, e_r = 0.01, e_p = 1.0]

[**Ans.** (*c*) e_a = 0.004, e_r = 0.008, e_p = 0.8]

[**Ans.** (*d*) e_a = 0.0044, e_r = 0.0099, e_p = 0.9]

10. Describe the possible causes of serious error in calculating $A = \dfrac{\sin x}{(1+\cos x)}$ for $\cos x \approx -1$

11. Find the percentage error if the number 5007932 is approximated to four significant figures. [**Ans.** 0.018%]

12. Compute the relative maximum error in the function $u = 7\dfrac{x^y}{x^2}$, when $x = y = z = 1$ and errors in x, y, z be 0.001. [**Ans.** 0.006]

13. Obtain a second-degree polynomial approximation to the function $f(x) = \dfrac{1}{1+x^2}, x \in [1, 2]$ using Taylor's series expansion about $x = 1$. Find a bound on the truncation error. [**Ans.** 0.25]

14. Find the number of terms in the series expansion of the function $f(x) = \cos x$, such that their sum gives the value of $\cos x$ correct to five decimal places for all values of x in the range $-\dfrac{\pi}{2} \le x \le +\dfrac{\pi}{2}$. Find also the truncation error. [**Ans.** $n = 6$, Trun. error = 0.020]

1.6 SOME IMPORTANT MATHEMATICAL PRELIMINARIES

There are some important mathematical preliminaries given below which would be useful in numerical computation.

(*a*) If $f(x)$ is continuous in $a \le x \le b$, and if $f(a)$ and $f(b)$ are of opposite sign, then $f(d) = 0$ for at least one number d such that $a < d < b$.

(*b*) **Intermediate Value Theorem:** Let $f(x)$ be continuous in $a \le x \le b$ and let any number between $f(a)$ and $f(b)$, then there exists a number d in $a < x < b$ such that $f(d) = l$.

(*c*) **Mean Value Theorem for Derivatives:** If $f(x)$ is continuous in $[a, b]$ and $f'(x)$ exists in (a, b) then $\exists$ at least one value of x, say d, between a and b $\ni$, $f'(d) = \dfrac{f(b) - f(a)}{b - a}, a < d < b$

(*d*) **Rolle's Theorem:** If $f(x)$ is continuous in $a \le x \le b$, $f'(x)$ exists in $a < x < b$ and $f(a) = f(b) = 0$ then $\exists$ at least one value of x, say d, $\ni f'(d) = 0$, $a < d < b$

(*e*) **Generalized Form of Rolle's Theorem:** If $f(x)$ is n times differentiable on $a \le x \le b$ and $f(x)$ vanishes at the $(n + 1)$ distinct points $x_0, x_1, x_2, \ldots\ldots\ldots x_n$ in (a, b), then there exists a number d in $a < x < b$ $\ni f^n(d) = 0$.

(*f*) **Taylor's Series for a Function of One Variable:** If $f(x)$ is continuous and possesses continuous derivatives of order n in an interval that includes $x = a$, then in that interval

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots\ldots\ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n(x)$$

where $R_n(x)$, the remainder term, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!}f^n(d), a < d < x.$$

(*g*) **Maclaurin's Expansion:** $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots.$

(*h*) **Taylor's Series for a Function of Two Variables:**

$f(x_1 + \Delta x_1, x_2 + \Delta x_2)$

$$= f(x_1, x_2) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1^2}(\Delta x_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}(\Delta x_2)^2\right] + \ldots..$$

This form can easily be generalized for function of several variables. Therefore

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots.., x_n + \Delta x_n) = f(x_1, x_2, x_3 \ldots.., x_n) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \ldots\ldots \frac{\partial f}{\partial x_n}\Delta x_n$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial {x_1}^2}(\Delta x_1)^2 + \frac{\partial^2 f}{\partial {x_n}^2}(\Delta x_n)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_n + \ldots\ldots + 2\frac{\partial^2 f}{\partial x_{n-1}\partial x_n}\Delta x_{n-1} \cdot \Delta x_n\right] + \ldots.$$

1.7 FLOATING POINT

The IEEE floating-point standards prescribe precisely how floating-point numbers should be represented, and the results of all operations on floating point numbers. There are two standards: IEEE 754 is for binary arithmetic, and IEEE 854 covers decimal arithmetic as well. The only IEEE 754, adopted almost universally by computer manufacturers. Unfortunately, not all manufacturers implement every detail of IEEE arithmetic the same way. Every one does indeed represent numbers with the same bit patterns and rounds results correctly (or tries to). But exceptional conditions (like 1/0, sqrt(–1) etc.), whose semantics are also specified in detail by the IEEE standards, are not always handled the same way. It turns out that many manufacturers believe (sometimes rightly and sometimes wrongly) that confirming to every detail of IEEE arithmetic would make their machines either a bit slower or a bit more expensive, enough so make them less attractive in the market place. The IEEE standards 754 for binary arithmetic specify 4 floating-point formats: single, single extended, double and double extended.

Floating point numbers are represented in the form + – significand *2^ (exponent), where the significand is a non-negative number. A normalized significand lies in the half open interval [1, 2), *i.e.*, it has no leading zero bits.

Macheps is the short for "machine epsilon", and is used below for round off error analysis. The distance between 1 and the next larger floating point number is 2*macheps. When the exponent has neither its largest possible value (a string of all once) nor its smallest value (a string of all zeros), then the significand is necessarily normalized, and lies in [1, 2). When the exponent has its largest possible value (all once), the floating-point number is +infinity, –infinity, or NAN (not-a-number). The largest finite floatingpoint number is called the overflow threshold.

When the exponent has its smallest possible value (all zero), the significand may have leading zeros, and is called either subnormal or de-normal (unless it is exactly zero). The subnormal floating-point numbers represent very tiny numbers between the smallest nonzero normalized floating-point number (the underflow threshold) and zero. An operation that underflows yield a subnormal number or possibly zero; this is called gradual underflow. The alternative, simply returning a zero, is called flush to zero. When the significand is zero, the floating-point number is + – 0.

The basic operations specified by IEEE arithmetic are first and foremost addition, subtraction, multiplication, and division. Square roots and remainders are also included. The default rounding for these operationsis "to nearest even". This means that the floating point result fl (*a* op *b*) of the exact operation (*a* op *b*) is the nearest floating point number to (*a* op *b*), breaking ties by rounding to the floating point number whose bottom bit is zero (the "even" one). It is also possible to round up, round down, or truncate (round towards zero). Rounding up and down are useful interval arithmetic, which can provide guaranteed error bounds; unfortunately most languages and/or compilers provide no access to the status flag which can select the rounding direction. When the result of floating point operation is not representable as a normalized floating point number, and exception occurs.

1.8 FLOATING POINT ARITHMETIC AND THEIR COMPUTATION

The computer performed five basic arithmetic operations such as addition, subtraction, multiplication and division. The decimal numbers are converted to machine numbers. The machine number consists of only the digit 0 and 1 with a base. It's base depending on the computer. If the base is two the number system is called the binary number system, if the base is eight it is called octal number system and if the base is sixteen it is called hexadecimal number system respectively. The decimal number system has the base 10. In numerical computation, there are mainly two types of arithmetic operations present in the system.

(*a*) Integer arithmetic, which deals with integer operands and

(*b*) Real or Floating-point arithmetic, which deals with fractional part of a number as operands.

Mostly computers carried out scientific calculations in floating point arithmetic to avoid the difficulty of keeping every number less than 1 in magnitude during computation. A floating point number is characterized by three parameters—the base b, the number of digit n and the exponent range (m, M).

An n-digit floating-point number with base b has the form:

$$x = \pm(0.d_1 d_2 ......d_n)_b b^e$$

where $d_1, d_2, d_3, ..., d_n$ are integers and satisfies $0 \le d, < b$ and the exponent e is such that $m \le e < M$. Also $(0, d_1d_2d_3 .... d_n)_b$ is a b-fraction called the mantissa, and it lies between +1 and –1. The number 0 is written as:

$$+ 0.000 .... 0 \times b^e$$

The floating-point number is said to be normalized if $d_1 \ne 0$ or else $d_1 = d_2 = ...$ $= d_n = 0$. If $d_1, d_n \ne 0$ the number is said to have an n significant digits.

There are two commonly used ways to translate any given real number x into an n b-digit floating-point number $f_p(x)$, rounding and chopping.

A floating-point number $x = \pm(0, d_1d_2 .... d_n)_b b^e$ is in n-digit mantissa standard form if it is normalized and its mantissa consists of exactly n-digit. If a number x can be represented by $x = (0.d_1d_2d_3 \quad ....d_nd_{n+1}.............)_b b^e$ then the floating-point number can be in chopping form and if it can be written as $f_p(x) = (0.d_1d_2d_3.............d_n)_n b^e$ then the floating point number is in rounding form. If it can be written as $f_p(x) = \left(0.d_1d_2.................d_nd_{n+1} + \frac{1}{2}b\right)$ where first n digits are used to write a floating-point number.

Example 1. *Digit normalized form of* $\frac{2}{3}$

Sol. $f_p(x) = f_p\left(\frac{2}{3}\right) = 0.6666667$; Result after rounding

$f_p(x) = f_p\left(\frac{2}{3}\right) = 0.6666666$; Result after chopping

In computers, each location called word in memory stores only a finite numbers of digits. If we assume computer memory store 6 digits in each location and also store one or more signs then to represent real number, computer assumed a fixed position for the decimal point and all numbers are stored after appropriate shifting with an assumed decimal point. For that, the maximum possible numbers are stored as 9999.99 and the minimum possible numbers are stored as 0000.01. These maximum and minimum limits for numbers are in magnitude. For this purpose, preserve the maximum number of significant digits in a real number and increase the range of values for that real number. This type of representation is called the normalized floating-point mode.

Example 2. *The number* 58.72×10^5 *is represented as* 0.5872×10^7 *or 0.5872e7.*

Sol. Here mantissa is 0.5872 and the exponent is 7. Also shifting of the mantissa to the left to its most significant digit, is nonzero, is called normalization.

1.8.1 Arithmetic Operations on Floating Point Numbers

Basically there are four arithmetic operations such as addition, subtraction, multiplication and division. These operations applied on floating point numbers as follows:

Example 3. *Add the following floating-point numbers 0.4546e3 and 0.5433e7.*

Sol. This problem contains unequal exponent. To add these floating-point numbers, take operands with the largest exponent as,

0.5433*e*7 + 0.0000*e*7 = 0.5433*e*7

(Because 0.4546*e*3 changes in the same operand as 0.0000*e*7).

Example 4. *Add the following floating-point numbers 0.6434e3 and 0.4845e3.*

Sol. This problem has an equal exponent but on adding we get 1.1279*e*3, that is, mantissa has 5 digits and is greater than 1, that's why it is shifted right one place. Hence we get the resultant value 0.l127*e*4.

Example 5. *Add the following floating-point numbers 0.6434e99 and 0.4845e99.*

Sol. In this example, mantissa is shifted right and exponent is increased by 1, resulting is a value of 100 for the exponent (because sum of mantissa exceeds by 1). This condition is called an **overflow condition** because exponent cannot store more than two digits.

Example 6. *Find the sum of 0.l23e3 and 0.456e2 and write the result in three digit mantissa form.*

Sol. Sum is = 0.123*e*3 + 0.456*e*2,

= 0. 123*e*3 + 0.0456*e*3 = 0.168*e*3 Result after chopping

Sum is = 0.123*e*3 + 0.456*e*2 ,

= 0.123*e*3 + 0.0456*e*3 = 0.169*e*3 Result after rounding.

Above examples (3 to 6) shows the addition of floating point numbers in different ways.

Example 7. *Subtract the floating-point number* 0.36132346×10^7 *from* 0.36143447×10^7.

Sol. The number 0.36132346×10^7 after subtracting from 0.36143447×10^7 gives 0.00011101×10^7. On shifting the fractional part three places to the left we have 0.11101×10^4 which is obviously a floating-point number. Also 0.00011101×10^7 is a floating-point number but not in the normalized form.

Example 8. *Subtract the following floating-point numbers:*

1. 0.5424e – 99 From 0.5452e – 99

2. 0.3862e – 7 From 0.9682e – 7

Sol. On subtracting we get 0.0028*e* – 99. Again this is a floating-point number but not in the normalized form. To convert it in normalized form, shift the mantissa to the left by 1. Therefore we get 0.028*e* – 100. This condition is called an **underflow condition**.

Similarly, after subtraction we get 0.5820*e – 7.*

Above examples (7 and 8) shows the subtraction of floating points numbers with underflow condition. Therefore we say that, if two numbers represented in normalized floating-point notation then for addition and subtraction it is required that the exponent of the numbers must be equal, if it is not then made be equal and shift the mantissa appropriately.

Example 9. *Multiply the following floating point numbers:*

1. *0.1111e74 and 0.2000e80*
2. *0.I234e – 49 and 0.1111e – 54*

Sol. 1. On multiplying 0.1111*e*74 × 0.2000*e*80 we have 0.2222*e*153. This Shows overflow condition of normalized floating-point numbers.

2. Similarly second multiplication gives 0.1370*e* – 104, which shows the underflow condition of floating-point number.

This example represent that two numbers are multiplied by multiplying the mantissa and by adding the exponent of given normalized floating-point representation. Similarly division is evaluated by division of mantissa of the numerator by that of the denominator and denominator exponent is subtracted from the numerator exponent. The resultant exponent is obtained by adjusting it appropriately and using previous results normalizes the quotient mantissa.

Example 10. *Calculate the sum of given floating-point numbers:*

1. *0.4546e5 and 0.5433e7*
2. *0.4546e5 and 0.5433e5*

Sol. 1. When the exponent is not equal, the operand is kept with large exponent number. That is 0.5433*e7* + 0.0045*e*7 = 0.5878*e*7.

2. Here mantissas are added because exponent numbers are equal. That is, 0.4546*e*5 + 0.5433*e*5 = 0.9979*e*5.

Example 11. *Subtract the floating-point number 0.5424e3 from 0.5452e3.*

Sol. While subtracting 0.5424*e*3 from 0.5452*e*3 we get 0.0028*e*3. It can also be written as 0.28el using normalized floating point representation because mantissa is greater than or equal to 0.1.

Example 12. *Calculate the value of e^x when x = 0.5250e1 and e = 2.7183. The expression for e^x is* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!}$.

Sol. We have

$$e^x = e^{0.5250e1} = e^5 \times e^{.25}$$

$$e^5 = (.2718e1) \times (.2718e1) \times (.27I8e1) \times (.27I8e1) \times (.2718e1)$$

$$= .1484e3$$

Also, we find $e^{.25}$.

Therefore $$e^{.25} = 1 + (.25) + \frac{(.25)^2}{2!} + \frac{(.25)^2}{3!}$$

$$= 1.25 + .03125 + .002604 = .1284e1$$

Hence $$e^{.5250e1} = (.1484e3) \times (.1284e1) = .1905e3$$

Example 13. *Compute the middle value of the number a = 4.568 and b = 6.762 using the four-digit arithmetic and compare the result by taking* $c = a + \left(\frac{b-a}{2}\right)$.

Sol. Since $a = .4568e1$, $b = .6762e1$ and c be the middle value of the numbers a and b, therefore $c = \frac{a+b}{2} = \frac{.4568e1 + .6762e1}{.2000e1} = \frac{.1133e2}{.2000e1} = .5665e1$.

If we use the formula $c = a + \left(\frac{b-a}{2}\right)$, we get $c = .4568e1 + \left(\frac{.6762e1 - .4568e1}{.2000e1}\right)$ or .4568e1 + .1097e1 = .5665e1 which is similar result as first result.

Example 14. *Evaluate 1 – cos x at x = 0.1396 radian. Assume cos(0.1396) = 0.9903 and compare it when evaluated 2* $\sin^2\frac{x}{2}$. *Also assumes in (0.0698) = 0.6794e – 1.*

Sol. Since $x = 0.1396$

Therefore $$1 - \cos(0.1396) = 0.1000e1 - 0.9903e0$$

$$= 0.1000e1 - 0.0990e1 = 0.1000e1 - 1$$

Now $$\sin\frac{x}{2} = \sin(0.0698) = 0.6974e - 1$$

$$2\sin^2\frac{x}{2} = (0.2000e1) \times (0.6974e - 1) \times (0.6974e - 1) = 0.9727e - 2$$

The value obtained by alternate formula is close to the true value 0.9728e – 2.

Example 15. *Evaluate the following floating-point numbers:*

1. *0.5334e9 × 0.l132e – 25*
2. *0.1111el0 × 0.1234e15*
3. *0.9998e – 5 ÷ 0.1000e98*
4. *0.1111e51 × 0.4444e50*
5. *0.1000e5 ÷ 0.9999e3*

6. *0.5543e12 × 0.4111e − 15*
7. *0.9998el + 0.l000e − 99*

Sol. 1. 0.5334*e*9 × 0.l132*e* – 25 = 0.6038*e* –17, this result shows the **underflow condition** of floating point numbers.

2. 0.1111*e*10 × 0.1234*e*15 = 0.1370*e*24
3. 0.9998*e* – 5 ÷ 0.1000*e*98 = 0.9998*e* – 104, this result shows the **underflow condition** of floating point numbers.
4. 0.1111*e*51× 0.4444*e*50 = 0.4937*e*100. Hence the resultant shows an **overflow condition.**
5. 0.1000*e*5 ÷ 0.9999*e*3 = 0.1000*e*2
6. 0.5543*e*12 × 0.411l*e* – 15 = 0.2278*e* – 3
7. 0.9998*e*1 ÷ 0.1000*e* – 99 = 0.9998*e*101, this shows an **overflow condition** of floating numbers.

Example 16. *For x = 0.4845 and y = 0.4800, calculate the value of* $\frac{x^2 - y^2}{x+y}$ *using normalized floating point arithmetic. Compare this with the value of (x − y).*

Sol. Since $x = 0.4845,\ y = 0.4800$

Hence $x + y = 0.4845e0 + 0.4800e0$ *or* 0.9645*e*0.

Again,

$$x^2 = (0.4845e0) \times (0.4845e0) = 0.2347e0$$
$$y^2 = (0.4800e0) \times (0.4800e0) = 0.2304e0$$
$$x^2 - y^2 = 0.2347e0 - 0.2304e0 = 0.0043e0$$

Therefore, $$\frac{x^2 - y^2}{x+y} = \frac{0.0043e0}{0.9645e0} = 0.4458e - 2$$

Also, $$x - y = 0.4845e0 - 0.4800e0 = 0.4500e - 2$$

Example 17. *Find the solution of the following equation using floating-point arithmetic with 4-digit mantissa* $x^2 - 1000x + 25 = 0$.

Sol. Given that, $x^2 - 1000x + 25 = 0$

$$\Rightarrow \quad x = \frac{1000 \pm \sqrt{10^6 - 10^2}}{2}$$

Now 10^6 = 0.000e7 and 10^2 = 0.1000e3

Therefore $10^6 - 10^2 = 0.1000e7 \Rightarrow \sqrt{10^6 - 10^2} = 0.1000e4$

Hence roots are: $\left(\frac{0.1000e4 + 0.1000e4}{2}\right)$ and $\left(\frac{0.1000e4 - 0.1000e4}{2}\right)$

which are 0.1000*e*4 and 0.0000*e*4 respectively. One of the roots becomes zero due to the limited precision allowed in computation. We know that in quadratic equation $ax^2 + bx + c$, the product of the roots is given by $\frac{c}{a}$, the smaller root may be obtained by dividing (*c/a*) by the largest root.

Therefore first root is given by 0.1000e4 and second root is as

$$\frac{25}{0.1000e4} = \frac{0.2500e2}{0.1000e4} = 0.2500e-1.$$

Example 18. *Associative and distributive laws are not always valid in case of normalized floating-point representation. Give example to prove this statement.*

Sol. According to the consequence of the normalized floating-point representation the associative and the distributive laws of arithmetic are not always valid. The example given below proves the above statement:

Let $a = 0.5555e1,\ b = 0.4545e1,\ c = 0.4535e1$ then

$$(b-c) = 0.0010e1 = 0.1000e-1$$
$$a(b-c) = (0.5555e1) \times (0.1000e-1)$$
$$= (0.0555e0) = 0.5550e-1$$
$$ab = (0.5555e1) \times (0.4545e1) = 0.2524e2$$
$$ac = (0.5555e1) \times (0.4535e1) = 0.2519e2$$

Therefore $ab - ac = 0.0005e2 = 0.5000e-1$

Thus, $a(b-c) \neq ab - ac$

This proves the non-distributivity of arithmetic.

Again let $a = 0.5665e1,\ b = 0.5556e-1,\ c = 0.5644e1$

Therefore $a + b = 0.5665e1 + 0.5556e-1$

$$= 0.5665e1 + 0.0055e1 = 0.5720e1$$
$$(a+b) - c = 0.5720e1 - 0.5644e1 = 0.0076e1 = 0.7600e-1$$
$$a - c = 0.5665e1 - 0.5644e1 = 0.0021e1 = 0.2100e-1$$
$$(a-c) + b = 0.2100e-1 + 0.5556e-1 = 0.7656e-1$$

Thus, $(a+b) - c \neq (a-c) + b$

This proves the non-associativity of arithmetic.

Example 19. *Calculate the smaller root of the equation $x^2 - 400x + 1 = 0$ using 4-digit arithmetic.*

Sol. Roots of the equation $ax^2 + bx + c = 0$ are $x_1 = \dfrac{b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \dfrac{b-\sqrt{b^2-4ac}}{2a}$

Here $b^2 >> |4ac|$ and product of roots are $\dfrac{c}{a}$.

Therefore smaller root is $\dfrac{c/a}{\left(\dfrac{b+\sqrt{b^2-4ac}}{2a}\right)}$ or $\dfrac{2c}{b+\sqrt{b^2-4ac}}$

According to the equation
$$a = 1 = 0.1000e1,$$
$$b = 400 = 0.4000e3,$$
$$c = 1 = 0.1000e1$$

Therefore $b^2 - 4ac = 0.1600e6 - 0.4000\ e1 = 0.1600e6$

or $\sqrt{b^2 - 4ac} = 0.4000e3$

Hence smaller root is $= \dfrac{2\times(0.1000e1)}{0.4000e3 + 0.4000e3} = \dfrac{0.2000e1}{0.8000e3} = 0.25e-2 = 0.0025$.

PROBLEM SET 1.2

1. Round off the following numbers to four significant figures:
38.46235,
0.70029,
0.0022218,
19.235101 [**Ans.** 38.46, 0.7003, 0.002222, 19.24]

2. Round off the following numbers to two decimal places:
48.21416,
2.385,
52.275,
81.255,
2.3742 [**Ans.** 48.21, 2.39, 52.28, 81.26, 2.37]

3. Obtain the range of values within which the exact value of $\dfrac{1.265(10.21-7.54)}{47}$ lies, if all the numerical quantities are rounded off. [**Hint.** on taking $e_a < 1\%$] [**Ans.** $0.06186 < x < 0.8186$]

4. Calculate the value of $\sqrt{102} - \sqrt{101}$ correct to four significant figures. [**Ans.** 0.04963]

5. Represent 44.85×10^6 in normalized floating-point mode. [**Ans.** 0.4485*e*8]

6. Explain Machine Epsilon in floating point arithmetic.

7. Calculate the value of $x^2 + 2x - 2$ and $(2x - 2) + x^2$ where $x = 0.7320e0$, using normalized point arithmetic and proves that they are not the same. Compare with the value of $(x^2 - 2) + 2x$. [**Ans.** –0.1000*e*–2, –0.2000*e*–3]

8. Find the value of sin $x \approx x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ for $x = 0.2000e0$ using normalized floating point arithmetic with 4-digit mantissa. [**Ans.** 0.1987*e*0 (taking $e_a = 0.005$)]

9. The following numbers are given in a decimal computer with a four digit normalized mantissa:

(*a*) 0.4523*e* – 4, (*b*) 0.2115*e* – 3, (*c*) 0.2583*e*1 .

Perform the following operations, and indicate the error in the result, assuming symmetric rounding:

1. (*a*) + (*b*) + (*c*) 2. (*a*) – (*b*) – (*c*) 3. (*a*)/(*c*)
4. (*a*)(*b*)/(*c*) 5. (*a*) – (*b*) 6. (*b*)/(*c*) (*a*)

[**Ans.** **1.** 0.2585*e*1 **2.** 0.2581*e*1 **3.** 1.7511*e*–8
4. 0.3717*e*–8 **5.** –0.1663*e*–3 **6.** 0.1823*e*3]

10. Give example to show that most of the laws of arithmetic fail to hold for floating-point arithmetic.

11. Find the root of smaller magnitude of the equation $x^2 + 0.4002e0x + 0.8e - 4 = 0$. Work in floating-point arithmetic using a four decimal place mantissa. [**Ans.** –0.2 *e*–3]

12. Give the normalized floating-point representation for the following:

1. 22/7
2. –22.75
3. 0.01
4. $9\frac{3}{8}$
5. $-\frac{3}{64}$
6. 3/6

[**Ans.** **1.** 0.3143*e*1 **2.** –0.2275*e*2 **3.** 1*e*–2 **4.** 0.9375*e*1 **5.** 0.5 *e*0 **6.** –0.4688*e*–1]

13. Using 5-digit arithmetic with rounding, calculate the sum of two numbers $x = 0.78596e-2$ and $y = 0.786327e1$. [**Ans.** 0.78712 *e*1]

14. Compute 403000 × 0.197 by 3-digit arithmetic with rounding. [**Ans.** 0.7939*e*5]

15. Evaluate $f(x) = \frac{1-\cos x}{x}$ for $x = 0.01$, using five-digit decimal arithmetic. [**Ans.** 0.1 *e*–1]

16. Calculate the value of the polynomial $P_3(x) = 2.75x^3 - 2.95x^2 + 3.16x - 4.67$ for $x = 1.07$ using both chopping and rounding off to three digits, proceeding through the polynomial term by term from left to right. [**Ans.** –0.133e1]

❑❑❑

CHAPTER 2

Algebraic and Transcendental Equation

2.1 INTRODUCTION

We have seen that expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

where a's are constant ($a_0 \neq 0$) and n is a positive integer, is called a polynomial in x of degree n, and the equation $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a transcendental equation. For example,

$$x^3 - 3x + 6 = 0, \; x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$$

are algebraic equations of third and fifth degree, whereas $x^2 - 3 \cos x + 1 = 0$, $xe^x - 2 = 0$, $x \log_{10} x = 1.2$ etc., are transcendental equations. In both the cases, if the coefficients are pure numbers, they are called numerical equations.

In this chapter, we shall describe some numerical methods for the solution of $f(x) = 0$ where $f(x)$ is algebraic or transcendental or both.

2.2 METHODS FOR FINDING THE ROOT OF AN EQUATION

Method for finding the root of an equation can be classified into following two parts:

(1) Direct methods

(2) Iterative methods.

2.2.1 Direct Methods

In some cases, roots can be found by using direct analytical methods. For example, for a quadratic equation $ax^2 + bx + c = 0$, the roots of the equation, obtained by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These are called closed form solution. Similar formulae are also available for cubic and biquadratic polynomial equations but we rarely remember them. For higher order polynomial equations and non-polynomial equations, it is difficult and in many cases impossible, to get closed form solutions. Besides this, when numbers are substituted in available closed form solutions, rounding errors reduce their accuracy.

2.2.2 Iterative Methods

These methods, also known as **trial** and **error** methods, are based on the idea of successive approximations, *i.e.*, starting with one or more initial approximations to the value of the root, we obtain the sequence of approximations by repeating a fixed sequence of steps over and over again till we get the solution with reasonable accuracy. These methods generally give only one root at a time.

For the human problem solver, these methods are very cumbersome and time consuming, but on other hand, more natural for use on computers, due to the following reasons:

(1) These methods can be concisely expressed as computational algorithms.

(2) It is possible to formulate algorithms which can handle class of similar problems. For example, algorithms to solve polynomial equations of degree n may be written.

(3) Rounding errors are negligible as compared to methods based on closed form solutions.

2.3 ORDER (OR RATE) OF CONVERGENCE OF ITERATIVE METHODS

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

The order of convergence of an iterative method is said to be kth order convergent if k is the largest positive real number such that

$$\lim_{i \to \infty} \left| \frac{e_{i+1}}{e_i^k} \right| \leq A$$

Where A, is a non-zero finite number called asymptotic error constant and it depends on derivative of $f(x)$ at an approximate root x. e_i and e_{i+1} are the errors in successive approximation.

In other words, the error in any step is proportional to the kth power of the error in the previous step. Physically, the kth order convergence means that in each iteration, the number of significant digits in each approximation increases k times.

2.4 BISECTION (OR BOLZANO) METHOD

This is one of the simplest iterative method and is strongly based on the property of intervals. To find a root using this method, let the function $f(x)$ be continuous between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then there is a root of $f(x) = 0$, lying between a and b. Let the first approximation be $x_1 = \frac{1}{2}(a + b)$ (*i.e.*, average of the ends of the range).

Now of $f(x_1) = 0$ then x_1 is a root of $f(x) = 0$. Otherwise, the root will lie between a and x_1 or x_1 and b depending upon whether $f(x_1)$ is positive or negative.

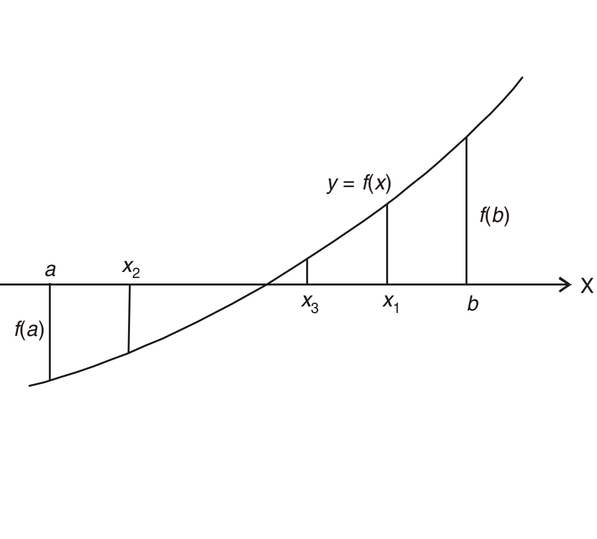

FIG. 2.1

Then, we bisect the interval and continue the process till the root is found to be desired accuracy. In the above figure, $f(x_1)$ is positive; therefore, the root lies in between a and x_1. The second approximation to the root now is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is negative as shown in the figure then the root lies in between x_2 and x_1, and the third approximation to the root is $x_3 = (x_2 + x_1)/2$ and so on.

This method is simple but slowly convergent. It is also called as Bolzano method or Interval halving method.

2.4.1 Procedure for the Bisection Method to Find the Root of the Equation *f*(*x*) = 0

Step 1: Choose two initial guess values (approximation) a and b (where $(a > b)$) such that $f(a) \,.\, f(b) < 0$.

Step 2: Evaluate the mid point x_1 of a and b given by $x_1 = \frac{1}{2}(a + b)$ and also evaluate $f(x_1)$.

Step 3: If $f(a) \,.\, f(x_1) < 0$, then set $b = x_1$ else set $a = x_1$. Then apply the formula of step 2.

Step 4: Stop evaluation when the difference of two successive values of x_1 obtained from step 2, is numerically less than the prescribed accuracy.

2.4.2 Order of Convergence of Bisection Method

In Bisection Method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper bound to the error.

In Bisection Method, $e_{i+1} = 0.5 e_i$ or $\frac{e_{i+1}}{e_i} = 0.5$

Here e_i and e_{i+1} are the errors in i^{th} and $(i + 1)^{th}$ iterations respectively. Comparing the above equation with

$$\lim_{i \to \infty} \left| \frac{e_{i+1}}{e_i^k} \right| \leq A$$

We get $k = 1$ and $A = 0.5$. Thus the Bisection Method is first order convergent or linearly convergent.

Example 1. *Find the root of the equation $x^3 - x - 1 = 0$ lying between 1 and 2 by bisection method.*

Sol. Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative

and $f(2) = 2^3 - 2 - 1 = 5$, which is positive

Therefore, $f(1)$ is negative and $f(2)$ is positive, so at least one real root will lie between 1 and 2.

First iteration: Now using Bisection Method, we can take first approximation

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Then, $f(1.5) = (1.5)^3 - 1.5 - 1$

$= 3.375 - 1.5 - 1 = 0.875$

$\therefore$ $f(1.5) > 0$ that is, positive

So root will now lie between 1 and 1.5.

Second iteration: The Second approximation is given by $x_2 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$.

Then, $f(1.25) = (1.25)^3 - 1.25 - 1$

$= 1.953 - 2.25 = -0.297 < 0$

$\therefore$ $f(1.25)$ is negative.

Therefore, $f(1.5)$ is positive and $f(1.25)$ is negative, so that root will lie between 1.25 and 1.5.

Third iteration: The third approximation is given by

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$x_3 = 1.375$

Now $f(1.375) = (1.375)^3 - 1.375 - 1$

$f(1.375) = 0.2246$

$\therefore$ $f(1.375)$ is positive.

$\therefore$ The required root lies between 1.25 and 1.375.

Fourth iteration: The fourth approximation is given by

$$x_4 = \frac{1.25 + 1.375}{2} = 1.313$$

Now $f(1.313) = (1.313)^3 - 1.313 - 1$

$f(1.313) = -0.0494$

Therefore, $f(1.313)$ is negative and $f(1.375)$ is positive. Thus root lies between 1.313 and 1.375.

Fifth iteration: The fifth approximation is given by

$$x_5 = \frac{1.313 + 1.375}{2} = 1.344$$

$\therefore$ $f(1.344) = (1.344)^3 - 1.344 - 1 = 0.0837$

$\therefore$ $f(1.313) > 0$

$\therefore f(1.313)$ is negative and $f(1.344)$ is positive, so root lies between 1.313 and 1.344.

Sixth iteration: The sixth approximation is given by

$$x_6 = \frac{1.313 + 1.344}{2} = 1.329$$

$\therefore f(1.329) = (1.329)^3 - 1.329 - 1 = 0.0183$

$\therefore$ $f(1.329) > 0$

$\therefore f(1.313)$ is negative and $f(1.329)$ is positive, so that the required root lies between 1.313 and 1.329.

Seventh iteration: The seventh approximation is given by

$$x_7 = \frac{1.313 + 1.329}{2} = 1.321$$

$\therefore$ $f(1.321) = (1.321)^3 - 1.321 - 1 = -0.0158$

$\therefore$ $f(1.321) < 0$

$\therefore f(1.321)$ is negative and $f(1.329)$ is positive, so that the required root lies between 1.321 and 1.329.

Eighth iteration: The eighth approximation is given by

$$x_8 = \frac{1.321 + 1.329}{2} = 1.325$$

From above iterations, the root of $f(x) = x^3 - x - 1 = 0$ up to three places of decimals is 1.325, which is of desired accuracy.

Example 2. *Find the root of the equation $x^3 - x - 4 = 0$ between 1 and 2 to three places of decimal by Bisection method.*

Sol. Given $f(x) = x^3 - x - 4$

We want to find the root lie between 1 and 2.

At $x_0 = 1 \Rightarrow f(x_0) = (1)^3 - 1 - 4 = -4$ negative

At $x_1 = 2 \Rightarrow f(x_1) = (2)^3 - 2 - 4 = 2$ positive

This implies that root lies between 1 and 2.

First iteration: Here, $x_0 = 1, x_1 = 2, \; x_2 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

Now, $f(x_0) = -4, f(x_1) = 2.$ Then, $f(x_2) = (1.5)^3 - 1.5 - 4 = -2.125$.

Since $f(1.5)$ is negative and $f(2)$ is positive.

So root will now lie between 1.5 and 2.

Second iteration: Here, $x_0 = 1.5, x_1 = 2, x_2 = \frac{1.5+2}{2} = 1.75$

Also, $f(x_0) = -2.125, f(x_1) = 2$ then, $f(x_2) = (1.75)^3 - 1.75 - 4 = -0.39062$

Since $f(1.75)$ is negative and $f(2)$ is positive, therefore the root lies between 1.75 and 2.

Third iteration: Here, $x_0 = 1.75, x_1 = 2, x_2 = \frac{1.75+2}{2} = 1.875$

Also, $f(x_0) = -0.39062, f(x_1) = 2$ then, $f(x_2) = (1.875)^3 - 1.875 - 4 = 0.71679$

Since $f(1.75)$ is negative and $f(1.875)$ is positive, therefore the root lies between 1.75 and 1.875.

Fourth iteration: Here, $x_0 = 1.75, x_1 = 1.875, x_2 = \frac{1.75+1.875}{2} = 1.8125$

Also, $f(x_0) = -0.39062, f(x_1) = 0.71679$ then, $f(x_2) = (1.8125)^3 - 1.8125 - 4 = 0.14184$

Since $f(1.75)$ is negative and $f(1.8125)$ is positive, therefore the root lies between 1.75 and 1.8125.

Fifth iteration: Here, $x_0 = 1.75, x_1 = 1.8125, x_2 = \frac{1.75+1.8125}{2} = 1.78125$

Also, $f(x_0) = -0.39062, f(x_1) = 0.14184$ then, $f(x_2) = (1.78125)^3 - 1.78125 - 4 = -0.12960$

Since $f(1.78125)$ is negative and $f(1.8125)$ is positive, therefore the root lies between 1.78125 and 1.8125.

Repeating the process, the successive approximations are

$x_6 = 1.79687, x_7 = 1.78906, x_8 = 1.79296, x_9 = 1.79491, x_{10} = 1.79589, x_{11} = 1.79638, x_{12} = 1.79613$

From the above discussion, the value of the root to three decimal places is 1.796.

Example 3. *Using Bisection Method determine a real root of the equation* $f(x) = 8x^3 - 2x - 1 = 0$.

Sol. It is given that $f(x) = 8x^3 - 2x - 1 = 0$.

Then $f(0) = 8(0)^3 - 2(0) - 1 = -1$

and $f(1) = 8(1)^3 - 2(1) - 1 = 5$

Therefore, $f(0)$ is negative and $f(1)$ is positive so that the root lies between 0 and 1.

First approximation: First approximation to the root is given by

$$x_1 = \frac{0+1}{2} = 0.5$$

$\therefore f(0.5) = 8(0.5)^3 - 2(0.5) - 1 = -1$, which is negative.

Thus $f(0.5)$ is negative and $f(1)$ is positive. Then the root lies between 0.5 and 1.

Second approximation: The second approximation to the root is given by

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$\therefore \quad f(0.75) = 8(0.75)^3 - 2(0.75) - 1$$

$$= 2.265 - 2.5 = 0.875, \text{ which is positive.}$$

Since $f(0.5)$ is negative, while $f(0.75)$ is positive. Therefore, the root lies between 0.5 and 0.75.

Third approximation: The third approximation to the root is given by

$$x_3 = \frac{0.5+0.75}{2} = 0.625$$

$$\therefore \quad f(0.625) = 8(0.625)^3 - 2(0.625) - 1$$

$$=1.935 - 2.25 = -0.297, \text{ which is negative.}$$

Therefore $f(0.75)$ is positive, while $f(0.625)$ is obtained negative. Therefore, the root lies between 0.625 and 0.75.

Fourth approximation: The fourth approximation to the root is given by

$$x_4 = \frac{0.625+0.75}{2} = 0.688$$

$$\therefore \quad f(0.688) = 8(0.688)^3 - 2(0.688) - 1$$

$$= 2.605 - 2.376 = 0.229, \text{ which is positive}$$

Therefore $f(0.688)$ is obtained positive, while $f(0.625)$ is negative. Therefore, the root lies between 0.625 and 0.688.

Fifth approximation: The fifth approximation to the root is given by

$$x_5 = \frac{0.625+0.688}{2} = 0.657$$

$$\therefore \quad f(0.673) = 8(0.657)^3 - 2(0.673) - 1$$

$$= 2.269 - 2.314 = -0.045, \text{ which is negative.}$$

Therefore $f(0.657)$ is negative and $f(0.688)$ is positive so the root lies between 0.657 and 0.688.

Sixth approximation: The sixth approximation to the root is given by

$$x_6 = \frac{0.657+0.688}{2} = 0.673$$

$$\therefore \quad f(0.673) = 8(0.673)^3 - 2(0.673) - 1$$

$$= 2.439 - 2.346 = 1.093, \text{ which is positive.}$$

Therefore $f(0.673)$ is positive and $f(0.657)$ is negative so the root lies between 0.657 and 0.673.

Seventh approximation: The seventh approximation to the root is given by

$$x_7 = \frac{0.657 + 0.673}{2} = 0.665$$

$\therefore$ $$f(0.665) = 8(0.665)^3 - 2(0.665) - 1$$

= 2.353 – 2.33 = 0.023, which is positive.

Therefore *f*(0.665) is positive and *f*(0.657) is negative so that the root lies between 0.657 and 0.665.

Eighth approximation: The eighth approximation to the root is given by

$$x_8 = \frac{0.657 + 0.665}{2} = 0.661$$

From last two approximations, *i.e.*, $x_7 = 0.665$ and $x_8 = 0.661$ it is observed that the approximate value of the root of *f*(*x*) = 0 up to two decimal places is 0.66.

Example 4. *Perform five interactions of Bisection method to obtain the smallest positive root of equation* $f(x) = x^3 - 5x + 1 = 0$.

Sol. Let *f*(2.1) = –ve, *f*(2.15) = +ve.

Therefore the root lies between 2.1 and 2.15.

First approximation to the root is

$$x_1 = \frac{2.1 + 2.15}{2} = 2.125$$

Now, $f(2.125) = -\text{ve}$

Therefore the root lies between 2.215 and 2.15.

Second approximation to the root is

$$x_2 = \frac{2.125 + 2.15}{2} = 2.1375$$

Now, $f(2.1375) = +\text{ve}$

Therefore the root lies between 2.125 and 2.1375.

Third approximation to the root is

$$x_3 = \frac{2.125 + 2.1375}{2} = 2.13125$$

Now, $f(2.13125) = +\text{ve}$

Therefore the root lies between 2.125 and 2.13125.

Fourth approximation to the root is

$$x_4 = \frac{2.125 + 2.13125}{2} = 2.1281$$

Now, $f(2.1281) = -\text{ve}$

Therefore the root lies between 2.1281 and 2.13125.

Fifth approximation to the root is

$$x_5 = \frac{2.1281 + 2.13125}{2} = 2.129$$

Hence the required root is 2.129.

Example 5. *Find the real root of equation* $x \log_{10} x = 1.2$ *by Bisection Method.*

Sol. Let $f(x) = x \log_{10} x - 1.2 = 0$

So that $f(1) = 1 \log_{10} 1 - 1.2 = -1.2 < 0$

and $f(2) = 2 \log_{10} 2 - 1.2 = 0.602 - 1.2$

$$= -0.598 < 0$$

and $f(3) = 3 \log_{10} 3 - 1.2$

$$= 3(0.4771) - 1.2 = 0.2313 > 0$$

Thus f (2) is negative and f (3) is positive, therefore, the root will lie between 2 and 3.

First approximation: The first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Again, $f(2.5) = 2.5 \log_{10} 2.5 - 1.2$

$$= 2.5\ (0.3979) - 1.2 = 0.9948 - 1.2 = -0.2052 < 0$$

Thus, f (2.5) is negative and f (3) is positive, therefore, the root lies between 2.5 and 3.

Second Approximation: The second approximation to the root is

$$x_2 = \frac{2.5+3}{2} = 2.75$$

Now, $f(2.75) = 2.75 \log_{10} 2.75 - 1.2$

$$= 2.75(0.4393) - 1.2 = 1.2081 - 1.2 = 0.0081 > 0$$

Thus, f (2.75) is positive and f (2.5) is negative, therefore, the root lies between 2.5 and 2.75.

Third approximation: The third approximation to the root is

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

Again, $f(2.625) = 2.625 \log_{10} 2.625 - 1.2$

$$= 2.625(0.4191) - 1.2 = 1.1001 - 1.2 = -0.0999 < 0$$

Thus, f(2.625) is found to be negative and f(2.75) is positive, therefore, the root lies between 2.625 and 2.75.

Fourth approximation: The fourth approximation to the root is

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

Again, $f(2.6875) = 2.6875 \log_{10} 2.6875 - 1.2$

$$= 2.6875(0.4293) - 1.2 = 1.1537 - 1.2 = -0.0463 < 0$$

Thus, $f(2.6875)$ is negative and $f(2.75)$ is positive, therefore, the root lies between 2.6875 and 2.75.

Fifth approximation: The fifth approximation to the root is

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.7188$$

Now, $f(2.7188) = 2.7188 \log_{10} 2.7188 - 1.2$

$= 2.7188\ (0.4344) - 1.2 = 0.1810 - 1.2 = -\ 0.019 < 0$

Thus, $f(2.7188)$ is negative and $f(2.75)$ is positive, therefore, the root lies between 2.7188 and 2.75

Sixth approximation: The sixth approximation to the root is

$$x_6 = \frac{2.7188 + 2.75}{2} = 2.7344$$

Again, $f(2.734) = 2.734 \log_{10} 2.734 - 1.2$

$= 2.734\ (0.4368) - 1.2 = 1.1942 - 1.2 = -\ 0.0058 < 0$

Thus, $f(2.734)$ is negative and $f(2.75)$ is positive, therefore, the root lies between 2.734 and 2.75

Seventh approximation: The seventh approximation to the root is

$$x_7 = \frac{2.734 + 2.75}{2} = 2.742$$

Again, $f(2.742) = 2.742 \log_{10} 2.742 - 1.2$

$= 2.742\ (0.4381) - 1.2 = 1.2012 - 1.2 = 0.0012 > 0$

Thus, $f(2.742)$ is positive and $f(2.734)$ is negative, therefore, the root lies between 2.734 and 2.742.

Eighth approximation: The eighth approximation to the root is

$$x_8 = \frac{2.734 + 2.742}{2} = 2.738$$

Hence, from the approximate value of the roots x_7 and x_8, we observed that, up to two places of decimal, the root is 2.74 approximately.

Example 6. *Using Bisection Method, find the real root of the equation* $f(x) = 3x - \sqrt{1 + \sin x} = 0$

Sol. The given equation

$f(x) = 3x - \sqrt{1 + \sin x} = 0$ is a transcendental equation.

Given $f(x) = 3x - \sqrt{1 + \sin x} = 0$ ...(1)

Then $f(0) = 0 - \sqrt{1 + \sin 0} = -1$

and $f(1) = 3 - \sqrt{1 + \sin 1} = 3 - \sqrt{1.8414}$

$= 3 - 1.3570 = 1.643 > 0$

Thus $f(0)$ is negative and $f(1)$ is positive, therefore, a root lies between 0 and 1.

First approximation: The first approximation to the root is given by

$$x_1 = \frac{0+1}{2} = 0.5$$

Now, $f(0.5) = 3(0.5) - \sqrt{1+\sin(0.5)}$

$= 1.5 - \sqrt{1.4794} = 1.5 - 1.2163 = 0.2837 > 0$

Thus, $f(0.5)$ is positive, while $f(0)$ is negative, therefore, a root lies between 0 and 0.5.

Second approximation: The second approximation to the root is given by

$$x_2 = \frac{0+0.5}{2} = 0.25$$

Again, $f(0.25) = 3(0.25) - \sqrt{1+\sin(0.25)}$

$= 0.75 - \sqrt{1.2474} = 0.75 - 1.1169 = -0.3669 < 0$

Thus, $f(0.25)$ is obtained to be negative and $f(0.5)$ is positive; therefore, a root lies between 0.25 and 0.5.

Third approximation: The third approximation to the root is given by

$$x_3 = \frac{0.25+0.5}{2} = 0.375$$

Now, $f(0.375) = 3(0.375) - \sqrt{1+\sin(0.375)}$

$= 1.125 - \sqrt{1.3663} = 1.125 - 1.1689 = -0.0439 < 0$

Thus, $f(0.375)$ is negative and $f(0.5)$ is positive, therefore, a root lies between 0.375 and 0.5.

Fourth approximation: The fourth approximation to the root is given by

$$x_4 = \frac{0.375+0.5}{2} = 0.4375$$

Now, $f(0.4375) = 3(0.4375) - \sqrt{1+\sin(0.4375)}$

$= 1.3125 - \sqrt{1.4237} = 1.3125 - 1.1932 = 0.1193 > 0$

Thus, $f(0.4375)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375 and 0.4375.

Fifth approximation: The fifth approximation to the root is given by

$$x_5 = \frac{0.375+0.4375}{2} = 0.4063$$

Again, $f(0.4063) = 3(0.4063) - \sqrt{1+\sin(0.4063)}$

$= 1.2189 - \sqrt{1.3952} = 1.2189 - 1.1812 - 0.0377 > 0$

Thus, $f(0.4063)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375 and 0.4063.

Sixth approximation: The sixth approximation to the root is given by

$$x_6 = \frac{0.375 + 0.4063}{2} = 0.3907$$

Again, $f(0.3907) = 3(0.3907) - \sqrt{1 + \sin(0.3907)}$

$= 1.1721 - \sqrt{1.3808} = 1.1721 - 1.1751 = -0.003 < 0$

Thus, f (0.3907) is negative, while f (0.4063) is positive, therefore, a root lies between 0.3907 and 0.4063.

Seventh approximation: The seventh approximation to the root is given by

$$x_7 = \frac{0.3907 + 0.4063}{2} = 0.3985$$

From the last two observations, that is, $x_6 = 0.3907$ and $x_7 = 0.3985$, the approximate value of the root up to two places of decimal is given by 0.39. Hence the root is 0.39 approximately.

Example 7. *Find a root of the equation $f(x) = x^3 - 4x - 9 = 0$, using the Bisection method in four stages.*

Sol. Given $f(x) = x^3 - 4x - 9 = 0$

Then $f(2) = 2^3 - 4(2) - 9 = -9$

and $f(3) = (3)^3 - 4(3) - 9 = 6$

Therefore, the root lies between 2 and 3.

First approximation: First approximation to the root is given by

$$x_1 = \frac{2+3}{2} = 2.5$$

Thus $f(2.5) = (2.5)^3 - 4(2.5) - 9$

$= 15.625 - 19 = -3.375$

Therefore, the root lies between 2.5 and 3.

Second approximation: Second approximation to the root is given by

$$x_2 = \frac{2.5+3}{2} = 2.75$$

Thus $f(2.75) = (2.75)^3 - 4(2.75) - 9$

$= 20.797 - 20 = 0.797$

Therefore, $f(2.75)$ is positive and $f(2.5)$ is negative. Thus the root lies between 2.5 and 2.75.

Third approximation: Third approximation to the root is given by

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

Now, $f(2.625) = (2.625)^3 - 4(2.625) - 9$

$= 18.088 - 19.5 = -1.412$

Therefore, $f(2.625)$ is negative while $f(2.75)$ is positive. Thus the root lies between 2.625 and 2.75.

Fourth approximation: Fourth approximation to the root is given by

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Hence, after the four steps the root is 2.6875 approximately.

PROBLEM SET 2.1

1. Find the smallest root lying in the interval (1, 2) up to four decimal places for the equation $x^6 - x^4 - x^3 - 1 = 0$ by Bisection Method **[Ans.** 1.4036]
2. Find the smallest root of $x^3 - 9x + 1 = 0$, using Bisection Method correct to three decimal places. **[Ans.** 0.111]
3. Find the real root of $e^x = 3x$ by Bisection Method. **[Ans.** 1.5121375]
4. Find the positive real root of $x - \cos x = 0$ by Bisection Method, correct to four decimal places between 0 and 1. **[Ans.** 0.7393]
5. Find a root of $x^3 - x - 11 = 0$ using Bisection Method correct to three decimal places which lies between 2 and 3. **[Ans.** 2.374]
6. Find the positive root of the equation $xe^x = 1$ which lies between 0 and 1. **[Ans.** 0.5671433]
7. Solve $x^3 - 9x + 1 = 0$ for the root between $x = 2$ and $x = 4$ by the method of Bisection. (*U.P.T.U.* 2005) **[Ans.** 2.94282]
8. Compute the root of $\log x = \cos x$ correct to 2 decimal places using Bisection Method. **[Ans.** 1.5121375]
9. Find the root of $\tan x + x = 0$ up to two decimal places which lies between 2 and 2.1 using Bisection Method. **[Ans.** 2.02875625]
10. Use the Bisection Method to find out the positive square root of 30 correct to 4 decimal places. **[Ans.** 5.4771]

2.5 FALSE POSITION METHOD (OR REGULA FALSI METHOD)

This method is essentially same as the bisection method except that instead of bisecting the interval.

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.

Consequently, $f(x_0)\, f(x_1) < 0$. Equation of the chord joining points $\{x_0, f(x_0)\}$ and $\{x_1, f(x_1)\}$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

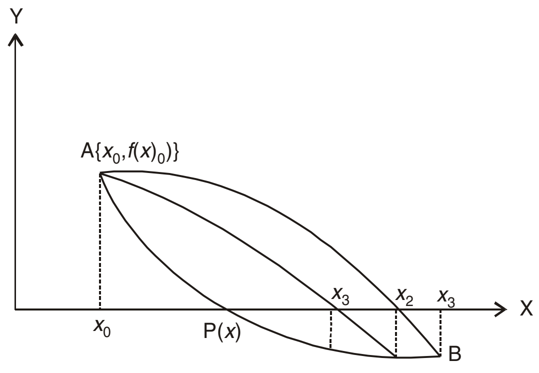

FIG. 2.2

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with X-axis as an approximation to the root.

So the abscissa of the point where chord cuts $y = 0$ is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

The value of x_2 can also be put in the following form:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general, the (i + 1)th approximation to the root is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

2.5.1 Procedure for the False Position Method to Find the Root of the Equation $f(x) = 0$

Step 1: Choose two initial guess values (approximations) x_0 and x_1 (where $x_1 > x_0$) such that $f(x_0).f(x_1) < 0$.

Step 2: Find the next approximation x_2 using the formula

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

and also evaluate $f(x_2)$.

Step 3: If $f(x_2)\, f(x_1) < 0$, then go to the next step. If not, rename x_0 as x_1 and then go to the next step.

Step 4: Evaluate successive approximations using the formula

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}, \text{ where } i = 2, 3, 4, \ldots$$

But before applying the formula for x_{i+1}, ensure whether $f(x_{i-1}).\, f(x_i) < 0$; if not, rename x_{i-2} as x_{i-1} and proceed.

Step 5: Stop the evaluation when $|x_i - x_{i-1}| < \varepsilon$, where ε is the prescribed accuracy.

2.5.2 Order (or Rate) of Convergence of False Position Method

The general iterative formula for False Position Method is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \ldots(1)$$

where x_{i-1}, x_i and x_{i+1} are successive approximations to the required root of $f(x) = 0$.

The formula given in (1), can also be written as:

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1}) f(x_i)}{f(x_i) - f(x_{i-1})} \quad \ldots(2)$$

Let α be the actual (true) root of $f(x) = 0$, *i.e.*, $f(\alpha) = 0$. If e_{i-1}, e_i and e_{i+1} are the successive errors in $(i - 1)$th, ith and $(i + 1)$th iterations respectively, then

$$e_{i-1} = x_{i-1} - \alpha,\ e_i = x_i - \alpha,\ e_{i+1} = x_{i+1} - \alpha$$

or

$$x_{i-1} = \alpha + e_{i-1},\ x_i = \alpha + e_i,\ x_{i+1} = \alpha + e_{i+1}$$

Using these in (2), we obtain

$$\alpha + e_{i+1} = \alpha + e_i - \frac{(e_i - e_{i-1}) f(\alpha + e_i)}{f(\alpha + e_i) - f(\alpha + e_{i-1})}$$

or
$$e_{i+1} = e_i - \frac{(e_i - e_{i-1}) f(\alpha + e_i)}{f(\alpha + e_i) - f(\alpha + e_{i-1})} \qquad ...(3)$$

Expanding $f(\alpha + e_i)$ and $f(\alpha + e_{i-1})$ in Taylor's series around α, we have

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + ...\right]}{\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + ...\right] - \left[f(\alpha) + e_{i-1} f'(\alpha) + \frac{e_{i-1}^2}{2} f''(\alpha) + ...\right]}$$

i.e., $$e_{i+1} = e_i - \frac{(e_i - e_{i-1})\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[(e_i - e_{i-1}) f'(\alpha) + \left(\frac{e_i^2 - e_{i-1}^2}{2}\right) f''(\alpha)\right]},$$ [on ignoring the higher order terms]

i.e. $$e_{i+1} = e_i = \frac{\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[f'(\alpha) + \left(\frac{e_i + e_{i-1}}{2}\right) f''(\alpha)\right]}$$

i.e. $$e_{i+1} = e_i - \frac{\left[e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[f'(\alpha) + \left(\frac{e_i + e_{i-1}}{2}\right) f''(\alpha)\right]}$$ [since $f(\alpha) = 0$]

i.e. $$e_{i+1} = e_i - \frac{\left[e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}\right]}{\left[1 + \left(\frac{e_i + e_{i-1}}{2}\right) \frac{f''(\alpha)}{f'(\alpha)}\right]},$$

[on dividing numerator and denominator by $f'(\alpha)$

i.e. $$e_{i+1} = e_i - \left[e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 + \left(\frac{e_i + e_{i-1}}{2}\right) \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

i.e. $$e_{i+1} = e_i - \left[e_i + \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 - \left(\frac{e_i + e_{i-1}}{2}\right) \frac{f''(\alpha)}{f'(\alpha)}\right]$$

i.e., $$e_{i+1} = e_i - \left[\frac{e_i(e_i + e_{i-1})}{2}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_i^2(e_i + e_{i-1})}{4}\left\{\frac{f''(\alpha)}{f'(\alpha)}\right\}^2\right]$$

i.e., $$e_{i+1} = e_i e_{i_1} - \frac{f''(\alpha)}{2f'(\alpha)} + 0\left(e_i^2\right)$$

If e_{i-1} and e_i are very small, then ignoring $0(e_i^2)$, we get

$$e_{i+1} = e_i e_{i-1}\frac{f''(\alpha)}{2f'(\alpha)} \quad ...(4)$$

which can be written as

$$e_{i+1} = e_i e_{i-1} M, \text{ where } M = \frac{f''(\alpha)}{2f'(\alpha)} \text{ and would be a constant} \quad ...(5)$$

In order to find the order of convergence, it is necessary to find a formula of the type

$$e_{i+1} = Ae_i^k \text{ with an appropriate value of } k. \quad ...(6)$$

With the help of (6), we can write

$$e_i = Ae_{i-1}^k \text{ or } e_{i-1} = (e_i/A)^{1/k}$$

Now, substituting the value of e_{i+1} and e_{i-1} in (5), we get

$$Ae_i^k = e_i.\left(\frac{e_i}{A}\right)^{1/k}.M$$

or $$e_i^k = MA^{-(1+1/k)}.e_i^{(1+1/k)} \quad ...(7)$$

Comparing the powers of e_i on both sides of (7), we get

$$k = 1 + (1/k)$$

or $$k^2 - k - 1 = 0 \quad ...(8)$$

From (8), taking only the positive root, we get $k = 1.618$

By putting this value of k in (6), we have

$$e_{i+1} = Ae_i^{1.618} \quad \text{or} \quad \frac{e_{i+1}}{e_i^{1.618}} = A$$

Comparing this with $\lim_{i\to\infty}\left(\frac{e_{i+1}}{e_i^k}\right) \le A$, we see that order (or rate) of convergence of false position method is 1.618.

Example 1. *Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by the method of false position up to three places of decimal.*

Sol. Given that $f(x) = x^3 - 2x - 5 = 0$

So that $f(2) = (2)^3 - 2(2) - 5 = -1$

and $f(3) = (3)^3 - 2(3) - 5 = 16$

Therefore, a root lies between 2 and 3.

First approximation: Therefore taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$, $f(x_1) = 16$, then by Regula-Falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3-2}{16+1}(-1) = 2 + \frac{1}{17} = 2.0588$$

Now, $f(x_2) = f(2.0588)$

$= (2.0588)^3 - 2\,(2.0588) - 5 = -\,0.3911$

Therefore, root lies between 2.0588 and 3.

Second approximation: Now, taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -\,0.3911$, $f(x_1) = 16$, then by Regula-Falsi method, we get

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0588 - \frac{3 - 2.0588}{16 + 0.3911}(-\,0.3911) = 2.0588 + 0.0225 = 2.0813$$

Now, $f(x_3) = f\,(2.0813)$

$= (2.0813)^3 - 2(2.0813) - 5 = -\,0.1468$

Therefore, root lies between 2.0813 and 3.

Third approximation: Taking $x_0 = 2.0813$ and $x_1 = 3$, $f(x_0) = -\,0.1468$, $f(x_1) = 16$. Then by Regula-Falsi method, we get

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0813 - \frac{3 - 2.0813}{16 + 0.1468}(-\,0.1468) = 2.0813 + 0.0084 = 2.0897$$

Now, $f(x_4) = f(2.0897)$

$= (2.0897)^3 - 2\,(2.0897) - 5$

$= 9.1254 - 9.1794 = -\,0.054$

Therefore, root lies between 2.0897 and 3.

Fourth approximation: Now, taking $x_0 = 2.0897$, $x_1 = 3$, $f\,(x_0) = -\,0.054$, $f\,(x_1) = 16$, then by Regula-Falsi method, we get

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0897 - \frac{3 - 2.0897}{16 + 0.054}(-0.054)$$

$$= 2.0897 + 0.0031 = 2.0928$$

Now, $f(x_5) = f(2.0928)$
$= (2.0928)^3 - 2(2.0928) - 5$
$= 9.1661 - 9.1856 = -0.0195$

Therefore, root lies between 2.0928 and 3.

Fifth approximation: Now, taking $x_0 = 2.0928$, $x_1 = 3$, $f(x_0) = -0.0195$, $f(x_1) = 16$, then we get

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0928 - \frac{3 - 2.0928}{16 + 0.0195}(-0.0195)$$

$$= 2.0928 + 0.0011 = 2.0939$$

Now, $f(x_6) = f(2.0939)$
$= (2.0939)^3 - 2\,(2.0939) - 5$
$= 9.1805 - 9.1879 = -0.0074$

Thus the root lies between 2.0939 and 3.

Sixth approximation: Now, taking $x_0 = 2.0939$, $x_1 = 3$, $f(x_0) = -0.0074$, $f(x_1) = 16$, then we get

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0939 - \frac{3 - 2.0939}{16 + 0.0074}(-0.0074)$$

$$= 2.0939 + 0.00042 = 2.0943$$

Now, $f(x_7) = f(2.0943)$
$= (2.0943)^3 - 2(2.0943) - 5$
$= 9.1858 - 9.1886 = -0.0028$

Therefore, root lies between 2.0943 and 3.

Seventh approximation: Taking $x_0 = 2.0943$, $x_1 = 3$, $f(x_0) = -0.0028$, $f(x_1) = 16$, then by Falsi position method, we get

$$x_8 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0943 - \frac{3 - 2.0943}{16 + 0.0028}(-0.0028)$$

$$= 2.0943 + 0.00016 = 2.0945$$

Hence, the root is 2.094 correct to three decimal places.

Example 2. *Find the real root of the equation $f(x) = x^3 - 9x + 1 = 0$ by Regula-Falsi method.*

Sol. Let $f(x) = x^3 - 9x + 1 = 0$ ...(1)

So that $f(2) = (2)^3 - 9(2) + 1 = -9$
$f(3) = (3)^3 - 9(3) + 1 = 1$

Since $f(2)$ and $f(3)$ are of opposite signs, therefore the root lies between 2 and 3, so taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -9$, $f(x_1) = 1$, then by Regula-Falsi method, we get

First approximation: $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$

$$= 2 - \frac{3-2}{1+9} \times (-9) = 2 + \frac{9}{10} = 2.9$$

Now, $f(x_2) = f(2.9)$

$$= (2.9)^3 - 9\,(2.9) + 1$$

$$= 24.389 - 25.1 = -0.711$$

Second approximation: The root lies between 2.9 and 3. Therefore, taking $x_0 = 2.9$, $x_1 = 3$, $f(x_0) = -0.711$, $f(x_1) = 1$. Then

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.9 - \frac{3-2.9}{1+0.711}(-0.711)$$

$$= 2.9 + 0.0416 = 2.9416$$

Now, $f(x_3) = f(2.9416)$

$$= (2.9416)^3 - 9(2.9416) + 1$$

$$= 25.4537 - 25.4744 = -0.0207$$

Third approximation: The root lies between 2.9416 and 3. Therefore, taking $x_0 = 2.9416$, $x_1 = 3$, $f(x_0) = -0.0207$, $f(x_1) = 1$. Then we get

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.9416 - \frac{3-2.9416}{1+0.0207}(-0.0207)$$

$$= 2.9416 + 0.0012 = 2.9428$$

Now, $f(x_4) = f\,(2.9428)$

$$= (2.9428)^3 - 9(2.9428) + 1$$

$$= 25.4849 - 25.\,4852 = -0.0003$$

Fourth approximation: The root lies between 2.9428 and 3. Therefore, taking $x_0 = 2.9428$, $x_1 = 3$, $f\,(x_0) = -0.0003$, $f\,(x_1) = 1$. Then by False Position method, we have

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.9428 - \frac{3-2.9428}{1+0.0003}(-0.0003)$$

$$= 2.9428 + 0.000017 = 2.942817$$

Hence, the root is 2.9428 correct to four places of decimal.

Example 3. *Using the method of False Position, find the root of equation $x^6 - x^4 - x^3 - 1 = 0$ up to four decimal places.*

Sol. Let

$$f(x) = x^6 - x^4 - x^3 - 1$$

$$f(1.4) = (1.4)^6 - (1.4)^4 - (1.4)^3 - 1 = -0.056$$

$$f(1.41) = (1.41)^6 - (1.41)^4 - (1.41)^3 - 1 = 0.102$$

Hence the root lies between 1.4 and 1.41.

Using the method of False Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.4 - \frac{1.41 - 1.4}{0.102 + 0.056}(-0.056)$$

$$= 1.4 + \left(\frac{0.01}{0.158}\right)(0.056) = 1.4035$$

Now,

$$f(1.4035) = (1.4035)^6 - (1.4035)^4 - (1.4035)^3 - 1$$

$$f(x_2) = -0.0016 \text{ (–ve)}$$

Hence the root lies between 1.4035 and 1.41.

Using the method of False Position,

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 1.4035 - \frac{1.41 - 1.4035}{0.102 + 0.0016}(-0.0016)$$

$$= 1.4035 + \left(\frac{0.0065}{0.1036}\right)(0.0016) = 1.4036$$

Now,

$$f(1.4036) = (1.4036)^6 - (1.4036)^4 - (1.4036)^3 - 1$$

$$f(x_3) = -0.00003 \text{ (–ve)}$$

Hence the root lies between 1.4036 and 1.41.

Using the method of False Position,

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 1.4036 - \frac{1.41 - 1.4036}{0.102 + 0.00003}(-0.00003)$$

$$= 1.4036 + \left(\frac{0.0064}{0.10203}\right)(0.00003) = 1.4036$$

Since, x_3 and x_4 are approximately the same upto four places of decimal, hence the required root of the given equation is 1.4036.

Example 4. *Find a real root of the equation $f(x) = x^3 - x^2 - 2 = 0$ by Regula-Falsi method.*

Sol. Let $f(x) = x^3 - x^2 - 2 = 0$

Then, $f(0) = -2$, $f(1) = -2$ and $f(2) = 2$

Thus, the root lies between 1 and 2.

First approximation: Taking $x_0 = 1, x_1 = 2, f(x_0) = -2$ and $f(x_1) = 2$. Then by Regula-Falsi method, an approximation to the root is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1 - \frac{2-1}{2+2}(-2) = 1 + \frac{1}{2} = 1.5$$

Now, $f(x_2) = f(1.5)$

$$= (1.5)^3 - (1.5)^2 - 2$$

$$= 3.375 - 4.25 = -0.875$$

Thus, the root lies between 1.5 and 2.

Second approximation: Taking $x_0 = 1.5, x_1 = 2, f(x_0) = -0.875$ and $f(x_1) = 2$. Then the next approximation to the root is given by

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.5 - \frac{2-1.5}{2+0.875}(-0.875)$$

$$= 1.5 + 0.1522 = 1.6522$$

Now, $f(x_3) = f(1.6522)$

$$= (1.6522)^3 - (1.6522)^2 - 2$$

$$= 4.5101 - 4.7298 = -0.2197$$

Thus, the root lies between 1.6522 and 2.

Third approximation: Taking $x_0 = 1.6522$, $x_1 = 2$, $f(x_0) = -0.2197$ and $f(x_1) = 2$. Then the next appoximation to the root is given by

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.6522 - \frac{2-1.6522}{2+0.2197}(-0.2197)$$

$$= 1.6522 + 0.0344 = 1.6866$$

Now, $f(x_4) = f(1.6866)$

$$= (1.6866)^3 - (1.6866)^2 - 2$$

$$= 4.7977 - 4.8446 = -0.0469$$

Thus, the root lies between 1.6866 and 2.

Fourth approximation: Taking $x_0 = 1.6866$, $x_1 = 2$, $f(x_0) = -0.046$ and $f(x_1) = 2$. Then the root is given by

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.6866 - \frac{-1.6866}{2 + 0.0469}(-0.0469)$$

$$= 1.6866 + 0.0072 = 1.6938$$

Now, $f(x_5) = f(1.6938)$

$$= (1.6938)^3 - (1.6938)^2 - 2$$

$$= 4.8594 - 4.8690 = -0.0096$$

Thus, the root lies between 1.6938 and 2.

Fifth approximation: Taking $x_0 = 1.6938$, $x_1 = 2$, $f(x_0) = -0.0096$ and $f(x_1) = 2$. Then the next approximation to the root is given by

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.6938 - \frac{2 - 1.6938}{2 + 0.0096}(-0.0096)$$

$$= 1.6938 + 0.0015 = 1.6953$$

Now, $f(x_6) = f(1.6953)$

$$= (1.6953)^3 - (1.6953)^2 - 2$$

$$= 4.8724 - 4.8740 = -0.0016$$

Therefore, the root lies between 1.6953 and 2.

Sixth approximation: Taking $x_0 = 1.6953$, $x_1 = 2$, $f(x_0) = -0.0016$ and $f(x_1) = 2$. Then the next approximation to the root is

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.6953 - \frac{2 - 1.6953}{2 + 0.0016}(-0.0016)$$

$$= 1.6953 + 0.0002 = 1.6955$$

Hence, the root is 1.695 correct to three places of decimal.

Example 5. *Find a real root of the equation $f(x) = xe^x - 3 = 0$, using Regula-Falsi method correct to three decimal places.*

Sol. We have $f(x) = xe^x - 3 = 0$

Then $f(1) = 1e^1 - 3 = -0.2817$

and $f(1.5) = (1.5)e^{(1.5)} - 3 = 3.7225.$

$\therefore$ The root lies between 1 and 1.5. Therefore, taking $x_0 = 1$, $x_1 = 1.5$, $f(x_0) = -0.2817$ and $f(x_1) = 3.7225$. The first approximation to the root is

First approximation:

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1 - \frac{1.5 - 1}{3.7225 + 0.2817} (-0.2817)$$

$$= 1 + \frac{0.14085}{4.0042} = 1.0352$$

Now, $f(x_2) = f(1.0352)$

$= 1.0352e^{1.0352} - 3$

$= 2.9148 - 3 = -0.0852.$

Thus, the root lies between 1.0352 and 1.5. Then taking $x_0 = 1.0352$, $x_1 = 1.5$, $f(x_0) = -0.0852$ and $f(x_1) = 3.7225$.

Second approximation: The next approximation to the root is

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.0352 - \frac{1.5 - 1.0352}{3.7225 + 0.0852} (-0.0852)$$

$$= 1.0352 + \frac{0.0396}{3.8077} = 1.0456$$

Now, $f(x_3) = f(1.0456)$

$= 1.0456e^{1.0456} - 3$

$= 2.9748 - 3 = -0.0252.$

Thus, the root lies between 1.0456 and 1.5. Then taking $x_0 = 1.0456$, $x_1 = 1.5$, $f(x_0) = -0.0252$ and $f(x_1) = 3.7225$.

Third approximation: The next approximation to the root is

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.0456 - \frac{1.5 - 1.0456}{3.7225 + 0.0252} (-0.0252)$$

$$= 1.0456 + \frac{0.0115}{3.7477} = 1.0487$$

Now, $f(x_4) = f(1.0487)$

$= (1.0487)e^{1.0487} - 3$

$= 2.9929 - 3 = -0.0071$

Thus, the root lies between 1.0487 and 1.5.

Fourth approximation: Taking $x_0 = 1.0487$, $x_1 = 1.5$, $f(x_0) = -0.0071$ and $f(x_1) = 3.7225$. Then the root becomes

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.0487 - \frac{1.5 - 1.0487}{3.7225 + 0.0071}(-0.0071)$$

$$= 1.0487 + \frac{0.0032}{3.7296} = 1.0496$$

Now, $f(x_5) = f(1.0496)$

$= (1.0496)\ e^{1.0496} - 3$

$= 2.9982 - 3 = -\ 0.0018.$

Thus the root lies between 1.0496, and 1.5.

Fifth approximation: Taking $x_0 = 1.0496$, $x_1 = 1.5$, $f(x_0) = -\ 0.0018$ and $f(x_1) = 3.7225$. Then the next approximation to the root is given by

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_{1)} - f(x_0)} f(x_0)$$

$$= 1.0496 - \frac{1.5 - 1.0496}{3.7225 + 0.0018}(-0.\ 0018)$$

$$= 1.0496 + \frac{0.00018}{3.7243} = 1.0498.$$

Hence, the root is approximately 1.0498 correct to three decimal places.

Example 6. *Find a real root of the equation $x^2 - \log_e x - 12 = 0$ using Regula-Falsi method correct to three places of decimals.*

Sol. Let $f(x) = x^2 - \log_e x - 12 = 0$

So that $f(3) = 3^2 - \log_e 3 - 12 = -\ 4.0986$ and $f(4) = 4^2 - \log_e 4 - 12$ and 2.6137

Therefore, $f(3)$ and $f(4)$ are of opposite signs. Therefore, a real root lies between 3 and 4. For the approximation to the root, taking

$$x_0 = 3,\ x_1 = 4,\ f(x_0) = -\ 4.0986 \text{ and } f(x_1) = 2.6137.$$

First approximation: By Regula-Falsi method, the root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3 - \frac{4 - 3}{2.6137 + 4.0986}(-4.0986)$$

$$= 3 + \frac{4.0986}{6.7123} = 3.6106$$

Now, $f(x_2) = f(3.6106)$

$= (3.6106)^2 - \log_e (3.6106) - 12$

$= 13.0364 - 13.2839 = -\ 0.2475.$

Second approximation: The root will lies between 3.6106 and 4. Therefore for next approximation, taking

$x_0 = 3.6106$, $x_1 = 4$, $f(x_0) = -\ 0.2475$ and $f(x_1) = 2.6137$. Then the root is

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3.6106 - \frac{4 - 3.6106}{2.6137 + 0.2475}(-0.2475)$$

$$= 3.6106 + \frac{0.0964}{2.8612} = 3.6443$$

Now,
$$f(x_3) = f(3.6443)$$
$$= (3.6443)^2 - \log_e (3.6443) - 12$$
$$= 13.2809 - 13.2932 = - 0.0123$$

Third approximation: The root lies between 3.6443 and 4. Therefore, taking $x_0 = 3.6443$, $x_1 = 4$, $f(x_0) = - 0.0123$ and $f(x_1) = 2.6137$. Then the root is given by

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3.6443 - \frac{4 - 3.6443}{2.6137 + 0.0123}(-0.0123)$$

$$= 3.6443 + \frac{0.0044}{2.626} = 3.6459$$

Now,
$$f(x_4) = f(3.6459)$$
$$= (3.6459)^2 - \log_e (3.6459) - 12$$
$$= 13.2926 - 13.2936 = - 0.001$$

Fourth approximation: The root lies between 3.6459 and 4. Therefore, taking $x_0 = 3.6459$, $x_1 = 4$, $f(x_0) = - 0.001$ and $f(x_1) = 2.6137$. Then the root is

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3.6459 - \frac{4 - 3.6459}{2.6137 + 0.001}(-0.001)$$

$$= 3.\ 6459 + \frac{0.00035}{2.6147} = 3.6460$$

Now,
$$f(x_5) = f(3.6460)$$
$$= (3.6460)^2 - \log_e (3.6460) - 12$$
$$= 13.2933 - 13.2936 = - 0.0003$$

Fifth approximation: The root lies between 3.6460 and 4. Then for next approximation, taking $x_0 = 3.6460$, $x_1 = 4$, $f(x_0) = - 0.0003$ and $f(x_1) = 2.6137$. Then the root is

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3.6460 - \frac{4 - 3.6460}{2.6137 + 0.0003}(-0.0003)$$

$$= 3.6460 + \frac{0.00011}{2.614} = 3.6461$$

Hence the root is approximated by 3.646 correct to three decimal places.

Example 7. *(1) Solve $x^3 - 5x + 3 = 0$ by using Regula-Falsi method.*

(2) Use the method of Falsi Position to solve $x^3 - x - 4 = 0$

Sol.

(1) Let $f(x) = x^3 - 5x + 3$

Since $f(0.65) = 0.024625$

and $f(0.66) = -0.012504$

Hence root lies between 0.65 and 0.66.

Let $x_0 = 0.65$ and $x_1 = 0.66$

Using method of Falsi Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.65 - \left(\frac{0.66 - 0.65}{-0.012504 - 0.024625}\right)(0.024625)$$

$$= 0.656632282$$

Now, $f(x_2) = -0.00004392$

Hence root lies between 0.65 and 0.656632282.

Using method of Falsi Position,

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.65 - \left(\frac{0.656632282 - 0.65}{-0.00004392 - 0.024625}\right)(0.024625)$$

$$= 0.656620474$$

Since x_2 and x_3 are same up to 4 decimal places hence the required root is 0.6566 correct up to four decimal places. Similarly the other roots of this equation are 1.8342 and –2.4909.

(2) Let $f(x) = x^3 - x - 4$

Since $f(1.79) = -0.054661$

and $f(1.80) = 0.032$

Hence root lies between 1.79 and 1.80.

Let $x_0 = 1.79$ and $x_1 = 1.80$

Using method of Falsi Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.79 - \frac{1.80 - 1.79}{0.032 + 0.054661}(-0.054661)$$

$$= 1.796307$$

Now, $f(x_2) = -\ 0.00012936$

Hence root lies between 1.796307 and 1.80.

Using method of Falsi Position,

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.796307 - \left(\frac{1.8 - 1.796307}{0.032 + 0.00012936}\right)(-0.00012936)$$

$$= 1.796321$$

Since x_2 and x_3 are same up to 4 decimal places hence the required root is 1.7963 correct up to four decimal places.

Example 8. *Find the root of the equation tan x + tan h x = 0 which lies in the interval (1.6, 3.0) correct to four significant digits using of Falsi Position.*

Sol. Let $f(x) = \tan x + \tan h\ x = 0$

Since $f(2.35) = -\ 0.03$

and $f(2.37) = 0.009$

Hence the root lies between 2.35 and 2.37.

Let $x_0 = 2.35$ and $x_1 = 2.37$.

Using method of Falsi Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.35 - \frac{2.37 - 2.35}{0.009 + 0.03}(-0.03)$$

$$= 2.35 + \frac{0.02}{0.039}(0.03) = 2.365$$

Now, $f(x_2) = -\ 0.00004$

Hence the root lies between 2.365 and 2.37.

Using method of Falsi Position,

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.365 - \frac{2.37 - 2.365}{0.009 + 0.00004}(-0.00004)$$

$$= 2.365 + \frac{0.005}{0.00904}(0.00004)$$

$$= 2.365$$

Hence the required root is 2.365 correct to four significant digits.

PROBLEM SET 2.2

1. Find the real root of the equation $x^3 - 2x - 5 = 0$ by the method of Falsi Position correct to three decimal places. **[Ans.** 2.094]
2. Find the real root of the equation $x \log_{10} x = 1.2$ by Regula-Falsi method correct to four decimal places. **[Ans.** 2.7406]
3. Find the positive root of $xe^x = 2$ by the method of Falsi Position. **[Ans.** 0.852605]
4. Apply Falsi Position method to find smallest positive root of the equation $x - e^{-x} = 0$ correct to three decimal places. **[Ans.** 0.567]
5. Find the real root of the equations:
   (*a*) $x = \tan x$ **[Ans.** 4.4934]
   (*b*) $x^2 - \log_e x - 12 = 0$ **[Ans.** 3.5425]
   (*c*) $3x = \cos x + 1$ **[Ans.** 0.6071]
6. Find the rate of convergence of Regula-Falsi method.
7. Find real cube root of 18 by Regula-Falsi method. **[Ans.** 2.62074]
8. Discuss method of Falsi Position.

2.6 ITERATION METHOD (METHOD OF SUCCESSIVE APPROXIMATION)

This method is also known as the direct substitution method or method of fixed iterations.

To find the root of the equation $f(x) = 0$ by successive approximations, we rewrite the given equation in the form

$$x = g(x) \qquad ...(1)$$

Now, first we assume the approximate value of root (let x_0) , then substitute it in $g(x)$ to have a first approximation x_1 given by

$$x_1 = g(x_0) \qquad ...(2)$$

Similarly, the second approximation x_2 is given by

$$x_2 = g(x_1) \qquad ...(3)$$

In general,

$$x_{i+1} = g(x_i) \qquad ...(4)$$

2.6.1 Procedure For Iteration Method To Find The Root of The Equation f(x) = 0

Step 1: Take an initial approximation as $x_{0.}$

Step 2: Find the next (first) approximation x_1 by using $x_1 = g(x_0)$

Step 3: Follow the above procedure to find the successive approximations x_{i+1} by using $x_{i+1} = g(x_i)$, $i = 1, 2, 3...$.

Step 4: Stop the evaluation where relative error $\le \varepsilon$, where ε is the prescribed accuracy.

Note 1: The iteration method $x = g(x)$ is convergent if $|g^1(x)| < 1$.

Note 2: When $|g^1(x)| > 1 \Rightarrow g^1(x) > 1$ or $g^1(x) < -1$, the iterative process is divergent.

2.6.2 Rate of Convergence of Iteration Method

Let $f(x) = 0$ be the equation which is being expressed as $x = g(x)$. The iterative formula for solving the equation is

$$x_{i+1} = g(x_i)$$

If a is the root of the equation $x = g(x)$ lying in the interval $]a,b[$, $\alpha = g(\alpha)$.

The iterative formula may also be written as

$$x_{i+1} = g\left(x + \overline{x_i - \alpha}\right)$$

Then by mean value theorem

$$x_{i+1} = g(\alpha) + (x_1 - \alpha)g'(c_i) \quad \text{Where } \alpha < c_i < b$$

But $\quad g(\alpha) = \alpha$

$\Rightarrow \quad x_{i+1} = \alpha + (x_i - \alpha)\, g'(c_i)$

$\Rightarrow \quad x_{i+1} - \alpha = (x_i - \alpha)\, g'(c_i)$ ...(1)

Now, if e_{i+1}, e_i are the error for the approximation x_{i+1} and x_i

Therefore, $\quad e_{i+1} = x_{i+1} - \alpha, \quad e_i = x_i - \alpha$

Using this in (1), we get

$$e_{i+1} = e_i\, g'(c_i)$$

Here $g(x)$ is a continuous function, therefore, it is bounded

$\therefore \quad |g'(c_i)| \le k$, where $k \in]a,b[$ is a constant.

$\therefore \quad e_{i+1} \le e_i k$

or $\quad \dfrac{e_{i+1}}{e_i} \le k$

Hence, by definition, the rate of convergence of iteration method is 1. In other words, iteration method converges linearly.

Example 1. *Find a real root correct upto four decimal places of the equation $2x - \log_{10} x - 7 = 0$ using iteration method.*

Sol. Here, we have $\quad f(x) = 2x - \log_{10} x - 7 = 0$

Now, we find that $\quad f(3) = -1.447 = -$ ve and $f(4) = 0.398 =$ +ve

Therefore, at least one real root of $f(x) = 0$ lies between $x = 3$ and $x = 4$.

Now, the given equation can be re-written as

$$x = \frac{1}{2}[\log_{10} x + 7] = g(x), \text{ say.}$$

Now, $\quad g'(x) = \dfrac{1}{2x}$, from which we clearly note that $|g^1(x)| < 1$ for all $x \in (3, 4)$.

Again since $|f(4)| < |f(3)|$, therefore, root is nearer to $x = 4$. Let the initial approximation be $x_0 = 3.6$ because $f(3.6)$ tends to zero. Then from the iterative formula $x_{i+1} = g(x_i)$, we obtain

$$x_1 = g(x_0) = \frac{1}{2}[\log_{10} x_0 + 7] = \frac{1}{2}[\log_{10} 3.6 + 7] = 3.77815$$

$$x_2 = g(x_1) = g(3.77815) = 3.78863$$

$$x_3 = g(x_2) = g(3.78863) = 3.78924$$
$$x_4 = g(x_3) = g(3.78924) = 3.78927$$

Hence, the root of the equation correct to the four places of decimal is 3.7892.

Example 2. *Solve $x = 0.21 \sin(0.5 + x)$ by iteration method starting with $x = 0.12$.*

Sol. Here, $x = 0.21 \sin(0.05 + x)$

$\Rightarrow$ $f(x) = 0.21 \sin(0.5 + x)$ ...(1)

Here we observe that $|f(x)| < 1$.

$\Rightarrow$ Method of iteration can be applied.

Now, first approximation of x is given by

$$x^{(1)} = 0.21 \sin(0.5 + 0.12) = 0.21 \sin(0.62)$$
$$= 0.21(0.58104) = 0.1220$$

The second approximation of x is given by

$$x^{(2)} = 0.21 \sin(0.5 + 0.122) = 0.21 \sin(0.622)$$
$$= 0.21(0.58267) = 0.1224$$

The third approximation of x is given by

$$x^{(3)} = 0.21 \sin(0.5+0.1224) = 0.21 \sin(0.6224)$$
$$= 0.21(0.58299) = 0.12243$$

The fourth approximation of x is given by

$$x^{(4)} = 0.21 \sin(0.5 + 0.12243) = 0.21 \sin(0.62243)$$
$$= 0.21(0.58301) = 0.12243$$

Here, we observe that $x^{(3)} = x^{(4)}$.

Hence, the required root is given by $x = 0.12243$.

Example 3. *The equation $\sin x = 5x - 2$ can be put as $x = \sin^{-1}(5x - 2)$ and also as $x = \frac{1}{5}(\sin x + 2)$ suggesting two iterating procedures for its solution. Which of these, if any, would succeed and which would fall to give root in the neighbourhood of 0.5.*

Sol. In First case, $\phi(x) = \sin^{-1}(5x - 2)$

$$\therefore \quad \phi(x) = \frac{5}{\sqrt{1-(5x-2)^2}}$$

Hence, $|\phi'(x)| > 1$ for all x for which $(5x - 2)^2 < 1$ or $x < 3/5$ or $x < 0.6$ in neighbourhood of 0.5. Thus the method would not give convergent sequence.

In Second case, $\phi(x) = \frac{1}{5}(\sin x + 2)$

$$\therefore \quad \phi'(x) = \frac{1}{5} \cos x$$

Hence, $|\phi'(x)| \le \frac{1}{5}$ for all x because $|\cos x| \le 1$

$\therefore$ $\phi'(x)$ will succeed.

Hence taking $x = \phi(x) = \frac{1}{5}$ (sin x + 2) and initial value $x_0 = 0.5$, we have the first approximation x_1 given by

$$x_1 = \frac{1}{5} (\sin 0.5 + 2) = 0.4017$$

$$x_2 = \frac{1}{5} (\sin 0.4017 + 2) = 0.4014$$

$$x_3 = \frac{1}{5} (\sin 0.4014 + 2) = 0.4014$$

Hence up to four places of decimal, the value of required root is 0.4014.

Example 4. *Find the real root of the equation cos x = 3x – 1 correct to three decimal places, using iteration method.*

Sol. Here, we have $f(x) = \cos x - 3x + 1$ ...(1)

We observe that $f(0) = 2 = +\text{ve}$ and $f(\pi/2) = -3(\pi/2) + 1 = -\text{ve}$

$\Rightarrow$ Roots lies between 0 and $\frac{\pi}{2}$.

Now, the given equation can be re-written as $x = \frac{1}{3} (\cos x + 1) = g(x)$ (say)

Then, we have $g'(x) = -\frac{\sin x}{3} = |g'(x)| < 1$ in $(0, \pi/2)$

Hence iteration method can be applied.

Take the first approximation $x_0 = 0$

Then we can find the successive approximation as:

$$x_1 = g(x_0) = \frac{1}{3} [\cos 0 + 1] = 0.667$$

$$x_2 = g(x_1) = \frac{1}{3} [\cos (0.667) + 1] = 0.5953$$

$$x_3 = g(x_2) = \frac{1}{3} [\cos (0.5953) + 1] = 0.6093$$

$$x_4 = g(x_3) = \frac{1}{3} [\cos (0.6093) + 1] = 0.6067$$

$$x_5 = g(x_4) = \frac{1}{3} [\cos (0.6067) + 1] = 0.6072$$

$$x_6 = g(x_5) = \frac{1}{3} [\cos (0.6072) + 1] = 0.6071$$

Now, x_5 and x_6 being almost same. Hence the required root is given by 0.607.

Example 5. *Find the real root of equation $f(x) = x^3 + x^2 - 1 = 0$ by using iteration method.*

Sol. Here, $f(0) = -1$ and $f(1) = 1$ so a root lies between 0 and 1. Now, $x = \dfrac{1}{\sqrt{1+x}}$ so that,

$$\phi(x) = \frac{1}{\sqrt{1+x}}$$

$$\therefore \quad \phi'(x) = -\frac{1}{2(1+x)^{3/2}}$$

We have, $|\phi'(x)| < 1$ for $x < 1$

Hence iterative method can be applied.

Take, $x_0 = 0.5$, we get

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{1.5}} = 0.81649$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{1.81649}} = 0.74196$$

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$$x_8 = 0.75487$$

Example 6. *Find the cube root of 15 correct to four significant figures by iterative method.*

Sol. Let $x = (15)^{1/3}$ therefore $x^3 - 15 = 0$

Real root of the equation lies in (2,3). The equation may be written as

$$x = \frac{15 + 20x - x^3}{20} = \phi(x)$$

Now, $\phi'(x) = 1 - \dfrac{3x^2}{20}$ therefore $|\phi'(x)| < 1$ (for $x \sim 2.5$)

Iterative formula is $$x_{i+1} = \frac{15 + 20x_i - x_i^3}{20} \quad ...(1)$$

Put $i = 0$, $x_0 = 2.5$, we get $x_1 = 2.47$

Put $i = 1$ in (1), $x_2 = 2.466$ (where $x_1 = 2.47$)

Similarly, $x_3 = 2.4661$

Therefore $\sqrt[3]{20}$ correct to 3 decimal places is 2.466.

Example 7. *Find the reciprocal of 41 correct to 4 decimal places by iterative formula $x_{i+1} = x_i(2 - 41x_i)$.*

Sol. Iterative formula is $x_{i+1} = x_i(2 - 41x_i)$ ...(1)

Putting $i = 0$, $x_1 = x_0(2 - 41x_0)$

Let $x_0 = 0.02$ then $x_1 = (0.02)(2 - 0.82) = 0.024$

Putting $i = 1$ in (1) $x_2 = (0.024)\{2 - (41 \times 0.024)\} = 0.0244$

Putting $i = 2$ in (1) $x_3 = 0.02439$

Therefore reciprocal of 41 is 0.0244.

Example 8. *Find the square root of 20 correct to 3 decimal places by using recursion formula*

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{20}{x_i}\right).$$

Sol. Put $i = 0, \quad x_1 = \frac{1}{2}\left(x_0 + \frac{20}{x_0}\right)$

Let $x_0 = 4.5$

$\therefore \quad x_1 = \frac{1}{2}\left(4.5 + \frac{20}{4.5}\right) = 4.47$

Put $i = 1, \quad x_2 = \frac{1}{2}\left(4.47 + \frac{20}{4.47}\right) = 4.472$

Put $i = 2, \; x_3 = \frac{1}{2}\left(4.472 + \frac{20}{4.472}\right) = 4.4721.$

Therefore $\sqrt{20} \approx 4.472$ correct to three decimal places.

Example 9. *Show that the following rearrangement of equation $x^3 + 6x^2 + 10x - 20 = 0$ does not yield a convergent sequence of successive approximations by iteration method near $x = 1$, $x = (20 - 6x^2 - x^3)/10$.*

Sol. Here, $x = \frac{20 - 6x^2 - x^3}{10} = f(x)$

Hence, $f'(x) = \frac{-12x - 3x^2}{10}$

Clearly, $f'(x) < -1$ in nbd of $x = 1$ Hence $|f'(x)| > 1$ and therefore the method and hence the sequence $< x_n >$ does not converge.

Example 10. *Find the smallest root of the equation*

$$\left\{1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0.\right\}$$

Sol. Written the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately.

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots = 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots = 1.3263$$

Similarly,
$$x_3 = \phi(x_2) = 1.38$$
$$x_4 = \phi(x_3) = 1.409$$
$$x_5 = \phi(x_4) = 1.425$$
$$x_6 = \phi(x_5) = 1.434$$
$$x_7 = \phi(x_6) = 1.439$$
$$x_8 = \phi(x_7) = 1.442$$

Values of x_7 and x_8 indicate that the root is 1.44 correct to two decimal places.

PROBLEM SET 2.3

1. Use the method of Iteration to find a positive root between 0 and 1 of the equation $xe^x - 1$. **[Ans.** 0.5671477]
2. Find the Iterative method, the real root of the equation $3x - \log_{10} x = 6$ correct to four significant figures. **[Ans.** 2.108]
3. Solve by Iteration method:
   (*a*) $x^3 + x + 1 = 0$ **[Ans.** –0.682327803]
   (*b*) $\sin x = \dfrac{x+1}{x-1}$ **[Ans.** –0.420365]
   (*c*) $x^3 - 2x^2 - 4 = 0$ **[Ans.** 2.5943]
4. By Iteration method, find $\sqrt{30}$. **[Ans.** 5.477225575]
5. If $f(x)$ is sufficiently differentiable and the iteration $x_{n+1} = F(x_n)$ converges, prove that the order of convergence is a positive integer.
6. The equation $f(x) = 0$, where $f(x) = 0.1 - x + \dfrac{x^2}{(2!)^2} - \dfrac{x^3}{(3!)^2} + \dfrac{x^4}{(4!)^2} - ...$ has one root in the interval (0,1). Calculate this root correct to 5 decimal places. **[Ans.** 0.10260]
7. The equation $x^2 + ax + b = 0$ has two real roots α and β. Show that the iteration method $x_{n+1} = -\left(\dfrac{x_n^2 + b}{a}\right)$ is convergent near $x = \alpha$ if $2|\alpha| < |a+\beta|$.

2.7 NEWTON–RAPHSON METHOD (OR NEWTON'S METHOD)

This method can be derived from Taylor's series as follows:

Let $f(x) = 0$ be the equation for which we are assuming x_0 be the initial approximation and h be a small corrections to x_0, so that

$$f(x_0 + h) = 0$$

Expanding it by Taylor's series, we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f'(x_0) + ... = 0$$

Since h is small, we can neglect second and higher degree terms in h and therefore, we get

$$f(x_0) + hf'(x_0) = 0$$

From which we have, $h = -\dfrac{f(x_0)}{f'(x_0)}$, where $[f'(x_0)^1\ 0]$.

Hence, if x_0 be the initial approximation, then next (or first) approximation x_1 is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The next and second approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is well known as Newton-Raphson formula.

The iterative procedure terminates when the relative error for two successive approximations becomes less than or equal to the prescribed tolerance.

2.7.1 Procedure for Newton Raphson Method to Find the Root of the Equation $f(x) = 0$

Step 1: Take a trial solution (initial approximation) as x_0. Find $f(x_0)$ and $f'(x_0)$.

Step 2: Find next (first) approximation x_1 by using the formula $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Step 3: Follow the above procedure to find the successive approximations x_{n+1} using the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $n = 1, 2, 3,...$

Step 4: Stop the process when $|x_{n+1} - x_n| < \varepsilon$, where ε is the prescribed accuracy.

2.7.2 Order (or Rate) of Convergence of Newton-Raphson Method

Let α be the actual root of equation $f(x) = 0$ *i.e.*, $f(a) = 0$. Let x_n and x_{n+1} be two successive approximations to the actual root α. If e_n and e_{n+1} are the corresponding errors we have, $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$. By Newton's-Raphson formula,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

$$e_{n+1} = e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \ldots} \quad \text{(By Taylor's expansion)}$$

$$e_{n+1} = e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \ldots} \qquad [\because f(\alpha) = 0]$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]} \quad \text{(On neglecting high powers of } e_n)$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)\left\{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right\}}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left\{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right\}^{-1}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left\{1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \ldots\ldots\right\}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2} \left\{\frac{f''(\alpha)}{f'(\alpha)}\right\}^2 + \ldots$$

or

$$\frac{e_{n+1}}{e_n^2} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n}{2} \left\{\frac{f''(\alpha)}{f'(\alpha)}\right\}^2 + \ldots..$$

$$\approx \frac{f''(\alpha)}{2f'(\alpha)} \quad \text{(Neglecting terms containing powers of } e_n)$$

Hence by definition, the order of convergence of Newton-Raphson method is 2 *i.e.*, Newton-Raphson method is **quadratic convergent**.

This also shows that subsequent error at each step is proportional to the square of the previous error and as such the convergence is quadratic.

Example 1. *Find the real root of the equation $x^2 - 5x + 2 = 0$ between 4 and 5 by Newton-Raphson's method.*

Sol. Let $f(x) = x^2 - 5x + 2$ ...(1)

Now, $f(4) = 4^2 - 5 \times 4 + 2 = -2$

and $f(5) = 5^2 - 5 \times 5 + 2 = 2$

Therefore, the root lies between 4 and 5.

From (1), we get $f'(x) = 2x - 5$ ...(2)

Now, Newton-Raphson's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 5x_n + 2}{2x_n - 5}$$

or

$$x_{n+1} = \frac{x_n^2 - 2}{2x_n - 5} \quad n = 0, 1, 2, \ldots \qquad \text{...(3)}$$

Let us take $x_0 = 4$ to obtain the approximation to the root by putting $n = 0, 1, 2...$ into (3), we get

First approximation:

$$x_1 = \frac{x_0^2 - 2}{2x_0 - 5} = \frac{4^2 - 2}{2(4) - 5} = \frac{14}{3} = 4.6667$$

Second approximation: The root is given by

$$x_2 = \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(4.6667)^2 - 2}{2(4.6667) - 5} = \frac{19.7781}{4.3334} = 4.5641$$

Third approximation: The root is given by

$$x_3 = \frac{x_2^2 - 2}{2x_2 - 5} = \frac{(4.5641)^2 - 2}{2(4.5641) - 5} = \frac{18.8310}{4.1282} = 4.5616$$

Fourth approximation: The root is given by

$$x_4 = \frac{x_3^2 - 2}{2x_3 - 5} = \frac{(4.5616)^2 - 2}{2(4.5616) - 5} = \frac{18.8082}{4.1232} = 4.5616$$

Since $x_3 = x_4$, hence the root of the equation is 4.5616 correct to four decimal places.

Example 2. *Solve $x^3 + 2x^2 + 10x - 20 = 0$ by Newton-Raphson's method*

Sol. Let $f(x) = x^3 + 2x^2 + 10x - 20$

$\Rightarrow$ $f'(x) = 3x^2 + 4x + 10$

Now, by Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 + 2x_n^2 + 10x_n - 20}{3x_n^2 + 4x_n + 10}$$

or $$x_{n+1} = \frac{2\left(x_n^3 + x_n^2 + 10\right)}{3x_n^2 + 4x_n + 10} \quad \text{...(1)}$$

Clearly, $f(1) = -7 < 0$ and $f(2) = 16 > 0$

Therefore root lies between 1 and 2.

Let $x_0 = 1.2$ be the initial approximation then

First approximation:

$$x_1 = \frac{2\left(x_0^3 + x_0^2 + 10\right)}{3x_0^2 + 4x_0 + 10}$$

$$= \frac{2((1.2)^3 + (1.2)^2 + 10)}{3(1.2)^2 + 4(1.2) + 10} = \frac{26.336}{19.12}$$

or $x_1 = 1.3774059$

Second approximation: $$x_2 = \frac{2\left(x_1^3 + x_1^2 + 10\right)}{3x_1^2 + 4x_1 + 10}$$

$$= \frac{2(1.3774059)^3 + (1.3774059)^2 + 10}{3(1.3774059)^2 + 1.3774059 + 10} = \frac{29.021052}{2.201364}$$

or $x_2 = 1.3688295$

Third approximation: $x_3 = \dfrac{2(x_2^3 + x_2^2 + 10)}{3x_2^2 + 4x_2 + 10}$

$$= \frac{2(1.3688295)^3 + (1.3688295)^2 + 10}{3(1.3688295)^2 + 4(1.3688295) + 10} = \frac{28.876924}{21.0964}$$

or $x_3 = 1.3688081$

Fourth approximation: $x_4 = \dfrac{2\left(x_3^3 + x_3^2 + 10\right)}{3x_3^2 + 4x_3 + 10}$

$$= \frac{2(1.3688081)^3 + (1.3688081)^2 + 10}{3(1.3688081)^2 + 4(1.3688081) + 10} = \frac{28.876567}{21.09614}$$

or $x_4 = 1.3688081$

Hence the required root is 1.3688081.

Example 3. *Find the real root of the equation $x \log_{10} x = 1.2$ by Newton-Raphson's method.*

Sol. Let $f(x) = x \log_{10} x - 1.2 = 0$ ...(1)

Then $f(1) = -1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314$

Therefore root lies between 2 and 3.

Let us take $x_0 = 2$, then from (1)

$$f'(x) = \log_{10} x + \frac{1}{x} \cdot x \log_{10} e = \log_{10} x + 0.4343 \quad ...(2)$$

Now, by Newton's-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.4343}$$

or $$x_{n+1} = \frac{0.4343\, x_n + 1.2}{\log_{10} x_n + 0.4343}, \quad n = 0, 1, 2, 3... \quad ...(3)$$

Putting $n = 0$ in (3), we get first approximation

First approximation: $x_1 = \dfrac{0.4343x_0 + 1.2}{\log_{10} x_0 + 0.4343}$

$$= \frac{0.4343(2) + 1.2}{\log_{10}(2) + 0.4343} = \frac{2.0686}{0.7353}$$

or $x_1 = 2.8133$

Putting n = 1 in (3), we get second approximation

Second approximation:

$$x_2 = \frac{0.4343\, x_1 + 1.2}{\log_{10} x_1 + 0.4343}$$

$$= \frac{0.4343(2.8133)+1.2}{\log_{10}(2.8133)+0.4343} = \frac{2.4128}{0.8835}$$

or $x_2 = 2.7411$

Putting $n = 2$ in (3), we get third approximation

Third approximation:

$$x_3 = \frac{0.4343\, x_2 + 1.2}{\log_{10} x_2 + 0.4343}$$

$$= \frac{0.4343(2.7411)+1.2}{\log_{10}(2.7411)+0.4343} = \frac{2.3905}{0.8722}$$

or $x_3 = 2.7408$

Putting $n = 3$ in (3), we get fourth approximation

Fourth approximation:

$$x_4 = \frac{0.4343\, x_3 + 1.2}{\log_{10} x_3 + 0.4343}$$

$$= \frac{0.4343(2.7408)+1.2}{\log_{10}(2.7408)+0.4343} = \frac{2.3903}{0.8721}$$

or $x_4 = 2.7408$

Since $x_3 = x_4$, hence the root of the equation is 2.7408 correct to four decimal places.

Example 4. *Find the real root of the equation $3x = \cos x + 1$ by Newton's method.*

Sol. Let $f(x) = 3x - \cos x - 1 = 0$ ...(1)

So that $f(0) = -2$

$f(1) = 3 - \cos 1 - 1 = 1.4597$

So the root lies between 0 and 1.

Let us take $x_0 = 0.6$

From (1) $f'(x) = 3 + \sin x$ ...(2)

Therefore the Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

or
$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad ...(3)$$

First approximation: Putting $n = 0$, in (3) we get first approximation

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{(0.6)(0.5646) + 0.8253 + 1}{3 + 0.5646} = \frac{2.16406}{3.5646}$$

or $x_1 = 0.6071$

Second approximation: Putting $n = 1$, in (3) we get second approximation

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{(0.6071) \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{(0.6071)(0.5705) + 0.8213 + 1}{3 + 0.5705} = \frac{2.1677}{3.5705}$$

or $x_2 = 0.6071$

Since $x_1 = x_2$ Therefore the root as 0.6071 correct to four decimal places.

Example 5. *Find the real root of the equation log x – cos x = 0 correct to three places of decimal by Newton-Raphson's method.*

Sol. Let $f(x) = \log x - \cos x = 0$ ...(1)

So that $f(1) = -0.5403$

$f(2) = 1.1092$

∴ The root lies between 1 and 2.

Also, $f(1.1) = \log 1.1 - \cos 1.1$

$= 0.0953 - 0.4535 = -0.3582$

$f(1.2) = \log 1.2 - \cos 1.2 = -0.18$

$f(1.3) = \log 1.3 - \cos 1.3 = -0.0051$

$f(1.4) = \log 1.4 - \cos 1.4 = 0.1665$

Thus the root lies between 1.3 and 1.4.

From (1) $f'(x) = \frac{1}{x} + \sin x$ ...(2)

Then by Newton's-Raphson method, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Using (1) and (2) $$x_{n+1} = x_n - \frac{\log x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n}$$

or $$x_{n+1} = \frac{x_n + x_n^2 \sin x_n - x_n \log x_n + x_n \cos x_n}{1 + x_n \sin x_n} \quad ...(3)$$

Let us take $x_0 = 1.3$

Now putting $n = 0$ into (3), we get first approximation

First approximation:

$$x_1 = \frac{x_0 + x_0^2 \sin x_0 - x_0 \log x_0 + x_0 \cos x_0}{1 + x_0 \sin x_0}$$

$$= \frac{(1.3) + (1.3)^2 \sin (1.3) - (1.3) \log (1.3) + (1.3) \cos (1.3)}{1 + (1.3) \sin (1.3)}$$

$$= \frac{1.3\,(1 + 1.2526 - 0.2623 + 0.2674)}{1 + 1.2526}$$

or $$x_1 = \frac{2.93501}{2.2526} = 1.3029$$

Now putting $n = 1$ into (3), we get second approximation

Second approximation:

$$x_2 = \frac{x_1 + x_1^2 \sin x_1 - x_1 \log x_1 + x_1 \cos x_1}{1 + x_1 \sin x_1}$$

$$= \frac{1.3029 + (1.3029)^2 \sin(1.3029) - (1.3029)\log(1.3029) + (1.3029)\cos(1.3029)}{1 + (1.3029)\sin(1.3029)}$$

$$= \frac{1.3029\,(1 + 1.2526 - 0.2645 + 0.2647)}{1 + 1.2564}$$

or $$x_2 = \frac{2.9401}{2.2564} = 1.3030$$

Hence the required root is 1.303 correct to three decimal places.

Example 6. *Evaluate $\sqrt{12}$ to four decimal places by Newton's iterative method.*

Sol. Let $x = \sqrt{12} \Rightarrow x^2 - 12 = 0$ ...(1)

Therefore Newton's Iterative formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 12}{2x_n} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right) \quad ...(2)$$

Now since $f(3) = -3$ (–ve) and $f(4) = 4$ (+ve)

Therefore the root lies between 3 and 4.

Take $x_0 = 3.5$, equation (2) gives

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right)$$

$$x_1 = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643$$

$$x_2 = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Since, $x_2 = x_3$ up to four decimal places. So we have $\sqrt{12} = 3.4641$.

Example 7. *Find a positive root of (17)$^{1/3}$ correct to four decimal places by Newton-Raphson's method.*

Sol. The iterative formula for the given equation is

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{k}{x_n^2}\right) \qquad ...(1)$$

Here $k = 17$. Take $x_0 = 2.5$ because $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$

Putting $n = 0$ in (1), we get

First approximation: $x_1 = \frac{1}{3}\left(2x_0 + \frac{17}{x_0^2}\right)$

$$x_1 = \frac{1}{3}\left(5 + \frac{17}{6.25}\right) = 2.5733$$

Putting $n = 1$ in (1), we get

Second approximation:

$$x_2 = \frac{1}{3}\left(2x_1 + \frac{17}{x_1^2}\right)$$

$$x_2 = \frac{1}{3}\left(5.1466 + \frac{17}{6.6220}\right) = 2.5713$$

Putting $n = 2$ in (1), we get

Third approximation:

$$x_3 = \frac{1}{3}\left(2x_2 + \frac{17}{x_2^2}\right)$$

$$x_3 = \frac{1}{3}\left(5.1426 + \frac{17}{6.61158}\right) = 2.57128$$

Putting $n = 3$ in (1), we get

Fourth approximation:

$$x_4 = \frac{1}{3}\left(2x_3 + \frac{17}{x_3^2}\right)$$

$$x_1 = \frac{1}{3}\left(5.14256 + \frac{17}{6.61148}\right) = 2.57138$$

Since x_3 and x_4 are accurate to four decimal places hence the required root is 2.5713.

Example 8. *Using Newton's iterative method, find the real root of x sin + cos x = 0, which is near x = π correct to 3 decimal places.*

Sol. Given, $f(x) = x \sin x + \cos x = 0$ therefore $f'(x) = x \cos x$

The iteration formula is, $x_{n+1} = x_n - \dfrac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$

With $x_0 = \pi$

First approximation:

The first approximation is given by

$$x_1 = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0}$$

$$x_1 = \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} = 2.8233$$

Similarly successive iterations are $x_2 = 2.7986$, $x_3 = 2.7984$, $x_4 = 2.7984$.

Since $x_3 = x_4$ hence the required root is 2.798 correct to three places of decimal.

Example 9. *Find the real root of the equation $x = e^{-x}$ using the Newton-Raphson's method.*

Sol. We have $f(x) = xe^x - 1$ then $f'(x) = (1 + x)e^x$

Let $x_0 = 1$ then,

First approximation:

$$x_1 = 1 - \left(\frac{e-1}{2e}\right) = \frac{1}{2}\left(1 + \frac{1}{e}\right) = 0.6839397$$

Now, $f(x_1) = 0.3553424$ and $f'(x_1) = 3.337012$

So that,

Second approximation:

$$x_2 = 0.6839397 - \left(\frac{0.3553424}{3.337012}\right) = 0.5774545$$

Third approximation:

$$x_3 = 0.5672297$$

Similarly, $x_4 = 0.5671433$

Hence the required root is 0.5671 correct to 4 decimal places.'

Example 10. *Using the starting value 2(1 + i), solve $x^4 - 5x^3 - 20x^2 - 40x + 60 = 0$ by Newton-Raphson's method given that all the roots of the equation are complex.*

Sol. Let $f(x) = x^4 - 5x^3 + 20x^2 - 40x + 60$

So that $f'(x) = 4x^3 - 15x^2 + 40x - 40$

Therefore Newton-Raphson method gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - 5x_n^3 + 20x_n^2 - 40x_n + 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}$$

$$x_{n+1} = \frac{3x_n^4 - 10x_n^3 + 20x_n^2 - 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}$$

Put $n = 0$, take $x_0 = 2\,(1 + i)$ by trial, we get

$$x_1 = 1.92(1 + i)$$

$$x_2 = 1.915 + 1.908i$$

Since, imaginary roots occur in conjugate pairs roots are 1.915 ± 1.908i upto 3 places of decimal. Assuming other pair of roots to be $\alpha \pm i\beta$, then

$$\text{Sum} = \begin{pmatrix} \alpha + i\,\beta + \alpha - i\beta \\ +1.915 + 1.908i \\ +1.915 - 1.908i \end{pmatrix} = 2\alpha + 3.83 = 5$$

$$\Rightarrow \qquad \alpha = 0.585$$

Also, products of roots are $(\alpha^2 + \beta^2)$ $[(1.915)^2 + (1.908)^2] = 60$

$$\Rightarrow \qquad \beta = 2.805$$

Hence other two roots are $0.585 \pm 2.805i$.

Example 11. *Apply Newton's formula to prove that the recurrence formula for finding the nth roots of a is*

$$x_{i+1} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

Hence, evaluate $(240)^{1/5}$.

Sol. Let $x = a^{1/n} \Rightarrow x^n = a$ or $x^n - a = 0$

Let $f(x) = x^n - a = 0$

$\Rightarrow$ $f'(x) = nx^{n-1}$.

Now, by Newtons's-Raphson method, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

or

$$x_{i+1} = \frac{(n-1)\,x_i^n + a}{nx_i^{n-1}} \qquad ...(1)$$

Now to find the value of $(240)^{1/5}$

We know that $(243)^{1/5} = (3^5)^{1/5} = 3$

Take $a = 240$ and $n = 5$ we get

$$x_{i+1} = \frac{4x_i^5 + 240}{5x_i^4} \qquad ...(2)$$

First approximation:

Let $i = 0$, $x_i = x_0 = 2.9$ (say), then from above equation (2), we get

$$x_1 = \frac{4x_0^5 + 240}{5x_0^4} = \frac{4(2.9)^5 + 240}{5(2.9)^4}$$

$$= \frac{4(205.111) + 240}{5 \times 70.7281} = \frac{1060.444}{353.6403} = 2.99$$

Second approximation:

Let $i = 1$, $x_i = x_1 = 2.99$ (say), then from above equation (2), we get

$$x_2 = \frac{4x_1^5 + 240}{5x_1^4} = \frac{4(2.99)^5 + 240}{5(2.99)^4}$$

$$= \frac{4(238.977) + 240}{399.627} = 2.9925$$

Hence, the required value of $(240)^{1/5}$ correct to three places of decimal is 2.993.

Example 12. *Determine the value of p and q so that rate of convergence of the iterative method* $x_{n+1} = px_n + q\,\frac{N}{x_n^2}$ *for computing* $N^{1/3}$ *becomes as high as possible.*

Sol. We have $x^3 = N$ therefore $f(x) = x^3 - N$.

Let α be the exact root, we have $\alpha^3 = N$.

Substituting $x_n = \alpha + e_n$, $x_{n+1} = \alpha + e_{n+1}$, $N = \alpha^3$ in $x_{n+1} = px_n + q\,\frac{N}{x_n^2}$, we have

$$\alpha + e_{n+1} = p(\alpha + e_n) + q\,\frac{\alpha^3}{(a + e_n)^2}$$

$$= p\,(\alpha + e_n) + q\,\frac{\alpha^3}{\alpha^2\left(1 + \frac{e_n}{\alpha}\right)^2}$$

$$= p(\alpha + e_n) + q\alpha\left(1 + \frac{e_n}{\alpha}\right)^{-2}$$

$$= p(\alpha + e_n) + q\alpha\left(1 - 2\frac{e_n}{\alpha} + 3\left(\frac{e_n}{\alpha}\right)^2 - \ldots\right)$$

$$= p(\alpha + e_n) + q\alpha - 2qe_n + 3q\frac{e_n^2}{\alpha} - \ldots$$

$$\Rightarrow \quad e_{n+1} = (p + q - 1)\,\alpha + (p - 2q)\,e_n + 0\,(e_n)^2 + \ldots$$

Now for the method to become of order as high as possible *i.e.*, of order 2, we must have $p + q = 1$ and $p - 2q = 0$ so that $p = 2/3$ and $q = 1/3$.

PROBLEM SET 2.4

1. Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$.
   (*U.P.T.U. 2005*) [**Ans.** 2.279]
2. Find the four places of decimal, the smallest root of the equation $e^{-x} = \sin x$. [**Ans.** 0.5885]
3. Find the cube root of 10. [**Ans.** 2.15466]
4. Show that the square roots of $N = AB$ is given by $\sqrt{N} \approx \frac{S}{4} + \frac{N}{S}$, where $S = A + B$.
5. Use Newton-Raphson method to obtain a root, correct to three decimal places of following equations:

   (*a*) $\sin x = \frac{x}{2}$ [**Ans.** 1.896]

   (*b*) $x + \log x = 2$ [**Ans.** 1.756]

   (*c*) $\tan x = x$ [**Ans.** 4.4934]

6. Using N–R method, obtain formula for $\sqrt{N}$ and find $\sqrt{20}$ correct to two decimal places. [**Ans.** 4.47]

7. Find cube root of 3 correct to three decimal places by Newton's iterative method. [**Ans.** 1.442]

8. Find the positive root of the equation $ex = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} e^{0.3x}$ correct to 6 decimal places. [**Ans.** 2.363376]

9. Apply Newton's formula to find the values of $(30)^{1/5}$. [**Ans.** 1.973]

10. Prove the Chebyshev formula $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\cdot\frac{[f(x_0)]^2 \cdot f''(x_0)}{[f'(x_0)]^3}$ for the roots of the equation $f(x) = 0$.

2.8 SECANT METHOD

The Secant method is similar to the Regula-Falsi method, except for the fact that we drop the condition that $f(x)$ should have opposite signs at the two points used to generate the next approximation. Instead, we always retain the last two points to generate the next. Thus, if x_{n-1} and x_n are two approximations to the root, then the next approximation x_{n+1} to root is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, ... \qquad ...\ (1)$$

Geometrically, in Secant method we replace the function $f(x)$ by a straight line passing through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ and take the point of intersection of the straight line with the x-axis as the next approximation to the root. In contrast to the Regula-Falsi method, the Secant iteration does not bracket the root and it is not even necessary to bracket the root to start the iteration. Hence, it is obvious that the iteration may not always coverage on the other hand, it generally converges faster. Thus, by dropping the necessity of bracketing the root, we improve the rate of convergence, however, in some cases, the iteration may not converge at all.

2.8.1 Procedure for Secant Method to Find the Root of $f(x) = 0$

Step 1: Choose the interval $[x_0, x_1]$ in which $f(x) = 0$ has a root, where $x_1 > x_0$.

Step 2: Find the next approximation x_2 of the required root using the formula

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1).$$

Step 3: Find the successive approximations of the required root using the formula

$$x_{n+1} = \varepsilon_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, ...$$

Step 4: Stop the process when the prescribed accuracy is obtained.

2.8.2 Rate or Order of Convergence of Secant Method

On substituting $x_n = \xi + \varepsilon_n$, etc. in (1) we obtain the error equation as

$$\varepsilon_{n+1} = \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1}) f(\xi + \varepsilon_n)}{f(\xi + \varepsilon_n) - f(\xi + \varepsilon_{n-1})} \qquad ...(2)$$

On expanding $f(\xi + \varepsilon_n)$ and $f(\xi + \varepsilon_{n-1})$ in Taylor's series about the point ξ in (2), and using $f(\xi) = 0$, we obtain

$$\varepsilon_{n+1} = \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1})\left[\varepsilon_n f'(\xi) + \frac{1}{2}\varepsilon_n^2 f''(\xi) + ...\right]}{\left[(\varepsilon_n - \varepsilon_{n-1}) f'(\xi) + \frac{1}{2}(\varepsilon_n^2 - \varepsilon_{n-1}^2) f''(\xi) + ...\right]}$$

On simplifying, and neglecting higher powers of ε_n, we get

$$\varepsilon_{n+1} = c\varepsilon_n \varepsilon_{n-1} \qquad ...(3)$$

where, $$c = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$$

Now, we seek a relation of the form

$$\varepsilon_{n+1} = k\varepsilon_n^p \qquad ...(4)$$

where constants k and p are to be determined.

We have from equation (4), $\varepsilon_n = k\varepsilon_{n-1}^p$ or $\varepsilon_{n-1} = k^{-1/p}\varepsilon_n^{1/p}$ ...(5)

Substituting ε_{n+1} from (4), and ε_{n-1} from (5), into the error equation (3), we get

$$\varepsilon_n^p = ck^{-(1+p)/p}\varepsilon_n^{(1+p)/p} \qquad ...(6)$$

On comparing the powers of ε_n, we have

$$p = \frac{1+p}{p}$$

or $$p^2 - p - 1 = 0$$

Roots of above equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Taking, $p = \frac{1+\sqrt{5}}{2} = 1.618$ and neglecting the other, we obtain from equation (4), the rate of convergence for the Secant method as $p = 1.618$. The constant k is determined from (6), and it is given by $k = c^{p/(p+1)}$.

The convergence of the secant method is superlinear

The purpose of this document is to show the following theorem:

Theorem 2.1: *Let* $\{x_k\}_k^\infty$ *be the sequence produced by the secant method. Assume the sequence converges to a root of f (x) = 0, i.e.,* $x_k \to x_\infty$, $f(x_\infty) = 0$. *Moreover, assume the root* x_∞ *is regular:* $f'(x_\infty) \neq 0$. *Denote the error in the kth step by* $E_k = x_k - x_\infty$. *Under these assumptions, we have*

$$E_{k+1} \approx CE_k^{(1+\sqrt{5})/2} \approx CE_k^{1.618}, \textit{ for some constant } C. \qquad ...(1)$$

The theorem is implied by three lemmas.

Lemma 2.1: *Under the assumptions and notations of the theorem:*

$$E_{k+1} \approx \frac{1}{2}\frac{f''(x_\infty)}{f'(x_\infty)}E_{k-1}E_k\,. \qquad ...(2)$$

Proof. Using the definition of x_{k+1}, we find

$$E_{k+1} = x_{k+1} - x_\infty = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} - x_\infty\,. \qquad ...(3)$$

We can replace x_{k+1} by $x_k + E_k$ and x_k by $x_{k-1} + E_{k-1}$, so that

$$E_{k+1} = x_\infty + E_k - f(x_\infty + E_k)\frac{x_\infty + E_k - x_\infty - E_{k-1}}{f(x_\infty + E_k) - f(x_\infty + E_{k-1})} - x_\infty\,. \qquad ...(4)$$

To simplify this expression, we apply the Taylor expansion of $f(x_\infty + E_k)$ and $f(x_\infty + E_{k-1})$ about x_∞:

$$f(x_\infty + E_k) = f(x_\infty) + f'(x_\infty)E_k + \frac{1}{2}f''(x_\infty)E_k^2 + O(E_k^3)\,, \qquad ...(5)$$

$$f(x_\infty + E_{k-1}) = f(x_\infty) + f'(x_\infty)E_{k-1} + \frac{1}{2}f''(x_\infty)E_{k-1}^2 + O(E_{k-1}^3)\,. \qquad ...(6)$$

Subtracting $f(x_\infty + E_{k-1})$ from $f(x_\infty + E_k)$:

$$f(x_\infty + E_{k-1}) - f(x_\infty + E_{k-1}) = f'(x_\infty)(E_k - E_{k-1}) + \frac{1}{2}f''(x_\infty)(E_k^2 - E_{k-1}^2) + O(E_k^3) - O(E_{k-1}^3) \qquad ....(7)$$

Since $O(E_k^3) - O(E_{k-1}^3)$ is of a smaller order than E_k and E_{k-1} we omit this term. Using $E_k^2 - E_{k-1}^2 = (E_k - E_{k-1})(E_k + E_{k-1})$, we organize the above expression as

$$f(x_\infty + E_k) - f(x_\infty + E_{k-1}) \approx (E_k - E_{k-1})(f'(x_\infty) + f''(x_\infty)(E_k + E_{k-1})\,. \qquad ...(8)$$

The left of (8) appears at the right of (4), we derive the following expression:

$$E_{k+1} \approx E_k - f(x_\infty + E_k)\frac{E_k - E_{k-1}}{(E_k - E_{k-1})(f'(x_\infty) + f''(x_\infty)(E_k + E_{k-1}))}\,. \qquad ...(9)$$

Using a Taylor expansion for $f(x_\infty + E_k)$ about x_∞ (recall $f(x_\infty) = 0$) we have

$$E_{k+1} \approx E_k - E_k\frac{f'(x_\infty) + \frac{1}{2}f''(x_\infty)E_k}{f'(x_\infty) + \frac{1}{2}f''(x_\infty)(E_k + E_{k-1})}. \qquad ...(10)$$

Now we put everything on the same denominator:

$$E_{k+1} \approx E_k\frac{f'(x_\infty) + \frac{1}{2}f''(x_\infty)(E_k + E_{k-1}) - f'(x_\infty) - \frac{1}{2}f''(x_\infty)E_k}{f'(x_\infty) + \frac{1}{2}f''(x_\infty)(E_k + E_{k-1})}\,. \qquad ...(11)$$

which can be simplified as

$$E_{k+1} \approx E_k\frac{\frac{1}{2}f''(x_\infty)E_{k-1}}{f'(x_\infty) + \frac{1}{2}f''(x_\infty)(E_k + E_{k-1})} \qquad ...(12)$$

Because $E_k \to 0$ as $k \to \infty, \frac{1}{2} f''(x_\infty)(E_k + E_{k-1})$ is negligible compared to $f'(x_\infty)$, so we omit the second term in the denominator, to find the estimate

$$E_{k+1} \approx \frac{1}{2} \frac{f''(x_\infty)}{f'(x_\infty)} E_k E_{k-1}. \qquad ...(13)$$

Lemma 2.2: *There exists a positive real number r such that:*

$$E_{k+1} \approx C E_{k-1} E_k \Rightarrow E_k^{1+1/r} \approx K E_k^r, \text{ for some constants } C \text{ and } K. \qquad ...(14)$$

Proof. Assuming the convergence rate is r, there exists some constant A, so we can write

$$E_{k+1} \approx A E_k^r \text{ and } E_k \approx A E_{k-1}^r \text{ or } \left(\frac{1}{A} E_k\right)^{1/r} \approx E_{k-1}. \qquad ...(15)$$

Now we can replace the expressions for E_k and E_{k-1} in the left hand side of (14):

$$E_{k+1} \approx C\left(\frac{1}{A}\right)^{1/r} E_k^{1/r} E_k \approx B E_k^{1+1/r}. \qquad ...(16)$$

Together with the assumption that $E_{k+1} \approx A E_k^r$, we obtain $E_k^{1+1/r} \approx \frac{A}{B} E_k^r$. So, we set $K = \frac{A}{B}$ and the lemma is proven. **[Q.E.D.]**

Lemma 2.3: *For the r of Lemma 2.1, we have*

$$E_k^{1+1/r} \approx C E_k^r \Rightarrow r = \frac{1+\sqrt{5}}{2}. \qquad ...(17)$$

Proof. r satisfies the following equation:

$$1 + \frac{1}{r} = r \Rightarrow r + 1 = r^2 \Rightarrow n^2 - r - 1 = 0. \qquad ...(18)$$

The roots of $n^2 - r - 1 = 0$ are $r = \frac{1 \pm \sqrt{5}}{2}$. We take the positive value for r.

The constant $r = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio.

Example 1. *A real root of the equation $f(x) = x^3 - 5x + 1 = 0$ lies in the interval (0.1). Perform four iterations of the Secant method.*

Sol. We have $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -3$

By Secant method.

First approximation:

First approximation is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1) = 0.25$$

$$f(x_2) = -0.234375$$

Second approximation:

Second approximation is given by

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2) = 0.186441$$

$$f(x_3) = 0.074276$$

Third approximation:

Third approximation is given by

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f(x_3) - f(x_2)}\right] f(x_3) = 0.201736$$

$$f(x_4) = -\ 0.000470$$

Fourth approximation:

Fourth approximation is given by

$$x_5 = x_4 - \left[\frac{x_4 - x_3}{f(x_4) - f(x_3)}\right] f(x_4) = 0.201640$$

Example 2. *Compute root of the equation $x^2 e^{-x/2} = 1$ in the interval [0.2] using Secant method. The root should be correct to three decimal places.*

Sol. We have, $x_0 = 1.42$, $x_1 = 1.43$, $f(x_0) = -\ 0.0086$, $f(x_1) = 0.00034$

By Secant method,

First approximation:

First approximation is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

$$x_2 = 1.43 - \left(\frac{1.43 - 1.42}{0.00034 + 0.0086}\right) (0.00034) = 1.4296$$

$$f(x_2) = -\ 0.000011$$

Second approximation: Second approximation is given by

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2)$$

$$x_3 = 1.4296 - \left(\frac{1.4296 - 1.42}{-0.000011 - 0.00034}\right) (-0.000011) = 1.4292$$

Since x_2 and x_3 agree up to three decimal places hence the required root is 1.429.

Example 3. *Find the root of the equation $x^3 - 5x^2 - 17x + 20 = 0$ by Secant method.*

Sol. Taking initial approximations as, $x_0 = 0$, $x_1 = 1$ and $f(x_0) = 20$, $f(x_1) = -\ 1$, then by Secant method the next approximation is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

$$x_2 = 1 - \left(\frac{1-0}{-1-20}\right)(-1) = 1 - 0.04762 = 0.95238$$

Hence, $f(x_2) = 0.13824$

Now the next approximation can be obtained by using x_1 and x_2 in Secant method.

Similarly, other approximations can be obtained by using two recent approximations in Secant method.

These are $x_3 = 0.95816, \quad f(x_3) = 0.00059$

$x_4 = 0.95818, \quad f(x_3) = 0.00011$

$x_5 = 0.95818.$

Thus the approximate root can be taken as 0.65818, which is correct up to five decimals.

Example 4. *Find the root of the equation $x^3 - 2x - 5$ by Regula-Falsi and Secant method.*

Sol. Solution by Regula-Falsi method:

Here $f(x) = x^3 - 2x - 5$ then $f(2) = -1$, $f(3) = 16$ and $f(2)\, f(3) < 0$

Therefore initial approximations are taken as $x_0 = 2$, $x_1 = 3$ and $f(x_0) = -1$, $f(x_1) = 16$

Then by Regula-Falsi method the next approximation is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

$$x_2 = 3 - \frac{(3-2)}{(16+1)} 16 = 3 - 0.9412$$

$$x_2 = 2.0588$$

Hence, $f(x_2) = -0.3911$ and $f(x_1)\, f(x_2) < 0$, therefore the next approximation to the root is obtained by using the values of x_1 and x_2 in Regula-Falsi method as

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_1)$$

$$x_3 = 2.0588 - \frac{(2.0588-3)}{(-0.3911-16)}(-0.3911) = 2.0588 + 0.0225$$

$$x_3 = 2.0813$$

Hence, $f(x_3) = -0.1468$ and $f(x_1)\, f(x_3) < 0$, therefore the next approximation is obtained by using the values of x_1 and x_3 in Regula-Falsi method.

Proceed in similar way to obtain the iterations as follows

$x_4 = 2.0899, \quad f(x_4) < 0$

$x_5 = 2.0928, \quad f(x_5) < 0$

$x_6 = 2.0939, \quad f(x_6) < 0$

$x_7 = 2.0943, \quad f(x_7) < 0$

$x_8 = 2.0945, \quad f(x_8) < 0$

$x_9 = 2.0945.$

Thus, the root can be taken as 2.0945 correct up to four decimals.

Solution by Secant method: Taking initial approximations as $x_0 = 2$, $x_1 = 3$ and $f(x_0) = -1$, $f(x_1) = 16$, then by Secant method, the next approximation is given by

$$x_2 = x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = 3 - \frac{(3-2)}{(16+1)} 16 = 3 - 0.9412$$

Hence, $x_2 = 2.0588$, $f(x_2) = -0.3911$

Now, the next approximation can be obtained by using the values of x_1 and x_2 in Secant method. Similarly, other approximations can be obtained by using two recent values of approximations. These are

$$x_3 = 2.0813, \quad f(x_3) = -0.1468$$
$$x_4 = 2.0948, \quad f(x_4) = -0.0028$$
$$x_5 = 2.0945, \quad f(x_5) = -0.0006$$
$$x_6 = 2.0945.$$

Thus, the root can be taken as 2.0945 correct to four decimals.

Example 5. *Find the root of the equation $f(x) = 4 \sin x + x^2 = 0$ by Secant method.*

Sol. In this method we neglect the condition $f(x_n) f(x_{n-1}) < 0$. Initially, take $x_0 = -1$, $x_1 = -2$ and $f(x_0) = -2.36588$, $f(x_1) = 0.36281$, the next approximation to the root by Secant method is given by

$$x_2 = x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = (-2) - \frac{(-2+1)(0.36281)}{[0.36281 + 2.36588]}$$

$$x_2 = -2 + 0.13296 = -1.86704$$

Hence, $x_2 = -1.86704$, $f(x_2) = -0.33992$

Now, the next approximation x_3 can be obtained by using the values of x_1 and x_2 in Secant method, which is given by,

$$x_3 = x_2 - \frac{(x_2 - x_1) f(x_2)}{f(x_2) - f(x_1)}$$

$$x_3 = (-1.86704) - \frac{(-1.86704 + 2)(-0.33992)}{[-0.33992 - 0.36281]}$$

$$x_3 = -1.86704 - 0.06431 = -1.93135$$

Hence, $x_3 = -1.93135$, $f(x_3) = -0.01269$

Now, the next approximation x_4 can be obtained by using the values of x_2 and x_3 in Secant method. Continuing this process and using two recent approximations, to get next approximation, in Secant method, we get

$$x_4 = -1.93384, \quad f(x_4) = 0.00045$$
$$x_5 = -1.93375, \quad f(x_5) = -0.00002$$

$$x_6 = -1.93375.$$

Thus, the approximation value to the root is –1.93375, correct up to five decimals.

2.9 METHODS FOR COMPLEX ROOTS

We now consider methods for determining complex roots of non-linear equations. Even if all coefficients of a non-linear equation are real, the equation can have complex roots. The iterative methods like the Secant method or the Newton-Raphson method are applicable to complex roots also, provided complex arithmetic is used. Starting with the complex initial approximation, if the iteration converges to a complex root, then the asymptotic convergence rate is the same as that for a real root.

The problem of finding a complex root of $f(z) = 0$, where z is a complex variable, is equivalent to finding real values x and y, such that

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) = 0$$

Where u and v are real functions.

This problem is equivalent to solving a system of two non-linear equations in two real unknowns x and y,

$$u(x, y) = 0, \; v(x, y) = 0$$

Which can be solved using the methods discussed in previous section.

Example 6. *Find all roots of the equation $f(x) = x^3 + 2x^2 - x + 5$ using Newton-Raphson method. Use initial approximations $x_0 = -3$ for real root and $x_0 = 1 + i$ for complex root.*

Sol. Given $f(x) = x^3 + 2x^2 - x + 5$

$$f'(x) = 3x^2 + 4x - 1$$

Newton-Raphson formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For real root: Taken initial approximation as $x_0 = -3$.

First approximation: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = -3 - \frac{(-1)}{14} = -2.928571429$$

Second approximation: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$x_2 = -2.928571429 + \frac{(0.035349848)}{13.01530612}$$

$$x_2 = -2.925855408$$

Third approximation: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$x_3 = -\,2.925855408 + \frac{(0.000050045)}{12.97846797}$$

$$x_3 = -\,2.925851552$$

Since the second and third approximations are same for five decimals hence the real root is –2.92585 correct up to five decimals.

For complex root: Initial approximation is $x_0 = 1 + i$

First approximation: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = 1 + i - \frac{(1+i)^3 + 2(1+i)^2 - (1+i) + 5}{3(1+i) + 4(1+i) - 1}$$

$$x_1 = \frac{53 + i(114)}{109} = 0.486238 + i\,(1.045871)$$

Thus 0.486238 + i (1.045871) is the first approximation value of the root. Proceeding similarly, we get next iterations as

$$x_2 = 0.448139 + i\,(1.23665)$$
$$x_3 = 0.462720 + i\,(1.22242)$$
$$x_4 = 0.462925 + i\,(1.22253)$$
$$x_5 = 0.462925 + i\,(1.22253)$$

Since the last two iterations are similar, we take 0.462925 + i(1.22253) as the value of the complex root.

2.10 MULLER'S METHOD

This method can also used to determine the both real and complex root of equation $f(x) = 0$. Let x_{i-2}, x_{i-1}, x_i be three distinct approximations to a root of $f(x) = 0$ and let y_{i-2}, y_{i-1}, y_i be the corresponding value of $y = f(x)$.

Let $p(x) = A(x - x_1)^2 + B(x - x_i) + y_i$ is a parabola passing through the points

$(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}), (x_i, y_i)$, we have

$$y_{i-1} = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) + y_i \quad \text{....(1)}$$

$$y_{i-2} = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) + y_i \quad \text{....(2)}$$

Equations (1) and (2) can be written as

$$A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) = y_{i-1} - y_i$$

$$A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) = y_{i-2} - y_i$$

Therefore

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

From A and B, the quadratic equation $p(x) = A(x - x_i)^2 + B(x - x_i) + y_i = 0$ gives the next approximation

$$x_{i+1} - x_i = \frac{-B \pm \sqrt{B^2 - 4Ay_i}}{2A} \qquad ...(3)$$

But direct solution from (3) lead to loss of accuracy therefore for maximum accuracy, equation (3) can be written as

$$x_{i+1} - x_i = -\frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

Note: If $B > 0$, we use +ve sign with square root of the equation and if $B < 0$, we use –ve sign with square root of the equation.

Example 7. *Find the root of the equation* $y(x) = x^3 - x^2 - x - 1 = 0$ *Muller's method, taking initial approximations as* $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

Sol. Let $x_i - 2 = 0$, $x_i - 1 = 1$, $x_i = 2$

Then $y_{i-2} = -1, y_{i-1} = -2, y_i = 1$

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)x_{i-2} - x_i)}$$

$$A = \frac{(0-2)(-2-1) - (1-2)(-1-1)}{(1-0)(1-2)(0-2)}$$

$A = 2$

$$B = \frac{(x_{i-2} - x_i)^2(y_{i-1} - y_i) - (x_{i-1} - x_i)^2(y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$B = \frac{(0-2)^2\ (-2-1) - (1-2)^2\ (-1-1)}{(0-1)\,(1-2)\,(0-2)}$$

$B = 5$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ayi}}$$

$$x_{i+1} = 2 - \frac{2 \times 1}{5 \pm \sqrt{25 - 4 \times 2 \times 1}} \quad \text{(taking +ve sign)}$$

$$x_{i+1} = 2 - \frac{2}{5 + 4.123106} = 1.780776$$

The procedure can now be repeated with three approximations as 1, 2, 1.780776.

Let $x_{i-2} = 1, x_{i-1} = 2, x_i = 1.780776$

Then $y_{i-2} = -2, y_{i-1} = 1, y_i = -0.304808$

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$A = \frac{(1-1.780776)(1+0.304808)-(2-1.780776)(-2+0.304808)}{(2-1)(2-1.780776)(1-1.780776)}$$

$$A = 3.780773$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$B = \frac{(1-1.780776)^2(1+0.304808)-(2-1.780776)^2(-2+0.304808)}{(1-2)(2-1.780776)(1-1.780776)}$$

$$B = 5.123098$$

The next approximation to the desired root is $x_{i+1} = x_i - \dfrac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$

$$x_{i+1} = 1.780776 - \frac{2\times(-0.304808)}{5.123098+\sqrt{(5.123098)^2 - 4\times 3.780773\times(-0.304808)}}$$

$$x_{i+1} = 1.837867$$

The procedure can repeated with three approximations as 2, 1.780776, 1.837867.

Let $\quad x_{i-2} = 2,\ x_{i-1} = 1.780776,\ x_i = 1.837867$

Then $\quad y_{i-2} = 1,\ y_{i-1} = -0.304808,\ y_i = -0.007757$

$$A = \frac{(x_{i-2}-x_i)(y_{i-1}-y_i)-(x_{i-1}-x_i)(y_{i-2}-y_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$A = \frac{(2-1.837867)(-0.304808+0.007757)-(1.780776-1.837867)(1+0.007757)}{(1.780776-2)(1.780776-1.837867)(2-1.837867)}$$

$$A = 4.619024$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$B = \frac{(2-1.837867)^2(-0.304808+0.007757)-(1.780776-1.837867)^2(1+0.007757)}{(2-1.780776)(1.780776-1.837867)(2-1.837867)}$$

$$B = 5.467225$$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$x_{i+1} = 1.837867 - \frac{2\times(-0.007757)}{5.467225+\sqrt{(5.467225)^2 - 4\times 4.619024-(0.007757)}}$$

$$x_{i+1} = 1.839284$$

The procedure can now be repeated with three approximations as 1.780776, 1.837867, and 1.839284.

Let $x_{i-2} = 1.780776,\ x_{i-1} = 1.837867,\ x_i = 1.839284$

Then $y_{i-2} = -\ 0.304808,\ \ y_{i-1} = -\ 0.007757,\ y_i = -\ 0.000015$

$$A = \frac{(x_{i-2}-x_i)(y_{i-1}-y_i)-(x_{i-1}-x_i)(y_{i-2}-y_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$A = \frac{(1.780776-1.839284)(-0.007757+0.000015)-(1.837867-1.839284)(-0.304808+0.000015)}{(1.837867-1.78.776)(1.837867-1.839284)(1.780776-1.839284)}$$

$A = 4.20000$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$B = \frac{(1.780776-1.839284)^2(-0.007757+0.000015)-(1.837867-1.839284)^2(-0.304808+0.000015)}{(1.780776-1.837867)(1.837867-1.839284)(1.780776-1.839284)}$$

$B = 5.20000$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$x_{i+1} = 1.839284 - \frac{2\times(-0.000015)}{5.2+\sqrt{(5.2)^2-4\times4.2\times(-0.000015)}}$$

$$x_{i+1} = 1.839287$$

Hence the required root is 1.839287

Example 8. *Using Muller's method, find the root of the equation* $y(x) = x^3 - 2x - 5 = 0$, *which lies between 2 and 3.*

Sol. Let $x_{i-2} = 1.9,\ x_{i-1} = 2,\ x_i = 2.1$

Then $y_{i-2} = -\ 1.941,\ y_{i-1} = -\ 1,\ y_i = 0.061$

$$A = \frac{(x_{i-2}-x_i)(y_{i-1}-y_i)-(x_{i-1}-x_i)(y_{i-2}-y_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$A = \frac{(-0.2)(-1.061)-(-0.1)(-2.002)}{(0.1)(-0.1)(-0.2)}$$

$$A = \frac{0.2122-0.2002}{0.002} = 6$$

$$B = \frac{(x_{i-2}-x_i)^2(y_i-1-y_i)-(x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

$$B = \frac{(-0.2)^2(-1.061)-(-0.1)^2(-2.002)}{(-0.1)(-0.1)(-0.2)}$$

$$B = \frac{-0.04244+0.02002}{-0.002} = 11.21$$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$x_{i+1} = 2.1 - \frac{2 \times (0.061)}{11.21 \pm \sqrt{(11.21)^2 - (24 \times 0.061)}}$$

$$x_{i+1} = 2.1 - \frac{0.122}{11.21 + 11.1445} = 2.094542$$

The procedure can now be repeated with three approximations as 2, 2.1 and 2.094542.

Let $x_{i-2} = 2,\ x_{i-1} = 2.1,\ x_i = 2.094542$

Then $y_{i-2} = -1,\ y_{i-1} = 0.061,\ y_i = -0.0001058$

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$A = \frac{(2 - 2.094542)(0.061 + 0.0001058) - (2.1 - 2.094542)(-1 + 0.0001058)}{(2.1 - 2)(2.1 - 2.094542)(2 - 2.094542)}$$

$A = 6.194492$

$$B = \frac{(x_{i-2} - x_i)^2 (y_{i-1} - y_i) - (x_{i-1} - x_i)^2 (y_{i-2} - y_i)}{(x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$B = \frac{(-0.094542)^2 (0.0611058) - (0.005458)^2 (-0.9998942)}{(-0.1)(0.005458)(-0.094542)}$$

$B = 11.161799$

The next approximation to the desired root is

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$x_{i+1} = 2.094542 - \frac{2 \times (-0.0001058)}{11.161799 \pm \sqrt{(11.161799)^2 - 4(6.194492)(-0.0001058)}}$$

$$x_{i+1} = 2.094542 + \frac{0.0002116}{11.161799 + 11.161916} = 2.094551$$

Hence the required root is 2.0945 correct up to 4 decimal places.

2.11 LIN BAIRSTOW METHOD

Let the polynomial equation be

$$p_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \ldots + a_{n-1} x + a_n = 0 \qquad \ldots(1)$$

where $a_0 \neq 0$ and all a_i' s are real.

For polynomials, if the coefficients are all real valued then the complex roots occurs in conjugate pair. Therefore we extract the quadratic factors that are the products of the pairs of complex roots, and then complex arithmetic can be avoided because such quadratic factors have real coefficients.

This method extracts a quadratic factor from polynomial given by equation (1), which gives a pair of complex roots or a pair of real roots.

Let us divide the given polynomial $p_n(x)$ by a quadratic factor $x^2 + px + q$, we obtain a quotient polynomial $Q_{n-2}(x)$ of degree $(n-2)$ and a linear remainder of the form $Rx + S$. Therefore

$$p_n(x) = (x^2 + px + q)Q_{n-2}(x) + (Rx + S) \qquad ...(2)$$

where $Q_{n-2}(x) = b_0x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + ... + b_{n-3}\,x + b_{n-2}$

If $(x^2 + px + q)$ is a factor of equation (1) then the remainder terms must vanish, therefore the problem is then to find p and q such that

$$R(p, q) = 0 \text{ and } S(p, q) = 0 \qquad ...(3)$$

If we regularly change the values of p and q, we can make the remainder zero or at least make its coefficient smaller, however this equation (3) will normally not be so, for the approximated values of p and q.

Since R and S are both functions of the two parameters p and q then the improved values are given by

$$R(p + \Delta p, q + \Delta q) = 0 \qquad ...(4)$$
$$S(p + \Delta p, q + \Delta q) = 0$$

Expand equation (4) by Taylor's series for a function of two variables, were the second and higher order terms are neglected. We get

$$R(p, q) + \frac{\partial R}{\partial p}\Delta p + \frac{\partial R}{\partial q}\Delta q = 0$$

$$S(p, q) + \frac{\partial S}{\partial p}\Delta p + \frac{\partial S}{\partial p}\Delta q = 0 \qquad ...(5)$$

On solving equation (5), we get

$$\Delta p = -\frac{R\left(\frac{\partial S}{\partial q}\right) - S\left(\frac{\partial R}{\partial q}\right)}{\left(\frac{\partial R}{\partial p}\right)\left(\frac{\partial S}{\partial q}\right) - \left(\frac{\partial R}{\partial q}\right)\left(\frac{\partial S}{\partial p}\right)}$$

$$\Delta q = -\frac{S\left(\frac{\partial R}{\partial p}\right) - S\left(\frac{\partial S}{\partial p}\right)}{\left(\frac{\partial R}{\partial p}\right)\left(\frac{\partial S}{\partial q}\right) - \left(\frac{\partial R}{\partial q}\right)\left(\frac{\partial S}{\partial p}\right)} \qquad ...(6)$$

Now the coefficients of b_i's, R and S can be obtained by comparing the like powers of x in (2). *i.e.*,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + ...\, a_{n-2}x^2 + a_{n-1}\,x + a_n = (x^2 + px + q)$$
$$\times\, (b_0x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + ...\, b_{n-4}x^2 + b_{n-3}x + b_{n-2}) + (Rx + S)$$

$$
\begin{aligned}
\Rightarrow \quad a_0 &= b_0 \\
a_1 &= b_1 + pb_0 \\
a_2 &= b_2 + pb_1 + qb_0 \\
a_3 &= b_3 + pb_2 + qb_1 \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
a_r &= b_r + pb_{r-1} + qb_{r-2} \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
a_{n-1} &= pb_{n-2} + qb_{n-3} + R \\
a_n &= S + qb_{n-2}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
b_0 &= a_0 \\
b_1 &= a_1 - pb_0 \\
b_2 &= a_2 - pb_1 - qb_0 \\
b_3 &= a_3 - pb_2 - qb_1 \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
b_r &= a_r - pb_{r-1} - qb_{r-2} \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
R &= a_{n-1} - pb_{n-2} - qb_{n-3} \\
S &= a_n - qb_{n-2} \qquad \ldots(7)
\end{aligned}
$$

Using $\quad b_r = a_r - pb_{r-1} - qb_{r-2},\ r = 1, 2, 3, \ldots n$ ...(8)

where $\quad b_0 = a_0,\ b_{-1} = 0$, Also from (8)

$$b_{n-1} = a_{n-1} - pb_{n-2} - qb_{n-3}$$

Therefore $\quad a_{n-1} = b_{n-1} + pb_{n-2} + qb_{n-3}$

and $\quad b_n = a_n - pb_{n-1} - qb_{n-2}$

Therefore $\quad a_n = b_n - pb_{n-1} - qb_{n-2}$

So equation (7) becomes $R = b_{n-1}$ and $S = b_n + pb_{n-1}$ ...(9)

For partial derivaties of R and S equation (8) can be used *i.e.*, differentiate equation (8) with respect to 'p' and 'q'.

$$-\frac{\partial b_r}{\partial p} = b_{r-1} + p\frac{\partial b_{r-1}}{\partial p} + q\frac{\partial b_{r-2}}{\partial p}$$

or

$$\frac{\partial b_0}{\partial p} = \frac{\partial b_{-1}}{\partial p} = 0 \qquad \ldots(10)$$

$$-\frac{\partial b_r}{\partial q} = b_{r-2} + p\frac{\partial b_{r-1}}{\partial q} + q\frac{\partial b_{r-2}}{\partial q}$$

or

$$\frac{\partial b_0}{\partial q} = \frac{\partial b_{-1}}{\partial q} = 0; \text{ where } r = 1, 2, 3, 4, \ldots \qquad \ldots(11)$$

Now, an equation (10) and (11) shows that $\frac{\partial b_{r+1}}{\partial q} = \frac{\partial b_r}{\partial p} = 0$ ...(12)

Now set, $-\frac{\partial b_r}{\partial p} = C_{r-1}$ so that $-\frac{\partial b_r}{\partial p} = C_{r-2}$; $r = 1, 2, 3,...$ ...(13)

Thus from (10) $$-\frac{\partial b_r}{\partial p} = b_{r-1} + p\frac{\partial b_{r-1}}{\partial p} + q\frac{\partial b_{r-2}}{\partial p}$$

$$C_{r-1} = b_{r-1} - pC_{r-2} - qC_{r-3}$$

or $$C_r = b_r - pC_{r-1} - qC_{r-2} \quad ...(14)$$

where $$C_{-1} = 0 \text{ and } C_0 = -\frac{\partial b_1}{\partial p} = -\frac{\partial}{\partial p}(a_1 - pb_0) = -\frac{\partial (pb_0)}{\partial p} \text{ or } C_0 = b_0 \quad ...(15)$$

From equations (9) and (15),

$$\frac{\partial R}{\partial p} = \frac{\partial}{\partial p}(b_{n-1}) \Rightarrow \frac{\partial R}{\partial p} = -C_{n-2}$$

$$\frac{\partial R}{\partial q} = \frac{\partial}{\partial q}(b_{n-1}) \Rightarrow \frac{\partial R}{\partial q} = -C_{n-3}$$

$$\frac{\partial S}{\partial p} = \frac{\partial b_n}{\partial p} + \frac{\partial}{\partial p}(pb_{n-1})$$

$\Rightarrow$ $$\frac{\partial S}{\partial p} = -C_{n-1} + p\frac{\partial b_{n-1}}{\partial p} + b_{n-1}\frac{\partial}{\partial p}(p)$$

$$= -C_{n-1}0 - pC_{n-2} + b_{n-1}$$

$$\frac{\partial S}{\partial q} = \frac{\partial b_n}{\partial q} + \frac{\partial}{\partial q}(pb_{n-1})$$

$$= -C_{n-2} - pC_{n-3}$$

On substituting these values in equation (6), we get

$$\Delta p = -\left[\frac{-R(C_{n-2} + pC_{n-3}) + SC_{n-3}}{C_{n-2}(C_{n-2} + pC_{n-3}) - [-C_{n-3}][-C_{n-1} - pC_{n-2} + b_{n-1}]}\right]$$

Since $R = b_{n-1}$ and $S = b_n + pb_{n-1}$, therefore

$$\Delta p = -\left[\frac{-b_{n-1}C_{n-2} - pb_{n-1}C_{n-3} + b_nC_{n-3} + pb_{n-1}C_{n-3}}{C_{n-2}\ C_{n-2} + pC_{n-2} + pC_{n-3}) - C_{n-3}C_{n-1} - pC_{n-2}C_{n-3} + b_{n-1}\ C_{n-3}]}\right]$$

$$\Delta p = -\frac{b_n\ C_{n-3} - b_{n-1}\ C_{n-2}}{C_{n-2}^2 - C_{n-3}\,(C_{n-1} - b_{n-1})} \quad ...(16)$$

Similarly $\Delta q = -\dfrac{b_{n-1}(C_{n-1} - b_{n-1}) - b_n C_{n-2}}{C_{n-2}^2 - C_{n-3}(C_{n-1} - b_{n-1})}$

Then the improved values of p and q are given by

$$p_1 = p + \Delta p \quad \text{and} \quad q_1 = q + \Delta q$$

The method for computation of b_r and C_r can be given by

	a_0	a_1	a_2	a_3	...	...	...	...	a_{n-2}	a_{n-1}	a_n
$-p$	–	$-pb_0$	$-pb_1$	$-pb_2$	...	...	...	...	$-pb_{n-3}$	$-pb_{n-2}$	$-pb_{n-1}$
$-q$	–	–	$-qb_0$	$-qb_1$	...	...	...	...	$-qb_{n-4}$	$-qb_{n-3}$	$-qb_{n-2}$
	b_0	b_1	b_2	b_3	...	...	...	...	b_{n-2}	b_{n-1}	b_n
$-p$	–	$-pC_0$	$-pC_1$	$-pC_2$	...	...	...	...	$-pC_{n-3}$	$-pC_{n-2}$	–
$-q$	–	–	$-qC_0$	$-qC_1$	...	...	...	...	$-qC_{n-4}$	$-qC_{n-3}$	–
	C_0	C_1	C_2	C_3	...	...	...	...	C_{n-2}	C_{n-1}	

The quotient polynomial $Q_{n-2}(x) = p_n(x) / (x^2 + px + q)$

$$= b_0 x^{n-2} + b_1 x^{n-3} + \ldots\ldots\ldots\ldots + b_{n-3}$$

can be obtained when p and q have been determined to the desired accuracy. This polynomial is called the defaulted polynomial. Another quadratic factor is of obtained using this default polynomial.

If the initial approximation of p and q are not known then the last three terms of given polynomial $a_{n-2}x^2 + a_{n-1} x + a_n = 0$ can be used to get approximations as $p_0 = \dfrac{a_{n-1}}{a_{n-2}}$, $q_0 = \dfrac{a_n}{a_{n-2}}$

Example 9. *Find the quadratic factor of the equation $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$ using Bairstow's method where $p_0 = -1.5$ and $q_0 = 1$. Also, find all the roots of the equation.*

Sol. Let the quadratic factor of the equation be $x^2 + px + q$. Using Bairstow's method we find the values of p and q.

First approximation: Let p_0 and q_0 be the initial approximations, then the first approximation can be obtained by $p_1 = p_0 + \Delta p$ and $q_1 = q_0 + \Delta q$. Because given equation is of the degree four then.

$$\Delta p = -\frac{b_4 C_1 - b_3 C_2}{C_2^2 - C_1(C_3 - b_3)} \qquad \ldots(1)$$

$$\Delta q = -\frac{b_3(C_3 - b_3) - b_4 C_2}{C_2^2 - C_1(C_3 - b_3)} \qquad \ldots(2)$$

Now to obtain the values of b_i and C_i we use the following procedure:

	1	–6	18	–24	16
1.5		1.5	–6.75	15.375	–6.1875
–1			–1	4.5	–10.25
	1	–4.5	10.25	–4.125	–0.4375
1.5		1.5	–4.5	7.125	9
–1			–1	3	–4.75
	1	–3	4.75	6	3.8125

Here from the table b_3 = –4.125, b_4 = –0.4375, C_1 = –3, C_2 = 4.75, C_3 = 6 therefore after substituting the values of b_i and C_i in equations (1) and (2) we get

$$\Delta p = -\frac{1.3125 + 19.594}{22.56 + 3(10.125)}$$

$$\Delta p = -0.3949$$

$$\Delta q = -\frac{(-39.6875)}{(52.935)}$$

$$\Delta q = 0.74974$$

Therefore the first approximation are given by

$$p_1 = p_0 + \Delta p = -1.5 + (-0.3949) = -1.8949$$

$$q_1 = q_0 + \Delta q = 1 + 0.74974 = 1.74974$$

Second approximation: Using p_1 = –1.8949 and q_1 = 1.74974 for second approximation, then $p_2 = p_1 + \Delta p$ and $q_2 = q_1 + \Delta q$.

Now to obtain the values of b_i and C_i we use the following procedure:

a_i	1	–6	18	–24	16
1.8949		1.8949	–7.7787	16.0526	–1.4487
–1.74974			–1.74974	7.1828	–14.8229
b_i	1	–4.1051	8.4715	–0.7645	–0.2716
1.8949		1.8949	–4.188	4.801	15.4044
–1.74974			–1.74974	3.8673	–4.433
C_i	1	–2.2102	2.5336	7.9038	10.6996

After substituting the values of b_i and C_i in equations (1) and (2) we get

$$\Delta p = \frac{2.53722752}{25.57781}$$

$$\Delta p = -0.09919$$

$$\Delta q = \frac{(-5.93879)}{25.57781}$$

$$\Delta q = 0.23218$$

Therefore the second approximations are given by

$$p_2 = p_1 + \Delta p = -1.99409$$
$$q_2 = q_1 + \Delta q = 1.98192$$

Third approximation: Using $p_2 = -1.99409$ and $q_2 = 1.98192$ for third approximation, then $p_3 = p_2 + \Delta p$ and $q_3 = q_2 + \Delta q$.

Now to obtain the values of b_i and C_i we use the following procedure:

a_i	1	−6	18	−24	16
1.99409		1.99409	−7.988	16.0124	−0.0962
−1.98192			−1.98192	7.9394	−15.9146
b_i	1	−4.0059	8.0299	−0.048226	−0.01082
1.99409		1.99409	−4.01173	4.06046	15.952
−1.98192			−1.98192	3.9873	−4.0357
C_i	1	−2.0118	0.03625	7.9995	11.9055

After substituting the values of b_i and C_i in equations (1) and (2) we get

$$\Delta p = \frac{0.11997}{20.3367}$$

$$\Delta p = -\ 0.005899$$

$$\Delta q = -\frac{(-0.36607)}{20.3367}$$

$$\Delta q = 0.01800$$

Therefore the third approximation are given by

$$p_3 = p_2 + \Delta p = -1.9882$$
$$q_3 = q_2 + \Delta q = 1.9999$$

Thus, we obtain $p = -1.9882$ and $q = 1.9999$. Hence quadratic factor of the given equation is $x^2 - 1.9882x + 1.9999 = 0$. Now if root of the quadratic factor is $\alpha \pm i\beta$, then

$$2\alpha = 1.9882 \Rightarrow \alpha = 0.9941,\ \alpha^2 + \beta^2 = 1.9999 \Rightarrow \beta = 1.0058$$

Hence a pair of roots is $0.9941 \pm 1.0058i$

Other roots can be obtained by using default polynomial.

Default polynomial is given by $b_0x^2 + b_1x + b_2 = 0$, where b_i are given by the same procedure, when p and q are of required accuracy.

a_1	1	–6	18	–24	16
1.9882		1.9882	–7.97626	15.95299	–0.047343
–1.9999			–1.9999	8.023198	–16.04687
b_i	1	–4.0118	8.02384	–0.023812	–0.094213

Thus $b_0 = 1$, $b_1 = -4.0118$, $b_2 = 8.02384$ and thus the polynomial becomes $1x^2 - 4.0118x + 8.02384 = 0$, whose roots are $\gamma = 2.0059$, $\delta = 2.00005$

Hence the pair of roots is $2.0059 \pm 2.00005i$

Example 10. *Solve $x^4 - 5x^3 + 20x^2 - 40x + 60 = 0$ given that all the roots are complex, by using Lin-Bairstow method. Take the values as $p_0 = -4$, $q_0 = 8$.*

Sol. Let the quadratic factor of the equation be $x^2 + px + q$. Using Bairstow's method we find the values of p and q:

First approximation: Let p_0 and q_0 be the initial approximations, then the first approximation can be obtained $p_1 = p_0 + \Delta p$ and $q_1 = q_0 + \Delta q$. Because given equation is of the degree four then

$$\Delta p = -\frac{b_4C_1 - b_3C_2}{C_2^2 - C_1(C_3 - b_3)} \quad ...(1)$$

$$\Delta q = -\frac{b_3(C_3 - b_3) - b_4C_2}{C_2^2 - C_1(C_3 - b_3)} \quad ...(2)$$

Now to obtain the values of b_i and C_i we use the following procedure:

a_i	1	–5	20	–40	60
4		4	–4	32	0
–8			–8	8	–64
b_i	1	–1	8	0	–4
4		4	12	48	96
–8			–8	–24	–96
C_i	1	3	12	24	–4

Here from the table $b_3 = 0$, $b_4 = -4$, $C_1 = 3$, $C_2 = 12$, $C_3 = 24$ therefore after substituting the values of b_i and C_i in equations (1) and (2), we get

$$\Delta p = -\frac{(-4)\times 3 - 0\times(12)}{(12)^2 - 3\times(24-0)}$$

$$\Delta p = 0.166666$$

$$\Delta q = -\frac{0\times(24-0) - (-4)\times 12}{(12)^2 - 3\times(24-0)}$$

$$\Delta q = -0.666666$$

Therefore the first approximation are given by

$$p_1 = p_0 + \Delta p = -4 + 0.166666 = -3.833334$$
$$q_1 = q_0 + \Delta q = 8 - 0.666666 = 7.333334$$

Second approximation: Using $p_1 = -3.833334$ and $q_1 = 7.333334$ for second approximation, then $p_2 = p_1 + \Delta p$ and $q_2 = q_1 + \Delta q$.

Now to obtain the values of b_i and C_i we use the following procedure:

a_i	1	–5	20	–40	60
3.833334		3.833334	–4.472220	31.412048	–0.124203
–7.333334			–7.333334	8.555551	–60.092609
b_i	1	–1.166666	8.194446	–0.032401	–0.216812
3.833334		3.8333334	10.222229	42.486147	87.7763681
–7.333334			–7.333334	–19.555567	–81.277841
C_i	1	2.666668	11.083341	22.898179	6.2817151

After substituting the values of b_i and C_i in equations (1) and (2), we get

$$\Delta p = -0.015192$$
$$\Delta q = -0.026908$$

Therefore the second approximation are given by

$$p_2 = p_1 + \Delta p = -3.833334 - 0.015192 = -3.848526$$
$$q_2 = q_1 + \Delta q = 7.333334 - 0.026908 = 7.306426$$

Third approximation: Using $p_2 = -\ 3.848526$ and $q_2 = 7.306426$ for third approximation, then $p_3 = p_2 + \Delta p$ and $q_3 = q_2 + \Delta q$.

Now to obtain the values of b_i and C_i we use the following procedure

a_i	1	–5	20	–40	60
3.848526		3.848526	–4.431477	31.796895	0.808398
–7.306426			–7.306426	8.413159	–60.366400
b_i	1	–1.151474	8.262097	0.210054	0.441998
3.848526		3.848526	10.379674	43.624369	92.859594
–7.306426			–7.306426	–19.705810	–82.820859
C_i	1	2.697052	11.335345	24.128613	10.480733

After substituting the values of b_i and C_i in equations (1) and (2), we get

$$\Delta p = 0.018582$$
$$\Delta q = -\ 0.0002186$$

Therefore the third approximation are given by

$$p_3 = p_2 + \Delta p = -3.848526 + 0.018582 = -3.829944$$

$$q_3 = q_2 + \Delta q = 7.306426 - 0.000218 = 7.306208$$

Thus, we obtain $p = -3.83$ and $q = 7.3062$. Hence quadratic factor of the given equation is $x^2 - 3.83x + 7.3062$. Now if root of the quadratic factor is $\alpha \pm i\beta^2$, then

$2\alpha = 3.83 \Rightarrow \alpha = 1.915$, $\alpha^2 + \beta^2 = 7.3062 \Rightarrow \beta = 1.9081$

Hence a pair of roots is $1.915 \pm 1.908li$

Other roots can be obtained by using default polynomial.

Default polynomial is given by $b_0x^2 + b_1x + b_2 = 0$, where b_i are given by the same procedure, when p and q are of required accuracy.

a_1	1	–5	20	–40	60
3.829944		8.829944	–4.481248	31.453583	0.008636
–7.306208			–7.306208	8.548672	–60.002554
b_i	1	–1.170056	8.212544	0.002255	0.00608185

Thus $b_0 = 1$, $b_1 = -1.17$, $b_2 = 8.2125$ and thus the polynomial becomes $1x^2 - 1.17x + 8.2125 = 0$, whose roots are $\gamma = 0.585$, $\delta = 2.8054$.

Hence the pair of roots is $0.585 \pm 2.8054i$.

2.12 QUOTIENT DIFFERENCE METHOD

This is a general method to obtain the approximate roots of the polynomial equations. The procedure is quite general and is illustrated here with a cubic polynomial. Let the given cubic equation be

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \quad ...(1)$$

and let x_1, x_2 and x_3 be its root such that $0 < |x_1| < |x_2| < |x_3|$

The roots can be obtained, directly by considering the transformed equation

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad ...(2)$$

Whose roots are the reciprocals of those of (1).

We then have $$\frac{1}{a_3x^3 + a_2x^2 + a_1x + a_0} = \sum_{i=0}^{\infty} \alpha_i x_i$$

So that $(a_3x^3 + a_2x^2 + a_1x + a_0)(\alpha_0 + \alpha_1x + \alpha_2x^2 + ...) = 1$ ...(3)

Comparing the coefficients of like powers of x on both sides of (3), we get

$$\alpha_0 = \frac{1}{a_0},\ \alpha_1 = -\frac{a_1}{a_0^2},\ \alpha_2 = \frac{-a_2}{a_0^2} + \frac{a_1^2}{a_0^3}$$

Hence, $$q_1^{(1)} = \frac{\alpha_1}{\alpha_0} = -\frac{a_1}{a_0}$$

$$q_1^{(2)} = \frac{\alpha_2}{\alpha_1} = \frac{a_2 a_0 - a_1^2}{a_0 a_1}$$

And so, $$\Delta_1^{(1)} = q_1^{(2)} - q^{(1)} = \frac{a_2}{a_1},\ \Delta_2^{(0)} = \frac{a_3}{a_2}$$

In general, $$\Delta_m^{(m)} = \frac{a_{m+1}}{a_m},\ m = 1, 2, 3,...(n-1)$$

$$q_m^{(1-m)} = 0, \quad m = 2, 3,..., n$$

That is $q_1^{(0)}$, $q_2^{(-1)}$, $q_3^{(-2)}$,..., top q's are 0.

We also set $\Delta_0^{(k)} = \Delta_n^{(k)} = 0$, for all k. [*i.e.,* first and last columns of Q-d table are zero].

The Quotient Difference table for a Cubic Equation

	$q_1^{(0)}$		$q_2^{(-1)}$		$q_3^{(-2)}$	
$\Delta_0^{(1)}$		$\Delta_1^{(0)}$		$\Delta_2^{(-1)}$		$\Delta_3^{(-2)}$
	$q_1^{(1)}$		$q_2^{(0)}$		$q_3^{(-1)}$	
$\Delta_0^{(2)}$		$\Delta_1^{(1)}$		$\Delta_2^{(0)}$		$\Delta_3^{(-1)}$
	$q_1^{(2)}$		$q_2^{(1)}$		$q_3^{(0)}$	
$\Delta_0^{(3)}$		$\Delta_1^{(2)}$		$\Delta_2^{(1)}$		$\Delta_3^{(0)}$

Remarks:

(1) If an Δ element is at the top of the rhombus, then the product of one pair is equal to that of the other pair. For example, in the rhombus

$$\begin{matrix} & \Delta_1^{(1)} & \\ q_1^{(2)} & & q_2^{(1)} \\ & \Delta_1^{(2)} & \end{matrix}$$

We have $\Delta_1^{(1)}.q_2^{(1)} = \Delta_1^{(2)}.q_1^{(2)}$

From which $\Delta_1^{(2)}$ can be computed since the other quantities are known.

(2) If a q-element is at the top, then the sum of one pair is equal to that of the other pair. For example, in the rhombus

$$\begin{matrix} & q_2^{(0)} & \\ \Delta_1^{(1)} & & \Delta_2^{(0)} \\ & q_2^{(1)} & \end{matrix}$$

We have $q_2^{(0)} + \Delta_2^{(0)} = q_2^{(1)} + \Delta_1^{(1)}$

From which $q_2^{(1)}$ can be computed when $q_2^{(0)}$, $\Delta_1^{(1)}$, $\Delta_2^{(0)}$ are known.

As the building up of table proceeds, the quantities $q_1^{(i)}$, $q_2^{(i)}$, $q_3^{(i)}$ tend to roots of cubic equations. The disadvantage of this method is that additional computation is also necessary. This method can be applied to find the complex roots and multiple roots of polynomials and also for determining the Eigen values of a matrix.

An important feature of the method is that it gives approximate values of all the roots simultaneously and this fact enables one to use this method to obtain the first approximations of all roots and then apply a rapidly convergent method such as the generalized Newton method to obtain the roots to the desired accuracy.

Example 11. Solve the following equation by using quotient-difference method $x^3 - 6x^2 + 11x - 6 = 0$.

Sol. To obtain the roots directly, we consider the transformed equation

$$-6x^3 + 11x^2 - 6x + 1 = 0$$

Here $a_3 = -6,\ a_2 = 11,\ a_1 = -6$ and $a_0 = 1$.

Therefore, we have $q_1^{(1)} = -\dfrac{a_1}{a_0} = 6$

$$q_1^{(2)} = -\frac{a_2 a_0 - a_1^2}{a_0 a_1} = \frac{11 - 36}{-6} = 4.167$$

$$\Delta_1^{(1)} = q_1^{(2)} - q_1^{(1)} = \frac{a_2}{a_1} = -1.833$$

Also $q_2^{(0)} = 0,\ q_3^{(-1)} = 0$ and $\Delta_2^{(0)} = \dfrac{a_3}{a_2} = -\dfrac{6}{11} = -0.5454$

First two rows containing starting values of

	$q_1^{(1)}$		$q_2^{(0)}$		$q_3^{(-1)}$	
$\Delta_0^{(2)}$		$\Delta_1^{(1)}$		$\Delta_2^{(0)}$		$\Delta_3^{(-1)}$
	6		0		0	
0		–1.833		–0.5454		0

The succeeding rows can be constructed as below:

Δ_0	q_1	Δ_1	q_2	Δ_2	q_3	Δ_3
	6		0		0	
0		–1.833		–0.545		0
	4.167		1.288		0.5454	
0		–0.5666		–0.2310		0
	3.600		1.624		0.7764	
0		–0.2556		–0.1105		0
	3.344		1.770		0.8869	
0		–0.1353		–0.0553		0
	3.209		1.8550		0.9422	
0		–0.0782		–0.0281		0
	3.131		1.9051		0.9703	
0		–0.0476		–0.0143		0
	3.083		1.9384		0.9846	
0		–0.0299		–0.0073		0
	3.053		1.961		0.9919	
0		–0.0192		–0.0037		0
	3.0338		1.976		0.9956	

It is evident that q_1, q_2, q_3 are gradually converging to the roots 3, 2 and 1 respectively.

PROBLEM SET 2.5

1. Use Secant method to determine the root of the equation $\cos x - xe^x = 0$.
[**Ans.** 0.5177573637]

2. Using Secant method, find the root of $x - e^{-x} = 0$ correct to three decimal places by taking $x_0 = 1$ and $x_1 = 1.5$. [**Ans.** 1.114]

3. Apply Muller's method to obtain the root of the equation $\cos x - xe^x = 0$ which lies between 0 and 1. [**Ans.** 0.518]

4. Solve by Muller's method $x^3 + 2x^2 + 10x - 20 = 0$ by taking $x = 0, x = 1, x = 2$ as initial approximation. [**Ans.** 1.368808108]

5. Find a quadratic factor of the polynomial $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$. Starting with $p_0 = 3$, $q_0 = -5$ by using Bairstow's Method. [**Ans.** $x^2 + 2.90255x - 4.91759$]

6. Solve the equation $x^4 - 8x^3 + 39x^2 - 62x + 50 = 0$. Starting with $p = 0, q = 0$.

7. Find the real roots of the equation $x^3 - 7x^2 + 10x - 2 = 0$ by using Quotient difference method. [**Ans.** 5.12487, 1.63668, 0.23845]

8. Solve the following equation $x^3 - 8x^2 + 17x - 10 = 0$ by using Quotient difference method.
[**Ans.** 5, 2.001, 0.9995]

9. Find all the roots of the equation $x^3 - 5x^2 - 17x + 20 = 0$ by using Quotient difference method. [**Ans.** 7.018, –2.974, 0.958]

❑❑❑

CHAPTER 3

Calculus of Finite Differences

3.1. INTRODUCTION

Finite differences: The calculus of finite differences deals with the changes that take place in the value of the dependent variable due to finite changes in the independent variable from this we study the relations that exist between the values, which can be assumed by function, whenever the independent variable changes by finite jumps whether equal or unequal.

The study of finite difference calculus has become very important due to its wide variety of application in routine life. It has been originated by Sir Issac Newton. It has been of great use for Mathematicians as well as Computer Scientists for solution of the Scientific, business and engineering problems. There it helps in reducing complex mathematical expressions like trigonometric functions in terms of simple arithmetic operations.

3.2 FINITE DIFFERENCES

Numerical methods are very important tools to provide practical methods for calculating the solution of problems to applied mathematics for a desired degree of accuracy.

If f is a function from x into y for $a \leq x \leq b$ such that $y = f(x)$, this means that one or more values of $y = f(x)$ exist corresponding to every value of x in the given range. However if the function f is not known, the value of y can be obtained, when a set of values of x is given. The method to find out such values is based on principle of finite differences provided the function is continuous.

3.3 ARGUMENT AND ENTRY

If $y = f(x)$ be a function assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ... corresponding to the values of x then each value of x is called **argument** and its corresponding values of y is called **entry.**

3.4 DIFFERENCES

Let $y = f(x)$ be a function tabulated for the equally spaced values or argument $a = x_0$, $x_0 + h$, $x_0 + 2h$, ..., $x_0 + 2h$, ...$x_0 + nh$, where h is the increment given to the independent variable of function $y = f(x)$. To determine the values of function $y = f(x)$ for given intermediate or argument values of x, three types of differences are useful:

3.4.1 Forward or Leading Differences

If we subtract from each value of y except y_0, the previous value of y, we have $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots, y_n - y_{n-1}$. These differences are called the first forward differences of y and is denoted by Δy. The symbol Δ denotes the forward difference operator. That is,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

Also it can be written as,

$$\Delta f(x) = f(x+h) - f(x)$$

where h is the interval of differencing.

Similarly for second and higher order differences,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

or

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Delta^3 y_{n-1} = \Delta^2 y_n - \Delta^2 y_{n-1}$$

In general, *nth* forward difference are given by

$$\Delta^n y_r = \Delta^{n-1} y_{r-1} - \Delta^{n-1} y_r, \text{ or}$$
$$\Delta^n f(x) = \Delta^{n-1} f(x + h) - \Delta^{n-1} f(x)$$

Forward Difference Table:

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
$x_0 + h$	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
$x_0 + 3h$	y_3		$\Delta^2 y_2$		
		Δy_3			
$x_0 + 4h$	y_4				

where $x_0 + h = x_1$, $x_0 + 2h = x_2$, ..., $x_0 + nh = x_n$

Example 1. *Construct a forward difference table for the following values:*

x	0	5	10	15	20	25
$f(x)$	7	11	14	18	24	32

Sol. Forward difference table for given data is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
5	11		–1			
		3		2		
10	14		1		–1	
		4		1		0
15	18		2		–1	
		6		0		
20	24		2			
		8				
25	32					

Example 2. *If $y = x^3 + x^2 - 2x + 1$, calculate values of y for x = 0, 1, 2, 3, 4, 5 and form the difference table. Also find the value of y at x = 6 by extending the table and verify that the same value is obtained by substitution.*

Sol. For x = 0, 1, 2, 3, 4, 5, we get the values of y are 1, 1, 9, 31, 73, 141. Therefore, difference table for these data is as:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		0		
1	1		8	
		8		6
2	9		14	
		22		6
3	31		20	
		42		6
4	73		26	
		68		6
5	141		32	
		100		
6	241			

Because third differences are zero therefore

$$\Delta^3 y_3 = 6 \Rightarrow \Delta^2 y_4 - \Delta^2 y_3 = 6$$

$$\Rightarrow \quad \Delta^2 y_4 - 26 = 6 \Rightarrow \Delta^2 y_4 = 32$$

Now, $$\Delta^2 y_4 = 32 \Rightarrow \Delta y_5 - \Delta y_4 = 32$$

$\Rightarrow \qquad \Delta y_5 - 68 = 32 \Rightarrow \Delta y_5 = 100$

Further, $\qquad \Delta y_5 = 100 \Rightarrow y_6 - y_5 = 100$

$\Rightarrow \qquad y_6 - 141 = 100 \Rightarrow y_6 = 241$

Verification: For given function $x^3 + x^2 - 2x + 1$, at $x = 6$, $y(6) = (6)^3 + (6)^2 - 2(6) + 1 = 241$

Hence Verified.

Example 3. *Given f(0) = 3, f(1) = 12, f(2) = 81, f(3) = 200, f(4) = 100 and f(5) = 8. From the difference table and find $\Delta^5 f(0)$.*

Sol. The difference table for given data is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	3					
		9				
1	12		60			
		69		–10		
2	81		50		–259	
		119		–269		755
3	200		–219		496	
		–100		227		
4	100		8			
		–92				
5	8					

Hence, $\qquad \Delta^5 f(0) = 755.$

Example 4. *Construct the forward difference table, given that:*

x	5	10	15	20	25	30
y	9962	9848	9659	9397	9063	8660

and point out the values of $\Delta^2 y_{10}$, $\Delta^4 y_5$.

Sol. For the given data, forward difference table is as:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962				
		–114			
10	9848		–75		
		–189		2	
15	9659		–73		–1
		–262		1	
20	9397		–72		2
		–334		3	
25	9063		–69		
		–403			
30	8660				

From the table, $\Delta^2 y_{10}, \Delta^4 y_5$ is as $\Delta^2 y_{10} = -73$ and $\Delta^4 y_5 = -1$.

Example 5. *Find f(6) given that f(0) = –3, f(1) = 6, f(2) = 8, f(3) = 12, the third differences being constant.*

Sol. For given data we construct the difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	–3			
		9		
1	6		–7	
		2		9
2	8		2	
		4		
3	12			

We have,

$$f(6) = f(0+6) = E^6 f(0) = (1+\Delta)^6 f(0)$$

$$= (1+6\Delta+15\Delta^2+20\Delta^3)f(0) \quad \text{[Higher differences being zero]}$$

$$= f(0)+6\Delta f(0)+15\Delta^2 f(0)+20\Delta^3 f(0)$$

$$= -3 + 6 \times 9 + 15 \times (-7) + 20 \times 9$$

$$= -3 + 54 - 105 + 180$$

$$= 126.$$

Example 6. *Prove that:*

(*a*) $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

(*b*) $f(4) = f(0) + 4\Delta f(0) + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$

Sol.

(*a*) We have,

$$f(4) - f(3) = \Delta f(3)$$

$$= \Delta[f(2) + \Delta f(2)] \quad \text{[Because } \Delta f(2) = f(3) - f(2)\text{]}$$

$$= \Delta f(2) + \Delta^2 f(2)$$

$$= \Delta f(2) + \Delta^2[f(1) + \Delta f(1)] \quad \text{[Because } \Delta f(1) = f(2) - f(1)\text{]}$$

$$= \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$$

Therefore,

$$f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$$

(*b*) We have,

$$f(4) = E^5 f(-1) = (1 + \Delta)^5 f(-1)$$

$$= \{1 + {}^5C_1\Delta + {}^5C_2\Delta^2 + {}^5C_3\Delta^3\} f(-1)$$

(On taking up to third differences)

$$= f(-1) + 5\Delta f(-1) + 10\Delta^2 f(-1) + 10\Delta^3 f(-1)$$

$$= [f(-1) + \Delta f(-1)] + 4[\Delta f(-1) + \Delta^2 f(-1)] + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$$

$$= [f(-1)+\Delta f(-1)]+4\Delta[f(-1)+\Delta f(-1)]+6\Delta^2 f(-1)+10\Delta^3 f(-1)$$

$$= f(0)+4\Delta f(0)+6\Delta^2 f(-1)+10\Delta^3 f(-1)$$

Because,

$$f(-1)+\Delta f(-1) = f(-1)+f(0)-f(-1) = f(0).$$

Example 7. *Find the function whose first difference is e^x.*

Sol. We know that $\Delta e^x = e^{x-h} - e^x = e^x(e^h - 1)$, where h is the interval of differencing.

Therefore, $e^x = \dfrac{1}{e^h - 1}\Delta e^x = \Delta\left(\dfrac{e^x}{e^h - 1}\right)$

Hence, required function is given by $\dfrac{e^x}{e^h - 1}$.

Example 8. *Find the first term of the series whose second and subsequent terms are 8, 3, 0, –1, and 0.*

Sol. If the interval of differencing is unity, then

$$\begin{aligned} f(1) &= E^{-1}f(2) \\ &= (1+\Delta)^{-1} f(2) \\ &= (1 - \Delta + \Delta^2 - \Delta^3 + \ldots\ldots\ldots\ldots)f(2). \end{aligned}$$

Since we have five observations, therefore the 4th differences will be constant and 5th differences will be zero.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	8		
		–5	
3	3	–3	2
4	0		2
		–1	
5	–1		2
		1	
6	0		

Hence, $f(1) = f(2) - \Delta f(2) + \Delta^2 f(2)$ [Higher order differences are 0]

$f(1) = 8 - (-5) + 2 = 15$

3.4.2 Backward or Ascending Differences

If we subtract from each value of y except y_0, the previous value of y, we get $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, ... $y_n - y_{n-1}$. These differences are called first backward differences of y and are denoted by ∇y. The symbol ∇ denotes the backward difference operator. That is,

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

....

$$\nabla y_n = y_n - y_{n-1}$$

Also it can be written as,

$$\nabla f(x+h) = f(x+h) - f(x)$$

Similarly, second forward difference is given by,

$$\nabla^2 f(x+h) = \nabla f(x+h) - \nabla f(x)$$

In general,

$$\nabla^n y_{r+1} = \nabla^{n-1} y_{r+1} - \nabla^{n-1} y_r \text{, or}$$
$$\nabla^n f(x+h) = \nabla^{n-1} f(x+h) - \nabla^{n-1} f(x)$$

Backward Difference Table:

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Example 9. *Construct the backward difference table for y = log x given that:*

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

and find the values of ∇^3 *log 40 and* ∇^4 *log 50.*

Sol. For the given data, backward difference table as:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1				
		0.3010			
20	1.3010		–0.1249		
		0.1761		0.0738	
30	1.4771		–0.0511		–0.0508
		0.1250		0.0230	
40	1.6021		–0.0281		
		0.0969			
50	1.6990				

Hence, $\nabla^3 \log 40 = 0.0738$ and $\nabla^4 \log 50 = -0.0508$

Example 10. *Given that:*

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Construct backward difference table and obtain $\nabla^4 f(8)$.

Sol. Backward difference table for given data is as:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		129		6	
7	343		42		
		169			
8	512				

Hence, $\nabla^4 f(8) = 0$.

Example 11. *Construct the backward difference table from the data:*
sin 30º = 0.5, sin 35º = 0.5736, sin 40º = 0.6428, sin 45º = 0.7071
Assuming third difference to be constant, find the value of sin 25°.
Sol. Backward difference table for given data is as:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
25	0.4225			
		0.0775		
30	0.5000		–0.0039	
		0.0736		–0.0005
35	0.5736		–0.0044	
		0.0692		–0.0005
40	0.6428		–0.0049	
		0.0643		
45	0.7071			

Since third differences are constant therefore

$$\nabla^3 y_{40} = -\ 0.0005$$

$$\Rightarrow \qquad \nabla^2 y_{40} - \nabla^2 y_{35} = 0.0005$$

$$\Rightarrow \qquad -0.0044 - \nabla^2 y_{35} = -0.0005$$

$\Rightarrow \quad \nabla^2 y_{35} = -0.0039$

Again, $\quad \nabla y_{35} - \nabla y_{30} = -0.0039$

$\Rightarrow \quad 0.0736 - \nabla y_{30} = -0.0039$

$\Rightarrow \quad \nabla y_{30} = 0.0775$

Again, $\quad y_{30} - y_{25} = 0.0775$

$\Rightarrow \quad 0.50 - y_{25} = 0.0775$

$\Rightarrow \quad y_{25} = 0.4225$

Therefore, $\quad \sin 25^\circ = 0.4225$

3.4.3 Central Differences

The central difference operator is denoted by the symbol δ and central differences is given by, $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ or

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}, \text{ or}$$

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1$$

....

$$\delta y_{n-\frac{1}{2}} = y_n - y_{n-1}$$

Central Difference Table:

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

3.4.4 Other Difference Operators

(a) The Operator E: The operator E is called **shift operator** or displacement or translation operator. It shows the operation of increasing the argument value x by its interval of differencing h so that.

$$Ef(x) = f(x+h) \text{ or } Ey_x = y_{x+h}$$

Similarly, $Ef(x+h) = f(x+2h)$

....

In general, $E^n y_x = y_{x+nh}$ or $E^n f(x) = f(x+nh)$

In the same manner, $E^{-1} f(x) = f(x-h)$

Also, $E^{-2} f(x) = f(x-2h)$

$$E^{-n} f(x) = f(x-nh)$$

This is called inverse of shift operator.

(b) Differential Operator D: The differential operator for a function $y = f(x)$ is defined by

$$Df(x) = \frac{d}{dx} f(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) \quad \text{... and so on.}$$

The operator Δ is an analogous to the operator D of differential calculus. In finite differences, we deal with ratio of simultaneous increments of mutually dependent quantities where as in differential calculus, we find the limit of such ratios when the increment tends to 0.

(c) The Unit Operator 1: The unit operator 1 has a property that $1.f(x) = f(x)$. It is also called identity operator.

(d) Averaging Operator μ: The operator μ is a averaging operator and is defined by,

$$\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right]$$

i.e.,

$$\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right]$$

3.4.5 Properties of Operators

1. The operators $\Delta, \nabla, E, \delta, \mu$ and D are all linear operators.

i.e., $\nabla\ (af(x+h) + b\phi(x+h) = [af(x+h) + b\phi(x+b)] - [af(x) + b\phi(x)]$

$$= a[f(x+h) - f(x)] + b[\phi(x+h) - \phi(x)]$$

$$= a\nabla f(x+h) + b\nabla \phi(x+h)$$

Hence, ∇ is a linear operator.

On substituting $a = 1$, $b = 1$, we get

$$\nabla [f(x+h) + \phi(x+h)] = \nabla f(x+h) + \nabla \phi(x+h)$$

Also on substituting $b = 0$, we get

$$\nabla [af(x+h)], = a\nabla f(x+h)$$

2. The operator is distributive over addition.
3. All the operators follows the law of indices. *i.e.,*

$$\Delta^p \Delta^q f(x) = \Delta^{p+q} f(x) = \Delta^q \Delta^p f(x)$$

Also, $\Delta[f(x) + \phi(x)] = \Delta[\phi(x) + f(x)]$

4. E and Δ are not commutative with respect to variables.
5. If $f(x) = 0$, then it does not mean that either $\Delta = 0$ or $f(x) = 0$.
6. Operators E and Δ cannot stand without operands.

3.4.6 Relation between Different Operators

There are few relations defined between these operators. Some of them are:

1. $\nabla = 1 - E^{-1}$ or $E = (1 - \nabla)^{-1}$
2. $\Delta = E - 1$ or $E = 1 + \Delta$
3. $E\nabla = \nabla E = \Delta$
4. $E = e^{hD} = 1 + \Delta$, where D is the differential operator.
5. $\delta = E^{1/2} - E^{-1/2}$
6. $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$
7. $\delta E^{1/2} = \Delta$

Proof:

3. $$(E\nabla) f(x) = E\{\nabla f(x)\} = E\{f(x) - f(x-h)\}$$
$$= Ef(x) - Ef(x-h)$$
$$= f(x+h) - f(x) = \Delta f(x) \quad ...(1)$$

Also, $$(\nabla E) f(x) = \nabla \{Ef(x)\} = \nabla f(x+h)$$
$$= f(x+h) - f(x) = \Delta f(x) \quad ...(2)$$

From (1) and (2), we get $E\nabla = \Delta$ and $\nabla E = \Delta$

$\Rightarrow$ $$E\nabla = \nabla E = \Delta.$$

4. $$Ef(x) = f(x+h)$$
$$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots\ldots\ldots \quad \text{(By using Taylor's theorem)}$$
$$= 1.f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots\ldots\ldots\ldots$$
$$= e^{hD} f(x)$$
$$Ef(x) = e^{hD} f(x) \text{ or } E = e^{hD}$$

Since, $E = 1 + \Delta$, therefore $\Delta = e^{hD} - 1$.

5. $$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}} = E^{1/2} y_x - E^{-1/2} y_x$$
$$= (E^{1/2} - E^{-1/2}) y_x$$

Therefore, $$\delta = E^{1/2} - E^{-1/2}$$

6. $$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right) = \frac{1}{2}(E^{1/2} y_x + E^{-1/2} y_x)$$
$$= \frac{1}{2}(E^{1/2} + E^{-1/2}) y_x$$

Therefore, $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

7. $\delta E^{1/2} y_x = \delta y_{x+h} = y_{x+\frac{h}{2}} - y_x = \Delta y_x$

Therefore, $\delta E^{1/2} = \Delta$.

Example 12. *Show that:*

(a) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = 2 + \Delta$

(b) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2/4}$

Sol. (*a*) Since $1 + \Delta = E$ therefore

$(E^{1/2} + E^{-1/2}) E^{1/2} = E + 1 = 1 + \Delta + 1 = \Delta + 2.$

(*b*) $\frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2/4}$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{1+\frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\left(\frac{E^{1/2} - E^{-1/2}}{2}\right)$$

$$= \frac{1}{2}(2E - 2) = E - 1 = \Delta$$

Example 13. *Prove that (1)* $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ *(2)* $(1 + \Delta)(1 - \nabla) \equiv 1$

Where Δ *and* ∇ *are forward and backward difference operators respectively.*

Sol. (1) $\left(\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}\right) y_x = \left(\frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}\right) y_x$

$$= \left(\frac{E-1}{\left[\frac{E-1}{E}\right]} - \frac{\left[\frac{E-1}{E}\right]}{E-1}\right) y_x = \left(E - \frac{1}{E}\right) y_x = (E - E^{-1}) y_x$$

$$= \{(1 + \Delta) - (1 - \nabla)\} y_x = (\Delta + \nabla) y_x$$

Hence, $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$.

(2) $(1 + \Delta)(1 - \nabla) y_x = (1 + \Delta)[y_x - \nabla y_x]$

$= (1 + \Delta)[y_x - \{y_x - y_{x-h}\}] = (1 + \Delta)[y_{x-h}]$

$= E(y_{x-h}) = EE^{-1} y_x = 1. y_x$ (the interval of differencing being 1)

Hence, $(1 + \Delta)(1 - \nabla) \equiv 1$.

Example 14. *Evaluate the following:*

I. $\Delta^2 (\cos 2x)$

II. $\Delta^2 (3e^x)$

III. $\Delta \tan^{-1} x$

IV. $\Delta(x + \cos x)$

the interval of differencing being h.

Sol. I. We have
$$\begin{aligned}
\Delta^2 (\cos 2x) &= (E-1)^2 \cos 2x \text{ because } \Delta = E - 1.\\
&= (E^2 - 2E + 1) \cos 2x\\
&= E^2 \cos 2x - 2E \cos 2x + \cos 2x\\
&= \cos(2x + 4h) - 2\cos(2x + 2h) + \cos 2x\\
&= \cos(2x + 4h) - \cos(2x + 2h) - \cos(2x + 2h) + \cos 2x\\
&= 2\sin(2x + 3h)\sin(-h) - 2\sin(2x + h)\sin h\\
&= -2\sin h\,[\sin(2x + 3h) - \sin(2x + h)]\\
&= -2\sin h\,[2\cos(2x + 2h)\sin h]\\
&= -4\sin^2 h \cos(2x + 2h).
\end{aligned}$$

II. We have
$$\begin{aligned}
\Delta(3e^x) = 3(\Delta e^x) &= 3(e^{x+h} - e^x)\\
&= 3e^x(e^h - 1)\\
\therefore \quad \Delta^2(3e^x) &= \Delta(\Delta 3e^x) = \Delta\{3e^x(e^h - 1)\}\\
&= 3(e^h - 1)(\Delta e^x) = 3(e^h - 1)(e^{x+h} - e^x)\\
&= 3(e^h - 1)e^x(e^h - 1) = 3e^x(e^h - 1)^2.
\end{aligned}$$

III. We have
$$\begin{aligned}
\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x &= \tan^{-1} \frac{(x+h) - x}{1 + (x+h)x}\\
&= \tan^{-1}\left[\frac{h}{1 + xh + x^2}\right].
\end{aligned}$$

IV. We have
$$\begin{aligned}
\Delta(x + \cos x) = \Delta x + \Delta \cos x &= \{(x + h) - x\} + \{\cos(x + h) - \cos x\}\\
&= h + 2\sin\frac{2x + h}{2}\sin\left(-\frac{h}{2}\right)\\
&= h - 2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}.
\end{aligned}$$

Example 15. *Evaluate* $\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E \sin(x+h)}$*, where h being the interval of differencing.*

Sol. To evaluate the given problem we use the operator property that is, $\Delta = E - 1$

Now
$$\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)} = \frac{(E-1)^2}{E}\sin(x+h) + \frac{(E-1)^2 \sin(x+h)}{\sin(x+2h)}$$

$$= (E - 2 + E^{-1}) \quad \sin(x + h) + \frac{(E^2 - 2E + 1)\sin(x+h)}{sin(x+2h)}$$

$$= [\sin(x + 2h) - 2\sin(x + h) + \sin x] + \left[\frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}\right]$$

$$= 2\sin(x + h)[\cos h - 1] + \frac{2\sin(x+2h)[\cos h - 1]}{\sin(x+2h)}$$

$$= 2(\cos h - 1)\{\sin(x + h) - 1\}.$$

Example 16. *Show that* $B(m + 1, n) = (-1)^m \Delta^m \left(\frac{1}{n}\right)$ *where m is a positive integer.*

Sol. We know that $\displaystyle\int_0^\infty e^{-nx}\, dx = \frac{1}{n}.$

Therefore, $\displaystyle\Delta^m \int_0^\infty e^{-nx}\, dx = \Delta^m \left(\frac{1}{n}\right)$

or $\displaystyle\int_0^\infty \Delta^m e^{-nx}\, dx = \Delta^m \left(\frac{1}{n}\right),$

where for $\Delta^m e^{-nx}$, n is to be regarded variable and x is to be regarded as constant.

Now,
$$\Delta^m e^{-nx} = \Delta^{m-1}[e^{-(n+1)x} - e^{-nx}]$$
$$= \Delta^{m-1} e^{-nx}(e^{-x} - 1) = (e^{-x} - 1)\,\Delta^{m-1} e^{-nx}$$
$$= (e^{-x} - 1)^2 \;\Delta^{m-2}\; e^{-nx} = \ldots\ldots$$
$$= (e^{-x} - 1)^m\; e^{-nx}$$

Therefore, $\displaystyle\int_0^\infty e^{-nx}\left(e^{-x} - 1\right)^m dx = \Delta^m \left(\frac{1}{n}\right)$

Put $e^{-x} = z$, so that $-e^{-x}\, dx = dz$ or $dx = -(1/z)\, dz$.

Then, $\displaystyle\int_1^0 z_n\, (z-1)^m (-1/z)\, dz = \Delta^m \left(\frac{1}{n}\right)$

or $\displaystyle(-1)^m \int_0^1 z^{n-1}\,(1 - z)^m\, dz = \Delta^m \left(\frac{1}{n}\right)$

or $\displaystyle\int_0^1 z^{n-1}\,(1 - z)^{(m+1)-1}\, dz = (-1)^m\, \Delta^m \left(\frac{1}{n}\right)$

or $\displaystyle B(m + 1, n) = (-1)^m\; \Delta^m \left(\frac{1}{n}\right)$

Example 17. *Show that* $e^x = \left(\frac{\Delta^2}{E}\right) e^x . \frac{Ee^x}{\Delta^2 e^x}$ *; the interval of differencing being h.*

Sol. Let $f(x) = e^x$, then $Ef(x) = f(x + h)$, therefore $Ee^x = e^{x+h}$.

Now, $\Delta f(x) = f(x + h) - f(x)$

$\therefore$ $\Delta e^x = e^{x+h} - e^x = e^x (e^h - 1)$

$\therefore$ $\Delta^2 e^x = \Delta (\Delta e^x) = \Delta\{e^x(e^h - 1)\}$

$\Delta^2 e^x = (e^h - 1) \Delta e^x = (e^h - 1)^2 e^x$

$$\therefore \quad \left(\frac{\Delta^2}{E}\right) e^x = (\Delta^2 \; E^{-1}) e^x = \Delta^2 (E^{-1} e^x) = \Delta^2 (e^{x-h})$$

$$= \Delta^2 (e^x e^{-h}) = e^{-h} \Delta_2 e^x = e^{-h} (e^h - 1)^2 e^x.$$

$$\therefore \quad \left(\frac{\Delta^2}{E}\right) e^x \frac{Ee^x}{\Delta^2 e^x} = e^{-h} (e^h - 1)^2 e^x \frac{e^{x+h}}{(e^h - 1)^2 e^x} = e^{-h} \; e^{x+h} = e^x.$$

Example 18. *Evaluate* $\Delta^2 \left(\frac{5x + 12}{x^2 + 5x + 6}\right)$*; the interval of differencing being unity.*

Sol. We have $\Delta^2 \left(\frac{5x + 12}{x^2 + 5x + 6}\right)$

Therefore,
$$\Delta^2 \left(\frac{5x + 12}{x^2 + 5x + 6}\right) = \Delta^2 \left\{\frac{5x + 12}{(x + 2)(x + 3)}\right\}$$

$$= \Delta^2 \left(\frac{2}{x + 2} + \frac{3}{x + 3}\right) = \Delta \left[\Delta \left(\frac{2}{x + 2}\right) + \Delta \left(\frac{3}{x + 3}\right)\right]$$

$$= \Delta \left[2 \left(\frac{1}{x + 3} - \frac{1}{x + 2}\right) + 3 \left(\frac{1}{x + 4} - \frac{1}{x + 3}\right)\right]$$

$$= -2 \Delta \left\{\frac{1}{(x + 2)(x + 3)}\right\} - 3\Delta \left\{\frac{1}{(x + 3)(x + 4)}\right\}$$

$$= -2 \left[\frac{1}{(x + 3)(x + 4)} - \frac{1}{(x + 2)(x + 3)}\right]$$

$$- 3 \left[\frac{1}{(x + 4)(x + 5)} - \frac{1}{(x + 3)(x + 4)}\right]$$

$$= \frac{4}{(x + 2)(x + 3)(x + 4)} + \frac{6}{(x + 3)(x + 4)(x + 5)}$$

$$= \frac{2(5x + 16)}{(x + 2)(x + 3)(x + 4)(x + 5)}.$$

Example 19. *Evaluate $\Delta^n e^{ax+b}$; where the interval of differencing taken to be unity?*

Sol. Given $\Delta^n e^{ax+b}$; which shows that $f(x) = e^{ax+b}$.

Now $\Delta f(x) = f(x+1) - f(x)$

$\therefore \quad \Delta(e^{a+bx}) = e^{a(x+1)+b} - e^{ax+b} = e^{ax+b}(e^a - 1)$

$\therefore \quad \Delta^2(e^{a+bx}) = \Delta(\Delta e^{a+bx}) = \Delta\{e^{ax+b}(e^a - 1)\}$

$$= (e^a - 1)(\Delta e^{ax+b})$$

$$= (e^a - 1)\, e^{ax+b}(e^a - 1)$$

$$= (e^a - 1)^2 e^{ax+b}.$$

Proceeding in the same way, we get

$$\Delta^n e^{ax+b} = (e^a - 1)^n e^{ax+b}$$

Example 20. *With usual notations, prove that,*

$$\Delta^n\left(\frac{1}{x}\right) = (-1)^n.\ \frac{n!h^n}{x(x+h)..........(x+nh)}$$

Sol.

$$\Delta^n\left(\frac{1}{x}\right) = \Delta^{n-1}\Delta\left(\frac{1}{x}\right) = \Delta^{n-1}\left[\frac{1}{x+h} - \frac{1}{x}\right] = \Delta^{n-1}\left\{\frac{-h}{x(x+h)}\right\}$$

$$= (-h)\Delta^{n-2}\Delta\left[\frac{1}{x(x+h)}\right] = (-1)\Delta^{n-2}\left[\Delta\left(\frac{1}{x} - \frac{1}{x+h}\right)\right]$$

$$= (-1)\Delta^{n-2}\left[\left(\frac{1}{x+h} - \frac{1}{x}\right) - \left(\frac{1}{x+2h} - \frac{1}{x+h}\right)\right]$$

$$= (-1)\Delta^{n-2}\left[\frac{2}{x+h} - \frac{1}{x} - \frac{1}{x+2h}\right] = (-1)\Delta^{n-2}\left[\frac{-2h^2}{x(x+h)(x+2h)}\right]$$

$$= -(1)^2\Delta^{n-2}\left[\frac{2!h^2}{x(x+h)(x+2h)}\right]$$

$$= (-1)^3\Delta^{n-3}\left[\frac{3!h^3}{x(x+h)(x+2h)(x+3h)}\right]$$

.
.
.
.
.

$$= (-1)^n \frac{n!h^n}{x(x+h)....(x+nh)}$$

Example 21. *Prove that:*

(a) $$\mu\left[\frac{f(x)}{g(x)}\right] = \frac{\mu f(x)\mu g(x) - \frac{1}{4}\delta f(x)\delta g(x)}{g(x-\frac{1}{2})g(x+\frac{1}{2})}$$

Here, interval of differencing being unity.

(b) $\nabla^2 = h^2 D^2 - h^3 D^3 + \frac{7}{12} h^4 D^4 - \ldots\ldots$

(c) $\nabla - \Delta = -\nabla \Delta$

(d) $1 + \left(\frac{\delta^2}{2}\right) = \sqrt{1+\delta^2\mu^2}$

(e) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Sol.

(a) Here R.H.S. is $= \dfrac{\mu f(x)\mu g(x) - \frac{1}{4}\delta f(x)\delta g(x)}{g(x-\frac{1}{2})g(x+\frac{1}{2})}$

Now numerator of R.H.S. is given by:

$= \frac{1}{2}[E^{1/2} + E^{-1/2}] f(x) \cdot \frac{1}{2}(E^{1/2} + E^{-1/2}) g(x) - \frac{1}{4}(E^{1/2} - E^{-1/2}) f(x) (E^{1/2} - E^{-1/2}) g(x)$

$= \frac{1}{4}[f(x + \frac{1}{2}) + f(x - \frac{1}{2})][g(x + \frac{1}{2}) + g(x - \frac{1}{2})] - \frac{1}{4}[f(x + \frac{1}{2}) - f(x - \frac{1}{2})][g(x + \frac{1}{2}) - g(x - \frac{1}{2})]$

$= \frac{1}{4}[f(x + \frac{1}{2})g(x + \frac{1}{2}) + f(x + \frac{1}{2})g(x - \frac{1}{2}) + f(x - \frac{1}{2})g(x + \frac{1}{2}) f(x - \frac{1}{2})g(x - \frac{1}{2})$
$- \frac{1}{4}[f(x + \frac{1}{2})g(x + \frac{1}{2}) - f(x + \frac{1}{2})g(x - \frac{1}{2}) - f(x - \frac{1}{2})g(x + \frac{1}{2}) + f(x - \frac{1}{2}) f(x - \frac{1}{2})]$

$= \frac{1}{2}[f(x + \frac{1}{2}) g(x - \frac{1}{2}) + f(x - \frac{1}{2}) g(x + \frac{1}{2})]$

Therefore right hand side is $= \dfrac{\frac{1}{2}\left[f\left(x+\frac{1}{2}\right)g\left(x-\frac{1}{2}\right)+f\left(x-\frac{1}{2}\right)g\left(x+\frac{1}{2}\right)\right]}{g\left(x-\frac{1}{2}\right)g\left(x+\frac{1}{2}\right)}$

$$= \frac{1}{2}\left[\frac{f\left(x+\frac{1}{2}\right)}{g\left(x+\frac{1}{2}\right)} + \frac{f\left(x-\frac{1}{2}\right)}{g\left(x-\frac{1}{2}\right)}\right] = \frac{E^{1/2}+E^{-1/2}}{2}\left[\frac{f(x)}{g(x)}\right] = \mu\left[\frac{f(x)}{g(x)}\right]$$

(b) We know $E = e^{hD}$ and $\nabla = 1 - E^{-1}$, therefore $\nabla^2 = (1 - e^{-hD})^2$.

$$= \left[1 - \left\{1 - hD + \frac{(hD)^2}{2!} - \frac{(hD)^3}{3!} + \frac{(hD)^4}{4!} - \ldots\ldots\right\}\right]^2$$

$$= \left\{hD - \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} - \frac{(hD)^4}{4!} + \ldots\ldots\right\}^2$$

$$= h^2D^2 \left[1 - \left\{\frac{hD}{2} - \frac{(hD)^2}{6} + \text{..........}\right\}\right]^2$$

$$= h^2D^2 \left[1 + \left\{\frac{hD}{2} - \frac{(hD)^2}{6} + \text{..........}\right\}\right]^2 - 2\left\{\frac{hD}{2} - \frac{(hD)^2}{6} + \text{..........}\right\}\Bigg]$$

$$= h^2D^2 \left[1 - hD + \left(\frac{1}{4} + \frac{1}{3}\right)(hD)^2 - \text{..........}\right]$$

$$= h^2D^2 \left(1 - hD + \frac{7}{12} h^2D^2 - \text{.........}\right) = h^2D^2 - h^3D^3 + \frac{7}{12} h^4D^4 - \text{..........}$$

(*c*) $\nabla - \Delta = (1 - E^{-1}) - (E - 1) = \left(\frac{E-1}{E}\right) - (E - 1) = (E - 1)(E^{-1} - 1)$

$= -(E - 1)(1 - E^{-1}) = -\nabla\Delta$

(*d*) L.H.S. $= \left\{1 + \left(\frac{\delta^2}{2}\right)\right\} y_x = \left\{1 + \frac{(E^{1/2} - E^{-1/2})^2}{2}\right\} y_x$

$$= \left\{1 + \left(\frac{E + E^{-1} - 2}{2}\right)\right\} y_x = \frac{1}{2}(E + E^{-1}) y_x.$$

R.H.S. $= \left(\sqrt{1 + \delta^2 \mu^2}\right) y_x$

$$= \left[1 + \left\{(E^{1/2} - E^{-1/2})^2 \cdot \frac{1}{4}(E^{1/2} + E^{-1/2})^2\right\}\right]^{1/2} y_x$$

$$= \left\{1 + \left(\frac{(E - E^{-1})^2}{4}\right)\right\}^{1/2} y_x = \left(\frac{E^2 + E^{-2} + 2}{4}\right)^{1/2} y_x = \left(\frac{E + E^{-1}}{2}\right) y_x$$

Hence, L.H.S. = R.H.S.

(*e*) $\mu\delta y_x = \mu(E^{1/2} - E^{-1/2}) y_x$

$$= \mu\left(y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}\right) = \mu\left(y_{x+\frac{h}{2}}\right) - \mu\left(y_{x-\frac{h}{2}}\right)$$

$$= \frac{1}{2}(E^{1/2} + E^{-1/2})\left(y_{x+\frac{h}{2}}\right) - \frac{1}{2}(E^{1/2} + E^{-1/2})\left(y_{x-\frac{h}{2}}\right)$$

$$= \frac{1}{2}(y_{x+h} + y_x) - \frac{1}{2}(y_x + y_{x-h}) = \frac{1}{2}(y_{x+h} - y_x) + \frac{1}{2}(y_x - y_{x-h})$$

$$= \frac{1}{2}(\Delta y_x) + \frac{1}{2}(\nabla y_x) = \frac{1}{2}(\Delta + \nabla) y_x$$

Hence, $\mu\delta = \frac{1}{2}(\Delta + \nabla)$.

Example 22. *Evaluate:* $\Delta^n [\sin (ax + b)]$

Sol. We know $\Delta f(x) = f(x + h) - f(x)$ therefore

$$\Delta \sin (ax + b) = \sin [a (x + h)+b] - \sin)(ax + b)$$

$$= 2 \sin \frac{ah}{2} \cos \left[a\left(x + \frac{h}{2}\right) + b\right] = 2 \sin \frac{ah}{2} \sin \left(ax + b + \frac{ah + \pi}{2}\right)$$

Therefore, $$\Delta^2 \sin (ax + b) = \Delta \left[2 \sin \frac{ah}{2} \sin \left(ax + b + \frac{ah + \pi}{2}\right)\right]$$

$$= (2 \sin \frac{ah}{2}) (2 \sin \frac{ah}{2}) \sin [ax + b + \frac{ah + \pi}{2} + \frac{ah + \pi}{2}]$$

$$= \left(2 \sin \frac{ah}{2}\right)^2 \sin \left[ax + b + 2\left(\frac{ah + \pi}{2}\right)\right]$$

On continuing in the same manner, we get

$$\Delta^3 \sin (ax + b) = \left(2 \sin \frac{ah}{2}\right)^3 \sin \left[ax + b + \left(\frac{3(ah + \pi)}{2}\right)\right]$$

....

....

$$\Delta^n \sin (ax + b) = \left(2 \sin \frac{ah}{2}\right)^n \sin \left[ax + b + \left(\frac{n(ah + \pi)}{2}\right)\right]$$

Example 23. *Show that:* $u_0 - u_1 + u_2 - .... = \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 - ....$

Sol. On taking left hand side

$$= u_0 - u_1 + u_2 - u_3 + ...$$

$$= u_0 - Eu_0 + E^2 u_0 - E^3 u_0 + ...$$

$$= (1 - E + E^2 - E^3 + ...) u_0$$

$$= \left[\frac{1}{1 - (-E)}\right] u_0 = \left(\frac{1}{1 + E}\right) u_0$$

$$= \left(\frac{1}{1 + 1 + \Delta}\right) u_0 = \left(\frac{1}{2 + \Delta}\right) u_0$$

$$= \frac{1}{2} \left(1 + \frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \frac{\Delta^3}{8} + ...\right] u_0$$

$$= \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 - \frac{1}{8} \Delta^2 u_0 - \frac{1}{16} \Delta^3 u_0 + ... \quad \text{(R.H.S.)}$$

Example 24. *Prove that:*

(1) $\delta[f(x)\,g(x)] = \mu f(x)\,\delta g(x) + \mu g(x)\,\delta f(x)$

(2) $\delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\mu g(x)\,\delta f(x) - \mu f(x)\,\delta g(x)}{g\left(x-\frac{1}{2}\right)g\left(x+\frac{1}{2}\right)}$

Interval of differencing is unity.

Sol.

(1) R.H.S. $= \mu f(x)\ \delta g(x) + \mu g(x)\ \delta f(x)$

$= \dfrac{E^{1/2}+E^{-1/2}}{2}\ f(x).\ (E^{1/2} - E^{-1/2})\ g(x) + \dfrac{E^{1/2}+E^{-1/2}}{2}\ g(x).\ (E^{1/2} - E^{-1/2})\ f(x)$

$= \frac{1}{2}\ [\{f(x + \frac{1}{2}) + f(x - \frac{1}{2})\}\ \{g\ (x + \frac{1}{2}) - g\ (x - \frac{1}{2})\} + \{g\ (x + \frac{1}{2})$

$+ g\ (x - \frac{1}{2})\}\ \{f\ (x + \frac{1}{2}) - f\ (x - \frac{1}{2})\}]$

$= \frac{1}{2}\ [\{f(x + \frac{1}{2})\ g\ (x + \frac{1}{2}) - f\ (x + \frac{1}{2})\ g\ (x - \frac{1}{2}) + f(x - \frac{1}{2})\ \ g\ (x + \frac{1}{2})$

$- f\ (x - \frac{1}{2})\ g\ (x - \frac{1}{2})\} + \{f(x + \frac{1}{2})\ g\ (x + \frac{1}{2}) + f(x + \frac{1}{2})\ \ g\ (x - \frac{1}{2})$

$- f\ (x - \frac{1}{2})\ \ g\ (x + \frac{1}{2})\ \ - f\ (x - \frac{1}{2})\ g\ (x - \frac{1}{2})\}]$

$= \frac{1}{4}\ f(x + \frac{1}{2})\ g(x + \frac{1}{2})\ \ - f\ (x - \frac{1}{2})\ \ g\ (x - \frac{1}{2})$

$= E^{1/2}\ f\ (x)\ g(x) - E^{-1/2} f(x)\ g(x) = (E^{1/2} - E^{-1/2})\ f\ (x)\ g(x) = \delta f\ (x)\ g(x).$

(2) R.H.S. $= \dfrac{\mu g(x)\,\delta f(x) - \mu f(x)\,\delta g(x)}{g\left(x-\frac{1}{2}\right)g\left(x+\frac{1}{2}\right)}$

Now first we solve the numerator of right hand side.

$= \dfrac{E^{1/2}+E^{-1/2}}{2}\ g(x)\ (E^{1/2} - E^{-1/2})\ f(x) - \dfrac{E^{1/2}+E^{-1/2}}{2}\ f(x)\ (E^{1/2} - E^{-1/2})\ g(x)$

$= \frac{1}{2}\ [\{g(x + \frac{1}{2}) + g(x - \frac{1}{2})\}\ \ \{f(x + \frac{1}{2}) - f(x - \frac{1}{2})\} - \{f(x + \frac{1}{2})$

$+ f(x - \frac{1}{2})\}\ \{g(x + \frac{1}{2}) - g(x - \frac{1}{2})\}]$

$= \frac{1}{2}[f(x+\frac{1}{2})g(x+\frac{1}{2}) + f(x+\frac{1}{2})g(x-\frac{1}{2}) - f(x-\frac{1}{2})g(x+\frac{1}{2}) - f\ (x-\frac{1}{2})\ g(x-\frac{1}{2})]$

$- \frac{1}{2}[f(x+\frac{1}{2})g(x+\frac{1}{2}) - f(x+\frac{1}{2})g(x-\frac{1}{2}) + f(x-\frac{1}{2})g(x+\frac{1}{2}) - f\ (x-\frac{1}{2})\ g(x-\frac{1}{2})]$

$= f\ (x + \frac{1}{2})\ g(x - \frac{1}{2})\ \ - f(x - \frac{1}{2})\ g(x + \frac{1}{2})$

Therefore right hand side as

$$= \frac{f(x+\frac{1}{2})\,g\,(x-\frac{1}{2}) - f(x-\frac{1}{2})\,g\,(x+\frac{1}{2})}{g(x-\frac{1}{2})\,g\,(x+\frac{1}{2})}$$

$$= \frac{f(x+\frac{1}{2})}{g(x+\frac{1}{2})} - \frac{f(x-\frac{1}{2})}{g(x-\frac{1}{2})}$$

$$= E^{1/2}\left[\frac{f(x)}{g(x)}\right] - E^{-1/2}\left[\frac{f(x)}{g(x)}\right] = (E^{1/2} - E^{-1/2})\left[\frac{f(x)}{g(x)}\right]$$

$$= \delta\left[\frac{f(x)}{g(x)}\right].$$

Example 25. *Evaluate: (a)* $\Delta\frac{2^x}{(x+1)!}$; *differencing 1. (b)* $\Delta(e^{ax}\log bx)$.

Sol.

(*a*) Let $f(x) = 2^x$, $g(x) = (x+1)!$, therefore

$\Delta f(x) = 2^{x+1} - 2^x = 2^x$ and $\Delta g(x) = (x+1+1)! - (x+1)! = (x+1)\,(x+1)!$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

$$= \frac{(x+1)!.2^x - 2^x.(x+1)(x+1)!}{(x+1+1)!(x+1)!} \quad \text{(Because } h = 1\text{)}$$

$$= \frac{2^x(x+1)\,(!(1-x-1)}{(x+2)\,!\,(x+1)!} = -\frac{x}{(x+2)!}2^x.$$

(*b*) Again let, $f(x) = e^{ax}$, $g(x) = \log bx$, therefore

$$\Delta f(x) = e^{a(x+h)} - e^{ax} = e^{ax}(e^{ah} - 1)$$

$$\Delta g(x) = \log b(x+h) - \log bx = \log(1+\frac{h}{x})$$

We know that $\Delta f(x)g(x) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

Therefore $$\Delta(e^{ax}\log bx) = e^{a(x+h))}\log\left(1+\frac{h}{x}\right) + (\log bx)e^{ax}(e^{ah} - 1)$$

$$\Delta(e^{ax}\log bx) = e^{ax}[e^{ah}\log\left(1+\frac{h}{x}\right) + (e^{ah}-1)\log bx].$$

Example 26. *Prove that, $hD = -\log(1-\nabla) = \sin h^{-1}$* $(\mu\delta)$.

Sol. Because, $E^{-1} = 1 - \nabla$ therefore,

$$hD = \log E = -\log(E^{-1}) = -\log(1-\nabla)$$

Also, $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\delta = E^{1/2} - E^{-1/2}$

Therefore, $\mu\delta = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sin h(hD)$

$hD = \sin h^{-1} (\mu\delta).$

Example 27. *Show that* $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$

Sol. L.H.S. $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{Ef(x)}{f(x)}\right]$$

$$= \log\left[\frac{(1+\Delta)f(x)}{f(x)}\right]$$

$$= \log\left[\frac{f(x) + \nabla f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Example 28. *Evaluate (1)* $\Delta^n\left(\frac{1}{x}\right)$ *(2)* Δ^n (ab^{cx}).

Sol.

(1) We have, $\Delta^n\left(\frac{1}{x}\right) = \Delta n^{-1}\, \Delta\left(\frac{1}{x}\right).$

Now, $\Delta\left(\frac{1}{x}\right) = \frac{1}{x+1} - \frac{1}{x} = \frac{x-(x+1)}{x(x+1)} = \frac{(-1)}{x(x+1)}$

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\,\Delta\left(\frac{1}{x}\right) = \Delta\left\{\frac{(-1)}{x(x+1)}\right\} = (-1)\ \Delta\left\{\frac{1}{x(x+1)}\right\}$$

$$= (-1)\ \left\{\frac{1}{(x+1)(x+2)} - \frac{1}{x(x+1)}\right\}$$

$$= (-1)\ \frac{x-(x+2)}{x(x+1)(x+2)} = \frac{(-1)(-2)}{x(x+1)(x+2)}$$

$$\Delta^3\left(\frac{1}{x}\right) = \frac{(-1)(-2)(-3)}{x(x+1)(x+2)(x+3)}$$

Similarly, $\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)(-2)(-3)........(-n)}{x(x+1)(x+2).......(x+n)}$

$$= \frac{(-1)^n n!}{x(x+1)(x+2)...(x+n)}$$

(2) Similarly, $\Delta\,(ab^{cx}) = a\,\Delta\,b^{cx} = a\{b^{c(x+1)} - b^{cx}\}$

$$= a\{b^{cx}\,b^c - b^{cx}\} = a(b^c-1)b^{cx}.$$

$$\Delta^2(ab^{cx}) = \Delta\Delta(ab^{cx}) = \Delta\{a(b^c-1)b^{cx}\}$$

$$= a(b^c-1)\Delta b^{cx} = a(b^c-1)^2 b^{cx}$$

Proceeding in the same manner, we get

$$\Delta^n(ab^{cx}) = a(b^c - 1)^n b^{cx}.$$

Example 29. *If p, q, r and s be the successive entries corresponding to equidistant arguments in a table, show that when third differences are taken into account, the entry corresponding to the argument half way between the arguments of q and r is* $A+\frac{1}{24}B$*, where A is the arithmetic mean of q, r and B is the arithmetic mean of 3q – 2p – s and 3r – 2s – p.*

Sol. On taking h being the interval of differencing the difference table is as:

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$
a	p			
		$q-p$		
$a+h$	q		$r-2q+p$	
		$r-q$		$s-3r+3q-p$
$a+2h$	r		$s-2r+q$	
		$s-r$		
$a+3h$	s			

The argument half way between the arguments of q and r is $\frac{1}{2}$ $(a + h + a + 2h)$ *i.e.*, $a+\frac{3}{2}h$.

Hence, the required entry is given by,

$$u_{a+(3/2)h} = E^{3/2}u_a = (1 + \Delta)^{3/2}\, u_a$$

$$= \left[1+\frac{3}{2}\Delta+\frac{3}{2}.\frac{1}{2}.\frac{1}{2!}\Delta^2+\frac{3}{2}.\frac{1}{2}.\left(-\frac{1}{2}\right)\frac{1}{3!}\Delta^3\right]u_a,$$

(Higher order differences being neglected).

Therefore $u_{a+(3/2)h} = u_a+\frac{3}{2}\Delta u_a+\frac{3}{8}\Delta^2 u_a-\frac{1}{16}\Delta^3 u_a$

$$= p+\frac{3}{2}(q-p)+\frac{3}{8}(r-2q+p)-\frac{1}{16}(s-3r+3q-p)$$

$$= p\left(1-\frac{3}{2}+\frac{3}{8}+\frac{1}{16}\right)+q\left(\frac{3}{2}-\frac{3}{4}-\frac{3}{16}\right)+r\left(\frac{3}{8}+\frac{3}{16}\right)-\frac{1}{16}s$$

$$= -\frac{1}{16}p+\frac{9}{16}q+\frac{9}{16}r-\frac{1}{16}s$$

$$= -\frac{1}{16}p+(q+r)\left(\frac{1}{16}+\frac{1}{2}\right)-\frac{1}{16}s$$

$$= \frac{1}{2}(q+r)+\frac{1}{16}(q+r-p-s) \qquad ...(1)$$

Again A = arithmetic mean of q and r = $\frac{1}{2}(q+r)$

B = Arithmetic mean of $3q - 2p - s$ and $3r - 2s - p$ is

$$= \frac{1}{2}[3q-2p-s+3r-2s-p] = \frac{3}{2}(q+r-s-p).$$

$$\therefore \qquad A+\frac{1}{24}B = \frac{q+r}{2}+\frac{1}{16}(q+r-s-p).$$

Substituting this value in (1), we get $u_{a+(3/2)h} = A + \frac{1}{24}B.$

Example 30. *Given u_0, u_1, u_2, u_3, u_4 and u_5. Assuming that, fifth order differences to be constant. Show that:* $u_{2\frac{1}{2}} = \frac{1}{2}c+\frac{25(c-b)+3(a-c)}{256}$. *where* $a = u_0 + u_5$, $b = u_1 + u_4$, $c = u_2 + u_3$

Sol. L.H.S. $\quad u_{2\frac{1}{2}} = E^{5/2}u_0 = (1 + \Delta)^{5/2}u_0$

$$= \left[1+\frac{5}{2}\Delta+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\Delta^2+.......+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\left(\frac{5}{2}-3\right)\left(\frac{5}{2}-4\right)}{5!}\Delta^5\right]u_0$$

$$= u_0+\frac{5}{2}\Delta u_0+\frac{15}{8}\Delta^2u_0+\frac{5}{16}\Delta^3u_0-\frac{5}{128}\Delta^4u_0+\frac{3}{256}\Delta^5u_0$$

$$= u_0+\frac{5}{2}(u_1-u_0)+\frac{15}{8}(u_2-2u_1+u_0)+\frac{5}{16}(u_3-3u_2+3u_1-u_0)+........$$

$$+\frac{3}{256}(u_5-5u_4+10u_3-10u_2+5u_1-u_0)$$

$$= \frac{3}{256}(u_0 + u_5) - \frac{25}{256}(u_1 + u_4) + \frac{75}{128}(u_2 + u_3)$$

$$= \frac{3}{256}a - \frac{25}{256}b + \frac{75}{128}c$$

$$= \frac{3}{256}a - \frac{25}{256}b + \left(\frac{1}{2} + \frac{11}{128}\right)c$$

$$= \frac{c}{2} + \frac{3(a-c)+25(c-b)}{256} \quad \text{(R.H.S.)}$$

Example 31. *Given:*

$u_0 + u_8 = 1.9243$, $u_1 + u_7 = 1.9590$, $u_2 + u_6 = 1.9823$, $u_3 + u_5 = 1.9956$. Find u_4.

Sol. Since 8 entries are given, therefore we have $\Delta^8 u_0 = 0$

i.e. $(E - 1)^8 u_0 = 0$

i.e. $(E^8 - {}^8C_1E^7 + {}^8C_2E^6 - {}^8C_3E^5 + {}^8C_4E^4 - {}^8C_5E^3 + {}^8C_6E^2 - {}^8C_7E^1 + 1)u_0 = 0$

i.e. $(E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1)u_0 = 0$

i.e. $u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$

i.e. $(u_8 + u_0) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0$

On putting the given values, we get

$1.9243 - 8(1.9590) + 28(1.9823) - 56\ (1.9956) + 70u_4 = 0$

or $\quad -69.9969 + 70u_4 = 0$

or $\quad u_4 = 0.9999557.$

Example 32. *Sum the following series $1^3 + 2^3 + 3^3 + ... + n^3$ using the calculus of finite differences.*

Sol. Let $1^3 = u_0$, $2^3 = u_1$, $3^3 = u_2$, ..., $u^3 = u_{n-1}$. Therefore sum is given by

$$S = u_0 + u_1 + u_2 + ... + u_{n-1}$$

$$= (1 + E + E^2 + E^3 + ... + E^{n-1})u_0$$

$$= \left(\frac{E^n - 1}{E - 1}\right) u_0 = \left[\frac{(1+\Delta)^n - 1}{\Delta}\right] u_0$$

$$= \frac{1}{\Delta}\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + ... + \Delta^n - 1\right] u_0$$

$$= n + \frac{n(n-1)}{2!}\Delta u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + ...$$

We know $\quad \Delta u_0 = u_1 - u_0 = 2^3 - 1^3 = 7.$

$\Delta^2 u_0 = u_2 - 2u_1 + u_0 = 3^3 - 2(2)^3 + 1^3 = 12.$

Similarly we have obtained $\Delta^3 u_0 = 6$ and $\Delta^4 u_0$, $\Delta^5 u_0$,... are all zero as $u_r = r^3$ is a polynomial of third degree.

$$\therefore\ S = n + \frac{n(n-1)}{2!}(7) + \frac{n(n-1)(n-2)}{6}\ 12 + \frac{n(n-1)(n-2)(n-3)}{24}(6)$$

$$= \frac{n^2}{4}\ (n^2 + 2n + 1) = \left[\frac{n(n+1)}{2}\right]^2$$

Example 33. *Prove that:* $\sum_{x=0}^{\infty} u_{2x} = \frac{1}{2}\sum_{x=0}^{\infty} u_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \ldots\right)u_0$.

Sol. Taking right hand side of the given expression

$$= \frac{1}{2}\sum_{x=0}^{\infty} u_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \ldots\ldots\right)u_0$$

$$= \frac{1}{2}\ (u_0 + u_1 + u_2 + u_3 + \ldots\ldots) + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\ (u_0 + Eu_0 ++ E^2u_0 + E^3u_0 + \ldots\ldots) + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\ (1 + E + E^2 + E^3 + \ldots\ldots)u_0 + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\ (1 - E)^{-1}u_0 + \frac{1}{2}\ (2 + \Delta)^{-1}\ u_0$$

$$= \frac{1}{2}\ (1 - E)^{-1}u_0 + \frac{1}{2}\ (1 + \Delta)^{-1}\ u_0$$

$$= \frac{1}{2}\ [(1 - E)^{-1} + (1 + E)^{-1}]u_0$$

$$= \frac{1}{2}\ .\ 2\ [1 + E^2 + E^4 + E^6 + \ldots\ldots]u_0$$

$$= u_0 + u_2 + u_4 + u_6 + \ldots\ldots$$

$$= \sum_{x=0}^{\infty} u_{2x} = \text{L.H.S.}$$

Example 34. *Given that* $u_0 = 3$, $u_1 = 12$, $u_2 = 81$, $u_3 = 200$, $u_4 = 100$, $u_5 = 8$. *Find the value of* $\Delta^5 u_0$.

Sol. We know $\Delta = E - 1$, therefore,

$$\begin{aligned} \Delta^5 u_0 &= (E - 1)^5 u_0 \\ &= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 \\ &= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 \\ &= 8 - 500 + 2000 - 810 + 60 - 3 \\ &= 755. \end{aligned}$$

PROBLEM SET 3.1

1. Form the forward difference table for given set of data:

X:	10	20	30	40
Y:	1.1	2.0	4.4	7.9

2. Construct the difference table for the given data and hence evaluate $\Delta^3 f(2)$.

X:	0	1	2	3	4
Y:	1.0	1.5	2.2	3.1	4.6

[**Ans.** 0.4]

3. Find the value of E^2x^2 when the values of x vary by a constant increment of 2. [**Ans.** $x^2 + 8x + 16$]

4. Evaluate $E^n e^x$ when interval of differencing is [**Ans.** $E^n e^x = e^{x+nh}$]

5. Evaluate $\Delta^3(1 - x)(1-2x)(1-3x)$; the interval of differencing being unity. [**Ans.** $\Delta^3 f(x) = -36$]

6. If $f(x) = \exp(ax)$, evaluate $\Delta^n f(x)$ [**Ans.** $\Delta^n e^{ax} = (e^{ah} - 1)^n e^{ax}$]

7. Evaluate $\left(\frac{\Delta^2}{E}\right)x^3$ [**Ans.** $6x$]

8. Find the value of $\Delta^2\left[\frac{a^{2x} + a^{4x}}{(a^2-1)^2}\right]$ [**Ans.** $a^{2x} + (a^2+1)^2 a^{4x}$]

9. Evaluate:

(*a*) $\Delta \cot 2^x$ [**Ans.** $-\text{Cosec } 2^{x+1}$]

(*b*) $\Delta \sin h\ (a + bx)$ [**Ans.** $2 \sin h \frac{b}{2} \cos h(a + \frac{b}{2} + bx)$]

(*c*) $\Delta \tan ax$ [**Ans.** $\frac{\sin a}{\cos ax \cos a(x+1)}$]

10. Prove that:

(*a*) $E^{1/2} = \mu = \frac{1}{2}\delta$

(*b*) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$

(*c*) $\delta = \nabla(1-\nabla)^{-1/2} = \Delta(1+\Delta)^{-1/2}$

(*d*) $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$

(*e*) $\nabla\Delta = \Delta\nabla = \delta^2$

(*f*) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

11. Show that:

(*a*) $\Delta \cot(a + bx) = \frac{-\sin b}{\sin(a+bx)\sin(a+b+bx)}$

(*b*) $\Delta^n \sin(a+bx) = (2\sin\frac{b}{2})^n \sin(a + bx + \frac{n((b+\pi)}{2}$

(*c*) $\Delta^n \cos(a+bx) = (2\sin\frac{b}{2})^n \cos(a + bx + \frac{n(b+\pi)}{2}$

12. What is the difference between $\left(\frac{\Delta}{E}\right)^2 u_x$ and $\left(\frac{\Delta^2 u_x}{E^2 u_x}\right)$. If $u_x = x^3$ and the interval of differencing is unity. Find out the expression for both. [**Ans.** $6h^2(3x - h)$, $\frac{6xh^2 + 6h^3}{(x+2h)^3}$]

13. If $f(x) = e^{ax}$, show that $f(0)$ and its leading differences form a geometrical progression.

14. A third degree polynomial passes through the points (0, –1), (1, 1), (2, 1) and (3, 2). Find the polynomial. [**Ans.** $-\frac{1}{6}(x^3 + 3x^2 - 16x + 6)$]

15. Prove that $\Delta \sin^{-1} x = [(x+1)\sqrt{1-x^2} - x\sqrt{1-(x+1)^2}]$.

3.5 FUNDAMENTAL THEOREM ON DIFFERENCES OF POLYNOMIAL

Statement: If $f(x)$ be the nth degree polynomial in x, then the n^{th} difference of $f(x)$ is constant and $\Delta^{n+1}f(x)$ and all higher differences are zero when the values of the independent variables are at equal interval.

Proof: Consider the polynomial $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ ...(1)

Where n is a positive integer and $a_0, a_1, a_2, ...a_n$ are constants.

We know $\Delta f(x) = f(x + h) - f(x)$.

On applying the operator Δ on equation (1), we get

$$\Delta f(x) = \Delta(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$$

$$\Rightarrow f(x + h) - f(x) = [a_0 + a_1(x + h) + a_2(x + h)^2 + ... + a_n(x + h)^n] - [a_0 + a_1x + a_2x^2 + ... + a_nx^n]$$

$$\Rightarrow a_1h + a_2[(x+h)^2 - x^2] + a_3[(x+h)^3 - x^3] + ... + a_n[(x+h)^n - x^n]$$

$$\Rightarrow a_1b + a_2[{}^2C_1xh + h^2] + a_3[{}^3C_1x^2h + {}^3C_2xh^2 + h^3] + ... + a_n[{}^nC_1x^{n-1}h + {}^nC_2x^{n-1}h^2 + ...{}^nC_nh^n]$$

$$\Rightarrow b_1 + b_2x + b_3x^2 + ... + b_{n-1}x^{n-2} + na_nhx^{n-1} \quad ...(2)$$

where $b_1, b_2, ...b_{n-1}$ are constant coefficients.

According to equation (2), we have the first difference of equation (1) is again a polynomial of degree $n - 1$.

From this we say that $\Delta f(x)$ is one degree less than the degree of original polynomial.

Again, on taking a difference of equation (2) *i.e.* second difference of equation (1), we get

$$\Delta^2 f(x) = C_2 + C_3x + C_4x^2 + ... + n(n-1)h^2a_nx^{n-2} \quad ...(3)$$

This is a polynomial of degree $n - 2$.

Thus, on continuing this process up to nth difference we get a polynomial of degree zero. Such that:

$$\Delta^n f(x) = n(n-1)(n-2)......1.h^na_nx^{n-n}$$

$$= n!h^na_nx^0$$

$$= n!h^na_n$$

Hence, we have *n*th difference of the polynomial is constant and so all higher differences are each zero. *i.e.*

$$\Delta^{n+1} f(x) = \Delta^{n+2} f(x) = ... = 0$$

3.6 ESTIMATION OF ERROR BY DIFFERENCE TABLE

Let y_0, y_1, y_2, ...y_n be the exact values of a function $y = f(x)$ corresponding to arguments x_0, x_1, x_2 , ..., x_n. Now to determine error in such a case and to correct the functional values, let an error δ is made in entering the value of y_3 in the table so that erroneous value of y_3 is $y_3 + \delta$.

x	y	Δy	$\Delta^2 y$
x_0	y_0		
x_1	y_1	Δy_0	
x_2	y_2	Δy_1	$\Delta^2 y_0$
x_3	$y_3 + \delta$	$\Delta y_2 + \delta$	$\Delta^2 y_1 + \delta$
x_4	y_4	$\Delta y_3 - \delta$	$\Delta^2 y_2 - 2\delta$
x_5	y_5	Δy_4	$\Delta^2 y_3 + \delta$
x_6	y_6	Δy_5	Δy_4

From the above difference table we noted that:

1. The error in column y affects two entries in column Δy, three entries in column $\Delta^2 y$ and so on. *i.e.* the error spreads in triangular form.
2. The error increases with the order of differences.
3. The coefficients of δ's are binomial coefficients with alternative signs +, –, ...
4. In various difference columns of the above table the algebraic sum of the errors is zero.,
5. The errors in the column $\Delta^i y$ are given by the coefficients of the binomial expansion $(1 - \delta)^i$.
6. In even differences columns of $\Delta^2 y$, $\Delta^4 y$, ..., the maximum error occurs in a horizontal line in which incorrct value of y lies.
7. In odd difference columns of $\Delta^1 y$, $\Delta^3 y$,......, the maximum error lies in the two middle terms and the incorrect value of y lies between these two middle terms.

Example 1. *Find the error and correct the wrong figure in the following functional values:*

Sol.

x	1	2	3	4	5	6	7
y	2	5	10	18	26	37	50

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	2			
		3		
2	5		2	
		5		1
3	10		3	
		8		–3
4	18 ⟵		0 ⟶	
		8		3
5	26		3	
		11		–1
6	37		2	
		13		
7	50			

Here the sum of all the third differences is zero and the adjacent values –3, 3 are equal in magnitude. Also horizontal line between –3 and 3 points out the incorrect functional value 18.

Therefore coefficient of first middle term on expansion of $(1 - p)^3 = -3$

$\Rightarrow \quad -3e = -3 \Rightarrow e = 1$

$\therefore$ Correct functional value = 18 – 1 = 17.

Example 2. *Find and correct by means of differences the error in the following table:*
20736, 28561, 38416, 50625, 65540, 83521, 104976, 130321, 160000

Sol. For the given data we form the following difference table:

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20736					
	7825				
28561		2030			
	9855		324		
38416		2354		28	
	12209		352		–20
50625		2706		8	
	14915		360		40
65540		3066		48	
	17981		408		–40
83521		3474		8	
	21455		416		20
104976		3890		28	
	25345		444		
130321		4334			
	29679				
160000					

From this table we have the third differences are quite iregular and the irregularity starts around the horizontal line corresponding to the value $y = 65540$.

Since the algebraic sum of the fifth differences is 0, therefore $-5\varepsilon = -20 \Rightarrow \varepsilon = 4$.

Therefore the true value of $y_5 = 65540 - 4 = 65536$.

Example 3. *Locate the error in following entries and correct it.*

1.203, 1.424, 1.681, 1.992, 2.379, 2.848, 3.429, 4.136

Sol. Difference table for given data is as follows:

$10^3 y$	$10^3 \Delta y$	$10^3 \Delta^2 y$	$10^3 \Delta^3 y$	$10^3 \Delta^4 y$
1203				
	221			
1424		36		
	257		18	
1681		54		4
	311		22	
1992		76		–16
	387		6	
2379 ←		82		→ 24
	469		30	
2848		112		–16
	581		14	
3429		126		
	707			
4136				

Sum of all values in column of fourth difference is –0.004 which is very small as compared to sum of values in other columns.

$\therefore \quad \Delta^4 y = 0$

Errors in this column are e, $-4e$, $6e$, $-4e$ and e.

Term of Maximum value = 24 $\Rightarrow$ $6e = 24$ $\Rightarrow$ $e = 4$.

Error lies in 2379.

Hence, required correct entry = 2379 – 4 = 2375.

Hence, correct value = 2.375

Example 4. *One number in the following is misprint. Correct it.*

1 2 4 8 16 26 42 64 93.

Sol. Difference table for given data it as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	1					
		1				
2	2		1			
		2		1		
3	4		2		1	
		4		2		–5
4	8		4		–4	
		8		–2		10
5	16		2		6	
		10		4		–10
6	26		6		–4	
		16		0		5
7	42		6		1	
		22		1		
8	64		7			
		29				
9	93					

In the above table, the fourth difference column have algebraic sum of all the values is 0. The middle term of this difference column is 6.

$\therefore$ $6e = 6$ or $e = 1$.

$\therefore$ Correct value is given by $16 - 1 = 15$.

Example 5. *Locate the error in the following table:*

x	1	2	3	4	5	6	7	8	9	10	11
y	1.0000	1.5191	2.0736	2.6611	3.2816	3.9375	4.6363	5.3771	6.1776	7.0471	8.0000

Sol. The difference table for the given table is:

x	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^3 y$	$10^4 \Delta^3 y$	$10^4 \Delta^4 y$	$10^4 \Delta^5 y$
1	10000					
		5191				
2	15191		354			
		5545		–24		
3	20736		330		24	
		5875		0		0
4	26611		330		24	
		6205		24		27
5	32816		354		51	
		6559		75		–135
6	39375		429		–84	
		6988		–9		270
7	46363		420		186	
		7408		177		–270
8	53771		597		–84	
		8005		93		135
9	61776		690		51	
		8695		144		
10	70471		834			
		9529				
11	80000					

Here, sum of fifth differences is small which may be neglected 0.270 and –270 are the adjacent values which are equal in magnitude and opposite in sign. Horizontal lines between these values point out the incorrect functional value 46363 coefficient of first middle term in $(1 - p)^5$ is +10.

$\therefore$ Error is given by $10e = 270 \;\Rightarrow\; e = 27$

Hence, correct functional value $\dfrac{46363 - 27}{10000} = 4.6336.$

3.7 TECHNIQUE TO DETERMINE THE MISSING TERM

Let, given a set of equidistant values of arguments and its corresponding value of $f(x)$. Suppose for $n + 1$ equidistant argument values $x = a, a + h, a + 2h, \ldots\ldots\ldots, a + nh$, are given.

$y = f(x) = f(x_0), f(x_1), f(x_2)$ ......, $f(a + nh)$. *i.e.*, $f(x_n)$.

Let one of the value of $f(x)$ is missing. Say it $f(i)$. To determine this missing value of $f(x)$, assume that $f(x)$ can be represented by a polynomial of degree $(n - 1)$ since n values of $f(x)$ are known.

Hence, $\Delta^{n-1} f(x)$ = constant and $\Delta^n f(x) = 0$

Therefore, $(E - I)^n f(x) = 0$ because $\Delta = E - 1$

$$\Rightarrow [E^n - {}^nC_1 E^{n-1} I + {}^nC_2\, E^{n-2} I - \text{..........} + (-1)^n\, E^{n-n}\, I^n]\, f(x) = 0$$

or $$E^n f(x) - {}^nC_1 E^{n-1} f(x) + {}^nC_2 E^{n-2} f(x) - \text{.........} + (-1)^n f(x) = 0$$

For first tabulated value of x, put $x = 0$

$$\Rightarrow E^n f(0) - {}^nE^{n-1} f(0) + \frac{n(n-1)}{2} E^{n-2} f(0) - \text{...........} + (-1)^n f(0) = 0$$

or $$f(n) - nf(n-1) + \frac{n(n-1)}{2} f(n-2) - \text{............} + (-1)^n f(0) = 0 \quad \text{...(1)}$$

In equation (1), except missing term, each term is known and hence from this way missing term can be obtained.

If two values of $f(x)$ are mssing then in that case only $(n - 1)$ values of $f(x)$ can be given by a polynomial of degree $(n - 2)$. *i.e.*, $\Delta^{n-1} f(x) = 0$ or $(E - 1)^{n-1} f(x) = 0$.

This gives for $x = 0$, (the first tabulated value) and for $x = 1$, (second tabulated value) and by solving these two we get the two missing values for given function $f(x)$. Similarly method proceeds to find three and more missing terms in given function $f(x)$.

Example 6. *Estimate the missing term in the following table:*

x	0	1	2	3	4
$y = f(x)$	1	3	9	?	81

Explain why values differ from 3^3 or 27.

Sol. Since we have given 4 values, therefore

$$\Delta^4 f(x) = 0, \forall x$$

i.e., $$(E-1)^4 f(x) = 0, \ \forall x$$

i.e., $$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0, \forall x$$

i.e., $$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0, \ \forall x$$

i.e., $$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0, \ \forall x$$

(on taking interval of differencing being 1)

On putting $x = 0$, we get

$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0 \quad \text{...(1)}$$

Substituting the value of $f(0)$, $f(1)$, $f(2)$, $f(4)$ in (1), we get

$$81 - 4f(3) + 6 \times 9 - 4 \times 3 + 1 = 0$$

i.e., $4f(3) = 124$

i.e., $f(3) = 31$

(function values are 3^n type and this is not a polynomial)

Example 7. *Find the missing value of the data:*

x	1	2	3	4	5
$f(x)$	7	?	13	21	37

Sol. Since 4 values are known, let us assume the fourth order differences being zero. Also since one value is unknown, we assume

$$\Delta^4 f(x) = 0, \forall x$$

i.e., $$(E-1)^4 f(x) = 0, \forall x$$

i.e $$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0, \forall x$$

i.e., $$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0, \ \forall x$$

i.e., $$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0, \forall x$$

(on taking interval of differencing being 1)

On putting $x = 0$, we get

$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0 \quad \text{...(1)}$$

Substituting the value of $f(0)$, $f(1)$, $f(2)$, $f(4)$ in (1), we get

$$37 - 4(21) + 6(13) - 4f(1) + 7 = 0$$

$$38 - 4f(1) = 0 \Rightarrow f(1) = 9.5$$

Hence, the required missing value is 9.5.

Example 8. *Find the missing values in the table:*

x	45	50	55	60	65
$f(x)$	3	–	2	–	–2.4

Sol. Difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	2		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3		$-0.4 - 2y_3$	
		$-2.4 - y_3$		
65	–2.4			

as only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$\therefore \qquad \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

$\Rightarrow \qquad 3y_1 + y_3 = 9$ and $y_1 + 3y_3 = 3.6$

On solving these, we get

$$y_1 = 2.925,\ y_2 = 0.225$$

Example 9. *Obtain the missing terms in the following table:*

x	1	2	3	4	5	6	7	8
f(x)	1	8	?	64	?	216	343	512

Sol. Here we have six known values, therefore sixth differences being zero.

i.e., $\Delta^6 f(x) = 0$ For all values of x

i.e., $$(E-1)^6 f(x) = 0,\ \forall x$$

i.e., $$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0,\ \forall x$$

i.e., $$E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6Ef(x) + f(x) = 0,\ \forall x$$

i.e., $$f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0,\ \forall x \qquad ...(1)$$

On putting $x = 1$ and $x = 2$ in equation (1), we get

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0 \qquad ...(2)$$

$$f(8) - 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0 \qquad ...(3)$$

Putting the value of $f(8)$, $f(7)$, $f(6)$, $f(4)$, $f(2)$, $f(1)$ in equation (1) and (2), we get

$$343 - 6 \times 216 + 15f(5) - 20 \times 64 + 15f(3) - 6 \times 8 + 1 = 0$$

Also $512 - 6 \times 343 + 15 \times 216 - 20f(5) + 15 \times 64 - 6f(3) + 8 = 0$

i.e., $15f(5) + 15f(3) = 2280$ and $20f(5) + 6f(3) = 2662$

i.e., $f(5) + f(3) = 152$ and $10f(5) + 3f(3) = 1331$

On solving these two, we get $f(3) = 27$ and $f(5) = 125$.

Example 10. *Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values:*

x	0	1	2	3	4	5	6	7
y	1	-1	1	–1	1	–	–	–

Sol. For the given data, difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		–2			
1	–1		4		
		2		–8	
2	1		–4		16
		–2		8	
3	–1		4		16
		2		$\Delta^3 y_2$	
4	1		$\Delta^2 y_3$		16
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since values of y belong to a polynomial of degree 4, fourth difference must be constant.

But $\Delta^4 y_0 = 16$

Therefore other fourth order differences will be 16.

Thus, $\Delta^4 y_1 = 16$,

$\therefore$ $\Delta^3 y_2 - \Delta^3 y_1 = 16$

$\Rightarrow$ $\Delta^3 y_2 = 24$

$\therefore$ $\Delta^2 y_3 - \Delta^2 y_2 = 24$

$\Rightarrow$ $\Delta^2 y_3 = 28$

$\Delta y_4 - \Delta y_3 = 28$

$\Rightarrow$ $\Delta y_4 = 30$

$y_5 - y_4 = 30$

$\Rightarrow$ $y_5 = 31$

Again, $\Delta^4 y_2 = 16$ then after solving, we get $y_6 = 129$ and $\Delta^4 y_3 = 16$ gives $y_7 = 351$.

PROBLEM SET 3.2

1. Locate the error in the following table and correct them?

x	3.60	3.61	3.62	3.63	3.64	3.65	3.66	3.67	3.68
$f(x)$	0.112046	0.120204	0.128350	0.136462	0.144600	0.152702	0.160788	0.168857	0.176908

[**Ans.** $f(3.63)$ = 0.136482]

2. Locate the error in the following: –1, 0, 7, 26, 65, 124, 215, 342, 511

[**Ans.** correct value = 63, Error = 2]

3. Obtain the missing term in the following table:

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	0.135	?	0.111	0.100	?	0.082	0.074

[**Ans.** $f(2.1) = 0.123$, $f(2.4) = 0.0900$]

4. Estimate the production for the year 1964 and 1966 from the following data:

Year	1961	1962	1963	1964	1965	1966	1967
Production	200	220	260	?	350	?	430

[**Ans.** $f(1964) = 306$ $f(1966 = 390)$]

5. Given, log 100 = 2, log 101 = 2.0043, log 103 = 2.0128, log 104 = 2.0170. find log 102.

[**Ans.** log 102 = 2.0086]

6. Estimate the missing term in the following:

x	1	2	3	4	5	6	7
y	2	4	8	?	32	64	128

Explain why the result differs from 16. [**Ans.** $f(4) = 16.1$]

7. Find the first term of the series whose second and subsequent terms are 8, 3, 0, –1, 0.

[**Ans.** First term is 15.]

8. Obtain the missing term in the following table:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	0.135	?	0.111	0.100	?	0.082	0.074

[**Ans.** $f(0.1)=0.123$, $f(0.4) = 0.090$]

9. Evaluate the production of wool in the year 1935 from the given data:

Year (x)	1931	1932	1933	1934	1935	1936	1937
Production (y)	17.1	13	14	9.6	?	12.4	18.2

[**Ans.** 6.6]

3.8 SEPARATION OF SYMBOLS

The relation $E = 1 + \Delta \Rightarrow E^n = (1 + \Delta)^n$ has been used to express $E^n y_x$ in terms of y_x and its differences. $(1+\Delta)^n$ has been expanded by binomial theorem without using y_x in it. Such methods of operations are known as method of separation of symbols. Point to be noted that the operations on symbols has no meaning without operand y_x *i.e.,*

$$\begin{aligned} y_{x+nh} = E^n y_x &= (1 + \Delta)^n y_x \\ &= (1 + {}^nC_1\Delta + {}^nC_2\Delta^2 + ...)y_x \\ &= y_x + {}^nC_1\Delta y_x + {}^nC_2\Delta^2 y_x + ... \end{aligned}$$

This type of operation in which we separate the operand from operator is called **separation of symbols.**

Example 1. *Show that* $\Delta^r y_k = \nabla^r y_{k+r}$

Sol. We know $\nabla = 1 - E^{-1}$.

Therefore

$$\begin{aligned} \nabla^r y_{k+r} &= (1 - E^{-1})^r y_{k+r} \\ &= \left(\frac{E-1}{E}\right)^r y_{k+r} \\ &= (E - 1)^r (E^{-r} y_{k+r}) \\ &= (E - 1)^r (E^{-r} y_{k+r}) \\ &= \Delta^r y_k \qquad [\because \Delta \equiv E - 1] \end{aligned}$$

Example 2. *Show that* $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

Sol.

$$\begin{aligned} \sum_{k=0}^{n-1} \Delta^2 f_k &= \sum_{k=0}^{n-1} (E-1)^2 f_k \\ &= \sum_{k=0}^{n-1} \left(E^2 - 2E + 1\right) f_k = \sum_{k=0}^{n-1} (f_{k+2} - 2 f_{k+1} + f_k) \\ &= f_2 - 2 f_1 + f_0 \\ &\quad + f_3 - 2 f_2 + f_1 \\ &\quad + f_4 - 2 f_3 + f_2 \\ &\quad + f_5 - 2 f_4 + f_3 \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + f_{n-1} - 2 f_{n-2} + f_{n-3} \end{aligned}$$

$+ f_n - 2f_{n-1} + f_{n-2}$
$+ f_{n+1} - 2f_n + f_{n-1}$
$= f_{n+1} - f_n + f_0 - f_1$, on adding and canceling the diagonal terms
$= (f_{n+1} - f_n) - (f_1 - f_0)$
$= \Delta f_n - \Delta f_0.$

Example 3. *Prove that* $y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_x + \Delta^2 y_{x+1})$; *Assuming that,* $\Delta^3 y_x = 0.$

Sol. $y_{x+\frac{1}{2}} = E^{1/2} y_x = (1 + \Delta)^{1/2} y_x = \left(1 + \frac{1}{2}\Delta - \frac{1}{8}\Delta^2\right) y_x$ ...(1)

Because $\Delta^3 y_x = 0$

Now, $\Delta^3 y_x = 0$

$\Rightarrow \Delta^2 y_{x+1} - \Delta^2 y_x = 0$

$\Rightarrow \Delta^2 y_{x+1} = \Delta^2 y_x$ and $\Delta y_x = y_{x+1} - y_x$

Therefore from equation (1), we have

$$y_{x+\frac{1}{2}} = y_x + \frac{1}{2}(y_{x+1} - y_x) - \frac{1}{8}\left(\frac{\Delta^2 y_x}{2} + \frac{\Delta^2 y_{x+1}}{2}\right)$$
$$= \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_x + \Delta^2 y_{x+1}).$$

Example 4. *Using the method of separation of symbol, show that*

$$u_0 - u_1 + u_2 - u_3 + u_4 - \ldots\ldots = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \ldots\ldots$$

Sol. On taking R. H. S of given identity

$$\frac{1}{2}\left[1 - \frac{1}{2}\Delta + \left(\frac{1}{2}\Delta\right)^2 - \left(\frac{1}{2}\Delta\right)^3 + \ldots..\right] u_0$$
$$= \frac{1}{2} \cdot \frac{1}{\left(1 + \frac{1}{2}\Delta\right)} u_0 = \frac{1}{2}\left(1 + \frac{1}{2}\Delta\right)^{-1} u_0$$
$$= (2 + \Delta)^{-1} u_0 = (1 + E)^{-1} u_0$$
$$= (1 - E + E^2 - E^3 + \ldots\ldots) u_0$$
$$= u_0 - u_1 + u_2 - u_3 + \ldots\ldots$$ **Hence proved.**

Example 5. *Prove by the method of separation of symbols, that*

$$u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \frac{u_3}{3!}x^3 + \frac{u_4}{4!}x^4 + \ldots\ldots = e^x\left[u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \ldots.\right]$$

Sol. L.H.S. of given identity

$$= u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \frac{u_3}{3!}x^3 + \frac{u_4}{4!}x^4 + \ldots.$$
$$= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2 u_0 + \frac{x^3}{3!}E^3 u_0 + \ldots..$$

$$= \left[1+\frac{x}{1!}E+\frac{x^2}{2!}E^2+\frac{x^3}{3!}E^3+......\right]u_0$$

$$= e^{xE}u_0 = e^{x(1+\Delta)}u_0 = e^x \cdot e^{x\Delta}u_0$$

$$= e^x\left[1+x\Delta+\frac{x^2}{2!}\Delta^2+\frac{x^3}{3!}\Delta^3+........\right]u_0$$

$$= e^x\left[u_0+x\Delta u_0+\frac{x^2}{2!}\Delta^2 u_0+\frac{x^3}{3!}\Delta^3 u_0+........\right]$$

= R.H.S.

Example 6. *Prove that:* $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... + \Delta^{n-1}u_{x-n} + \Delta^n u_{x-n}$

Sol. To prove $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... + \Delta^{n-1}u_{x-n} + \Delta^n u_{x-n}$, shift the last term of right hand side and then solve left hand side.

$$u_x - \Delta^n u_{x-n} = (1 - \Delta^n E^{-n})u_x$$

$$= \left[1-\left(\frac{\Delta}{E}\right)^n\right] u_x = \frac{1}{E^n}(E^n - \Delta^n)\, u_x = \frac{1}{E^n}\left(\frac{E^n-\Delta^n}{E-\Delta}\right)u_x$$

$$\because\ 1 + \Delta = E \Rightarrow E - \Delta = 1$$

$$= \frac{1}{E^n}\,[E^{n-1} + \Delta E^{n-2} + \Delta^2 E^{n-3} + ...+ \Delta^{n-1}]u_x$$

$$= (E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + ...+ \Delta^{n-1}E^{-n})u_x$$

$$= u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ...+ \Delta^{n-1}u_{x-n}$$

Example 7. *Show that:* $u_{2n} - {}^nC_1 2u_{2n-1} + {}^nC_2 2^2 u_{2n-2} - ...... + (-2)^n u_n = (-1)^n (c - 2an)$
Where $u_n = an^2 + bn + c$

Sol. L.H.S $= u_{2n} - {}^nC_1 2u_{2n-1} + {}^nC_2\, 2^2\, u_{2n-2} - ... + (-2)^n u_n$

$= E^n u_n - {}^nC_1 \cdot 2E^{n-1}u_n + {}^nC_2 . 2^2\, E^{n-2}\, u_n - ... + (-2)^n u_n$

$= (E^{n-n}\ {}^nC_1 . 2E^{n-1} + {}^nC_2 . 2^2\, E^{n-2} - ...+(-2)^n\, u_n$

$= (E - 2)^n\, u_n = (\Delta - 1)^n\, u_n = (-1)^n\, (1 - \Delta)^n u_n$

$$= (-1)^n\left[1-n\Delta+\frac{n(n-1)}{1.2}\Delta^2\right]u_n$$

(On neglecting higher order differences as u_n is a polynomial of second degree)

$$= (-1)^n\left[u_n - n\Delta u_n + \frac{n(n-1)}{2}\Delta^2 u_n\right]$$

$$= (-1)^n\left[\left(an^2+bn+c\right)-n\Delta\left(an^2+bn+c\right)+\frac{n^2-n}{2}\Delta^2\left(an^2+bn+c\right)\right]$$

$$= (-1)^n\left[\left(an^2+bn+c\right)-n\left\{a\Delta n^2+b\Delta n\right\}+\frac{n^2-n}{2}\left(a\Delta^2 n^2\right)\right]$$

$$= (-1)^n \left[\left(an^2 + bn + c\right) - n\left\{a\left(n+1\right)^2 - an^2\right\} - bn\left(n+1-n\right) + \frac{n^2-n}{2} a\Delta\left\{\left(n+1\right)^2 - n^2\right\}\right]$$

$$= (-1)^n \left[\left(an^2 + bn + c\right) - n\left(2an + a + b\right) + \frac{n^2-n}{2} \cdot a\Delta\left(2n+1\right)\right]$$

$$= (-1)^n \left[\left(an^2 + bn + c\right) - n\left(2an + a + b\right) + \frac{n^2-n}{2} \cdot a\{2\left(n+1\right) - 2n\}\right]$$

$$= (-1)^n \left[\left(an^2 + bn + c\right) - n\left(2an + a + b\right) + a\left(n^2 - n\right)\right]$$

$= (-1)^n\ [c\ -2an] =$ R.H.S.

Example 8. *Using the method of separation of symbols, show that:*

$$\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2} u_{x-2} - \ldots\ldots + (-1)^n u_{x-n}$$

Sol. R.H.S. $= u_x - nu_{x-1} + \dfrac{n(n-1)}{2} u_{x-2} - \ldots\ldots + (-1)^n u_{x-n}$

$$= u_x - nE^{-1}u_x + \frac{n(n-1)}{2}E^{-2}u_x - \ldots.. + (-1)^n E^{-n}u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} - \ldots\ldots.. + (-1)E^{-n}u_x\right]u_x$$

$$= (1 - E^{-1})^n u_x$$

$$= \left(1 - \frac{1}{E}\right)^n u_x = \left(\frac{E-1}{E}\right)^n u_x = \frac{\Delta^n}{E^n}u_x$$

$$= \Delta^n E^{-n} u_x = \Delta^n u_{x-1} = \text{L.H.S.}$$

Example 9. *Use the method of separation of symbols to prove the following identities:*

1. $u_0 + {}^xC_1\Delta u_1 + {}^xC_2\Delta u_2 + \ldots = u_x + {}^xC_1\Delta^2 u_{x-1} + {}^xC_2\Delta^4 u_{x-2} + \ldots..$

2. $u_x - u_{x+1} + u_{x+2} - u_{x+3} + \ldots.$

$$= \frac{1}{2}\left[u_{x-(1/2)} - \frac{1}{8}\Delta^2 u_{x-(3/2)} + \frac{1.3}{2!}\left(\frac{1}{8}\right)^2 \Delta^4 u_{x-(5/2)} - \frac{1.3.5}{3!}\left(\frac{1}{8}\right)^3 \Delta^6 u_{x-(7/2)} + \ldots.\right]$$

3. $u_0 + {}^nC_1u_1x + {}^nC_2u_2x^2 + {}^nC_3u_3x^3 + \ldots = (1+x)^n u_0 + {}^nC_1(1+x)^{n-1} x\Delta u_0$
$+ {}^nC_2(1+x)^{n-2} x^2\Delta^2 u_0 + \ldots$

Sol.

1. R.H.S. $= u_x + {}^xC_1\Delta^2 u_{x-1} + {}^xC_2\Delta^4 u_{x-2} + \ldots..$

$$= u_x + {}^xC_1\Delta^2 E^{-1}u_x + {}^xC_2\Delta^4 E^{-2}u_x + \ldots.$$

$$= [1 + {}^xC_1\Delta^2 E^{-1} + {}^xC_2\Delta^4 E^{-2} + \ldots.]u_x$$

$$= (1+\Delta^2 E^{-1})^x u_x = \left(\frac{E+\Delta^2}{E}\right)^x u_x = \left[\frac{E+(E-1)^2}{E}\right]^x u_x$$

$$= \left(\frac{E^2-E+1}{E}\right)^x u_x = [1+E(E-1)]^x E^{-x} u_x$$

$$= (1 + \Delta E)^x\ u_0 = (1 + {}^xC_1\Delta E + {}^xC_2\Delta^2E^2 + ...)u_0$$

$$= u_0 + {}^xC_1\Delta E u_0 + {}^xC_2\Delta^2 E^2 u_0 + ...$$

$$= u_0 + {}^xC_1\Delta u_1 + {}^xC_2\Delta^2 u_2 + ... = \text{L.H.S.}$$

2. R.H.S. $= \frac{1}{2}\left[u_{x-(1/2)} - \frac{1}{8}\Delta^2 u_{x-(3/2)} + \frac{1.3}{2!}\left(\frac{1}{8}\right)^2 \Delta^4 u_{x-(5/2)} - \frac{1.3.5}{3!}\left(\frac{1}{8}\right)^3 \Delta^6 u_{x-(7/2)} + ...\right]$

$$= \frac{1}{2}\left[E^{-1/2}u_x - \frac{1}{2}\cdot\frac{1}{4}\Delta^2 E^{-3/2}u_x + \frac{(1/2)(3/2)}{2!}\left(\frac{1}{4}\right)^2 \Delta^4 E^{-5/2}u_x - \frac{(1/2)(3/2)(5/2)}{3!}\left(\frac{1}{4}\right)^3 \Delta^6 E^{-7/2}u_x + ...\right]$$

$$= \frac{1}{2}E^{-1/2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{4}\Delta^2E^{-1}\right) + \frac{(-1/2)(-3/2)}{2!}\left(\frac{1}{4}\Delta^2E^{-1}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(\frac{1}{4}\Delta^2E^{-1}\right)^3 + ...\right]u_x$$

$$= \frac{1}{2}E^{-1/2}[1+\frac{1}{4}\Delta^2E^{-1}]^{-1/2}u_x$$

$$= \frac{1}{2}E^{-1/2}\left[\frac{4E+\Delta^2}{4E}\right]^{-1/2} u_x = \frac{1}{2}E^{-1/2}2E^{1/2}[4(1+\Delta)+\Delta^2]^{-1/2}u_x$$

$$= [(2+\Delta)^2]^{-1/2}u_x = (2+\Delta)^{-1}u_x = (1+E)^{-1}u_x$$

$$= [1-E+E^2-E^3+E^4-E^5+....]u_x$$

$$= u_x - u_{x+1} + u_{x+2} - u_{x+3} + u_{x+4} - u_{x+5} + ... = \text{L.H.S.}$$

3. R.H.S. $= (1 + x)^n\ u_0 + {}^nC_1\ (1 + x)^{n-1}\ x\,\Delta u_0 + {}^nC_2\ (1 + x)^{n-2}\ x^2\ \Delta^2 u_0 + ...$

$$= \{(1 + x) + x\Delta\}^n\ u_0 = (1 + x\ (1 + \Delta)\}^n\ u_0$$

$$= (1 + xE)^n\ u_0 = [1 + {}^nC_1\ xE + {}^nC_2x^2E^2 + {}^nC_3x^3E^3 + ...\ ...\ ]u_0$$

$$= u_0 + {}^nC_1u_1\ x + {}^nC_2u_2x^2 + {}^nC_3\ u_3x^3 + ...\ ...\ = \text{L.H.S.}$$

Example 10. *Prove that* $\Delta x^n - \frac{1}{2}\Delta^2 x^n + \frac{1.3}{2.4}\Delta^3 x^n - \frac{1.3.5}{2.4.6}\Delta^4 x^n + \ldots = \left(x+\frac{1}{2}\right)^n - \left(x-\frac{1}{2}\right)^n.$

Sol. L.H.S. $= \Delta\left[1 - \frac{1}{2}\Delta + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\Delta^2 + \ldots\right]x^n$

$$= \Delta(1+\Delta)^{-1/2}\, x^n = \Delta E^{-1/2} x^n = \Delta\left(x - \frac{1}{2}\right)^n$$

$$= \left(x+1-\frac{1}{2}\right)^n - \left(x-\frac{1}{2}\right)^n = \left(x+\frac{1}{2}\right)^n - \left(x-\frac{1}{2}\right)^n = \text{R.H.S.}$$

3.9 FACTORIAL NOTATIONS

The product of n consecutive factors each at a constant difference and the first factor being x is called a factorial function or a factorial polynomial of degree n and is defined by

$$x^{(n)} = x(x-h)(x-2h)(x-3h)\ldots(x-(n-1)),\ n > 0$$

If interval of differencing being unity then

$$x^{(n)} = x(x-1)(x-2)(x-3)\ldots(x-(n-1)),\ n > 0$$

Because of their properties, this function play an important role in the theory of finite differences and also it helps in finding the various order differences of a polynomial directly by simple rule of differentiation.

Example 11. *Obtain the function whose first difference is* $9x^2 + 11x + 5$.

Sol. Let $f(x)$ be the required function so that

$$\Delta f(x) = 9x^2 + 11x + 5$$

Let $9x^2 + 11x + 5 = 9[x]^2 + A[x] + B$

$$= 9x(x-1) + Ax + B$$

On substitution $x = 0$, we get $B = 5$ and for $x = 1$, we get $A = 20$.

Therefore we have $\Delta f(x) = 9[x]^2 + 20[x] + 5$

On integrating, we have

$$f(x) = 9\frac{[x]^3}{3} + 20\frac{[x]^2}{2} + 5[x] + c$$

$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + c$$

$= 3x^3 + x^2 + x + c$, where c is the constant of integration.

Example 12. *Find a function* f_x *for which* $\Delta f_x = x(x-1)$.

Sol. We have $\Delta f_x = x(x-1) = x^{(2)}$

Therefore $f_x = \frac{x^{(3)}}{3} + C$, where C is an arbitrary constant

$$= \frac{1}{3}x(x-1)(x-2) + C$$

Example 13. *Prove that if m is a positive integer then* $\dfrac{(x+1)^{(m)}}{m!} = \dfrac{x^{(m)}}{m!} + \dfrac{x^{(m-1)}}{(m-1)!}$.

Sol. To prove this we start from R.H.S. and use factorial notations to simplify.

$$\frac{x^{(m)}}{m!} + \frac{x^{(m-1)}}{(m-1)!} = \frac{x(x-1)(x-2)\ldots(x-m+1)}{m!} + \frac{x(x-1)(x-2)\ldots(x-m+2)}{(m-1)!}$$

$$= \frac{x(x-1)(x-2)\ldots(x-m+2)}{m!}\left[(x-m+1)+m\right]$$

$$= \frac{(x+1)x(x-1)(x-2)\ldots(x-m+2)}{m!}$$

$$= \frac{(x+1)^{m}}{m!}.$$

Example 14. *Prove that* $\Delta^2 x^{(m)} = m(m-1)x^{(m-2)}$; *where m is a positive integer and interval of differencing being unity.*

Sol. We know, $x^{(m)} = x(x-1)(x-2)\ldots(x-\overline{m-1})$.

Therefore, $\Delta x^{(m)} = \left\{(x+1)x(x-1)(x-2)\ldots\ldots(x+1-\overline{m-1})\right\} - \left\{x(x-1)\ldots\ldots(x-\overline{m-1})\right\}$

$= x(x-1)(x-2)\ldots\ldots(x-\overline{m-2})\left\{x+1-(x-\overline{m-1})\right\}$

$= mx^{(m-1)}$

Also, $\Delta^2 x^{(m)} = \Delta(\Delta x^{(m)} = \Delta\left\{mx^{(m-1)}\right\}$

$= m\Delta x^{(m-1)} = m(m-1)x^{(m-2)}$. **Hence proved**

Example 15. *Evaluate* $\Delta^n(ax^n + bx^{n-1})$

Sol. $\Delta^n(ax^n + bx^{n-1}) = a\Delta^n(x^n) + b\Delta^n(x^{n-1})$

$= an! + b(0) = an!$

Example 16. *Denoting* $\binom{x}{n} = \dfrac{x(x-1)\ldots(x-n+1)}{n!}$, *prove that for any polynomial* $\phi(x)$ *of degree* k, $\phi(x) = \sum_{i=0}^{k}\binom{x}{i}\Delta^i\phi(0)$.

Sol. We have $E^n f(a) = f(a + nh)$

$= f(a) + {}^nC_1\Delta f(a) + {}^nC_2\Delta^2 f(a) + \ldots + {}^nC_n\Delta^n f(a)$

Substitute $a = 0$, $n = x$ we have for $h = 1$

$$f(x) = f(0) + {}^xC_1\Delta f(0) + {}^xC_2\Delta^2 f(0) + \ldots + {}^xC_x\Delta^x f(0)$$

Again $f(x) = \phi(x)$ is the given polynomial of degree k therefore we have $\Delta^k\phi(x) = L$ (constant). Therefore higher order differences become zero.

$$\therefore\ \phi(x) = \phi(0) + {}^xC_1\Delta\phi(0) + {}^xC_2\Delta^2(0) + \ldots + {}^xC_k\Delta^k\phi(0)$$

$$= \sum_{i=1}^{k} \binom{x}{i}\Delta^i\phi(0)$$

3.10 RECIPROCAL FACTORIAL NOTATION

The reciprocal factorial is denoted by $x^{(-n)}$ and is defined by

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\ldots(x+nh)} = \frac{1}{(x+nh)^{(n)}}$$

If the interval of differencing being unity, then

$$x^{(-n)} = \frac{1}{(x+1)(x+2)\ldots(x+n)} = \frac{1}{(x+n)^{(n)}}$$

Example 17. *Prove that* $\Delta^2 x^{(-m)} = m(m+1)x^{(-m-2)}$

Sol. We have, $$x^{(-m)} = \frac{1}{(x+1)(x+2)\ldots(x+m)}$$

Therefore, $$\Delta x^{(-m)} = \frac{1}{(x+2)(x+3)\ldots.(x+m+1)} - \frac{1}{(x+1)(x+2)\ldots(x+m)}$$

$$= \frac{1}{(x+2)(x+3)\ldots.(x+m)}\left[\frac{1}{(x+m+1)} - \frac{1}{(x+1)}\right]$$

$$= \frac{-m}{(x+1)(x+2)\ldots..(x+m)(x+m+1)}$$

$$= -mx^{[-(m+1)]} = -mx(^{-m-1})$$

Therefore, $$\Delta^2 x^{(-m)} = \Delta(\Delta x^{(-m)}) = \Delta(-mx^{(-m-1)})$$

$$= -m\Delta x^{(-m-1)} = -m(-m-1)x^{(-m-2)}$$

$$= m\ (m + 1)\ x^{(-m-2)}.$$

Example 18. *Express* $f(x) = \dfrac{x-1}{(x+1)(x+3)}$ *in terms of negative factorial polynomials.*

Sol. Here, $$f(x) = \frac{x-1}{(x+1)(x+3)}$$

Multiply and divide by $(x + 2)$

$$= \frac{(x-1)(x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{1}{x+1} - \frac{4}{(x+1)(x+2)} + \frac{4}{(x+1)(x+2)(x+3)}$$

$$= x^{(-1)} - 4x^{(-2)} + 4x^{(-3)}$$

Corollary:

To show that, $\Delta^n x^{(n)} = n!\, h^n$ and $\Delta^{n+1} x^{(n)} = 0$

We know that, $\Delta x^{(n)} = (x + h)^{(n)} - (x)^{(n)}$ ...(1)

Since, $(x + h)^{(n)} = (x + h) .x. (x - h) (x - 2h)... \{x - (n-2)h\}$

$(x)^{(n)} = x(x - h) (x - 2h)... \{x - (n - 1)h\}$

Therefore from (1), we have

$\Rightarrow \quad (x + h)^{(n)} - (x)^{(n)} = (x + h) .x. (x - h) (x - 2h)... \{x - (n - 2)h\}$
$\qquad - x (x - h) (x - 2h)... \{x - (n - 1)h\}$

or $\quad (x + h)^{(n)} - (x)^{(n)} = x (x - h) (x - 2h) ... \{x - (n - 2)h\} [(x + h) -\{x - (n - 1)h\}]$

or $\quad (x + h)^{(n)} - (x)^{(n)} = x^{(n-1)}\, nh$ ...(*a*)

This is a polynomial of degree $(n - 1)$ in factorial notation.

Similarly, $\Delta^2 x^{(n)} = \Delta\left[\Delta x^{(n)}\right] = \Delta\left[nhx^{(n-1)}\right]$ ...[using (*a*)]

or $\quad nh\,\Delta x^{(n-1)} = nh\left[(x+h)^{(n-1)} - x^{(n-1)}\right]$

$= nh[(x + h) x (x - h)... \{x - (n - 3)h\} - x (x - h)...\{x - (n - 2)h\}]$

$= nh[x(x-h)...\{x - (n - 3) h\}] (nh - h)$

$= nh (n - 1) h [x (x - h) (x - 2h)... \{x - (n - 3)h\}]$

$= n(n - 1) h^2 x^{(n-2)}$

This is a polynomial of degree (n – 2) in factorial notation. Proceeding in the same way, we get

$\Delta^n x^{(n)} = n (n - 1) (n - 2)... 3.2.1.h^n x^{n-n}$

$= n!\, h^n x^0 = n!\, h^n$. This term is a constant

Again, $\Delta^{n+1} x^{(n)} = \Delta\left[\Delta^n x^{(n)}\right]$

$= \Delta\left[n!h^n\right]$

$= 0.$ Because $n!h^n$ is constant

3.11 METHOD OF REPRESENTING POLYNOMIAL IN FACTORIAL NOTATIONS

3.11.1 Direct Method

Example 19. *Express $2x^3 - 3x^2 + 3x - 10$ and its differences in factorial notation, consider the interval of differencing being unity.*

Sol. Let $2x^3 - 3x^2 + 3x - 10 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$

$= Ax (x - 1) (x - 2) + Bx (x - 1) + Cx + D$ ...(1)

Where A, B, C and D are constants to be found.

Now substitute $x = 0$ on both the sides of equation (1), we get $D = - 10$.

Again, substituting $x = 1$ on both sides of equation (1), we get

$2 - 3 + 3 - 10 = C + D$ *i.e.*, $C = 2$

Again substituting $x = 2$, we get

$$16 - 12 + 6 - 10 = 2B + 2C + D$$

or $0 = 2B - 6$ *i.e.*, $B = 3$.

By equating the coefficient of x^2 on both sides of (1), we get $A = 2$

Hence, the required polynomial in factorial notation will be

$$f(x) = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$$

Again by the simple rule of differentiation, we have

$$\Delta f(x) = 6x^{(2)} + 6x + 2$$

$$\Delta^2 f(x) = 12x + 6$$

$$\Delta^3 f(x) = 12$$

Example 20. *Find the relation between* α, β *and* γ *in order that* $\alpha + \beta x + \gamma x^2$ *may be expressible in one term in the factorial notation.*

Sol. Let, $f(x) = \alpha + \beta x + \gamma x^2 = (a + bx)^{(2)}$ where a and b are certain unknown constants.

Now, $$(a + bx)^{(2)} = (a + bx)[a + b\,(x - 1)]$$

$$= (a + bx)(a - b + bx) = (a + bx)^2 - ab - b^2 x$$

$$= (a^2 - ab) + (2ab - b^2)x + b^2 x^2 = \alpha + \beta x + \gamma x^2$$

Comparing the coefficients of various powers of x we get

$$\alpha = a^2 - ab, \beta = 2ab - b^2, \gamma = b^2$$

Eliminating a and b from the above equations, we get

$$\gamma^2 + 4\alpha\gamma = \beta^2$$

This is the required relation.

3.11.2 Method of Synthetic Division

It is also called the method of detached coefficient.

Example 21. *Represent* $2x^3 - 3x^2 + 3x - 10$ *in factorial notation using synthetic division method.*

Sol. Let, $$f(x) = 2x^3 - 3x^2 + 3x - 10 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

$$= Ax(x - 1)(x - 2) + Bx(x - 1) + Cx + D$$

In this case we divide the function $f(x)$ by x. Then the remainder will be –10 and the quotient will be $2x^2 - 3x + 3$.

$\therefore$ $D = -10$

Now divide the quotient by (*x* – 1) *i.e.,*

$$\begin{array}{r|r} & 2x-1 \\ \hline (x-1) & 2x^2-3x+3 \\ & 2x^2-2x \\ \hline & -x+3 \\ & -x+1 \\ \hline & 2 \end{array}$$

In this case quotient is (2*x* – 1) and remainder is 2 = *C*. Again divide (2*x* – 1) by (*x* – 2),

$$\begin{array}{r|r} & 2 \\ \hline (x-2) & 2x-1 \\ & 2x-4 \\ \hline & 3 \end{array}$$

Therefore the quotient 2 = *A* and the remainder are 3 = *B*. Hence the required polynomial will be

$$2x^{(3)}+3x^{(2)}+2x-10$$

We can also simplify the above in the following way:

Taking the coefficients of various powers of *x* in *f*(*x*), we have

1	2	–3	3	–10 = *D*	.....(*a*)
	0	2	–1		
2	2	–1	2 = *C*		....(*b*)
	0	4			
3	2	3 = *B*			....(*c*)
	0				
	2 = *A*				

The following steps are to be followed in the method of synthetic division:

1. Put the coefficients of different powers of *x* in order beginning with the coefficients of higher power of *x*.
2. Put 1 in the left hand side column and write zero below the coefficients of highest power of *x* (in this case we have written zero below 2 which is coefficient of x^3).
3. Now multiply 2 by 1 and 0 by 1, add them to get the sum 2, put 2 below –3 as given in (*a*). Now multiply –3 by 1 and 2 by 1 and add them to get the sum –1 which is to be written below 3. The remainder –10 is the value of *D*.
4. Add the terms of corresponding columns of (*a*) and get 2, –1 and 2 of (*b*)
5. Now again apply the steps (1) and (3): in this way we get (2 × 2) + (0 × 2) = 4 which is to be written below –1. The remainder 2 of (*b*) is equal to *C*.
6. Apply step (4) on (*b*) and get 2 and 3 of (*c*).
7. Again apply the steps (2), (3) and (4) to get 2 which will be equal to A and remainder 3 of (*c*) which will be equal to *B*.

Example 22. *Express $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$, and its successive differences in factorial notation. Hence show that $\Delta^5 f(x) = 0$.*

Sol. Let $f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x] + E.$

Use method of synthetic division, we divide by x, $x - 1$, $x - 2$, $x - 3$ etc. successively, then

1	1	–12	24	–30	9 = *E*
		1	–11	13	
2	1	–11	13	–17 = *D*	
		2	–18		
3	1	–9	–5 = *C*		
		3			
4	1	–6 = *B*			
	1 = *A*				

Hence, $f(x) = [x]^4 - 6[x]^3 - 5[x]^2 - 17[x] + 9$

$\therefore$ $\Delta f(x) = 4[x]^3 - 18[x]^2 - 10[x] - 17$

$\Delta^2 f(x) = 12[x]^2 - 36[x] - 10$

$\Delta^3 f(x) = 24[x] - 36 \Rightarrow \Delta^4 f(x) = 24$

And $\Delta^5 f(x) = 0.$

Example 23: *Find the lowest degree polynomial which takes the following values:*

x	0	1	2	3	4	5
$f(x)$	0	3	8	15	24	35

Sol. We know that

$$f(a + nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \ldots + {}^nC_n \Delta^n f(a) \quad \ldots(1)$$

Putting $a = 0$, $h = 1$, $n = x$, we get

$$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(0) + {}^xC_3 \Delta^3 f(0) \ldots$$

$$= f(0) + x^{(1)} \Delta f(0) + \frac{x^{(2)}}{2!} \Delta^2 f(0) + \frac{x^{(3)}}{3!} \Delta^3 f(0) + \ldots \quad \ldots(2)$$

Now form the difference table for the given data to find $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$ etc.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0			
		3		
1	3		2	
		5		0
2	8		2	
		7		0
3	15		2	
		9		0
4	24		2	
		11		
5	35			

Substituting the values of $f(0), \Delta f(0), \Delta^2 f(0), \Delta^3 f(0)$ in (2), we get

$$f(x) = 0 + 3x^{(1)} + 2 \cdot \frac{x^{(2)}}{2!} + 0$$

$$= 3x + x(x-1) = x^2 + 2x.$$

Example 24. *A second degree polynomial passes through the points (0,1), (1,3), (2,7), (3,13). Find the polynomial.*

Sol. Let $f(x) = Ax^2 + Bx + C$.

Now form the difference table for given points:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	1		
		2	
1	3		2
		4	
2	7		2
		6	
3	13		

Since $\Delta f(x) = A\Delta x^2 + B\Delta x + \Delta C$

Therefore, $\Delta f(x) = A\{(x+1)^2 - x^2\} + B(x+1-x) + 0$

$= A(2x+1) + B$

Put $x = 0$, $\Delta f(0) = A + B \Rightarrow A + B = 2$

Again, $\Delta^2 f(x) = 2A \Rightarrow \Delta^2 f(0) = 2 = 2A \Rightarrow A = 1$

Also, we have $B = 1$

Therefore polynomial for the given point is $f(x) = x^2 + x + 1$.

Example 25. *Write down the polynomial of lowest degree which satisfies the following set of numbers 0, 7, 26, 63, 124, 215, 342, 511.*

Sol. We know $f(x) = f(0) + x^{(1)}\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(0) + \frac{x^{(3)}}{3!}\Delta^2 f(0) + \ldots$

By forming the difference table we calculate $f(0), \Delta f(0), \Delta^2 f(0), \Delta^3 f(0)$ etc.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		0
		61		6	
4	124		30		0
		91		6	
5	215		36		0
		127		6	
6	342		42		
		169			
7	511				

Now substituting the values of $f(0), \Delta f(0), \Delta^2 f(0), \Delta^3 f(0)$ from the difference table, we obtain

$$\begin{aligned} f(x) &= 0 + 7x^{(1)} + \frac{12}{2!}x^{(2)} + \frac{6}{3!}x^{(3)} \\ &= 7x + 6x(x-1) + x(x-1)(x-2) \\ &= 7x + 6x^2 - 6x + x^3 - 3x^2 + 2x \\ &= x^3 + 3x^2 + 3x \end{aligned}$$

3.12 ERRORS IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$ having $(n + 1)$ points (x_i, y_i) for $i = 0, 1, 2, ...n$, be continuous and differentiable. Let this function be approximated by a polynomial $p_n(x)$ of degree not exceeding n such that

$$p_n(x_i) = y_i; \quad i = 0, 1, 2, ..., n \qquad ...(a)$$

Now, to obtain approximate values of $y(x)$ at some points other than those given by (a), let $y(x) - p_n(x)$ vanishes for

$$x = x_0, x_1, x_2, ..., x_n; \text{ and } y(x) - p_n(x) = L\pi_{n+1}(x) \qquad ...(b)$$

Where $\pi_{n+1}(x) = (x - x_0)(x - x_{1).....(x-x_n)}$ and L is to be determined such that equation (b) holds for any intermediate values of x, say $x = x'$, $x_0 < x'\ x_n$

$$\text{Therefore } L = \frac{y(x') - p_n(x')}{\pi_{n+1}(x')} \qquad ...(c)$$

Now, construct a function $F(x)$ such that

$$F(x) = y(x) - p_n(x) - L\pi_{n+1}(x) \qquad ...(d)$$

Where L is given by equation (c). it is clear that:

$$F(x_0) = F(x_1) = F(x_2) = ... = F(x_n) = F(x') = 0$$

i.e., $F(x)$ vanishes $(n + 2)$ times in the interval $x_0 \le x \le x_n$

According to Rolle's Theorem, $F'(x)$ must vanish $(n + 1)$ times, $F''(x)$ must vanish n times, etc, in the interval $x_0 \le x \le x_n$. In particular, $F^{n+1}(x)$ must vanish once in the interval. Let this part be $x = \eta$, $x_0 < \eta < x_n$. On differentiating equation (d) $(n + 1)$ times with respect to x and substituting $x = \eta$. We get

$$0 = y^{n+1}(\eta) - L(n+1)!$$

$$\Rightarrow L = \frac{y^{n+1}(\eta)}{(n+1)!} \qquad ...(e)$$

On comparing (c) and (e) we get,

$$y(x') - p_n(x') = \frac{y^{n+1}(\eta)}{(n+1)!}\pi_{n+1}(x')$$

Dropping the prime on x',

$$y(x) - p_n(x) = \frac{y^{n+1}(\eta)}{(n+1)!}\pi_{n+1}(x), x_0 \le x \le x_n$$

This is required expression for polynomial interpolation.

3.13. DIFFERENCES OF ZEROS

Let $y = x^m$ be a function of x, where m be a positive integer. If $x = 0, 1, 2, 3...$, we get $0^m, 1^m, 2^m, 3^m$, ... respectively. If we constructed difference table for this data we get leading differences is as: $\Delta 0^m$, $\Delta^2 0^m$, $\Delta^3 0^m$, and these differences are called differences of zeros.

To find differences of zeros,

$$\begin{aligned}\Delta^n x^m &= (E-1)^n x^m \\ &= [E^n - {}^nC_1 E^{n-1} I + {}^nC_2 E^{n-2} I^2 - ..... + (-1)^n I^n]\, x^m \\ &= [E^n x^m - {}^nC_1 E^{n-1} I\, x^m + {}^nC_2 E^{n-2} I^2 x^m - ..... + (-1)^n I^n x^m] \\ &= [x+n]^m - {}^nC_1 [x+n-1]^m + {}^nC_2 [x+n-2]^m - ..... + (-1)^n x^m\end{aligned}$$

(Taking interval of differencing $h = 1$)

This expression for $x = 0$, becomes

$$[\Delta^n x^m]_{x=0} = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - ...... + {}^nC_{n-1} (-1)^{n-1} \qquad ...(a)$$

The expression (a) for $[\Delta^n x^m]_{x=0}$ is written as $\Delta^n 0^m$ and is known as differences of zero.

If we substitute $n = 1$, $m = 1$, then $\Delta^1 0^1 = 1$. If, $n = 2$ and $m = 2$ then $\Delta^2 0^2 = 2!$ and so on.

Proceeding in the same way we get $\Delta^n 0^n = n!$. Also, $\Delta^{n+1} 0^n = 0, \Delta^n 0^m = 0$ for $n > m$.

Remark: $\Delta^n 0^m = n(\Delta^{n-1} 0^{m-1} + \Delta^n 0^{m-1})$

Example 26. *Sum the series using differences of zero:*

$n^2 + {}^nC_1 (n-1)^2 + {}^nC_2 (n-2)^2 + ...$, n being a positive integer.

Sol. Given

$$\begin{aligned} & n^2 + {}^nC_1 (n-1)^2 + {}^nC_2 (n-2)^2 + ... \\ &= [(x+n)^2 + {}^nC_1 (x+n-1)^2 + {}^nC_2 (x+n-2)^2 + ...]_{x=0} \\ &= [E^n x^2 + {}^nC_1 E^{n-1} x^2 + {}^nC_2 E^{n-2} x^2 + {}^nC_3 E^{n-3} x^2 + ...]_{x=0} \\ &= [\{E^n + {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} + {}^nC_3 E^{n-3} + ...\} x^2]_{x=0} \\ &= [(E+1)^n x^2]_{x=0} = [(2+\Delta)^n x^2]_{x=0} \\ &= [\{2^n + {}^nC_1 2^{n-1} \Delta + {}^nC_2 2^{n-2} \Delta^2 + ...\} x^2]_{x=0} \\ &= [2^n x^2 + {}^nC_1 2^{n-1} \Delta x^2 + {}^nC_2 2^{n-2} \Delta^2 x^2 + ...]_{x=0} \\ &= 0 + {}^nC_1 2^{n-1} \Delta 0^2 + {}^nC_2 2^{n-2} \Delta^2 0^2 + {}^nC_3 2^{n-3} \Delta^3 0^2 + ... \\ &= n.2^{n-1}.1 + \frac{n(n-1)}{1.2} 2^{n-2}\, 2! + 0 + 0 + ...\end{aligned}$$

Because the differences of 0^m of orders higher than m are all zero.

$$= n.2^{n-1} + n(n-1).2^{n-2}$$

$$= n2^{n-2}[2+n-1] = n(n+1).2^{n-2}$$

Example 27. *Prove that:* $\Delta^n 0^{n+1} = \dfrac{n(n+1)}{2} \Delta^n 0^n$.

Sol. Using the relation $\Delta^n 0^m = n[\Delta^{n-1} 0^{m-1} + \Delta^n 0^{m-1}]$, we get

$$\Delta^n 0^{n+1} = n[\Delta^{n-1} 0^n + \Delta^n 0^n]$$

$$\Delta^{n-1} 0^n = (n-1)[\Delta^{n-2} 0^{n-1} + \Delta^{n-1} 0^{n-1}]$$

$$\Delta^{n-2} 0^{n-1} = (n-2)[\Delta^{n-3} 0^{n-2} + \Delta^{n-2} 0^{n-2}]$$

....

$$\Delta^2 0^3 = 2[\Delta 0^2 + \Delta^2 0^2]$$

$$\Delta 0^2 = 1[\Delta^0 0^1 + \Delta^1 0^1].$$

By back substitution of these values, we get

$$\Delta^n 0^{n+1} = n\Delta^n 0^n + n(n-1)\Delta^{n-1} 0^{n-1} + n(n-1)(n-2)\Delta^{n-2} 0^{n-2} + ... + n(n-1)(n-2)....3.2.1\Delta^1 0^1$$

$$nn! + n(n-1)(n-1)! + n(n-1)(n-2)(n-2)! + ... + n(n-1)(n-2)....3.2.1!$$

$$= n!\{n + (n-1) + (n-2) + ... + 3 + 2 + 1\}$$

$$= n!.\frac{n(n+1)}{2} = \frac{n(n+1)}{2}.\Delta^n 0^n \text{ ; Because } \Delta^n 0^n = n!$$

Example 28. *Sum to n terms the series* $1.2\Delta x^n - 2.3\Delta^2 x^n + 3.4\Delta^3 x^n - 4.5\Delta^4 x^n + ...$

Sol. Let $u_x = x^n$. Then nth difference of u_x will be constant and the higher order differences will be zero. Thus the given expression will contain terms up to $\Delta^n x^n$ as the higher order terms will be zero. Hence the sum of the above series to n terms is the same as up to ∞. Therefore

$$1.2\Delta x^n - 2.3\Delta^2 x^n + 3.4\Delta^3 x^n - 4.5\Delta^4 x^n + ... = 2\Delta[1 - 3\Delta + 3.2\Delta^2 - 2.5\Delta^3 + ...]x^n$$

$$= 2\Delta\left[1 - 3\Delta + \frac{3.4}{2}\Delta^2 - \frac{3.4.5}{2.3}\Delta^3 + ...\right]x^n$$

$$= 2\Delta[1+\Delta]^{-3} x^n = 2(E-1)E^{-3} x^n$$

$$= 2(E^{-2} - E^{-3})x^n$$

$$= 2[E^{-2} x^n - E^{-3} x^n]$$

$$= 2[(x-2)^n - (x-3)^n].$$

PROBLEM SET 3.3

1. Use the method of separation of symbols to prove the following identities:

(*a*) $u_1 x + u_2 x^2 + u_3 x^3 + ... = \frac{x}{(1-x)} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + ...$

(*b*) $\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + ... + (-1)^n u_x$

(*c*) $u_x - \frac{1}{8}\Delta^2 u_{x-1} + \frac{1.3}{8.16}\Delta^4 u_{x-2} - \frac{1.3.5}{8.16.24}\Delta^6 u_{x-3} + ... = u_{x+\frac{1}{2}} - \frac{1}{2}\Delta u_{x+\frac{1}{2}} + \frac{1}{4}\Delta^2 u_{x+\frac{1}{2}} - \frac{1}{8}\Delta^3 u_{x+\frac{1}{2}} + ...$

(*d*) $u_x + {}^xC_1 \Delta^2 u_{x-1} + {}^xC_2 \Delta^4 u_{x-2} + ... = u_0 + {}^xC_1 \Delta u_1 + {}^xC_2 \Delta^2 u_2 + ...$

(*e*) $u_0 + Eu_0 + E^2 u_0 + ... + E^n u_0 = {}^{n+1}C_1 u_0 \; {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + ... + \Delta^n u_0$

(*f*) $u_n + {}^xC_1 \Delta E^{-1} u_n + {}^{x+1}C_2 \Delta^2 E^{-2} u_n + {}^{x+2}C_3 \Delta^3 E^{-3} u_n + ... = u_{n+x}$

2. Show that $(x\Delta)^{(n)} u_x = (x+n-1)^{(n)} \Delta^n u_x$.

3. Prove that if m is a positive integer, then $x^{(m+1)} + mx^{(m)} = x.x^{(m)}$

4. Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ and its differences in factorial notation, the interval of differencing being one. [**Ans.** $\Delta f(x) = 6x^{(2)} + 6x + 2, \Delta^2 f(x) = 12x + 6, \Delta^3 f(x) = 12$]

5. Find the successive differences of $x^4 - 12x^3 + 42x^2 - 30x + 9$, where h, the interval of differencing is unity. [**Ans.** $\Delta f(x) = 4x^3 - 30x^2 + 52x + 1, \Delta^2 f(x) = 12x^2 - 48x + 26, \Delta^3 f(x)$
$= 24x - 36, \Delta^4 f(x) = 24$]

6. Represent the following polynomials and its successive differences in factorial notation.

(*a*) $11x^4 + 5x^3 + x - 15$ [**Ans.** $11[x]^4 + 71[x]^3 + 92[x]^2 + 17[x] - 15$]

(*b*) $2x^3 - 3x^2 + 3x + 10$ [**Ans.** $2[x]^3 + 3[x]^2 + 2[x] + 10$]

7. Show that:

(*a*) $(n+1)\Delta^n 0^n = 2[\Delta^{n-1} 0^n + \Delta^n 0^n]$

(*b*) $n! = n^{n-n} c_1 (n-1)^n +^n c_2 (n-2)^n - ....$

8. u_x is a function of x for which differences are constant and $u_1 + u_7 = -786$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$. Find the value of u_4. [**Ans.** 570.9]

❑❑❑

CHAPTER 4

Interpolation with Equal Interval

4.1 INTRODUCTION

Interpolation is the method of estimating unknown values with the help of given set of observations. According to Theile Interpolation is, "The art of reading between the lines of the table". Also interpolation means insertion or filling up intermediate terms of the series. It is the technique of obtaining the value of a function for any intermediate values of the independent variables *i.e.,* of argument when the values of the function corresponding to number of values of argument are given.

The process of computing the value of function outside the range of given values of the variable is called **extrapolation**.

Interpolation is based on the following assumption:

1. The values of the function should be in an increasing or decreasing order *i.e.,* there are no sudden change (Jumps or falls) in the values during the period of under consideration.
2. The jumps and falls in the values should be uniform, this implies that the changes in the values of the observations should be in a uniform pattern.
3. When we apply calculus of finite differences on this, we assume that given set of observations are capable of being expressed in a polynomial form.

There are three methods to interpolate the certain quantity.

(1) Graphical method: According the Graphical Method, take a suitable scale for the values of x and y *i.e.,* for arguments and entries values, and plot the different points on the Graph for these values of x and y. Draw the Curve passes through the different plotted points and find a point on the curve corresponding to argument x and obtain the corresponding entry value y. This method is not reliable most of the cases.

(2) Method of curve fitting: This method can be used only in those cases in which the form of the function in known. This method is not exact and becomes more complicated when the number of observations is sufficiently large. The only merit of this method lies in the fact that it gives closer approximation than the graphical method.

(3) Use of the calculus of finite differences: The study of finite differences for the purpose of interpolation can be divided into three parts.

(*a*) The technique of interpolation with equal intervals.

(*b*) The technique of interpolation with unequal intervals.

(*c*) The technique of central differences.

These methods are less approximate than the others and do not assume the form of the function to be known. On using the method of calculus of finite difference, the calculations remain simple even if some additional observations are included in the given data.

4.2 NEWTON'S GREGORY FORMULA FOR FORWARD INTERPOLATION

Let $y = f(x)$ be a given function of x which takes the value $f(a), f(a+h), f(a+2h) \ldots f(a+nh)$ for $(n+1)$ equally spaced values $a, a+h, a+2h, \ldots a+nh$ of the independent variable x. Assume that $f(x)$ be a polynomial of nth degree, given by

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \ldots A_n(x-a)\ldots\left(x-a-\left(\overline{n-1}\right)h\right) \quad \ldots(1)$$

Where $A_0, A_1, A_2, \ldots A_n$ are to be determined. Now to find the values of $A_0, A_1, A_2, \ldots A_n$, put $x = a, a+h, a+2h, \ldots a+nh$ in equation (1) successively.

Therefore for $x = a$, $f(a) = A_0$ ...(2)

for $x = a+h$, $f(a+h) = A_0 + A_1(a+h-a)$

$$f(a+h) = A_0 + hA_1$$

$$f(a+h) - f(a) = hA_1 \quad \text{[from (2)]}$$

$$\Rightarrow \quad A_1 = \frac{\Delta f(a)}{h}$$

For $x = a + 2h$

$$f(a+2h) = A_0 + A_1(2h) + A_2(2h)h$$

$$= f(a) + 2h\left\{\frac{\Delta f(a)}{h}\right\} + 2h^2A_2.$$

$$\Rightarrow \quad 2h^2A_2 = f(a+2h) - 2f(a+h) + f(a) = \Delta^2 f(a)$$

$$\Rightarrow \quad A_2 = \frac{\Delta^2 f(a)}{2!h^2}$$

Similarly, $A_3 = \dfrac{\Delta^3 f(a)}{3!h^3}$

$$\therefore \quad A_n = \frac{\Delta^n f(a)}{n!h^n}$$

Put the values of $A_0, A_1, A_2, A_3 \ldots A_n$, in the equation (1), we get

$$f(x) = f(a) + (x-a)\frac{\Delta f(a)}{h} + (x-a)(x-a-h)\frac{\Delta^2 f(a)}{2!h^2} + \ldots$$

$$(x-a)\ldots(x-a-\overline{n-1h})\frac{\Delta^n f(a)}{n!h^n}$$

Again, put $x = a + hu \Rightarrow u = \dfrac{x-a}{h}$, we have

$$f(a+hu) = f(a) + hu\frac{\Delta f(a)}{h} + \frac{(hu)(hu-h)}{2!h^2}\Delta^2 f(a) + \ldots \frac{(hu)(hu-h)(hu-2h)\ldots\left(hu-\overline{n-1}h\right)}{n!h^n}$$

$$\Rightarrow f(a + hu) = f(a) + \Delta uf(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n f(a)$$

which is required formula for forward interpolation.

This formula is useful for interpolating the values of $f(x)$ near the starting of the set of values given. h is called interval of differencing while Δ is forward difference operator. This is applicable for equally spaced argument only.

Example 1. *The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.*

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when x = 218 ft.

Sol. Let us form the difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.40					
150	13.03		–0.39				
		2.01		0.15			
200	15.04		–0.24		–0.07		
		1.77		0.08		0.02	
250	16.81		–0.16		–0.05		0.02
		1.61		0.03		0.04	
300	18.42		–0.13		–0.01		
		1.48		0.02			
350	19.9		–0.11				
		1.37					
400	21.27						

Here, $h = 50$, $a = 100 = x_0$

$$\therefore\ u = \frac{x - x_0}{h} = \frac{218 - 100}{50} = \frac{118}{50} = 2.36$$

Now, applying Newton's forward difference formula.

$$\therefore\ f(218) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \ldots + \frac{u(u-1)\ldots(u-5)}{6!}\Delta^6 f(a)$$

$$= 10.63 + 2.36 \times 2.40 + \frac{2.36(1.36)}{2} \times (-0.39) + \frac{2.36(1.36)(0.36)}{6} \times (0.15)$$

$$+ \frac{2.36\,(1.36)\,(0.36)\,(-0.64)}{24} \times (-0.07) + \frac{2.36\,(1.36)\,(0.36)\,(-0.64)\,(-1.64)}{120} \times (0.02)$$

$$+ \frac{2.36\,(1.36)\,(0.36)\,(-0.64)(-1.64)\,(-2.64)}{720} \times 0.02$$

$= 10.63 + 5.664 - 0.625872 + 0.0288864 + 0.002156 + 0.00020212776 - 0.00008893621$

$= 16.3252453777 - 0.62596093621$

$= 15.69928$ (Approx.)

Example 2. *Ordinates f(x) of a normal curve in terms of standard deviation x are given as:*

x	1.00	1.02	1.04	1.06	1.08
$f(x)$	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation x = 1.025.

Sol. Let us first form the difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4
1.00	0.2420				
		–0.0049			
1.02	0.2371		0.0001		
		–0.0048		–0.0001	
1.04	0.2323		0		0.001
		–0.0048		0	
1.06	0.2275		0		
		–0.0048			
1.08	0.2227				

Here, $h = 0.02$, $a = 1.00$, $x = 1.025$

$$\therefore \quad u = \frac{1.025-1.00}{0.02} = 1.25$$

$$\therefore f(1.025) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

Now on putting values of various functions, we get

$$0.2420 + 1.25 \times (-0.0049) + \frac{1.25(0.25)}{2} \times (0.0001) + \frac{1.25(0.25)(-0.75)}{6} \times (0.0001)$$

$$+ \frac{1.25(0.25)(-0.75)(-1.75)}{24} \times (0.0001)$$

$$= 0.2420 - 0.006125 + 0.000015625 + 0.000003906 + 0.000001708$$

$$= 0.242021239 - 0.006125 = 0.235896239 \text{ (Approx.)}$$

Example 3. *Find the cubic polynomial which takes the following data:*

x	0	1	2	3
$f(x)$	1	0	1	10

Sol. Let us first form the difference table:

x	y	Δ	Δ^2	Δ^3
0	1			
		−1		
1	0		+2	
		1		0.6
2	1		8	
		9		
3	10			

$$\therefore f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) \quad ...(1)$$

Now, $x = a + hu$

$$u = \frac{x-a}{h} = \frac{x}{1}$$

$\Rightarrow$ $x = u$

Therefore from (1)

$$y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$= 1 + (-1)x + \frac{x(x-1)}{2} \times (2) + \frac{x(x-1)(x-2)}{6} \times 6$$

$$= 1 - x + x^2 - x + x^3 - 2x^2 - x^2 + 2x$$

$$= x^3 - 2x^2 + 1$$

Example 4. *Find the number of students from the following data who secured marks not more than 45.*

Marks range	*30 – 40*	*40 – 50*	*50 – 60*	*60 – 70*	*70 – 80*
No. of students	*35*	*48*	*70*	*40*	*22*

Sol. Let us form a difference table as,

x	y	Δ	Δ^2	Δ^3	Δ^4
40	35				
		48			
50	83		22		
		70		–52	
60	153		–30		64
		40		12	
70	193		–18		
		22			
80	215				

Here, $h = 10$, $a = 40$, $x = 45$

$$\therefore u = \frac{x-a}{h} = \frac{45-40}{10} = 0.5$$

$$\therefore f(40+5\times 0.5) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$= 35 + 0.5\times 48 + \frac{0.5(-0.5)}{2}\times 22 + \frac{0.5(-0.5)(-1.5)}{6}\times(-52) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}\times(64)$$

$= 35 + 24 - 2.75 - 3.25 -2.5$

$= 59 - 8.5$

$= 50.5 \cong 51$

$\therefore$ No. of students who secured not more than 45 marks are 51.

Example 5. *Use Newton's Method to find a polynomial p(x) of lowest possible degree such that p(n) = 2^n = 0, 1, 2, 3, 4*

Sol. Since $P(n) = 2^n$ for $n = 0, 1, 2, 3, 4$

$A = 0, h = 0.1$

$\therefore \quad x = a + hu$

or $\quad u = x$

n	$P(n)$	Δ	Δ^2	Δ^3	Δ^4
0	1				
		1			
1	2		1		
		2		1	
2	4		2		1
		4		2	
3	8		4		
		8			
4	16				

From Newton's formula,

$$P(n) = P(0)+x\Delta P(0)+\frac{x(x+1)}{2!}\Delta^2 P(0)+\frac{x(x-1)(x-2)}{3!}\Delta^3 P(0)+\frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 P(0)$$

$$=1+x+\frac{x(x-1)}{2}+\frac{x(x-1)(x-2)}{6}+\frac{x(x-1)(x-2)(x-3)}{24}$$

$$=1+x+\frac{x^2}{2}-\frac{x}{2}+\left(\frac{x^3}{6}-\frac{3x^2}{6}+\frac{x}{3}\right)+\left(\frac{x^4}{24}-\frac{x^3}{4}+\frac{11x^2}{24}-\frac{x}{4}\right)$$

$$=\frac{x^4}{24}-\frac{x^3}{12}+\frac{11}{12}x^2+\frac{7}{12}x+1.$$ **Ans.**

Example 6. *Find the value of sin 52° from the given table:*

θ	45°	50°	55°	60°
$\sin\theta$	0.7071	0.7660	0.8192	0.8660

Sol. Here $a = 45°$, $h = 5$, $x = 52$ therefore $u = 1.4$

$x°$	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	7071			
		589		
50°	7660		–57	
		532		–7
55°	8192		–64	
		468		
60°	8660			

By forward interpolation formula

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \ldots\ldots$$

$$\Rightarrow \qquad 10^4 f(x) = 7071 + (1.4)589 + \frac{(1.4)(0.4)}{2}(-57) + \frac{(1.4)(0.4)(-0.6)}{6}(-7)$$

$$= 7880$$

$$\therefore \qquad f(52) = 0.7880 \quad \text{or} \quad \sin 52^\circ = 0.7880$$

Example 7. *Following are the marks obtained by 492 candidates in a certain examination:*

Marks	0 – 40	40 – 45	45 – 50	50 – 55	55 – 60	60 – 65
No. of Candidates	210	43	54	74	32	79

Find out

(a) No. of candidates, if they secure more than 48 but less than 50 marks.

(b) Less than 48 but not less than 45 marks.

Sol. First we form the difference table as follows:

Marks less than	*No. of Candidates*	Δ	Δ^2	Δ^3	Δ^4	Δ^5
40	210					
		43				
45	253		11			
		54		9		
50	307		20		–71	
		74		–62		222
55	381		–42		151	
		32		89		
60	413		47			
		79				
65	492					

Here, $h = 5$,

For (a), $x = 48$, $a = 40$, $\therefore u = \frac{48-40}{5} = 1.6$

Now on applying Newton's forward difference formula, we have

$$f(48) = 210 + 1.6 \times 4.3 + \frac{1.6 \times 0.6}{2} \times 11 + \frac{1.6 \times (0.6)(-0.4)}{6} \times 9 + \frac{1.6 \times (0.6)(-0.4)(-1.4)}{24} \times (-71)$$

$$+ \frac{1.6 \times (0.6)(-0.4)(-1.4)(-2.4)}{120} \times 222$$

$$= 210 + 68.8 + 5.28 - 1.5904 - 0.576 - 2.38694$$

$$= 284.08 - 4.553344 = 279.52 \cong 280$$

∴ No. of students secured more than 48 marks but less than 50 marks = 307 – 280 = 27

for (b), $x = 48$, $a = 40$, ∴ $u = 1.6$

$$f(48) = 210 + 1.6 \times 4.3 + \frac{1.6 \times 0.6}{2} \times 11 + \frac{1.6 \times (0.6)(-0.4)}{6} \times 9 + \frac{1.6 \times (0.6)(-0.4)(-1.4)}{24} \times (-71)$$

$$+ \frac{1.6 \times (0.6)(-0.4)(-1.4)(-2.4)}{120} \times 222$$

$$= 279.52 \cong 280$$

∴ No. of students secured more than 45 marks but less than 48 marks = 280 –253 = 27

Example 8. *Use Newton's forward difference formula to obtain the interpolating polynomial f(x) satisfying the following data:*

x	1	2	3	4
$f(x)$	26	18	4	1

If another point x = 5, f(x) = 26 is added to the above data, will the interpolating polynomial be the same as before or different. Explain why?

Sol. The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
1	26				
		–8			
2	18		–6		
		–14		17	
3	4		11		0
		–3		17	
4	1		28		
		25			
5	26				

As we seen here that its third difference is being constant either on taking fifth entry or on not taking fifth entry (*i.e.,* 26). So the interpolating polynomial should be same as before, and no change will occur.

Here, $h = 1,\ x = 5,\ a = 1$

∴ $u = \frac{x-a}{h} = 4$

$$f(x) = 26 + (x-1)(x-8) + \frac{(x-1)(x-2)}{2!} \times (-6) + \frac{(x-1)(x-2)(x-3)}{3!} \times 17$$

$$= 26 - 8x + x - 3x^2 + 9x - 6 - \left(x^3 - 6x^2 + 11x - 6\right)\frac{17}{6}$$

$$= 28 + x - 3x^2 + \frac{17}{6}x^3 - 17x^2 + \frac{187}{6}x - 17$$

$$= \frac{17}{6}x^3 - 20x^2 + \frac{193}{6}x + 11$$

which is the required interpolating polynomial.

Example 9. *If p , q, r, s be the successive entries corresponding to equidistant arguments in a table. Show that when third differences are taken into account, the entry corresponding to the argument half way between the arguments at q and r is* $A + \frac{B}{24}$, *where A is the arithmetic mean of q and r and B is arithmetic mean of 3q − 2p − s and 3r − 2s − p.*

Sol. Given A is the arithmetic mean of q and r

$$\Rightarrow \qquad A = \frac{q + r}{2}$$

$$\Rightarrow \qquad q + r = 2A$$

Also, B is the arithmetic means of $3q - 2p - s$ and $3r - 2s - p$.

$$\Rightarrow \qquad \text{B} = \frac{3q - 2p - s + 3r - 2s - p}{2} = \frac{3q - 3p - 3s + 3r}{2}$$

$$\text{B} = \frac{3(q + r)}{2} - \frac{3(p + s)}{2}$$

Let quantities p, q, r and s corresponds to argument a, $a + h$, $a + 2h$ and $a + 3h$ respectively then the value of the argument lying half way between $a + h$ and $a + 2h$ will be

$$a + h + \frac{h}{2} \text{ i.e., } a + \frac{3h}{2}$$

Hence, $\qquad a + mh = a + \frac{3}{2}h$

$$\Rightarrow \qquad m = \frac{3}{2}$$

Now, difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	p			
		$q - p$		
$a + h$	q		$r - 2q + p$	
		$r - q$		$s - 3r + 3q - p$
$a + 2h$	r		$s - 2r + q$	
		$s - r$		
$a + 3h$	s			

Using Newton's forward interpolation formula up to third difference only and taking $m = 3/2$, we get

$$f\left(a+\frac{3}{2}h\right) = f(a)+\frac{3}{2}\Delta f(a)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\Delta^2 f(a)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{6}\Delta^3 f(a)$$

$$= p+\frac{3}{2}(q-p)+\frac{3}{8}(r-2q+p)-\frac{1}{16}(s-3r+3q-p)$$

$$= \frac{1}{16}(-p+9q+9r-s)=\frac{9}{16}(q+r)-\left(\frac{p+s}{16}\right)$$

$$= \frac{9}{16}(2A)-\frac{2}{3}\left(\frac{3A-B}{16}\right)=\frac{9}{8}A-\frac{1}{8}A+\frac{B}{24}$$

$$= A+\frac{B}{24}$$

Example 10. *The table below shows value of tan x for 0.10* $\le x \le$ 0.30.

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Evaluate tan 0.12 using Newton's forward difference table.

Sol. Let us, first form the difference table.

x	$\tan x$	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

Here, $h = 0.05$

$a = 0.10,\ x = 0.12$

$$\therefore\ u = \frac{x-a}{h}=\frac{0.12-0.10}{0.05}=\frac{0.02}{0.05}=0.4$$

$\therefore$ By using Newton's forward difference formula, we get

$$f(u) = \tan(0.12) = f(a) + \Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$= 0.1003 + 0.4 \times 0.0508 + \frac{0.4(-0.6)}{2} \times (0.0008) + \frac{0.4(-0.6)(-1.6)}{6} \times (0.0002)$$

$$+ \frac{0.4(-0.6)(-1.6)(-2.6)}{24} \times (0.0002)$$

$= 0.1003 + 0.02032 - 0.000096 + 0.0000128 - 0.00000832$

$= 0.1206328 - 0.00010432$

$= 0.12052848$ Approx.

PROBLEM SET 4.1

1. The population of a town in the decimal census was as given below. Estimate the population for the year 1895.

Year x	1891	1901	1911	1921	1931
Population y (in thousands)	46	66	81	93	101

[**Ans.** 54.8528 thousands]

2. If l_x represents the number of persons living at age x in a life table, find as accurately as the data will permit l_x for values of x = 35, 42 and 47. Given $l_{20} = 512$, $l_{30} = 390$, $l_{40} = 360$, $l_{50} = 243$.

[**Ans.** 394, 326, 274]

3. From the following table, find the value of $e^{0.24}$

x	0.1	0.2	0.3	0.4	0.5
e^x	1.10517	1.22140	1.34986	1.49182	1.64872

[**Ans.** $e^{0.24}=1.271249$]

4. From the table, Estimate the number of students who obtained marks between 40 and 45.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

[**Ans.** 17]

5. Find the cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

[**Ans.** $2x^3-7x^2+6x+1$]

6. Find the number of men getting wages between Rs. 10 and Rs.15 from following table:

Wages (in Rs.)	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	9	30	35	42

[**Ans.** 15]

7. The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45-50 and 50-55.

Age	25 – 35	35 – 45	45 – 55	55 – 65
Deaths	13229	18139	24225	31496

[**Ans.** 11278, 12947]

8. The following table give the marks secured by 100 students in Mathematics:

Range of marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	25	35	22	11	7

Use Newton's forward difference interpolation formula to find

(1) the number of students who got more than 55 marks.

(2) the number of students who secured marks in the range from 36 to 45.

[**Ans.** (i) 28, (ii) 36]

9. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46.

Age	45	50	55	60	65
Premium (in Rupees)	114.84	96.16	83.32	74.48	64.48

[**Ans.** Rs. 110.52]

4.3 NEWTON'S GREGORY FORMULA FOR BACKWARD INTERPOLATION

Let $y = f(x)$ be a function of x which takes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ... $f(a+ nh)$ for $(n + 1)$ equally spaced values a, $a + h$, ..., $a + nh$ of the independent variable x. Let us assume $f(x)$ be a nth degree polynomial given by

$$f(x) = A_0 + A_1(x - a - nh) + A_2(x - a - nh)(x - a - \overline{n-1}h) + ...A_n(x - a - hn)(x - a - \overline{n-1}h) \\ ... (x - a - h) \quad ...(1)$$

Where A_0, A_1, A_2, ...A_n are to be determined.

Put $\quad x = a + nh$, $a + \overline{n-1}h$,... a in (1) respectively.

Put $\quad x = a + nh$, then $f(a + nh) = A_0 \quad ...(2)$

Put $\quad x = a + (n - 1)h$, then

$$f(a + \overline{n-1}h) = A_0 - hA_1 = f(a + nh) - hA_1 \text{ (from 2)}$$

$$\Rightarrow \quad A_1 = \frac{\nabla f(a + nh)}{h} \quad ...(3)$$

Put $\quad x = a + (n - 2)h$, then

$$f(a + \overline{n-2}h) = A_0 - 2hA_1 + (-2h)(-h)A_2$$

$$\Rightarrow \quad 2!h^2A_2 = f(a + \overline{n-2}h) - f(a + nh) + 2\nabla f(a + nh) = \nabla^2 f(a + nh)$$

$$A_2 = \frac{\nabla^2 f(a + nh)}{2!h^2} \quad ...(4)$$

Proceeding in similar way, $A_n = \dfrac{\nabla^n f(a+nh)}{n!h^n}$ ...(5)

Substituting the vaues in (1), we get

$$\left[f(x) = f(a+nh) + (x-a-nh)\frac{\nabla f(a+nh)}{h} + ... + (x-a-nh)(x-a-\overline{n-1}h)\right.$$

$$\left. ....(x-a-h)\frac{\nabla^n f(a+nh)}{n!h^n}\right] \quad ...(6)$$

Put $x = a + nh + uh$, then $x - a - nh = uh$

and $x - a - (n-1)h = (u+1)h$

$$x - a - h = \left(u + \overline{n-1}\right)h$$

$\therefore$ Equation (6) becomes

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + ... + uh(u+1)h \ ...(u+\overline{n-1}h)\nabla^n \frac{f(a+nh)}{n!h^n}$$

or $$f(a+nh+uh) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + ... + \frac{u(u+1)(u+\overline{n-1}h)}{n!}\nabla^n f(a+nh)$$

Which in required Newton's Gregory formular for backward interpolation. This formula is used when we want to interpolate the value near the end of the table.

Example 1. *The table below gives the value of tan x for 0.10 ≤ x ≤ 0.30*

x	:	0.10	0.15	0.20	0.25	0.30
$\tan x$	:	0.1003	0.1511	0.2027	0.2553	0.3093

Find (a) tan 0.50 (b) tan 0.26 (c) tan 0.40

Sol. First of all we construct the difference table:

x	$\tan x$	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

(*a*) Here, $h = 0.05,\ a = 0.30,\ x = 0.50$

$$\therefore \qquad u = \frac{0.50 - 0.30}{0.05} = 4$$

$$\tan(0.50) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$$= 0.3093 + 4 \times 0.0540 + \frac{4\times5}{2}\times0.0014 + \frac{4\times5\times6}{6}\times0.0004 + \frac{4\times5\times6\times7}{24}\times0.0002$$

$$= 0.3093 + 0.216 + 0.014 + 0.008 + 0.007$$

$$= 0.5543$$

(*b*) Here, $h = 0.05,\ a = 0.30,\ x = 0.26$

$$\therefore \qquad u = \frac{0.26 - 0.30}{0.05} = -0.8$$

$$= 0.3093 + (-0.8)\times0.054 + \frac{(-0.8)\times(0.2)}{2}\times0.0014 + \frac{(-0.8)\times(0.2)\times(1.2)}{6}\times0.0004$$

$$+ \frac{(-0.8)\times(0.2)\times(1.2)\times(2.2)}{24}\times0.0002$$

$$= 0.3093 - 0.0432 - 0.000112 - 0.0000128 - 0.00000352$$

$$= 0.3093 - 0.04332882 \cong 0.2662$$

(*c*) Here, $h = 0.05,\ a = 0.30,\ x = 0.40$

$$\therefore \qquad u = \frac{0.40 - 0.30}{0.05} = 2$$

$$f(0.40) = \tan(0.40) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a)$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

On putting the subsequent values, we get

$$f(0.40) = 0.3093 + 2 \times 0.054 + \frac{2\times3}{2}\times0.0014 + \frac{2\times3\times4}{6}\times0.0004 + \frac{2\times3\times4\times5}{24} = 0.0002$$

$$= 0.3093 + 0.108 + 0.0042 + 0.0016 + 0.0001$$

$$= 0.4241.$$

Example 2. *Using Newton's backward difference formula find the value of* $e^{-1.9}$ *from the following table of value of* e^{-x}.

x	1	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

Sol. Difference table for the given data as follows:

x	e^{-x}	∇	∇^2	∇^3	Δ^4
1	0.3679				
		–0.0814			
1.25	0.2865		0.0180		
		–0.0634		–0.0039	
1.50	0.2231		0.0141		0.0006
		–0.0493		–0.0033	
1.75	0.1738		0.0108		
		–0.0385			
2.00	0.1353				

Here, $u = \dfrac{1.9-2}{0.25} = -0.4$

using Newton's backward difference formula

$$f(e^{-x}) = f(a)+u\nabla f(a)+\frac{u(u+1)}{2!}\nabla^2 f(a)+\frac{u(u+1)(u+2)}{3!}\nabla^3 f(a)+\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

On putting the subsequent values, we get

$$f(e^{-x}) = 0.1353 + (-0.4) \times (-0.0385) + \frac{(-0.4)\times(-0.441)}{2}\times 0.0108$$

$$+\frac{(-0.4)\times(-0.4+1)(-0.4+2)}{6}\times(0.0033) + \frac{(-0.4)\times(-0.4+1)(-0.4+2)(-0.4+3)}{24}\times 0.0006$$

$$= 0.1353+0.0154 - 0.001296 + 0.0002112 + 0.000024$$

$$= 0.14959$$

Example 3. *In the following table, values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series.*

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Sol. The difference table for the given date is as follows:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	4.8			
		3.6		
4	8.4		2.5	
		6.1		0.5
5	14.5		3	
		9.1		0.5
6	23.6		3.5	
		12.6		0.5
7	36.2		4	
		16.6		0.5
8	52.8		4.5	
		21.1		
9	73.9			

To obtain first term, we use Newton's forward interpolation formula,

Here, $a = 3$, $h = 1$, $x = 1$ $\therefore$ $u = -2$

Hence we have

$$f(1) = y_3 + u\Delta y_3 + \frac{u(u-1)}{2!}\Delta^2 y_3 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_3$$

On putting the subsequent values, we get

$$f(1) = 4.8 + (-2)\times 3.6 + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5) = 3.1$$

Similarly, to obtain tenth term, we use Newton's backward interpolation formula. So

$$a + nh = 9, \quad h = 1, \; a + nh + uh = 10$$

$$\therefore \quad u = 1$$

$$\Rightarrow \quad f(10) = y_9 + u\nabla y_9 + \frac{u(u+1)}{2!}\nabla^2 y_9 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_9$$

$$= 73.9 + 21.1 + 4.5 + 0.5 = 100$$

Example 4. *Find the value of an annuity at* $5\frac{3}{8}\%$, *given the following table:*

Rate	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
Annuity Value	172.2903	162.888	153.7245	145.3375	137.6483

Sol. Difference table,

Rate	*Annuity Values*	∇	∇^2	∇^3	∇^4
4	172.2903				
		–9.4014			
$4\frac{1}{2}$	162.888		0.237		
		–9.1644		0.537	
5	153.7245		0.774		–0.6132
		–8.387		–0.0762	
$5\frac{1}{2}$	145.3375		0.6978		
		–7.6892			
6	137.6483				

$$x = 5\frac{3}{8} = \frac{43}{8}, a = 6, n = \frac{1}{2}$$

$$y = y(6) + (-1.25)\nabla y(6) + \frac{(-1.25)(-0.25)}{2!}\nabla^2 y(6) + \frac{(-1.25)(-0.25)(0.75)}{3!}\nabla^3 y(6)....$$

$$= 137.6483 + (-1.25)(-7.6892) + \frac{(-1.25)(-0.25)}{2!}(0.6978)$$

$$+ \frac{(-1.25)(-0.25)(0.75)}{3!}(-0.0762) + \frac{(-1.25)(-0.25)(0.75)(1.75)}{4!}(0.6132)....$$

$= 147.2251$ Approx.

Example 5. *In an examination, the number of candidates who obtained marks between certain limits are as follows:*

Marks	0 – 19	20 – 39	40 – 59	60 – 79	80 – 99
No. of candidates	41	62	65	50	17

Find no. of candidates who obtained fewer than 70 marks.

Sol. First, we form the difference table.

Marks less than (x)	*No.of candidates* (y)	∇	∇^2	∇^3	∇^4
19	41				
		62			
39	103		3		
		65		–18	
59	168		–15		0
		50		–18	
79	218		–33		
		17			
99	235				

Here, we have $h = 20$, $a = 99$

$$\therefore\ u = \frac{70-99}{20} = -1.45$$

Now on applying 'Newton's backward difference formula, we get

$$f(70) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$$= 235 + (-1.45)(17) + \frac{(-1.45)(-0.45)}{2}\times(-33) + \frac{(-1.45)(-0.45)(0.55)}{6}\times(-18)$$

$$= 235 - 24.65 - 10.76625 - 1.076625$$

$$= 235 - 36.492875$$

$$= 198.507 \cong 198$$

$\therefore$ Total no. of candidates who obtained fewer than 70 marks are 198.

Example 6. *The area A of a circle of diameter d is given for the following values:*

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find A for 105.

Sol. First of all we form the difference table as follow:

d (x)	A $f(x)$	∇	∇^2	∇^3	∇^4
80	5026				
		648			
85	5674		40		
		688		–2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Here, $h = 5$, $a = 100$, $x = 105$

$$\therefore\ u = \frac{105-100}{5} = 1$$

Now on applying Newton's backward difference formula, we have

$$f(105) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$$= 7854 + 1\times766 + \frac{40\times2}{2} + \frac{1\times2\times3}{6}\times2 + \frac{1\times2\times3\times4}{24}\times4$$

= 7854 + 766 + 46

= 8666

Which is the required area for the given diameter of circle.

Example 7. *The probability integral* $P = \sqrt{\frac{2}{\pi}}\int_0^{\pi} e^{-\frac{1}{2}t^2}\, dt$ *has the following values:*

x	1.00	1.05	1.10	1.15	1.20	1.25
P	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Calculate P for x = 1.235

Sol. First we form the difference table

x	$f(x)$	∇	∇^2	∇^3	∇^4	∇^5
1.00	0.682689					
		0.023593				
1.05	0.706282		–0.001207			
		0.022386		0.000009		
1.10	0.728668		–0.001198		0.000006	
		0.021188		0.000015		–0.000004
1.15	0.749856		–0.001183		0.000002	
		0.020005		0.000017		
1.20	0.769861		–0.001166			
		0.018839				
1.25	0.788700					

Here, $h = 0.05$ $a = 1.20$

$$\therefore\ u = \frac{1.235 - 1.20}{0.05} = -0.3$$

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a) + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f(a)$$

$$= 0.788700 + (-0.3)(0.018839) + \frac{(-0.3)(0.7)}{2}(-0.001166) + \frac{(-0.3)(0.7)(1.7)}{6}(0.000017) + \frac{(-0.3)(0.7)(1.7)(2.7)}{24}(0.000002) + \frac{(-0.3)(0.7)(1.7)(2.7)(3.7)}{120}(0.000004)$$

$= 0.788700 - 0.0056517 + 0.0012243 - 0.0000010115 - 0.00000008032$

$= 0.7888225488 - 0.00566189532$

$= 0.78316065356$

Example 8: *Calculate the value of tan 48°15′ from the following table:*

$x°$	45°	46°	47°	48°	49°	50°
tan x	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

Sol. Given that $a + nh = 50$

$h = 1$

$a + nh + uh = 48°15' = 48.25°$

$\therefore \quad 50 + u = 48.25$

$\Rightarrow \quad u = -1.75$

The difference table for given data is as follows:

$x°$	$10^5 y$	$10^5 \nabla y$	$10^5 \nabla^2 y$	$10^5 \nabla^3 y$	$10^5 \nabla^4 y$	$10^5 \nabla^5 y$
45°	100000					
		3553				
46°	103553		131			
		3684		9		
47°	107237		140		3	
		3824		12		–5
48°	111061		152		–2	
		3976		10		
49°	115037		162			
		4138				
50°	119175					

$$y(a+nh) = y_{a+nh} + u\nabla y_{a+nh} + \frac{u(u+1)}{2!}\nabla^2 y_{a+nh} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{a+nh} + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{a+nh} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{a+nh}$$

$$10^5 y_{48.25} = 119175 + (-1.75)\times 4138 + \frac{(-1.75)(-0.75)}{2}\times 162 + \frac{(-1.75)(-0.75)(0.25)}{6}\times 10 + \frac{(-1.75)(-0.75)(0.25)(1.25)}{24}\times(-2) + \frac{(-1.75)(-0.75)(0.25)(1.25)(2.25)}{120}\times(-5)$$

$$10^5 y_{48.25} = 112040.2867$$

$$\Rightarrow \quad y_{48.25} = \tan 48°15' = 1.120402867.$$

PROBLEM SET 4.2

1. The population of a town is as follows:

Year	1921	1931	1941	1951	1961	1971
Population (in lakhs)	20	24	29	36	46	51

Estimate the increase in population during the period 1955 to 1961

[**Ans.** 621036.8 lakhs.]

2. From the following table find the value of tan 17°.

θ	0	4	8	12	16	20	24
$\tan\theta$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4402

[**Ans.** 0.3057]

3. From the given table find the value of log 5875

x	40	45	50	55	60	65
$\log x$	1.60206	1.65321	1.69897	1.74036	1.77815	1.81291

[**Ans.** 3.7690058]

4. From the following table, find y when $x = 1.84$ and 2.4

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

[**Ans.** 6.36, 11.02]

5. From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policy maturing at the age of 63:

Age	45	50	55	60	65
Premium (in Rs.)	114.84	96.16	83.32	74.48	68.48

[**Ans.** 70.585152]

6. The values of annuities are given for the following ages. Find the value of annuity at the age of $27\frac{1}{2}$.

Age	25	26	27	28	29
Annuity	16.195	15.919	15.630	15.326	15.006

[**Ans.** 15.47996]

7. Show that Newton's Gregory interpolation formula can be written in the form as

$$u_x = u_0 + x\Delta u_0 - xa\Delta^2 u_0 + xab\Delta^3 u_0 - xabc\Delta^4 u_0 + ...$$

where $a = 1 - \frac{1}{2}(x+1)$, $b = 1 - \frac{1}{3}(x+1)$, $c = 1 - \frac{1}{4}(x+1)$ etc.

4.4 CENTRAL DIFFERENCE FORMULAE

As earlier we study formulae for leading terms and differences. These formulae are fundamental and are applicable to nearly all cases of interpolation, but they do not converge as rapidly as central difference formulae. The main advantage of central difference formulae is that they give more accurate result than other method of interpolation. Their disadvantages lies in complicated calculations and tedious expression, which are rather difficult to remember. These formulae are used for interpolation near the middle of a argument values. In this category we use the following formulae:

4.4.1 Gauss Forward Difference Formula

We know Newton's Gregory forward difference formula is given by

$$f(a+hu) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)+\ldots \quad \ldots(1)$$

Substitute $a = 0$, $h = 1$ in (1), we get

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(0)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(0)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0)+\ldots \quad \ldots(2)$$

Now obtain the values of $\Delta^2 f(0), \Delta^3 f(0), \Delta^4 f(0)\ldots$

To get these values,

$$\Delta^3 f(-1) = \Delta^2 f(0)-\Delta^2 f(-1)$$

$$\Rightarrow \quad \Delta^2 f(0) = \Delta^3 f(-1)+\Delta^2 f(-1)$$

Also, $\Delta^4 f(-1) = \Delta^3 f(0)-\Delta^3 f(-1)$

$$\Rightarrow \quad \Delta^3 f(0) = \Delta^4 f(-1)+\Delta^3 f(-1)$$

$$\Delta^5 f(-1) = \Delta^4 f(0)-\Delta^4 f(-1)$$

$$\Rightarrow \quad \Delta^4 f(0) = \Delta^5 f(-1)+\Delta^4 f(-1)$$

$$\Delta^6 f(-1) = \Delta^5 f(0)-\Delta^5 f(-1)$$

$$\Rightarrow \quad \Delta^5 f(0) = \Delta^6 f(-1)+\Delta^5 f(-1)$$ .... and so on.

Substituting these values in equation (2)

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\left[\Delta^3 f(-1)+\Delta^2 f(-1)\right]+\frac{u(u-1)(u-2)}{3!}[\Delta^4 f(-1)+\Delta^3 f(-1)]$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^5 f(-1)+\Delta^4 f(-1)\right]+\frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\left[\Delta^6 f(-1)+\Delta^5 f(-1)\right]+\ldots.$$

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2}\Delta^2 f(-1)+\frac{u(u-1)}{2!}\left\{1+\frac{(u-2)}{3}\right\}\Delta^3 f(-1)$$
$$+\frac{u(u-1)(u-2)}{6}\left\{1+\frac{(u-3)}{4}\right\}\Delta^4 f(-1)+\frac{u(u-1)(u-2)(u-3)}{24}\left\{1+\frac{(u-4)}{5}\right\}\Delta^5 f(-1)$$
$$+\frac{u(u-1)(u-2)(u-3)(u-4)}{120}\Delta^6 f(-1)+\ldots.$$

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1)$$
$$+\frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 f(-1)+.... \quad ...(3)$$

But $\Delta^5 f(-2) = \Delta^4 f(-1) - \Delta^4 f(-2)$

$\Rightarrow$ $\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$

and $\Delta^6 f(-2) = \Delta^5 f(-1) + \Delta^5 f(-2)$

$\Rightarrow$ $\Delta^5 f(-1) = \Delta^5 f(-2) + \Delta^6 f(-2)$

The equation (3) becomes

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^5 f(-2)+\frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 f(-2)$$
$$+\frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2)$$

$$\therefore f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2)+\frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2)+....$$

This formula is known as **Gauss forward difference formula.**

This formula is applicable when u lies between $0 \text{ and } \frac{1}{2}$

4.4.2 Gauss Backward Difference Formula

This formula is also solved by using Newton's forward difference formula.

Now, we know Newton's formula for forward interpolation is

$$f(a+hu) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)+.... \quad ...(1)$$

Put $a = 0$, and $h = 1$, in equation (1), we get

$$f(u) = f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(0)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(0)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0)+... \quad ...(2)$$

Now $\Delta f(0) = \Delta f(-1)+\Delta^2 f(-1)$

$\Delta^2 f(0) = \Delta^2 f(-1)+\Delta^3 f(-1)$

$\Delta^3 f(0) = \Delta^3 f(-1)+\Delta^4 f(-1)$

$\Delta^4 f(0) = \Delta^4 f(-1)+\Delta^5 f(-1)$ ... and so on.

On substituting these values in (2), we get

$$f(u) = f(0)+u\left[\Delta f(-1)+\Delta^2 f(-1)\right]+\frac{u(u-1)}{2!}\left[\Delta^2 f(-1)+\Delta^3 f(-1)\right]+\frac{u(u-1)(u-2)}{3!}\left[\Delta^3 f(-1)+\Delta^4 f(-1)\right]$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 f(-1)+\Delta^5 f(-1)\right]+....$$

$$\therefore\ f(u) = f(0) + u\Delta f(-1) + u\Delta^2 f(-1)\left[1 + \frac{(u-1)}{2}\right] + \frac{u(u-1)}{2}\Delta^3 f(-1)\left[1 + \frac{(u-2)}{3}\right]$$
$$+ \frac{u(u-1)(u-2)}{3}\Delta^4 f(-1)\left[1 + \frac{(u-3)}{4}\right] + \ldots$$
$$= f(0) + u\Delta f(-1) + \frac{(u+1)}{2!}u\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1) + \ldots \quad \ldots(3)$$

Again $\Delta^3 f(-1) = \Delta^3 f(-2) + \Delta^4 f(-2)$

$\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$

$\Delta^5 f(-1) = \Delta^5 f(-2) + \Delta^6 f(-2)$ ... and so on.

Therefore, equation (3) becomes

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\Delta^3 f(-2) + \Delta^4 f(-2)\right]$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\left[\Delta^4 f(-2) + \Delta^5 f(-2)\right] + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\left[\Delta^5 f(-2) + \Delta^6 f(-2)\right] + \ldots$$
$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+1)u(u-1)}{3!}\left[1 + \frac{(u-2)}{4}\right]\Delta^4 f(-2)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\left[1 + \frac{(u-3)}{5}\right]\Delta^5 f(-2) + \ldots$$
$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}$$
$$\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \ldots$$

This is known as **Gauss Backward difference formula** and useful when u lies between $-\frac{1}{2}$ and 0.

4.4.3 Stirling's Formula

This is another central difference formula and useful when $|u| < \frac{1}{2}\, or -\frac{1}{2} < u < \frac{1}{2}$. It gives best estimation when $-\frac{1}{4} < u \frac{1}{4}$. This formula is obtained by taken mean of Gauss forward and Gauss backward difference formula.

Gauss forward formula for interpolating central difference is,

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2) + \ldots \quad \ldots(1)$$

Gauss Backward difference is,

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}$$

$$\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + ... \quad ...(2)$$

Take mean of Equation (1) and (2)

$$f(u) = f(0) + \frac{u\left[\Delta f(0) + \Delta f(-1)\right]}{2} + \frac{u}{2!}\Delta^2 f(-1)\left[\frac{(u-1)+(u+1)}{2}\right]$$

$$+\frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right] + \frac{(u+1)u(u-1)}{3!}\Delta^4 f(-2)\left[\frac{(u-2)+(u+2}{2}\right]$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + ...$$

$$f(u) = f(0) + u\left(\frac{\Delta f(0) + \Delta f(-1)}{2}\right) + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left(\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right)$$

$$+\frac{(u+1)u^2(u-1)}{3!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + ...$$

This is called Stirling's formula.

4.4.4 Bessel's Interpolation Formula

This is one of the another type of central difference formula and obtained by (1) shifting the origin by 1 in Gauss backward difference and then (2) replacing u by $(u - 1)$, (3) take mean of this equation with Gauss forward formula.

Gauss backward difference formula is,

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!} + \Delta^5 f(-2) + ...$$

Now shift the origin by one, we get

$$f(u) = f(1) + u\Delta f(0) + \frac{(u+1)u}{2!}\Delta^2 f(0) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-1)$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-1) + ...$$

On replacing u by $(u - 1)$

$$f(u) = f(1) + (u-1)\Delta f(0) + \frac{(u-1)u}{2!}\Delta^2 f(0) + \frac{(u-1)u(u-2)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1)$$

$$+\frac{(u+1)((u-1)u(u-2)(u-3)}{5!}\Delta^5 f(-1) + ... \quad ...(1)$$

Gauss forward difference formula is,

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2) + ... \quad ...(2)$$

Take the mean of equation (1) and (2)

$$f(u)=\frac{f(0)+f(1)}{2}+\left\{\frac{(u-1)+u}{2}\right\}\Delta f(0)+\frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(0)+\Delta^2 f(-1)}{2}\right\}$$
$$+\frac{u(u-1)}{3!}\left\{\frac{u-2)+(u+1)}{2}\right\}\Delta^3 f(-1)+\frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 f(-1)+\Delta^4 f(-2)}{2}\right)$$
$$+\frac{(u+1)u(u-1)(u-2)}{5!}\left\{\frac{(u-3)\Delta^5 f(-1)+(u+2)\Delta^5 f(-2)}{2}\right\}+\ldots$$

$$f(u)=\frac{f(0)+f(1)}{2}+\left\{u-\frac{1}{2}\right\}\Delta f(0)+\frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(0)+\Delta^2 f(-1)}{2}\right\}+\frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 f(-1)$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 f(-1)+\Delta^4 f(-2)}{2}\right)+\frac{(u+1)u(u-1)(u-2)(u-1/2)}{5!}\Delta^5 f(-2)+\ldots.$$

This formula is very useful when $u=\frac{1}{2}$ and gives best result when $\frac{1}{4}<u<\frac{3}{4}$.

4.4.5 Laplace-Everett's Formula

Gausss forward formula is given by

$$f(u)=f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+\frac{(u+2)(u+1)u(u-1)(u\_2)}{5!}\Delta^5 f(-2)+\frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2)+\ldots \quad \ldots(1)$$

We know

$$\Delta f(0)=f(1)-f(0)$$
$$\Delta^3 f(-1)=\Delta^2 f(0)-\Delta^2 f(-1)$$
$$\Delta^5 f(-2)=\Delta^4 f(-1)-\Delta^4 f(-2)$$

Therefore, using this in equation (1), we get

$$f(u)=f(0)+u[f(1)-f(0)]+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}(\Delta^2 f(0)-\Delta^2 f(-1))$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\left(\Delta^4 f(-1)-\Delta^4 f(-2)\right)+\ldots$$
$$=(1-u)f(0)+uf(1)+\frac{u(u-1)}{2!}\Delta^2 f(-1)-\frac{(u+1)u(u-1)}{3!}\Delta^2 f(-1)$$
$$+\frac{(u+1)u(u-1)}{3!}\Delta^2 f(0)+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1)$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)-\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-2)\ldots..$$

$$= (1-u)f(0) + uf(1) + \frac{u(u-1)(2-u)}{3!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0)$$

$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1) + \frac{(u+1)u(u-1)(u-2)(3-u)}{5!}\Delta^4 f(-2) + \ldots$$

$$f(u) = \left\{uf(1) + \frac{u(u+1)(u-1)}{3!}\Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1) + \ldots\right\}$$

$$+ \left\{(1-u)f(0) + \frac{u(u-1)(2-u)}{3!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)(u-2)(3-u)}{5!}\Delta^4 f(-2) + \ldots\right\} \quad \ldots(2)$$

Substitute $1 - u = w$ in second part of equation (2)

$$f(u) = \left\{uf(1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1) + \ldots\right\}$$

$$+ \left\{wf(0) + \frac{(w-1)w(w+1)}{3!}\Delta^2 f(-1) + \frac{(w+2)(w+1)w(w-1)(w-2)}{5!}\Delta^4 f(-2) + \ldots\right\}$$

This is called Laplace-Everett's formula. It gives better estimate value when $u > \frac{1}{2}$.

Example 1. *From the following table, find the value of* $e^{1.17}$ *using Gauss forward formula:*

x	1	1.05	1.10	1.15	1.20	1.25	1.30
$e^x = f(x)$	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Sol. The difference table is as given:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1	2.7183						
		0.1394					
1.05	2.8577		0.0071				
		0.1465		0.0004			
1.10	3.0042		0.0075		0		
		0.154		0.0004		0	
1.15	3.1582		0.0079		0		0.0001
		1.619		0.0004		0.0001	
1.20	3.3201		0.0083		0.0001		
		0.1702		0.0005			
1.25	3.4903		0.0088				
		0.179					
1.30	3.6693						

Now, let taking origin at 1.15, here $h = 0.05$

Then, $u = \dfrac{1.17 - 1.15}{0.05} = 0.4$

On applying Gauss's forward interpolation, we have

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + ...$$

$$f(0.4) = 3.1582 + 0.4 \times (0.1619) + \frac{(0.4)(-0.6)}{2} \times (0.0079) + \frac{(1.4)(0.4)(-0.6)}{6} \times (0.0004)$$
$$+0 + \frac{(1.4)(0.4)(-0.6)(-1.6)(2.4)}{120} \times 0.0001$$

$$f(0.4) = 3.1582 + 0.06476 - 0.000948 - 0.0000224 + 0.0000010752$$

$$f(0.4) = 3.22199 \text{ (Approx.)}$$

Example 2. *Given that*

x	25	30	35	40	45
$\log x$	1.39794	1.47712	1.54407	1.60206	1.65321

log 3.7 = ?

Sol.

x	$\log x$	$\Delta \log x$	$\Delta^2 \log x$	$\Delta^3 \log x$	$\Delta^4 \log x$
x_{-2} 25	1.39794				
		0.07918			
x_{-1} 30	1.47712		–0.01223		
		0.06695		0.00327	
x_0 35	1.54407		–0.00896		–0.00115
		0.05799		0.00212	
x_1 40	1.60206		–0.00684		
		0.05115			
x_2 45	1.65321				

$$x = a + hu,\ x = 37,\ a = 35,\ h = 5$$

$$u = \frac{x-a}{h} = \frac{37-35}{5} = 0.4$$

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1)$$

$$= 1.54407 + 0.4 \times 0.05798 + \frac{(0.4)(0.4-1)}{2} \times (-0.00896) + \frac{(0.4)(0.4-1)(0.4+1)}{6} \times (0.00212)$$

$$+ \frac{(0.4)(0.4-1)(0.4+1)(0.4-2)}{24} \times (-0.00115)$$

$$f(u) = 1.54407 + 0.023192 + 0.0010752 - 0.00011872 - 0.00002576 = 1.56819272$$

Since log 3.7 = $\log \frac{3.7 \times 10}{10} = \log \frac{37}{10}$

$\Rightarrow$ log 3.7 = log 37 – log 10 = 1.56819272 – 1 = 0.56819272. **Ans.**

Example 3. *From the following table find y when x = 1.45.*

x	1.0	1.2	1.4	1.6	1.8	2.0
y	0.0	–0.112	–0.016	+0.336	+0.992	2.0

Sol.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	0.0				
		–0.112			
1.2	–0.112		0.208		
		0.096		0.048	
1.4	–0.016		0.256		0
		0.352		0.048	
1.6	0.336		0.304		0
		0.656		0.048	
1.8	0.992		0.352		
		1.008			
2.0	2.0				

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$u = \frac{1.45 - 1.4}{0.2} = 0.25$$

$$f(u) = -0.016 + 0.25 \times 0.352 + \frac{(0.25)(-0.75)}{2} \times 0.256 + \frac{(0.25)(-0.75)(1.25)}{6} \times 0.048$$

$$= 0.047875$$

Example 4. *Use Gauss's forward formula to find a polynomial of degree four which takes the following values of the function f(x):*

x	1	2	3	4	5
$f(x)$	1	–1	1	–1	1

Sol. Taking center at 3 *i.e.*, $x_0 = 3$ and $h = 1$

$$u = \frac{x - x_0}{h}$$

$\Rightarrow$ $\qquad u = x - 3$

Now, for the given data difference table becomes:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	1				
		–2			
2	–1		4		
		2		–8	
3	1		–4		16
		–2		8	
4	–1		4		
		2			
5	1				

Gauss forward formula is

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$= 1 + (x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-3)(x-4)(x-2)}{6}(8) + \frac{(x-2)(x-3)(x-4)(x-5)}{24}(16)$$

$$= 1 - 2x + 6 - 2x^2 + 14x - 24 + \frac{4}{3}x^3 - 12x^2 - \frac{104}{3}x - 32 + \frac{2}{3}x^4 - \frac{28}{3}x^3 + \frac{142}{3}x^2 - \frac{308}{3}x + 80$$

$$\therefore f(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31. \quad \textbf{Ans.}$$

Example 5. *Use Gauss's forward formula to find* y_{30} *for the following data.*

y_{21}	y_{25}	y_{29}	y_{33}	y_{37}
18.4708	17.8144	17.1070	16.3432	15.5154

Sol. Let us take the origin at $x = 29$

then $$u = \frac{30 - 29}{4} = \frac{1}{4} = 0.25$$

Now, for the given data difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	18.4708				
		–0.6564			
25	17.8144		–0.0510		
		–0.7074		–0.0054	
29	17.1070		–0.0564		–0.0022
		–0.7638		–0.0076	
33	16.3432		–0.0640		
		–0.8278			
37	15.5154				

Putting these values in Gauss forward interpolation formula, we have

$$y_{0.25} = 17.1070 + (0.25) \times (-0.7638) + \frac{(0.25)(-0.750)}{2} \times (-0.0564)$$

$$+ \frac{(0.25)(-0.750)(1.25)}{6} \times (-0.0076) + \frac{(1.25)(0.25)(-0.750)(-1.75)}{24} \times (-0.0022)$$

$y_{0.25}$ = 17.1070 – 0.19095 + 0.0052875 + 0.00002968 – 0.00000375

$y_{0.25}$ = 16.9216. **Ans.**

PROBLEM SET 4.3

1. The values of e^{-x} at x = 1.72 to x = 1.76 are given in the following table:

x	1.72	1.73	1.74	1.75	1.76
$f(x)$	0.17907	0.17728	0.17552	0.17377	0.17204

Find the values of $e^{-1.7425}$ using Gauss forward difference formula

[**Ans.** 0.1750816846]

2. Apply Gauss's forward formula to find the value of $f(x)$ at x = 3.75 from the table:

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

[**Ans.** 19.407426]

3. Apply Central difference formula to obtain $f(32)$. Given that :
$f(25)$ = 0.2707, $f(30)$ = 0.3027, $f(35)$ = 0.3386, $f(40)$ = 0.3794. [**Ans.** 0.316536]

4. Apply Gauss forward formula to find the value of U_9, if
$u(0)$ = 14, $u(4)$ = 24, $u(8)$ = 32, u (12) = 35, $u(16)$ = 40 [**Ans.** 33.1162109]

5. Apply Gauss forward formula to find a polynomial of degree three which takes the values of y as given on next page:

x	2	4	6	8	10
y	–2	1	3	8	20

$$\left[\textbf{Ans. } 3+\frac{17}{6}x+\frac{3}{2}x^2+\frac{2}{3}x^3\right]$$

6. Use Gauss's forward formula to find the annuity value for 27 years from the following data:

Year	15	20	25	30	35	40
Annuity	10.3797	12.4622	14.0939	15.3725	16.3742	17.1591

[**Ans.** 14.643]

7. Use Gauss's forward formula to find the value of $f(x)$, when $x = \frac{1}{2}$, given that:

x	2	1	0	–1
$f(x)$	100	108	105	110

GAUSS BACKWARD

Example 1. *Given that*

$x°$	50	51	52	53	54
$\tan x°$	1.1918	1.2349	1.2799	1.3270	1.3764

Using Gauss's backward formula, find the value of tan 51° 42′

Sol. Take the origin at 52° and given $h = 1$

$$\therefore \qquad u = \frac{x-a}{h} = x - a = 51°\ 42' - 52° = -18' = -0.3°$$

Now using Gauss backward formula

$$f(u) = f(0) + u\,\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1)$$
$$+ \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)$$

Difference table for given data is:

$x°$	$tan\ x°$	Δ	Δ^2	Δ^3
50	1.1918			
		0.0431		
51	1.2349		0.0019	
		0.045		0.0002
52	1.2799		0.0021	
		0.0471		0.0002
53	1.3270		0.0023	
		0.0494		
54	1.3764			

From equation (1)

$$f(-0.3^{\circ}) = 1.2799 + (-0.3)\ (0.045) + \frac{(-0.3)\,(0.7)}{2} \times 0.0021 + \frac{(-0.3)\,(0.7)\,(1.7)}{6} \times 0.0002$$

$$= 1.2799 - 0.0135 - 0.0002205 - 0.0000119$$

$$= 1.266167 \text{ (Approx.)}$$

Example 2. *Apply Gauss backward formula to find sin 45° from the following table*

$\theta°$	20	30	40	50	60	70	80
Sin $\theta°$	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

Sol. Difference table for given data is:

$\theta°$	$sin\ \theta°$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
20	0.34202					
		0.15998				
30	0.502		–0.01919			
		0.14079		0.00165		
40	0.64274		–0.01754		–0.00737	
		0.12325		–0.00572		0.01002
50	0.76604		–0.02326		0.00265	
		0.09999		–0.00307		–0.00179
60	0.86603		–0.02633		0.00086	
		0.07366		–0.00221		
70	0.93969		–0.02854			
		0.04512				
80	0.98481					

$$\therefore \qquad u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

Now using Gauss backward formula

$$f(u) = f(0) + u\,\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \ldots$$

$$f(0.5) = 0.64279 + 0.5 \times 0.14079 + \frac{0.5\times 1.5}{2} \times (-0.01754) + \frac{0.5\times 1.5\,(-0.5)}{6} \times (0.00165)$$
$$+ \frac{0.5\times 1.5\,(-0.5)\,(2.5)}{24} \times (-0.00737)$$
$$= 0.64279 + 0.070395 - 0.0065775 - 0.000103125 + 0.00028789$$
$$= 0.706792.$$ **Ans.**

Example 3. *Apply Gauss backward formula to find the value of* $(1.06)^{19}$ *if*
$(1.06)^{10} = 1.79085$, $(1.06)^{15} = 2.39656$, $(1.06)^{20} = 3.20714$, $(1.06)^{25} = 4.29187$, $(1.06)^{30} = 5.74349$

Sol. The difference table is given by

x	y	Δ	Δ^2	Δ^3	Δ^4
10	1.79085				
		0.60571			
15	2.39656		0.20487		
		0.81058		0.06928	
20	3.20714		0.27415		0.02346
		1.08473		0.09274	
25	4.29187		0.36689		
		1.45162			
30	5.74349				

$$\therefore \qquad u = \frac{x-a}{h} = \frac{19-20}{5} = -0.2$$

From Gauss backward formula

$$f(u) = f(0) + u\,\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \ldots$$

$$f(u) = 3.20714 - 0.2 \times 0.81058 - \frac{0.2(0.8)}{2} \times 0.27415 - \frac{0.2(0.8)(-1.2)}{6} \times 0.06928$$
$$+ \frac{0.2\,(0.8)(-1.2)(1.8)}{24} \times 0.02346$$

$f(u) = 3.20714 - 0.162116 - 0.021932 + 0.002216 + 0.00033782$

$= 3.0256458$ (Approx.)

Example 4. *Using Gauss backward formula, Estimate the no. of persons earning wages between Rs. 60 and Rs. 70 from the following data:*

Wages (*Rs.*)	*Below* 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of Persons (*in thousands*)	250	120	100	70	50

Sol. Difference table for the given data is as:

Wages below	*No. of Persons*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		–20		
		100		–10	
80	470		–30		20
		70		10	
100	540		–20		
		50			
120	590				

$$\therefore \; u = \frac{x-a}{h} = \frac{70-80}{20} = \frac{-10}{20} = -0.5$$

From Gauss backward formula

$$f(0.5) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \ldots$$

$$= 470 + (-0.5)\times(+100) + \frac{(-0.5)(0.5)}{2}\times(-30) + \frac{(-0.5)(0.5)(-1.5)}{6}\times(-10) + \frac{(-0.5)(0.5)(-1.5)(1.5)}{24}\times(20)$$

$$= 470 - 50 + 3.75 - 0.625 + 0.46875$$

$$= 423.59375$$

Hence No. of Persons earning wages between Rs. 60 to 70 is 423.59375 – 370 = 53.59375 or 54000. (Approx.)

Example 5. *If f(x) is a polynomial of degree four find the value of f(5.8) using Gauss's backward formula from the following data:*

$f(4) = 270, \quad f(5) = 648, \quad \Delta f(5) = 682, \quad \Delta^3 f(4) = 132.$

Sol. Given $\Delta f(5) = 682$

$$\begin{aligned}
\Rightarrow \quad f(6) - f(5) &= 682 \\
\Rightarrow \quad f(6) &= 682 + 648 \\
\Rightarrow \quad f(6) &= 1330 \\
\text{Also,} \quad \Delta^3 f(4) &= 132 \\
\Rightarrow \quad (E - 1)^3 f(4) &= 132 \\
\Rightarrow \quad f(7) - 3f(6) + 3f(5) - f(4) &= 132 \\
\Rightarrow \quad f(7) &= 3 \times 1330 - 3 \times 648 + 270 + 132 \\
f(7) &= 2448
\end{aligned}$$

Now form difference table as:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
4	270			
		378		
5	648		304	
		682		132
6	1330		436	
		1118		
7	2448			

Take $a = 6$, $h = 1$ $a + hu = 5.8$

$\therefore \quad u = -0.2$

From Gauss backward formula

$$f(-0.2) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1))}{3!}\Delta^3 f(-2)$$

$$= 1330 + (-0.2) \times 682 + \frac{(-0.2)(0.8)}{2} \times 436 + \frac{(-0.2)(0.8)(-1.2)}{6} \times 132$$

$$= 1330 - 136.4 - 34.88 + 4.224$$

$$= 1162.944$$

Hence $f(5.8) = 1162.944$.

Example 6. *Using Gauss backward interpolation formula, find the population for the year 1936. Given that*

Year	1901	1911	1921	1931	1941	1951
Population (in thousands)	12	15	20	27	39	52

Sol. Here h = 10. Take origin at 1941 to evaluate population in 1936

$$\Rightarrow \quad u = \frac{x-a}{h} = \frac{1936-1941}{10} = \frac{-5}{10} = -0.5$$

Difference table for given data is as:

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$	$\Delta^5 f(u)$
–4	12					
		3				
–3	15		2			
		5		0		
–2	20		2		3	
		7		3		–10
–1	27		5		–7	
		12		–4		
0	39		1			
		13				
1	52					

Gauss backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)$$

$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-3) + \ldots.$$

$$= 39 + (-0.5)\times 12 + \frac{(-0.5)(0.5)}{2}\times 1 + \frac{(-0.5)(0.5)(-1.5)}{6}\times(-4)$$

$= 39 - 6.0 - 0.125 - 0.25$

$= 32.625$ thousands

Hence, the population in 1936 is 32625 thousand.

PROBLEM SET 4.4

1. Given that $\sqrt{12500}$ = 111.803399, $\sqrt{12510}$ = 111.848111, $\sqrt{12520}$ = 111.892806, $\sqrt{12530}$ = 111.937483. Using Gauss's backward formula show that $\sqrt{12516}$ = 111.8749301

2. Find the value of cos 51°42[1] by Gauss's backward formula from the following data:

x	50°	51°	52°	53°	54°
cos x	0.6428	0.6293	0.6157	0.6018	0.5878

[**Ans.** 0.61981013]

3. The population of a town in the years are as follows:

Year	1931	1941	1951	1961	1971
Population (in thousands)	15	20	27	39	52

Find the population of the town in 1946 by applying Gauss's backward formula.

[**Ans.** 22898]

4. Interpolate by means of Gauss's backward formula, the population of a town KOSIKALAN for the year 1974, given that:

Year	1939	1949	1959	1969	1979	1989
Population (in thousands)	12	15	20	27	39	52

[**Ans.** 32.345 thousands appros]

5. Use Gauss interpolation formula to find y_{41} from the following data:

y_{30} = 3678.2, y_{35} = 2995.1, y_{40} = 2400.1, y_{45} = 1876.2, y_{50} = 1416.3

[**Ans.** y_{41} = 2290.1]

6. Use Gauss's backward formula to find the value of y when x = 3.75, given the following table:

x	2.5	3.0	3.5	4.0	4.5	5.0
y_x	24.145	22.043	20.225	18.644	17.262	16.047

[**Ans.** 19.704]

Stirling's Formula

Example 1. *Evaluate sin (0.197) from the table given:*

x	0.15	0.17	0.19	0.21	0.23
$\sin x$	0.14944	0.16918	0.18886	0.20846	0.22798

Sol. The difference table is given by

x	x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
–2	0.15	0.15	0.14944				
				0.01974			
–1	0.17	0.17	0.16918		–0.00006		
				0.01968		–0.00002	
0	0.19	0.19	0.18886		–0.00008		0.00002
				0.0196		0	
1	0.21	0.21	0.20846		–0.00008		
				0.01952			
2	0.23	0.23	0.22798				

$$\therefore \qquad u = \frac{x-a}{h} = \frac{0.197-0.19}{0.02} = 0.35$$

From Stirling formula, we have

$$f(u) = f(0) + \frac{u[\Delta f(0) + \Delta f(-1)]}{2} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right]$$

$$+ \frac{(u+1)u^2(u-1)}{3!}\Delta^4 f(-2)$$

$$= 0.18886 + 0.35\left[\frac{0.0196 + 0.01968}{2}\right] + \frac{(0.35)^2}{2} \times (-0.00008) + \frac{(0.35+1)(0.35)(0.35-1)}{6}$$

$$\times \left(\frac{-0.00002}{2}\right) + (0.35)^2\left[\frac{(0.35)^2 - 1}{24}\right] \times 0.00002$$

$$= 0.18886 + 0.0068741 - 0.0000049 + 0.0000005 - 0.00000009$$

$$= 0.19573 \text{ (Approx.)}$$

Example 2. *Find the value of e^x = at $x = 0.644$ using by Stirling's formula. The following data given below:*

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
$y = e^x$	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

Sol. For the given data difference table is as:

	x	e^x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
–3	0.61	1.840431						
			0.018497					
–2	0.62	1.858928		0.000185				
			0.018682		0.000004			
–1	0.63	1.877610		0.000189		–0.000004		
			0.018871		0		0.000006	
0	0.64	1.896481		0.000189		0.000002		–0.000007
			0.01906		0.000002		–0.000001	
1	0.65	1.915541		0.00091		0.000001		
			0.019251		0.000003			
2	0.66	1.934792		0.000194				
			0.019445					
3	0.67	1.954237						

Here $h = 0.01$

$\therefore$ $$u = \frac{x-a}{h} = \frac{0.644-0.64}{0.01} = 0.4$$

By stirling formula, we get,

$$f(u) = f(0) + \frac{u[\Delta f(0)+\Delta f(-1)}{2} + \frac{u^3}{2!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 f(-1)+\Delta^3 f(-2)}{2}\right]$$

$$+\frac{(u+1)u^2(u-1)}{3!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + ...$$

$$= 1.896481 + 0.4\left[\frac{0.01906+0.018871}{2}\right] + \frac{(0.4)^2}{2}\times 0.000189 + \frac{(1.4)(0.4)(-0.6)}{6}\left[\frac{0.000004}{2}\right]$$

$$+\frac{(0.4)^2\left[(0.4)^2-1\right]}{24}\times 0.000002 + \frac{(1.4)(2.4)(0.4)(-0.6)(-1.6)}{120}\left[\frac{0.000006-0.000001}{2}\right]$$

$$+\frac{(0.4+2)(1.4)(0.4)(-0.6)(-0.4-2)^2}{720}\times(-0.000007)$$

$= 1.896481 + 0.0075862 + 0.00001512 – 0.000000112 – 0.0000000112 + 0.000000026 – 0.0000002$

$= 1.904081.$ (Approx.)

Example 3. *Employ Stirling's formula to evaluate $y_{12.2}$ from the data given below ($y_x = 1 + \log_{10} \sin x$).*

$x°$	10	11	12	13	14
$10^5 y_x$	23967	28060	31788	35209	38368

Sol. For the given data difference table is as:

$x°$	$10^5 y_x$	Δ	Δ^2	Δ^3	Δ^4
10	23967				
		4093			
11	28060		–365		
		3728		58	
12	31788		–307		–13
		3421		45	
13	35209		–262		
		3159			
14	38368				

$\therefore\ u = 12.2 - 12 = 0.2$

From Stirling formula

$$f(u) = 31788 + \frac{0.2}{2}[3421 + 3728] + \frac{(0.2)^2}{2}(-307) + \frac{0.2\{(0.2)^2 - 1\}}{6}\frac{1}{2}(45 + 58) + \frac{(0.2)^2\{(0.2)^2 - 1\}}{4!}(-13)$$

$$= 31788 - 714.9 - 6.14 - 1.648 + 0.0208$$

$$y_{12.2} = 32495$$

$$10^5 y_{12.2} = 32495$$

$$\therefore\ y_{12.5} = 0.32495.\ \textbf{Ans.}$$

Example 4. *Apply Stirling's formula to find a polynomial of degree three which takes the following values of x and y:*

x	2	4	6	8	10
y	–2	1	3	8	20

Sol. Let $u = \frac{x-6}{2}$. Now, we construct the following difference table:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
2	–2	–2				
			3			
4	–1	1		–1		
			2		4	
6	0	3		3		0
			5		4	
8	1	8		7		
			12			
10	2	20				

Stirling's formula is

$$f(u) = f(0) + \frac{u[\Delta f(0) + \Delta f(-1)]}{2} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right] + \frac{(u+1)u^2(u-1)}{3!}\Delta^4 f(-2)$$

$$y_u = 3 + u\left[\frac{2+5}{2}\right] + \frac{u^2}{2}\times 3 + \frac{u(u^2-1)}{6}\left(\frac{4+4}{2}\right) + 0$$

$$= 3 + \frac{7}{2}u + \frac{3}{2}u^2 + \frac{2}{3}(u^3 - u)$$

$$= 3 + \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{7}{2}u - \frac{2}{3}u$$

$$= 3 + \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u$$

$$= \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3 = \frac{2}{3}\left(\frac{x-6}{2}\right)^3 + \frac{3}{2}\left(\frac{x-6}{2}\right)^2 + \frac{17}{6}\left(\frac{x-6}{2}\right) + 3$$

$$= 0.0833x^3 - 1.125x^2 + 8.9166x - 19.$$

Example 5. *Apply Stirling's formula to find the value of f(1.22) from the following table which gives the value of* $f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{\frac{x^2}{2}} dx$ *at intervals of x = 0.5 from x = 0 to 2.*

x	0	0.5	1.0	1.5	2.0
$f(x)$	0	0.191	0.341	0.433	0.477

Sol. Let the origin be at 1 and $h = 0.5$

$\therefore\ x = a + hu,\ u = \frac{x-a}{h} = \frac{1.22 - 1.00}{0.5} = 0.44$

Applying Stirling's formula

$$f(u) = f(0) + \frac{u[\Delta f(0) + \Delta f(-1)]}{2} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right] + \frac{(u+1)u^2(u-1)}{3!}\Delta^4 f(-2)$$

$$\therefore\quad f(0.44) = f(0) + (0.44)\frac{1}{2}[\Delta f(0) + \Delta f(-1)] + \frac{(0.44)^2}{2}\Delta^2 f(-1) + \frac{(0.44)\{(0.44)^2 - 1\}}{6}\frac{1}{2}\left[\Delta^3 f(-1) + \Delta^3 f(-2)\right] + \frac{(0.44)^2\{(0.44)^2 - 1\}}{24}\Delta^4 f(-2)$$

$$= f(0) + (0.22)\ [\Delta f(0) + \Delta f(-1)] + 0.0968\ \Delta^2 f(-1) - 0.029568\ [\Delta^3 f(-1) + \Delta^3 f(-2)] - 0.06505\Delta^4 f(-2) \quad \text{...(1)}$$

The difference table is as follows:

u	x	$10^3 f(x)$	$10^3 \Delta f(x)$	$10^3 \Delta^2 f(x)$	$10^3 \Delta^3 f(x)$	$10^3 \Delta^4 f(x)$
–2	0	0				
			191			
–1	0.5	191		–41		
			150		–17	
0	1	341		–58		27
			92		10	
1	1.5	433		–48		
			44			
2	2	477				

$f(0)$ and the differences are being multiplied by 10^3

$\therefore \quad 10^3 f(0.44) \approx 341 + 0.22 \times (150 + 92) + 0.0968 \times (-58)$

$$- 0.029568 \times (-17 + 10) - 0.006505 \times 27$$

$$\approx 341 + 0.22 \times 242 - 0.0968 \times 58 + 0.029568 \times 7 - 0.006505 \times 27$$

$$\approx 341 + 53.24 - 5.6144 + 0.206276 - 0.175635$$

$$\approx 388.66$$

$f(0.44) = 0.389$ at $x = 1.22$. **Ans.**

Example 6. *Use Stirling's formula to find* y_{28} *given.*

$y_{20} = 49225, \quad y_{25} = 48316, \quad y_{30} = 47236,$

$y_{35} = 45926, \quad y_{40} = 44306$

Sol. Let the origin be at 30 and $h = 5$

$$a + hu = 28$$

$\Rightarrow \quad 30 + 5u = 28 \quad \Rightarrow \quad u = -0.4$

The difference table is as follows:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	20	49225				
			–909			
–1	25	48316		–171		
			–1080		–59	
0	30	47236		–230		–21
			–1310		–80	
1	35	45926		–310		
			–1620			
2	40	44306				

By Stirling's formula,

$$f(-.4) = 47236 + (-.4)\left(\frac{-1080 - 1310}{2}\right) + \frac{(-0.4)^2}{2!}(-230)$$

$$+ \frac{(.6)(-.4)(-1.4)}{3!}\left(\frac{-59 - 80}{2}\right) + \frac{(-.4)^2\{(-4.4)^2 - 1\}}{4!}(-21)$$

Hence, $\quad y_{28} = 47691.8256.$

PROBLEM SET 4.5

1. Use Stirling's formula to find y_{32} from the following table:

x	20	25	30	35	40	35
y	14.035	13.674	13.257	12.734	12.089	11.309

[**Ans.** 13.062]

2. Use the following table to evaluate tan 16° by Stirling's formula:

θ°	0°	5°	10°	15°	20°	25°	30°
tan θ°	0	0.0875	0.1763	0.2679	0.364	0.4663	0.5774

[**Ans.** 0.2866980499]

3. Use Stirling's formula to find the value of $f(1.22)$ from the following data:

x	$f(x)$
1.0	0.84147
1.1	0.89121
1.2	0.93204
1.3	0.96356
1.4	0.98545
1.5	0.99749
1.6	0.99957
1.7	0.99385
1.8	0.97385

[**Ans.** 0.9391002]

4. Find $f(0.41)$ using Stirling's formula if,

x	0.30	0.35	0.40	0.45	0.50
$f(x)$	0.1179	0.1368	0.1554	0.1736	0.1915

[**Ans.** 0.15907168]

5. Use Stirling's formula to find y_{35}, data being:
$y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, and $y_{50} = 243$,

[**Ans.** 394.6875]

6. From the following table find the value of $f(0.5437)$ by Stirling's formula:

x	0.51	0.52	0.53	0.54	0.55	0.56	0.57
$f(x)$	0.529244	0.537895	0.546464	0.554939	0.663323	0.571616	0.579816

[**Ans.** 0.558052]

7. Apply Stirling's formula to find a polynomial of degree four which takes the values of y as given below:

x	1	2	3	4	5
y	1	–1	1	–1	1

[**Ans.** $\frac{2}{3}u^4 - \frac{8}{3}u^2 + 1$]

8. Apply Stirling's formula to interpolate the value of y at x = 1.91 from the following data:

x	1.7	1.8	1.9	2.0	2.1	2.2
y	5.4739	6.0496	6.6859	7.3851	8.1662	9.0250

[**Ans.** 6.7531]

4.5 BESSEL'S

Example 1. *Using Bessel's formula find the value of y at x = 3.75 for the data given below:*

x	*2.5*	*3.0*	*3.5*	*4.0*	*4.5*	*5.0*
y	*24.145*	*22.043*	*20.225*	*18.644*	*17.262*	*16.047*

Sol. Difference table for the given data is as:

	x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
–2	2.5	24.145					
			–2.102				
–1	3.0	22.043		0.284			
			–1.818		–0.047		
0	3.5	20.225		0.237		0.009	
			–1.581		–0.038		–0.003
1	4.0	18.644		0.199		0.006	
			–1.382		–0.032		
2	4.5	17.262		0.167			
			–1.215				
3	5.0	16.047					

Here $h = 0.5$

$$\therefore \quad u = \frac{x-a}{h} = \frac{3.75-3.5}{0.5} = 0.5$$

Now from Bessel's formula, we have

$$f(u) = \frac{f(0)+f(1)}{2} + \left\{u-\frac{1}{2}\right\}\Delta f(0) + \frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(0)+\Delta^2 f(-1)}{2}\right\} + \frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 f(-1)$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 f(-1)+\Delta^4 f(-2)}{2}\right)+\frac{(u+1)u(u-1)(u-2)(u-1/2)}{5!}\Delta^5 f(-2)+\ldots$$

$$=\frac{1}{2}[18.644+20.225]+(0.5-0.5)(-1.581)+\frac{(0.5)(0.5)}{2}\times\frac{1}{2}[1.99+2.37]$$

$$+0+(0.5+1)(0.5)(-0.5)(2.5)\times\left(\frac{16+9}{2}\right)+0$$

= 19.407. (Approx.)

Example 2. *Following table gives the values of e^x for certain equidistant values of x. Find the value of e^x at x = 0.644 using Bessel's formula:*

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^x	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

Sol. Given h = 0.01, take it origin as 0.64

$$\Rightarrow u = \frac{x-a}{h} = \frac{0.644-0.64}{0.01} = \frac{0.004}{0.01} = 0.4$$

u = 0.4

Difference table for the given data is as:

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0.61	1.840431						
		0.018497					
0.62	1.858928		–0.000165				
		0.018682		0.000004			
0.63	1.877610		0.000189		–0.000004		
		0.18871		0		0.000006	
0.64	1.896481		0.000189		0.000002		–0.000007
		0.1906		0.000002		–0.000001	
0.65	1.915541		0.000191		0.000001		
		0.019251		0.000003			
0.66	1.934792		0.000144				
		0.019445					
0.67	1.954237						

Bessel's formula

$$f(u) = \frac{f(0)+f(1)}{2} + \left\{u - \frac{1}{2}\right\}\Delta f(0) + \frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(0) + \Delta^2 f(-1)}{2}\right\} + \frac{u(u-1)\left(-\frac{1}{2}\right)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 f(-1) + \Delta^4 f(-2)}{2}\right) + \frac{(u+1)u(u-1)(u-2)(u-1/2)}{5!}\Delta^5 f(-2) + ...$$

$$= \left(\frac{1.896481 + 1.915541}{2}\right) + (-0.1)(0.01906) + \frac{0.4(-0.6)}{2}\left[\frac{0.000191 + 0.000189}{2}\right]$$

$$+ \frac{(0.4)(-0.6)(-0.1)(0.000002)}{6} + \frac{(1.4)(0.4)(-0.6)(-1.6)}{24}\left[\frac{0.000001 + 0.000002}{2}\right]$$

$$+ \frac{(1.4)(0.4)(-0.6)(-0.1)(-1.6)}{120}(-0.000001)$$

$= 1.906011 - 0.001906 - 0.0000228 + 0.000000008 + 0.000000033 + 1.68 \times 10^{-10}$

$= 1.904082$

Example 3. *Find the value of y_{25} from the following data using Bessel's formula. Data being y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992.*

Sol. The difference table for the data is as:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
24	3162		74	
		382		–8
28	3544		66	
		448		
32	3992			

$$u = \frac{25-24}{4} = 0.25$$

$$f(0.25) = \frac{(3162+3544)}{2} + (-0.25)\times 382 + (0.25)(-0.75)\left[\frac{66+74}{2}\right] + \frac{(0.25)(-0.75)(-0.25)(-8)}{6}$$

$= 3353 - 95.5 - 6.5625 - 0.0625$

$= 3250.875$

Example 4. *The pressure p of wind corresponding to velocity v is given by following data. Estimate pressure when v = 25.*

v	10	20	30	40
P	1.1	2	4.4	7.9

Sol. The difference table for the given data is as:

v	P	Δ	Δ^2	Δ^3
10	1.1			
		0.9		
20	2		1.5	
		2.4		–0.4
30	4.4		1.1	
		3.4		
40	7.9			

Let origin = 20, h = 10,

$$u = \frac{25-20}{10} = 0.5$$

Bessel's formula for interpolation is:

$$P(u) = \frac{1}{2}(P_0 + P_1) + \left(u - \frac{1}{2}\right)\Delta P_0 + \frac{u(u-1)}{2!}\left[\Delta^2 P_{-1} + \frac{\Delta^2}{2} P_0\right] + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 P_{-1}$$

$$= \frac{1}{2}(2+4.4) + \left(0.5 - \frac{1}{2}\right)\times 2.4 + \frac{0.5(0.5-1)}{2}\left[\frac{1.5+1.1}{2}\right] + \left(0.5 - \frac{1}{2}\right)\frac{0.5(0.5-1)}{6}\times -0.4$$

$$= \frac{1}{2}\times 6.4 + 0 - 0.16250 + 0$$

$$= 3.2 - 0.16250$$

$$P_{25} = 3.03750$$

Example 5. *Probability distribution function values of a normal distribution are given as follows:*

x	0.2	0.6	1.0	1.4	1.8
$p(x)$	0.39104	0.33322	0.24197	0.14973	0.07895

Find the value of p(x) for x = 1.2.

Sol. Taking the origin at 1.0 and h = 0.4

$$x = a + uh \Rightarrow 1.2 = 1.0 + u \times 0.4$$

$$u = \frac{1.2 - 1.0}{0.4} = \frac{1}{2}$$

The difference table is:

u	$10^5 f(u)$	$10^5 \Delta f(u)$	$10^5 \Delta^2 f(u)$	$10^5 \Delta^3 f(u)$	$10^5 \Delta^4 f(u)$
–2	39104				
		–5782			
–1	33322		–3343		
		–9125		3244	
0	24197		–99		–999
		–9224		2245	
1	14973		2146		
		–7078			
2	7895				

Bessel's formula is

$$f(u) = \frac{f(0)+f(1)}{2} + \left\{u - \frac{1}{2}\right\}\Delta f(0) + \frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(0) + \Delta^2 f(-1)}{2}\right\} + \frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 f(-1)$$

$$10^5 f(0.5) = \left(\frac{24197 + 14973}{2}\right) + 0 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{2146 - 99}{2}\right) + 0$$

$$= 19457.0625$$

$\therefore \quad f(0.5) = 0.194570625$

Hence, $p(1.2) = 0.194570625$.

Example 6. *Given $y_0, y_1, y_2, y_3, y_4, y_5$ (fifth difference constant), prove that*

$$y_{2\frac{1}{2}} = \frac{1}{2}c + \frac{25(c-b)+3(a-c)}{256}$$

where $a = y_0 + y_5$, $b = y_1 + y_4$, $c = y_2 + y_3$.

Sol. Put $u = \frac{1}{2}$ in bessel's formula, we get

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{16}\left(\Delta^2 y_0 + \Delta^2 y_{-1}\right) + \frac{3}{256}\left(\Delta^4 y_{-1} + \Delta^4 y_{-2}\right)$$

Shifting origin to 2, we have

$$y_{2\frac{1}{2}} = \frac{1}{2}(y_2 + y_3) - \frac{1}{16}\left(\Delta^2 y_2 + \Delta^2 y_1\right) + \frac{3}{256}(\Delta^4 y_1 + \Delta^4 y_0)$$

$$= \frac{c}{2} - \frac{1}{16}\left(y_3 - 2y_2 + y_1 + y_4 - 2y_3 + y_2\right) + \frac{3}{256}(y_5 - 3y_4 + 2y_3 + 2y_2 - 3y_1 + y_0)$$

$$y_{2\frac{1}{2}} = \frac{c}{2} - \frac{1}{16}(y_4 - y_3 - y_2 + y_1) + \frac{3}{256}(a - 3b + 2c)$$

$$= \frac{c}{2} - \frac{1}{16}(b - c) + \frac{3}{256}(a - 3b + 2c)$$

$$y_{2\frac{1}{2}} = \frac{c}{2} + \frac{1}{256}[25(c - b) - 3(a - c)]$$

Example 7. *If third differences are constant, prove that*

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x).$$

Sol. Put $u = \frac{1}{2}$ in Bessel's formula, we get

$$y_{\frac{1}{2}} = \frac{1}{2}(y_0 + y_1) - \frac{1}{16}(\Delta^2 y_0 + \Delta^2 y_{-1})$$

Shifting the origin to x.

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_x + \Delta^2 y_{x-1})$$

Example 8. *Given that:*

x	4	6	8	10	12	14
$f(x)$	3.5460	5.0753	6.4632	7.7217	8.8633	9.8986

Apply Bessel's formula to find the value of f(9).

Sol. Taking the origin at 8, h = 2,

$$9 = 8 + 2u \text{ or } u = \frac{1}{2}$$

The difference table is:

u	$10^4 y_u$	$10^4 \Delta y_u$	$10^4 \Delta^2 y_u$	$10^4 \Delta^3 y_u$	$10^4 \Delta^4 y_u$	$10^4 \Delta^5 y_u$
–2	35460					
		15293				
–1	50753		–1414			
		13879		120		
0	64632		–1294		5	
		1258		125		–24
1	77217		–1169		–19	
		11416		106		
2	88633		–1063			
		10353				
3	98986					

Bessel's formula is

$$y_u = \frac{1}{2}(y_1 + y_0) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)^{\frac{1}{2}}}{2!}(\Delta^2 y_0 + \Delta^2 y_{-1}) + \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!} \times \frac{1}{2}(\Delta^4 y_{-3} + \Delta^4 y_{-2}) + \frac{(u-2)(u-1)\left(-\frac{1}{2}\right)u(u+1)}{5!}\Delta^5 y_{-2}$$

$$10^4 y_{1/2} = \frac{1}{2}(77217 + 64632) + 0 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}.\frac{1}{2}(-1169 - 1294) + 0 + \frac{\frac{3}{2}.\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{24}.\frac{1}{2}(-19 + 15) + 0$$

$\Rightarrow\ 10^4 y_{1/2} = 71078.27344$

$\therefore\quad y_{1/2} = 7.107827344$

Hence, $f(9) = 7.107827344$

Example 9. *Find a polynomial for the given data using Bessel's formula f(2) = 7, f(3) = 9, f(4) = 12, f(5) = 16.*

Sol. Let us take origin as 3 therefore,

$u = x - 3$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	7			
		2		
3	9		1	
		3		0
4	12		1	
		4		
5	16			

Bessel's formula:

$$y_u = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right) \times \Delta y_0 + \frac{P(P-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$= \left(\frac{9+12}{2}\right) + \left(u - \frac{1}{2}\right) \times 3 + \frac{u(u-1)}{2}$$

$$= \frac{21}{2} + 3u - \frac{3}{2} + \frac{u^2}{2} - \frac{u}{2}$$

$$= \frac{u^2}{2} + \frac{5}{2}u + 9$$

Put $u = x - 3$

$$y = \frac{(x-3)^2}{2} + \frac{5}{2}(x-3) + 9$$

$$= \frac{1}{2}\left(x^2 - 6x + 9\right) + \frac{5}{2}(x-3) + 9$$

$$y = \frac{1}{2}x^2 - \frac{1}{2}x + 6. \textbf{ Ans.}$$

PROBLEM SET 4.6

1. Find y (0.543) from the following values of x and y:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$y(x)$	2.631	3.328	4.097	4.944	5.875	6.896	8.013

[**Ans.** 6.303]

2. Apply Bessel's formula to find the value of $y_{2.73}$ from the data given below:

$y_{2.5} = 0.4938,\quad y_{2.6} = 0.4953,\quad y_{2.7} = 0.4965,\quad y_{2.8} = 0.4974$

$y_{2.9} = 0.4981,\quad y_{3.0} = 0.4987.$

[**Ans.** 0.496798]

3. Find y_{25} by using Bessel's interpolation formula from the data:

$y_{20} = 24,\quad y_{24} = 32,\quad y_{28} = 35,\quad y_{32} = 40$

[**Ans.** 32.9453125]

4. Apply Bessel's formula to evaluate $y_{62.5}$ from the data:

x	60	61	62	63	64	65
y_x	7782	7853	7924	7993	8062	8129

[**Ans.** 7957.1407]

5. Apply Bessel's formula to obtain the value of f (27.4) from the data:

x	25	26	27	28	29	30
$f(x)$	4.000	3.846	3.704	3.571	3.448	3.333

[**Ans.** 3.649678336]

6. Apply Bessel's formula to find the value of f(12.2) from the following data:

x	0	5	10	15	20	25	30
$f(x)$	0	0.19146	0.34634	0.43319	0.47725	0.49379	0.49865

[**Ans.** 0.39199981]

7. Apply Bessel's interpolation formula, show that tan160° = 0.2867, Given that:

x	0°	5°	10°	15°	20°	25°	30°
$\tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

8. Apply Bessel's formula to find a polynomial of degree 3 from the data:

x	4	6	8	10
y	1	3	8	20

[**Ans.** $\frac{2}{9}u^3 + \frac{13}{6}u^2 + \frac{47}{18}u + 3; u = \frac{x-6}{2}$]

9. From the following table find the value of f (0.5437) by Gauss and Bessel's formula:

x	0.51	0.52	0.53	0.54	0.55	0.56	0.57
$f(x)$	0.529244	0.537895	0.546464	0.554939	0.663323	0.571616	0.579816

[**Ans.** 0.558052]

10. Apply Bessel's formula to obtain a polynomial of degree three:

x	7	8	9	10	11
$f(x)$	14	17	19	22	25

[**Ans.** $-\frac{x^3}{6} + 5x^2 - \frac{281}{6}x + 157$]

4.6 LAPLACE EVERETTS

Example 1. *Find the value of f(27.4) from the following table:*

u	25	26	27	28	29	30
$f(x)$	4.000	3.846	3.704	3.571	3.448	3.333

Sol. Here, $u = \frac{27.4-27.0}{1} = 0.4,$ origin is at 27.0 $h = 1$

Also, $w = 1 - u = 0.6$

Difference table is:

u	$10^3 f(u)$	$10^3 \Delta f(u)$	$10^3 \Delta^2 f(u)$	$10^3 \Delta^3 f(u)$	$10^{3\Delta^4} f(u)$
–2	4000				
		–154			
–1	3846		12		
		–142		–3	
0	3704		9		4
		–133		1	
1	3571		10		–3
		–123		–2	
2	3448		8		
		–115			
3	3333				

By Laplace Everett's formula,

$$f(0.4) = \left\{(0.4)(3571) + \frac{(1.4)(0.4)(-0.6)}{3!}(10) + \frac{(2.4)(1.4)(0.4)(-0.6)(-1.6)}{5!}(-3) + \ldots\right\}$$

$$+\left\{(0.6)(3704) + \frac{(1.6)(0.6)(-0.4)}{3!}(9) + \frac{(2.6)(1.6)(0.6)(-0.4)(-1.4)}{5!}(4)\ldots\right\}$$

$= 3649.678336.$

Hence $f(27.4) = 3649.678336.$

Example 2. *Using Laplace Everett's formula, find f(30), if f(20) = 2854, f(28) = 3162, f(36) = 7088, f(44) = 7984.*

Sol. Take origin at 28, $h = 8$

$\therefore \quad a + hu = 30$

$\Rightarrow \quad 28 + 8u = 30 \Rightarrow u = 0.25$

Also, $w = 1 - u = 1 - .25 = 0.75$

Difference table is:

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$
–1	2854			
		308		
0	3162		3618	
		3926		–6648
1	7088		–3030	
		896		
2	7984			

By Everett's formula,

$$f(.25) = \left\{(0.25)(7088) + \frac{(1.25)(0.25)(-0.75)}{3!}(-3030)\ldots\right\} + \left\{(0.75)(3162) + \frac{(1.75)(0.75)(-0.25)}{5!}(3618) + \ldots\right\}$$

$= 4064$

Hence $f(30) = 4064$.

Example 3. *Apply Laplace Everett's formula to find the value of log 2375 from the data given below:*

x	21	22	23	24	25	26
$\log x$	1.3222	1.3424	1.3617	1.3802	1.3979	1.4150

Sol. Here $h = 1$

We take origin at 23.

Now difference table is given by

	x	$\log x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
–2	21	1.3222					
			0.0202				
–1	22	1.3424		–0.0009			
			0.0193		0.0001		
0	23	1.3617		–0.0008		–0.0001	
			0.0185		0		0.0003
1	24	1.3802		–0.0008		0.0002	
			0.0171		0.0002		
2	25	1.3979		–0.0006			
			0.0171				
3	26	1.4150					

Here $h = 1$

$$\therefore \quad u = \frac{x-a}{h} = \frac{23.75-23}{1} = 0.75$$

$$w = 1 - 0.75 = 0.25$$

From Laplace Everett formula, we have

$$f(u) = \left\{ uf(1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1) + \ldots \right\}$$

$$+ \left\{ wf(0) + \frac{(w+1)w(w-1)}{3!}\Delta^2 f(-1) + \frac{(w+2)(w+1)w(w-1)(w-2)}{5!}\Delta^4 f(-2) + \ldots \right\}$$

$$= \left[0.75 \times 1.3802 + \frac{(1.75)(0.75)(-0.25)}{6} \times (-0.0008) + \frac{(1.75)(2.75)(0.75)(-0.25)(-1.25)}{120} \times (0.0002) \right.$$

$$\left. + \ 0.25 + 1.3617 + \frac{0.25(1.25)(-0.75(-0.0008)}{6} + \frac{(2.25)(1.25)(0.25)(-0.75)(-1.75)}{120} \times (-0.0001) \right]$$

$$= 1.035419 + 0.340455$$

$$= 1.375874$$

$$\log 2375 = \log (23.75 \times 100) = \log 23.75 + \log 100$$

$$\Rightarrow \log 2375 = 1.375872 + 2 = 3.375872$$

Example 4. *Find the value of e^{-x} when $x = 1.748$ from the following data:*

x	1.72	1.73	1.74	1.75	1.76	1.77
e^{-x}	0.1790	0.1773	0.1755	0.1738	0.1720	0.1703

Sol. Here $h = 0.01$, take origin as 1.74.

The difference table for the given data is as:

x	e^{-x}	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.72	0.1790					
		–0.0017				
1.73	0.1773		–0.0001			
		–0.0018		0.0002		
1.74	0.1755		0.0001		–0.0004	
		–0.0017		–0.0002		0.0008
1.75	0.1738		–0.0001		0.0004	
		–0.0018		–0.0002		
1.76	0.1720		0.0001			
		–0.0017				
1.77	0.1703					

$$u = \frac{1.748 - 1.74}{0.01} = 0.8$$

$w = 0.2$

$$f(0.8) = 0.8(0.1738) + (0.8)(1.8)(-0.2)(-0.00017) + \frac{(2.8)(1.8)(0.8)(-0.2)(-1.2)(0.0004)}{120}$$

$$+0.2(0.1755) + \frac{(-0.8)(0.2)(1.2)}{6} \times (0.0001) + \frac{(1.2)(2.2)(0.2)(-0.8)(-1.8)}{120} \times (-0.0004)$$

$= 0.13904 + 0.0000816 + 0.000003225 + 0.0351 - 0.0000032 + 0.000002534$

$= 0.174224.$

Example 5. *Prove that if third differences are assumed to be constant*

$$y_x = xy_1 + \frac{x(x^2-1)}{3!}\Delta^2 y_0 + uy_0 + \frac{u(u^2-1)}{3!}\Delta^2 y_{-1}$$ *where $u = 1 - x$. Apply this formula to find the value of y_{11} and y_{16} if $y_0 = 3010$, $y_5 = 2710$, $y_{10} = 2285$, $y_{15} = 1860$, $y_{20} = 1560$, $y_{25} = 1510$, $y_{30} = 1835$.*

Sol. $x_1 = \frac{11-10}{5} = 0.2, \qquad x_2 = \frac{16-15}{5} = 0.2.$

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^5 y_x$
0	3010			
		–300		
5	2710		–125	
		–425		125
10	2285		0	
		–425		125
15	1860		125	
		–300		125
20	1560		250	
		–50		125
25	1510		375	
		325		
30	1835			

Using given formula,

$$y_x = xy_1 + \frac{x(x^2-1)}{3!}\Delta^2 y_0 + uy_0 + \frac{u(u^2-1)}{3!}\Delta^2 y_{-1}$$

$$y_{11} = (0.2)(1860) + \frac{(1.2)(0.2)(-0.8)}{6}(125) + (0.8)(2285) + \frac{(0.8)(0.64-1)}{6}(10)$$

$= 2196$

$$y_{16} = (0.2)(1560) + \frac{(1.2)(0.2)(-0.8)}{6}(250) + (0.8)(1860) + \frac{(0.8)(-0.2)(1.8}{6}(125)$$

$= 1786$

Example 6. *Find the compound interest on the sum of Rs. 10,000/- at 7% for the period 16 years if,*

x	5	10	15	20	25	30
$(1.07)^n$	1.40255	1.96715	2.75903	3.86968	5.42743	7.61236

Sol. The difference table can be formed as:

	x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
–2	5	1.40255					
			0.5646				
–1	10	1.96715		0.22728			
			0.79188		0.09149		
0	15	2.75903		0.31877		–0.35901	
			1.11065		–0.26752		1.60246
1	20	3.86968		0.05125		1.24345	
			1.5775		0.97593		
2	25	5.42743		1.02718			
			2.18493				
3	30	7.61236					

Here, $h = 5$

$\therefore \quad u = \frac{x-a}{h} = \frac{16-15}{5} = 0.2$

$\therefore \quad w = 1-u = 1-0.2 = 0.8$.

On applying Laplace Everett formula, we have

$$f(u) = \left[0.2\times 3.86968 + \frac{0.2(0.2+1)(0.2-1)}{3!}\times 0.05125 + \frac{0.2(0.2+2)(0.2+1)(0.2-1)(0.2-2)}{5!}\times 1.24345\right]$$

$$+\left[\frac{0.8\times 2.75903 + 0.8(0.8+1)(0.8-1}{6}\times 0.31877 + \frac{0.8(0.8+2)(0.8+1)(0.8-1)(0.8-2)}{120}\times(-0.35901)\right]$$

$= 0.776593 + 2.189027 = 2.96595$ (Approx.)

Example 7. *The values of the ellipitc integral* $k(m) = \int_0^{\pi/2} \left(1 - m\sin^2\theta\right)^{-1/2} d\theta$ *for certain equidistant values of m are given below. Use Everett's formula to determine k (0.25).*

m	0.20	0.22	0.24	0.26	0.28	0.30
$k(m)$	1.659624	1.669850	1.680373	1.691208	1.702374	1.713889

Sol. Here h = 0.02, take origin as 0.24

$$u = \frac{x-a}{h} = \frac{0.25-0.24}{0.02} = 0.5$$

$$w = 1 - u = 0.5$$

m	$k(m)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0.20	1.659624					
		0.010226				
0.22	1.669850		0.000297			
		0.010523		0.000015		
0.24	1.680373		0.000312		0.000004	
		0.010835		0.000019		–0.000005
0.26	1.691208		0.000331		–0.000001	
		0.011166		0.000018		
0.28	1.702374		0.000349			
		0.011515				
0.30	1.713889					

$$f(u) = \left\{uf(1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0)....\right\} + \left\{wf(0) + \frac{(w+1)w(w-1)}{3!}\Delta^2 f(-1)....\right\}$$

$$= \left\{(0.5)(1.691208) + \frac{(0.5)(1.5)(-0.5)}{6}(0.000331) + \frac{(-1.5)(-.5)(.5)(1.5)(2.5)}{120}(-0.000001)\right\}$$

$$+ (0.5)(1.680073) + \frac{(.5)(1.5)(-.5)}{6}(0.000312) + \frac{(.5)(1.5)(-.5)(2.5)(-1.5)}{120} \times (.000004)$$

$= 0.845604 - 0.00002069 - 0.00000014 + 0.84018650 - 0.0000195 + 0.00000005$

$= 1.685750$

Example 8. *Find the value of log 337.5 by using Laplace Everett's formula. Given that:*

x	310	320	330	340	350	360
$\log x$	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Sol. Here h = 10, take origin as 340

$$u = \frac{x-a}{h} = \frac{337.5-340}{10} = -0.25$$

$w = 1 - u = 1 + 0.25 = 1.25$

For the given data difference table is as:

x		$\log x$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
–3	310	2.49136					
			0.01379				
–2	320	2.50515		–0.00043			
			0.01336		0.00004		
–1	330	2.51851		–0.00039		–0.00005	
			0.01297		0.00001		–0.00004
0	340	2.53148		–0.00038		0.00001	
			0.01259		0.00002		
1	350	2.54407		–0.00036			
			0.01223				
2	360	2.55630					

$$f(u) = \left\{uf(1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 f(-1)\right\}$$

$$+\left\{wf(0) + \frac{(w+1)w(w-1)}{3!}\Delta^2 f(-1) + \frac{(w+2)(w+1)w(w-1)(w-2)}{5!}\Delta^4 f(-2)\right\}$$

$$= (-0.25)\times 2.54407 + \frac{(-1.25)(-0.25)(0.75)}{6}\times(-0.00036) + 1.25\times 2.53148$$

$$+\frac{(0.25)(1.25)(2.25)}{6}(-0.00038) + \frac{(3.25)(2.25)(1.25)(0.25)(-0.75)}{120}(0.00001)$$

$= -0.6360175 - 0.0000140625 + 3.16435 - 0.00004453125 - 0.0000001428$

$= 2.528273$ (Approx.)

PROBLEM SET 4.7

1. Eliminate odd difference from the Gauss Forward formula to drive Everett's formula:

$$y_u = (1-u) f_0 + u\delta_1 = \frac{u(u-1)(u-2)}{2!} S^2\delta_0 + \frac{(u+1)u(u-1)}{3!} S^2\delta_1 + ...$$

where $u = \frac{x - x_0}{h}$

2. From the following table of values of x and $y = e^x$, interpolate the value of y when $x = 1.91$:

x	1.7	1.8	1.9	2.0	2.1	2.2
$y = e^x$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

[**Ans.** 6.7531]

3. From the following present value annuity a_n table:

x	20	25	30	35	40
a_n	11.4699	12.7834	13.7648	14.4982	15.0463

Find the present value of the annuity a_{31}, a_{33}.

[**Ans.** 13.9186, 14.2306]

4. Find the value of $x^{1/3}$ when $x = 51$ to 54 from the data:

x	40	45	50	55	60	65
$x^{1/3}$	3.4200	3.3569	3.6840	3.8030	3.9149	4.0207

[**Ans.** 3.7084096, 3.7325079, 3.7563005, 3.7797956]

5. From the following data, find the value of $f(31)$, $f(32)$,
$f(20) = 3010$, $f(25) = 3979$, $f(30) = 4771$, $f(35) = 5441$,
$f(40) = 6021$, $f(45) = 6532$

[**Ans.** 4913, 5052]

6. Apply Everett's formula to find the value of $f(26)$ and $f(27)$ from the data given below:

x	15	20	25	30	35	40
$f(x)$	12.849	16.351	19.524	22.396	24.999	27.356

[**Ans.** 20.121431, 20.707077]

7. The following table gives the values of e^x for certain equidistant values of x:

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^x	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

Find the value of e^x when $x = 0.644$ using Everett's formula.

[**Ans.** 1.904082]

8. Apply Everett's formula to find the values of e^{-x} for $x = 3.2, 3.4, 3.6, 3.8$ if,

x	1	2	3	4	5	6
e^{-x}	0.36788	0.13534	0.04979	0.01832	0.00674	0.00248

[**Ans.** 0.04087, 0.03354, 0.02749, 0.02248]

9. Use Everett's formula to find the present value of the annuity of $n = 36$ from the table:

x	25	30	35	40	45	50
a_x	12.7834	13.7648	14.4982	15.0463	15.4558	15.7619

[**Ans.** 14.620947]

10. Obtain the value of y_{25}, given that:

$y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$ [**Ans.** 3250.875]

❑❑❑

CHAPTER 5

Interpolation with Unequal Interval

5.1 INTRODUCTION

The interpolation formulae derived before for forward interpolation, Backward interpolation and central interpolation have the disadvantages of being applicable only to equally spaced argument values. So it is required to develop interpolation formulae for unequally spaced argument values of x. Therefore, when the values of the argument are not at equally spaced then we use two such formulae for interpolation.

1. Lagrange's Interpolation formula
2. Newton's Divided difference formula.

The main advantage of these formulas is, they can also be used in case of equal intervals but the formulae for equal intervals cannot be used in case of unequal intervals.

5.2 LAGRANGE'S INTERPOLATION FORMULA

Let $f(x_0), f(x_1)... f(x_n)$ be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x_0, x_1, x_2,... x_0$. So that

The polynomial $f(x)$ may be written as

$$f(x) = A_0 (x - x_1)(x - x_2)... (x - x_n) + A_1 (x - x_1)(x - x_2) ... (x - x_n) + ... + A_n (x - x_1)(x - x_2)...(x - x_{n-1}) \quad ...(1)$$

where $A_0, A_1,... A_n$, are constants to be determined.

Putting $x = x_0, x_1, x_2,..., x_n$ in (1) successively, we get

For $\boxed{x = x_0}$, $f(x_0) = A_0 (x_0 - x_1)(x_0 - x_2) ... (x_0 - x_n)$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)} \quad ...(2)$$

For $\boxed{x = x_1}$, $f(x_1) = A_1 (x_1 - x_0)(x_1 - x_2) ... (x_1 - x_n)$

$$\Rightarrow A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)...(x_0 - x_n)} \quad ...(3)$$

Similarly,

For $\boxed{x = x_n}$, $f(x_n) = A_n (x_n - x_0)(x_n - x_1) ... (x_n - x_{n-1})$

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_2)...(x_n - x_{n-1})} \quad ...(4)$$

Substituting the values of $A_0, A_1, ...A_n$, in equation (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} f(x_1)$$

$$+ ... + \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} f(x_n) \quad ...(5)$$

This is called Lagrange's interpolation formula. In equation (5), dividing both sides by $(x - x_0)(x - x_1)... (x - x_n)$, Lagrange's formula may also to written as

$$\frac{f(x)}{(x-x_0)(x-x_1)...(x-x_n)} = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)(x-x_0)}$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} \frac{1}{(x-x_1)} + ... + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} \frac{1}{(x-x_n)}$$

Corollary. Show that Lagrange's formula can be put in the form

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x-x_r)\phi'(x_r)}$$

where $\phi(x) = \prod_{r=0}^{n} (x - x_r)$ and $\phi'(x_r) = \left[\frac{d}{dx}\{\phi(x)\}\right]_{x=xr}$

Sol. We have, $$P_n(x) = \sum_{r=0}^{n} \frac{(x-x_0)(x-x_1)...(x-x_{r-1})(x-x_{r+1})...(x-x_n)}{(x_r-x_0)(x_r-x_1)...(x_r-x_{r-1})(x-x_{r+1})...(x_r-x_n)} f(x_r)$$

$$= \sum_{r=0}^{n} \left\{\frac{\phi(x)}{(x-x_r)}\right\} \frac{f(x_r)}{(x_r-x_0)(x_r-x_1)...(x_r-x_{r-1})(x_r-x_{r+1})...(x_r-x_n)} \quad ...(1)$$

Now, $\phi(x) = \prod_{r=0}^{n} (x - x_r)$ (given)

Therefore, $\phi(x) = (x-x_0)(x-x_1)(x-x_2)...(x-x_{r-1})(x-x_r)(x-x_{r+1})...(x-x_n)$

$\therefore$ $\phi'(x) = (x-x_1)(x-x_2)...(x-x_r)...(x-x_n) + (x-x_0)(x-x_2)...(x-x_r)...(x-x_n)$

$+ (x-x_0)(x-x_1)...(x-x_r)(x-x_{r+1})...(x-x_n) + (x-x_0)(x-x_1)...(x-x_r)...(x-x_{n-1})$

$\Rightarrow$ $\phi'(x) = [\phi'(x)]_{x = xr} = (x_r - x_0)(x_r - x_1)... (x_r - x_{r-1})(x_r - x_{r+1})... (x_r - x_n)$ ...(2)

Hence from equation (1)

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x-x_r)\phi'(x_r)} \quad \text{[Using (2)]}$$

Example 1. *Using Lagrange's formula, find the value of*

(i) y_x *if* $y_1 = 4, y_3 = 120, y_4 = 340, y_5 = 2544,$

(ii) y_0 *if* $y_{-30} = 30, y_{-12} = 34, y_3 = 38, y_{18} = 42,$

Sol. (*i*) Here, $x_0 = 1, x_1 = 3, x_2 = 4, x_3 = 5,$

$f(x_0) = 4, f(x_1) = 120, f(x_2) = 340, f(x_3) = 2544,$

Now using Lagrange's interpolation formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_1)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f(x) = \frac{(x-3)(x-4)(x-5)}{(1-3)(1-4)(1-5)} \times 4 + \frac{(x-1)(x-4)(x-5)}{(3-1)(3-4)(3-5)} \times 120$$

$$+ \frac{(x-1)(x-3)(x-5)}{(4-1)(4-3)(4-5)} \times 340 + \frac{(x-1)(x-3)(x-4)}{(5-1)(5-3)(5-4)} \times 2544$$

$$y_x = f(x) = -\frac{1}{6}(x-3)(x-4)(x-5) + 30(x-1)(x-4)(x-5)$$

$$-\frac{340}{3}(x-1)(x-3)(x-5) + 318(x-1)(x-3)(x-4)$$

(*ii*) Here, $x_0 = -30, x_1 = -12, x_2 = 3, x_3 = 18,$

$y_0 = 30, y_1 = 34, y_2 = 38, y_3 = 42,$

Now from Lagrange's interpolation formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$y_x = \frac{(x+12)(x-3)(x-18)}{(-30+12)(-30-3)(-30-18)} \times 30 + \frac{(x+30)(x-3)(x-18)}{(-12+30)(-12-3)(-12-18)} \times 34$$

$$+ \frac{(x+30)(x+12)(x-18)}{(3+30)(3+12)(3-18)} \times 38 + \frac{(x+30)(x+12)(x-3)}{(18+30)(18+12)(18-3)} \times 42$$

$$y_x = -0.001052188(x+12)(x-3)(x-18) + 0.00419753(x+30)(x-3)(x-18)$$

$$-0.005117845(x+30)(x+12)(x-18) + 0.001944444(x+30)(x+12)(x-3)$$

for $x = 0$

$$y_0 = -0.001052188(12)(-3)(-18) + 0.00419753(30)(-3)(-18)$$

$$-0.005117895(30)(12)(-18) + 0.001944444(30)(12)(-3)$$

$$y_0 = -\ 0.681817824 + 6.7999986 + 33.1636356 - 2.09999952$$
$$y_0 = 39.9636546 - 2.781817344$$
$$y_0 = 37.1818. \quad \textbf{Ans.}$$

Example 2. *If $y_0, y_1, y_2, y_3 \dots y_9$ are consecutive terms of a series. Prove that* $y_5 = \frac{1}{70}$ *$[56(y_4 + y_6) - 28\ (y_3 + y_7) + 8\ (y_2 + y_8) - (y_1 + y_9)]$*

Sol. Here, the arguments are 1, 2, 3,... 9 so for these values Lagrange's formula is given by

$$\frac{y_x}{(x-1)(x-2)(x-3)(x-4)(x-6)(x-7)(x-8)(x-9)}$$

$$= \frac{y_1}{(x-1)(-1)(-2)(-3)(-5)(-6)(-7)(-8)} + \frac{y_2}{(x-2)1(-1)(-2)(-4)(-5)(-6)(-7)}$$
$$+ \frac{y_3}{(x-3)2(1)(-1)(-3)(-4)(-5)(-6)} + \frac{y_4}{(x-4)3.2.1(-2)(-3)(-4)(-5)}$$
$$+ \frac{y_6}{(x-6)5.4.3.2(-1)(-2)(-3)} + \frac{y_7}{(x-7)6.5.4.3.1(-2)(-1)} + \frac{y_8}{(x-8)7.6.5.4.2.1(-1)}$$
$$+ \frac{y_9}{(x-9)8.7.6.5.3.2.1} \frac{y_x}{(x-1)(x-2)(x-3)(x-4)(x-6)(x-7)(x-8)(x-9)}$$

$$= \frac{y_1}{-(x-1)(10080)} + \frac{y_2}{(x-2)(1680)} + \frac{y_3}{-720(x-3)} + \frac{y_4}{720(x-4)} + \frac{y_6}{-720(x-6)}$$
$$+ \frac{y_7}{720(x-7)} + \frac{y_8}{-1680(x-8)} + \frac{y_9}{10080(x-9)}$$

Now for y_5, put $x = 5$

$$\frac{y_5}{4.3.2.1(-1)(-2)(-3)(-4)} = \frac{y_1}{-4\times 10080} + \frac{y_2}{3\times 1680} + \frac{y_3}{(-720)\times 2}$$
$$+ \frac{y_4}{720\times 1} + \frac{y_6}{(-1)\times(-720)} + \frac{y_7}{(-2)\times 720} + \frac{y_8}{(-1680)\times(-3)} + \frac{y_9}{(-4)\times(-10080)}$$

$$\frac{y_5}{576} = -\frac{y_1}{40320} + \frac{y_2}{5040} - \frac{y_3}{1440} + \frac{y_4}{720} + \frac{y_6}{720} - \frac{y_7}{1440} + \frac{y_8}{5040} - \frac{y_9}{40320}$$

$$\frac{y_5}{576} = \frac{1}{720}(y_4 + y_6) - \frac{1}{1440}(y_3 - y_7) + \frac{1}{5040}(y_2 + y_8) - \frac{1}{40320}(y_1 + y_9)$$

$$y_5 = \frac{576}{720}(y_4 + y_6) - \frac{576}{1440}(y_3 + y_7) + \frac{576}{5040}(y_2 + y_8) - \frac{576}{40320}(y_1 - y_9)$$

$$y_5 = \frac{1}{70}\left[\frac{70\times 576}{720}(y_4 + y_6) - \frac{576\times 70}{1440}(y_3 + y_7) + \frac{576\times 70}{5040}(y_2 + y_8) - \frac{576\times 70}{40320}(y_1 + y_9)\right]$$

$$y_5 = \frac{1}{70}\ [56\ (y_4 + y_6) - 28\ (y_3 + y_7) + 8\ (y_2 + y_8) - (y_1 + y_9)]. \quad \textbf{Proved.}$$

Example 3. *Find f(x) as a polynomial of x if*

x	:	–1	0	3	6	7
$f(x)$	:	3	–6	39	822	1611

Sol. Now from Lagrange's interpolation formula,

$$P(x) = f(x) = \frac{(x-0)(x-3)(x-6)(x-7)}{(-1-0)(-1-3)(-1-6)(-1-7)} \times 3 + \frac{(x+1)(x-3)(x-6)(x-7)}{(0+1)(0-3)(0-6)(0-7)} \times (-6)$$

$$+ \frac{(x+1)(x-0)(x-6)(x-7)}{(3+1)(3-0)(3-6)(3-7)} \times 39 + \frac{(x+1)(x-0)(x-3)(x-7)}{(6+1)(6-0)(6-3)(6-7)} \times 822$$

$$+ \frac{(x+1)(x-0)(x-6)(x-3)}{(7+1)(7-0)(7-3)(7-6)} \times 1611$$

$$P(x) = \frac{3}{224}\, x(x-3)(x-6)(x-7) + \frac{6}{126}\,(x+1)(x-3)(x-6)(x-7)$$

$$+ \frac{39}{144}\,(x+1)x(x-6)(x-7) - \frac{822}{126}\,(x+1)x(x-3)(x-7) + \frac{1611}{224}\,(x+1)x(x-6)(x-3)$$

$$P(x) = \frac{3}{224}\,(x^4 - 16x^3 + 81x^2 - 126x) + \frac{39}{144}\,(x^4 - 12x^3 + 29x^2 + 42x)$$

$$+ \frac{6}{126}\,(x^4 - 15x^3 + 23x^2 - 45x - 126)$$

$$- \frac{822}{126}\,(x^4 - 9x^3 + 11x^2 + 21x) + \frac{1611}{224}\,(x^4 - 8x^3 + 9x^2 + 18x)$$

$$P(x) = x^4\left(\frac{3}{224} + \frac{39}{144} + \frac{6}{126} - \frac{822}{126} + \frac{1611}{224}\right) + x^3\left(\frac{-16\times 3}{224} - \frac{12\times 39}{144} - \frac{15\times 6}{126} + \frac{822\times 9}{126} - \frac{1611\times 8}{224}\right)$$

$$+ x^2\left(\frac{3\times 81}{224} + \frac{39\times 29}{144} + \frac{23\times 6}{126} - \frac{822\times 11}{126} + \frac{9\times 1611}{224}\right)$$

$$+ x\left(\frac{-45\times 6}{126} - \frac{126\times 3}{224} + \frac{39\times 42}{144} - \frac{822\times 21}{126} - \frac{1611\times 18}{224}\right) - 6$$

$$P(x) = x^4(1) + x^3(-3) + x^2(5) - 6$$

$$P(x) = x^4 - 3x^3 + 5x^2 - 6$$

which is the required polynomial.

Example 4. *Value of f(x) are given at a, b, c. Show that the maximum is obtained by*

$$x = \frac{f(a)(b^2 - c^2) + f(b)(c^2 - a^2) + f(c)(a^2 - b^2)}{[f(a)(b-c) + f(b)(c-a) + f(c)(a-b)]}$$

Sol. Here, $x_0 = a,\ x_1 = b,\ x_2 = c,$

$f(x_0) = f(a),\ f(x_1) = f(b),\ f(x_2) = f(c)$

By applying Lagrange's formula, we have

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

$$f(x) = \frac{x^2-(b+c)\,x+bc}{(a-b)\,(a-c)}\,f(a)+\frac{x^2-(a+c)\,x+ac}{(b-a)\,(b-c)}\,f(b)+\frac{x^2-(a+b)x+ab}{(c-a)\,(c-b)}\,f(c)$$

For maximum, we have

$$f'(x) = 0$$

i.e.,

$$f'(x) = \frac{2x-(b+c)\,f(a)}{(a-b)\,(a-c)}+\frac{2x-(a+c)\,f(b)}{(b-a)(b-c)}+\frac{2x-(a+b)\,f(c)}{(c-a)(c-b)}$$

$$= \{2x-(b+c)\}\,(b-c)\,f(a)+\{2x-(a+c)\}\,(c-a)\,f(b)+\{2x-(a+b)\}(a-b)\,f(c)=0$$

$$= 2x\,[(b-c)\,f(a)+(c-a)\,f(b)+(a-b)\,f(c)] = (b^2-c^2)\,f(a)+(c^2-a^2)\,f(b)+(a^2-b^2)\,f(c)$$

$(\because 2 \neq 0)$

$$\therefore \quad x = \frac{\left(b^2-c^2\right)f(a)+\left(c^2-a^2\right)f(b)+\left(a^2-b^2\right)f(c)}{f(a)(b-c)+f(b)(c-a)+f(c)(a-b)}.$$ **Proved.**

Example 5. *Applying Lagrange's formula, find a cubic polynomial which approximate the following data*

x	:	–2	–1	2	3
y(x)	:	–12	–8	3	5

Sol. Now, using Lagrange's formula, we have

$$f(x) = \frac{(x+1)\,(x-2)(x-3)\times(-12)}{(-2+1)(-2-2)\,(-2-3)}+\frac{(x+2)\,(x-2)\,(x-3)\times(-8)}{(-1+2)\,(-1-2)\,(-1-3)}$$

$$+\ \frac{(x+2)\,(x+1)\,(x-3)\times 3}{(2+2)\,(2+1)\,(2-3)}+\frac{(x+2)\,(x+1)(x-2)\times 5}{(3+2)\,(3+1)\,(3-2)}$$

$$f(x) = \frac{3}{5}\left[x^3-4x^2+x-6\right]-\frac{2}{3}\left[x^3-3x^2-4x+12\right]-\frac{1}{4}\left[x^3-7x-6\right]+\frac{1}{4}\left[x^3+x^2-4x-4\right]$$

$$f(x) = x^3\left[\frac{3}{5}-\frac{2}{3}-\frac{1}{4}+\frac{1}{4}\right]+x^2\left[-\frac{12}{5}+2+\frac{1}{4}\right]+x\left[\frac{3}{5}+\frac{8}{3}+\frac{7}{4}-1\right]+\frac{18}{5}-\frac{24}{3}+\frac{3}{2}-1$$

$$\therefore f(x) = -\frac{1}{15}x^3-\frac{3}{20}x^2+\frac{241}{60}x-3.9$$

Example 6. *(i) Determine by Lagrange's formula, the percentage number of criminals under 35 years:*

Age	***% no. of Criminals***
Under 25 year	52
Under 30 years	67.3
Under 40 years	84.1
Under 50 years	94.4

(ii) Find a Lagrange's interpolating polynomial for the given data:

$x_0 = 1$, $x_1 = 2.5$, $x_2 = 4$, *and* $x_3 = 5.5$

$f(x_0) = 4$, $f(x_1) = 7.5$, $f(x_2) = 13$ *and* $f(x_3) = 17.5$

also, find the value of f(5)

Sol. (*i*) Here $x_0 = 25$, $x_1 = 30$, $x_2 = 40$ and $x_3 = 50$

$f(x_0) = 52$, $f(x_1) = 67.3$, $f(x_2) = 84.1$ and $f(x_3) = 94.4$

By using Lagrange's interpolation formula, we have

$$f(x) = \frac{(x-30)(x-40)(x-50)}{(25-30)(25-40)(25-50)} \times 52 + \frac{(x-25)(x-40)(x-50)}{(30-25)(30-40)(30-50)} \times 67.3$$

$$+ \frac{(x-30)(x-25)(x-50)}{(40-25)(40-30)(40-50)} \times 84.1 + \frac{(x-25)(x-30)(x-40)}{(50-25)(50-30)(50-40)} \times 94.4$$

Now for $f(35)$, put $x = 35$

$$f(35) = \frac{(5)(-5)(-15)}{(-5)(-15)(-25)} \times 52 + \frac{(10)(-5)(-15)}{(5)(-10)(-20)} \times 67.3 + \frac{(5)(10)(-15)}{(15)(10)(-10)} \times 84.1 + \frac{(10)(5)(-5)}{(25)(20)(10)} \times 94.4$$

$f(35) = -10.5 + 50.475 + 42.05 + 4.72$

$f(35) = 77.405$

(*ii*) By using Lagrange's formula, we have

$$f(x) = \frac{(x-2.5)(x-4)(x-5.5)}{(1-2.5)(1-4)(1-5.5)} \times 4 + \frac{(x-1)(x-4)(x-5.5)}{(2.5-1)(2.5-4)(2.5-5.5)} \times 7.5$$

$$+ \frac{(x-1)(x-2.5)(x-5.5)}{(4-1)(4-2.5)(4-5.5)} \times 13 + \frac{(x-1)(x-2.5)(x-4)}{(5.5-1)(5.5-2.5)(5.5-4)} \times 17.5$$

Put $x = 5$

$$f(5) = \frac{(2.5)(1)(-0.5)}{(-1.5)(-3)(-4.5)} \times 4 + \frac{(4)(1)(-0.5)}{(1.5)(-1.5)(-3)} \times 7.5 + \frac{(4)(2.5)(-0.5)}{(3)(1.5)(-1.5)} \times 13 + \frac{(4)(2.5)(1)}{(4.5)(3)(1.5)} \times 17.5$$

$$f(5) = \frac{5}{20.25} - \frac{15}{6.75} + \frac{65}{6.75} + \frac{175}{20.25} = 0.246913 - 2.2222 + 9.62962 + 8.641975$$

$f(5) = 18.51850831 - 2.222222 = 16.296296.$ **Ans.**

$$f(x) = \frac{-4}{20.25}(x-2.5)(x-4)(x-5.5) + \frac{7.5}{6.75}(x-1)(x-4)(x-5.5)$$

$$+ \frac{(-13)}{6.75}(x-1)(x-2.5)(x-5.5) + \frac{17.5}{20.25}(x-1)(x-2.5)(x-4)$$

$$P(x) = -0.1481x^3 + 1.5555x^2 - 1.6666x + 4.2592$$

which is a required polynomial

At $x = 5$, $f(x) = 16.3012$ (Approx.)

Example 7. *By means of Lagrange's formula, show that*

$$(i)\ y_0 = \frac{1}{2}(y_1 - y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

(ii) $y_3 = 0.05\ (y_0 + y_6) - 0.3\ (y_1 + y_5) + 0.75\ (y_2 + y_4)$

(iii) $y_1 = y_3 - 0.3\ (y_5 - y_{-3}) + 0.2\ (y_{-3} - y_{-5})$

Sol. (*i*) For the arguments – 3, –1, 1, 3, the Lagrange's formula is

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(x+3)(x-1)(x-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(x+3)(x+1)(x-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(x+3)(x+1)(x-1)}{(3+3)(3+1)(3-1)} y_3$$

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-48)} y_{-3} + \frac{(x+3)(x-1)(x-3)}{16} y_{-1}$$

$$+ \frac{(x+3)(x+1)(x-3)}{(-16)} y_1 + \frac{(x+3)(x+1)(x-1)}{48} y_3 \quad ...(1)$$

Putting $x = 0$ in (1), we get

$$y_0 = -\frac{1}{16}\ y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}\ (y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

(*ii*) For the arguments 0, 1, 2, 4, 5, 6, the Lagrange's formula is

$$y_x = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} y_0 + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} y_1$$

$$+ \frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} y_2 + \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} y_4$$

$$+ \frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} y_5 + \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} y_6 \quad ...(2)$$

Putting $x = 3$ in (2), we get

$$y_3 = 0.05\ y_0 - 0.3\ y_1 + 0.75\ y_2 + 0.75\ y_4 - 0.3\ y_5 + 0.05 y_6$$

$$y_3 = 0.05\ (y_0 + y_6) - 0.3\ (y_1 + y_5) + 0.75\ (y_2 + y_4)$$

(*iii*) For the arguments –5, –3, 3, 5, the Lagrange's formula is

$$y_x = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(5+3)(3+3)(3-5)} y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} y_5 \quad ...(3)$$

Putting $x = 1$ in equation (3), we get

$$y_1 = (-0.2)\ y_{-5} + 0.5\ y_{-3} + y_3 - 0.3 y_5$$

$$= y_3 - 0.3\ (y_5 - y_{-3}) + 0.2\ (y_{-3} - y_{-5})$$

Example 8. *Prove that Lagrange's formula can be expressed in the*

$$\begin{vmatrix} P_n(x) & | & x & x^2 \ldots x^n \\ f(x_0) & | & x_0 & x_0^2 \ldots x_0^n \\ f(x_1) & | & x_1 & x_1^2 \ldots x_1^n \\ \ldots & \ldots & \ldots & \ldots \\ f(x_n) & | & x_n & x_n^2 \ldots x_n^n \end{vmatrix} = 0$$

where $P_n(x) = f(x)$.

Sol. Let $\quad P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \qquad \ldots(1)$

Given that $P_n(x) = f(x)$

$\Rightarrow \quad P_n(x_i) = f(x_i);\ i = 0, 1, 2 \ldots, n \qquad \ldots(2)$

Substitute $x = x_0, x_1, x_2, \ldots x_n$ Successively in equation (2)

$$\Rightarrow \quad f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n$$

$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n$$

Now Eliminating $a_0, a_1, a_2 \ldots, a_n$ from above equations, we get

$$\begin{vmatrix} -P_n(x) & | & x & x^2 \ldots x^n \\ -f(x_0) & | & x_0 & x_0^2 \ldots x_0^n \\ -f(x_1) & | & x_1 & x_1^2 \ldots x_1^n \\ \ldots & \ldots & \ldots & \ldots \\ -f(x_n) & | & x_n & x_n^2 \ldots x_n^n \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} P_n(x) & | & x & x^2 \ldots x^n \\ f(x_0) & | & x_0 & x_0^2 \ldots x_0^n \\ f(x_1) & | & x_1 & x_1^2 \ldots x_1^n \\ \ldots & \ldots & \ldots & \ldots \\ f(x_n) & | & x_n & x_n^2 \ldots x_n^n \end{vmatrix} = 0$$

Example 9. *Four equidistant values u_{-1}, u_0, u_1 and u_2 being given, a value is interpolated by Lagrange's formula show that it may be written in the form.*

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0 \text{ where } x + y = 1.$$

Sol. $\Delta^2 u_1 = (E-1)^2 u_{-1} = (E^2 - 2E + 1)\, u_{-1} = u_1 - 2u_0 + u_{-1}$

$\Delta^2 u_0 = (E_2 - 2E + 1)\, u_0 = u_2 - 2u_1 + u_0$

$$\text{R.H.S.} = (1-x)\, u_0 + xu_1 + \frac{(1-x)\left\{(1-x)^2 - 1\right\}}{3!}(u_1 - 2u_0 + u_{-1})$$

$$+ \frac{x(x^2-1)}{3!}(u_2 - 2u_1 + u_0) \text{ where } y = 1 - x$$

$$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \frac{(x+2)(x-1)(x-2)}{2}u_0 + \frac{(x+1)x(x-2)}{2}u_1$$

$$+ \frac{(x+1)x(x-1)}{6}u_2 \qquad \ldots(1)$$

Applying Lagrange's formula for the arguments – 1, 0, 1 and 2.

$$u_x = \frac{x(x-1)(x-2)}{(-1)(-2)(-3)} u_{-1} + \frac{(x-2)(x-1)(x+1)}{(1)(-1)(-2)} u_0 + \frac{(x+1)x(x-2)}{(2)(1)(-1)} u_1 + \frac{(x+1)x(x-1)}{(3)(2)(1)} u_2$$

$$= -\frac{x(x-1)(x-2)}{6} u_{-1} + \frac{(x-2)(x-1)(x+1)}{2} u_0 - \frac{(x+1)x(x-2)}{2} u_1 + \frac{(x+1)x(x-1)}{6} u_2 \quad ...(2)$$

From (1) and (2), we observe that R.H.S. = L.H.S.

Hence the result.

Example 10. *Find the cubic Lagrange's interpolating polynomial from the following data*

x	:	0	1	2	5
$f(x)$	:	2	3	12	147

Sol. Here, $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 5,$

$f(x_0) = 2, \quad f(x_1) = 3, \quad f(x_2) = 12, \quad f(x_3) = 147,$

Lagrange's formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{-1}{5}(x-1)(x-2) + \frac{3}{4}x(x-2)(x-5) - 2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$$

$$= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10) - 2(x^3 - 6x^2 + 5x)$$

$$+ \frac{49}{20}(x^3 - 3x^2 + 2x)$$

$\Rightarrow \; f(x) = x^3 + x^2 - x + 2$

which is the required Lagrange's interpolating polynomial.

Example 11. *The function $y = f(x)$ is given at the points (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for $x = 9.5$ using Lagrange's interpolation formula.*

Sol. We are given

x	:	7	8	9	10
$f(x)$	:	3	1	1	9

Here,
$$x_0 = 7, \quad x_1 = 8, \quad x_2 = 9, \quad x_3 = 10,$$
$$f(x_0) = 3, \quad f(x_1) = 1, \quad f(x_2) = 1, \quad f(x_3) = 9,$$

Lagrange's interpolation formula is

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 + x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$
$$= \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)}(3) + \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)}(1) + \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)}(1)$$
$$+ \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)}(9)$$
$$= -\frac{1}{2}(x-8)(x-9)(x-10) + \frac{1}{2}(x-7)(x-9)(x-10) - \frac{1}{2}(x-7)(x-8)(x-10)$$
$$+ \frac{3}{2}(x-7)(x-8)(x-9) \quad \text{...(1)}$$

Putting $x = 9.5$ in eqn. (1), we get

$$f(9.5) = -\frac{1}{2}(9.5-8)(9.5-9)(9.5-10) + \frac{1}{2}(9.5-7)(9.5-9)(9.5-10)$$
$$- \frac{1}{2}(9.5-7)(9.5-8)(9.5-10) + \frac{3}{2}(9.5-7)(9.5-8)(9.5-9)$$
$$= 3.625$$

Example 12. *Find the unique polynomial P(x) of degree 2 such that:*

P(1) = 1 P(3) = 27 P(4) = 64

Use Lagrange's method of interpolation.

Sol. Here,
$$x_0 = 1, \quad x_1 = 3, \quad x_2 = 4,$$
$$f(x_0) = 1, \quad f(x_1) = 27, \quad f(x_2) = 64,$$

Lagrange's interpolation formula is

$$P(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_2 - x_1)} f(x_2)$$
$$= \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$
$$= \frac{1}{6}(x^2 - 7x + 12) - \frac{27}{2}(x^2 - 5x + 4) + \frac{64}{3}(x^2 - 4x + 3)$$
$$= 8x^2 - 19x + 12$$

Hence the required polynomial is,

$$P(x) = 8x^2 - 19x + 12$$

PROBLEM SET 5.1

1. Using Lagrange's interpolation formula, find $y(10)$ from the following table.

X	:	5	6	9	11
Y	:	12	13	14	16

[**Ans.** 14.6666667]

2. Use Lagrange's interpolation formula to fit a polynomial for the data

x	:	–1	0	2	3
u_x	:	–8	3	1	12

Hence or otherwise find the value of u_1. [**Ans.** $u_x = 2x^3 - 6x^2 + 3x + 3$, $u_1 = 2$]

3. Compute the value of $f(x)$ for $x = 2.5$ from the following data:

x	:	1	2	3	4
$f(x)$	:	1	8	27	64

Using Lagrange's interpolation method. [**Ans.** 15.625]

4. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$ and $y(6) = 132$, find the four point Lagrange's interpolation polynomial which takes the same values as the function y at the given points.

[**Ans.** $x^3 - 3x^2 + 5x - 6$]

5. Find the polynomial of degree three which takes the values given below:

x	:	0	1	2	4
y	:	1	1	2	5

[**Ans.** $-\dfrac{1}{12}x^3 + \dfrac{3}{4}x^2 - \dfrac{2}{3}x + 1$]

6. Find $f(x)$ by using Lagrange's interpolation formula:

x	:	0	2	3	4
$f(x)$	:	659	705	729	804

Also find maximum value of $f(x)$ [**Ans.** $\dfrac{151}{24}x^3 - \dfrac{249}{8}x^2 + \dfrac{721}{12}x + 659$]

[No real value of x exist for which $f(x)$ is max.]

7. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$. Find $\log_{10} 656$ by Lagrange's interpolation formula [**Ans.** 2.8169]

8. Compute Sin 15° by Lagrange's Method from the data given below:

x	:	0°	30°	45°	60°	90°
y	:	0.0000	0.50000	0.70711	0.86603	1.0000

[**Ans.** 0.25859]

9. The percentage of Criminals for different age group are given below:

Age less than	:	25	30	40	50
Percentage of Criminals	:	52	67	84	94

Apply Lagrange's formula to find the percentage of criminals under 35 years of age.

[**Ans.** 77]

10. If $f(x) = \dfrac{1}{a-x}$ show that

$$f(x_0, x_1, x_2, x_3, \ldots x_n) = \frac{1}{(a-x_0)(a-x_1)\ldots(a-x_n)} \text{ and } f(x_0, x_1, x_2, x_3, \ldots x_n, x)$$

$$= \frac{1}{(a-x_0)(a-x_1)\ldots(a-x_n)(a-x)}$$

11. Certain corresponding values of x and $\log_{10} x$ are given as

x	:	300	304	305	307
$\log_{10} x$	:	2.4771	2.4829	2.4843	2.4871

Find the $\log_{10}301$ by Lagrange's formula. [**Ans.** 2.4786]

12. The following table gives the normal weights of babies during the first 12 months of life:

Age in Months	:	0	2	5	8	10	12
Weight in lbs	:	7.5	10.25	15	16	18	21

Find the weight of babies during 5 to 5.6 months of life. [**Ans.** 15.67]

13. Find the value of tan 33° by Lagrange's formula if tan 30° = 0.5774, tan 32° = 0.6249, tan 35° = 0.7002, tan 38° = 0.7813. [**Ans.** 0.64942084]

14. Apply Lagrange's formula to find $f(5)$ and $f(6)$ given that $f(2) = 4$, $f(1) = 2$, $f(3) = 8$, $f(7) = 128$. Explain why the result differs from those obtained by completing the series of powers of 2? [**Ans.** 38.8, 74; 2^x is not a polynomial]

5.3 ERRORS IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$, defined by the $(n + 1)$ points $(x_i, y_i) i = 0, 1, 2, \ldots n$ be continuous and differentiable $(n + 1)$ times, and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = yi,\ i = 1, 2, \ldots n. \qquad \ldots(1)$$

Now use $\phi_n(x)$ to obtain approximate value of $y(x)$ at some points other than those defined by (1). Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, \ldots x_n$, we put

$$y(x) - \phi_n(x) = L\,\pi_{n+1}(x) \qquad \ldots(2)$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$ ...(3)

and L is to be determined such that equation (2) holds for any intermediate value of x, say $x = x'$, $x_0 < x' < x_n$ clearly,

$$L = \frac{y(x') - \phi_n(x')}{\pi_{n+1}(x')} \qquad \ldots(4)$$

We construct a function $F(x)$ such that

$$F(x) = y(x) - \phi_n(x) - L.\,\pi_{n+1}(x) \qquad \ldots(5)$$

where L is given by (4).

It is clear that

$$F(x_0) = F(x_1) = \ldots F(x_n) = F(x') = 0$$

that is $F(x)$ vanished $(n + 2)$ times in the interval $x_0 \le x \le x_n$; consequently, by the repeated application of Rolle's theorem, $F^{|}(x)$ must vanish $(n + 1)$ times, $F^{||}(x)$ must vanish n times, etc., in the interval $x_0 < x < x_n$;. In particular, $F^{(n+1)}(x)$ must vanish once in the interval. Let this point be given by $x = \xi$, $x_0 < \xi < x_n$. On differentiating (5) $(n + 1)$ times with respect to x and putting $x = \xi$, we obtain

$$0 = y^{(n+1)}(\xi) - L.\,(n + 1)!$$

so that

$$L = \frac{y^{(n+1)}(\xi)}{(n+1)!} \qquad \ldots(6)$$

On comparison of (4) and (6), we get

$$y(x') - \phi_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\,\pi_{n+1}(x')$$

Dropping the prime on x', we obtain

$$y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}\,y^{(n+1)}(\xi),\ x_0 < \xi < x_n \qquad \ldots(7)$$

Which is the required Expression for error.

5.3.1 Error in Lagrange's interpolation formula

To estimate the error of Lagrange's interpolation formula for the class of functions which have continuous derivatives of order upto $(n + 1)$ on $[a, b]$ we use above equation 7 (Art. 5.3).

Therefore we have,

$$y(x) - L_n(x) = R_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}\,y^{(n+1)}(\xi), \qquad a < \xi < b$$

and the quantity E_L where

$$E_L = \max_{[a,b]} |R_n(x)|$$

may be taken as an estimate of error. Further, if we assume that

$$\left|y^{(n+1)}(\xi)\right| \le M_{n+1}, \qquad a \le \xi \le b$$

then

$$E_L \le \frac{M_{n+1}}{(n+1)!}\max_{[a,b]}\left|\pi_{n+1}(x)\right|$$

Example 1. *Find the Value of Sin*$\left(\frac{\pi}{16}\right)$ *from the data given using by Lagrange's interpolation formula. Hence estimate the error in the solution.*

x	:	0	$\pi/4$	$\pi/2$
$y = \sin x$	:	0	0.70711	1.0

Sol. $$\sin\left(\frac{\pi}{6}\right) = \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711) + \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)} \quad ...(1)$$

$$= \frac{8}{9}(0.70711) - \frac{1}{9} = \frac{4.65688}{9} = 0.51743$$

Now, $y(x) = \sin x$, $y'(x) = \cos x$, $y''(x) = \sin x$, $y'''(x) = -\cos x$,

Hence, $|y'''(\xi)| < 1$

When $x = \pi/6$

$$|R_n(x)| \le \left| \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{3!} \right| = 0.02392$$

where agrees with the actual error in problem.

Example 2. *Show that the truncation error of quadratic interpolation in an equidistant table is bounded by* $\frac{h^3}{9\sqrt{3}}$ *max* $|f''' \xi|$ *where h is the step size and f is the tabulated function.*

Sol. Let x_{i-1}, x_i, x_{i+1} denote three consecutive equispaced points with step size h. The truncation error of the quadratic Lagrange interpolation is bounded by

$$|E_2(f\,;x)| \le \frac{M_3}{6} \max |(x-x_{i-1})(x-x_i)(x-x_{i+1})|$$

where $x_{i-1} \le x \le x_{i+1}$ and $M_3 = \max\limits_{-1 \le x \le 1} |f'''(x)|$

Substitute $t = \frac{x - x_i}{h}$ then,

$$x - x_{i-1} = x - (x_i - h) = x - x_i + h = th + h = (t+1)\,h$$
$$x - x_{i+1} = x - (x_i + h) = x - x_i - h = th - h = (t-1)\,h$$

and $$(x - x_{i-1})(x - x_i)(x - x_{i+1}) = (t+1)\,t(t-1)h^3 = t(t^2-1)\,h^3 = g(t)$$

Setting $g'(t) = 0$, we get

$$3t^2 - 1 = 0 \Rightarrow t = \pm \frac{1}{\sqrt{3}}$$

For both these values of t, we obtain

$$\max |(x-x_{i-1})(x-x_i)(x-x_{i+1})| = h^3 \max_{-1 \le t \le 1} |t\,(t^2-1)| = \frac{2h^3}{3\sqrt{3}}$$

Hence, the truncation error of the quadratic interpolation is bounded by

$$|E_2(f\,;x)| \le \frac{h^3}{9\sqrt{3}} M_3$$

$$|E_2(f\,;x)| \le \frac{h^3}{9\sqrt{3}} \max |f'''(\xi)|$$

Example 3. *Determine the step size that can be used in the tabulation of f(x) = sin x in the interval* $\left[0, \frac{\pi}{4}\right]$ *at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than* 5×10^{-8}.

Sol. From Example 2, we know

$$|E_2(f\,;x)| \le \frac{h^3}{9\sqrt{3}} M_3$$

For $f(x) = \sin x$, we get $f'''(x) = -\cos x$ and $M_3 = \max\limits_{0 \le x \le \pi/4} |\cos x| = 1$

Hence the step size h is given by

$$\frac{h^3}{9\sqrt{3}} \le 5 \times 10^{-8} \text{ or } h = 0.009$$

5.3.2 Inverse Interpolation

We know different formulae for obtaining y corresponding to argument value of x (for equal and unequal spaced argument). On the other hand the process of Estimating the value of x for a entry value of y (which is not in the table) is called **Inverse Interpolation.** In this case when the values of x are unequally spaced, we use Lagrange's method and when x are equally spaced, then Iterative method should be employed.

(a) Lagrange's method for inverse interpolation: The only difference of this formula from Lagrange's method is that x is assumed to be expressible as a polynomial in y. So on interchanging x and y in the Lagrange's formula we have,

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)} x_1 + \ldots$$
$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})} x_n$$

Which is the inverse interpolation formula.

(b) Iterative method: Newton's forward interpolation formula is

$$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

From this

$$u = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u(u-1)}{2!}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 - \ldots\right] \qquad \ldots(1)$$

On neglecting the second and higher order differences, we get first approximation to u as

$$u_1 = \frac{y_u - y_0}{\Delta y_0} \qquad \ldots(2)$$

To find second approximation, retaining the term with second difference in (1) and replace u by u_1, we get

$$u_2 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1-1)}{2!}\Delta^2 y_0\right]$$

Similarly,
$$u_3 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_2(u_2-1)}{2!}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{3!}\Delta^3 y_0\right]$$

This process is continued till two successive approximations of u agree with desired accuracy. This technique can equally be also applied by starting with any other interpolation formula. This method is a powerful iterative procedure for finding the roots of an equation to a good degree.

Example 4. *Using Inverse interpolation find the real root of the equation $x^3 + x - 3$ which is closed to 1.2.*

Sol.

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	1	–1				
			0.431			
–1	1.1	–0.569		0.066		
			0.497		0.006	
0	1.2	–0.072		0.072		0
			0.569		0.006	
1	1.3	0.497		0.078		
			0.647			
2	1.4	1.144				

Let the origin be at 1.2. Using Stirling's formula

$$y = y_0 + u\left(\frac{\Delta y_0 + \Delta_{y-1}}{2}\right) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

we have,
$$0 = -0.072 + u\left(\frac{0.569 + 0.497}{2}\right) + \frac{u^2}{2}(0.072) + \frac{u(u^2-1)}{6}\left(\frac{0.006+0.006}{2}\right)$$

or $\quad 0 = -0.072 + 0.533u + 0.036u^2 + u(u^2 - 1)(0.001)$

or $\quad 0 = -0.072 + 0.532u + 0.036u^2 + 0.001u^3 \quad$ ...(1)

From equation (1)

$$u = 0.1353 - 0.0675u^2 - 0.0019u^3 \quad ...(2)$$

Neglecting all terms beyond the R.H.S of (2), we get

$$u^{(1)} = 0.1353$$

Substitute $u^{(1)}$ for u in (2), we get second approximation

$$u^{(2)} = 0.1341$$

$u^{(1)}$ and $u^{(2)}$ are nearly equal up to third decimal place so the required root is

$$x = uh + x_0 = 1.2 + 0.1 \times 0.134. \qquad \text{Since } u = \frac{x - x_0}{h} = 1.2134$$

Example 5. *Values of elliptic integral* $F(\theta) = \sqrt{2}\int_0^{\theta} \frac{d\theta}{\sqrt{1+\cos^2\theta}}$ *are given below:*

θ	:	21°	23°	25°
$F(\theta)$	:	0.3706	0.4068	0.4433

Find θ *for which* $F(\theta) = 0.3887$.

Sol. By inverse interpolation formula

$$\theta = \frac{(F-F_1)(F-F_2)}{(F_0-F_1)(F_0-F_2)}\theta_0 + \frac{(F-F_0)(F-F_2)}{(F_1-F_0)(F_1-F_2)}\theta_1 + \frac{(F-F_0)(F-F_1)}{(F_2-F_0)(F_2-F_1)}\theta_2$$

$$= \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}(0.3706) + \ldots + \ldots$$

$$= 7.884 + 17.20 - 3.087 = 22°$$

Example 6. *From the given table*

x	:	20	25	30	35
$y(x)$	:	0.342	0.423	0.5	0.65

Find the value of x for y(x) = 0.390.

Sol. By inverse interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

$$= \frac{(.39-.423)(.39-.5)(.39-.65)}{(.342-.423)(.342-.5)(.342-.65)}(20) + \frac{(.39-.342)(.39-.5)(.39-.65)}{(.423-.342)(.423-.5)(.423-.65)}(25)$$

$$+ \frac{(.39-.342)(.39-.423)(.39-.65)}{(.5-.342)(.5-.423)(.5-.65)}(30) + \frac{(.39-.342)(.39-.423)(.39-.5)}{(.65-.342)(.65-.423)(.65-.5)}(35)$$

$= 22.84057797.$ **Ans.**

Example 7. *Find* the value of x correct to one decimal place for which $y = 7$

Given

x	:	1	3	4
y	:	4	12	19

Sol. Here we use Lagrange's inverse interpolation formula *i.e.,*

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(7-12)(7-19)}{(4-12)(4-19)} \times 1 + \frac{(7-4)(7-19)}{(12-4)(12-19)} \times 3 + \frac{(7-4)(7-12)}{(19-4)(19-12)} \times 4$$

$$= 0.5 + 1.9286 - 0.5714$$

$$x = 1.8572$$

Example 8. *Tabulate* $y = x^3$ *for* $x = 2, 3, 4, 5$ *and calculate the cube root of 10 correct to three decimal places.*

Sol. For $x = 2, y = 8$
$x = 3, y = 27$
$x = 4, y = 64$
$x = 5, y = 125$ respectively.

Here $h = 1$ so form forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

The first approximation is given by (using Newton's Forward formula)

$$u_1 = \frac{1}{\Delta y_0} (y_u - y_0) = \frac{1}{19} (10 - 8) = 0.1$$

The second approximation is

$$u_2 = \frac{1}{\Delta y_0} \left(y_u - y_0 - \frac{(u_1 - 1)}{2!} \Delta^2 y_0 \right)$$

$$= \frac{1}{19} \left[10 - 8 - \frac{(0.1)(0.1-1)}{2} 18 \right]$$

$$= 0.15$$

The third approximation is

$$u_3 = \frac{1}{\Delta y_0}\left(y_u - y_0 - \frac{u_2(u_2-1)}{2!}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{3!}\Delta^3 y_0\right)$$

$$= \frac{1}{19}\left[10 - 8 - \frac{0.15(0.15-1)}{2}18 - \frac{0.15(0.15-1)(0.15-2)}{6}6\right]$$

$$= 0.1532$$

Similarly fourth approximation is

$$u_4 = \frac{1}{\Delta y_0}$$

$$= \left(y_u - y_0 - \frac{u_3(u_3-1)}{2!}\Delta^2 y_0 - \frac{u_3(u_3-1)(u_3-2)}{3!}\Delta^3 y_0 - \frac{u_3(u_3-1)(u_3-2)(u_3-3)}{4!}\Delta^4 y_0\right)$$

$$= \frac{1}{19}\left[10 - 8 - \frac{0.1532(0.1532-1)}{2}18 - \frac{0.1532(0.1532-1)(0.1532-2)}{6}6\right]$$

$$= 0.1541$$

and fifth approximation is

$$u_5 = 0.1542$$

Hence $u_4 \approx u_5$ (correct to 3 places of decimal)

We have to find cube root of 10. Since 10 lies between the value of y corresponding to $x = 2$ and $x = 3$, therefore the required value of $\sqrt[3]{10}$ is $x = x_0 + uh$

$$= 2 + 0.1541 \times 1$$

$$x = 2.154.$$ **Ans.**

5.3.3 Expression of Function as a Sum of Partial Fractions

Example 9. *Let* $f(x) = \dfrac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$

Sol. Consider $\phi(x) = x^2 + x - 3$ and tabulate its values for $x = 1, -1, 2$, we get

x	1	–1	2
$x^2 + x - 3$	–1	–3	3

Using Lagrange's formula, we get

$$f(x) = \frac{(x+1)(x-2)}{(1+1)(1-2)}(-1) + \frac{(x-1)(x-2)}{(-1-1)(-1-2)}(-3) + \frac{(x-1)(x+1)}{(2-1)(2+1)}(3)$$

$$= \frac{1}{2}(x+1)(x-2) - \frac{1}{2}(x-1)(x-2) + (x-1)(x+1)$$

$$\Rightarrow \quad f(x) = \frac{\phi(x)}{(x-1)(x+1)(x-2)}$$

$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x-2}.$$ **Ans.**

Example 10. *Show that the sum of Lagrangian co-efficients is unity.*

Sol. Let $\prod(x) = (x-x_0)(x-x_1)\dots(x-x_n)$

The reciprocal of $\prod(x) \Rightarrow \dfrac{1}{\Pi(x)} = \dfrac{1}{(x-x_0)(x-x_1)\dots(x-x_n)}$

Let $$\frac{1}{(x-x_0)(x-x_1)\dots(x-x_n)} = \frac{A_0}{x-x_0} + \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \dots + \frac{A_n}{x-x_n}$$

$$= \sum_{i=0}^{n} \frac{A_i}{x-x_i} \quad \text{(This can be expressible as partial fractions.)}$$

Now, $$A_i = \frac{1}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} \quad \text{...(1)}$$

is obtained by taking L.C.M. of (1) and setting $x = x_i$

A_i can be written as $$A_i = \frac{1}{\Pi'(x)_{at\,(x-x_i)}} = \frac{1}{\Pi'(x_i)}$$

From (1) $$\frac{1}{\Pi(x)} = \sum_{i=0}^{n} \frac{1}{\Pi'(x_i)(x-x_i)}$$

Multiply both sides by $\prod(x)$, we get

$$1 = \sum_{i=0}^{n} \frac{\Pi(x)}{\Pi'(x_i)(x-x_i)} = \sum_{i=0}^{n} L_i(x). \quad \textbf{Proved.}$$

PROBLEM SET 5.2

1. Given that $y_{10} = 1754$, $y_{15} = 2648$, $y_{20} = 3564$, find the value of x for $y = 3000$ by using, iterative method of inverse interpolation. **[Ans.** 16.935]
2. Given that

x	:	1.8	2.0	2.2	2.4	2.6
y	:	2.9	3.6	4.4	5.5	6.7

   Find x when $y = 5$ using iterative interpolation formula.
3. Using inverse interpolation find the real root of the equation $x^3 - 15x + 4 = 0$ close to 0.3 correct upto 4 decimal places. **[Ans.** 0.2679]
4. Find the value of θ if $f(\theta) = 0.3887$ from the table given below:

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

   [Ans. 22°]
5. Find x when $f(x) = 14$ for the following data using Lagrange's inverse interpolation formula.

x	0	5	10	15
$f(x)$	16.35	14.88	13.59	12.46

   [Ans. 8.337]

6. Using Lagrange's interpolation formula express the function $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sums of partial fractions.

$$\left[\textbf{Ans. } \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}\right]$$

7. Express the function $\dfrac{x^2+6x+1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial factions using Lagrange's interpolation formula.

$$\left[\textbf{Ans. } \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}\right]$$

8. From the following data find the value of x corresponding to $y = 12$ using Lagrange's technique of inverse interpolation.

x	:	1.2	2.1	2.8	4.1	4.9	6.2
y	:	4.2	6.8	9.8	13.4	15.5	19.6

[**Ans.** 3.55]

9. Obtain the values of t when $A = 85$ from the following table using Lagrange's Method of inverse interpolation.

t	:	2	5	8	14
A	:	94.8	87.9	81.3	68.7

[**Ans.** 6.5928]

5.4 DIVIDED DIFFERENCE

When the values of the argument are given at unequal spaced interval, then the various differences will also be affected by the changes in the values of the argument. The differences defined by taking into consideration the changes in the values of argument are known as **divided differences** where as the difference defined earlier are called ordinary differences. Lagrange's interpolation formula has the disadvantage that if any other interpolation point were added, the interpolation co-efficient will have to be recomputed. So an interpolation polynomial, which has the property that a polynomial of higher degree may be derived from it by simply adding new terms, in Newton's divided difference formula.

Let (x_0, y_0), (x_1, y_1), (x_2, y_2)... (x_n, y_n) be given $(n + 1)$ points. Let $y_0, y_1, y_2, \ldots y_n$ be the values of the function corresponding to the values of argument $x_0, x_1, x_2, \ldots x_n$ which are not equally spaced. Since the difference of the function values with respect to the difference of the arguments are called divided differences, so the first divided difference for the arguments x_0, x_1, is given by

$$f(x_0, x_1) = \underset{x_1}{\Delta} y_1 = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly $\quad f(x_1, x_2) = \underset{x_2}{\Delta} y_1 = [x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and so on.

The second divided difference for x_0, x_1, x_2 is given by

$$f(x_0, x_1, x_2) = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is given by

$$f(x_0, x_1, x_2, x_3) = [x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

$$f(x_0, x_1, x_2 ... ,x_n) = [x_0, x_1, x_2 ... ,x_n] = \frac{[x_1, x_2, ..., x_n] - [x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$$

5.4.1 Properties of Divided Differences

1. Divided differences are symmetric with respect to the arguments *i.e.,* independent of the order of arguments.

   i.e., $[x_0, x_1] = [x_1, x_0]$

   Also, $[x_0, x_1, x_2] = [x_2, x_0, x_1]$ or $[x_1, x_2, x_0]$

2. The n^{th} divided differences of a polynomial of n^{th} degree are constant.

   Let $f(x) = A_0x^n + A_1x^{n-1} + ... + A_{n-1}\, x + A_n$ by a polynomial of degree n provided $A_0 \neq 0$ and arguments be equally spaced so that

   $$x_1 - x_0 = x_2 - x_1 = ... x_n - x_{n-1} = h$$

   Then first divided difference

   $$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

   Second divided difference

   $$[x_0, x_1, x_2] = \frac{1}{2!h^2} = \Delta^2 y_0$$

   $$[x_0, x_1, x_2 ... x_{n-1}, x_n] = \frac{1}{n!h^n} = \Delta^n y_0$$

   Since, function is a n^{th} degree polynomial

   Therefore, $\Delta^n y_0 = \text{constant}$

   $\therefore$ n^{th} divided difference will also be constant.

3. The n^{th} divided difference can be expressed as the quotient of two determinants of order $(n + 1)$. *i.e.,*

   $$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0}$$

   $$\Rightarrow \quad [x_0, x_1] = \begin{vmatrix} y_1 & y_0 \\ 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & x_0 \\ 1 & 1 \end{vmatrix}$$

   Similarly, $$[x_0, x_1, x_2] = \begin{vmatrix} y_0 & y_1 & y_2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} \text{ ... and so on.}$$

4. The n^{th} divided difference can be expressed as the product of multiple integral.

5.4.2. Relation Between Divided Differences and Ordinary Differences

Let the arguments $x_0, x_1, x_2, \ldots x_n$, be equally spaced such that

$$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$$

$$\therefore \quad x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n = x_0 + nh$$

Now, first divided difference for arguments (x_0, x_1) be given by

$$\underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h} \qquad \ldots(1)$$

Similarly, second divided difference be given as

$$\underset{x_1 x_2}{\Delta^2} f(x_0) = \frac{1}{x_2 - x_0}\left[f(x_1, x_2) - f(x_0, x_1)\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{1}{2h}\left[\frac{f(x_0 + 2h) - f(x_0 + h)}{h} - \frac{f(x_0 + h) - f(x_0)}{h}\right]$$

$$= \frac{1}{2h^2}\left[f(x_0 - 2h) - 2f(x_0 + h) + f(x_0)\right] = \frac{\Delta^2 f(x_0)}{2!h^2} \qquad \ldots(2)$$

$$\underset{x_1 x_2 x_3}{\Delta} f(x_0) = \frac{1}{x_2 - x_0}\left[f(x_1, x_2, x_3) - f(x_0, x_1, x_2)\right]$$

$$= \frac{1}{3h}\left[\frac{\Delta^2 f(x_1)}{2h^2} - \frac{\Delta^2 f(x_0)}{2h^2}\right] = \frac{\Delta^2 f(x_1) - \Delta^2 f(x_0)}{6h^3} \quad \text{[From (1)}$$

$$= \frac{\Delta^3 f(x_0)}{3!h^3}$$

In general, we have

$\underset{x_1 \ldots x_n}{\Delta^n} f(x_0) = \dfrac{\Delta^n f(x_0)}{n!h^n}$ which shows the relation between divided difference and ordinary difference.

5.5 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots y_n$, be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$, then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So, that,

$$y = y_0 + (x - x_0)\,[x, x_0] \qquad \ldots(1)$$

Again,

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives, $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$ ...(2)

From (1) and (2), $y = y_0 + [x - x_0]\ (x_0, x_1) + (x - x_0)(x - x_1)[x, x_0, x_1]$ ...(3)

Also $[x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

which gives, $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$ ...(4)

From (3) and (4)

$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)\ [x_0, x_1] + (x - x_0)\ (x - x_1)\ [x_0, x_1, x_2]$$
$$+ (x - x_0)\ (x - x_1)\ (x - x_2)\ [x_0, x_1, x_2, x_3] + \ldots + (x - x_0)\ (x - x_1)\ (x - x_2)$$
$$\ldots (x - x_{n-1})\ [x_0, x_1, \ldots x_n] + (x - x_0)\ (x - x_1) \ldots (x - x_n)\ [x, x_0, x_1, x_2, \ldots x_n]$$

This is called Newton's general interpolation formula with divided differences, the last term being the remainder term after $(n + 1)$ terms.

Newton's divided difference formula can also be written as

$$y = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)\ (x - x_1)\,\Delta^2 y_0 + (x - x_0)\ (x - x_1)\ (x - x_2)\,\Delta^3 y_0$$
$$+ (x - x_0)\ (x - x_1)\ (x - x_2)\ (x - x_3)\,\Delta^4 y_0 + \ldots + (x - x_0)\ (x - x_1)\ldots\ (x - x_{n-1})\ \Delta^n y_0$$

Example 1. *Apply Newton's divided difference formula to find the value of f(8) if*

f(1) = 3, f(3) = 31, f(6) = 223, f(10) = 1011, f(11) = 1343,

Sol. The divided difference table is given by

x	$f(x)$	Δ	Δ^2	Δ^3
1	3			
		28/2 = 14		
3	31		50/5 = 10	
		192/3 = 64		9/9 = 1
6	223		133/7 = 19	
		788/4 = 197		8/8 = 1
10	1011		135/5 = 27	
		332/1 = 332		
11	1343			

On, applying Newton's divided difference formula, we have

$f(x)$ or $y_x = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)\ (x - x_1)\,\Delta^2 y_0 + (x - x_0)\ (x - x_1)\ (x - x_2)\,\Delta^3 y_0 + \ldots$

$f(x)$ or $y_x = 3 + (x - 1) \times 14 + (x - 1)\ (x - 3) \times 10 + (x - 1)\ (x - 3)\ (x - 6) \times 1$

for $f(8)$, we put $x = 8$ in above equation, we get

$$f(8) = 3 + (7)\ (14) + (7)\ (5)\ (10) + (7)\ (5)\ (2)$$

$f(8) = 3 + 98 + 350 + 70$

$f(8) = 521.$ **Ans.**

Example 2. *Find the function u_x in powers of $x - 1$ given that*

$u_0 = 8,\ u_1 = 11,\ u_4 = 68,\ u_5 = 123,$

Sol. Here,

$x_0 = x_0,\ x_1 = 1,\ x_2 = 4,\ x_3 = 5,$

$y_0 = 8,\ y_1 = 11,\ y_2 = 68,\ y_3 = 123,$

The divided difference table is given by

x	$f(x)$	Δ	Δ^2	Δ^3
0	8			
		3		
1	11		4	
		19		1
4	68		9	
		55		
5	123			

From Newton's divided difference formula, we have

$y_x = 8 + (x - 0)\times (3) + (x - 0)(x-1)\times (4) + (x - 0)(x - 1)(x - 4)\times 1$

$y_x = 8 + 3x + 4x^2 - 4x + x(x^2 - 5x + 4)$

$y_x = x^3 - x^2 + 3x + 8$

In order to express it in powers of $(x - 1)$, we use synthetic division method as,

1	1	–1	+3	8
		1	0	3
1	1	0	3	11
		1	1	
1	1	1	4	
		1		
	1	2		

$x^3 - x^2 + 3x + 8 = (x - 1)^3 + 2(x - 1)^2 + 4(x - 1) + 11.$ **Ans.**

Example 3. *Find the Newton's divided difference interpolation polynomial for:*

x	:	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$	:	1.625	5.875	31.0	131.0	282.125	521.0

Sol. The divided difference table is given as, if here

x	:	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$	:	1.625	5.875	31.0	131.0	282.125	521.0

The table for divided difference is as:

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0.5	1.625				
		4.25			
1.5	5.875		5		
		16.75		1	
3.0	31.0		9.5		0
		50		1	
5.0	131.0		14.5		0
		100.75		1	
6.5	282.125		19.5		
		159.25			
8.0	521.0				

By using Newton's divided difference formula, we have

$$y_x = 1.625 + (x - 0.5)\times 4.25 + (x - 0.5)(x - 1.5)\times 5 + (x - 0.5)(x - 1.5)(x - 3)\times 1$$

$$y_x = 1.625 + 4.25x - 2.125 + 5x^2 - 10x + 3.75 + (x - 0.5)(x^2 - 4.5x + 4.5)$$

$$y_x = 1.625 + 4.25x - 2.125 + 5x^2 - 10x + 3.75 + x^3 - 4.5x^2 + 4.5x - 0.5x^2 + 2.25x - 3.25$$

$$\therefore \quad y_x = x^3 + 11x - 10x + 1$$

$$y_x = x^3 + x + 1.$$ **Ans.**

Example 4. *Construct a divided difference table for the following:*

x	:	1	2	4	7	12
$f(x)$	:	22	30	82	106	216

Sol. The divided difference table is given as,

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-36-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{0.535+1.6}{12-1}=0.194$
		$\frac{106-82}{7-4}=8$		$\frac{1.75+3.6}{12-2}=0.535$	
7	106		$\frac{22-8}{12-4}=1.75$		
		$\frac{216-106}{5}=22$			
12	216				

Example 5. *(i) Prove that* $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

(ii) Show that the nth divided differences $[x_0, x_1, ...x_n]$ *for* $u_x = \frac{1}{x}$ *is* $\left[\frac{(-1)^n}{x_0, x_1, ...x_n}\right]$

Sol. (*i*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a}$			
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a}=-\frac{1}{ba}$		
b	$\frac{1}{b}$		$(-1)^2\frac{1}{abc}$	
		$\frac{\frac{1}{c}-\frac{1}{b}}{c-b}=-\frac{1}{bc}$		$(-1)^3\frac{1}{abcd}$
c	$\frac{1}{c}$		$(-1)^2\frac{1}{bcd}$	
		$\frac{\frac{1}{d}-\frac{1}{c}}{d-c}=-\frac{1}{dc}$		
d	$\frac{1}{d}$			

From the table, we observe that $\underset{bcd}{\overset{3}{\Delta}}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$ ...(1)

(*ii*) From (1), we see that

$$\underset{bcd}{\overset{3}{\Delta}}\left(\frac{1}{a}\right) = -\frac{1}{abcd} = (-1)^3 f(a, b, c, d)$$

In general, $$\underset{x_0, x_1, \ldots x_n}{\overset{n}{\Delta}}\left(\frac{1}{x_0}\right) = (-1)^n f(x_0, x_1, x_2, \ldots x_n) = \frac{(-1)^n}{[x_0, x_1, x_2, \ldots x_n]}$$

Example 6. *(i) Find the third divided difference with arguments 2, 4, 9, 10 of the function* $f(x) = x^3 - 2x$.

(ii) if $f(x) = \frac{1}{x^2}$, *find the first divided differences f(a, b), f(a, b, c,), f (a, b, c, d)*

(iii) If f(x) = g(x) h(x), prove that

$$f(x_1, x_2) = g(x_1)\, h(x_1, x_2) + g\,(x_1, x_2)\, h\,(x_2)$$

Sol. (*i*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

Hence third divided difference is 1.

(*ii*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a^2}$			
		$\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}=-\left(\frac{a+b}{a^2b^2}\right)$		
b	$\frac{1}{b^2}$		$\left(\frac{ab+bc+ca}{a^2b^2c^2}\right)$	
		$-\left(\frac{b+c}{b^2c^2}\right)$		$-\left(\frac{abc+acd+abd+bcd}{a^2b^2c^2d^2}\right)$
c	$\frac{1}{c^2}$		$\left(\frac{bc+cd+bd}{b^2c^2d^2}\right)$	
		$-\left(\frac{c+d}{c^2d^2}\right)$		
d	$\frac{1}{d^2}$			

From the above divided difference table, we observe that first divided differences,

$$f(a,\ b) = -\left(\frac{a+b}{a^2b^2}\right)$$

$$f(a,\ b,\ c) = \left(\frac{ab+bc+ca}{a^2b^2c^2}\right)$$

and

$$f(a,\ b,\ c,\ d) = -\left(\frac{abc+acd+abd+bcd}{a^2b^2c^2d^2}\right)$$

(*iii*) R.H.S. $= g(x_1)\ \dfrac{h(x_2)-h(x_1)}{x_2-x_1} + \dfrac{g(x_2)-g(x_1)}{x_2-x_1}h(x_2)$

$$= \frac{1}{x_2-x_1}\ [\{g\ (x_1)h\ (x_2) - g(x_1)\ h(x_1)\} + \{g(x_2)\ h\ (x_2) - g(x_1)\ h\ (x_2)\}]$$

$$= \frac{g(x_2)\,h(x_2)-g(x_1)\,h(x_1)}{x_2-x_1} = \underset{x_2}{\Delta}\ g\ (x_1)\ h\ (x_1) = \underset{x_2}{\Delta} f(x_2)\ f(x_1,\ x_2) = \text{L.H.S.}$$

Hence the result.

Example 7. *The following are the mean temperatures (°F) on three days, 30 days apart round the pds. of summer and winter. Estimate the app. dates and values of max. and min. temperature.*

Day	*Summer*		*Winter*	
	Date	*Temp.*	*Date*	*Temp.*
0	15 June	58.8	16 Dec.	40.7
30	15 July	63.4	15 Jan	38.1
60	14 August	62.5	14 Feb.	39.3

Sol. Divided difference table for summer is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	58.8		
		4.6	
1	63.4		–2.75
		–0.9	
2	62.5		

$$\therefore \quad f(x) = 58.8 + (x - 0)(4.6) + (x - 0)(x - 1)(-2.75)$$

$$= -2.75x^2 + 7.35x + 58.8$$

For maximum and minimum of $f(x)$, we have

$$f'(x) = 0$$

$$\Rightarrow \quad -5.5x + 7.35 = 0 \Rightarrow x = 1.342$$

Again, $\quad f''(x) = -5.5 < 0$

$\therefore$ $f(x)$ is maximum at $x = 1.342$

Since unit ≡ 30 days

$$1.342 = 30 \times 1.342 = 40.26 \text{ days}$$

$\therefore$ Maximum temperature was on 15 June + 40 days *i.e.*, on 25 July and value of maximum temperature is

$$[f(x)]_{max} = [f(x)]_{1.342} = 63.711°F \text{ approximately.}$$

Divided difference table for winter is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	40.7		
		–2.6	
1	38.1		1.9
		1.2	
2	39.3		

$\therefore \qquad f(x) = 40.7 + (x - 0)(-2.6) + x(x - 1)(1.9)$

$= 1.9x^2 - 4.5x + 40.7$

For $f(x)$ to be maximum or minimum, we have $f'(x) = 0$

$$3.8x - 4.5 = 0 \Rightarrow x = 1.184$$

Again, $f''(x) = 3.8 > 0$

$\therefore$ $f(x)$ is minimum at $x = 1.184$

Again, unit 1 ≡ 30 days

$\therefore$ 1.184 = 30 × 1.184 = 35.52 days

$\therefore$ Minimum temperature was on 16 Dec + 35.5 days *i.e.,* on the mid night of 20th Jan. and its value can be obtained similarly.

$$[f(x)]_{\min} = [f(x)]_{1.184} = 63.647\ °F \text{ approximately.}$$

Example 8. *The mode of a certain frequency Curve y = f (x) is very near to x = 9 and the values of frequency density f(x) for x = 8.9, 9.0 and 9.3 are respectively equal to 0.30, 0.35 and 0.25. Calculate the approximate value of mode.*

Sol. Divide difference table for given frequency density is as

x	$100f(x)$	$100\Delta f(x)$	$100\Delta^2 f(x)$
8.9	30		
		$\frac{50}{9}$	
9.0	35		$-\frac{3500}{36}$
		$-\frac{100}{3}$	
9.3	25		

Applying Newton's divided difference formula

$$100\ f(x) = 30 + (x - 8.9) \times \frac{50}{9} + (x - 8.9)(x - 9)\left(-\frac{3500}{36}\right)$$

$$= -97.222x^2 + 1745.833x - 1759.7217$$

$$\therefore \qquad f(x) = -0.9722x^2 + 17.45833x - 17.597217$$

$$f'(x) = -1.9444x + 17.45833$$

Putting $f'(x) = 0$, we get

$$x = \frac{17.45833}{1.9444} = 8.9788$$

Also, $f''(x) = -1.9444$ *i.e.,* (–) ve

$\therefore$ $f(x)$ is maximum at $x = 8.9788$

Hence, mode is 8.9788.

Example 9. *Using Newton's divided difference formula, prove that*

$$f(x) = f(0) + x\,\Delta f(-1) + \frac{(x+1)x}{2!}\,\Delta^2 f(-1) + \frac{(x+1)x(x-1)}{3!}\,\Delta^3 f(-2) + \ldots$$

Sol. Taking the arguments, 0, –1, 1, –2,... the Newton's divided difference formula is

$$f(x) = f(0) + x\underset{-1}{\Delta} f(0) + x(x+1)\underset{-1,1}{\Delta^2} f(0) + x(x+1)(x-1)\underset{-1,1-2}{\Delta^3} f(0) + \ldots \quad \ldots(1)$$

$$f(x) = f(0) + x\underset{0}{\Delta} f(-1) + x(x+1)\underset{0,1}{\Delta^2} f(-1) + x(x+1)(x-1)\underset{0,-1,1}{\Delta^3} f(-2) + \ldots$$

Now, $\underset{0}{\Delta} f(-1) = \dfrac{f(0) - f(-1)}{0-(-1)} = \Delta f(-1)$

$$\underset{0,1}{\Delta^2} f(-1) = \frac{1}{1-(-1)} = \left[\underset{1}{\Delta} f(0) - \underset{0}{\Delta} f(-1)\right]$$

$$= \frac{1}{2}\left[\Delta f(0) - \Delta f(-1)\right] = \frac{1}{2}\Delta^2 f(-1)$$

$$\underset{-1,0,1}{\Delta^3} f(-2) = \frac{1}{1-(-2)}\left[\underset{0,1}{\Delta^2} f(-1) - \underset{-1,0}{\Delta^2} f(-2)\right]$$

$$= \frac{1}{3}\left[\frac{\Delta^2 f(-1)}{2} - \frac{\Delta^2 f(-2)}{2}\right]$$

$$= \frac{\Delta^3 f(-2)}{3\times 2} = \frac{\Delta^3 f(-2)}{3!} \text{ and so on.}$$

Substituting these values in (1)

$$f(x) = f(0) + x\,\Delta f(-1) + \frac{(x+1)x}{2!}\,\Delta^2 f(-1) + \frac{(x+1)x(x-1)}{3!}\,\Delta^3 f(-2) + \ldots$$

Example 10. *Using Newton's divided difference formula, calculate the value of f(6) from the following data:*

x	:	1	2	7	8
$f(x)$	:	1	5	5	4

Sol. The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		–1		
8	4			

Applying Newton's divided difference formula,

$$f(x) = 1 + (x-1)(4) + (x-1)(x-2)\left(-\frac{2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$$f(6) = 1 + 20 + (5)(4)\left(-\frac{2}{3}\right) + (5)(4)(-1)\left(\frac{1}{14}\right)$$

$$= 6.2381$$

Example 11. *Find the value of $\log_{10} 656$ using Newton's divided difference formula from the data given below:*

x	:	654	658	659	661
$\log_{10} x$	:	2.8156	2.8182	2.8189	2.8202

Sol. Divided difference table for the given data is as:

x	$10^5 f(x)$	$10^5 \Delta f(x)$	$10^5 \Delta^2 f(x)$	$10^5 \Delta^3 f(x)$
654	281560			
		$\frac{260}{4} = 65$		
658	281820		$\frac{70 - 65}{5} = 1$	
		$\frac{70}{1} = 70$		$\frac{-1.66 - 1}{7} = 10.38$
659	281890		$\frac{65 - 70}{3} = -1.66$	
		$\frac{130}{2} = 65$		
661	282020			

For the given argument, the divided difference formula is,

$$f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$10^5 f(x) = 281560 + (x - 654)\,65 + (x - 654)(x - 658).1 + (x - 654)(x - 658)(x - 659)(0.38)$$

On Substituting $x = 656$, we get

$$10^5 f(x) = 281560 + 2 \times 65 + (-4) + (-4)(-3) \times (0.38)$$

$$= 281560 + 130 - 4 + 4.56$$

$$10^5 f(x) = 281690.\ 56$$

$$\Rightarrow \quad f(x) = 2.8169056$$

$$\Rightarrow \quad \log_{10} 656 = 2.8169056$$

PROBLEM SET 5.3

1. By means of Newton's divided difference formula,
Find the value of $f(8)$ and $f(15)$ from the following table:

x	:	4	5	7	10	11	13
$f(x)$	:	48	100	294	900	1210	2028

[**Ans.** 448, 3150]

2. Using Newton's divided difference formula, find a polynomial function satisfying the following data:

x	:	–4	–1	0	2	5
$f(x)$	:	1245	3	5	9	1335

Hence find $f(3)$ [**Ans.** $f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$, $f(3) = 89$]

3. From the given data, find $f(x)$ as a polynomial in powers of $(x - 5)$

x	:	0	2	3	4	7	9
$f(x)$	:	4	26	58	112	466	922

[**Ans.** $(x - 5)^3 + 17(x - 5)^2 + 98(x - 5) + 194$]

4. Find the value of $\log_{10} 33$ from the data given below:
$\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, [**Ans.** $\log_{10} 33 = 1.5184$]

5. Find $f'(10)$ from the given data:

x	:	3	5	11	27	34
$f(x)$	:	–13	23	899	17315	35606

[**Ans.** $f'(10) = 233$]

6. Find approximately the real root of the equation $x^3 - 2x - 5 = 0$ [**Ans.** 2.0945595]

7. Find the value of $f(x)$ at point $x = 2$ and 5 from the data:

x	:	1.5	3	6
$f(x)$	:	0.25	2	20

[**Ans.** $f(2) = 0$, $f(5) = 12$]

8. Evaluate $f(9)$ using Newton's divided difference formula.

x	:	5	7	11	13	17
$f(x)$	:	150	392	1452	2366	5202

[**Ans.** $f(9) = 810$]

9. Use Newton's divided difference formula to find $f(7)$ if $f(3) = 24$, $f(5) = 120$, $f(8) = 504$ $f(9) = 720$, and $f(12) = 1716$ **[Ans.** $f(7) = 328$]

10. There is a data be given

x	:	0	1	2	5
$f(x)$	:	2	3	12	147

What is the form of the function? **[Ans.** $x^3 + x^2 - x + 2$]

11. Find yx in powers of $x - 4$ where $y_0 = 8$, $y_1 = 11$, $y_4 = 68$, $y_5 = 125$

$$\left[\textbf{Ans. } \frac{1}{10}\left[11(x-4)^2 + 117(x-4)^2 - 447(x-4) + 680\right]\right]$$

12. Express the function as a sums of partial functions.

$$\frac{x^2 + 6x - 1}{(x^2 - 1)(x-4)(x-6)}$$

$$\left[\textbf{Ans. } \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}\right]$$

13. If $f(x) = u(x)\, v(x)$, find the divided difference $f(x_0, x_1)$ in terms of $u(x_0)$, $v(x_1)$ and the divided differences $u(x_0, x_1)$, $v(x_0, x_1)$

14. Find $f(3)$, using Newton's divided difference formula from the given data

x	:	0	1	2	4	5	6
$f(x)$	:	1	14	15	5	6	19

[Ans. 10]

5.6. HERMITE'S INTERPOLATION FORMULA

Hermite's interpolation is similar to as Lagrange's interpolation. The difference is that in the Lagrange's interpolation the interpolating polynomial consider with $f(x)$ at the interpolation points $x_0, x_1, x_2, \ldots x_n$ where as Hermite's interpolation formula interpolate both the function as well as its derivative at each of the points. Sometimes it is also called osculating interpolation formula.

Let the set of data points (x_i, y_i, y'_i), $0 \le i \le n$ be given. A polynomial of the least degree say $H(x)$ is to be determined such that

$$H(x_i) = y_i \text{ and } H'(x_i) = y'_i;\ i = 0, 1, 2, \ldots n \qquad \ldots(1)$$

$H(x)$ is called Hermite's interpolating polynomial

Since there are $2n + 2$ conditions to be satisfied, $H(x)$ must be a polynomial of degree $\le 2n + 1$

The required polynomial may be written as

$$H(x) = \sum_{i=0}^{n} u_i(x)\, y_i + \sum_{i=0}^{n} v_i(x) y'_i \qquad \ldots(2)$$

Where $u_i(x)$ and $vi(x)$ are polynomials in x of degree $\le 2n + 1$ and satisfy.

$$(i)\ u_i(x_j) = \begin{bmatrix} 0, i \ne j \\ 1, i = j \end{bmatrix} \qquad \ldots(3a)$$

(*ii*) $v_i(x_j) = = 0 \;\forall\, i, j$ ...(3*b*)

(*iii*) $u'_i(x_j) = 0 \;\forall\, i, j$ ...(3*c*)

(*iv*) $v'_i(x_j) = \begin{bmatrix} 0, i \neq j \\ 1, i = j \end{bmatrix}$ ...(3*d*)

Using the Lagrange fundamental polynomials $L_i(x)$, we choose

and
$$\left.\begin{aligned} u'_i(x) &= A_i(x)\left[L_i(x)\right]^2 \\ v_i(x) &= B_i(x)\left[L_i(x)\right]^2 \end{aligned}\right\} \quad ...(4)$$

where $L_i(x)$ is defined as

$$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_n)}$$

Since $L_i^2(x)$ is a polynomial of degree $2n$, $A_i(x)$ and $B_i(x)$ must be linear polynomials.

Let $A_i(x) = a_i x + b_i$ *and* $B_i(x) = c_i x + d_i$

therefore from (4),
$$\left.\begin{aligned} u_i(x) &= (a_i x + b_i)[L_i(x)]^2 \\ v_i(x) &= (c_i x + d_i)\left[L_i(x)\right]^2 \end{aligned}\right\} \quad ...(5)$$

using conditions (3*a*) and (3*b*) in (5), we get

$$a_i x + b_i = 1 \quad ...(6a)$$

and $c_i x + d_i = 0$ (6*b*) Since $[L_i(x_i)]^2 = 1$

Again, using conditions (3*c*) and (3*d*) in (5), we get

$$a_i + 2L'_i(x_i) = 0 \quad ...(6c)$$

and $c_i = 1$ ...(6*d*)

From equations 6(*a, b, c,* & *d*), we deduce

$$\left.\begin{aligned} a_i &= -2L'_\lambda(x_i) \\ b_i &= 1 + 2x_i L'_i(x_i) \\ c_i &= 1 \end{aligned}\right\}$$

and $d_i = -x_i$ ...(7)

Hence, from (5)

$$\begin{aligned} u_i(x) &= [-2xL_i'(x_i) + 1 + 2x_i L_i'(x_i)]\,[L_i(x)]^2 \\ &= [1 - 2(x - x_i)\,L_i'(x_i)]\,[L_i(x)]^2 \end{aligned}$$

and $v_i(x) = (x - x_i)\,[L_i(x)]^2$

Therefore from (2),

$$H(x) = \sum_{i=0}^{n} \{1 - 2(x - x_i)\,L_i'(x_i)\}\,[L_i(x)]^2\, y_i + \sum_{i=0}^{n} (x - x_i)\left[L_i(x)\right]^2 y_i'$$

Which is the required Hermite's interpolation formula.

Example 1. *Apply Hermite's interpolation formula to obtain a polynomial of degree 4 from the following data.*

x_i	:	0	1	2
y_i	:	1	0	9
y_i'	:	0	0	24

Sol. Using *h* Hermite's interpolation formula

$$H(x) = \sum_{i=0}^{2} \left[1 - 2(x - x_i)L_i'(x_i)\right] [L_i(x)]^2 y_i + \sum_{i=0}^{2} (x - x_i)[L_i(x)]^2 y_i'$$

We have,

$$H(x) = \{1 - 2(x - x_0) L_0'(x_0)\} [L_0(x)]^2 y_0 + \{1 - 2(x - x_1) L_1'(x_1)\} [L_1(x)]^2 y_1$$
$$\{1 - 2(x - x_2) L_2'(x_2)\} [L_2(x)]^2 y_2 + (x - x_0) [L_0(x_0)]^2 y'_0$$
$$+ (x - x_1) [L_1(x)]^2 y'_1 + (x - x_2) [L_2(x)]^2 y'_2$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}\left(x^2 - 3x + 2\right)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x-0)(x-2)}{(1-0)(0-2)} = 2x - x^2$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_1)(x_2 - x_0)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}\left(x^2 - x\right)$$

$$L_0'(x) = \frac{2x - 3}{2},\ L_1'(x) = 2 - 2x,\ L_2'(x) = \frac{2x - 1}{2},$$

$$L_0'(x_0) = -\frac{3}{2},\ L_1'(x_1) = 0,\ L_2'(x_2) = \frac{3}{2},$$

Using these values in equation, we get

$$H(x) = [1 - 2(x - 0)(-3/2)]\frac{1}{4}(x^2 + 3x + 2)^2 \times 1$$
$$+ [1 - 2(x - 2)(3/2)]\frac{1}{4}(x^2 - x)^2 \times 9 + (x - 2)\frac{1}{4}(x^2 - x)^2 \times 24$$

$$H(x) = (1 + 3x)\frac{1}{4}(x^2 + 3x + 2)^2 + (1 - 3x + 6)\frac{9}{4}(x^2 - x)^2 + (x - 2)6(x^2 - x)^2$$

$\therefore \qquad H(x) = x^4 - 2x^2 + 1$

Example 2. *Apply Hermite's interpolation formula to find a cubic polynomial which meets the following specifications.*

x_i	y_i	y_i'
0	0	0
1	1	1

Sol. Hermite's interpolation formula is

$$H(x) = \sum_{i=0}^{1}\left[1-2(x-x_i)L_i'(x_i)\right]\left[L_i(x)\right]^2 y_i + \sum_{i=0}^{1}(x-x_i)\left[L_i(x)\right]^2 y_i'$$

$$H(x) = [1-2(x-x_0)L_0'(x_0)][L_0(x)]^2 y_0 + [1-2(x-x_1)L_i'(x_1)][L_1(x)]^2 y_1 + (x-x_0)[L_0(x)]^2 y'_0 + (x-x_1)[L_i(x)]^2 y_1' \quad ...(1)$$

Now, $L_0(x) = \dfrac{x-x_0}{x_0-x_1} = \dfrac{x-0}{0-1} = 1-x$

$$L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-0}{1-0} = x$$

$\therefore$ $L_0'(x) = -1$ and $L_1'(x) = 1$

Hence, $L_0'(x) = -1$ and $L_1'(x_1) = 1$

$\therefore$ From (1),

$$H(x) = [1-2(x-0)(-1)](1-x^2)(0) + [1-2(x-1)(1)]x^2(1) + (x-0)(1-x)^2(0) + (x-1)x^2(1)$$

$$= x^2 - 2x^2(x-1) + x^2(x-1)$$

$$= x^2 - x^2(x-1) = x^2(2-x)$$

$$= 2x^2 - x^3$$

Example 3. *Apply Hermite interpolation to find the value of sin (1.05) from the following data*

x_i	1.00	1.10
Sin x	0.84147	0.89121
Cos x	0.54030	0.45360

Sol. Here, $f(x) = \sin x$, $f'(x) = \cos x$, $x_0 = 1$ & $x_1 = 1.10$

$$L_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-1.10}{1-1.10} = -10x + 11$$

$$L'_0(x) = -10$$

and $L_1(x) = \dfrac{x-x_0}{x_1-x_0} = \dfrac{x-1.00}{1.10-1.00} = 10x - 10$

$$L'_1(x_1) = 10$$

$$f(x_0) = 0.84147,\ f(x_1) = 0.54030$$

$$f(x_2) = 0.89121,\ f(x_3) = 0.45360$$

Now, using Hermite's formula, we have

$$H(x) = \sum_{k=0}^{1}\left[1-2L^1_k(x_k)(x-x_k)\right]\left[L_k(x)\right]^2 f(x_k) + \sum_{k=0}^{1}(x-x_k)\left[L_k(x)\right]^2 f'(x_k)$$

$$H(x) = \{1-2L_0^1(x_0)(x-x_0)\}[L_0(x)]^2 f(x_0) + \{1-2L_1^1(x_1)(x-x_1)\}[L_1(x)]^2 f(x_1)$$
$$+ (x-x_0)[L_0(x)]^2 f'(x_0) + (x-x_1)[L_1(x)]^2 f'(x_1)$$

$$H(x) = \{1 + 20\,(x-1)\}\,(-10x+11)^2\,(0.84147) + \{1 - 20(x-1.10)\}\,(10x-10)^2\,(0.89121)$$
$$+ (x-1.00)\,(-10x+11)^2\,(0.54030)^2\,(0.54030) + (x-1.10)\,(10x-10)^2\,(0.45360)$$

Put, $x = 1.05$, so we get

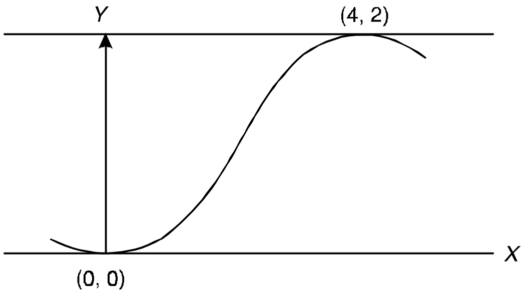
$$H\,(1.05) = \sin\,(1.05) = \{1 + 20(1.05-1)\}\,(-10(1.05)+11)^2\,(0.84147)$$
$$+ \{1 - 20((1.05) - 1.10)\}\,(10(1.05)-10)^2\,(0.89121) + ((1.05) - 1.00)$$
$$(-10(1.05)+11)^2\,(0.54030) + ((1.05) - 1.10)\,(10(1.05)-10)^2\,(0.45360)$$

$$H(x) = 2(0.25)\,(0.84147) + 2(0.\,25)\,(0.89122) + (0.05)\,(0.25)\,(0.54030) - (0.5)\,(0.25)\,(0.45360)$$

$$\sin\,(1.05) = 0.420735 + 0.445605 + 0.00675375 - 0.00567$$

$$\sin\,(1.05) = 0.86742.$$ **Ans.**

Example 4. *A switching path between parallel railroad tracks is to be a cubic polynomial joining positions (0,0) and (4,2) and tangent to the lines y = 0 and y = 2 as shown in the figure. Apply Hermite's interpolation formula to obtain this polynomial*



Sol. Since tangents are parallel to X-axis,

$y' = 0$ in both the cases.

∴ We have the table of values.

x	y	y'
0	0	0
4	2	0

Hermite interpolation formula is

$$H(x) = \sum_{i=0}^{1}\left[1-2L_i'(x_i)(x-x_i)\right][L_i(x)]^2 f(x_i) + \sum_{i=0}^{1}(x-x_i)[L_i(x)]^2 f'(x_i) \quad ...(1)$$

Now, $L_0(x) = \dfrac{x-x_1}{x_0-x_1} = \dfrac{x-4}{0-4} = 1 - \dfrac{x}{4}$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{4 - 0} = \frac{x}{4}$$

$$\therefore \quad L_0'(x) = -\frac{1}{4} \text{ and } L_1'(x) = \frac{1}{4}$$

Hence, $L_0'(x_0) = -\frac{1}{4}$ and $L_1'(x_1) = \frac{1}{4}$.

Therefore from (1)

$$H(x) = \left[1 - 2(x-0)\left(-\frac{1}{4}\right)\right]\left(1 - \frac{x}{4}\right)^2 \times 0 + \left[1 - 2(x-4)\left(\frac{1}{4}\right)\right]\left(\frac{x}{4}\right)^2 \times 2$$

$$+ (x-0)\left(1 - \frac{x}{4}\right)^2 \times 0 + (x-4)\left(\frac{x}{4}\right)^2 \times 0$$

$$H(x) = \left[1 - \left(\frac{x-4}{2}\right)\right]\frac{x^2}{8} = \frac{(6-x)x^2}{16}$$

$$H(x) = \frac{1}{16}(6x^2 - x^3). \quad \textbf{Ans.}$$

Example 5. *Using Hermite interpolation formula, estimate the value of* 1_n *(3.2) from the following table:*

x	$y = l_n(x)$	$y' = \frac{1}{x}$
3	1.09861	0.33333
3.5	1.25276	0.28571
4.0	1.38629	0.25000

Sol. Using Hermite's interpolation formula, we have

$$H(x) = \sum_{i=0}^{2}\left[1 - 2L_i(x_i)(x - x_i)\right][l_i(x)]^2 y_i + \sum_{i=0}^{2}(x - x_i)\left[L_i(x)\right]^2 y_i'$$

$$H(x) = [1 - 2(x - x_0)L_0'(x_0)]\,[L_0(x)]^2 y_0 + [1 - 2(x - x_1)\,L_1'(x_1)]\,[L_1(x)]^2 y_1$$
$$+ [1 - 2(x - x_2)\,L_2'(x_2)]\,[L_2(x)]^2 y_2 + (x - x_0)\,[L_0(x)]^2 y_0'$$
$$+ (x - x_1)\,[L_1(x)]^2 y_1' + (x - x_2)\,[L_2(x)]^2 y_2'$$

Now, $$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 3.5)(x - 4)}{(3 - 3.5)(3 - 4)}$$
$$= 2(x^2 - 4x - 3.5x + 14)$$
$$= 2(x^2 - 7.5x + 14)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-3)(x-4)}{(3.5-3)(3.5-4)}$$

$$= -4(x^2 - 7x + 12)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-3)(x-3.5)}{1\times 0.5}$$

$$= 2(x^2 - 6.5x + 10.5)$$

$\therefore$ $L_0'(x) = 4x - 15$, $L_1'(x) = -8x + 28$,

$L_0'(x_0) = -3$, $L_1'(x_1) = 0$

$L_2'(x) = 4x - 13$

i.e., $L_2'(x_2) = 3$

$$\therefore \quad H(x) = [1-2(x-3)(-3)]\left[2\left(x^2-7.5x+14\right)\right]^2 \times (1.09861) + [1-2(x-3.5)(0)]$$
$$\times\left[-4\left(x^2-7x+12\right)\right]^2 \times (1.25276) + [1-2(x-4)(3)]\left[2\left(x^2-6.5x+10.5\right)\right]^2$$
$$\times (1.38629) + (x-3)\left[2\left(x^2-7.5x+14\right)\right]^2 \times (0.33333) + (x-3.5)$$
$$\times\left[-4\left(x^2-7x+12\right)\right]^2 \times (0.28571) + (x-4)\left[2(x^2-6.5x+10.5)\right]^2 \times (0.25000)$$

On Putting $x = 3.2$, we get

$$H(3.2) = [1-2(0.2)(-3)]\left[2(3.2)^2-7.5(3.2)+14\right]^2 \times (1.09861) + 0 + [1-2(-1.2)(3)]$$
$$\times\left[2((3.2)^2-6.5(3.2)+10.5)\right]^2 \times (1.386929) + (0.2)\left[2((3.2)^2-7.5(3.2)+14)\right]^2$$
$$\times (0.33333) + (-0.3)\left[-4((3.2)^2-7(3.2)+12)\right]^2 \times (0.28571) + (-1.2)$$
$$\times\left[2((3.2)^2-6.5(3.2)+10.5)\right]^2 \times (0.25000)$$

$$l_n(3.2) = (2.2\ \{4(0.0576)\} \times (1.09861) + 0(8.2)[4(0.0036)] \times (1.38629) + (0.2)$$
$$\times [4(99.2016)](0.33333) - (0.3)$$
$$\times [+46\{0.0256\}](0.28571) - (1.2)[4\{0.0036\}]\ (0.25000)$$

$$l_n(3.2) = 1.6314$$

Example 6. *Show that* $f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2} + \frac{(b-a)[f'(a)-f'(b)]}{8}$

Sol. By Hermite's interpolation formula.

x	$f(x)$	$f'(x)$
a	$f(a)$	$f'(a)$
b	$f(b)$	$f'(b)$

Hermite's interpolation formula is

$$H(x) = \sum_{i=0}^{1}[1-2(x-x_i)L_i'(x)]\,L_i\,[(x)]^2\,y_i + \sum_{i=0}^{1}(x-x_i)[L_i\,(x)]^2\,y_i'$$

$$= [1-2\,(x-x_0)\,L_0'\,(x_0)]\,[L_0\,(x)]^2\,y_0 + [1-2(x-x_i)\,L_1'\,(x_1)]\,[L_1(x)]^2 y_1$$

$$+ (x-x_0)\,[L_0\,(x)]^2\,y'_0 + (x-x_1)\,[L_1\,(x)]^2\,y_1' \quad ...(1)$$

Now, $$L_0\,(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-b}{a-b}$$

$$L_1\,(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-a}{b-a}$$

$\therefore$ $$L_0'(x) = \frac{1}{a-b} \text{ and } L_1'\,(x) = \frac{1}{b-a}$$

Hence, $$L_0'(x_0) = \frac{1}{a-b} \text{ and } L_1'\,(x_1) = \frac{1}{b-a}$$

Therefore from equation (1)

$$H(x) = \left[1-2\frac{(x-a)}{a-b}\right]\left(\frac{x-b}{a-b}\right)^2 f(a) + \left[1-2\frac{(x-b)}{b-a}\right]\left(\frac{x-a}{b-a}\right)^2 f(b)$$

$$+(x-a)\left(\frac{x-b}{a-b}\right)^2 f'(a) + (x-b)\left(\frac{x-a}{b-a}\right)^2 f'(b)$$

$$H\left(\frac{a+b}{2}\right) = \left[1-2\frac{\left(\frac{a+b}{2}-a\right)}{a-b}\right]\left(\frac{\frac{a+b}{2}-b}{a-b}\right)^2 f(a) + \left[1-2\frac{\left(\frac{a+b}{2}-b\right)}{b-a}\right]\left(\frac{\frac{a+b}{2}-a}{b-a}\right)^2 f(b)$$

$$+\left(\frac{a+b}{2}-a\right)\left(\frac{\frac{a+b}{2}-b}{a-b}\right)^2 f'(a) + \left(\frac{a+b}{2}-b\right)\left(\frac{\frac{a+b}{2}-a}{b-1}\right)^2 f'(b)$$

$$= \frac{1}{2}\,f(a) + \frac{1}{2}\,f(b) + \frac{(b-a)}{8}\,f'\,(a) - \frac{(b-a)}{8}\,f'(b)$$

$$= \frac{f(a)+f(b)}{2} + \frac{(b-a)[f'(a)-f'(b)]}{8}.$$ **Hence Proved.**

PROBLEM SET 5.4

1. Apply Hermite formula to find a polynomial which meets the following specifications:

x_k	0	1	2
y_k	0	1	0
y_k'	0	0	0

[**Ans.** $x^4 - 4x^3 + 4x^2$]

2. Apply Hermite's interpolation to find $f(1.05)$ given:

x	f	f'
1	1.0	0.5
1.1	1.04881	0.47673

[**Ans.** 1.02470]

3. Apply Hermite's interpolation to find log 2.05 given that:

x	$\log x$	$\frac{1}{x}$
2.0	0.69315	0.5
2.1	0.74194	0.47619

[**Ans.** 0.71784]

4. Determine the Hermite polynomial of degree 5 which fits the following data and hence find an approximate value of $\log_e 2.7$

x	$y + \log_e x$	$y' = \frac{1}{x}$
2.0	0.69315	0.5
2.5	0.91629	0.4
3.0	1.09861	0.33333

[**Ans.** 0.993252]

5. Find $y = f(x)$ by Hermite's interpolation from the table:

x_i	y_i	y_i'
–1	1	–5
0	1	1
1	3	7

Computer y_2 and y_2'. [**Ans.** $1 + x - x^2 + 2x^4$, $y_2 = 31$, $y_2' = 61$]

6. Compute $\sqrt{e}$ by Hermite's formula for the function $f(x) = e^x$ at the points 0 and 1. Compare the value with the value obtained by using Lagrange's interpolation.

[**Ans.** $(1 + 3x)(1 - x)^2 + (2 - x)\,ex^2$; 1.644, 1.859]

7. Apply Hermite's formula to find a polynomial which meets the following specifications:

x_i	y_i	y_i'
–1	–1	0
0	0	0
1	1	0

$\left[\textbf{Ans. } \frac{1}{2}\left(5x^3 - 3x^5\right)\right]$

8. Apply osculating interpolation formula to find a polynomial which meets the following requirements:

x_i	y_i	y_i'
0	1	0
1	0	0
2	9	0

[**Ans.** $x^4 - 4x^3 + 4x^2$]

9. Apply Hermite's interpolation formula to find $f(x)$ at $x = 0.5$ which meets the following requirements:

x_i	$f(x_i)$	$f'(x_i)$
–1	1	–5
0	1	1
1	3	7

Also find $f(-0.5)$. $\left[\textbf{Ans. } 2x^4 - x^2 + x + 1;\ \frac{11}{8}, \frac{3}{8}\right]$

10. Construct the Hermite interpolation polynomial that fits the data:

x	$f(x)$	$f'(x)$
1	7.389	14.778
2	54.598	109.196

Estimate the value of $f(1.5)$. [**Ans.** $29.556x^3 - 85.793x^2 + 97.696x - 34.07$; 19.19125]

11. (*i*) Construct the Hermite interpolation polynomial that fits the data:

x	$f(x)$	$f'(x)$
0	0	1
0.5	0.4794	0.8776
1.0	0.8415	0.5403

Estimate the value of $f(0.75)$.

(*ii*) Construct the Hermite interpolation polynomial that fits the data

x	$y(x)$	$y'(x)$
0	4	–5
1	–6	–14
2	–22	–17

Interpolate $y(x)$ at $x = 0.5$ and 1.5.

12. Obtain the unique polynomial $p(x)$ of degree 3 or less corresponding to a funcion $f(x)$ where $f(0) = 1, f'(0) = 2, f(1) = 5, f'(1) = 4$.

13. (*i*) Construct the Hermite interpolation polynomial that fits the data

x	$f(x)$	$f'(x)$
2	29	50
3	105	105

Interpolate $f(x)$ at $x = 2.5$

(*ii*) Fit the cubic polynomial $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ to the data given in problem 13 (*i*). Are these polynomials same?

5.7. SOME RELATED TERMS

5.7.1 Some Remarkable Points about Chosen Different Interpolation Formulae

We have derived some central difference interpolation formulae. Obviously here a question arise that which one of these formulae gives the most accurate or approximate, nearest result?

(1) If interpolation is required near the beginning or end of a given data, there is only alternative to apply Newton's Forward and backward difference formulae.

(2) For interpolation near the center of a given data, Stirling's formula gives the best or most accurate result for $-\frac{1}{4} < u < \frac{1}{4}$ and Bessel's formula is most efficient near $u = \frac{1}{2}$ or $\frac{1}{4} \le u \le \frac{3}{4}$.

(3) But in the case where a series of calculations have to be made, it would be inconvenient to use both these (Stirling's & Bessel's) formulae *i.e.,* the choice depends on the order of the highest differences that could be neglected so that contributions from it and further differences would be less than half a unit in the last decimal place. If this highest difference is of odd order. Stirling's formula is recommended; if it is even order Bessel's formula might be preferred.

(4) It is known from algebra that the n^{th} degree polynomial which passes through $(n + 1)$ points is unique. Hence the various interpolaion formulae derived here are actually, only different forms of the same polynomial. Therefore all the interpolation formulae should give the same functional value.

(5) Here we discussed several interpolation formulae for equispaced argument value. The most important thing about these formulae is that, the co-efficients in the central difference formulae are smaller and converges faster than those in Newton's formulae. After a few terms, the co-efficients in the stirling's formula decrease more rapidly than those of the Bessel's formulae and the co-efficient of Bessel's formula decreases more rapidly than those of Newton's formula. Therefore, whenever possible central difference formulae should be used in preference to Newton's formulae. However, the right choice depends on the position of the interpolated value in the given pairs of values.

The Zig-Zag paths for various formulae

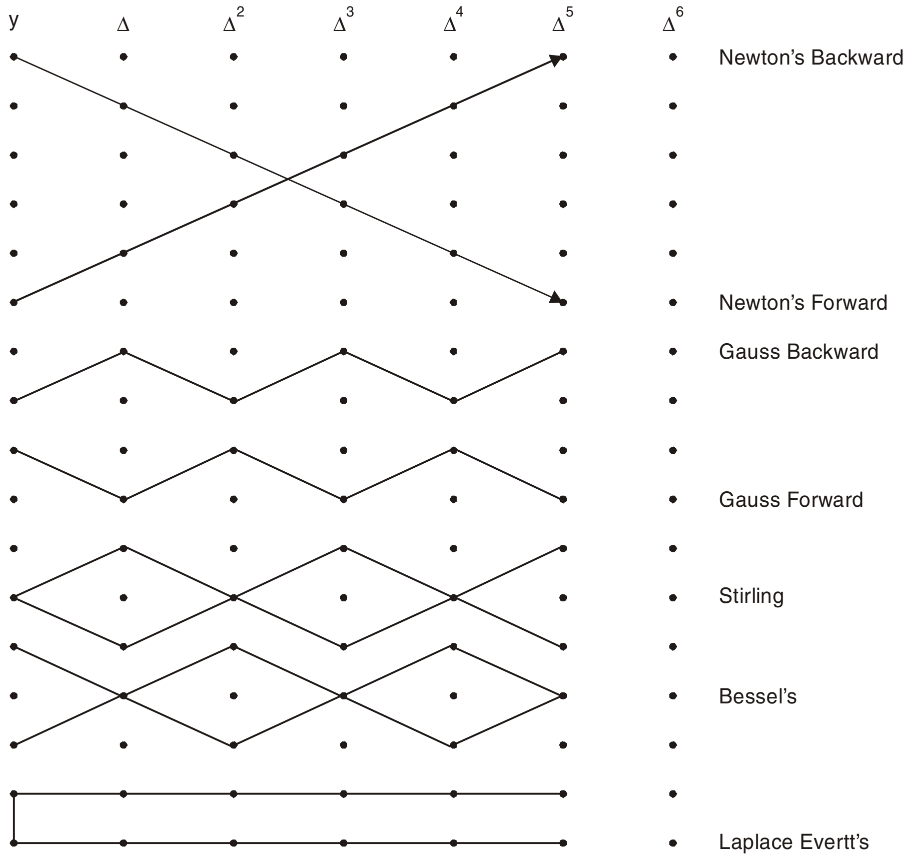


FIG. 5.1

5.7.2 Approximation of Function

To evaluate most mathematical functions, we must first produce computable approximations to them. Functions are defined in a variety of ways in applications, with integrals and infinite series being the most common types of formulas used for the definition. Such a definition is useful in establishing the properties of the function, but it is generally not an efficient way to evaluate the function. In this part we examine the use of polynomials as approximation to a given function.

For evaluating a function $f(x)$ on a computer it is generally more efficient of space and time to have an analytic approximation to $f(x)$ rather than to store a table and use interpolation *i.e.*, function evaluation through interpolation techniques over stored table of values has been found to be quite costlier when compared to the use of efficient function approximations. It is also desirable to use the lowest possible degree of polynomial that will give the desired accuracy in approximating $f(x)$. The amount of time and effort expended on producing an approximation should be directly proportional to how much the approximation will be used. If it is only to be used a few times, a truncated Taylor series will often suffice. But if an approximation is to be used millions of times by many people, then much care should be used in producing the approximation. There are forms of approximating function other than polynomials.

Let $f_1, f_2 ... f_n$ be the values of given function and $\phi_1, \phi_2 ... \phi_n$ be the corresponding values of the approximating function. Then the error vector is e where the components of e are given by $e_i = f_i - \phi_i$. So the approximation may be chosen in two ways. One is, to find the approximation such that the quantity $\sqrt{e_1^2 + e_2^2 + ... e_n^2}$ is minimum. This leads us to the least square approximation. Second is, choose the approximation such that the maximum components of e is minimized. This leads to Chebyshev polynomials which have found important applications in the approximation of functions.

(*i*) **Approximation of function by Taylor's series method:** Taylor's series approximation is one of the most useful series expressions of a function. If a function $f(x)$ has upto $(n + 1)^{th}$ derivatives in an interval $[a, b]$, near $x = x_0$ then it can be expressed as,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + ... f^n(x_0)\frac{(x - x_0)^n}{n!} + f^{n+1}(s) \times \frac{(x - x_0)^{n+1}}{(n+1)!} \quad ...(1)$$

In the above expansion $f'(x_0), f''(x_0)$ etc., are the first, second derivatives of $f(x)$ evaluated at x_0.

The term $$\frac{f^{n+1}(s)(x - x_0)^{n+1}}{(n+1)!}$$

is called the **remainder term.** The quantity s is a number which is a function of x and lies between x and x_0. The remainder term gives the **truncation error** if only the first n terms in the Taylor series are used to represent the function. The truncation error is thus:

$$\text{Truncation error} = \frac{\left|f^{n+1}(s)\right|\left|(x - x_0)^{n+1}\right|}{(n+1)!} \quad ...(2)$$

or

$$T_\in = \frac{\left|(x - x_0)^{n+1}\right|}{(n+1)!} M \quad ...(3)$$

where $M = \max.\left|f^{n+1}(s)\right|$ for x in $[a, b]$.

Obviously, the Taylor's series is a polynomial with base function 1, $(x - x_0)$, $(x - x_0)^2$,... $(x - x_0)^n$. The co-efficients are constants given by $f(x_0)$, $f'(x_0)$, $f''(x_0)$ $f''(x_0)/2!$ etc. Thus the series can be written in the rested form.

(*ii*) **Approximation of function by Chebyshev polynomial:** The polynomials are linear combination of the monomials 1, x, $x_2 ... x_n$. An examination of the monomial in the interval (–1, + 1) shows that each achieves its maximum magnitude 1 at $x = \pm 1$ and minimum magnitude O at $x = 0$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

dropping the higher order terms or modification of the co-efficients $a_1, a_2 ... a_n$ will produce little error for small x near zero. But probably substantial error near the ends of the interval

(x near ± 1). In particular, it seems reasonable to look for other sets of simple related functions that have their extreme values well distributed on their interval (–1, 1). We want to find approximations which are fairly easy to generate and which reduce the maximum error to minimum value. The cosine functions $\cos\theta$, $\cos 2\theta, \ldots \cos n\theta$ appear to be good candidates. The set of polynomials $T_n(x) = \cos n\theta$, $n = 0, 1\ldots$ generates from the sequence of cosine functions using the transformation.

$$\theta = \cos^{-1} x$$

is known as Chebyshev polynomial. These polynomial are used in the theory of approximation of function.

Chebyshev polynomials: Chebyshev polynomial $T_n(x)$ of the first kind of degree n over the interval [–1, 1] is defined by the relation

$$T_n(x) = \cos[n \cos^{-1}(x)] \qquad \ldots(1)$$

Let $\cos^{-1} x = \theta$, so that $x = \cos\theta$

$$\Rightarrow \qquad T_n(x) = \cos n\theta$$

for $\quad n = 0,\ T_0(x) = 1$

for $\quad n = 1,\ T_1(x) = x$

The Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \qquad \ldots(2)$$

which can be obtained easily using the following trigonometric identity.

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta \cos n\theta$$

Above recurrence relation can be used to generate successively all $T_n(x)$, as well as to express the powers of x in terms of the Chebyshev polynomials. Some of the Chebyshev polynomials and the expansion for powers of x in terms of $T_n(x)$ are given as follows:

$T_0(x) = 1, \qquad 1 = T_0, (x),$

$T_1(x) = x, \qquad x = T_1(x),$

$T_2(x) = 2x^2 - 1, \qquad x^2 = \frac{1}{2}(T_0(x) + T_2(x))$

$T_3(x) = 4x^3 - 3x, \qquad x^3 = \frac{1}{4}(3T_1(x) + T_3(x)),$

$T_4(x) = 8x^4 - 8x^2 + 1, \qquad x^4 = \frac{1}{8}(3T_0(x) + 4T_2(x) + T_4(x)),$

$T_5(x) = 16x^5 - 20x^3 + 5x, \qquad x^5 = \frac{1}{16}(10T_1(x) + 5T_3(x) + T_5(x)),$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1, \qquad x^6 = \frac{1}{32}(10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x)),$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x,$

$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1,$

$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x,$

$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \qquad \ldots(3)$

Note that the co-efficient of x^n in $T_n(x)$ is always 2^{n-1}, and expression for x^n *i.e.,* 1, x, $x^2...x^n$ will be useful in the economization of power series.

Further, these polynomials satisfy the differential equation.

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

where, $y = T_n(x)$,

We also have $|T_n(x)| \le 1, x \in [-1,1]$ ...(4)

Also, the Chebyshev polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx = \begin{cases} 0, & \text{if } m \ne n; \\ \pi, & \text{if } m = n = 0; \\ \frac{\pi}{2}, & \text{if } m = n \ne 0 \end{cases} \quad ...(5)$$

Another important property of these polynomials, is that, of all polynomials of degree n where the co-efficient of x^n is unity, the polynomial $2^{1-n}\ T_n(x)$ has the smallest least upper bound to its magnitude in the interval [–1, 1], *i.e.,*

$$\max_{-1\le x\le 1} \left|2^{1-n}\ T_n(x)\right| \le \max_{-1\le x\le 1} |P_n(x)| \quad ...(6)$$

This is called the minimax property.

Here, $p_n(x)$ is any polynomial of degree n with leading co-efficient unity, $T_n(x)$ is defined by

$$T_n(x) = \cos(n \cos^{-1} x) = 2^{n-1} x^n - ... \quad ...(7)$$

Because the maximum magnitude of $T_n(x)$ is one, the upper bound referred to is $1/2^{n-1}$ *i.e.,* 2^{1-n}. This is important because we will be able to write power-series representations of functions whose maximum errors are given in terms of this upper bound.

Thus in Chebyshev approximation, the maximum error is kept down to a minimum. This is called as minimax principle and the polynomial $P_n(x) = 2^{1-n}\ T_n(x)$; $(n \ge 1)$ is called the minimax polynomial. By this process we can obtain the best lower order approximation called the minimax approximation.

Example 1. *Find the best lower-order approximation to the cubic* $2x^3 + 3x^2$.

Sol. Using the relation gives in equation (3), we have

$$2x^3 + 3x^2 = 2\left[\frac{1}{4}\{3T_1(x) + T_3(x)\}\right] + 3x^2$$

$$= 3x^2 + \frac{3}{2}\ T_1(x) + \frac{1}{2}\ T_3(x)$$

$$= 3x^2 + \frac{3}{2}x + \frac{1}{2}\ T_3(x), \text{ Since } T_1(x) = x$$

The polynomial $3x^2 + \frac{3}{2}x$ is the required lower order approximation to the given cubic with a maximum error $\pm\frac{1}{2}$ in the range [–1, 1].

Example 2. *Obtain the best lower degree approximation to the cubic ($x^3 + 2x^2$), on the interval [–1, 1].*

Sol. We write

$$x^3 + 2x^2 = \frac{1}{4}\,[3\,T_1(x) + T_3(x)] + 2x^2,$$

$$= 2x^2 + \frac{3}{4}\,T_1(x) + \frac{1}{4}\,T_3(x),$$

$$= 2x^2 + \frac{3}{4}x + \frac{1}{4}\,T_3(x).$$

Hence, the polynomial $\left(2x^2 + \frac{3}{4}x\right)$ is the required lower order approximation to the given cubic.

The error of this approximation on the interval [–1, 1] is

$$\max_{-1 \le x \le 1} \frac{1}{4}|T_3(x)| = \frac{1}{4}.$$

Example 3. *Use Chebyshev polynomials to find the best uniform approximation of degree 4 or less to x^5 on [–1, 1].*

Sol. x^5 in terms of Chebyshev polynomials can be written as

$$x^5 = \frac{5}{8}T_1 + \frac{5}{16}T_3 + \frac{1}{16}T_5$$

Now T_5 being polynomial of degree five therefore we omit the term $\frac{T_5}{16}$ and approximate $f(x) = x^5$ by $\left(\frac{5}{8}T_1 + \frac{5}{16}T_3\right)$.

Thus the uniform polynomial approximation of degree four or less to x^5 is given by

$$x^5 = \frac{5}{8}\,T_1 + \frac{5}{16}\,T_3 = \frac{5}{8}x + \frac{5}{16}\,[4x^3 - 3x]$$

$$= \left(-\frac{5}{16}x + \frac{5}{4}x^3\right)$$

and the error of this approximation on [–1, 1] is $\max_{-1 \le x \le 1}\left(\frac{T_5}{16}\right) = \frac{1}{16}$.

Example 4. *Find the best lower order approximation to the polynomial.*

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120},\ x \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Sol. On substituting $x = \frac{\xi}{2}$, we get

$$y(\xi) = 1 + \frac{\xi}{2} + \frac{\xi^2}{8} + \frac{\xi^3}{48} + \frac{\xi^4}{384} + \frac{\xi^5}{3840},\ -1 < \xi < 1.$$

Above equation can be written in Chebyshev polynomals

$$y(\xi) = T_0(\xi) + \frac{1}{2}T_1(\xi) + \frac{1}{16}[T_0(\xi) + T_2(\xi)] + \frac{1}{192}[3T_1(\xi) + T_3(\xi)]$$

$$+ \frac{1}{3072}[3T_0(\xi) + 4T_2(\xi) + T_4(\xi)] + \frac{1}{61440}[10T_1(\xi) + 5T_3(\xi) + T_5(\xi)]$$

$$= 1.063477T_0(\xi) + 0.515788T_1(\xi) + 0.063802T_2(\xi) + 0.00529T_3(\xi)$$

$$+ 0.000326T_4(\xi) + 0.000052T_5(\xi)$$

dropping the term containing $T_5(\xi)$, we get

$$y(\xi) = 1 + 0.4999186\xi + 0.125\xi^2 + 0.0211589\xi^3 + 0.0026041\xi^4.$$

Hence, $y(x) = 1 + 0.999837x + 0.5x^2 + 0.0211589x^3 + 0.041667x^4$.

Properties of Chebyshev polynomial $T_n(x)$

1. $T_n(x)$ is a polynomial of degree n.
2. $T_n(-x) = (-1)^n T_n(x)$,

which show that $T_n(x)$ is an odd function of x if n is odd and an even function of x if n is even.

3. $|T_n(x)| \le, x \in [-1, 1]$
4. $T_n(x)$ assumes extreme values at $(n + 1)$ points

$$x_m = \cos(m\pi), m = 0, 1, 2, ..., n$$

and the extreme value of x_m is $(-1)^m$.

5. $$\int_{-1}^{1} T_m(x) T_n(x)\, dx = \begin{cases} 0, \text{ if } m \ne n \\ \pi/2, \text{ if } m = n \ne 0 \\ \pi, \text{ if } m = n = 0 \end{cases}$$

which can be proved easily by putting $x = \cos\theta$.

Also $T_n(x)$ are orthogonal on the interval [–1, 1] with respect to the weight function

$$w(x) = 1/\sqrt{(1 - x^2)}$$

6. If $P_n(x)$ is a monic polynomial of degree n then $\max_{-1 \le x \le 1} |2^{1-n} T_n(x)| \le \max_{-1 \le x \le 1} |P_n(x)|$ is known as minimax property since $|T_n(x)| \le 1$.

Chebyshev polynomial approximation: Let $f(x)$ be a continuous function defined on the interval [–1, 1] and let $B_0 + B_1x + Bx^2 + ... B_nx^n$ be the required minimax polynomial approximation for $f(x)$.

Suppose $f(x) = \frac{a_0}{2} \sum_{i=1}^{\infty} a_i T_i(x)$ is Chebyshev series expansion for $f(x)$. Then the truncated series of the partial sum.

$$P_n(x) = \frac{a_0}{2} \sum_{i=1}^{n} a_i \; T_i\,(x) \qquad ...(1)$$

is very near solution to the problem.

$$\max_{-1 \le x \le 1} \left| f(x) - \sum_{i=0}^{n} B_i x^i \right| = - \min_{-1 \le x \le 1} \left| f(x) - \sum_{i=0}^{n} B_i x^i \right|$$

i.e., the partial sum (1) is closely the best approximation to $f(x)$.

Chebyshev polynomial approximation

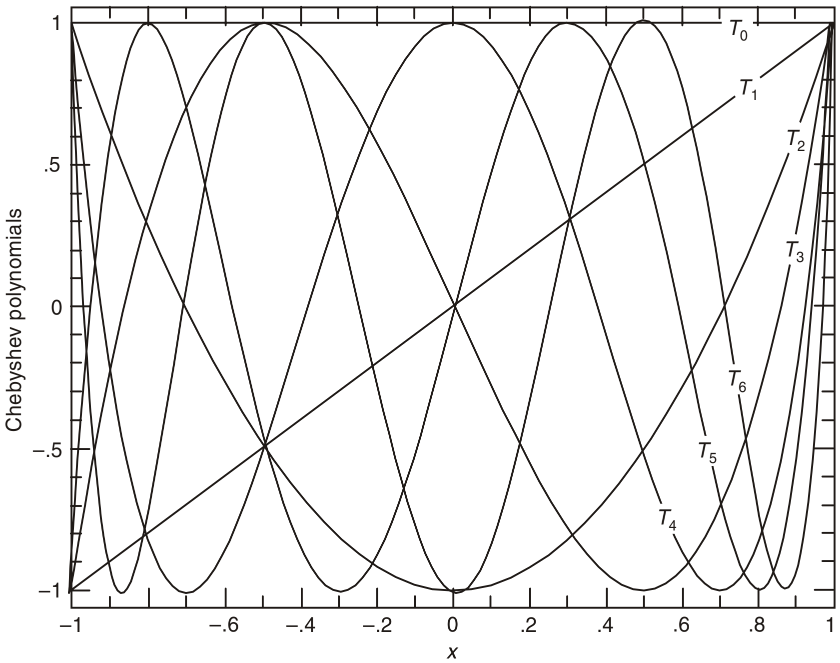


FIG. 5.2. Chebbyshev polynomiats $T_0(x)$ through $T_6(x)$. Note that T_j has j roots in the interval (–1,1) and that all the polynomials are bounded between ± 1.

(*iii*) **Economization of power series:** To describe the process of economization, which is essential due to Lanczos, we first express the given function as a power series in x. Let power series expansion of x is.

$$f(x) = A_0 + A_1 x + A_2 x^2 + ... + A_n x^n; \quad -1 \le x \le 1 \qquad ...(1)$$

Now convert each term in the power series, in terms of Chebyshev polynomials. Thus we obtain the Chebyshev series expansion of the given continuous function $f(x)$ on the interval [–1, 1]. *i.e.,*

$$P_n(x) = \sum_{i=0}^{n} B_i T_i(x) \qquad ...(2)$$

or $$P_n\;(x) = B_0 + B_1 T_1(x) + B_2 T_2(x) + ... + B_n T_n(x)$$

Now, if the truncated Chebyshew expansion is taken by (2) then

$$\max_{-1\le x\le}|f(x)-P_n(x)| \le |B_{n+1}|+|B_{n+2}|+ \ldots\ldots \le \in$$

and Hence P_n (x) is a good uniform approximation to $f(x)$ in which the number of terms retained depends on the given tolerance of $\in$. However for a large number of functions, an expansion as in (2), converges more rapidly than the initial power series for the given function.

This process is known as 'economization of the power series', which is essentially due to Lanczos. Replacing each Chebyshev polynomial $T_i(x)$ by its polynomial form and rearranging the terms, we get the required economized polynomial approximation. We have, thus economized the initial power series in the sense of using fewer terms to achieve almost the same accuracy.

Example 5. *Economize the power series.*

$$\sin x = x - \frac{x^2}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots$$

to 3 significant digit accuracy.

Sol. Here, we have $\sin x = x - \frac{x^2}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots$

Now it is required to compute sin x correct to 3 significant digits. So truncating after 3 terms as the truncation error after 3 terms of the given series is $\le \frac{1}{5040} = 0.000198$. Thus,

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

Now converting the powers of x in to Chebyshev polynomials.

$$\sin x \approx T_1(x) - \frac{1}{24}\,[3T_1(x) + T_3(x)] + \frac{1}{1920}\left[10T_1(x) + 5T_3(x) + T_5(x)\right]$$

$$\Rightarrow \qquad \sin x \approx \frac{169}{192}T_1(x) - \frac{5}{128}T_3(x) + \frac{1}{1920}T_5(x)$$

Again, since the truncation error after two terms of the series is $\le \frac{1}{1920} = 0.00052$. Thus we have

$$\sin x \approx \frac{169}{192}T_1(x) - \frac{5}{128}T_3(x)$$

Now, to get the economized series, we put basic values of T_1 and T_3

$$\sin x \approx \frac{169}{192}(x) - \frac{5}{128}\left(4x^3 - 3x\right)$$

$$\Rightarrow \qquad \sin x = \frac{383}{384}x - \frac{5}{32}x^3$$

$$\Rightarrow \qquad \sin x = 0.9974x - 0.1562x^3$$

Which gives sin x to 3 significant digit accuracy and therefore, it is the economized series.

Example 6. *Prove that* $\sqrt{1-x^2\, T_n}\,(x) = U_{n+1}(x) - xU_n(x)$.

Sol. If $x = \cos\theta$, we get

$$T_n(\cos\theta) = \cos n\theta$$

and $U_n(\cos\theta) = \sin n\theta$ [here $U_n(\cos\theta) = \sin n\theta$]

is Chebyshev polynomial of second kind of degree n over the interval [–1, 1].

Then we have to prove,

$$\sin\theta\ \cos n\theta = \sin(n+1)\theta - \cos\theta\ \sin n\theta$$

Now, $$\text{R.H.S} = \sin n\theta\ \cos\theta + \cos n\theta\ \sin\theta - \cos\theta \sin n\theta$$
$$= \sin\theta\ \cos n\theta = \text{L.H.S.}$$

Example 7. *Find a uniform polynomial approximation of degree four or less to* $\sin^{-1} x$ *on [–1, 1], using Lanczos economization with an error tolerance of 0.05*

Sol. We have,

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{15}{336}x^7 + \ldots$$

Since the error is required to be less than 0.05 and we see that if the given series is truncated after three terms then the truncation error.

$$\left|\frac{15}{336}x^7\right| < 0.044643$$

On [–1, 1] hence, we retain three terms and write

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5$$
$$= T_1 + \frac{1}{24}[3T_1 + T_3] + \frac{3}{640}[10T_1 + 5T_3 + T_5]$$
$$= \frac{75}{64}T_1 + \frac{25}{384}T_3 + \frac{3}{640}T_5$$

Now, the co-efficient of $T_5 = \frac{3}{640} = 0.0046875$ and as $|T_5| \le 1$

For all $x \in [-1, 1]$, we have

$$\left|\frac{3}{640}T_5\right| < 0.0046875$$

Therefore, we omit this term and this omission will not affect the desired accuracy, because the total error

$$= 0.044642857 + 0.0046875$$
$$= 0.04933 < 0.05$$

Hence, required expension for $\text{Sin}^{-1} x$ is

$$\sin^{-1} x = \frac{75}{64}T_1 + \frac{25}{384}$$
$$= \frac{125}{128}x + \frac{25}{96}x^3$$

Example 8. *Find a uniform polynomial approximation of degree 4 or less to e^x in [–1, 1], using lanczos economization with a tolerance of $\in$ = 0.02.*

Sol. Since $f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$

Since, $\frac{1}{120} = 0.00833\ldots$

Therefore, we take $f(x)$ up to $\frac{x^4}{24}$ with a tolerance of $\in = 0.02$

S.t. $$f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \quad \ldots(1)$$

Changing each power of x in (1) in terms of Chebyshev polynomials, we get

$$e^x = \frac{81}{64} T_0 + \frac{9}{8} T_1 + \frac{13}{48} T_2 + \frac{1}{24} T_3 + \frac{1}{192} T_4$$

Neglecting the last term because its magnitude 0.005 is less than 0.02. Hence, the required economized polynomial approximation for e^x is given by

$$e^x \approx \frac{81}{64} T_0 + \frac{9}{8} T_1 + \frac{13}{48} T_2 + \frac{1}{24} T_3 = \frac{x^3}{6} + \frac{13}{24} x^2 + x + \frac{191}{192}$$

(*iv*) **Least square approximation:** To obtain a polynomial approximation to the given function $f(x)$ on the interval $[a, b]$ using least square approximation, with weight function $w(x)$.

Let $P_n(x) = a_0 + a_1x^2 + a_2x^2 + \ldots + a_nx^n$ ...(1)

be a polynomial of degree n. Where $a_0, a_1, a_2, \ldots a_n$, are arbitrary constant we then have,

$$S(a_0, a_1 \ldots a_n) = \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i x^i \right]^2 dx \quad \ldots(2)$$

where $w(x) > 0$ is a weight function.

The necessary conditions for S to be minimum, are given by

$$\frac{\partial S}{\partial a_0} = -2 \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i x^i \right] dx = 0,$$

$$\frac{\partial S}{\partial a_1} = -2 \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i x^i \right] x dx = 0,$$

$$\frac{\partial S}{\partial a_2} = -2 \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i x^i \right] x^2 dx = 0,$$

....

$$\frac{\partial S}{\partial a_n} = -2 \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i x^i \right] x^n dx = 0,$$

after simplication, we get

$$a_0\int_a^b w(x)\,dx + a_1\int_a^b xw(x)\,dx + ... + a_n\int_a^b x^n w(x)\,dx = \int_a^b w(x)f(x)dx$$

$$a_0\int_a^b xw(x)\,dx + a_1\int_a^b x^2 w(x)\,dx + ... + a_n\int_a^b x^{n+1} w(x)\,dx = \int_a^b xw(x)f(x)dx$$

$$a_0\int_a^b x^n w(x)\,dx + a_1\int_a^b x^{n+1} w(x)\,dx + ... + a_n\int_a^b x^{2n} w(x)\,dx = \int_a^b x^n w(x)f(x)dx$$

which are normal equations for $P_n(x)$. These are $(n + 1)$ equations in $(n + 1)$ unknowns and are solved to obtain $a_0, a_1, a_2, ... a_n$.

Example 9. *Obtain a least-square quadratic approximation to the function* $y(x) = \sqrt{x}$ *on [0, 1] w.r.t. weight function w(x) = 1.*

Sol. Let $y = a_0 + a_1x + a_2x^2$ be required quadratic approximation

then, $S(a_0, a_1, a_2) = \int_0^1 \left[x^{1/2} - a_0 - a_1x - a_2x^2\right]dx$ = minimum

The normal equations are

$$\left.\begin{aligned}\frac{\partial S}{\partial a_0} &= -2\int_0^1 \left[x^{1/2} - a_0 - a_1x - a_2x^2\right]dx = 0\\ \frac{\partial S}{\partial a_1} &= -2x\int_0^1 \left[x^{1/2} - a_0 - a_1x - a_2x^2\right]dx = 0\\ \frac{\partial S}{\partial a_2} &= -2x^2\int_0^1 \left[x^{1/2} - a_0 - a_1x - a_2x^2\right]dx = 0\end{aligned}\right\}$$

or

$$\int_0^1 x^{1/2}\,dx = a_0\int_0^1 dx + a_1\int_0^1 xdx + a_2\int_0^1 x^2dx$$

$$\int_0^1 x^{3/2}\,dx = a_0\int_0^1 xdx + a_1\int_0^1 x^2dx + a_2\int_0^1 x^3dx$$

$$\int_0^1 x^{5/2}\,dx = a_0\int_0^1 x^2\,dx + a_1\int_0^1 x^3\,dx + a_2\int_0^1 x^4dx$$

or Simplifying above equations, we get

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} = \frac{2}{3}$$

$$\frac{a_0}{2} + \frac{a_1}{3} + \frac{a_2}{4} = \frac{2}{5}$$

$$\frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} = \frac{2}{7}$$

On solving the equations, we get

$$a_0 = \frac{6}{35},\ a_1 = \frac{48}{35},\ a_2 = -\frac{20}{35}$$

Hence the required quadratic approximation to $y = \sqrt{x}$ on [0, 1] is

$$y = \frac{6}{35} + \frac{48}{35}x - \frac{20}{35}x^2$$

or

$$y = \frac{1}{35}\left(6 + 48x - 20x^2\right). \quad \textbf{Ans.}$$

Example 10. *Using the Chebyshev polynomials, obtain the least square approximation of second degree for $f(x) = x^4$ on [–1, 1].*

Sol. Let $f(x) \approx P(x) = C_0T_0(x) + C_1T_1(x) + C_2T_2(x)$

We have, $$S(C_0, C_1, C_2) = \int_{-1}^{1}\left(x^4 - C_0T_0(x) - C_1T_1(x) - C_2T_2(x)\right)^2 dx$$

which is to be minimum when $\frac{\partial S}{\partial c_0} = \frac{\partial S}{\partial c_1} = \frac{\partial S}{\partial c_2} = 0$

Now, $$\frac{\partial S}{\partial c_0} = 0 \quad \Rightarrow \int_{-1}^{1}\left(x^4 - C_0T_0(x) - C_1T_1(x) - C_2T_2(x)\right)\frac{T_0(x)}{\sqrt{1-x^2}}dx = 0$$

$$\Rightarrow c_0 = \frac{1}{\pi}\int_{-1}^{1}\frac{x^4T_0(x)}{\sqrt{1-x^2}}dx = \frac{3}{8}$$

Similarly, $$\frac{\partial S}{\partial c_1} = 0 \quad \Rightarrow \int_{-1}^{1}\left(x^4 - C_0T_0(x) - C_1T_1(x) - C_2T_2(x)\right)\frac{T_1(x)}{\sqrt{1-x^2}}dx = 0$$

$$\Rightarrow c_1 = \frac{2}{\pi}\int_{-1}^{1}\frac{x^4\,T_1(x)}{\sqrt{1-x^2}}dx = 0$$

and $$\frac{\partial S}{\partial c_2} = 0 \quad \Rightarrow \int_{-1}^{1}\left(x^4 - C_0T_0(x) - C_1T_1(x) - C_2T_2(x)\right)\frac{T_2(x)}{\sqrt{1-x^2}}dx = 0$$

$$\Rightarrow c_2 = \frac{2}{\pi}\int_{-1}^{1}\frac{x^4\,T_2(x)}{\sqrt{1-x^2}}dx = \frac{1}{2}$$

Hence the required approximation is $f(x) = \frac{3}{8}T_0 + \frac{1}{2}T_2$.

Example 11. *The function f is defined by*

$$f(x) = \frac{1}{x}\int_0^x \frac{1 - e^{-t^2}}{t^2}dt$$

Approximate f by a polynomial P(x) = a + bx + cx² such that

$$\underset{|x| \le 1}{may} |f(x) - P(x)| \le 5 \times 10^{-3}$$

Sol. The given function

$$f(x) = \frac{1}{x}\int_0^x \left(1 - \frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{24} + \frac{t^8}{120} - \frac{t^{10}}{720} + \ldots\right)dt$$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{30} - \frac{x^6}{168} + \frac{x^8}{1080} - \frac{x^{10}}{7920} + \ldots \quad \ldots(1)$$

given that $\in = 5 \times 10^{-3}$

$\Rightarrow \quad \in = 0.005$

Now, truncating the series (1) at x^8, we have

$$P(x) = 1 - \frac{x^2}{6} + \frac{x^4}{30} - \frac{x^6}{168} + \frac{x^8}{1080}$$

$$= T_0 - \frac{1}{12}(T_2 + T_0) + \frac{1}{240}(T_4 + 4T_2 + 3T_0) - \frac{1}{5376}(T_6 + 6T_4 + 15T_2 + 10T_0)$$

$$+ \frac{1}{138240}(T_8 + 8T_6 + 28T_4 + 56T_2 + 35T_0)$$

$$= 0.92755973T_0 - 0.06905175T_2 + 0.003253T_4 - 0.000128T_6 + 0.000007T_8 \quad \ldots(2)$$

Truncate the equation (2) at T_2, to get required polynomial

$$P(x) = 0.92755973T_0 - 0.06905175T_2$$

$$= 0.99661148 - 0.13810350x^2$$

or $\quad P(x) = 0.9966 - 0.1381\, x^2$. **Ans.**

Example 12. *Obtain a linear polynomial approximation to the function y(x) = x³ on [0, 1] using the least squares approximation with respect to weight function w(x) = 1.*

Sol. Let $y = a_0 + a_1x$ be the required linear approximation

Then, $\quad S(a_0, a_1) = \int_0^1 \left[x^3 - a_0 - a_1x\right]^2 dx$ = minimum

$$\Rightarrow \quad \frac{\partial S}{\partial a_0} = -2\int_0^1 \left[x^3 - a_0 - a_1x\right]dx = 0$$

$$\Rightarrow \quad a_0\int_0^1 dx + a_1\int_0^1 xdx = \int_0^1 x^3dx \quad \ldots(1)$$

Similarly, $\quad \frac{\partial S}{\partial a_1} = -2\int_0^1 \left[x^3 - a_0 - a_1x\right]xdx = 0$

$$\Rightarrow \quad a_0\int_0^1 xdx + a_1\int_0^1 x^2d_x = \int_0^1 x^4dx \quad \ldots(2)$$

From (1) and (2)

$$a_0 + \frac{a_1}{2} = \frac{1}{4}$$

$$\frac{a_0}{2} + \frac{a_1}{3} = \frac{1}{5}$$

$$\Rightarrow \qquad a_0 = \frac{9}{10}, \; a_1 = -\frac{1}{5}.$$

Hence the required linear approximation to $y(x) = x^3$ on [0, 1] is $y = \frac{9}{10} - \frac{1}{5}x^1$.

Example 13. *Using the Chebyshev polynomials obtain the least square approximation of second degree for* $x^3 + x^2 + 3$ *on the interval [–1, 1].*

Sol. Let $f(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x)$

So, $$S(a_0, a_1, a_2) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[(x^3 + x^2 + 3) - (a_0T_0(x) + a_1T_1(x) + a_2T_2(x)) \right] dx$$

For S to be minimum $\frac{\partial S}{\partial a_0} = \frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = 0$

Therefore, we have

$$\int_{-1}^{1} \left[x^3 + x^2 + 3 - a_0T_0(x) - a_1T_1(x) - a_2T_2(x) \right] \frac{T_0(x)}{\sqrt{1-x^2}} dx = 0$$

$$\int_{-1}^{1} \left[x^3 + x^2 + 3 - a_0T_0(x) - a_1T_1(x) - a_2T_2(x) \right] \frac{T_1(x)}{\sqrt{1-x^2}} dx = 0$$

$$\int_{-1}^{1} \left[x^3 + x^2 + 3 - a_0T_0(x) - a_1T_1(x) - a_2T_2(x) \right] \frac{T_2(x)}{\sqrt{1-x^2}} dx = 0$$

Using the orthogonality conditions, we have

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{(x^3 + x^2 + 3)T_0(x)}{\sqrt{1-x^2}} dx = \frac{7}{2}$$

$$a_1 = \frac{2}{\pi} \int_{-1}^{1} \frac{(x^3 + x^2 + 3)T_1(x)}{\sqrt{1-x^2}} dx = \frac{3}{4}$$

$$a_2 = \frac{2}{\pi} \int_{-1}^{1} \frac{(x^3 + x^2 + 3)T_2(x)}{\sqrt{1-x^2}} dx = \frac{1}{2}$$

Hence, the required least-square approximation is,

$$f(x) = \frac{7}{2}T_0(x) + \frac{3}{4}T_1(x) + \frac{1}{2}T_2(x)$$

Minimax polynomial approximation: Let $f(x)$ be continuous on $[a, b]$ and it is approximated by the polynomial $P_n(x) = a_0 + a_1x + ... + a_nx^n$, then the minimax polynomial approximation problem is to determine the constants $a_0, a_1, a_2 ..., a_n$ such that

$$\max_{a \le x \le b} |\in(x)| = \min_{a \le x \le b} |\in(x)| \qquad ...(1)$$

where, $\in(x) = f(x) - P_n(x)$. ...(2)

If $P_n(x)$ is the best uniform approximation in the sense of eqn. (2) and

$$E_n = \max_{a \le x \le b} |f(x) - P_n(x)|$$

then there are at least $(n + 2)$ points $a = x_0 < x_1 < x_2 < ... x_n < x_{n+1} = b$ where error must alternate in signs, and

(*i*) $\in (x_i) = \pm E_n$, $i = 0, 1, 2, ... n + 1$

(*ii*) $\in (x_i) = -(x_{i+1})$, $i = 0, 2, ..., n$

(*iii*) $\in^1 (x_i) = 0$ for $i = 1, 2, ..., n$

Example 14. *Obtain the Chebyshev linear polynomial approximation (Uniform approximation) to the function $f(x) = x^2$, on [0, 1].*

Sol. Let $P_1(x) = a_0 + a_1x$ and $x_0 = 0$, $x_1 = \alpha$, $x_2 = 1$

Therefore, $\in(x) = x^2 - a_0 - a_1x$

Thus, $\in(x_0) = -\in(x_1)$

$\Rightarrow \in(x_0) + \in(x_1) = 0$ or $\in(0) + \in(\alpha) = 0$ ...(1)

and $\in(x_1) = -\in(x_2) \Rightarrow \in(x_1) + \in(x_2) = 0$

$$\in(\alpha) + \in(1) = 0 \qquad ...(2)$$

and $\in^1(x) = 2x - a_1 = 0$ ...(3)

Hence from (1),

$$-a_0 + \alpha^2 - a_0 - a_1\alpha = 0$$

$$\Rightarrow \alpha^2 - a_1\alpha - 2a_0 = 0 \qquad ...(4)$$

Similarly from (2)

$$\alpha^2 - a_0 - a_1\alpha + 1 - a_0 - a_1 = 0 \qquad ...(5)$$

$$\Rightarrow \alpha^2 - (1 + \alpha)\alpha_1 - 2a_0 + 1 = 0 \qquad ...(6)$$

From (3), $2\alpha - a_1 = 0$

From eq. (4), (5), and (6) we get

$$a_0 = -\frac{1}{8}, \ \alpha = \frac{1}{2}, \ a_1 = 1$$

Therefore the required Chebyshev linear approximation is

$$P(x) = -\frac{1}{8} + x. \quad \textbf{Ans.}$$

Example 15. *Determine the best minimax approximation to* $f(x) = \dfrac{1}{x^2}$ *on [1, 2] with a straight line* $y = a_0 + a_1x$. *Calculate the constants* a_0 *and* a_1 *, correct to two decimals.*

Sol. Given $y = a_0 + a_1x$

Therefore, $\in(x) = \dfrac{1}{x^2} - a_0 - a_1x$ and $x_0 = 1,\ x_1 = \alpha,\ x_2 = 2$

We have,

$$\left.\begin{aligned} &\in(1) + \in(\alpha) = 0 \\ &\in(\alpha) + \in(2) = 0 \\ &\in'(x) = -\frac{2}{x^3} - a_1 \end{aligned}\right\} \quad \ldots(1)$$

Thus, from (1), we have

$$1 - 2a_0 + \frac{1}{\alpha^2} - (1 + \alpha)a_1 = 0$$

$$\frac{1}{4} - 2a_0 + \frac{1}{\alpha^2} - (2 + \alpha)a_1 = 0$$

$$\frac{2}{\alpha^3} + a_1 = 0$$

On solving these equations, we get $a_0 = 1.66$ and $a_1 = -0.75$

Hence, the best minimax approximation is $y = 1.66 - 0.75x$.

5.7.3 Spline Interpolation

Sometimes the problem of interpolation can be solved by dividing the given range of points by subintervals and use low order polynomial to interpolate each subintervals. Such types of polynomial are called piecewise polynomial.

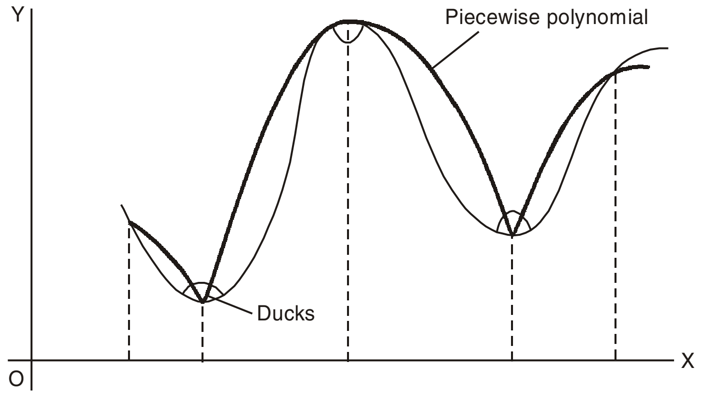


FIG. 5.3

In the above figure piecewise polynomial exhibit discontinuity at some points. If it is possible to construct piecewise polynomial that prevent these discontinuities at the connecting points. Such piecewise polynomial are called spline function. According to the idea of draftsman spline, it is required that both $\dfrac{dy}{dx}$ and the curvature $\dfrac{d^2y}{dx^2}$ are the same for the pair of cubics that join at each point. The spline have possess the given properties.

1. $S(x_i) = f(x_i)$; $i = 0, 1, 2,...n$
2. On each subinterval $[x_{i-1},x_i]$, $1 \le i \le n$, $S(x)$ is a polynomial in n of degree at most n .
3. $S(x)$ and its $(n-1)$ derivatives are continuous on $[a, b]$.
4. $S(x)$ is a polynomial of degree one for $x < a$ and $x > b$.

The process of constructing such type of polynomial is called spline interpolation.

5.7.4 Cubic Spline Interpolation for Equally and Unequally Spaced Values

According to the idea of draftsman spline, it is required that both $\frac{dy}{dx}$ and the curvature $\frac{d^2y}{dx^2}$ are the same for the pair of cubic that join at each point. The cubic spline have possess the following properties:

1. $S(x_i) = f_i$, $i = 0, 1, 2,...,n.$
2. The cubic and their first and second derivatives are continuous *i.e.,* $S(x)$, $S^{I}(x)$ and $S^{II}(x)$ and continuous on $[a, b]$
3. On each subintervals $[x_{i-1}, x_i]$ $1 \le i \le n$, $S(x)$ is a third degree polynomial.
4. The third derivatives of the cubics usually have jumps discontinuities at the ducks or the junction points.

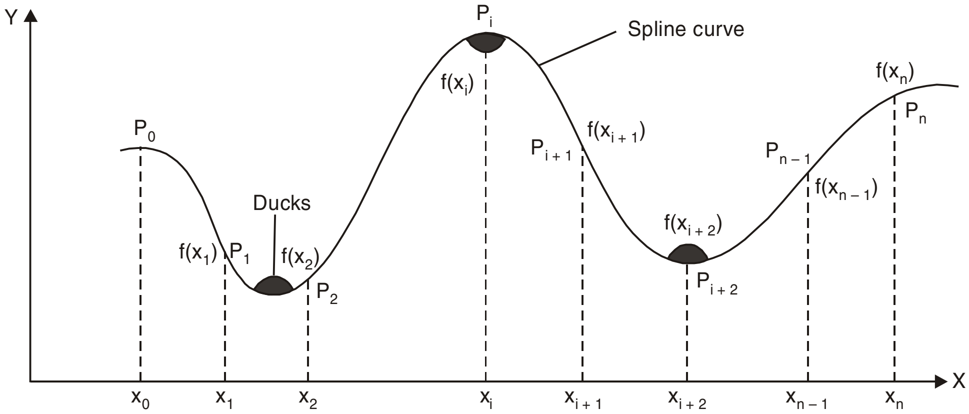


FIG. 5.4

Where x_i = for $i = 0, 1, 2..., n$ may or may not be equally spaced.

Let a cubic polynomial for the i^{th} interval is

$$S(x_i) = a_i (x - x_i)^3 + b_i (x - x_i)^2 + c_i (x - x_i) + d_i \quad ...(1)$$

Since this polynomial is valid for both the points x_i and x_{i+1} therefore,

$$S(x_i) = a_i (x_i - x_i)^3 + b_i (x_i - x_i)^2 + c_i (x_i - x_i) + d_i \quad ...(2)$$

$$\Rightarrow \quad S(x_i) = d_i$$

$$S(x_{i+1}) = a_i (x_{i+1} - x_i)^3 + b_i (x_{i+1} - x_i)^2 + c_i (x_{i+1} - x_i) + d_i$$

$$\Rightarrow \quad S\left(x_{i+1}\right) = a_i h_{i+1}^3 + b_i h^2{}_{i+1} + c_i h_{i+1} + d_i \quad ...(3)$$

where $h_{i+1} = x_{i+1} - x_i$.

Now, Twice differentiate Equation (1) we get,

$$S'(x_i) = 3a_i (x - x_i)^2 + 2b_i (x - x_i) + c_i \quad ...(4)$$

$$S''(x_i) = 6a_i (x - x_i)^4 + 2b_i \quad ...(5)$$

Now, Let $P_i = S''(x_i)$ then equation (5) becomes

$$P_i = 6a_i (x - x_i) + 2b_i$$

at $x = x_i$,

$$P_i = 2b_i \Rightarrow b_i = \frac{P_i}{2} \quad ...(6)$$

at $x = x_{i+1}$,

$$P_{i+1} = 6a_i (x_{i+1} - x_i) + 2b_i$$

$$P_{i+1} = 6a_i (x_{i+1} - x_i) + P_i \quad \text{[using (6)]}$$

$$P_{i+1} = 6a_i h_{i+1} + P_i$$

$$a_i = \frac{P_{i+1} - P_i}{6h_{i+1}} \quad ...(7)$$

Now substituting the values of d_i, a_i and b_i from (2), (6) and (7) in (3)

$$S(x_{i+1}) = \frac{1}{6h_{i+1}} (P_{i+1} - P_i) h^3_{i+1} + \frac{P_i}{2} h^2{}_{i+1} + c_i h_{i+1} + s(x_i)$$

$$S(x_{i+1}) = (P_{i+1} - P_i) \frac{h^2{}_{i+1}}{6} + \frac{P_i}{2} h^2{}_{i+1} + c_i h_{i+1} + s(x_i)$$

$$S(x_{i+1}) = S(x_i)\left(\frac{P_{i+1} - P_i}{6} + \frac{P_i}{2}\right) h^2{}_{i+1} + c_i h_{i+1}$$

$$S(x_{i+1}) = S(x_i) = \frac{h^2{}_{i+1}}{6} (P_{i+1} - P_i + 3P_i) + c_i h_{i+1}$$

$$c_i = \frac{S(x_{i+1}) - S(x_i)}{h_{i+1}} - \frac{h^2{}_{i+1}}{6} [P_{i+1} - P_i + 3P_i]$$

$$c_i = \frac{S(x_{i+1}) - S(x_i)}{h_{i+1}} - \frac{h_{i+1}}{6} [P_{i+1} + 2P_i] \quad ...(8)$$

Now, the slope at the point x_i (because the curve has equal slope at the point $[x_i, S(x_i)]$ hence from equation (4).

$$S'(x_i) = 3a_i (x_i - x_i)^2 + 2b_i (x_i - x_i) + c_i \Rightarrow S'(x_i) = c_i \quad ...(9)$$

$$c_i = S' = \frac{S(x_{i+1}) - S(x_i)}{h_{i+1}} - \frac{h_{i+1}}{6} \left[P_{i+1} + 2P_i\right] \quad ...(10)$$

But $S'(x_i)$ for the last subinterval is,

$$S'(x_i) = 3a_{i-1} h^2{}_i + 2b_{i-1} h_i + c_{i-1}. \quad ...(11)$$

and after using a_{i-1}, b_{i-1}, and c_{i-1}

$$S'(x_i) = 3 \frac{1}{6h_i} [P_i - P_{i-1}] h_i^2 + 2.\frac{1}{2} P_{i-1} h_i + \frac{S(x_1) + S(x_{i-1})}{h_i} - \frac{h_i}{6} [P_i - 2P_{i-1}]$$

$$S'(x_i) = \frac{S(x_i)+S(x_{i-1})}{h_i} + \frac{2P_i + P_{i-1}}{6} h_i \qquad ...(12)$$

For equation (9) and (10)

$$c_i = 3a_{i-1} = h_i^2 + 2b_{i-1}\, h_i + c_{i-1}$$

On substituting the values of a_{i-1}, b_{i-1}, c_{i-1} and c_i

$$\frac{S(x_{i+1})-S(x_i)}{h_{i+1}} - \frac{h_{i+1}}{6}(P_{i+1}+2P_i) = \frac{S(x_i)-S(x_{i-h})}{h_i} + \frac{2P_i h_i}{6} + \frac{P_{i-1}}{6}$$

$$\frac{S(x_{i+1})-S(x_i)}{h_{i+1}} - \frac{S(x_i)-S(x_{i-1})}{h_i} = \frac{h_{i+1}}{6}(P_{i+1}+2P_i) + \frac{2P_i h_i}{6} + \frac{P_{i-1}h_i}{6}$$

$$\frac{S(x_i+1)-S(x_i)}{h_{i+1}} - \frac{S(x_i)-S(x_{i-1})}{h_i} = \frac{h_{i+1}\,P_{i+1}}{6} + \frac{(h_{i+1}+h_i)P_i}{3} + \frac{h_i P_{i-1}}{6}$$

for $i = 1, 2, \ldots n - 1$

$$\Rightarrow \quad h_{i+1}P_{i+1} + 2(h_{i+1}+h_i)P_i + h_i\, P_{i-1} = 6\left[\frac{S(x_{i+1})-S(x_i)}{h_{i+1}} - \frac{S(x_i)-S(x_{i-1})}{h_i}\right] \qquad ...(13)$$

Now for equally spaced argument *i.e.,* $h_i = h$ Equation (13) becomes

$$h[P_{i+1} + 4P_i + P_{i-1}] = \frac{6}{4}[S(x_{i+1}) - 2S(x_i) + S(x_{i-1})]$$

or

$$P_{i+1} + 4P_i + P_{i-1} = \frac{6}{h^2}[S(x_{i+1}) - 2S(x_i) + S(x_{i-1})] \qquad ...(14)$$

while the $S(x)$ for equally spaced becomes.

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 P_{i-1} + (x - x_{i-1})^3 P_i\right] + \frac{1}{h}(x_i - x)\left[S\left(x_i - 1) - \frac{h^2}{6}P_{i-1}\right)\right]$$
$$+ \frac{1}{h}(x - x_{i-1})\left[S(x_i) - \frac{h^2}{6}P_i\right] \qquad ...(15)$$

Equation (15) gives cubic spline interpolation while equation (14) gives the condition for P_i.

Remarks:

(1) If $P_0 = P_n = 0$; it is called free boundary conditions and the spline curve for this condition is called the natural spline because the splines are assumed to take their natural straight line shape outside the interval of approximation.

(2) If $P_0 = P_n$, $P_{n+1} = P_1$; $f_0 = f_n$, $f_1 = f_{n+1}$, $h_1 = h_{n+1}$ then spline is called periodic splines.

(3) For a non-periodic spline we use.

$$f'(a) = f_0', f'(b) = f_n'$$

$$\Rightarrow \quad 2P_0 + P_1 = \frac{6}{h}\left(\frac{f_i - f_0}{h_i} - f_0'\right)$$

$$\Rightarrow \quad P_{n-1} + 2P_n = \frac{6}{h_n}\left(f_n' - \frac{f_n - f_{n-1}}{h_n}\right)$$

Example 1. *Obtain cubic spline for every subinterval, given in the tubular form.*

x	0	1	2	3
$f(x)$	1	2	33	244

With the end conditions $M_0 = 0 = M_3$

Sol. Here, we have equal spaced intervals as $h_1 = h_2 = h_3 = 1$, hence the condition for M_i becomes.

$$M_{i-1} + 4M_i + M_{i+1} = 6\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right]\ i = 1, 2$$

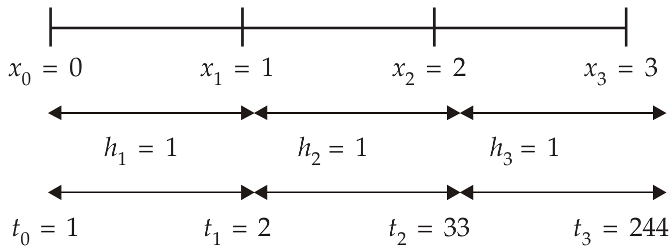
$$\Rightarrow \quad M_0 + 4M_1 + M_2 = 6\left[f(x_2) - 2f(x_1) + f(x_0)\right]$$

$$\Rightarrow \quad M_1 + 4M_2 + M_3 = 6\left[f(x_3) - 2f(x_2) + f(x_1)\right]$$

Now, after substituting the values of $f(x_i)$ and $M_0 = 0 = M_3$ we get

$$4M_1 + M_2 = 180 \text{ and } M_1 + 4M_2 = 1080$$

$$M_1 = -24 \text{ and } M_2 = 276$$



Now, the corresponding cubic spline can be obtained by having

$$f(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)$$

$$\left[f(x_{i-1}) - \frac{h_2}{6} M_{i-1}\right] + \frac{1}{h}(x - x_{i-1})\left[f(x_i) - \frac{h^2}{6} M_i\right], i = 1, 2, 3$$

Now, for $i = 1$ (the interval is [0, 1]), $f(x) = -4x^3 + 5x + 1$

Similarly, for [1, 2], $f(x) = 50x^3 - 162x^2 + 167x - 53$ and for [2, 3],

$$f(x) = -46x^3 + 414x^2 - 985x + 715.$$

Example 2: *Find the cube splines for following data:*

x	:	0	1	2	3
$f(x)$	:	1	2	5	11

with the end condition $M_0 = 0 = M_3$ *and also calculate* $f(2.5)$ *and* $f'(2.5)$.

Sol. Here intervals are equally spaced with difference 1 and $n = 3$. Now, the condition for M_i is

$$M_{i-1} + 4M_i + M_{i+1} = 6\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right]\ i = 1, 2$$

$$\Rightarrow \quad M_0 + 4M_1 + M_2 = 6\left[f(x_0) - 2f(x_1) + f(x_2)\right]$$

$\Rightarrow \qquad M_1 + 4M_2 + M_3 = 6\left[f(x_1) - 2f(x_2) + f(x_3)\right]$

but $\quad M_0 = 0\ M_3$ then it becomes

$$4M_1 + M_2 = 12 \text{ and } M_1 = 4M_2 = 18$$

$$M_1 = 2 \text{ and } M_2 = 4$$

Now, the corresponding cubic spline can be obtained by having

$$f(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + (x_i - x)$$

$$\left[f(x_{i-1}) - \frac{M_{i-1}}{6} + (x - x_{i-1})\right]\left[f(x_i) - \frac{M_i}{6}\right], i = 1, 2, 3$$

Now, for $i = 1$ (the interval is [0, 1])

$$f(x) = \frac{1}{3}\left(x^3 + 2x + 3\right)$$

Similarly, for [1, 2] $f(x) = \frac{1}{3}$ $(x^3 + 2x + 3)$ and for [2, 3], $f(x) = \frac{1}{3}\left(-2x^3 + 18x^2 - 34x + 27\right)$

Now, $f(2.5) = 7.66$ and $f'(2.5) = 6.16$

Example 3. *Obtain the cubic spline for the following data:*

x	:	0	1	2	3
$f(x)$	:	2	−6	−8	2

Sol. Take initial conditions $M_0 = 0 = M_3$ for $i = 1, 2 \ldots n$

$$h^2 \left[M_{i-1} + 4M_i + M_{i+1}\right] = 6 \left[f_{i+1} - 2f_i + f_{i-1}\right]$$

Here, $h = 1$;

$$\therefore \qquad M_0 + 4M_1 + M_2 = 6\left(f_2 - 2f_1 + f_0\right) \text{ for } 0 \le x \le 1$$

$$M_1 + 4M_2 + M_3 = 6\left(f_3 - 2f_2 + f_1\right) \text{ for } 1 \le x \le 2$$

$$\left.\begin{array}{l} M_2 + 4M_1 + M_0 = 36 \\ M_3 + 4M_2 + M_1 = 72 \end{array}\right\}$$

Using initial conditions, we get

$$M_1 = 4.8, \quad M_2 = 16.8$$

Hence for $0 \le x \le 1$ spline is given by

$$S(x) = \frac{1}{6}\left[(1-x)^3 M_0 + (x-0)^3 M_1 + (1-x)(6f_0 - M_0) + (x-0)(6f_1 - M_1)\right]$$

$$= \frac{1}{6}\left[x^3 (4.8) + (1-x)(12) + x(-36 - 4.8)\right]$$

$$= 0.8\, x^3 - 8.8\, x + 2$$

Hence for $1 \le x \le 2$ spline is given by

$$S(x) = 2x^3 - 3.\ 6x^2 - 5.2x + 0.8$$

Similarly, for $2 \le x \le 3$

$$S(x) = -2.8x^3 + 25.2x^2 - 62.8x + 39.2.$$ **Ans.**

Example 4. *Estimate the function value f at x = 7 using cubic splines from the following data: Given* $P_2 = P_0 = 0$.

i	0	1	2
x_i	4	9	16
f_i	2	3	4

Sol.

$$h_1 = x_1 - x_0 = 9 - 4 = 5$$
$$h_2 = x_2 - x_1 = 16 - 9 = 7$$

$$\frac{h_1}{6} P_0 + \frac{h_1 + h_2}{3} P_1 + \frac{h_2}{3} P_2 = \frac{1}{h_2}(f_2 - f_1) - \frac{1}{h_1}(f_1 - f_0)$$

$$\Rightarrow \qquad P_1 = -\frac{1}{70} = -0.0143$$

Since, $n = 3$ therefore, there are two cubic splines given by

$S_1(x) = x_0 \le x \le x_1$

$S_2(x) = x_1 \le x \le x_2$

$$S(x) = \frac{1}{6h_i}\left\{(x_i - x)^3 P_{i-1} + (x - x_{i-1})^3 P_i\right\} + \frac{1}{h_i}\left\{(x_i - x)\left(f_{i-1} - \frac{h_i^2}{6} P_{i-1}\right)\right\} + \frac{1}{h_i}\left\{(x - x_{i-1})\left(f_i - \frac{h_i^2}{6} P_i\right)\right\}$$

For $i = 1$

$$S(x) = \frac{1}{6h_1}\left\{(x_1 - x)^3 P_0 + (x - x_0)^3 P_1\right\} + \frac{1}{h_1}\left\{(x_1 - x)\left(f_0 - \frac{h_1^2}{6} P_0\right)\right\} + \frac{1}{h_1}\left\{(x - x_0)\left(f_1 - \frac{h_1^2}{6} P_1\right)\right\}$$

$$S(x) = \frac{1}{30}\left[27\,(-0.0143)\right] + \frac{4}{5} + \frac{3}{5}(3 + 25 \times 0.0024)$$

$S(7) = 2.64862.$ **Ans.**

PROBLEM SET 5.5

1. Using the Chebyshev polynomials $T_n(x)$, obtain the least square approximation of degree eleven for $f(x) = \cos^{-1} x$.

$$\left[\textbf{Ans. } f(x) = \frac{\pi}{2} T_0(x) - \frac{4}{\pi} T_1(x) - \frac{4}{9\pi} T_3(x) - \frac{4}{25\pi} T_5(x) - \frac{4}{49\pi} T_7(x) - \frac{4}{81\pi} T_9(x) - \frac{4}{121\pi} T_{11}(x)\right]$$

2. Find the linear least-squares polynomial approximation to the function $f(x) = 5 + x^2$ on the interval [0, 1].

$$\left[\textbf{Ans. } y = \frac{1}{6}(29 + 6x)\right]$$

3. Find the quadratic least squares polynomial approximation to the function $f(x) = x^{3/2}$ on the interval [0, 1].

$$\left[\textbf{Ans. } y = \frac{1}{105}(-2 + 48x + 60x^2)\right]$$

4. Using the Chebyshev polynomials $T_n(x)$, obtain the least squares approximation of second degree for $f(x) = 4x^3 + 2x^2 + 5x - 2$ on the interval [1, 1].

[**Ans.** $f(x) = -T_0(x) + 8T_1(x) + T_2(x)$]

5. Find the best lower-order approximation to the cubic $9x^3 + 7x^2$, $-1 \le x \le 1$.

$$\left[\textbf{Ans. } 7x^2 + \frac{27}{4}x, \text{ max.error} = \frac{9}{4} \text{ in } [-1,1]\right]$$

6. Economize the series $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \frac{x^{12}}{720} + \ldots$ on the interval [–1, 1] allowing for a tolerance of 0.05.

[**Ans.** $e^{x^2} = 1.0075 + 0.869x^2 + 0.8229x^4$]

7. Economize the series $x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040}$ on the interval [–1, 1], allowing for a tolerance of 0.0005.

$$\left[\textbf{Ans. } \sin hx = \frac{383}{384}x + \frac{17}{96}x^3\right]$$

8. Find a uniform polynomial approximation of degree 1 to $(2x - 1)^3$ on the interval [0, 1] so that the maximum norm of the error function is minimized, using Lanczos economization. Also calculate the norm of the error function.

Hint: Put $x = \frac{t+1}{2}$, linear approximation $= \frac{3}{4}(2x-1)$, maximum error $= \frac{1}{4}$

9. Find the lowest order polynomial which approximates the function $f(x) = 1 - x + x^2 - x^3 + x^4$, $0 \le x \le 1$ with an error less than 0.1.

$$\left[\textbf{Ans. } f(x) = \frac{160}{128}x^2 - \frac{160}{128}x + \frac{131}{128}\right]$$

10. Obtain an approximation in the sense of the principle of least squares in the form of a polynomial of second degree to the function $f(x) = \frac{1}{1+x^2}$ in the range $-1 \le x \le 1$.

[**Ans.** $P(x) = \frac{3}{4}(2\pi - 5) + \frac{15}{4}(3 - p)x^2$]

11. Find the polynomial of second degree, which is the best approximation in maximum norm to $\sqrt{x}$ on the point set $\left\{0, \frac{4}{9}, \frac{1}{9}, 1, 0\right\}$.

$$\left[\textbf{Ans. } P(x) = \frac{1}{16} + 2x - \frac{9}{8}x^2\right]$$

12. Find a polynomial $P(x)$ of degree as low as possible such that $\max_{|x| \le 1} \left| e^{x^2} - P(x) \right| \le 0.05$

[**Ans.** $1.0075 + 0.8698x^2 + 0.82292x^4$]

13. Prove that $x^2 = \frac{1}{2}\left[T_0(x) + T_2(x)\right]$.

14. Express $T_0(x) + 2T_1(x) + T_2(x)$ as polynomials in x. [**Ans.** $2x + 2x^2$]

15. Economize the series $f(x) = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16}$.

16. Economize the series $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$.

17. Prove that $T_n(x)$ is a polynomial in x of degree n

18. Find the best lower order approximation to the cubic $5x^3 + 4x^2$ in the closed interval [–1, 1]. [**Ans.** $4x^2 + \frac{5}{4}x$]

19. Find cubic spline for the following data:

x	0	1	2	3
$f(x)$	1	2	5	11

with end conditions $P_0 = 0 = P_3$ and also calculate $f(2.5)$, $f'(2.5)$.

20. Estimate the function value f at $x = 7$ using cubic splines from the following data:

i	0	1	2
x_i	4	9	16
f_i	2	3	4

[**Ans.** S_1 (7) = 2.6229]

21. Fit the following points by the cubic spline:

x	1	2	3	4
$f(x)$	1	5	11	8

By using the conditions $M_0 = 0 = M_3$. Hence find $f(1.5)$ and $f'(2)$

$$\left[\textbf{Ans. } f(x) = \frac{1}{15}\left[17x^3 - 51x^2 + 94x - 45\right] 1 \le x \le 2\right.$$

$$f(x) = \frac{1}{15}\left[-55x^3 - 381x^2 - 770x + 53\right] 2 \le x \le 3$$

$$\left. f(x) = \frac{1}{15}\left[38x^3 - 456x^2 + 1741x - 1980\right] 3 \le x \le 4 \quad f(1.5) = \frac{103}{40},\ f'(2) = \frac{94}{15}\right]$$

22. Find the cubic spline corresponding to the interval [2, 3] which means the following representation:

x	1	2	3	4	5
$f(x)$	30	15	32	18	25

with the end condition $M_1 = 0 = M_5$ and also compute $f(2.5)$, $f'(3)$

$$\left[\textbf{Ans. } f(x) = \frac{1}{16}\left[-142.9x^3 + 1058.4x^2 - 2475.2x + 1950\right] f(2.5) = -24.03 \;\&\; f'(3) = 2.817\right]$$

23. Fit the following points by Cubic spline and obtain $y(1.5)$:

x	1	2	3
y	–8	–1	18

[**Ans.** $3(x-1)^3 + 4x - 12$

$y(1.5) = -5.625$]

24. Obtain cubic spline approximation valid in the interval [3, 4],
Given that

x	1	2	3	4
y	3	10	29	65

Under the natural spline conditions $M(1) = 8 = M(4)$.

$$\left[\textbf{Ans. } S(x) = \frac{1}{15}\left\{-56x^3 + 72x^2 - 2092x + 2175\right\}\right.$$

$$\left. m_2 = \frac{62}{5},\ m_3 = \frac{112}{5}\right]$$

❑❑❑

CHAPTER 6

Numerical Differentiation and Integration

6.1 INTRODUCTION

The differentiation and integration are losely linked processes which are actually inversely related. For example, if the given function $y(t)$ represents an objects position as a function of time, its differentiation provides its velocity,

$$v(t) = \frac{d}{dt} y(t)$$

On the other hand, if we are provided with velocity $v(t)$ as a function of time, its integration denotes its position.

$$y(t) = \int_0^t v(t) dt$$

There are so many methods available to find the derivative and definite integration of a function. But when we have a complicated function or a function given in tabular form, they we use numerical methods. In the present chapter, we shall be concerned with the problem of numerical differentiation and integration.

6.2 NUMERICAL DIFFERENTIATION

The method of obtaining the derivatives of a function using a numerical technique is known as numerical differentiation. There are essentially two situations where numerical differentiation is required.

They are:

1. The function values are known but the function is unknown, such functions are called tabulated function.
2. The function to be differentiated is complicated and, therefore, it is difficult to differentiate.

The choice of the formula is the same as discussed for interpolation if the derivative at a point near the beginning of a set of values given by a table is required then we use Newton forward formula, and if the same is required at a point near the end of the set of given tabular values, then we use Newton's backward interpolation formula. The central difference formula (Bessel's and Stirling's) used to calculate value for points near the middle of the set of given tabular values. If the values of x are not equally spaced, we use Newton's divided difference interpolation formula or Lagrange's interpolation formula to get the required value of the derivative.

6.2.1 Derivation Using Newton's Forward Interpolation Formula

Newton's forward interpolation is given by

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 .... \quad ...(1)$$

where $$u = \frac{x - x_0}{h}$$

Differentiating equation (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 .... \quad ...(2)$$

Now $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h} \cdot \frac{dy}{du}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{4!}\Delta^4 y_0 .....\right] \quad ...(3)$$

As $x = x_0,\ u = 0$, therefore, putting $u = 0$ in (3), we get

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 ......\right]$$

Differentiating equation (3) again w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx} = \frac{1}{h} \times \frac{d}{du}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2 - 18u + 11}{12}\Delta^4 y_0 - ......\right] \quad ...(4)$$

Putting $u = 0$ in (4), we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^2 y_0 - ......\right] \quad ...(5)$$

Similarly,

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + ...\right] \text{ and so on.}$$

Aliter: We know that

$$I + \Delta = E = e^{hD}, D = \frac{d}{dx}$$

$\therefore$

$$hD = \log(1 + \Delta)$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \frac{\Delta^6}{6} + ...$$

or

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \frac{\Delta^6}{6} + ...\right] \qquad ...(1)$$

$\Rightarrow$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \frac{\Delta^6}{6} + ...\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 ......\right] \qquad ...(2)$$

also

$$D^3 = \frac{1}{h^3}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + ...\right]^3$$

$$= \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + ...\right] \qquad ...(3)$$

Applying equations (1), (2) and (3) for $y_{0,}$ we get

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^2 y_0 - ......\right]$$

$$\left[\frac{d^{3y}}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + ...\right]$$

6.2.2 Derivatives Using Newton's Backward Difference Formula

Newton's backward interpolation formula is given by

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n ...... \qquad ...(1)$$

where $u = \frac{x - x_n}{n}$.

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2 + 6u + 2}{3!}\nabla^3 y_n + ...\right] \qquad ...(2)$$

At $x = x_n, u = 0$. Therefore putting $u = 0$ in (2), we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + ...\right] \quad ...(3)$$

Again differentiating both sides of equation (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\nabla^3 y_n + \frac{12u^2+36u+22}{24}\nabla^4 y_n + ...\right]$$

At $x = x_n, u = 0$. Therefore putting $u = 0$ in (3), we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right] \quad ...(4)$$

Similarly,

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right] \text{ and so on.} \quad ...(5)$$

Aliter: We know that

$$1 - \nabla = E^{-1} = e^{-hD}$$

$$\therefore \quad -hD = \log(1-\nabla)$$

$$= -\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + ...\right]$$

or

$$D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + ...\right]$$

and

$$D^2 = \frac{1}{h^2}\left[\nabla^2 + \frac{1}{2}\nabla^3 + \frac{11}{12}\nabla^4 ......\right]$$

and

$$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + ...\right]$$

Applying these identities to y_n, we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n ......\right]$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

and

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right]$$

6.2.3 Derivatives Using Stirling's Formula

If we want to determine the values of the derivatives of the function near the middle of the given set of arguments. We may apply any central difference formula. Therefore using Stirling's formula, we get.

$$y_n = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots \quad \ldots(1)$$

where $u = \frac{x - x_0}{h}$.

Now, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{(3u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{(4u^3-2u)}{4!}\Delta^4 y_{-2} + \ldots\ldots\right]\frac{du}{dx}$$

Since $\quad u = \frac{x - x_0}{h}$

$\therefore \quad \frac{du}{dx} = \frac{1}{h}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3-2u}{4!}\Delta^4 y_{-2} + \ldots\ldots\right] \quad \ldots(2)$$

At $x = x_0, u = 0$, therefore, putting $u = 0$ in (2), we get

$$\left[\frac{dy}{dx}\right]_{x-x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\ldots\right]$$

Again differentiating, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{6u}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{12u^2-2}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\right] \quad \ldots(3)$$

At $x = x_0, u = 0$ therefore, putting $u = 0$ in (3), we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots\ldots\ldots\right] \text{ and so on.}$$

6.2.4 Derivative Using Newton's Divided Difference Formula

Newton's divided difference formula for finding the successive differentiation at the given value of x. Let us consider a function $f(x)$ of degree n, then

$$y = f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$
$$+ \ldots.. + (x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})\,\Delta^n f(x_0)$$

Differentiate this equation w.r.t. 'x' as many times as we require and put $x = x_i$, we get the required derivatives.

Example 1. *Find* $\frac{dy}{dx}$ *at x = 0.1 from the following table:*

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.9604

Sol. Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		–0.0075		
0.2	0.9900		-0.0049	
		–0.0124		0.0001
0.3	0.9776		-0.0048	
		–0.0172		
0.4	0.9604			

Here, $x_0 = 0.1, h = 0.1$ and $y_0 = 0.9975$ we know that, Newton's forward difference formula.

$$\left[\frac{dy}{dx}\right]_{x=0.1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$= \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right] = -0.050167. \quad \textbf{Ans.}$$

Example 2. *Using following table.*

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at x = 1.1.*

Sol. Since the values are at equidistant and we want to find the value of y at x = 1.1. Therefore, we apply Newton's forward difference formula.

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		–0.036				
		0.378		0.006			
1.2	8.781		–0.030		–0.002		
		0.348		0.004		0.001	
1.3	9.129		–0.026		–0.001		–0.002
		0.322		0.003		–0.001	
1.4	9.451		–0.023		–0.002		
		0.299		0.005			
1.5	9.750		–0.018				
		0.281					
1.6	10.031						

We have,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 .....\right]$$

Putting $x_0 = 1.1, \Delta y_0 = 0.378, \Delta^2 y_0 = 0.030,$ and so on and $h = 0.1$, we get

$$\left[\frac{dy}{dx}\right]_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.030) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001)\right]$$

$$= 10[0.378 + 0.015 + 0.0013 + 0.00025 - 0.0002]$$

$$= 10[0.39435] = 3.9435$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 ....\right]$$

$\Rightarrow$

$$\left[\frac{d^2y}{dx^2}\right]_{1.1} = \frac{1}{(0.1)^2}\left[-0.030 - 0.004 - \frac{11}{12}\times 0.001 + \frac{5}{6}\times 0.001\right]$$

$$= 100[-0.030 - 0.004 - 0.0009 + 0.0008] = 100(-0.0341)$$

$$= -0.341.$$ **Ans.**

Example 3. *Using the given table, find dy/dx at x = 1.2*

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

We have

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 .....\right]$$

Here, $x_0 = 1.2, \Delta y_0 = 0.7351, \Delta^2 y_0 = 0.1627$, and so on and $h = 0.2$, we get

$$\left[\frac{dy}{dx}\right]_{1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 5[0.7351 - 0.08135 + 0.0120 - 0.002 + 0.00028]$$

$$= 3.32015.$$ **Ans.**

Example 4. *Find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *of* $y = x^{1/3}$ *at* $x = 50$ *from the following table:*

x	50	51	52	53	54	55	56
$y = x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Sol.

x	$y = x^{1/3}$	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		–0.0003	
		0.0241		0
52	3.7325		–0.0003	
		0.0238		0
53	3.7563		–0.0003	
		0.0235		0
54	3.7798		–0.0003	
		0.0232		0
55	3.8030		–0.0003	
		0.0229		
56	3.8259			

Here, $x_0 = 50, h = 1.$ Then, we have Newton's forward difference formula.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - ......\right]$$

$$\left(\frac{dy}{dx}\right)_{50} = \frac{1}{1}\left(0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right)$$

$$= (0.0244 + 0.00015) = 0.02455$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + ...\right)$$

$$= \frac{1}{(1)^2}[(-0.0003)]$$

$$= \left(\frac{d^2y}{dx^2}\right)_{50} = -0.0003.$$ **Ans.**

Example 5. *The table given below reveals the velocity v of a body during the time t. Find its, acceleration at t = 1.1.*

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

Sol. Difference table:

t	v	Δ	Δ^2	Δ^3	Δ^4
1.0	43.1				
		4.6			
1.1	47.7		–0.2		
		4.4		0.1	
1.2	52.1		–0.1		0.1
		4.3		0.2	
1.3	56.4		–0.1		
		4.4			
1.4	60.8				

We have, $t_0 = 1.1, v_0 = 4.77$ and $h = 0.1$

Then the acceleration at $t = 1.1$ is given by

$$\left(\frac{dv}{dt}\right)_{t=t_0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right)$$

$$= \frac{1}{0.1}\left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2)\right]$$

$$= 1 - \left(4.4 + 0.5 + \frac{0.2}{3}\right)$$

$$= 45.167 \text{ (approx.)}$$ **Ans.**

Example 6. *Find f′(1.1) and f″(1.1) from the following table :*

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.1280	0.5440	1.2960	2.4320	4.0000

Sol. Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0				
		0.1280			
1.2	0.1280		0.288		
		0.4160		0.048	
1.4	0.5440		0.336		0
		0.7520		0.048	
1.6	1.2960		0.384		0
		1.1360		0.048	
1.8	2.4320		0.432		
		1.5680			
2.0	4.0000				

Here we have to find the derivatives at x=1.1 which lies between given arguments 1.0 and 1.2. So apply Newton's forward formula, we have

$$u = \frac{x - x_0}{h} = \frac{x-1}{0.2} = 5(x-1) \qquad ...(1)$$

$$f(x) = f(x)_0 + u\Delta f(x_0) + \frac{4(4-1)}{21}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)....... \qquad ...(2)$$

Differentiating w.r.t. x, we get

$$f'(x) = \left[\Delta f(x_0) + \frac{(2u-1)}{2!}\Delta^2 f(x_0) + \frac{3u^2 - 6u + 2}{3!}\Delta^3 f(x_0)\right]\frac{du}{dx}$$

Also, from (1) $\dfrac{du}{dx} = 5$

$$\therefore \qquad f'(x) = 5\left[\Delta f(x_0) + \frac{(2u-1)}{2}\Delta^2 f(x_0) + \frac{3u^2 - 6u + 2}{6}\Delta^3 f(x_0)\right]$$

At x =1.1, u = 5(1.1 – 1) = 0.5

$$\therefore \qquad f'(1.1) = 5\left[0.128 + \frac{2(0.5)-1}{2}(0.288) + \frac{3(0.5)^2 - 6(0.5) + 2}{6}(0.048)\right]$$

$$= 5[0.128 + 0 - 0.002]$$

$f'(1.1) = 0.63.$ **Ans.**

Differentiating equation (2) again w.r.t. 'x', we get

$$f''(x) = 5\left[\Delta^2 f(x_0) + \frac{6u-6}{6}\Delta^3 f(x_0)\right]\frac{du}{dx}$$

$$\therefore \qquad f''(1.1) = 25[0.288 + (0.5-1)\times 0.0481]_5 = 25[0.288 - 0.024]$$

Hence, $f''(1.1) = 6.6.$ **Ans.**

Example 7. *Find f′(1.5) from the following table:*

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.3989	0.3521	0.2420	0.1245	0.0540

Sol. Here we want to find the derivatives of $f(x)$ at x = 1.5, which is near the end of the arguments. Therefore, apply Newton's backward formula.

Difference Table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.0	0.3989				
		–0.0468			
0.5	0.3521		–0.0633		
		–0.1101		0.0609	
1.0	0.2420		–0.0024		–0.0215
		–0.1125		0.0394	
1.5	0.1295		0.037		
		0.0755			
2.0	0.0540				

Here, $x_n = 1.5$, $h = 0.5$

Therefore, $\left(\dfrac{dy}{dx}\right)_{x=x_n} = \dfrac{1}{h}\left[\nabla y_n + \dfrac{1}{2}\nabla^2 y_n + \dfrac{1}{3}\nabla^3 y_n + \dfrac{1}{4}\nabla^4 y_n + ...\right]$

$$f'(1.5) = \frac{1}{0.5}\left[-0.1125 + \frac{1}{2}(-0.0024) + \frac{1}{4}(0.0609)\right]$$

$$= 2[-0.1125 - 0.0012 + 0.0152251] = 2[-0.098475]$$

$\Rightarrow$ $\qquad f'(1.5) = -0.19695.$ **Ans.**

Example 8. *From the following table, find the values of* $\dfrac{dy}{dx}$ *and* $\dfrac{d^2y}{dx^2}$ *at* $x = 2.03$.

x	1.96	1.98	2.00	2.02	2.04
y	0.7825	0.7739	0.7651	0.7563	0.7473

Sol. Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.96	0.7825				
		–0.0086			
1.98	0.7739		–0.0002		
		–0.0088		0.0002	
2.00	0.7651		0		–0.0004
		–0.0088		–0.0002	
2.02	07563		–0.0002		
		–0.0090			
2.04	0.7473				

Here, $x_n = 2.04, h = 0.02\, u = \dfrac{x - x_n}{h} \Rightarrow \dfrac{du}{dx} = \dfrac{1}{h}$ at $x = 2.03$

$\therefore \quad u = \dfrac{2.03 - 2.04}{0.02} = -\dfrac{0.01}{0.02} = -\dfrac{1}{2}$

Then, by Newton's backward formula, we have

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2}\nabla^2 y_n + \frac{u(u+1)(u+2)}{6}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{24}\nabla^3 y_n + ... \quad ...(1)$$

Differentiating w.r.t. x, we have

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + ...\right] \quad ...(2)$$

$$y'(2.03) = \frac{1}{0.02}\left[-0.0090 + 0 + \frac{3\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 2}{6}(-0.0002) + \frac{4\left(-\frac{1}{2}\right)^3 + 18\left(-\frac{1}{2}\right)^2 + 22\left(-\frac{1}{2}\right) + 6}{24}(-0.0004)\right]$$

$$= 50[-0.0090 + 0.000008 + 0.000017] = -0.44875 \quad \textbf{Ans.}$$

Again differentiating equation (2) w.r.t. x,

$$y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\nabla^3 y_n + \frac{12u^2+36u+22}{24}\nabla^4 y_n + ...\right]$$

$$y''(2.03) = \frac{1}{(0.02)^2}\left[-0.0002 + \left(-\frac{1}{2}+1\right)(-\,0.0002) + \frac{12\left(-\frac{1}{2}\right)^2 + 36\left(-\frac{1}{2}\right) + 22}{24}(-0.0004)\right]$$

$$= 2500[-\,0.0002 - 0.0001 - 0.00012]$$

$$y''(2.03) = -1.05. \quad \textbf{Ans.}$$

Example 9. *Find $f'(5)$ from the following table:*

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

Sol. Here the arguments are not equally spaced. So we use Newton's divide difference formula.

Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		1			
2	1		$\frac{1}{3}$		
		2		0	
4	5		$\frac{1}{3}$		$-\frac{1}{144}$
		4		$-\frac{1}{16}$	
8	21		$-\frac{1}{6}$		
		3			
10	27				

Newton's divided difference formula is given by:

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)$$

$$\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 f(x_0) \quad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = \Delta f(x_0) + (2x - x_0 - x_1)\Delta^2 f(x_0) + [(x - x_1)(x - x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1)]\Delta^3 f(x_0) + ...$$

At $x = 5$

$$f'(5) = 1 + (10 - 1 - 2)\frac{1}{3} + [(5 - 2)(5 - 4) + ((5 - 1)(5 - 4) + (5 - 1)(5 - 2)] \times 0$$

$$+[(5-2)(5-4)(5-8) + (5-1)(5-4)(5-8) + (5-1)(5-2)(5-8) + (5-1)(5-2)(5-4)]\frac{-1}{144}$$

$$f' = 1 + 7\left(\frac{1}{3}\right) - \frac{1}{144}[-9 - 12 - 36 + 12] = 1 + \frac{7}{3} + \frac{45}{144}$$

Hence $f'(5) = 3.6458.$ **Ans.**

Example 10. *Find* $f'''(5)$ *from the data given below:*

x	2	4	9	13	16	21	29
$f(x)$	57	1345	66340	402052	1118209	4287844	21242820

Sol. Here, the arguments are not equally spaced and therefore we shall apply Newton's divided difference formula.

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)$$

$$\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 f(x_0) \quad ...(1)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
2	57						
		644					
4	1345		1765				
		12999		556			
9	66340		7881		45		
		83928		1186		1	
13	402052		22113		64		0
		238719		2274		1	
16	1118209		49401		89		
		633927		4054			
21	4287844		114265				
		2119372					
29	21242820						

Substituting values in eqn. (1), we get

$$f(x) = 57 + (x-2)(644) + (x-2)(x-4)(1765) + (x-2)(x-4)(x-9)(556)$$

$$+(x-2)(x-4)(x-9)(x-13)(45) + (x-2)(x-4)(x-9)(x-13)(x-16)(1)$$

$$= 57 + 644(x-2) + 1765(x^2 - 6x + 8) + 556(x^3 - 15x^2 + 62x - 72)$$

$$+ 45(x^4 - 28x^3 + 257x^2 - 878x + 936) + x^5 - 44x^4 + 705x^3 - 4990x^2$$

$$+ 14984x - 14976$$

$$f'(x) = 644 + 1765(2x-6) + 556(3x^2 - 30x + 62) + 45(4x^3 - 84x^2 + 514x - 878)$$

$$+5x^4 - 176x^3 + 2115x^2 - 9980x + 14984$$

$$f''(x) = 3530 + 556(6x-30) + 45(12x^2 - 168x + 514) + 20x^3 - 528x^2 + 4230x - 9980$$

$$f'''(x) = 3336 + 45(24x - 168) + 60x^2 - 1056x + 4230$$

$$= 60x^2 + 24x + 6$$

Where $x = 5$;

$$f'''(5) = 60(5)^2 + 24(5) + 6 = 1626.$$ **Ans.**

Example 11. *Find f′(4) from the following data:*

x	0	2	5	1
$f(x)$	0	8	125	1

Sol. Though this problem can be solved by Newton's divided difference formula, we are giving here, as an alternative, Lagrange's method. Lagrange's polynomial, in this case, is given by

$$f(x) = \frac{(x-2)(x-5)(x-1)}{(0-2)(0-5)(0-1)}(0) + \frac{(x-0)(x-5)(x-1)}{(2-0)(2-5)(2-1)}(8)$$

$$+ \frac{(x-0)(x-2)(x-1)}{(5-0)(5-2)(5-1)}(125) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(1)$$

$$= -\frac{4}{3}(x^3 - 6x^2 + 5x) + \frac{25}{12}(x^3 - 3x^2 + 2x) + \frac{1}{4}(x^3 - 7x^2 + 10x) = x^3$$

$\therefore \quad f'(x) = 3x^2$

When $x = 4$, $f'(4) = 3(4)^2 = 48$. **Ans.**

Example 12. *Find* $f'(0.6)$ *and* $f''(0.6)$ *from the following table:*

x	0.4	0.5	0.6	0.7	0.8
$f(x)$	1.5836	1.7974	2.0442	2.3275	2.6510

Sol. Here, the derivatives are required at the central point $x = 0.6$, so we use Stirling's formula.

Difference table

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–2	0.4	1.5836				
			0.2138			
–1	0.5	1.7974		0.0330		
			(0.2468		(0.0035	
0	0.6	(2.0442)		(0.0365)		(0.0002)
			0.2833)		0.0037)	
1	0.7	2.3275		0.0402		
			0.3235			
2	0.8	2.6510				

Here we have $x_0 = 0.6, h = 0.1, u = \dfrac{x - x_0}{h}$ at $x = 0.6$, $u = 0$

Stirling's formula is

$$f(x) = y = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3 - u}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^4 - u^2}{24}\Delta^4 y_{-2} \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3 - 2u}{24}\Delta^4 y_{-2} + ...\right]$$

Using difference table, we have

$$\Delta y_0 = 0.2468,\ \Delta y_{-1} = 0.2833,\ \Delta^2 y_{-1} = 0.0365,\ \Delta^3 y_{-1} = 0.0035,$$

$$\Delta^3 y_{-2} = 0.0037,\ \Delta^4 y_{-2} = 0.0002,$$

$$f'(0.6) = \left[\left(\frac{0.2468 + 0.2833}{2}\right) + 0 - \frac{1}{6}\left(\frac{0.0035 + 0.0037}{2}\right) + 0\right]\frac{1}{0.1}$$

$$= 10[0.26505 - 0.0006]$$

$$= f'(0.6) = 2.6445. \qquad \textbf{Ans.}$$

Again differentiating (1), we get

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{12u^2 - 2}{24}\Delta^4 y_{-2} + \ldots\right]$$

$$f''(0.6) = \frac{1}{(0.1)^2}\left[0.0365 + 0 - \frac{1}{12} \times 0.0002\right]$$

$$= 100[0.0365 - 0.000016]$$

Hence, $f''(0.6) = 3.6484$ **Ans.**

Example 13. *Find* $f'(93)$ *from the following table:*

x	60	75	90	105	120
$f(x)$	28.2	38.2	43.2	40.9	37.7

Sol. Difference table

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–2	60	28.2				
			10	–5		
–1	75	38.2				
			(5	–7.3	(–2.3	
0	90	(43.2)				(8.7)
			–2.3)	–0.9	6.4)	
1	105	40.9				
			–3.2			
2	120	37.7				

Here we have $x_0 = 90, x = 93, h = 15$

$$\therefore \qquad u = \frac{x - x_0}{h} = \frac{93 - 90}{15} = \frac{3}{15} = \frac{1}{5} = 0.2$$

Now using Stirling's formula

$$f(x) = y = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3 - u}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^4 - u^2}{24}\Delta^4 y_{-2} + ... \quad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3 - 2u}{24}\Delta^4 y_{-2} + ...\right]$$

Putting the values of $x = 93$, $u = 0.2$, $h = 15$ and

$$\Delta y_0 = 5, \Delta y_{-1} = -2.3,\ \Delta^2 y_{-1} = -7.3\ \ \Delta^3 y_{-1} = -2.3,\ \Delta^3 y_{-2} = 6.4,\ \Delta^4 y_{-2} = 8.7$$

We get,

$$f'(93) = \frac{1}{15}\left[\frac{5 - 2.3}{2} + 0.2(-7.3) + \frac{3(0.2)^2 - 1}{6}\left(\frac{-2.3 + 6.4}{2}\right) + \frac{4(0.2)^3 - 2(0.2)}{24}(8.7)\right]$$

$$= \frac{1}{15}\left[\frac{2.7}{2} - 1.46 - \frac{3.608}{6 \times 2} - \frac{3.2016}{24}\right]$$

$$= \frac{1}{15}\left[1.35 - 1.46 - \frac{0.6013}{2} - 0.1334\right]$$

$$= \frac{1}{15}[1.35 - 1.46 - 0.30065 - 0.1334]$$

$f'(93) = -0.03627.$ **Ans.**

Example 14. *Find x for which y is maximum and find this value of y*

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Sol. The Difference table is as follows:

x	u	Δ	Δ^2	Δ^3	Δ^4
1.2	0.9320				
		0.0316			
1.3	0.9636		–0.0097		
		0.0219		–0.0002	
1.4	0.9855		–0.0099		0.0002
		0.0120		0	
1.5	0.9975		–0.0099		
		0.0021			
1.6	0.9996				

Let $y_0 = 0.9320$ and $a = 1.2$

By Newton's forward difference formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + ...$$

$$= 0.9320 + 0.0316u + \frac{u(u-1)}{2}(-0.0097) \text{ (Neglecting higher differences)}$$

$$\frac{dy}{du} = 0.0316 + \left(\frac{2u-1}{2}\right)(-0.0097)$$

At a maximum,

$$\frac{dy}{du} = 0$$

$$\Rightarrow \quad 0.0316 = \left(u - \frac{1}{2}\right)(0.0097) \Rightarrow u = 3.76$$

$$\therefore \quad x = x_0 + hu = 1.2 + (0.1)(3.76) = 1.576$$

To find $y_{max,}$ we use backward difference formula,

$$x = x_n + hu$$

$$\Rightarrow \quad 1.576 = 1.6 + (0.1)u \Rightarrow u = -0.24$$

$$y(1.576) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$= 0.9996 - (0.24 \times 0.0021) + \frac{(-0.24)(1-0.24)}{2}(-0.0099)$$

$$= 0.9999988 = 0.9999 \text{ nearly}$$

$\therefore$ Maximum y = 0.9999. (Approximately) **Ans.**

Example 15. *From the following table, for what value of x, y is minimum. Also find this value of y.*

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.260	0.250	0.224

Sol. Difference table

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		–0.016	
		0.019		0.000
5	0.259		–0.016	
		0.003		0.001
6	0.262		–0.015	
		–0.012		0.001
7	0.250		–0.014	
		–0.026		
8	0.224			

Now taking $x_0 = 3$, we have $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$ and $\Delta^3 y_0 = 0$.
Therefore, Newton's forward interpolation formula gives

$$y = 0.205 + u(0.035) - \frac{u(u-1)}{2}(-0.016) \quad ...(1)$$

Differentiating (1), w.r.t. u, we get

$$\frac{dy}{du} = 0.035 + \frac{2u-1}{2}(-0.016)$$

For y to be minimum put $\frac{dy}{du} = 0$

$\Rightarrow$ $0.035 - 0.008(2u-1) = 0$

$\Rightarrow$ $u = 2.6875$

Therefore, $x = x_0 + uh$

$= 3 + 2.6875 \times 1 = 5.6875$

Hence, y is minimum when $x = 5.6875$.
Putting $u = 2.6875$ in (1), we get the minimum value of y given by

$$= 0.205 + 2.6875 \times 0.035 + \frac{1}{2}(2.6875 \times 1.6875)(-0.016)$$

$= 0.2628.$ **Ans.**

PROBLEM SET 6.1

1. Find $f'(6)$ from the following table:

x	0	1	3	4	5	7	9
$f(x)$	150	108	0	–54	–100	–144	–84

[**Ans.** –23]

2. Use the following data to find $f'(3)$:

x	3	5	11	27	34
$f(x)$	–13	28	899	17315	35606

[**Ans.** 1.8828]

3. Use the following data to find $f'(5)$:

x	2	4	9	10
$f(x)$	4	56	711	980

[**Ans.** 2097.69]

4. From the table, find $\frac{dy}{dx}$ at $x = 1$, $x = 3$ and $x = 6$:

x	0	1	2	3	4	5	6
$f(x)$	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

[**Ans.** 0.3952, 0.3341, 0.2719]

5. Find $f'(5)$ and $f''(5)$ from the following data:

x	2	4	9	13
$f(x)$	57	1345	66340	402052

6. Using Newton's Divided Difference Formula, find $f'(10)$ from the following data:

x	3	5	11	27	34
$f(x)$	–13	23	99	17315	35606

[**Ans.** 232.869]

7. From the table below, for what value of x, y is minimum? Also find this value of y.

x	3	4	5	6	7	8
4	0.205	0.240	0.259	0.262	0.250	0.224

[**Ans.** 5.6875, 0.2628]

8. A slider in a machine moves along a fixed straight rod. Its distance x (in cm.) along the rod is given at various times t (in secs.)

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.28	31.43	32.98	33.54	33.97	33.48	32.13

Evaluate $\frac{dx}{dt}$ at $t = 0.1$ and at $t = 0.5$ [**Ans.** 32.44166 cm/sec.; –24.05833 cm/sec.]

9. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t (seconds).

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and acceleration of the rod when $t = 0.6$ sec.

[**Ans.** (*i*) 3.82 radians/sec. (*ii*) 6.75 radians/sec^2]

10. The table given below reveals the velocity 'v' of a body during the time 't' specified. Find its acceleration at $t = 1.1$.

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

[**Ans.** 44.92]

6.3 NUMERICAL INTEGRATION

Like numerical differentiation, we need to seek the help of numerical integration techniques in the following situations:

1. Functions do not possess closed from solutions. Example:

$$f(x) = C\int_0^x e^{-t^2}\,dt.$$

2. Closed form solutions exist but these solutions are complex and difficult to use for calculations.
3. Data for variables are available in the form of a table, but no mathematical relationship between them is known as is often the case with experimental data.

6.4 GENERAL QUADRATURE FORMULA

Let $y = f(x)$ be a function, where y takes the values $y_0, y_1, y_2, \ldots\ldots y_n$ for $x = x_0, x_1, x_2, \ldots\ldots x_n$. We want to find the vaule of $I = \int_a^b f(x)dx$.

Let the interval of integration (a, b) be divided into n equal subintervals of width $h = \left(\frac{b-a}{n}\right)$ so that $x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2hT, \ldots\ldots\ldots\ldots, x_n = x_0 + nh = b.$

$$\therefore \qquad I = \int_a^b f(x)dx = \int_{x_0}^{x_{0+nh}} f(x)dx \qquad \ldots(1)$$

Newton's forward interpolation formula is given by

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

Where $u = \frac{x - x_0}{h}$

$$\therefore \qquad du = \frac{1}{h}dx \;\Rightarrow\; dx = hdu$$

$\therefore$ Equation (1) becomes,

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right]du$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots + \text{up to}(n+1)\text{ terms}\right]$$

$$\therefore \qquad \int_{x_0}^{x_n} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots + \text{up to }(n+1)\text{ terms}\right] \qquad \ldots(2)$$

This is called general quadrature formula.

6.5 TRAPEZOIDAL RULE

Putting n =1 in equation (2) and taking the curve $y = f(x)$ through (x_0, y_0) and (x_0, y_0) as a polynomial of degree one so that differences of order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}\left[2y_0 + (y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly, for the next sub interval $(x_0 + h,\ x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}(y_1 + y_2)\ldots\ldots \quad \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_n + y_0) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]}$$

which is known as Trapezoidal rule.

6.6 SIMPSON'S ONE-THIRD RULE

Putting n = 2 in equation (2) and taking the curve through $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}\left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 - y_0)\right] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4), \ldots\ldots.$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

which is known as Simpson's one-third rule.

Note: Using the formula, the given interval of integration must be divided into an even number of sub-intervals.

6.7 SIMPSON'S THREE-EIGHT RULE

Putting $n = 3$ in equation (2) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) as a polynomial of degree three so that differences of order higher than three vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6), \ldots\ldots\ldots$$

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[(y_0 + y_{n)} + 3(y_1 + y_2 + y_4 + y_5 \ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

which is known as Simpson's three-eighth rule.

Note: Using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

6.8 BOOLE'S RULE

Putting $n = 4$ in equation (2) and neglecting all differences of order higher than four, we get

$$\int_{x_0}^{x_0+4h} f(x)dx = h\int_0^4\left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0\right]dr$$

(By Newton's forward interpolation formula)

$$= 4h\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12} - \Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!}\right]_0^4$$

$$= 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right]$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly, $\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8)$ and so on.

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_{8....})$$

This is known as Boole's rule.

Note: Using Boole's rule, the number of sub-intervals should be taken as a multiple of 4.

6.9 WEDDLE'S RULE

Putting $n = 6$ in equation (2) and neglecting all differences of order higher than six, we get

$$\int_{x_0}^{x_0+6h} f(x)dx = h\int_0^6 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 \right.$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$\left. + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 \right] dr$$

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 \right.$$

$$+ \frac{1}{24}\left(\frac{r^5}{5} - \frac{3r^4}{2} + \frac{11r^3}{3} - 3r^2\right)\Delta^4 y_0 + \frac{1}{120}\left(\frac{r^6}{6} - 2r^5 + \frac{35r^4}{4} - \frac{50r^3}{3} + 12r^2\right)\Delta^5 y_0$$

$$\left. + \frac{1}{720}\left(\frac{r^7}{7} - \frac{5r^8}{2} + 17r^5 - \frac{225r^4}{4} + \frac{274r^3}{3} - 60r^2\right)\Delta^6 y_0 \right]_0^6$$

$$= 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0 \right]$$

$$= \frac{6h}{20}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \frac{41}{42}\Delta^6 y_0 \right]$$

$$= \frac{3h}{10}\left[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0) + 80(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$+ 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$

$$+ (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)] \qquad \left[\because \frac{41}{42} \approx 1\right]$$

$$= \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly, $\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12})$

....

....

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)dx = \frac{3h}{10}(y_{n-6} + 5y_{n-5} + y_{y-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + ...)$$

which is known as Weddle's rule. Here n must be a multiple of 6.

Example 1. *Use Trapezoidal rule to evaluate* $\int_0^1 \frac{1}{1+x} dx.$

Sol. Let $h = 0.125$ and $y = f(x) = \frac{1}{1+x}$, then the values of y are given for the arguments which are obtained by dividing the interval [0,1] into eight equal parts are given below:

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = \frac{1}{1+x}$	1.0 y_0	0.8889 y_1	0.8000 y_2	0.7273 y_3	0.6667 y_4	0.6154 y_5	0.5714 y_6	0.5333 y_7	0.5 y_8

Now by Trapezoidal rule

$$\int_0^1 \frac{1}{1+x} dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{0.125}{2}[1 + 2(0.8889 + 0.800 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333) + 0.5]$$

$$= \frac{0.125}{2}[1.5 + 2(4.803)] = \frac{0.125}{2}[11.106] = 0.69413.$$ **Ans.**

Example 2. *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

(i) Simpson's $\frac{1}{3}$ *rule taking* $h = \frac{1}{4}$

(ii) Simpson's $\frac{3}{8}$ *rule taking* $h = \frac{1}{6}$

(iii) Weddle's rule taking $h = \frac{1}{6}$

Hence compute an approximate value of π *in each case.*

Sol. (*i*) The values of $f(x)=\frac{1}{1+x^2}$ at $x=0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1$ are given below:

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$	1	$\frac{16}{17}$	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpon's $\frac{1}{3}$ rule

$$\int_0^1 \frac{dx}{1+x^2}=\frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

$$=\frac{1}{12}\left[(1+0.5)+4\left\{\frac{16}{17}+.64\right\}+2(0.8)\right]=0.78539215$$

Also, $$\int_0^1 \frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_0^1=\tan^{-1}1=\frac{\pi}{4}$$

$\therefore$ $$\frac{\pi}{4}=0.785392156 \Rightarrow \pi \approx 3.1415686.$$ **Ans.**

(*ii*) The values of $f(x)=\frac{1}{1+x^2}$ at $x=0,\ \frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},1$ are given below:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule

$$\int_0^1 \frac{dx}{1+x^2}=\frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$=\frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right)+3\left\{\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right\}+2\left(\frac{4}{5}\right)\right] = 0.785395862$$

Also, $$\int_0^1 \frac{dx}{1+x^2}=\frac{\pi}{4}$$

$\therefore$ $$\frac{\pi}{4}=0.785395862 \Rightarrow \pi = 3.141583.$$ **Ans.**

(*iii*) By Weddle's rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3\left(\frac{1}{6}\right)}{10}\left[1+5\left(\frac{36}{37}\right)+\frac{9}{10}+6\left(\frac{4}{5}\right)+\frac{9}{13}+5\left(\frac{36}{61}\right)+\frac{1}{2}\right] = 0.785399611$$

Since $$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

$\therefore$ $$\frac{\pi}{4} = 0.785399611$$

$\Rightarrow$ $$\pi = 3.141598$$

Example 3. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using*

(i) Trapezoidal Rule
(ii) Simpson's one-third rule
(iii) Simpson's three-eighth rule
(iv) Weddle's rule.

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$.

The value of $f(x) = \frac{1}{1+x^2}$ are given below:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(0.5+0.2+0.1+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$= 1.410798581.$ **Ans.**

(*ii*) By Simpson's one-third rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{37}\right)+4\left(0.5+0.1+\frac{1}{26}\right)+2\left(0.2+\frac{1}{17}\right)\right] = 1.366173413.$$ **Ans.**

(*iii*) By Simpson's three-eighth rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{1}{37}\right)+3\left(0.5+0.2+\frac{1}{17}+\frac{1}{26}\right)+2(0.1)\right]$$

$= 1.357080836.$ **Ans.**

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3}{10}\left[1+5(0.5)+0.2+6(0.1)+\frac{1}{17}+5\left(\frac{1}{26}\right)+\frac{1}{37}\right]$$

$= 1.373447475.$ **Ans.**

Example 4. *Using Simpson's one-third rule, find* $\int_0^6 \frac{dx}{(1+x)^2}$. *(B. Tech. 2002)*

Sol. Divide the interval [0,6] into 6 equal parts with $h = \frac{6-0}{6} = 1$. The values of $y = \frac{1}{(1+x)^2}$ at each points of sub-divisions are given by

x	0	1	2	3	4	5	6
$y = \frac{1}{(1+x)^2}$	1 y_0	0.25 y_1	0.11111 y_2	0.0625 y_3	0.04 y_4	0.02778 y_5	0.02041 y_6

By Simpson's one-third rule, we get

$$\int_0^6 \frac{dx}{(1+x)^2} = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$$

$$= \frac{1}{3}[1 + 4(0.25 + 0.0625 + 0.02778) + 2(0.11111 + 0.04) + 0.02041]$$

$$= \frac{1}{3}[1.02041 + 4(0.34028) + 2(0.15111)]$$

$$= \frac{1}{3}[1.02041 + 1.36112 + 0.30222]\frac{1}{3}(2.68375)$$

$= 0.89458.$ **Ans.**

Example 5. *Evaluate* $x\int_0^4 \frac{dx}{1+x^2}$ *using Boole's rule taking*

(i) $h = 1$ *(ii)* $h = 0.5$

Compare the rusults with the actual value and indicate the error in both.

Sol. (*i*) Dividing the given interval into 4 equal subintervals (*i.e.* $h = 1$), the table is as below:

x	0	1	2	3	4
y	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4

Using Boole's rule,

$$\int_0^4 ydx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

$$= \frac{2(1)}{45}\left[7(1) + 32\left(\frac{1}{2}\right) + 12\left(\frac{1}{5}\right) + 32\left(\frac{1}{10}\right) + 7\left(\frac{1}{17}\right)\right] = 1.289412 \text{ (approx.)}$$

$$\therefore \quad \int_0^4 \frac{dx}{1+x^2} = 1.289412. \quad \textbf{Ans.}$$

(*ii*) Dividing the given interval into 8 equal subintervals (*i.e.* $h = 0.5$), the table is as below:

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1.0	0.8	0.5	$\frac{4}{13}$	0.2	$\frac{4}{29}$	0.1	$\frac{4}{53}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Boole's rule,

$$\int_0^4 ydx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{1}{45}\left[7(1) + 32(8) + 12(5) + 32\left(\frac{4}{13}\right) + 7(.2) + 7(.2) + 32\left(\frac{4}{29}\right) + 12(.1) + 32\left(\frac{4}{53}\right) + 7\left(\frac{1}{17}\right)\right]$$

$$= 1.326373$$

$$\therefore \int_0^4 \frac{dx}{1+x^2} = 1.326373$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^4 = \tan^{-1}(4) = 1.325818$$

$$\text{Error in I result} = \left(\frac{1.325818 - 1.289412}{1.325818}\right) \times 100 = 2.746\%$$

$$\text{Error in II result} = \left(\frac{1.325818 - 1.326373}{1.325818}\right) \times 100 = -0.0419\%$$

Example 6. *Evaluate the integral* $\int_0^6 \frac{dx}{1+x^3}$ *by using Weddle's rule.*

Sol. Divide the interval [0,6] into 6 equal parts each of width $h = \frac{6-0}{6} = 1$. The value of $y = \frac{1}{1+x^3}$ at each points of sub-divisions are given below:

x	0	1	2	3	4	5	6
y	1.0000	0.5000	0.1111	0.0357	0.0153	0.0079	0.0046
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's Rule, we get

$$\int_0^6 \frac{dx}{1+x^3} = \frac{3h}{10}[y_0 + 5(y_1 + y_5) + y_2 + y_4 + 6y_3 + y_6]$$

$$= \frac{3}{10}[1.0000 + 5(0.5000 + 0.0079) + 0.1111 + 0.0153 + 6(0.357) + 0.0046]$$

$$= \frac{3}{10}[1.131 + 5(0.5079) + 6(0.0357)]$$

$$= \frac{3}{10}[1.131 + 2.5395 + 0.2142]$$

$$= \frac{3}{10}(3.8847) = 1.1654.$$ **Ans.**

Example 7. *Evaluate the integral* $\int_0^{1.5} \frac{x^3}{e^x - 1} dx$ *by using Weddle's rule.*

Sol. Dividing the interval [0, 1, 5] into 6 equal parts of each of width $h = \frac{1.5-0}{6} = 0.25$ and the values of $y = \frac{x^3}{e^x - 1}$ at each points of sub-interval are given by

x	0	0.25	0.50	0.75	1.00	1.25	1.50
y	0	0.0549	0.1927	0.3777	0.5820	0.7843	0.9694
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now by Weddle's rule, we get

$$\int_0^{1.5} \frac{x^3}{e^x - 1} dx = \frac{3h}{10}[y_0 + 5(y_1 + y_5) + y_2 + y_4 + 6y_3 + y_6]$$

$$= \frac{3(0.25}{10}[0 + 5(0.0549 + 0.7843) + 0.1927 + 0.5820 + 0.9694 + 6(0.3777)]$$

$$= 0.075[1.7441 + 5(0.8392) + 6(0.3777)]$$

$$= 0.075[1.7441 + 4.196 + 2.2662] = 0.075(8.2063)$$
$$= 0.6155.$$ **Ans.**

Example 8. *Evaluate the integral* $\int_4^{5.2} \log x dx$, *using Weddle's rule.*

Sol. Divide the interval [4, 5.2] into 6 equal sub-interval of each width $= \frac{5.2-4}{6} = 0.2$ and values of $y = \log x$ are given below:

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3862	1.4350	1.4816	1.5261	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's Rule, we get

$$\int_4^{5.2} \log x dx = \frac{3h}{10}[y_0 + 5(y_1 + y_5) + y_2 + y_4 + 6y_3 + y_6]$$

$$= \frac{3(0.2)}{10}[1.3862 + 5(1.4350 + 1.6094) + 1.4816 + 1.5686 + 6(1.5261) + 1.6486]$$

$$= \frac{0.6}{10}[1.3862 + 5(3.0444) + 1.4816 + 1.5686 + 6(1.526) + 1.6486]$$

$$= \frac{0.6}{10}[6.085 + 5(3.0444) + 6(1.5261)]$$

$$= \frac{0.6}{10}[6.085 + 15.222 + 9.1566]$$

$$= \frac{0.6}{10}(30.4636) = 1.8278$$ **Ans.**

Example 9. *A river is 80 m wide. The depth y of the river at a distance 'x' from one bank is given by the following table:*

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the approximate area of cross section of the river using

(i) Boole's rule.

(ii) Simpson's one-third rule.

Sol. The required area of the cross-section of the river.

$$= \int_0^{80} y dx \qquad ...(1)$$

Here no. of sub intervals is 8

(*i*) Boole's rule,

$$\int_0^{80} y dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{2(10)}{45}[7(0)+32(4)+12(7)+32(9)+7(12)+32(15)+12(14)+32(8)+7(3)]$$
$$= 708$$

Hence the required area of the cross-section of the river = 708 sq. m. **Ans.**

(*ii*) By Simpson's one-third rule,

$$\int_0^{80} y dx = \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$
$$= \frac{10}{3}[(0+3)+4(4+9+15+8)+2(7+12+14)]$$
$$= 710$$

Hence the required area of the cross-section of the river = 710 sq. m. **Ans.**

Example 10. *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *by dividing the interval of integration into 8 equal parts. Hence find* $\log_e 2$ *approximately.*

Sol. Since the interval of integration is divided into an even number of subintervals, we shall use Simpson's one-third rule.

Here, $y = \frac{1}{1+x} = f(x)$

$y_0 = f(0) = \frac{1}{1+0} = 1$, $y_1 = f\left(\frac{1}{8}\right) = \frac{1}{1+\frac{1}{8}} = \frac{8}{9}$, $y_2 = f\left(\frac{2}{8}\right) = \frac{4}{5}$,

$y_3 = f\left(\frac{3}{8}\right) = \frac{8}{11}$, $y_4 = f\left(\frac{4}{8}\right) = \frac{2}{3}$, $y_5 = f\left(\frac{5}{8}\right) = \frac{8}{13}$,

$y_6 = f\left(\frac{6}{8}\right) = \frac{4}{7}$, $y_7 = f\left(\frac{7}{8}\right) = \frac{8}{15}$, and $y_8 = f(1) = \frac{1}{2}$.

Hence,

x	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1
y	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's one-third rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$
$$= \frac{1}{24}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{8}{9}+\frac{8}{11}+\frac{8}{13}+\frac{8}{15}\right)+2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)\right] \quad \text{(Hence, } h = 1/8)$$
$$= 0.69315453.$$ **Ans.**

Since $$\int_0^1 \frac{dx}{1+x} = [\log_e(1+x)]_0^1 = \log 2$$

$$\log_e 2 = 0.69315452. \qquad \textbf{Ans.}$$

6.10 EULER-MACLAURIN'S FORMULA

This formula is based on the expansion of operators. Suppose $\Delta F(x) = f(x)$, then an operator Δ^{-1}, called inverse operator, is defined as

$$F(x) = \Delta^{-1} f(x)$$

Again we have $\Delta F(x) = f(x_0)$

$$\Rightarrow \quad F(x_1) - F(x_0) = f(x_0)$$

$$F(x_2) - F(x_1) = f(x_1)$$

....

$$F(x_n) - F(x_{n-1}) = f(x_{n-1})$$

Adding all these, we get

$$F(x_n) - F(x_0) = \sum_{i=0}^{n-1} f(x_i) \qquad ...(1)$$

where $x_{0,}\ x_{1,}\ x_n$, are the (n+1) equidistant values of x with interval h.

Now $$F(x) = \Delta^{-1} f(x) = (E-1)^{-1} f(x)$$

$$= \left(e^{hD} - 1\right)^{-1} f(x)$$

$$= \left\{\left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...\right) - 1\right\}^{-1} f(x)$$

$$= \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...\right)^{-1} f(x)$$

$$= (hD)^{-1}\left\{1 + \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right)^{-1}\right\} f(x)$$

$$= \frac{1}{h} D^{-1}\left\{1 - \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right) + \frac{(-1)(-2)}{2!}\left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right)^2 + ...\right\} f(x)$$

$$= \frac{1}{h} D^{-1}\left(1 - \frac{hD}{2!} + \frac{h^2D^2}{12} - \frac{h^4D^4}{720} + ...\right) f(x)$$

$$= \frac{1}{h}\int f(x)dx - \frac{1}{2} f(x) + \frac{h}{12} f'(x) - \frac{h^3}{720} f'''(x) \qquad ...(2)$$

Between limits $x = x_0$ and $x = x_n$ from equation (2), we have

$$F(x_n) - F(x_0) = \frac{1}{h}\int_{x_0}^{x_n} f(x)dx - \frac{1}{2}\{f(x_n) - f(x_0)\} + \frac{h}{12}\{f'(x_n) - f'(x_0)\}$$

$$- \frac{h^3}{720}\{f'''(x_n) - f'''(x_0)\} + ... \quad ...(3)$$

From eqs. (1) and (3), we have

$$\sum_{i=1}^{n-1} f(x_i) = \frac{1}{h}\int_{0}^{x_n} f(x)dx - \frac{1}{2}\{f(x_n) - f(x_0)\} + \frac{h}{12}\{f'(x_n) - f'(x_0)\}$$

$$- \frac{h^3}{720}\{f'''(x_n) - f'''(x_0)\} + ...$$

But $\sum_{i=0}^{n-1} f(x_i) = \sum_{i=0}^{n} f(x_i) - f(x_n)$ and $x_n = x_0 + nh$. Then the above relation reduces to

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)dx = \sum_{i=0}^{n} f(x_i) - \frac{1}{2}\{f(x_n) + f(x_0)\} - \frac{h}{12}\{f'(x_0 + nh) - f'(x)\}$$

$$- \frac{h^3}{720}\{f'''(x_0 + nh) - f'''(x_0)\} + ... \quad ...(4)$$

$$\Rightarrow \quad \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\{f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\}$$

$$- \frac{h^2}{12}\{f'(x_n) - f'(x_0)\} + \frac{h^4}{720}\{f''(x_n) - f''(x_0)\} + ...$$

$$\Rightarrow \quad \int_{x_0}^{x_n} ydx = \frac{h}{2}(y_0 + y_1 + 2y_2 + ... + 2y_{n-1} + y_n) - \frac{h^2}{12}(y'_n - y'_0) + \frac{h^4}{720}(y''''_n - y''''_0) + ...$$

which is known as Euler's Maclaurin's summation formula.

Example 11. *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *to five places of decimal, using Euler-Maclaurin' formula.*

Sol. Let $y = \frac{1}{1+x}$

Here, we have $x_0 = 0$, $n = 10$ and $h = 0.1$

Then we want to evaluate $\int_{x_0}^{x_1} ydx$

where $y' = \frac{1}{(1+x)^2}$ and $y''' = -\frac{6}{(1+x)^4}$

Using Euler-Maclaurin formula, we get

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + ... + y_n) - \frac{h^2}{12}(y'_n - y'_0) + \frac{h^4}{720}(y''''_n - y''''_0) + ...$$

$$= \frac{0.1}{2}\left[\frac{1}{1}+\frac{2}{1.01}+\frac{2}{1.02}+\frac{2}{1.03}+\frac{2}{1.04}+\frac{2}{1.05}+\frac{2}{1.06}+\frac{2}{1.07}+\frac{2}{1.08}+\frac{2}{1.09}+\frac{1}{2}\right]$$

$$-\frac{(0.1)^2}{1^2}\left[-\frac{1}{2^2}+\frac{1}{1^2}\right]+\frac{(0.1)^4}{720}\left[-\frac{6}{2^4}+\frac{6}{1^4}\right]$$

= 0.693773 – 0.000625+0.000001 = 0.693149. **Ans.**

Example 12. *Using Euler-Maclaurin's formula, obtain the value of* $\log_e 2$ *from* $\int_0^1 \frac{dx}{1+x}$.

Sol. Let $y = f(x) = \frac{1}{1+x}$

$$y = \frac{1}{1+x}, x_0 = 0, n = 10, h = 0.1, x_n = 1.$$

$$\Rightarrow \quad y' = -\frac{1}{(1+x^2)}, y'' = \frac{2}{(1+x)^3} \text{ and } y''' = -\frac{6}{(1+x)^4}$$

Then by Euler-Maclaurins's formula, we get

$$\int_0^1 y\,dx = \frac{1}{2}\left[\frac{1}{1+0}+\frac{2}{1+0.1}+\frac{2}{1+0.2}+\frac{2}{1+0.3}+\frac{2}{1+0.4}+\frac{2}{1+0.5}+\frac{2}{1+0.6}+\ldots+\frac{2}{1+1.0}\right]$$

$$-\frac{(0.1)^2}{12}\left[\frac{1}{(1+1)^2}\frac{-1}{(1+0)^2}\right]+\frac{(0.1)^4}{720}\left[-\frac{6}{(1+1)^4}+\frac{6}{(1+0)^4}\right]$$

= 0.693773 – 0.000625 + 0.0000010

= 0.631149. **Ans.**

Example 13. *Find the sum of the series using Euler-Maclaurin formula.*

$$\frac{1}{51^2}+\frac{1}{53^2}+\frac{1}{55^2}+\ldots+\frac{1}{99^2}.$$

Sol. Here, we have $y = \frac{1}{x^2}, x_0 = 51, n = 24, h = 2$

Then $y^1 = \frac{-2}{x^3}, y''' = -\frac{24}{x^5}$ and so on.

Using Euler-Maclaurin's formula, we get

$$\int_{51}^{99} \frac{1}{x^2}\,dx = \frac{h}{2}(y_0+2y_1+2y_2+\ldots+2y_{23}+y_{24})-\frac{h^2}{12}(y'_{24}-y'_0)+\frac{h^4}{720}(y'''_{24}-y'''_0)+\ldots$$

$$=\left[\frac{1}{51^2}+\frac{2}{53^2}+\frac{2}{55^2}+\ldots+\frac{2}{97^2}+\frac{1}{99^2}\right]-\frac{4}{12}\left[-\frac{2}{99^3}+\frac{2}{51^3}\right]+\frac{16}{720}\left[-\frac{24}{99^5}+\frac{24}{51^5}\right]+\ldots$$

which gives

$$\frac{1}{51^2}+\frac{2}{53^2}+\frac{2}{55^2}+\ldots+\frac{2}{97^2}+\frac{1}{99^2}=\int_{51}^{99}\frac{1}{x^2}\,dx+\frac{2}{3}\left[\frac{1}{51^3}-\frac{1}{99^3}\right]-\frac{8}{15}\left[\frac{1}{51^5}-\frac{1}{55^5}\right]+\ldots.$$

$$\Rightarrow 2\left[\frac{1}{51^2}+\frac{1}{53^2}+\frac{1}{55^2}+\ldots+\frac{1}{99^2}\right]=\int_{51}^{99}\frac{1}{x^2}\,dx+\left[\frac{1}{51^2}+\frac{1}{99^2}\right]+\frac{2}{3}\left[\frac{1}{51^3}-\frac{1}{99^3}\right]-\frac{8}{15}\left[\frac{1}{51^5}-\frac{1}{55^5}\right]+\ldots.$$

$$\Rightarrow \frac{1}{51^2}+\frac{1}{53^2}+\frac{1}{55^2}+...+\frac{1}{99^2}=\frac{1}{2}\left[-\frac{1}{x}\right]_{51}^{99}+\frac{1}{2}\left[\frac{1}{51^2}+\frac{1}{99^2}\right]+\frac{1}{3}\left[\frac{1}{51^3}-\frac{1}{99^3}\right]-\frac{4}{15}\left[\frac{1}{51^5}-\frac{1}{99^5}\right]+....$$

$$= 0.00475 + 0.00024 + 0.000002 + ...$$

$$= 0.00499.$$ **Ans.**

Example 14. *Use Euler-Maclaurin's formula to prove that*

$$\sum_{i}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sol. By Euler-Maclaurin's formula,

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}(y_0+2y_i+2y_2+......+2y_{n-1}+y_n)-\frac{h^2}{12}(y_n'-y_0')+\frac{h^4}{720}(y_n'''-y_0''')-\frac{h^6}{30240}(y_n^v-y_0^v)+......$$

$$\Rightarrow \frac{1}{2}y_0+y_1+y_2+......+y_{n-1}+\frac{1}{2}y_n$$

$$=\frac{1}{h}\int_{x_0}^{x_n} ydx+\frac{h}{12}(y_n'-y_n')-\frac{h^3}{720}(y_n'''-y_0''')+\frac{h^5}{30240}(y_n^v-y_0^v)-...... \quad ...(1)$$

Here, $y(x)=x^2, y'(x)=2x$ and $h=1$

$\therefore$ From (1),

$$\text{Sum} = \int_1^n x^2dx+\frac{1}{2}(n^2+1)+\frac{1}{12}(2n-2) \qquad \left(\because \frac{1}{2}y_0=\frac{1}{2}, \frac{1}{2}y_n=\frac{n^2}{2}\right)$$

$$=\frac{1}{3}(n^3-1)+\frac{1}{2}(n^2+1)+\frac{1}{6}(n-1)=\frac{n(n+1)(2n+1)}{6}.$$ **(Proved)**

PROBLEM SET 6.2

1. Use Trapezoidal rule to evaluate $\int_0^1 x^3dx$ consisting five sub-intervals. [**Ans.** 0.26]
2. Calculate an approximate value of integral $\int_0^{\pi/2} \sin x\, dx$, by using Trapezoidal rule. [**Ans.** 0.99795]
3. Evaluate the integral $\int_0^4 e^x dx$ by Simpson's one-third rule. [**Ans.** 53.87]
4. Using Simpson's $\frac{3}{8}$ rule, evalute $\int_0^1 \frac{1}{1+x}dx$ with $h=\frac{1}{6}$. [**Ans.** 0.69319]
5. Use Boole's rule to compute $\int_0^{\pi/2} \sqrt{\sin x\, dx}$. [**Ans.** 1.18062]
6. Using Weddle's rule to evaluate $\int_0^5 \frac{dx}{4x+5}$. [**Ans.** 0.4023]
7. Evaluate the integral $\int_0^{\pi/2} \sqrt{\cos\theta\, d\theta}$ by dividing the interval into six parts. [**Ans.** 1.1873]

8. Evaluate using Trapezoidal rule

(*i*) $\int_0^{\pi} t \sin t \, dt$ (*ii*) $\int_{-2}^{2} \frac{t dt}{5+2t}$ **[Ans.** (*i*) 3.14, (*ii*) –0.747]

9. Use Simpson's rule dividing the range into ten equal parts to show that

$\int_0^1 \frac{\log(1+x^2)}{1+x^2} dx = 0.173.$

10. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using simpson's one-third rule, find the velocity of the rocket at t = 80 seconds.

t (sec)	0	10	20	30	40	50	60	70	80
$f\left(\text{cm/sec}^2\right)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

[Ans. 30.87 m/sec.]

11. Find by Weddle's rule the value of the integral I = $\int_{0.4}^{1.6} \frac{x}{\sin hx} dx$ by taking 12 sub-intervals.

[Ans. 1.0101996]

12. Evaluate $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x) dx$ approximately using Weddle's rule correct to four decimals.

[Ans. 4.051]

13. Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$ using Weddle's rule. **[Ans.** 0.52359895]

14. Using $\frac{3}{8}$ th Simpson's rule. Evaluate $\int_0^6 \frac{dx}{1+x^4}$. **[Ans.** 1.019286497]

15. Evaluate $\int_0^1 \frac{x^2+2}{x^2+1} dx$, using Weddle's rule correct to four places of decimals. **[Ans.** 1.7854]

16. Evaluate $\int_0^1 \sqrt{\sin x + \cos x \, dx}$ correct to two decimal places using seven ordinates. **[Ans.** 1.14]

17. Evaluate $\int_0^{\pi/2} \sin x \, dx$ using the Euler-Maclaurin's formula. **[Ans.** 1.000003]

18. Prove that $\sum_1^n x^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ applying Euler-Maclaurin's formula.

19. Find the sum of the fourth powers of the first n natural numbers by means of the Euler-Maclaurin's formula. $\left[\textbf{Ans. } \frac{n^2}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right]$

20. Using Euler-Maclaurin's formula, sum the following series.

(*i*) $\frac{1}{400} + \frac{1}{402} + ... + \frac{1}{498} + \frac{1}{505}$

(*ii*) $\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \frac{1}{104}$ **[Ans.** (*i*) 0.11382114 (*ii*) 0.0490291]

❑❑❑

CHAPTER 7

Numerical Solution of Ordinary Differential Equation

7.1 INTRODUCTION

In the fields of Engineering and Science, we come across physical and natural phenomena which, when represented by mathematical models, happen to be differential equations. For example, simple harmonic motion, equation of motion, deflection of a beam etc., are represented by differential equations. Hence, the solution of differential equation is a necessity in such studies. There are number of differential equations which we studied in Calculus to get closed form solutions. But, all differential equations do not possess closed form of finite form solutions. Even if they possess closed form solutions, we do not know the method of getting it. In such situations, depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the numerical solutions of the differential equations have become easy for manipulations. Hence, we present below some of the methods of numerical solutions of the ordinary differential equations. No doubt, such numerical solutions are approximate solutions. But, in many cases approximate solutions to the required accuracy are quite sufficient.

7.2 TAYLOR'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{...(1)}$$

with the initial condition $y(x_0) = y_0$

If $y(x)$ is the exact solution of (1) then $y(x)$ can be expanded into a Taylor's series about the point $x = x_0$ as

$$y(x) = y_0 + (x - y_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + ... \qquad \text{...(2)}$$

where dashed denote diferentiation w.r.t. 'x'

Differentiating (1) successively w.r.t x, we get

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right) f \qquad \text{...(3)}$$

$$\therefore \qquad y''' = \frac{\partial}{\partial x}(y'') = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + f^2\frac{\partial^2 f}{\partial y^2} \qquad ...(4)$$

and so on.

Putting $x = x_0$ and $y = y_0$ in the expressions for y', y'', y'''.......... and substituting them in equation (2), we can obtain the solution of (1).

Note: Taylor's series method has advantages that it is derived in any order and values of $y(x)$ are easily obtained. However, the method suffers from time consumed in computing higher derivatives.

Example 1. *Solve the differential equation*

$$\frac{dy}{dx} = x + y \text{ with } y(0) = 1, \; x \in [0,1]$$

by Taylor series expansion to obtain y for x = 0.1.

Sol. Here, $x_0 = 0, \; y_0 = 1$

$y' = (x+y), \qquad y'_0 = 0+1 = 1$

$y'' = (1+y'), \qquad y''_0 = 1+1 = 2$

$y''' = (0+y''), \qquad y'''_0 = 0+2 = 2$

Using Taylor series expansions about $x_0 = 0$ is given by

$$y(x) = y_0 + (x-0)y'_0 + \frac{(x-0)^2}{2!}y''_0 + \frac{(x-0)^3}{3!}y'''_0 + ...$$

at $x = 0.1$

$$y(0.1) = 1 + 0.1 + \frac{2(0.1)^2}{2!} + \frac{2(0.1)^3}{3!} + \frac{2(0.1)^4}{4!} + ...$$

$= 1 + 0.1 + 0.1 + 0.000333 + 0.0000083$

$= 1.11033.$ **Ans.**

Example 2. *Using Taylor series for y(x), find y(0.1) correct to four decimal places if y(x) satisfies* $y' = x + (-y^2)$, $y(0) = 1$ *where x(0) = 0.*

Sol. Here, $x_0 = 0$, $y_0 = 1$

$y' = x - y^2, \qquad y'_0 = 0 - 1 = -1$

$y'' = 1 - 2yy', \qquad y''_0 = 1 - 2\times1\times(-1) = 3$

$y''' = -2yy'' - 2y'^2 \qquad y'''_0 = -8$

$y'''' = -2yy''' - 6y'y'', \qquad y''''_0 = 34$

$y''''' = -2yy'''' - 8y'y''' - 6y''^2, \qquad y'''''_0 = -186$

The Taylor sereis expansion about $x_0 = 0$ is given by

$$y(x) = y_0 + (x - y)y_0' + \frac{(x-0)^2}{2!}y_0'' + \frac{(x-0)^3}{3!}y_0''' + \frac{(x-0)^4}{4!}y_0'''' + \frac{(x-0)^5}{5!}y_0''''' + \ldots\ldots.$$

at $x = 0.1$,

$$y(0.1) = 1 + 0.1(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \frac{(0.1)^4}{4!}(34) + \frac{(0.1)^5}{5!}(-186) + \ldots\ldots$$

$= 0.91379$

$= 0.9138.$ **Ans.**

Example 3. *Using Taylor's series, find the solution of the differential equation* $xy' = x - y$, *y(2) = 2 at x = 2.1 correct to five decimal places.*

Sol. Here, $x_0 = 2$, $y_0 = 2$.

Also, $y' = 1 - \frac{y}{x}$, $\quad y_0' = 0$

$$y'' = -\frac{y'}{x} + \frac{y}{x^2}, \qquad y_0'' = -0 + \frac{2}{4} = \frac{1}{2}$$

$$y''' = -\frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3}, \qquad y_0''' = -\frac{3}{4}$$

$$y'''' = -\frac{y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4}, \qquad y_0'''' = \frac{3}{4}$$

Using Taylor series expansion, we obtain

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0'''' + \ldots\ldots.$$

at $x = 2.1$,

$$y(2.1) = 2 + (2.1-2)(0) + \frac{(2.1-2)^2}{2!} \times \frac{1}{2} + \frac{(2.1-2)^3}{3!}\left(-\frac{3}{4}\right) + \frac{(2.1-2)^4}{4!}\left(\frac{3}{2}\right)$$

$= 2.00238.$ (correct to five decimal places) **Ans.**

Example 4. *Using Taylor's series Expansion tabulate the solution* $x = 4$ *to* $x = 4.4$ *in steps of 0.1 of differential equation.*

$$5xy' + y'' - 2 = 0$$

with $\qquad y(4) = 1$

Sol. Differentiating successively the differential equation, we obtain

$$5xy'' + 5y' + 2yy' = 0$$

$$5xy''' + 10y'' + 2yy'' + 2y'^2 = 0$$

$$5xy'''' + 15y''' + 2yy''' + 6y'y'' = 0$$

$$5xy''''' + 20y'''' + 2yy'''' + 8y'y''' + 6y''^2 = 0$$

The values of various derivatives at $x_0 = 4, y = 1$ are

$y_0' = 0.05$, $\quad y_0'' = -0.0175$, $\quad y_0''' = .01025$, $\quad y_0'''' = -.00845$, $\quad y_0''''' = .008998125$.

Then by Taylor series, we obtain

$$y(x)=1+0.05(x-4)-.00875(x-4)^2+.0017083(x-4)^3+(-.0003521)(x-4)^4$$
$$+.00007498(x-4)^5+\ldots\ldots$$

Tabulating from $x=4$ to $x=4.4$ we obtain

$y(4.3)=1.014256$ and $y(4.4)=1.018701$. **Ans.**

Example 5. *Solve the equation* $y'=2xy+1$, *given that* $y=0$, *at* $x=0$, *by the use of Taylor series, taking* $h=0.2$ *and going as far as* $x=4$.

Sol. The first few derivatives and their values at $x_0=0$, $y_0=0$, are

$$y'=2xy+1, \qquad y_0'=1$$
$$y''=2(xy'+y), \qquad y_0''=0$$
$$y'''=2(xy''+y'+y'), \qquad y_0'''=4$$
$$y''''=2(xy'''+y''+y''+y''), \qquad y_0''''=0$$
$$y'''''=2(xy''''+y'''+y'''+y'''+y'''), \qquad y_0'''''=32$$

Now by Taylor's series, we have

$$y(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\frac{(x-x_0)^4}{4!}y_0''''+\frac{(x-x_0)^5}{5!}y_0'''''+\ldots.$$

Substituting the values of y_0 and its derivatives, we obtain

$$y(.2)=0+(.2)(1)+(.2)^2(0)+\frac{(.2)^3}{3!}(4)+\frac{(.2)^4}{4!}(0)+\frac{(.2)^5}{5!}(32)$$
$$= .2 + .00533 + .00002133$$
$$= .20535466$$
$$= .21$$

Now, with $x_1=.2$, $y_1=.21$, we compute $y(.4)$. So, we have

$$y_1'=2(.2)(.21)+1=1.084$$
$$y_1''=2\{.2(1.084)+.21\}=.8536$$
$$y_1'''=2\{.2(.8536)+2\times1.084\}=4.67744$$
$$y_1''''=2\{.2(4.67744)+3\times.8536\}=6.992576$$
$$y_1'''''=2\{.2(6.992576)+4\times4.67744\}=40.21655$$

Substituting the values of y_1 and its derivativies in Taylor series expansion, we obtain

$$y(x)=y_1+(x_1-x_0)y_1'+\frac{(x_1-x_0)^2}{2!}y_1''+\frac{(x_1-x_0)^3}{3!}y_1'''+\ldots.$$

$$y(.4) = .21 + .2(1.084) + \frac{(.2)^2}{2}(.8536) + \frac{(.2)^3}{6}(4.67744) + \frac{(.2)^4}{24}(6.992576) + \frac{(.2)^5}{120}(40.21655)$$

$= .45068 = .451.$ **Ans.**

7.3 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Consider first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{...(1)}$$

with the initial condition

$$y = y_0 \text{ at } x = x_0$$

Integrating (1) with respect to x between x_0 and x, we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$$

or

$$y = y_0 \int_{x_0}^{x} (x, y)\,dx \quad \text{...(2)}$$

Now, we solve (2) by the method of successive approximation to find out the solution of (1). The first approximate solution (approximation) y_1 of y is given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similarly, the second approximation y_2 is given by

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

for the nth aproximation y_n is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{...(3)}$$

with $y(x_0) = y_0$.

Hence, this method gives a sequence of approximation $y_1, y_2 \ldots\ldots\ldots\ldots\ldots\ldots y_n$ and it can be proved $f(x, y)$ is bounded in some regions containing the point (x_0, y_0) and if $f(x, y)$ satisfies the Lipchitz condition, namely

$\left|f(x, y) - f(x, \bar{y})\right| \le k\left|y - \bar{y},\right|$ where k is a constant. Then the sequence

$y_1, y_2 \ldots\ldots\ldots\ldots\ldots\ldots$ converges to the sol. (2).

Example 6. *Use Picard's method to obtain* y *for x = 0.2. Given*

$$\frac{dy}{dx} = x - y \text{ with initial condition } y = 1 \text{ when } x = 0.$$

Sol. Here, $f(x,y)=x-y, \quad x_0=0, \quad y_0=1$

We have first approximation,

$$y_1=y_0+\int_0^x f(x,y_0)\,dx=1+\int_0^x (x-1)\,dx$$

$$= 1 - x + \frac{x^2}{2}$$

Second approximation,

$$y_2=y_0+\int_0^x f(x,y_1)\,dx=1+\int_0^x (x-y_1)\,dx$$

$$=1+\int_0^x \left(x-1+x-\frac{x^2}{2}\right)dx$$

$$=1-x+x^2-\frac{x^3}{6}$$

Third approximation,

$$y_3 = y_0+\int_0^x f(x,y_2)\,dx = 1+\int_0^x (x-y_2)\,dx$$

$$= 1+\int_0^x \left(x-1+x-x^2+\frac{x^3}{6}\right)dx$$

$$= 1-x+x^2-\frac{x^3}{6}+\frac{x^4}{24}$$

Fourth approximation,

$$y_4 = y_0+\int_0^x f(x,y_3)\,dx = 1+\int_0^x (x-y_3)\,dx$$

$$= 1+\int_0^x \left(x-1+x-x^2+\frac{x^3}{3}-\frac{x^4}{24}\right)dx$$

$$= 1-x+x^2-\frac{x^3}{3}+\frac{x^4}{12}-\frac{x^5}{120}$$

Fifth Approximation,

$$y_5 = y_0+\int_0^x f(x,y_4)\,dx = 1+\int_0^x (x-y_4)\,dx$$

$$= 1+\int_0^x \left(x-1+x-x^2+\frac{x^3}{3}-\frac{x^4}{12}+\frac{x^5}{120}\right)dx$$

$$= 1-x+x^2-\frac{x^3}{3}+\frac{x^4}{12}-\frac{x^5}{60}+\frac{x^6}{720}$$

When $x=0.2$, we get

$y_1=.82, \quad y_2=.83867, \quad y_3=.83740$

$y_4=.83746, \quad y_5=.83746$

Thus, $y=.837$ when $x=.2$. **Ans.**

Example 7. *Find the solution of* $\frac{dy}{dx} = 1 + xy, y(0) = 1$ *which passess through (0, 1) in the interval (0, 0.5) such that the value of* y *is correct to three decimal places (use the whole interval as one interval only) Take h = 0.1.*

Sol. The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = 1 + xy; \; y(0) = 1$$

i.e., $y = y_0 = 1$ at $x = x_0 = 0$

Here, $y_1 = 1 + x + \frac{x^2}{2}$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y_4 = y_3 + \frac{x^7}{105} + \frac{x^8}{384}$$

When $x = 0, y = 1.000$

$x = 0.1, y_1 = 1.105, y_2 = 1.1053.......$

$\therefore$ $y = 1.105$

$x = 0.2, y_1 + 1.220, y_2 = 1.223 = y_3$ (correct up to 3 decimals)

$\therefore$ $y = 1.223$

$x = 0.3, \; y = 1.355$ as $y_2 = 1.355 = y_3$ (correct up to 3 decimals)

$x = 0.4, \; y = 1.505$

$x = 0.5, \; y = 1.677$ as $y_4 = y_3 = 1.677$

Thus,

x	0	0.1	0.2	0.3	0.4	0.5
y	1.000	1.105	1.223	1.355	1.505	1.677

We have numerically solved the given differential equation for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5.

Example 8. *Using Picard's method of successive approximation to with y(0) = 0, obtain y(.25), y(.3) and y(1) correct to 3 decimal places.*

Sol. Here, $f(x, y) = \frac{x^2}{y^2 + 1}$, $x_0 = 0$ and $y_0 = 0$.

The first approximation y_1 of y is given by

$$y_1 = 0 + \int_0^x \frac{x^2}{0 + 1} dx = \frac{x^3}{3}$$

Similarly, the second approximation y_2 of y is given by

$$y_2 = 0 + \int_0^x \frac{x^2}{1+\left(x^3/3\right)^2}\, dx = \tan^{-1}\frac{x^3}{3}$$

$$= \frac{x^3}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right)^3 + ... \approx \frac{x^3}{3}.$$

We see that y_1 and y_2 agree to first term, namely $\frac{x^3}{3}$. Neglecting $\frac{x^9}{81}$, we obtain the range in which the result is correct to 3 decimal places, *i.e.,* we put

$$\frac{1}{81}x^9 \leq .0005$$

which yield $x \leq .7$.

Hence, we obtain $y(.25) = \frac{1}{3}(.25)^3 = .005$

$$y(.5) = \frac{1}{3}(.5)^3 = .042$$

$$y(1) = \frac{1}{3}(1)^3 - \frac{1}{3}\left(\frac{1}{3}\right)^3 = .321. \quad \textbf{Ans.}$$

Example 9. *Use Picard's method to obtain* y *for x = 0.1. Given that* $\frac{dy}{dx} = 3x + y^2; y = 1$ *at x = 0.*

Sol. Here $f(x, y) = 3x + y^2;\ x_0 = 0, y_0 = 1$

First approximation, $y_1 = y_0 + \int_0^x f(x, y_0)\, dx$

$$= 1 + \int_0^x (3x+1)\, dx$$

$$= 1 + x + \frac{3}{2}x^2$$

Second approximation, $y_2 = 1 + x + \frac{5}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5$

Third approximation, $y_3 = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{1157}{1260}x^7$

$$+ \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

When, $x = 0.1$, we have

$y_1 = 1.115$, $y_2 = 1.1264$, $y_3 = 1.12721$

Thus, $y = 1.127$ when $x = 0.1$. **Ans.**

Example 10. *Obtain y when x = 0.1, x = 0.2 Given that* $\frac{dy}{dx}=x+y$; *y(0) = 1, Check the result with exact value.*

Sol. We have, $\frac{dy}{dx}=f(x,y)=x+y; x_0=0$ and $y_0=1$

Now First approximation,

$$y_1=1+\int_0^x (1+x)\,dx=1+x+\frac{x^2}{2}$$

Second approximation,

$$y_2=1+\int_0^x \left(x+1+x+\frac{x^2}{2}\right)dx=1+x+x^2+\frac{x^3}{6}$$

Third approximation,

$$y_3=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{24}$$

When $x=.1,\ y_1=1.105$

$y_2=1.11016$

$y_3=1.11033$ (closer appr.)

When $x=.2$

$y_3=1.2427$

We can continue further to get the better approximations. Now we shall obtain exact value.

$\frac{dy}{dx}-y=x$ is the given differential equation. General sol. is

$$ye^{-x}=-e^{-x}(1+x)+c \qquad (IF=e^{-x})$$

Putting $y=1, x=0$ we obtain, $c=2$.

$\therefore$ $y=-x-1+2e^x$

When $x=0.1, y=1.11034$

and $x=0.2,\ y=1.24281$

These results reveal that the approximations obtained for $x=0.1$ is correct to four decimal places while that for $x=0.2$ is correct to 3 decimal places.

Example 11. *If* $\frac{dy}{dx}=\frac{y-x}{y+x}$. *Find the value of* y *at x = 0.1 using Picard's method. Given that y(0) = 1.*

Sol. First approximation,

$$y_1=y_0+\int_0^x \frac{y_0-x}{y_0+x}\,dx=1+\int_0^x \left(\frac{1-x}{1+x}\right)dx$$

$$=1+\int_0^x \left(\frac{2}{1+x}-1\right)dx$$

$$=1-x+2\log(1+x)$$

Second approximation,

$$y_2 = 1 + x - 2\int_0^x \frac{xdx}{1 = 2\log(1+x)}$$

which is difficult to integrate.

Thus, when, $x = 0.1, y_1 = 1 - 0.1 + 2\log(1.1) = 0.9828$. **Ans.**

Here in this example, only I approximation can be obtained and so it gives that approximate value of y for $x = 0.1$

Example 12. *Find the series expansion that gives* y *as a function of* x *in the neighbourhood of* $x = 0$*, when* $\frac{dy}{dx} = x^2 + y^2$*, with* $y(0) = 0$.

Sol. Here, $f(x, y) = x^2 + y^2, x_0 = 0$ and $y_0 = 0$. The nth approximation y_n of y is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

As an initial approximation, it is given that $y_0 = 0$. Then, the first approximation y_1 is given by

$$y_1 = 0 + \int_0^x (x^2 + 0)dx = \frac{x^3}{3}\cdot$$

Similarly, the second approximation y_2 is given by

$$y_2 = 0 + \int_0^x \left(x^2 + \left(\frac{x^3}{3}\right)^2\right)dx = \frac{x^3}{3} + \frac{x^7}{63}\cdot$$

Likewise, the higher order approximations are given as

$$y_3 = 0 + \int_0^x \left(x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63}\right)^2\right)dx = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535} + \ldots\ldots$$

$$y_4 = 0 + \int_0^x \left(x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}\right)^2\right)dx$$

$$= \frac{1}{3}x^3 + \frac{1}{63}x^7 + \frac{2}{2079}x^{11} + \frac{13}{218295}x^{15} + \ldots$$

If the series is truncated after the third term and used to approximate y to 4 decimal places, then using the first neglected term, namely $\frac{13}{218295}x^{15}$ as an approximation of the error, we have.

$$\frac{13}{218295}x^{15} \le .00005.$$

Taking logarithm, we obtain

$$15 \log x \le \log \frac{(.00005)(218295)}{13}$$

or $\qquad x \le .988.$

Thus, $\frac{x^3}{3}+\frac{x^7}{63}+\frac{2}{2079}x^{11}$ represents y correct to 4 decimal places. In the range $|x| \le .988$. *i.e.* $-0.988 \le x \le 0.988$.

Example 13. *Integrate the differential equation* $\frac{dy}{dx} = x \sin \pi y$ *with* $y = \frac{1}{2}$ *at* $x = 0$, *by Picard's method of successive approximations.*

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx. \quad ...(1)$$

Sol. The first approximation y_1 of y is obtained by substituting $y = \frac{1}{2}$ In the right hand member of (1) *i.e.,* we have

$$y_1 = \frac{1}{2} + \int_0^x x \sin\left(\pi.\frac{1}{2}\right)dx = \frac{1}{2}+\frac{x^2}{2}.$$

Similarly, the second and third approximation y_2 and y_3 are given as

$$y^2 = \frac{1}{2} + \int_0^x x \sin\left\{\pi.\frac{(1+x^2)}{2}\right\}dx = \frac{1}{2} + \int_0^x x\cos\frac{\pi x^2}{2}dx$$

$$= \frac{1}{2} + \int_0^x x\left(1 - \frac{\pi^2 x^4}{8} + \ldots\right)dx$$

$$= \frac{1}{2} + \frac{x^2}{2} - \frac{\pi^2 x^6}{48} + \ldots$$

and

$$y_3 = \frac{1}{2} + \int_0^x x\sin\pi\left\{\frac{1}{2}+\frac{x^2}{2}-\frac{\pi^2x^6}{48}+\frac{1}{2}+\int_0^x x\cos\pi\left(\frac{x^2}{2}-\frac{\pi^2x^6}{8}\right)dx\right\}$$

$$= \frac{1}{2} + \int_0^x x\left\{1-\frac{1}{2}\left(\frac{\pi x^2}{2}-\frac{\pi^2x^6}{48}\right)^2+\ldots\right\}dx$$

$$= \frac{1}{2} + \frac{x^2}{2} - \frac{\pi^2 x^6}{48} + \ldots$$

We observe that y_2 agree with y_3 upto and including term in x^6.

We can use the relation $y = \frac{1}{2}+\frac{x^2}{2}$ with the knowledge that the error is approximately $\frac{-x^6}{5}$. Thus, we can find y_1 and y_2 correct to 4 decimal places with $h = 0.1$.

7.4 EULER'S METHOD

The oldest and simplest method was derived by Euler. In this method, we determine the change Δy is y corresponding to small increment in the argument x. Consider the differential equation.

$$\frac{dy}{dx}=f(x,y) \quad ...(1)$$

with the initial condition $y(x_0)=y_0$.

Integrating (1) w.r.t. x between x_0 and x_1, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx \qquad \text{...(2)}$$

Now, replacing $f(x, y)$ by the approximation $f(x_0, y_0)$, we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0, y_0) dx$$

$$= y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + hf(x_0, y_0) \qquad (\because x_1 - x_0 = \Delta x = h)$$

This is the formula for first approximation y_1 of y.

Similarly, second approximation y_2 is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

In general,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

7.5 EULER'S MODIFIED METHOD

Instead of approximating $f(x, y)$ by $f(x_0, y_0)$ in equation (2). Let the integral is approximated by Trapezoidal rule to botain.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1 y_1)]$$

We obtain the iteration formula,

$$y_1^{n+1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right] \qquad n = 0, 1, 2......$$

where, $y_1^{(n)}$ is the nth approximation to y_1.

The above iteration formula can be started by $y_1^{(1)}$ from Euler's method.

$$y_1^{(0)} = y_0 + h(x_0, y_0)$$

Example 14. *Using Euler's method, compute y(0.5) for differential equation*

$$\frac{dy}{dx} = y^2 - x^2, \text{ with } y = 1 \text{ when } x = 0$$

Sol. Let $\qquad h = \frac{0.5}{5} = 0.1$

$$x_0 = 0, y_0 = 1, f(x, y) = y^2 - x^2$$

Using Euler's method we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

But considering $n=0,1,2,........$in succession, we get

$$y_1=y_0+hf(x_0,y_0)$$
$$=1+0.1(1^2-0)=1.10000$$
$$y_2=y_1+hf(x_1,y_1)$$
$$=1.10000+0.1\left[(1.10000)^2-(0.1)^2\right]=1.22000$$
$$y_3=y_2+hf(x_2,y_2)$$
$$=1.22000+0.1\left[(1.22)^2-(0.2)^2\right]=1.36484$$
$$y_4=y_3+hf(x_3,y_3)$$
$$=1.36484+0.1[(1.36484)^2\_(0.3)^2]=1.54212$$
$$y_5=y_4+hf(x_4,y_4)$$
$$=1.54212+0.1\left[(1.54212)^2-(0.4)^2\right]=1.76393$$

Hence, the value of y at $x=0.5$ is 1.76393. **Ans.**

Example 15. *Using Euler's method, compute y(0.04) for the differential equation.*

$y^1 = -y$ *with* $y(0) = 1$ *(Take* $h = 0.01$*)*

Sol. Using Euler's method

$$y_{n+1}=y_n+hf(x_n,y_n)$$

By considering $n=0,1,2,.........$ in succession, we obtain

$$y_1=y_0+hf(x_0,y_0)$$
$$=1+0.01(-1)=0.99$$
$$y_2=y_1+hf(x_1,y_1)$$
$$=0.99+0.01(-0.99)=0.9801$$
$$y_3=0.9801+0.01(-0.9801)=0.970299$$
$$y_4=0.970299+0.01(-0970299)=0960596$$

Hence, the value of $y(0.04)$ is 0.960596. **Ans.**

Example 16. *Find the solution of differential equation*

$$\frac{dy}{dx}=xy \text{ with } y(1)=5$$

in the interval 1,1.5] using h = 0.1.

Sol. As per given we have

$$x_1=1, \quad y_0=5, \quad f(xy)=xy$$

Using Euler's method

$$y_{n+1}=y_n+hf(x_n,y_n)$$

Now, by considering $n=0,1,2......$ in succession, we get

For $n=0$ $\quad y_1=y_0+0.1f(x_0,y_0)$

$$=5+0.1(1\times5)=5.5$$

For $n=1$ $\quad y_2=y_1+0.1f(x_1,y_1)$

$$=5.5+0.1(1.1\times5.5)=6.105$$

For $n=2$ $\quad y_3=y_2+0.1f(x_2,y_2)$

$$=6.105+0.1(1.2\times6.105)=6.838$$

For $n=3$ $\quad y_4=y_3+0.1f(x_3,y_3)$

$$=6.838+0.1(1.3\times6.838)=7.727$$

For $n=4$ $\quad y_5=y_4+0.1\,f(x_4,y_4)$

$$=7.727+0.1(1.4\times7.727)=8.809$$

Hence, the value of $y(1.5)$ is 8.809. **Ans.**

Example 17. *Given* $y'=\dfrac{y-x}{y+x}$ *with* y_0 *= 1 find* y *for x = 0.1 in four steps by Euler's method.*

Sol. Let $\quad h=\dfrac{0.1}{4}=0.025$, given $y_0=1$, where $x=0$

We know that

$$y_{n+1}=y_n+hf(x_n,y_n)$$

By putting $n=0,1,2,3,$ we obtain

$$y_1=y_0+hf(x_0,y_0)$$

$$=1+0.025\frac{(1-0)}{(1+0)}=1.025$$

$\Rightarrow$ $\quad y_1=1.025$

Again, $\quad y_2=y_1+hf(x_1,y_1)$

$$=1.025+0.025\frac{(1.025-0.025)}{(1.025+0.025)}$$ (where $x_1=x_0+h=0+0.025\Rightarrow x_1=0.025$)

$$=1.0488$$

$\Rightarrow$ $\quad y_2=1.0488$

Now again $\quad y_3=y_2+hf(x_2,y_2)$ (where $x_2=x_0+2h=0+2\times0.025=0.05$)

$$=1.0488+0.025\frac{(1.0488-0.05)}{(1.0488+0.05)}=1.07152$$

$\Rightarrow$ $\quad y_3=1.07152$

$$y_4=y_3+hf(x_3,y_3)$$ (where $x_3=x_0+3h=0+3\times0.025=0.075$)

$$=1.07152+0.025\frac{(1.07152-0.075)}{(1.07152+0.075)}=1.09324$$

$\Rightarrow$ $y_4=1.09324$ at $(x_4=x_0+4h=0+4\times 0.025=0.1)$

Hence, $y_{(0.1)}=1.0932$. **Ans.**

Example 18. *Given* $\frac{dy}{dx}=x+y$ *with initial condition y(0) = 1. Find y(.05) and y(.1) correct to 6 decimal places.*

Sol. Using Euler's method, we obtain

$$y_1^{(0)}=y_1=y_0+hf(x_0,y_0) = 1+.05(0+1)=1.05$$

We improve y_1 by using Euler's modified method

$$y_1^{(1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^0\right)\right]$$

$$=1+\frac{.05}{2}\left[(0+1)+(.05+1.05)\right]$$

$$=1.0525.$$

$$y_1^{(2)}=1+\frac{.05}{2}\left[(0+1)+(.05+1.0525)\right]$$

$$=1.0525625$$

$$y_1^{(3)}=1+\frac{.05}{2}\left[(0+1)+(.05+1.0525625)\right]$$

$$=1.052564$$

$$y_1^{(4)}=1+\frac{.05}{2}\left[(0+1)+(.05+1.052564)\right]$$

$$=1.0525641.$$

Since, $y_1^{(3)}=y_1^{(4)}=1.052564$ correct to 6 decimal places. Hence we take $y_1=1.052564$ i.e., we have $y(.05)=1.052564$

Again, using Euler's method, we obtain

$$y_2^{(0)}=y_2=y_1+hf(x_1,y_1)$$

$$=1.052564+.05(1.052564+.05)$$

$$=1.1076922.$$

We improve y_2 by using Euler's modified method

$$y_2^{(1)}=1.052564+\frac{.05}{2}\left[(1.052564+.05)+(1.1076922+.1)\right]$$

$$=1.1120511.$$

$$y_2^{(2)}=1.052564+\frac{.05}{2}\left[(1.052564+.05)+(1.1120511+.1)\right]$$

$$=1.1104294.$$

$$y_2^{(3)} = 1.052564 + \frac{.05}{2}\left[(1.052564 + .05) + (1.1104294 + .1)\right]$$
$$= 1.1103888.$$
$$y_2^{(4)} = 1.052564 + \frac{.05}{2}\left[(1.052564 + .05) + (1.1103888 + .1)\right]$$
$$= 1.1103878.$$
$$y_2^{(5)} = 1.052564 + \frac{.05}{2}\left[(1.052564 + .05) + (1.1103878 + .1)\right]$$
$$= 1.1103878.$$

Since, $y_2^{(4)} = y_2^{(5)} = 1.1103878$, correct to 7 decimal places. Hence, we take $y_2 = 1.1103878$. Therefore, we have $y(.1) = 1.110388$, correct to 6 decimal places. **Ans.**

Example 19. *Find y(2.2) using Euler's method for*

$$\frac{dy}{dx} = -xy^2, \text{ where } y(2) = 1. \qquad (\text{Take } h = .1)$$

Sol. By Euler's method, we obtain,

$$y_1^{(0)} = y_1 = y_0 + hf(x_0, y_0) = 1 + .1(-2)(-1)^2 = .8.$$

This value of y_1 is improved by using Euler's modified method

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left\{-2(1)^2 + (-2.1)(.8)^2\right\}$$
$$= .8328$$

Similarly
$$y_1^{(2)} = 1 + \frac{0.1}{2}\left\{-2(1)^2 + (-2.1)(.8328)^2\right\}$$
$$= .8272$$
$$y_1^{(3)} = 1 + \frac{0.1}{2}\left\{-2(1)^2 + (-2.1)(.8272)^2\right\}$$
$$= .8281$$
$$y_1^{(4)} = 1 + \frac{0.1}{2}\left\{-2(1)^2 + (-2.1)(.8281)^2\right\}$$
$$= .8280$$
$$y_1^{(5)} = 1 + \frac{0.1}{2}\left\{-2(1)^2 + (-2.1)(.8280)^2\right\}$$
$$= .8280$$

Since $y_1^{(4)} = y_1^{(5)} = 0.8280$. Hence, we take $y_1 = .828$ at $x_1 = 2.1$

Now, if y_2 is the value of y at $x = 2.2$. Then, we apply Euler's method to compute $y(2.2)$, *i.e.*, we obtain

$$y_2^{(0)} = y_2 = y_1 + hf(x_1, y_1)$$
$$= .828 + .1(-2.1)(.828)^2 = .68402$$

Now, using Euler's modified formula, we obtain

$$y_2^{(1)} = .828 + \frac{0.1}{2}\left[(-2.1)(.828)^2 + (-2.2)(.68402)^2\right]$$
$$= .70454$$

$$y_2^{(2)} = .828 + \frac{0.1}{2}\left[(-2.1)(.828)^2 + (-2.2)(.70454)^2\right]$$
$$= .70141.$$

$$y_2^{(3)} = .828 + \frac{0.1}{2}\left[(-2.1)(.828)^2 + (-2.2)(.70141)^2\right]$$
$$= .70189.$$

$$y_2^{(4)} = .828 + \frac{0.1}{2}\left[(-2.1)(.828)^2 + (-2.2)(.70189)^2\right]$$
$$= .70182$$

$$y_2^{(5)} = .828 + \frac{0.1}{2}\left[(-2.1)(.828)^2 = (-2.2)(.70182)^2\right]$$
$$= .70183$$

Since, $y_2^{(4)} = y_2^{(5)} = .7018,$ correct to 4 decimal places.

Hence, we have $y(2.2) = .7018.$ **Ans.**

Example 20. *Find y(.2) and y(.5) Given*

$$\frac{dy}{dx} = log_{10}(x+y)$$

with initial condition y = 1 for x = 0.

Sol. Let $\frac{dy}{dx} = f(x, y) = \log_{10}(x+y)$ and $h = .2$

By Euler's formula, we have

$$y_1^{(0)} = y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + .2\log(0+1) = 1.$$

Now, we improve this value by using Euler's modified formula and thus we obtain

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$
$$= 1 + \frac{0.2}{2}\{\log(0+1) = \log(.2+1)\}$$
$$= 1.0079$$

$$y_1^{(2)} = 1 + \frac{0.2}{2}\{\log(0+1) + \log(.2+1.0079)\}$$
$$= 1.0082$$

$$y_1^{(3)} = 1 + \frac{0.2}{2}\{\log(0+1) + \log(.2+1.0082)\}$$
$$= 1.0082$$

Since, $y_1^{(2)} = y_1^{(3)} = 1.0082$. Hence, we take $y_1 = 1.0082$ at $x = .2$, i.e., $y(.2) = 1.0082$. **Ans.**
Again using Euler's formula, we obtain

$$y_2^{(0)} = y_2 = y_1 + hf(x_1 + y_1)$$
$$= 1.0082 + 0.3\left[\log(0.2 + 1.0082)\right] = 1.0328$$

To improve y_2, we use Euler's modified formula and we obtain

$$y_2^{(1)} = 1.0082 + \frac{0.3}{2}\left[\log(1.0328 + .5) + \log(.2 + 1.0082)\right]$$
$$= 1.0483$$
$$y_2^{(2)} = 1.0082 + \frac{0.3}{2}\left[\log(1.0483 + .5)\ \log(.2 + 1.0082)\right]$$
$$= 1.049$$
$$y_2^{(3)} = 1.0082 + \frac{0.3}{2}\left[\log(1.049 + .5) + \log(.2 + 1.0082)\right]$$
$$= 1.0490$$

Since $y_2^{(3)} = y_2^{(2)} = 1.0490$, we take $y_2 = 1.0490$, i.e., $y(.5) = 1.0490$. **Ans.**

Example 21. *Using Euler's modified method, compute y(0.1) correct to six decimal figures, where*

$$\frac{dy}{dx} = x^2 + y \quad \textit{with } y = .94 \textit{ when } x = 0.$$

Sol. By Euler's method, we have

$$y_1^{(0)} = y_1 = y_0 + hf(x_0, y_0)$$
$$= .94 + .1(0 = .94)$$
$$= 1.034$$

Now, we improve y_1 by using Euler's modified formula and we obtain

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1 y_1^{(0)}\right)\right]$$
$$= .94 + \frac{0.1}{2}\left[(0 + .94) + \left\{(1)^2 + 1.034\right\}\right]$$
$$= 1.0392$$
$$y_1^{(2)} = .94 + \frac{0.1}{2}\left[(0 + .94) + \left\{(.1)^2 + 1.0392\right\}\right]$$
$$= 1.03946$$
$$y_1^{(3)} = .94 + \frac{0.1}{2}\left[(0 + .94) + \left\{(.1)^2 + 1.03946\right\}\right]$$
$$= 1.039473$$
$$y_1^{(4)} = .94 + \frac{0.1}{2}\left[(0 + .94) + \left\{(.1)^2 + 1.039473\right\}\right]$$
$$= 1.0394737$$

Since, $y_1^{(3)} = y_1^{(4)} = 1.039473,$ correct to 6 decimal places

Hence, we have $y(.1) = 1.039473.$ **Ans.**

Example 22. *Using Euler's modified method, solve numerically the equation* $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$ *with y(0) = 1 for 0 ≤ x ≤ 0.6, in steps of 0.2.*

Sol. The interval $h = 0.2$

By Euler's method, we obtain $y_1^{(0)} = y_1 = y_0 + h(x_0, y_0)$

$$= 1 + 0.2\left(0 + \left|\sqrt{1}\right|\right) = 1.2.$$

The value of $y_1^{(0)}$, thus obtained is improved by modified method.

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$$

By considering $n = 0$, we obtain

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \left|\sqrt{1.2}\right|\right)\right]$$

$$= 1.2295$$

By considering $n = 1$, we obtain

$$y_1^{(2)} = 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \left|\sqrt{1.2295}\right|\right)\right]$$

$$= 1.2309.$$

By considering $n = 2$, we obtain

$$y_1^{(3)} = 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \left|\sqrt{1.2309}\right|\right)\right]$$

$$= 1.2309.$$

Since $y_1^{(2)} = y_1^{(3)} == 1.2309$. Hence, we take $y_1 = 1.2309$ at $x = 0.2$ and proceed to compute y at $x = 0.4$.

Again, applying Euler's method, we obtain

$$y_2^{(0)} = y_2 = 1.2309 + 0.2\left(0.2 + \left|\sqrt{1.2309}\right|\right) = 1.49279$$

Now, we apply modified method for more accurate approximations and we obtain

$$y_2^{(1)} = 1.2309 + \frac{.2}{2}\left[\left(2 + \left|\sqrt{1.2309}\right|\right) + \left(0.4 + \left|\sqrt{1.49279}\right|\right)\right]$$

$$= 1.52402$$

$$y_2^{(2)} = 1.2309 + \frac{.2}{2}\left[\left(2 + \left|\sqrt{1.2309}\right|\right) + \left(0.4 + \left|\sqrt{1.52402}\right|\right)\right]$$

$$= 1.525297$$

$$y_2^{(3)} = 1.2309 + \frac{.2}{2}\left[\left(2 + \left|\sqrt{1.2309}\right|\right) + \left(0.4 + \left|\sqrt{1.525297}\right|\right)\right]$$
$$= 1.52535$$
$$y_2^{(4)} = 1.2309 + \frac{.2}{2}\left[\left(.2\left|\sqrt{1.2309}\right|\right) + \left(0.4 + \left|\sqrt{1.52535}\right|\right)\right]$$
$$= 1.52535$$

Since, $y_2^{(3)} = y_2^{(4)} = 1.52535$. Hence, we take $y_2 = 1.52535$ at $x = 0.4$. To find the value of $y(= y_3)$ for $x = 0.6$, we apply Euler's method to have

$$y_3^{(0)} = y_3 = 1.52535 + .2\left[\left(4 + \left|\sqrt{1.52535}\right|\right)\right] = 1.85236$$

For better approximations, we use Euler's modified formula and we obtain

$$y_3^{(1)} = 1.52535 + \frac{.2}{2}\left[\left(.4 + \left|\sqrt{1.52535}\right|\right) + \left(0.6 + \left|\sqrt{1.85236}\right|\right)\right]$$
$$= 1.88496$$
$$y_3^{(2)} = 1.52535 + \frac{.2}{2}\left[\left(.4 + \left|\sqrt{1.52535}\right|\right) + \left(0.6 + \left|\sqrt{1.88496}\right|\right)\right]$$
$$= 1.88615$$
$$y_3^{(3)} = 1.52535 + \frac{.2}{2}\left[\left(.4 + \left|\sqrt{1.52535}\right|\right) + \left(0.6 + \left|\sqrt{1.88615}\right|\right)\right]$$
$$= 1.88619$$
$$y_3^{(4)} = 1.52535 + \frac{.2}{2}\left[\left(.4 + \left|\sqrt{1.52535}\right|\right) + \left(0.6 + \left|\sqrt{1.88619}\right|\right)\right]$$
$$= 1.88619, \text{ correct to 5 decimal places.}$$

Since, $y_3^{(3)} = y_3^{(4)} = 1.88619$. Hence, we take $y = 1.88619$ at $x = 0.6$. **Ans.**

PROBLEM SET 7.1

1. Solve by Taylor's method, $y' = x'' + y'', y(0) = 1$. Computer $y(0.1)$. [**Ans.** 1.11146]
2. Solve by Taylor's method, $\frac{dy}{dx} = y - \frac{2x}{y}; y(0) = 1$. Also compute $y(0.1)$. [**Ans.** 1.0954]
3. Given differential equation $\frac{dy}{dx} = \frac{1}{x^2 + y}$ with $y(4) = 4$. Obtain $y(4.1)$ and $y(4.2)$ by Taylor's series method. [**Ans.** 4.005. 4.0098]
4. Apply Picard's method to find the third approximation of the solution of $\frac{dy}{dx} = x + y^2$ with the condition $y(0) = 1$. [**Ans.** $= 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \ldots$]
5. Using Picard'*d* method, obtain the solution of $\frac{dy}{dx} = x\left(1 + x^3 y\right); y(0) = 3$ Compute the value of $y(0.1)$ and $y(0.2)$. [**Ans.** 3.005, 3.020]

6. Solve the following initial value problem by Picard method

$\frac{dy}{dx} = xe^y$ with $y(0) = 0$, compute $y(0.1)$. [**Ans.** 0.0050125]

7. Use Picard's method to approximate y when $x = 0.2$, given that $\frac{dy}{dx} = x - y$ with $y(0) = 1$. [**Ans.** 0.0837]

8. Use Picard's method to approximate the value of y when $x = 0.1$, given that $\frac{dy}{dx} = 3x + y^2$ with $y(0) = 1$. [**Ans.** 1.12721]

9. Solve by Euler's method $\frac{dy}{dx} - 2y = 0, y(0) = 1, h = 0.1$ and $x \in [0, 0.3]$. [**Ans.** $y(0.3)$ = 0512]

10. Apply Euler's method to find the approximate solution of $\frac{dy}{dx} = x + y, y(0) = 1, h = 0.1$ and $x \in [0, 1]$. [**Ans.** 3.1874]

11. Obtain by Euler's modified method for the numerical solution for $y(1)$ of $\frac{dy}{dx} = \frac{-y}{1+x}$ with $y(3) = 2$ and $h = 0.1$. [**Ans.** $y(1)$ = 0.94771]

12. Using Euler's modified method, solve $\frac{dy}{dx} = 1 - y$ with $y\ (0) = 0$ in the range $0 \le x \le 0.2$ (take $h = 0.1$). [**Ans.** $y(0.1)$ = 0.09524, $y(0.2)$ = 0.1814076]

7.6 RUNGE-KUTTA METHOD

The method is very simple. It is named after two german mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). These methods are well-known as Runge-Kutta Method. They are distinguished by their orders in the sense that they agree with Taylor's series solution upto terms of h^r where r is the order of the method.

It was developed to avoid the computation of higher order derivations which the Taylors' method may involve. In the place of these derivatives extra values of the given function $f(x, y)$ are used.

(*i*) First order Runge-Kutta method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \qquad \text{...(1)}$$

By Euler's method, we know that

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad \text{...(2)}$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + y_0' + \frac{h^2}{2!} y_0'' + \qquad \text{...(3)}$$

It follows that Euler's method agrees with Taylor's series solution upto the terms in h. Hence Euler's method is the first order Runge-Kutta method.

(*ii*) **Second order Runge-Kutta method**

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$$

Let h be the interval between equidistant values of x. Then the second order Runge-Kutta method, the first increment in y is computed from the formulae

$$\left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf(x_0 + h, y_0 + k_1) \\ \Delta y &= \frac{1}{2}(k_1 + k_2) \end{aligned}\right\}$$

Then, $$x_1 = x_0 + h$$

$$y_1 = y_0 + \Delta y = y_0 + \frac{1}{2}(k_1 + k_2)$$

Similarly, the increment in y for the second interval is computed by the formulae,

$$\left.\begin{aligned} k_1 &= hf(x_1, y_1) \\ k_2 &= hf(x_1 + h, y_1 + k_1) \\ \Delta y &= \frac{1}{2}(k_1 + k_2) \end{aligned}\right\}$$

and similarly for other intervals.

(*iii*) **Third order Runge-Kutta method**

This method agrees with Taylors' series solution upto the terms in h^3. The formula is as follows:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3); x_1 = x_0 + h$$

where, $$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

Similarly for other intervals.

(*iv*) **Fourth order Runge-Kutta method**

This method coincides with the Taylor's series solution upto terms of h^4.

Consider the differential equation

$\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x. Then the first increment in y is computed from the formulae.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Then $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $x_1 = x_0 + h$

Similarly, the increment in y for the second interval is computed by

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

Then, $$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $$x_2 = x_1 + h$$

and similarly for the next intervals.

Runge-Kutta Method for Simultaneous First Order Equations

Consider the simultaneous equations

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dy}{dx} = f_2(x, y, z)$$

With the initial condition $y(x_0) = y_0$ and $z(x_0) = z_0$. Now, starting from (x_0, y_0, z_0), increments k and l in y and z are given by the following formulae:

$$k_1 = hf_1(x_0, y_0, z_0); \qquad l_1 = hf_2(x_0, y_0, z_0);$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right); \qquad l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right); \qquad l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3); \qquad l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence, $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); \qquad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2, z_2, we simply replace (x_0, y_0, z_0) by (x_1, y_1, z_1) in the above formulae.

Example 1. *Apply Runge-Kutta Method to solve.*

$$\frac{dy}{dx} = xy^{1/3}, y(1) = 1 \text{ to } \textit{obtain } y(1.1).$$

Sol. Here, $x_0 = 1, y_0 = 1$ and $h = 0.1$. Then, we can find

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(1)(1)^{1/3} = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right)^{1/3} = 0.10672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.10672}{2}\right)^{1/3} = 0.10684$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1(1 + 0.1)(1 + 0.10684)^{1/3} = 0.11378 \quad = 0.11378$$

$$\therefore \qquad y(1.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.10672 + 2 \times 1.0684 + 0.11378)$$

$$= 1 + 0.10682 = 1.10682. \quad \textbf{Ans.}$$

Example 2. *The unique solution of the problem*

$$y' = -xy \text{ with } y_0 = 1 \text{ is } y = e^{-x^2/2}.$$

Find approximately the value of $y_{(0.2)}$ *using one application of Runge-Kutta method of order four.*

Sol. Let $h = 0.2$, we have $y_0 = 1$ when $x_0 = 0$.

$$k_1 = hf(x_0, y_0)$$

$$= 0.2[(0)1] = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\left[-\left(0 + \frac{0.2}{2}\right)(1 + 0)\right] = -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\left[-\left(0 + \frac{0.2}{2}\right)\left(1 - \frac{0.02}{2}\right)\right] = -0.0198$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2(0 + 0.2)(1 - 0.0198) = 0.039208$$

$$\therefore \qquad y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.0198) + (-0.039208)]$$

$$= 1.0000 - 0.198013$$

$$= 0.9801986 \cong 0.9802$$

The exact value of $y(0.2)$ is 0.9802.

Example 3. *Solve the equation* $y' = (x + y)$ *with* $y_0 = 1$ *by Runge-Kutta rule from* $x = 0$ *to* $x = 0.4$ *with* $h = 0.1$.

Sol. Here $f(x, y) = x + y, h = 0.1$, given $y_0 = 1$ when $x_0 = 0$.

We have,

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1(0.1 + 1.1105) = 0.12105$$

$$y_1 = y_{(x=0.1)} = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) = 1.11034$$

Similarly for finding $y_2 = y(x = 0.2)$, we get

$$k_1 = hf(x_1 y_1 = 0.1)[(0.1) + 1.11034] = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1[0.15 + 1.11034 + 0.660517] = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1[0.15 + 1.11034 + 0.06604] = 0.13208$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$

$$= 0.1[0.20 + 1.11034 + 0.13263] = 0.14263$$

$$\therefore \qquad y_2 = y_{(x=0.2)} = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.11034 + \frac{1}{6}\left[0.121034 + 2(0.13208 + 0.13263 + 0.14429)\right] = 1.2428$$

Similarly, for finding $y_3 = y(x = 0.3)$, we get

$$k_1 = hf\left(x_2 y_2\right) = 0.1\left[(0.2) + 1.2428\right] = 0.14428$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= 0.1[0.25 + 132428 + 0.07214] = 0.15649$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= 0.1[0.25 + 1.2428 + 0.07824] = 0.15710$$

$$k_4 = hf\left(x_2 + h, y_2 + k_3\right)$$

$$= 0.1[0.30 + 1.2428 + 0.15710] = 0.16999$$

$$\therefore \qquad y_3 = y_{(x=0.3)} = y_2 + \frac{1}{6}(k_1 + 2K_2 + 2K_3 + k_4)$$

$$= 0.13997$$

Similarly, for finding $y_4 = y(x = 0.4)$, we get

$$k_1 = (0.1)[0.3 + 1.3997] = 0.16997 \qquad \Rightarrow k_1 = 0.16997$$

$$k_2 = (0.1)[0.35 + 1.3997 + 0.08949] = 0.18347 \qquad \Rightarrow k_2 = 0.18347$$

$$k_3 = (0.1)[0.35 + 1.3997 + 0.9170] = 0.18414 \qquad \Rightarrow k_3 = 0.18414$$

$$k_4 = (0.1)[0.4 + 1.3997 + 0.18414] = 0.19838 \qquad \Rightarrow k_1 = 0.19838$$

$$\therefore \qquad y_4 = 1.3997 + \frac{1}{6}\left[0.16997 + 2(0.18347 + 0.18414 + 0.19838)\right]$$

$y_4 = 1.5836.$ **Ans.**

Example 4. *Given* $\frac{dy}{dx} = y - x$ *with y(0) = 2, find y(0.1) and y(0.2) correct to 4 decimal places.*

Sol. We have $x_0 = 0,\ y_0 = 2,\ h = 0.1$

Then, we get

$$k_1 = hf\left(x_0, y_0\right)$$

$$= 0.1(2 - 0) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left[2 + \frac{0.2}{2} - \left(0 + \frac{0.1}{2}\right)\right] = 0.205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left[2 + \frac{0.205}{2} - \left(0 + \frac{0.1}{2}\right)\right] = 0.20525$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$= 0.1[2 + 0.20525 - (0 + 0.1)] = 0.210525$$

Therefore,

$$y = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 2 + 0.2051708 = 2.2051708$$

$\Rightarrow$ $y(0.1) = 2.2052$ Corect to 4 decimal places.

For $y(0.2)$, we have $x_0 = 0.1$, $y_0 = 2.2052$, we get

$$k_1 = hf\left(x_0, y_0\right)$$

$$= 0.1(2.2052 - 0.1) = 0.21052$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left[2.2052 + \frac{0.21052}{2} - \left(0.1 + \frac{0.1}{2}\right)\right] = 0.216046$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left[2.2052 + \frac{0.216046}{2} - \left(0.1 + \frac{0.1}{2}\right)\right] = 0.2163223$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$= 0.1\left[2.2052 + 0.2163223 - (0.1 + 0.1)\right] = 0.22215223$$

Hence, $y(0.2) = 2.2052 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$= 2.2052 + 0.2162348$$

$$= 2.4214.$$ **Ans.**

Example 5. *Solve* $\frac{dy}{dx} = -2xy^2$ *with y(0) = 1 and h = 0.2 on the interval [0.1] using Runge-Kutta fourth order method.*

Sol. As per given, we have $x_0 = 0,\ \ y_0 = 1,\ h = 0.2$

$$k_1 = hf(x_0, y_0)$$
$$= -2(0.2)(0)(1)^2 = 0$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= -2(0.2)\left(\frac{0.2}{2}\right)(1)^2 = -0.4$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= -2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2 = -0.38416$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= -22(0.2)(0.2)(0.961584)^2 = -0.0739715$$

Hence,

$$y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 1 + \frac{1}{6}[0 - 0.08 - 0.076832 - 0.0739715]$$
$$= 0.9615328$$

Now, we have

$$x_1 = 0.2, y_1 = 0.9615328, h = 0.2, \text{ we get}$$
$$k_1 = hf(x_1, y_1)$$
$$= -2(0.2)(0.2)(0.9615328)^2 = -0.0739636$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= -2(0.2)(0.3)(0.924551)^2 = 0.1025754$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= -2(0.2)(0.3)(0.9102451)^2 = 0.0994255$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= -2(0.2)(0.4)(0.8621073)^2 = -0.1189166$$

Thus, $\left[y(0.4) = y_1 + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4\right]$

$$= 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051508 - 0.1988510 - 0.1189166]$$
$$= 0.8620525$$

Similarly, we can obtained

$$y(0.6) = 0.7352784$$

$$y(0.8) = 0.6097519$$

$$y(1.0) = 0.500073. \qquad \textbf{Ans.}$$

Example 6. *Solve* $\frac{dy}{dx} = yz + x, \quad \frac{dz}{dx} = xz + y;$

Given that y(0) = 1, z(0) = –1 for y(0.1), z(0.1).

Sol. Here, $f_1(x, y, z) = yz + x$

$$f_2(x, y, z) = xz + y$$

$$h = 0.1, x_0 = 0, y_0 = 1, z_0 = -1$$

$$k_1 = hf_1(x_0, y_0, z_0) = h(y_0 z_0 + x_0) = -0.1$$

$$l_1 = hf_2(x_0, y_0, z_0) = h(x_0 z_0 + y_0) = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_1(0.05, 0.95, -0.95) = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_2(0.05, 0.95, -0.95) = -0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_1(0.05, 0.957375, -0.954875) = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_2(0.05, 0.957375, -0.954875 = -0.0864173$$

$$k_4 = hf_1(x_0 + h, y_0 + h_3, z_0 + l_3) = -0.073048$$

$$l_4 = hf_2(x_0 + h, y_0 + h_3, z_0 + l_3) = +0.0822679$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0860637$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0.0907823$$

$$y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$$

$$z_1 = z(0.1) = z_0 + k = -1 + 0.0907823 = -0.9092176$$

7.7 MILNE'S PREDICTOR-CORRECTOR METHOD

Predictor-Corrector Methods

Predictor-Corrector formulae are easily derived but require the previous evaluation of y and $y_1 = f(x, y)$ at a certain number of evenly spaced pivotal point (discrete points of x_1 of x-axis) in the neighbourhood of x_0.

In general the Predictor-Corrector methods are the methods which require the values of y at x_n, x_{n-1}, x_{n-2}............For computing the value of y at x_{n+1}. A Predictor formula is used to predict the value of y_{n+1}. Now we discuss Milne's method which is known as Predictor-Corrector methods.

Milne's Method

The method is a simple and eresonable accurate method of solving the ordinary first order differential equation numerically. To solve the differential equation

$$\frac{dy}{dx} = y' = f(x, y)$$

by this method we first obtain the approximate value of y_{n+1} by Predictor formula and then improve the value of y_{n+1} by means of a corrector formula. Both these formulas can be derived from the Newton forward interpolation formula as follows:

From Newton's formula, we have

$$f(x) = f(x_0 + uh) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{1\times 2}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 f(x_0) + ... \quad ...(1)$$

where $u = \frac{x - x_0}{h}$, or $x = x_0 + uh$.

Putting $y' = f(x)$ and $y'_0 = f(x_0)$ in the above formula, we get

$$y' = y'_0 + u\Delta y'_0 + \frac{u(u-1)}{1\times 2}\Delta^2 y'_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 y'_0 + \frac{u(u-1)(u-2)(u-3)}{1\times 2\times 3\times 4}\Delta^4 y'_0 \quad ...(2)$$

Intergrating (2) from x_0 to $x_0 + 4h$ i.e. from $u = 0$ to $u = 4$, we get

$$\int_{x_0}^{x_{0+4h}} y'dx = h\int_0^4 y'_0 + u\Delta y'_0 + \frac{u(u-1)}{2}\Delta^2 y'_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y'_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y'_0 ......]\, du$$

($\because$ $hdu = dx$) which gives

$$y_{x_0+4h} - y_{x_0} = h\left[4h'_0 + 8\Delta y'_0 + \frac{20}{3}\Delta^2 y'_0 + \frac{8}{3}\Delta^3 y'_0 + \frac{28}{90}\Delta^4 y'_0\right]$$

[considering upto fourth differences only]

Using $\Delta = E - 1$, we get

$$y_4 - y_0 = h\left[4y'_0 + 8(E-1)y'_0 + \frac{20}{3}(E-1)^2 y'_0 + \frac{8}{3}(E-1)^3 y'_0 + \frac{14}{45}\Delta^4 y'_0\right]$$

$$\Rightarrow \quad y_4 - y_0 = \frac{4h}{3}[2y_1' - y_2' + 2y_3'] + \frac{14}{45}\Delta^2 y_0' \qquad ...(3)$$

This is known as Milne's predictor formula. The corrector formula is obtained by integrating (2) from x_0 to $x_0 + 2h$ *i.e.,* from $u = 0$ to $u = 2$.

$$\int_{x_0}^{x_0+2h} y\,dx = h\int_0^2 \left(y_0' + u\Delta y_0' + \frac{u(u-1)}{2}\Delta^2 y_0' + \ldots \right) du$$

$$\Rightarrow \quad y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' + \frac{1}{3}\Delta^2 y_0' - \frac{1}{90}\Delta^4 y_0'\right]$$

Using $\Delta = E - 1$, and simplifying we get

$$y_2 = y_0 + \frac{h}{3}[y_0' + 4y_1' + y_2'] - \frac{h}{90}\Delta^4 y_0' \qquad ...(4)$$

Expression (4) is called Milne's corrector formula.

The general forms of equations (3) and (4) are

$$y_{n+1} = y_{n-3} + \frac{4h}{3}[2y_{n-2}' + y_{n-1}' + 2y_n'] \qquad ...(5)$$

and
$$y_{n+1} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}'] \qquad ...(6)$$

i.e.,
$$\bar{y}_{n+1} - y_{n-3} = \frac{4h}{3}[2y_{n-2}' + y_{n-1}' + 2y_n'] \qquad ...(7)$$

and
$$y_{n+1} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}'] \qquad ...(8)$$

In terms of f the Predictor formula is

$$\bar{y}_{n+1} = y_{n-3} + \frac{4h}{3}[2f_{n-2} - f_{n-1} + 2f_n] \qquad ...(9)$$

and the corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + f_{n+1}] \qquad ...(10)$$

Example 7. *Tabulate by Milne's method the numerical solution of* $\frac{dy}{dx} = x + y$ *with initial conditions* $x_0 = 0$, $y_0 = 1$, *from* $x = 0.20$ *to* $x = 0.30$.

Sol. Here $y' = x + y$

$$y'' = 1 + y', y''' = y'', y'''' = y''', y''''' = y'''', ...$$

Hence,
$$y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y_0'' = 1 + y_0' + = 1 + 1 = 2$$

$$y_0''' = y_0'' = 2$$

$$y_0'''' = 2, y_0''''' = 2$$

Now taking $h = 0.05$, we get

$$y_1 = 1.1026, \quad y_2 = 1.2104, \quad y_3 = 1.3237$$

Using Milne's Predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.05)}{3}[2.2052 - 1.2104 + 2.6474] = 1.2428$$

$$\overline{y'}_4 = x_4 + y_4 = 0.2 + 1.2428 = 1.4428$$

Using corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.1104 + \frac{(0.05)}{3}[1.2104 + 5.2948 + 1.4428] = 1.2428$$

which is the same as the predicted value.

$\therefore \quad y_4 = y_{0.20} = 1.2428$

and $\quad y_4' = 1.4428$

and putting $n = 4$, $h = 0.05$, we get

$$y_5 = y_1 + \frac{4h}{3}[2y_2' - y_3' + 2y_4']$$

$$= 1.0526 + \frac{4(0.05)}{3}[2.4208 - 1.3237 + 2.8856] = 1.3181$$

$$y_5' = x_5 + y_5 = 0.25 + 1.3181 = 1.5681.$$

Using Milne's corrector formula, we get

$$y_5 = y_3 + \frac{h}{3}\left[y_3' + 4y_4' + \overline{y'}_5\right]$$

$$= 1.1737 + \frac{(0.05)}{3}[1.3237 + 5.7712 + 1.5681] = 1.3181$$

which is the same as the predicted value.

$\therefore \quad y_5 = y_{0.25} = 1.3181$

and $\quad y_5' = 1.5681$

Again putting $n = 5$, $h = 0.05$ and using Milne's predictor formula, we get

$$y_6 = y_2 + \frac{4h}{3}[2y_3' - y_4' + 2y_5']$$

$$= 1.1104 = \frac{4(0.05)}{3}[2.6474 - 1.4428 + 3.1362] = 1.3997$$

$$\overline{y'}_6 = 0.3 + 1.39972 = 1.6997$$

which is corrected by

$$y_6 = y_4 + \frac{h}{3}[y_4' + 4y_5' + y_6']$$

$$= 1.2428 + \frac{(0.05)}{3}[1.4428 + 6.2724 + 1.6997] = 1.3997$$

which is same as the predicted value.

$\therefore \quad y_6 = y_{0.30} = 1.3997$

and $\quad y'_6 = 1.6997$

The result can be put in the tabular form

x	0.20	0.25	0.30
y	1.2428	1.3181	1.3997
y'	1.4428	1.5681	1.6997

Example 8. *Compute y(2) if y(x) is the solution of* $\frac{dy}{dx} = \frac{1}{2}(x+y)$ *using Milne Predictor-corrector method. Given* $y(0) = 2, y(0.5) = 2.6336, y(1.0) = 3.595, y(1.5) = 4.968.$

Sol. Here, we have

$x_0 = 0, y_0 = 2 \qquad f_0 = \frac{1}{2}(0+2) = 1$

$x_1 = -0.5, y_1 = 2.636 \qquad f_1 = \frac{1}{2}(0.5+2.636) = 1.568$

$x_2 = 1, y_2 = 3.595 \qquad f_2 = \frac{1}{2}(1+3.595) = 2.2975$

$x_3 = 1.5, y_3 = 4.968 \qquad f_3 = \frac{1}{2}(1.5+4.968) = 3.234$

Using Predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 2.0 + \frac{4 \times 0.5}{3}(2 \times 1.568 - 2.2975 + 2 \times 3.234) = 6.871$$

and $\quad f_4 = x_4 + y_4 = \frac{1}{2}(2+6.871) = 4.4335$

Using corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 3.595 + \frac{0.5}{3}(2.2975 + 4 \times 3.234 + 4.4355)$$

$$= 6.873166 \approx 6.8732$$

Thus, corrected $\quad f_4 = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2.0 + 6.8732) = 4.4366$

Again, using Corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 3.595 + \frac{0.5}{3}(2.2975 + 4 \times 3.234 + 4.4366)$$

$$= 6.87335 \approx 6.8734$$

Example 9. *The differential equation* $\frac{dy}{dx} = 1 + y^2$ *satisfies the following sets of values of x and y.*

x	0	0.2	0.4	0.6
y	0	0.2027	0.4228	0.6871

Sol. Firstly, we calculate the following

$$f_0 = 1 + y_0^2 = 1$$

$$f_1 = 1 + y_1^2 = 1.0411$$

$$f_2 = 1 + y_2^2 = 1.1787$$

$$f_3 = 1 + y_3^2 = 1.4681$$

Using Predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + \frac{4 \times 0.2}{3}[2(1.0411) - 1.1787 + 2(1.4681)] = 1.0239$$

$$\Rightarrow \qquad f_4 = 1 + y_4^2 = 1 + (1.0239)^2 = 2.0480$$

Using Corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.4228 + \frac{0.2}{3}(1.1787 + 4 \times 1.4681 + 2.0480) = 1.0294$$

$$\Rightarrow \qquad y(0.8) = 1.0294$$

The corrected value of

$$f_4 = 1 + y_4^2 = 2.0597$$

Now, to find $f(1)$ we use predictor formula such that

$$y_5 = y_1 + \frac{4h}{3}(2f_1 - f_3 + 2f_4)$$

$$= 0.2027 + \frac{4 \times 0.2}{3}[2(1.1787) - 1.4681 + 2(2.0597)] = 1.5384$$

and $$f_5 = 1 + y_5^2 = 1 + (1.5384)^2 = 3.3667$$

Finally using corrector formula, we get

$$y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

$$= 0.6841 + \frac{0.2}{3}(1.4681 + 4(2.0597) + 3.3667)$$

$$= 1.5556733 = 1.5557.$$

Example 10. *Solve $y' = 2e^x - y$ at $x = 0.4$ and $x = 0.5$ by Milne's method, given their values of the four points.*

x	0	0.1	0.2	0.3
y	2	2.010	2.040	2.090

Sol. Here we find the value of $f_1, f_2, f_3.$

$$f_1 = 2e^{0.1} - 2.010 = 0.2003$$

$\Rightarrow \quad f_1 = 0.2003$

$$f_2 = 2e^{0.2} - 2.040 = 0.4028$$

$\Rightarrow \quad f_2 = 0.4028$

$$f_3 = 2e^{0.3} - 2.090 = 0.6097$$

$\Rightarrow \quad f_3 = 0.6097$

By Milne' predictor formula, we have

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 2 + \frac{4\times 0.1}{3}[2(0.2003) - (0.4028) + 2(0.6097)]$$

$$= 2.162293 \approx 2.1623$$

$$y_4 = 2.1623$$

Now $\quad f_4 = 2e^{0.4} - 2.1623 = 0.8213494 = 0.8213$

$\Rightarrow \quad f_4 = 0.8213$

Now again by Corrector formula, we get

$$y_4 = y_2 \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.04 + \frac{0.1}{3}[.4028] - 4(0.6097) + 2(0.8213)$$

$$= 2.162096 = 2.1621$$

$\Rightarrow \quad y_4 = 2.1621$

Again by using Predictor formula

$$y_5 = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

$$= 2.10 + \frac{4\times 0.1}{3}[2(0.4028) - 0.6097 + 2(0.8215)]$$

$$= 2.2551867 = 2.2552$$

$\Rightarrow \quad y_5 = 2.2552$

then $f_5 = 2e^{0.5} - 2.2552$

$=1.0422425=1.0422$

$\Rightarrow \quad f_5 = 1.0422$

By Corrector formula, we get

$$y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

$$= 2.090 + \frac{0.1}{3}[0.6097 + 4(0.8215) + 1.0422]$$

$$= 2.2545967 = 2.255$$

$\Rightarrow \quad y_5 = 2.255.$ **Ans.**

Example 11. *Apply Milne's method to find a solution of the differential equation* $\frac{dy}{dx} = x - y^2$ *in the range 0 ≤ x ≤ 1 with y(0) = 0.*

Sol. Here, we use Picard' method to compute y_1, y_2, and y_3.

Picard's successive approximations are given by

$$y_n = y_0 + \int_0^x f(x, y_{n-1})dx$$

for $n = 1$, we have

$$y_1 = 0 + \int_0^x x dx = \frac{x^2}{2}$$

for $n = 2$,

$$y_2 = 0 + \int_0^x \left[x - \left(\frac{x^2}{2}\right)^2\right]dx$$

$$= \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, $y_3 = 0 + \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx$

$$= \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}$$

Let us take $y = \frac{x^2}{2} - \frac{x^5}{20}$ for finding the various values of $y_i's$ and $f_i's$

$y_1 = y(0.2) = 0.019987 = 0.02, \qquad f_1 = 0.1996$

$y_2 = y(0.4) = 0.079488 = 0.0795, \qquad f_2 = 0.3937$

$y_3 = y(0.6) = 0.176112 = 0.176, \qquad f_3 = 0.5690$

Now, using Predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + \frac{4 \times 0.2}{3}\left[2(0.1996) - 0.3937 + 2(0.5690)\right]$$

$$= 0.3049333 = 0.3049$$

Further, using corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4 \times 0.5690 + 0.7070) = 0.3046$$

$$\left(\because f_4 = x_4 + y_4 = 0.8 - (0.3049)^2 = 0.7070359 = 0.7070\right)$$

Hence, $y_4 = 0.3046$ at $x = 0.8$ and corrected $f_4 = 0.8 - (0.3046)^2 = 0.7072$

Again, using Predictor formula, we get,

$$y_5 = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

$$= 0.2 + \frac{4 \times 0.2}{3}(2 \times 0.3937 - 0.5690 + 2 \times 0.7072)$$

$$= 0.4554133 = 0.4554$$

Now, $\left(\because f_5 = x_5 + y_5 = 1 - (0.4554)^2 = .792610 = 0.7926\right)$

Using corrector formula, we get

$$y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

$$= 0.176 + \frac{0.2}{3}[0.5690 + 4 \times 0.7072 + 0.7926]$$

$$= 0.45536 = 0.4554$$

Hence, $y(1) = 0.4554$. **Ans.**

Example 12. *Given* $\frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)y^2$ *and y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21. Evaluate by Milne' predictor—Corrector method y(0.4).*

Sol. Milne's predictor formula is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

Putting $n = 3$ in the above formula, we get

$$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \quad \text{...(1)}$$

We have

$$y_0 = 1, y_1 = 1.06, y_2 = 1.12, y_3 = 1.21, \text{ and } h = 0.1$$

The given differential equation is

$$y' = \frac{1}{2}\left(1 + x^2\right)y^2$$

$$y_1' = \frac{1}{2}\left(1+x_1^2\right) y_1^2 = \frac{1}{2}\left[1+(0.1)^2\right]\times(1.06)^2$$

$$= 0.505\times(1.06)^2 \ = \ 0.5674$$

$$y_2' = \frac{1}{2}\left(1+x_2^2\right) y_2^2 = \frac{1}{2}\left[1+(0.2)^2\right]\times(1.12)^2$$

$$= 0.52\times(1.12)^2 \ = \ 0.6522$$

$$y_3' = \frac{1}{2}\left(1+x_3^2\right) y_3^2 = \frac{1}{2}\left[1+(0.3)^2\right]\times(1.21)^2$$

$$= 0.545\times(1.21)^2 \ = \ 0.7980$$

Substituting these values in (1), we get

$$\bar{y}_4 = 1+\frac{4\times(0.1)}{3}[2\times0.5674-0.6522+2\times0.7980]$$

$$= 1.27715 \ = 1.2772 \qquad \text{...(2)}$$

(correct to 4 decimal places)

∴ We get

$$y_4' = \frac{1}{2}\left[\left(1+x_4^2\right) y_4^2\right]$$

$$= \frac{1}{2}\left[1+(0.4)^2\right]\times(1.2772)^2 = 0.9458$$

Milne's corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(2y'_{n-1}+4y'_n+\bar{y}'_{n+1}\right) \qquad \text{...(3)}$$

Putting n = 3 in (3), we get

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + \bar{y}'_4\right]$$

$$= 1.12+\frac{0.1}{3}[0.6522+4\times0.798+0.9458] = 1.2797 \qquad \text{(correct to 4 decimal places)}$$

∴ $y(0.4) = 1.2797.$ **Ans.**

7.8 AUTOMATIC ERROR MONITORING

Error Analysis: The numerical solutions of differential equations certainly differs from their exact solutions. The difference between the computed value y_i and the true value $y(x_i)$ at any stage is known as the total error. The total error at any stage is comprised of **truncation error and round-off error**.

The most important aspect of numerical methods is to minimize the errors and obtain the solutions with the least errors. It is usually not possible to follow error development quite closely. We can make only rough estimates. That is why our treatment of error analysis at times, has to be somewhat intuitive.

In any method, the truncation error can be reduced by taking smaller sub-intervals. The round-off error cannot be controlled easily unless the computer used has the double precision arithmetic facility. In fact, this error has proved to be more **elusive** that the **truncation error.**

The truncation error in Euler's method is $\frac{1}{2}h^2y_n''$ *i.e.,* $O(h^2)$ while that of modified Euler's method is $\frac{1}{2}h^3y_n'''$ *i.e.,* $O(h^3)$.

Similarly, in the fourth order Range-Kutta method, the truncation error is of $O(h^5)$

In the Milne's method, the truncation error,

(*i*) due to predictor formula $\frac{14}{45}h^5y_n'''''$ and

(*ii*) due to corrector formula $= -\frac{1}{90}h^5y_n'''''$

i.e., the truncation error in Milne's method is also of $O(h^5)$.

The **relative error** of an approximate solution is the ratio of the total error to the exact value. It is of greater importance than the error itself for if the true value becomes larger than a larger, error may be acceptable. If the true value diminishes, then the error must also diminish otherwise the computed results may be absurd.

Convergence of a Method

Any numerical method for solving a differential equation is said to be convergent if the approximate solution y_n approaches the exact solution $y(x_n)$ at h tends to zero provided the rounding error arising from the initial conditions approach zero. This means that as a method is continually refined by taking smaller and smaller step-sizes, the sequence of approximate solutions must converge to the exact solution.

Taylor's series method is convergent provided $f(x, y)$ possesses enough continuous derivatives. The Runge-Kutta methods are also convergent under similar conditions. Predictor-corrector methods are convergent if $f(x, y)$ satisfies Lipschitz condition, *i.e.,*

$$\left|f(x,y) - f(x,\bar{y})\right| \le k(y - \bar{y})$$

k being constant, then the sequence of approximations to the numerical solution converges to the exact solution.

7.9 STABILITY IN THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

There is a limit to which the steps-size h can be reduced for controlling the truncation error, beyond which a further reduction in h will result in the increase of round-off error and hence increase in the total error.

A method is said to be **stable** if it produces a bounded solution which imitates the exact solution. Otherwise it is said to be **unstable**. If a method is stable for all values of the parameter, it is said to be absolutely or unconditionally stable. If it is stable for some values of the parameter, it is said to be conditionally stable.

Euler's method and Runge-Kutta method are conditionally stable. The Milne's method is, however, unstable since when the parameter is negative, each of the error is magnified while the exact solution decays.

Two types of stability considerations in the solution of ordinary differential equations.

(*i*) Inherent stability

(*ii*) Numerical stability

Inherent stability is determined by the mathematical formulations of the problem and is dependent on the eigen values of Jacobian Matrix of the differential equation.

Numerical stability is a function of the error propagation in the numerical method. Three types of errors occur in the application of numerical integration methods:

(*a*) Truncation error

(*b*) Round-off error

(*c*) Propagation error

Example 13. *Applying Euler's method to the equation.*

$$\frac{dy}{dx} = \lambda y, \text{ given } y(x_0) = y_0$$

determine its stability zone.

What would be the range of stability when $\lambda = -1$.

Sol. Here, $y = \lambda y, y(x_0) = y_0$ ...(1)

By Euler's method, we have

$$y_n = y_{n-1} + hy'_{n-1} = y_{n-1} + \lambda h y_{n-1}$$

$$= (1 + \lambda y) y_{n-1}$$

$$y_{n-1} = (1 + \lambda y) y_{n-2}$$

....

$$y_2 = (1 + \lambda y) y_1$$

$$y_1 = (1 + \lambda y) y_0$$

Multiplying all these equations, we obtain ...(2)

$$y_n = (1 + \lambda y)^n y_0$$

Integrating (1), we get $y = ce^{\lambda x}$

Using $y(x_0) = y_0$, we obtain $y_0 = ce^{\lambda x_0}$

Hence, we have $y = y_0 e^{\lambda(x - x_0)}$

In particular, the exact solution through $(x_n, y_{n)}$ is

$$y = y_0 e^{\lambda(x_n - x_0)} = y_0 e^{\lambda nh} \qquad [x_n = x_0 + nh]$$

or

$$y = y_0 \left(e^{\lambda h}\right)^n = y_0 \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + ...\right)^n \qquad ...(3)$$

Clearly, the numerical solution (2), agrees with exact solution (3) for small values of *h*. The solution (2), increases if $|1 + \lambda h| > 1$.

Hence, $|1 + \lambda h| < 1$ defines *a* stable zone.

When λ is real, then the method is stable if $|1 + \lambda h| <$ *i.e.,* $-2 < \lambda h < 0$.

When λ is complex $(= a + ib)$, then it is stable if

$$[1 + (a = ib)h] < 1$$

i.e., $(1 + ah)^2 + (bh)^2 < 1$

i.e., $(x + 1)^2 + y^2 < 1$ where $x = ah$, $y = bh$.

i.e., λh lies within the unit circles shown in Fig. 7.1

When λ is imaginary $(= ib)$, $|1 + \lambda h| = 1$, then we have a periodic stability.

Hence, Euler's method is absolutely stable if and only if

(*i*) real $\lambda : -2 < \lambda h < 0$.

(*ii*) Complex λ, λh lies within the unit circle (Fig. 7.1) *i.e.* Euler's method is conditionally convergent.

(*iii*) When $\lambda = -1$, the solution is stable in the range

$$-2 < -h < 0, \text{ i.e., } 0 < h < 2.$$

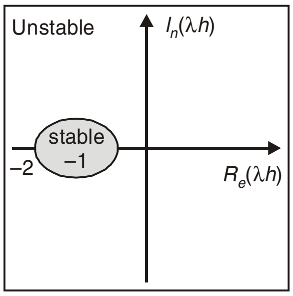


FIG. 7.1

PROBLEM SET 7.2

1. Apply Runge-Kutta method find the solution of the differential equation $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y_0 = 1$ at $x = 0.1$ **[Ans.** 1.066652421875]

2. Given $\frac{dy}{dx} = 1 + y^2$ where $y = 0$ when $x = 0$, find $y_{(0.2)}$, $y_{(0.4)}$ and $y_{(0.6)}$, using Runge-Kutta formula of order four. **[Ans.** $y_{(0.2)} = 0.2027$, $y_{(0.4)} = 0.4228$, $y_{(0.6)} = 0.6841$]

3. Use classical Runge-Kutta method of fourth order to find the numerical solution at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2, y(1) = 0$. Assume step size $h = 0.2$. **[Ans.** $y(1.2)$= 0.246326, $y(1.4)$=0.622751489]

4. Using Runge-Kutta method to solve $10\frac{dy}{dx} = x^2 + y^2$ $y_{(0)} =$ for the interval $0 < x \le 0.4$ with $h = 0.1$. **[Ans.** 1.0101, 1.0207, 1.0318, 1.0438]

5. Solve the differential equation $\frac{dy}{dx} = \frac{2x-1}{x^2}y + 1$ where $x_0 = 1$, $y_0 = 2$, $h = 0.2$. Obtain $y_{(1.2)}$ and $y_{(1.4)}$ using Runge-Kutta method. **[Ans.** 2.658913 and 3.432851]

6. Using Runge-Kutta method solve simultaneous differential equation $\frac{dy}{dx} = f(x, y, t) = xy + t$ and $\frac{dx}{dx} = ty + x = g(x, y, t)$ where $t_0 = 0$, $x_0 = 1$, $y_0 = -1$, $h = 0.2$. **[Ans.** $y_{(0.2)} = -0.8341$]

7. By Milne's method solve $\frac{dy}{dx} = 2 - xy^2$ with $y_{(0)} = 1$ for $x = 1$ taking $h = 0.2$. **[Ans.** 1.6505]

8. Apply Milne's method to solve the differential equation $\frac{dy}{dx} = -xy^2$ at $x = 0.8$ given that $y_{(0)} = 2, y_{(0.2)} = 1.923, y_{(0.4)} = 1.724, y_{(0.6)} = 1.471.$ **[Ans.** $y_{(0.8)} = 1.219$]

9. Given that $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2$ and $y(0) = 0$, $y(0.1) = 1.06$, $y(0.2) = 1.12, y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's predictor corrector method. **[Ans.** $y_{(0.2)}$ =1.2797]

10. By Milne's method solve $y(0.3)$ from $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-0.1)$ and $y(0.1)$, $y(0.2)$ from the Taylor's series method. **[Ans.** 1.4392]

❑❑❑

CHAPTER 8

Solution of Simultaneous Linear Equation

8.1 INTRODUCTION

Let us consider a system of non-homogeneous linear equations in n unknowns $x_1, x_2, ..., x_{n_3}$,

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

which we can solve by matrix method.

Apart from this method we have some other direct methods to find out the solution of a system of equations like Gauss-Elimination method, Gauss-Jordan's method etc. The drawback with Cramer's and matrix inversion method is having the too much calculations. If we consider a system of 10 equations (or variables) then Cramer's rule contains 7,00,00,000 multiplication while in matrix inversion method, evaluation of A^{-1} by cofactors became very complicated.

Now, we shall describe a few direct methods to solve a system of linear equations:

8.2 GAUSS-ELIMINATION METHOD

This is one of the most widely used method. This emthod is a systematic process of eliminating unknowns from the linear equations. This method is divided into two parts: (*i*) Triangularization, (*ii*) Back substitution.

Let the system of equations in three unknowns, x, y, z be

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \qquad ...(1)$$

We multiply the first equation by $\frac{a_{21}}{a_{11}}$ and subtract it from second equations to eliminate x from second equation. Similarly, we multiply the first equation by $\frac{a_{31}}{a_{11}}$ and subtract it from third equation to eliminate x from third equation. Then the above system (1) becomes,

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{22}^1 y + a_{23}^1 z &= b_2^1 \\ a_{32}^1 y + a_{33}^1 z &= b_3^1 \end{aligned}\right\} \qquad ...(2)$$

Again, to eliminate y from the third equation of the system (2), we multiply the second equation by $\frac{a_{32}}{a_{22}^1}$ and subtract it from the third equation. Then, the above system of equations (2) becomes:

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{22}^1 y + a_{23}^1 z &= b_2^1 \\ a_{33}^{11} z &= b_2^{11} \end{aligned}\right\} \qquad ...(3)$$

from which the values of $x, y, z,$ can be obtained by back substitution. The values of z, y and x can be obtained from third, second and first equations respectively.

The Gauss-Elimination method can be generalized to find the solutions of n simultaneous equations in n-unknowns.

Example 1. *Solve the system of equation by Gauss-Elimination method.*

$$\left.\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ 2x - 3y + 2x &= 2 \end{aligned}\right\} \qquad ...(1)$$

Sol. To eliminate x from the second equation of the system (1), we multiply the first equation by 2 and subtract it from the second equation and obtain.

$$-2y - z = -7$$

or

$$2y + z = 7$$

Similarly, to eliminate x from the third equation of the system (1) we subtract first equation from the third equation and obtain.

$$-6y + 3z = -3$$

Now, the system of equation (1) becomes.

$$\left.\begin{aligned} 2x + 3y - z &= 5 \\ 2y + z &= 7 \\ -6y + 3z &= -3 \end{aligned}\right\} \qquad ...(2)$$

Now, to eliminate y from the third equation of the system (2) we multiply the second equation by 3 and add it to third equation of the system (2) and obtain

$$6z = 18$$

Thus, the system of equation (2) becomes.

$$\left.\begin{aligned} 2x + 3y - z &= 5 \\ 2y + z &= 7 \\ 6z &= 18 \end{aligned}\right\}$$

By back substitution, gives the solution

$$z = 3,\ y = 2 \text{ and } x = 1$$

Example 2. *Solve the following system of equations by Gauss-Elimination method.*

$$\left.\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}\right\} \quad ...(1)$$

Sol. To eliminate x from the second equation of the system (1), we multiply the first equation by $\frac{3}{2}$ and subtract it from the second equation and obtain.

$$y + 3z = 6$$

Similarly, to eliminate x from the third equations of the system (1), we multiply the first euqaiton by $\frac{1}{2}$ and subtract it from the third equation and obtain.

$$7y + 17z = 22$$

Now, the system of equation (1), becomes

$$\left.\begin{aligned} 2x + y + z &= 10 \\ y + 3z &= 6 \\ 7y + 17z &= 22 \end{aligned}\right\} \quad ...(2)$$

Now, to eliminate y from the third equation of the system (2), we multiply the second equation by 7 and subtract it from the third equation of the system (2) and obtain

$$4z = 20$$

Thus, the system of equation (2) becomes

$$2x - y + z = 10;\ y + 3z = 6;\ 4z = 20 \quad ...(3)$$

Back substitution gives the solution.

$$z = 5,\ y = -9 \quad \text{and } x = 7.$$ **Ans.**

8.3 GAUSS-ELIMINATION WITH PIVOTING METHOD

Pivoting: One of the ways around this problem is to ensure that small values (especially zeros) do not appear on the diagonal and, if they do, to remove them by rearranging the matrix and vectors.

Partial Pivoting: If zero element is found in diagonal position *i.e.*, a_{ij} for $i = j$ which is called pivot element interchange the corresponding elements of two rows such that new diagonal element i if non-zero and having maximum value in that corresponding column. The process can be explained in following steps. In the first step the largest coefficient of x_1 (may be positive or negative) is selected from all the equations. Now we interchange the first equation with the equation having largest coefficient of x_i. In the second step, the numerically largest coefficient of x_2 is selected from the remaining equations. In this step we will not consider the first equations now interchange the second equation with the equation having largest coefficient of y. We continue this process till last equation. This procedure is known as partial pivoting. In general, the rearrangement of equation is done even if pivot element is non-zero to improve the accuracy of solution by reducing the round off errors involved in elimination process, by getting a larger determinant, which is done by finding a largest element of the row as the pivotal element.

Complete Pivoting: If the order of elimination of $x_1, x_2, x_3, ...$ is not important, then we may choose at each stage the largest coefficient of the whole matrix of coefficients. We may search the largest value, not only in rows but also in columns. After searching largest value, we bring at the diagonal position. This method of elimination is known as complete pivoting.

The superiority of this method is that it gives the solution of a system, provided its determinant does not vanish in finite number of steps.

8.4 ILL-CONDITIONED SYSTEM OF EQUATIONS

A system of equations A X = B is said to be ill-conditioned or unstable if it is highly sensitive to small changes in A and B *i.e.,* small change in A or B causes a large change in the solution of the system. On the other hand if small changes in A and B give small changes in the solution, the system is said to be stable, or, well conditioned. Thus in a ill-conditioned system, even the small round off errors effect the solutions very badly. Unfortunately it is quite difficult to recognize an ill-conditioned system.

For example, consider the following two almost identical systems.

$$x_1 - x_2 = 1 \qquad x_1 - x_2 = 1$$

$$x_1 - 1.00001 \quad x_2 = 0 \quad \text{and} \quad x_1 - 0.99999 \quad x_2 = 0$$

Respective solutions are:

$$(100001,\ 100000) \quad \text{and} \quad (-99999,\ -100000)$$

obviously the two solutions differ very widely. Therefore the system is ill conditioned.

Example 3. *Show that the following system of linear equations is ill-conditioned.*

$$7x - 10y = 1$$

$$5x + 7y = 0.7$$

Sol. On solving the given equations we get $x = 0$ and $y = 0.1$.

Now, we make slight changes in the given system of equations. The new system becomes.

$$7x + 10y = 1.01$$

$$5x + 7y = 0.69$$

Here we get $x = -0.17$ and $y = 0.22$.

Hence the given system is ill-conditioned.

8.5 ITERATIVE REFINEMENT OF THE SOLUTION BY GAUSS ELIMINATION METHOD

The solution of system of equations will have some rounding error, we will discuss a technique called as 'iterative refinement' which leads to reduced rounding errors and often a reasonable solution for some ill-conditioned problems is obtained.

Consider the system of equations:

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \quad ...(1)$$

Let x', y', z' be an approximate solution, Substituting these values on the left-hand sides, we get new values of d_1, d_2, d_3 as d'_1, d'_2, d'_3, so their new system becomes;

$$\left.\begin{aligned} a_1x'+b_1y'+c_1z'=d_1' \\ a_2x'+b_2y'+c_2z'=d_2' \\ a_3x'+b_3y'+c_3z'=d_3' \end{aligned}\right\} \quad ...(2)$$

Subtracting each equation in (2) from the corresponding equations in (1), we get

$$\left.\begin{aligned} a_1x_e+b_1y_e+c_1z_e=k_1 \\ a_2x_e+b_2y_e+c_2z_e=k_2 \\ a_3x_e+b_3y_e+c_3z_e=k_3 \end{aligned}\right\} \quad ...(3)$$

where, $x_e = x - x,\ y_e = y - y',\ z_e = z - z'$ and $k_i = d_i - d_i'$

We now solve the system (3) for x_e, y_e, z_e giving $z = x' + x_e,\ \ y = y' + y_e,\ \ z = z' + z_e$ which will be better approximations for $x, y, z.$ We can repeat the process for improving the accuracy.

Example 4. *An approximate solution of the system 2x + 2y − z = 6, x − y + 2z = 8; − x + 3y + 2z = 4 is given by x = 2.8, y′ = 1, z = 1.8. Using the iterative method improve this solution.*

Sol. Substituting the approximate value $x' = 2.8$, $y' = 1$, $z' = 1.8$ in the given equations, We get

$$\left.\begin{aligned} 2(2.8)+2(1)-1.8=5.8 \\ 2.8+1+2(1.8)=7.4 \\ -2.8+3(1)+2(1.8)=3.8 \end{aligned}\right\} \quad ...(1)$$

Subtracting each equation in (1) from the corresponding given equations, we get

$$\left.\begin{aligned} 2x_e+2y_e-z_e=0.2 \\ x_e+y_e+2z_e=0.6 \\ -x_e+3y_e-2z_e=0.2 \end{aligned}\right\} \quad ...(2)$$

where $x_e = x - 2.8$, $y_e = y - 1$, $z_e = z - 1.8$

Solving the equations (2), we get $x_e = 0.2$, $y_e = 0$, $z_e = 0.2$

This gives the better solution $x = 3,\ \ y = 1,\ \ z = 2,$ which incidentally is the exact solution.

Ans.

8.6 ITERATIVE METHOD FOR SOLUTION OF SIMULTANEOUS LINEAR EQUATION

All the previous methods seen in solving the system of simultaneous algebraic linear equations are direct methods. Now we will see some indirect methods or iterative methods.

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. When this condition is satisfied, the system will be solvable by the iterative method. The system,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{32}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$

will be solvable by this method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist (iterating will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system AX = B are greater than the sum of absolute values of the other coefficients of that row. The condition is sufficient but not necessary.

Under the category of iterative method, we shall describe the following two methods:

(*i*) Jacobi's method (*ii*) Gauss-Seidel method.

8.6.1 Jacobi's Method or Gauss-Jacobi Method

Let us consider the system of simultaneous equations.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad ...(1)$$

such that $a_1, b_2,$ and c_3 are the largest coefficients of *x,y,z,* respectively. So that convergence is assured. Rearranging the above system of equations and rewriting in terms of $x, y, z,$ as:

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad ...(2)$$

let x_0, y_0, z_0 be the initial approximations of the unknowns x, y and z. Then, the first approximation are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Similarly, the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

Proceeding in the same way, if x_n, y_n, z_n are the nth iterates then

$$x_{n+1} = \frac{1}{a_1}(d_1 - b_1y_n - c_1z_n)$$

$$y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_n - c_2z_n)$$

$$z_{n+1} = \frac{1}{c_3}(d_3 - a_3x_n - b_3y_n)$$

The process is continued till convergency is secured.

Note: In the absence of any better estimates, the initial approximations are taken as $x_0 = 0,\ y_0 = 0,$ $z_0 = 0.$

Example 5. *Solve the following system of equation using Jacobi's method*

$$5x - y + z = 10$$

$$2x + 4y = 12$$

$$x + y + 5z = -1$$

Start with the solution (2, 3, 0).

Sol. Given system of equation can be written in the folliwng form, if we assume, x_0, y_0, z_0 as initial approximation:

$$x_1 = \frac{1}{5}\{10 + y_0 + z_0\}$$

$$y_1 = \frac{1}{4}\{12 - 2x_0\}$$

$$z_1 = \frac{1}{5}\{-1 - x_0 - y_0\}$$

Now if $x_0 = 2,\ y_0 = 3,\ z_0 = 0$, then

First approximation: $x_1 = \frac{1}{5}\{10 + 3 - 0\} = 2.6$

$$y_1 = \frac{1}{4}\{12 - 4\} = 2.0$$

$$z_1 = \frac{1}{5}\{-1 - 2 - 3\} = -1.2$$

Second approximation: $$x_2 = \frac{1}{5}\{10 + 2 + 1.2\} = 2.64$$

$$y_2 = \frac{1}{4}\{12 - 5.2\} = 1.70$$

$$z_2 = \frac{1}{5}\{-1 - 2.6 - 2\} = -1.12$$

Third approximation: $$x_3 = \frac{1}{5}\{10 + 1.7 + 1.12\} = 2.564$$

$$y_3 = \frac{1}{4}\{12 - 5.28\} = 1.680$$

$$z_3 = \frac{1}{5}\{-1 - 2.64 - 1.7\} = -1.068$$

Fourth approximation: $$x_4 = \frac{1}{5}\{10 + 1.68 + 1.068\} = 2.5496$$

$$y_4 = \frac{1}{4}\{12 - 5.128\} = 1.7180$$

$$z_4 = \frac{1}{5}\{-1 - 2.564 - 1.68\} = -1.0488.$$

Fifth approximation: $$x_5 = \frac{1}{5}\{10 + 1.718 + 1.0428\} = 2.553$$

$$y_4 = \frac{1}{4}\{12 - 5.0992\} = 1.725,$$

$$z_5 = \frac{1}{5}\{-1 - 2.5496 - 1.718\} = -1.054$$

Hence, approximating solution after having some other approximations is (up to 3 decimal places)

$$x = 2.556$$
$$y = 1.725$$
$$z = -1.055.$$ **Ans.**

Example 6. *Solve the following system of equations by Jacobi iteration method.*

$$3x + 4y + 15z = 54.8,\ x + 12y + 3z = 39.66 \text{ and } 10x + y - 2z = 7.74.$$

Sol. The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations, as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now, we write the equations in the form

$$\left.\begin{aligned} x &= \frac{1}{10}(7.74 - y + 2z) \\ y &= \frac{1}{12}(39.66 - x - 3z) \\ z &= \frac{1}{15}(54.8 - 3x - 4y) \end{aligned}\right\} \quad ...(1)$$

We start from an approximation = $x_0 = y_0 = z_0 = 0$

Substituting these on RHS of (1), we get

First approximation:

$$x_1 = \frac{1}{10}[7.74 - 0 + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12}[39.66 - 0 - 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15}[54.8 - 3(0) - 4(0)] = 3.6533333$$

Second approximation:

$$x_2 = \frac{1}{10}[7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12}[39.66 - 0.744 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15}[54.8 - 3(0.744) - 4(1.1383333)] = 3.1949778$$

Third approximation:

$$x_3 = \frac{1}{10}[7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12}[39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15}[54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

Fourth approximation:

$$x_4 = \frac{1}{10}[7.74 - 2.3903528 + 2(2.7545889)] = 1.0858825$$

$$y_4 = \frac{1}{12}[39.66 - 1.1802789 - 3(2.7545889)] = 2.5179962$$

$$z_4 = \frac{1}{15}[54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Fifth approximation:

$$x_5 = \frac{1}{10}\left[7.74 - 2.5179962 + 2(2.7798501)\right] = 1.0781704$$

$$y_5 = \frac{1}{10}\left[39.66 - 1.0858825 - 3(2.7798501)\right] = 2.5195473$$

$$z_5 = \frac{1}{15}\left[54.8 - 3(1.0858825) - 4(2.5179962)\right] = 2.7646912$$

Sixth approximation:

$$x_6 = \frac{1}{10}[7.74 - 2.5195473 + 2(2.7646912)] = 1.0749835$$

$$y_6 = \frac{1}{12}[39.66 - 1.0781704 - 3(2.7646912)] = 2.5239797$$

$$z_6 = \frac{1}{15}[54.8 - 3(1.0781704) - 4(2.5195473)] = 2.76582$$

Seventh approximation:

$$x_7 = \frac{1}{10}[7.74 - 2.5239797 + 2(2.76582)] = 1.074766$$

$$y_7 = \frac{1}{12}[39.66 - 1.0749835 - 3(2.76582)] = 2.523963$$

$$z_7 = \frac{1}{15}[54.8 - 3(1.0749835)] - 4(2.5239797) = 2.7652754$$

From the sixth and seventh approximations:

$x = 1.075,\ \ y = 2.524$ and $z = 2.765$ correct to three decimals. **Ans.**

8.6.2 Guass-Seidel Method

This is a modification of Gauss-Jacobi method. As before, the system of the linear equations.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \qquad ...(1)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \qquad ...(2)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \qquad ...(3)$$

and we start with the intial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in Eqn. (1), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Now substituting $x = x_1, z = z_0$ in Eqn. (2), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Substituting $x = x_1,\ \ y = y_1$ in Eqn. (3), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is continued till the value of $x, y, z,$ are obtained to the desired degree of accuracy. In general, kth iteration can be written as

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - c_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - c_2 z_k)$$

$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

The rate of convergence of Gauss-Seidel method is roughly twice that of Gauss-Jacobi method.

Example 7. *Solve by Gauss-Seidel iteration method the system of equations*

$$8x - 3y + 2z = 20; \ 6x + 3y + 12z = 35 \text{ and } 4x + 11y - z = 33.$$

Sol. From the given equations, we have

$$x = \frac{1}{8}(20 + 3y - 2z) \quad \text{...(1)}$$

$$y = \frac{1}{11}(33 - 4x + z) \quad \text{...(2)}$$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad \text{...(3)}$$

Putting $y = 0, \quad z = 0$ in RHS of (1), we get $x = \frac{20}{8} = 2.5$

Putting $x = 2.5, \quad z = 0$ in RHS of (2), we get

$$y = \frac{1}{11}[33 - 4(2.5)] = 2.0909091$$

Putting $x = 2.5, \ y = 2.0909091$ in RHS of (3), we get

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

For the **second approximation:**

$$x_2 = \frac{1}{8}(20 + 3y_1 - 2z_1)$$

$$= \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1]$$

$$= \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2]$$

$$= \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

Third approximation:

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

Fourth approximation:

$$x_4 = \frac{1}{8}[20 + 3(1.9825163)] - 2(0.9077262) = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.90777262] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.9120088$$

Fifth approximation:

$$x_5 = \frac{1}{8}[20 + 3(1.9856071) - 2(0.9120088)] = 3.0166005$$

$$y_5 = \frac{1}{11}[33 - 4(3.0166005) + 0.9120088] = 1.9859643$$

$$z_5 = \frac{1}{12}[35 - 6(3.0166005) - 3(1.9859643)] = 0.9118753$$

Sixth approximation:

$$x_6 = \frac{1}{8}[20 + 3(1.9859643) - 2(0.9118753)] = 3.0167568$$

$$y_6 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118753] = 1.9858913$$

$$z_6 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858913] = 0.9118099$$

Seventh approximation:

$$x_7 = \frac{1}{18}[20 + 3(1.9858913) - 2(0.9118099)] = 3.0167568$$

$$y_7 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118099] = 1.9858894$$

$$z_7 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858894)] = 0.9118159$$

Since at the sixth and seventh approximations, the values of $x, y, z,$ are the same, correct to four decimal places, we can stop the iteration process.

$\therefore \quad x = 3.0167, \quad y = 1.9858, \quad z = 0.9118.$

We find that 12 iteration are necessary in Gauss-Jacobi Method to get the same accuracy as achieved by 7 iterations in Gauss-Seidel method.

Example 8. *Solve the following system of equations using Gauss-Seidel method:*

$$10x + y + 2z = 44$$

$$2x + 10y + z = 51$$

$$x + 2y + 10z = 61$$

Sol. Given system of equations can be written as:

$$x = \frac{1}{10}(44 - y - 2z)$$

$$y = \frac{1}{10}(51 - 2x - z)$$

$$z = \frac{1}{10}(61 - x - 2y)$$

If we start by assuming $y_0 = 0 = z_0$ then, we obtain

$$x_1 = \frac{1}{10}(44 - 0 - 0) = 4.4$$

Now we susbtitute $x = 4.4$ and $z_0 = 0$ for y_1 and we obtain

$$y_1 = \frac{1}{10}(51 - 8.8 - 0) = 4.22$$

Similarly, we obtain $z_1 = \frac{1}{10}(61 - 4.4 - 2 \times 4.22) = 4.816$

Now for **second approximation,** we obtain

$$x_2 = 4.0154$$

$$y_2 = 3.0148$$

$$z_2 = 5.0955$$

Third approximation is given by

$$x_3 = 3.0794$$

$$y_3 = 3.9746$$

$$z_3 = 4.9971$$

Similarly, if we proceed up to eighth approximation, then, we obtain

$$x_8 = 3.00$$

$$y_8 = 4.00$$

$$z_8 = 5.00$$

PROBLEM SET 8.1

1. Apply Gauss-Elimination method to solve the system of equations.

$$x + 4y - z = -5$$

$$z + y - 6z = -12$$

$$3x - y - z = 4$$

[Ans. $x = \frac{117}{71}$, $y = \frac{-81}{71}$, $z = \frac{148}{71}$ **]**

2. Solve the following system of equations using Gauss-Elimination method:

(*a*) $x - y + z = 1$

$-3x + 2y - 3z = -6$

$2x - 5y + 4z = 5$ **[Ans.** –2, 3, 6]

(*b*) $x + 3y + 6z = 2$

$x - 4y + 2z = 7$

$3x - y + 4z = 9$ **[Ans.** 2, –1, $\frac{1}{2}$]

(*c*) $5x + y + z + u = 4$

$z + 7y + z + u = 12$

$x + y + 6z + u = -5$

$x + y + z + u = -6$ **[Ans.** 1, 2, –1, –2]

3. What do you understand by ill-conditioned equations? Consider the following system of equations:

$$100x - 200y = 100$$

$$-200x + 401y = 100$$

Determine, whether given system is ill-conditioned or not.

4. Solve the following system of equations by Jacobi's iterations method:

(*a*) $2x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$ **[Ans.** 1, –1, 1]

(*b*) $5x + 2y + z = 12$

$x + 4y + 2z = 15$

$x + 2y + 5z = 20$ **[Ans.** 1.08, 1.95, 3.16]

5. Using Gauss-Seidel method, solve the following system of equations:

(*a*) $10x + y + z = 12$

$2x + 10y + z = 13$

$2x + 2y + 10z = 14$ **[Ans.** 1, 1, 1]

(*b*) $2x - y + z = 5$

$2 + 3y - 2z = 7$

$x + 2y + 3z = 10$ **[Ans.** 3, 2, 1]

(*c*) $20x + y - 2z = -17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$ **[Ans.** 1, –1, 1]

❑❑❑

CHAPTER 9

Curve Fitting

9.1 INTRODUCTION

In many branches of applied mathematics and engineering sciences we come across experiments and problems, which involve two variables. For example, it is known that the speed v of a ship varies with the horsepower p of an engine according to the formula $p = a + bv^3$. Here a and b are the constants to be determined. For this purpose we take several sets of readings of speeds and the corresponding horsepowers. The problem is to find the best values for a and b using the observed values of v and p. Thus the general problem is to find a suitable relation or law that may exist between the variables x and y from a given set of observed values (x_i, y_i), $i = 1, 2, \ldots, n$. Such a relation connecting x and y is known as empirical law.

The process of finding the equation of the curve of best fit, which may be most suitable for predicting the unknown values, is known as curve fitting. Therefore, curve fitting means an exact relationship between two variables by algebraic equations. There are following methods for fitting a curve:

I. Graphic method
II. Method of group averages
III. Method of moments
IV. Principle of least square.

Out of above four methods, we will only discuss and study here principle of least square.

9.2 PRINCIPLE OF LEAST SQUARES

The method of least square is probably the most systematic procedure to fit a unique curve through the given data points.

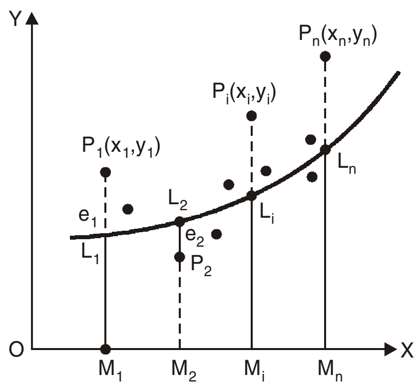


FIG. 9.1

Let the curve

$$y = a + bx + cx^2 + ...kx^{m-1} \quad ...(1)$$

be fitted to the set of n data points (x_1, y_1), (x_2, y_2), (x_3, y_3),......., (x_n, y_n). At ($x = x_i$) the observed (or experimental) value of the ordinate is $y_i = p_i m_i$ and the corresponding value on the fitting curve (i) is $a + bx_i + cx_i^2 + ...kx_i^m = L_i M_i$ which is the expected or calculated value. The difference of the observed and the expected value is $P_i M_i - L_i M_i = e_i$ (*say*) this difference is called error at ($x = x_i$) clearly some of the error $e_1, e_2, e_3, ..., e_i ...., e_n$ will be positive and other negative. To make all errors positive we square each of the errors *i.e.,* $S = e_1^2 + e_2^2 + e_3^2 + ... + e_i^2 + ... + e_n^2$ the curve of best fit is that for which $e's$ are as small as possible *i.e. S,* the sum of the square of the errors is a minimum this is known as the principle of least square.

9.2.1 Fitting of Straight Line

Let a straight line $y = a + bx$ ...(1)

Which is fitted to the given date points $(x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)$.

Let y_λ be the theoretical value for x_1 then $e_1 = y_1 - y_\lambda$

$$\Rightarrow \quad e_1 = y_1 - (a + bx_1)$$

$$\Rightarrow \quad e_1^2 = (y_1 - a - bx_1)^2$$

Now we have $S = e_1^2 + e_2^2 + e_3^2 + ... + e_n^2$

$$S = \sum_{i=1}^{n} e_i^1$$

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

By the principle of least squares, the value of S is minimum therefore

$$\frac{\partial S}{\partial a} = 0 \quad ...(2)$$

And

$$\frac{\partial S}{\partial b} = 0 \quad ...(3)$$

On solving equations (2) and (3), and dropping the suffix, we have

$$\sum y = na + b\sum x \quad ...(4)$$

$$\sum xy = a\sum x + b\sum x^2 \quad ...(5)$$

The equation (3) and (4) are known as normal equations.

On solving equations (3) and (4), we get the value of a and b. Putting the value of a and b in equation (1), we get the equation of the line of best fit.

9.2.2 Fitting of Parabola

Let a parabola $y = a + bx + cx^2$ ...(1)

which is fitted to a given date (x_1, y_1), (x_2, y_2), (x_3, y_3),..........., (x_n, y_n)

Let y_λ be the theoretical value for x_1 then $e_1 = y_1 - y_\lambda$

$$\Rightarrow \qquad e_1 = y_1 - \left(a + bx_1 + cx_1^2\right)$$

$$\Rightarrow \qquad e_1^2 = \left(y_1 - a - bx_1 - cx_1^2\right)^2$$

Now we have
$$S = \sum_{i=1}^{n} e_i^2$$

$$S = \sum_{i=1}^{n} \left(y_1 - a - bx_1 - cx_i^2\right)^2$$

By the principle of least squares, the value of S is minimum therefore,

$$\frac{\partial S}{\partial a} = 0, \ \frac{\partial S}{\partial b} = 0 \ \text{and} \ \frac{\partial S}{\partial c} = 0 \qquad ...(2)$$

Solving equation (2) and dropping suffix, we have

$$\sum y = na + b\sum x + c\sum x^2 \qquad ...(3)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad ...(4)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad ...(5)$$

The equation, (3), (4) and (5) are known as normal equations.

On solving equations (3), (4) and (5), we get the value of a, b and c. Putting the value of a, b and c in equation (1), we get the equation of the parabola of best fit.

9.2.3 Change of Scale

When the magnitude of the variable in the given data is large number then calculation becomes very much tedious then problem is further simplified by taking suitable scale when the value of x are given at equally spaced intervals.

Let h be the width of the interval at which the values of x are given and let the origin of x and y be taken at the point x_0, y_0 respectively, then putting

$$u = \frac{(x - x_0)}{h} \qquad \text{and} \qquad v = y - y_0$$

If m is odd then,
$$u = \frac{x - (\text{middle term})}{\text{interval } (h)}$$

But if m is even then,
$$u = \frac{x - (\text{mean of two middle term})}{\frac{1}{2}(\text{interval})}$$

Example 1. *Find the best-fit values of a and b so that* $y = a + bx$ *fits the data given in the table.*

x	*0*	*1*	*2*	*3*	*4*
y	*1*	*1.8*	*3.3*	*4.5*	*6.3*

Sol. Let the straight line is $y = a + bx$ ...(1)

x	y	xy	x^2
0	1	0	0
1	1.8	1.8	1
2	3.3	6.6	4
3	4.5	13.5	9
4	6.3	25.2	16
$\sum x = 10$	$\sum y = 16.9$	$\sum xy = 47.1$	$\sum x^2 = 30$

Normal equations are: $\sum y = na + b\sum x$ ...(2)

$$\sum xy = a\sum x + b\sum x^2 \quad \text{...(3)}$$

Here $n = 5$, $\sum x = 10$, $\sum y = 16.9$, $\sum xy = 47.1$ $\sum x^2 = 30$

Putting these values in normal equations we get,

$$16.9 = 5a + 10b$$

$$47.1 = 10a + 30b$$

On solving these two equations we get, $a = 0.72$, $b = 1.33$.

So required line $y = 0.72 + 1.33x$. **Ans.**

Example 2. *Fit a straight line to the given data regarding x as the independent variable.*

x	*1*	*2*	*3*	*4*	*5*	*6*
y	*1200*	*900*	*600*	*200*	*110*	*50*

Sol. Let the straight line obtained from the given data by $y = a + bx$ ...(1)

Then the normal equations are $\sum y = na + b\sum x$ ...(2)

$$\sum xy = a\sum x + b\sum x^2 \quad \text{...(3)}$$

x	y	x^2	xy
1	1200	1	1200
2	900	4	1800
3	600	9	1800
4	200	16	800
5	110	25	550
6	50	36	300
$\sum x = 21$	$\sum y = 3060$	$\sum x^2 = 91$	$\sum xy = 6450$

Putting all values in the equations (2) and (3), we get

$$3060 = 6a + 21b$$
$$6450 = 21a + 91b$$

Solving these equations, we get

$$a = 1361.97 \quad \text{and} \quad b = -243.42$$

hence the fitted equation is

$$y = 1361.97 - 243.42x. \qquad \textbf{Ans.}$$

Example 3. *Find the least square polynomial approximation of degree two to the data.*

x	0	1	2	3	4
y	–4	–1	4	11	20

also compute the least error.

Sol. Let the equation of the polynomial be $y = a + bx + cx^2$ ...(1)

x	y	xy	x^2	x^2y	x^3	x^4
0	–4	0	0	0	0	0
1	–1	–1	1	–1	1	1
2	4	8	4	16	8	16
3	11	33	9	99	27	81
4	20	80	16	320	64	256
$\sum x = 10$	$\sum y = 30$	$\sum xy = 120$	$\sum x^2 = 30$	$\sum x^2y = 434$	$\sum x^3 = 100$	$\sum x^4 = 354$

The normal equations are :

$$\sum y = na + b\sum x + c\sum x^2 \qquad \text{...(2)}$$
$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \text{...(3)}$$
$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \text{...(4)}$$

Here $n = 5$, $\sum x = 10$, $\sum y = 30$, $\sum xy = 120$, $\sum x^2 = 30$, $\sum x^2y = 434$, $\sum x^3 = 100$, $\sum x^4 = 354$.

Putting all these values in (2), (3) and (4), we get

$$30 = 5a + 10b + 30c \qquad \text{...(5)}$$
$$120 = 10a + 30b + 100c \qquad \text{...(6)}$$
$$434 = 30a + 100b + 354c \qquad \text{...(7)}$$

On solving these equations, we get $a = -4$, $b = 2$, $c = 1$. Therefore required polynomial is $y = -4 + 2x + x^2$, errors = 0. **Ans.**

Example 4. *Fit a second-degree parabola to the following data taking x as the indenpendent variable.*

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Sol. The equation of second-degree parabola is given by $y = a + bx + cx^2$ and the normal equations are :

$$\left.\begin{aligned} \sum y &= na + b\sum x + c\sum x^2 \\ \sum xy &= a\sum x + b\sum x^2 + c\sum x^3 \\ \sum x^2 y &= a\sum x^2 + b\sum x^3 + c\sum x^4 \end{aligned}\right\} \quad ...(1)$$

Here $n = 9$. The various sums are appearing in the table as follows:

x	y	xy	x^2	x^2y	x^3	x^4
1	2	2	1	2	1	1
2	6	12	4	24	8	16
3	7	21	3	63	27	81
4	8	32	16	128	64	256
5	10	50	25	250	125	625
6	11	66	36	396	216	1296
7	11	77	49	539	343	2401
8	10	80	64	640	512	4096
9	09	81	81	729	729	6561
$\sum x = 45$	$\sum y = 74$	$\sum xy = 421$	$\sum x^2 = 279$	$\sum x^2y = 2771$	$\sum x^3 = 2025$	$\sum x^4 = 15333$

Putting these values of $\sum x$, $\sum y$, $\sum x^2$, $\sum xy$, $\sum x^2y$, $\sum x^3$, and $\sum x^4$, in equation (1) and solving the equations for a, b and c; we get

$$a = -0.923; \quad b = 3.520; \quad c = -0.267.$$

Hence the fitted equation is

$$y = -0.923 + 3.53x - 0.267x^2. \quad \textbf{Ans.}$$

Example 5. *Show that the line of fit to the following data is given by $y = 0.7x + 11.28$.*

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Sol. Here $m = 6$ (even)

Let $x_0 = 12.5$, $h = 5$, $y_0 = 20$ (say)

Then, $u = \dfrac{x-12.5}{2.5}$ and $v = y-20$, we get

x	y	u	v	uv	u^2
0	12	–5	–8	40	25
5	15	–3	–5	15	9
10	17	–1	–3	3	1
15	22	1	2	2	1
20	24	3	4	12	9
25	30	5	10	50	25
		$\sum u = 0$	$\sum v = 0$	$\sum uv = 122$	$\sum u^2 = 70$

The normal equations are :

$$0 = 6a + 0 \times b \quad \Rightarrow a = 0$$

$$122 = 0 \times a + 70b \Rightarrow b = 1.743$$

Thus line of fit is $v = 1.743u.$

or
$$y - 20 = (1.743)\left(\frac{x-12.50}{2.5}\right) = 7x - 8.175$$

or
$$y = 0.7x + 11.285. \quad \textbf{Ans.}$$

Example 6. *Fit a second-degree parabola to the following data by least square method.*

x	*1929*	*1930*	*1931*	*1932*	*1933*	*1934*	*1935*	*1936*	*1937*
y	*352*	*356*	*357*	*358*	*360*	*361*	*361*	*360*	*359*

Sol. Taking $x_0 = 1933$, $y_0 = 357$ then $u = \dfrac{(x-x_0)}{h}$

x	$u = x-1933$	y	$v = y-357$	uv	u^2	u^2v	u^3	u^4
1929	–4	352	–5	20	16	–80	–64	256
1930	–3	360	–1	3	9	–9	–27	81
1931	–2	357	0	0	4	0	–8	16
1932	–1	358	1	–1	1	1	–1	1
1933	0	360	3	0	0	0	0	0
1934	1	361	4	4	1	4	1	1
1935	2	361	4	8	4	16	8	16
1936	3	360	3	9	9	27	27	81
1937	4	359	2	8	16	32	64	256
Total	$\sum u = 0$		$\sum v = 11$	$\sum uv = 51$	$\sum u^2 = 60$	$\sum u^2v = -9$	$\sum u^3 = 0$	$\sum u^4 = 708$

Here $h = 1$

Taking $u = x - x_0$ and $v = y - y_0$, therefore $u = x - 1933$ and $v = y - 357$

Then the equation $y = a + bx + cx^2$ is transformed to $v = A + Bu + Cu^2$ ... (1)

Normal equations are:

$$\sum v = 9A + B\sum u + C\sum u^2 \quad \Rightarrow \quad 11 = 9A + 60C$$

$$\sum uv = A\sum u + B\sum u^2 + C\sum u^3 \quad \Rightarrow \quad B = 17/20$$

$$\sum u^2 v = A\sum u^2 + B\sum u^3 + C\sum u^4 \quad \Rightarrow \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \dfrac{694}{231}$, $B = \dfrac{17}{20}$ and $C = -\dfrac{247}{924}$

$$\therefore \qquad v = \frac{694}{231} + \frac{17}{20}u - \frac{247}{924}u^2$$

$$\Rightarrow \quad y - 357 = \frac{694}{231} + \frac{17}{20}(x - 1933) - \frac{247}{924} + (x - 1933)^2$$

$$\Rightarrow \quad y - 357 = \frac{694}{231} + \frac{17}{20}x - \frac{32861}{20} - \frac{247}{924}x^2 - \frac{247}{924}(-3866x) - \frac{247}{924} + (1933)^2$$

$$\Rightarrow \quad y = \frac{694}{231} - \frac{32861}{20} - \frac{247}{924}(1933)^2 + \frac{17}{20}x + \frac{247 \times 3866}{924}x - \frac{247}{924}x^2$$

$$\Rightarrow \quad y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$$

$$\Rightarrow \quad y = -1000106.41 + 1034.29x - 0.267x^2. \quad \textbf{Ans.}$$

Example 7. *Fit second degree parabola to the following*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Sol. Here $m = 5$ (odd) therefore $x_0 = 2$

Now let $u = x - 2$, $v = y$ and the curve of fit be $v = a + bu + cu^2$.

x	y	u	v	uv	u^2	u^2v	u^3	u^4
0	1	–2	1	–2	4	4	–8	16
1	1.8	–1	1.8	–1.8	1	1.8	–1	1
2	1.3	0	1.3	0	0	0	0	0
3	2.5	1	2.5	2.5	1	2.5	1	1
4	6.3	2	6.3	12.6	4	25.2	8	16
Total		0	12.9	11.3	10	33.5	0	34

Hence the normal equations are:

$$\sum v = 5a + b\sum u + c\sum u^2$$

$$\sum uv = a\sum u + b\sum u^2 + c\sum u^3$$

$$\sum u^2 v = a\sum u^2 + b\sum u^3 + c\sum u^4$$

On putting the values of $\sum u$, $\sum v$ etc. from the table in these, we get

$$12.9 = 5a + 10c, \quad 11.3 = 10b, \quad 33.5 = 10a + 34c.$$

On solving these equations, we get

$$a = 1.48, \; b = 1.13 \text{ and } c = 0.55$$

Therefore the required equation is $v = 1.48 + 1.13u + 0.55u^2$.

Again substituting $u = x - 2$ and $v = y$, we get

$$y = 1.48 + 1.13(x-2) + 0.55(x-2)^2$$

or $$y = 1.42 - 1.07x + 0.55x^2.$$ **Ans.**

9.2.4 Fitting of an Exponential Curve

Suppose an exponential curve of the form

$$y = ae^{bx}$$

Taking logarithm on both the sides, we get

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

i.e., $$Y = A + Bx \quad \text{...(1)}$$

where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$.

The normal equations for (1) are,

$$\sum Y = nA + B\sum x$$

$$\sum xY = A\sum x + B\sum x^2$$

On solving above two equations, we get A and B.

then $$a = anti \log A, \; b = \frac{B}{\log_{10} e}$$

9.2.5 Fitting of the Curve $y = ax + bx^2$

Error of estimate for i th point (x_i, y_i) is

$$e_i = \left(y_i - ax_i - bx_i^2\right)$$

We have, $$S = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} \left(y_i - ax_i - bx_i^2\right)^2$$

By the principle of least square, the value of S is minimum

$$\therefore \quad \frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0$$

Now $$\frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2\left(y_i - ax_i - bx_i^2\right)(-x_i) = 0$$

or $$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 \qquad ...(1)$$

and $$\frac{\partial S}{\partial b} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2\left(y_i - ax_i - bx_i^2\right)\left(-x_i^2\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^4 \qquad ...(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are,

$$\sum xy = a\sum x^2 + b\sum x^3$$
$$\sum x^2 y = a\sum x^3 + b\sum x^4$$

9.2.6 Fitting of the Curve $y = ax + \frac{b}{x}$

Error of estimate for *i*th point (x_i, y_i) is

$$e_i = \left(y_i - ax_i - \frac{b}{x_i}\right)$$

We have, $$S = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n}\left(y_i - ax_i - \frac{b}{x_i}\right)^2$$

By the principle of least square, the value of S is minimum

$$\therefore \quad \frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0$$

Now $$\frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2\left(y_i - ax_i - \frac{b}{x_i}\right)\left(-\frac{1}{x_i}\right) = 0$$

or $$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + nb \qquad ...(1)$$

and $$\frac{\partial S}{\partial b} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2\left(y_i - ax_i - \frac{b}{x_i}\right)\left(-\frac{1}{x_i}\right) = 0$$

or $$\sum_{i=1}^{n} \frac{y_i}{x_i} = na + b\sum_{i=1}^{n} \frac{1}{x_i^2} \qquad ...(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are :

$$\sum xy = a\sum x^2 + nb$$

$$\sum \frac{y}{x} = na + b\sum \frac{1}{x^2}$$

Where n is the number of pair of values of x and y.

9.2.7 Fitting of the Curve $y = \frac{c_0}{x} + c_1\sqrt{x}$

Error of estimate for i th point (x_i, y_i) is

$$e_i = \left(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}\right)$$

We have, $$S = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} \left(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}\right)^2$$

By the principle of least square, the value of S is minimum

$$\therefore \quad \frac{\partial S}{\partial c_0} = 0 \text{ and } \frac{\partial S}{\partial c_1} = 0$$

Now $$\frac{\partial S}{\partial c_0} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}) - \frac{1}{x_i} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \frac{y_i}{x_i} = c_0\sum_{i=1}^{n} \frac{1}{x_i^2} + c_1\sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} \qquad ...(1)$$

and $$\frac{\partial S}{\partial c_1} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2\left(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}\right)(-\sqrt{x_i}) = 0$$

or $$\sum_{i=1}^{n} y_i \sqrt{x_i} = c_0 \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} + c_1 \sum_{i=1}^{n} x_i \qquad ...(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are,

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}}$$

$$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sqrt{x}$$

Example 8. *Find the curve of best fit of the type* $y = ae^{bx}$ *to the following data by the method of least sqaures:*

$x:$	1	5	7	9	12
$y:$	10	15	12	15	21

Sol. The curve to be fitted is $y = ae^{bx}$ or $Y + A + Bx$, where $Y = \log_{10} y$, $A = \log_{10} a$, and $B = b\log_{10} e$.

Therefore the normal equations are:

$$\sum Y = 5A + B\sum x$$

$$\sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
1 5 7 9 12	10 15 12 15 21	1.0000 1.1761 1.0792 1.1761 1.3222	1 25 49 81 144	1 5.8805 7.5544 10.5849 15.8664
$\sum x = 34$		$\sum Y = 5.7536$	$\sum x^2 = 300$	$\sum xY$

Substituting the values of $\sum x$, etc. and calculated by means of above table in the normal equations. We get,

$$5.7536 = 5A + 34B$$

and $$40.8862 = 34A + 300B$$

On solving these equations we obtain, $A = 0.9766$; $B = 0.02561$

Therefore $a = \text{anti} \log_{10} A = 9.4754$; $b = \dfrac{B}{\log_{10} e} = 0.059$

Hence the required curve is $y = 9.4754e^{0.059x}$. **Ans.**

Example 9. *For the given data below, find the equation to the best fitting exponential curve of the form* $y = ae^{bx}$.

x	1	2	3	4	5	6
y	1.6	4.5	13.8	40.2	125	300

Sol. $y = ae^{bx}$

On taking log both the sides, log y = log a + bx log e, which is of the form Y = A + Bx, where Y = log y, A = log a and B = b log e.

x	y	$Y = \log y$	x^2	xY
1	1.6	0.2041	1	0.2041
2	4.5	0.6532	4	1.3064
3	13.8	1.1399	9	3.4197
4	40.2	1.6042	16	6.4168
5	12.5	2.0969	25	10.4845
6	300	2.4771	36	14.8626
$\sum x = 21$		$\sum Y = 8.1754$	$\sum x^2 = 91$	$\sum xY = 36.6941$

Normal equations are: $$\sum Y = 6A + B\sum x$$

$$\sum xY = A\sum x + B\sum x^2$$

Therefore from these equations, we have,

$$8.1754 = 6A+21B$$
$$36.6941 = 21A+91B$$

$\Rightarrow$ $A = -0.2534,\ B = 0.4617$

Therefore, $a = anti\log A = anti\log(-0.2534) = anti\log(\bar{1}.7466) = 0.5580$

and $$b = \frac{B}{\log e} = \frac{0.4617}{0.4343} = 1.0631$$

Hence required equation is $y = 0.5580e^{1.0631x}$. **Ans.**

Example 10. *Given the following experimental values:*

x	0	1	2	3
y	2	4	10	15

Fit by the method of least squares a parabola of the type $y = a + bx^2$.

Sol. Error of estimate for i th point (x_i, y_i) is $e_i = (y_i - a - bx_i^2)$

By the principle of least squares, the values of a and b are such that

$$S = \sum_{i=1}^{4} e_i^2 = \sum_{i=1}^{4}\left(y_i - a - bx_i^2\right)^2 \text{ is minimum.}$$

Therefore normal equations are given by

$$\frac{\partial S}{\partial a} = 0 \Rightarrow \sum y = na + b\sum x^2 \quad ...(1)$$

$$\frac{\partial S}{\partial b} = 0 \Rightarrow \sum x^2 y = a\sum x^2 + b\sum x^4 \quad ...(2)$$

x	y	x^2	x^2y	x^4
0	2	0	0	0
1	4	1	4	1
2	10	4	40	16
3	15	9	135	81
Total	$\sum y = 31$	$\sum x^2 = 14$	$\sum x^2y = 179$	$\sum x^4 = 98$

Here $n = 4$

From (1) and (2), $31 = 4a + 14b$ and $179 = 14a + 98b$

Solving for a and b, we get $a = 2.71$ and $b = 1.44$

Hence the required curve is $y = 2.71 + 1.44x^2$

Example 11. *By the method of least square, find the curve* $y = ax + bx^2$ *that best fits the following data:*

x	*1*	*2*	*3*	*4*	*5*
y	*1.8*	*5.1*	*8.9*	*14.1*	*19.8*

Sol. Error of estimate for ith point (x_i, y_i) is $e_i = \left(y_i - ax_i - bx_i^2\right)$

By the principle of least squares, the values of a and b are such that

$$S = \sum_{i=1}^{5} e_i^2 = \sum_{i=1}^{5} \left(y_i - ax_i - bx_i^2\right)^2 \text{ is minimum.}$$

Therefore normal equations are given by

$$\frac{\partial S}{\partial a} = 0 \Rightarrow \sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i^2 + b\sum_{i=1}^{5} x_i^3 \quad \text{and} \quad \frac{\partial S}{\partial b} = 0 \Rightarrow \sum_{i=1}^{5} x_i^2 y_i = a\sum_{i=1}^{5} x_i^3 + b\sum_{i=1}^{5} x_i^4$$

Dropping the suffix i, normal equations are :

$$\sum xy = a\sum x^2 + b\sum x^3 \qquad ...(1)$$

$$\sum x^2y = a\sum x^3 + b\sum x^4 \qquad ...(2)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.8	1	1	1	1.8	1.8
2	5.1	4	8	16	10.2	20.4
3	8.9	9	27	81	26.7	80.1
4	14.1	16	64	256	56.4	225.6
5	19.8	25	125	625	99	495
Total		$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 194.1$	$\sum x^2y = 822.9$

Substituting these values in equations (1) and (2), we get

$$194.1 = 55a + 225b \text{ and } 822.9 = 225a + 979b$$

$$\Rightarrow \quad a = \frac{83.85}{55} \approx 1.52$$

and $$b = \frac{317.4}{664} \approx 0.49$$

Hence required parabolic curve is $y = 1.52x + 0.49x^2$. **Ans.**

Example 12. *Fit an exponential curve of the form* $y = ab^x$ *to the following data:*

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Sol. $y = ab^x$ takes the form $Y = A + Bx$, where $Y = \log y$; $A = \log a$ and $B = \log b$.

Hence the normal equations are given by

$$\sum Y = nA + B\sum x \text{ and } \sum xY = A\sum x + \sum x^2$$

x	y	$Y = \log y$	xY	x^2
1	1.0	0.0000	0.000	1
2	1.2	0.0792	0.1584	4
3	1.8	0.2553	0.7659	9
4	2.5	0.3979	1.5916	16
5	3.6	0.5563	2.7815	25
6	4.7	0.6721	4.0326	36
7	6.6	0.8195	4.7365	49
8	9.1	0.9590	7.6720	64
$\sum x = 36$	$\sum y = 30.5$	$\sum Y = 3.7393$	$\sum xY = 22.7385$	$\sum x^2 = 204$

Putting the values in the normal equations, we obtain

$$3.7393 = 8A + 36B \text{ and } 22.7385 = 36A + 204B \Rightarrow B = 0.1406 \text{ and } A = 1.8336$$

$$\Rightarrow \quad b = anti\log B = 1.38 \text{ and } a = anti\log A = 0.68.$$

Thus the required curve of best fit is $y = (0.68)(1.38)x$. **Ans.**

Example 13. *Fit a curve* $y = ab^x$ *to the following data:*

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

Sol. Given equation $y = ab^x$ reduces to $Y = A + Bx$ where $Y = \log y$, $A = \log a$ and $B = \log b$.

The normal equations are:

$$\sum \log y = n\log a + \log b \sum x$$

$$\sum x\log y = \log a \sum x + \log b \sum x^2$$

The calculations of $\sum x$, $\sum \log y$, $\sum x^2$ and $\sum x\log y$ are substitute in the following tabular form.

x	y	x^2	$\log y$	$x \log y$
2	144	4	2.1584	4.3168
3	172.8	9	2.2375	6.7125
4	207.4	16	2.3168	9.2672
5	248.8	25	2.3959	11.9795
6	298.5	36	2.4749	14.8494
20		90	11.5835	47.0254

Putting these values in the normal equations, we have

$$11.5835 = 5\log a + 20\log b$$

$$47.1254 = 20\log a + 90\log bx^2.$$ **Ans.**

Solving these equations and taking antilog, we have $a = 100$, b = 1.2 approximate. Therefore equation of the curve is $y = 100(1.2)x$.

Example 14. *Derive the least square equations for fitting a curve of the type y = ax² + (b/x) to a set of n points. Hence fit a curve of this type to the data.*

x	1	2	3	4
y	–1.51	0.99	3.88	7.66

Sol. Let the n points are given by (x_1, y_1), (x_2, y_2), (x_3, y_3) ,............., (x_n, y_n). The error of estimate for the *i*th point (x_i, y_i) is $e_i = [y_i - ax_i^2 - (b/x_i)]$.

By the principle of least square, the values of a and b are such so that the sum of the square of error S, *viz.,*

$$S = \sum_{i=1}^{n} e_i^2 = \sum \left(y_i - ax_i^2 - \frac{b}{x_i} \right)^2 \quad \text{is minimum.}$$

Therefore the normal equations are given by

$$\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$$

or $$\sum_{i=1}^{n} y_i x_i^2 = a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i \quad \text{and} \quad \sum_{i=1}^{n} \frac{y_i}{x_i} = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} \frac{1}{x_i^2}$$

These are the required least square equations.

x	y	x^2	x^4	$\frac{1}{x}$	$\frac{1}{x^2}$	yx^2	$\frac{y}{x}$
1	–1.51	1	1	1	1	–1.51	–1.51
2	0.99	4	16	0.5	0.25	3.96	0.495
3	3.88	9	81	0.3333	0.1111	34.92	1.2933
4	3.66	16	256	0.25	0.0625	122.56	0.9150
10			354		1.4236	159.93	1.1933

Putting the values in the above least square equations, we get

$$159.93 = 354a + 10b \quad \text{and} \quad 2.1933 = 10a + 1.4236b.$$

Solving these, we get $a = 0.509$ and $b = -2.04$.

Therefore the equation of the curve fitted to the above data is $y = 0.509x^2 - \frac{2.04}{x}$. **Ans.**

Example 15. *Fit the curve* $pv^{\gamma} = k$ *to the following data:*

$p(kg/cm^2)$	*0.5*	*1*	*1.5*	*2*	*2.5*	*3*
$v(litres)$	*1620*	*1000*	*750*	*620*	*520*	*460*

Sol. Given $\quad pv^{\gamma} = k$

$$v = \left(\frac{k}{p}\right)^{1/\gamma} = k^{1/\gamma} p^{-1/\gamma}$$

On taking log both the sides, we get

$$\log v = \frac{1}{\gamma} \log k - \frac{1}{\gamma} \log p$$

This is of the form $\quad Y = A + BX$

Where $Y = \log v$, $X = \log p$, $A = \frac{1}{\gamma} \log k$ and $B = -\frac{1}{\gamma}$

p	v	X	Y	XY	X^2
0.5	1620	–0.30103	3.20952	–0.96616	0.09062
1	1000	0	3	0	0
1.5	750	0.17609	2.87506	0.50627	0.03101
2	620	0.30103	2.79239	0.84059	0.09062
2.5	520	0.39794	2.716	1.08080	0.15836
3	460	0.47712	2.66276	1.27046	0.22764
Total		$\sum X = 1.05115$	$\sum Y = 17.25573$	$\sum XY = 2.73196$	$\sum X^2 = 0.59825$

Here $n = 6$

Normal equations are,

$$17.25573 = 6A + 1.05115B$$

$$2.73196 = 1.05115A + 0.59825B$$

On solving these, we get

$$A = 2.99911 \quad \text{and} \quad B = -0.70298$$

$$\therefore \qquad \gamma = -\frac{1}{B} = \frac{1}{0.70298} = 1.42252$$

Again $$\log k = \gamma A = 4.26629$$

$$\therefore \qquad k = anti\log(4.26629) = 18462.48$$

Hence the required curve is $pv^{1.42252} = 18462.48.$ **Ans.**

Example 16. *For the data given below, find the equation to the best fitting exponential curve of the form* $y = ae^{bx}$.

x	1	2	3	4	5	6
y	1.6	4.5	13.8	40.2	125	300

Sol. Given $y = ae^{bx}$, taking log we get $\log y = \log a + bx \log_{10} e$ which is of the $Y = A + Bx$, where $Y = \log y$, $A = \log a$ and $B = \log_{10} e$.

Put the values in the following tabular form, also transfer the origin of x series to 3, so that $u = x - 3$.

x	y	$\log y = Y$	$x - 3 = u$	uY	u^2
1	1.6	0.204	–2	–408	4
2	4.5	0.653	–1	–653	1
3	13.8	1.140	0	0	0
4	40.2	1.604	1	1.604	1
5	125.0	2.094	2	4.194	4
6	300	2.477	3	7.431	9
Total		8.175	3	12.168	19

In case $Y = A' + B'.u$, then normal equations are given by

$$\sum Y = nA' + B'\sum u \qquad \Rightarrow 8.175 = 6A' + 3B' \qquad ...(1)$$

$$\sum uY = A'\sum u + B'\sum u^2 \Rightarrow 12.168 = 3A' + 19B' \qquad ...(2)$$

Solving (1) and (2), we get

$$A' = 1.13 \text{ and } B' = 0.46$$

This equation is $Y = 1.13 + 0.46u$, *i.e.*, $Y = 1.13 + 0.46(x - 3)$

or $$Y = 0.46x - 0.25$$

which gives log a = –25 *i.e., anti* log(–25) = anti log(1.75) = 0.557

$$b = \frac{B}{\log_{10} e} = \frac{64}{0.4343} = 1.06$$

Hence the required equation of the curve is $y = (0.557)e^{1.06}x$. **Ans.**

PROBLEM SET 9.1

1. Fit a straight line to the given data regarding x as the independent variable:

x	1	2	3	4	6	8
y	2.4	3.1	3.5	4.2	5.0	6.0

[**Ans.** $y = 2.0253 + 0.502x$]

2. Fit a straight line $y = a + bx$ to the following data by the method of least square:

x	0	1	3	6	8
y	1	3	2	5	4

[**Ans.** $1.6 + 0.38x$]

3. Find the least square approximation of the form $y = a + bx^2$ for the data:

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.01	0.99	0.85	0.81	0.75

[**Ans.** $y = 1.0032 - 1.1081x^2$]

4. Fit a second degree parabola to the following data:

x	0.0	1.0	2.0
y	1.0	6.0	17.0

[**Ans.** $y = 1 + 2x + 3x^2$]

5. Fit a second degree parabola to the following data:

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

[**Ans.** $y = 1.04 - 0.193x + 0.243x^2$]

6. Fit a second degree parabola to the following data by the least square method:

x	1	2	3	4	5
y	1090	1220	1390	1625	1915

[**Ans.** $y = 27.5x^2 + 40.5x + 1024$]

7. Fit a parabola $y = a + bx + cx^2$ to the folliwng data:

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

[Ans. $y = 0.34 - 0.78x + 0.99x^2$]

8. Determine the constants a and b by the method of least squares such that $y = ae^{bx}$ fits the following data:

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

[Ans. $y = 1.49989e^{0.50001x}$]

9. Fit a least square geometric curve $y = ax^b$ to the following data:

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

[Ans. $y = 0.5012x^{1.9977}$]

10. A person runs the same race track for five consecutive days and is timed as follows:

Day (x)	1	2	3	4	5
Time (y)	15.3	15.1	15	14.5	14

Make a least square fit to the above data using a function $a + \frac{b}{x} + \frac{c}{x^2}$.

[Ans. $y = 13.0065 + \frac{6.7512}{x} + \frac{4.4738}{x^2}$]

11. Use the method of least squares to fit the curve $y = \frac{c_0}{x} + c_1\sqrt{x}$ to the following table of values:

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

[Ans. $y = \frac{1.97327}{x} + 3.28182\sqrt{x}$]

12. Using the method of least square to fit a parabola $y = a + bx + cx^2$ in the following data:

$(x, y) : (-1, 2), (0, 0), (0, 1), (1, 2)$ **[Ans.** $y = \frac{1}{2} + \frac{3}{2}x^2$]

13. The pressure of the gas corresponding to various volumes V is measured, given by the following data:

$V\left(cm^3\right)$	50	60	70	90	100
$p\left(kgcm^{-2}\right)$	64.7	51.3	40.5	25.9	78

Fit the data to the equation $pV^{\gamma} = c.$

[**Ans.** $pV^{0.28997} = 167.78765$]

9.3 REGRESSION

We know that in a functional relation between two variables, if we know the value of one variable, then the corresponding value of the other variable can be determined exactly.

But, in a statistical relationship between the two variables, when the value of one variable is known, we can simply estimate the corresponding value of another variable.

Regression analysis is the method used for estimating the unknown values of one variable corresponding to the known values of another variable.

9.3.1 Dependent and Independent Variables

Suppose there is a relation between two variables. The variable, whose values are known, is known as independent variable, while another one is called the dependent variable.

9.3.2 Line of Regression

Let $\{x_i, y_i\} : 1 \le i \le n$ and $1 \le j \le n\}$ be a bivariate distribution. If we plot the corresponding values of x and y, taking the values of x along x-axis and the values of y along y-axis, we obtain a collection of dots, called the scatter-diagram.

If the scatter diagram indicates some relationship between x and y, then the dots of the scatter diagram will be concentrated round a line, called the line of regression or the line of best fit.

9.3.3 Regression Line of *y* on *x*

If we have to predict the values of y from given values of x, then the line of regression has an equation of the form $y = a + bx$. This is called the regression line of y on x.

9.3.4 Regression Line of x on y

If we have to predict the values of x from given values of y, then the line of regression has an equation of the form $x = a + by$. This is called the regression line of x on y.

9.3.5 To obtain the Equation of Line of Regression of *y* on *x*

Suppose that the line approximating the set of point $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\ldots, (x_n, y_n)$ has the equation:

$$y = a + bx \qquad \ldots(1)$$

Then, $y_i = a + bx_i$ and $x_i y_i = ax_i + bx_i^2$ for each $i = 1, 2, ..., n$ therefore

$$\sum y_i = na + b\sum x_i \quad ...(2)$$

and
$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad ...(3)$$

Equations (2) and (3) are normal equations for this line.

Solving (2) and (3) for a and b and putting these values in (1), we obtain the required equation of the line of regression of y on x.

9.3.6 To obtain the Equation of Line of Regression of x on y

Suppose that the line approximating the set of points $(x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)$ has the equation:

$$x = a + by \quad ...(1)$$

Then, $x_i = a + by_i$ and $x_i y_i = ay_i + by_i^2$ for each $i = 1, 2, ..., n$ therefore

$$\sum x_i = na + b\sum y_i \quad ...(2)$$

and
$$\sum x_i y_i = a\sum y_i + b\sum y_i^2 \quad ...(3)$$

Equations (2) and (3) are normal equations for this line.

Solving (2) and (3) for a and b and putting these values in (1), we obtain the required equation of the line of regression of x on y.

Example 1. *Find the line of regression of y on x for the following data:*

x	10	9	8	7	6	4	3
y	8	12	7	10	8	9	6

Sol. Here $n = 7$. Now form the table given below:

x_i	y_i	x_i^2	$x_i y_i$
10	8	100	80
9	12	81	108
8	7	64	56
7	10	49	70
6	8	36	48
4	9	16	36
3	6	9	18
$\sum x_i = 47$	$\sum y_i = 60$	$\sum x_i^2 = 352$	$\sum x_i y_i = 416$

Let the required equation be $y = a + bx$ ...(1)

Then, $y_i = a + bx_i$ and $x_i y_i = ax_i + bx_i^2$ for each i.

Therefore the normal equations are:

$$\sum y_i = na + b\sum x_i \qquad ...(2)$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \qquad ...(3)$$

Putting the values from the table in (2) and (3), we get

$$60 = 7a + 47b$$

$$416 = 47a + 355b$$

Solving these equations, we get $a = 8.582$ and $b = 1.094$.

Putting these values in (1) the required equation is $y = 8.582 + 1.094x$ **Ans.**

Example 2. *Find the line of regression of x on y for the following data:*

x	*6*	*2*	*10*	*4*	*8*
y	*9*	*11*	*5*	*8*	*7*

Sol. Here $n = 5$. Now, form the table given below :

x_i	y_i	y_i^2	$x_i y_i$
6	9	81	54
2	11	121	22
10	5	25	50
4	8	64	32
8	7	49	56
$\sum x_i = 30$	$\sum y_i = 40$	$\sum y_i^2 = 340$	$\sum x_i y_i = 214$

Let the required line be, $x = a + by$ ...(1)

Then $x_i = a + by_i$ and $x_i y_i = ay_i + by_i^2$ for each i.

Therefore the normal equations are:

$$\sum x_i = na + b\sum y_i \qquad ...(2)$$

$$\sum x_i y_i = a\sum y_i + b\sum y_i^2 \qquad ...(3)$$

Putting the values from the table in (2) and (3), we get

$$30 = 5a + 40b \quad \Rightarrow \quad a + 8b = 6$$

$$214 = 40a + 340b \Rightarrow 20a + 170b = 107$$

On solving these equations we get $a = 16.4$ and $b = -1.3$.

Therefore the requried equation is, $x = 16.4 - 1.3y$. **Ans.**

Example 3. *Prove that arithmetic mean of the coefficient of regression is greater than the coefficient of correlation.*

Sol. Coefficients of regression are $r\frac{\sigma_y}{\sigma_x}$, $r\frac{\sigma_x}{\sigma_y}$

We have to prove that $A.M. > r$.

or $$\frac{1}{2}\left[r\frac{\sigma_y}{\sigma_x}+r\frac{\sigma_x}{\sigma_y}\right] > 2 \quad \text{or} \quad \frac{1}{2}\left[\frac{\sigma_y}{\sigma_x}+\frac{\sigma_x}{\sigma_y}\right] > 1$$

or $$\frac{\sigma_y}{\sigma_x}+\frac{\sigma_x}{\sigma_y}-2 > 0 \quad \text{or} \quad \frac{1}{\sigma_x\sigma_y}[\sigma_x^2+\sigma_y^2-2\sigma_x\sigma_y] > 0$$

or $$\frac{1}{\sigma_x\sigma_y}[\sigma_x-\sigma_y]^2$$ which is true. **Proved.**

Example 4. *Find the regression line of y on x for the following data:*

x	*1*	*3*	*4*	*6*	*8*	*9*	*11*	*14*
y	*1*	*2*	*4*	*4*	*5*	*7*	*8*	*9*

Estimate the value of y, when $x = 10$.

Sol.

S.No.	*x*	*y*	*xy*	x^2
1	1	1	1	1
2	3	2	6	9
3	4	4	16	16
4	6	4	24	36
5	8	5	40	64
6	9	7	63	81
7	11	8	88	121
8	14	9	126	196
Total	56	40	364	524

Let $y = a + bx$ be the line of regression of y on x. Therefore normal equations are :

$$\sum y_i = na + b\sum x_i \quad \Rightarrow \quad 40 = 8a + 56b \qquad ...(1)$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad \Rightarrow \quad 364 = 56a + 524b \qquad ...(2)$$

On solving (1) and (2) we get

$$a = \frac{6}{11} \text{ and } b = \frac{7}{11}$$

The equation of the required line is

$$y = \frac{6}{11} + \frac{7}{11}x \quad \text{or} \quad 7x - 11y + 6 = 0$$

If $$x = 0,\ y = \frac{6}{11} + \frac{7}{11}(10) = \frac{76}{11} = 6\frac{10}{11}.$$ **Ans.**

Example 5. *In a study between the amount of rainfall and the quantity of air pollution removed the following data were collected.*

Daily Rainfall in 0.01cm	*4.3*	*4.5*	*5.9*	*5.6*	*6.1*	*5.2*	*3.8*	*2.1*
Pollution Removed (mg/m^3*)*	*12.6*	*12.1*	*11.6*	*11.8*	*11.4*	*11.8*	*13.2*	*14.1*

Find the regression line of y on x.

Sol.

$x\,(metre)$	y	xy	x^2
4.3	12.6	54.18	18.49
4.5	12.1	54.45	20.25
5.9	11.6	68.44	34.81
5.6	11.8	66.08	31.36
6.1	11.4	69.54	37.21
5.2	11.8	61.36	27.04
3.8	13.2	50.16	14.44
2.1	14.1	29.61	4.41
37.5	98.6	453.82	188.01

Let $y = a + bx$ be the equation of the line of regression of y on x.

$\therefore$ Normal equations are:

$$\sum y_i = na + b\sum x_i \;\Rightarrow\; 98.6 = 8a + 37.5b$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \;\Rightarrow\; 453.82 = 37.5a + 188.01b$$

After solving these normal equations we get $a = 15.49$ and $b = -0.675$.

The equation of the line of regression is $y = 15.49 - 0.675x$. **Ans.**

9.3.7 Another Form of Equations of Lines of Regression

Theorem 1: Show that the equation of the line of regression of y on x is given by $y - \bar{y} = r.\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$, where $\bar{x}$ and $\bar{y}$ are the means of x-series and y-series respectively; r is the coefficient of correlation between x and y; σ_x and σ_y are the standard deviations of x-series and the y-series respectively.

Proof: Suppose that the line approximating the set of points $(x_1, y_1), (x_2, y_2), \ldots\ldots\ldots, (x_n, y_n)$ has the equation

$$y = a + bx \qquad \ldots(1)$$

Then $y_i = a + bx_i$ and $x_i y_i = ax_i + bx_i^2$ for each $i = 1, 2, \ldots\ldots\ldots\ldots, n$.

$$\therefore \qquad \sum y_i = na + b\sum x_i \qquad \ldots(2)$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad ...(3)$$

From (2), we have $$\frac{\sum y_i}{n} = a + b\frac{\sum x_i}{n} \text{ or } \bar{y} = a + b\bar{x} \quad ...(4)$$

Thus, it follows that $(\bar{x}, \bar{y})$ lies on the line.

Shifting the origin to $(\bar{x}, \bar{y})$ (2) becomes

$$\sum (y_i - \bar{y}) = na + b\sum (x_i - \bar{x}) \quad \text{or} \quad a = 0$$

$$\left[\because \sum (x_i - \bar{x}) = \sum (y_i - \bar{y}) = 0\right]$$

Shifting the origin to $(\bar{x}, \bar{y})$ and taking $a = 0$,

(1) becomes $(y - \bar{y}) = b(x - \bar{x})$ ...(5)

(3) becomes $\sum (x_i - \bar{x})(y_i - \bar{y}) = b\sum (x_i - \bar{x})^2$ ...(6)

From (6), we have

$$b = \frac{\sum (x_i \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\sum (dx_i\, dy_i)}{\sum (dx_i)^2}$$

$$= \frac{\sum (dx_i . dy_i)}{n(\sigma_x)^2} = r\frac{\sigma_y}{\sigma_x} \quad \because \quad r = \frac{\sum (dx_i dy_i)}{n(\sigma_x, \sigma_y)}$$

Putting this values of b in (5), the required equation of the line if regression of y on x i

$$\boxed{(y - \bar{y}) = r.\frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

Coefficient of Regression of *y* on *x*: The real number $b = r.\frac{\sigma_y}{\sigma_x}$ is called the coefficient of regression of y on x and is denoted by b_{yx}. Thus $b_{yx} = r\frac{\sigma_y}{\sigma_x}$.

Theorem 2: The equation of the line of regression of x on y is given by

$$\boxed{(x - \bar{x}) = r.\frac{\sigma_x}{\sigma_y}(y - \bar{y})}$$

Proof: Proceed as in theorem 1.

Coefficient of Regression of *x* on *y*: The real number $b = r.\frac{\sigma_x}{\sigma_y}$ is called the coefficient of regression of x on y and is denoted by b_{xy}. Thus $b_{xy} = r.\frac{\sigma_x}{\sigma_y}$.

Theorem 3: Prove that:

(*i*) $$b_{yx} = \frac{\sum x_i y_i - \frac{-\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}\right\}}$$

(*ii*) $$b_{xy} = \frac{\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}\right\}}$$

Proof: (*i*) By definition, we have

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = r.\frac{\sigma_y \sigma_x}{(\sigma_x)^2}$$

$$= \frac{\text{cov}(x, y)}{(\sigma_x)^2} \qquad [\because r\sigma_y\sigma_x = \text{cov}(x, y)]$$

$$= \frac{\left[\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}\right]}{\left\{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}\right\}}$$

Similarly, (*ii*) can be proved.

Example 6. *Find the regression coefficient* b_{yx} *between* x *and* y *for the following data:* $\sum x = 24$, $\sum y = 44$, $\sum xy = 306$, $\sum x^2 = 164$, $\sum y^2 = 574$ *and* $n = 4$.

Sol. The given data may be written as $\sum x_i = 24$, $\sum y_i = 44$, $\sum x_i y_i = 306$, $\sum x_i^2 = 164$, $\sum y_i^2 = 574$ and $n = 4$.

$$b_{yx} = \frac{\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}\right\}} = \frac{306 - \frac{24 \times 44}{4}}{164 - \frac{(24)^2}{4}}$$

$$= \frac{(306 - 264)}{164 - 144} = \frac{42}{20} = 2.1. \quad \textbf{Ans.}$$

Example 7. *Find the regression coefficient* b_{xy} *between* x *and* y *for the following data:* $\sum x = 30$, $\sum y = 42$, $\sum xy = 199$, $\sum x^2 = 184$, $\sum y^2 = 318$ *and* $n = 6$.

Sol. The given data may be given as under: $\sum x_i = 30$, $\sum y_i = 42$, $\sum x_i y_i = 199$, $\sum x_i^2 = 184$, $\sum y_i^2 = 318$ and $n = 6$.

$$\therefore \quad b_{xy} = \frac{\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}\right\}}$$

$$= \frac{\left(199 - \frac{30 \times 42}{6}\right)}{\left(318 - \frac{42 \times 42}{6}\right)} = \left(\frac{199 - 210}{318 - 294}\right) = \frac{-11}{24} = -0.46. \quad \textbf{Ans.}$$

Example 8. *For the following observations (x, y), find the regression coefficient b_{yx} and b_{xy} and hence find the correlation coefficient between x and y: (1, 2), (2, 4), (3, 8), (4, 7), (5, 10), (6, 5), (7, 14), (8, 16), (9, 2), (10, 20).*

Sol. Here n = 10. We may prepare the table, given below:

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	2	1	4	2
2	4	4	16	8
3	8	9	64	24
4	7	16	49	28
5	10	25	100	50
6	5	36	25	30
7	14	4	196	98
8	16	64	256	128
9	2	81	4	18
10	20	100	400	200
$\sum x_i = 55$	$\sum y_i = 88$	$\sum x_i^2 = 385$	$\sum y_i^2 = 1114$	$\sum x_i y_i = 586$

$$b_{yx} = \frac{\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}\right\}} = \frac{586 - \frac{55 \times 88}{10}}{385 - \frac{(55)^2}{10}} = \frac{102}{82.5} = 1.24$$

And
$$b_{xy} = \frac{\sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}}{\left\{\sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}\right\}} = \frac{586 - \frac{(55 \times 88)}{10}}{1114 - \frac{(88)^2}{10}} = \frac{102}{339.6} = 0.30$$

Now, $b_{yx} \cdot b_{xy} = \left(r \cdot \dfrac{\sigma_y}{\sigma_x} \right)\left(r \cdot \dfrac{\sigma_x}{\sigma_y} \right) = r^2,$ where r is the coefficient of correlation.

$\therefore \quad r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{1.24 \times 0.30} = 0.609.$

Thus, $b_{yx} = 1.24,\quad b_{xy} = 0.30$ and $r = 0.609$. **Ans.**

9.3.8 Some Properties of Regression Coefficients

Let, the regression coefficient of y on x is b_{yx}; the regression coefficient of x on y is b_{xy}; and, the correlation coefficient between x and y is r. Then, we have the following results.

Theorem 1: Prove that $r = \sqrt{b_{yx} \cdot b_{xy}}$.

Proof: We have: $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$ and $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$. Therefore, $b_{yx}\ b_{xy} = r^2$ or $r = \sqrt{b_{yx} . b_{xy}}$.

Remark: Clearly we can say that, correlation coefficient is the geometric mean between the two regression coefficients.

Theorem 2: Prove that r, b_{yx} and b_{xy} are of the same sign.

Proof: We know that $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$ and $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$. Since σ_x and σ_y are both positive, it follows from the two equations, given above that b_{yx} and b_{xy} have the same sign as r.

Hence r, b_{yx} and b_{xy} are always of the same sign.

Theorem 3: Prove that the arithmetic mean of regression coefficient is greater than the correlation coefficient.

Proof: Clearly, the required result is true,

If $\quad \dfrac{1}{2}\left(b_{yx} + b_{xy}\right) > r$ *i.e.,* if $\quad \dfrac{1}{2}\left[r.\dfrac{\sigma_y}{\sigma_x} + r\dfrac{\sigma_x}{\sigma_y} \right] > r$

i.e., if $\quad \sigma_y^2 + \sigma_x^2 > 2\sigma_x\sigma_y$

i.e., if $\quad (\sigma_y^2 - \sigma_x^2)^2 - 2\sigma_x\sigma_y > 0$

i.e., if $\quad (\sigma_y - \sigma_x)^2 > 0,$ which is true.

Hence the required result is true. **Proved**

Theorem 4: Let θ be the angle between the regression line of y on x and the regression line of x on y. Then, prove that $\tan\theta = \left\{ \dfrac{\left(1 - r^2\right)}{r} . \dfrac{\sigma_x , \sigma_y}{\left(\sigma_x^2 + \sigma_y^2\right)} \right\}.$

Proof: The equation of the line of regression of x on y is

$$\left(x - \bar{x}\right) = r.\frac{\sigma_x}{\sigma_y}\left(y - \bar{y}\right) \qquad ...(i)$$

And, the equation of the line of regression of y on x is

$$\left(y-\bar{y}\right)=r.\frac{\sigma_y}{\sigma_x}\left(x-\bar{x}\right) \qquad ...(ii)$$

Let m_1 and m_2 be the slopes of (i) and (ii) respectively.

Then, $m_1 = \frac{\sigma_y}{r.\sigma_x}$ and $m_2 = \frac{r.\sigma_y}{\sigma_x}$.

Therefore,
$$\tan\theta = \frac{(m_1 - m_2)}{1+m_1 m_2}$$

$$= \frac{\frac{\sigma_y}{r.\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1+\frac{(\sigma_y)^2}{(\sigma_x)^2}} = \left\{\frac{(1-r)^2}{r}.\frac{\sigma_x\sigma_y}{\left(\sigma_x^2+\sigma_y^2\right)}\right\}. \quad \textbf{Proved.}$$

Example 9. *The lines of regression of x on y and y on x are respectively x = 19.13 – 0.87y and y = 11.64 – 0.50x. Find:*

(a) The mean of x - series;

(b) The mean of y- series;

(c) The correlation coefficient between x and y.

Sol. Let the mean of x-series is $\bar{x}$ and that of y-series be $\bar{y}$.

Since the lines of regression pass through $(\bar{x},\bar{y})$, we have:

$$\bar{x} = 19.13 - 0.87\,\bar{y} \quad \text{or} \quad \bar{x} + 0.87\,\bar{y} = 19.13 \qquad ... (1)$$

and
$$\bar{y} = 11.64 - 0.50\bar{x} \quad \text{or} \quad 0.50\bar{x} + \bar{y} = 11.64 \qquad ... (2)$$

On solving (1) and (2), we get

$$\bar{x} = 15.94 \text{ and } \bar{y} = 3.67.$$

Therefore, mean of x-series = 15.94

And mean of y-series = 3.67

Now, the line of regression of y on x is:

$$y = 11.64 - 0.50x \qquad \therefore\ b_{yx} = -0.50$$

Also, the line of regresson x on y is:

$$x = 19.13 - 0.87y \qquad \therefore\ b_{xy} = -0.87$$

$$\therefore \qquad r = \sqrt{b_{yx}b_{xy}} = \sqrt{(-0.50)(-0.87)} = \sqrt{0.435} = -0.66$$

Clearly, r is taken as negative, since each one of b_{yx} and b_{xy} is negative.

Example 10. *Out of the following two regression lines, find the line of regression of x on y:*

2x + 3y = 7 and 5x + 4y = 9.

Sol. Let $2x+3y=7$ be the regression line of x on y.

Then, $5x + 4y = 9$ is the regression line of y on x.

Therefore $2x+3y=7$ and $5x+4y=9$

$$\Rightarrow \quad x=-\frac{3}{2}y+\frac{7}{2} \quad \text{and} \quad y=-\frac{5}{4}x+\frac{9}{4} \quad \Rightarrow \quad b_{xy}=-\frac{3}{2} \text{ and } b_{yx}=-\frac{5}{4}$$

$$\Rightarrow \quad r=\sqrt{b_{xy}b_{yx}}=-\sqrt{\left(-\frac{3}{2}\right)\left(-\frac{5}{4}\right)} \qquad [\because\ r, b_{xy}, b_{yx} \text{ have the same sign}]$$

$$=-\sqrt{\frac{15}{8}}<-1, \text{ which is impossible.}$$

Therefore our choice of regression line is incorrect.

Hence, the regression line of x on y is $5x + 4y = 9$. **Ans.**

Example 11. *Find the correlation coefficient between x and y, when the lines of regression are: $2x - 9y + 6 = 0$ and $x - 2y + 1 = 0$.*

Sol. Let the line of regression of x on y be $2x - 9y + 6 = 0$

Then, the line of regression of y on x is $x-2y+1=0$.

Therefore $2x-9y+6=0$ and $x-2y+1=0$

$$\Rightarrow \quad x=\frac{9}{2}y-3 \quad \text{and} \quad y=\frac{1}{2}x+\frac{1}{2}$$

$$\Rightarrow \quad b_{xy}=\frac{9}{2} \quad \text{and} \quad b_{yx}=\frac{1}{2}$$

$$\Rightarrow \quad r=\sqrt{b_{xy}\cdot b_{yx}}=\sqrt{\left(\frac{9}{2}\times\frac{1}{2}\right)}=\frac{3}{2}>1, \text{ which is impossible.}$$

So, our choice of regression line is incorrect.

Therefore, the regression line of x on y is $x-2y+1=0$.

And, the regression line of y on x is $2x-9y+6=0$.

$$\Rightarrow \quad x=2y-1 \quad \text{and} \quad y=\frac{2}{9}x+\frac{2}{3}$$

$$\Rightarrow \quad b_{xy}=2 \quad \text{and} \quad b_{yx}=\frac{2}{9}$$

$$\Rightarrow \quad r=\sqrt{b_{xy}.b_{yx}}=\sqrt{\left(2\times\frac{2}{9}\right)}=\frac{2}{3}$$

Hence, the correlation coefficient between x and y is $\frac{2}{3}$. **Ans.**

Example 12. *The equations of two lines of regression are: $3x + 12y = 19$ and $3y + 9x = 46$. Find* *(i) the mean of x-series*

(ii) the mean of y-series

(iii) Regression coefficient b_{xy} *and* b_{yx},

(iv) Correlation coefficient between x *and* y.

Sol. Let the mean of *x-series* be $\bar{x}$ and that of *y-series* be $\bar{y}$. Then, each of the given lines passes through $(\bar{x}, \bar{y})$.

Therefore $$3\bar{x} + 12\bar{y} = 19 \qquad ...(1)$$

And $$9\bar{x} + 3\bar{y} = 46 \qquad ...(2)$$

On solving (1) and (2), we get $\bar{x} = 5$ and $\bar{y} = \frac{1}{3}$.

Therefore mean of x-series is 5 and mean of y-series is $\frac{1}{3}$.

Now, let the line of regression of x on y be $3x + 12y = 19$

Then, the line of regression of y on x is $3y + 9x = 46$.

Therefore $3x + 12y = 19$ and $3y + 9x = 46$

$$\Rightarrow \quad x = -4y + \frac{19}{3} \text{ and } y = -3x + \frac{46}{3} \Rightarrow b_{xy} = -4 \text{ and } b_{yx} = -3$$

$$\Rightarrow \quad r = -\sqrt{(-4)(-3)} = -2\sqrt{3} < -1, \text{ which is impossible.}$$

$\therefore$ Our choice of regression line is incorrect.

Consequently, the regression line of x on y is $3y + 9x = 46$.

And, the regression line of y on x is $3x + 12y = 19$.

Therefore $3y + 9x = 46$ and $3x + 12y = 19$

$$\Rightarrow \quad x = -\frac{1}{3}y + \frac{46}{9} \quad \text{and} \quad y = -\frac{1}{4}x + \frac{19}{12}$$

$$\Rightarrow \quad b_{xy} = -\frac{1}{3}, b_{yx} = -\frac{1}{4} \text{ and } r = \sqrt{\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)} = \frac{-1}{2\sqrt{3}} = \frac{-\sqrt{3}}{6}$$

(Because r, b_{xy} and b_{yx} have the same sign).

Example 13. *You are given the following data:*

Series	x	y
Mean	*18*	*100*
standard deviation	*14*	*20*

Correlation coefficient between x *and* y *is 0.8. Find the two regression lines.*

Estimate the value of y, *when* x *is 70.*

Estimate the value of x, *when* y *is 90.*

Sol. Given that $\bar{x}=18,\ \bar{y}=100,\ \sigma_x=14,\ \sigma_y=20$ and $r=0.8$.

Therefore the line of regression y on x is :

$$y-\bar{y}=r.\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

or $$(y-100)=\left(\frac{0.8\times 20}{14}\right)(x-18)$$

or $$y=1.14x+79.41$$

When $x=70$, we have: $y=(1.14\times 70+79.41)=159.21$

And, the line of regression of x on y is:

$$x-\bar{x}=r.\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

or $$(x-18)=0.8\times\frac{14}{20}(y-100)$$

or $$x=0.56y-38$$

When $y=90$, we have $x=(0.56\times 90-38)=12.4$. **Ans.**

To Find b_{yx} and b_{xy} Using Assumed Mean: Let the assumed means of x-series and y-series be A and B respectively. Then, taking $dx_i=(x_i-A)$ and $dy_i=(y_i-B)$, we have

$$b_{yx}=\frac{\sum(dx_i\cdot dy_i)-\frac{\left(\sum dx_i\right)\left(\sum dy_i\right)}{n}}{\left\{\sum(dx_i)^2-\frac{\left(\sum dx_i\right)^2}{n}\right\}}$$

And, $$b_{xy}=\frac{\sum(dx_i\cdot dy_i)-\frac{\left(\sum dx_i\right)\left(\sum dy_i\right)}{n}}{\left\{\sum(dy_i)^2-\frac{\left(\sum dy_i\right)^2}{n}\right\}}$$

Example 14. *Find the regression coefficients and hence the equations of the two lines of regression from the following data:*

Age of husband (x)	25	22	28	26	35	20	22	40	20	18
Age of wife (y)	18	15	20	17	22	14	16	21	15	14

Hence estimate

(i) The age of wife, when the age of husband is 30.

(ii) The age of husband, when the age of wife is 19.

Sol. We have

$$\bar{x} = \frac{\sum x_i}{x} = \frac{256}{10} = 25.6 \text{ and } \bar{y} = \frac{\sum y_i}{n} = \frac{172}{10} = 17.2$$

Let the assumed mean of x- series and y- series be 26 and 17 respectively. Then, we may prepare the table given below:

x_i	y_i	$dx_i = (x_i - 26)$	$dy_i = (y_i - 17)$	$(dx_i)^2$	$(dy_i)^2$	$dx_i \times dy_i$
25	18	–1	1	1	1	–1
22	15	–4	–2	16	4	8
28	20	2	3	4	9	6
26	17	0	0	0	0	0
35	22	9	5	81	25	45
20	14	–6	–3	36	9	18
22	16	–4	–1	16	1	4
40	21	14	4	196	16	56
20	15	–6	–2	36	4	12
18	14	–8	–3	64	9	24
$\sum x_i = 256$	$\sum y_i = 17$	$\sum dx_i = -4$	$\sum dy_i = 2$	$\sum (dx_i)^2 = 450$	$\sum (dy_i)^2 = 78$	$\sum dx_i dy_i = 172$

Therefore,

$$b_{yx} = \frac{\sum (dx_i . dy_i) - \frac{\left(\sum dx_i\right)\left(\sum dy_i\right)}{n}}{\left\{\sum (dx_i)^2 - \frac{\left(\sum dx_i\right)^2}{n}\right\}} = \frac{172 - \frac{(-4)(2)}{10}}{\left\{450 - \frac{(-4)^2}{10}\right\}}$$

$$b_{yx} = \frac{(172 + 0.8)}{(450 - 1.6)} = \frac{172.8}{448.4} = 0.385$$

$$b_{xy} = \frac{\sum (dx_i . dy_i) - \frac{\left(\sum dx_i\right)\left(\sum dy_i\right)}{n}}{\left\{\sum (dy_i)^2 - \frac{\left(\sum dy_i\right)^2}{n}\right\}} = \frac{172 - \frac{(-4)(2)}{10}}{\left\{78 - \frac{2^2}{10}\right\}}$$

$$b_{xy} = \frac{(172 + 0.8)}{(78 - 0.4)} = \frac{172.8}{77.6} = 2.23$$

Therefore the equation of the line of regression of y on x is:

$$(y - \bar{y}) = b_{yx} . (x - \bar{x}) \text{ or } (y - 17.2) = (0.385)(x - 25.6)$$

Now, when $x = 30$, we get

$$y - 17.2 = (0.385)(30 - 25.6) \text{ or } y = 19 \text{ (approximately).}$$

$\therefore$ When the age of husband is 30 years, the estimated age of husband is 19 years.

Again, the equation of the line of regression of x on y is:

$$\left(x - \bar{x}\right) = b_{xy}\left(y - \bar{y}\right) \text{ or } (x - 25.6) = (2.23)(y - 17.2)$$

Thus, when $y = 19$, we get $x = 30$ (approximately).

So, when the age of wife is 19 years, the estimated age of husband is 30 years. **Ans.**

9.4 ERROR OF PREDICTION

The deviation of the predicted value from the observed value is known as the standard error of prediction. It is given by

$$E_{yx} = \sqrt{\frac{\sum\left(y - y_p\right)^2}{n}},$$

where y is the actual value and y_p the predicted value.

Theorem: Prove that:

(1) $E_{yx} = \sigma_y \cdot \sqrt{\left(1 - r^2\right)}$, (2) $E_{xy} = \sigma_x \cdot \sqrt{\left(1 - r^2\right)}$

Proof: (1) The equation of the line of regression of y on x is

$$y - \bar{y} = r.\frac{\sigma_y}{\sigma_x}\left(x - \bar{x}\right)$$

$$\therefore \quad y_p = \bar{y} + r.\frac{\sigma_y}{\sigma x}\left(x - \bar{x}\right) \quad \text{... (1)}$$

So,

$$E_{yx} = \frac{\sqrt{\sum\left(y - y_p\right)^2}}{n} = \left[\frac{1}{n}\sum\left\{y - \bar{y} - r\frac{\sigma_y}{\sigma_x}\left(x - \bar{x}\right)\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{n}\sum\left\{(y - \bar{y})^2 + \frac{r^2.\sigma_y^2.\left(x - \bar{x}\right)^2}{\sigma_x^2} - \frac{2r.\sigma_y}{\sigma_x}.\left(x - \bar{x}\right)\left(y - \bar{y}\right)\right\}\right]^{1/2}$$

$$= \left\{\frac{\sum\left(y - \bar{y}\right)^2}{n} + \frac{r^2.\sigma_y^2}{\sigma_x^2}.\frac{\sum\left(x - \bar{x}\right)^2}{n} - \frac{2r.\sigma_y}{\sigma_x}.\frac{\sum\left(x - \bar{x}\right)\left(y - \bar{y}\right)}{n}\right\}^{1/2}$$

$$= \left\{\sigma_y^2 + \frac{r^2.\sigma_y^2}{\sigma_x^2}.\sigma_x^2 - \frac{2r.\sigma_y}{\sigma_x}.r.\sigma_x.\sigma_y\right\}^{1/2}$$

$$= \left(\sigma_y^2 - r^y\sigma_y^2\right)^{1/2} = \sigma_y \cdot \sqrt{\left(1 - r^2\right)}.$$

(2) Similarly, (2) may be proved.

Example 15. *For the data given below, find the standard error of estimate of* y *on* x.

x	1	2	3	4	5
y	2	5	3	8	7

Sol. We leave it to the reader to find the line of regression of y on x.

This is: $y = 1.3x + 1.1$. So, $y_p = 1.3x + 1.1$

Now form the table for given data:

x	y	$y_p = 1.3x + 1.1$	$(y - y_p)$	$(y - y_p)^2$
1	2	2.4	–0.4	0.16
2	5	3.7	1.3	1.69
3	3	5	–2	4
4	8	6.3	1.7	2.89
5	7	7.6	0.6	0.36
				$\sum (y - y_p)^2 = 9.10$

Therefore $E_{yx} = \sqrt{\dfrac{\sum (y - y_p)^2}{n}} = \sqrt{\dfrac{9.10}{5}} = \sqrt{1.82} = 1.349$. **Ans.**

9.5 MULTIPLE LINEAR REGRESSION

There are a number of situations where the dependent variable is a function of two or more independent variables either linear or non-linear. Here, we shall discuss an approach to fit the experimental data where the variable under consideration is linear function of two independent variables.

Let us consider a two-variable linear function given by

$$y = a + bx + cz \qquad ...(1)$$

The sum of the squares of the errors is given by

$$S = \sum_{i=1}^{n} (y_i - a - bx_i - cz_i)^2 \qquad ...(2)$$

Differentiating S partially w.r.t. a, b, c, we get

$$\frac{\partial S}{\partial a} = 0 \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i - cz_i)(-1) = 0$$

$$\frac{\partial S}{\partial b} = 0 \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i - cz_i)(-x_i) = 0$$

and $$\frac{\partial S}{\partial c} = 0 \Rightarrow 2\sum_{i=1}^{n}(y_i - a - bx_i - cz_i)(-z_i) = 0$$

which on simplification and omitting the suffix i, yields.

$$\Sigma y = ma + b\Sigma x + c\Sigma z$$
$$\Sigma xy = a\Sigma x + b\,\Sigma x^2 + c\Sigma xz$$
$$\Sigma yz = a\Sigma z + b\Sigma xz + c\Sigma z^2$$

Solving the above three equations, we get values of a, b, and c. Consequently, we get the linear function $y = a + bx + cz$ called regression plane.

Example 16. *Obtain a regression plane by using multiple linear regression to fit the data given below :*

x :	*1*	*2*	*3*	*4*
y :	*0*	*1*	*2*	*3*
z :	*12*	*18*	*24*	*30*

(U.P.TU. 2002)

Sol. Let $y = a + bx + cz$ be required regression plane where a, b, c are the constants to be determined by following equations :

$$\left.\begin{aligned}\sum y &= ma + b\sum x + c\sum z\\ \sum xy &= a\sum x + b\sum x^2 + c\sum xz\\ \text{and} \quad \sum yz &= a\sum z + b\sum zx + c\sum z^2\end{aligned}\right\} \quad ...(1)$$

Here, $m = 4$

x	z	y	x^2	z^2	xy	xz	yz
1	0	12	1	0	12	0	0
2	1	18	4	1	36	2	18
3	2	24	9	4	72	6	48
4	3	30	16	9	120	12	90
$\sum x = 10$	$\sum z = 6$	$\sum y = 84$	$\sum x^2 = 30$	$\sum z^2 = 14$	$\sum xy = 240$	$\sum xz = 20$	$\sum yz = 156$

From table, equation (1) can be written as

$$84 = 4a + 10b + 6c$$
$$240 = 10a + 30b + 20c$$
and $$156 = 6a + 20b + 14c$$

Solving, we get $a = 10,\ b = 2,\ c = 4$

Hence the required regression plane is

$$y = 10 + 2x + 4z.$$ **Ans.**

PROBLEM SET 9.2

1. Find the equation of the lines of regression on the basis of the data:

x:	4	2	3	4	2
y:	2	3	2	4	4

[**Ans.** $y = 3.75 - 0.25x, x = 3.75 - 0.25y$]

2. Find the regression coefficient b_{yx} for the data:

$\sum x = 55,\quad \sum y = 88,\quad \sum x^2 = 385,\quad \sum y^2 = 1114,\quad \sum xy = 586,$ and $n = 10$

[**Ans.** 1.24]

3. The following data regarding the heights (y) and weights (x) of 100 college students are given:

$\sum x = 15000,\quad \sum x^2 = 2272500,\quad \sum y = 6800,\quad \sum y^2 = 463025,$ and $\sum xy = 1022250$.

[**Ans.** $y = 0.1x + 53$]

4. Find the coefficient of correlation when two regression equations are:

$x = -0.2y + 4.2$ and $y = -0.80x + 8.4$. [**Ans.** $r = -0.4$]

5. Find the standard error of estimate of y on x for the data given below:

x:	1	3	4	6	8	9	11	14
y:	1	2	4	4	5	7	8	9

[**Ans.** $E_{yx} = 0.564$]

6. If two regression coefficients are 0.8 and 0.2, what would be the value of coefficient of correlation? [**Ans.** $r = 0.4$]

7. x and y are two random variables with the same standard deviation and correlation coefficient r. Show that the coefficient of correlation between x and $x + y$ is $\sqrt{\frac{1+r}{2}}$.

8. Show that the geometric mean of the coefficients of regression is the coefficient of correlation.

❑❑❑

CHAPTER **10**

Time Series and Forecasting

10.1 INTRODUCTION

Business executives, economists, and government officials are often faced with problems that require forecast such as future sales, future revenue and expenditures, and the total business activity for the next decade. Time series analysis is a statistical method, which helps the businessman to understand the past behaviour of economic variables based on collection of observations taken at different time intervals. Having recognized the behaviour or movements of a time series, the businessman tries to forecast the future of economic variables on the assumption that the time series of such an economic variable will continue to behave in the same fashion as it had in the past. Thus analyzing information for the previous time periods is the subject of time series analysis.

Thus the statistical data, which are collected, observed or recorded at successive intervals of time or arranged chronologically are said to form a time series.

"A time series a set of observations taken at specified times, usually (but not always) at equal intervals". Thus a set of data depending on time, which may be year, quarter, month, week, days etc. is called a time series.

Examples:

1. The annual production of Rice in India over the last 15 years.
2. The daily closing price of a share in the Calcutta Stock Exchange.
3. The monthly sales of an Iron Industry for the last 6 months.
4. Hourly temperature recorded by the meteorological office in a city.

Mathematically, a time series is defined by the value $y_1, y_2, \ldots\ldots\ldots\ldots$, of a variable y (closing price of a share, temperature etc.) at time t_1, t_2, t_3, ... Thus y is a function of t and given by

$$y = f(t)$$

10.2 TIMES SERIES GRAPH

A time series involving a variable y is represented pictorially by constructing a graph of y verses t.

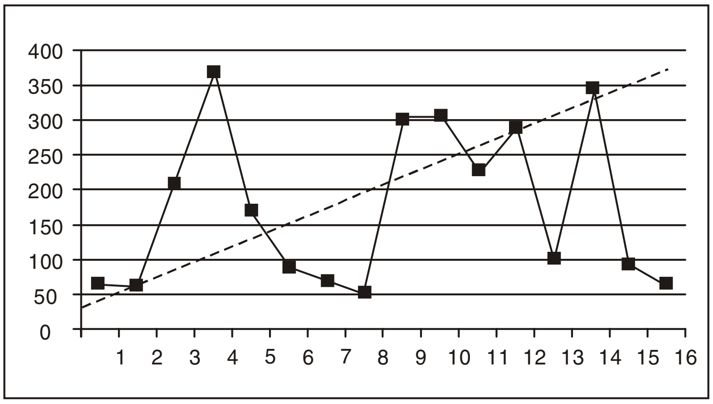


FIG. 10.1

10.3 COMPONENT OF TIME SERIES

The analysis of Time Series consists of the description and measurement of various changes or movements as they appear in the series during a period of time. These changes or movements are called the components of elements of time series. Fluctuations in a time series are mainly due to four basic types of variations (or movements). These four types of component are:

1. Secular Trend or Long Term Movement (T)
2. Seasonal Variation or Seasonal Movement (S)
3. Cyclical Fluctuation or Cyclic Variation (C)
4. Residual, Irregular of Random Movement (I)

(1) Secular Trend: In Business, Economics and in our daily conversation the term Secular Trend or simply trend is popularly used. Where we speak of rising trend of population or prices, we mean the gradual increase in population or prices over a period of Time. Similarly, by declining trend of production or sales, we mean gradual decrease in production or sales over a period of time. The concept of trend does not include short range Oscillations, but refers to the steady movement over a long period time.

"Secular trend is the smooth, regular and long term movement of a series showing continuous growth stagnation or decline over a long period of time. Graphically it exhibits general direction and shape of time series". The trend movement of an economic time series may be upward or downward. The upward trend may be due to population growth, technological advances, improved methods of Business Organization and Management, etc. Similarly, the downward trend may be due to lack of demand for the product, storage of raw materials to be used in production, decline in death rate due to advance in medical sciences, etc.

(2) Seasonal Variation: Seasonal variation is a short-term periodic movement, which occurs more or less regularly within a stipulated period of one year or shorter. The major factors that cause seasonal variations are climate and weather conditions, customs and habits of people, religious festivals, etc. For instance, the demand for electric fans goes up in summer season, the sale of Ice-cream increases very much in summer and the sale of woolen cloths goes up in winter. Also the sales of jewelleries and ornaments go up in marriage seasons, the sales and profits of departmental stores go up considerably during festivals like Id, Christmas, etc.

Although the period of seasonal variations refers to a year in business and economics, it can also be taken as a month, week, day, hour, etc. depending on the type of data available. Seasonal variation gives a clear idea about the relative position of each season and on this basis, it is possibe to plan for the season.

(3) **Cyclical Fluctuations:** These refer to the long term oscillations, or swings about a trend line or curve. These cycles, as they are some times called, may or may not be periodic that is they may or may not follow exactly similar patterns after equal intervals of time. In business and economic activities, moments are considered cyclic only if they recur after intervals of more than one year. The ups and downs in business, recurring at intervals of times are the effects of cyclical variations. A business cycle showing the swing from prosperity through recession, depression, recovery and back again to prosperity. This movement varies in time, length and intensity.

(4) **Residual Irregular or Random Movement:** Random movements are the variations in a time series which are caused by chance factors or unforeseen factors which cannot be predicted in advance. For example, natural calamities like flood, earthquake etc, may occur at any movement and at any time. They can be neither predicted nor controlled. But the occurrence of these events influences business activities to a great extent and causes irregular or random variations in time series data.

10.4 ANALYSIS OF TIME SERIES

Time series analysis consists of a description (generally mathematical) of the component movements present. To understand the procedures involved in such a description consider graph (A), which shows Ideal Time Series, Graph (A1) shows the graph of long-term, or secular, trend line. Graph (A_2) shows this long-term trend line with a superimposed cyclic movement (assumed to be periodic) and graph (A_3) shows a seasonal movement superimposed of graph (A2). The concept in graph (A) suggests a technique for analyzing time series.

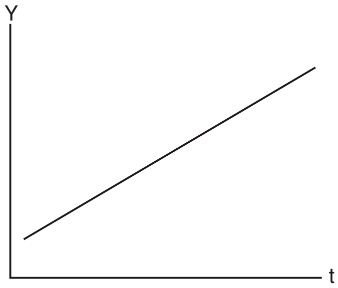


FIG. 10.2 (Graph A1)

In Traditional or classical time series analysis, it is normally assumed that there is multiplicative relationship between the four components. Symoblically.

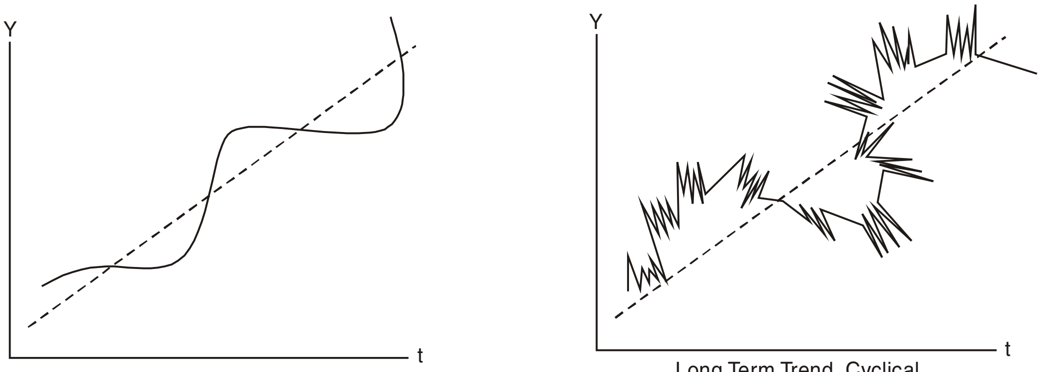
$$y = T \times S \times C \times I \quad ...(1)$$

$y =$ Result of the Four Components (or original data)

It is assumed that trend has no effect on seasonal component. Also it is assumed that the business cycle has no effect on the seasonal component. Instead of Multiplicative model (1) some statisticians may prefer an additive model.

$$y = T + S + C + I \qquad \text{...(2)}$$

where y is the sum of the four components.



Long Term Trend and Cyclical Movement

Long Term Trend, Cyclical and Seasonal Movement

FIG. 10.3

10.4.1 Analysis of Trend or Secular Trend

In time series analysis the analysis of secular trend is very important. It helps us to predict or forecast future results. There are four methods used in analyzing trend in time series analysis. They are:

(*a*) Method of Free Hand Curve (or graphic)

(*b*) Method of Semi Averages

(*c*) Method of Moving Averages

(*d*) Method of Least Square

(*a*) Free Hand Method: It is simplest method for studying trend. In this graphic method, the time series data are first plotted on the graph paper taking time on the x-axis and observed values of the other variable on y-axis. Then points obtained are joined by a free hand smooth curve of first degree. The line so obtained is called the trend curve and it shows the direction of the trend. The vertical distical of this line from x-axis gives the trend value for each time period. This method should be used only when a quick approximate idea of the trend is required.

Example 1. *Fit a trend line to the following data by the free hand grpahical method.*

Year	*2000*	*2001*	*2002*	*2003*	*2004*	*2005*	*2006*
Sales	*52*	*54*	*56*	*53.5*	*57*	*54.5*	*59*

Sol.

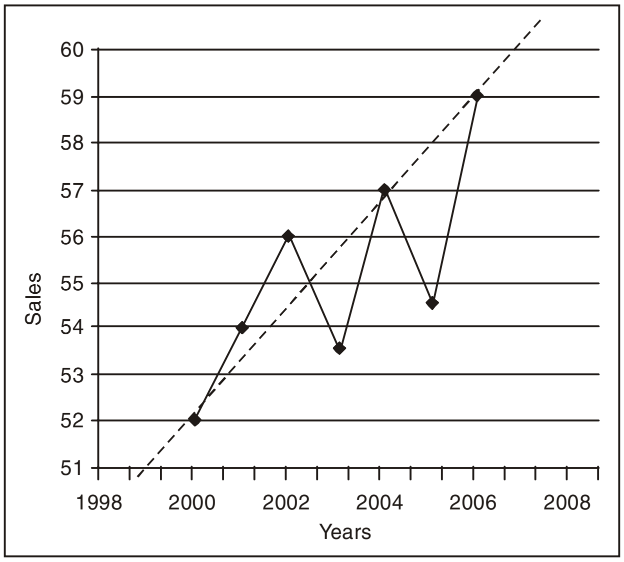


FIG. 10.4

(*b*) Method of Semi Averages: This method is very simple and gives greater accuracy than the method of free hand or graphical. In this method, the given data is first divided into two parts and an average for each part is found. Then these two averages are plotted on a graph paper with respect to the midpoint of the two respective time intervals. The line obtained on joining these two points is the required trend line and may be extend both ways to estimate intermediate values.

Remark: If given data is in odd number, then divide the whole series into two equal parts ignoring the middle period.

Example 2. *Fit a trend line to the following data by the method of semi averages.*

Year	*1991*	*1992*	*1993*	*1994*	*1995*	*1996*	*1997*	*1998*	*1999*	*2000*	*2001*	*2002*	*2003*
Bank Clearance	*53*	*79*	*76*	*66*	*69*	*94*	*105*	*87*	*79*	*104*	*97*	*92*	*101*

Sol. Here $n = 13$ *i.e.,* odd no. of data

Now divide the given data into two equal parts (by omitting 1997)

Year	*Clearenc e*	*Semi Total*	*Semi Avarage*
1991	53		
1992	79		
1993	76	53+79+76+66+69+94	437/6=72.8333
1994	66		
1995	69		
1996	94		
1997	105		
1998	87		
1999	79		
2000	104	87+79+104+97+92+101	560/6=93.333
2001	97		
2002	92		
2003	101		

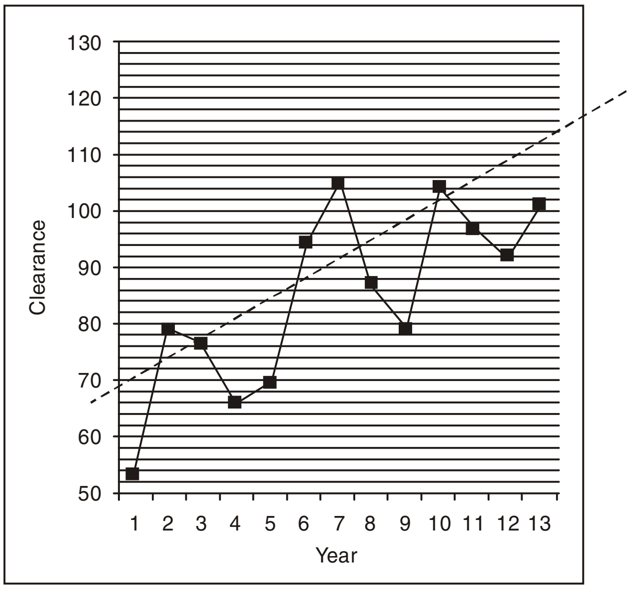


FIG. 10.5

Now the two semi averages 72.833 and 93.333 are plotted against the middle of their respective periods.

Example 3. *Draw the trend line by semi-average method using the given data*

Year	*1998*	*1999*	*2000*	*2001*	*2002*	*2003*
Production (In Tons)	*253*	*260*	*255*	*266*	*259*	*264*

Sol. Here $n = 6$

Year	Pr*oduction*	*Semi Total*	*Semi Average*
1998	253	$253+260+255$	$768/3 = 256$
1999	260		
2000	255		
2001	266		
2002	259	$266+259+264$	$789/3 = 263$
2003	264		

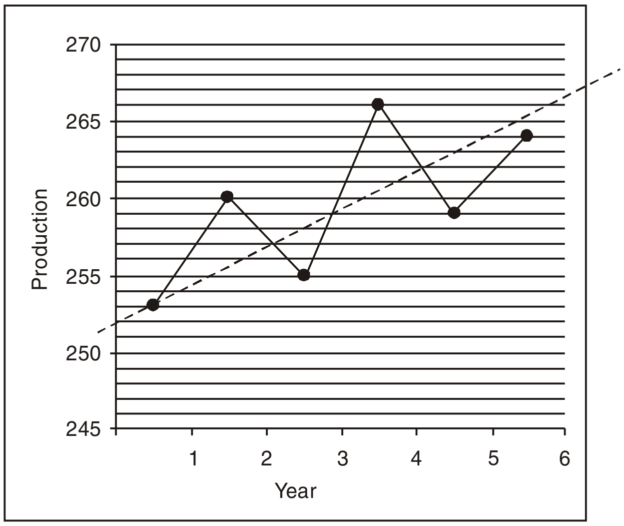


FIG. 10.6

(*c*) Method of Moving Averages: In the moving average method, the trend is described by smoothing out the fluctuations of the data by means of a moving average.

Let $(t_1, y_1), (t_2, y_2), \ldots\ldots\ldots\ldots, (t_n, y_n)$ be the given time series, where $t_1, t_2, t_3, \ldots\ldots\ldots\ldots, t_n$ denote the time periods and $y_1, y_2, y_3, \ldots\ldots\ldots\ldots, y_n$ denote the corresponding values of the variable.

The p-period moving totals (or sums) are defined as $y_1, y_2, y_3, \ldots\ldots\ldots\ldots, y_p$, $y_2, y_3, \ldots\ldots\ldots\ldots, y_{p+1.}$ $y_3, y_4, y_5, \ldots\ldots\ldots\ldots, y_{p+2}$, and so on.

The *p*-period moving averages are defined as

$$\frac{y_1 + y_2 + \ldots\ldots\ldots + y_p}{p}, \frac{y_2 + y_3 + \ldots\ldots\ldots + y_{p+1}}{p}, \frac{y_3 + y_4 + \ldots\ldots\ldots\ldots + y_{p+2}}{p} \text{ etc.}$$

The p period moving totals (or sums) and moving averages be also called moving totals (or sums) of order p and moving averages of order p respectively. These moving averages are also called the trend values.

When we estimate the trend, we should select the order or period of the moving average (such as 3 yearly moving average, 5 yearly moving average, 4 yearly moving average, 8 yearly moving average, etc.). This order should be equal to the length of cycles in the time series. The method of moving averages is the most frequently used approach for determining the trend because it is definitely simpler process of fitting a polynomial.

(1) Calculation of Moving Averages when the Period is Odd: In the case of odd period we would obtain the trend values and trend line as follows:

(***a***) In the case of 3–yearly period, first of all calculate the following moving totals (or sums) $y_1 + y_2 + y_3$, $y_2 + y_3 + y_4$, $y_3 + y_4 + y_5$, ... etc.

In the case of 5 yearly period, calculate the following moving total (or sums)

$y_1 + y_2 + y_3 + y_4 + y_5$, $y_2 + y_3 + y_4 + y_5 + y_6$, $y_3 + y_4 + y_5 + y_6 + y_7$, ... etc.

A period may be a year, a week, a day, etc.

(***b***) Place the moving totals at the centres of three respective time.

(*c*) Calculte the corresponding moving averages for 3 yearly or 5 yearly periods by dividing the moving totals by 3 or 5 respectively. Place these at the centre of the respective time.

(*d*) If required we can plot these moving averages or trend values against the periods and obtain the trend line (or curve) from which we can determine the increasing or decreasing trend of the data.

(*e*) It is more convenient to calculate the moving averages when the period is odd than when it is even, because there is only one middle period when the peirod is odd so and the moving average can be easily centred.

(2) Calculation of Moving Averages when the Period is Even: When an even number of data is included in the moving averages (as 4 years), the centre point of the group will be between two years. It is therefore necessary to adjust or shift (known technically as centre) these averages so that they concide with the years. The 4 yearly moving total and the 4 yearly moving average may be obtained by the methods already outlined for the odd period average. To centre the values, a 2 yearly moving average is taken of the even period moving average.

A 2 yearly moving average is taken of the 4 yearly moving average. The resulting average is located between the two 4 yearly moving average values and, therefore, coincides with the years. The end results (*i.e.*, a 2 yearly moving average of 4 yearly moving average) are known as the 4 yearly moving average centred.

We shall follow the steps given below in calculating the moving average when the order is even, say 4.

(*i*) We calculate the following moving totals:

$$y_1 + y_2 + y_3 + y_4, \qquad y_2 + y_3 + y_4 + y_5, \qquad y_3 + y_4 + y_5 + y_6, \; \ldots\ldots\ldots\ldots \text{ etc.}$$

(*ii*) Place these moving totals at the centres of the respective time spans. In the case or 4 yearly time period, there are two middle terms viz. 2nd and 3rd. Hence, place this moving total against the centre of these two middle terms. Similarly, place other moving totals at the centres of 3rd and 4th periods, 4th and 5th periods and so on.

(*iii*) Calculate the corresponding moving averages for 4 yearly periods by dividing the moving totals by 4. Place these at the centres of the time spans. *i.e.*, against the corresponding moving totals. (Note that the moving averages so placed do not coincide with the original time period.)

(*iv*) We take the total of 4 yearly moving averages taking two terms at a time starting from the first and place the sum at the middle of these two terms. The same procedure is repeated for other averages.

(*v*) Finally, we take the two-period averages of the above moving averages by dividing each by 2. These are the required trend values. This process is called centreing of moving averages. If required, we can plot these moving averages or trend values and obtain trend line or curve.

The major disadvantage of this method is that some trend values at the beginning and end of the series cannot be determined.

Example 4. *Calculate 3 yearly moving averages or trend values for the following data.*

Year (t)	*1998*	*1999*	*2000*	*2001*	*2002*	*2003*	*2004*	*2005*
Value(y)	*3*	*5*	*7*	*10*	*12*	*14*	*15*	*16*

Sol. 3 yearly moving averages means that there are three values induced in a group.

Calculation of 3 yearly moving averages

Year (*t*)	*Value* (*y*)	*3 Yearly Moving Total*	*3 Yearly Moving Total (Trend Valuea)*
1998	3		
1999	5	15	5.00
2000	7	22	7.33
2001	10	29	9.67
2002	12	36	12.00
2033	14	41	13.67
2004	15	45	15.00
2005	16		

Hence the trend values are 5.00, 7.33, 9.67, 12.00, 13.67, 15.00

Example 5. *Compute the 4 yearly moving averages from the following data:*

Year	*1991*	*1992*	*1993*	*1994*	*1995*	*1996*	*1997*	*1998*
Annual sales (Rs. In crores)	*36*	*43*	*43*	*34*	*44*	*54*	*34*	*24*

Sol. **Calculation of 4 yearly moving averages**

Year	*Annual Sales (Rs in Crores)*	*4 Yearly Moving Totals*	*4 Yearly Moving Average*	*2 Yearly Total of col. 4 (Centred)*	*4 Yearly Centred Moving Average (Trend values)*
1991	36				
1992	43				
1993	43	156	39		
1994	34	164	41	80	40
1995	44	175	43.75	84.75	42.375
1996	54	166	41.50	85.25	42.625
1997	34	156	39	80.50	40.25
1998	24				

Hence the trend values are 40, 42.375, 42.625, 40.25.

Example 6. *Assuming 5 yearly moving averages, calculate trend value from the data given below and plot the results on a graph paper.*

Year	*1971*	*1972*	*1973*	*1974*	*1975*	*1976*	*1977*	*1978*	*1979*	*1980*	*1981*	*1982*	*1983*
Production (Thousand)	*105*	*107*	*109*	*112*	*114*	*116*	*118*	*121*	*123*	*124*	*125*	*127*	*129*

Sol. Calculation of 5 yearly moving averages

Year	*Value*	*5 Yearly Moving Total*	*5 Yearly Moving Averages (trend value)*
1971	105	–	–
1972	107	–	–
1973	109	547	109.4
1974	112	558	111.6
1975	114	569	113.8
1976	116	581	116.2
1977	118	592	118.4
1978	121	602	120.4
1979	123	611	122.2
1980	124	620	124.0
1981	125	628	125.6
1982	127	–	–
1983	129	–	–

Hence the trend values are 109.4, 111.6, 113.8, 116.2, 118.4, 120.4, 122.2, 124.0, and 125.6.

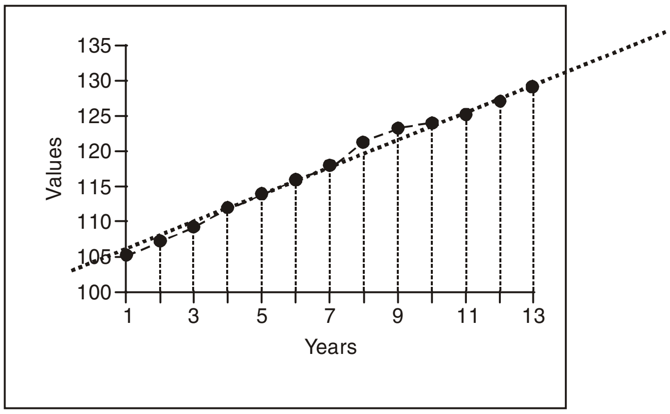


FIG. 10.7

(*d*) Method of Least Square: This method is widely used for the measurement of trend. In method of least square we minimize the sum of the squares of the deviation of observed values from their expected values with respect to the constants.

Let $T = a + bx$ be the required trend line

By the principal of least square, the line of the best fit is obtained when the sum of the squares of the differences, S_i is minimum *i.e.*

$$S_i = \sum (T_i - a - bx_i)^2 \text{ is minimum}$$

When S_i is minimum, we obtain normal equations as

$$\sum T = na + b\sum x$$

$$\sum Tx = a\sum x + b\sum x^2$$

On solving these two equations, we get

$$a = \frac{\sum T}{n}, \quad b = \frac{\sum Tx}{\sum x^2}$$

Remark: If we take the midpoint in time as the origin, the negative values in the first half of the series balance out the positive values in the second half so

$$\sum x = 0$$

Example 7. *Determine the equation of a straight-line which best fits the following data*

Year	*1974*	*1975*	*1976*	*1977*	*1978*
Sales (in Rs. 000)	*35*	*56*	*79*	*80*	*40*

Compute the trend values for all the years from 1974 to 1978.

Sol. Let the equation of the straight-line of best fit, with the origin at the middle year 1976 and unit of x as 1 year, be

$$y = a + bx$$

By the method of least squares, the values of a and b given by

$$a = \frac{\sum y}{N}$$

and $$b = \frac{\sum xy}{x^2}$$

Here N = number of years = 5

Calculations for the line of best fit

Year	*Sale (Rs '000) y*	*x*	x^2	*xy*
1974	35	–2	4	–70
1975	56	–1	1	–56
1976	79	0	0	0
1977	80	1	1	80
1978	40	2	4	80
1979	$\sum y = 290$	0	$\sum x^2 = 10$	$\sum xy = 34$

Using (2), $a = \frac{\sum y}{N} = \frac{290}{5} = 58$ and $b = \frac{\sum xy}{\sum x^2} = \frac{34}{10} = 3.4$

From (1), the required equation of the best fitted straight-line is $Y = 58 + 3.4x$.

Year	*x*	*Trend Values (y = 58 + 3.4x)*
1974	–2	58 + 3.4 × (–2) = 51.2
1975	–1	58 + 3.4 × (–1) = 54.6
1976	0	58 + 3.4 × 0 = 58.0
1977	1	58 + 3.4 × 1 = 64.4
1978	2	58 + 3.4 × 2 = 64.8

Example 8. *Fit a straight-line trend equation by the method of least square and estimate the trend value.*

Year	*1961*	*1962*	*1963*	*1964*	*1965*	*1966*	*1967*	*1968*
Values	*80*	*90*	*92*	*83*	*94*	*99*	*92*	*104*

Sol. Here N = Number of years = 8, which is even

Let the straight line trend equation by the method of least squares with the origin at the mid point of 1964 and 1965, and unit of x as 1/2 year be

$$y = a + bx \quad \text{...(1)}$$

Then a and b are given by

$$a = \frac{\sum y}{N} \text{ and } b = \frac{\sum xy}{\sum x^2}$$

Calculations for fitting the straight-line trend

Year	*Value y*	*x*	x^2	*xy*
1961	80	–7	49	–560
1962	90	–5	25	–450
1963	92	–3	9	–276
1964	83	–1	1	–83
1965	94	1	1	94
1966	99	3	9	297
1967	92	5	25	460
1968	104	7	49	728
Total	$\sum Y = 734$	0	$\sum x^2 = 168$	$\sum XY = 210$

Using (2), $a = \frac{\sum y}{N} = \frac{734}{8} = 91.75$, and $b = \frac{\sum xy}{\sum x^2} = \frac{210}{168} = 1.25$

∴ From (1), the required equation of the straight-line trend is

$$y = 91.25 + 1.25x$$

Year	*X*	*Trend Value* *(y = 91.75 + 1.25x)*
1961	–7	91.75 + 1.25 × –7 = 83.0
1962	–5	91.75 + 1.25 × –5 = 85.5
1963	–3	91.75 + 1.25 × –3 = 88.0
1964	–1	91.75 + 1.25 × -1 = 90.5
1965	1	91.75 + 1.25 × 1 = 93.0
1966	3	91.75 + 1.25 × 3 = 95.5
1967	5	91.75 + 1.25 × 5 = 98.0
1968	7	91.75 + 1.25 × 7 = 100.5

Note: If the number of years is even, there is no middle year and in this case the midpoint, which is take as the origin, lies midway between the two middle years. In example 8, the midpoint (*i.e.*, the origin) lies midway between July 1, 1964 and July 1, 1965, which is January 1, 1965 (or December 31, 1964). To avoid fractions, the units of x are taken as $1/2$ year (or 6 months).

10.4.2 Analysis of Seasonal Variation

Seasonal variations are short term fluctuations in recorded values due to different circumstances, which affect results at different times of the year, on different days of the week, at different times of day or whatever.

Seasonal variations are measured through their indices called the seasonal indices. The measurement of seasonal variations requires determining the seasonal component s_t which indicates how a times series from quarter to quarter, month to month, or week to week, etc. through out a year. A series of numbers showing relative values of a variable during the quarters or months or weeks etc. of the year is called seasonal index for the variable.

If Rs. x be the average quarterly (or monthly) sales in a year and I be the Seasonal index of that quarter (or month), then

Sale for the Qaurter or Month =1% of $x = \frac{1}{100} \times x.$

The following methods are commonly used for measuring seasonal variations.

(*a*) Method of Averages (Quarterly, Monthly or Weekly)

(*b*) Moving Average Method

(*c*) Ratio to Trend Method

(*d*) Link Relative Method

(*a*) Method of Averages: This method is used when trend and cyclical fluctuations, if any, have little effect on the time series.

If quarterly data are given, first find quarterly totals for each quarter and the averages for the four-quarter of the years. To find these averages, we divide the quarterly totals by the number of the years for which the data are given. Then we find grand average of the 4 quarterly averages.

$$\text{Grand average } G = \frac{\overline{x_1} + \overline{x_2} + \overline{x_3} + \overline{x_4}}{4}$$

If we use multiplicative identity, then the seasonal indices are the 4 quarterly averages expressed as percentages of the grand average G.

$$i.e., \quad \frac{\overline{x_1}}{G} \times 100, \quad \frac{\overline{x_2}}{G} \times 100, \quad \frac{\overline{x_3}}{G} \times 100, \quad \frac{\overline{x_4}}{G} \times 100$$

Similarly, if we use additive model, then seasonal variations for the 4 quarters are

$$\overline{x_1} - G, \ \overline{x_2} - G, \ \overline{x_3} - G, \ \overline{x_4} - G$$

When monthly or weekly data are given, we find monthly (or weekly) averages for the 12 months or (52 weeks).

Example 9. *Calculate Seasonal indices for each quarter from the following percentages of wholesale price indices to their moving averages.*

Year	*Quarter*			
	I	*II*	*III*	*IV*
1996	–	–	11.0	11.0
1997	12.5	13.5	15.5	14.5
1998	16.8	15.2	13.1	15.3
1999	11.2	11.0	12.4	13.2
2000	10.5	13.3	–	–

Sol. Calculation for Seasonal Indices

Year	*Quarter*			
	I	*II*	*III*	*IV*
1996	–	–	11.0	11.0
1997	12.5	13.5	15.5	14.5
1998	16.8	15.2	13.1	15.3
1999	11.2	11.0	12.4	13.2
2000	10.5	13.3	–	–
Total	51.0	53.0	52.0	54.0
Averages	12.75	13.25	13.0	13.5

$$\text{Grand Average } (G) = \frac{12.75 + 13.25 + 13.0 + 13.5}{4}$$

$$= \frac{52.5}{4} = 13.125$$

Using Multiplicative model, *Seasonal index* $= \dfrac{Average\ (\overline{x_i})}{Grand\ \ Average\ (G)} \times 100$

$\therefore$ Seasonal indices for the first, second, third, and fourth quarters are respectively

$$\frac{12.75}{13.125} \times 100 = 97.14$$

$$\frac{13.25}{13.125} \times 100 = 100.95$$

$$\frac{13.0}{13.125} \times 100 = 99.95$$

$$\frac{13.5}{13.125} \times 100 = 102.86$$

Example 10. *Compute the Seasonal Index for the following data:*

	Qaurters			
Year	*I*	*II*	*III*	*IV*
2001	*75*	*60*	*54*	*59*
2002	*86*	*65*	*63*	*80*
2003	*90*	*72*	*66*	*85*
2004	*100*	*78*	*72*	*93*

Sol. Let Cyclical Fluctuations and Trend are absent in the given data

Year	*Quarters*			
	I	*II*	*III*	*IV*
2001	75	60	54	59
2002	86	65	63	80
2003	90	72	66	85
2004	100	78	72	93
Total	351	275	255	317
Averages $(\bar{x}_i)$ (Total/4)	87.75	68.75	63.75	79.25

Grand Average (G) = $\dfrac{87.75 + 68.75 + 63.75 + 79.25}{4} = \dfrac{299.50}{4} = 74.875$

Now, using Multiplicative model,

$$\text{Seasonal index } \frac{\text{Average } (\bar{x}_i)}{\text{Grand Average } (G)} \times 100$$

Hence Seasonal indices for the 1st, 2nd, 3rd and 4th quarters are respectively

$$1\text{st} = \frac{87.75}{74.875} \times 100 = 117.20$$

$$2\text{nd} = \frac{68.75}{74.875} \times 100 = 91.82$$

$$3\text{rd} = \frac{63.75}{74.875} \times 100 = 85.14$$

$$4\text{th} = \frac{79.25}{74.875} \times 100 = 105.84$$

Hence the sum of Seasonal indices is 400.

Similarly, we can obtain Seasonal indices using additive model. Using additive model,

$$\text{Seasonal index} = \text{Average } (\bar{x}_i) - \text{Grand average (G)}$$

Therefore 1st, 2nd, 3rd and 4th quarters are respectively

12.875, – 6.125, –11.125, 4.375

Hence the sum of seasonal indices for the four quarters is

12.875 + 4.375 – 11.125 – 6.125 = 0. **Ans.**

(*b*) Moving Average Method (or ratio to moving averages): This is a improved method over method of averages and is widely used for measuring seasonal variation. According to this method, if monthly data are given, we find 12-month centred moving averages, if quarterly data are given, we find 4 quarter centred moving averages and so on. This represent trend and then eliminate the effect of trend by using either additive model or multiplicative model.

Case 1: If multiplicative model is used, then express the original data as percentage of the corresponding moving averages expressed as percentage that is ratio to moving averages expressed as percentage. These percentages for corresponding months or quarters are then averaged by the method of averages, gives the required seasonal indices. This method is known as **Ratio to Moving Average Method.**

Example 11. *Calculate Seasonal indices by the ratio to moving average method from the following data.*

Iron Prices (In Rupees Per Kg.)

Quarter Year	*2001*	*2002*	*2003*	*2004*
Quarter 1	75	86	90	100
Quarter 2	60	65	72	78
Quarter 3	54	63	66	72
Quarter 4	59	80	85	93

Calculation of Ratio to moving averages

Year/Quarter		Prices	4 Quarter moving total	2 Point Moving total	4 Quarter Average (2-pt. Moving ÷ 8)	Ratio to Average
2001	Q_1	75				
	Q_2	60				
	Q_3	54	248			
	Q_4	59	259	507	63.375	$\frac{54}{63.375}\times 100 = 85.21$
2002	Q_1	86	264	523	65.375	$\frac{59}{65.375}\times 100 = 90.25$
	Q_2	65	273	537	67.125	$\frac{86}{67.125}\times 100 = 128.12$
	Q_3	63	294	567	70.875	$\frac{65}{70.875}\times 100 = 91.71$
	Q_4	80	298	592	74.000	$\frac{63}{74.000}\times 100 = 85.14$
2003	Q_1	90	305	603	75.375	$\frac{80}{75.375}\times 100 = 106.41$
	Q_2	72	308	613	76.625	$\frac{90}{76.625}\times 100 = 117.46$
	Q_3	66	313	621	77.625	$\frac{72}{77.625}\times 100 = 92.75$
	Q_4	85	323	636	79.500	$\frac{66}{79.500}\times 100 = 83.02$
2004	Q_1	100	329	652	81.500	$\frac{85}{81.500}\times 100 = 104.29$
	Q_2	78	335	664	83.000	$\frac{100}{83.000}\times 100 = 120.48$
	Q_3	72	343	678	84.750	$\frac{78}{84.750}\times 100 = 92.04$
	Q_4	93				

Calculation for Seasonal Indices

	Quarter			
Year	Q_1	Q_2	Q_3	Q_4
2001	--	--	85.21	90.25
2002	128.12	91.71	85.14	106.14
2003	117.46	92.75	83.02	104.29
2004	120.48	92.04	--	--
Total	366.06	276.50	253.37	300.68
Averages	122.02	92.17	84.46	100.23

$$\text{Grand Average } (G) = \frac{122.02 + 92.17 + 84.46 + 100.23}{4} = 99.72$$

∵ Seasonal indicies for 4 quarters are respectively

$$Q_1 = \frac{122.02}{99.72} \times 100 = 122.36$$

$$Q_2 = \frac{92.17}{99.72} \times 100 = 92.43$$

$$Q_3 = \frac{84.46}{99.72} \times 100 = 84.70$$

$$Q_4 = \frac{100.23}{99.72} \times 100 = 100.51$$

Hence sum of seasonal indices is 400. **Ans.**

Case 2: If the additive model is used, to eliminate trend subtract the moving averages from the original data and also obtain deviations from trend. Now apply the method of averages to these deviations to obtain required seasonal variations.

Example 12. *Obtain Seasonal Fluctuation from the following time series using moving averages method.*

	Quaterly output for 4 years			
Year	*I*	*II*	*III*	*IV*
1998	*65*	*58*	*56*	*61*
1999	*68*	*63*	*63*	*67*
2000	*70*	*59*	*56*	*52*
2001	*60*	*55*	*51*	*58*

Sol. Calculation of moving averages and deviations from trend

Year/Quarter		Output	4-Quarter Moving total	2-Point Moving total	4-Quarter Moving Average	Deviation from Trend
1998	Q_1	65				
	Q_2	58				
	Q_3	56	240			
	Q_4	61	243	483	483/8 = 60.38	56 – 60.38 = –4.38
1999	Q_1	68	248	491	491/8 = 61.38	61 – 61.38 = –0.38
	Q_2	63	255	503	503/8 = 62.88	68 – 62.88 = 5.12
	Q_3	63	261	516	516/8 = 64.50	63 – 64.50 = –1.50
	Q_4	67	263	524	524/8 = 65.25	63 – 65.50 = –2.50
2000	Q_1	70	259	522	522/8 = 65.25	67 – 65.25 = 1.75
	Q_2	59	252	511	511/8 = 63.88	70 – 63.88 = 6.12
	Q_3	56	237	489	489/8 = 61.13	59 – 61.13 = –2.13
	Q_4	52	227	464	464/8 = 58.00	56 – 58.00 = –2.00
2001	Q_1	60	223	450	450/8 = 56.25	52 – 56.25 = –4.25
	Q_2	55	218	441	441/8 = 55.13	60 – 55.13 = 4.87
	Q_3	51	224	442	442/8 = 55.25	55 – 55.25 = –0.25
	Q_4	58				

Calculate the Seasonal Fluctuations

	Deviation from Trend			
Year	*Quarter-I*	*Quarter-II*	*Quarter-III*	*Quarter-IV*
1998	---	---	–4.38	–0.38
1999	5.12	–1.50	–2.50	1.75
2000	6.12	–2.13	–2.00	–4.25
2001	4.87	–0.25	---	---
Total	16.11	–3.88	–8.88	–2.88
Average $\bar{x}_i$	5.37	–1.29	2.96	–0.96

Grand Average (G) = 5.37 + (–1.29) + (2.96) + (–0.96) = 0.16 ÷ 4 = 0.04

Therefore the seasonal functions are $\left(\bar{x}_i - G\right)$ *i.e.,*

$$5.37 - 0.04 = 5.33$$
$$-1.29 - 0.04 = -1.33$$
$$2.96 - 0.04 = 2.92$$
$$-0.96 - 0.04 = -1.00 \text{ respectively}$$

(*c*) Ratio to Trend Method: In this method trend values are first determined by the method of least squares fitting a mathematical curve and the given data are expressed as percentage of the corresponding trend values. Using the multiplicative identity these percentages are then averaged by the method of averages.

(*d*) **Link Relative Method:** According to this method for given data for each quarter or m089onth are expressed as percentage of data for the preceding quarter or month. These percentages are known as Link relatives. The link relative for the first quarter (or 1st month) of the year cannot be determined. An appropriate average (Arithmetic Mean or Median) of the link relatives for each quarter (or month) is then found. From these average link relatives, we find the chain relative with respect to 1st quarter (or 1st month) for which the chain relative is taken as 100. If Q_1, Q_2, Q_3, Q_4 denotes 4 quarters respectively and chain relative represents by C.R., or link relative represents by L.R. then

C.R. for Q_2 = (Average L.R. for Q_2 × C.R. for Q_1) ÷ 100

C.R. for Q_3 = (Average L.R. for Q_3 × C.R. for Q_2) ÷ 100

C.R. for Q_4 = (Average L.R. for Q_4 × C.R. for Q_3) ÷ 100

and 2nd C.R. for Q_1 = (Average L.R. for Q_1 × C.R. for Q_4) ÷ 100

Generally, the 2nd C.R. for Q_1 will be either higher or lower than the first C.R. 100 for Q_1 depending on the presence of an increase or decrease in trend.

If d = 2nd C.R. for Q_1–100, *i.e.,* the difference between 1st and 2nd C.R. for Q_1 then we subtract

$(1 \div 4)\ (2 \div 4)d,\ (3 \div 4)\ (4 \div 4)d$

From the chain relatives for Q_2, Q_3 and Q_4 and the 2nd C.R. for Q_1 respectively to obtain the adjusted chain relatives. These adjusted chain relatives expressed as percentages of their A.M. gives the required Seasonal indices.

Example 13. *Calculate seasonal indices by method of link relatives from the data given in Example 12.*

Sol. **Calculate of Average Link Relatives**

		Link Relatives		
Year/Quarter	*Quarter-I*	*Quarter-II*	*Quarter-III*	*Quarter-IV*
1998	---	89.23	96.55	108.93
1999	111.48	92.65	100.00	106.35
2000	104.48	84.29	94.92	92.86
2001	115.38	91.67	92.73	113.73
Total	331.34	357.84	384.20	421.87
Averages (A.M.)	110.45	89.46	96.05	105.47

The link relative (L.R.) for the 1st quarter Q_1 of the first year 1998 cannot be determined. For the other three quarters of 1998,

L.R. for Q_2 = $(58 \div 65) \times 100$,

L.R. for Q_3 = $(56 \div 58) \times 100$,

L.R. for Q_4 = $(61 \div 56) \times 100$,

Similarly, we determine the other link relative.

From the average link relatives obtained in the last row of the above table, we now find chain relatives, taking 100 as the chain relative (C.R.) for Q_1.

C.R. for Q_1 = 100

C.R. for Q_2 = (89.46 × 100) ÷ 100 = 89.46

C.R. for Q_3 = (96.05 × 89.46) ÷ 100 = 85.93

C.R. for Q_4 = (105.47 × 85.93) ÷ 100 = 90.63

2nd C.R. for Q_1 = (110.45 × 90.63) ÷ 100 = 100.10

$\therefore \quad d = 100.10 - 100 = 0.10;$

$$\therefore \quad \frac{1}{4}d = 0.025, \quad \frac{2}{4}d = 0.050, \quad \frac{3}{4}d = 0.075, \quad \frac{4}{4}d = 0.10.$$

The adjusted chain relatives are respectively

	100,	89.46 – 0.025,	85.93 – 0.05,	90.63 – 0.075,
i.e.	100,	89.435,	85.88,	90.555
or	100,	89.44,	85.88,	90.56.

A.M. of the adjusted chain relative = (100 + 89.44 + 85.88 + 90.58) ÷ 4 = 91.47

The required seasonal indices are $\frac{100}{91.47} \times 100, \frac{89.44}{91.47} \times 100, \frac{85.88}{91.47} \times 100, \frac{90.56}{91.47} \times 100$

i.e., 109.33, 97.78, 93.89, 99.00. **Ans.**

10.4.3 Analysis of Cyclical Fluctuation or Cyclic Variations

To analyse Cyclical fluctuation, we first find trend (T) and seasonal Variation (S) by any suitable method *i.e.,* by the method of moving averages or other method and then eliminate them from the original data by using additive or multiplicative identity. Irregular movement is removed by using moving average of appropriate period depending average duration of irregular movement, leaving only cyclical fluctuation.

10.4.4 Analysis of Irregular of Random Movements

Irregular movements are obtained by eliminating trend (T), seasonal variation (S) and cyclical fluctuation (C) from the original data. Normally irregular movements are found to be of small magnitude.

10.5 IMPORTANCE OF TIME SERIES

The time series analysis is of great importance not only to a businessman, scientist or economist, but also to people working in various disciplines in natural, social and physical sciences. Some of its uses are:

1. It enables us to study the past behaviour of the phenomenon under consideration, *i.e.,* to determine the type and nature of the variations in the data.
2. The segregation and study of the various components is of paramount importance to a businessman in the planning of future operations and in the formulation of executive and policy decisions.
3. It enables us to predict or estimate or forecast the behaviour of the phenomenon in future, which is very essential for business planning.
4. It helps us to compare the changes in the values of different phenomenon at different times or places, etc.

PROBLEM SET 10.1

1. Fit a trend line to the following data by the free hand method.

Year	1990	1991	1992	1993	1994	1995	1996	1997
Sales (in million Rs.)	62	64	66	63.5	67	64.5	69	67

2. Draw a trend line by semi average method using the following data.

Year	1991	1992	1993	1994	1995	1996	1997	1998
Production (in tons)	36	43	43	34	44	54	34	24

[**Ans.** Semi Average 39.30]

3. Obtain the 5 yearly moving averages for the following series of observations.

Year	1997	1998	1999	200	2001	2002	2003	2004
Annual Sales (Rs'0000)	3.6	4.3.	4.3	3.4	4.4	5.4	3.4	2.4

[**Ans.** 5 yearly moving averages are 4, 4.36, 4.18 and 3.80]

4. Find the trend from the following series using a three year weighted moving average with weight 1, 2 and 1.

Year	1	2	3	4	5	6	7
Values	2	4	5	7	8	10	13

[**Ans.** Trend values 3.75, 5.25, 6.75, 8.25 and 10.25]

5. For the following series of observations, verify that 4 year centred moving average is equivalent to a 5 year weighted moving average with weight 1, 2, 2, 2, 1 respectively.

Year	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994
Sales (Rs. '000)	2	6	1	5	3	7	2	6	4	8	3

6. Represent the following data graphically and show the trend of the series on the basis of three year moving averages.

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979
Birthrate	30.9	30.2	29.1	31.4	33.4	30.2	30.4	31.0	29.0
Year	1980	1981	1982	1983	1984	1985	1986	1987	1988
Birthrate	27.9	27.7	26.4	24.7	24.1	23.1	27.7	22.6	23.6
Year	1989	1990	1991						
Birthrate	23.0	22.0	22.6						

[**Ans.** Trend values are: 30.7, 30.2, 31.3, 31.7, 30.5, 30.1, 29.3, 28.2, 27.3, 26.3, 25.6, 24.0, 25.0, 23.8, 24.6, 23.6, 22.9, and 22.5.]

7. The revenue from sales Tax in U.P. during 1948–99 to 1952–53 is shown in the following table. Fit a straight-line trend by the method of least square.

Years	*Revenue (Rs. Lakhs)*
1948–49	427
1949–50	612
1950–51	521
1951–52	195
1952–53	490

[**Ans.** Trend values are 311.2, 410.1, 509.0, 607.9, and 706.8.]

8. Find the seasonal indices by the method of moving averages from the series observations.

	Sales of Woollen Yarn ('000 Rs.)			
Quarter	*1976*	*1977*	*1978*	*1979*
I	97	100	106	100
II	88	93	96	101
III	76	79	83	88
IV	94	98	103	106

[**Ans.** Seasonal Indices 10.12, 0.13, –14.08, 3.83.]

9. Calculate the seasonal index from the following data using the average method.

Year	*1st Quarter*	*2nd Quarter*	*3rd Quarter*	*4th Quarter*
1995	72	68	80	70
1996	76	70	82	74
1997	74	66	84	80
1998	76	74	84	78
1999	78	74	86	82

[**Ans.** 96.4, 92.1, 106.9, 100.5]

10. Using 4-Quarterly moving averages find seasonal indices using ratio to moving average method from the given data

	Quarter			
Year	*I*	*II*	*III*	*IV*
1998	101	93	79	98
1999	106	96	83	103
2000	110	101	88	106

[**Ans.** 110.9, 99.9, 84.9, 104.3]

10.6 FORECASTING

The method and principles of Time series are used in the important work of the forecasting. Forecasting is an art of making an estimate of future conditions on a systematic basis using prior available information. On another way we say that the forecasting is the projection of the past data into future and therefore it has varity of applications. Forecasting is done on specified assumption and is always made with probability ranges. The need for forecasting arises because future is characterized by uncertainty. Successful business activity demands a reasonably accurate forecasting of future business conditions upon which decisions regarding production, inventories, price fixation, etc. depend. To estimate guesswork modern statistical methods are employed as a very useful tool of forecasting.

10.7 FORECASTING MODES

The time series analysis essentially involves decomposition of the time series into its four components for forecasting. The main purpose is to estimate and separate the four types of variations and to bring out the relative impact of each on the over all behaviour of the time series. For the purpose of forecasting these will be two-model decomposition of time series.

10.7.1 Additive Model

This model is used when it is assumed that the four components of time series are independent of one another. Thus, if M_t is taken represent the magnitude of time series then,

$$M_t = T_t + S_t + C_t + I_t$$

where T_t = Trend Variation at time t

S_t = Seasonal Varaition at time t

C_t = Cyclical Variation at time t

I_t = Irregular or random Variation at time t

When the time series data are recorded against years, the seasonal component of time series vanish and therefore we have.

$$M_t = T_t + C_t + I_t$$

10.7.2 Multiplicative Model

This model is used when it is assumed that the forces giving rise to the four types of variations of time series are interdependent. *i.e.*

$$M_t = T_t \times S_t \times C_t \times I_t$$

Similarly to additive model, if the time series data are recorded against years then S_t vanish and we have

$$M_t = T_t \times C_t \times I_t$$

by taking logarithm on both sides,

$$\log M_t = \log T_t + \log C_t + \log I_t$$

This implies the four components of time series are essentially additive, in additive as well as multiplicative models.

Note: The multiplicative model is better than the additive model for forecasting when the trend is increasing or decreasing over time. In such circumstances, seasonal variations are likely to be increasing or decreasing too. The additive model simply adds absolute and unchanging seasonal variations to the trend figures where as the multiplicative model, by multiplying increasing or decreasing trend values by a constant seasonal variation factor, takes account of changing seasonal variations.

10.8 TYPES OF FORECASTING AND FORECASTING METHODS

Forecasting are of two types:

(*a*) **Qualitative Forecasting:** Qualitative forecasting is used when past data is not available.

(*b*) **Quantitative Forecasting:** Quantitative forecasting is used if historical or past data are available.

Quantitative forecasting are two types. One is **Time Series Forecasting** and another is **Casual Forecasting**. In casual forecasting methods, factors relating to the variable whose values are to be predicted are determined and in time series forecasting method, projection of the future values of a variable is indicated depending on the past and the present movements of the variable. Different forecasting methods using time series are given in the following.

1. **Mean Forecast:** It is the simplest forecasting method. According to this method the mean $\bar{y}$ of the time series is taken as a forecast or predicted value for the value of y_t of the series for the time period t *i.e.*, $\hat{y}_t = \bar{y}$.
2. **Naive Forecast:** In this method, recent past is considered for the predication of immediate future. If there exist high correlation between the pair of values in the time series then the value y_t for the time period t is the forecast of the value y_{t+1} for the time period $(t+1)$ *i.e.*, $\hat{y}_{t+1} = y_t$.
3. **Linear Trend Forecast:** In this method, the equation of the trend line $y = a + bx$ for the given time series is first determined by the method of least squares. Then the forecast for the period t is found from the relation $\bar{y}_t = a + bx$, where x is obtained from the value of t.
4. **Non Linear Trend Forecast:** In this method a parabolic or non-linear relationship between the time and the response value (time series observation) is first determined by the method of least squares. Then the forecast for the period t is found from the relation $\hat{y}_t = a + bx + cx^2$, where x is obtained from value of t.

10.9 SMOOTHING OF CURVE

Smoothing techniques are used to reduce irregularities (random fluctuations) in time series data. They provide a clearer view of the underlying behaviour of the series. In some, time series, seasonal variation is so strong it obscures any trends or cycles, which are very important for the understanding of the process being observed. Smoothing can remove seasonality and makes long-term fluctuations in the series stand out more clearly. The most common type of smoothing technique is moving average smoothing although others do exist. Since the type of seasonality will vary from series to series, so must the type of smoothing.

(*a*) Exponential Smoothing: Exponential smoothing is a smoothing technique used to reduce irregularities (random fluctuations) in time series data, thus providing a clearer view of the true underlying behaviour of the series. It also provides an effective means of predicting future values of the time series (forecasting).

(*b*) Moving Average Smoothing: A moving average is a form of average, which has been adjusted to allow for seasonal or cyclical components of a time series. Moving average smoothing is a smoothing technique used to make the long-term trends of a time series clearer. When a variable, like the number of unemployed, or the cost of strawberries, is graphed against time, there are likely to be considerable seasonal or cyclical components in the variation. These may make it difficult to see the underlying trend. These components can be eliminated by taking a suitable moving averages. By reducing random fluctuations, moving average smoothing makes long term trends clearer.

(*c*) Running Medians Smoothing: Running medians smoothing is a smoothing technique analogous to that used for moving averages. The purpose of the technique is the same, to make a trend clearer by reducing the effects of other fluctuations.

❑❑❑

CHAPTER 11

Statistical Quality Control

11.1 INTRODUCTION

The important, appealing and easily understood method of presenting the statistical data is the use of diagrams and graphs. They are nothing but geometrical figures like points, lines, bars, squares, rectangles, circles, cubes etc., pictures, maps or charts. Diagrammatic and graphic representation has a number of advantages. Some of them are given below:

1. Diagrams are generally more attractive and impressive than the set of numerical data. They are more appealing to the eye and leave a much lasting impression on the mind as compared to the uninteresting statistical figures.
2. Diagrams and graphs are visuals aids, which give a bird's eye view of a given set of numerical data. They present the data in simple, readily comprehensible form.
3. They register a meaning impression on the mind almost before we think. They also save lot of time, as very little effort is required to grasp them and draw meaningful inferences from them.
4. The technique of diagrammatic representation is made use of only for purpose of comparison. It is not to be used when comparison is either not possible or is not necessary.
5. When properly constructed, diagrams and graphs readily show information that might otherwise be lost a mid the detail of numerical tabulations. They highlight the salient features of the collected data, facilitate comparisons among two or more sets of data and enable use to study the relationship between them more readily.

11.1.1 Difference between Diagrams and Graphs

There are no certain method to distinguish between diagrams and graphs but some points of difference may be observed

1. Generally graph paper is used in the construction of the graph, which helps us to study the mathematically relationship between the two variables, whereas diagrams are generally constructed on a plain paper and used for comparison only not for studying the relationship between two variables.
2. In graphic mode of representation points or lines (dashes, dot, dot-dashes) of different kinds are used to represent the data while in diagrammatic representation data are presented by bars, rectangles, circles, squares, cubes, etc.

3. Diagrams funish only approximate information. They do not add anything to the meaning of the data and therefore, are not of much use to a statistician or researcher for further statistical analysis. On the other hand graphs are more obvious, precise and accurate than the diagrams and are quite helpful to the mathematician for the study of slopes, rate of change and estimation *i.e.,* interpolation and extrapolation, whenever possible.
4. Construction of graphs is easier as compared to the construction of diagrams. Diagrams are useful in depicting categorical and geographical data but it fails to present data relating to frequency distributions and time series.

11.1.2 Types of Diagrams

A variety of diagrammatic devices are used commonly to present statistical data.

(*a*) One Dimensional Diagrams *i.e.,* line diagrams and bar diagrams.

(*b*) Two Dimensional Diagrams *i.e.,* rectangle, squares, circles and pie diagrams.

(*c*) Three Dimensional Diagrams *i.e.,* cubes, spheres, prisms, cyclinders etc.

(*d*) Pictograms.

(*e*) Cartograms.

11.1.3 Rules for Drawing Diagrams

1. The first and the most important thing is the selection of a proper scale. No definite rules can be laid down as regards the selection of scale. But as a guiding principle the scale should be selected consistent with the size of the paper and the size of the observations to be displayed so that the diagram obtained is neither too small nor too large.
2. The vertical and horizontal scales should be clearly shown on the diagram itself. The former on the left hand side and the latter at the bottom of the diagram.
3. Neatness should be strictly being written on the top in bold letter and should be very explanatory. If necessary the footnotes may be given at the left hand bottom of the diagram to explain certain points of facts.

11.2 LINE DIAGRAM

This is the simplest of all the diagrams. It consists in drawing vertical lines, each vertical line being equal to the frequency. The variate values are presented on a suitable scale along the X-axis and the corresponding frequencies are presented on a suitable scale along Y-axis.

Example 1. *Draw line diagram for the following data:*

No. of rooms	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
No. of houses	*170*	*183*	*191*	*146*	*105*	*75*	*42*	*30*	*25*

Sol.

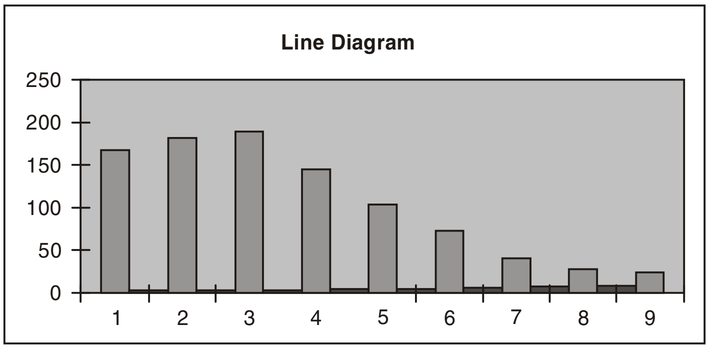


FIG. 11.1

11.3 BAR DIAGRAM

The terms 'bar' is used for a thick wide line. The width of the bar diagram shows merely to make the diagram more explanatory. Bar diagrams are one of the easiest and the commonly used diagram of presenting most of the business and economics data. They consist of a group of equidistant rectangles one for each group or category of the data in which the length or height of the rectangles represents the values or the magnitudes, the width of the rectangles being arbitrary. There are various types of bar diagrams.

(a) Simple Bar Diagram: It is used for comparative study of two or more items or values of a single variable or category of data.

Example 2. *Birth rate of a few countries of the World during the year 1934.*

Country	*India*	*Germany*	*Irish Free State*	*Soviet Russia*	*New Zealand*	*Swedon*
Birth Rate	*33*	*16*	*20*	*40*	*30*	*15*

Sol.

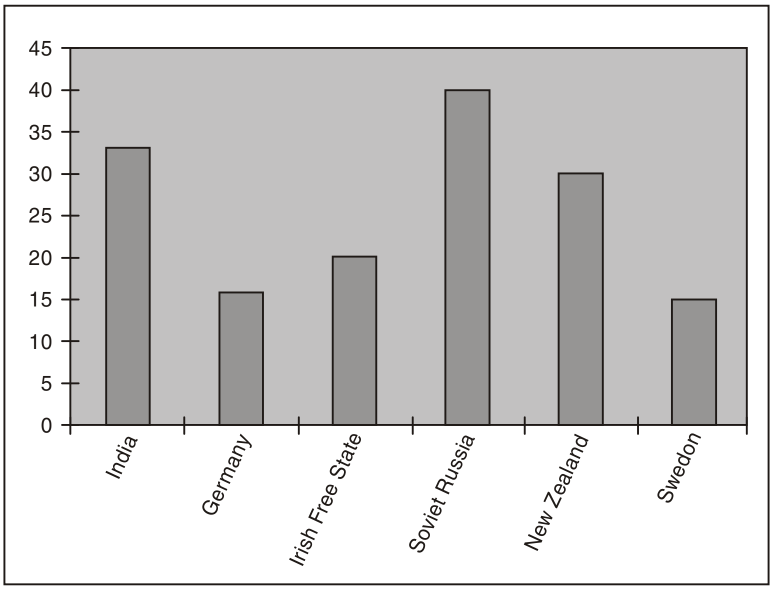


FIG. 11.2

(*b*) **Subdivided Bar Diagram:** If a magnitude is capable of being broken into component parts or if there are independent quantities which form the subdivisions of the total, in either of these cases, bars may be subdivided into the ratio of the various components to show the relationship of the parts to the whole.

Example 3. *Represent the following data by sub-divided bar diagram.*

	Family A	*Family B*
	Income Rs. 500	*Income Rs. 300*
Food	150	150
Clothing	125	60
Education	25	50
Miscellaneous	190	70
Saving or Deficit	10	(–)30

Sol.

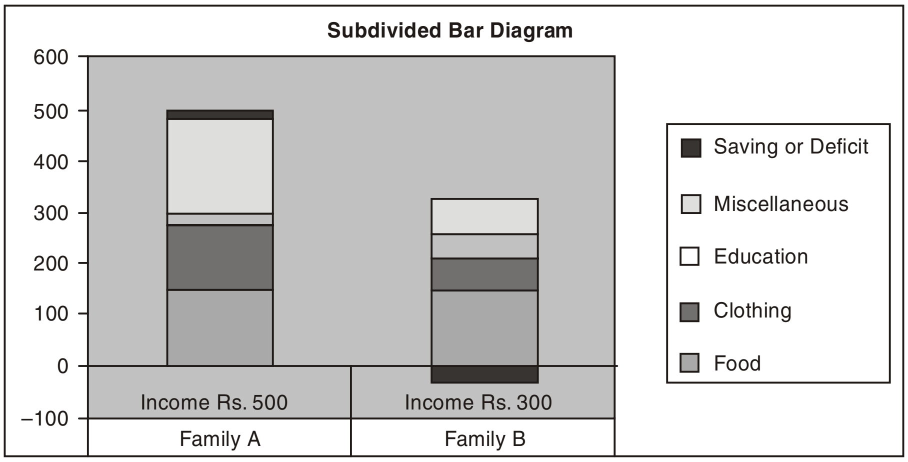


FIG. 11.3

(*c*) **Percentage Bar Diagram:** Subdivided bar diagrams presented graphically on percentage basis give percentage bar diagrams. They are especially useful for the diagrammatic portrayal of the relative changes in the data.

Example 4. *Draw a bar chart for the following data showing the percentage of the total population in villages and towns.*

	Percentage of total Villages	*Population in Towns*
Infants and Young Children	13.7	12.9
Boys and Girls	25.1	23.2
Young men and women	32.3	36.5
Middle aged men and women	20.4	20.1
Elderly person	8.5	7.3

Sol.

	%	*Villages* *Cumulative %*	%	*Towns* *Cumulative %*
Infants and young children	13.7	13.7	12.9	12.9
Boys and Girls	25.1	38.5	23.2	36.1
Young men and women	32.3	71.1	36.5	72.6
Middle aged men and women	20.4	91.5	20.1	92.7
Elderly persosn	8.5	100	7.3	100

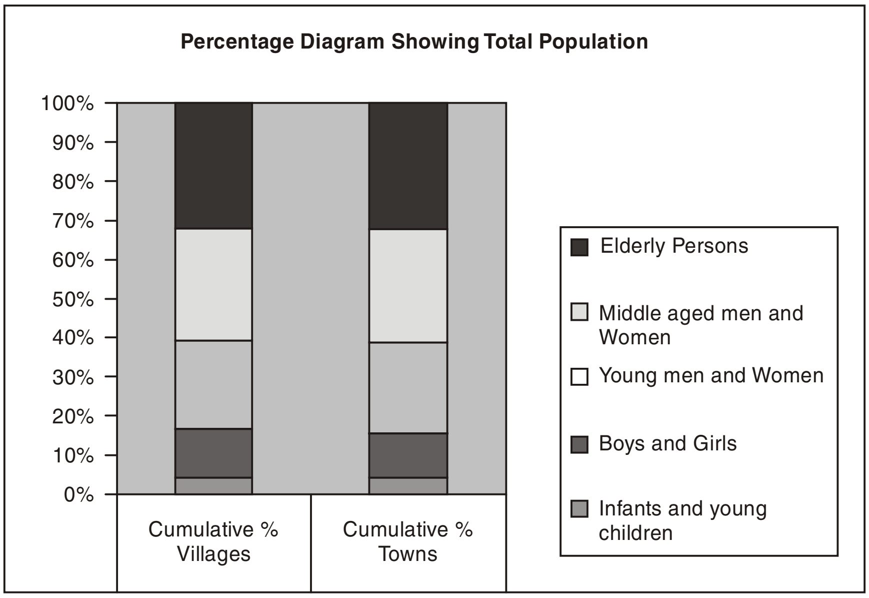


FIG. 11.4

Some other bar diagrams are multiple bar diagram, Deviation bar, Broken bars etc. In a multiple bar diagram two or more sets of interrelated data are represented. The method of drawing multiple bar diagram is the same as that of simple bar diagram. Deviation bars are popularly used for representing net quantities excess or deficit, i.e., net loss, net profit etc. Such types of bars have both positive and negative values. Obviously positive values are shown above the base line and negative values below the base line.

Example 5. *Draw a multiple bar diagram from the following data.*

Year	*Sales ('000 Rs.)*	*Gross Profit ('000 Rs.)*	*Net Profit ('000 Rs.)*
1992	*120*	*40*	*20*
1993	*135*	*45*	*30*
1994	*140*	*55*	*35*
1995	*150*	*60*	*40*

Sol.

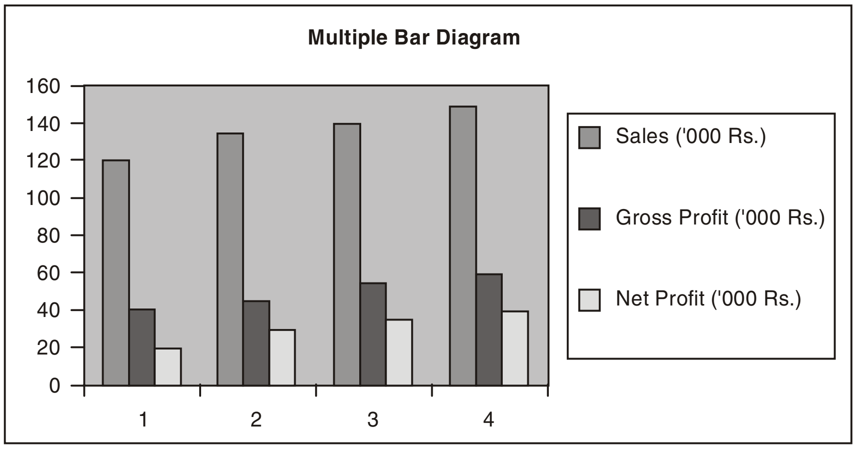


FIG. 11.5

Example 6. *Present the following data by a suitable diagram showing the sales and net profits of private industrial companies.*

Year	*Sales*	*Net Profits*
1995–1996	*14%*	*49%*
1996–1997	*10%*	*–25%*
1997–1998	*13%*	*–1%*

Sol.

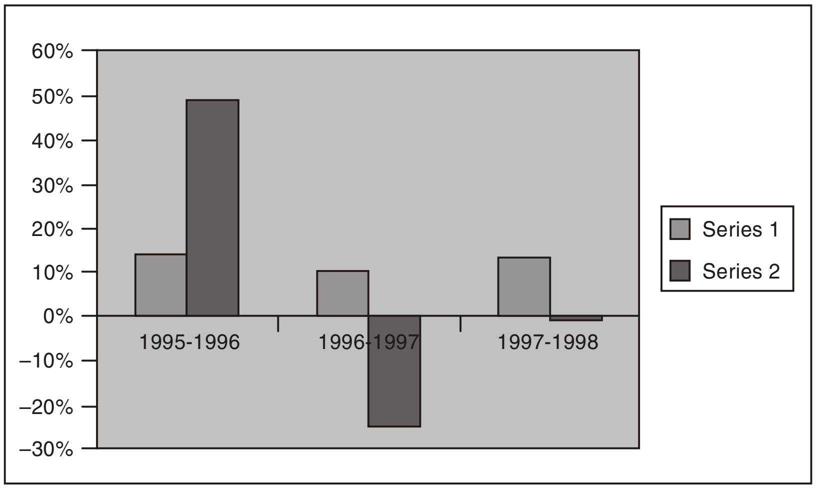


FIG. 11.6

11.4 ONE DIMENSIONAL DIAGRAM

In one dimensional diagram magnitude of the observations are represented by only one of the dimension. *i.e.*, height (length) of the bars while the widths of the bars is arbitrary and uniform.

11.5 TWO DIMENSIONAL DIAGRAMS

In two dimensional diagrams, the magnitude of given observations are represented by the area of the diagram. Thus the length as well as width of the bars will have to be considered. It is also known as are diagram or surface diagram. Some two dimensional diagrams are

(*a*) **Rectangles Diagram:** A rectangle is a two dimensional diagram because area of rectangle is given by the product of its length and widths. *i.e.*, length and width of the bars is taken into consideration.

Example 7. *Represent the following data on detail of cost of the two commodities by the rectangular diagram.*

Details	*Commodity A*	*Commodity B*
Price per unit	*Rs. 4*	*Rs. 5*
Quantity sold	*40 units*	*30 units*
Value of raw material	*Rs. 52*	*Rs. 50*
Other expenses of production	*Rs. 64*	*Rs. 60*
Profits	*Rs. 44*	*Rs. 40*

Sol. Let us calculate the cost of material, other expenses and profit per unit.

	Commodity A		*Commodity B*	
	40 units		*30 units*	
Items	*Total (Rs.)*	*Per Unit (Rs.)*	*Total (Rs.)*	*Per Unit (Rs.)*
Value of raw material	52	1.3	50	1.6
Other expenses of production	64	1.6	60	2.0
Profits	44	1.1	40	1.4

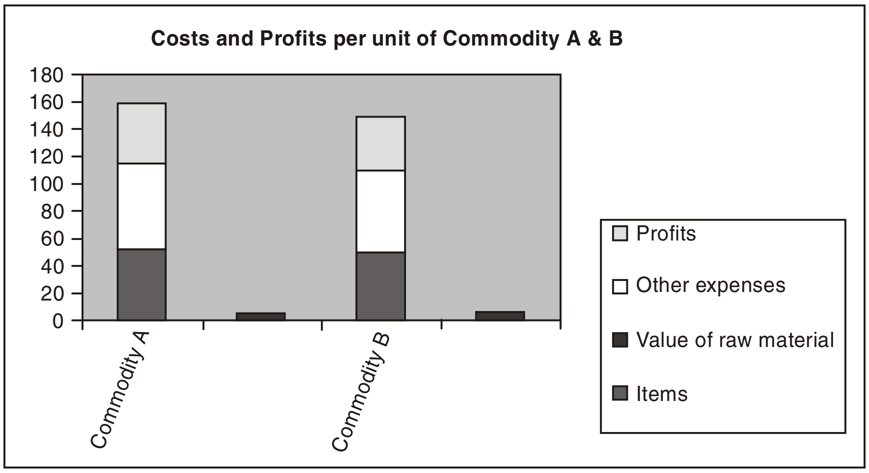


FIG. 11.7

(*b*) **Square Diagram:** It is specially useful, if it is desired to compare graphically the values or quantities which differ widely from one another. The method of drawing a square diagram is very simple. First of all take the square root of the values of the given observations and then squares are drawn with sides proportional to these square roots, on an appropriate scale, which must be satisfied.

Example 8. *Draw a square diagram to represent the following data.*

Country	*A*	*B*	*C*
Yield in (kg) per hectare	*350*	*647*	*1,120*

Sol. First find out the square root of the quantities.

Country	*A*	*B*	*C*
Yield in (kg)	350	647	1,120
Square root	18.7083	25.4362	33.4664
Ratio of the sides of the square	1	$\frac{25.4362}{18.7083} = 1.36$	$\frac{33.4664}{18.7083} = 1.79$

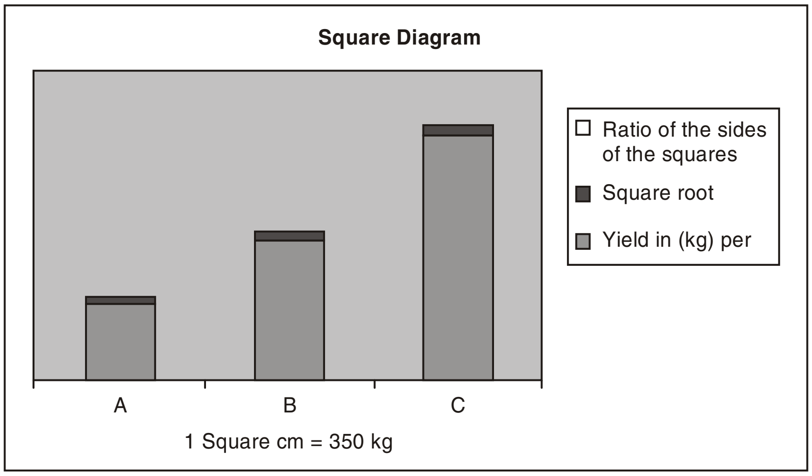


FIG. 11.8

(*c*) **Circle Diagram:** Circle diagrams are alternative to square diagrams and are used for the same purpose. The area of circle, which represents the given values, is given πr^2, where $\pi = \frac{22}{7}$ and r is the radius of circle. That is the area of circle is proportional to the square of its radius and consequently, in the construction of the circle diagram the radius of circle is a value proportional to the square root of the given magnitude. The scale to be used for constructing circle diagrams can be calculated as:

For a given magnitude '*a*', Area = πr^2 square units = a

$$\Rightarrow 1 \text{ square unit} = \frac{a}{\pi r^2}$$

Example 9. *Represent the data of example 8 by a circle diagram.*

Sol. Above example shows as follows.

$$\text{Scale } 1 \text{ square cm.} = \frac{350}{\pi} = \frac{2450}{22} = 111.36 \text{ kg.}$$

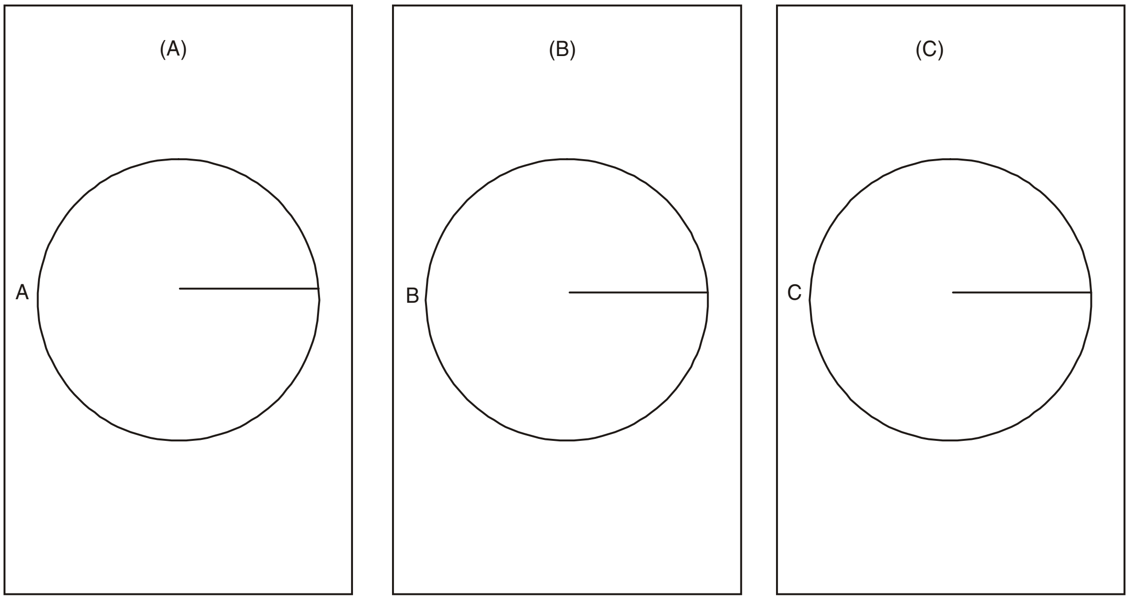


FIG. 11.9

(*d*) Pie diagram: Pie diagram are also called circular diagrams. For the construction of pie diagram,

1. Each of the component values expressed by a percentage of the respective total.
2. Since the angle at the center of the circle is 360°, the total magnitude of various components is taken to be equal to 360° and each component part is to be expressed proportionally in degrees.
3. Since 1 per cent of the total value is equal to $\frac{360}{100} = 3.6^\circ$, the percentage of the component parts obtained in step 1 can be converted to degrees by multiplying each of them by 3.6.
4. Draw a circle of appropriate radius using an appropriate scale depending on the space available.
5. The degrees represented by the various component parts of given magnitude can be obtained directly without computing their percentage to the total values.

$$\text{Degree of any component part} = \frac{\text{component value}}{\text{Total value}} \times 360^\circ$$

Example 10. *Draw a pie diagram to represent the following data.*

Items	*A*	*B*	*C*	*D*
Proposed Expenditure (in million Rs.)	*4,200*	*1,500*	*1,000*	*500*

Sol. Following table gives proposed expenditure in angle form

Items	*Proposed Expenditure*	*Angle at the centre*
A	4,200	$\frac{42}{72} \times 360° = 210°$
B	1,500	$\frac{15}{72} \times 360° = 75°$
C	1,000	$\frac{10}{72} \times 360° = 50°$
D	500	$\frac{10}{72} \times 360° = 25°$
Total	7,200	360°

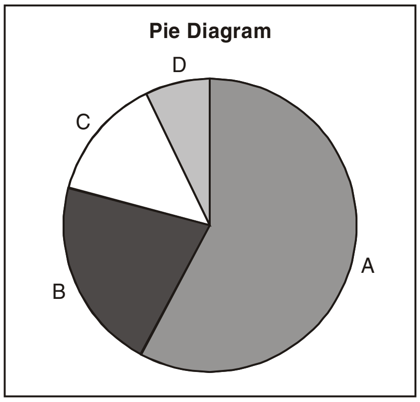


FIG. 11.10

11.6 THREE DIMENSIONAL DIAGRAMS

Three dimensional diagrams are also known as volume diagrams, consists of cubes, cylinders spheres etc. length, width and height have to be taken into account. Such diagrams are used where the range of difference between the smallest and the largest value is very large. Of the various three dimensional diagrams, 'cubes' are the smallest and most commonly used devices of diagrammatic presentation of the data.

11.7 PICTOGRAMS

Pictograms is the technique of presenting statistical data through appropriate pictures and is one of very important key particularly when the statistical facts are to be presented to a layman without any mathematical background. Pictograms have some limitations also. They are difficult to construct and time consuming. Besides, it is necessary to one symbol to represent a fixed number of units, which may create difficulties. It gives only an overall picture, not give minute details.

11.8 CARTOGRAMS

Cartograms or statistical maps are used to give quantitative information on a geographical basis. Cartograms are simple and elementary forms of visual presentation and are easy to understand. Normally it is used when the regional or geographical comparisons are to be required to highlight.

11.9 GRAPHIC REPRESENTATION OF DATA

Graphs is used to study the relationship between the variables. Graphs are more obvious, precise and accurate than diagrams and can be effectively used for further statistical analysis, viz., to study slopes, forecasting whenever possible. Graphs are drawn on a special type of paper known as graph paper. Graph paper has a finite network of horizontal and vertical lines; the thick lines for each division of a centimeter or an inch measure and thin lines for small parts of the same. Graphs are classified in two parts.

1. Graphs of frequency distribution
2. Graphs of time series

11.9.1 Graphs of Frequency Distribution

The so-called frequency graphs are designed to reveal clearly the characteristic features of a frequency data. The most commonly graph for charting a frequency distribution of the data are:

(*a*) **Histogram:** A frequency density diagram is a histogram. According to Opermann, "A histogram is a bar chart or graph showing the frequency of occurrence of each value of the variable being analyzed". In another way we say that, a histogram is a set of vertical bars whose areas are proportional to the frequencies represented. While constructing histogram the variable is always taken on the x-axis and the frequencies depending on it on the y-axis. It applies in general or when class intervals are equal. In each case the height of the rectangle will be proportional to the frequencies.

When class intervals are unequal, a correction for unequal class intervals is required. For making the correction we take that class which has lowest class interval and adjust the frequencies of other classes. If one class interval is twice as wide as the one having lowest class interval we divide the height of its rectangle by two, if it is three times more we divide the height of its rectangle by three and so on.

Example 11. *Represent the following data by a histogram.*

Marks	*No. of Students*	*Marks*	*No. of Students*
0–10	*8*	*50–60*	*60*
10–20	*12*	*60–70*	*52*
20–30	*22*	*70–80*	*40*
30–40	*35*	*80–90*	*30*
40–50	*40*	*90–100*	*05*

Sol. Since the class intervals are equal throughout no adjustment in frequencies are required.

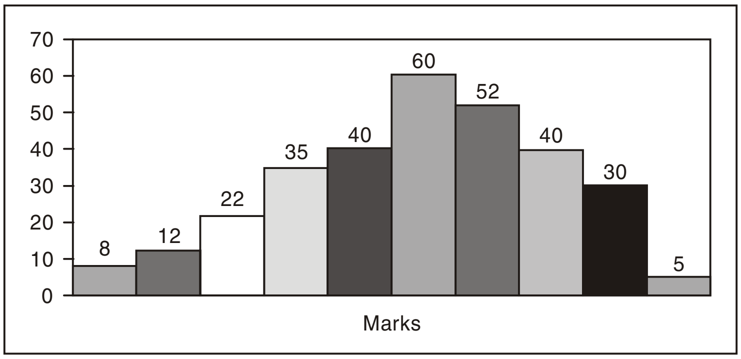


FIG. 11.11

Example 12. *Represent the following data by a histogram.*

Weekly Wages in Rs.	*No. of Workers*
10–15	7
15–20	19
20–25	27
25–30	15
30–40	12
40–60	12
60–80	08

Sol. Since class intervals are unequal, frequencies are required to adjust. The adjustment is done as follows. The lowest class interval is 5 therefore the frequencies of class 30–40 shall be divided by two since the class interval is double, that of 40–60 by 4 etc.

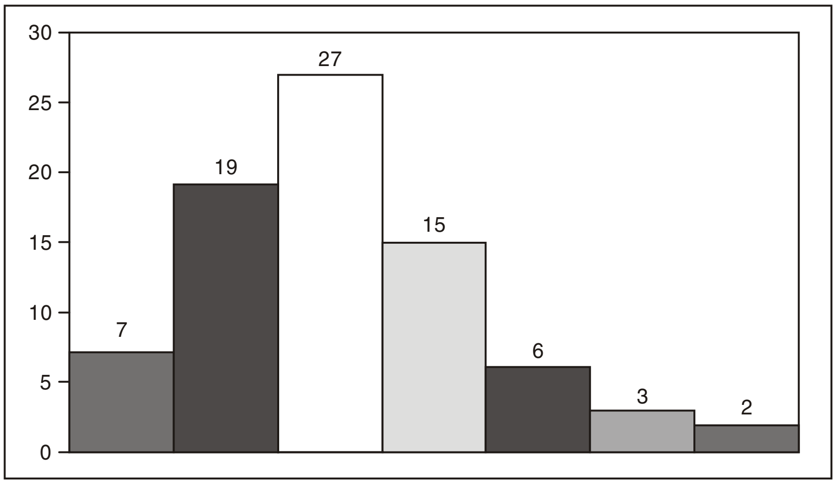


FIG. 11.12

(*b*) **Frequency Polygon:** 'Polygon' literally means 'many-angled' diagram. A frequency polygon is a graph of frequency distribution. It is particularly effective in comparing two or more frequency distribution. There are two ways for constructing frequency polygon.

1. Draw a histogram for a given data and then join by straight lines the midpoints of the upper horizontal sides of each rectangle with the adjacent once. The figure so formed is called frequency polygon. To close the polygon at both ends of the distribution, extending them to the base line.
2. Take midpoints of the various class-intervals and then plot the frequency corresponding to each point and to join all these points by a straight lines. The figure obtained would exactly be the same as obtained by method no. 1. The only difference is that here we have not to construct a histogram.

Example 13. *Draw a frequency polygon from the following data.*

Marks	*0–10*	*10–20*	*20–40*	*40–50*	*50–60*	*60–70*	*70–90*	*90–100*
No. of students	*4*	*6*	*14*	*16*	*14*	*8*	*16*	*5*

Sol. Since class intervals are unequal, so we have to adjust the frequencies. The class 20-40 would be divided into two parts 20–30 and 30–40 with frequency of 7 each class.

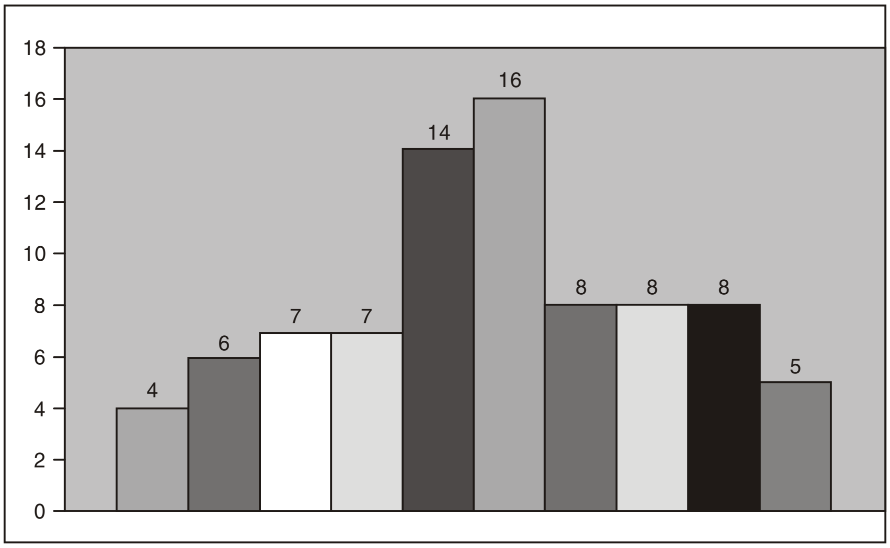


FIG. 11.13

(*c*) **Frequency Curve:** A frequency curve is a smooth free hand curve drawn through the vertices of a frequency polygon. The area enclosed by the frequency curve is same as that of the histogram or frequency polygon but its shape is smooth one and not with sharp edges. Smoothing should be done very carefully so that the curve looks as regular as possible and sudden and sharp turns should be avoided. Though different types of data may give rise to a variety of frequency curves.

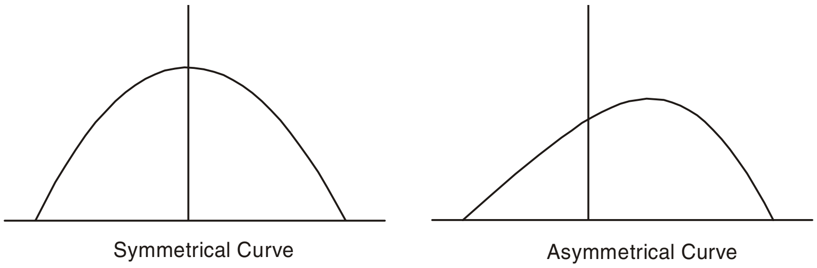


FIG. 11.14

1. **Symmetrical Curve:** In this type of curve, the class frequencies first rise steadily, reach a maximum and then fall in the same identical manner.
2. **Asymmetrical (skewed) frequency curves:** A frequency curve is said to be skewed if it is not symmetrical.
3. **U-Curve:** The frequency distributions in which the maximum frequency occurs at the extremes (i.e., both ends) of the range and frequency keeps on falling symmetrically (about the middle), the minimum frequency being attained at the centre, give rise to a U-shaped curve.

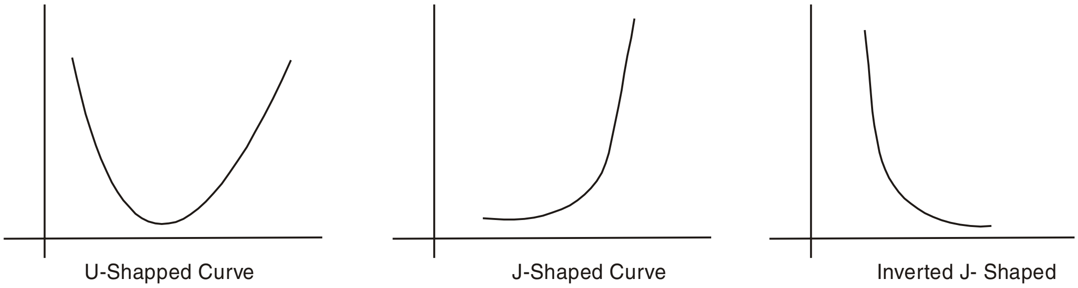


FIG. 11.15

4. **J-Shaped Curve:** In a J-shaped curve the distribution starts with low frequencies in the lower classes and then frequencies increase steadily as the variable value increases and finally the maximum frequency is attained in the last class. Such curves are not regular but become unavoidable in certain situations.

(*d*) Cumulative frequency curve or Ogive: Ogive, pronounced Ojive, is a graphic presentation of the cumulative frequency distribution. There are two types of cumulative frequency distributions. One is 'less than' ogive and second is 'more than' ogive. The curve obtained by plotting cumulative frequencies (less than or more than) is called a cumulative frequency curve of an ogive.

1. **'Less than' method**: In this method we start with the upper limits of the classes and go on adding the frequencies. When these frequencies are plotted we get a rising curve.
2. **'More than' method:** In this method we start with the lower limits of the classes and from the frequencies we subtract the frequency of each class. When these frequencies are plotted we get a declining curve.

11.9.2 Graphs of Time-Series

A time series is an arrangement of statistical data in a chronological order i.e., with respect to occurrence of time. The time series data are represented geometrically by means of **time series graph,** which is also known as Historigram. The various types of time series graphs are

1. Horizontal line graph or historigrams
2. Net balance graphs
3. Range or variation graphs
4. Components or band graphs.

11.10 STATISTICAL QUALITY CONTROL

Statistical quality control abbreviated as SQC involves the statistical analysis of the inspection data, which is based on sampling and the principles involved in normal curve. The origin of

Statistical Quality Control is only recent. Walter A. Shewhart and Harold F. Dodge of the Bell Laboratories (U.S.A) introduced it after the First World War. They used probability theory to developed methods for predicting the quality of the products by conducting tests of the quality on samples of products turned out from the factory. During the Second World War these methods were used for testing war equipment. Today the methods of SQC are used widely in production, storage, aircraft, automobile, textile, plastic, petroleum, electrical equipment, telephones, transportation, chemical, medicine and so on. In fact, it is impossible to think of any industrial field where statistical techniques are not used. Also it has become an integral and permanent part of management controls.

The makers of the product normally set the quality standards. The quality consciousness amongst producer is always more than there is competition from rival producers. Also when consumers are quality conscious. The need for quality control arises because of the fact that even after the quality standards have been specified some variation in quality is unavoidable.

Further, the SQC is only diagnostic. It can only indicate whether the standard is being maintained. The re-medical action rests with the technician. It is therefore remarked, *"Quality control is achieved most efficiently, of course, not by the inspection operation itself, but by getting at causes".* **—Dodge and Roming)**

Statistician's role is there because the analysis is probabilistic. There is use of sampling and rules of statistical inference. Also SQC refers to the statistical techniques employed for the maintenance of uniform quality in a continous flow of manufactured products.

"SQC is an effective system for co-ordinating the quality maintenance and quality improvement efforts of the various graphs in an organization so as to enable production at the most economical levels which allow for a full customer satisfaction". **—A.V. Feigenbaum**

Advantages and Uses of SQC: SQC is a very important technique, which is used to assess the causes of variation in the quality of the manufactured product. It enables us to determine whether the quality standards are being met without inspecting every unit produced in the process. It primarily aims at the isolation of the chance and assignable causes of variation and consequently helps in the detection, identification and elimination of the assignable causes of erratic fluctuations whenever they are present.

"A production process is said to be in a state of statistical control if it is operating in the presence of chance causes only and is free from assignable causes of variation".

There are some advantages, when a manufacturing process is operating in a state of statistical control.

1. The important use and advantage of SQC is the control, maintenance and improvement in the quality standards.
2. Since only a fraction of output is inspected, costs of inspection are greatly reduced.
3. SQC have greater efficiency because much of the boredom is avoided, the work of inspection being considerable reduced.
4. An excellent feature of quality control is that it is easy to apply. One the system is established person who have not had extensive specialized training can operate it.
5. It ensures an early detection of faults and hence a minimum waste of rejects production.
6. From SQC charts one can easily detach whether or not a change in the production process results in a significant change in quality.
7. The diagnosis of the assignable causes of variation gives us an early and timely warningabout the occurrence of defects. These are help in reduction in, waste and scrap, cost per unit etc.

8. The presence of an SQC scheme in any manufacturing concern has a very healthy effect as it creates quality consciousness among their personal. Such ways keep the staff and the worker on their alert they are by increasing their efficiency.

11.10.1 Causes of Variation

In every manufacturing concern, it is intended that all the products produced should be exactly same quality and should confirm to same prescribed specification. However refined and accurate the manufacturing process is, some amount of variation among manufactured products is always noticed which is mainly due to two types of causes:

(*a*) Chance Causes: Variation, which results from many minor causes, that behaves in a random manner. This type of variation is permissible and indeed inevitable, in manufacturing. There is no way in which it can completely be eliminated when the variability present in a production process is confined to chance variation only, the process is said to be in a state of statistical control. These type of causes are also known as random causes. These small variations, which are natural to and inherent in the manufacturing process, are also called allowable variations as they cannot be removed or prevented altogether in any way. The allowable variation is also sometimes known as natural variation, as it cannot be eliminated and one has to allow for such variation in the process.

(*b*) Assignable Causes: These are some variations which are neither natural nor inherent in the manufacturing process and they can be assigned as well as prevented if the causes of such variations are detached. These variations are generally caused by the defects and faults in the production design and manufacturing process.

11.10.2 Types of Quality Control

The control refers to action (or inaction) designed to change a present condition or causes it to remain unchanged; and quality refers to a level or standard which is turn, depends on manpower, materials, machines and management. The main purpose of any production process is to control and maintain a satisfactory quality level for produced product and also it should be ensured that the product conforms to specified quality standards *i.e.,* it should not contain a large number of defective items. The quality of a product manufactured in any factory may be controlled by two ways.

(*a*) Process Control: The first way for controlling the quality is process control which is concerned with controlling the quality during the process of production *i.e.,* the control of a process during manufacture. Also, when statistical techniques are employed during manufacturing period for controlling the quality by detecting the systematic causes of variation as soon as they occur then it is called process control. Process control is achieved by the technique of control charts pioneered by W.A. Shewhart in 1924.

(*b*) Product Control: This is concerned with the inspection of goods already produced whether these are fit to be dispatched. On the other hand by product control we mean controlling the quality of the product by critical examination at strategic points and this is achieved through 'Sampling inspection plans' pioneered by Dodge and Romig.

Process Under Control: A production process is said to be under control when there is no evidence of the presence of assignable causes (or these causes have been detached and removed) and it is governed by the chance causes of variations alone.

Tolerance of the Specification Limits: The manufactures of the manufactured goods often standards to which these product must confirm if they are to be considered of good quality. These standards generally specify the desirable process average together with the limits above and below this process average. These upper and lower limits are called the specification limits or the tolerance limits.

11.11 CONTROL CHARTS

A control chart is a statistical device principally used for the study and control of repetitive process. A control chart is essentially a graphic device for presenting data so as to directly reveal the frequency and extent of variations from established standards of goals. Control charts are simple to construct and easy to interpret and they tell the user at a glance whether or not the process is in control i.e., with in the tolerance limits.

Walter A. Shewhart of Bell Telephone laboratories made the discovery and development of the control charts in 1924. A control chart is an indispensable tool for bringing a process under statistical control. The Shewhart's control charts provides a very simple but powerful graphic method of obtaining if a process is in statistical control or not. Its construction is based on 3– σ limits and a sequence of suitable sample statistics e.g., *mean*$(\bar{x})$, *Range*$(\bar{R})$, *Standard deviation(S), fraction defective(p)* etc. Computed from independent samples drawn at random from the product of the process.

These sample points depict the frequency and extent of variations from specified standards. A control chart consist of three horizontal lines:

(1) Upper Control Limit

(2) Lower Control Limit

(3) Central Limit

together with a number of sample points. In the control chart UCL and LCL are usually plotted as dotted lines and the CL is plotted as a bold line.

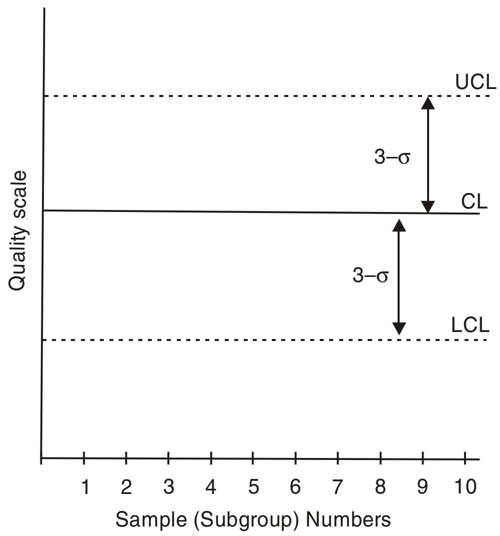


FIG. 11.16

Remarks:

1. A central limit representing the average value of the quality characteristics, or desired standard or level of the control process.
2. An upper control limits (UCL) and lower control limits (LCL) indicates the upper and lower tolerance limit.
3. Ordinarily UCL and LCL are at equal distance from central line, this common distance being equal to three items the standard deviation σ (called "standard error' in sampling theory) of the sample characteristics for which the control chart is prepared.
4. If t is the underlying statistic, then UCL and LCL depends on the sampling distribution of t and are given by

$$\text{UCL} = E(t) + 3S.E.(t)$$
$$\text{LCL} = E(t) - 3S.E(t)$$
$$\text{CL} = E(t)$$

11.12 3–σ CONTROL LIMITS

Control chart is based on the fundamental property of area under the normal distribution. The standard normal probability curve is given by the equation

$$P(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}; -\infty \le t \le \infty, \quad \text{where} \quad t = \frac{x-\mu}{\sigma}$$

and x is normally distributed with mean $\mu = \bar{x}$ and standard deviation σ. Therefore by the property of normal distribution

$$P[\mu - 3\sigma < x < \mu + 3\sigma] = 0.9973 \qquad ...(1)$$

the probability statement in (1) states that if x is normal with μ and standard deviation σ, then the probability or chance that a randomly selected value of x will lie outside the limit μ + 3σ is $1.0 - 0.9973 = 0.0027$, *i.e.* very small, only 27 out of 10,000. In veiw of this, 3-σ limits, are termed as LCL and UCL for quality characteristic x.

In other words $\mu \pm 3\sigma$ covers 99.73 percent of the sample. Hence if points fall outside 3-σ limits, they indicate the presence of some assignable cause all is not due to random causes. It should be noted that if points fall outside 3-σ limits, there is a good reason for believing that they point to some factor contributing to quality variation that can be identified.

11.13 TYPES OF CONTROL CHART

There are two main types of control charts.

1. Control charts for variable ($\bar{x}$, *R*, σ *chart*)
2. Control charts for attributes (*p*, *pn* and *C-chart*)

11.13.1 Control Chart for Variable

The chart used for characteristics on which the actual measurements in numerical forms are possible to be made, *i.e.,* whose samples are subjected to quantitative measurements such as weight, length, diameter, volume etc. are called control for variables.

The charts used for qualitative characteristics as 'defective' or 'non-defective', as 'good' or 'bad', as 'better' or 'worst', are called control charts for attributes.

Control Chart for Mean ($\bar{x}$) **:** The mean chart is used to show the quality averages of the samples drawn from a given process. Before a ($\bar{x}$) chart is constructed the following values must be obtained.

1. **Obtain the Mean of Each Sample:** Let $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, ..., $\bar{x}_i$, be the means of sample on the 1st, 2nd, 3rd,................., *ith* sample observations respectively and let R_1, R_2, R_3, ..., R_i be the values of corresponding ranges for the *i*th samples. Thus for the *j*th sample (j = 1, 2, 3,..........., *i*)

$$\bar{x}_j = \textit{Mean of observations on the jth sample}$$

$$\Rightarrow \qquad \bar{x}_j = \frac{1}{n}\,[\textit{Sum of observations on the jth sample}]$$

where n is the sample size.

$R_j = R_{max} - R_{min}$, (where $R_{\max}$ is the largest and R_{min} is the lowest observation in the *jth* sample.)

2. **Obtain the Mean of Sample Mean:** Now, $\bar{\bar{x}}$, the mean of i sample mean and $\bar{R}$, the mean of the i sample ranges are given by

$$\bar{\bar{x}} = \frac{1}{i}\left(\bar{x}_1 + \bar{x}_2 + \ldots\ldots\ldots\ldots + \bar{x}_i\right) = \frac{\sum \bar{x}}{i}$$

$$\bar{R} = \frac{1}{i}(R_1 + R_2 + \ldots\ldots\ldots\ldots + R_i) = \frac{\sum R}{i}$$

3. **Setting of Control Limits for $\bar{x}$:** From sampling theory, we know that if μ be the process mean and σ be the process standard deviation then sample mean $\bar{x}$ is normally distributed with mean μ and stadnard deviation on $\frac{\sigma}{\sqrt{n}}$ *i.e.,*

$E(\bar{x}) = \mu$ and $S.E.(\bar{x}) = \frac{\sigma}{\sqrt{n}}$, where n is the sample size. Hence the 3-σ control limits for $\bar{x}$ chart are:

$$\left.\begin{array}{l} E(\bar{x}) \pm 3S.E.(\bar{x}) \\ \text{or} \quad \mu \pm \frac{3\sigma}{\sqrt{n}} \Rightarrow \mu \pm A\sigma \end{array}\right\} \qquad \ldots(1)$$

i.e.,
$$\left.\begin{aligned} UCL &= \mu + A\sigma \\ CL &= \mu \\ LCL &= \mu - A\sigma \end{aligned}\right\} \quad ...(2)$$

where $A = \frac{3}{\sqrt{n}}$ is a constant and obtained for different values of n. Equation (2) are applicable in those situations in which values of μ and σ are known.

Remark: If μ and σ are not known then we use their estimates provided by the *i*-given samples. Now $\overline{\overline{x}}$ provides an unbiased estimate of the population mean μ, while process standard deviation σ is estimate by $\frac{\overline{R}}{d_2}$; where d_2 is a constant depending on the given sample size n. Thus

$$\mu = \overline{\overline{x}} \text{ and } \sigma = \frac{\overline{R}}{d_2} \text{ (where } d_2 \text{ is correlation factor)}$$

Substitute these values in (1) to obtain 3-σ control limits for $\overline{x}$-chart.

$$= \overline{\overline{x}} \pm 3\left(\frac{\overline{R}}{d_2}\right) \times \frac{1}{\sqrt{n}} \Rightarrow \overline{\overline{x}} \pm \left(\frac{3}{d_2\sqrt{n}}\right)\overline{R}$$

$$= \overline{\overline{x}} = \pm A_2 \overline{R}$$

where $A_2 = \frac{3}{d_2\sqrt{n}}$ is a constant depending on sample size n.

$$\therefore \quad \left.\begin{aligned} UCL &= \overline{\overline{x}} + A_2\overline{R} \\ CL &= \overline{\overline{x}} \\ LCL &= \overline{\overline{x}} - A_2\overline{R} \end{aligned}\right\}$$

4. **Construction of ($\overline{x}$) Chart:** The control chart for mean is drawn by taking the sample number along the horizontal line (x-axis) and the statistic ($\overline{x}$) along the vertical line (y-axis). The sample points are plotted as points or dots against the corresponding sample number. These points may or may not be joined. The central line is drawn as a bold and UCL or LCL are plotted as dotted horizontal lines at the compute values.
5. **Control Chart for Range (*R*):** The R chart is used to show the variability or dispersion of the quality produced by a given process. The R chart is generally presented along with the ($\overline{x}$) chart and procedure for constructing the R chart is similar to that for ($\overline{x}$) chart. The required values for constructing R chart are
   1. The range of each sample R
   2. The mean of the sample ranges ($\overline{R}$)
   3. Setting of control limits for R

The three sigma limits for R-chart if process standard deviation is known, are given by

$$\left.\begin{aligned} UCL &= \mu_R + 3\sigma_R \\ CL &= \mu_R \\ LCL &= \mu_R - 3\sigma_R \end{aligned}\right\} \quad ...(1)$$

Now if quality characteristics x is normally distributed with mean μ and standard deviation σ then

$$\mu_R = E(R) = d_2\sigma, \qquad \text{where } E(R) = \text{expected mean for } R$$

$$\sigma_R = \sqrt{v(R)} = d_3\sigma, \qquad \text{where } v(R) = \text{Variance of } R$$

Therefore equation (1) becomes

$$UCL = d_2\sigma + 3d_3\sigma = (d_2 + 3d_3)\sigma = D_2\sigma$$

$$CL = d_2\sigma$$

$$LCL = d_2\sigma - 3d_3\sigma = (d_2 - 3d_3)\sigma = D_1\sigma \quad ...(2)$$

where d_2, D_1 and D_2 are constants depending on sample size n and have been computed for different values of n from 2 to 25. Since range can never be negative so if it comes out to be negative, it is taken as zero. Equation (2) is used when σ is known.

Remark: When standard deviation are not known *i.e.,* σ is unknown then in this case σ is obtained by

$$\sigma = \frac{\bar{R}}{d_2}$$

Therefore from equation (2)

$$UCL = D_2\frac{\bar{R}}{d_2} = D_4\bar{R}$$

$$CL = d_2\frac{\bar{R}}{d_2} = \bar{R}$$

$$LCL = D_1\frac{\bar{R}}{d_2} = D_3\bar{R}$$

i.e.,

$$UCL = D_4\bar{R}$$

$$CL = \bar{R}$$

$$LCL = D_3\bar{R}$$

D_3 and D_4 also depends on sample size n and tabulated for different values of n from 2 to 25.

1. **Construction of ($\bar{R}$) Control Chart:** The control chart for mean is drawn by taking the sample number along the horizontal line (*x-axis*) and the statistic ($\bar{R}$) along the vertical line (*y-axis*). The sample points are plotted as points against the corresponding

sample number. The central line is drawn as a bold and UCL or LCL are plotted as dotted horizontal lines at the computed values. Point to be noted that the use of R chart is recommended only for relatively small samples sizes (near 12 to 15 units). For the large sample sizes ($n > 12$) the σ chart is to be recommended.

2. **Standard Deviation σ Chart:** The variability in the quality characteristic is controlled by σ chart (when $n >= 10$). Control limits of σ charts are given by

$$\left.\begin{aligned} \text{UCL} &= \mu_S + 3\sigma_s \\ \text{CL} &= \mu_S \\ \text{LCL} &= \mu_S - 3\sigma_s \end{aligned}\right\} \qquad \text{...(1)}$$

For normally distributed variable x, μ_s and σ_s is given by

$$\mu_s = E(S) = C_2\sigma$$

and

$$\sigma_s = \sqrt{v(S)} = \sigma\sqrt{\frac{(n-1)}{n} - C_2^2} \qquad \text{...(2)}$$

Therefore from (1) control limits are:

$$\left.\begin{aligned} \text{UCL} &= C_2\sigma + 3\sigma\sqrt{\frac{(n-1)}{n} - C_2^2} \\ \text{LCL} &= C_2\sigma - 3\sigma\sqrt{\frac{(n-1)}{n} - C_2^2} \end{aligned}\right\}$$

$$\Rightarrow \qquad \left.\begin{aligned} \text{UCL} &= B_2\sigma \\ \text{CL} &= C_2\sigma \\ \text{LCL} &= B_1\sigma \end{aligned}\right\} \qquad \text{...(3)}$$

These control limits are employed when standards (σ is given) are given. If standards are not given then in that case σ is estimated by

$$\sigma = \frac{\bar{S}}{C_2}, \qquad \text{where } \bar{S} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \frac{\text{Sum of sample standard deviations}}{\text{Number of samples}}$$

Using value of σ in equation (3) we get control limit as

$$\text{UCL} = \frac{B_2}{C_2}\bar{S} = B_4\bar{S}$$

$$\text{CL} = \frac{C_2}{C_2}\bar{S} = \bar{S}$$

$$\text{LCL} = \frac{B_1}{C_2}\bar{S} = B_3\bar{S}$$

11.13.2 Control Chart for Attributes

The chart used for qualitative characteristic are called control charts for attributes. When we deal with quantity characteristic which cannot be measured quantitatively, in such cases the inspection of units is accompanied by classifying them as acceptable or non-acceptable, defective or non-defective. Here we use two words 'defect' and 'defective'. Any instance of a characteristic or unit not conforming to specification (required standards) is known as a defect. A defective is a unit which contains more than allowable number (usually one) of defects. Control chart for attributes are:

1. Control chart for fraction defectives, *i.e., p*-chart
2. Control chart for number of defectives, *i.e., np*-chart
3. Control chart for number of defects per unit, *i.e., c*-chart

1. **Control Chart for Fraction Defectives (*p*-Chart):** Control chart for fraction defective is used, when sample unit as a whole is classified as defective or non-defective, or good or bad.

$$\text{Fraction defective} = \frac{\text{Total no. of defective units}}{\text{Total no. of units}}$$

i.e., $$p = \frac{d}{n}$$

therefore sampling distribution of the statistic 'p' is given by

$$E(p) = p$$

$$S.E.(p) = \sqrt{\frac{pQ}{n}}; \text{ where } Q = 1 - p$$

(*i*) 3-σ Control Limits for *p*-Chart: 3-σ control limits for p-chart are given by $E(p) \pm 3S.E.(p)$.

i.e., $$p \pm 3\sqrt{\frac{pQ}{n}}$$

Therefore if p is known then

$$\text{UCL} = p + 3\sqrt{\frac{pQ}{n}}$$

$$\text{CL} = p$$

$$\text{LCL} = p - 3\sqrt{\frac{pQ}{n}} \qquad \text{where } Q = 1 - p$$

Again if p is not given, then p is denoted by $\bar{p}$ and is obtained as

$$\bar{p} = \frac{\text{Total no. of defective in all samples}}{\text{Total no. of units in all samples}}$$

(when k sample is used out of n)

i.e., $$\bar{p} = \frac{\sum d}{\sum n} = \frac{\sum d}{n}$$

Therefore control limits of p-chart are

$$\text{UCL} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$\text{CL} = \bar{p}$$

$$\text{LCL} = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

It can also be written as

$$\text{UCL} = \bar{p} + A\sqrt{\bar{p}(1-\bar{p})}$$

$$\text{CL} = \bar{p}$$

$$\text{LCL} = \bar{p} - A\sqrt{\bar{p}(1-\bar{p})}$$

Where $A = \frac{3}{\sqrt{n}}$ is obtained for different values of n from the table.

(*ii*) Construction of $\bar{p}$ Chart: To construct $\bar{p}$-chart, take the sample number along the horizontal scale and the statistic 'p' along the vertical scale. Then the sample fraction defectives p_1, p_2, p_3, ... p_i are plotted against the corresponding sample numbers as points (dots). The central line as a dark horizontal line and UCL_p and LCL_p are plotted as dotted horizontal lines at the computed values. Since p cannot be negative so if LCL_p obtained as negative, it is taken as zero for control chart.

2. Control Chart for Number of Defectives (np-Chart): If the sample size is constant for all the samples, say n then the sampling distribution of the statistic

d = No. of defectives in the sample = np

is given by $E(d) = np$ and $S.E.(d) = \sqrt{npQ}$

Hence the 3-σ limits for np-chart are given by

$$E(d) \pm 3\, S.E.(d)$$

$$= np \pm 3\sqrt{npQ} = np \pm 3\sqrt{np(1-p)}\ ; \quad \text{where } Q = 1-p$$

Hence

$$\text{UCL}_{np} = np + 3\sqrt{np(1-p)}$$

$$\text{CL}_{np} = np$$

$$\text{LCL}_{np} = np - 3\sqrt{np(1-p)}$$

If $\bar{p}$ is not known, then p is obtained by sample values and given by

$$\bar{p} = \frac{\text{Total no. of defectives in all sample inspected}}{\text{Total no. of samples inspected}}$$

$$\bar{p} = \frac{\sum_{i=1}^{k} p_1}{k}$$

Thus control limits for *np*-chart are

$$\text{UCL} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$$

$$\text{CL} = n\bar{p}$$

$$\text{LCL} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$$

3. **Control Chart for Number of Defects per Unit (*c*-Chart):** The statistical basis for the control *c*-chart is the poisson distribution. If we regard the statistic *c* distributed as a poisson variate with parameter λ then,

$$E(c) = \lambda \text{ and } S.E.(c) = \sqrt{\lambda}$$

where λ is the average number of defects in all the inspection units. Hence the 3-σ control limits are given by

$$E(c) \pm 3S.E.(c)$$

$$= \lambda \pm 3\sqrt{\lambda}$$

i.e.,

$$\text{UCL} = \lambda + 3\sqrt{\lambda}$$

$$\text{CL} = \lambda$$

$$\text{LCL} = \lambda - 3\sqrt{\lambda}$$

If λ is unknown then, $\lambda = \bar{c} = \dfrac{\sum c}{k} = \dfrac{c_1 + c_2 + \ldots\ldots\ldots\ldots\ldots + c_k}{k}$

where $c_1, c_2, \ldots\ldots\ldots\ldots, c_k$ are the numbers of defects observed in *k*th sample observation.

Hence,

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}}$$

$$\text{CL} = \bar{c}$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}}$$

Since c, the number of defects per unit cannot be negative so if LCL_c is obtained from above formula as negative then it is taken as zero.

Construction of $\bar{c}$-Chart: The sample points $c_1, c_2, \ldots\ldots\ldots\ldots, c_k$ are plotted as dots by taking sample statistic *c* along the vertical scale and the sample number along the horizontal scale. The central line (CL) is drawn as bold horizontal line at λ or $\bar{c}$ and UCL_c and LCL_c are plotted as dotted lines at the computed values.

Example 1. *There are given the values of sample mean $\bar{x}$ and range (R) for ten samples of size 5 each. Draw Mean and Range charts and comment on the state of control of the process.*

Sample No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\bar{x}$	*43*	*49*	*37*	*44*	*45*	*37*	*51*	*46*	*43*	*47*
R	*5*	*6*	*5*	*7*	*7*	*4*	*8*	*6*	*4*	*6*

Given for $n = 5$, $A_2 = 0.58$, $D_3 = 0$, $D_4 = 2.115$

Sol. Mean Chart

Mean of 10 sample mean $X = \frac{\sum \bar{x}}{10} = \frac{442}{10} = 44.2$

Mean Range of 10 sample ranges $\bar{R} = \frac{\sum R}{10} = \frac{58}{10} = 5.8$

As we have, for $n = 5$, $A_2 = 0.58$, $D_3 = 0$, $D_4 = 2.115$
3-σ control limits for $\bar{x}$ chart are:

$$\text{UCL}_{\bar{x}} = X + A_2\bar{R}$$
$$= 44.2 + 0.58 \times 5.8 = 47.567$$
$$\text{LCL}_{\bar{x}} = X - A_2\bar{R}$$
$$= 44.2 - 0.58 \times 5.8 = 40.836$$
$$\text{CL}_{\bar{x}} = X = 44.2$$

Range Chart: 3-σ Control Limits for R chart are:

$$\text{UCL}_R = D_4\bar{R} = 2.115 \times 5.8 = 12.267$$
$$\text{LCL}_R = D_3\bar{R} = 0 \times 5.8 = 0$$
$$\text{CL}_R = \bar{R} = 5.8$$

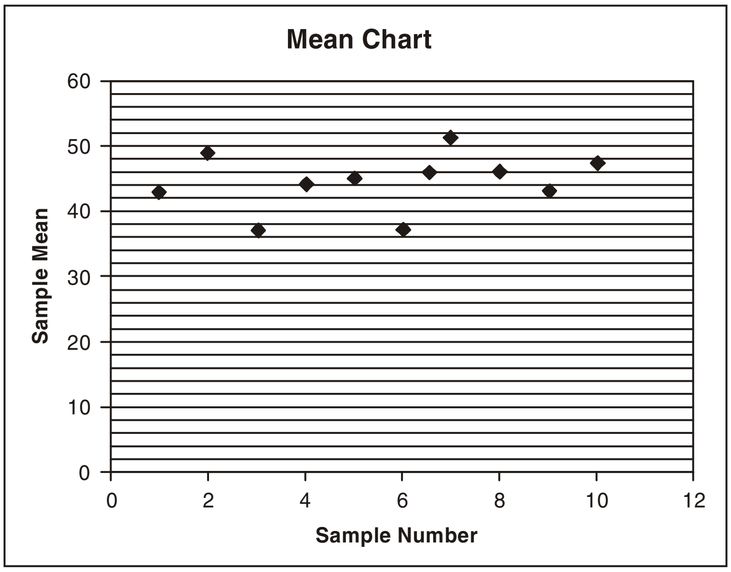


FIG. 11.17

From mean chart we see that 2nd, 3rd, 6th and 7th samples lies outside the control limits. Hence the process is out of control. This shows that some assignable causes of variation are operating which should be detected and removed.

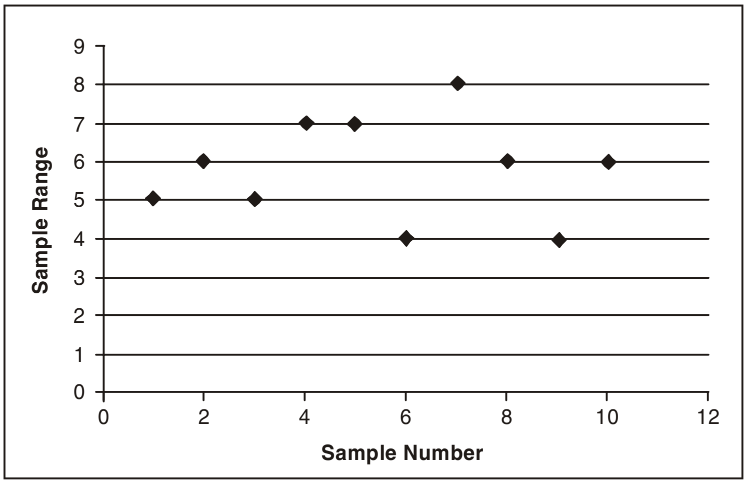


FIG. 11.18

Since all the points with in the control limits. Hence the process is in statistical control.

Example 2. *The following are the mean lengths and ranges of lengths of a finished product from 10 samples each of size 5. The specification limits for length are* 200 ± 5 cm. *Construct* $\bar{x}$ *and R-chart and examine whether the process is under control and state your recommendation.*

Sample No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\bar{x}$	*201*	*198*	*202*	*200*	*203*	*204*	*199*	*196*	*199*	*201*
R	*5*	*0*	*7*	*3*	*4*	*7*	*2*	*8*	*5*	*6*

Assume for $n = 5$, $A_2 = 0.577$, $D_3 = 0$, $D_4 = 2.115$.

Sol. In given problem specification limits for length are given 200 ± 5 cm. Hence standard deviation is unknown.

(1) Control Limits for $\bar{x}$-chart are:

Central limit, $CL_{\bar{x}} = \mu = 200$

$$UCL_{\bar{x}} = \mu + A_2\bar{R} = 200 + 0.577 \times 4.7$$

$$= 202.712; \quad \bar{R} = \frac{\sum R}{10} = \frac{47}{10}$$

$$LCL_{\bar{x}} = \mu - A_2\bar{R} = 200 - 0.577 \times 4.7 = 197.288 \qquad \bar{R} = 4.7$$

(2) Control limits for *R*-Chart are:

$$UCL_R = 9.941 = D_4\bar{R} = 2.115 \times 4.7$$

$$CL_R = 4.7 = \bar{R}$$

$$LCL_R = 0 = D_3\bar{R} = 0 \times 4.7$$

from control charts for mean and range, the process is in statistical control in $\bar{R}$-Chart because all points lies with in the control limits where as in $\bar{x}$-chart, process is out of control because sample 5, 6 and 8 lies outside the control limits. The process therefore should be halted to check whether there are any assignable causes. If assignable causes found, the process should be re-adjusted to remove assignable cause.

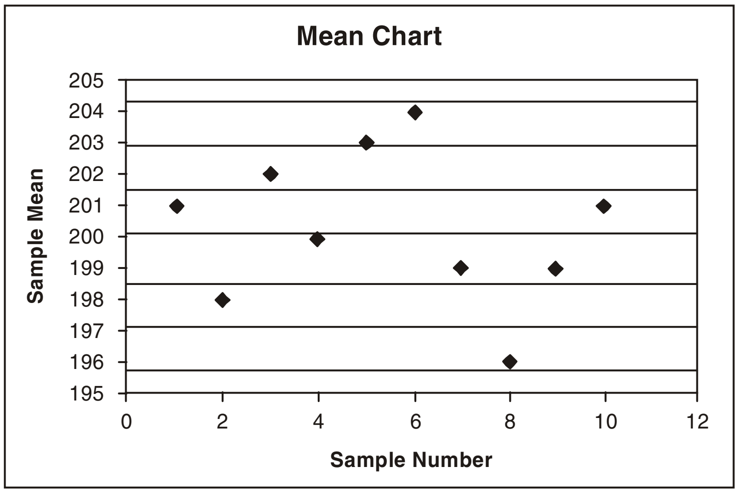


FIG. 11.19

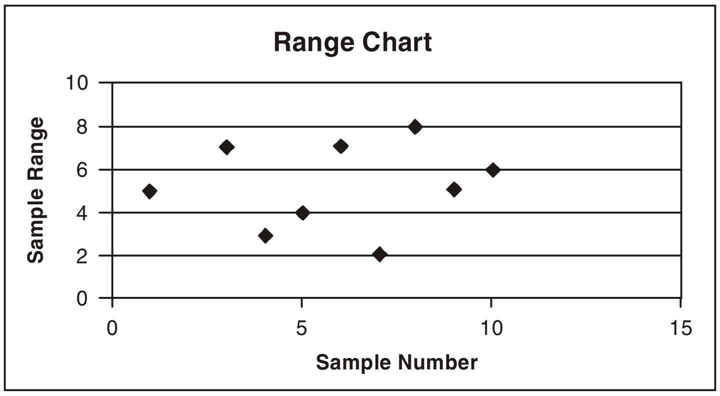


FIG. 11.20

Example 3. *In a glass factory, the task of quality control was done with the help of mean ($\bar{x}$) and standard deviation σ charts. 18 samples of 10 items each were chosen and then values ΣX and ΣS were found to be 595.8 and 8.28 respectively. Determine the 3-σ limits for mean and standard deviation chart. Given that $n = 10$, $A_1 = 1.03$, $B_3 = 0.28$, $B_4 = 1.72$, $\Sigma S = 8.28$.*

Sol.

No. of samples 18

$$\bar{S} = \frac{\Sigma S}{18} = \frac{8.28}{18} = 0.46$$

hence, 3-σ control limits for standard deviation chart are:

$$\text{UCL}_{\bar{S}} = B_4.\bar{S} = 1.72 \times 0.48 = 0.7912$$

$$\text{LCL}_{\bar{S}} = B_3.\bar{S} = 0.28 - 0.46 = 0.1288$$

$$\text{CL}_{\bar{S}} = 0.46$$

3-σ control limits for mean chart ($\bar{x}$) are:

$$\bar{X} = \frac{\Sigma \bar{x}}{18} = \frac{595.8}{18} = 33.1$$

$$\text{UCL}_{\bar{X}} = \bar{x} + A_1\sigma$$
$$= 33.1 + 1.03 \times 0.46$$
$$\text{UCL}_{\bar{X}} = 33.57$$
$$\text{LCL}_{\bar{X}} = \bar{x} - A_1\sigma$$
$$= 33.1 - 1.03 \times 0.46$$
$$\text{LCL}_{\bar{X}} = 32.63$$
$$\text{CL}_{\bar{X}} = 33.1.$$

Example 4. *If the average fraction defective of a large sample of a product is 0.1537, calculate the control limits when subgroup size is 2,000.*

Sol. Here, Sample size n = 2,000 for each sample

Average fraction defective = 0.1537 i.e., $P = 0.1537$

$\Rightarrow \quad Q = 1 - P = 1 - 0.1537$

$Q = 0.8463$

Hence, 3–σ control limits for P-Chart are :

$$P \pm 3\sqrt{\frac{PQ}{n}}$$

$$\text{UCL}_p = 0.1537 + 3\sqrt{\frac{0.1537 \times 0.8463}{2{,}000}}$$

$$\text{UCL}_p = 0.1537 + 0.02418 = 0.17788$$

$$\text{LCL}_p = 0.1537 - 3\sqrt{\frac{0.1537 \times 0.8463}{2{,}000}}$$

$$\text{LCL}_P = 0.1537 - 0.02418 = 0.12952$$

$$\text{CL}_P = 0.5137.$$

Example 5. *The following data gives the number of defectives in 10 independent samples of varying sizes from a production process.*

Sample no.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Sample size	*2000*	*1500*	*1400*	*1350*	*1250*	*1760*	*1875*	*1955*	*3125*	*1575*
No. of defectives	*425*	*430*	*216*	*341*	*225*	*322*	*280*	*306*	*337*	*305*

Draw the control chart for fraction defective.

Sol. (In problem 4 sample size is fixed whereas in this problem sample size is variable) Since it is a problem of variable sample size so control chart for fraction defective can be drawn in two ways.

(1) By first way, we set up two sets of control limits, one based on the maximum sample size, $n = 3125$ and the second based on minimum sample size $n = 1{,}250$.

(*a*) For $n = 3{,}125$; $\quad \text{UCL} = 0.200, \quad \text{LCL} = 0.159$

(*b*) For $n = 1{,}250$; $\quad \text{UCL} = 0.212, \quad \text{LCL} = 0.147$

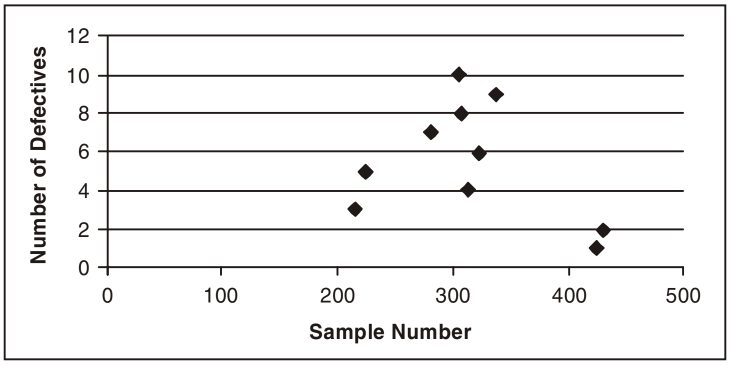


FIG. 11.21 Control Chart for Fraction Defective

Since there are 4 points lies outside (based on minimum sample size) of control limits, so process is of out of control.

(2) By second way, 3-σ limit for each sample separately obtained by using formula

$$\bar{P} \pm 3\sqrt{\frac{\bar{P}\bar{Q}}{n}}$$

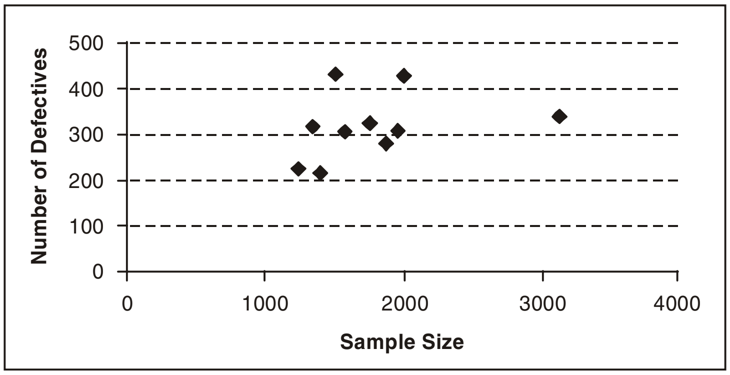
where $$\bar{P} = \frac{\text{Total no. of defectives}}{\text{Total sample size}} = \frac{\sum d}{\sum n}$$

and n is corresponding sample size.

$$\bar{P} = \frac{\Sigma d}{\Sigma n} = \frac{3187}{17790} = 0.1791 \Rightarrow \bar{Q} = 1 - \bar{P} = 0.8209$$

$\therefore$ $(\bar{P}\,\bar{Q})$ = 0.1791 × 0.8209 = 0.1470231

n	d	$P = d/n$	$1/n$	$\frac{\bar{P}\bar{Q}}{n}$	$\sqrt{\frac{\bar{P}\bar{Q}}{n}}$	$3 \times \sqrt{\frac{\bar{P}\bar{Q}}{n}}$	UCL	LCL
2000	425	0.2125	0.0005	0.000735	0.008573	0.025719	0.205	0.153
1500	430	0.2867	0.00066	0.000098	0.009899	0.029698	0.209	0.149
1400	216	0.1543	0.00071	0.000105	0.010247	0.030741	0.210	0.148
1350	341	0.2526	0.00074	0.000109	0.010440	0.031321	0.210	0.148
1250	225	0.1800	0.00080	0.000118	0.010863	0.032588	0.212	0.147
1760	322	0.1829	0.00057	0.000084	0.009138	0.027413	0.207	0.152
1875	280	0.1495	0.00053	0.000078	0.008854	0.026562	0.206	0.153
1995	306	0.1565	0.00051	0.000075	0.008672	0.026015	0.205	0.153
3125	337	0.1078	0.00032	0.000047	0.006856	0.020567	0.200	0.159
1575	305	0.1937	0.00063	0.000093	0.009659	0.028977	0.0208	0.150
17790	3187							



Sample points corresponding to sample no. 1, 2, 4, 7 and 9 lie outside the control limits. Hence, process is out of control.

FIG. 11.22

Example 6. *A daily sample of 30 items was taken over a period of 14 days in order to establish attributes control limits. If 21 defectives were found, what should be upper and lower control limits of the proportion of defectives?*

Sol. Since a sample of 30 items is taken daily over a period of 14 days.

Total No. of items inspected = 30 × 14 = 420

No. of defective found = 21

$n = 30$

$\therefore$ Average fraction defective $\bar{P} = \frac{21}{420} = 0.05$

$$\therefore \quad UCL_p = \bar{P} + 3\sqrt{\frac{\bar{P}\bar{Q}}{n}} \quad \text{where } \bar{Q} = 1 - \bar{P}$$

$$= \bar{P} + 3\sqrt{\frac{\bar{P}(1-\bar{P})}{n}} = 0.05 + 3\sqrt{\frac{(0.05)\times 0.95}{30}}$$

$$UCL_p = 0.05 + 3 \times 0.0398$$

$$UCL_p = 0.1694$$

$$LCL_p = \bar{P} - 3\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}$$

$$= 0.05 - 0.1194 < 0 \text{ (negative)}$$

$$\therefore \quad LCL_p = 0.$$

Example 7. *The past record of a factory using quality control melthods show that on the average 4 articles produced are defective out of a batch of 100. What is the maximum number of defective articles likely to be encountered in the batch of 100, when the production process is in a state of control?*

Sol. n = Sample size = 400

P = Process fraction defective = $\frac{4}{100} = 0.04$

$Q = 1 - P = 0.96$

Let d be the number of defectives in a sample size of n. *i.e.*, np. The 3–σ limit for number of defectives are given by

$$E(d) \pm 3s.E(d)$$

or
$$np \pm 3\sqrt{nPQ}$$

$$= 400 \times 0.04 \pm 3\sqrt{400 \times 0.04 \times 0.96}$$

$$= 16 \pm 3\sqrt{15.36} = 16 \pm 3 \times 3.9192$$

$$= 16 \pm 11.7576 = (4.2424, 27.7576)$$

Therefore if the production process is in a statistical control, the number of defective items to be encountered in a batch of 400 should lie within the control limits, *viz.* (4.2424, 27.7576), *i.e.*, (4, 28). Hence the maximum number of defective items in this batch is 28.

Example 8. *In a blade manufacturing factory, 1000 blades are examined daily. Following information shows number of defective blades obtained there. Draw the np-chart and give your comment?*

Date	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No. of Defective	9	10	12	8	7	15	10	12	10	8	7	13	14	15	16

Sol. Here $n = 10000$, $k = 15$ (sample no.)

If $\overline{P}$ denotes the fraction defectives produced by the entire process then

$$\overline{P} = \frac{\Sigma P}{kn} = \frac{166}{15 \times 1000} = 0.011$$

∴ $$n\overline{p} = 1000 \times 0.011 = 11$$

Hence control limits are

$$\text{CL} = n\overline{p} = 11$$

$$\text{UCL} = n\overline{p} = 3\sqrt{n\overline{p} - (1 - \overline{p})}$$

$$= 11 + 3\sqrt{11 - (1 - 0.011)}$$

$$\text{UCL} = 20.894$$

$$\text{LCL} = n\overline{p} - 3\sqrt{n\overline{p} - \left(1 - \overline{p}\right)}$$

$$= 11 - 3\sqrt{11 - (1 - 0.011)}$$

$$\text{LCL} = 1.106$$

Since all the 15 points lies within the control limits, the process is under control.

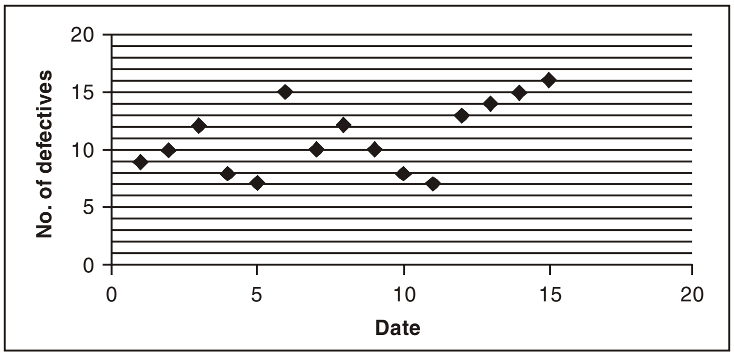


FIG. 11.23

Example 9. *The number of mistakes made by an accounts clerk is given below:*

Week	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of Mistakes	1	0	2	0	1	0	1	0	1	2	3	3	1	0	0	7	1	0	1	0

Establish a suitable control chart and state how it should be used in future in order to control the mistakes of the clerk.

Sol. The control chart to be used for the given problem is the number of defects chart *i.e., C*-chart.

Average no. of mistakes.

$$\bar{c} = \frac{\Sigma C}{20} = \frac{24}{20} = 1.2$$

Thus the control limits for $\bar{c}$-chart are;

(*i*) $\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 1.2 + 3\sqrt{1.2} = 4.49$

(*ii*) $\text{CL} = \bar{c} = 1.2$

(*iii*) $\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 1.2 - 3\sqrt{1.2} = 2.09 \approx 0$

∵ The number of mistakes during the 16th week lies outside the UCL the process is not under control.

Now to establish the suitable control chart for future, we homogenize the data for future control by eliminating the data corresponding to the 16th week.

$\bar{C}_{new} = \frac{17}{19} = 0.895$. Hence the revised control limits for $\bar{c}$ chart are:

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 0.895 + 3\sqrt{0.895} = 3.73$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 0.895 - 3\sqrt{0.895} = -1.94 \cong 0$$

$$\text{CL} = \bar{C} = \frac{17}{19} = 0.895.$$

So the revised C-chart for revised control limit is in statistical control, *i.e.*, all the points lies within the control limits.

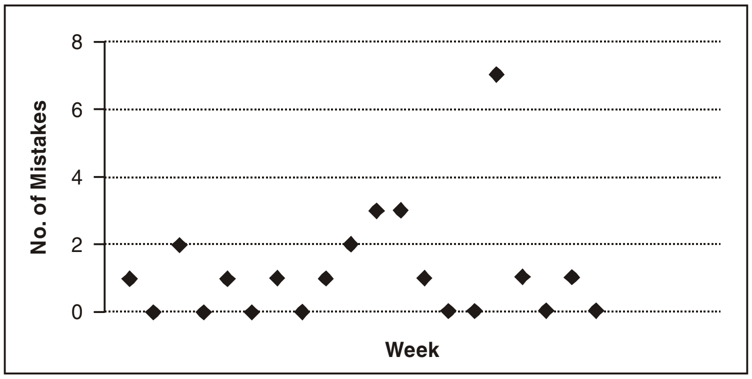


FIG. 11.24

Example 10. *During the examination of equal length of cloth, the following are the number of defects observed.*

2 3 4 0 5 6 7 4 3 2

Draw a control chart for the number of defects and comment whether the process is under control or not?

Sol. Let the no. of defects per unit (equal length) be denoted by *c*.

The average no. of defects in 10 samples

$$\bar{c} = \frac{\Sigma c}{20} = \frac{36}{10} = 3.6$$

Hence 3–σ limit for c-chart are:

$$\bar{c} \pm 3\sqrt{\bar{c}}$$

$$= 3.6 \pm 3\sqrt{3.6}$$

$$= 3.6 \pm 3 \times 1.8974$$

$$= 3.6 \pm 5.6922$$

$$UCL_{\bar{c}} = 3.6 + 5.6922 = 9.2922$$

$$LCL_{\bar{c}} = 3.6 - 5.6922 = -2.0922 \approx 0$$

$$CL_{\bar{c}} = 3.6$$

($LCL_{\bar{c}}$ = 0 because no. of defects per unit cannot be negative)

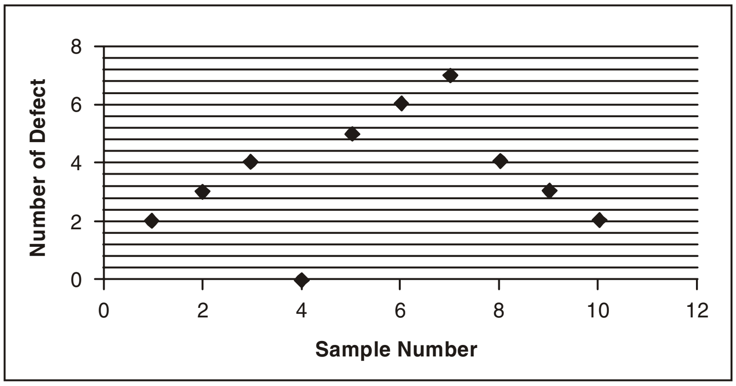


FIG. 11.25

Since all the points are within the control limits therefore the process is in statistical control.

Example 11. *An automobile producer wishes to control the number of defects per automobile. The data for 16 such automobiles is shown below:*

Sample No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
No. of defects	*2*	*4*	*3*	*2*	*1*	*8*	*1*	*0*	*5*	*2*	*3*	*1*	*3*	*4*	*1*	*2*

1. *Set up the control lmits for c-charts.*
2. *Do these data come from a controlled process ? If not, calculated the revised control charts limits.*

Sol. Here $k = 16$

Average no. of defects in 16 units

$$\bar{c} = \frac{1}{k}\Sigma C = \frac{42}{16} = 2.625$$

Thus, the control limits for c-chart are:

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 2.625 + 3\sqrt{2.625} = 2.625 + 4.861 = 7.486$$

$$\text{CL} = \bar{c} = 2.625$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 2.625 - 3\sqrt{2.625} = 2.625 - 4.861 = -2.236 \approx 0$$

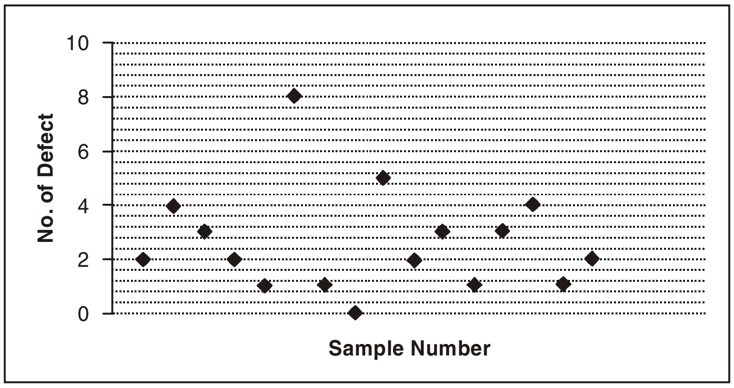


FIG. 11.26

Negative LCL being taken as zero. Also for drawing the control chart we mark the sample No.'s along the horizontal axis and control limits and central line marked along the vertical axis. Finally the number of defects. (c_i) per inspection units are marked in the c-chart.

From the control chart, we observe that the point corresponding to 6th inspection unit goes beyond UCL showing a out-of-control situation. So for computing revised control limits we omit this unit and use the remaining 15 inspection units for the purpose. The average number of defects in the remaining 15 units is

$$\bar{c} = \frac{1}{15}\sum_{i=1}^{15} c_i = \frac{34}{15} = 2.27$$

so the revised limits for c-chart are:

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 2.27 + 3\sqrt{2.27} = 2.27 + 4.52 = 6.79$$

$$\text{CL} = \bar{c} = 2.27$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 2.27 - 3\sqrt{2.27} = 2.27 - 4.52 = -2.25 \approx 0$$

Negative LCL being as zero.

Example 12. *A food company puts mango juice into cans advertised as containing 10 ounces of the juice. The weights of the juice drained from cans immediately after filling for 20 samples are taken by a random method (at an interval of every 30 minutes). Each of the samples includes 4 cans. The samples are tabulated in the following table. The weights in the table are given in units of 0.01 ounces in excess of 10 ounces. For example, the weight of juice drained from the first can of the sample is 10.15 ounces whch is in excess of 10 ounces being 0.15 ounces (10.15 – 10 = 0.15) since the unit in the table is 0.01 ounce, the excess is recorded as 15 units in the table. Construct an* $\bar{x}$*-chart to control the weights of mango juice for the filling.*

Sample Number	*Weight of each can (4 cans in each sample, x, n = 4)*			
	x_1	x_2	x_3	x_4
1	15	12	13	20
2	10	8	8	14
3	8	15	17	10
4	12	17	11	12
5	18	13	15	4
6	20	16	14	20
7	15	19	23	17
8	13	23	14	16
9	9	8	18	5
10	6	10	24	20
11	5	12	20	15
12	3	15	18	18
13	6	18	12	10
14	12	9	15	18

15	*15*	*15*	*6*	*16*
16	*18*	*17*	*8*	*15*
17	*13*	*16*	*5*	*4*
18	*10*	*20*	*8*	*10*
19	*5*	*15*	*10*	*12*
20	*6*	*14*	*12*	*14*

Sol.

Sample Number	*Weight of each can (4 cans in each sample, x, n = 4)*				*Total weight of 4 cans*	*Sample Mean*	*Sample Range*
	x_1	x_2	x_3	x_4	Σx	$\bar{x} = \frac{\Sigma x}{4}$	$R = x_{max} - x_{min}$
1	15	12	13	20	60	15.0	8
2	10	8	8	14	40	10.0	6
3	8	15	17	10	50	12.5	9
4	12	17	11	12	52	13.0	6
5	18	13	15	4	50	12.5	14
6	20	16	14	20	70	17.5	6
7	15	19	23	17	74	18.5	8
8	13	23	14	16	66	16.5	10
9	9	8	18	5	40	10.0	13
10	6	10	24	20	60	15.0	18
11	5	12	20	15	52	13.0	15
12	3	15	18	18	54	13.5	15
13	6	18	12	10	46	11.5	12
14	12	9	15	18	54	13.5	9
15	15	15	6	16	52	13.0	10
16	18	17	8	15	58	14.5	10
17	13	16	5	4	38	9.5	12
18	10	20	8	10	48	12.0	12
19	5	15	10	12	42	10.5	10
20	6	14	12	14	46	11.5	8
Total						$\Sigma \bar{x} = 263.0$	$\Sigma R = 211$

$$\text{UCL} = \bar{\bar{x}} + A_2 \bar{R}$$

$$= 13.15 + 0.729 \times 10.55 \quad (A_2 = 0.729 \text{ for } n = 4)$$

$$\text{UCL} = 20.84095$$

$$\text{CL} = \bar{\bar{x}} = 13.15$$

$$\text{LCL} = \bar{\bar{x}} - A_2 \bar{R}$$

$$= 13.15 - 0.729 \times 10.55$$

$$= 5.46$$

The values in above computation are expressed in units of 0.01 ounces in excess of 10 ounces. The actual value of UCL = 10.2084, and LCL = 10.0546 ounces. Since all points are falling with in control limits the process is in a statistical control.

Now since standards are not given calculating

1. The mean of the sample mean $\bar{\bar{x}}$ is given by

$$\bar{x} = \frac{\sum x}{20} = \frac{263}{20} = 13.15$$

2. The mean of the Range values $\bar{R}$ is given by

$$\bar{R} = \frac{\sum R}{20} = \frac{211}{20} = 10.55$$

3. Trial control limits for $\bar{x}$-chart

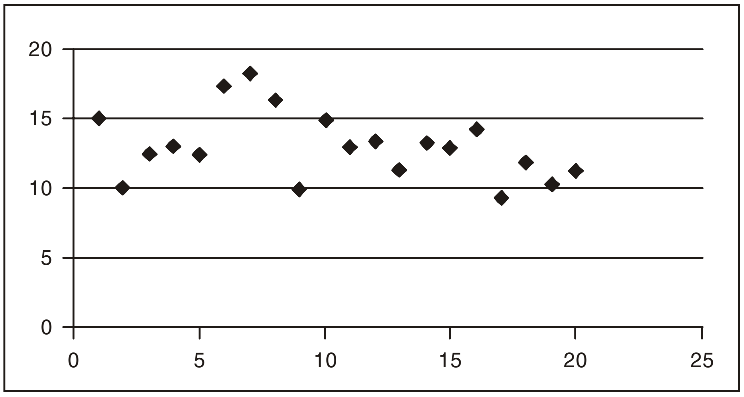


FIG. 11.27

PROBLEM SET 11.1

1. A machine is set to deliver packets of a given weight 10 samples of size 5 each were recorded. Data being given below:

Sample no.	1	2	3	4	5	6	7	8	9	10
Mean $\bar{x}$	15	17	15	18	17	14	18	15	17	16
Range R	7	7	4	9	8	7	12	4	11	5

Calculate the values for the central line and control limits of mean chart and the range and then comment on the state of control.

Given for $n = 5$, $A_2 = 0.58$, $D_3 = 0$, $D_4 = 2.115$.

Ans. $\left\{ \begin{array}{ll} UCL_{\bar{R}} = 15.614 & UCL_{\bar{x}} = 20.492 \\ LCL_{\bar{R}} = 0 & LCL_{\bar{x}} = 11.908 \\ CL_{\bar{R}} = 7.4 & CL_{\bar{x}} = 16.2 \end{array} \right\}$

2. The data below give the number of defective bearing in samples of size 150. Construct p-chart for these data and state your comment.

Sample no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of defective	12	7	5	4	1	5	9	0	15	6	7	4	1	3	6	8	10	5	2	7

Compute control limits for p-chart. [**Ans.** UCL = 0.08650, CL = 0.03905, LCL = 0]

3. A process produces rubber belts in lots of size 2300. Inspection of the last 20 lots reveals the following data:

Lot no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of defective belts	308	342	311	285	327	230	346	221	435	230	407	221	269	131	414	198	331	285	394	456

Compute control limits for p-chart.

[**Ans.** UCL_p = 0.1548, CL_p = 0.1335, LCL_p = 0.1122]

4. The following figure give the number of defectives in 20 samples, each sample containing 2,000 items.

425	430	216	341	225	322	280	306	337	305
356	402	216	264	126	409	193	326	280	389

Calculate the control limits for fraction defective chart (p-chart). Draw the p-chart and state the comment. [**Ans.** UCL_p = 0.178, CL_p = 0.154, LCL_p = 0.130]

5. An inspection of 10 samples of size 400 each from 10 lots revealed the following no. of defective units;

17, 15, 14, 26, 9, 4, 19, 12, 9, 15.

Calculate control limits for the no. of defective units. Plot the control limits and the observations and state whether the process is under control or not.

[**Ans.** UCL_{np} = 25.02679, CL_{np} = 14, LCL_{np} = 2.97231]

6. The following data refer to visual defects found during inspection of the first 10 samples of size 100 each. Use them to obtain upper and lower control limits for percentage defective in sample of 100.

Sample no.	1	2	3	4	5	6	7	8	9	10
No. of defective	4	8	11	3	11	7	7	16	12	6

[**Ans.** UCL_{np} = 16.87, CL_{np} = 8.5, LCL_{np} = 0.13]

7. The pieces of cloth out of the different rolls of equal length contained the following number of defects:

3 0 2 8 4 2 1 3 7 1

prepare a *c*-chart and state whether the process is in a statistical control?

[**Ans.** $UCL_c = 8.38$, $CL_c = 3.1$, $LCL_c = 0$]

8. The following table gives the no. of defects in carpets manufactured by a company.

Carpet serial no.	1	2	3	4	5	6	7	8	9	10
No. of defective	3	4	5	6	3	3	5	3	6	2

Determine the control line and the control limits for *c*-chart.

9. The following data relate to the number or break downs in the rubber covered wires in 24 successive lengths of 10,000 feet each.

8	1	1	3	7	1	2	6	1	1
10	5	0	19	16	20	1	6	12	4
2	3	7	5						

Draw *c*-chart and state your comment.

[**Ans.** $UCL_c = 13.0715$, $CL_c = 5.875$, $LCL_c = 0$ (Process out of control)]

10. A drilling machine bores holes with a mean diameter of 0.5230 cm. and a standard deviation of 0.0032 cm. Calculate the 2-sigma and 3-sigma upper and lower control limits for mean of sample of 4.

$$\textbf{Ans.}\ 2\text{–sigma} \begin{cases} UCL = 0.5262 \text{ cm} \\ LCL = 0.5198 \text{ cm} \\ CL = 0.5230 \text{ cm} \end{cases} \quad 3\text{-sigma} \begin{cases} UCL = 0.5278 \text{ cm} \\ LCL = 0.5182 \text{ cm} \\ CL = 0.5230 \text{ cm} \end{cases}$$

❑❑❑

FACTORS USEFUL IN THE CONSTRUCTION OF CONTROL CHARTS

Mean Chart				*Standard Deviation Chart*					*Range Chart*				
Sample size	*Factors for control limits*			*Factors for central line*	*Factors for control limits*				*Factors for central line*	*Factors for control limit*			
n	*A*	A_1	A_2	C_2	B_1	B_2	B_3	B_4	d_2	D_1	D_2	D_3	D_4
2	2.121	3.760	1.880	0.5642	0	1.843	0	3.267	1.128	0	3.686	0	3.267
3	1.732	2.394	1.023	0.7236	0	1.858	0	2.568	1.693	0	4.358	0	2.575
4	1.500	1.880	0.729	0.7979	0	1.808	0	2.266	2.059	0	4.698	0	2.282
5	1.342	1.596	0.577	0.8407	0	1.756	0	2.089	2..326	0	4.918	0	2.115
6	1.225	1.410	0.483	0.8686	0.026	1.711	0.030	1.970	2.534	0	5.078	0	2.004
7	1.134	1.277	0.419	0.8882	0.105	1.672	0.118	1.882	2.704	0.205	5.203	0.076	1.924
8	1.061	1.175	0.373	0.9027	0.167	1.638	0.185	1.815	2.847	0.387	5.307	0.136	1.864
9	1.000	1.094	0.337	0.9139	0.219	1.609	0.239	1.761	2.970	0.546	5.394	0.184	1.816
10	0.949	1.028	0308	0.9227	0.262	1.584	0.284	1.716	3.078	0.687	5.469	0.223	1.777
11	0.905	0.973	0.285	0.9300	0.299	1.561	0.321	1.679	3.173	0.812	5.534	0.256	1.744
12	0.866	0.925	0.266	0.9359	0.331	1.541	0.354	1.646	3.258	0.924	5.592	0.284	1.716
13	0.832	0.884	0.249	0.9410	0.359	1.523	0.382	1.618	3.336	1.026	5.646	0.308	1.692
14	0.802	0.848	0.235	0.9443	0.384	1.507	0.406	1.594	3.407	1.121	5.693	0.329	1.671
15	0.775	0.816	0.223	0.9490	0.406	1.492	0.428	1.572	3.472	1.207	5.737	0.348	1.652
16	0.750	0.788	0.212	0.9523	0.427	1.478	0.448	1.552	3.532	1.285	5.779	0.364	1.636
17	0.728	0.762	0.203	0.9551	0.445	1.465	0.466	1.534	3.588	1.359	5.817	0.379	1.621
18	0.707	0.738	0.194	0.9576	0.461	1.454	0.482	1.518	3.640	1.426	5.854	0.392	1.608
19	0.688	0.717	0187	0.9599	0.477	1.443	0.497	1.503	3.689	1.490	5.888	0.404	1.596
20	0.671	0.697	0.180	0.9619	0.491	1.433	0.510	1.490	3.735	1.548	5.922	0.414	1.586
21	0.655	0.679	0.173	0.9638	0.504	1.424	0.523	1.477	3.778	1.606	5.950	0.425	1.575
22	0.640	0.662	0.167	0.9655	0.516	1.415	0.534	1.466	3.819	1.659	5.979	0.434	1.566
23	0.626	0.647	0.162	0.9670	0.527	1.407	0.545	1.455	3.858	1.710	6.006	0.443	1.557
24	0.612	0.632	0.157	0.9684	0.538	1.399	0.555	1.445	3.395	1.759	6.031	0.452	1.548
25	0.600	0.619	0.153	0.9696	0.548	1.392	0.565	1.435	3.931	1.804	6.058	0.459	1.541

CHAPTER **12**

Testing of Hypothesis

12.1 INTRODUCTION

Suppose some business concern has an average sale of Rs. 10000/- daily estimated over a long period. A salesman claims that he will increase the average sales by Rs. 700/- a day. The concern is interested in an increased sale no doubt, but how to know whether the claim of the man is justified or not? For this some such a mathematical model for the population of increased sales is assumed which agrees to the maximum with the practical observations. In the example given, let us assume that the claim of the girl about her sales is justified and that the increase in sales is normally distributed with mean μ = 700 and variance σ^2. This assumption is called statistical hypothesis. Thereafter the suitability of the assumed model is examined on the basis of the sale observations made. This procedure is called testing of hypothesis.

A statistical hypothesis is some statement or assertion about a population or equivalently about the probability distribution characterising a population which we want to verify on the basis of information available from a sample. If the statistical hypothesis specifies the population completely then it is termed as a simple statistical hypothesis, otherwise it is called a composite statistical hypothesis.

Example: If $X_1, X_2, ..., X_n$ is a random sample of size n from a normal population with mean μ and variance σ^2, then the hypothesis.

$$H_0: \mu = \mu_0, \sigma^2 = \sigma_0^2$$

is a simple hypothesis, whereas each of the following hypothesis is a composite hypothesis:

(1) $\mu = \mu_0$

(2) $\sigma^2 = \sigma_0^2$

(3) $\mu = \mu_0, \sigma^2 < \sigma_0^2$

(4) $\mu < \mu_0, \sigma^2 > \sigma_0^2$

(5) $\mu < \mu_0, \sigma^2 = \sigma_0^2$

(6) $\mu = \mu_0, \sigma^2 > \sigma_0^2$

(7) $\mu > \mu_0, \sigma^2 = \sigma_0^2$

A hypothesis which does not specify completely 'r' parameters of a population is termed as a composite hypothesis with r degrees of freedom.

12.2 SOME IMPORTANT DEFINITIONS

Test of a Statistical Hypothesis: A test of a statistical hypothesis is a two action decision problem after the experimental sample values have been obtained, the two actions being the acceptance or rejection of the hypothesis under consideration.

Null Hypothesis: The statistical hypothesis tested under the assumption that it is true is called null hypothesis. It is tested on the basis of the sample observations and is liable to be rejected as well, depending upon the outcome of the statistical test applied. There are many occasions where null hypothesis is formulated for the sole purpose of rejecting it.

In other words, null hypothesis is statement of zero or no change. If the original claim includes equality (< =, =, or > =), it is the null hypothesis. If the original claim does not include equality (<, not equal, >) the null hypothesis is the complement of original claim. The null hypothesis always includes the equal sign. The decision is based on the null hypothesis. The null hypothesis is denoted by H_0.

Alternative Hypothesis: Statement which is true if the null hypothesis is false is known as alternative hypothesis. In other words a possible or the acceptable alternative to the null hypothesis called alternative hypothesis, and is denoted by H_1. It testing if H_0 is rejected, then H_1 is accepted. The type of test (left, right, or two tail) is based on the alternative hypothesis.

Type I Error and Type II Error: When a null hypothesis H_0 is tested against an alternative H_1, then there can be either of the following two types of errors:

(*a*) Rejecting the null hypothesis H_0 when actually it is true

(*b*) Failing to reject the null hypothesis when it is false

These are called errors of Type I and Type II and denoted by α and β respectively.

The other two possible outcomes of testing are:

(*c*) Rejection of H_0 when it was wrong and

(*d*) Acceptance of H_0 when it was true.

	H_0 is true	H_1 is true (H_0 is false)
Accept H_0	Correct decision	Type II error (β)
Accept H_1 (reject H_0)	Type I error (α)	Correct decision

Alpha: The probability of rejecting H_0, when it was true = The probability of committing type I error = The size of type I error = α.

Beta: The probability of accepting H_0, when it was wrong = The probability of committing type II error = The size of type II error = β.

Level of Significance: Alpha, the probability of type I error is known as the level of significance of the test. It is also called the size of the critical region. In other words, the maximum value of type I error which we would be willing to risk is called level of significance of the test. In general, 0.05 and 0.01 are the commonly accepted values of the levels of the significance. When the level of significance is 0.05, it simply means that on the average in 5 chances out of 100 we are likely to reject a correct H_0.

Probability (*P*-Value) Value: The probability of getting the results obtained if the null hypothesis is true. If this probability is too small (smaller than the level of significance), then we reject the null hypothesis. If the level of significance is the area beyond the critical values, then the probability value is the area beyond the test statistic.

Test Statistic: "Sample statistic used to decide whether to reject or fail to reject the null hypothesis".

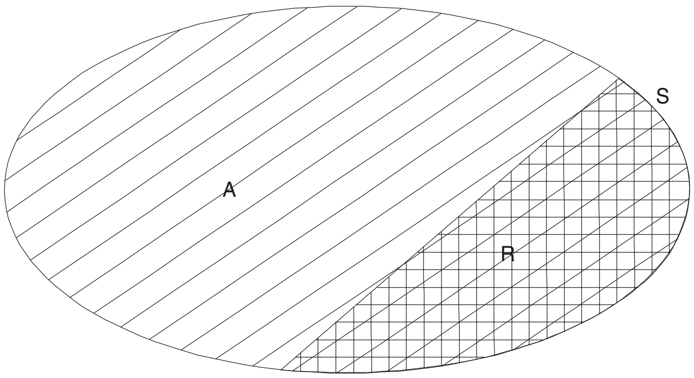
Critical Region: Set of all values which would cause us to reject H_0. Suppose the sample values $x_1, x_2, ... x_n$ determine a point E on the n-dimensional sample space S which would be the set of the various sample points corresponding to the all possible outcomes of the experiment.

The testing of statistics hypothesis is made on the basis of the division of this sample space into two mutually exclusive regions:

(1) Acceptance region

(2) Rejection (critical region) region of H_0

The null hypothesis H_0 is rejected as soon as the sample points falls in the critical region of the sample space S. The region of rejection is denoted either by R or by C.



Critical region $R \subseteq S$
acceptance region $A \subset S$
$A + R = S$

FIG. 12.1

The null hypothesis is accepted as soon as the sample point falls in the acceptance region, which is denoted by A. The values which separates the critical region from the non-critical region is known as critical values. The critical values are determined independently of the sample statistics.

Decision: Decision is a statement based upon the null hypothesis. It is either "reject the null hypothesis" or "fail to reject the null hypothesis" we will never accept the null hypothesis.

Conclusion: Conclusion is a statement which indicates the level of evidence (sufficient or insufficient), at what level of significance, and whether the original claim is rejected (null) or supported (alternative).

Unbiased Critical Region: A critical region is said to be unbiased if the size of type II error β comes out to be less than the size of type I error.

12.3 UNDERSTANDING THE TYPE OF TEST

The type of test is determined by the Alternative Hypothesis (H_1). The following way explain how to determine if the test is a left tail, right tail, or two tail test.

(*a*) Left Tailed Test

H_1: Parameter < value

Notice that the inequality points to the left.

Decision Rule: Reject H_0 if t.s.< c.v.

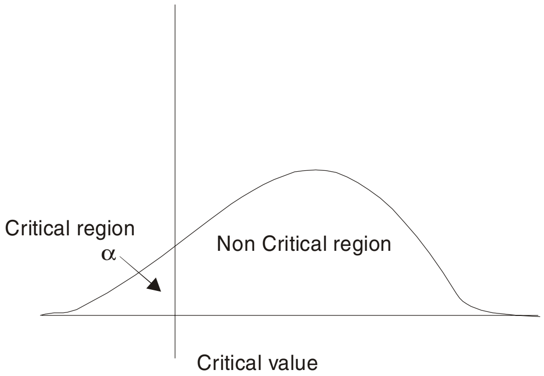


FIG. 12.2

(*b*) Right Tailed Test

H_1: Parameter > value

Notice that the inequality points to the right.

Decision Rule: Reject H_0 if t.s. > c.v.

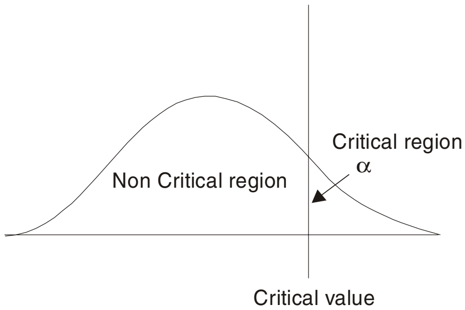


FIG. 12.3

(*c*) Two-Tailed Test

H_1: Parameter not equal value another way to write not equal is < or >

Notice that the inequality points to both sides.

Decision Rule: Reject H_0 if t.s. < c.v. (left) or t.s. > c.v. (right)

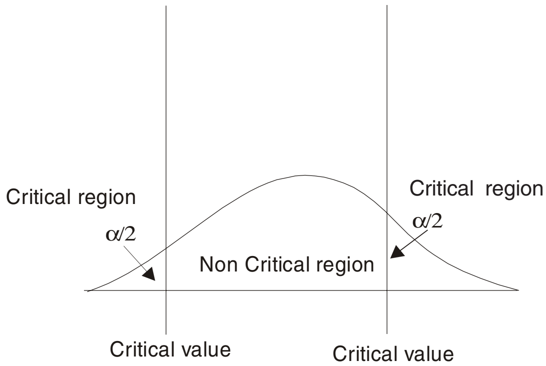


FIG. 12.4

The decision rule can be summarized as: Reject H_0 if the test statistic falls in the critical region (Reject H_0 if the test statistic is more extreme than the critical value).

12.4 PROCEDURE FOR TESTING OF HYPOTHESIS

(1) Null Hypothesis: Set up the Null Hypothesis H_0.

(2) Alternative Hypothesis: Set up the Alternative Hypothesis H_1. This would decide whether we have to use a one tailed test or two tailed test.

(3) Level of Significance: Choose the appropriate level of significance α.

(4) Test Statistic: Compute the test statistic.

$$Z = \frac{t - E(t)}{S.E.(t)}$$

under the null Hypothesis

(5) Conclusion: We compare Z the computed value of Z in above step 4 with the significant value (tabulated value) Z_α, at the given level of significance 'α'.

(*a*) If $|Z| < Z_\alpha$, we say that it is not significant *i.e.*, there is no significant difference and we accept H_0.

(*b*) If $|Z| > Z_\alpha$, we say that it is significant and the null hypothesis H_0 is rejected at level of significance α.

12.5 STANDARD ERROR

The standard error is defined as the standard deviation of the sampling distribution of a statistic. This is denoted by *S.E.* The standard error (*S.E.*) plays a very important role in the large sample theory and forms the basis of the testing of hypothesis. If *t is* any statistic, for large sample.

$$Z = \frac{t - E(t)}{S.E.(t)}$$

is normally distributed with $N(0, 1)$ *i.e.*, mean 0 and variance unity.

For large samples, the standard errors of some of the well known statistic are given below.

	Statistic	*Standard Error*
1.	Sample mean: $\bar{x}$	$\sigma/\sqrt{n}$
2.	Observed sample proportion: 'P'	$\sqrt{PQ/n}$
3.	Sample standard deviation: S	$\sqrt{\sigma^2/2n}$
4.	Sample variance: S^2	$\sigma^2\sqrt{2/n}$
5.	Difference of two sample means: $(\bar{x}_1 - \bar{x}_2)$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

6.	Difference of two sample $S.D.$'s $(S_1 - S_2)$	$\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$
7.	Difference of two sample proportions	$\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$

Standard error of a statistic may be reduced by increasing the sample size, but this results in corresponding increase in cost, time, labour, etc.

Now in the similar manner we given below the critical values of Z at commonly used levels of significance α for both two tailed and single tailed tests. These values have been obtained on using normal probabilities for equations.

$$p\left(|Z| > Z_\alpha\right) = \alpha;\ p(Z > Z_\alpha) = \alpha;\ p(Z < -Z_\alpha) = \alpha$$

Critical Values (Z_α) of Z

Critical values (Z_α)	*Level of significance* (α)		
	1%	*5%*	*10%*
Left tailed test	$Z_\alpha = -2.33$	$Z_\alpha = -1.645$	$Z_\alpha = -1.28$
Right tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
Two tailed test	$\lvert Z_\alpha \rvert = 2.58$	$\lvert Z_\alpha \rvert = 1.96$	$\lvert Z_\alpha \rvert = 1.645$

12.6 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

If sample size is large (For $n > 30$) the number of trials, almost all the distribution *i.e.*, poisson, binomial, negative binomial etc. are very closely approximated by normal distribution. Therefore we use the normal test which is based on the fundamental area property of the normal probability curve. The test of significance for large samples follows some important tests to test the significance:

(A) Testing of Significance for Single Proportion: This test is used to find significant difference between the population and proportion of the sample. If X is the number of successes in n independent trials with constant probability P of success for each trial then

$$E(X) = nP,\ V(X) = nPQ;\ \text{where } Q = 1 - P = \text{Prob. of failure}$$

In the sample of size n, let X be the number of persons possessing the given attributes then

$\frac{X}{n} = p$(say) is called the observed proportion of success.

$$\therefore \quad E(p) = E\left(\frac{X}{n}\right) = \frac{1}{n} E(X) = \frac{1}{n}.nP = P$$

$$\Rightarrow \qquad \boxed{E(p) = P}$$

Thus the sample proportion '*p*' gives an unbiased estimate of the population proportion.

Similarly
$$V(p) = V\left(\frac{X}{n}\right) = \frac{1}{n}V(X)$$
$$= \frac{nPQ}{n^2} = \frac{PQ}{n}$$
$$\boxed{V(p) = \frac{PQ}{n}}$$

and
$$S.E.(p) = \sqrt{\frac{PQ}{n}}$$
$$Z = \frac{p - E(P)}{S.E.(p)} = \frac{p - P}{S.E.(p)}$$
$$= \frac{p - P}{\sqrt{PQ/n}} \sim N(0, 1)$$

This Z is called test statistic. This is used to test the significant difference of sample and population proportion.

The General pattern of test statistic is,

$$\text{Test Statistic} = \frac{\text{observed} - \text{expected}}{\text{standard deviation}}$$

Remarks: (1) Since the probable limit for a normal variate X are $E(X) \pm 3\sqrt{V(X)}$, the probable limits for the observed proportion of success are

$$E(p) \pm 3\ S.E.(p)$$

i.e.,
$$P \pm 3\sqrt{\frac{PQ}{n}}$$

Hence this shows the confidence limits for observed proportion p.

(2) If P is not known then taking p (the sample proportion) as an estimate of P, the confidence limits for the population proportion are

$$p \pm \sqrt{\frac{pq}{n}}$$

(3) The probable limit for the observed proportion of successes, at the level of significance α are given by

$$p \pm Z_\alpha \sqrt{\frac{pq}{n}}$$

where Z_α is the significant value of Z.

Example 1. *A bag contains defective switch, the exact number of which is not known. A sample of 100 from the bag gives 10 defective switches. Find the limits for the proportion of defective switch in the bag.*

Sol. We have p = proportion of defective switch = $\frac{10}{100}$ = 0.1; $q = 1 - p = 1 - 0.1 = 0.9$

Since the confidence limit is not given, we assume it is 95%. $\therefore$ level of significance is 5% $Z_\alpha = 1.96$.

Again, the proportion of population P is not given. To get the confidence limit, we use p and it is given by $p \pm Z_\alpha \sqrt{\frac{pq}{n}} = 0.1 \pm 1.96 \sqrt{\frac{0.1 \times 0.9}{100}} = 0.1 \pm 0.0588 = 0.1588, 0.0412.$

Hence 95% confidence limits for defective switch in the bag are (0.1588,0.0412).

Example 2. *A manufacturer claims that only 4% of his products supplied by him are defective. A random sample of 600 products contained 36 defectives. Test the claim of the manufacturer.*

Sol. (*i*) P = observed proportion of success.

i.e., P = proportion of defective in the sample = $\frac{36}{600} = 0.06$

p = Proportion of defectives in the population = 0.04

H_0: $p = 0.04$ is true.

i.e., the claim of the manufacturer is accepted.

H_1: (*i*) $p \neq 0.04$ (two tailed test)

(*ii*) If we want to reject, only if $p > 0.04$ then (right tailed).

Under H_0, $Z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.06-0.04}{\sqrt{\dfrac{0.04 \times 0.96}{600}}} = 2.5$

Conclusion: Since $|Z| = 2.5 > 1.96$, we reject the hypothesis H_0 at 5% level of significance two tailed.

If H_1 is taken as $p > 0.04$ we apply right tailed test.

$|Z| = 2.5 > 1.645$ (Z_α) we reject the null hypothesis here also.

In both cases, manufacturer's claim is not acceptable.

Example 3. *A machine is producing bolts of which a certain fraction is defective. A random sample of 400 is taken from a large batch and is found to contain 30 defective bolt. Does this indicate that the proportion of defectives is larger than that claimed by the manufacturer where the manufacturer claims that only 5% of his product are defective. Find 95% confidence limits of the proportion of defective bolts in batch ?*

Sol. Null hypothesis H_0: The manufacturer claim is accepted

i.e., $P = \frac{5}{100} = 0.05$

$Q = 1 - P = 1 - 0.05 = 0.95$

Alternative hypothesis: $p > 0.05$ (Right tailed test).

p = observed proportion of sample

$$= \frac{30}{400} = 0.075$$

Under H_0 the test statistic $Z = \dfrac{p-P}{\sqrt{PQ/n}}$

$$\therefore \quad Z = \frac{0.075 - 0.05}{\sqrt{\dfrac{0.05 \times 0.95}{400}}} = 2.2941.$$

Conclusion: The tabulated value of Z at 5% level of significance for right tailed test is

$$Z_\alpha = 1.645,$$

Since $|Z| = 2.2941 > 1.645,$

Therefore, H_0 is rejected at 5% level of significance. *i.e.,* the proportion of defective is larger than the manufacturer claim.

Also,to find 95% confidence limits of the proportion.

It is given by $p \pm Z_\alpha \sqrt{PQ/n}$

$$= 0.05 \pm 1.96 \sqrt{\frac{0.05 \times 0.95}{400}} = 0.05 \pm 0.02135 = 0.07136,\ 0.02865$$

Hence 95% confidence limits for the proportion of defective bolts are (0.07136, 0.02865).

Example 4. *Twenty people were attacked by a disease and only 18 survived. Will you reject the hypothesis that the survival rate, if attacked by this disease, is 85% in favour of the hypothesis that it is more, at 5% level. (Use Large Sample Test).*

Sol. Given that

n = 20

X = Number of persons who survived after attack by a disease = 18

p = Proportion of persons survived in the sample

$$= \frac{18}{20} = 0.90$$

Null Hypothesis, H_0: $P = 0.85$, *i.e.,* the proportion of persons survived after attack by a disease in the lot is 85%.

Alternative Hypothesis, H_1:$P > 0.85$ (Right-tail alternative).

Test Statistic. Under H_0 the test statistic is:

$$Z = \frac{p-P}{\sqrt{PQ/n}} \sim N(0, 1), \text{ (since sample is large).}$$

Now, $$Z = \frac{0.90 - 0.85}{\sqrt{0.85 \times 0.15/20}} = \frac{0.05}{0.079} = 0.633$$

Conclusion: Since the alternative hypothesis is one-sided (right-tailed), we shall apply right-tailed test for testing significance of Z. The significant value of Z at 5% level of significance for right-tail test is + 1.645. Since computed value of $Z = 0.633$ is less than 1.645, it is not significant and we may accept the null hypothesis at 5% level of significance.

Example 5. *A dice is thrown 9,000 times and a throw of 3 or 4 is observed 3,240 times. Show that the dice cannot be regarded as an unbiased one and find the limits between which the probability of a throw of 3 or 4 lies.*

Sol. If the coming of 3 or 4 is called a success, then in usual notations we are given

$$n = 9{,}000;\ X = \text{Number of successes} = 3{,}240$$

Under the null hypothesis (H_0) that the dice is an unbiased one, we get

$$P = \text{Probability of success} = \text{Probability of getting a 3 or 4}$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Alternative hypothesis, H_1: $\quad p \neq \frac{1}{3}$, (*i.e.,* dice is biased).

We have $\quad Z = \dfrac{X - nP}{\sqrt{nQP}} \sim N(0, 1)$, since n is large.

Now, $\quad Z = \dfrac{3240 - 9000 \times 1/3}{\sqrt{9000 \times (1/3) \times (2/3)}} = \dfrac{240}{\sqrt{2000}} = \dfrac{240}{44.73} = 5.36$

Since $|Z| > 3$, H_0 is rejected and we conclude that the dice is almost certainly biased.

Since dice is not unbiased, $P \neq \frac{1}{3}$. The probable limits for 'P' are given by:

$$\hat{P} \pm 3\sqrt{\hat{P}\hat{Q}/n} = p \pm 3\sqrt{pq/n};$$

where $\hat{P} = P = \dfrac{3240}{9000} = 0.36$ and $\hat{Q} = q = 1 - p = 0.64$

Hence the probable limits for the population proportion of successes may be taken as

$$\hat{P} \pm 3\sqrt{\hat{P}\hat{Q}/n} = 0.36 \pm 3\sqrt{\frac{0.36 \times 0.64}{9000}}$$

$$= 0.36 \pm 3 \times \frac{0.6 \times 0.8}{30\sqrt{10}}$$

$$= 0.360 \pm 0.015 = 0.345 \text{ and } 0.375.$$

Hence the probability of getting 3 or 4 almost certainly lies between 0.345 and 0.375.

Example 6. *A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Show that the S.E. of the proportion of bad ones in a sample of this size is 0.015 and deduce that the percentage of bad pineapples in the consignment almost certainly lies between 8.5 and 17.5.*

Sol. Here we are given $n = 500$

X = Number of bad pineapples in the sample = 65

p = Proportion of bad pineapples in the sample = $\frac{65}{500} = 0.13$

$\therefore \quad q = 1 - p = 0.87$

Since P, the proportion of bad pineapples in the consignment is not known, we may take (as in the last example)

$$\hat{P} = p = 0.13,$$

$$\hat{Q} = q = 0.87$$

$$\text{S.E. of proportion} = \sqrt{\hat{P}\hat{Q}/n} = \sqrt{0.13 \times 0.87/500} = 0.015$$

Thus, the limits for the proportion of bad pineapples in the consignment are:

$$\hat{P} \pm 3\sqrt{\hat{P}\hat{Q}/n} = 0.130 \pm 3 \times 0.015 = 0.130 \pm 0.045 = (0.085, 0.175)$$

Hence the percentage of bad pineapples in the consignment lies almost certainly between 8.5 and 17.5.

Example 7. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased.*

Sol. H_0: The coin is unbiased *i.e.,* $P = 0.5$

H_1: The coin is not unbiased (biased); $P \neq 0.5$

$n = 400$; X = No. of success = 216

p = proportion of success in the sample $\frac{X}{n} = \frac{216}{400} = 0.54$

Population proportion $= 0.5 = P$; $Q = 1 - P = 1 - 0.5 = 0.5$.

$$\text{Under } H_0, \text{ test statistic } Z = \frac{p - P}{\sqrt{PQ/n}}$$

$$|Z| = \left| \frac{0.54 - 0.5}{\frac{\sqrt{0.5 \times 0.5}}{400}} \right| = 1.6$$

we use two tailed test.

Conclusion: Since $|Z| = 1.6 < 1.96$

i.e., $|Z| < z_\alpha$, z_α is the significant value of z at 5% level of significance.

i.e., the coin is unbiased if $P = 0.5$

Example 8. *A random sample of 500 apples was taken from a large consignment and 60 were found to be bad. Obtain the 98% confidence limits for the percentage number of bad apples in the consignment.*

$$\left[\int_0^{2.33} \phi(t)\, dt = 0.49 \text{ nearly}\right]$$

Sol. We have:

$$p = \text{Proportion of bad apples in the sample} = \frac{60}{500} = 0.12$$

Since the significant value of Z at 98% confidence coefficient (level of significance 2%) is given to be 2.33, 98% confidence limits for population proportion are:

$$p \pm 2.33\sqrt{pq/n} = 0.12 \pm 2.33\sqrt{0.12 \times 0.88/500}$$

$$= 0.12 \pm 2.33 \times \sqrt{0.0002112} = 0.12 \pm 2.33 \times 0.01453$$

$$= 0.12000 \pm 0.03385 = (0.08615,\ 0.15385)$$

Hence 98% confidence limits for percentage of bad apples in the consignment are (8.61, 15.38).

Example 9. *In a sample of 1,000 people in Maharashtra, 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance ?*

Sol. We have:

$$n = 1000$$

$$X = \text{Number of rice eaters} = 540$$

$$\therefore \quad p = \text{sample proportion of rice eaters} = \frac{X}{n}$$

$$= \frac{540}{1000} = 0.54$$

Null Hypothesis, H_0: Both rice and wheat are equally popular in the state so that

$$P = \text{Population proportion of rice eaters in Maharashtra} = 0.5$$

$$\Rightarrow \quad Q = 1 - P = 0.5$$

Alternative Hypothesis, H_1: $P \neq 0.5$ (two-tailed alternative).

Test Statistic. Under H_0, the test statistic is:

$$Z = \frac{p - P}{\sqrt{PQ/n}} \sim N(0, 1), \text{ (since } n \text{ is large)}.$$

Now,

$$Z = \frac{0.54 - 0.50}{\sqrt{0.5 \times 0.5/1000}} = \frac{0.04}{0.0138} = 2.532$$

Conclusion: The significant or critical value of Z at 1% level of significance for two-tailed test is 2.58. Since computed $Z = 2.532$ is less than 2.58, it is not significant at 1% level of significance. Hence the null hypothesis is accepted and we may conclude that rice and wheat are equally popular in Maharashtra State.

(B) Testing of Significance for Difference of Proportions: Let X_1, X_2 be the two samples of sizes n_1 and n_2 respectively from the two populations respectively. Thus sample proportions are given by

$$P_1 = \frac{X_1}{n_1},$$

$$P_2 = \frac{X_2}{n_2}$$

If P_1 and P_2 are the populations proportions, then

$$E(p_1) = P_1$$

$$E(p_2) = P_2$$

and

$$V(p_1) = \frac{P_1 Q_1}{n}$$

$$V(p_2) = \frac{P_2 Q_2}{n}$$

Under the null hypothesis H_0, there is no significant difference between the sample proportions *i.e.,* $P_1 = P_2$. Also for large samples p_1 and p_2 are asymptotically normally distributed therefore their difference $p_1 - p_2$ is also normally distributed.

Then

$$Z = \frac{(p_1 - p_2) - E(p_1 - p_2)}{\sqrt{V(p_1 - p_2)}} \sim N(0, 1)$$

Since $E(p_1 - p_2) = E(p_1) - E(p_2) = P_1 - P_2 = 0$

Also, $V(p_1 - p_2) = V(p_1) + V(p_2)$

the covariance term $COV(p_1, p_2)$ vanishes, because sample proportions are independent

$$\therefore \quad V(p_1 - p_2) = PQ\left(\frac{1}{n_1} - \frac{1}{n_2}\right)$$

Since under null hypothesis

$$H_0: P_1 = P_2 = P,$$

$$Q_1 = Q_2 = Q$$

Therefore,

$$\boxed{Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}} \sim N(0, 1)$$

where $P = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$ and $Q = 1 - P$.

Remark: Suppose population proportions P_1 and P_2 are given to be distinctly different *i.e.,* $P_1 \neq P_2$ and we want to test if the difference $(P_1 - P_2)$ in population proportions is likely to be hidden in simple samples of sizes n_1 and n_2 from the two populations respectively *i.e.,*

$$|Z| = \frac{|P_1 - P_2|}{\sqrt{\frac{P_1Q_1}{n} + \frac{P_2Q_2}{n}}} \sim N(0, 1)$$

Example 10. *A cigarette manufacturing firm claims that its brand A of the cigarettes outsells its brand B by 8%. If it is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another random sample of 100 smokers prefer brand B, test whether the 8% difference is a valid claim. (Use 5% level of significance).*

Sol. We are given:

$$n_1 = 200,\ X_1 = 42 \Rightarrow p_1 = \frac{X_1}{n_1} = \frac{42}{200} = 0.21$$

$$n_2 = 100,\ X_2 = 18 \Rightarrow p_2 = \frac{X_2}{n_2} = \frac{18}{100} = 0.18.$$

We set up the Null Hypothesis that 8% difference in the sale of two brands of cigarettes is a valid claim,

i.e., $H_0: P_1 - P_2 = 0.08.$

Alternative Hypothesis:

$H_1: P_1 - P_2 \neq 0.08$ (Two-tailed).

Under H_0, the test statistic is (since samples are large)

$$Z = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

where $$\hat{P} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{42 + 18}{200 + 100} = \frac{60}{300} = 0.20$$

$$\Rightarrow \quad \hat{Q} = 1 - \hat{P} = 0.80$$

Therefore, $$Z = \frac{(0.21 - 0.18) - 0.08}{\sqrt{0.2 \times 0.8\left(\frac{1}{200} + \frac{1}{100}\right)}} = \frac{-0.05}{\sqrt{0.16 \times 0.015}} = \frac{-0.05}{0.04899} = -1.02$$

Since $|Z| = 1.02 < 1.96$, it is not significant at 5% level of significance.

Hence null hypothesis may be retained at 5% level of significance. Also we can say that a difference of 8% in the sale of two brands of cigarette is a valid claim by the firm.

Example 11. *Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty?*

Sol. Given that

$$n_1 = 800,\ n_2 = 1200$$

$$p_1 = \frac{X_1}{n_1} = \frac{800}{1000} = \frac{4}{5};$$

$$p_2 = \frac{X_2}{n_2} = \frac{800}{1200} = \frac{2}{3}$$

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2}$$

$$= \frac{800 + 800}{1000 + 1200} = \frac{8}{11};\ Q = \frac{3}{11}$$

Also, *Null hypothesis* H_0: $p_1 = p_2$

*i.e.,*there is no significant difference in the consumption of tea before and after increase of excise duty.

$$H_1: p_1 > p_2 \text{ (right tailed test)}$$

The test statistic,
$$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.8 - 0.6666}{\sqrt{\frac{8}{11} \times \frac{3}{11}\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = 6.842$$

Conclusion: Since the calculated value of $|Z| > 1.645$ also $|Z| > 2.33$, both the significant values of z at 5% and 1% level of significance. Hence H_0 is rejected *i.e.,* there is a significant decrease in the consumption of tea due to increase in excise duty.

Example 12. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations.*

Sol. P_1 = proportion of fair haired people in the first population = 30% = 0.3; P_2 = 25% = 0.25; Q_1 = 0.7; Q_2 = 0.75.

Here H_0: Sample proportions are equal *i.e.,* the difference in population proportions is likely to be hidden in sampling.

$$H_1 : P_1 \neq P_2$$

$$Z = \frac{P_1 - P_2}{\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}} = \frac{0.3 - 0.25}{\sqrt{\frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}}} = 2.5376.$$

Conclusion: $|Z| > 1.96$ the significant value of Z at 5% level of significance. H_0 is rejected. However $|Z| < 2.58$, the significant value of Z at 1% level of signficance, H_0 is accepted. At 5% level, these samples will reveal the difference in the population proportions.

Example 13. *500 articles from a factory are examined and found to be 2% defective, 800 similar articles from a second factory are found to have only 1.5% defective. Can it reasonably be concluded that the product of the first factory are inferior to those of second?*

Sol.
$n_1 = 500$
$n_2 = 800$
p_1 = proportion of defective from first factory = 2% = 0.02
p_2 = proportion of defective from second factory = 1.5% = 0.015

H_0: There is no significant difference between the two products *i.e.,* the products do not differ in quality.

$$H_1 : P_1 < p_2 \text{ (one tailed test)}$$

$$\text{Under } H_0 : z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$= \frac{0.02(500) + (0.015)(800)}{500 + 800}$$

$$= 0.01692$$

$$Q = 1 - P = 1 - 0.01692$$

$$= 0.9830$$

$$Z = \frac{0.02 - 0.015}{\sqrt{0.01692 \times 0.983\left(\frac{1}{500} + \frac{1}{800}\right)}}$$

$$= 0.68$$

Conclusion: As $|Z| < 1.645$, the significant value of Z at 5% level of significance, H_0 is accepted *i.e.,* the products do not differ in quality.

Example 14. *Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal, are same against that they are not, at 5% level.* **[Agra Univ. M.A., 1992]**

Sol. Null Hypothesis H_0: $P_1 = P_2 = P$, (say), *i.e.,* there is no significant difference between the opinion of men and women as far as proposal of flyover is concerned.

Alternative Hypothesis, H_1:$P_1 \neq P_2$ (two-tailed).

We are given:

$n_1 = 400$, X_1 = Number of men favouring the proposal = 200
$n_2 = 600$, X_2 = Number of women favouring the proposal = 325

∴ p_1 = Proportion of men favouring the proposal in the sample

$$= \frac{X_1}{n_1} = \frac{200}{400} = 0.5$$

$$p_2 = \text{Proportion of women favouring the proposal in the sample}$$

$$= \frac{X_2}{n_2} = \frac{325}{600} = 0.541$$

Test Statistic. Since samples are large, the test statistic under the Null Hypothesis, H_0 is:

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

where
$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2}$$

$$= \frac{200 + 325}{400 + 600} = \frac{525}{1000} = 0.525$$

$$\Rightarrow \quad \hat{Q} = 1 - \hat{P} = 1 - 0.525 = 0.475$$

$$\therefore \quad Z = \frac{0.500 - 0.541}{\sqrt{0.525 \times 0.475 \times \left(\frac{1}{400} + \frac{1}{600}\right)}}$$

$$= \frac{-0.041}{\sqrt{0.525 \times 0.475 \times \left(\frac{5}{1200}\right)}}$$

$$\therefore \quad = \frac{-0.041}{\sqrt{0.001039}} = \frac{-0.041}{0.0323} = -1.269$$

Conclusion: Since $|Z| = 1.269$, which is less than 1.96, it is not significant at 5% level of significance. Hence H_0 may be accepted at 5% level of significance and we may conclude that men and women do not differ significantly as regards proposal of flyover is concerned.

Example 15. *A machine produced 16 defective bolts in a batch of 500. After overhauling it produced 3 defectives in a batch of 100. Has the machine improved ?*

Sol. We have,

$$p_1 = \frac{16}{500} = 0.032; \; n_1 = 500$$

$$p_2 = \frac{3}{100} = 0.03; \; n_2 = 100$$

Null Hypothesis H_0: The machine has not improved due to overhauling

$$H_0 : p_1 = p_2.$$

i.e.,
$$H_1 : p_1 > p_2 \text{ (right tailed)}$$

$$\therefore \qquad P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{19}{600} \cong 0.032$$

Under H_0, the test statistic

$$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.032 - 0.03}{\sqrt{(0.032)(0.968)\left(\frac{1}{500} + \frac{1}{100}\right)}} = 0.104$$

Conclusion: The calculated value of $|Z| < 1.645$, the significant value of Z at 5% level of significance, H_0 is accepted *i.e.,* the machine has not improved due to overhauling.

Example 16. *A company has the head office at Calcutta and a branch at Bombay. The personnel director wanted to know if the workers at the two places would like the introduction of a new plan of work and a survey was conducted for this purpose. Out of a sample of 500 workers at Calcutta, 62% favoured the new plan. At Bombay out of a sample of 400 workers, 41% were against the new plan. Is there any significant difference between the two groups in their attitude towards the new plan at 5% level ?*

Sol. In the usual notations, we are given:

$$n_1 = 500, \; p_1 = 0.62 \text{ and } n_2 = 400, \; p_2 = 1 - 0.41 = 0.59$$

Null hypothesis, $H_0 : P_1 = P_2$, *i.e.,* there is no significant difference between the two groups in their attitude towards' the new plan.

Alternative hypothesis, $H_1 : P_1 \neq P_2$ (Two-tailed)

Test Statistic. Under H_0, the test statistic for large samples is:

$$Z = \frac{p_1 - p_2}{S.E.(p_1 - p_2)} = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

where,
$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{500 \times 0.62 + 400 \times 0.59}{500 + 400} = 0.607$$

and
$$\hat{Q} = 1 - \hat{P} = 0.393$$

$$\therefore \qquad Z = \frac{0.62 - 0.59}{\sqrt{0.607 \times 0.393 \times \left(\frac{1}{500} + \frac{1}{400}\right)}}$$

$$= \frac{0.03}{\sqrt{0.00107}} = \frac{0.03}{0.0327} = 0.917.$$

Critical Region: At 5% level of significance, the critical value of Z for a two-tailed test is 1.96. Thus the critical region consists of all values of $Z \geq 1.96$ or $Z \leq -1.96$.

Conclusion: Since the calculated value of $|Z| = 0.917$ is less than the critical value of Z (1.96), it is not significant at 5% level of significance. Hence the data do not provide us any evidence against the null hypothesis which may be accepted, and we conclude that there is no significant difference between the two groups in their attitude towards the new plan.

Example 17. *On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30 per cent and the remaining 70 per cent. Consider the first question of this examination. Among the first group, 40 had the correct answer, whereas among the second group, 80 had the correct answer. On the basis of these results, can one conclude that the first question is no good at discriminating ability of the type being examined here?*

Sol. Here, we have

n = Total number of candidates = 200

n_1 = The number of candidates in the upper 30% group

$$= \frac{30}{100} \times 200 = 60$$

n_2 = The number of candidates in the remaining 70% group

$$= \frac{70}{100} \times 200 = 140$$

X_1 = The number of candidates, with correct answer in the first group = 40

X_2 = The number of candidates, with correct answer in the second group = 80

$$\therefore \quad p_1 = \frac{X_1}{n_1} = \frac{40}{60} = 0.6666 \text{ and } p_2 = \frac{X_2}{n_2} = \frac{80}{140} = 0.5714.$$

Null Hypothesis, H_0: There is no significant difference in the sample proportions, *i.e.*, $P_1 = P_2$, *i.e.*, the first question is no good at discriminating the ability of the type being examined here.

Alternative Hypothesis,

$$H_1 : P_1 \neq P_2$$

Test Statistic: Under H_0 the test statistic is:

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1) \quad \text{(since samples are large).}$$

where,
$$\hat{P} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{40 + 80}{60 + 140} = 0.6$$

$$\hat{Q} = 1 - \hat{P} = 0.4$$

$$\therefore \quad Z = \frac{0.6666 - 0.5714}{\sqrt{0.6 \times 0.4\left(\frac{1}{60} + \frac{1}{140}\right)}} = \frac{0.0953}{0.0756} = 1.258$$

Conclusion: Since $|Z| < 1.96$, the data are consistent with the null hypothesis at 5% level of significance. Hence we conclude that the first question is not good enough to distinguish between the ability of the two groups of candidates.

(C) Testing of Significance for Single Mean: Let $x_1, x_2, ... x_n$ be a random sample of size n from a large population $X_1, X_2, ... X_N$ (of size N) with mean μ and variance σ^2. Then sample mean $(\overline{X})$ and variance (S^2) are

$$\bar{x} = -\frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Also, the standard error of mean of a random sample of size n from a population with variance σ^2 is $\sigma/\sqrt{n}$.

i.e.,

$$S.E.(\bar{x}) = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$$

For large samples, the standard normal variate corresponding to $\overline{X}$ is

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

where σ is the standard deviation of the population.

Under the null hypothesis, H_0 that the sample has been drawn from a population with mean μ and variance σ^2, *i.e.,* there is no signficiant difference between the sample mean $(\bar{x})$ and population mean (μ), for large samples the test statistic is

$$\boxed{Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}}$$

If the population standard deviation σ is not known

$$\boxed{Z = \frac{\overline{X} - \mu}{S/\sqrt{n}}}$$

where S is the standard deviation of the sample.

Confidence Limits for μ

1. If the level of significance is α and Z_α is the the critical value then

$$-Z_\alpha < \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| < Z_\alpha$$

The limit of the population mean μ are given by

$$\bar{x} - Z_\alpha \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_\alpha \frac{\sigma}{\sqrt{n}}$$

2. 95% confidence interval for μ at 5% level of significance are

$$\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}$$

Similarly, 99% confidence limits for μ at 1% level of significance are

$$\bar{x} - 2.58 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.58 \frac{\sigma}{\sqrt{n}}$$

3. In sampling from a finite population of size N, the corresponding 95% and 99% confidence limits for μ are

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

and

$$\bar{x} \pm 2.58 \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

4. The confidence limits for any parameters (P, μ etc.) are known as its fiducial limits.

Example 18. *A sample of 900 members has a mean 3.5 cms, and S.D. 2.61 cms. Is the sample from a large population of mean 3.25 cms, and S.D. 2.61 cms?*

If the population is normal and its mean is unknown. Find the 95% and 98% fiducial limits of true mean.

Sol. *Null hypothesis*, (H_0): The sample has been drawn from the population with mean

$$\mu = 3.25 \text{ cms. and S.D. } \sigma = 2.61 \text{ cms.}$$

Alternative Hypothesis, H_1: $\mu \neq 3.25$ (Two-tailed)

Test Statistic. Under H_0, the test statistic is:

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1), \text{ (since } n \text{ is large)}$$

Here, we are given $\bar{x} = 3.4$ cms., $n = 900$ cms., $\mu = 3.25$ cms. and $\sigma = 2.61$ cms.

$$Z = \frac{3.40 - 3.25}{2.61/\sqrt{900}} = \frac{0.15 \times 30}{2.61} = 1.73$$

Since $|Z| < 1.96$, we conclude that the data don't provide us any evidence against the null hypothesis (H_0), which may, therefore, be accepted at 5% level of significance.

95% *fiducial limits* for the population mean μ are:

$$\bar{x} \pm 1.96 \, \sigma/\sqrt{n} \Rightarrow 3.40 \pm 1.96 \times 2.61/\sqrt{900}$$

$$\Rightarrow \quad 3.40 \pm 0.1705, \quad \textit{i.e.,} \quad 3.5705 \text{ and } 3.2295$$

98% *fiducial limits* for μ are given by:

$$\bar{x} \pm 2.33 \frac{\sigma}{\sqrt{n}}, \quad \textit{i.e.,} \quad 3.40 \pm 2.33 \times \frac{2.61}{30}$$

$$\Rightarrow \quad 3.40 \pm 0.2027, \quad \textit{i.e.,} \quad 3.6027 \text{ and } 3.1973$$

Remark: 2.33 is the value z_1 of Z from standard normal probability intergrals, such that

$$P\left(|Z| > z_1\right) = 0.98 \quad \Rightarrow \quad P\left(|Z| > z_1\right) = 0.49.$$

Example 19. *The average marks in Mathematics of a sample of 100 students was 51 with a S.D. of 6 marks. Could this have been a random sample from a population with average marks 50?*

Sol. Given

$n = 100$, $\bar{x} = 51$, $s = 6$, $\mu = 50$ (σ is unknown in this problem)

H_0 : The sample is drawn from a population with mean 50, $\mu = 50$

$H_1 : \mu \neq 50$

Under H_0,
$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{51 - 50}{6/\sqrt{100}} = \frac{10}{6} = 1.6666.$$

Conclusion: Since $|Z| = 1.666 < 1.96$, Z_α the significant value of Z at 5% level of significance, H_0 is accepted.

Example 20. *An insurance agent has claimed that the average age of policyholders who insure through him is less than the average for all agents, which is 30.5 years.*

A random sample of 100 policyholders who had insured through him gave the following age distribution:

Age last birthday	*No. of persons*
16–20	*12*
21–25	*22*
26–30	*20*
31–35	*30*
36–40	*16*

Calculate the arithmetic mean and standard deviation of this distribution and use these values to test his claim at the 5% level of significance. You are given that Z(1.645) = 0.95.

Sol. *Null Hypothesis*, H_0, $\mu = 30.5$ years, *i.e.*, the sample mean $(\bar{x})$ and population mean (μ) do not differ significantly.

Alternative Hypothesis, $H_1 : \mu < 30.5$ years (Left-tailed alternative).

Calculations for Sample Mean and S.D.

Age last birthday	*No. of persons (f)*	*Mid-point x*	$d = \frac{x - 28}{5}$	*fd*	fd^2
16–20	12	18	–2	–24	48
21–25	22	23	–1	–22	22
26–30	20	28	0	0	0
31–35	30	33	1	30	30
36–40	16	38	2	32	64
Total	$N = 100$			$\Sigma fd = 16$	$\Sigma fd^2 = 164$

$$\bar{x} = 28 + \frac{5 \times 16}{100} = 28.8 \text{ years}$$

$$s = 5 \times \sqrt{\frac{164}{100} - \left(\frac{16}{100}\right)^2} = 6.35 \text{ years}$$

Since the sample is large, $\hat{\sigma} \simeq s = 6.35$ years.

Test Statistic. Under H_0, the test statistic is:

$$Z = \frac{\bar{x} - \mu}{\sqrt{s^2 / n}} \sim N(0, 1), \text{ (since sample is large).}$$

Now $$Z = \frac{28.8 - 30.5}{6.35 / \sqrt{100}} = \frac{-1.7}{0.635} = -2.681.$$

Conclusion: Since computed value of $Z = -2.681 < -1.645$ or $|Z| = 2.681 > 1.645$, it is significant at 5% level of significance. Hence we reject the null hypothesis H_0 (Accept H_1) at 5% level of significance and conclude that the insurance agent's claim that the average age of policyholders who insure through him is less than the average for all agents, is valid.

Example 21. *A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference significant?*

Sol. H_0 : There is no significant difference between $\bar{x}$ and μ.

H_1 : There is significant difference between $\bar{x}$ and μ.

Given $\mu = 6.8$ $\sigma = 1.5$ $\bar{x} = 6.75$ and $n = 400$

$$|z| = \left| \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right| = \left| \frac{6.75 - 6.8}{1.5/\sqrt{900}} \right| = |-0.67| = 0.67$$

Conclusion. As the calculated value of $|Z| < Z_\alpha = 1.96$ at 5% level of significance. H_0 is accepted therefore there is no significant difference between $\bar{x}$ and μ.

Example 22. *The mean muscular endurance score of a random sample of 60 subjects was found to be 145 with a S.D. of 40. Construct a 95% confidence interval for the true mean. Assume the sample size to be large enough for normal approximation. What size of sample is required to estimate the mean within 5 of the true mean with a 95% confidence?* **[Calicut University B.Sc. (Main State) 1989]**

Sol. We are given: $n = 60$, $\bar{x} = 145$ and $s = 40$.

95% confidence limits for true mean (μ) are:

$$\bar{x} \pm 1.96\ s/\sqrt{n} \quad (\sigma^2 = s^2, \text{ since sample is large})$$

$$= 145 \pm \frac{1.96 \times 40}{\sqrt{60}} = 145 \pm \frac{78.4}{7.75} = 145 \pm 10.12 = 134.88,\ 155.12$$

Hence 95% confidence interval for μ is (134.88, 155.12). we know that

$$n = \left(\frac{Z_\alpha . \sigma}{E}\right)^2 = \left(\frac{1.96 \times 40}{5}\right)^2$$

$$[\because Z_{0.05} = 1.96,\ \hat{\sigma} = s = 40 \text{ and } |\bar{x} - \mu| < 5 = E]$$

$$= (15.68)^2 = 245.86 \simeq 246.$$

Example 23. *A random sample of 900 members has a mean 3.4 cms. Can it be reasonably regarded as a sample from a large population of mean 3.2 cms and S.D. 2.3 cms?*

Sol. We have:

$$n = 900,\ \bar{x} = 3.4,\ \mu = 3.2,\ \sigma = 2.3$$

H_0 : Assume that the sample is drawn from a large population with mean 3.2 and S.D. = 2.3

H_1 : $\mu \neq 3.25$ (Apply two-tailed test)

Under H_0 :
$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{3.4 - 3.2}{2.3\sqrt{900}} = 0.261$$

Conclusion: As the calculated value of $|Z| = 0.261 < 1.96$, the significant value of Z at 5% level of significance, H_0 is accepted therefore the sample is drawn from the population with mean 3.2 and S.D. = 2.3.

Example 24. *The mean weight obtained from a random sample of size 100 is 64 gms. The S.D. of the weight distribution of the population is 3 gms. Test the statement that the mean weight of the population is 67 gms at 5% level of significance. Also set up 99% confidence limits of the mean weight of the population.*

Sol. We have:

$$n = 100,\ \mu = 67,\ \bar{x} = 64,\ \sigma = 3$$

H_0 : There is no significant difference between sample and population mean.

i.e., $\mu = 67$, the sample is drawn from the population with $\mu = 67$.

H_1 : $\mu \neq 67$ (Two-tailed test).

Under H_0 :
$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{64 - 67}{3\sqrt{100}} = -10 \quad \therefore |Z| = 10.$$

Conclusion: Since the calculated value of $|Z| > 1.96$, the significant value of Z at 5% level of significance, H_0 is rejected *i.e.,* the sample is not drawn from the population with mean 67. To find 99% confidence limits. It is given by $\bar{x} \pm 2.58 \dfrac{\sigma}{\sqrt{n}}$

$$= 64 \pm 2.58 \times \frac{3}{\sqrt{100}} = 64.774,\ 63.226$$

(D) Test of Significance for Difference of means of two large samples: The test statistic is given by, in this case

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

where $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2.

$\bar{x}_2$ be the mean of an independent sample of size n_2 from population with mean μ_2 and variance σ_2^2.

Remarks: 1. Under the null hypothesis H_0: $\mu_1 = \mu_2$, *i.e.,* there is no significant difference between the sample means therefore $\sigma_1^2 = \sigma_2^2 = \sigma^2$ *i.e.,* if the sample have been drawn from the populations with common standard deviation σ then

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

2. If $\sigma_1^2 \neq \sigma_2^2$ and σ_1 and σ_2 are not known, then test statistic estimated from sample values. *i.e.,*

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{S_1^2}{n_1}\right) + \left(\frac{S_2^2}{n_2}\right)}}$$

3. If σ is not known, then its test statistic based on the sample variances is used.

If $\sigma_1 = \sigma_2$, we use $\sigma^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2}$ to evaluate σ.

Test statistic
$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

(E) Test of Significance for the Difference of standard Deviations: If S_1 and S_2 are the standard deviations of two independent samples, then under the null hypothesis, $H_0 : \sigma_1 = \sigma_2$ (the sample *S.D.* do not differ significantly), the test statistic is given by

$$Z = \frac{S_1 - S_2}{S.E.(S_1 - S_2)} \qquad \text{(For large samples)}$$

but the difference of the sample standard deviation is given by

$$S.E.\ (S_1 - S_2) = \sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$$

$$\therefore \qquad Z = \frac{S_1 - S_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}}$$

when σ_1^2 and σ_2^2 are not known (*i.e.,* population S.D. are not known) then the test statistic reduces to

$$Z = \frac{S_1 - S_2}{\sqrt{\frac{S_1^2}{2n_1} + \frac{S_2^2}{2n_2}}}$$

Example 25. *Intelligence tests were given to two groups of boys and girls*

	Mean	*S.D.*	*Size*
Girls	*75*	*8*	*60*
Boys	*73*	*10*	*100*

Examine if the difference between mean scores is significant.

Sol. *Null hypothesis* H_0: There is no significant difference between mean scores i.e., $\overline{x}_1 = x_2$.

$$H_1 : \overline{x}_1 = \overline{x}_2$$

Under the null hypothesis $Z = \dfrac{\overline{x}_1 - \overline{x}_2}{\sqrt{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)}} = \dfrac{75 - 73}{\sqrt{\dfrac{8^2}{60} + \dfrac{10^2}{100}}} = 1.3912$

Conclusion: As the calculated value of $|Z| < 1.96$, the significant value of Z at 5% level of significance, H_0 is accepted.

Example 26. *The means of two single large samples of 1000 and 2000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn from the same population of standard deviation 2.5 inches? (Test at 5% level of significance).*

Solution: Given:

$$n_1 = 1000$$
$$n_2 = 2000$$
$$\overline{x}_1 = 67.5 \text{ inches}$$
$$\overline{x}_2 = 68.0 \text{ inches}$$

Null hypothesis: H_0: $\mu_1 = \mu_2$ and $\sigma = 2.5$ inches

i.e., the samples have been drawn from the same population of standard deviation 2.5 inches.

Alternative hypothesis: H_1: $\mu_1 \neq \mu_2$ (Two-tailed)

Test statistic: Under H_0, the test statistic (For large samples)

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\sigma^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

$$Z = \frac{67.5 - 68.0}{2.5\sqrt{\left(\dfrac{1}{1000} + \dfrac{1}{2000}\right)}} = \frac{-0.5}{2.5 \times 0.0387}$$

$$Z = -5.1$$

Conclusion: Since $|Z| > 3$, the value is highly significant and we reject the null hypothesis and conclude that samples are certainly not from the same population with standard deviation 2.5.

Example 27. *The average income of persons was Rs. 210 with a S.D. of Rs. 10 in sample of 100 people of a city. For another sample of 150 persons, the average income was Rs. 220 with standard deviation of Rs. 12. The S.D. of incomes of the people of the city was Rs.11. Test whether there is any significant difference between the average incomes of the localities.*

Sol. Given that $n_1=100$, $n_2 = 150$, $\bar{x}_1 = 210$, $\bar{x}_2 = 220$, $S_1 = 10$, $S_2 = 12$.

Null Hypothesis: The difference is not significant. *i.e.,* there is no difference between the incomes of the localities.

$$H_0 : \bar{x}_1 = \bar{x}_2,\ H_1 : \bar{x}_1 \neq \bar{x}_2$$

Under H_0,

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{210 - 220}{\sqrt{\frac{10^2}{100} + \frac{12^2}{150}}} = -7.1428 \therefore |Z| = 7.1428$$

Conclusion: As the calculated value of $|Z| > 1.96$, the significant value of Z at 5% level of significance, H_0 is rejected *i.e.,* there is significant difference between the average incomes of the localities.

Example 28. *In a survey of buying habits, 400 women shoppers are chosen at random in super market 'A' located in a certain section of the city. Their average weekly food expenditure is Rs. 250 with a standard deviation of Rs. 40. For 400 women shoppers chosen at random in super market 'B' in another section of the city, the average weekly food expenditure is Rs. 220 with a standard deviation of Rs. 55. Test at 1% level of significance whether the average weekly food expenditure of the two populations of shoppers are equal.*

Sol. We have: $n_1 = 400$, $n_2 = 400$, $\bar{x}_1 =$ Rs. 250, $\bar{x}_2 =$ Rs. 220, $S_1 =$ Rs. 40, $S_2 =$ Rs. 55

Null hypothesis, $H_0: \mu_1 = \mu_2$

i.e., the average weekly food expenditures of the two populations of shoppers are equal.

Alternative Hypothesis, $H_1: \mu_1 \neq \mu_2$ (Two-tailed)

Test Statistic: Since samples are large, under H_0 then

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}}$$

Since σ_1 and σ_2 are not known then we use

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$Z = \frac{250 - 220}{\sqrt{\frac{(40)^2}{400} + \frac{(55)^2}{400}}} = 8.82 \text{ (Approx.)}$$

Conclusion: Since $|Z|$ is much greater than 2.58, the null hypothesis ($\mu_1 = \mu_2$) is rejected at 1% level of significance and we conclude that the average weekly expenditures of two populations of shoppers in market *A* and *B* differ significantly.

Example 29. *In a certain factory there are two independent processes manufacturing the same items. The average weight in a sample of 250 items produced from one process is found to be 120 ozs. with a standard deviation of 12 ozs. While the corresponding figures in a sample of 400 items from the other process are 124 and 14. Obtain the standard error of difference between the two sample means; Is this* *difference significant? Also find the 99% confidence limits for the difference in the average weights of items produced by the two processes respectively.*

Sol. Given: $n_1 = 250,\ \bar{x}_1 = 120$ ozs, $S_1 = 12\ ozs = \sigma_1$

$n_2 = 400,\ \bar{x}_2 = 124$ ozs, $S_2 = 14\ ozs = \sigma_2$

$$S.E.\ (\bar{x}_1 - \bar{x}_2) = \sqrt{\left(\frac{\sigma_1^2}{n_1}\right) + \left(\frac{\sigma_2^2}{n_2}\right)} = \sqrt{\left(\frac{S_1^2}{n_1}\right) + \left(\frac{S_2^2}{n_2}\right)} \text{ (Samples are large)}$$

$$= \sqrt{\left(\frac{144}{250} + \frac{196}{400}\right)} = \sqrt{0.576 + 0.490} = 1.034$$

Null Hypopthesis, H_0: $\mu_1 = \mu_2$ (*i.e.,* the sample means do not differ significantly)

Altnerative Hypothesis, $H_1 = \mu_1 \neq \mu_2$ (Two-tailed)

Test Statistic: Under H_0, the test statistic is given by

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{S.E.(\bar{x}_1 - \bar{x}_2)} = \frac{120 - 124}{1.034}$$

$$\therefore \quad |Z| = \frac{4}{1.034} = 3.87$$

Conclusion: Since $|Z| > 3$, the null hypothesis is rejected and we can say that there is significant difference between the sample means. 99% confidence limits for $|\mu_1 - \mu_2|$ is

$$|\bar{x}_1 - \bar{x}_2| \pm 2.58\ S.E.\ |\bar{x}_1 - \bar{x}_2|$$

$$= 4 \pm 2.58 \times 1.034$$

$$= 4 \pm 2.67 \text{ (Approx.)}$$

$= 6.67$ (on taking $+ve$ sign) and 1.33 (on taking $-ve$ sign).

$$\therefore \quad 1.33 < |\mu_1 - \mu_2| < 6.67$$

Example 30. *Two populations have their means equal, but S.D. of one is twice the other. Show that in the samples of size 2000 from each drawn under simple sampling conditions, the difference of means will, in all probability not exceed 0.15σ, where σ is the smaller S.D. what is the probability that the difference will exceed half this amount ?*

Sol. Let standard deviations of the two populations be σ and 2σ respectvely and let μ be the mean of each of two populations.

Given $n_1 = n_2 = 2000$

If $\bar{x}_1$ and $\bar{x}_2$ be two sample means then

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - E(\bar{x}_1 - \bar{x}_2)}{S.E.(\bar{x}_1 - \bar{x}_2)}$$

Now $E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu - \mu = 0$

Also $$S.E\ (\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma^2}{n_1} + \frac{(2\sigma)^2}{n_2}}$$

$$= \sigma\sqrt{\frac{1}{2000} + \frac{4}{2000}} = 0.05\sigma$$

$$\therefore \quad Z = \frac{\bar{x}_1 - \bar{x}_2}{S.E.(\bar{x}_1 - \bar{x}_2)} \sim N\ (0,\ 1)$$

Under simple sampling conditions, we should in all probability have

$$|Z| < 3$$

$$\Rightarrow \quad |\bar{x}_1 - \bar{x}_2| < 3\ \text{S.E.}\ (\bar{x}_1 - \bar{x}_2)$$

$$\Rightarrow \quad |\bar{x}_1 - \bar{x}_2| < 0.15\ \sigma$$

which is the required result.

We want $p = P\left[|\bar{x}_1 - \bar{x}_2| > \frac{1}{2} \times 0.15\sigma\right]$

$$\therefore \quad p = P\ [0.05\ \sigma\ |Z| > 0.075\ \sigma] \qquad \left(\because Z = \frac{\bar{x}_1 - \bar{x}_2}{0.05\sigma}\right)$$

$$= P\ [|Z| > 1.5]$$
$$= 1 - P\ [|Z| \le 1.5]$$
$$= 1 - 2P\ (0 \le Z \le 1.5)$$
$$= 1 - 2 \times 0.4332 = 0.1336.$$ **Ans.**

Example 31. *Random samples drawn from two countries gave the following data relating to the heights of adult males:*

	Country A	*Country B*
Males height (in inches)	*67.42*	*67.25*
Standard deviation	*2.58*	*2.50*
Number in samples	*1000*	*1200*

(i) Is the difference between the means significant?

(ii) Is the difference between the standard deviations significant?

Sol. Given: $n_1 = 1000$, $n_2 = 1200$, $\bar{x}_1 = 67.42$; $\bar{x}_2 = 67.25$, $s_1 = 2.58$, $s_2 = 2.50$

Since the samples size are large we can take

$$\sigma_1 = S_1 = 2.58;$$
$$\sigma_2 = S_2 = 2.50.$$

(*i*) **Null Hypothesis:** $H_0 : \mu_1 = \mu_2$ *i.e.,* sample means do not differ significantly.
Alternative hypothesis: $H_1 : \mu_1 \neq \mu_2$ (two-tailed test)

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.42 - 67.25}{\sqrt{\frac{(2.58)^2}{1000} + \frac{(2.50)^2}{1200}}} = 1.56$$

Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

(*ii*) We set up the null hypothesis.

H_0: $\sigma_1 = \sigma_2$ *i.e.,* the sample S.D.'s do not differ significantly.

Alternative hypothesis: $H_1 : \sigma_1 \neq \sigma_2$ (two-tailed)

∴ The test statistic is

$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} \quad (\because \sigma_1 = s_1, \sigma_2 = s_2 \text{ for large samples})$$

$$= \frac{2.58 - 2.50}{\sqrt{\frac{(2.58)^2}{2 \times 1000} + \frac{(2.50)^2}{2 \times 1200}}} = \frac{0.08}{\sqrt{\frac{6.6564}{2000} + \frac{625}{2400}}} = 1.0387.$$

Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

PROBLEM SET 12.1

1. 325 men out of 600 men chosen from a big city were found to be smokers. Does this information support the conclusion that the majority of men in the city are smokers?
[Ans. H_0 rejected at 5% level]

2. A sample of size of 600 persons selected at random from a large city shows that the percentage of males in the sample is 53. It is believed that the ratio of males to the total population in the city is 0.5. Test whether the belief is confirmed by the observation.
[Ans. H_0 accepted at 5% level]

3. In a city a sample of 1000 people were taken and out of them 540 are vegetarian and the rest are non-vegetarian. Can we say that the both habits of eating are equally popular in the city at (*i*) 5% level of significance (*ii*) 1% level of significance.
[Ans. H_0 rejected at 5% level
H_0 accepted at 1% level]

4. In a hospital 475 female and 525 male babies were born in a week. Do these figures confirm the hypothesis that males and females are born in equal number?
[Ans. H_0 accepted at 5% level]

5. A random sample of 500 bolts was taken from a large consignment and 65 were found to be defective. Find the percentage of defectives bolts in the consignment.
[Ans. Between 17.51 and 8.49]

6. In a town A, there were 956 births of which 52.5% were males while in towns A and B combined, this proportion in total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns? **[Ans.** H_0: Rejected]

7. 1,000 apples are taken from a large consignment and 100 are found to be bad. Estimate the percentage of bad apples in the consignment and assign the limits within which the percentage lies.

8. In a referendum submitted to the students body at a university, 850 men and 560 women voted. 500 men and 320 women voted yes. Does this indicate a significant difference of opinion between men and women on this matter at 1% level?

[Ans. H_0: Accepted]

9. A manufacturing firm claims that its brand *A* product outsells its brand *B* product by 8%. If it is found that 42 out of a sample of 200 persons prefer brand *A* and 18 out of another sample of 100 persons prefer brand *B*. Test whether the 8% difference is a valid claim. **[Ans.** H_0: Accepted]

10. In a large city *A*, 25% of a random sample of 900 school boys had defective eye-sight. In another large city *B*, 15.5% of a random sample of 1,600 school boys had the same defect. Is this difference between the two proportions significant?

[Ans. Not Significant]

11. A sample of 1000 students from a university was taken and their average weight was found to be 112 pounds with a S.D. of 20 pounds. Could the mean weight of students in the population be 120 pounds? **[Ans.** H_0: Rejected]

12. A sample of 400 male students is found to have a mean height of 160 cms. Can it be reasonably regarded as a sample from a large population with mean height 162.5 cms and standard deviation 4.5 cms? **[Ans.** H_0: Accepted]

13. A random sample of 200 measurements from a large population gave a mean value of 50 and a S.D. of 9. Determine 95% confidence interval for the mean of population?

[Ans. 48.8 and 51: 2]

14. The guaranteed average life of certain type of bulbs is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. What must be the minimum size of the sample? **[Ans.** n = 4]

15. The heights of college students in a city are normally distributed with S.D. 6 cms. A sample of 1000 students has mean height 158 cms. Test the hypothesis that the mean height of college students in the city is 160 cms.

[Ans. H_0: Rejected at both level 1% and 5%]

16. A normal population has a mean of 0.1 and standard deviation of 2.1. Find the probability that mean of a sample of size 900 will be negative? **[Ans.** 0.0764]

17. Intelligence tests on two groups of boys and girls gave the following results. Examine if the difference is significant.

	Mean	*S.D.*	*Size*
Girls	70	10	70
Boys	75	11	100

[Ans. Not a significant difference]

18. Two random samples of sizes 1000 and 2000 farms gave an average yield of 2000 kg and 2050 kg respecitvely. The variance of wheat farms in the country may be taken as 100 kg. Examine whether the two samples differ significantly in yield.

[Ans. Highly significant]

19. A random sample of 200 measurements from a large population gave a mean value of 50 and S.D. of 9. Determine the 95% confidence interval for the mean of the population?

[Ans. 49.58, 50.41]

20. The means of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms? **[Ans.** Not significant]

21. A sample of heights of 6400 soldiers has a mean of 67.85 inches and a S.D. of 2.56 inches. While another sample of heights of 1600 sailors has a mean of 68.55 inches with S.D. of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers? **[Ans.** Highly significant]

22. The yield of wheat in a random sample of 1000 farms in a certain area has a S.D. of 192 kg. Another random sample of 1000 farms gives a S.D. of 224 kg. Are the standard deviations significantly different?

[Ans. $Z = 4.851$ and standard deviations are significantly different]

12.7 TEST OF SIGNIFICANCE FOR SMALL SAMPLES

Generally when the size of the sample is less than 30, it is called small sample. For small sample size we use *t*-test, *f*-test, *z*-test and chi-square (χ^2) test for testing of hypothesis. Chi-square test is flexible for small sample size problem as well as large sample size.

For small sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

12.7.1 Chi-Square (χ^2) Test

χ^2 test is one of the simplest and general known test. It is applicable to a very large number as well as small number of problems in general practice under the following headings.

(*i*) As a test of goodness of fit.

(*ii*) As a test of independence of attributes.

(*iii*) As a test of homogenity of independent estimates of the population variance.

(*iv*) As a test of the hypothetical value of the population variance σ^2.

(*v*) To test the homogeneity of independent estimates of the population correlation coefficient.

The quantity χ^2 describes the magnitude of discrepancy between theory and observations. If $\chi = 0$, the expected and the observed frequencies completely coincide.

The greater the discrepancy between the observed and expected frequencies, the greater is the value of χ^2. Thus χ^2 **affords a measure of the correspondence between theory and observation**.

If O_i ($i = 1, 2, \ldots\ldots., n$) is a set of observed (experimental) frequencies and E_i ($i = 1, 2,\ldots., n$) is the corresponding set of expected (theoretical or hypothetical) frequencies, then, χ^2 **is defined as**

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

where $\Sigma O_i = \Sigma E_i = N$ (total frequency) and degrees of freedom (*d.f.*) = $(n - 1)$.

Remarks

(*i*) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.

(*ii*) If $\chi^2 > 0$, they do not agree exactly.

Degrees of Freedom (*d.f.*): The number of independent variates which make up the statistic χ^2 is known as the degrees of freedom (*d.f.*) and is denoted by ν (Greek alphabet Nu).

In other way, the number of degrees of freedom, is the total number of observations less the number of independent constraints imposed on the observations.

i.e., $\nu = n - k$

where n = no. of observations

k = the number of independent constraints in a set of data of n observations.

Thus in a set of n observations the *d.f.* for χ^2 are $(n - 1)$ generally, one *d.f.* being lost because of linear constraints.

$$\sum_i O_i = \sum_i E_i = N, \text{ on the frequencies.}$$

For a $p \times q$ contingency table, $\nu = (p - 1)(q - 1)$; where (p columns and q rows)

Also, in case of a contingency table, the expected frequency of any class

$$= \frac{\text{Total of rows in which it occurs} \times \text{Total of columns in which it occurs}}{\text{Total no. of observations}}$$

Conditions For the Validity of χ^2 Test: χ^2 test is an approximate test for large values of n. For the validity of chi-square test of 'goodness of fit' between theory and experiment, the following conditions must be satisfied.

1. The sample observations should be independent.
2. The constraints on the cell frequencies, if any, should be linear. *e.g.*

$$\Sigma O_i = \Sigma E_i$$

3. N, the total number of frequencies should be large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we can say that ***N* should be atleast 50,** however, few the cells.

4. No theoretical cell-frequency should be small. Also it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and **10 is better.** If small theoretical frequencies occur (*i.e.*, < 10), the difficulty is overcome by grouping two or more classes together before calculating $(O - E)$. **It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.**

5. χ^2 test depends only on the set of observed and expected frequencies and on *d.f.* It does not make any assumptions regarding the parent population from which the observations are

taken. Since χ^2 does not involve any population parameters it is termed as a statistic and the test is known as Non-parametric test or Distribution-Free test.

Remark: The probability function of χ^2 distribution is given by

$$f(\chi^2) = c\left(\chi^2\right)^{\left(\nu/2 - 1\right)} \cdot e^{-x^2/2}$$

where e = 2.71828,

ν = degree of freedom

c = a constrant depending only on ν.

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as the chi-square distribution.

If the data is given in a series of "n" numbers then degrees of freedom = n –1

In the case of Binomial distribution $d.f.$ = n – 1

In the case of Poisson distribution $d.f.$ = n – 2

In the case of Normal distribution $d.f.$ = n – 3.

(*i*) **Chi-Square test For Population Variance:** Under the null hypothesis that the population variance is $\sigma^2 = \sigma_0^2$ the statistic

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{\sigma_0^2}$$

$$= \frac{1}{\sigma_0^2}\left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\Sigma x_i\right)^2}{n}\right]$$

$$\chi^2 = \frac{nS^2}{\sigma_0^2}$$

follows chi-square distribution with (n –1) $d.f.$

This test can be applied only if the population from which sample is drawn is normal.

If the sample size n is large (n > 30) then we can use Fisher's approximation

i.e., $$Z = \sqrt{2\chi^2} - \sqrt{2n-1}$$

and apply Normal test.

Example 1. *Test the hypothesis that σ = 10, given that S = 15 for a random sample of size 50 from a normal population.*

Sol. *Null Hypothesis,*

$$H_0: \sigma = 10$$

We are given n = 50, S = 15

$$\therefore \quad \chi^2 = \frac{nS^2}{\sigma^2}$$

$$= \frac{50 \times 225}{100}$$

$$= 112.5$$

Since n is large, the test statistic is

$$Z = \sqrt{2\chi^2} - \sqrt{2n-1} \sim N(0, 1)$$

Now, $$Z = \sqrt{225} - \sqrt{99} = 15 - 9.95 = 5.05$$

Since $|Z| > 3$, it is significant at all levels of significance and hence H_0 is rejected and we conclude that $\sigma \neq 10$.

Example 2. *It is believed that the precision (as measured by the variance of an instrument is no more than 0.16. Write down the null and alternative hypothesis for testing this belief. Carry out the test at 1% level, given 11 measurements of the same subject on the instrument:*

2.5, 2.3, 2.4, 2.3, 2.5, 2.7, 2.5, 2.6, 2.6, 2.7, 2.5

[B.U. (2006), Kanpur (2007)]

Sol. *Null Hypothesis*, H_0: $\sigma^2 = 0.16$

Alternative Hypothesis, H_1: $\sigma^2 > 0.16$

Computation of Sample Variance

X	$X - \overline{X}$	$(X - \overline{X})^2$
2.5	− 0.01	0.0001
2.3	− 0.21	0.0441
2.4	− 0.11	0.0121
2.3	− 0.21	0.0441
2.5	− 0.01	0.0001
2.7	+ 0.19	0.0361
2.5	− 0.01	0.0001
2.6	+ 0.09	0.0081
2.6	+ 0.09	0.0081
2.7	+ 0.19	0.0361
2.5	− 0.01	0.0001
$\overline{X} = \frac{27.6}{11} = 2.51$		$\Sigma(X - \overline{X})^2 = 0.1891$

Under the null hypothesis H_0: σ^2=0.16, the test statistic is:

$$\chi^2 = \frac{nS^2}{\sigma^2} = \frac{\Sigma(X-\overline{X})^2}{\sigma^2} = \frac{0.1891}{0.16} = 1.182$$

which follows χ^2-distribution with *d.f.* $(11 - 1) = 10$.

Since the calculated value of χ^2 is less than the tabulated value 23.2 of χ^2 for 10 *d.f.* at 1% level of significance, it is not significant. Hence H_0 may be accepted and we conclude that the data are consistent with the hypothesis that the precision of the instrument is 0.16.

(*ii*) **Chi-Square Test of Goodness of Fit:** χ^2 test is an approximate test for large values of *n*. χ^2 test enables us to ascertain how well the theoretical distributions fit empirical distributions or distribution obtained from sample data. If the calculated value of chi-square is less than the table value at a specified level of significance the fit is considered to be good. Generally we take significance at 5% level. Similarly if the calculated value of χ^2 is greater than the table value, the chi-square fit is considered to be poor.

Example 3. *The following table shows the distribution of digits in numbers chosen at random from a telephone directory:*

Digits	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Frequency	*1026*	*1107*	*997*	*996*	*1075*	*933*	*1107*	*972*	*964*	*853*

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Null Hypothesis H_0: The digits taken in the directory occur equally frequently. Therefore there is no significant difference between the observed and expected frequency.

Under H_0, the expected frequency is given by $= \dfrac{10{,}000}{10} = 1000.$

To find the value of χ^2

O_i	1026	1107	997	996	1075	1107	933	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
$(O_i - E_i)^2$	676	11449	9	1156	5625	11449	4489	784	1296	21609

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{58542}{1000} = 58.542.$$

Conclusion. The tabulated value of χ^2 at 5% level of of significance for 9 *d.f.* is 16.919. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

i.e., there is significant difference between the observed and theoretical frequency.

i.e., the digits taken in the directory do not occur equally frequently.

Example 4. *The following table gives the number of aircraft accidents that occurs during the various days of the week. Find whether the accidents are uniformly distributed over the week*

Days	...	*Sun.*	*Mon.*	*Tues.*	*Wed.*	*Thus.*	*Fri.*	*Sat.*
No. of accidents	...	*14*	*16*	*8*	*12*	*11*	*9*	*14*

(Given: The values of chi-square significant at 5, 6, 7, d.f. are respecitvely 11.07.,12.59, 14.07 at the 5% level of significance.

Sol. Here we set up the null hypothesis that the accidents are uniformly distributed over the week.

Under the null hypothesis, the expected frequencies of the accidents on each of the days would be:

Days	...	Sun.	Mon.	Tues.	Wed.	Thus.	Fri.	Sat.	Total
No. of accidents	...	12	12	12	12	12	12	12	84

$$\chi^2 = \frac{(14-12)^2}{12} + \frac{(16-12)^2}{12} + \frac{(8-12)^2}{12} + \frac{(12-12)^2}{12} + \frac{(11-12)^2}{12} + \frac{(9-12)^2}{12} + \frac{(14-12)^2}{12}$$

$$= \frac{1}{12}(4 + 16 + 16 + 0 + 1 + 9 + 4) = \frac{50}{12}$$

$$= 4.17$$

The number of degrees of freedom

= Number of observations – Number of independent constraints.

= 7 – 1 = 6

The tabulated $\chi^2_{0.05}$ for *6 d.f.* = 12.59

Since the calculated χ^2 is much less than the tabulated value, it is highly insignificant and we accept the null hypothesis. Hence we conclude that the accidents are uniformly distributed over the week.

Example 5. *Records taken of the number of male and female births in 800 families having four children are as follows:*

No. of male births	*0*	*1*	*2*	*3*	*4*
No. of female births	*4*	*3*	*2*	*1*	*0*
No. of families	*32*	*178*	*290*	*236*	*94*

Test whether the data are consistent with the hypothesis that the Binomial law holds and the chance of male birth is equal to that of female birth, namely p = q = 1/2.

Sol. H_0: The data are consistent with the hypothesis of equal probability for male and female births, *i.e.*, $p = q = 1/2$.

We use Binomial distribution to calculate theoretical frequency given by:

$$N(r) = N \times P(X = r)$$

where N is the total frequency. $N(r)$ is the number of families with r male children:

$$P(X = r) = {}^nC_r\, p^r q^{n-r}$$

where p and q are probability of male and female births, n is the number of children.

$$N(0) = \text{No. of families with 0 male children} = 800 \times {}^4C_0 \left(\frac{1}{2}\right)^4 = 800 \times 1 \times \frac{1}{2^4} = 50$$

$$N(2) = 800 \times {}^4C_1 \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = 200;\ N(2) = 800 \times {}^4C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 300$$

$$N(4) = 800 \times {}^4C_3 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(4) = 800 \times {}^4C_4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 50$$

Observed frequency O_i	32	178	290	236	94
Expected frequency E_i	50	200	300	200	50
$(O_i - E_i)^2$	324	484	100	1296	1936
$\frac{(O_i - E_i)^2}{E_i}$	6.48	2.42	0.333	6.48	38.72

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 54.433$$

Conclusion. Table value of χ^2 at 5% level of significance for 5 – 1 = 4 *d.f.* is 9.49.

Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

i.e., the data are not consistent with the hypothesis that the Binomial law holds and that the chance of a male birth is not equal to that of a female birth.

Since the fitting is Binomial, the degrees of freedom $\nu = n - 1$ *i.e.*, $\nu = 5 - 1 = 4$

Example 6. *A survey of 320 families with 5 children each revealed the following distribution:*

No. of boys	5	4	3	2	1	0
No. of girls	0	1	2	3	4	5
No. of families	14	56	110	88	40	12

Is this result consistent with the hypothesis that male and female births are equally probable ?

Sol. Let us set up the *null hypothesis* that *the data are consistent with the hypothesis of equal probability for male and female births*. Then under the null hypothesis:

$$p = \text{Probability of male birth} = \frac{1}{2} = q$$

$$p(r) = \text{Probability of 'r' male births in a family of 5}$$

$$= \binom{5}{r} p^r q^{5-r} = \binom{5}{r}\left(\frac{1}{2}\right)^5$$

The frequency of r male births is given by:

$$f(r) = N.\, p(r) = 320 \times \binom{5}{r} \times \left(\frac{1}{2}\right)^5$$

$$= 10 \times \binom{5}{r} \quad ...(1)$$

Substituting $r = 0, 1, 2, 3, 4$ successively in (1), we get the expected frequencies as follows :

$$f(0) = 10 \times 1 = 10, \quad f(1) = 10 \times {}^5C_1 = 50$$
$$f(2) = 10 \times {}^5C_2 = 100, \quad f(3) = 10 \times {}^5C_3 = 100$$
$$f(4) = 10 \times {}^5C_4 = 50, \quad f(5) = 10 \times {}^5C_5 = 10$$

Calculations for χ^2

Observed Frequencies (O)	*Expected Frequencies (E)*	$(O - E)^2$	$(O - E)^2/E$
14	10	16	1.6000
56	50	36	0.7200
110	100	100	1.0000
88	100	144	1.4400
40	50	100	2.0000
12	10	4	0.4000
Total 320	320		7.1600

$$\therefore \qquad \chi^2 = \sum\left[\frac{(O-E)^2}{E}\right] = 7.16$$

Tabulated $\chi^2_{0.05}$ for $6 - 1 = 5$ *d.f.* is 11.07.

Calculated value of χ^2 is less than the tabulated value, it is not significant at 5% level of significance and hence the null hypothesis of equal probability for male and female births may be accepted.

Example 7. *Fit a Poisson distribution to the following data and test the goodness of fit:*

X:	*0*	*1*	*2*	*3*	*4*	*5*	*6*
f:	*275*	*72*	*30*	*7*	*5*	2	1

Sol. Mean of the given distribution is:

$$\overline{X} = \frac{\sum_i f_i x_i}{N} = \frac{189}{392} = 0.482$$

In order to fit a Poisson distribution to the given data, we take the mean (parameter) m of the Poisson distribution equal to the mean of the given distribution, *i.e.,* we take

$$m = \overline{X} = 0.482$$

The frequency of r successes is given by the Poisson law as:

$$f(r) = Np(r) = 392 \times \frac{e^{-0.482}(0.482)^r}{r!}\ ; \ r = 0, 1, 2, \ldots\ldots, 6$$

Now,
$$f(0) = 392 \times e^{-0.482} = 392 \times \text{Antilog } [-\ 0.482 \log e]$$
$$= 392 \times \text{Antilog } [-\ 0.482 \times \log 2.7183] \qquad [\because e = 2.7183]$$

$$= 392 \times \text{Antilog}\ [-\ 0.482 \times 0.4343]$$
$$= 392 \times \text{Antilog}\ [-\ 0.2093]$$
$$= 392 \times \text{Antilog}\ [\bar{1}.7907] = 392 \times 0.6176 = 242.1$$
$$f(1) = m \times f(0) = 0.482 \times 242.1 = 116.69$$
$$f(2) = \frac{m}{2} \times f(1) = 0.241 \times 116.69 = 28.12$$
$$f(3) = \frac{m}{3} \times f(2) = \frac{0.482}{3} \times 28.12 = 4.518$$
$$f(4) = \frac{m}{4} \times f(3) = \frac{0.482}{4} \times 4.518 = 0.544$$
$$f(5) = \frac{m}{5} \times f(4) = \frac{0.482}{5} \times 0.544 = 0.052$$
$$f(6) = \frac{m}{6} \times f(5) = \frac{0.482}{6} \times 0.052 = 0.004$$

Hence the theoretical Poisson frequencies correct to one decimal place are as given below:

X	0	1	2	3	4	5	6	Total
Expected Frequency	242.1	116.1	28.1	4.5	0.5	0.1	0	392

CALCULATIONS FOR CHI-SQUARE

Observed Frequency (O)	*Expected Frequency (E)*	*(O – E)*	*$(O - E)^2$*	*$(O - E)^2/E$*
275	242.1	32.9	1082.41	4.471
72	116.7	44.7	1998.09	17.121
30	28.1	1.9	3.61	0.128
7, 5, 2, 1 } 15	4.5, 0.5, 0.1, 0 } 5.1	9.9	98.01	19.217
392	392.0			40.937

$$\therefore \qquad \chi^2 = \frac{\Sigma(O-E)^2}{E} = 40.937$$

degree of freedom = 7 – 1 – 1 – 3 = 2

Tabulated value of χ^2 for 2 degree of freedom at 5% level of significance is 5.99.

Conclusion: Since calculated value of χ^2 (40.937) is much greater than 5.99, it is therefore highly significant. Hence we say that poisson distribution is not a good fit to the given data.

Example 8. *A die is thrown 270 times and the results of these throws are given below:*

No. appeared on the die	1	2	3	4	5	6
Frequency	40	32	29	59	57	59

Test whether the die is biased or not.

Sol. Null Hypothesis H_0: Die is unbiased.

Under this H_0, the expected frequencies for each digit is $\frac{276}{6} = 46$.

To find the value of χ^2,

O_i	40	32	29	59	57	59
E_i	46	46	46	46	46	46
$(O_i - E_i)^2$	36	196	289	169	121	169

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{980}{46} = 21.30.$$

Conclusion: Tabulated value of χ^2 at 5% level of significance for (6 – 1= 5) *d.f.* is 11.09. Since the calculated value of $\chi^2 = 21.30 > 11.07$ the tabulated value, H_0 is rejected.

i.e., die is not unbiased or die is biased.

Example 9. *The theory predicts the proportion of beans in the four groups, G_1, G_2, G_3, G_4 should be in the ratio 9: 3: 3: 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory.*

Sol. H_0: The experimental result support the theory, *i.e.,* there is no significant difference between the observed and theoretical frequency under H_0, the theoretical frequency can be calculated as follows:

$$E(G_1) = \frac{1600 \times 9}{16} = 900;$$

$$E(G_2) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_3) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_4) = \frac{1600 \times 1}{16} = 100.$$

To calculate the value of χ^2

Observed frequency O_i	882	313	287	118
Expected frequency E_i	900	300	300	100
$\frac{(O_i - E_i)^2}{E_i}$	0.36	0.5633	0.5633	3.24

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 4.7266.$$

Conclusion: Table value of χ^2 at 5% level of significance for 3 *d.f.* is 7.815. Since the calculated value of χ^2 is less than that of the tabulated value. Hence H_0 is accepted *i.e.,* the experimental result support the theory.

(*iii*) **χ^2 test as a test of Attributes:** Let us consider two attributes A and B, A divided into r classes $A_1, A_2,\ldots\ldots\ldots, A_r$ and B divided into S classes $B_1, B_2, \ldots\ldots\ldots B_S$, such a classification in which attributes are divided into more than two classes is known as manifold classification. The various cell frequencies can be expressed in the following table known as $r \times s$ manifold contingency table. Here (A_i) is the number of persons possessing the attributes and (B_j) is the number of persons possessing the attributes (B_j) and (A_iB_j) is the number of persons possessing both the attributes

A_i and B_j for $[i = 1, 2, \ldots.., r; j = 1, 2, \ldots\ldots..S]$

$$\sum_{i=1}^{r} A_i = \sum_{j=1}^{s} B_j = N, \text{ is the total frequency.}$$

The contingency table for $r \times s$ is given below:

A \ B	A_1	A_2	A_3	$...A_r$	Total
B_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	$...(A_1B_1)$	B_1
B_2	(A_1B_2)	(A_2B_2)	(A_3B_2)	$...(A_rB_2)$	B_2
B_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	$...(A_rB_3)$	B_3
....	....	....	....	....	....
....	....	....	....	....	....
B_s	(A_1B_s)	(A_2B_s)	(A_3B_s)	$...(A_rB_s)$	(B_s)
Total	(A_1)	(A_2)	(A_3)	$...(A_r)$	N

The problem is to test if two attributes A and B under consideration are independent or not.

Under the null hypothesis, both the attributes are independent, the theoretical cell frequencies are calculated as follows.

$P(A_i)$ = Probability that a person possesses the attribute $A_i = \dfrac{(A_i)}{N}$ $i = 1, 2, \ldots, r$

$P(B_j)$ = Probability that a person possesses the attribute $B_j = \dfrac{(B_j)}{N}$

$P(A_iB_j)$ = Probability that a person possesses both attributes A_i and $B_j = \dfrac{(A_iB_j)}{N}$

If $(A_iB_j)_0$ is the expected number of persons possessing both the attributes A_i and B_j

$$(A_iB_j)_0 = N.P\ (A_iB_j) = NP\ (A_i)(B_j)$$

$$= N\ \frac{(A_i)}{N}\frac{(B_j)}{N} = \frac{(A_i)(B_j)}{N}$$

(Since A and B are independent)

Therefore $$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\frac{\left[(A_iB_j)-(A_iB_j)_0\right]^2}{(A_iB_j)_0}$$

which is distributedd as a χ^2 variate with $(r-1)(S-1)$ *d.f.*

Some Remarkable points:

1. For a 2 × 2 contingency table where the frequencies are $\dfrac{a/b}{c/d}$, χ^2 can be calculated from independent frequencies as $\chi^2 = \dfrac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(b+d)(a+c)}$

2. If the contingency table is not 2 × 2, then the above formula for calculating χ^2 cannot be used. Hence, we have another formula for calculating the expected frequency $(A_iB_j)_0$ $= \dfrac{(A_i)(B_j)}{N}$ *i.e.*, expected frequency in each cell is $= \dfrac{\text{Product of column total and row total}}{\text{whole total}}$

3. If $\dfrac{a/b}{c/d}$ is the 2 × 2 contingency table with two attributes, $Q = \dfrac{ad-bc}{ad+bc}$ is called the coefficient of association. If the attributes are independent then $\dfrac{a}{b} = \dfrac{c}{d}$.

Remark: Yates's Correction: In a 2 × 2 table, if the frequencies of a cell is small, we make Yates's correction to make χ^2 continuous.

Decrease by $\dfrac{1}{2}$ those cell frequencies which are greater than expected frequencies, and increase by $\dfrac{1}{2}$ those which are less than expectation. This will not affect the marginal columns. This correction is known as Yates's correction to continuity.

After Yates's correction $$\chi^2 = \frac{N\left(bc - ad - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when} \quad ad - bc < 0$$

$$\chi^2 = \frac{N\left(ad - bc - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when} \quad ad - bc > 0$$

Example 10. *(2 × 2 contingency table). For the 2 × 2 table,*

a	b
c	d

prove that chi-square test of independence gives

$$\chi^2 = \frac{N(ad-bc)^2}{(a+c)(b+d)(a+b)(c+d)}, \; N = a + b + c + d.... \qquad ...(1)$$

[Guwahati Univ. B.Sc., 2002]

Sol. Under the hypothesis of independence of attributes,

$$E(a) = \frac{(a+b)(a+c)}{N}$$

$$E(b) = \frac{(a+b)(b+d)}{N}$$

$$E(c) = \frac{(a+c)(c+d)}{N}$$

and $$E(d) = \frac{(b+d)(c+d)}{N}$$

a	b	$a+b$
c	d	$c+d$
$a+c$	$b+d$	N

$\therefore$ $$\chi^2 = \frac{[a-E(a)]^2}{E(a)} + \frac{[b-E(b)]^2}{E(b)} + \frac{[c-E(c)]^2}{E(c)} + \frac{[d-E(d)]^2}{E(d)} \qquad ...(2)$$

$$a - E(a) = a - \frac{(a+b)(a+c)}{N}$$

$$= \frac{a(a+b+c+d) - (a^2 + ac + ab + bc)}{N} = \frac{ad - bc}{N}$$

Similarly, we will get

$$b - E(b) = -\frac{ad - bc}{N} = c - E(c);\ d - E(d) = \frac{ad - bc}{N}$$

Substituting in (2), we get

$$\chi^2 = \frac{(ad - bc)^2}{N^2}\left[\frac{1}{E(a)} + \frac{1}{E(b)} + \frac{1}{E(c)} + \frac{1}{E(d)}\right]$$

$$= \frac{(ad - bc)^2}{N}\left[\left\{\frac{1}{(a+b)(a+c)} + \frac{1}{(a+b)(b+d)}\right\} + \left\{\frac{1}{(a+c)(c+d)} + \frac{1}{(b+d)(c+d)}\right\}\right]$$

$$= \frac{(ad - bc)^2}{N}\left[\frac{b+d+a+c}{(a+b)(a+c)(b+d)} + \frac{b+d+a+c}{(a+c)(c+d)(b+d)}\right]$$

$$= (ad - bc)^2\left[\frac{c+d+a+b}{(a+b)(a+c)(b+d)(c+d)}\right]$$

$$= \frac{N(ad - bc)^2}{(a+b)(a+c)(b+d)(c+d)}$$

Example 11. *From the following table regarding the colour of eyes of father and son test if the colour of son's eye is associated with that of the father.*

Eye colour of son

		Light	*Not light*
Eye colour of father	*Light*	471	51
	Not light	148	230

Sol. *Null Hypothesis* H_0 : The colour of son's eye is not associated with that of the father *i.e.*, they are independent.

Under H_0, we calculate the expected frequency in each cell as

$$= \frac{\text{Product of column total and row total}}{\text{whole total}}$$

Expected frequencies are:

Eye colour of father \ *Eye colour of son*	*Light*	*Not light*	*Total*
Light	$\frac{619 \times 522}{900} = 359.02$	$\frac{289 \times 522}{900} = 167.62$	522
Not Light	$\frac{619 \times 378}{900} = 259.98$	$\frac{289 \times 378}{900} = 121.38$	378
Total	619	289	900

$$\chi^2 = \frac{(471-359.02)^2}{359.02} + \frac{(51-167.62)^2}{167.62} + \frac{(148-259.98)^2}{259.98} + \frac{(230-121.38)^2}{121.38}$$

$$= 261.498.$$

Conclusion: At 5% level for 1 *d.f.*, χ^2 is 3.841 (tabulated value)

Since tabulated value of χ^2 < calculated value of χ^2. Hence H_0 is rejected.

Example 12. *The following table gives the number of good and bad parts produced by each of the three shifts in a factory:*

	Good parts	*Bad parts*	*Total*
Day shift	*960*	*40*	*1000*
Evening shift	*940*	*50*	*990*
Night shift	*950*	*45*	*995*
Total	*2850*	*135*	*2985*

Test whether or not the production of bad parts is independent of the shift on which they were produced.

Sol. *Null Hypothesis* H_0 : The production of bad parts is independent of the shift on which they were produced.

i.e., the two attributes, production and shifts are independent.

Under H_0,
$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{3} \left[\frac{\left[(A_iB_j)_0 - (A_iB_j) \right]^2}{(A_iB_j)_0} \right]$$

Calculation of expected frequencies

Let A and B be the two attributes namely production and shifts. A is divided into two classes A_1, A_2 and B is divided into three classes B_1, B_2, B_3.

$$(A_1B_1)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850)\times(1000)}{2985} = 954.77;$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850)\times(990)}{2985} = 945.226$$

$$(A_1B_3)_0 = \frac{(A_1)(B_3)}{N} = \frac{(2850)\times(995)}{2985} = 950;$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{(135)\times(1000)}{2985} = 45.27$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{(135)\times(990)}{2985} = 44.773;$$

$$(A_2B_3)_0 = \frac{(A_2)(B_3)}{N} = \frac{(135)(995)}{2985} = 45.$$

To calculate the value of χ^2.

Class	O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
(A_1B_1)	960	954.77	27.3529	0.02864
(A_1B_2)	940	954.226	27.3110	0.02889
(A_1B_3)	950	950	0	0
(A_2B_1)	40	45.27	27.7729	0.61349
(A_2B_2)	50	44.773	27.3215	0.61022
(A_2B_3)	45	45	0	0
				1.28126

Conclusion: The tabulated value of χ^2 at 5% level of significance for 2 degrees of freedom $(r - 1)(s - 1)$ is 5.991. Since the calculated value of χ^2 is less than the tabulated value, we accept H_0. *i.e.,* the production of bad parts is independent of the shift on which they were produced.

12.7.2 Student's *t*-distribution

The *t*-distribution is used when sample size is less than equal to 30 (≤ 30) and the population standard deviation is unknown.

Let X_i, $i = 1, 2, \ldots\ldots\ldots, n$ be a random sample of size n from a normal population with mean μ and variance σ^2. Then student's t is defined by

$$t = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t\ (n - 1\ d.f.)$$

where $$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ is the sample mean}$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

is an unbiased estimate of the population variance σ^2.

The t-distribution has different values for each *d.f.* and when the *d.f.* are infinitely large, the t-distribution is equivalent to normal distribution.

Example 13. *The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5 ?*

Sol. H_0: $\mu = 47.5$

i.e., there is no significant difference between the sample and population mean.

H_1: $\mu \neq 47.5$ (two tailed test): Given: $n = 9$, $\mu = 47.5$

X	45	47	50	52	48	47	49	53	51
$X - \overline{X}$	– 4.1	– 2.1	0.9	2.9	–1.1	–2.1	–0.1	3.9	1.9
$\left(X - \overline{X}\right)^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\overline{X} = \frac{\Sigma x}{n} = \frac{442}{9} = 49.11; \; \Sigma\left(X - \overline{X}\right)^2 = 54.89;$$

$$s^2 = \frac{\Sigma\left(X - \overline{X}\right)^2}{(n-1)} = 6.86 \qquad \therefore\ s = 2.619$$

Applying t-test $$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{49.1 - 47.5}{2.619/\sqrt{8}} = \frac{(1.6)\sqrt{8}}{2.619} = 1.7279$$

$$t_{0.05} = 2.31 \text{ for } \gamma = 8.$$

Conclusion: Since $|t| < t_{0.05}$, the hypothesis is accepted *i.e.,* there is no significant difference between their mean.

Example 14. *A random sample of 10 boys had the following I. Q'. s: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q. of 100 ? Find a reasonable range in which most of the mean I.Q. values of samples of 10 boys lie.*

Sol. Null hypothesis, H_0: The data are consistent with the assumption of a mean I.Q. of 100 in the population, *i.e.,* $\mu = 100$.

Alternative hypothesis: $H_1 : \mu \neq 100$

Test Statistic. Under H_0, the test statistic is:

$$t = \frac{(\bar{x} - \mu)}{\sqrt{S^2/n}} \sim t_{(n-1)}$$

where $\bar{x}$ and S^2 are to be computed from the sample values of I.Q.'s.

Calculation for Sample Mean and S.D.

X	$(X - \bar{x})$	$(X - \bar{x})^2$
70	–27.2	739.84
120	22.8	519.84
110	12.8	163.84
101	3.8	14.44
88	–9.2	84.64
83	–14.2	201.64
95	–2.2	4.84
98	0.8	0.64
107	9.8	96.04
100	2.8	7.84
Total 972		1833.60

Hence $n = 10,\ \bar{x} = \frac{972}{10} = 97.2$ and $S^2 = \frac{1833.60}{9} = 203.73$

$$\therefore \qquad |t| = \frac{|97.2 - 100|}{\sqrt{203.73/10}} = \frac{2.8}{\sqrt{20.37}} = \frac{2.8}{4.514} = 0.62$$

Tabulated $t_{0.05}$ for (10 – 1) *i.e.,* 9 *d.f.* for two-tailed test is 2.262.

Conclusion: Since calculated t is less than tabulated $t_{0.05}$ for 9 *d.f.*, H_0 may be accepted at 5% level of significance and we may conclude that the data are consistent with the assumption of mean I.Q. of 100 in the population.

The 95% confidence limits within which the mean I.Q. values of samples of 10 boys will lie are given by

$$\bar{x} \pm t_{0.05}\ S/\sqrt{n} = 97.2 \pm 2.262 \times 4.514 = 97.2 \pm 10.21 = 107.41 \text{ and } 86.99$$

Hence the required 95% confidence intervals is [86.99, 107.41]

Example 15. *The mean weekly sales of soap bars in departmental stores was 146.3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a standard deviation of 17.2. Was the advertising campaign successful?*

Sol. We are given: $n = 22,\ \bar{x} = 153.7,\ s = 17.2.$

Null Hypothesis: The advertising campaign is not successful, *i.e.,*

$$H_0: \mu = 146.3$$

Alternative Hypothesis: H_1: $\mu > 146.3$. (Right-tail).

Test Statistic: Under the null hypothesis, the test statistic is:

$$t = \frac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{22-1} = t_{21}$$

Now $$t = \frac{153.7 - 146.3}{\sqrt{(17.2)^2/21}} = \frac{7.4 \times \sqrt{21}}{17.2} = 9.03$$

Conclusion: Tabulated value of t for 21 *d.f.* at 5% level of significance for single-tailed test is 1.72. Since calculated value is much greater than the tabulated value, therefore it is highly significant. Hence we reject the null hypothesis.

Example 16. *A machinist is making engine parts with axle diameters of 0.700 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.040 inch. Compute the statistic you would use to test whether the work is meeting the specifications. Also state how you would proceed further.*

Sol. Here we are given:

$$\mu = 0.700 \text{ inches}, \ \bar{x} = 0.742 \text{ inches}, \ s = 0.040 \text{ inches and } n = 10$$

Null Hypothesis, H_0: $\mu = 0.700$, *i.e.*, the product is conforming to specifications.

Alternative Hypothesis, H_1: $\mu \neq 0.700$

Test Statistic : Under H_0, the test statistic is:

$$t = \frac{\bar{x} - \mu}{\sqrt{s^2/n}} = \frac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{(n-1)}$$

Now,

$$t = \frac{\sqrt{9}(0.742 - 0.700)}{0.040} = 3.15$$

Here the test statistic 't' follows Student's t-distribution with $10 - 1 = 9$ *d.f.* We will now compare this calculated value with the tabulated value of t for 9 *d.f.* and at certain level of significance, say 5%. Let this tabulated value be denoted by t_0.

(*i*) If calculated 't' viz., $3.15 > t_0$, we say that the value of t is significant. This implies that $\bar{x}$ differs significantly from μ and H_0 is rejected at this level of significance and we conclude that the product is not meeting the specifications.

(*ii*) If calculated $t < t_0$, we say that the value of t is not significant, *i.e.*, there is no significant difference between $\bar{x}$ and μ. In other words, the deviation $(\bar{x} - \mu)$ is just due to fluctuations of sampling and null hypothesis H_0 may be retained at 5% level of significance, *i.e.*, we may take the product conforming to specifications.

Example 17. *A random sample of size 16 has 53 as mean. The sum of squares of the derivation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean ? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. H_0: There is no significant difference between the sample mean and hypothetical population mean.

$$H_0: \mu = 56; \ H_1: \mu \neq 56 \text{ (Two tailed test)}$$

$$t : \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t(n - 1 \ d.f.)$$

Given:

$$\bar{X} = 53, \ \mu = 56, \ n = 16, \ \Sigma(X - \bar{X})^2 = 135$$

$$s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3; \ t = \frac{53 - 56}{3/\sqrt{16}} = -4$$

$$|t| = 4, \ d.f. = 16 - 1 = 15.$$

Conclusion: $t_{0.05} = 1.753$.

Since $|t| = 4 > t_{0.05} = 1.753$ *i.e.,* the calculated value of t is more than the table value. The hypothesis is rejected. Hence, the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$= \overline{X} \pm \frac{s}{\sqrt{n}} t_{0.05} = 53 \pm \frac{3}{\sqrt{16}} (1.725) = 51.706;\ 54.293$$

99% confidence limits of the population mean

$$= \overline{X} \pm \frac{s}{\sqrt{n}} t_{0.01,} = 53 \pm \frac{3}{\sqrt{16}} (2.602) = 51.048;\ 54.951.$$

(*i*) *t*-Test of Significance for Mean of a Random Sample: To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0: There is no significant difference between the sample mean $\bar{x}$ and the population mean μ *i.e.,* we use the statistic.

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} \text{ where } \overline{X} \text{ is mean of the sample.}$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 \text{ with degrees of freedom } (n-1).$$

At given level of significance α_1 and degrees of freedom $(n - 1)$. We refer to t-table t_α (two tailed or one tailed). If calculated t value is such that $|t| < t_\alpha$ the null hypothesis is accepted and for $|t| > t_\alpha$, H_0 is rejected.

(*ii*) *t*-Test For Difference of Means of Two Samples: This test is used to test whether the two samples $x_1, x_2, \ldots\ x_{n_1}, y_1, y_2, \ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1 and μ_2 respectively under the assumption that the population variance are equal. $(\sigma_1 = \sigma_2 = \sigma)$.

H_0: The samples have been drawn from the normal population with means μ_1 and μ_2 *i.e.,* H_0: $\mu_1 \neq \mu_2$.

Let $\overline{X}, \overline{Y}$ be their means of the two samples.

Under this H_0 the test of statistic t is given by $t = \dfrac{\left(\overline{X} - \overline{Y}\right)}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} - t(n_1 + n_2 - 2\ d.f.)$

Also, if the two sample's standard deviations s_1, s_2 are given then we have $s^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

And, if $n_1 = n_2 = n$, $t = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$ can be used as a test statistic.

If the pairs of values are in some way associated (correlated) we can't use the test statistic as given in Note 2. In this case, we find the differences of the associated pairs of values and apply for single mean *i.e.*, $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ with degrees of freedom $n - 1$.

The test statistic is $t = \dfrac{\overline{d}}{s/\sqrt{n}}$ or $t = \dfrac{\overline{d}}{s/\sqrt{n-1}}$, where $\overline{d}$ is the mean of paired difference.

i.e.,
$$d_i = x_i - y_i$$
$$\overline{d}_i = \overline{X} - \overline{Y}, \text{ where } (x_i, y_i) \text{ are the paired data } i = 1, 2, \ldots\ldots, n.$$

Example 18. *Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.*

Sol. $H_0: \mu_1 = \mu_2$, *i.e.*, the means of the two populations are the same.

$H_1: \mu_1 \neq \mu_2$.

Given $\overline{X} = 20.3,\ \overline{X}_2 = 18.6;\ n_1 = 10,\ n_2 = 14,\ s_1 = 3.5,\ s_2 = 5.2$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2} = 22.775. \quad \therefore s = 4.772$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{20.3 - 18.6}{\left(\sqrt{\dfrac{1}{10} + \dfrac{1}{14}}\right)4.772} = 0.8604$$

The value of t at 5% level for 22 *d.f.* is $t_{0.05} = 2.0739$.

Conclusion: Since $|t| = 0.8604 < t_{0.05}$ the hypothesis is accepted *i.e.*, there is no significant difference between their means.

Example 19. *Two samples of sodium vapour bulbs were tested for length of life and the following results were got:*

	Size	*Sample mean*	*Sample S.D.*
Type I	8	*1234 hrs*	*36 hrs*
Type II	7	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalise that Type I is superior to Type II regarding length of life ?

Sol. $H_0: \mu_1 = \mu_2$, *i.e.*, two types of bulbs have same lifetime.

$H_1: \mu_1 > \mu_2$ *i.e.*, type I is superior to type II.

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{8(36)^2 + 7(40)^2}{8+7-2} = 1659.076. \qquad \therefore\ s = 40.7317$$

The t-statistic
$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{1234 - 1036}{40.7317\sqrt{\frac{1}{8} + \frac{1}{7}}} = 18.1480 \sim t(n_1 + n_2 - 2d.f)$$

$t_{0.05}$ at $d.f.$ 13 is 1.77 (one tailed test).

Conclusion: Since calculated $|t| > t_{0.05}$, H_0 is rejected *i.e.,* H_1 is accepted.

$\therefore$ Type I is definitely superior to Type II.

where $\overline{X} = \sum_{i=1}^{n_1} \frac{X_i}{n_i}$, $\overline{Y} = \sum_{j=1}^{n_2} \frac{Y_j}{n_2}$; $s^2 = \frac{1}{n_1 + n_2 - 2}\left[E\left(X_i - \overline{X}\right)^2 + \left(Y_j - \overline{Y}\right)^2\right]$ is an unbiased estimate of the population variance σ^2.

t follows t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Example 20. *The following figures refer to observations in live independent samples:*

Sample I	25	30	28	34	24	20	13	32	22	38
Sample II	40	34	22	20	31	40	30	23	36	17

Analyse whether the samples have been drawn from the populations of equal means.

Sol. H_0: The two samples have been drawn from the population of equal means. *i.e.,* there is no significant difference between their means.

i.e., $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

Given n_1 = Sample I size = 10; n_2 = Sample II size = 10

To calculate the two sample mean and sum of squares of deviation from mean. Let X_1 be the Sample I and X_2 be the Sample II.

X_1	25	30	28	34	24	20	13	32	22	38
$X - \overline{X}_1$	– 1.6	3.4	1.4	7.4	–2.6	–6.6	–13.6	5.4	4.6	11.4
$\left(X_1 - \overline{X}_1\right)^2$	2.56	11.56	1.96	54.76	6.76	43.56	184.96	29.16	21.16	129.96
X_2	40	34	22	20	31	40	30	23	36	17
$X_2 - \overline{X}_2$	10.7	4.7	– 7.3	– 9.3	1.7	10.7	0.7	– 6.3	6.7	–12.3
$\left(X_2 - \overline{X}_2\right)^2$	114.49	22.09	53.29	86.49	2.89	114.49	0.49	39.67	44.89	151.29

$$\overline{X}_1 = \sum_{i=1}^{10} \frac{X_1}{n_1} = 26.6 \quad \overline{X}_2 = \sum_{i=1}^{10} \frac{X_2}{n_2} = \frac{293}{10} = 29.3$$

$$\Sigma\left(X_1 - \overline{X}_1\right)^2 = 486.4 \qquad \Sigma\left(X_2 - \overline{X}_2\right)^2 = 630.08$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\left[\Sigma\left(X_1 - \overline{X}_1\right)^2 + \Sigma\left(X_2 - \overline{X}_2\right)^2\right]$$

$$= \frac{1}{10+10-2}[486.4 + 630.08] = 62.026. \qquad \therefore S = 7.875$$

Under H_0 the test statistic is given by

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{26.6 - 29.3}{7.875\sqrt{\frac{1}{10} + \frac{1}{10}}} = -\,0.7666\ -t(n_1 + n_2 - 2\ d.f)$$

$$|t| = 0.7666.$$

Conclusion: The tabulated value of t at 5% level of significance for 18 *d.f.* is 2.1. Since the calculated value $|t| = 0.7666 < t_{0.05}$. H_0 is accepted.

i.e., there is no significant difference between their means.

i.e., the two samples have been drawn from the populations of equal means.

Applications of *t*-Distribution: The *t*-distribution has a wide number of applications in statistics, some of them are:

1. To test if the sample mean $(\overline{X})$ differs significantly from the hypothetical value μ of the population mean.
2. To test the significance between two sample means.
3. To test the significance of observed partial and mutiple correlation coefficients.
4. To test the significance of an observed sample correlation co-efficient and sample regression coefficient. Also, the critical value or significant value of t at level of significance α and degree of freedom ν for two tailed test are given by

$$P[|t| > t_\nu(\alpha)] = \alpha$$

$$\Rightarrow \quad P[|t| \le t_\nu(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance 'α' for a single tailed test can be obtained from those of two tailed test by considering the values at level of significance '2α'.

12.7.3 Snedecor's Variance Ratio Test or F-test

Suppose we want to test (*i*) whether two independent samples x_i and y_j For $i = 1, 2\ldots.., n_1$ and $j = 1, 2, \ldots., n_2$ have been drawn from the normal populations with the same variance σ^2, (say) or (*ii*) whether two independent estimates of the population variance are homogenous or not.

Under the null hypothesis H_0, (*i*) $\sigma_x^2 = \sigma_y^2 = \sigma^2$ *i.e.*, the population variances are equal or (*ii*) two independent estimates of the population variances are homogeneous, then the statistic F is given by

$$F = \frac{S_x^2}{S_y^2}$$

where

$$S_x^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} \left(x_i - \bar{x}\right)^2$$

$$S_y^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} \left(y_i - \bar{y}\right)^2$$

It follows Snedecor's F-distribution with *d.f.* $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$. Also greater of two variances S_x^2 and S_y^2 is to be taken in the numerator and n_1 corresponds to the greater variance.

The critical values of F for left tail test H_0: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 < \sigma_2^2$ are given by

$$F < F_{n_1-1,\, n_2-1}\,(1-\alpha)$$

and for the two tailed test, H_0: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \neq \sigma_2^2$ are given by

$$F < F_{n_1-1,\, n_2-2}\left(\frac{\alpha}{2}\right) \text{ and } F < F_{n_1-1,\, n_2-2}\left(1-\frac{\alpha}{2}\right).$$

12.7.4 Fisher's Z-test

To test the significance of an observed sample correlation coefficient from an uncorrelated bivariate normal population, t-test is used. But in random sample of size n_i from a normal bivariate population in which $P \neq 0$ it is proved that the distribution of 'r' is by no means normal and in the neighbourhood of $\rho = \pm 1$, its probability curve is extremely skewed even for large n. If $\rho \neq 0$ Fisher's suggested the transformation.

$$Z = \frac{1}{2} \log_e \frac{1+r}{1-r} = \tan h^{-1} r$$

and proved that for small samples, the distribution of Z is approximately normal with mean

$$\zeta_n = \frac{1}{2} \log_e \frac{1+\rho}{1-\rho} = \tan h^{-1} \rho$$

and variance $1/(n - 3)$ and for large values of n, $(n > 50)$ the approximation is very good.

Example 21. *Two independent sample of sizes 7 and 6 had the following values:*

Sample A	*28*	*30*	*32*	*33*	*31*	*29*	*34*
Sample B	*29*	*30*	*30*	*24*	*27*	*28*	

Examine whether the samples have been drawn from normal populations having the same variance.

Sol. H_0: The variance are equal. *i.e.,* $\sigma_1^2 = \sigma_2^2$.

i.e., the samples have been drawn from normal populations with same variance.

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Under null hypothesis, the test statistic $F = \dfrac{s_1^2}{s_2^2}$ $(s_1^2 > s_2^2)$

Computations for s_1^2 and s_2^2

X_1	$X - \bar{X}_1$	$(X_1 - \bar{X}_1)^2$	X_2	$X_2 - \bar{X}_2$	$(X_2 - \bar{X}_2)^2$
28	– 3	9	29	1	1
30	– 1	1	30	2	4
32	1	1	30	2	4
33	2	4	24	– 4	16
31	0	0	27	– 1	1
29	– 2	4	28	0	0
34	3	9			
		28			26

$$\bar{X}_1 = 31,\ n_1 = 7;\ \Sigma(X_1 - \bar{X}_1)^2 = 28$$

$$\bar{X}_2 = 28,\ n_2 = 6;\ \Sigma(X_2 - \bar{X}_2)^2 = 26$$

$$s_1^2 = \frac{\Sigma(X_1 - \bar{X}_1)^2}{n_1 - 1} = \frac{28}{6} = 4.666;\ s_2^2 = \frac{\Sigma(X_2 - \bar{X}_2)^2}{n_2 - 1} = \frac{26}{5} = 5.2$$

$$F = \frac{s_2^2}{s_1^2} = \frac{52}{4.666} = 1.1158. \qquad (\because s_2^2 > s_1^2)$$

Conclusion: The tabulated value of F at $\nu_1 = 6 - 1$ and $\nu_2 = 7 - 1$ *d.f.* for 5% level of significance is 4.39. Since the tabulated value of F is less than the calculated value, H_0 is accepted *i.e.,* there is no significant difference between the variance *i.e.,* the samples have been drawn from the normal population with same variance.

Example 22. *The two random samples reveal the following data:*

Sample no.	*Size*	*Mean*	*Variance*
I	*16*	*440*	*40*
II	*25*	*460*	*42*

Test whether the samples come from the same normal population.

Sol. A normal population has two parameters namely the mean μ and the variance σ^2. To test whether the two independent samples have been drawn from the same normal population, we have to test

(*i*) the equality of means (*ii*) the equality of variance.

Since the *t*-test assumes that the sample variance are equal, we first apply F-test.

F-test: Null hypothesis: $\sigma_1^2 = \sigma_2^2$

The population variance do not differ significantly.

Alternative hypothesis: $\sigma_1^2 \neq \sigma_2^2$

Under the null hypothesis, the test statistic is given by $F = \frac{s_1^2}{s_2^2}$, $(s_1^2 > s_2^2)$

Given: $n_1 = 16,\ n_2 = 25;\ s_1^2 = 40,\ s_2^2 = 42$

$$\therefore \quad F = \frac{s_1^2}{s_2^2} = \frac{\frac{n_1 s_1^2}{n_1 - 1}}{\frac{n_2 s_2^2}{n_2 - 1}} = \frac{16 \times 40}{15} \times \frac{24}{25 \times 42} = 0.9752.$$

Conclusion: The calculated value of F is 0.9752. The tabulated value of F at 16 –1, 25 –1 *d.f.* for 5% level of significance is 2.11.

Since the calculated value is less than that of the tabulated value, H_0 is accepted, *i.e.*, the population variance are equal.

t-test: Null hypothesis: H_0 ; $\mu_1 = \mu_2$ *i.e.*, the population means are equal.

Alternative hypothesis: H_1: $\mu_1 \neq \mu_2$

Given: $n_1 = 16,\ n_2 = 25,\ \overline{X}_1 = 440,\ \overline{X}_2 = 460$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{16 \times 40 + 25 \times 42}{16 + 25 - 2} = 43.333. \quad \therefore s = 6.582$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{440 - 460}{6.582\sqrt{\frac{1}{16} + \frac{1}{25}}} = -9.490 \text{ for } (n_1 + n_2 - 2) \ d.f.$$

Conclusion: The calculated value of $|t|$ is 9.490. The tabulated value of t at 39 *d.f.* for 5% level of significance is 1.96.

Since the calculated value is greater than the tabulated value, H_0 is rejected.

i.e., there is significant difference between means, *i.e.*, $\mu_1 \neq \mu_2$.

Since there is significant difference between means, and no significant difference between variance, we conclude that the samples do not come from the same normal population.

PROBLEM SET 12.2

1. The following table gives the number of accidents that took place in an industry during various days of the week. Test if accidents are uniformly distributed over the week.

Day	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
No. of accidents	14	18	12	11	15	14

[**Ans.** H_0 is accepted]

2. Verify whether Poisson distribution can be assumed from the data given below:

No. of defects	0	1	2	3	4	5
Frequency	6	13	13	8	4	3

[**Ans.** H_0 is accepted; Poisson distribution provides a good fit to the given data]

3. A survey of 320 families with 5 children shows the following distribution.

No. of boys	5	4	3	2	1	0	*Total*
No. of girls	0	1	2	3	4	5	
Families	18	56	110	88	40	8	320

Given that values of χ^2 of 5 *d.f.* are 11.1 and 15.1 at 0.05 and 0.01 significance level respectively, test the hypothesis that male and female births are equally probable.

[**Ans.** H_0 is accepted at 1% level of significance and rejected at 5% level of significance]

4. The following table gives the frequency of occupance of the digits 0, 1, ..., 9 in the last place in four logarithm of numbers 10-99. Examine if there is any peculiarity.

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	6	16	15	10	12	12	3	2	9	5

[**Ans.** No]

5. The sales in a supermarket during a week are given below. Test the hypothesis that the sales do not depend on the day of the week, using a significant level of 0.05.

Days	:	Mon	Tues	Wed	Thurs	Fri	Sat
Sales (in 1000 Rs.)	:	65	54	60	56	71	84

[**Ans.** Accepted at 0.05 significant level]

6. A die is thrown 90 times with the following results:

Face	:	1	2	3	4	5	6	Total
Frequency	:	10	12	16	14	18	20	90

Use χ^2-test to test whether these data are consistent with the hypothesis that die is unbiased. Given $\chi^2_{0.05} = 11.07$ for 5 degrees of freedom.

[**Ans.** Accepted at 0.05 significant level]

7. 4 coins were tossed at a time and this operation is repeated 160 times. It is found that 4 heads occur 6 times, 3 heads occur 43 times, 2 heads occur 69 times, one head occur 34 times. Discuss whether the coin may be regarded as unbiased. [**Ans.** Unbiased]

8. A sample analysis of examination results of 500 students, it was found that 280 students have failed, 170 have secured a third class, 90 have secured a second class and the rest, a first class. Do these figures support the general belief that above categories are in the ratio 4 : 3 : 2 : 1 respectively? [**Ans.** Yes, these figures support]

9. In the accounting department of bank, 100 accounts are selected at random and estimated for errors. The following results were obtained:

No. of errors	:	0	1	2	3	4	5	6
No. of accounts	:	35	40	19	2	0	2	2

Does this information verify that the errors are distributed according to the Poisson probability law? [**Ans.** May be]

10. Fit a Poisson distribution to the following data and best the goodness of fit:

x	:	0	1	2	3	4
f	:	109	65	22	3	1

[**Ans.** Poisson lawfits the data]

11. What are the expected frequencies of 2 × 2 contigency tables given below

(*i*)

a	b
c	d

(*ii*)

2	10
6	6

Ans. (*i*)

$\frac{(a+c)(a+b)}{a+b+c+d}$	$\frac{(b+d)(a+b)}{a+b+c+d}$
$\frac{(a+c)(c+d)}{a+b+c+d}$	$\frac{(b+d)(c+d)}{a+b+c+d}$

(*ii*)

4	8
4	8

12. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below:

		Education		
		Middle	*High school*	*College*
Sex	Male	10	15	25
	Female	25	10	15

Based on this information can you say the education depends on sex. [**Ans.** Yes]

13. The following data is collected on two characters:

	Smokers	*Non smokers*
Literature	83	57
Illiterate	45	68

Based on this information can you say that there is no relation between habit of smoking and literacy. **[Ans.** No]

14. In an experiment on the immunisation of goats from anthrax, the following results were obtained. Derive your inferences on the efficiency of the vaccine.

	Died anthrax	*Survived*
Inoculated with vaccine	2	10
Not inoculated	6	6

[Ans. No]

15. The lifetime of electric bulbs for a random sample of 10 from a large consignment gave the following data:

Item	1	2	3	4	5	6	7	8	9	10
Life in '000' hrs.	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6

Can we accept the hypothesis that the average lifetime of bulb is 4000 hrs ?

[Ans. Accepted]

16. A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units. **[Ans.** H_0 is rejected]

17. The following values gives the lengths of 12 samples of Egyptian cotton taken from a consignment: 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46. **[Ans.** Accepted]

18. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.

[Ans. Rejected]

19. A filling machine is expected to fill 5 kg of powder into bags. A sample of 10 bags gave the following weights: 4.7, 4.9, 5.0, 5.1, 5.4, 5.2, 4.6, 5.1, 4.6 and 4.7. Test whether the machine is working properly. **[Ans.** Accepted]

20. Memory capacity of 9 students was tested before and after a course of meditation for a month. State whether the course was effective or not from the data given below.

Before	10	15	9	3	7	12	16	17	4
After	12	17	8	5	6	11	18	20	3

[Ans. H_0 is accepted]

21. A certain stimulus administered to each of 12 patients resulted in the following increase of blood pressure: 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure?

[**Ans.** H_0 is rejected]

22. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between means of the two samples? [**Ans.** Accepted]

23. The height of 6 randomly chosen sailors in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers. [**Ans.** H_0 is accepted]

❑❑❑

CHAPTER 13

Computer Programming in 'C' Language

13.1 INTRODUCTION

At its most basic level, programming a computer simply means telling it what to do, and this vapid-sounding definition is not even a joke. There are no other truly fundamental aspects of computer programming; everything else we talk about will simply be the details of a particular, usually artificial, mechanism for telling a computer what to do. Sometimes these mechanisms are chosen because they have been found to be convenient for programmers (people) to use; other times they have been chosen because they're easy for the computer to understand. The first hard thing about programming is to learn, become comfortable with, and accept these artificial mechanisms, whether they make 'sense' to you or not.

Many computer programming mechanisms are quite arbitrary, and were chosen not because of any theoretical motivation but simply because we needed an unambiguous way to say something to a computer. C is sometimes referred to as a "high-level assembly language".

Elements of Real Programming Languages

There are several elements which programming languages, and programs written in them, typically contain. These elements are found in all languages, not just C.

1. There are *variables* or *objects*, in which you can store the pieces of data that a program is working on. Variables are the way we talk about memory locations (data). Variables may be *global* (that is, accessible anywhere in a program) or *local* (that is, private to certain parts of a program).
2. There are *expressions,* which compute new values from old ones.
3. There are *assignments* which store values (of expressions, or other variables) into variables.
4. There are *conditionals* which can be used to determine whether some condition is true, such as whether one number is greater than another. In some languages, including C, conditionals are actually expressions which compare two values and compute a 'true' or 'false' value.
5. Variables and expressions may have *types*, indicating the nature of the expected values.
6. There are *statements* which contain instructions describing what a program actually does. Statements may compute expressions, perform assignments, or call functions.
7. There are *control flow constructs* which determine what order statements are performed in. A certain statement might be performed only if a condition is true. A sequence of several statements might be repeated over and over, until some condition is met; this is called a *loop.*

8. An entire set of statements, declarations, and control flow constructs can be lumped together into a *function* (also called *routine, subroutine, or procedure*) which another piece of code can then *call* as a unit.
9. A set of functions, global variables, and other elements makes up a *program*.
10. In the process of specifying a program in a form suitable for a compiler, there are usually a few logistical details to keep track of. These details may involve the specification of compiler parameters or interdependencies between different functions and other parts of the program.

Computer Representation of Numbers

Most computers represent integers as binary numbers with a certain number of bits. A computer with 16-bit integers can represent integers from 0 to 65,535 or if it chooses to make half of them negative, from –32,767 to 32,767. A 32-bit integer can represent values from 0 to 4,294,967,295, or + –2,147,483,647. Most of today's computers represent real (*i.e.,* fractional) numbers using exponential notation. The advantage of using exponential notation for real numbers is that it lets you trade off the range and precision of values in a useful way. Since there's an infinitely large number of real numbers, it will never be possible to represent.

Characters, Strings, and Numbers

One fundamental component of a computer's handling of alphanumeric data is its *character set.* A character set is, not surprisingly, the set of all the characters that the computer can process and display. (Each character generally has a key on the keyboard to enter it and a bitmap on the screen which displays it.) A character set consists of letters, numbers, and punctuation.

A *character* is, well, a single character. If we have a variable which contains a character value, it might contain the letter 'A', or the digit '2', or the symbol '&'. A *string* is a set of zero or more characters. For example, the string "and" consists of the characters 'a', 'n', and 'd'.

Compiler Terminology

C is a *compiled* language. This means that the programs we write are translated, by a program called a compiler, into executable machine-language programs which we can actually run. Executable machine-language programs are self-contained and run very quickly. A compiler is a special kind of progam: it is a program that builds other programs. The main alternative to a compiled computer language or program is an interpreted one, such as BASIC . In other words, for each statement that you write, a compiler translates into a sequence of machine language instructions which does the same thing, while an interpreter simply does it.

Example

Program to print "hello, world" or display a simple string, and exit.

```
# include < stdio.h>
main()
{
printf ("Hello, word!\n");
return 0;
}
```

Printf is a library function which prints formatted output. The parentheses surround printf's argument list: the information which is handed to it which it should act on. The semicolon at the end of the line terminates the statement.

The second line in the main function is

return 0;

In general, a function may return a value to its caller, and main is no exception. When main returns (that is, reaches its end and stops functioning), the program is at its end, and the return value from main tells the operating system whether it succeeded or not. By convention, a return value of 0 indicates success.

Basic Data Types and Operators

The *type* of a variable determines what kinds of values it may take on. An *operator* computes new values out of old ones. An *expression* consists of variables, constants, and operators combined to perform some useful computation.

There are only a few basic data types in C.

- char a character
- int an integer, in the range – 32,767 to 32,767
- long int a larger integer (up to +–2,147,483,647)
- float a floating-point number

double a floating-point number, with more precision and perhaps greater range than float.

Constant: A constant is just an immediate, absolute value found in an expression. The simplest constants are decimal integers *e.g.,* 0, 1, 2, 123. Occasionally it is useful to specify constants in base 8 or base 16 (octal or hexadecimal).

A constant can be forced to be of type long int by suffixing it with the letter L. A constant that contains a decimal point or the letter e (or both) is a floating-point constant: 3. 14, .01, 123e4, 123.456e7. The e indicates multiplication by a power of 10; 123.456e7 is 123.456 times 10 to the 7th, or 1,234, 560,000.

A character constant is simply a single character between single quotes: 'A', '.', '%'. The numeric value of a character constant is, naturally enough, that character's value in the machine's character set. Characters enclosed in double quotes: "apple", "hello, world", "this is a test". Within character and string constants, the backslash character \ is special, and is used to represent characters not easily typed on the keyboard or for various reasons not easily typed in constants. The most common of these "character escapes" are:

\n a "newline" character

\b a backspace

\r a carriage return (without a line feed)

\' a single quote (*e.g.,* in a character constant)

\" a double quote (*e.g.,* in a string constant)

\\ a single backslash

Declarations: Informally, a variable (also called an object) is a place where computer can store a value. So that they can refer to it unambiguously, a variable needs a name. A declaration tells the compiler the name and type of a variable we'll be using in our program. In its simplest form, a declaration consists of the type, the name of the variable, and a terminating semicolon:

char c;

int i;

int i1, i2.

it is on line.

Variable Names: Variable names (the formal term is "identifiers") consist of letters, numbers, and underscores. For our purposes, names must begin with a letter. The capitalization of names in C is significant.

Arithmetic Operators

The basic operators for performing arithmetic are the same in many computer languages:

+ addition

– subtraction

* multiplication

/ division

% modulus (remainder)

The operator can be used in two ways: to subtract two numbers (as in a – b), or to negate one number (as in –a + b or a + –b).

When applied to integers, the division operator/discards any remainder, so 1/2 is 0 and 7/4 is 1. But when either operand is a floating-point quantity (type float or double), the division operator yields a floating–point result, with a potentially non-zero fractional part. So 1/2.0 is 0.5, and 7.0/4.0 is 1.75.

The *modulus* operator % gives you the remainder when two integers are divided: 1 % 2 is 1; 7 % 4 is 3. (The modulus operator can only be applied to integers.)

An additional arithmetic operation you might be wondering about is exponentiation. Some languages have an exponentiation operator (typically ^ or).

Multiplication, division, and modulus all have higher *precedence* than addition and subtraction. The term "precedence" refers to how "tightly" operators bind to their operands. All of these operators "group" from left to right, which means that when two or more of them have the same precedence and participate next to each other in an expression, the evaluation conceptually proceeds from left to right.

Assignment Operators

The assignment operator = assigns a value to a variable. For example,

x = 1

sets x to 1, and

a = b

sets a to whatever b's value is. The expression

i = i + 1

is, as we've mentioned elsewhere, the standard programming idiom for increasing a variable's value by 1.

Function Calls

Any function can be called by mentioning its name followed by a pair of parentheses. If the function takes any arguments, you place the arguments between the parentheses, separated by commas. These are all function calls:

```
printf("Hello, world!\n"), printf("%d\n", i), sqrt(144), getchar()
```

The arguments to a function can be arbitrary expressions.

Statements And Control Flow

Statements are the "steps" of a program. Most statements compute and assign values or call functions. By default, statements are executed in sequence, one after another. We can, however, modify that sequence by using control flow constructs which arrange that a statement or group of statements is executed only if some condition is true or false, or executed over and over again to form a *loop*.

"A statement is an element within a program which we can apply control flow to; control flow is how we specify the order in which the statements in our program are executed".

Expression Statements: Most of the statements in a C program are *expression statements*. An expression statement is simply an expression followed by a semicolon. The lines

```
i = 0;
i = i + 1;
```

and

```
printf("Hello, world!\n");
```

are all expression statements. The semicolon is a statement terminator; all simple statements are followed by semicolons.

If Statements: The simplest way to modify the control flow of a program is with an if statement, which in its simplest form looks like this:

```
if(x > max)
   max = x;
```

More generally, the syntax of an if statement is:

```
if (expression)
statement
```

Where *expression* is any expression and *statement* is any statement.

If

```
if(expression)
{
statement < sub > 1 </sub >
statement < sub > 2 </sub >
statement < sub > 3 </sub >
}
```

An if statement may also optionally contain a second statement, the "else clause", which is to be executed if the condition is not met. Example:

```
If(n > 0)
average = sum/n;
else {
```

```
printf("can't compute average\n");
average = 0;
}
```

The first statement or block of statements is executed if the condition is true, and the second statement or block of statements is executed if the condition is *not true*.

Boolean Expressions

An if statement like

```
    if(x > max)
        max = x;
```

is perhaps deceptively simple. Conceptually, we say that it checks whether the condition x > max is "true" or "false". The study of mathematics involving only two values is called Boolean algebra. In C, "false" is represented by a value of 0 (zero), and "true" is represented by value 1.

The **relational operators** such as <, <=, >, and >= are in fact operators, just like +, –, *, and /. The relational operators take two values "return" a value of 1 or 0 depending on whether the tested relation was true or false. The complete set of relational operators in C is:

< less than

<= less than or equal

> greater than

>= greater than or equal

== equal

!= not equal

While Loops: Loops generally consist of two parts: one or more *control expressions* which control the execution of the loop, and the *body*, which is the statement or set of statements which is executed over and over. The most basic *loop* in C is the while loop. A while loop has one control expression, and executes as long as that expression is true.

Example:

```
int x = 2;
while(x < 1000)
    {
    printf("%d\n", x);
    x = x * 2;
    }
```

The general syntax of a while loop is

while (*expression*)

statement

A while loop starts out like an if statement: if the condition expressed by the *expression* is true, the *statement* is executed. However, after executing the statement, the condition is tested again, and if it's still true, the statement is executed again.

For Loops: More generally, the syntax of a for loop is

```
for(expr < sub > 1 </sub >; expr < sub >2 </sub >; expr < sub > 3 </sub >)
        statement
```

(Here we see that the for loop has three control expressions. As always, the *statement* can be a brace-enclosed block.)

```
for (i = 0; i < 10; i = i + 1)
        printf("i is %d\n", i);
```

The three expressions in a for loop encapsulate these conditions: *expr ₁* sets up the initial condition, *expr < sub > 2 </sub >* tests whether another trip through the loop should be taken, and *expr < sub > 3 </sub >* increments or updates things after each trip through the loop and prior to the next one. All three expressions of a for loop are optional.

Break and Continue: Sometimes, due to an exceptional condition, we need to jump out of a loop early, that is, before the main controlling expression of the loop causes it to terminate normally. Other times, in an elaborate loop, we may want to jump back to the top of the loop without playing out all the steps of the current loop. The break and continue statements allow to do these two things.

here is a program for printing prime numbers between 1 and 100:

```
#include<stdio.h>
#include<math.h>
main()
{
int i, j;
printf("%d\n", 2);
for(i = 3; i <= 100; i = i + 1)
{
for(j = 2; j < i; j = j + 1)
  {
        if(i% j == 0)
        break;
        if(j > sqrt(i))
              {
              printf("%d\n", i);
              break;
              }
                    }
              }
        return 0;
        }
```

Arrays

The declaration

```
int i;
```

declares a single variable, named i, of type int. It is also possible to declare an *array* of several elements.

The declaration

```
nt a[10];
```

declares an array, named a, consisting of ten elements, each of type int. an array is a variable that can hold more than one value. We can represent the array a above with a picture like this:

a:

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]

In C, arrays are *zero-based:* the ten elements of a 10-element array are numbered from 0 to 9. The subscript which specifies a single element of an array is simply an integer expression in square brackets. The first element of the array is a[0], the second element is a[1], etc.

Array Initialization

Although it is not possible to assign to all elements of an array at once using an assignment expression, it is possible to initialize some or all elements of an array when the array is defined. The syntax looks like this:

```
int a[10] = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9};
```

The list of values, enclosed in braces {}, separated by commas, provides the initial values for successive elements of the array.

If there are fewer initializers than elements in the array, the remaining elements are automatically initialized to 0. For example,

```
int a[10] = {0, 1, 2, 3, 4, 5, 6};
```

would initialize a[7], a[8], and a[9] to 0.

In the case of arrays of char, the initializer may be a string constant:

```
char s1[7] = "Hello";
char s2[10] = "there";
char s3[] = "world!";
```

Arrays of Arrays ("Multidimensional" Arrays)

The declaration of an array of arrays looks like this:

```
int a2[5][7];
```

illustration of the use of multidimensional arrays

```
int i, j;
for (i = 0; i < 5; i = i + 1)
   {
   for (j = 0; j < 7; j = j + 1)
        a2[i] [j] = 10 * i + j;
```

Functions and Program Structure

A function is a "black box" that we've locked part of our program into. The idea behind a function is that it *compartmentalizes* part of the program, and in particular, that the code within the function has some useful properties:

1. It performs some well-defined task, which will be useful to other parts of the program.

2. It might be useful to other programs as well; that is, we might be able to reuse it (and without having to rewrite it).
3. The rest of the program doesn't have to know the details of how the function is implemented. This can make the rest of the program easier to think about.
4. The function performs its task *well.* It may be written to do a little more than is required by the first program that calls it, with the anticipation that the calling program (or some other program) may later need the extra functionality or improved performance.

Function Basics

It has a *name* that you call it by, and a list of zero or more *arguments* or *parameters* that you hand to it for it to act on or to direct its work; it has a *body* containing the actual instructions (statements) for carrying out the task the function is supposed to perform; and it may give you back a *return value,* of a particular type.

print f **:** printf's name comes from **print** formatted. It generates output under the control of a *format string* (its first argument) which consists of literal characters to be printed and also special character sequences--*format specifers*--which request that other arguments be fetched, formatted, and inserted into the string. There are quite a number of format specifiers for printf. Here are the basic ones:

%d	print an int argument in decimal
%1d	print a long int argument in decimal
%c	print a character
%s	print a string
%f	print a float or double argument
%e	same as % f, but use exponential notation
%g	use %e or %f, whichever is better
%o	print an int argument in octal (base 8)
%x	print an int argument in hexadecimal (base 16)
%%	print a single %

It is also possible to specify the width and precision of numbers and strings as they are inserted.

Character Input and Output

The most basic way of reading input is by calling the function getchar. Getchar reads one character from the "standard input", which is usually the user's keyboard, but which can sometimes be redirected by the operating system. Getchar returns (rather obviously) the character it reads, or, if there are no more characters available, the special value EOF ("end of file").

A campanion function is putchar, which writes one character to the "standard output".

Assignment Operators

The first and more general way is that any time you have the pattern

$$v = v \ op \ \mathrm{e}$$

where v is any vairable (or anything like a[i]), *op* is any of the binary arithmetic operators, and e is any expression.

For example, replace the expressions

```
i = i + 1
j = j – 10
k = k* (n + 1)
a[i] = a[i] /b
```

with

```
i + = 1
j – = 10
k * = n + 1
a[i]/ = b
```

Increment and Decrement Operators

C provides another set of shortcuts: the *autoincrement* and *autodecrement* operators. In their simplest forms, they look like this:

```
++i        add 1 to i
––j        subtract 1 from j
```

The ++ and–– operators apply to one operand (they're *unary* operators). The expression ++i adds 1 to i, and stores the incremented result back in i. This means that these operators don't just compute new values; they also modify the value of some variable.

Strings

Strings in C are represented by arrays of characters. The end of the string is marked with a special character, the *null character,* which is simply the character with the value 0. The null or string-terminating character is represented by another character escape sequence, \0.

The C Preprocessor

Conceptually, the "preprocessor" is a translation phase that is applied to your source code before the compiler proper gets its hands on it. Generally, the preprocessor performs textual substitutions on your source code, in three sorts of ways:

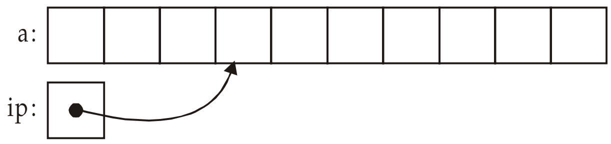
- File inclusion: Inserting the contents of another file into your source file, as if you had typed it all in there.
- Macro substitution: Replacing instances of one piece of text with another.
- Conditional compilation: Arranging that, depending on various circumstances, certain parts of your source code are seen or not seen by the compiler at all.

Pointers and Arrays

Pointers do not have to point to single variables. They can also point at the cells of an array. For example, we can write

```
int *ip;
int a[10];
ip = & a[3];
```

and we would end up with ip pointing at the fourth cell of the array a (remember, arrays are 0-based, so a[0] is the first cell). We could illustrate the situation like this:



* ip gives us what ip points to, which in this case will be the value in a[3].

Once we have a pointer pointing into an array, we can start doing *pointer arithmetic.*

Null Pointers

A null pointer is a special pointer value that is known not to point anywhere.

13.2 ALGORITHM FOR BISECTION METHOD

Step 1. Start of the program to compute the real root of the equation

Step 2. Input the value of x_1 and x_2

Step 3. Check $f(x_1) \times f(x_2) < 0$

Step 4. If no, print 'Error' and exit

Step 5. If yes, compute $x_0 = \frac{x_1 + x_2}{2}$

Step 6. Compute $f(x_0)$

Step 7. Again, if $f(x_0) \times f(x_1) < 0$

Step 8. Set $x_2 = x_0$

Step 9. Else, set $x_1 = x_0$

Step 10. Continue the process step 5 to step 9 till to get required accuracy.

Step 11. Print output

Step 12. End of the program.

13.3 PROGRAMMING FOR BISECTION METHOD

(1) Find the Real Root of the Equation $x^3 - x - 1 = 0$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void main()
      {
        void bisec(float, float);
        float i, j;
        float at n, x0, sum = 0, sum 1 = 0, a, b;
        clrscr();
```

```
                    prinf("Enter the range:");
                    scanf("%f", & n);
                    for(i = 0; i < = n; i++)
        {
            sum = pow(i, 3) –i –1;
            for(j = i + 1; j < = n; j++)
        {
            sum1 = pow(j, 3) –j –1;
            if(sum < 0 && sum1 > 0 || sum > 0 && sum1 < 0)
        {
            a = i;
            b = j;
            bisec(a, b);
            break;
            }
        }
        }
getch();
}
void bisec(float a, float b)
{
int i;
float x1, sum;
for(i = 1; i <= 20; i++)
        {
            x1 = (a + b)/2;
            sum = pow(x1, 3)–x1–1;
            if(sum < 0)
            a = x1;
            else
            b = x1;
        }
x1 = (a + b)/2;
printf("%f",x1);
}
```

The root of the given equation is 1.3247.

(2) Find the Real Root of the Given Equation $e^x - 3x = 0$

```
#include<conio.h>
```

```
#include<stdio.h>
#include<math.h>
void main()
    {
        void bisec(float, float);
        float i, j, e = 2.718;
        float n, x0, sum = 0, sum 1 = 0, a, b;
        clrscr();
    prinf("Enter the range:");
    scanf("%f", &n);
    for(i = 0; i < n; i++)
    {
          sum = pow(e, i) – (3*i);
          for(j = i + 1; j <= n; j++)
    {
        sum1 = pow(e, j) – (3*j);
        if(sum < 0 && sum1 > 0 || sum > 0 && sum1 < 0)
    {
        a = i;
        b = j;
    bisec(a, b);
    break;
    }
    }
}
    getch();
}
    void bisec(float a, float b)
    {
        int i;
        float x1, sum; e = 2.718;
        for i(i = 1; i <= 20; i++)
    {
        x1 = (a + b)/2;
        sum = pow (e, x1) – (3*x1);
        if(sum < 0)
        a = x1;
        else
```

```
        b = x1;
      }
        x1 = (a + b)/2;
        printf("%f",x1);
}
```

The root of the given equation is 1.5121.

13.4 ALGORITHM FOR FALSE POSITION METHOD

Step 1. Start of the program to compute the real root of the equation
Step 2. Input the value of x_0, x_1 and e
Step 3. Check $f(x_0) \times f(x_1) < 0$
Step 4. If no, print "Error" and exit
Step 5. If yes, compute $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$
Step 6. Compute $f(x_2)$
Step 7. Again, if $f(x_2) \times f(x_0) < 0$
Step 8. Set $x_1 = x_2$
Step 9. Else, set $x_0 = x_2$
Step 10. Continue the process step 5 to step 9 till to get required accuracy
Step 11. Print output
Step 12. End of the program.

13.5 PROGRAMMING FOR FALSE POSITION METHOD

(1) Find the Real Root of the Given Equation $x^3 - 2x - 5 = 0$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void false(float, float);
void main()
{
    float x0 = 3, x1 = 4;
    clrscr();
    false(x0, x1);
    getch();
}
void false(float x0, float x1)
{
```

```
    int i;
    float x2 = 0, a = 0, b = 0, c = 0;
    for(i = 0; i < 12; i++)
    {
        a = pow(x0, 3)–2*x0–5;
        b = pow(x1, 3)–2*x1–5;
        x2 = x0–(x1–x0)/(b–a)*a);
        c = pow(x2, 3)–2*x2–5;
        if(c < 0)
        x0 = x2;
    else
        x1 = x2;
    }
        printf("%f", x2);
    }
```

The root of the given equation is 2.094.

(2) Find the Real Root of the Given Equation $3x + \sin x - e^x = 0$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
float flase(float, float);
void main()
{
    float a, x0 = 0, x1 = 1, b, x2;
    clrscr();
    a = false(x0, x1);
    b = false(x2);
    printf("%f", b);
    getch();
}
    float false(float x0, float x1)
{
    float x2;
    int i;
    for(i = 1; i <= 13; i++)
{
    y0 = 3*x0 + sin(x0)–pow(2.7187, x0);
    y1 = 3*x1 + sin(x1)–pow(2.7187, x1);
    x2 = x0–(x1–x0)/(y1–y0)*y0;
    y2 = 3*x2 + sin(x2) – pow(2.7187, x2);
    if(y2 < 0)
```

```
        x0 = x2;
        else
        x1 = x2;
    }
    return (x2);
    }
```

The root of the given equation is 36042.

13.6 ALGORITHM FOR ITERATION METHOD

Step 1. Start of the program to compute the real root of the equation

Step 2. Input the value of x_0 (initial guess)

Step 3. Input the value of required alllowed error e

Step 4. Input the total iteration to be allowed n

Step 5. Compute $\phi(x_0)$, $x_1 \leftarrow \phi(x_0)$

(step 7 to 8 are repeated until the procedure converges to a root)

Step 6. For i = 1 to n, in step 2 to step 4 do

Step 7. $x_0 \leftarrow x_1$, $x_1 \leftarrow \phi(x_0)$

Step 8. If $\left|\frac{x_1 - x_0}{x_1}\right| \le e$ then GOTO step 11

end for

Step 9. Print "does not converge to a root", x_0, x_1

Step 10. Stop

Step 11. Print "converge to a roof ", i, x_1

Step 12. End of the program.

13.7 PROGRAMMING FOR ITERATION METHOD

(1) Find the Real Root of the Given Equation $xe^x = 1$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void main()
{
void iterat(float);
    int i, j;
    float x0, a;
    clrscr();
```

```
    a = 0.5;
    x0 = a;
    iterat(x0);
    getch();
}
void iterat(float x0)
{
    int i;
    float x1, x2;
    for(i = 1; i <= 12; i++)
{
    x1 = 1/pow(e, x0);
    x2 = 1/pow(e, x1);
    x0 = x2;
}
print("The result is: %f", x2);
}
```

The root of the given equation is 0.5671.

(2) Find the Real Root of the Given Equation $2x - \log_{10} x = 7$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void main()
{
void iterat(float);
    int i, j;
    float x0, a;
    clrscr();
    a = 3.7;
    x0 = a;
    iterat(x0);
    getch();
}
void iterat(float x0)
{
    int i;
    float x1, x2;
    for(i = 1; i <= 12; i++)
{
```

```
x1 = (7 + log(x0)/2;
x2 = (7 + log(x1)/2;
x0 = x2;
}
print("The result is: %f", x2);
}
```

The root of the given equation is 4.2199.

(3) Find the Real Root of the Given Equation x sin x = 1

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void main()
{
void iterat(float);
    int i, j;
    float x0, a;
    clrscr();
    a = 1.5;
    x0 = a;
    iterat(x0);
    getch();
}
void iterat (float x0)
    {
       int i;
       float x1, x2;
       for (i = 1; i <= 12; i++)
    {
       x1 = 1/sin (x0);
       x2 = 1/sin (x1);
       x0 = x2;
    }
printf("The result is: %f", x2);
}
```

The root of the given equation is 1.114.

(4) Find the Real Root of the Given Equation $2x - \log_{10} x = 7$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
```

```
float iteration(float);
void main()
{
    float a;
    float x = 3.7;
    clrscr();
    a = iteration(x);
    printf("%f", a);
    getch();
}
    float iteration(float x)
    {
        int i;
        float s = 0;
        for(i = 0; i < 15; i++)
    {
        s = 0.5*(7 + log(x));
        x = s;
    }
return(s);
    }
```

The root of the given equation is 4.2199.

13.8 ALGORITHM FOR NEWTON'S RAPHSON METHOD

Step 1. Start of the program to compute the real root of the equation

Step 2. Input the value of x_0, n and e

Step 3. For i = 1 and repeat if i < = n

Step 4. $f0 = f(x_0)$

Step 5. $df0 = df(x_0)$

Step 6. Compute $x1 = x_0 - (f0/df0)$

Step 6a. If $\left|\frac{x_1 - x_0}{x_1}\right| < e$

Step 6b. Print "convergent"

Step 6c. Print x1, f(x1), i

Step 7. End of the program

Step 8. Else, Set $x_0 = x1$

Step 9. Repeat the process until to get required accuracy

Step 10. End of the program.

13.9 PROGRAMMING FOR NEWTON RAPHSON METHOD

(1) Find the Real Root of the Given Equation $x^2 = 12$

```
//program-netwon raphson
#include<conio.h>
#include<stdio.h>
#include<math.h>
float f(float x)
{
    return((x*x)–(12));
}
float d(float x)
{
    return((2*x));
}
void main()
{
    float x, y, s;
    int i, c = 0;
    clrscr();
    for(i = 0; ; i++)
    {
        if(f(i) > 0)
        break;
    }
    x = i;
    aa:
    {
        ++c;
        y = x–(f(x)/d(x));
        x = y;
        s = (y*10000);
        printf("\nthe position of iteration %d", c);
        printf("\nthe root is %f", y);
        y = (y*10000);
        if(y != s)
        goto aa;
    }
        printf("\nreal root is %f",y);
    getch();
}
```

The root of the given equation is 3.4641.

(2) Find the Real Root of the Given Equation $x^2 - 5x + 2 = 0$

```
//program-netwon raphson
#include<conio.h>
#include<stdio.h>
#include<math.h>
float f(float x)
{
    return((x*x) – (5*x) + 2;
}
float d(float x)
{
    return((2*x) – 5);
}
void main()
{
    float x, y = 0;
    int i;
    clrscr();
    for(i = 0;; i++)
    {
        if(f(i) > 0)
        break;
    }
    x = i;
    for(i = 0; i < 10; i++)
    {
        y = x–f(x)/d(x);
        x = y;
        printf("\nreal root is %f", y);
    }
    getch();
}
```

The root of the given equation is 0.438447.

13.10 PROGRAMMING FOR MULLER'S METHOD

(1) Find the Real Root of the Given Equation $x^3 - x^2 - x - 1 = 0$

```
#include<conio.h>
#include<stdio.h>
```

```
#include<math.h>
void main()
{
        int i, j;
        flat x0, x1, x2, x3, y0, y1, y2, a, b;
        float val(float);
clrscr();
        x0 = 1.9;
        x1 = 2.0;
        x2 = 2.1;
                for(i = 0; i < 3; i++)
                {
                        y0 = val(x0);
                        y1 = val(x1);
                        y2 = val(x2);
    a = ((x0–x1)*(y1–y2)–(x1–x2)) (y0–y2))/((x1–x0)*(x1–x2)*(x0–x2));
    b = (pow((x0–x1), 2)*(y1–y2)–pow((x1–x2), 2)*(y0–y2))/((x0–x1)*(x1–x2)*(x0–x2));
    x3 = x2–((2*y2)/(b + pow((pow(b, 2)–4*a*y2), .5)));
        x0 = x1;
        x1 = x2;
        x2 = x3;
        }
printf("%f", x3);
getch();
}
float val(float x)
{
float y;
    y = pow(x, 3) – pow(x, 2) – x – 1;
    return(y);
}
```

The root of the given equation is 1.8382067.

(2) Find the Real Root of the Given Equation $x^3 - 3x - 5 = 0$

```
#include<conio.h>
#include<stdio.h>
#include<math.h>
void main()
{
    int i, j;
    float x0, x1, x2, x3, y0, y1, y2, a, b;
```

```
    float val(float);
clrscr();
    x0 = 1.9;
    x1 = 2.0;
    x2 = 2.7;
        for(i = 0; i < 3; i++)
    {
        y0 = val(x0);
        y1 = val(x1);
        y2 = val(x2);
    a = ((x0–x1)*(y1–y2)–(x1–x2)*(y0–y2))/((x1–x0)*(x1–x2)*(x0–x2));
    b = (pow((x0–x1), 2)*(y1–y2)–(pow(x1–x2), 2)*(y0–y2))/((x0–x1)*(x1–x2)*(x0–x2));
    x3 = x2–((2*y2)/(b + pow((pow(b, 2)–4*a*y2),. 5)));
        x0 = x1;
        x1 = x2;
        x2 = x3;
    }
    printf("%f", x3);
    getch();
}
    float val(float x)
{
    float y;
       y = pow(x, 3)–(3* x) –5;
       return(y);
}
```

The root of the given equation is 2.417728.

13.11 ALGORITHM FOR NEWTON'S FORWARD INTERPOLATION METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1] – ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. For j = 2; j <= 4; j++

Step 10. For i = 0; i < n–j; i++
Step 11. diff[i][j] = diff[i+1] [j-1] – diff[i] [j–1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i]<x
Step 16. i = i + 1
Step 17. i = i – 1
Step 18. p = (x – ax[i])/h
Step 19. y1 = p * diff [i–1] [1]
Step 20. y2 = p * (p + 1) * diff [i – 1] [2]/2
Step 21. y3 = p * (p + 1) * (p–1) * diff [i – 2] [3]/6
Step 22. y4 = p * (p + 1) * (p + 2) * (p–1) * diff [i – 3] [4]/24
Step 23. Print the output x, y
Step 24. End of the program.

13.12 PROGRAM FOR CONSTRUCTING DIFFERENCE TABLE

```
//program for newton forward difference table
#include<conio.h>
#include<stdio.h>
#include<math.h>
int fact(int a)
{
    if(a==0)
    return 1;
    else
    return (a*fact(a–1));
}
void main()
{
    float x[60], y, diff[5][5], fx[60], u, h, temp = 1.00, sum;
    int n, i = 0, j = 0, k = 0;
    clrscr();
    printf("enter the no. of values");
    scanf("%d",&n);
    printf("\n\n enter the values of x having constant difference between them \n");
    for (i = 0; i < n; i++)
    scanf("%f", & x[i]);
    printf("\n enter the values of y = f(x)\n");
```

```
    for(i = 0; i < n; i++)
    scanf("%f", & fx[i]);
    for(i = 0; i<n–1; i++)
    diff[0][i] = fx[i+1]–fx[i];
    for(i = 1; i < n–1; i++)
    for(j = 0; j < n–1; j++)
    diff[i][j] = diff[i–1][j+1]–diff[i–1][j];
    printf("n\n\t newton forward difference table is:\n");
    printf("\nX            Y   –Y");
    for(k = 2; k < n; k++)
    printf("–^%dY",k);
    printf("\n");
    for(i = 0; i < n; i++)
    {
        printf("\n");
        printf("%f %f", x[i], fx[i]);
        for(j = 0; j < n–1–i; j++)
        printf("%f ",diff[j][i]);
    }
    getch();
}
```

13.13 PROGRAMMING FOR NEWTON'S FORWARD INTERPOLATION METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
int fac(int a)
{
    if(a==0)
    return (1);
    else
    return (a*fac(a–1));
}
void main()
{
    int x[60], X;
    float dif[5][5], fx[60];
    float u, h, sum, temp = 1.00;
```

```
    nt n, i = 0, j = k = 0;
    clrscr();
print("Enter the no. of values");
scanf("%d",&n);
printf("Enter the values of x having constant diff b/w them\n");
    for(i = 0; i < n; i++)
    scanf("%d", &x[i]);
    printf("Enter the values of Y = f(x)\n");
    for(i = 0; i < n; i++)
    scanf("%f", & fx[i]);
        for(i = 0; i < n–1; i++)
        dif[0][i] = fx[i+1]–fx[i];
        for(i = 1; i < n–1; i++)
        for(j = 0; j < n–1–i; j++)
        dif[i][j] = dif[i–1][j+1]–dif[i–1][j];
    printf("\n\t The Newton Forward Difference Table is given by:-\n");
    printf("\nX\tY\t\t–Y");
         for(k = 2; k < n; k++)
         printf("\t–^%dY", k);
         printf("\n");
for(i = 0; i < n; i++)
    {
    printf("\n");
    printf("%d\t%f\t", x[i], fx[i]);
    for(j = 0; j < n–1–i; j++)
    printf("%f ", dif[j][i]);
printf("\n\n\tEnter the value of X for which u want F(X)\t");
scanf("0%d", & X);
h = (x[1]–x[0]);
u = ((X–x[0])/h);
sum = fx[0];
         for(i = 0; i < n; i++)
{
         for(j = 0; j <= i; j++)
         temp* = (u–j);
         sum+ = ((temp/fac(i+1))*dif[i][0]);
         temp = 1;
        }
printf("\n\n\tThe value of F(%d) is %f\t", X, sum);
getch();
}
```

13.14 ALGORITHM FOR NEWTON'S BACKWARD INTERPOLATION METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1]–ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff[i+1] [j–1] – diff[i][j–1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. x0 = mx[i]
Step 18. Sum = 0, y0 = my[i]
Step 19. fun = 1
Step 20. p = (x – x0)/h
Step 21. Sum = 0
Step 22. for k = 1; k <= 4; k++
Step 23. fun = (fun * (p – (k – 1)))/k
Step 24. sum = sum + fun* diff[i] [k]
Step 25. End of the loop k
Step 26. Print the output x, sum
Step 27. End of the program.

13.15 PROGRAMMING FOR NEWTON'S BACKWARD INTERPOLATION METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
int fac(int a)
{
    if(a==0)
```

```
    return (1);
    else
    return (a*fac(a–1));
}
void main()
{
        int x[60], X;
        float dif[5][5], fx[60];
        float u, h, sum, temp = 1.00;
        int n, i = 0, j = 0, k = 0;
        clrscr();
        printf("Enter the no. of values");
        scanf("%d", &n);
printf("Enter the values of x having constant diff b/w them\n");
        for(i = 0; i < n; i++)
         scanf("%d", & x[i]);
         printf("Enter the values of Y = f(x)\n");
         for(i = 0; i < n; i++)
         scanf("%f", & fx[i]);
         for(i = 0; i < n–1; i++)
         dif[0][i] = fx[i+1]–fx[i];
         for(i = 1; i < n–1; i++)
         for(j  =0; j < n–1–i; j++)
         dif[i][j] = dif[i–1][j+1]–dif[i–1][j];
        printf("\n\tThe Newton Backward Difference Table is given by:-\n");
        printf("\nX\tY\t\tY");
         for(k = 2; k < n; k++)
         printf("\t ^%dY", k);
         printf("\n");
        for(i = 0; i < n; i++)
{
        printf("\n");
        printf("%d\t%f\t", x[i], fx[i]);
        for(j = 0; j < n–1–i; j++)
        printf("%f ", dif[j][i]);
}
        printf("\n\n\t Enter the value of X for which u want F(X)\t");
        scanf("%d", & X);
        h = (x[1]–x[0]);
        u = ((X–x[n–1])/h);
```

```
        sum = fx[n–1];
        for(i = 0; i < n; i++)
{
        for(j = 0; j <= i; j++)
        temp* = (u + j);
        sum+ = ((temp/fac(i+1)*dif[i][n–2–i]);
        temp = 1;
}
        printf("\n\n\tThe value of F(%d) is %f\t", X, sum);
        getch();
}
```

13.16 ALGORITHM FOR GAUSS FORWARD INTERPOLATION METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1] –ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff[i+1] [j-1] – diff[i][j-1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. i = i–1
Step 18. p = (x – ax[i])/h
Step 19. y1 = p *diff[i] [1]
Step 20. y2 = p * (p–1)* diff[i–1] [2]/2
Step 21. y3 = p * (p+1)*(p–1)* diff[i–2] [3]/6
Step 22. y4 = p * (p+1) * (p–1) * (p–2) * diff[i–3][4]/24
Step 23. y = ay [i] + y1 + y2 + y3 + y4
Step 24. Print the output x, y
Step 25. End of the program.

13.17 PROGRAMMING FOR GAUSS FORWARD INTERPOLATION METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
#include<process.h>
void main()
{
int n;
int i, j;
float ax[10];
float ay[10];
float x;
float nr, dr;
float h;
float p;
float diff[20][20];
float y1, y2, y3, y4;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the value in the form of x–");
for(i = 0; i < n; i++)
{
printf(enter the value of x%d", i+1);
scanf("%f", & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
{
printf("enter the value of y%d", i+1);
scanf("%f", & ay[i]);
}
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f",%x);
h = ax[1] –ax[0];
for(i = 0; i < n–1; i++)
```

```
{
diff[i][1] = ay[i + 1]–ay[i];
}
for(j = 2; j <= 4; j++)
{
for(i = 0; i < n–j; i++)
{
diff[i][j] = diff[i+1][j–1]–diff[i][j–1];
}
}
i = 0;
do
{
i++;
}while(ax[i] < x);
i– –;
p = (x–ax[i])/h;
y1 = p*diff[i][1];
y2 = p*(p–1)*diff[i–1][2]/2;
y3 = (p+1)*p*(p–1)*diff[i–2][3]/6;
y4 = (p+1)*p*(p–1)*(p-2)*diff[i–3][4]/24
y = ay[i] + y1 + y2 + y3 + y4;
printf("when x = %6.4f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –7
Enter the value in form of x–
Enter the value of x1 – 1.00
Enter the value of x2 – 1.05
Enter the value of x3 – 1.10
Enter the value of x4 – 1.15
Enter the value of x5 – 1.20
Enter the value of x6 – 1.25
Enter the value of x7 – 1.30
Enter the value in the form of y–
Enter the value of y1 – 2.7183
Enter the value of y2 – 2.8577
```

Enter the value of y3 – 3.0042
Enter the value of y4 – 3.1582
Enter the value of y5 – 3.3201
Enter the value of y6 – 3.4903
Enter the value of y7 – 3.6693
Enter the value of x for
Which you want the value of y – 1.17
When x = 1.17, y = 3.2221
Press enter to exit.

13.18 ALGORITHM FOR GAUSS BACKWARD INTERPOLATION METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of *x*
Step 4. Input the array ay for data of *y*
Step 5. Compute h = ax[1] –ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff[i+1] [j–1] – diff[i][j–1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. i = i–1
Step 18. p = (x – ax[i])/h
Step 19. y1 = p *diff[i–1] [1]
Step 20. y2 = p * (p+1)* diff[i – 1] [2]/2
Step 21. y3 = p * (p+1) * (p–1)* diff [i – 2] [3]/6
Step 22. y4 = p * (p+1) * (p+2) * (p–1) * diff[i – 3][4]/24
Step 23. Print the output x, y
Step 25. End of the program.

13.19 PROGRAMMING FOR GAUSS BACKWARD INTERPOLATION METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
```

```
#include<string.h>
#include<process.h>
void main()
{
int; n;
int i, j;
float ax[10];
float ay[10];
float x;
float y = 0;
float h;
float p;
float diff[20][20];
float y1, y2, y3, y4;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the value in the form of x–");
for(i = 0; i < n; i++)
{
printf(enter the value of x%d", i+1);
scanf(%f', & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
{
printf("enter the value of y%d ", i+1);
scanf("%f", & ay[i];
}
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f", %x);
h = ax[1] –ax[0];
for(i = 0; i < n–1; i++)
{
diff[i][1] = ay[i+1]–ay[i];
}
for(j = 2; j < = 4; j++)
{
```

```
for(i = 0; i < n–j; i++)
{
diff[i][j] = diff[i+1][j–1]–diff[i][j–1];
}
}
i = 0;
do
{
i++;
}while(ax[i]<x);
i– –;
p = (x–ax[i])/h;
y1 = p*diff[i–1][1];
y2 = p*(p+1)*diff[i–1][2]/2;
y3 = (p+1)*p*(p–1)*diff[i–2][3]/6;
y4 = (p+1)*p*(p-1)*(p+2)*diff[i–3][4]/24
y = ay[i] + y1 + y2 + y3 + y4;
printf("when x = %6.4f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –7
Enter the value in form of x–
Enter the value of x1 – 1.00
Enter the value of x2 – 1.05
Enter the value of x3 – 1.10
Enter the value of x4 – 1.15
Enter the value of x5 – 1.20
Enter the value of x6 – 1.25
Enter the value of x7 – 1.30
Enter the value in the form of y–
Enter the value of y1 – 2.7183
Enter the value of y2 – 2.8577
Enter the value of y3 – 3.0042
Enter the value of y4 – 3.1582
Enter the value of y5 – 3.3201
Enter the value of y6 – 3.4903
Enter the value of y7 – 3.6693
```

Enter the value of x for
Which you want the value of y – 1.35
When x = 1.35, y = 3.8483
Press enter to exit

13.20 ALGORITHM FOR STIRLING'S METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1] – ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff [i+1] [j–1] – diff [i] [j–1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. i = i–1
Step 18. p = (x – ax[i])/h
Step 19. y1 = p * (diff [i][1] + diff [i – 1][1])/2
Step 20. y2 = p * p * diff[i – 2][2]/2
Step 21. y3 = p * (p *p–1)* (diff[i – 1][3] + diff[i–2][3])/6
Step 22. y4 = p * p * (p * p–1) * diff [i – 2] [4]/24
Step 23. y = ay[i] + y1 + y2 + y3 + y4
Step 24. Print the output x, y
Step 25. End of the program.

13.21 PROGRAMMING FOR STIRLING'S METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
#include<process.h>
```

```
void main()
{
int n;
int i, j;
float ax[10];
float ay[10];
float h;
float p;
float x, y;
float diff[20][20];
float y1, y2, y3, y4;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the no. in the form of x–");
for(i = 0; i < n; i++)
{
printf(enter the value of x%d", i+1);
scanf("%f", & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
}
printf("enter the value of y%d ", i+1);
scanf("%f", & ay[i];
}
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f", %x);
h = ax[1] –ax[0];
for(i = 0; i < n–1; i++)
{
diff[i][1] = ay[i+1]–ay[i];
}
for(j = 2; j <= 4; j++)
{
for(i = 0; i < n–j; i++)
{
diff[i][j] = diff[i+1][j–1]–diff[i][j–1];
}
}
```

```
i = 0;
do
{
i++;
}while(ax[i] < x);
i– –;
p = (x–ax[i])/h;
y1 = p*(diff[i][1]+diff[i–1][1]/2;
y2 = p*(p)*diff[i–1][2]/2;
y3 = p*(p*p–1)*(diff[i–1][3]+diff[i–2][3])/6;
y4 = p*p*(p*p–1)*diff[i–2][4]/24;
y = ay[i] + y1 + y2 + y3 + y4;
printf("when x = %6.4f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –7
Enter the value in form of x
Enter the value of x1 – .61
Enter the value of x2 – .62
Enter the value of x3 – .63
Enter the value of x4 – .64
Enter the value of x5 – .65
Enter the value of x6 – .66
Enter the value of x7 – .67
Enter the value in the form of y–
Enter the value of y1 – 1.840431
Enter the value of y2 – 1.858928
Enter the value of y3 – 1.877610
Enter the value of y4 – 1.896481
Enter the value of y5 – 1.915541
Enter the value of y6 – 1.934792
Enter the value of y7 – 1.954237
Enter the value of x for
Which you want the value of y – 0.6440
When x = 0.6440, y = 1.90408230
Press enter to continue.
```

13.22 ALGORITHM FOR BESSEL'S METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the value of n (number of terms)
Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1] – ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff [i+1] [j–1] – diff [i] [j–1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. i = i–1
Step 18. p = (x – ax[i])/h
Step 19. y1 = p * (diff [i][1])
Step 20. y2 = p * (p–1) * (diff [i] [2] + diff [i – 1] [2]/4
Step 21. y3 = p * (p-1) * (p–0.5) * (diff [i – 1] [3])/6
Step 22. y4 = p * (p+1) * (p–2) * (p–1) * (diff [i – 2] [4] + diff [i – 1][4])/48
Step 23. y = ay[i] + y1 + y2 + y3 + y4
Step 24. Print the output x, y
Step 25. End of the program.

13.23 PROGRAMMING FOR BESSEL'S METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
#include<process.h>
void main()
{
int n;
int i, j;
```

```
float ax[10];
float ay[10];
float h;
float p;
float x, y;
float diff[20][20];
float y1, y2, y3, y4;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the no. in the form of x–");
for(i = 0; i < n; i++)
{
printf(enter the value of x%d", i+1);
scanf("%f', & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
{
printf("enter the value of y%d ", i+1);
scanf("%f", & ay[i]);
}
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f",%x);
h = ax[1]–ax[0];
for(i = 0; i < n–1; i++)
{
diff[i][1] = ay[i+1]–ay[i];
}
for(j = 2; j <= 4; j++)
{
for(i = 0; i < n–j; i++)
{
diff[i][j] = diff[i+1][j–1]–diff[i][j–1];
}
}
i=0;
do
{
```

```
i++;
}while(ax[i] < x);
i– –;
p = (x–ax[i])/h;
y1 = p*(diff[i][1]);
y2 = p*(p–1)*(diff[i][2]+diff[i–1][2])/4/2;
y3 = p*(p–1)*(p–.5)*(diff[i–1][3])/6;
y4 = (p+1)*p*(p–1)*(p–2)*(diff[i–2][4]+diff[i–1][4])/48;
y = ay[i] + y1 + y2 + y3 + y4;
printf("when x = %6.4f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –7
Enter the value in form of x–
Enter the value of x1 – .61
Enter the value of x2 – .62
Enter the value of x3 – .63
Enter the value of x4 – .64
Enter the value of x5 – .65
Enter the value of x6 – .66
Enter the value of x7 – .67
Enter the value in the form of y–
Enter the value of y1 – 1.840431
Enter the value of y2 – 1.858928
Enter the value of y3 – 1.877610
Enter the value of y4 – 1.896481
Enter the value of y5 – 1.915541
Enter the value of y6 – 1.934792
Enter the value of y7 – 1.954237
Enter the value of x for
Which you want the value of y – 0.6440
When x = 0.6440, y = 1.90408230
Press enter to continue.
```

13.24 ALGORITHM FOR LAPLACE EVERETT METHOD

Step 1. Start of the program to interpolate the given data

Step 2. Input the value of n (number of terms)

Step 3. Input the array ax for data of x
Step 4. Input the array ay for data of y
Step 5. Compute h = ax[1] – ax[0]
Step 6. For i = 0; i < n–1; i++
Step 7. diff[i] [1] = ay[i+1]–ay[i]
Step 8. End of the loop i
Step 9. for j = 2; j <= 4; j++
Step 10. for i = 0; i < n–j; i++
Step 11. diff [i][j] = diff [i+1] [j-1] – diff [i] [j-1]
Step 12. End of the loop i
Step 13. End of the loop j
Step 14. i = 0
Step 15. Repeat step 16 until ax[i] < x
Step 16. i = i+1
Step 17. i = i–1
Step 18. p = (x –ax[i])/h
Step 19. q = 1–p
Step 20. y1 = q * (ay [i])
Step 21. y2 = q * (q * q–1) * (diff [i–1] [2])/6
Step 22. y3 = q * (q * q–1) * (q *q–4) * (diff [i–2] [4])/120
Step 23. py 1 = p * ay [i + 1]
Step 24. py2 = ay[i] + y1 + y2 + y3 + y4
Step 25. Print the output x, y
Step 26. End of the program.

13.25 PROGRAMMING FOR LAPLACE EVERETT METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
#include<process.h>
void main()
{
int n;
int i, j;
float ax[10];
float ay[10];
float h;
```

```
float p, q;
float x, y = 0;
float nr, dr;
float diff[20][20];
float y1, y2, y3, y4;
float py1, py2, py3, py4;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the value in the form of x–");
for(i = 0; i < n; i++)
{
printf("enter the value of x%d", i+1);
scanf("%f", & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
{
printf("enter the value of y%d ", i+1);
scanf("%f",& ay[i]);
{
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f", %x);
h = ax[1]–ax[0];
for(i = 0; i < n–1; i++)
{
diff[i][1] = ay[i+1]–ay[i];
}
for(j = 2; j < = 4; j++)
{
for(i = 0; i < n–j; i++)
{
diff[i][j] = diff[i+1][j–1]–diff[i][j–1];
}
}
i = 0;
do
{
i++;
```

```
}while(ax[i] < x);
i– –;
p = (x–ax[i])/h;
q = 1–p;
y1 = q*(ay[i]);
y2 = q*(q*q–1)*(diff[i–1][2])/6;
y3 = q*(q*q–1)*(q*q–4)*(diff[i–2][4])/120;
py1 = p*ay[i+1];
py2 = p*(p*p–1)*diff[i][2]/6;
py3 = p*(p*p–1)*(p*p–4)*(diff[i–1][4])/120
y = y1 + y2 + y3 + y4 + py1 + py2 + py3;
printf("when x = %6.4f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –7
Enter the value in form of x–
Enter the value of x1 – 1.72
Enter the value of x2 – 1.73
Enter the value of x3 – 1.74
Enter the value of x4 – 1.75
Enter the value of x5 – 1.76
Enter the value of x6 – 1.78
Enter the value of x7 – 1.79
Enter the value in the form of y–
Enter the value of y1 – .1790661479
Enter the value of y2 – .1772844100
Enter the value of y3 – .1755204006
Enter the value of y4 – .1737739435
Enter the value of y5 – .1720448638
Enter the value of y6 – .1703329888
Enter the value of y7 – .1686381473
Enter the value of x for
Which you want the value of y – 1.7475
When x = 1.7475, y = .17420892
Press enter to exit.
```

13.26 ALGORITHM FOR LAGRANGE'S INTERPOLATION METHOD

Step 1. Start of the program to interpolate the given data
Step 2. Input the number of terms n
Step 3. Input the array ax and ay
Step 4. For i = 0; i < n; i++
Step 5. nr = 1
Step 6. dr = 1
Step 7. for j = 0; j < n; j++
Step 8. if j !=1
(a) nr = nr * (x – ax[j])
(b) dr * (ax [i] – ax [j])
Step 9. End loop j
Step 10. y + =(nr / dr)*ay[i]
Step 11. end loop i
Step 12. print output x, y
Step 13. End of the program.

13.27 PROGRAMMING FOR LAGRANGE'S INTERPOLATION METHOD

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
#include<process.h>
void main()
{
int n;
int i, j;
float ax[100];
float ay[100];
float h;
float p;
float nr, dr;
float x = 0, y = 0;
clrscr();
printf("enter the no. of term–");
scanf("%d", & n);
printf("enter the value in the form of x–");
for(i = 0; i < n; i++)
```

```
{
printf("enter the value of x%d", i+1);
scanf("%f", & ax[i]);
}
printf("enter the value in the form of y");
for(i = 0; i < n; i++)
{
printf("enter the value of y%d ", i+1);
scanf("%f", & ay[i]);
}
printf("enter the value of x for");
printf("which you want the value of y");
scanf("%f", %x);
for(i = 0; i < n; i++)
{
nr = 1;
dr = 1;
for(j = 0; j < n; j++)
{
if(j != i)
{
nr = nr*(x–ax[j]);
dr = dr*(ax[i]–ax[j]);
}
y = y+(nr/dr)*ay[i];
}
printf("when x = %5.2f, y = %6.8f", x, y);
printf("press enter to exit");
getch();
}
```

OUTPUT

```
Enter the no. of term –5
Enter the value in form of x–
Enter the value of x1 – 5
Enter the value of x2 – 7
Enter the value of x3 – 11
Enter the value of x4 – 13
Enter the value of x5 – 17
Enter the value in the form of y–
Enter the value of y1 – 150
```

Enter the value of y2 – 392
Enter the value of y3 – 1452
Enter the value of y4 – 2366
Enter the value of y5 – 5202
Enter the value of x for
Which you want the value of y – 9.0
When x = 9.0, y = 810.00
Press enter to exit.

13.28 ALGORITHM FOR TRAPEZOIDAL RULE

Step 1. Start of the program for numerical integration
Step 2. Input the upper and lower limits a and b
Step 3. Obtain the number of subinterval by h = (b–a)/n
Step 4. Input the number of subintervals
Step 5. sum = 0
Step 6. sum = func(a) + func(b)
Step 7. for i = 1; i < n; i++
Step 8. sum += 2 * func(a + i)
Step 9. End loop i
Step 10. Result = sum *h/2
Step 11. Print output
Step 12. End of the program and start of section func
Step 13. temp = 1/(1+(x * x))
Step 14. Return temp
Step 15. End of section func.

13.29 PROGRAMMING FOR TRAPEZOIDAL RULE

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
float h, a, b, s1 = 0, s2 = 0, s3 = 0, s = 0, c, f, y;
int i;
clrscr ( );
printf("Integrate the equation 1/(1+x^2) with limit 0 to 2");
printf("\n enter the initial and final limits =:");
```

```
scanf("%f %f", &a, &b);
printf("\n enter the interval = ");
scanf("%f", &h);
c = 0, y = 0;
for (i = 0; y <= b; i++)
    {
    s1 = 0;
    s2 = 0;
    c = h;
    f = 1/ (1+(y*y));
if (i == a :: y == b)
    s1 = f/2;
else
    s2 = f;
    s3 = s3+_(s1 + s2);
y = y + c;
}
s = s3*h;
printf("the exact value of the function = %f", s);
getch();
}
```

OUTPUT

Enter the interval = 0.5

The exact value of the function = 1.103846.

13.30 ALGORITHM FOR SIMPSON'S 1/3 RULE

Step 1. Start of the program for numerical integration
Step 2. Input the upper and lower limits a and b
Step 3. Obtain the number of subinterval by h = (b–a)/n
Step 4. Input the number of subintervals
Step 5. sum = 0
Step 6. sum = func(a) + 4 * func(a + h) + func(b)
Step 7. for i = 3; i < n; i += 2
Step 8. sum += 2 * func(a + (i–1) * h) + 4 * func(a + i * h)
Step 9. End loop i
Step 10. Result = sum * h/3
Step 11. Print output
Step 12. End of the program and start of section func

Step 13. temp = 1/ (1+(x * x))

Step 14. Return temp

Step 15. End of section func.

13.31 PROGRAMMING FOR SIMPSON'S 1/3 RULE

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
float h, a, b, s1 = 0, s2 = 0, s3 = 0, s4 = 0, s = 0, c, f, y;
int i = 0;
clrscr ( );
printf("Integrate the equation 1/(1+x^2) with limit 0 to 2");
printf("\n enter the initial and final limits =:");
scanf("%f %f", &a, &b);
printf("\n enter the interval = ");
scanf("%f", &h);
c = 0, y = 0;
while (y <= b)
    {
    s1 = 0;
    s2 = 0;
    s4 = 0;
    c = h;
    f = 1/(1+(y*y));
if (i == a :: y == b)
    s1 = f;
else
    if ((i! = a :: i! = b) && (i%2 == 1))
    s2 = 4*f;
else if((i! = a :: i! = b) && (i%2 == 0))
    s4 = 2*f;
    s3 = s3 + (s1 + s2 + s4);
y = y + c;
i++;
}
```

```
s = s3*(h/3);
printf("the exact value of the function = %f", s);
getch();
}
```

OUTPUT

Enter the interval = 1

The exact value of the function = 1.066667.

13.32 ALGORITHM FOR SIMPSON'S 3/8 RULE

Step 1. Start of the program for numerical integration

Step 2. Input the upper and lower limits a and b

Step 3. Obtain the number of subinterval by h = (b–a)/n

Step 4. Input the number of subintervals

Step 5. sum = 0

Step 6. sum = func(a) + func(b)

Step 7. for i = 1; i < n; i++

Step 8. if i%3 = 0

Step 9. sum += 2 * func(a + i * h)

Step 10. else

Step 11. sum += 3 * func(a + (i) * h)

Step 12. End loop i

Step 13. Result = sum * 3 * h/8

Step 14. Print output

Step 15. End of the program and start of section func

Step 16. temp = 1/(1+(x * x))

Step 17. Return temp

Step 18. End of section func.

13.33 PROGRAMMING FOR SIMPSON'S 3/8 RULE

```
#include<stdio.h>
#include<conio.h>
float sim(float);
void main()
{
float res, a, b, h, sum;
int i, j, n;
```

```
clrscr ( );
printf("Enter the Range\n");
printf("\n Enter the Lower limit a=");
scanf("%f", &a);
printf("\n Enter the Upper limit b=");
scanf("%f", &b);
printf("\n Enter the number of sub-intervals= ");
scanf("%d", &n);
h = (b–a)/n;
sum = 0;
res = 1;
sum = sim(a) + sim(b);
   for (i = 1; i < n; i++)
   {
      if (i%3 == 0)
      sum += 2*sim(a + i*h);
      else.
      sum += 3*sim(a + i*h);
   }
res = sum*3*h/8;
printf("\n value of the integral is: % .4f", res);
getch();
}
float sim(float x)
{
      float temp;
      temp = 1/(1+(x*x));
      return temp;
}
```

OUTPUT

```
Enter the Range
Lower limit a = 0
Upper limit b = 6
Enter the number of subintervals = 6
Value of the integral is: 1.3571
```

13.34 ALGORITHM FOR FITTING A STRAIGHT LINE OF THE FORM Y = a + bX

Step 1. Start of the program for fitting straight line
Step 2. Input the number of terms observe
Step 3. Input the array ax and ay
Step 4. for i = 0 to observe
Step 5. sum1 + = x[i]
Step 6. sum2 + = y[i]
Step 7. xy[i] = x[i] * y[i]
Step 8. sum3 += xy[i]
Step 9. End of the loop i
Step 10. for i = 0 to observe
Step 11. x2[i] = x[i] * x[i]
Step 12. sum4 += x2[i]
Step 13. End of the loop i
Step 14. temp1 = (sum2 * sum4) – (sum3 * sum1)
Step 15. a = temp1/((observe * sum4) – (sum1 * sum1)
Step 16. b = (sum2 – observe * a)/sum1
Step 17. Print output a, b and line "a + bx"
Step 18. End of the program.

❑❑❑